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# Identification for port-controlled Hamiltonian systems

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## **THÈSE**

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des Systèmes LCIS EA – 3747 Grenoble INP -UPMF**  
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## **Identification des systèmes hamiltoniens à ports**

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*I hope that within the current thesis, we opened at least a bit the way to the possibilities offered to modelling and control, by developing a system identification theory of Port Controlled Hamiltonian systems.*

*I believe that everything is possible with a lot of patience, ambition, optimism and belief in yourself.*

# Résumé

*L'Objectif de cette thèse a été de développer une théorie spécifique d'identification pour les systèmes Hamiltonien à ports, en raison de leurs remarquables propriétés de conservation de puissance et stabilité sous la préservation d'énergie d'interconnexion (e.g. interconnexions parallèle, série ou feedback). Par rapport à l'identifiabilité structurelle entrée/sortie déjà connu, les systèmes Hamiltonien à ports, peut être analysée pour identifiabilité, par utiliser les énergies de stockage, commande ou dissipation correspondant aux ports, qui peut être représentée en relation avec les paramètres inconnus du système. Par utiliser des informations en temps réel pour les énergies de puissance, c'est possible de déterminer les paramètres du système, pour une représentation dans l'état d'espace. Avec le nouveau formalisme d'identification par énergies de puissance, une nouvelle classe d'algorithmes d'identification peut être développée à cote de l'approche classique sur l'état d'espace. Une autre raison pour développer une théorie d'identification des systèmes Hamiltoniens à ports, c'est leur utilité et avantages pour une commande basée sur le modèle. A cet égard, le modèle doit être adaptée et assez représentatif pour les systèmes dynamiques sélectionnée. C'est le cas des modèles Hamiltoniens dans plusieurs applications en temps réel, ou ces modèles inclure l'information nécessaire pour la solution de commande, mais avec différents degrés de complexité. Comme la thèse présente, les systèmes Hamiltoniens à ports peut être analysée pour identifiabilité structurelle ou pratique, par utiliser un large set des techniques dont certains peut être prolongée pour les applications en temps réel. Un modèle de perturbation c'est proposée pour les systèmes Hamiltoniens à ports, par introduire une paire entrée/sortie par le port d'interaction du système avec l'environnement externe. Comme pour le cas non perturbé, c'est prouvé que les propriétés générales des systèmes Hamiltoniens sont aussi préservé dans ce cas. Une autre raison pour développer cette théorie, c'est donné par la possibilité de développer des algorithmes d'identification dans la boucle ouverte ou ferme adaptée pour les systèmes Hamiltoniens à ports. La discrétisation symplectique en temps représente un autre problème important en vue du développement d'une théorie pour les systèmes Hamiltoniens à ports, par introduire un cadre générale dans le cas non linéaire ou linéaire, qui combine des différentes règles de discrétisation pour les fluxés et efforts. Ainsi des algorithmes d'identification peuvent être développés en relation avec les méthodes de discrétisation symplectique. Un Hamiltonien d'erreur discrète peut être introduit entre les systèmes en temps continu et en temps discrète, qui préserve en même temps les propriétés structurelles des systèmes Hamiltoniens à ports. Par utiliser des projections orthogonales, obliques dans un contexte sous-espace, des algorithmes d'identification classique ou nouveaux (énergie de puissance) peuvent être analysé ou développée en vue d'une théorie d'identification des systèmes.*

*La première partie de la thèse 'Introduction' est divisé en deux parties: une qui présente certaines notions de base, d'identification des systèmes et une deuxième partie qui*

introduit un nouveau formalisme pour la modélisation et commande des systèmes Hamiltoniens à ports.

Ainsi l'identification des systèmes est définie comme une partie d'ingénierie de commande, qui construit des modèles mathématiques en utilisant des données observées. Pour un système dynamique, on peut associer deux types de signaux: signaux observables (sorties) et signaux contrôlables (entrées).

A côté de cela, dans les applications en temps réel, des perturbations peuvent être rencontrées entre le système et l'environnement externe. Les modèles mathématiques pour les systèmes dynamiques sont utiles dans la plupart de la recherche scientifique et peut avoir différentes formes: des équations différentielles, équations dans l'état d'espace ou fonctions de transfert.

L'une des approches de modélisation mathématique, c'est basée sur l'utilisation des modèles mathématiques, par utiliser les lois de la physique pour un système dynamique. Cette approche a des points faibles, comme la grande complexité du modèle ou la difficulté d'estimation grâce à plus de paramètres. Ces problèmes peuvent être résolus, par utiliser des techniques d'identification, qui permet la simplification de la description physique de modèles, en utilisant des modèles à complexité réduite. Les systèmes dynamiques sont en général décrits en temps continu, par utiliser des équations différentielles, mais les procédures d'identification des systèmes sont réalisées en général en temps discret, en utilisant des données échantillonnées.

Les sources principales des perturbations rencontrées pour un système dynamique sont: erreurs de mesure, perturbations de processus non mesurées, inadéquation du modèle ou une combinaison de tous. Les modèles mathématiques sont utiles dans une grande classe des applications pour simulation ou prévision, mais aussi dans des domaines non-technique: économie, écologie ou biologie. Dans le domaine d'identification des systèmes sont rencontrées une grande classe des modèles, utilisés pour différentes applications de commande.

Ces modèles peuvent être classifiés comme: modèles analytiques et expérimentales; modèles linéaires ou non linéaires; modèles paramétriques ou non paramétriques; modèles SISO, SIMO, MISO, MIMO; modèles variables et non variables avec le temps; modèles en temps discret et en temps continu; modèles avec des paramètres distribués et localisés; modèles en domaine de temps et fréquence. Les modèles mathématiques peuvent être déterminés de deux manières différentes. Une première manière est connue comme modélisation, qui divise le système global en petits sous-systèmes, dont les propriétés sont bien connues à partir des connaissances antérieures. Ces petits sous-systèmes sont connectés par relations mathématiques, pour construire le système global. Cette technique de modélisation nécessite pas des expérimentations avec le système. La deuxième approche pour la modélisation mathématique est appelée identification des systèmes, et est représentée par l'expérimentation pratique avec le système, utilisant les signaux entrée/sortie pour l'analyse des données et la construction du modèle.

Les principaux sujets dans l'identification des systèmes sont: l'identification de la structure du modèle, l'identification du retard pur de temps, l'identification des modèles de bruit en temps continu, l'identification des systèmes multi-variables, l'identification dans le domaine fréquentiel, logiciel d'identification des modèles. Une notion importante pour l'identification des systèmes, c'est le concept de vrai système, qui c'est une représentation

*abstrait pour un système dynamique, différent de l' modèle mathématique, ou des modèles estimés par identification.*

*La **problème d'identification** est définie comme suit: Détermine un modèle en temps continu pour le système continu originel, à partir de  $M$  mesures échantillonnées de l'entrée et sortie. Les entités de base d'une procédure d'identification des systèmes sont: l'ensemble de données; l'ensemble des modèles candidate; une méthodologie utilisée pour l'évaluation du modèle, par utiliser un ensemble des données.*

*Une procédure d'identification c'est composé de quatre étapes principales: la première étape représente une méthodologie d'identification, ou l'utilisateur peut choisir les entrées, signaux mesurée ou pas de temps, afin de donner suffisamment informations sur le comportement du système réel basé sur quelques contraintes; pour la deuxième étape un ensemble des modèles peut être identifiée à partir de laquelle le plus représentatif doit être sélectionnée. Celle-ci est considéré l'étape plus difficile, comme les informations a priori doit être combinées avec des propriétés spécifiques du modèle et l'expérience en ingénierie.*

*La troisième étape de la procédure d'identification évalue les modèles en rapport avec les mesures réelles, par utiliser un algorithme d'identification, pour obtenir un modèle qui approxime les données entrée/sortie basée sur un critérium sélectionné.*

*Pour la quatrième étape, une validation de modèle est réalisée, qui montre que le modèle est suffisamment représentatif pour le système dynamique sélectionné. Le modèle finale est seulement utile pour une application spécifique, et pas une dernière description du système. Le modèle obtenu par identification peut être inefficace pour diverses raisons: la procédure numérique échoué à trouver le meilleur modèle estimé, selon le critère sélectionne; le critère d'estimation n'est pas approprié pour l'application; l'ensemble des modèles sélectionnées n'a pas contenu une bonne description pour le système actuel; la base des données n'a pas été assez informative, afin de sélectionner les modèles appropriés.*

*La deuxième partie d'introduction, présente l'approche de modélisation et commande pour les systèmes Hamiltoniens à ports, qui c'est une modélisation basée sur le port. Cette approche de modélisation crée un cadre unifié pour la modélisation des systèmes de différentes branches physiques (mécanique, électrique, hydraulique et d'autres).*

*Dans la modélisation basée sur le port, le système physique est considéré comme l'interconnexion de trois types de composants idéaux: éléments de stockage d'énergie; éléments de dissipation d'énergie et éléments de routage d'énergie. L'élément central d'un système Hamiltonien à ports c'est la structure d'interconnexion Dirac, pour lequel est associée la lettre **D**. Une propriété centrale des structures Dirac, c'est que la composition de deux ou plus structures Dirac, c'est aussi une structure Dirac. Cette propriété montre que la puissance est préservée par l'interconnexion des systèmes Hamiltoniens à ports, au moyen de ports externes.*

*La modélisation basée sur le port, inclure aussi un ensemble des notations graphique, pour représenter la structure de l'système physique, comme un ensemble des éléments idéaux interconnectée par des bords, qui capture les fluxes d'énergie entre eux. Le nom introduit pour ces bords c'est liaisons et le graphique global de l'système physique s'appelle Graph de liaison. Par utiliser des connexions avec la théorie des circuits électriques, le flux d'énergie le long des liens, est représentée par paires des variables, dont le produit est toujours égal à*

la puissance. Quelques exemples de ces paires sont les voltages et courants dans le domaine électrique, vitesses, et forces dans la mécanique translationnelle, fluxes et pressions en hydraulique, etc.

Les interconnexions dans les systèmes Hamiltonien à ports sont réalisées par utilisation de paires des fluxes et d'efforts dénoté par  $(f, e)$ . Les systèmes Hamiltoniens à ports sont décrits aussi par la loi de conservation de puissance, qui montre que la puissance totale qui entre (ou part) le système est conservée. À côté de ça, les systèmes Hamiltoniens à ports peuvent être décrits par la forme bilinéaire, qui est positif semi-définie et non dégénère, mais aussi par utilisation de la forme implicite de représentation. Un autre domaine dans la construction de la théorie des systèmes Hamiltoniens à ports c'est la commande, où les systèmes dynamiques sont considérés comme ouverts pour interaction avec l'environnement et aussi pour l'interaction de commande.

Les développements pour les systèmes de commande sont connectés avec la théorie de la synthèse des réseaux électriques. Par exemple, il existe plusieurs méthodes pour façonnage de l'énergie/puissance ou commande basé sur la passivité, pour les ports de stockage et dissipation, ou des techniques qui reposent sur la représentation du système de commande en tant que système Port-Hamiltonien. La représentation de type Port-Hamiltonien des systèmes dynamiques multi-physiques sont aussi utiles pour des calculs numériques, par utilisation de différents logiciels.

Quelques exemples des systèmes Hamiltoniens à ports sont introduits dans la dernière partie de la section, représentée par un circuit LC, microphone à condensateur circuit ou un moteur à courant continu. Ces trois exemples sont considérés dans tous les chapitres comme exemplification pour les définitions et propositions proposées. Les dernières deux sections de l'introduction présentent les objectifs principaux de la thèse et un résumé sur chapitres.

La deuxième partie de la thèse '**Identifiabilité structurelle des systèmes Hamiltoniens à ports**', fait une analyse théorique ou structurelle des systèmes Hamiltoniens à ports, pour explorer la structure de modèle et vérifier si les paramètres de système ont une solution unique globalement ou localement, par utilisation de la connaissance des signaux d'entrée et sortie. Certaines techniques classiques d'identifiabilité structurelle sont: l'analyse d'observabilité et contrôlabilité, le test direct, la transformation de Laplace, l'expansion en série de puissance, ou la transformation de similarité.

Bien que les systèmes Hamiltoniens soient des modèles basés sur la connaissance qui peut être décrit par un nombre réduit des paramètres physiques, peu d'études existent sur l'analyse d'identifiabilité, comme l'identifiabilité structurelle. Il devrait être mentionné dans la littérature une exception pour le cas particulier d'un échangeur de chaleur.

Dans la deuxième partie de ce chapitre ils sont rappelés certaines définitions (observabilité, contrôlabilité, identifiabilité globale/locale ou identifiabilité fort locale) avant de présenter l'analyse d'identifiabilité des systèmes Hamiltoniens à ports. L'analyse d'identifiabilité pour les systèmes Hamiltoniens à ports, débute par montrer que il peut être converti dans une forme canonique observable, préservant dans le même temps la structure Hamiltonien.

Ensuite, un résultat général est montré pour les systèmes Hamiltoniens à ports dans le cas lossy et les conditions nécessaires sont écrites pour l'analyse

*d'observabilité/contrôlabilité. Un autre analyse d'identifiabilité pour les systèmes Hamiltoniens à ports est réalisé par utiliser le test direct, qui utilise les définitions classique d'identifiabilité globale/locale appliqué pour la forme sur l'état d'espace. Pour l'identifiabilité forte locale des systèmes Hamiltoniens à ports, c'est proposé le test de l'expansion en série de puissance, qui détermine une solution unique pour les paramètres du système, par calculer les dérivés de la sortie dans un point d'intérêt.*

*L'analyse est réalisé dans le cas lossless ou lossy des systèmes Hamiltoniens à ports, montrant aussi quel est le numéro minimum d'équations nécessaire à résoudre. A cote des approches classiques pour l'analyse d'identifiabilité structurelle, c'est proposé une approche en énergie de puissance, qui utilisé les entrée et les énergies de puissances générés, pour déterminer l'identifiabilité structurelle des paramètres.*

*Ce résultat est dû au fait que entre l'entrée et les énergies de puissances, une fonction de transfert existe et aussi une modèle, nécessaire dans l'analyse d'identifiabilité. Ainsi est montré que les énergies de puissances peuvent être utilisées pour l'analyse d'identifiabilité des paramètres et nouvelles définitions sont proposées pour l'identifiabilité locale et globale.*

*Aussi est introduit le concept de port identifiabilité, qui fait une analyse pour l'identifiabilité des paramètres correspondant à un seul port. Après, nouvelles définitions sont proposé pour l'identifiabilité globale ou locale.*

*L'analyse d'identifiabilité basée sur la puissance, est réalisée à partir de l'observation que les énergies de puissance associées à chaque port sont liées avec les paramètres inconnus. De plus, les relations entre les énergies des ports et les paramètres inconnus ont une structure triangulaire (partiellement découplé) et l'analyse d'identifiabilité peut être réalisée séparément pour chaque port de l système. L'approche d'identifiabilité basée sur la puissance peut être appliquée aussi pour le cas non linéaire, où le modèle est plus complexe, mais la structure d'interconnexion est aussi une structure Dirac.*

*En avantage important de l'approche par énergies de puissances, c'est que l'analyse d'identifiabilité peut être réalisée pour chaque port, pour sous-ensembles des paramètres, par introduire le concept d'identifiabilité du port. La dernière partie de cette section est dédiée à la présentation des exemples pour les solutions proposées, par trois modèles simples (circuit LC, microphone à condensateur circuit, moteur à courant continu) présenté dans le chapitre précédent.*

*La troisième partie de la thèse '**Identifiabilité pratique des systèmes Hamiltoniens à ports**', réalise une analyse d'identifiabilité pratique pour les systèmes Hamiltoniens à ports, par utiliser l'analyse d'observabilité/contrôlabilité, dans la présence d'un modèle de perturbation. L'identifiabilité pratique est réalisé après l'analyse d'identifiabilité structurelle qui donne un modèle théoriquement identifiable.*

*Dans la littérature scientifique, plusieurs méthodes ont été proposées et étudiées à partir de laquelle certains des plus importants sont: méthodes pour l'analyse des fonctions de sensibilité, la méthode de simulation Monte Carlo, la méthode de matrice de corrélation, la méthode de Weijers et Vanrolleghem, méthodes basées sur la commande et autres. Les techniques d'analyse pour l'identifiabilité pratique ont été appliquées pour plusieurs types de modèles linéaires ou non linéaires, dont certains sont: modèles dynamiques HIV, modèles de*



réseaux de réactions biologiques, modèles de réseau métabolique, modèles de fermentation d'acide acétique, modèles biochimiques de la rivière, modèles de thérapie photo dynamique, modèles de cristallisation de crème glacée et autres.

Le modèle de perturbation est défini et introduit par une paire entrée/sortie sur le port d'interaction, parce qu'il représente l'interaction de système avec l'environnement externe. Après, un système Hamiltonien à ports perturbée est décrit sous la forme de l'état d'espace, pour lequel une forme équivalent est trouvé et montré à préserver les propriétés des systèmes Hamiltoniens à ports dans le cas lossless ou lossy. Un résultat général est formulé pour l'observabilité de l'modèle Hamiltonien à ports perturbé, comme pour le cas non perturbé.

La dernière partie de cette section fait une analyse d'identifiabilité pratique pour les trois exemples proposée dans le premier chapitre de la thèse. Pour le modèle de circuit LC, il a été prouvé que il est identifiable pratiquement pour une modèle de perturbation de type bruit blanc, par utiliser l'analyse d'observabilité/contrôlabilité. Pour le cas lossy il a été prouvé que le microphone à condensateur circuit n'est pas observable et ainsi non-identifiable pratiquement, par utiliser l'analyse d'observabilité/contrôlabilité. Pour le modèle de moteur à courant continu il a été prouvé avec cette approche, que il est identifiable pratiquement dans la présence d'un modèle de perturbation de type bruit blanc.

La quatrième partie de la thèse '**Intégration en temps symplectique et Hamiltonien d'erreur discrète**', présente une cadre de discrétisation en temps pour les systèmes Hamiltoniens à ports dans le cas linéaire ou non-linéaire et aussi il propose un Hamiltonien d'erreur discrète pour différentier les systèmes en temps continu et discrète avec le formalisme Hamiltonien à ports.

La commande de processus assisté par ordinateur, est réalisée en temps discret en échantillonnant le signal continu. Ainsi, plusieurs techniques de discrétisation ont été désigné pour approximer avec précision, les modèles du processus ou control. Parmi ceux-ci, lorsqu'on considère les modèles d'intégration de bas niveau, on pourrait citer les plus importants: la règle **Étape Invariant**, la règle **Pole-Zero Apparié**, règles d'**Euler implicite/explicite**, la règle de **Tustin**, la règle **Transformation Bilinéaire Généralisée**, la règle du point milieu ou la règle **Runge-Kutta**. Pour techniques d'intégration plus élevées du processus ou control, des méthodes de discrétisation comme **Boxer-Thaler** et **Madwed**, ont été introduit, avec une précision plus élevée pour le modèle estimé, mais avec une implémentation pratique complexe et faible possibilité pour l'analyse de stabilité. Les systèmes Hamiltoniens à ports ont des propriétés remarquables comme la conservation d'énergie de puissance et stabilité sous l'interconnexion qui préserve la puissance (interconnexions parallèles, séries ou feedback). L'interconnexion en temps par préserver la structure des systèmes Hamiltoniens, c'est un problème actuel de recherche et développement.

Pour la discrétisation en temps des systèmes Hamiltoniens à port, c'est proposé une discrétisation combiné pour les fluxes et les efforts, pour que les propriétés principales (loi de conservation de puissance, stabilisation de puissance, etc.) sont préservé en temps discrète. Après, les relations nécessaires sont dérivé en temps discrète pour le cas linéaire ou non-linéaire des systèmes Hamiltoniens à ports, pour préserver les propriétés caractéristiques.

Comme règles de discrétisation pour les fluxes et les efforts, quatre règles ont été proposées: Euler implicite/explicite, la règle du point milieu, ou la règle implicite trapézoïdale. Pour le cas non linéaire il est prouvé, que par combiner n'importe quelle règles du premier ordre (Euler implicite/explicite) pour les fluxes et efforts, les conditions nécessaires sont préservée pour le système Hamiltonien en temps discrète. Pour le cas linéaire il a été prouvé que par combiner une règle de première degré pour les fluxes avec une règle de première ou deuxième degré pour les efforts, les relations nécessaires sont satisfait pour le système Hamiltonien à ports discrète. L'analyse est réalisée dans le cas linéaire par utiliser le cadre proposé et quatre règles de discrétisation, développant ainsi douze schèmes de discrétisation dans le cas lossless ou lossy, à partir de laquelle certains sont prouvée de pas préserver le formalisme proposé. Pour simulations, seulement trois schèmes ont été sélectionnée dans le cas lossless ou lossy (Euler implicite/explicite pour les fluxes et les efforts, Euler implicite pour les fluxes et implicite trapézoïdal pour les efforts, implicite trapézoïdal règle pour les fluxes et implicite point milieu règle pour les efforts), à partir de laquelle la dernière a été prouvé par calcul et simulation de pas préserver les conditions nécessaires dans le cas lossy.

Comme exemple pour simulations, un circuit LC (cas lossless) et un circuit de microphone à condensateur (cas lossy) ont été considérés, pour lequel plusieurs simulations ont été effectuées sur les sorties du système, énergies de puissance, logarithme d'erreur, loi de conservation de puissance ou Hamiltonien d'erreur discrète.

Par discrétisation en temps, la structure et règles d'interconnexion Dirac ( $\mathbf{D}$ ) sont préservé comme pour le système en temps continu. La deuxième partie de ce chapitre, définit un Hamiltonien d'erreur discrète pour les cas lossless ou lossy, qui représent l'erreur de discrétisation entre les systèmes en temps continu et discrète avec une structure Hamiltonien à ports. Cela est un outil nécessaire pour l'analyse de l'erreur de discrétisation par utiliser une représentation de type Hamiltonien, mais aussi pour les algorithmes d'identification, pour vérifier les performances des modelés estimées, par l'approche dans l'énergie de puissance.

Le cadre de discrétisation courant peut être prolongé à des lois de discrétisation plus générale (Runge Kutta, collocation, etc.) dans le cas non linéaire ou linéaire.

La cinquième partie de la thèse, '**Identification des systèmes Hamiltonien à ports sur l'état d'espace**', présente une analyse et développement des algorithmes d'identification déterministe pour les systèmes Hamiltonien à ports, par utiliser une approche classique ou nouveau (énergie de puissance). Les algorithmes d'identification de type sous-espaces, sont basée sur concepts de la théorie des systèmes, algèbre linéaire ou statistique.

Les principales nouveautés conceptuelles dans les algorithmes d'identification sous-espace sont: l'état du système est central pour l'identification des systèmes par rapport à l'approche classique, qui utilise le contexte entrée/sortie; l'approche d'identification sous-espace est basée sur concepts et algorithmes de l'algèbre linéaire numérique (décomposition QR, décomposition SVD et ses généralisations avec des angles entre sous-espaces; l'approche d'identification sous-espace offre une cadre géométrique où différents modèles sont traités de manière unifiée; les concepts théorique se traduisent par des implémentations logicielles conviviales (il n'y a pas de besoin explicite de paramétrages et

*donc pas de problèmes très techniques et théoriques pour l'utilisateur, comme des paramétrisations canoniques).*

*Par rapport aux approches classique, certains des principaux points introduits par les algorithmes d'identification de type sous-espaces sont:*

*- paramétrisations: l'approche d'identification de type sous-espace utilise seulement l'ordre du système. Avec les approches classiques, une partie de recherche a été concentré sur modèles canoniques (formes canoniques incontrôlable et observable avec un nombre minimal des paramètres). Les points faibles de cette approche sont: ils peuvent conduire à des problèmes numériquement mal conditionnés, avec des résultats sensible à petites perturbations; il est nécessaire de chevaucher les paramétrages, car aucun des paramètres existants ne couvrira tous les systèmes dynamiques; en practice des modèles minimales de type sous-espace are désirable. En cas de modes incontrôlables mais observables, des paramétrages spéciaux sont nécessaires. Tous ces problèmes sont éliminés dans le cas d'identification de type sous-espaces.*

*- convergence: Les algorithmes d'identification sous-espace sont plus rapides que l'approche classique, car ils ne sont pas itératifs et aucun problème de convergence n'existe. Ainsi, le manque de convergence, la convergence lente ou l'instabilité numérique sont éliminés.*

*- Réduction du modèle: Dans l'identification de type sous-espace, le modèle réduit du système dynamique résulte directement par les données d'entrée/sortie, sans avoir à calculer à l'avance le modèle de haut niveau.*

*Pour l'analyse d'identification, un schème simple de discrétisation de l'chapitre précédent a été sélectionnée, qui utilise une règle Euler explicit pour les fluxes et une règle Euler implicite pour les efforts, qui a aussi une structure similaire avec l'approche classique. Dans le même temps, un contexte générale déterministe a été défini pour tous les schèmes de discrétisation, présenté dans le chapitre 4, par utiliser des projections orthogonales ou obliques, comme on peut le voir aussi dans l'annexe F.*

*Par utiliser le schème de discrétisation symplectique sélectionnée, pour deux algorithmes d'identification classique, plusieurs simulations ont été réalisé pour les entrée/sorties du système, énergies de puissance et loi de conservation de puissance.*

*Dans la deuxième partie du chapitre, une cadre d'identification déterministe de type énergie de puissance est défini théoriquement, par utiliser la loi de conservation de puissance et les projections orthogonales ou obliques. Par utiliser ce contexte, trois modèles équivalents sous une forme d'état d'espace peut être obtenu pour chaque port, comme conséquence aux définitions et propositions proposé dans le deuxième chapitre, pour l'identifiabilité globale/locale ou port-identifiabilité. Ainsi c'est prouvé et défini une cadre d'identification déterministe pour l'approche de l'énergie de puissance, à partir de l'approche classique. Dans ce nouveau contexte il est supposé un certain nombre de mesures connues de l'entrée et énergies de puissance et il est désirée à déterminer l'ordre du système inconnu et les paramètres du système, donné par les matrices du système  $A, B, C$ . Comme cette approche utilise la loi de conservation, c'est nécessaire d'avoir d'informations au moins sur deux énergies du port, pour préserver la structure Hamiltonien. Les hypothèses initiales, pour le contexte proposée suivez les mêmes étapes comme dans l'approche classique pour une entrée bien sélectionnée, intersection entre les futures entrées et les états passés ou les*

matrices de pondération. Ensuite, sont définies les projections obliques correspondant aux ports, pour lesquels une décomposition SVD est réalisée. Ainsi, l'ordre du système dynamique peut être déterminé correspondant à chaque port et une décomposition des projections obliques peuvent être prouvés et réalisés en utilisant les matrices d'observabilité étendues et les états du système. En utilisant la décomposition SVD, nous pouvons déterminer séparément les matrices d'observabilité étendues correspondant aux ports avec les états associés.

Aussi est proposé un algorithme d'identification déterministe de type énergie de puissance, qui utilise le nouveau cadre d'identification et détermine comme prévu trois modèles équivalents sur l'état d'espace pour les systèmes Hamiltonien à ports.

La dernière partie de chapitre est dédié à simulations avec un algorithme d'identification déterministe de type énergie de puissance, pour un modèle de moteur à courant continu, sur lequel plusieurs simulations ont été réalisé pour la réponse du système, correspondant à modelés estimé pour les trois ports (commande, stockage, dissipation). Ainsi le cadre générale déterministe de type énergie de puissance est prouvé, et aussi les définitions et propositions de deuxième chapitre. La dernière partie de chapitre, fait une analyse pour les performances des modèles estimées par utiliser un index de validation, dans la présence d'un modèle de perturbation proposé dans le troisième chapitre, qui est plutôt une modèle déterministe. En conclusion, les modèles des systèmes Hamiltonien à ports peut être estimé par utiliser des algorithmes d'identification sous une forme d'état d'espace, dans une approche classique ou nouveau (énergie de puissance), qui utilise un schème de discrétisation symplectique.

Les principales contributions de la thèse sont: l'analyse du utilité, nécessite ou possibilités de développement d'une théorie d'identification des systèmes, en vue des solutions améliorées en temps réel de modélisation et commande; l'analyse de la structure de modèle des systèmes Hamiltonien à ports, nécessaire pour l'identification des systèmes; la sélection des techniques approprié pour l'analyse d'identifiabilité structurelle des systèmes Hamiltonien à ports, à partir de la théorie d'identifiabilité classique; l'analyse, preuve et formulation des propositions et définitions spécifique pour les systèmes Hamiltonien à ports dans le cas lossless ou lossy; le développement des nouveaux (spécifique) techniques d'identifiabilité structurelle pour les systèmes Hamiltonien à ports, par utiliser les énergies de puissance correspondant à ports; l'introduction d'un nouveau concept '**port identifiabilité**', avec des définitions spécifique pour l'identifiabilité locale ou globale correspondant a un port; l'adoption d'un modèle Hamiltonien perturbé, par introduire une paire entrée/sortie correspondant à la port d'interaction de l'système avec l'environnement; l'analyse, preuve et formulation des nouveaux propositions pour les systèmes Hamiltonien à ports, pour prouver que par convertir à la forme canonique observable dans la présence d'un modèle de perturbation, les principales propriétés sont conservé; la sélection d'une cadre de discrétisation symplectique pour les systèmes Hamiltonien à ports, par discrétisation combinée des fluxes et efforts; la sélection de quatre règles de discrétisation (Euler implicite/explicite, implicite point milieu ou trapézoïdale) pour exemplification; l'analyse du cadre symplectique discrète dans le cas linéaire ou non linéaire et la formulation des propositions nécessaires ou règles générales; la simulation de l'cadre proposé avec un microphone à condensateur circuit pour trois schèmes, pour montrer la conservation ou non

conservation des propriétés pour les systèmes Hamiltonien à ports aussi en temps discrète; l'introduction d'un Hamiltonien d'erreur discrète, utile dans l'analyse de l'erreur, pour la méthode de discrétisation pour les systèmes Hamiltonien à ports dans le cas lossless ou lossy; La simulation de l' nouveau concept proposé (Hamiltonien d'erreur discrète) par utiliser un modèle de microphone à condensateur circuit pour les schèmes de discrétisation sélectionnée; le développement d'un logiciel de simulation spécifique pour le cadre de discrétisation symplectique proposée avec douze schèmes de discrétisation pour le cas lossless ou lossy; la formulation générale des équations matricielles entrée/sortie pour les schèmes de discrétisation proposée; la formulation d'une cadre d'identification déterministe pour chaque schème symplectique; la sélection d'un schème de discrétisation pour l'analyse des algorithmes d'identification; la sélection des algorithmes d'identification classique approprié de type d'état d'espace pour les systèmes Hamiltonien à ports; l'analyse, preuve et formulation d'un nouveau (énergie de puissance) cadre d'identification déterministe pour les systèmes Hamiltonien à ports; la proposition d'un nouvel algorithme spécifique (énergie de puissance) déterministe pour le nouveau cadre d'identification; l'analyse en simulation des algorithmes d'identification déterministe de type classique ou nouveau (énergie de puissance) par utiliser le modèle d'un moteur à courant continu sous forme Hamiltonien à ports; l'analyse de précision pour le modèle estimé, par introduire une index de validation entre les modèles estimées et réels dans la présence de l' modèle de perturbation proposée dans le troisième chapitre.

Les principales perspectives à la fin de la thèse sont: l'analyse d'identifiabilité structurelle ou pratique des systèmes Hamiltonien à ports par utiliser des autres techniques ou la proposition des autres, nouveau; le développement des applications améliorées d'identification et commande en temps réel, par utiliser l'approche de modélisation et commande des systèmes Hamiltonien à ports pour les systèmes dynamiques complexes; le développement des nouveau (spécifiques) notions et algorithmes d'identification, pour l'approche énergie de puissance proposé dans la thèse; le développement des plateformes de test en temps réel dans le laboratoire pour tester et analyser les algorithmes d'identification et théories de commande; le développement des autres schèmes symplectique de discrétisation en temps pour les systemes Hamiltonien à ports par utiliser des règles de discrétisation plus complexe (Runge-Kutta, collocation, etc.); le développement d'une théorie d'identification et commande nécessaire pour le cas stochastique des systèmes Hamiltonien à ports, qui contient aussi une modèle de perturbation plus proche de la réalité; l'analyse des schèmes de discrétisation symplectique proposée, aussi pour solutions de commande connu dans la littérature (Modèle de Commande Prédictif, Commande  $H_2, H_\infty$ , Commande Quadratique Linéaire, etc.); le développement des modules d'identification, commande et discrétisation des systèmes, pour être utilise à une plus grande échelle; l'introduction des solutions d'identification et commande dans une environnement industrielle.

## Publications

**S., Medianu, D.**, Stefanoiu, “Study on the identification of a system with distributed parameters represented by a heat exchanger using open-loop recursive techniques”, *2<sup>nd</sup> International Conference on Systems and Computer Science*, pp.26-31, 26-27 August, Villeneuve d'Ascq, France 2013.

**S., Medianu, C.**, Lupu, D., Popescu, L., Lefevre, “Optimal closed-loop identification solution using direct and indirect methods for a flexible transmission system”, *2<sup>nd</sup> International Conference on Systems and Computer Science*, pp.74-79, 26-27 August, Villeneuve d'Ascq, France, 2013.

**S., Medianu, D.**, Popescu, L., Lefevre, C., Lupu, “Identification of a distributed parameters system for thermo-energetic applications”, *17<sup>th</sup> International Conference on System Theory, Control and Computing*”, pp.349-355, 11-13 October, Sinaia, Romania, 2013.

**S., Medianu, L.**, Lefevre, D., Stefanoiu, “Identifiability of linear lossless port-controlled Hamiltonian systems”, *2<sup>nd</sup> International Conference on Systems and Computer Science*, pp.56-61, 26-27 August, Villeneuve d'Ascq, France, 2013.

**S., Medianu, L.**, Lefevre, C., Lupu, D., Popescu, “Structural Identifiability of linear lossy Port Controlled Hamiltonian Systems”, *20<sup>th</sup> International Conference on System Theory, Control and Computing*, pp.738-743, 13-15 October, Sinaia, Romania, 2016.

**S., Medianu, C.**, Lupu, L., Lefevre, D., Popescu, “Solution for the model adequacy verification in indirect adaptive control”, *Journal of Control Engineering and Applied informatics*, Vol. 18, No.1, pp.51-58, 2016.

L., Lefevre, **S., Medianu**, “Symplectic Discretization of Port Controlled Hamiltonian Systems”, *The 20<sup>th</sup> World Congress of the International Federation of Automatic Control*, Vol. 50, Issue 1, pp.3629-3634, Toulouse, 2017.

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## Notations

In this section we will present the main notations introduced for each chapter of the thesis, for a clear understanding of the interested reader.

In *chapter 2* of the thesis we have denoted with  $\theta \in \Theta$ , the unknown parameters ( $\theta$ ) from the parameter space ( $\Theta$ ), that are generally searched by system identification techniques. We can have unknown parameters corresponding to the system matrices  $A, B, C$  ( $\theta_{HA}, \theta_{HB}, \theta_{HC}$ ) or Port Hamiltonian system matrices  $R, Q, B$  ( $\theta_R, \theta_Q, \theta_B$ ).

For the power energy approach proposed in section 2.5, the unknown parameters corresponding to the storage, control or dissipation ports are defined by  $\theta_{PS}, \theta_{PC}, \theta_{PR}$ , while for the *port-identifiability* concept proposed, the unknown parameters are defined by  $\theta_T$  where  $T$  represents one of the system ports. For the local identifiability analysis, the unknown parameters in a specific point of interest are defined by  $\theta^*$ , which become  $\theta_{PS}^*, \theta_{PC}^*, \theta_{PR}^*$  for the power energy approach or  $\theta_T^*$  for the port-identifiability case. For the observable canonical form representation of Port Hamiltonian systems, we will introduce the *obs* index for the necessary terms in the state-space representation. By  $T \in R^{n \times n}$  we have denoted a non-singular transformation matrix corresponding to the Port-Hamiltonian representation. For the power series expansion identifiability analysis, we have denoted with  $u^{(k)}(t_0), y^{(k)}(t_0)$  the  $k^{th}$  order derivate of the system inputs ( $u$ ) or outputs ( $y$ ) at a specific point of interest  $t_0$ .

With  $A_{loss}$  respectively  $A_{res}$  we have denoted the lossless and respectively the dissipative parts of the matrix  $A'' = A_{loss} + A_{res}$  defined for lossy Port Hamiltonian systems.  $R_{ID}$  was introduced as notation for the identifiability matrix, necessary for the observability/controllability test analysis.

In *chapter 3* of the thesis, *obs* index represents the observable canonical form terms corresponding to Port Hamiltonian systems in the lossless or lossy case. For an equivalent observable canonical form state-space representation of Port-Hamiltonian systems, we have used the notation  $obsT$  for the terms. With  $T_1, T_2, T_{obs} \in R^{n \times n}$  we have denoted non-singular transformation matrices applied to the state-space system. By  $\gamma_{1k}, \gamma_{2k}$  were defined the terms corresponding to the equivalent observable canonical form representation of the input ( $B$ ) and perturbation ( $K$ ) matrices.

$TF_1$  and  $TF_2$  represent transfers functions from the input to the output and from the perturbation to the output. By  $R_{ID}$  it was defined the identifiability matrix used for the analysis of the observability and controllability of Port Hamiltonian systems in the lossless or lossy case.  $C_k$  ( $k = 1..n$ ) represents the terms which compose the identifiability matrix  $R_{ID}$  in an equivalent observable canonical form.

With  $A_{loss}$  and  $A_{res}$  we have defined the lossless respectively lossy parts of the matrix  $A'' = A_{loss} + A_{res}$  defined in the lossy case of Port Hamiltonian systems similar to chapter 2.

In *chapter 4* of the thesis, we have introduced  $F_{di}, E_{di}$  as notations for the flow and efforts discretization rules at discrete-time step  $i$ . For all the discrete-time Port-Hamiltonian systems terms, we have used the hat notation ( $\hat{\cdot}$ ) to differentiate from the continuous-time ones.  $\hat{e}_S, \hat{f}_S, \hat{e}_C, \hat{f}_C$  and  $\hat{e}_R, \hat{f}_R$  defines the discrete-time efforts and flows corresponding to the storage (**S**), control (**C**) or dissipation (**R**) ports.  $F_{di} * E_{di}$  represents the composed time-discretization of the flows and efforts rules applied to a continuous-time term, which in our case is the effort  $\frac{\partial H}{\partial x}(x)$ . The differentiation term of the flow ( $\frac{dx}{dt}$ ) becomes  $\frac{\Delta x}{\Delta t}$ , the energy balance ( $\frac{dH}{dt}$ ) translates to  $\frac{\Delta H}{\Delta t}$  and the effort  $\frac{\partial H}{\partial x}$  becomes  $\frac{\Delta H}{\Delta x}$  for discrete-time Port-Hamiltonian systems. Beside this notations, were introduced the terms  $\bar{x}_k$  or  $\overline{\bar{x}}_k$  to define the medium value of two successive discrete-time terms ( $x_k, x_{k+1}$  or  $\bar{x}_k, \overline{\bar{x}}_{k+1}$ ) for the states ( $x_k$ ) or inputs ( $u_k$ ).

For the difference between two Port-Hamiltonian systems defined in section 4.4, we have used the index  $A$  or  $B$  for the terms corresponding to one or another systems. For the terms representing the Port-Hamiltonian system corresponding to the difference between systems  $A$  and  $B$ , we have used the notation  $\overline{w}_{A \perp B}$ , where  $w$  represents one of the Port-Hamiltonian system elements. For the discrete-time simulation analysis, we have denoted with  $N$  the number of discrete-time steps,  $\Delta t$  sampling-time of discretization,  $t_S$  the time-interval and  $o(\Delta t^k)$  the  $k^{th}$  order of convergence of the discretization method.

In *chapter 5* of the thesis, a series of notations have been realized for the subspace identification analysis performed on Port-Hamiltonian systems, using the classic or new power energy approach.

$U_{0/2i-1}, Y_{0/2i-1}$  were used to define the input and output *block Hankel matrices*, which can be expressed using past ( $U_P, Y_P$ ) or future ( $U_f, Y_f$ ) informations. When augmenting or decreasing one row from the past or future inputs or outputs, the notations use a  $+$  ( $U_P^+, Y_P^+$ ) or  $-$  ( $U_f^-, Y_f^-$ ). The state sequences are defined by  $X_i^d \in R^{n \times j}$  where  $X_P^d$  stands for the past states ( $X_0^d$ ) and  $X_f^d$  stands for the future states.  $\Pi_L$  is used to define the *projector operator* which projects the row space of a matrix onto the row space of the matrix  $L \in R^{q \times j}$ .  $M^+$  defines the *Moore-Penrose pseudo-inverse* of a matrix  $M \in R^{n \times m}$ . We define the *orthogonal projection* by  $G/L$ , which represents the projection of the row space of the matrix  $G \in R^{p \times j}$  on the row space of the matrix  $L$ . An important concept for the subspace

identification algorithms is the *oblique projection*, by which a matrix  $G \in R^{p \times j}$  can be decomposed as linear combinations of two non-orthogonal matrices ( $L \in R^{q \times j}$  and  $F \in R^{q \times j}$ ) and the orthogonal complement of  $L$  and  $F$ . For the subspace identification approach presented in this chapter the oblique projection will be denoted by  $o_i$  and will be calculated using the future inputs and outputs ( $Y_f, U_f$ ) and the *Willems matrix* ( $w_p$ ) of the past inputs/outputs ( $U_p, Y_p$ ). For the power energy approach, the oblique projections corresponding to the ports will be denoted by  $o_{iS}$  (storage),  $o_{iC}$  (control) or  $o_{iR}$  (dissipation).  $w_1 \in R^{l \times l}$  and  $w_2 \in R^{j \times j}$  define two weighting matrices, which are multiplied to the oblique projection and influence the result of the SVD decomposition.

The **SVD** notation represents the **Singular Value Decomposition** of a matrix  $M \in R^{n \times m}$ . For the power energy approach, the matrix elements corresponding to the SVD decomposition will have indexes representing the ports ( $U_S, S_S, V_S$  - storage,  $U_C, S_C, V_C$  - control,  $U_R, S_R, V_R$  - dissipation).  $\bar{A}, \bar{B}, \bar{C}$  have been used to define the discrete-time system matrices corresponding to the continuous-time Port Hamiltonian model ( $A, B, C$ ).  $T \in R^{n \times n}$  represents a non-singular similarity transformation, which become  $T_S, T_C, T_R$  for the storage control or dissipation ports in the power energy approach.

In the power energy approach, indexes corresponding to the ports (**S** – storage, **C**- control, **R**- dissipation) have been introduced to designate the system states ( $x_f^d$ ), system outputs ( $Y_f$ ), extended observability matrix ( $\Gamma_i$ ), Henkel matrix ( $w_p$ ) or system degree

( $n$ ). With  $c_{\mathbf{B}} \binom{k}{n}$  have been denoted the binomial coefficients or combinatorial numbers,

which is the coefficient of  $x^k$  in the polynomial expansion  $(1+x)^n$ .  $\Gamma_{iH}, C_{iH}$  have been used as notations for the observability and controlability matrices of an equivalent continuous-time PCH system.

$w_O, w_C$  represent the observability and controlability grammians of a Port Hamiltonian system.  $H_S, H_C, H_R$  and  $Y_S, Y_C, Y_R$  were used to define the transfer functions and system outputs corresponding to the system ports (**S** – storage, **C** – control, **R** – dissipation).

$v\_index$  is a notation used for the validation index between the estimated data and real data, which is used in system identification to see the accuracy of the estimated model.

In Appendix F, several notations have been adopted to distinguish between the current time-discretization scheme parameters and the approach proposed in chapter 5, but also for a more compact writing of the equations.

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# Introduction

*The first chapter of the thesis makes a general presentation of the system identification domain, introducing notions like modelling, types of models, modelling and identification procedure. A second part of the chapter makes a general presentation of Port-Hamiltonian systems with their special properties of power energy conservation and Dirac interconnection structure, that permits the interconnection of multiple sub-systems from different domains of physics or chemistry. Three classic examples of Port-Controlled-Hamiltonian systems represented by an LC circuit, capacitor microphone circuit or DC motor are introduced for the lossless and lossy cases, to be used in the next chapters.*

*The last two sections of this chapter, introduce the main objectives of the thesis together with a summary of the chapters.*

## 1.1 Systems and system identification

This subsection of the introduction presents some basic definitions necessary in system identification and makes connections with a dynamic system represented in input/output form and also introduces disturbances.

The physical models represented by hypotheses, laws of nature, may be of more or less formal in character, but they have a basic property, that they connect the system observations together into a general pattern. *System identification* represents a control engineering branch, which deals with building mathematical models for dynamical systems, using observed data. A *dynamic system* can be associated with a global object, where variables of different kinds interact and produce observable signals. A dynamic system corresponds to two types of signals: observable signals called *outputs* and controllable signals called *inputs*. Beside these two signals, in real applications *disturbances* are encountered between the system and external world. These are classified as: directly measurable disturbances and disturbances observed on the outputs. The notion of a system is central in modern science and many problems are solved in a system-oriented framework. These systems can be dynamic, which means that the current output value depends not only on the current external stimuli, but also on their earlier values. The outputs of dynamic systems whose external stimuli are not observed, are called *time series* [1,2].

### 1.1.1 Models

When interacting with a system, it is necessary to have a mathematical description that connects its variables. A model can be generally defined using the relationship between the observed signals. These models may come in different forms, with different degrees of mathematical formalism. Depending on the control necessities, a complexity of the model will be chosen [1,2].

The use of mathematical models, is necessary in a large class of domains of engineering and physics. Mathematical models of dynamical systems are useful in most areas of scientific

enquiry and can take various forms, such as: differential equations, difference equations, state-space equations or transfer functions. A possible approach for mathematical modelling, is based on the use of mathematical equations which describe the physical laws that govern the process. This approach may have drawbacks such as the high complexity of the obtained model or the difficulty of estimation due to over parameterization. These drawbacks extend also to process control solutions [8]. To overcome this modelling issues, identification techniques were introduced [1-8], to make a simpler physical description of the systems, using reduced complexity models. The dynamic systems encountered in real world, are generally described in *continuous time*, using differential equations. Most part of the system identification techniques, make use of *discrete time* models, also known as sampled data models. Some approaches using continuous time techniques, were realized in [9,10,11].

A linear time-invariant continuous-time system with input  $u$  and output  $y$ , can be described as:

$$y(t) = H(p)u(t) + v(t) \quad (1)$$

where  $H$  is the transfer function,  $p$  the time-domain differential operator and the additive term  $v(t)$  represents errors or disturbances of all natures. The main sources for  $v(t)$  can be measurement errors, unmeasured process disturbances, model inadequacy or a combination of all of these. It is assumed that the input signal  $\{u(t), t_1 < t < t_M\}$  is applied to the system, with  $u(t)$ ,  $y(t)$  both sampled at discrete times  $t_1, \dots, t_M$ . The sampled signals are denoted with  $\{u(t_k); y(t_k)\}$ . Mathematical models are useful in a broad class of applications for simulation or forecasting, but also in non-technical fields like: economy, ecology or biology [1,2].

### 1.1.2 Model classification

In system identification, are encountered a broad class of models, used for different control applications. These models can be classified as [3]: analytical models and experimental models; linear and nonlinear models; parametrical and non-parametrical models; SISO, SIMO, MISO, MIMO models; variable and invariable models with time; continuous-time and discrete-time models; models with lumped and distributed parameters; time-domain and frequency-domain models.

### 1.1.3 Procedure for building mathematical models

A physical model, can be determined on the basis of real observed data from the system's response, by interaction with external stimuli. Mathematical models can be determined in two different ways. A first way is known as *modelling*, which splits up the global system into small subsystems, whose properties are well known from previous knowledge. These small subsystems are connected through mathematical relations, to construct the global system. This modelling technique doesn't necessitate any experiments with the system. The second approach for mathematical modeling is called *system identification* and is represented by practical experimentation with the system, using

input/output signals for data analysis and model construction [1,2]. Important topics in system identification are: the identification of the model structure; the identification of the pure time-delay; the identification of continuous-time noise models; identification of multivariable systems; identification in closed-loop; identification in frequency-domain; software for model identification [8].

#### 1.1.4 The true dynamic system

The so-called *true system* is an entity distinct from the mathematical models obtained by physical modeling, using equations. Between these two descriptions of a dynamic system, certain relations can be realized but never an exact connection. The mathematical models that describe a physical system need to be connected with their practical usefulness, not with the true system model. Even so, the necessity of a true system, it is necessary to define an ideal context, for identification techniques and their properties [1,2].

### 1.2 System identification procedure

**The identification problem:** *Determine a continuous-time model for the original continuous-time system, from  $M$  sampled measurements of the input and output,  $Z^M = \{u(t_k), y(t_k)\}_{k=1}^M$  (as for example the model in equation (1)) [8].* The basic entities of a *system identification procedure* are: the data set; the set of candidate models; a methodology used for the model evaluation, using a data set. The *first step* of the identification procedure, is represented by a specific identification methodology, realized using input/output data from the real system, where the user may choose the inputs, measured signals or time steps. The objective of this identification step, is to give enough informations on the behaviour of the real system, subject to some constraints. At the *second step*, a set of possible models might be identified, from which the most representative model can be selected for the real system. This is considered the most important and difficult step of the identification procedure, where apriori knowledge of the system behaviour and engineering experience, need to be combined together with specific properties of the models [1-7]. The *third step* of the identification procedure, is represented by the application of an identification algorithm, where the models are evaluated in rapport with the real measurements. At the end of these steps, a model that best suits the input/output data is obtained, using a certain proposed criterion. The *last step* of the identification procedure, is represented by the model validation, which shows that the model is representative enough for the selected dynamic system. At this step, certain procedures can be performed to see if the model is confident for further use, or it has to be rejected and replaced with a new one. The final model that is validated after the identification procedure, is only useful for a particular application and not a last and real description of the system [1-7]. The model obtained by identification, can be unefficient for a variety of reasons: the numerical procedure failed to find the best estimated model according to the selected criterion; the estimation criterion is not appropriate to the application; the selected model set did not contain a good description for the current system; the data set was not informative enough in order to select appropriate models. A general diagram which may be used for the identification procedure is presented on Figure 1.1 hereafter [1-7].

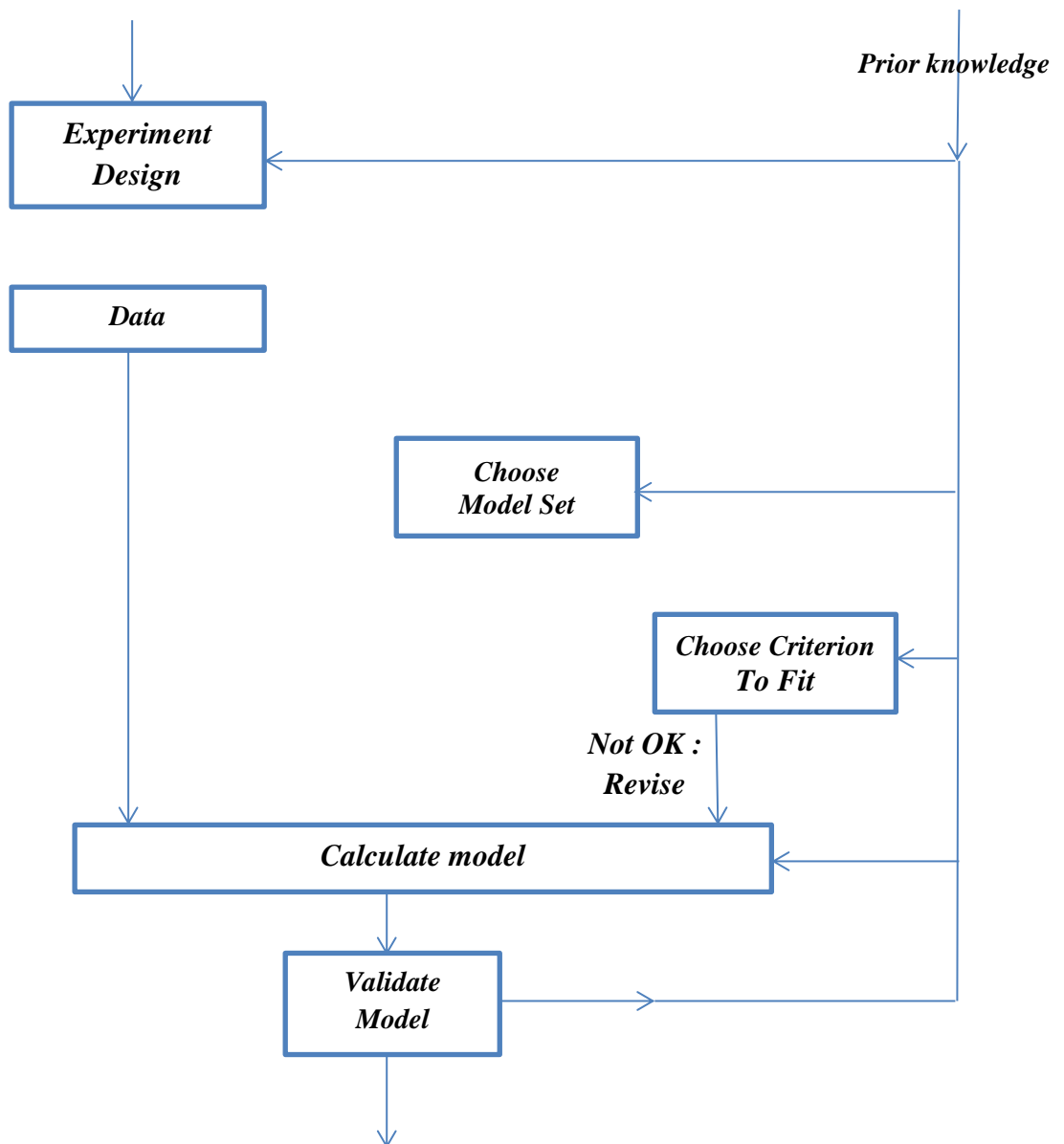


Figure 1.1. System identification loop.

### 1.3 Port-Controlled-Hamiltonian systems

Port-Controlled-Hamiltonian systems, represent a new generation of nonlinear systems for network modelling of complex physical systems, from different energy domains [12-19]. This new modelling formulation, can be seen as a *re-thinking* of standard control techniques, opening a broader way for the formulation of control problems [19]. From a modelling perspective, this theory starts from the *port-based modelling* theory developed by Henry Paynter [20,23] and Breedveld [21,22,23]. This modelling approach creates a unified framework for modelling of systems from different physical branches (mechanical, electrical, hydraulic, thermal and others). Central in port-based modeling is the *energy*, which makes the connection between the physical branches and the identification of the system's components

which capture the main physical characteristics, given by the *energy-storage*, *energy-dissipation* and *energy-routing*. Port-based modelling includes also a set of graphical notations, to represent the structure of the physical system, as a set of ideal elements connected by edges, that capture the energy flows between them. The name introduced for these edges is *bonds* and the global graph of the physical system is called a *bond graph*. Using connections with electrical circuit theory, the energy flow along the bonds is represented by pairs of variables, whose product always equals power. Some examples of such pairs are voltages and currents in the electrical domain, velocities and forces in translational mechanics, flows and pressures in hydraulics, etc. A Port-Hamiltonian formulation of bond graph theory can be found in [24]. Port-based modelling is also regarded as an abstraction of the theory of *across* and *through* variables [25]. Another origin of Port-Hamiltonian systems theory is *geometric mechanics* [26-30]. The basic idea of geometric mechanics is to represent the Hamiltonian dynamics in a coordinate-free manner, using a state-space representation (the phase space of the system) given by a symplectic or Poisson structure, together with a Hamiltonian function for the energy. This approach created a path for the analysis of Hamiltonian systems behaviour, showing their basic features like symmetries or conserved quantities. This geometric approach was also successfully extended to infinite-dimensional Port-Hamiltonian systems [31]. Another inspiring domain in the construction of Port-Hamiltonian systems theory is *control*, where the dynamical systems are regarded as being opened for interaction with the environment and also for control interaction. The control systems developments are connected with electrical network synthesis theory. The geometric formulation of this theory starts with [12, 29, 30, 32-38], mainly for the analysis and control of nonlinear mechanical systems. One important difference, between Port-Hamiltonian systems and Hamiltonian mechanics, is that for Port-Hamiltonian systems, the interconnection is not symplectic in the phase space, but in the augmented Bond space which includes both port variables for the system components and “external” interaction ports. In this way, the theory of Port-Hamiltonian systems, is a connection between *geometry* and *network theory*. The geometric element that makes a generalization for symplectic and Poisson structures, is called a *Dirac structure* [39-41]. Dirac structures were introduced in [16, 42,43] and used for mechanical systems with constraints resulting in Hamiltonian *differential-algebraic equations*. The theory could also be extended to distributed-parameter case [44]. A central property of Dirac structures, is that the composition of two or more Dirac structures, is again a Dirac structure. This property shows that the power is conserved by the interconnection of Port-Hamiltonian systems, by means of external ports. A new extension of Port-Hamiltonian systems theory in rapport with the geometric mechanics, is the inclusion of *energy-dissipating* elements. Together the port-based modelling and Port-Hamiltonian systems, can be seen as a general theory for the modelling of complex physical systems from different areas of engineering. Because this theory is based on two major concepts of *energy* and *power*, it is ideal for mathematical modelling of *multi-physics systems*. Another important aspect of this theory, is that it offers a clear view also for control solutions, mostly for the nonlinear case, by means of appropriate theoretical concepts and tools. For instance, there exists many methods for energy/power shaping or passivity-based control designs, of the storage and dissipation ports, or techniques which rely on the representation of the controller system as

port-Hamiltonian system itself. Port-Hamiltonian representation of dynamical multi-physics systems are also useful for numerical computations, for instance using different software suits the 20-sim (based on bond graphs) or Modelica/Dymola environments.

In port-based modelling, a physical system is regarded as the interconnection of three types of ideal components: *energy-storing* elements; *energy-dissipating* (resistive) elements and *energy-routing* elements. Some classical examples of such Port-Hamiltonian system components are:

- springs, inductors, capacitors, ideal inductors for the *energy-storing* components;
- dampers or resistors for the *energy-dissipating* components;
- transformers, gyrators and ideal constraints for the *energy-routing* elements.

For simplicity of notation, each port of the Hamiltonian system will be assigned a letter as follows: **S** for the *storage port*, **R** for the *energy-dissipating* components. The central component of a Port-Hamiltonian system, the Dirac structure is assigned the letter **D**. A graphical representation of Port-Hamiltonian systems, using the above notations, is given in Figure 1.2.

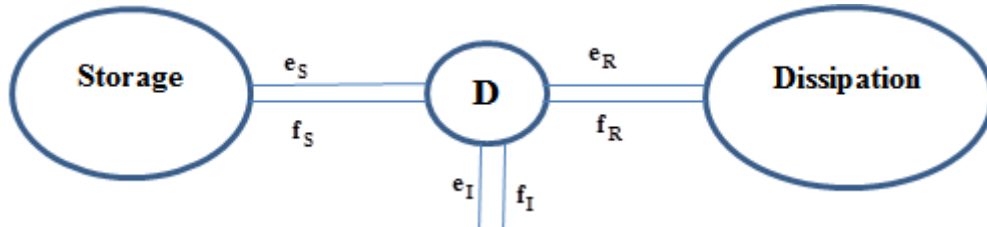


Figure 1.2. Port-Hamiltonian system.

The interconnections within the Port-Hamiltonian system are realized using pairs of *flow and effort variables*, usually denoted  $(f, e)$ . One such pair of vectors of flow and effort variables forms an abstract entity called *port*, while the complete set of flow and effort variables, forms the *port variables*. As the notations show in Figure 1.2, the flow and effort pairs, are associated to the ports as follows:  $(f_S, e_S)$  for the storage power-port;  $(f_R, e_R)$  for the dissipation power-port and  $(f_I, e_I)$  for the external power-port. Associated to each port, there were defined scalar entities, called energies (powers)  $(e_S^T f_S, e_R^T f_R$  and  $e_I^T f_I)$  that are transmitted through the connections [19,23]. As was already presented in Figure 1.2, the central component of a Port-Hamiltonian system, the Dirac structure has an important property of *power conservation*: the Dirac structure connects the various port elements (flows and efforts), in order to conserve the total power associated to the system [19,23]. The power port elements are defined as follows: the flows are defined on a finite-dimensional linear space of flows  $\mathbf{F}$ , while the efforts are defined on the dual linear space  $\mathbf{E} := \mathbf{F}^*$ . The *total space* of flow and effort variables is  $\mathbf{F} \times \mathbf{E}$  and represents the space of *port variables* or *Bond space*. As a consequence, the total power entering (or going out, according to the sign convention) is expressed as [19,23]:

$$P = \langle e | f \rangle, \langle f, e \rangle \in \mathbf{F} \times \mathbf{E} \quad (1.1)$$

where  $\langle e|f \rangle$  corresponds to the *duality product* for  $e \in \mathbf{E}$  acting on  $f \in \mathbf{F}$ . In vector representation, (1.1) becomes:

$$\langle e|f \rangle = e^T f \quad (1.2)$$

where  $f \in R^k$  and  $e \in (R^k)^*$ . In the following lines, will be introduced a definition for a Dirac structure [19,23].

**Definition 1.1 (Dirac structure) [23]**

Consider a finite-dimensional linear space  $F$ , with  $\mathbf{E} = \mathbf{F}^*$ . A subspace  $D \subset \mathbf{F} \times \mathbf{E}$  is a Dirac structure if:

1.  $\langle e|f \rangle = 0$ , for all  $(f, e) \in D$
2.  $\dim D = \dim F$

In the above definition, *property 1* corresponds to *power-conservation* and expresses the fact that the total power entering or (leaving) a Dirac structure is zero. This definition can be generalized to the *infinite-dimensional* linear space  $\mathbf{F}$ , defining in this way a infinite-dimensional Dirac structure. This definition is essential for the class of Port-Hamiltonian systems called *distributed-parameter systems*. In connection to the power of Port-Hamiltonian systems, it was defined a bilinear form  $\langle\langle \cdot, \cdot \rangle\rangle$  on the space  $\mathbf{F} \times \mathbf{E}$ , as:

$$\langle\langle (f^a, e^a), (f^b, e^b) \rangle\rangle := \langle e^a | f^b \rangle + \langle e^b | f^a \rangle \quad (1.3)$$

with  $(f^a, e^a), (f^b, e^b) \in \mathbf{F} \times \mathbf{E}$ . This form is *positive semi definite*. It is also *non-degenerate*, meaning that  $\langle\langle (f^a, e^a), (f^b, e^b) \rangle\rangle = 0$  for all  $(f^b, e^b)$  implies that  $(f^a, e^a) = 0$  [19,23].

Therefore it is symmetric positive-definite and may be used as an inner product. This inner product in turn may be used to give a geometric characterization of Dirac structures, as in the proposition below.

**Proposition 1.1 (geometric characterization of Dirac structures [40,41])**

A Dirac structure on  $\mathbf{F} \times \mathbf{E}$  is a subspace  $D \subset \mathbf{F} \times \mathbf{E}$ , such that:

$$D = D^\perp \quad (1.4)$$

where  $\perp$  denotes the orthogonal companion with respect to the bilinear form  $\langle\langle \cdot, \cdot \rangle\rangle$ .

Some simple examples of Dirac structures  $D \subset \mathbf{F} \times \mathbf{E}$  are given below:

- a) Let  $J : E \rightarrow F$  be a skew-symmetric linear mapping, that is  $J = -J^*$ , where  $J^* : E \rightarrow E^* = F$  is the adjoint mapping. Then:

$$\text{graph } J := \{(f, e) \in \mathbf{F} \times \mathbf{E} \mid f = Je\} \quad (1.5)$$

is a Dirac structure.

- b) Let  $\omega : \mathbf{F} \rightarrow \mathbf{E}$  be a skew-symmetric linear mapping. Then:

$$\text{graph } \omega := \{(f, e) \in \mathbf{F} \times \mathbf{E} \mid e = \omega f\} \quad (1.6)$$

is a Dirac structure.

c) Let  $K \subset F$  be any subspace. Define:

$$K^\perp = \{e \in E \mid \langle e \mid f \rangle = 0, \text{ for all } f \in K\} \quad (1.7)$$

then  $K \times K^\perp \subset F \times E$  is a Dirac structure [19,23].

An important property of Port-Hamiltonian systems, is their stability under interconnection. The *interconnection* of Port-Hamiltonian systems results in a new Port-Hamiltonian system with power continuity, which means that the resulting Hamiltonian will be the sum of the sub-systems Hamiltonians and, similarly, the total dissipation will be the sum of the sub-systems dissipations. As an example, we consider the interconnection of two Dirac structures,  $D_A$  and  $D_B$  as in the following figure.

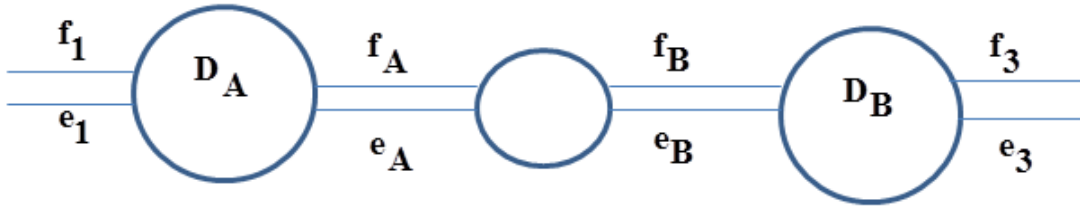


Figure 1.3. The interconnection of two Dirac structures.

In Figure 1.3, two Dirac structures were considered  $D_A \subset F_1 \times F_2 \times E_1 \times E_2$  with  $E_i = F_i^*$  ( $i = 1, 2$ ) and  $D_B \subset F_2 \times F_3 \times E_2 \times E_3$  with  $E_3 = F_3^*$ . In this representation,  $F_2$  is the space of shared flow variables and  $E_2 = F_2^*$  the space of shared effort variables. The interconnection of the two Dirac structures is defined with the following constraints:

$$f_A = -f_B \in F_2, e_A = e_B \in E_2 \quad (1.8)$$

Thus the composition of two Dirac structures, results in:

$$D_A \circ D_B := \left\{ \begin{array}{l} (f_1, e_1, f_3, e_3) \in F_1 \times E_1 \times F_3 \times E_3 \mid \exists (f_2, e_2) \in F_2 \times E_2 \text{ s.t.} \\ (f_1, e_1, f_2, e_2) \in D_A \text{ and } (-f_2, e_2, f_3, e_3) \in D_B \end{array} \right\} \quad (1.9)$$

For the composition of two Dirac structures, the following theorem was proposed. [16,23,45,46]

### Theorem 1.1

Let  $D_A \subset F_1 \times F_2 \times E_1 \times E_2$  and  $D_B \subset F_2 \times F_3 \times E_2 \times E_3$  be Dirac structures.

Then  $D_A \circ D_B \subset F_1 \times F_3 \times E_1 \times E_3$  is a Dirac structure.



Below we will introduce some examples of Dirac structures.

### 1.3.1 Dirac structures examples

One example of Dirac structure, is a transformer [20,21,23], which links two scalar bonds with flow and effort variables  $(f_1, e_1) \in R^2$  and  $(f_2, e_2) \in R^2$  using the following relation:

$$\begin{aligned} f_2 &= \alpha f_1 \\ e_1 &= -\alpha e_2 \end{aligned} \quad (1.10)$$

where  $\alpha$  is a constant, called *transformer ratio*. The generalized vectorial form for (1.10) is:

$$\begin{aligned} f^b &= A f^a \\ e^a &= -A^T e^b \end{aligned} \quad (1.11)$$

with  $(f^a, e^a)$  and  $(f^b, e^b)$  pairs of column vectors for the flow and effort variables and  $A$  an invertible matrix of appropriate dimension. Another example of Dirac structure is a *gyrator*, which can be defined using the following relations:

$$\begin{aligned} f_1 &= \beta e_2 \\ \beta e_1 &= -f_2 \end{aligned} \quad (1.12)$$

For  $\beta = 1$ , the structure from (1.12) is called a *symplectic gyrator*. The multi-dimensional case for (1.12), is given below:

$$\begin{aligned} f^a &= G e^b \\ G^T e^a &= -f^b \end{aligned} \quad (1.13)$$

where  $G$  is an invertible matrix of appropriate dimensions. Other examples of Dirac structures are ideal  $k$ -dimensional interconnection structures such as the  $\mathbf{0}$  junction (common effort or parallel connection in electricity) or the  $\mathbf{1}$  junction (common flow or serial connection in electricity) [20,21,23]. Port-Hamiltonian systems are made of the connection of Dirac structures with energy storing and energy dissipating elements. Those are described in the next to sub-sections.

### 1.3.2 Energy storing elements

The energy-storing multi-port element ( $\mathbf{S}$ ), is introduced as the union of all energy-storing elements within the system. The port variables for this case are represented by  $(f_S, e_S)$ ,

where  $f_S$  and  $e_S$  are vectors of equal dimension, where  $e_S^T f_S$  denotes the total power within the Dirac structure for the energy storing elements. The complete storage energy is defined using a state space  $X$ , together with a Hamiltonian function  $H : X \rightarrow R$  representing the energy [23]. The rate  $\dot{x}$  represents the vector of flow variables for the energy storing multi-port element. For any state  $x \in X$ , the flow vector  $\dot{x}$  is assumed to be an element of the

linear space  $T_X X$ , corresponding to the tangent space of  $X$  at  $x \in X$ . Using local coordinates for the states  $x = (x_1, \dots, x_n)^T$ , the flow variables become  $\dot{x} = (\dot{x}_1, \dots, \dot{x}_n)^T$ . The vector of effort variables is given by the gradient vector  $\partial H / \partial x(x) \in T_X^* X$ , the dual space of the tangent space  $T_X X$ . Using the states coordinates  $x = (x_1, \dots, x_n)^T$  of  $X$ , the effort variables can be vectorially represented using the vector  $\partial H / \partial x(x)$  of partial derivatives of the energy  $H$  with respect to the states. This gives rise to the following *power-balance equation* for the energy-storing multi-port element:

$$\frac{d}{dt} H = \left\langle \frac{\partial H}{\partial x}(x) \middle| \dot{x} \right\rangle = \frac{\partial^T H}{\partial x}(x) \dot{x} \quad (1.14)$$

The interconnection of the energy-storing elements with the storage port  $(f_S, e_S)$  of the Dirac structure, is realized through the notations:

$$f_S = -\dot{x}, \quad e_S = \frac{\partial H}{\partial x}(x) \quad (1.15)$$

Thus, the *power balance* for the storage port is:

$$\frac{d}{dt} H = \frac{\partial^T H}{\partial x}(x) \dot{x} = -e_S^T f_S \quad (1.16)$$

The minus sign convention, was introduced due to the fact that  $\frac{\partial^T H}{\partial x}(x) \dot{x}$  is the power flowing into the energy-storing elements and  $e_S^T f_S$  the power flowing into the Dirac structure.

### 1.3.3 Energy dissipating elements

As was already introduced above, the second multi-port element  $\mathbf{R}$ , is related to internal energy dissipation caused by friction, resistance or others. For this port, the following port variables  $(f_R, e_R)$  are introduced together with a static relation  $\mathbf{R}$ . This relation (or its graph) satisfies:

$$R \subset F_R \times E_R \quad (1.17)$$

and:

$$\left\langle e_R \middle| f_R \right\rangle = e_R^T f_R \leq 0 \quad (1.18)$$

for all  $(f_R, e_R) \in R$ . The static relation  $\mathbf{R}$  is called an *energy-dissipating relation* or *resistive structure*. Without any power supplied to the external port, the power balance relation for the whole system becomes:

$$e_S^T f_S + e_R^T f_R = 0 \quad (1.19)$$

Using (1.16) and (1.19), we can write:

$$\frac{d}{dt} H = -e_S^T f_S = e_R^T f_R \leq 0 \quad (1.20)$$

A special case for the energy-dissipating relations, appears when the resistive relation can be expressed using an *input-output* mapping:

$$f_R = -F(e_R) \quad (1.21)$$

where  $F : R^m \rightarrow R^m$  satisfies  $e_R^T F(e_R) \leq 0$  for all  $e_R \in R^m$  ( $m$  the number of energy dissipating elements). In case of linear resistive elements, (1.21) can be expressed as:

$$f_R = -R e_R \quad (1.22)$$

with a positive semi-definite symmetric matrix  $R = R^T \geq 0$ .

### 1.3.4 External ports

An external port  $(f_I, e_I)$  can be introduced within a Dirac structure interconnection, for the interaction of the system with the environment. This interaction of the system with its environment can be realized through a *controller* or an *interaction port*. A possible example of an interaction port, is given by a controlled robotic system interacting with its environment. Another example of external ports is represented by sources, in electrical circuits where the input can be the voltage while the current through the source is then the output variable.

$$e_S^T f_S + e_R^T f_R + e_I^T f_I = 0 \quad (1.23)$$

Then (1.20) becomes:

$$\frac{d}{dt} H = e_R^T f_R + e_I^T f_I \leq e_I^T f_I \quad (1.24)$$

due to  $e_R^T f_R \leq 0$ . This relation can be translated into the fact that the increase of the internal energy or the Hamiltonian, has as an upper bound which is the externally supplied power.

### 1.3.5 Port-Hamiltonian dynamics

The dynamic behaviour of a Port-Hamiltonian system, can be introduced using the following definition.

**Definition 1.3** (*implicit definition of Port-Hamiltonian systems*)

Consider a state space  $x$  and a Hamiltonian:

$$H : X \rightarrow R \quad (1.25)$$

defining the energy-storage. A Port-Hamiltonian system on  $X$  is defined by a Dirac structure:

$$D \subset T_X \times X^T \times X^F \times R^E \times R^F \times I^E \times I^F$$

having energy-storing port  $(f_S, e_S) \in T_X \times X^T$  and a resistive structure:

$$R \subset F_R \times R^E$$

corresponding to an energy-dissipating port  $(f_R, e_R) \in F_R \times R^E$ . Its dynamics is specified by:

$$\left( \begin{array}{l} -\dot{x}(t), \quad \frac{\partial H}{\partial x}(x(t)) \quad f_R(t) \quad e_R(t) \quad f_I(t) \quad e_I(t) \end{array} \right) \in D(x(t)), \quad (1.26)$$

$$(f_R(t), e_R(t)) \in R(x(t)), t \in R$$

### 1.3.6 Input-state-output Port-Hamiltonian systems

A particular class of Port-Hamiltonian systems, is the one of the *input-state-output port-Hamiltonian systems*, for which the dynamics is explicitly described by explicit state-space equations and where the flow and effort variables describing the resistive, control or interaction ports are splitted into conjugated input-output pairs. This class of dynamic systems, can be defined using the following system of equations:

$$\Sigma : \left\{ \begin{array}{l} \dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u + k(x)d \\ y = g^T(x) \frac{\partial H}{\partial x}(x) \\ z = k^T(x) \frac{\partial H}{\partial x}(x) \end{array} \right. , x \in X \quad (1.27)$$

where  $(u, y)$  are the input-output pairs related to the control port  $C$ , while  $(d, z)$  corresponds to the input-output pairs of the interaction port  $I$ . In (1.27)  $y^T u$  and  $z^T d$  correspond to the power energies of respectively the control and interaction ports. Here the matrix  $J(x)$  (interconnection matrix), is skew-symmetric and the dissipation matrix  $R(x) = R(x)^T \geq 0$  positive semi-definite. For a linear resistive relation  $f_R = -\tilde{R}e_R$  with  $\tilde{R} = \tilde{R}^T \geq 0$  and  $g_R$  an input matrix associated to the dissipation port, a Dirac interconnection structure associated to the explicit input-state-output form may be defined as:

$$\left[ \begin{array}{cccc} -J(x) & -g_R(x) & -g(x) & -k(x) \\ g_R^T(x) & 0 & 0 & 0 \\ g^T(x) & 0 & 0 & 0 \\ k^T(x) & 0 & 0 & 0 \end{array} \right] \quad (1.28)$$

This structure represents a *modulated Dirac structure* where the matrices  $J, g_R, g$  and  $k$  depend on the energy variables  $x$ .

### 1.3.7 Port-Hamiltonian systems examples

This section introduces some examples of Port-Hamiltonian systems, for the lossless or lossy cases, which will also serve as running examples in the thesis.

#### 1.3.7.1 Lossless Port-Hamiltonian systems

A first example of a Port-Hamiltonian system, is represented by an LC circuit, as depicted in Figure 1.4.

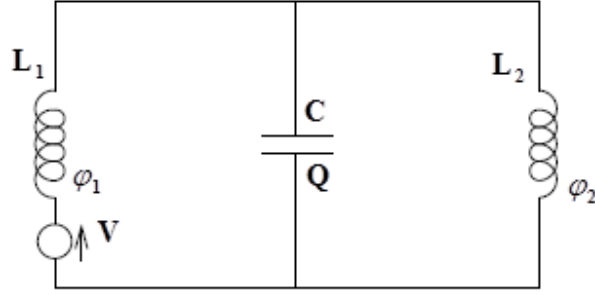


Figure 1.4. Controlled LC circuit.

This example considers an LC-circuit with two inductors of magnetic energies  $H_1(\varphi_1)$  and  $H_2(\varphi_2)$  ( $\varphi_1$  and  $\varphi_2$  being the magnetic flux linkages) and a capacitor with electric energy  $H_3(Q)$  ( $Q$  being the charge). In the linear case, the following relations are used:

$$H_1(\varphi_1) = \frac{1}{2L_1}\varphi_1^2, \quad H_2(\varphi_2) = \frac{1}{2L_2}\varphi_2^2, \quad H_3(Q) = \frac{1}{2C}Q^2 \quad (1.29)$$

Considering a voltage source  $v = u$  and applying the Kirchoff's laws, the following input-state-output port-Hamintonian representation for the considered system results:

$$\begin{bmatrix} \dot{Q} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial \varphi_1} \\ \frac{\partial H}{\partial \varphi_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \quad (1.30)$$

$$y = \frac{\partial H}{\partial \varphi_1} \quad (\text{current through the first inductor})$$

The total energy of the system is:

$$H(Q, \varphi_1, \varphi_2) = H_1(\varphi_1) + H_2(\varphi_2) + H_3(Q) \quad (1.31)$$

### 1.3.7.2 Lossy Port-Hamiltonian systems

For the lossy case, a capacitor microphone circuit is considered as in Figure 1.5.

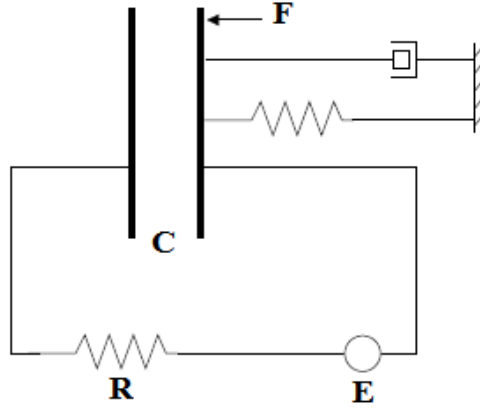


Figure 1.5. Capacitor microphone.

In this case, the capacitance  $C(q)$  has a nonlinear model as it is dependent on the displacement  $q$  of the right plate (mass  $m$ ). Attached to the right plate is a spring (with spring constant  $k$ ) and a damper (with constant  $c > 0$ ) and also a mechanical force  $F$  is considered for the air pressure due to the sound. The dynamical equations for the displacement of this system are:

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{Q} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & \frac{1}{R} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial Q} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} F + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{R} \end{bmatrix} E \quad (1.32)$$

$$y_1 = \frac{\partial H}{\partial p} = \dot{q}$$

$$y_2 = \frac{1}{R} \frac{\partial H}{\partial Q} = I$$

In (1.32),  $p$  denotes the momentum,  $R$  the resistance of the resistor and  $I$  the current through the voltage source. The total energy of the system, representing the Hamiltonian is given by the following relation:

$$H(q, p, Q) = \frac{1}{2m} p^2 + \frac{1}{2} k(q - \bar{q})^2 + \frac{1}{2C(q)} Q^2 \quad (1.33)$$

where  $\bar{q}$  is associated to the equilibrium position of the spring. Here it should be remarked that  $F\dot{q}$  is the mechanical power, while  $EI$  the electrical power of the system. For the measurement of the mechanical force  $F$ , the voltage over the resistor will be used.

Another example of lossy PCH system is represented by a DC motor, as depicted in Figure 1.6, where six interconnected sub-systems may be distinguished: two energy-storing elements: an ideal inductor  $L$  with state energy variable  $\varphi$  (flux-linkage) and a rotational inertia  $J_E$  with state energy variable  $p$  (angular momentum); two energy-dissipating

elements: the electrical resistor  $R$  and the viscous mechanical friction  $b$  ; a gyrator  $K$  ; an ideal voltage source  $V$  .

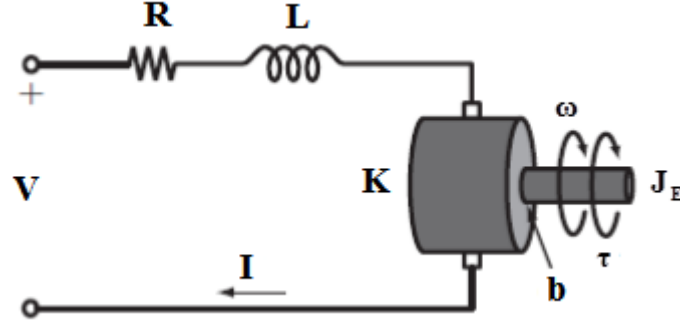


Figure 1.6. DC motor.

The Port-Hamiltonian model for the interconnection of these six sub-systems is is:

$$\begin{bmatrix} \dot{\varphi} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -R & -K \\ K & -b \end{bmatrix} \begin{bmatrix} \frac{\varphi}{L} \\ \frac{p}{J_E} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V \quad (1.34)$$

$$I = \begin{bmatrix} \frac{\varphi}{L} \\ \frac{p}{J_E} \end{bmatrix}$$

and the corresponding total Hamiltonian function is:

$$H(p, \varphi) = \frac{1}{2L} \varphi^2 + \frac{1}{2J_E} p^2 \quad (1.35)$$

## 1.4 Motivations and necessity for a Port-Controlled-Hamiltonian systems identification theory

This subsection, puts in discussion the usefulness and necessity of a *system identification theory* for Port-Controlled-Hamiltonian systems, starting from the theoretical concepts introduced in this chapter. The motivations for a specific *identification theory* of Port-Controlled-Hamiltonian systems comes from their remarkable property of *power energy conserving* for the complete system, which connects the storage, control or interaction ports, through a Dirac structure. Compared with the classic input/output formulation for the structural identifiability analysis, the identifiability of Port-Controlled-Hamiltonian systems can be analysed (as will be proved in chapter 2), also in terms of storage, dissipation or control power energies, possibly associated to sub-systems from different domains of engineering (mechanical, electrical, thermal, etc.). Thus it is possible, by having real-time informations on the power energies, to determine the dynamic system parameters, given in state-space form. Using this power energy approach, a new class of identification algorithms can be proposed in the future, besides the classic known state-space identification techniques.

Another important issue of an *identification theory* for PCH systems, is their utility and advantages for model-based control. In this application, the control model needs to be

representative and suitable enough for the considered dynamical systems. This is the case of PCH models in many real applications, where these models include enough “physical” information for the control design, but still are of tractable complexity. As it will be shown in the thesis, PCH systems can be easily adapted, to be used for structural or practical identifiability analysis, using a broad class of techniques, some being adapted for real-time applications. A perturbation model can be proposed for these systems (see chapter 3), by introducing an interaction port of the system with the environment, preserving in this way the general model structure. It might also be possible to propose in the future, efficient open-loop or closed-loop identification algorithms specifically dedicated to PCH systems.

Symplecticity (i.e. conservation of the power-pairing form) of PCH systems allows to identify their parameters using identification techniques in discrete-time form, provided symplectic discretization schemes are used. For instance, no numerical dissipation (due to the sampling/discretization) will affect the value of the identified parameters for the dissipation relation. It will be shown in chapter 4 how continuous-time PCH systems may be converted to discrete-time form, using a broad class of discretization schemes for the flows and efforts spaces. A *discretization error Hamiltonian* can be introduced between the continuous and discrete-time systems, preserving the properties and structure of the system, which may be used to get structured information on the power error energies. Due to their remarkable structure, identification procedures and algorithms can be developed for PCH systems, to identify only the parameters with a physical meaning.

## 1.5 Thesis objectives

The main goal of this thesis is to check the possibility to develop a specific *identification theory* for PCH systems. Thus the main objectives proposed for the thesis are: the structural and practical identifiability analysis of PCH systems using classic or new (specific) proposed techniques; the selection of proper symplectic time-discretization techniques which combine the flow and effort discretization and conserve the main characteristic properties of PCH systems; the introduction of a *discretization error Hamiltonian* between the continuous and discrete-time PCH systems for the error analysis; the description of discrete-time PCH systems using matrix I/O equations, orthogonal or oblique projections for the selected symplectic discretization schemes, suitable for state-space identification algorithms; the selection, analysis and development of state-space identification algorithms for PCH systems in the deterministic or stochastic cases.

## 1.6 Summary of the thesis

The introduction was dedicated to a general presentation of *system identification* and *Port-Controlled-Hamiltonian systems theories*, as well as for the motivations to develop an *identification theory* for PCH systems. It ends with the main objectives of the thesis and a summary on chapters.

In *chapter 2*, a structural identifiability analysis of PCH systems is proposed using several classical or new (specific) techniques. As classic techniques were selected: the observability/controllability analysis, the direct test or the power series expansion. A new



proposed identifiability analysis is realized using the power energies corresponding to PCH systems ports and thus, new specific definitions and propositions are proposed for the global and local identifiability. Also a new concept of *port identifiability* is introduced, specific to the unknown parameters associated to one port of a PCH system. For each identifiability analysis technique, examples are proposed for the lossless and lossy cases using the LC circuit, the capacitor microphone circuit or the DC motor models. A complete set of proofs for the results of chapter 2 may be found in Appendix A to C.

In *chapter 3*, a practical identifiability analysis of PCH systems is proposed using the observability/controllability concepts, in the presence of a proposed perturbation model. As the system perturbation is caused by the interaction of the PCH system with the environment, it is introduced a perturbed model, by an input/output pair corresponding to the interaction port. Thus the states and the system outputs have a perturbed model, as for classic state-space systems. In this case, it is proved the conservation of basic PCH systems laws, when converting to the observable canonical form for identifiability analysis in the presence of a perturbation model. Again, the LC circuit, capacitor microphone circuit and DC motor examples are used as examples to illustrate the practical identifiability analysis.

*Chapter 4* introduces a symplectic time-discretization framework for PCH systems, using different discretization rules for the flows and efforts. As discretization rules for the flows and efforts of PCH systems were selected the implicit/explicit Euler rule, the implicit mid-point rule or the implicit trapezoidal rule. The second part of the chapter introduces the difference between two PCH systems and its application to the definition of a *discretization error Hamiltonian* system between the continuous and discrete-time systems. The LC circuit and the capacitor microphone circuit are considered for the lossless or lossy case as examples. In appendix E can be found a complete set of symplectic discretization schemes using the selected discretization rules for the flows and efforts, which are proved to conserve (or not) the basic PCH systems properties in discrete-time.

*Chapter 5*, makes an analysis of the subspace identification algorithms for PCH systems using the time-discretization schemes of chapter 4. Thus matrix I/O representations, orthogonal or oblique projections are proved and defined for the selected schemes. For the analysis using the identification algorithms, it is selected a simple scheme that uses Euler rules for the flows and efforts and gives a similar state-space structure as the classic one. First the analysis is realized using classic state-space algorithms and then a new context is proposed for a *power energy* approach, that makes use of the inputs/power energies knowledge. The system identification analysis is performed using a DC motor model, to illustrate the results. Thus it is proved that performing the estimation algorithms, equivalent PCH systems can be found, which preserve the PCH systems structure.

*Chapter 6* draws the main conclusions that result at the end of the thesis on each subject proposed for study throughout the thesis. Also are introduced the personal contributions and perspectives for further work on identification of Port-Controlled-Hamiltonian systems.

## 2. Structural Identifiability of Port-Controlled-Hamiltonian Systems

*This chapter presents a theoretical and practical analysis of Port-Controlled-Hamiltonian systems structural identifiability, using general concepts from classic identification theory and also introducing a new approach making use of the power-port energies. Classical observability/controllability or direct tests for structural identifiability are investigated. For each of these tests, a general rule is found to be used for Port-Controlled-Hamiltonian systems. Using the power energies associated to the ports, new definitions are introduced for the global/local identifiability of Port-Controlled-Hamiltonian systems and also a new concept of **port-identifiability** is proposed. Three examples are considered: a LC circuit for the lossless case and a capacitor microphone circuit and DC motor for the lossy case.*

## 2.1. Introduction

The system identification field uses statistical methods to build mathematical models for dynamic systems on the basis of measured datas. Among the main issues in system identification are the optimal design of experiments, necessary to generate efficient informative datas for fitting such models as well as model reduction [1,4,44].

Related to the field of system identification, the following notions have been introduced in the literature: *theoretical* or *structural identifiability*, *practical identifiability* (being used when experimental datas are perturbed with noise) or *sensitivity analysis*, which is used in mathematical modelling, to evaluate the sensitivity of the output variables, to parameter values and input variables [45]. This chapter makes a theoretical and practical analysis of Port-Controlled-Hamiltonian systems structural identifiability, which form a class of nonlinear systems very useful for multiphysics systems modelling and control problems [23]. The concept of structural identifiability was first proposed by Bellman and Astrom [46], where identifiability analysis is performed by exploring the model structure and verifying whether the system parameters have a unique solution, globally or locally, using the knowledge of the input-output signals (see [46,47,48, 49] for application-oriented papers or [50,51,52,53,54] for theoretical contributions on structural identifiability). Some of the main structural identifiability techniques for linear systems are: the *Laplace transform* [45, 46, 55, 56], the *power series expansion* proposed by Pohjapalo [45,46,55, 56,57] or the *similarity transformation*, proposed by Walter and Lecourtier [45, 55, 56, 58]. Another useful approach for structural identifiability is the *direct test*. This test, represents a direct analysis of the parameters identifiability, using either the global or local identifiability definitions, analytically [45,59] or numerically [45, 60]. Although PCH systems are basically knowledge based models which may be described using only a limited number of physical parameters, quite surprisingly enough very few studies investigate issues related to their identifiability, even the structural identifiability analysis. It should be mentioned the exception of [61] where the very particular application case of a heat exchanger is studied. The aim of this chapter, is therefore to explore the structural identifiability of linear PCH systems. In *section 2.2*, the identifiability definitions which will be used for the structural identifiability analysis are reiterated. *Section 2.3* proposes a transformation of linear PCH systems to the observable canonical form. In *section 2.4* the structural identifiability of PCH systems is analyzed, using either the observability/controllability concepts or the direct test. *Section 2.5* proposes a new approach for the structural identifiability analysis of PCH systems, by introducing a new set of definitions for the global/local identifiability, using the powers associated to each port, together with a new concept of '*port-identifiability*'. *Section 2.6* is dedicated to examples for the indentifiability characterizations developed in sections 2.3, 2.4 and 2.5 using an LC circuit for the lossless case or a a capacitor microphone circuit or a DC motor, for the lossy case.

## 2.2 Definitions

Let us consider the dynamical system:

$$\dot{x}(t) = g(t, x(t), u(t), \theta) \quad (2.1)$$

$$y(t) = h(x(t), u(t), \theta) \quad (2.2)$$

where  $x(t) \in R^m$  is a vector of state-variables,  $y(t) \in R^d$  represents the measurement or output vector,  $u(t) \in R^p$  the known system input vector and  $\theta \in R^q$  the parameter vector [1,4, 45].

### **Definition 2.1: Controllability** [45,62]

For the dynamical system(2.1) - (2.2), a (nonzero) state  $\bar{x} \in \mathbf{X}$  is controllable to the zero state, if there exists an input function  $\bar{u}(t)$  and a time  $\bar{T} < \infty$ , such that  $\Phi(\bar{u}, \bar{x}, \bar{T}) = 0$  (where  $\Phi(\bar{u}, \bar{x}, \bar{T})$  is the trajectory with initial condition  $x(0) = \bar{x}$  and input  $u(t), t \in [0, \bar{T}]$ ).

### **Definition 2.2: Observability** [45,62]

A state  $\bar{x} \in \mathbf{X}$  is unobservable if  $y(t) = \mathbf{C}\Phi(0, \bar{x}, t) = 0$  for all  $t \geq 0$ , i.e., if  $\bar{x}$  is undistinguishable from the zero state for all  $t \geq 0$ . The unobservable subspace  $\mathbf{x}^{unobs}$  of  $\mathbf{X}$ , is the set of all unobservable states of the dynamical system (2.1) - (2.2). The system (2.1) - (2.2) is completely observable if  $\mathbf{x}^{unobs} = \{0\}$ .

The controllability and observability definitions, represent two basic concepts in system identification, due to the connections realized between the inputs, states and outputs of a dynamic system represented in state-space form [1,4,45,62] and also, may indicate when the system is identifiable. For example, the observable canonical (minimal) form of a system, is identifiable when it satisfies also the controllability condition. This however doesn't give any information on the identifiability of the parameters from the original  $A, B, C$  realization (which may be non-minimal, for instance). The observability/controllability will be considered as necessary conditions for (global or local) structural identifiability as defined hereafter. In 2.4 the observability and controllability will be used for the identifiability analysis of PCH systems written in a minimal (observable canonical) form. Next are defined two central concepts of *global* and *local identifiability* introduced by Glad and Ljung [1,4,45,62].

### **Definition 2.3: Global Identifiability**

The dynamical system (2.1) - (2.2) is said to be globally identifiable, if for any admissible input  $u(t)$  and any two sets of parameter vectors  $\theta_1$  and  $\theta_2$ , from the parameter space  $\Theta$ , the following equality  $y(u, \theta_1) = y(u, \theta_2)$  holds if and only if  $\theta_1 = \theta_2$ .

### **Definition 2.4: Local identifiability**

A dynamic system defined by (2.1) - (2.2) is said to be locally identifiable, if for any  $\theta$  within an open neighborhood of some point  $\theta^*$  in the parameter space  $\Theta$ , the following relation  $y(u, \theta_1) = y(u, \theta_2)$  holds if and only if  $\theta_1 = \theta_2$ .

The above two definitions are useful in section 2.4, 2.5 and 2.6 for the direct test identifiability analysis of PCH systems parameters and for the introduction of new definitions using the power port energies and also for identifiability examples. Another concept in identification is given assuming an initial known state, or *local strong identifiability*, also termed as  $x_0$  *identifiability* [45,70,71].

### **Definition 2.5: Local strong identifiability ( $x_0$ identifiability)**

For an admissible input  $u(t)$  in the time range of interest  $[t_0, t_1]$  and for a given initial state  $x_0 = x(t_0)$ , which is independent of  $\theta$  and not an equilibrium point, if there exist an open set  $\Theta^0$  within the parameter space  $\Theta$  such that, for any two different parameter vectors  $\theta_1, \theta_2 \in \Theta^0$ , the solutions  $x(t, \theta, u)$  exist on  $[t_0, t_0 + e]$  ( $t_0 < e \leq t_1 - t_0$ ) for both  $\theta_1$  and  $\theta_2$ , and  $y(t, \theta_1, x_0, u(t)) \neq y(t, \theta_2, x_0, u(t))$  on  $[t_0, t_0 + e]$ , the system structure is said to be locally strong identifiable (or  $x_0$  identifiable).

This definition is introduced, as it is related to the power series expansion identifiability test of PCH systems presented in section 2.4.3. In some sense it is a local identifiability analysis, which is local both for the states space and time, being in that sense very similar to the related concept of local observability for non-linear systems. Other definitions for identifiability have been introduced for practical or theoretical reasons. Some of them may be found in [46, 64, 65, 66].

## **2.3. Observable canonical form representation of PCH systems**

This section, presents a proof for the conversion of PCH systems to observable canonical forms, conserving the basic properties of PCH systems, by which they are defined. This result will be used in section 2.4 to prove structural identifiability result for PCH systems.

**Remark:** *The observable canonical form representation can be applied only for SISO PCH systems.*

In the linear time-invariant lossless case, a PCH system reduces to:

$$\begin{cases} \dot{x}(t) = JQx(t) + Bu(t) \\ y(t) = B^T Qx(t) \end{cases} \quad (2.3)$$

where  $(u, y)$  is the input-output pair corresponding to the control port,  $J$  is a skew-symmetric interconnection matrix and the Hamiltonian(energy), is  $H = x^T Q x / 2$  where  $Q$  is assumed to be symmetric positive definite. The system (2.3) may be written in the usual state-space form:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad (2.4)$$

with  $A = JQ \in R^{n \times n}$ ,  $B \in R^{n \times m}$  and  $C = B^T Q \in R^{p \times n}$ . In order to express this system in the observable canonical form, a transformation matrix is applied:

$$x_{obs} = T_{obs} x \quad (2.5)$$

where:

$$T_{obs}^T = \begin{bmatrix} C^T & (CA)^T & \dots & (CA^{n-1})^T \end{bmatrix}$$

is the usual observability matrix. The obtained observable canonical form reads then:

$$\begin{cases} \dot{x}_{obs} = T_{obs} J Q T_{obs}^{-1} x_{obs} + T_{obs} B u \\ y = C T_{obs}^{-1} x_{obs} \end{cases} \quad (2.6)$$

with:

$$T_{obs} J Q T_{obs}^{-1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix}, T_{obs} B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \dots \\ b_n \end{bmatrix}, \quad (2.7)$$

$$C T_{obs}^{-1} = [1 \quad 0 \quad 0 \quad 0 \quad 0]$$

To prove that (2.6) still has the PCH form, it is sufficient to introduce an equivalent form:

$$\begin{cases} \dot{x}_{obs} = J_{obs} Q_{obs} x_{obs} + B_{obs} u \\ y = C_{obs} x_{obs} \end{cases} \quad (2.8)$$

with:

$$\begin{cases} T_{obs} J Q T_{obs}^{-1} = J_{obs} Q_{obs} \\ C T_{obs}^{-1} = C_{obs} \\ T_{obs} B = B_{obs} \end{cases} \quad (2.9)$$

Using standard matrix computations and properties of the the matrices applied to the original system (2.3), one then obtains the following results: (see appendix A.1 for details)

$$\begin{cases} J_{obs} = -J_{obs}^T = T_{obs} J T_{obs}^T \\ Q_{obs} = Q_{obs}^T = T_{obs}^{-T} Q T_{obs}^{-1} \\ C_{obs} = B_{obs}^T Q_{obs} = B^T Q T_{obs}^{-1} \end{cases} \quad (2.10)$$

which prove that (2.8), still has the usual explicit PCH form. Applying the same ideas for the lossy case:

$$\begin{cases} \dot{x} = (J - R)Qx + Bu \\ y = B^T Qx \end{cases} \quad (2.11)$$

where  $R$  is some positive semi-definite ( $R = R^T \geq 0$ ) dissipation matrix, results in the observable canonical form:

$$\begin{cases} \dot{x}_{obs} = T_{obs} (J - R)Q_{obs}^{-1} x_{obs} + T_{obs} Bu \\ y = CT_{obs}^{-1} x_{obs} \end{cases} \quad (2.12)$$

Equivalently this may be written:

$$\begin{cases} \dot{x}_{obs} = (J_{obs} - R_{obs})Q_{obs} x_{obs} + B_{obs} u \\ y = C_{obs} x_{obs} \end{cases} \quad (2.13)$$

with:

$$\begin{cases} T_{obs} (J - R)Q_{obs}^{-1} = (J_{obs} - R_{obs})Q_{obs} \\ CT_{obs}^{-1} = C_{obs} \\ B_{obs} = T_{obs} B \end{cases} \quad (2.14)$$

Using standard matrix computations together with the properties of the  $J$ ,  $R$  and  $Q$  matrices, one gets:

$$\begin{cases} Q_{obs} = Q_{obs}^T = T_{obs}^{-T} Q_{obs}^{-1} \geq 0 \\ J_{obs} = -J_{obs}^T = T_{obs} J_{obs}^T \\ R_{obs} = R_{obs}^T = T_{obs} R_{obs}^T \geq 0 \end{cases} \quad (2.15)$$

This allows to conclude that (2.13) has the usual explicit PCH form (see Appendix A.2 for detailed computations).

## 2.4 Structural identifiability of PCH systems

This section presents an analysis of PCH systems structural identifiability, using either the controllability/observability definitions, direct test or the power series expansion. For each proposed identifiability test, both the lossless and the lossy cases are considered. As was also presented in section 2.2, two basic conditions which must be satisfied for a state-space system to be identifiable (using a minimal realization) are the observability and controllability of the system. As a consequence the direct test or power series expansion will be applied to prove respectively the identifiability, assuming first that the system is observable and controllable. Omitting this necessary conditions for the identifiability tests, situations may be encountered when the parameters are identifiable according to the selected test, but the system being unobservable or uncontrollable, the transfer function degree reduces by simplifications and some parameters of the original ( $A, B, C$ ) state-space realization might not be analyzed [1,67]. All identifiability tests analyzed in this work may be considered as roughly equivalent, since they all search for a unique solution of the unknown parameters. However the observability/controllability identifiability test, the direct test or the power port test are related

with the global and local identifiability definitions, while the power series expansion test is related to the local strong identifiability (see section 2.2).

#### 2.4.1 Observability and controllability analysis

This part of section 2.4 proves the structural identifiability of PCH systems in the lossless and lossy cases, using the observability/controllability concepts introduced in section 2.2, for a minimal realization representation of the dynamic system. For the observable canonical form representation of PCH systems (see section 2.3), it is only necessary to prove that:

$$R_{ID} = \begin{bmatrix} B_{obs} & A_{obs} B_{obs} & \dots & \dots & A_{obs}^{n-2} B_{obs} & A_{obs}^{n-1} B_{obs} \end{bmatrix} \quad (2.16)$$

is full rank, which means that the pair  $(A_{obs}, B_{obs})$  is controllable. Using basic properties of matrix multiplication, transposition, inverse, together with PCH systems properties, it may be found that (see appendix B.1.1, formulas (B.1)-(B.21), for details):

$$R_{ID} = Q \begin{bmatrix} B & -AB & \dots & (-1)^{n-1} (A^{n-1} B) \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{n-1} B \end{bmatrix} \quad (2.17)$$

Then, the following proposition from [69] may be used:

#### **Proposition 2.1:**

*Consider the linear lossless Port-Controlled-Hamiltonian system from (2.3)*

- a) *If (2.3) is observable, then  $\det Q \neq 0$  and (2.3) is controllable.*
- b) *Assume  $\det Q \neq 0$ , then (2.3) is observable if (2.3) is controllable.*

Therefore a minimal condition, for the identifiability of linear lossless PCH systems, is that the dynamic system (2.3) is observable. In the lossy case, the structural identifiability matrix has the form:

$$R_{ID} = \begin{bmatrix} T_{obs} B & T_{obs} (J - R) Q T_{obs}^{-1} T_{obs} B & \dots & \left( T_{obs} (J - R) Q T_{obs}^{-1} \right)^{n-1} T_{obs} B \end{bmatrix} \quad (2.18)$$

where  $T_{obs}$  is the usual observability matrix, given in the case of lossy PCH systems as (see appendix B.1.2, equations B.22-42, for details):

$$T_{obs} = Q \begin{bmatrix} (-1)^0 B & (-1)^1 (A^n B) & \dots & (-1)^{n-1} (A^n)^{n-1} B \end{bmatrix} \quad (2.19)$$

with:

$$A^n \stackrel{def}{=} (J + R) Q \quad (2.20)$$

The following generalization of proposition 2.1 may be proved (see appendix B.1.2) also in the lossy case.



**Proposition 2.2 [193]:**

Consider the linear lossy PCH system (2.11), defining  $A'' = (J + R)Q$  :

- a) If (2.11) is observable, then  $\det Q \neq 0$  and the pair  $(A'', B)$  is controllable;
- b) Assume  $\det Q \neq 0$ , then (2.11) is observable if  $(A'', B)$  is controllable.

Using the result from (2.18) and (2.19), the identifiability matrix, can be also expressed as follows (see (B.43-49) in Appendix B.1.2):

$$R_{ID} = Q \begin{bmatrix} (-1)^0 B & (-1)^1 (A''B) & \dots & (-1)^{n-1} (A''^{n-1} B) \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{n-1} B \end{bmatrix} \quad (2.21)$$

Therefore, it may be concluded, for instance, that a lossy PCH system is identifiable if it is observable (then according to proposition 2.2,  $\det Q \neq 0$  and the pair  $(A'', B)$  is controllable) and controllable (i.e. the pair  $(A, B) = ((J - R)Q, B)$  is controllable).

**2.4.2 Direct identifiability**

The direct structural identifiability test, relies on the global or local identifiability definitions (see section 2.2) which states that the output equality:

$$y(u, \theta_1) = y(u, \theta_2) \quad (2.22)$$

holds if and only if  $\theta_1 = \theta_2$  (where  $\theta_1, \theta_2$  are parameter vectors from  $\Theta$  ). A linear lossless PCH system may be represented either using the unstructured (usual) state-space realization:

$$\begin{cases} \dot{x} = A(\theta_{HA})x + B(\theta_{HB})u \\ y = C(\theta_{HC})x \end{cases} \quad (2.23)$$

where  $\theta_{HA}, \theta_{HB}, \theta_{HC} \in \Theta$  represent the unknown parameters, included in the  $A, B, C$  matrices, or using the structured realization:

$$\begin{cases} \dot{x} = (JQ)(\theta_{HA})x + B(\theta_{HB})u \\ y = (B^T Q)(\theta_{HC})x \end{cases} \quad (2.24)$$

where  $J = -J^T$  is a skew-symmetric matrix and  $Q = Q^T > 0$  is a positive definite matrix. In the case of the structured realization for PCH systems, the interconnection matrix  $J = -J^T$  is generally known. It reflects the interconnection relations (such as Kirchoff's laws in electrical circuits or balance equations in the general case). It is very often built only with elements in  $\{-1, 0, +1\}$  (or any other non-parametric finite set of numbers), even in the hybrid (switching) case where  $J = J^T(t)$ . Therefore, the  $\theta_{HA}, \theta_{HB}, \theta_{HC} \in \Theta$  parameters in the state-space form (2.23) are usually the elements from the  $Q$  (and possibly  $B$ ) matrix. Parameters in the  $Q$  matrix typically characterize the constitutive energy storing equations. Using the direct identifiability test applied for the PCH systems (2.23), we can write the following relation:

$$f(x, u, \theta_1) = f(x, u, \theta_2) \quad (2.25)$$

where:

$$f \stackrel{def}{=} \begin{pmatrix} \dot{x} \\ y \end{pmatrix}$$

and  $\theta_1, \theta_2 \in \Theta$  represent two set of unknown parameters corresponding to two distincts sets of  $A, B, C$  matrices which characterize the PCH system. It is easily seen that the relations:

$$\begin{cases} \theta_{HA 1} = \theta_{HA 2} \\ \theta_{HB 1} = \theta_{HB 2} \\ \theta_{HC 1} = \theta_{HC 2} \end{cases} \quad (2.26)$$

need to be satisfied to obtain structurally identifiable parameters  $\theta_{HA}, \theta_{HB}, \theta_{HC} \in \Theta$  (see Appendix B.2.1, for detailed computations). In (2.26)  $\theta_{HA 1}, \theta_{HB 1}, \theta_{HC 1} \in \Theta$  and  $\theta_{HA 2}, \theta_{HB 2}, \theta_{HC 2} \in \Theta$  represent two sets of unknown parameters included in the state-space matrices  $A, B, C$  of a lossless PCH system. With the assumptions that  $J = -J^T$  is skew-symmetric and  $Q = Q^T > 0$ , the lossless system (2.24) is thus necessarily structurally identifiable for this test. In the lossy case, a supplementary dissipation matrix  $R = R^T \geq 0$  arises from the dissipative constitutive equations. In this case (2.23) becomes:

$$\begin{cases} \dot{x} = ((J - R)Q)(\theta_{HA})x + B(\theta_{HB})u \\ y = (B^T Q)(\theta_{HC}) \end{cases} \quad (2.27)$$

and similar computations (see Appendix B.2.2), allow to conclude that the following relations need to be satisfied:

$$\begin{cases} \theta_{HA 1} = \theta_{HA 2} \\ \theta_{HB 1} = \theta_{HB 2} \\ \theta_{HC 1} = \theta_{HC 2} \end{cases} \quad (2.28)$$

to prove the structural identifiability of the lossy PCH system (2.27) parameters  $\theta_{HA}, \theta_{HB}, \theta_{HC} \in \Theta$ . In (2.28)  $\theta_{HA 1}, \theta_{HB 1}, \theta_{HC 1} \in \Theta$  and  $\theta_{HA 2}, \theta_{HB 2}, \theta_{HC 2} \in \Theta$  represent two sets of unknown parameters included in the state-space matrices  $A, B, C$ , which define a lossy PCH system, for which the direct identifiability test is applied. In case we don't obtain direct relations between the parameters, as in (2.26) and (2.28), we can get equality of reports between the unknown parameters and thus obtain unidentifiable parameters.

### 2.4.3 Power series expansion identifiability

In this paragraph we will test linear PCH systems for *local strong identifiability* ( $x_0$  identifiability). A general mathematical formulation will be found and demonstrated for the lossless and lossy cases, using the power series expansion approach to test the structural identifiability. Starting from the explicit state-space form of a PCH system (2.23), the following general relations:

$$a_k(t_0) = y^{(k)}(t_0) \quad (2.29)$$

may be computed. Step by step computations (see appendix B.3.1, formulas (B.58-85)), in the lossless case, lead to:

$$a_k(t_0) = B^T(\theta_{HB}) \left\{ (-1)^k \left( A^T(\theta_{HA}) \right)^k \mathcal{Q}(\theta_{HA})x + \sum_{i=0}^{k-1} (-1)^i \left( A^T(\theta_{HA}) \right)^i \mathcal{Q}(\theta_{HA})B(\theta_{HB})\mu^{(k-1-i)} \right\} \quad (2.30)$$

for at least:  $k \geq \frac{n^2 + n}{2} + nm$  relations, corresponding to the unknown parameters. These relations have to be solved for the unknown parameters included in the matrices  $\mathcal{Q}$  (with  $\left(\frac{n^2 + n}{2}\right)$  parameters) and  $B$  (with  $nm$  parameters). In the lossy case, the general state-space representation of linear PCH systems is:

$$\begin{cases} \dot{x} = ((J - R)\mathcal{Q})(\theta_{HA})x + B(\theta_{HB})\mu \\ y = C(\theta_{HC})x \end{cases} \quad (2.31)$$

The  $k^{th}$  order term of the power series expansion (2.29), has then the following form:

$$a_k(t_0) = \left( B^T \mathcal{Q} \right) (\theta_{HC}) \left\{ A^k(\theta_{HA})x + \sum_{i=0}^{k-1} A^i(\theta_{HA})B(\theta_{HB})\mu^{(k-i-1)} \right\} \quad (2.32)$$

Using PCH systems properties and step by step computations, similar to the ones for the lossless case (see appendix B.3.2, (B.86-B.101)), one obtains:

$$a_k(t_0) = B^T(\theta_{HB}) \left\{ (-1)^k \left( (A_{loss} + A_{res})^T(\theta_{HA}) \right)^k \mathcal{Q}(\theta_{HA})x + \sum_{i=0}^{k-1} (-1)^i \left( (A_{loss} + A_{res})^T(\theta_{HA}) \right)^i \mathcal{Q}(\theta_{HA})B(\theta_{HB})\mu^{(k-1-i)} \right\} \quad (2.33)$$

for at least:

$$k \geq \frac{n^2 + n}{2} + nm$$

relations, with:

$$(A_{loss} + A_{res})^T \stackrel{def}{=} ((J + R)\mathcal{Q})^T = \mathcal{Q}(R - J)$$

It may be noticed that, for the lossy case, the minimal number of equations to be solved using the power series expansion test is given by the number unknown parameters included in the matrices  $\mathcal{Q}$  (with  $\left(\frac{n^2 + n}{2}\right)$  parameters),  $R$  (with  $\left(\frac{n^2 + n}{2}\right)$  parameters) and  $B$  (with  $nm$  parameters). If the solution of (2.30) (lossless case) or (2.33) (lossy case), for the unknown parameters is unique, then the PCH structure is considered to be locally strong identifiable or  $x_0$  identifiable. Using the general results from (2.30) or (2.33), the following proposition can be formulated for local strong identifiability of PCH systems.

### Proposition 2.3

A linear Port-Controlled-Hamiltonian system is local strong identifiable or  $x_0$  identifiable, when the system (2.30) or (2.33) of equations, representing the lossless and lossy cases, has a unique solution of the parameters.

This proposition represents a generalization of the results for the lossless or lossy PCH systems, when applying identifiability tests in a local point of interest on the system's trajectory.

### 2.5. Energy based identifiability

This section investigates local and global structural identifiability of linear PCH systems from known informations on powers through the ports, associated to the unknown parameters to be identified, introducing a new concept of *port-identifiability*. From (2.11), the unknown parameters considered for PCH systems in the general lossy case, are represented by the positive semi-definite symmetric matrices  $Q \in R^{n \times n}$  and  $R \in R^{n \times n}$  and by the input matrix  $B \in R^{n \times m}$ , while the skew-symmetric interconnection matrix  $J \in R^{n \times n}$  is generally known. The energy based identifiability analysis, is realized starting from the observation that the power energies associated to each port are related with the considered unknown parameters. Moreover, the relations between powers in the ports and unknown parameters have a triangular (partially decoupled) structure and the identifiability analysis may be realized separately for each port of the PCH system, as it will be shown later. The energy based approach proposed in this section for the identifiability analysis of port-Hamiltonian systems can be applied also in the nonlinear case, where the model is more complex but the interconnection structure is still a Dirac structure. Obviously, in the nonlinear case, unidentifiable parameters may also result from the non-injectivity of the interconnected subsystems constitutive equations. Similarly to previous sections, we assume that the PCH system is observable and controllable before performing this identifiability analysis. Using the usual input/output pairs of conjugate variables (efforts and flows), the power associated to each port of a PCH system may be explicitly computed. These power variables are quite often the variables which can be effectively measured. This justifies identifiability definitions specific for PCH systems. Let us consider the general lossy case:

$$\begin{cases} \dot{x} = (J - R)Qx + Bu \\ y = B^T Qx \end{cases} \quad (2.34)$$

Introducing in (2.34), the storage ( $s$ ) and control ( $c$ ) port variables  $f_S = -\dot{x}$ ,  $e_S = Qx$ , and  $e_C = y$ , one gets:

$$\begin{cases} -f_S = (J - R)e_S + Bf_C \\ e_C = B^T e_S \end{cases} \quad (2.35)$$

From (2.35), we observe that  $f_S, e_S$  and  $e_C$  may be computed from the unknown parameters  $\theta_Q$  (storage),  $\theta_B$  (control) and  $\theta_R$  (dissipation) parameters, solving a triangular system of the form:

$$\begin{cases} f_S = F(\theta_R, \theta_Q, \theta_B) \\ e_C = F(\theta_B, \theta_Q) \\ e_S = F(\theta_Q) \end{cases} \quad (2.36)$$

As previously, the interconnection matrix  $J$  is assumed with constant known parameters (very often the  $J$  interconnection matrix is considered as symplectic and built only from  $+1$  or  $-1$  values). In (2.31),  $\theta_R$  includes (at most)  $(n^2 + n)/2$  unknown real parameters from the semi-positive definite dissipation matrix  $R$ ,  $\theta_Q$  the  $(n^2 + n)/2$  (at most) unknown real parameters from the positive definite  $Q$  matrix and  $\theta_B$  (at most) the  $nm$  unknown real parameters included in the input matrix  $B$ . Using the definition of port conjugate variables (from the power pairing (1.1)), the power associated to the storage port may be written as:

$$P_S = e_S^T f_S = (Qx)^T (- (J - R)Qx - Bu) \quad (2.37)$$

The powers corresponding to the control and dissipative ports are:

$$P_C = e_C^T f_C = \left( B^T Qx \right)^T u \quad (2.38)$$

$$P_R = e_R^T f_R = e_R^T (-R)e_R \quad (2.39)$$

We will now adopt the direct identifiability test approach from section 2.4.2 and consider two independent sets of parameters for the PCH system in (2.34). The storage port powers can be written then:

$$P_{S1} = (Q_1 x)^T (- (J - R_1)Q_1 x - B_1 u) \quad (2.40)$$

$$P_{S2} = (Q_2 x)^T (- (J - R_2)Q_2 x - B_2 u) \quad (2.41)$$

Since the powers  $P_{S1}$  and  $P_{S2}$  represent known variables, using the direct identifiability test, we may assume  $P_{S1} = P_{S2}$ , that is:

$$(Q_1 x)^T (- (J - R_1)Q_1 x - B_1 u) = (Q_2 x)^T (- (J - R_2)Q_2 x - B_2 u) \quad (2.42)$$

The above relation can be further developed:

$$\begin{aligned} & -x^T Q_1 J Q_1 x + x^T Q_1 R_1 Q_1 x - x^T Q_1 B_1 u = \\ & = -x^T Q_2 J Q_2 x + x^T Q_2 R_2 Q_2 x - x^T Q_2 B_2 u \end{aligned} \quad (2.43)$$

Moving the right elements to the left in (2.43), the following relation results:

$$\begin{aligned} & -x^T (Q_1 J Q_1 - Q_2 J Q_2)x + x^T (Q_1 R_1 Q_1 - Q_2 R_2 Q_2)x \\ & - x^T (Q_1 B_1 - Q_2 B_2)u = 0 \end{aligned} \quad (2.44)$$

Due to the skew-symmetry property of the interconnection matrix  $J$ , the 1<sup>st</sup> term of (2.44) is zero and we can write then:

$$x^T (Q_1 R_1 Q_1 - Q_2 R_2 Q_2)x - x^T (Q_1 B_1 - Q_2 B_2)u = 0 \quad (2.45)$$

Then we can write the following relations:

$$\begin{cases} Q_1 R_1 Q_1 = Q_2 R_2 Q_2 \\ Q_1 B_1 = Q_2 B_2 \end{cases} \quad (2.46)$$

in order for (2.45) to hold. In order to solve (2.46), we will consider further two sets of unknown parameters, associated to the dissipative port:

$$P_{R1} = e_R^T (-R_1) e_R \quad (2.47)$$

$$P_{R2} = e_R^T (-R_2) e_R \quad (2.48)$$

Using the direct identifiability test applied for the power of the dissipative port, the following relation results:

$$e_R^T (-R_1) e_R = e_R^T (-R_2) e_R \quad (2.49)$$

or:

$$e_R^T (-R_1 + R_2) e_R = 0 \quad (2.50)$$

From (2.50), we can deduce:

$$R_1 = R_2 \quad (2.51)$$

which means that unknown parameters matrices  $R_1, R_2$  associated to the power energies of the dissipative port, for two independent sets of parameters are equal. In this case, it must be underlined, that the direct identifiability test, using the power energy associated to the dissipative port, takes in consideration only a subset of parameters, given by the resistive matrix  $R$  and also that the test, can be applied independently of the direct identifiability result obtained for the storage port or control port. Replacing the result from (2.51) in (2.46), we can write the following relation:

$$\begin{cases} Q_1 R_1 Q_1 = Q_2 R_1 Q_2 \\ Q_1 B_1 = Q_2 B_2 \end{cases} \quad (2.52)$$

Multiplying further the first relation in (2.52), with the inverse  $Q_1^{-1}$  to the left and right, it results:

$$R_1 = Q_1^{-1} Q_2 R_1 Q_2 Q_1^{-1} \quad (2.53)$$

where  $Q_1$  is assumed non-singular. This relation is satisfied if and only if:

$$\begin{cases} Q_1^{-1} Q_2 = I \\ Q_2 Q_1^{-1} = I \end{cases} \quad (2.54)$$

that is:

$$Q_2 = Q_1 \quad (2.55)$$

Replacing the result from (2.55) in (2.52), it results:

$$\begin{cases} Q_1 R_1 Q_1 = Q_1 R_1 Q_1 \\ Q_1 B_1 = Q_1 B_2 \end{cases} \quad (2.56)$$

Multiplying the 2<sup>nd</sup> equation in (2.56) with  $Q_1^{-1}$  to the left it results:

$$B_1 = B_2 \quad (2.57)$$

From (2.51),(2.55) and (2.57) it results that the unknown parameters  $Q_1, B_1, R_1$  and  $Q_2, R_2, Q_2$ , corresponding to the storage port power energies from (2.40) and (2.41) are equal to:

$$\begin{cases} Q_1 = Q_2 \\ R_1 = R_2 \\ B_1 = B_2 \end{cases} \quad (2.58)$$

when applying the direct identifiability test. From these results, we may conclude that knowing the powers associated to the storage and dissipation ports are sufficient to analyse the identifiability of all the parameters included in the  $Q, R, B$  matrices, which define the general structured representation of a lossy PCH system as in (2.34). It should be remarked that for the lossless case, the first relation in (2.46) disappears and it might be possible not to characterize all the parameters for identifiability. When measurements are available from the control port, the following relation between the control power and system's parameters will be used (see (2.35) and (2.36)):

$$P_C = e_C^T f_C = \left( B^T Q x \right)^T u \quad (2.59)$$

For two sets of unknown parameters, the powers associated to the control port have the following form:

$$P_{C1} = \left( B_1^T Q_1 x \right)^T u \quad (2.60)$$

$$P_{C2} = \left( B_2^T Q_2 x \right)^T u \quad (2.61)$$

Applying the direct identifiability test and assuming  $P_{C1} = P_{C2}$ , we have:

$$x^T Q_1 B_1 u = x^T Q_2 B_2 u \quad (2.62)$$

Moving to the left the right elements, the relation becomes:

$$x^T \left( Q_1 B_1 - Q_2 B_2 \right) u = 0 \quad (2.63)$$

From (2.63), the following relation can be written, between the unknown parameters included in the power energies associated to the control port:

$$Q_1 B_1 = Q_2 B_2 \quad (2.64)$$

In this case, the direct identifiability test, using the power associated to the control port, applies only for a subset of parameters given by the  $Q, B$  matrices. This test might give unidentifiable parameters when it is applied independently from the results obtained using the power energy associated to the storage port and dissipation port. However, when all the power energies are known, since the unknown parameters  $Q_1, B_1$  and  $Q_2, B_2$  are also parameters used in (2.58), then we may conclude from (2.64) that:

$$\begin{cases} Q_1 = Q_2 \\ B_1 = B_2 \end{cases} \quad (2.65)$$

When the  $B$  matrix comes from structural relations (or is perfectly known), the control port power may be tested independently from the storage and dissipation ports and (2.64) allows to analyse for identifiability only a subspace of the storage parameters (parameters from the  $\mathcal{Q}$  matrix). Another observation is that the control and dissipative power energies allow the identifiability analysis of only a subset of the parameters given in the  $\mathcal{Q}$ ,  $B$  and  $R$  matrices, while the storage port power allows the analysis of all system's parameters. In any case, it may be noticed from the previous developments that a PCH system can be analyzed for structural identifiability using only the powers associated to the ports (a single variable for each port), and not necessarily using the usual input/output pairs (used in the general identifiability definitions of section 2) which are required for *classical* systems. The resulting structural identifiability analysis may be splitted into sub-problems using the triangular form of system (2.36), each sub-problem corresponding to power energies for the storage, control and dissipation ports of the PCH system. Using these observations and the results from (2.58) and (2.65), we will now reformulate the definitions for global and local identifiability, specifically for PCH systems. Then we will define a notion of *port-identifiability* associated to each independent port.

**Proposition 2.4: Global Identifiability**

A lossy Port-Controlled-Hamiltonian system represented in state-space form (2.34), is said to be **globally identifiable** if, for any energy variables  $x(t)$  in the finite-dimensional state-space manifold  $X$ , any input  $u(t)$ , any dissipative port effort variable  $e_R(t)$  in the total Dirac structure  $D$  and any two sets of parameter vectors  $\theta_{PS1}, \theta_{PS2}, \theta_{PC1}, \theta_{PC2}, \theta_{PR1}, \theta_{PR2}$  in the parameter space  $\Theta$ , corresponding respectively to the storage, control and dissipative ports, the following relations:

$$\begin{cases} P_S(x, u, \theta_{PS1}) = P_S(x, u, \theta_{PS2}) \\ P_C(x, u, \theta_{PC1}) = P_C(x, u, \theta_{PC2}) \\ P_R(e_R, \theta_{PR1}) = P_R(e_R, \theta_{PR2}) \end{cases}$$

hold between the power energies, if and only if  $\theta_{PS1} = \theta_{PS2}, \theta_{PC1} = \theta_{PC2}, \theta_{PR1} = \theta_{PR2}$ .

**Proposition 2.5: Local Identifiability**

A lossy Port-Controlled-Hamiltonian system represented in state-space form (2.34) is said to be **locally identifiable** if for any: energy variables  $x(t)$  in the finite-dimensional state-space manifold  $X$ , any input  $u(t)$  any dissipative port effort variable  $e_R(t)$  in the total Dirac structure  $D$  and any two sets of parameter vectors  $\theta_{PS1}, \theta_{PS2}, \theta_{PC1}, \theta_{PC2}, \theta_{PR1}, \theta_{PR2}$  within an open neighborhood of some points  $\theta_{PS}^*, \theta_{PC}^*, \theta_{PR}^*$  in the parameter space  $\Theta$ , corresponding respectively to the storage, control and dissipative ports, the following relations:



$$\begin{cases} P_S(x, u, \theta_{PS1}) = P_S(x, u, \theta_{PS2}) \\ P_C(x, u, \theta_{PC1}) = P_C(x, u, \theta_{PC2}) \\ P_R(e_R, \theta_{PR1}) = P_R(e_R, \theta_{PR2}) \end{cases}$$

hold between the power energies, if and only if  $\theta_{PS1} = \theta_{PS2}, \theta_{PC1} = \theta_{PC2}, \theta_{PR1} = \theta_{PR2}$ .

In the above two definitions,  $\theta_{PS1}, \theta_{PS2}, \theta_{PC1}, \theta_{PC2}$  and  $\theta_{PR1}, \theta_{PR2}$  represent two sets of unknown parameter vectors, from the parameter space  $\Theta$ , corresponding to the storage power port  $P_S$ , control-port power port  $P_C$  and dissipative-port power port  $P_R$ , respectively given in (2.37), (2.38) and (2.39). Starting from the above definitions, more appropriate definitions can be formulated only for a particular port, corresponding to a Port-Hamiltonian system, using the power energy formulation, introducing in this way, a new concept of **port-identifiability**.

**Definition 2.1: Global port-identifiability**

Let consider the **port T** of a lossy Port-Controlled-Hamiltonian system represented in state-space form (2.34). Let define  $\tilde{x}_T := (x, u)$  when  $T$  is a storage or control port or  $\tilde{x}_T := e_R$  when  $T$  is a dissipation port. Then, the **port T** is said to be **globally identifiable**, if for any  $\tilde{x}_T$  any two sets of parameter vectors  $\theta_{T1}, \theta_{T2}$  in the parameter space  $\Theta$  associated to port  $T$ , the following relation  $P_T(\tilde{x}_T, \theta_{T1}) = P_T(\tilde{x}_T, \theta_{T2})$  holds between the power energies if and only if  $\theta_{T1} = \theta_{T2}$ .

**Definition 2.2: Local port-identifiability**

The **port T** of a lossy Port-Controlled-Hamiltonian system represented in state-space form in (2.34), is said to be **locally identifiable**, if for any  $\tilde{x}_T$  (as in definition 2.1) and any  $\theta_{T1}, \theta_{T2}$  within an open neighborhood of some point  $\theta_T^*$  in the parameter space  $\Theta$  associated to port  $T$ ,  $P_T(\tilde{x}_T, \theta_{T1}) = P_T(\tilde{x}_T, \theta_{T2})$  holds between the power energies, if and only if  $\theta_{T1} = \theta_{T2}$ .

In the above two definitions, depending on the selected port for identifiability analysis,  $\tilde{x}_T$  represents the pair  $(x, u)$  corresponding to the parameters included in the power of the storage port  $P_S$  (defined in (2.37)), control port  $P_C$  (defined in (2.38)) or alternatively  $\tilde{x}_T = e_R$  stands for the effort associated to the dissipative port,  $P_R$  (defined in (2.39)). In any case,  $\theta_{T1}, \theta_{T2}$  represents two sets of unknown parameter vectors from the parameter space  $\Theta$ , which describe the chosen PCH system port's power  $P_T$ . When compared to the previously selected structural identifiability tests for PCH systems, the power based energy formulation has the advantage of splitting the whole structural identifiability analysis into sub-parts,

corresponding to the system's ports, using the power of these ports as known variables. In this way, it is possible to observe that a sub-set of parameters, corresponding to one of the system's port are identifiable, without performing the whole identifiability analysis.

## 2.6 Structural identifiability examples

This section presents some structural identifiability analysis examples, where the results from sections 2.3, 2.4 and 2.5 are applied to PCH systems, using the observability/controllability definitions, direct test, power series expansion or the new propositions and definitions based on power energies, presented in section 2.5. The systems considered as examples are a lossless LC circuit, a lossy capacitor microphone circuit or a DC motor (see section 1.3.7).

### 2.6.1 Observability and Controllability analysis

In the case of the lossless PCH systems represented by an *LC* circuit, it can be proved that the determinant of the identifiability matrix  $R_{ID}$  from (2.17), has the following form (see detailed computations in Appendix C.1.1):

$$\det R_{ID} = - \frac{1}{C^3 L_1^5 L_2} \quad (2.66)$$

The above determinant is different from zero ( $\det R_{ID} \neq 0$ ), when:

$$\begin{cases} C \neq \pm\infty \\ L_1 \neq \pm\infty \\ L_2 \neq \pm\infty \end{cases} \quad (2.67)$$

When (2.66) is satisfied, the identifiability matrix  $R_{ID}$  is full rank and the *LC* circuit model is observable and controllable, from which the *LC* circuit system is structurally identifiable in any minimal realization form. In the case of the lossy capacitor microphone circuit, it can be proved that the system is not observable (see Appendix C.1.2 formulas (C.6-8)), because the observable transformation matrices  $T_{obs_1}, T_{obs_2}$  corresponding to the inputs  $B_1, B_2$  are rank deficient. Due to the fact that the capacitor microphone circuit is not observable, it is not structurally identifiable in a minimal realization form. In this particular case, before applying some estimation methods for the parameters, it is necessary to have a closer view on the model structure to obtain an observable/controllable dynamic system or to propose or apply other identifiability analysis techniques. Therefore the DC motor example will be considered as an illustration for the lossy case. Indeed it satisfies the observability/controllability conditions assumed initially for the structural identifiability analysis. For simplicity of the computations we will assume that the interconnection matrix  $J$  contains only +1/-1 values, which means that the gyrator transformation coefficient is considered to be  $\kappa = 1$  for the DC motor. In this case, it can be proved that the determinant of the identifiability matrix from (2.21), has the following value:

$$\det R_{ID} = - \frac{1}{R^3 J_E} \quad (2.68)$$

The necessary conditions for this determinant to be different from zero are:

$$\begin{cases} R \neq + / - \infty \\ J_E \neq + / - \infty \end{cases} \quad (2.69)$$

A complete proof of the result (2.68) can be found in Appendix C.1.2, formulas (C.9-13).

### 2.6.2 Direct identifiability

As was explained in section 2.4, this identifiability analysis assumes that the necessary conditions of observability/controllability are satisfied. The direct test examples presented below are performed using the global identifiability definition presented in section 2.2.

Using the direct identifiability analysis in the case of a lossless PCH system, represented by an LC circuit (see section 1.3.7), the following relations can be written:

$$\begin{cases} \begin{bmatrix} 0 & \frac{1}{L_{11}} & -\frac{1}{L_{21}} \\ -\frac{1}{C_{11}} & 0 & 0 \\ \frac{1}{C_{11}} & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u = \begin{bmatrix} 0 & \frac{1}{L_{12}} & -\frac{1}{L_{22}} \\ -\frac{1}{C_{12}} & 0 & 0 \\ \frac{1}{C_{12}} & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \\ \begin{bmatrix} 0 & \frac{1}{L_{11}} & 0 \end{bmatrix} x = \begin{bmatrix} 0 & \frac{1}{L_{12}} & 0 \end{bmatrix} x \end{cases} \quad (2.70)$$

considering two sets of unknown parameters  $L_{11}, L_{21}, C_{11} \in \Theta$  and  $L_{12}, L_{22}, C_{12} \in \Theta$ , that describe the dynamic system. From simple computations, the following result is obtained:

$$\begin{cases} L_{11} = L_{12} \\ C_{11} = C_{12} \\ L_{21} = L_{22} \end{cases} \quad (2.71)$$

from which, it can be concluded that the LC circuit is global identifiable, and the parameters  $C, L_1, L_2 \in \Theta$  identifiable for this identifiability test. A complete proof of the results (2.71) is given in Appendix C.2.1 (formulas (C.14-16)). Due to the fact that the capacitor microphone circuit is an unobservable system and the proposed initial conditions (observability/controllability) for the identifiability analysis using this test are not satisfied, the analysis is not performed further for this example. For the lossy case, we shall rather consider the DC motor example, which satisfies the observability/controllability conditions. For this example, we can write the following relations:

$$\begin{cases} \begin{bmatrix} -\frac{R_{11}}{L_{11}} & -\frac{1}{J_{E11}} \\ \frac{1}{L_{11}} & -\frac{b_{11}}{J_{E11}} \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u = \begin{bmatrix} -\frac{R_{12}}{L_{12}} & -\frac{1}{J_{E12}} \\ \frac{1}{L_{12}} & -\frac{b_{12}}{J_{E12}} \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ \begin{bmatrix} \frac{1}{L_{11}} & 0 \end{bmatrix} x = \begin{bmatrix} \frac{1}{L_{12}} & 0 \end{bmatrix} x \end{cases}$$

(2.72)

where:

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{J_E} \\ \frac{1}{L} & -\frac{b}{J_E} \end{bmatrix}, \quad C = B^T Q = \begin{bmatrix} \frac{1}{L} & 0 \end{bmatrix}$$

Using standard algebraic computations, we conclude that:

$$\begin{cases} R_{11} = R_{12} \\ L_{11} = L_{12} \\ J_{E11} = J_{E12} \\ b_{11} = b_{12} \end{cases} \quad (2.73)$$

The DC motor model is therefore globally identifiable for the parameters  $R, L, J_E, b \in \Theta$ . A complete proof of this results, can be found in Appendix C.2 between (C.17) and (C.21). The above examples, can be also realized using the local identifiability definition from section 2.2 for the identifiability analysis.

### 2.6.3 Power series expansion identifiability

As was already stated in the previous sections, a necessary condition for the identifiability analysis using this test, is the observability/controllability condition. In order to test the local strong identifiability of a lossless PCH system represented by a LC circuit, using the power series expansion, a simple example is considered, with the input  $u(t)$  and output data  $y(t)$ , given as follows:

$$\begin{cases} y(t) = t \\ u(t) = 2 \end{cases} \quad (2.74)$$

A time  $t_0 = 1$ , a state-space vector  $x(t) = [t \quad t \quad t]^T$ , with the value  $x(t_0) = [1 \quad 1 \quad 1]^T$  are considered for testing. Using (2.30), the power series expansion elements, are computed for the first eight elements, deducing the following relation:

$$a_k(t_0) = y^{(k)}(t_0) = 0 \quad (2.75)$$

between the  $k^{\text{th}}$  order derivate of the output  $y(t_0)$  and the  $k^{\text{th}}$  order element of the power series expansion  $a_k(t_0)$ , when  $k > 1$ . This result is due to the fact that the product between  $B^T Q$  and the sum of the elements from the paranthesis, of the form  $A^k x(t_0) + A^{k-1} B u(t_0)$ , for the  $k^{\text{th}}$  order element  $a_k(t_0)$ , is always zero for the LC circuit, when  $k > 1$ . The complete proof of the result from (2.75) is given in Appendix C.3 between (C.22) and (C.34) formulas. From the above results, it can be concluded that the solution of the unknown parameters for the LC circuit:

$$\begin{cases} C = 1 \\ L_1 = 1 \\ L_2 = 1 \end{cases} \quad (2.76)$$

is unique and the system is said to be local strong identifiable or  $x_0$  identifiable. For the case of the capacitor microphone circuit, due to the fact that the initial conditions of observability/controllability are not satisfied, the local strong identifiability test is not further performed.

#### 2.6.4 Energy based identifiability

As for the previous tests, the necessary conditions to be satisfied before the energy based identifiability analysis is performed, are the observability/controllability conditions. The examples considered below for the power based identifiability analysis have been investigated considering both the global identifiability of the PCH system or global port-identifiability, using the power energy at some specific port. In the case of the lossless LC circuit, the following relations can be written:

$$\begin{bmatrix} \frac{1}{C_{11}} & 0 & 0 \\ 0 & \frac{1}{L_{11}} & 0 \\ 0 & 0 & \frac{1}{L_{21}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{C_{12}} & 0 & 0 \\ 0 & \frac{1}{L_{12}} & 0 \\ 0 & 0 & \frac{1}{L_{22}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (2.77)$$

using relation (2.46) for the power energy associated to the storage port, where  $B_1 = B_2 = [0 \ 1 \ 0]^T$ . In the above relations,  $C_{11}, L_{11}, L_{21} \in \Theta$  and  $C_{12}, L_{12}, L_{22} \in \Theta$  represent two sets of unknown of parameters, which describe the LC circuit. Using (2.77), it results:

$$L_{11} = L_{12} \quad (2.78)$$

In this case, we cannot give any information on the identifiability of the parameters  $C, L_2$  and thus the LC circuit cannot be said to be global port identifiable for the storage port, with  $L_1, C, L_2$  as unknown parameters. In the case of the control port, the following relation:

$$\begin{bmatrix} \frac{1}{C_{11}} & 0 & 0 \\ 0 & \frac{1}{L_{11}} & 0 \\ 0 & 0 & \frac{1}{L_{21}} \end{bmatrix} B_1 = \begin{bmatrix} \frac{1}{C_{12}} & 0 & 0 \\ 0 & \frac{1}{L_{12}} & 0 \\ 0 & 0 & \frac{1}{L_{22}} \end{bmatrix} B_2 \quad (2.79)$$

may be written relating two sets of unknown parameters  $C_{11}, L_{11}, L_{21} \in \Theta$  and  $C_{12}, L_{12}, L_{22} \in \Theta$ , by assuming  $P_{C1} = P_{C2}$  (see relation (2.64)). Assuming  $B_1 = B_2 = B$ , since  $B$  doesn't include any unknown parameters in this example, it results:

$$\begin{bmatrix} 0 \\ \frac{1}{L_{11}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{L_{12}} \\ 0 \end{bmatrix} \quad (2.80)$$

and  $L_{11} = L_{12}$ . It results that the  $LC$  circuit cannot be said to be global port identifiable for the control port, as only the  $L_1 \in \Theta$  unknown parameter can be analysed. In this case, we observe that the parameters  $C, L_2 \in \Theta$  from (2.79) cannot be tested for identifiability using the control port power energy, as the constant  $B$  matrix includes null elements. Since the  $B$  matrix is structural in this example (i.e. does not include any numerical parameters), the identifiability analysis for the control port may be performed independently from the results obtained for the storage port, with  $L_1 \in \Theta$  as unknown parameter. From (2.78) and (2.80), it results that the  $LC$  circuit, is not global identifiable, using the identifiability proposition formulated in section 2.5, as only the  $L_1$  parameter is found to be identifiable and  $(C, L_2)$  parameters are not. In the case of the capacitor microphone circuit, due to the fact that the system doesn't satisfy the observability/controllability conditions, the identifiability analysis is not further performed. For the lossy DC motor example, using the power energy identifiability formulation from section 2.5, the following relations can be written:

$$\begin{cases} \begin{bmatrix} \frac{1}{L_{11}} \\ \frac{1}{L_{12}} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{L_{11}} \\ \frac{1}{L_{12}} \\ 0 \end{bmatrix} \\ \begin{bmatrix} -\frac{R_{11}}{L_{11}^2} & 0 \\ 0 & -\frac{b_{11}}{J_{11}^2} \end{bmatrix} = \begin{bmatrix} -\frac{R_{12}}{L_{12}^2} & 0 \\ 0 & -\frac{b_{12}}{J_{12}^2} \end{bmatrix} \end{cases} \quad (2.81)$$

for the storage port (using relation (2.46)), by applying the direct identifiability test for two sets  $L_{11}, J_{E11}, R_{11}, b_{11} \in \Theta$  and  $L_{12}, J_{E12}, R_{12}, b_{12} \in \Theta$  of unknown parameters. From (2.81) it results the following relation:

$$\begin{cases} L_{11} = L_{12} \\ R_{11} = R_{12} \end{cases} \quad (2.82)$$

between the unknown parameters  $L_{11}, L_{12}$  and  $R_{11}, R_{12}$ . At the next state, we apply the identifiability analysis of the dissipative port power energy and we can write:

$$\begin{bmatrix} R_{11} & 0 \\ 0 & b_{11} \end{bmatrix} = \begin{bmatrix} R_{12} & 0 \\ 0 & b_{12} \end{bmatrix} \quad (2.83)$$

using (2.51), for two sets  $R_{11}, b_{11} \in \Theta$  and  $R_{12}, b_{12} \in \Theta$  of unknown parameters. From (2.83) it results:

$$\begin{cases} R_{11} = R_{12} \\ b_{11} = b_{12} \end{cases} \quad (2.84)$$

The identifiability analysis using the dissipative port power energy can be applied independently from the analysis performed for the storage port or control port, as was also explained in section 2.5. From (2.84) it results that the DC motor model, is global port identifiable for the dissipative port power energy associated to the unknown parameters  $R, b \in \Theta$ . Using (2.82), (2.84) in (2.81), it results:

$$\begin{cases} L_{11} = L_{12} \\ J_{E11} = J_{E12} \\ R_{11} = R_{12} \\ b_{11} = b_{12} \end{cases} \quad (2.85)$$

when the storage port power energies satisfy:

$$P_{S1} = P_{S2} \quad (2.86)$$

for two sets  $L_{11}, J_{E11}, R_{11}, b_{11} \in \Theta$  and  $L_{12}, J_{E12}, R_{12}, b_{12} \in \Theta$  of unknown parameters. It results then, that the DC motor PCH model is global port identifiable for the storage port. For the control port power energy, we can write the following relations (using (2.64)):

$$\begin{bmatrix} \frac{1}{L_{11}} & 0 \\ L_{11} & 0 \\ 0 & \frac{1}{J_{E11}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{L_{12}} & 0 \\ L_{12} & 0 \\ 0 & \frac{1}{J_{E12}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (2.87)$$

for two sets of unknown parameters  $L_{11}, J_{E11} \in \Theta$  and  $L_{12}, J_{E12} \in \Theta$  associated to the control port power energy. Equation (2.87) is equivalent to:

$$\begin{bmatrix} \frac{1}{L_{11}} \\ L_{11} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{L_{12}} \\ L_{12} \\ 0 \end{bmatrix} \quad (2.88)$$

or  $L_{11} = L_{12}$ . Due to the fact that  $B$  is non-parametric and with structurally zero elements, the parameter  $J_E$  cannot be analysed for identifiability using only the control port power energy. Again, because the  $B$  matrix is non-parametric, the identifiability analysis of the  $L \in \Theta$  parameter may be performed independently from the storage port and dissipative port identifiability analysis results. From (2.88) it results that the DC motor PCH model is not global port identifiable for the control port power energy, as only the parameter  $L \in \Theta$  results as identifiable using this approach. From (2.84), (2.85) and (2.88), it results that the DC motor PCH system is global identifiable using the power energy identifiability formulation from section 2.5. In consequence the parameters  $L, J_E, R, b \in \Theta$  are identifiable using the power based propositions and definitions from section 2.5. The previous examples can be also performed for the local identifiability of the PCH system or a specified selected port.

## Conclusions

This chapter proposed an analysis of the structural identifiability of linear PCH systems in connection with the known classical theory. The appropriate structural identifiability tests, selected for the study of PCH systems, were the following ones: the observability/controlability based test (which give necessary conditions for a system to be identifiable in a minimal realization form), the direct test and the power series expansion test. Each of these tests were applied to linear PCH systems (in the lossless and lossy cases) and corresponding general characterizations for the structural identifiability were found. Section 2.5 of the chapter, introduced new propositions for the global and local identifiability of PCH systems, using the power energies associated to the ports, as each power port energy corresponds to a set of unknown parameters. A new concept, of *port-identifiability* was proposed, which is more appropriate to PCH systems power energy form, making an explicit use of their power-conserving Dirac interconnection structure. This new port-identifiability concept, gives the opportunity to make identifiability analysis of PCH systems only for a subset of the parameters. The identifiability analysis tests and definitions, were applied to some illustrative examples: a linear lossless PCH system (a *LC* circuit) and both a capacitor microphone circuit and a DC motor for the lossy case. The introduction of new propositions and definitions for the identifiability analysis of PCH systems, using the power port energies, permits also future work on estimation algorithms using this concepts. An important advantage of the power port energy approach, it was proved to be the possibility to split the identifiability analysis on different ports, representing a subset of the parameters by introducing the *port-identifiability* concept. Taken together the power energies corresponding to the ports can give informations on the global/local identifiability of all system's parameters contained in the  $Q, B$  or  $R$  matrices, where the analysis can be performed in a triangular fashion (as explained in 2.36). Also it was proved that using this approach, some parameters cannot be analysed for identifiability (see the LC circuit example) since the input matrix ( $B$ ) is structural with null elements. An open topic of research starting from the results of this chapter, is the selection, development and verification of practical identifiability methods, which are more suitable to be used for PCH systems, in the presence of a perturbation model.



### 3. Practical Identifiability of Port-Controlled-Hamiltonian systems

*This chapter aims to propose a practical identifiability methodology for Port-Controlled-Hamiltonian systems starting from the classical state-space theory, which makes use of the observability and controllability concepts. It is proved the conservation of the basic properties for Port-Controlled-Hamiltonian systems, when converted to the observable canonical form, in the presence of an external perturbation model, in the lossless and lossy cases. The perturbation model for Port-Controlled-Hamiltonian systems is defined by introducing a new input/output pair for the interaction port. It is proposed and demonstrated a methodology for converting the Port-Controlled-Hamiltonian systems into an equivalent observable canonical form, by applying a similarity transformation matrix to the system. The last two sub-sections, are dedicated to the formulation of a general context for the practical identifiability analysis of Port-Controlled-Hamiltonian systems, using the observability and controllability concepts, a definition useful for observability and controllability and three examples with the proposed methodology. As examples are considered a lossless Port-Controlled-Hamiltonian system, represented by an LC circuit and two lossy Port-Controlled-Hamiltonian systems, represented by a capacitor microphone circuit and a DC motor.*

#### 3.1 Introduction

While structural identifiability makes a theoretical analysis of a system, in order to obtain unique values of the parameters after the model structure parameters have been selected, the practical identifiability takes in consideration also the experimental conditions, with the quality and quantity of measurements [119-145].

Practical identifiability is performed after the structural identifiability analysis gives a theoretically identifiable model [45]. In the scientific literature [119-162], several practical identifiability analysis methods have been proposed and studied, from which some of the most relevant are: methods for the analysis of the sensitivity functions, Monte Carlo simulation method, correlation matrix method, Weijers and Vanrolleghem method, control based methodologies or others. Practical identifiability analysis techniques, have been applied for diverse kinds of linear or nonlinear models of interest, from which some are [119-145]: HIV dynamic models, biological reaction networks models, metabolic network models, acid acetic fermentation models, biochemical models of the river, photodynamic therapy models, ice-cream cristalization models or others. In this chapter the practical identifiability of Port-Hamiltonian systems is adressed, using the observability and controllability concepts known in classical system theory [2,4]. A perturbation model specific for this systems is proposed using the interaction port of Port-Hamiltonian systems, which is used later to find a general formulation of the practical identifiability methodology. *Section 3.2* demonstrates the conservation of the Port-Controlled-Hamiltonian system properties, when converted to the observable canonical form in the presence of a perturbation model, for the lossless and lossy cases and defines the perturbation model by means of the interaction port.

*Section 3.3* presents a generalization for the conversion of the observable canonical form of Port-Hamiltonian systems into an equivalent form, used for the practical identifiability

analysis. *Section 3.4* demonstrates and formulates some general results for the proposed methodology for Port-Hamiltonian systems, using the classical theoretical notions of observability and controllability, for the lossless and lossy cases. *Section 3.5* presents some examples for the practical identifiability analysis of Port-Hamiltonian systems, using the general results presented in the previous sections, for a lossless Port-Hamiltonian system represented by an *LC* circuit and two lossy Port-Hamiltonian systems given by a capacitor microphone circuit and DC motor.

## 3.2 Observable canonical form of Port-Hamiltonian systems

In this section, it is generally proved the conservation of the basic properties of Port-Hamiltonian systems, when converted to the observable canonical form, in the presence of a perturbation model, in the lossless and lossy case. The perturbation model is defined with the help of the interaction port of the system with the environment.

### 3.2.1 Lossless PCH systems

A lossless Port-Hamiltonian system represented in the explicit form, in the presence of a perturbation model, can be written as follows:

$$\begin{cases} \dot{x} = JQx + Bu + Ke \\ y = B^T Qx + K^T Qx \\ z = K^T Qx \end{cases} \quad (3.1)$$

In this representation of lossless Port-Hamiltonian systems,  $(u, y)$  represents the input-output pair, corresponding to the control port  $C$ , while  $(e, z)$  represents the input-output pair, corresponding to the interaction port  $I$ , necessary for the perturbation model. The perturbation model  $Ke(t)$  proposed in (3.1), for the system states represents the fact that the states are perturbed by interaction with the environment. The output  $z$  of the interaction port  $I$ , is added over the system outputs  $(y)$  and is associated to the same perturbation at the interaction port. In (3.1),  $K \in R^{n \times m}$  represents the perturbation model matrix and  $e(t)$  represents a white noise sequence, acting over the system states. The state-space representation from (3.1), can be equivalently written:

$$\begin{cases} \dot{x} = JQx + Bu + Ke \\ y = (C + E)x \\ z = Ex \end{cases} \quad (3.2)$$

where,  $B \in R^{n \times m}$ ,  $C = B^T Q \in R^{p \times n}$  and  $E = K^T Q \in R^{m \times n}$ . In order to convert the state-space representation from (3.2), to the observable canonical form, the observable transformation matrix  $T_{obs}$  must be applied to the state-space system as follows:

$$x_{obs} = T_{obs} x \quad (3.3)$$

where  $T_{obs}$  has the following structure:

$$T_{obs} = \begin{bmatrix} C + E \\ (C + E)A \\ \dots \\ (C + E)A^{n-1} \end{bmatrix} \quad (3.4)$$

**Remark:** The observable canonical form applies only to SISO PCH systems.

Using (3.3), (3.2) can be expressed as follows:

$$\begin{cases} T_{obs}^{-1} \dot{x}_{obs} = JQT_{obs}^{-1} x_{obs} + Bu + Ke \\ y = (CT_{obs}^{-1} + ET_{obs}^{-1})x_{obs} \end{cases} \quad (3.5)$$

Multiplying with  $T_{obs}$  to the left in (3.5), it results:

$$\begin{cases} \dot{x}_{obs} = T_{obs} JQT_{obs}^{-1} x_{obs} + T_{obs} Bu + T_{obs} Ke \\ y = (CT_{obs}^{-1} + ET_{obs}^{-1})x_{obs} \end{cases} \quad (3.6)$$

In order to prove that the state-space representation (3.6) still has an explicit Port-Hamiltonian form, we write it equivalently as:

$$\begin{cases} \dot{x}_{obs} = J_{obs} Q_{obs} x_{obs} + B_{obs} u + K_{obs} e \\ y = (C_{obs} + E_{obs})x_{obs} \end{cases} \quad (3.7)$$

Further we introduce a proposition for the observable canonical representation of lossless PCH systems, which is proved to be generally valuable.

**Proposition 3.1**

A lossless Port-Controlled-Hamiltonian system (3.1), can be equivalently transformed to the observable canonical form (3.7), when the following relations hold:

$$\begin{cases} J_{obs} = -J_{obs}^T = T_{obs} J^T T_{obs} \\ Q_{obs} = Q_{obs}^T = T_{obs}^{-T} Q^T T_{obs}^{-1} \geq 0 \\ C_{obs} = B_{obs}^T Q_{obs} = B^T Q^T T_{obs}^{-1} \\ E_{obs} = K_{obs}^T Q_{obs} = K^T Q^T T_{obs}^{-1} \end{cases}$$

In the following lines we will give a complete proof of **Proposition 3.1**.

**Proof:**

Using (3.1), we can write the following relations:

$$\begin{cases} T_{obs} JQT_{obs}^{-1} = J_{obs} Q_{obs} \\ B^T Q^T T_{obs}^{-1} = B_{obs}^T Q_{obs} = C_{obs} \\ K^T Q^T T_{obs}^{-1} = K_{obs}^T Q_{obs} = E_{obs} \end{cases} \quad (3.8)$$

Replacing  $B_{obs} = T_{obs} B$  and  $E_{obs} = T_{obs} K$  from (3.6) in (3.8), the following relations result:

$$\begin{cases} T_{obs} J Q_{obs}^{-1} = J_{obs} Q_{obs} \\ B^T Q_{obs}^{-1} = (T_{obs} B)^T Q_{obs} \\ K^T Q_{obs}^{-1} = (T_{obs} K)^T Q_{obs} \end{cases} \quad (3.9)$$

Using the matrix transpose properties, the 2<sup>nd</sup> and 3<sup>rd</sup> relations from (3.9) can be expressed as:

$$\begin{cases} B^T Q_{obs}^{-1} = B^T T_{obs}^T Q_{obs} \\ K^T Q_{obs}^{-1} = K^T T_{obs}^T Q_{obs} \end{cases} \quad (3.10)$$

Multiplying (3.10), to the left with the matrix pseudo-inverses  $T_B^{-T}$  and  $T_K^{-T}$  of  $B$  and  $K$ , the following relations result:

$$\begin{cases} Q_{obs}^{-1} = T_{obs}^T Q_{obs} \\ Q_{obs}^{-1} = T_{obs}^T Q_{obs} \end{cases} \quad (3.11)$$

In (3.11) it can be observed, that both relations can be multiplied with  $T_{obs}^{-T}$  to the left, in which case the following relation result:

$$Q_{obs} = T_{obs}^{-T} Q_{obs}^{-1} \quad (3.12)$$

Replacing the value of  $Q_{obs}$  from (3.12) in (3.9), the following relation results:

$$T_{obs} J Q_{obs}^{-1} = J_{obs} T_{obs}^{-T} Q_{obs}^{-1} \quad (3.13)$$

(3.13) can be multiplied to the right with  $T_{obs}$ ,  $Q_{obs}^{-1}$  and  $T_{obs}^T$ , in which case the following relation results for  $J_{obs}$ :

$$J_{obs} = T_{obs} J T_{obs}^T \quad (3.14)$$

For the next step  $Q_{obs}^T$  is computed using (3.12), in order to prove that  $Q_{obs} = Q_{obs}^T \geq 0$ , and the following relations:

$$Q_{obs}^T = \left( T_{obs}^{-T} Q_{obs}^{-1} \right)^T = T_{obs}^{-T} \left( T_{obs}^{-T} Q_{obs} \right)^T = T_{obs}^{-T} Q_{obs}^T T_{obs}^{-1} \quad (3.15)$$

can be written for  $Q_{obs}^T$  using the matrix transpose properties. Using the property  $Q = Q^T$  of a PCH system, (3.15) becomes:

$$Q_{obs}^T = T_{obs}^{-T} Q_{obs}^{-1} \quad (3.16)$$

From (3.12) and (3.16), it results:

$$Q_{obs} = Q_{obs}^T = T_{obs}^{-T} Q_{obs}^{-1} \quad (3.17)$$

The matrix  $Q$  is semi-positive definite ( $Q = Q^T \geq 0$ ), in which case the following relation:

$$z^T Q z \geq 0 \quad (3.18)$$

is always satisfied for any real vector  $z \in R^n$ . Using (3.17) and (3.18), it can be observed that  $Q_{obs}$ , is composed by positive semi-definite elements, given by the matrix  $T_{obs}^{-1}$ , in which case we can write:

$$Q_{obs} = Q_{obs}^T \geq 0 \quad (3.19)$$

Computing  $J_{obs}^T$  using (3.14), in order to prove that  $J_{obs} = -J_{obs}^T$ , the following relation can be written:

$$J_{obs}^T = \left( T_{obs} J_{obs}^T \right)^T = T_{obs} \left( T_{obs} J \right)^T \quad (3.20)$$

using the matrix transpose properties. (3.20) can be further expressed as:

$$J_{obs}^T = T_{obs} J^T T_{obs}^T \quad (3.21)$$

Using the skew-symmetry property ( $J = -J^T$ ) of a Port-Hamiltonian system, (3.21) becomes:

$$J_{obs}^T = -T_{obs} J_{obs}^T \quad (3.22)$$

From (3.14) and (3.22), it results that:

$$J_{obs} = -J_{obs}^T = T_{obs} J_{obs}^T \quad (3.23)$$

Using (3.6), (3.7) and (3.17), the following relation:

$$C_{obs} = B_{obs}^T Q_{obs} \quad (3.24)$$

can be equivalently written:

$$C_{obs} = \left( T_{obs} B \right)^T T_{obs}^{-T} Q_{obs}^{-1} \quad (3.25)$$

Using matrix transpose properties, (3.25) can be expressed as follows:

$$C_{obs} = B^T T_{obs}^T T_{obs}^{-T} Q_{obs}^{-1} \quad (3.26)$$

Performing the simplifications, (3.26) becomes:

$$C_{obs} = B^T Q_{obs}^{-1} \quad (3.27)$$

Using (3.6), (3.7) and (3.27), it results:

$$C_{obs} = C_{obs}^{-1} = B_{obs}^T Q_{obs} \quad (3.28)$$

Using (3.6),(3.7) and (3.17), the following relation:

$$E_{obs} = K_{obs}^T Q_{obs} \quad (3.29)$$

can be expressed as follows:

$$E_{obs} = \left( T_{obs} K \right)^T T_{obs}^{-T} Q_{obs}^{-1} \quad (3.30)$$

Using the matrix transpose properties, (3.30) becomes:

$$E_{obs} = K^T T_{obs}^T T_{obs}^{-T} Q_{obs}^{-1} \quad (3.31)$$

Performing the simplifications, (3.31) becomes:

$$E_{obs} = K^T Q_{obs}^{-1} \quad (3.32)$$

From (3.6),(3.7) and (3.32), it results:

$$E_{obs} = K_{obs}^T Q_{obs} = K^T Q_{obs}^{-1} \quad (3.33)$$

From (3.12),(3.15),(3.14),(3.19), (3.23), (3.28) and (3.33), it results that the observable canonical form state-space representation from (3.6) of lossless PCH systems, conserves the basic properties and thus the proof of **Proposition 3.1** is complete. More generally, the observable canonical form of lossless PCH system in the presence of a perturbation model is:

$$\begin{cases} \dot{x}_{obs} = J_{obs} Q_{obs} x_{obs} + B_{obs} u + K_{obs} e \\ y = (C_{obs} + E_{obs}) x_{obs} \end{cases} \quad (3.34)$$

where the relations from **Proposition 3.1** were proved to hold. Using **Proposition 3.1** we can write:

$$\begin{cases} A_{obs} = J_{obs} Q_{obs} = T_{obs} J Q T_{obs}^{-1} \\ B_{obs} = T_{obs} B \\ K_{obs} = T_{obs} K \\ C_{obs} = B^T Q T_{obs}^{-1} \\ E_{obs} = K^T Q T_{obs}^{-1} \end{cases} \quad (3.35)$$

where  $A_{obs} \in R^{n \times n}$ ,  $B_{obs} \in R^{n \times m}$ ,  $C_{obs} \in R^{l \times n}$ ,  $K_{obs} \in R^{n \times m}$ ,  $E_{obs} \in R^{l \times n}$  represent the system matrices in the observable canonical form of PCH systems.

### 3.2.2 Lossy PCH systems

A general lossy PCH system, in state-space form representation, can be written as follows:

$$\begin{cases} \dot{x} = (J - R)Qx + Bu + Ke \\ y = B^T Qx + K^T Qx \\ z = K^T Qx \end{cases} \quad (3.36)$$

Similarly with the lossless case,  $(e, z)$  represents an input-output pair corresponding to the interaction port  $I$ , which is used for the perturbation model over the states and outputs.

**Remark:** The perturbation model is introduced by means of the interaction port  $I$ , due to the reason that the PCH system is perturbed when interacting with the external environment. Applying the perturbation model (as it is usually done in the classic case) might not preserve the power balance of the system, hence its Port Hamiltonian structure.

The following definition for the perturbation model is proposed both for the lossless and lossy cases.

#### Definition 3.1

The perturbation model of a lossy Port-Controlled-Hamiltonian system (3.36), is defined by an input-output pair  $(e, z)$ , corresponding to the interaction port  $I$ , where:

- i) the input  $e(t)$ , represents a white-noise sequence of perturbation, corresponding to the system states perturbation model  $K \in R^{n \times m}$ ;
- ii) the output  $z(t)$ , represents the output perturbation, acting over the system output  $y(t)$ .

Similarly with the lossless case, applying the state-space transformation from (3.3) to a lossy PCH system from (3.36), in order to convert it to the observable canonical form, the following relations result:

$$\begin{cases} T_{obs}^{-1} \dot{x}_{obs} = (J - R)QT_{obs}^{-1} x_{obs} + Bu + Ke \\ y = CT_{obs}^{-1} x_{obs} + ET_{obs}^{-1} x_{obs} \\ z = ET_{obs}^{-1} x_{obs} \end{cases} \quad (3.37)$$

with  $T_{obs}$  the observable canonical form transformation matrix. An equivalent representation of (3.37), can be obtained by multiplying the 1<sup>st</sup> relation with  $T_{obs}$  to the left, in which case it results:

$$\begin{cases} \dot{x}_{obs} = T_{obs} (J - R)QT_{obs}^{-1} x_{obs} + T_{obs} Bu + T_{obs} Ke \\ y = (CT_{obs}^{-1} + ET_{obs}^{-1})x_{obs} \end{cases} \quad (3.38)$$

In order to prove that the state-space representation (3.38) has the usual Port Hamiltonian form, we write it equivalently as:

$$\begin{cases} \dot{x}_{obs} = (J_{obs} - R_{obs})Q_{obs} x_{obs} + B_{obs} u + K_{obs} e \\ y = (C_{obs} + E_{obs})x_{obs} \end{cases} \quad (3.39)$$

and we introduce a proposition as for the lossless case, which is proved later.

### **Proposition 3.2**

*A lossy Port-Controlled-Hamiltonian system (3.36), can be equivalently transformed to the observable canonical form (3.39), when the following relations hold:*

$$\begin{cases} J_{obs} = -J_{obs}^T = T_{obs} J T_{obs}^T \\ Q_{obs} = Q_{obs}^T = T_{obs}^{-T} Q T_{obs}^{-1} \geq 0 \\ R_{obs} = R_{obs}^T = T_{obs} R T_{obs}^T \\ C_{obs} = B_{obs}^T Q_{obs} = B_{obs}^T Q T_{obs}^{-1} \\ E_{obs} = K_{obs}^T Q_{obs} = K_{obs}^T Q T_{obs}^{-1} \end{cases}$$

**Proof:**

Using (3.38) and (3.39), the following two relations:

$$\begin{cases} B^T Q T_{obs}^{-1} = (T_{obs} B)^T Q_{obs} \\ K^T Q T_{obs}^{-1} = (T_{obs} K)^T Q_{obs} \end{cases} \quad (3.40)$$

result. By means of matrix transpose properties, (3.40) can be further written as:

$$\begin{cases} B^T Q_{obs}^{-1} = B^T T_{obs}^T Q_{obs} \\ K^T Q_{obs}^{-1} = K^T T_{obs}^T Q_{obs} \end{cases} \quad (3.41)$$

Putting in correspondence the pseudo-inverse matrix of  $B$  as  $T_B \in R^{m \times n}$  and of  $K$  as  $T_K \in R^{m \times n}$ , we can write further:

$$\begin{cases} Q_{obs}^{-1} = T_{obs}^T Q_{obs} \\ Q_{obs}^{-1} = T_{obs}^T Q_{obs} \end{cases} \quad (3.42)$$

Applying also the inverse of  $T_{obs}$ , results in:

$$T_{obs}^{-T} Q_{obs}^{-1} = Q_{obs} \quad (3.43)$$

For the next step, we prove the basic PCH systems property  $Q_{obs} = Q_{obs}^T \geq 0$ . Using (3.43),  $Q_{obs}^T$  can be written as:

$$Q_{obs}^T = \left( T_{obs}^{-T} Q_{obs}^{-1} \right)^T \quad (3.44)$$

Equivalently we can write (3.44) as follows:

$$Q_{obs}^T = T_{obs}^{-T} \left( T_{obs}^{-T} Q_{obs}^{-1} \right)^T = T_{obs}^{-T} Q_{obs}^T T_{obs}^{-1} \quad (3.45)$$

Using the PCH systems property  $Q = Q^T$ , (3.45) can be further expressed as:

$$Q_{obs}^T = T_{obs}^{-T} Q_{obs}^{-1} \quad (3.46)$$

From (3.43) and (3.46), the following relation is obvious:

$$Q_{obs} = Q_{obs}^T = T_{obs}^{-T} Q_{obs}^{-1} \quad (3.47)$$

As the matrix  $Q$  is semi-positive definite, the following relation hold:

$$z^T \cdot Q \cdot z \geq 0 \quad (3.48)$$

for every real vector  $z \in R^n$ . From (3.47),  $Q_{obs}$  is formed by positive semi-definite elements, from the columns of  $T_{obs}^{-1}$ , in which case  $Q_{obs}$  is then positive semi-definite and the following relation:

$$Q_{obs} = Q_{obs}^T \geq 0 \quad (3.49)$$

holds. For the next step, the properties  $J_{obs} = -J_{obs}^T$  and  $R_{obs} = R_{obs}^T \geq 0$ , which define a PCH system are proved. Using (3.38), (3.39) and (3.47), the following relation can be written:

$$T_{obs} (J - R) Q_{obs}^{-1} = \left( J_{obs} - R_{obs} \right) T_{obs}^{-T} Q_{obs}^{-1} \quad (3.50)$$

to determine  $J_{obs}$  and  $R_{obs}$ . Multiplying to the right with  $T_{obs}$ ,  $Q_{obs}^{-1}$  and  $T_{obs}^T$ , (3.50) becomes:

$$T_{obs} (J - R) T_{obs}^T = \left( J_{obs} - R_{obs} \right) \quad (3.51)$$

Developing further the left part of (3.51), it results:



$$T_{obs}^T J T_{obs}^T - T_{obs}^T R T_{obs}^T = J_{obs} - R_{obs} \quad (3.52)$$

From (3.52), it can be observed that the equality holds when:

$$\begin{cases} T_{obs}^T J T_{obs}^T = J_{obs} \\ R_{obs} = T_{obs}^T R T_{obs}^T \end{cases} \quad (3.53)$$

Using the 1<sup>st</sup> relation from (3.53),  $J_{obs}^T$  can be computed as follows:

$$J_{obs}^T = \left( T_{obs}^T J T_{obs}^T \right)^T \quad (3.54)$$

Using the matrix transpose properties, (3.54) can be also expressed as:

$$J_{obs}^T = T_{obs} J^T T_{obs}^T \quad (3.55)$$

Using the skew-symmetry property  $J = -J^T$  of a PCH system, (3.55) can be also written as:

$$J_{obs}^T = T_{obs} (-J) T_{obs}^T \quad (3.56)$$

From (3.53) and (3.56), the following relation results:

$$J_{obs} = -J_{obs}^T = T_{obs} \cdot J \cdot T_{obs}^T \quad (3.57)$$

Using (3.53),  $R_{obs}^T$  can be calculated as follows:

$$R_{obs}^T = \left( T_{obs}^T R T_{obs}^T \right)^T \quad (3.58)$$

and becomes further:

$$R_{obs}^T = T_{obs} R^T T_{obs}^T \quad (3.59)$$

using the matrix transpose properties. Using the property ( $R = R^T$ ) of a PCH system, (3.59) can be expressed as:

$$R_{obs}^T = T_{obs} R T_{obs}^T \quad (3.60)$$

From (3.53) and (3.60), it results that:

$$R_{obs} = R_{obs}^T = T_{obs} R T_{obs}^T \quad (3.61)$$

The matrix  $R$  is semi-positive definite ( $R \geq 0$ ), in which case the following relation can be written:

$$z^T R z \geq 0 \quad (3.62)$$

for any real vector  $z \in R^n$ . Using (3.61) and (3.62),  $R_{obs}$  is formed by positive semi-definite elements, given by the columns of  $T_{obs}^T$ . In this case, it can be concluded that  $R_{obs}$  is also positive semi-definite and the following relation:

$$R_{obs} = R_{obs}^T \geq 0 \quad (3.63)$$

is satisfied. For the next step it is necessary to prove that  $C_{obs} = B_{obs}^T Q_{obs}$ , for the state-space system from (3.38). Replacing  $B_{obs}$  from (3.38) and  $Q_{obs}$  from (3.47), the following relation results:

$$C_{obs} = \left( T_{obs}^T B \right)^T T_{obs}^{-T} Q_{obs}^{-1} \quad (3.64)$$

Using the matrix transpose properties, (3.64) can be written as:

$$C_{obs} = B^T T_{obs}^T T_{obs}^{-T} Q_{obs}^{-1} \quad (3.65)$$

Performing the possible simplifications, (3.65) becomes:

$$C_{obs} = B^T Q_{obs}^{-1} \quad (3.66)$$

Using (3.38), (3.39) and (3.66), the following relation results:

$$C_{obs} = B^T Q_{obs}^{-1} = C_{obs}^{-1} = B_{obs}^T Q_{obs} \quad (3.67)$$

In order to prove that  $E_{obs} = K_{obs}^T Q_{obs}$ ,  $K_{obs}$  is replaced from (3.38) and  $Q_{obs}$  from (3.47), in which case the following relation for  $E_{obs}$ :

$$E_{obs} = (T_{obs}^T K)^T T_{obs}^{-T} Q_{obs}^{-1} \quad (3.68)$$

is written. Using the matrix transpose properties, (3.68) can be expressed as:

$$E_{obs} = K^T T_{obs}^T T_{obs}^{-T} Q_{obs}^{-1} \quad (3.69)$$

Performing the simplifications, (3.69) becomes:

$$E_{obs} = K^T Q_{obs}^{-1} \quad (3.70)$$

Using (3.38),(3.39) and (3.70), the following relation:

$$E_{obs} = E_{obs}^{-1} = K^T Q_{obs}^{-1} = K_{obs}^T Q_{obs} \quad (3.71)$$

holds. From (3.38), (3.39), (3.47), (3.49), (3.53), (3.57), (3.61), (3.63),(3.67) and (3.71) it results that the observable canonical form representation of the lossy PCH system from (3.38), conserves the basic properties and thus the proof of **Proposition 3.2** is completed. We can write the following general form:

$$\begin{cases} \dot{x}_{obs} = (J_{obs} - R_{obs}) Q_{obs} x_{obs} + B_{obs} u + K_{obs} e \\ y = (C_{obs} + E_{obs}) x_{obs} \\ z = E_{obs} x_{obs} \end{cases} \quad (3.72)$$

for which the following relations hold:

$$\begin{cases} A_{obs} = T_{obs} (J - R) Q_{obs}^{-1} \\ B_{obs} = T_{obs} B \\ C_{obs} = B_{obs}^T Q_{obs} = C_{obs}^{-1} \\ K_{obs} = T_{obs} K \\ E_{obs} = K_{obs}^T Q_{obs} = E_{obs}^{-1} \\ Q_{obs} = Q_{obs}^T = T_{obs}^{-T} Q_{obs}^{-1} \\ J_{obs} = -J_{obs}^T = T_{obs} J_{obs}^T \\ R_{obs} = R_{obs}^T = T_{obs} R_{obs}^T \end{cases} \quad (3.73)$$

### 3.3 Equivalent representation of the observable canonical form

This section presents a technique of conversion into an equivalent observable canonical form of lossless and lossy PCH systems. This canonical form representation is necessary for introducing a general rule for the practical identifiability analysis of Port-Hamiltonian systems in section 3.4. This way we introduce the following proposition:

#### Proposition 3.3

*A Port-Controlled-Hamiltonian system in the observable canonical form for the lossless (3.34) or lossy (3.72) cases, can be equivalently transformed to:*

$$A_{obsT} \stackrel{def}{=} \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix}, B_{obsT} \stackrel{def}{=} \begin{bmatrix} \gamma_{11} \\ \gamma_{12} \\ \dots \\ \gamma_{1n-1} \\ \gamma_{1n} \end{bmatrix}, K_{obsT} \stackrel{def}{=} \begin{bmatrix} \gamma_{21} \\ \gamma_{22} \\ \dots \\ \gamma_{2,n-1} \\ \gamma_{2n} \end{bmatrix}$$

$$C_{obsT} \stackrel{def}{=} [1 \ 0 \ \dots \ \dots \ 0]$$

using the following notations:

$$\left\{ \begin{array}{l} \stackrel{def}{sx_1} = \gamma_{1,1} U(s) + \gamma_{21} E(s) - a_1 Y(s) + X_2(s) \\ \stackrel{def}{sx_2} = \gamma_{1,2} U(s) + \gamma_{22} E(s) - a_2 Y(s) + X_3(s) \\ \dots \\ \stackrel{def}{sx_{n-1}} = \gamma_{1,n-1} U(s) - a_{n-1} Y(s) + X_{n-1}(s) \\ \stackrel{def}{sx_n} = \gamma_{1,n} U(s) + \gamma_{2,n} E(s) - a_n Y(s) \\ \stackrel{def}{Y(s)} = X_1(s) \\ \stackrel{def}{\gamma_{lk}} = \sum_{\substack{i>0, j<k \\ i=k, \\ j=0 \\ i=i-1 \\ j=j+1}} b_i a_j; l=1,2 \end{array} \right.$$

**Proof:**

The proof of this proposition is presented further step by step. The Port-Hamiltonian systems represented in state-space form (3.34), for the lossless case, or (3.39), for the lossy case, can be also written as follows:

$$\begin{cases} \dot{x}_1 = x_2 + b_1 u + k_1 e \\ \dot{x}_2 = x_3 + b_2 u + k_2 e \\ \dots \\ \dot{x}_{n-1} = x_n + b_{n-1} u + k_{n-1} e \\ \dot{x}_n = -a_n x_1 - \dots - a_1 x_n + b_n u + k_n e \\ y = x_1 \end{cases} \quad (3.74)$$

That is the usual  $(A_{obs}, B_{obs}, C_{obs}, K_{obs})$  observable canonical state space form with:

$$\begin{aligned} A_{obs} &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \\ B_{obs} &= [b_1 \quad b_2 \quad \dots \quad b_n]^T \\ C_{obs} &= [1 \quad 0 \quad \dots \quad 0] \\ K_{obs} &= [k_1 \quad k_2 \quad \dots \quad k_n] \end{aligned} \quad (3.75)$$

From (3.75), the following relations result:

$$\begin{cases} x_2 = \dot{x}_1 - b_1 u - k_1 e \\ x_3 = \dot{x}_2 - b_2 u - k_2 e \\ \dots \\ x_n = \dot{x}_{n-1} - b_{n-1} u - k_{n-1} e \\ y = x_1 \end{cases} \quad (3.76)$$

Replacing  $x_1$  with  $y$  in (3.76), we can write:

$$\begin{cases} x_2 = \dot{y} - b_1 u - k_1 e \\ x_3 = \ddot{y} - b_1 \dot{u} - k_1 \dot{e} - b_2 u - k_2 e \\ x_4 = y^{(3)} - b_1 \ddot{u} - b_2 \dot{u} - b_3 u - k_1 \ddot{e} - k_2 \dot{e} - k_3 e \\ \dots \\ x_n = y^{(n-1)} - b_1 u^{(n-2)} - \dots - b_{n-2} \dot{u} - b_{n-1} u - k_1 e^{(n-2)} - \dots - k_{n-1} e \\ \dot{x}_n = -a_n y - \dots - a_1 x_n + b_n u + k_n e \end{cases} \quad (3.77)$$

Using the last two relations from (3.77), it results:

$$\begin{aligned}
& y^{(n)} - b_1 u^{(n-1)} - \dots - b_{n-2} \ddot{u} - b_{n-1} \dot{u} - k_1 e^{(n-1)} - \dots - k_{n-2} \ddot{e} - k_{n-1} \dot{e} = \\
& - a_n y - a_{n-1} (\dot{y} - b_1 u - k_1 e) - a_{n-2} (\ddot{y} - b_1 \dot{u} - b_2 u - k_1 \dot{e} - k_2 e) \dots \\
& - a_1 \left( y^{(n-1)} - b_1 u^{(n-2)} - \dots - b_{n-2} \dot{u} - b_{n-1} u - k_1 e^{(n-2)} - \dots - k_{n-2} \dot{e} - k_{n-1} e \right) \\
& + b_n u + k_n e
\end{aligned} \tag{3.78}$$

The transfer function from (3.78), can be splitted in two separate transfer functions corresponding to the plant and perturbation model, as follows:  $TF_1(s)$  represents the transfer function acting between the input  $U(s)$  and output  $Y(s)$  of the system, while  $TF_2(s)$  the transfer function acting between the perturbation and the output  $Y(s)$  of the system. Converting (3.78) to the continuous form, using the Laplace transform, we can write:

$$\begin{aligned}
& s^n Y(s) - b_1 s^{n-1} U(s) - \dots - b_{n-2} s^2 U(s) - b_{n-1} s U(s) - k_1 s^{n-1} E(s) - \dots - k_{n-2} s^2 E(s) - k_{n-1} s E(s) = \\
& - a_n Y(s) - a_{n-1} (s Y(s) - b_1 U(s) - k_1 E(s)) - a_{n-2} (s^2 Y(s) - b_1 s U(s) - b_2 U(s) - k_1 s E(s) - \\
& k_2 E(s)) - \dots - a_2 (s^{n-2} Y(s) - b_1 s^{n-3} U(s) - \dots - b_{n-3} s U(s) - b_{n-2} U(s) - k_1 s^{n-3} E(s) - \dots - k_{n-3} s E(s) - \\
& k_{n-2} E(s)) - a_1 \left( s^{n-1} Y(s) - b_1 s^{n-2} U(s) - \dots - b_{n-2} s U(s) - b_{n-1} U(s) - k_1 s^{n-2} E(s) - \dots - k_{n-2} s E(s) - \right) + \\
& b_n U(s) + k_n E(s)
\end{aligned} \tag{3.79}$$

Equivalently (3.79) can be written as:

$$\begin{aligned}
& Y(s) \left( s^n + a_n + a_{n-1} s + a_{n-2} s^2 \dots + a_1 s^{n-1} \right) = U(s) (b_1 s^{n-1} + s^{n-2} (b_2 + a_1 b_1) + s^{n-3} (b_3 + a_2 b_1 + a_1 b_2) \dots \\
& + s (b_{n-1} + a_1 b_{n-2} + a_2 b_{n-3} + \dots + a_{n-2} b_1) + a_{n-1} b_1 + a_{n-2} b_2 + \dots + a_1 b_{n-1} + b_n) + \\
& E(s) \left( k_1 s^{n-1} + s^{n-2} (k_2 + a_1 k_1) + s^{n-3} (k_3 + a_2 k_1 + a_1 k_2) + \dots \right. \\
& \left. + s (k_{n-1} + a_1 k_{n-2} + a_2 k_{n-3} + \dots + a_{n-2} k_1) + a_{n-1} k_1 + a_{n-2} k_2 + \dots + a_1 k_{n-1} + k_n \right)
\end{aligned} \tag{3.80}$$

Using (3.80), the plant transfer function  $TF_1(s)$ , has the form:

$$TF_1(s) = \frac{\left( \sum_{k=0}^{n-1} a_k b_{n-k} + \sum_{k=1}^{n-1} s^{n-k} \sum_{\substack{i \geq 1, j < k \\ i=k \\ j=0 \\ i=i-1 \\ j=j+1}} b_i a_j \right)}{\sum_{k=0}^n s^k a_{n-k}} \tag{3.81}$$

An equivalent expression for (3.81) is:

$$TF_1(s) = \frac{\sum_{k=n}^1 s^{n-k} \gamma_{1k}}{\sum_{k=0}^n s^k a_{n-k}} = \frac{\sum_{k=0}^{n-1} s^k \gamma_{1n-k}}{\sum_{k=0}^n s^k a_{n-k}} \quad (3.82)$$

In (3.82)  $\gamma_{1k}$  is defined as follows:

$$\gamma_{1k} = \sum_{\substack{i>0, j<k \\ i=k, \\ j=0 \\ i=i-1 \\ j=j+1}} b_i a_j \quad (3.83)$$

In (3.83) if  $j = 0$ ,  $a_0 = 1$ . The transfer function from (3.82), can be also written as:

$$TF_1(s) = \frac{\sum_{k=1}^n s^{n-k} \gamma_{1k}}{\sum_{k=0}^n s^k a_{n-k}} \quad (3.84)$$

Taking the common factor at the numerator and denominator  $s^n$  in (3.84), the transfer function becomes:

$$TF_1(s) = \frac{s^n \sum_{k=1}^n \gamma_{1k} s^{-k}}{s^n \sum_{k=0}^n s^{k-n} a_{n-k}} = \frac{\sum_{k=1}^n \gamma_{1k} s^{-k}}{\sum_{k=0}^n s^{k-n} a_{n-k}} \quad (3.85)$$

In a similar way with the transfer function of the plant model from (3.85), the transfer function of the perturbation model is:

$$TF_2(s) = \frac{s^n \sum_{k=1}^n \gamma_{2k} s^{-k}}{s^n \sum_{k=0}^n s^{k-n} a_{n-k}} = \frac{\sum_{k=1}^n \gamma_{2k} s^{-k}}{\sum_{k=0}^n s^{k-n} a_{n-k}} \quad (3.86)$$

where the term  $\gamma_{2k}$  is defined similarly with  $\gamma_{1k}$  as:

$$\left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \gamma_{2k} = \sum_{\substack{i>0, j<k \\ i=k, \\ j=0 \\ i=i-1 \\ j=j+1}} k_i a_j \quad (3.87)$$

Using (3.85) and (3.86), (3.80) becomes:

$$Y(s) \left( \sum_{k=0}^n s^{k-n} a_{n-k} \right) = U(s) \left( \sum_{k=1}^n \gamma_{1k} s^{-k} \right) + E(s) \left( \sum_{k=1}^n \gamma_{2k} s^{-k} \right) \quad (3.88)$$

(3.88) can be equivalently written as follows:

$$Y(s) \left( s^{-n} a_n + s^{1-n} a_{n-1} + \dots + a_1 s^{-1} + 1 \right) = U(s) \left( \sum_{k=1}^n \gamma_{1k} s^{-k} \right) + E(s) \left( \sum_{k=1}^n \gamma_{2k} s^{-k} \right) \quad (3.89)$$

From (3.89), the value of the output  $Y(s)$  becomes:

$$Y(s) = \sum_{k=1}^n s^{-k} \left( \gamma_{1k} U(s) + \gamma_{2k} E(s) - a_k Y(s) \right) \quad (3.90)$$

Making the following notations [163,164]:

$$\left\{ \begin{array}{l} \overset{def}{sx_1} = \gamma_{1,1} U(s) + \gamma_{21} E(s) - a_1 Y(s) + X_2(s) \\ \overset{def}{sx_2} = \gamma_{1,2} U(s) + \gamma_{22} E(s) - a_2 Y(s) + X_3(s) \\ \dots \\ \overset{def}{sx_{n-1}} = \gamma_{1,n-1} U(s) - a_{n-1} Y(s) + X_{n-1}(s) \\ \overset{def}{sx_n} = \gamma_{1,n} U(s) + \gamma_{2,n} E(s) - a_n Y(s) \\ \overset{def}{Y(s)} = X_1(s) \end{array} \right. \quad (3.91)$$

an equivalent state-space representation of (3.74) is:

$$\left\{ \begin{array}{l} \dot{x}_1 = -a_1 x_1 + x_2 + \gamma_{11} u + \gamma_{21} e \\ \dot{x}_2 = -a_2 x_1 + x_3 + \gamma_{12} u + \gamma_{22} e \\ \dots \\ \dot{x}_{n-1} = -a_{n-1} x_1 + x_n + \gamma_{1,n-1} u + \gamma_{2,n-1} e \\ \dot{x}_n = -a_n x_1 + \gamma_{1,n} u + \gamma_{2,n} e \end{array} \right. \quad (3.92)$$

(3.92) can be equivalently written as:

$$\dot{x} = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \gamma_{11} \\ \gamma_{12} \\ \dots \\ \gamma_{1,n-1} \\ \gamma_{1,n} \end{bmatrix} u + \begin{bmatrix} \gamma_{21} \\ \gamma_{22} \\ \dots \\ \gamma_{2,n-1} \\ \gamma_{2n} \end{bmatrix} e \quad (3.93)$$

$$y = [1 \ 0 \ 0 \ \dots \ 0] x_1$$

We introduce the following notations for the equivalent state-space representation of the observable canonical form:

$$\overset{def}{A_{obsT}} = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix}, \overset{def}{B_{obsT}} = \begin{bmatrix} \gamma_{11} \\ \gamma_{12} \\ \dots \\ \gamma_{1,n-1} \\ \gamma_{1n} \end{bmatrix}, \overset{def}{K_{obsT}} = \begin{bmatrix} \gamma_{21} \\ \gamma_{22} \\ \dots \\ \gamma_{2,n-1} \\ \gamma_{2n} \end{bmatrix} \quad (3.94)$$

$$\overset{def}{C_{obsT}} = [1 \ 0 \ \dots \ \dots \ 0]$$

$B_{obsT}$  from (3.94) can be also expressed as:

$$B_{obsT} = \begin{bmatrix} \gamma_{11} \\ \gamma_{12} \\ \dots \\ \gamma_{1n-1} \\ \gamma_{1n} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 + a_1 b_1 \\ b_3 + a_1 b_2 + a_2 b_1 \\ \dots \\ b_n + a_1 b_{n-1} + \dots + a_{n-1} b_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_{n-1} \\ b_n \end{bmatrix} + \begin{bmatrix} 0 \\ a_1 b_1 \\ \dots \\ a_1 b_{n-2} \\ a_1 b_{n-1} \end{bmatrix} + \dots + \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ a_{n-1} b_1 \end{bmatrix} \quad (3.95)$$

In the following lines we will find an equivalent representation of  $B_{obsT}$ . Using (3.95) we can write:

$$\begin{bmatrix} 0 \\ a_1 b_1 \\ \dots \\ \dots \\ a_1 b_{n-1} \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 \\ \dots & \dots & \dots & a_1 & \dots \\ 0 & 0 & 0 & 0 & a_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_{n-1} \\ b_n \end{bmatrix} \quad (3.96)$$

$$\begin{bmatrix} 0 \\ a_2 b_1 \\ a_2 b_2 \\ \dots \\ a_2 b_{n-2} \end{bmatrix} = \begin{bmatrix} a_2 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 \\ \dots & \dots & \dots & a_2 & \dots \\ 0 & 0 & 0 & 0 & a_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_{n-1} \\ b_n \end{bmatrix} \dots \quad (3.97)$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \dots \\ a_{n-1} b_1 \end{bmatrix} = \begin{bmatrix} a_{n-1} & 0 & 0 & \dots & 0 \\ 0 & a_{n-1} & 0 & \dots & 0 \\ 0 & 0 & a_{n-1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_{n-1} \\ b_n \end{bmatrix} \quad (3.98)$$

$$\begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_{n-1} \\ b_n \end{bmatrix} \quad (3.99)$$

Using (3.96)-(3.99),  $B_{obsT}$  can be further expressed as:

$$B_{obsT} = \begin{bmatrix} [1 & 0 & 0 & \dots & 0] \\ [0 & 1 & 0 & \dots & 0] \\ [0 & 0 & 1 & \dots & 0] \\ \dots \\ [0 & 0 & 0 & \dots & 1] \end{bmatrix} + \begin{bmatrix} [0 & 0 & 0 & \dots & 0] \\ [a_1 & 0 & 0 & \dots & 0] \\ [0 & a_1 & 0 & \dots & 0] \\ \dots \\ [0 & 0 & 0 & \dots & 0] \end{bmatrix} + \dots + \begin{bmatrix} [0 & 0 & 0 & \dots & 0] \\ [0 & 0 & 0 & \dots & 0] \\ [0 & 0 & 0 & \dots & 0] \\ \dots \\ [a_{n-1} & 0 & 0 & 0 & 0] \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_{n-1} \\ b_n \end{bmatrix} \quad (3.100)$$



from which it results:

$$B_{obsT} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ a_1 & 1 & 0 & \dots & 0 \\ a_2 & a_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_{n-1} \\ b_n \end{bmatrix} \quad (3.101)$$

More compact we can express  $B_{obsT}$  as follows:

$$B_{obsT} = T_2 A_{obs} B_{obs} \quad (3.102)$$

where  $T_2 A_{obs}$  is defined as follows:

$$T_2 A_{obs} \stackrel{def}{=} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ a_1 & 1 & 0 & \dots & 0 \\ a_2 & a_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & 1 \end{bmatrix} \quad (3.103)$$

and  $T_2 \in \mathbb{R}^{n \times n}$  represents a non-singular matrix. Replacing  $B_{obs}$  and  $A_{obs}$  from (3.35) or (3.73) for the lossless or lossy case of Port-Hamiltonian systems in (3.102), it results:

$$B_{obsT} = T_2 T_{obs} A_{obs}^{-1} T_{obs} B \quad (3.104)$$

Performing the possible simplifications in (3.104), it results:

$$B_{obsT} = T_2 T_{obs} AB \quad (3.105)$$

Similarly with  $B_{obsT}$ , we can proceed for  $K_{obsT}$  and the following relation can be written:

$$K_{obsT} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ a_1 & 1 & 0 & \dots & 0 \\ a_2 & a_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \dots \\ k_{n-1} \\ k_n \end{bmatrix} \quad (3.106)$$

$K_{obsT}$  from (3.106), can be then expressed as:

$$K_{obsT} = T_2 A_{obs} K_{obs} \quad (3.107)$$

where  $T_2 A_{obs}$  has the form from (3.103). Replacing  $K_{obs}$  and  $A_{obs}$  from (3.35) or (3.73) for the lossless or lossy PCH systems in (3.107), it results:

$$K_{obsT} = T_2 T_{obs} A_{obs}^{-1} T_{obs} K \quad (3.108)$$

This can be further expressed as follows:

$$K_{obsT} = T_2 T_{obs} AK \quad (3.109)$$

after simplifications.  $A_{obsT}$  can be equivalently written as:

$$A_{obsT} = T_1 A_{obs} \quad (3.110)$$

where  $T_1 \in R^{n \times n}$  is a non-singular square matrix. Using (3.38) or (3.78), (3.115) can be also expressed as:

$$A_{obsT} = T_1 T_{obs} A T_{obs}^{-1} \quad (3.111)$$

### 3.4. Practical identifiability analysis

This section presents a general formulation of the proposed practical identifiability methodology, using the observability and controlability concepts, for PCH systems in the lossless or lossy cases.

#### 3.4.1 Lossless PCH systems

Using the general results obtained in section 3.3, the extended identifiability matrix used for the practical identifiability analysis, for the lossless case of Port-Hamiltonian systems, has the following form:

$$R_{ID} = \left[ \begin{array}{cc} [B_{obsT} & K_{obsT}] & A_{obsT} [B_{obsT} & K_{obsT}] & \dots & A_{obsT}^{n-1} [B_{obsT} & K_{obsT}] \end{array} \right] \quad (3.112)$$

Replacing  $A_{obsT}$ ,  $B_{obsT}$  and  $K_{obsT}$  from (3.111), (3.105) and (3.109), (3.112) becomes:

$$R_{ID} = \left[ \begin{array}{cc} [T_2 T_{obs} AB & T_2 T_{obs} AK] & \dots & \left( T_1 T_{obs} A T_{obs}^{-1} \right)^{n-1} [T_2 T_{obs} AB & T_2 T_{obs} AK] \end{array} \right] \quad (3.113)$$

Further we can write (3.113) as:

$$R_{ID} = \left[ \begin{array}{cc} [T_2 T_{obs} AB & T_2 T_{obs} AK] & [T_1 T_{obs} A T_{obs}^{-1} T_2 T_{obs} AB & T_1 T_{obs} A T_{obs}^{-1} T_2 T_{obs} AK] & \dots & \left( T_1 T_{obs} A T_{obs}^{-1} \right)^{n-1} [T_2 T_{obs} AB & T_2 T_{obs} AK] \end{array} \right] \quad (3.114)$$

More compactly we can write:

$$R_{ID} = \left[ \begin{array}{cccc} [T_2 T_{obs} A & 0 & 0 & \dots & 0 \\ 0 & T_1 T_{obs} A T_{obs}^{-1} T_2 T_{obs} A & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \left( T_1 T_{obs} A T_{obs}^{-1} \right)^{n-1} T_2 T_{obs} A \end{array} \right] \left[ \begin{array}{c} [B \quad K] \\ [B \quad K] \\ \dots \\ \dots \\ [B \quad K] \end{array} \right]^T \quad (3.115)$$

Using (3.110),  $A_{obs}$  can be determined by multiplying  $A_{obsT}$  to the left with  $T_1^{-1}$ , from which results the following relation:

$$T_1^{-1} A_{obsT} = A_{obs} \quad (3.116)$$

When the system is observable, it results that  $A_{obs}$  is of full rank, with  $\det(A_{obs}) = n$ . Using the following property of a determinant:

$$\det(A_1 B_1) = \det(A_1) \det(B_1) \quad (3.117)$$

it results that:

$$\det(A_{obs}) = \det(T_1^{-1}) \det(A_{obsT}) \quad (3.118)$$

Using the property:

$$\det(T_1^{-1}) = \frac{1}{\det(T_1)} \quad (3.119)$$

(3.118) can be also expressed as:

$$\det(A_{obs}) = \frac{1}{\det(T_1)} \det(A_{obsT}) \quad (3.120)$$

As  $A_{obs}$  is of full rank for a lossless Port-Hamiltonian system in the observable canonical form, it results that  $\det(A_{obs}) \neq 0$  when  $\det(T_1) \neq 0$  and  $\det(A_{obsT}) \neq 0$ . Hence  $T_1$  and  $A_{obsT}$  are full rank. Using (3.102), the following relation can be written:

$$B_{obsT} = T_2 A_{obs} T_{obs} B \quad (3.121)$$

Multiplying to the left with  $T_2^{-1}$  in (3.121), the following relation results:

$$T_2^{-1} B_{obsT} = A_{obs} T_{obs} B \quad (3.122)$$

For the  $B$  matrix, a pseudo-inverse matrix  $T_B \in R^{m \times n}$  exists, which satisfies:

$$B T_B = I \quad (3.123)$$

where  $I \in R^{n \times n}$  is the identity matrix. Multiplying (3.122) to the right with  $T_B$ , the following relation is obtained:

$$T_2^{-1} B_{obsT} T_B = A_{obs} T_{obs} \quad (3.124)$$

Multiplying further (3.124) to the right with  $T_{obs}^{-1}$ , the following relation results:

$$T_2^{-1} B_{obsT} T_B T_{obs}^{-1} = A_{obs} \quad (3.125)$$

Due to the fact that  $\det(A_{obs}) \neq 0$  when the system is observable, it results that  $\det(T_2^{-1} B_{obsT} T_B T_{obs}^{-1})$  must be also different from 0. Using the properties of the determinant (3.117) and (3.119),  $\det(A_{obs})$  can be also expressed as follows:

$$\det(A_{obs}) = \frac{1}{\det(T_2)} \det(B_{obsT} T_B) \frac{1}{\det(T_{obs})} \quad (3.126)$$

Because  $\det(A_{obs}) \neq 0$  when the system is observable, it results that  $\det(T_2)$  and  $\det(B_{obsT} T_B)$  must be also different from zero and thus the matrices  $T_2$ ,  $B_{obsT} T_B$  and

$T_{obs}$  are full rank with  $rank(T_2) = rank\left(\begin{matrix} B_{obsT} & T_B \end{matrix}\right) = rank(T_{obs}) = n$ . The following mathematical notations:

$$\begin{cases} \overset{def}{C_1} = T_2 T_{obs} A \\ \overset{def}{C_2} = T_1 T_{obs} A T_{obs}^{-1} T_2 T_{obs} A \\ \dots \\ \overset{def}{C_n} = \left(T_1 T_{obs} A T_{obs}^{-1}\right)^{n-1} T_2 T_{obs} A \end{cases} \quad (3.127)$$

can be introduced, to get a compact expression for (3.115). Using the properties of the determinant from (3.117) and (3.119), and the results from (3.120) and (3.126) for the rank of  $T_1, T_2, T_{obs}, A_{obsT}$  and  $B_{obsT}, T_B$ , it can be deduced that:

$$\det(C_k) \neq 0, k = 1..n \quad (3.128)$$

from which it results that the matrices from (3.127), are also full rank:

$$rank(C_k) = n \quad (3.129)$$

In order to prove that the identifiability matrix  $R_{ID}$  from (3.115) is full rank, a minor of  $R_{ID}$  of the form:

$$R' = [R_1 \quad R_2 \quad R_3 \quad \dots \quad R_n] \quad (3.130)$$

is selected, where  $R_k$  represents column elements from  $C_k B$  or  $C_k K$  and  $R_1, R_2, \dots, R_n$  form an  $n \times n$  dimension matrix. A first condition for  $R'$  to be of maximal rank, is that  $rank(R_k)$  is maximal. In order to prove that  $rank(R_k)$  is maximal,  $R_k$  can be expressed as a product of the form:

$$R_k = A' B' \quad (3.131)$$

where  $A'$  represents a matrix  $C_k$  from (3.127) and  $B'$  represents a column from  $B$  or  $K$ .

Using the following relation for matrix ranks:

$$rank A + rank B - n \leq rank(AB) \leq \min(rank(A), rank(B)) \quad (3.132)$$

we can write:

$$rank A' + rank B' - n \leq rank(A'B') \leq \min(rank(A'), rank(B')) \quad (3.133)$$

Using (3.129), (3.133) becomes:

$$n + rank B' - n \leq rank(R_k) \leq \min(n, rank(B')) \quad (3.134)$$

It is known that  $rank(B') \leq n$ , as  $B' \in R^{n \times 1}$  and (3.134) becomes:

$$rank B' \leq rank(R_k) \leq rank(B') \quad (3.135)$$

From (3.135), it results that  $rank(R_k) = rank B'$  and is thus maximal, when  $rank(B') = 1$ . The following relation can be written in this case:

$$rank(R_k) = rank(B') = 1 \quad (3.136)$$

An equivalent condition for  $R'$  to have full rank as in (3.136), is that  $\ker(R') = nullity(R')$  with  $R'$  defined in (3.130) (the columns of  $R'$  linearly independent). This statement, can be

also expressed as follows: for a vector  $x \in R^n$ , the unique solution to the equation  $R'x = 0$  is the null vector  $x = 0_{n,1}$ . We can write also the relation:

$$\text{rank}(R') + \text{nullity}(R') = n \quad (3.137)$$

### 3.4.2 Lossy PCH systems

The practical identifiability matrix in this case, has the following form:

$$R_{ID} = \begin{bmatrix} [B_{obsT} & K_{obsT}] & A_{obsT} & [B_{obsT} & K_{obsT}] & \dots & (A_{obsT})^{n-1} [B_{obsT} & K_{obsT}] \end{bmatrix} \quad (3.138)$$

Equivalently this relation can be written as:

$$R_{ID} = \begin{bmatrix} \left[ \begin{array}{cccc} (T_1 A_{obs})^0 T_2 T_{obs} A & 0 & 0 & \dots & 0 \\ 0 & (T_1 A_{obs})^1 T_2 T_{obs} A & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & (T_1 A_{obs})^{n-1} T_2 T_{obs} A \end{array} \right] \left[ \begin{array}{c} [B \ K] \\ [B \ K] \\ \dots \\ \dots \\ [B \ K] \end{array} \right]^T \end{bmatrix} \quad (3.139)$$

replacing the matrix elements  $B_{obsT}$ ,  $K_{obsT}$ ,  $A_{obsT}$  from (3.105), (3.109) and (3.110). In the following lines we will formulate and prove a proposition for the observability of lossy PCH systems, considering also the presence of the perturbation model.

#### **Proposition 3.4:**

Consider the linear lossy PCH system (3.36), defining  $A'' = (J + R)Q$ :

- a) If (3.36) is observable, then  $\det Q \neq 0$  and the pair  $(A'', B + K)$  is controllable;
- b) Assume  $\det Q \neq 0$ , then (3.36) is observable if  $(A'', B + K)$  is controllable.

#### **Proof:**

For the next step we search for a simplified form of the similarity transformation  $T_{obs}$  defined in (3.4), for the lossy case of Port-Hamiltonian systems.

In this case:

$$C + E = (B^T + K^T)Q = (B^T + K^T)Q^T = (Q(B + K))^T \quad (3.140)$$

$$(C + E)A = (B^T + K^T)Q((J - R)Q) = (B^T + K^T)Q^T \left( \left[ \begin{array}{c} -J^T - R^T \\ \end{array} \right] Q \right) \quad (3.141)$$

can be written using the PCH systems properties  $Q = Q^T$  and  $J = -J^T$ . For a more compact expression in (3.141), we introduce the notations:

$$\begin{cases} \stackrel{def}{A_{loss}} = JQ \\ \stackrel{def}{A_{res}} = RQ \end{cases} \quad (3.142)$$

representing the lossless and the dissipative part of  $A$  matrix. Using (3.142), (3.141) becomes:

$$(C + E)A = \begin{pmatrix} B^T + K^T \end{pmatrix} \begin{pmatrix} -A_{loss}^T & -A_{res}^T \end{pmatrix} Q \quad (3.143)$$

Equivalently we can write:

$$(C + E)A = \begin{pmatrix} B^T + K^T \end{pmatrix} \begin{pmatrix} -A_{loss} & -A_{res} \end{pmatrix}^T Q \quad (3.144)$$

Using the property  $Q = Q^T$  of PCH systems, (3.144) becomes:

$$(C + E)A = \begin{pmatrix} B^T + K^T \end{pmatrix} (-1)^1 \left( Q \begin{pmatrix} A_{loss} & A_{res} \end{pmatrix} \right)^T = \begin{pmatrix} B^T + K^T \end{pmatrix} (-1)^1 \begin{pmatrix} A_{loss} & A_{res} \end{pmatrix}^T Q \quad (3.145)$$

Equivalently we can write:

$$(C + E)A = (-1)^1 \left( Q \begin{pmatrix} A_{loss} & A_{res} \end{pmatrix} \begin{pmatrix} B + K \end{pmatrix} \right)^T \quad (3.146)$$

Using (3.145), we compute the term  $(C + E)A^2$  as follows:

$$(C + E)A^2 = (-1)^1 \begin{pmatrix} B^T + K^T \end{pmatrix} \begin{pmatrix} A_{loss} & A_{res} \end{pmatrix}^T Q \left( (J - R)Q \right) \quad (3.147)$$

Using the properties of PCH systems, (3.147) becomes:

$$(C + E)A^2 = (-1)^1 \begin{pmatrix} B^T + K^T \end{pmatrix} \begin{pmatrix} A_{loss} & A_{res} \end{pmatrix}^T Q \left( \begin{pmatrix} -J^T & -R^T \end{pmatrix} Q \right) \quad (3.148)$$

Using (3.142) we can write equivalently:

$$(C + E)A^2 = (-1)^2 \begin{pmatrix} B^T + K^T \end{pmatrix} \begin{pmatrix} A_{loss} & A_{res} \end{pmatrix}^T \begin{pmatrix} A_{loss} & A_{res} \end{pmatrix}^T Q \quad (3.149)$$

(3.149) can be equivalently written as:

$$(C + E)A^2 = (-1)^2 \begin{pmatrix} B^T + K^T \end{pmatrix} \left( \begin{pmatrix} A_{loss} & A_{res} \end{pmatrix}^T \right)^2 Q \quad (3.150)$$

The next element  $(C + E)A^3$  of the transformation matrix  $T_{obs}$  is:

$$(C + E)A^3 = (-1)^3 \begin{pmatrix} B^T + K^T \end{pmatrix} \left( \begin{pmatrix} A_{loss} & A_{res} \end{pmatrix}^T \right)^2 Q (J - R)Q \quad (3.151)$$

From PCH systems properties, (3.151) can be expressed further as:

$$(C + E)A^3 = (-1)^3 \begin{pmatrix} B^T + K^T \end{pmatrix} \left( \begin{pmatrix} A_{loss} & A_{res} \end{pmatrix}^T \right)^2 Q^T \begin{pmatrix} -J^T & -R^T \end{pmatrix} Q \quad (3.152)$$

Using (3.142), we can write:

$$(C + E)A^3 = (-1)^3 \begin{pmatrix} B^T + K^T \end{pmatrix} \left( \begin{pmatrix} A_{loss} & A_{res} \end{pmatrix}^T \right)^3 Q \quad (3.153)$$

From (3.145), (3.150) and (3.153), we can deduce the following general relation:

$$(C + E)A^k = (-1)^k \begin{pmatrix} B^T + K^T \end{pmatrix} \left( \begin{pmatrix} A_{loss} & A_{res} \end{pmatrix}^T \right)^k Q \quad (3.154)$$

for the terms of  $T_{obs}$  in (3.4). For the next step we prove the equivalence between

$$\left( \begin{pmatrix} A_{loss} & A_{res} \end{pmatrix}^T \right)^k \text{ and } \left( \begin{pmatrix} A_{loss} & A_{res} \end{pmatrix}^k \right)^T.$$

We can write then:

$$\left( (A_{loss} + A_{res})^T \right)^k = (A_{loss} + A_{res})^T \dots (A_{loss} + A_{res})^T \quad (3.155)$$

using matrix properties. For the term  $\left( (A_{loss} + A_{res})^k \right)^T$ , we can write:

$$\left( (A_{loss} + A_{res})^k \right)^T = \left( (A_{loss} + A_{res}) (A_{loss} + A_{res}) \dots (A_{loss} + A_{res}) \right)^T \quad (3.156)$$

Using matrix transpose properties, (3.156) can be further expressed as:

$$\left( (A_{loss} + A_{res})^k \right)^T = (A_{loss} + A_{res})^T \left( (A_{loss} + A_{res}) \dots (A_{loss} + A_{res}) \right)^T \quad (3.157)$$

Applying  $k - 2$  times the matrix transpose properties, (3.157) results as:

$$\left( (A_{loss} + A_{res})^k \right)^T = (A_{loss} + A_{res})^T (A_{loss} + A_{res})^T \dots (A_{loss} + A_{res})^T \quad (3.158)$$

From (3.155) and (3.158), we conclude that:

$$\left( (A_{loss} + A_{res})^T \right)^k = \left( (A_{loss} + A_{res})^k \right)^T \quad (3.159)$$

Using (3.159), (3.154) becomes:

$$(C + E)A^k = (-1)^k (B^T + K^T) \left( (A_{loss} + A_{res})^k \right)^T Q \quad (3.160)$$

Using (3.160), the similarity matrix  $T_{obs}$ , has the form:

$$T_{obs} = \begin{bmatrix} (B^T + K^T)Q \\ (B^T + K^T)(-1)^1 (A_{loss} + A_{res})^T Q \\ (B^T + K^T)(-1)^2 \left( (A_{loss} + A_{res})^2 \right)^T Q \\ \dots \\ (B^T + K^T)(-1)^{n-1} \left( (A_{loss} + A_{res})^{n-1} \right)^T Q \end{bmatrix} \quad (3.161)$$

Equivalently we can write:

$$T_{obs} = \begin{bmatrix} (B^T + K^T)Q^T \\ (B^T + K^T)(-1)^1 (A_{loss} + A_{res})^T Q^T \\ \dots \\ (B^T + K^T)(-1)^{n-1} \left( (A_{loss} + A_{res})^{n-1} \right)^T Q^T \end{bmatrix} \quad (3.162)$$

More compactly we can write:

$$T_{obs} = \begin{bmatrix} (Q(B + K))^T \\ (-1)^1 (B^T + K^T) (Q(A_{loss} + A_{res}))^T \\ \dots \\ (-1)^{n-1} (B^T + K^T) (Q(A_{loss} + A_{res})^{n-1})^T \end{bmatrix} \quad (3.163)$$

or:

$$T_{obs} = \begin{bmatrix} (-1)^0 (Q(B+K))^T \\ (-1)^1 (Q(A_{loss} + A_{res})(B+K))^T \\ \dots \\ (-1)^{n-1} (Q(A_{loss} + A_{res})^{n-1}(B+K))^T \end{bmatrix} \quad (3.164)$$

We can write also as:

$$T_{obs} = \begin{bmatrix} (-1)^0 (Q(B+K)) \\ (-1)^1 (Q(A_{loss} + A_{res})(B+K)) \\ \dots \\ (-1)^{n-1} (Q(A_{loss} + A_{res})^{n-1}(B+K)) \end{bmatrix}^T \quad (3.165)$$

From (3.165), the transformation matrix  $T_{obs}$  can be equivalently written as:

$$T_{obs} = \begin{bmatrix} (-1)^0 (Q(B+K)) & (-1)^1 (Q(A_{loss} + A_{res})(B+K)) & \dots & (-1)^{n-1} (Q(A_{loss} + A_{res})^{n-1}(B+K)) \end{bmatrix} \quad (3.166)$$

Also we can write:

$$T_{obs} = Q \begin{bmatrix} (-1)^0 (B+K) & (-1)^1 (A_{loss} + A_{res})(B+K) & \dots & (-1)^{n-1} (A_{loss} + A_{res})^{n-1}(B+K) \end{bmatrix} \quad (3.167)$$

Using the matrix notation:

$$A'' \stackrel{def}{=} (A_{loss} + A_{res}) \quad (3.168)$$

we can write equivalently:

$$T_{obs} = Q \begin{bmatrix} (-1)^0 (B+K) & (-1)^1 (A''(B+K)) & \dots & (-1)^{n-1} (A'')^{n-1}(B+K) \end{bmatrix} \quad (3.169)$$

Thus a lossy Port-Hamiltonian system in the presence of a known perturbation model is observable, when  $\det Q \neq 0$  and the pair  $(A'', (B+K))$  is controllable.

### 3.5 Practical identifiability examples

This section presents three representative examples of practical identifiability analysis using the observability and controllability concepts, and the general results from sections 3.2 to 3.4.

#### 3.5.1 Lossless PCH systems

For the practical identifiability analysis of the LC circuit, it was considered a Gaussian White noise perturbation model, generated by software of the following form:



$$K = \begin{bmatrix} 8.77 \\ 7.36 \\ 0.04 \end{bmatrix} \quad (3.170)$$

In order to analyse the practical identifiability property of the LC circuit, the following values have been considered for the parameters of the real system:

$$\begin{cases} L_1 = L_2 = 1 \\ C = \frac{1}{2} \end{cases} \quad (3.171)$$

Thus the semi-positive definite matrix  $Q$  becomes:

$$Q = \begin{bmatrix} \frac{1}{C} & 0 & 0 \\ 0 & \frac{1}{L_1} & 0 \\ 0 & 0 & \frac{1}{L_2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.172)$$

A necessary condition to apply the practical identifiability analysis, is first to test the observability of the system without the presence of the perturbation model (or structural identifiability analysis). The observable transformation matrix in this case is:

$$T_{obs} = \begin{pmatrix} 0 & \frac{1}{L_1} & 0 \\ -\frac{1}{L_1 C} & 0 & 0 \\ 0 & -\frac{1}{L_1^2 C} & \frac{1}{L_1 L_2 C} \end{pmatrix} \quad (3.173)$$

Using (3.173) it can be easily proved that the system is observable and the practical identifiability analysis can be further realized. Using (3.170) and (3.172), the matrix  $E$  corresponding to the perturbation model, has the following form:

$$E = K^T Q = [17.54 \quad 7.36 \quad 0.04] \quad (3.174)$$

while the matrix  $C$  has the following form:

$$C = B^T Q = [0 \quad 1 \quad 0] \quad (3.175)$$

Computing the transformation matrix  $T_{obs}$  from (3.4) for the observable canonical form of the system, the following result is obtained:

$$T_{obs} = \begin{bmatrix} C + E \\ (C + E)A \\ (C + E)A^2 \end{bmatrix} = \begin{bmatrix} 17.54 & 8.36 & 0.04 \\ -16.64 & 17.54 & -17.54 \\ -70.16 & -16.64 & 16.64 \end{bmatrix} \quad (3.176)$$

Computing the  $A$ ,  $B$ ,  $C'$  and  $K$  matrices in the observable canonical form, the following results are obtained using (3.35):

$$A_{obs} = T_{obs} A T_{obs}^{-1} = \begin{bmatrix} 0 & 1 & -1.19 \cdot 10^{-7} \\ 0 & 4.8 \cdot 10^{-7} & 1 \\ 0 & -3.99 & 4.8 \cdot 10^{-7} \end{bmatrix} \cong \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{bmatrix} \quad (3.177)$$

$$B_{obs} = T_{obs} B = \begin{bmatrix} 8.36 \\ 17.54 \\ -16.64 \end{bmatrix} \quad (3.178)$$

$$C'_{obs} = (C + E) T_{obs}^{-1} = \begin{bmatrix} 0.99 & -1.2 \cdot 10^{-7} & -3.5 \cdot 10^{-6} \end{bmatrix} \cong [1 \ 0 \ 0] \quad (3.179)$$

$$K_{obs} = T_{obs} K = \begin{bmatrix} 215.357 \\ -17.54 \\ -737.108 \end{bmatrix} \quad (3.180)$$

In (3.179),  $C'$  represents the matrix  $C + E$  corresponding to the output perturbation applied over the system output as in (3.2). Using (3.94), (3.95) and (3.106), the equivalent observable canonical form of the LC circuit is:

$$A_{obsT} = \begin{bmatrix} 4.8 \cdot 10^{-7} & 1 & -1.19 \cdot 10^{-7} \\ -3.99 & 4.8 \cdot 10^{-7} & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (3.181)$$

$$B_{obsT} = \begin{bmatrix} 8.36 \\ 17.53 \\ 16.71 \end{bmatrix} \quad (3.182)$$

$$C_{obsT} = [1 \ 0 \ 0] \quad (3.183)$$

$$K_{obsT} = \begin{bmatrix} 8.77 \\ 7.35 \\ 35.03 \end{bmatrix} \quad (3.184)$$

Using (3.112), (3.181), (3.182), (3.183) and (3.184), the practical identifiability matrix, is:

$$R_{ID} = \begin{bmatrix} 8.36 & 8.77 & 17.53 & 7.35 & -16.80 & -0.30 \\ 17.53 & 7.35 & -16.64 & 0.03 & -69.98 & -29.36 \\ 16.71 & 35.03 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.185)$$

In order to determine the controllability property of the system in the presence of a perturbation model, the matrix rank of  $R_{ID}$  is determined by selecting for instance the 2<sup>nd</sup>, 4<sup>th</sup> and 6<sup>th</sup> columns in (3.185), for which the determinant has the following value:

$$\det R'_{ID} = \det \begin{bmatrix} 8.77 & 7.35 & -0.30 \\ 7.35 & 0.03 & -29.36 \\ 35.03 & 0 & 0 \end{bmatrix} = -7570.89 \quad (3.186)$$

Thus it can be concluded that the identifiability matrix  $R_{ID}$  is of full rank and the LC circuit is said to be practically identifiable in the presence of the selected perturbation model. In case

the transformation matrix  $T_{obs}$  in (3.176) is rank deficient, the LC circuit is not practically identifiable for the corresponding perturbation model.

### 3.5.2 Lossy PCH systems

For the case of the capacitor microphone circuit, the observable transformation matrix has the following general form:

$$T_{obs} = [T_{obs 1} \quad T_{obs 2}] \quad (3.187)$$

where:

$$T_{obs 1} = \begin{bmatrix} 0 & \frac{1}{m} & 0 \\ -\frac{k}{m} & -\frac{c}{m^2} & 0 \\ \frac{ck}{m^2} & -\frac{k}{m^2} + \frac{c^2}{m^3} & 0 \end{bmatrix} \quad (3.188)$$

represents the transformation matrix corresponding to the first input  $B_1 = [0 \quad 1 \quad 0]^T$  and:

$$T_{obs 2} = \begin{bmatrix} 0 & 0 & \frac{1}{Rc} \\ 0 & 0 & -\frac{1}{(Rc)^2} \\ 0 & 0 & \frac{1}{(Rc)^3} \end{bmatrix} \quad (3.189)$$

From (3.188) and (3.189), it can be observed, that the observable transformation matrices  $T_{obs 1}, T_{obs 2}$  corresponding to the inputs  $B_1, B_2$  are rank deficient and in conclusion, the system is not observable to perform identifiability analysis in the presence of the perturbation model. For the lossy case a second example is therefore considered, represented by a DC motor, which will be proved to be observable. The observable transformation matrix  $T_{obs}$  from (3.4) when no perturbation is applied over the system, has the following form:

$$T_{obs} = \begin{bmatrix} \frac{1}{L} & 0 \\ -\frac{R}{L^2} & -\frac{1}{LJ_E} \end{bmatrix} \quad (3.190)$$

for a gyrator term  $\kappa_{DC} = 1$ . Considering a particular case for the real parameters with  $L = R = 1$  and  $J_E = b = 1/2$ , it is easy to see that the determinant of  $T_{obs}$  is different from 0 and the system is then observable. In this case, we consider a Gaussian White noise perturbation model of the form:

$$K = \begin{bmatrix} 1.7545 \\ 1.47 \end{bmatrix} \quad (3.191)$$

for the matrix  $K$ , corresponding to the perturbation model. The matrix  $E = K^T Q$  corresponding to the perturbation model, has the following form:

$$E = [1.755 \quad 2.94] \quad (3.192)$$

while the matrix  $C = B^T Q$  has the form:

$$C = [1 \quad 0] \quad (3.193)$$

The observable transformation matrix from (3.4) becomes:

$$T_{obs} = \begin{bmatrix} C + E \\ (C + E)A \end{bmatrix} = \begin{bmatrix} 2.755 & 2.94 \\ 0.185 & -8.45 \end{bmatrix} \quad (3.194)$$

Computing the  $A$ ,  $B$ ,  $C'$  and  $K$  matrices for the observable canonical form using (3.73), we get:

$$A_{obs} = T_{obs} A T_{obs}^{-1} = \begin{bmatrix} -0.002 & 1.003 \\ -3.001 & -1.999 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \quad (3.195)$$

$$B_{obs} = T_{obs} B \quad (3.196)$$

$$C'_{obs} = (C + E) T_{obs}^{-1} = [1.002 \quad -0.002] = [1 \quad 0] \quad (3.197)$$

$$K_{obs} = T_{obs} K = \begin{bmatrix} 9.155 \\ -12.097 \end{bmatrix} \quad (3.198)$$

As for the lossless case,  $C'$  represents the matrix  $C + E$  corresponding to the perturbation applied over the system output as in (3.36). Using (3.94), (3.95) and (3.106), we can write the equivalent observable canonical form as follows:

$$A_{obsT} = \begin{bmatrix} -1.999 & 1.003 \\ -3.001 & -0.002 \end{bmatrix} \quad (3.199)$$

$$B_{obsT} = \begin{bmatrix} 2.755 \\ 5.695 \end{bmatrix} \quad (3.200)$$

$$C_{obsT} = [1 \quad 0] \quad (3.201)$$

$$K_{obsT} = \begin{bmatrix} 9.155 \\ 6.213 \end{bmatrix} \quad (3.202)$$

Using (3.138), (3.199), (3.200), (3.201) and (3.202) the practical identifiability matrix for the equivalent observable canonical form is:

$$R_{ID} = \begin{bmatrix} 2.755 & 9.155 & 0.203 & -12.069 \\ 5.695 & 6.213 & -8.279 & -27.487 \end{bmatrix} \quad (3.203)$$

In order to determine the controllability of the system, it is necessary to prove that the matrix  $R_{ID}$  is not rank deficient. We select then a minor formed with the 1<sup>st</sup> two columns for which we check the determinant:

$$\det R'_{ID} = \det \begin{bmatrix} 2.755 & 9.155 \\ 5.695 & 6.213 \end{bmatrix} = -35.021 \quad (3.204)$$

Using the result from (3.204), it results that the DC motor is practically identifiable for the selected perturbation model and parameters. As for the lossless case example, the DC motor is not practically identifiable when the transformation matrix  $T_{obs}$  in (3.194) is rank deficient.

## Conclusions

The present chapter proposed a practical identifiability methodology for Port-Hamiltonian systems, starting from classic theoretical aspects of state-space systems, which combine the basic known observability and controllability concepts. Section 2 proved that Port-Hamiltonian systems can be converted to the observable canonical form, in the presence of a perturbation model, conserving the basic properties of PCH systems. Also it was introduced a definition for the perturbation model of Port-Hamiltonian systems, using the interaction port of the system with the environment. Section 3 was dedicated to the conversion of Port-Hamiltonian systems into an equivalent observable canonical form, which is used for the practical identifiability analysis in section 4 and 5. The last two sections proposed some general formulations for the proposed practical identifiability methodology in the lossless or lossy cases in the presence of a perturbation model and presented some examples of a LC circuit, a capacitor microphone circuit and a DC motor. The results of this chapter show that the LC circuit and DC motor are identifiable in the presence of a perturbation model, while the capacitor microphone circuit is not observable and thus not practical identifiable.

## 4. Symplectic time integration and discretization error Hamiltonian

*A general framework of time-discretization is proposed in this chapter for PCH systems, in the nonlinear and linear case, by combining different time-discretization rules for the flows and efforts, preserving thus the structure. A set of symplectic time-discretization schemes is proposed using classic first or second order rules for the flows and efforts. For each time-discretization scheme, the symplecticity with respect to the natural power pairing (bilinear form) is proved and the resulting conservation of the total energy is checked both for the lossless or lossy cases. The differentiation of two PCH systems, is then introduced and applied in order to define a **discretization error Hamiltonian**, for which the power energies of the ports, are given as the difference between the continuous and discrete-time systems.*

### 4.1 Introduction

Computer aided process control, is realized in discrete-time by sampling the continuous-time signal. Therefore, many discretization techniques have been designed to approximate accurately, the process and controller models [71-80]. Among these, when considering low order integration models, one could mention among the most important ones: **Step Invariant** or **Zero Order Hold (ZOH)** rule, **Matched Pole-Zero** rule, forward and backward **Euler** rules, **Tustin** rule, **Generalized Bilinear Transformation** rule, mid-point rule or **Runge-Kutta** rule. For higher integration techniques of the process or controller, discretization methods like **Boxer-Thaler** and **Madwed** [81-91], have been introduced, with higher accuracy for the approximated model, but with complex practical implementation and low possibility for stability analysis. PCH systems have some remarkable properties, among others power-energy conservation and stability under power preserving interconnection (e.g. parallel, series or feedback interconnections) [91]. Structure preserving time-integration (also termed as geometric integration) of Hamiltonian or PCH systems is thus a natural question and indeed a current research concern [9-106]. The aim of this chapter, is to develop discrete-time (control or simulation) models which inherits the structural properties of the continuous-time model (e.g. conservativeness, dissipativity or passivity, stability, controllability, etc.). In this chapter, we propose a methodology to generate symplectic time-discretization schemes for PCH systems, combining distinct discretization rules for the flow and effort variables. Some combinations of classical rules (explicit and implicit Euler, trapezoidal or mid-point rules) are investigated as an illustration of the methodology. For each of these combinations, the symplecticity is proved and the power conservation law checked. Then we introduce a similar approach to the backward error analysis, which has been previously developed for closed Hamiltonian systems in [96]. This generalization is based on a differentiation technique between two PCH systems, which defines a new PCH system with respect to the power-energy difference between ports of the two original PCH systems. A **discretization error Hamiltonian system** is then built using this difference between the continuous-time (original model) and the discrete-time model, obtained by the previously defined symplectic integration schemes. The chapter is organized as follows:

*Section 4.2* presents a general framework for symplectic time-discretization in the nonlinear and linear case, discussing the necessary conditions to preserve the power conservation law and also introduces the power pairing bilinear form, as well as explicit and implicit forms for discrete-time PCH systems.

Starting from this general framework of symplectic time-discretization, *section 4.3* presents a set of three symplectic time-discretization schemes, using four first or second order rules for the flows and efforts, by writing the associated power conservation law, bilinear form and the explicit and implicit discrete-time PCH systems.

*Section 4.4* introduces a new concept to differentiate two PCH systems with respect to the power energies associated to the ports, as a new Port-Hamiltonian system. This formulation is then used to define a ***discretization error Hamiltonian system*** between the continuous-time and discrete-time PCH systems.

*Section 4.5* presents a set of simulations for the selected symplectic time-discretization schemes to prove the theoretical results, by drawing specific curves like output responses, power energies, error logarithm for the continuous-time, discrete-time systems or ***discretization error Hamiltonian***.

## **4.2 Structural preserving discretization of PCH systems**

This section presents a symplectic time-discretization methodology for PCH systems, assuming that two different discretization rules are used for the flows and efforts, in order to obtain a discrete-time PCH system. The first part of this section presents a general framework for time-discretization in the nonlinear and linear case, with necessary conditions which must be satisfied to preserve the PCH systems basic properties. The second part presents a set of three symplectic time-discretization schemes, using four first or second order time-discretization rules (explicit and implicit Euler, implicit mid-point rule and implicit trapezoidal rule) for the flows and efforts.

### **4.2.1 General framework of symplectic discretization**

This section introduces a general framework for symplectic time-discretization in the nonlinear and linear cases, which considers two different discretization rules for the flows and efforts. The power conservation law is derived for the discretized system together with all other characteristic properties of PCH systems. The linear case of the proposed methodology is easily derived from the nonlinear one in section 4.2.1.1, 4.2.1.2 and used further for time-discretization of three symplectic schemes in section 4.3. By time-discretization it is proved that a discrete-time PCH system results, for which the flows and efforts of the continuous-time system ports, are approximated with discrete-time flows and efforts, while the interconnection structure ( $D$ ) is conserved. In the sequel, we will consider general approximations denoted:

$$\begin{cases} \hat{e}_S(i) \stackrel{def}{=} Ed_i \left( \frac{\partial H}{\partial x}(x) \right), \hat{f}_S(i) \stackrel{def}{=} Fd_i(-\dot{x}) \\ \hat{e}_C(i) \stackrel{def}{=} Ed_i \left( B^T(x) \cdot \left( \frac{\partial H}{\partial x}(x) \right) \right), \hat{f}_C(i) \stackrel{def}{=} Fd_i(u) \\ \hat{e}_R(i) \stackrel{def}{=} Ed_i \left( \frac{\partial H}{\partial x}(x) \right), \hat{f}_R(i) \stackrel{def}{=} Ed_i \left( -R(x) \cdot \frac{\partial H}{\partial x}(x) \right) \end{cases} \quad (4.1)$$

respectively for the discrete-time flows and efforts associated to the storage  $(\hat{e}_S, \hat{f}_S)$ , control  $(\hat{e}_C, \hat{f}_C)$  and dissipation  $(\hat{e}_R, \hat{f}_R)$  ports.

**Remark:** The discrete-time flow  $\hat{f}_R(i)$  of the dissipation port, is not considered for discretization similarly to the storage port flow  $(\hat{f}_S)$ , as the continuous-time flow  $(f_R)$  can be generally expressed in relation to the effort  $(e_R)$  using an input-output mapping  $f_R = -F(e_R)$  [23].

We introduce the discrete-time power conservation law as:

$$\hat{e}_S^T(i)\hat{f}_S(i) + \hat{e}_C^T(i)\hat{f}_C(i) + \hat{e}_R^T(i)\hat{f}_R(i) = 0 \quad (4.2)$$

which must be satisfied also in discrete-time. For the discrete-time flows and efforts introduced in (4.1), we define the discrete-time finite-dimensional spaces  $\hat{F}$  and its dual  $\hat{E} := \hat{F}^*$ . Using (4.1) and (4.2) we can derive the following relation:

$$\begin{aligned} & \hat{e}_S^T(i)\hat{f}_S(i) + \hat{e}_C^T(i)\hat{f}_C(i) + \hat{e}_R^T(i)\hat{f}_R(i) = \\ & - \left( Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \right)^T \cdot Fd_i(J(x)) \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x}(x) \right) + \\ & + \left( Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \right)^T \cdot Fd_i(R(x)) \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x}(x) \right) - \\ & - \left( Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \right)^T \cdot Fd_i(B(x)) \cdot Fd_i(u) + \\ & + \left( Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \right)^T \cdot Ed_i(B(x)) \cdot Fd_i(u) - \\ & - \left( Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \right)^T \cdot Ed_i(R(x)) \cdot Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \end{aligned} \quad (4.3)$$

which we observe to be satisfied for:



$$\begin{cases}
\left( Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \right)^T \cdot Fd_i(J(x)) \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x}(x) \right) = 0 \\
\left( Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \right)^T \cdot Fd_i(R(x)) \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x}(x) \right) - \\
- \left( Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \right)^T \cdot Ed_i(R(x)) \cdot Ed_i \left( \frac{\partial H}{\partial x}(x) \right) = 0 \\
\left( Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \right)^T \cdot Ed_i(B(x)) \cdot Fd_i(u) - \\
- \left( Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \right)^T \cdot Fd_i(B(x)) \cdot Fd_i(u) = 0
\end{cases} \quad (4.4)$$

**Remark:** The composition between the flow ( $Fd_i$ ) and effort ( $Ed_i$ ) time-discretization rules, is defined by  $Fd_i * Ed_i$ , where the effort and flow discretization rules, are applied successively on the continuous-time effort model. When performing the time-discretization of the storage port flow  $f_s$  in the nonlinear case, it is necessary to discretize the interconnection matrix ( $J(x)$ ) and dissipation matrix ( $R(x)$ ) separately, before performing the product with the discrete-time effort  $Ed_i \left( \frac{\partial H}{\partial x}(x) \right)$ , as it is not possible to multiply a continuous-time value with a discrete-time one. Separate time-discretization is considered thus, for all the continuous-time parameters.

The necessary conditions for (4.4) to hold are then:

$$\begin{cases}
Ed_i \left( \frac{\partial H}{\partial x}(x) \right) = Fd_i * Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \\
Ed_i(R(x)) = Fd_i(R(x)) \\
Ed_i(B(x)) = Fd_i(B(x))
\end{cases} \quad (4.5)$$

**Remark:** The dissipation flow  $f_R$  can be discretized also as  $\hat{f}_R = -Fd_i(R(x))Fd_i * Ed_i \left( \frac{\partial H}{\partial x}(x) \right)$ , in which case the first two relations in (4.5) are always satisfied and thus, the necessary conditions are reduced, making possible to generate a broader class of discretization schemes. In order to generate also non-symplectic time-discretization schemes as counter examples, we will consider further the case of (4.1) to discretize the dissipation flow  $f_R$ . It can easily be observed in (4.5), that using the same Euler time-discretization rule (explicit or implicit) for the flows and efforts with a nonlinear model, will always satisfy all three relations. Also it can be proved, that using the same second order time-discretization rules (implicit mid-point or implicit trapezoidal rule) for the flows and efforts, will satisfy the second and third relations of (4.5), but never the first one for a nonlinear model.

For the linear case which is considered in the next sections, the necessary conditions for (4.4) to be satisfied become:

$$\begin{cases} \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T \cdot J \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right) = 0 \\ \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T \cdot R \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right) = \\ \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T R \cdot Ed_i \left( \frac{\partial H}{\partial x} \right) \end{cases} \quad (4.6)$$

In this case, the last relation from (4.4) disappears, as the input matrix  $B$  doesn't have a nonlinear model. Using the necessary conditions proposed in (4.4) for the preservation of the conservation law (4.2) in discrete-time, we propose the following proposition for the preservation of the *power conservation law*:

**Proposition 1[194]**

*The PCH system (1.27), preserves the power conservation law (4.2) by symplectic time-discretization using the discretization rules (4.1) for the power port flows and efforts, when (4.5) holds.*

**Remark:** *This proposition can be also used for the linear case analysis of PCH systems, when (4.6) holds. As will be seen in the next sections, situations may occur in which one of the conditions in (4.5) or (4.6) is satisfied but not the others and thus, the preservation of PCH systems structural properties in discrete-time is not satisfied.*

We introduce further the discrete-time equivalent for the energy balance (Hamiltonian) using (1.24) and (4.1), by:

$$\begin{aligned} H(t + \Delta t) &\stackrel{def}{=} H(t) + \Delta t \cdot \left( Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \right)^T \cdot \\ &\cdot Fd_i \left( \begin{array}{l} (J(x) - R(x)) \cdot Ed_i \left( \frac{\partial H}{\partial x}(x) \right) + \\ + B(x) \cdot u \end{array} \right) \end{aligned} \quad (4.7)$$

which is equivalent to:

$$H(t + \Delta t) \stackrel{def}{=} H(t) + \Delta t \cdot \left( \hat{e}_C^T \hat{f}_C + \hat{e}_R^T \hat{f}_R \right) \quad (4.8)$$

by straightforward computation when (4.5) or (4.6) holds and thus, (1.24) is also preserved in discrete-time. An equivalent description of a Dirac structure can be realized by introducing the discrete-time bilinear form [23]:

$$\hat{B}_F(i) \stackrel{def}{=} \ll \left( \hat{f}^a, \hat{e}^a \right), \left( \hat{f}^b, \hat{e}^b \right) \gg = \left\langle \hat{e}^a \middle| \hat{f}^b \right\rangle + \left\langle \hat{e}^b \middle| \hat{f}^a \right\rangle \quad (4.9)$$

on the discrete-time space of flows and efforts  $(\hat{F}_x \hat{F}^*)$  with  $(\hat{f}^a, \hat{e}^a), (\hat{f}^b, \hat{e}^b) \in \hat{F}_x \hat{F}^*$ .

Replacing the discrete-time flows and efforts from (4.1) in (4.9), we can write equivalently [117]:

$$\begin{aligned}
\hat{B}_F(i) &\stackrel{def}{=} \left\langle \left( Ed_i \left( \frac{\partial H}{\partial x}(x) \right), Ed_i \left( B^T(x) \cdot \frac{\partial H}{\partial x}(x) \right), Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \right) \right\rangle^a \Bigg| \\
&\left( Fd_i(-\dot{x}), Fd_i(u), Ed_i \left( -R(x) \cdot \left( \frac{\partial H}{\partial x}(x) \right) \right) \right) \Bigg\rangle^b > + \\
&\left\langle \left( Ed_i \left( \frac{\partial H}{\partial x}(x) \right), Ed_i \left( B^T(x) \cdot \frac{\partial H}{\partial x}(x) \right), Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \right) \right\rangle^b \\
&\left( Fd_i(-\dot{x}), Fd_i(u), Ed_i \left( -R(x) \cdot \left( \frac{\partial H}{\partial x}(x) \right) \right) \right) \Bigg\rangle^a >
\end{aligned} \tag{4.10}$$

The discrete-time bilinear form from (4.10) is related to the definition of discrete-time power conservation law defined in (4.2), for which (4.5) must be satisfied. We also introduce the implicit discrete-time dynamics form for the PCH system using (1.26), as:

$$\left\{ \begin{array}{l} Fd_i(-\dot{x}), Ed_i \left( \frac{\partial H}{\partial x}(x) \right), Fd_i(u), Ed_i \left( B^T(x) \cdot \frac{\partial H}{\partial x}(x) \right), \\ Ed_i \left( -R(x) \cdot \left( \frac{\partial H}{\partial x}(x) \right) \right), Ed_i \left( \frac{\partial H}{\partial x}(x) \right) \end{array} \right\} \in D \tag{4.11}$$

**Remark:** *The Dirac interconnection (D) structure and rules are preserved by time-discretization, as they serve only for power port interconnection. This structural preserving time-discretization methodology, can be adapted to more general classes of time-discretization rules (see Runge-Kutta methods, collocation methods, etc.) [96, 114].*

#### 4.2.1.1 Lossless PCH systems

The general continuous-time explicit form of the linear lossless PCH systems, for which we apply symplectic discretization is:

$$\left\{ \begin{array}{l} \dot{x} = J \frac{\partial H}{\partial x} + Bu \\ y = B^T \cdot \frac{\partial H}{\partial x} \end{array} \right. \tag{4.12}$$

A general form of the symplectic time-discretization schemes for the lossless case, when using different discretization rules for the flows and efforts as defined in (4.1), can be written as follows:

$$\left\{ \begin{array}{l} x_{k+1} = x_k + \Delta t \cdot Fd_i \left( J \cdot \frac{\partial H}{\partial x} + B \cdot u \right) \\ y_i = B^T \cdot Ed_i \left( \frac{\partial H}{\partial x} \right) \end{array} \right. \tag{4.13}$$

**Remark:** In the above relation,  $F_{di}$  and  $E_{di}$ , represent the time-discretization rules (explicit or implicit form) applied for the continuous-time flows and efforts at step  $i$  of discretization as defined in (4.1).

More compactly, we can write (4.13) as:

$$\begin{cases} \frac{\Delta x}{\Delta t}(i) = J \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right) + B \cdot Fd_i(u) \\ y_i = B^T Ed_i \left( \frac{\partial H}{\partial x} \right) \end{cases} \quad (4.14)$$

where  $i$  corresponds to the current discretization time-step for the selected rules. In the lossless case, the discrete-time power conservation law (4.2) becomes:

$$\hat{e}_S^T(i) \hat{f}_S(i) + \hat{e}_C^T(i) \hat{f}_C(i) = 0 \quad (4.15)$$

Using the general time-discretization form from (4.3) in the linear case, the right side part of (4.14) becomes:

$$\left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T \left( -J \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right) - B \cdot Fd_i(u) \right) + \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T B \cdot Fd_i(u) \quad (4.16)$$

Performing the possible simplifications in (4.16), it results that the total discrete-time power-conservation law is:

$$\hat{e}_S^T(i) \hat{f}_S(i) + \hat{e}_C^T(i) \hat{f}_C(i) = - \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T \cdot J \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right) \quad (4.17)$$

From (4.17) it results that the power conservation law is satisfied in discrete-time, when the right part of the equality is zero.

**Remark:** For the linear lossless case presented in this section, the only relation that needs to be satisfied in (4.6) is the first one, in order to preserve the discrete-time power conservation law (4.1). This happens when the discret-time terms  $Ed_i \left( \frac{\partial H}{\partial x} \right)$  and  $Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right)$  have common matrix factors to the left and right of the skew-symmetric matrix  $J$ .

In this case, the right part of the product in (4.17) is zero and the power conservation law holds in discrete-time. Using (4.7) we define the discrete-time energy (Hamiltonian) as:

$$\frac{\Delta H}{\Delta t}(i) \stackrel{def}{=} \left( \frac{\Delta H}{\Delta x} \right)^T \frac{\Delta x}{\Delta t} = \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T B \cdot Fd_i(u) = y_i^T u_i \quad (4.18)$$

when (4.15) holds. Then it is obvious from (4.8), that the discretized (approximated) Hamiltonian energy function  $\hat{H}(x)$  is also conserved in discrete-time form. Using (4.9) and (4.10), the expression of the discrete-time bilinear form, for lossless PCH systems can be written as follows:

$$\begin{aligned} \hat{B}_F(i) = & \left\langle \left( Ed_i \left( \frac{\partial H}{\partial x} \right), B^T \cdot Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^a, (Fd_i(-\dot{x}), Fd_i(u))^b \right\rangle + \\ & \left\langle \left( Ed_i \left( \frac{\partial H}{\partial x} \right), B^T \cdot Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^b, (Fd_i(-\dot{x}), Fd_i(u))^a \right\rangle \end{aligned} \quad (4.19)$$

where  $i$  corresponds to the current time-discretization step for the selected symplectic scheme. Using (4.11), we introduce the discrete-time implicit form in the lossless linear case as:

$$\left( Fd_i(-\dot{x}), Ed_i\left(\frac{\partial H}{\partial x}\right), Fd_i(u), B^T \cdot Ed_i\left(\frac{\partial H}{\partial x}\right) \right) \in D \quad (4.20)$$

or more generally:

$$\left( \hat{f}_S(i), \hat{e}_S(i), \hat{f}_C(i), \hat{e}_C(i) \right) \in D \quad (4.21)$$

**Remark:** *The interconnection rules for the Dirac structure ( $D$ ) remain in the discretized Bond Space, which is basically the same (the efforts and flows at discrete times have values in the same vector spaces as for the continuous case).*

#### 4.2.1.2 Lossy PCH systems

The general explicit form of a linear lossy PCH system, for which we apply the symplectic time-discretization, is:

$$\begin{cases} \dot{x} = (J - R) \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (4.22)$$

A general expression of the symplectic time-discretization schemes for lossy PCH systems, when applying two different discretization rules for the flows and efforts, is the following:

$$\begin{cases} x_{k+1} = x_k + \Delta t \cdot (J - R) \cdot Fd_i * Ed_i\left(\frac{\partial H}{\partial x}\right) + \Delta t \cdot B \cdot Fd_i(u) \\ y_i = B^T \cdot Ed_i\left(\frac{\partial H}{\partial x}\right) \end{cases} \quad (4.23)$$

**Remark:**  $F_{di}$  and  $E_{di}$  represent two different discretization rules applied for the flows and efforts at step  $i$  (in explicit or implicit form) of time-discretization.

More compactly, we can write (4.23) as:

$$\begin{cases} \frac{\Delta x}{\Delta t}(i) = (J - R) \cdot Fd_i * Ed_i\left(\frac{\partial H}{\partial x}\right) + B \cdot Fd_i(u) \\ y_i = B^T \cdot Ed_i\left(\frac{\partial H}{\partial x}\right) \end{cases} \quad (4.24)$$

**Remark:** *This symplectic time-discretization methodology, can be adapted and applied for any two time-discretization rules (even the same), applied to the flows and efforts.*

Using the continuous-time power conservation law introduced in (1.23), we can write the discrete-time equivalent as:

$$\hat{e}_S^T(i) \hat{f}_S(i) + \hat{e}_C^T(i) \hat{f}_C(i) + \hat{e}_R^T(i) \hat{f}_R(i) = 0 \quad (4.25)$$

Using the discrete-time flows and efforts defined in (4.1), we can write further:

$$\begin{aligned} & \left( Ed_i\left(\frac{\partial H}{\partial x}\right) \right)^T \left( -(J - R) \cdot Fd_i * Ed_i\left(\frac{\partial H}{\partial x}\right) - B \cdot Fd_i(u) \right) + \left( Ed_i\left(\frac{\partial H}{\partial x}\right) \right)^T B \cdot Fd_i(u) \\ & - \left( Ed_i\left(\frac{\partial H}{\partial x}\right) \right)^T R \cdot Ed_i\left(\frac{\partial H}{\partial x}\right) \end{aligned} \quad (4.26)$$

for the discrete-time power conservation law. Performing the possible simplifications in (4.26), the power conservation law becomes:

$$\begin{aligned} \hat{e}_S^T(i) \hat{f}_S(i) + \hat{e}_C^T(i) \hat{f}_C(i) + \hat{e}_R^T(i) \hat{f}_R(i) = & - \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T J \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right) + \\ & + \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T R \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right) - \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T R \cdot Ed_i \left( \frac{\partial H}{\partial x} \right) \end{aligned} \quad (4.27)$$

As for the nonlinear case, the necessary condition to be satisfied in discrete-time for power conservation law becomes then:

$$\begin{aligned} & - \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T J \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right) + \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T R \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right) \\ & - \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T R \cdot Ed_i \left( \frac{\partial H}{\partial x} \right) = 0 \end{aligned} \quad (4.28)$$

We observe that this relation holds for:

$$\begin{cases} \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T \cdot J \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right) = 0 \\ \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T \cdot R \cdot Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right) = \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T R \cdot Ed_i \left( \frac{\partial H}{\partial x} \right) \end{cases} \quad (4.29)$$

Then we can derive a condition for (4.29) to be satisfied as:

$$Ed_i \left( \frac{\partial H}{\partial x} \right) = Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right) \quad (4.30)$$

Using the continuous-time relation (1.24) for the energy balance (Hamiltonian), we define the discrete-time equivalent as for the nonlinear case in (4.7):

$$\frac{\Delta H}{\Delta t}(i) \stackrel{def}{=} \left( Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^T Fd_i \left( (J - R) \cdot Ed_i \left( \frac{\partial H}{\partial x} \right) + B \cdot u \right) = \hat{e}_C^T \hat{f}_C + \hat{e}_R^T \hat{f}_R \quad (4.31)$$

The right part of (4.31) results by straightforward computation replacing the discrete-time flows and efforts as defined in (4.1), when (4.30) is satisfied. The expression of the linear discrete-time bilinear form as defined in (4.10) for the nonlinear case becomes:

$$\begin{aligned} \hat{B}_F(i) = & \left\langle \left( Ed_i \left( \frac{\partial H}{\partial x} \right), B^T \cdot Ed_i \left( \frac{\partial H}{\partial x} \right), Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^a, \left( Fd_i(-\dot{x}), Fd_i(u), -R \cdot Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^b \right\rangle + \\ & + \left\langle \left( Ed_i \left( \frac{\partial H}{\partial x} \right), B^T \cdot Ed_i \left( \frac{\partial H}{\partial x} \right), Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^b, \left( Fd_i(-\dot{x}), Fd_i(u), -R \cdot Ed_i \left( \frac{\partial H}{\partial x} \right) \right)^a \right\rangle \end{aligned} \quad (4.32)$$

The implicit form of the discrete-time linear lossy PCH system is then:

$$\left( Fd_i(-\dot{x}), Ed_i \left( \frac{\partial H}{\partial x} \right), Fd_i(u), B^T \cdot Ed_i \left( \frac{\partial H}{\partial x} \right), -R \cdot Ed_i \left( \frac{\partial H}{\partial x} \right), Ed_i \left( \frac{\partial H}{\partial x} \right) \right) \in D \quad (4.33)$$

or more compactly:

$$\left( \hat{f}_S(i), \hat{e}_S(i), \hat{f}_C(i), \hat{e}_C(i), \hat{f}_R(i), \hat{e}_R(i) \right) \in D \quad (4.34)$$

where the Dirac structure  $D$  conserves its structure by time-discretization as was already explained.

### 4.3 Symplectic discretization schemes

A set of three symplectic time-discretization schemes for the lossless and lossy PCH systems are considered here as examples in the linear case, using different (first or second order) discretization rules for the flows and efforts. As discretization rules are proposed: explicit and implicit Euler rule, implicit mid-point rule and implicit trapezoidal rule. For each combined time-discretization scheme, it is checked and proved the power conservation law of the system, the bilinear form is expressed together with the explicit and implicit forms for the discrete-time PCH system, using the general framework of 4.2.1.

#### 4.3.1 Lossless PCH systems

In the case of lossless PCH systems, the following three symplectic discretization schemes were proposed to check the general framework introduced in section 4.2.1: explicit Euler rule for the flows and implicit Euler rule for the efforts; implicit Euler rule for the flows and implicit trapezoidal rule for the efforts; implicit trapezoidal rule for the flows and implicit mid-point rule for the efforts. We have selected these simple symplectic time-discretization schemes as examples for the general framework of 4.2.1, to see if the PCH systems properties are satisfied in discrete-time and also to point out their performances.

**Remark:** *Non-symplectic time-discretization schemes for lossless PCH systems can be obtained when the first relation of (4.6) doesn't hold.*

##### 4.3.1.1 Symplectic discretization scheme I

For the first symplectic time-discretization scheme, it was selected the explicit Euler rule for the flows and implicit Euler rule, for the efforts. Combining the explicit Euler rule for the flows and implicit Euler rule for the efforts using (4.3), the following relations result for the explicit form of the discrete-time PCH system:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k) = J \cdot \frac{\Delta H}{\Delta x}(k) + B \cdot u_k \\ \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1}, y_k = B^T \cdot Qx_k \end{cases} \quad (4.35)$$

**Remark:** *The composed time discretization  $Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right)$  of the effort becomes  $Qx_k$ , by applying the flow discretization over  $\frac{\Delta H}{\Delta x}$ .*

The second relation in (4.35) for the discrete-time flow results from the fact that the total energy (Hamiltonian) of the system, can be generally written as:

$$H(x) = \frac{1}{2} x^T Q x \quad (4.36)$$

**Remark:** In all time-discretization schemes that follow, we have adopted the notation " $\Delta$ ", to designate the discrete-time differentiator of the flows and efforts, while the current discretization time-step of the flow and efforts, is represented by  $k$  or  $k+1$ , corresponding also to the explicit or implicit forms of the discretization rules. The discrete-time forms of PCH systems from the general framework, will borrow the explicit or implicit form as the flow discretization rules selected.

Using (4.35) the discrete-time power energy associated to the storage port at step  $k$  of time-discretization, can be defined as:

$$\hat{P}_S(k) \stackrel{def}{=} \hat{e}_S^T(k) \cdot \hat{f}_S(k) = \left( \frac{\Delta H}{\Delta x}(k) \right)^T \left( -\frac{\Delta x}{\Delta t}(k) \right) = (Q x_k)^T \left( -J \cdot \frac{\Delta H}{\Delta x}(k) - B \cdot u_k \right) \quad (4.37)$$

where  $\hat{e}_S(k)$  and  $\hat{f}_S(k)$  represent the discrete-time forms of the effort and flow defined in (4.1). From straight-forward computations, we obtain the following general relation:

$$\hat{P}_S(k) \stackrel{def}{=} -\hat{e}_S^T(k) \cdot B \cdot \hat{f}_C(k) \quad (4.38)$$

using discrete-time flows and efforts. Similarly for the control port power energy, we can write:

$$\hat{P}_C(k) \stackrel{def}{=} \hat{e}_C^T(k) \hat{f}_C(k) = y_k^T u_k = \left( B^T Q x_k \right)^T u_k \quad (4.39)$$

From simple computations, we can write (4.39) as:

$$\hat{P}_C(k) \stackrel{def}{=} \hat{e}_S^T(k) \cdot B \cdot \hat{f}_C(k) \quad (4.40)$$

using the discrete-time flows and efforts of the control and storage ports. Using (4.38) and (4.40), the total power energy at step  $k$  of discretization is conserved, since we can write:

$$\hat{P}_S(k) + \hat{P}_C(k) = -\hat{e}_S^T(k) \cdot B \cdot \hat{f}_C(k) + \hat{e}_S^T(k) \cdot B \cdot \hat{f}_C(k) = 0 \quad (4.41)$$

Using (4.18), we define the discrete-time energy (Hamiltonian) as:

$$\frac{\Delta H}{\Delta t}(k) \stackrel{def}{=} -\hat{e}_S^T(k) \hat{f}_S(k) = y_k^T u_k \quad (4.42)$$

Using (4.38) and (4.40), it becomes further:

$$\frac{\Delta H}{\Delta t}(k) \stackrel{def}{=} \hat{e}_S^T(k) B \hat{f}_C(k) \quad (4.43)$$

The discrete-time bilinear defined in (4.19) becomes in this case:

$$\hat{B}_F(k) \stackrel{def}{=} \left\langle \left( \hat{e}_S(k), \hat{e}_C(k) \right)^a, \left( \hat{f}_S(k), \hat{f}_C(k) \right)^b \right\rangle + \left\langle \left( \hat{e}_S(k), \hat{e}_C(k) \right)^b, \left( \hat{f}_S(k), \hat{f}_C(k) \right)^a \right\rangle \quad (4.44)$$

which becomes further for this scheme:

$$\hat{B}_F(k) \stackrel{def}{=} \left\langle \left( (Q x_k), (B^T \cdot Q \cdot x_k) \right)^a, \left( -\frac{\Delta x}{\Delta t}(k), u_k \right)^b \right\rangle + \left\langle \left( (Q x_k), (B^T \cdot Q \cdot x_k) \right)^b, \left( -\frac{\Delta x}{\Delta t}(k), u_k \right)^a \right\rangle \quad (4.45)$$



by replacing the discrete-time efforts and flows of each port as defined in (4.1). Using (4.20), the implicit form of the discrete-time linear lossless PCH system at step  $k$  of discretization, is then:

$$\left( \hat{f}_S(k), \hat{e}_S(k), \hat{f}_C(k), \hat{e}_C(k) \right) \in D \quad (4.46)$$

where the Dirac structure ( $D$ ) is described by the same linear equations as the original continuous-time one.

#### 4.3.1.2 Symplectic discretization scheme II

For the second symplectic time-discretization scheme, the implicit Euler rule and the implicit trapezoidal rule, were selected for the flows and efforts as defined in (4.1). Applying the selected discretization rules, to the continuous-time form of the lossless PCH system from (4.12) using (4.4), it results the following explicit form of the linear lossless discrete-time PCH system:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = JQ \cdot \bar{x}_{k+1} + B \cdot u_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Q \cdot \bar{x}_{k+1}, y_{k+1} = B^T \cdot Q \cdot \bar{x}_{k+1} \end{cases} \quad (4.47)$$

where the following notation was adopted for simplicity:

$$\bar{x}_{k+1} \stackrel{def}{=} \left( \frac{x_k + x_{k+1}}{2} \right) \quad (4.48)$$

**Remark:** The composed time- discretization  $Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right)$  of the effort becomes

$Q \left( \frac{x_k + x_{k+1}}{2} \right)$  by applying the flow discretization over  $\frac{\Delta H}{\Delta x}$  at step  $k+1$ .

At the next step, the power conservation law (4.15) is checked by time-discretization. The power energy associated to the storage port at step  $k+1$  of time-discretization, can be defined as follows using (4.47):

$$\hat{P}_S(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) \hat{f}_S(k+1) = \left( Q \cdot \bar{x}_{k+1} \right)^T \left( -J \cdot Q \cdot \bar{x}_{k+1} - B \cdot u_{k+1} \right) \quad (4.49)$$

From simple computations, (4.49) can be written more compact as:

$$\hat{P}_S(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1) \cdot B \cdot \hat{f}_C(k+1) \quad (4.50)$$

The discrete-time power energy of the control port, can be defined using the following formula by replacing the discrete-time flows and efforts:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_C^T(k+1) \hat{f}_C(k+1) = \left( B^T \cdot Q \cdot \bar{x}_{k+1} \right)^T u_{k+1} \quad (4.51)$$

Equivalently, we can write this relation as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) \cdot B \cdot \hat{f}_C(k+1) \quad (4.52)$$

Using (4.50) and (4.52), it results that the discrete-time power conservation law (4.15) is conserved:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) = -\hat{e}_S^T(k+1) \cdot B \cdot \hat{f}_C(k+1) + \hat{e}_S^T(k+1) \cdot B \cdot \hat{f}_C(k+1) = 0 \quad (4.53)$$

From (4.18), we define the energy (Hamiltonian) as:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1) \hat{f}_S(k+1) = y_{k+1}^T u_{k+1} \quad (4.54)$$

which becomes further:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (4.55)$$

Using (4.19), the discrete-time bilinear form is defined as:

$$\begin{aligned} \hat{B}_F(k+1) &\stackrel{def}{=} \left\langle \left( \hat{e}_S(k+1), \hat{e}_C(k+1) \right)^a, \left( \hat{f}_S(k+1), \hat{f}_C(k+1) \right)^b \right\rangle + \\ &+ \left\langle \left( \hat{e}_S(k+1), \hat{e}_C(k+1) \right)^b, \left( \hat{f}_S(k+1), \hat{f}_C(k+1) \right)^a \right\rangle \end{aligned} \quad (4.56)$$

which becomes further by replacing the discrete-time flows and efforts defined in (4.1):

$$\begin{aligned} \hat{B}_F(k+1) &\stackrel{def}{=} \left\langle \left( Q \bar{x}_{k+1}, B^T \cdot Q \cdot \bar{x}_{k+1} \right)^a, \left( -\frac{\Delta x}{\Delta t}(k+1), u_{k+1} \right)^b \right\rangle + \\ &+ \left\langle \left( Q \cdot \bar{x}_{k+1}, B^T \cdot Q \cdot \bar{x}_{k+1} \right)^b, \left( -\frac{\Delta x}{\Delta t}(k+1), u_{k+1} \right)^a \right\rangle \end{aligned} \quad (4.57)$$

for the selected symplectic time-discretization scheme. Using (4.21), the implicit form of the discrete-time lossless PCH system, has the following form:

$$\left( \hat{f}_S(k+1), \hat{e}_S(k+1), \hat{f}_C(k+1), \hat{e}_C(k+1) \right) \in D \quad (4.58)$$

where the the Dirac structure ( $D$ ) is conserved by time-discretization.

### 4.3.1.3 Symplectic discretization scheme III

For the last symplectic time-discretization scheme in the linear lossless case, it was selected the implicit trapezoidal rule for the flows and implicit mid-point rule for the efforts, which are two second order discretization rules. Applying these time-discretization rules using (4.1), for the continuous-time lossless PCH system from (4.12), it results the following explicit form for the discrete-time system:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = J \cdot Q \cdot \bar{x}_{k+1} + B \cdot \bar{u}_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Q \cdot \bar{x}_{k+1}, y_{k+1} = B^T \cdot Q \cdot \bar{x}_{k+1} \end{cases} \quad (4.59)$$

where the following notations:

$$\begin{cases} \bar{x}_{k+1} & \stackrel{def}{=} \left( \frac{x_k + x_{k+1}}{2} \right) \\ \bar{u}_{k+1} & \stackrel{def}{=} \left( \frac{u_k + u_{k+1}}{2} \right) \\ \bar{x}_{k+1} & \stackrel{def}{=} \frac{\bar{x}_k + \bar{x}_{k+1}}{2} \end{cases} \quad (4.60)$$

were adopted for simplicity.

**Remark:** The composed time-discretization  $Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right)$  of the effort becomes  $Q \bar{x}_{k+1}$ , by applying the flow discretization over  $\frac{\Delta H}{\Delta x}$  at step  $k+1$ .

Next we determine if the power conservation law (4.15) is conserved by time-discretization for the selected discretization rules. The discrete-time power energy, associated to the storage port, can be defined as follows:

$$\hat{P}_S(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) \hat{f}_S(k+1) = (Q \bar{x}_{k+1})^T (-J \cdot Q \cdot \bar{x}_{k+1} - B \cdot \bar{u}_{k+1}) \quad (4.61)$$

by replacing the discrete-time flows and efforts as defined in (4.1).

Then it is straightforward that:

$$\hat{P}_S(k+1) = -\hat{e}_S^T(k+1) \cdot B \cdot \hat{f}_C(k+1) \quad (4.62)$$

The discrete-time control port power energy, is defined as follows:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_C^T(k+1) \hat{f}_C(k+1) = (B^T \cdot Q \cdot \bar{x}_{k+1})^T \bar{u}_{k+1} \quad (4.63)$$

and it can be also expressed as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) \cdot B \cdot \hat{f}_C(k+1) \quad (4.64)$$

It is obvious then that the total power energy is conserved in discrete-time:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) = -\hat{e}_S^T(k+1) \cdot B \cdot \hat{f}_C(k+1) + \hat{e}_S^T(k+1) \cdot B \cdot \hat{f}_C(k+1) = 0 \quad (4.65)$$

Using (4.18), the energy (Hamiltonian) of this scheme is defined as:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1) \hat{f}_S(k+1) = y_{k+1}^T u_{k+1} \quad (4.66)$$

Using (4.62) and (4.64) it results further:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (4.67)$$

Hence, the discrete-time bilinear form (4.19) for this symplectic time-discretization scheme is defined as:

$$\begin{aligned} \hat{B}_F(k+1) & \stackrel{def}{=} \left\langle \left( \hat{e}_S(k+1), \hat{e}_C(k+1) \right)^a, \left( \hat{f}_S(k+1), \hat{f}_C(k+1) \right)^b \right\rangle + \\ & + \left\langle \left( \hat{e}_S(k+1), \hat{e}_C(k+1) \right)^b, \left( \hat{f}_S(k+1), \hat{f}_C(k+1) \right)^a \right\rangle \end{aligned} \quad (4.68)$$

which becomes:

$$\begin{aligned} \hat{B}_F(k+1) &\stackrel{def}{=} \left\langle \left( Q \cdot \bar{x}_{k+1}, B^T \cdot Q \cdot \bar{x}_{k+1} \right)^a, \left( -\frac{\Delta x}{\Delta t}(k+1), \bar{u}_{k+1} \right)^b \right\rangle + \\ &+ \left\langle \left( Q \cdot \bar{x}_{k+1}, B^T \cdot Q \cdot \bar{x}_{k+1} \right)^b, \left( -\frac{\Delta x}{\Delta t}(k+1), \bar{u}_{k+1} \right)^a \right\rangle \end{aligned} \quad (4.69)$$

when replacing the discrete-time flows and efforts using (4.1) and (4.59).

Using (4.21), the implicit form of the discrete Dirac structure, that results for the lossless PCH system of the selected symplectic time-discretization scheme, is the following:

$$\left( \hat{f}_S(k+1), \hat{e}_S(k+1), \hat{f}_C(k+1), \hat{e}_C(k+1) \right) \in D \quad (4.70)$$

**Remark:** *The implicit form of the discrete-time PCH systems remains basically the same and thus, in the proposed time-discretization structural preserving framework, we have used the explicit PCH form, power conservation law and bilinear form to describe a meaningful discrete-time system. For the selected time-discretization rules (explicit/implicit Euler, implicit mid-point and implicit trapezoidal) it can be easily proved that the linear lossless PCH systems always conserve the structural properties defined in the general framework of (4.2.1.1) for any combination of time-discretization rules of the flows and efforts.*

### 4.3.2. Lossy PCH systems

As for the lossless case, in the linear lossy case are selected the same three symplectic time-discretization schemes, for which it is also proved using the general framework from (4.2), the preservation of PCH systems basic properties in discrete-time.

**Remark:** *Non-symplectic time-discretization rules for lossy PCH systems, can be obtained when (4.30) doesn't hold.*

#### 4.3.2.1 Symplectic discretization scheme I

The first symplectic time-discretization scheme selected for linear lossy PCH systems, is represented by an explicit Euler rule for the flows and implicit Euler rule for the efforts. Applying the general symplectic time-discretization framework from (4.13) for the selected discretization rules, it results:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k) = (J - R) \cdot Q \cdot x_k + B \cdot u_k \\ \frac{\Delta H}{\Delta x}(k) = Q \cdot x_k \cdot y_k = B^T \cdot Q \cdot x_k \end{cases} \quad (4.71)$$

**Remark:** *It is possible to select the same time-discretization rules for the flows and effort in the general framework of 4.2 on lossless or lossy cases, maintaining the same conditions for the preservation of PCH systems power conservation law. The composed time-discretization*

$Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right)$  of the effort becomes  $(J - R)Qx_k$ , by applying the flow discretization over  $\frac{\Delta H}{\Delta x}$ .

For the next step, the preservation of the discrete-time power conservation law is checked. Using (4.1) and (4.71), the power energy associated to the storage port at step  $k$  of discretization, can be defined as:

$$\hat{P}_S(k) \stackrel{def}{=} \hat{e}_S^T(k) \hat{f}_S(k) = (Q \cdot x_k)^T \left( -(J - R) \cdot Q \cdot x_k - B \cdot u_k \right) \quad (4.72)$$

Using simple relations, it is straightforward that:

$$\hat{P}_S(k) \stackrel{def}{=} \hat{e}_S^T(k) \cdot R \cdot \hat{e}_S(k) - \hat{e}_S^T(k) \cdot B \cdot \hat{f}_C(k) \quad (4.73)$$

The discrete-time control port power energy, can be defined as:

$$\hat{P}_C(k) \stackrel{def}{=} \hat{e}_C^T(k) \cdot \hat{f}_C(k) = \left( B^T \cdot Q \cdot x_k \right)^T \cdot u_k \quad (4.74)$$

Equivalently this relation can be written as:

$$\hat{P}_C(k) \stackrel{def}{=} \hat{e}_S^T(k) \cdot B \cdot \hat{f}_C(k) \quad (4.75)$$

using simple mathematical relations. For the dissipation port, it results:

$$\hat{P}_R(k) \stackrel{def}{=} \hat{e}_R^T(k) \cdot \hat{f}_R(k) \quad (4.76)$$

for the power energy. Using simple relations, (4.76) can be equivalently written as:

$$\hat{P}_R(k) \stackrel{def}{=} - (Qx_k)^T R (Qx_k) = - \hat{e}_S^T(k) \cdot R \cdot \hat{e}_S(k) \quad (4.77)$$

Using (4.73), (4.75) and (4.77), the total power energy in discrete-time form is:

$$\hat{P}_S(k) + \hat{P}_C(k) + \hat{P}_R(k) = \hat{e}_S^T R \hat{e}_S(k) - \hat{e}_S^T(k) \cdot B \cdot \hat{f}_C(k) + \hat{e}_S^T(k) B \hat{f}_C(k) - \hat{e}_S^T(k) R \hat{e}_S(k) = 0 \quad (4.78)$$

which satisfies the power conservation law (4.25) and thus the necessary conditions from (4.30). Using (4.31), the discrete-time energy (Hamiltonian) can be defined as:

$$\frac{\Delta H}{\Delta t}(k) \stackrel{def}{=} - \hat{e}_S^T(k) \hat{f}_S(k) = y_k^T u_k + \hat{e}_R^T(k) \hat{f}_R(k) \quad (4.79)$$

which is equivalent to:

$$\frac{\Delta H}{\Delta t}(k) \stackrel{def}{=} - \left( \hat{e}_S^T(k) R \hat{e}_S(k) - \hat{e}_S^T(k) B \hat{f}_C(k) \right) = \hat{e}_S^T(k) B \hat{f}_C(k) - \hat{e}_S^T(k) R \hat{e}_S(k) \quad (4.80)$$

using (4.73), (4.75) and (4.77).

The discrete-time bilinear form, can be defined using (4.32) as:

$$\begin{aligned} \hat{B}_F(k) &\stackrel{def}{=} \left\langle \left( \hat{e}_S(k), \hat{e}_C(k), \hat{e}_R(k) \right)^a, \left( \hat{f}_S(k), \hat{f}_C(k), \hat{f}_R(k) \right)^b \right\rangle + \\ &+ \left\langle \left( \hat{e}_S(k), \hat{e}_C(k), \hat{e}_R(k) \right)^b, \left( \hat{f}_S(k), \hat{f}_C(k), \hat{f}_R(k) \right)^a \right\rangle \end{aligned} \quad (4.81)$$

Using (4.1) and (4.71), we can write then:

$$\begin{aligned} \hat{B}_F(k) &\stackrel{def}{=} \left\langle \left( Q \cdot x_k, B^T \cdot Q \cdot x_k, Qx_k \right)^a, \left( -\frac{\Delta x}{\Delta t}(k), u_k, -R \cdot Qx_k \right)^b \right\rangle + \\ &+ \left\langle \left( Qx_k, B^T \cdot Qx_k, Qx_k \right)^b, \left( -\frac{\Delta x}{\Delta t}(k), u_k, -R \cdot Qx_k \right)^a \right\rangle \end{aligned} \quad (4.82)$$

Using (4.34), the implicit form of the discrete-time Dirac structure at step  $k$  of discretization, is then:

$$\left( \hat{f}_S(k), \hat{e}_S(k), \hat{f}_C(k), \hat{e}_C(k), \hat{f}_R(k), \hat{e}_R(k) \right) \in D \quad (4.83)$$

where the Dirac structure  $D$ , is conserved by time-discretization.

### 4.3.2.2 Symplectic discretization scheme II

For the second symplectic time-discretization scheme of the linear lossy PCH systems, it was proposed the implicit Euler rule for the flows and implicit trapezoidal rule for the efforts. Applying this symplectic time-discretization scheme using (4.1) and (4.24) to the linear lossy PCH system from (4.22), it results:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = (J-R)Q\bar{x}_{k+1} + B \cdot u_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Q\bar{x}_{k+1}, y_{k+1} = B^T \cdot Q\bar{x}_{k+1} \end{cases} \quad (4.84)$$

where we have adopted the notation:

$$\bar{x}_{k+1} \stackrel{def}{=} \left( \frac{x_k + x_{k+1}}{2} \right) \quad (4.85)$$

**Remark:** The composed time discretization  $Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right)$  of the effort becomes

$$(J-R)Q \left( \frac{x_k + x_{k+1}}{2} \right), \text{ by applying the flow discretization over } \frac{\Delta H}{\Delta x} \text{ at step } k+1.$$

for a compact representation. For the next step, the power conservation law, is checked in discrete-time using (4.25), (4.27), (4.30) and (4.84). The storage port power energy can be defined as:

$$\hat{P}_S(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) \hat{f}_S(k+1) = (Q\bar{x}_{k+1})^T \left( -(J-R)Q\bar{x}_{k+1} - B \cdot u_{k+1} \right) \quad (4.86)$$

Using simple relations, we can write then:

$$\hat{P}_S(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) R \hat{e}_S(k+1) - \hat{e}_S^T(k+1) \cdot B \cdot \hat{f}_C(k+1) \quad (4.87)$$

For the control port power energy, the discrete-time form is defined as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_C^T(k+1) \hat{f}_C(k+1) = \left( B^T Q\bar{x}_{k+1} \right)^T u_{k+1} \quad (4.88)$$

Equivalently, this relation can be written as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (4.89)$$

The discrete-time dissipation port power energy, can be expressed as:

$$\hat{P}_R(k+1) \stackrel{def}{=} \hat{e}_R^T(k+1)\hat{f}_R(k+1) \quad (4.90)$$

Using simple relations, which describe PCH systems, we can write equivalently:

$$\hat{P}_R(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1)\mathbf{R}\hat{e}_S(k+1) \quad (4.91)$$

Using (4.87), (4.89) and (4.91), the discrete-time power conservation law is:

$$\begin{aligned} \hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) &= \hat{e}_S^T(k+1)\mathbf{R}\hat{e}_S(k+1) - \hat{e}_S^T(k+1)\mathbf{B}\hat{f}_C(k+1) \\ &+ \hat{e}_S^T(k+1)\mathbf{B}\hat{f}_C(k+1) - \hat{e}_S^T(k+1)\mathbf{R}\hat{e}_S(k+1) = 0 \end{aligned} \quad (4.92)$$

Thus the discrete-time power conservation law defined in (4.25) is satisfied for the selected scheme as (4.30) holds. Using (4.31), the discrete-time energy(Hamiltonian) is defined as:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1)\hat{f}_S(k+1) = y_{k+1}^T u_{k+1} + \hat{e}_R^T(k+1)\hat{f}_R(k+1) \quad (4.93)$$

Which using (4.87), (4.89) and (4.91) is equivalent to:

$$\begin{aligned} \frac{\Delta H}{\Delta t}(k+1) &\stackrel{def}{=} -\left(\hat{e}_S^T(k+1)\mathbf{R}\hat{e}_S(k+1) - \hat{e}_S^T(k+1)\mathbf{B}\hat{f}_C(k+1)\right) = \\ &= \hat{e}_S^T(k+1)\mathbf{B}\hat{f}_C(k+1) - \hat{e}_S^T(k+1)\mathbf{R}\hat{e}_S(k+1) \end{aligned} \quad (4.94)$$

The discrete-time bilinear form, can be defined as:

$$\begin{aligned} \hat{B}_F(k+1) &\stackrel{def}{=} \left\langle \left(\hat{e}_S(k+1), \hat{e}_C(k+1), \hat{e}_R(k+1)\right)^a, \left(\hat{f}_S(k+1), \hat{f}_C(k+1), \hat{f}_R(k+1)\right)^b \right\rangle + \\ &+ \left\langle \left(\hat{e}_S(k+1), \hat{e}_C(k+1), \hat{e}_R(k+1)\right)^b, \left(\hat{f}_S(k+1), \hat{f}_C(k+1), \hat{f}_R(k+1)\right)^a \right\rangle \end{aligned} \quad (4.95)$$

Replacing the discrete-time flows and efforts as defined in (4.1) for the selected symplectic time-discretization scheme, it results:

$$\begin{aligned} \hat{B}_F(k+1) &\stackrel{def}{=} \left\langle \left(\mathcal{Q}\bar{x}_{k+1}, \mathbf{B}^T \mathcal{Q}\bar{x}_{k+1}, \mathcal{Q}\bar{x}_{k+1}\right)^a, \left(-\frac{\Delta x}{\Delta t}(k+1), u_{k+1}, -R\mathcal{Q}\bar{x}_{k+1}\right)^b \right\rangle + \\ &+ \left\langle \left(\mathcal{Q}\bar{x}_{k+1}, \mathbf{B}^T \mathcal{Q}\bar{x}_{k+1}, \mathcal{Q}\bar{x}_{k+1}\right)^b, \left(-\frac{\Delta x}{\Delta t}(k+1), u_{k+1}, -R\mathcal{Q}\bar{x}_{k+1}\right)^a \right\rangle \end{aligned} \quad (4.96)$$

Using (4.34), the implicit form of the discrete-time linear lossy PCH system for this scheme, is the following:

$$\left(\hat{f}_S(k+1), \hat{e}_S(k+1), \hat{f}_C(k+1), \hat{e}_C(k+1), \hat{f}_R(k+1), \hat{e}_R(k+1)\right) \in D$$

(4.97)

where the Dirac structure conserves its structure by time-discretization.

### 4.3.2.3 Symplectic discretization scheme III

For the third symplectic time-discretization scheme of the linear lossy PCH systems, it was selected the implicit trapezoidal rule for the flows and implicit mid-point rule for the efforts,

which are two second order time-discretization rules. Applying this time-discretization scheme using the general framework defined using (4.1) and (4.24), to the linear lossy PCH system from (4.22), the following explicit form is derived for the discrete-time system:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = (J-R)Q\bar{x}_{k+1} + B\bar{u}_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Q\bar{x}_{k+1} \cdot y_{k+1} = B^T \cdot Q\bar{x}_{k+1} \end{cases} \quad (4.98)$$

where we have used the notations:

$$\begin{cases} \bar{x}_{k+1} \stackrel{def}{=} \left( \frac{x_k + x_{k+1}}{2} \right) \\ \bar{u}_{k+1} \stackrel{def}{=} \left( \frac{u_k + u_{k+1}}{2} \right) \\ \bar{x}_{k+1} \stackrel{def}{=} \left( \frac{\bar{x}_k + \bar{x}_{k+1}}{2} \right) \end{cases} \quad (4.99)$$

for simplicity.

**Remark:** The composed time discretization  $Fd_i * Ed_i \left( \frac{\partial H}{\partial x} \right)$  of the effort becomes  $(J-R)Q\bar{x}_{k+1}$ , by applying the flow discretization over  $\frac{\Delta H}{\Delta x}$  at step  $k+1$ .

At the next step, the power conservation law of the PCH system is checked in discrete-time. The discrete-time power energy associated to the storage port at time-step  $k+1$ , can be defined as follows for the selected time-discretization rules:

$$\hat{P}_S(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) \hat{f}_S(k+1) = (Q\bar{x}_{k+1})^T \left( -(J-R)Q\bar{x}_{k+1} - B \cdot \bar{u}_{k+1} \right) \quad (4.100)$$

From simple mathematical computations, the following relation results:

$$\hat{P}_S(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) R \hat{e}_S(k+1) - \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (4.101)$$

In (4.101), the following notation has been introduced:

$$\hat{e}_S(k+1) \stackrel{def}{=} \frac{1}{2} (\hat{e}_S(k) + \hat{e}_S(k+1)) \quad (4.102)$$

The discrete-time control port power energy, can be defined as follows:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_C^T(k+1) \hat{f}_C(k+1) = \left( B^T Q\bar{x}_{k+1} \right)^T \bar{u}_{k+1} \quad (4.103)$$

Equivalently this relation can be written as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) \cdot B \cdot \hat{f}_C(k+1) \quad (4.104)$$

The discrete-time dissipation port power energy, is defined as:

$$\hat{P}_R(k+1) \stackrel{def}{=} \hat{e}_R^T(k+1) \hat{f}_R(k+1) \quad (4.105)$$

From simple mathematical relations, we can write (4.105) equivalently:

$$\hat{P}_R(k+1) = -\hat{e}_S^T(k+1) R \hat{e}_S(k+1) \quad (4.106)$$



Using (4.101), (4.104) and (4.106), we can write the power conservation law in discrete-time form:

$$\begin{aligned} \hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) &= \hat{e}_S^T(k+1) \cdot R \cdot \hat{e}_S^T(k+1) - \hat{e}_S^T(k+1) \cdot B \cdot \hat{f}_C(k+1) \\ &+ \hat{e}_S^T(k+1) \cdot B \cdot \hat{f}_C(k+1) - \hat{e}_S^T(k+1) \cdot R \cdot \hat{e}_S(k+1) \end{aligned} \quad (4.107)$$

Making the possible simplifications in (4.107), it results that:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) = \hat{e}_S^T(k+1) \cdot R \cdot (\bar{e}_S(k+1) - \hat{e}_S(k+1)) \quad (4.108)$$

which shows that the general condition of (4.30) is not satisfied. Then we conclude that this symplectic discretization scheme, is not appropriate for lossy PCH systems time-discretization.

**Remark:** For the selected time-discretization rules (explicit/implicit Euler, implicit trapezoidal and implicit mid-point) and general framework from 4.2 in the linear lossy case, it can be easily proved that symplectic time-discretization schemes (4.30 satisfied) can be obtained when using a first order rule (explicit/implicit Euler) for the flows and any other rules for the efforts.

#### 4.4 Difference between two PCH systems

This section, introduces and defines a new concept to differentiate two PCH systems  $A$  and  $B$ , in the lossless or lossy case, on the basis of the difference between the power energies associated to the ports. Particularly this concept is useful also to make a differentiation between the continuous-time PCH system (system  $A$ ) and a discrete-time PCH system (system  $B$ ), by introducing a **time-discretization error Hamiltonian** system, that has as power energy corresponding to the ports, the difference between the power energies of the continuous-time and discrete-time systems. A schematic overview of the proposed idea, is given in figure 4.1 below, for the general case which includes also dissipation terms.

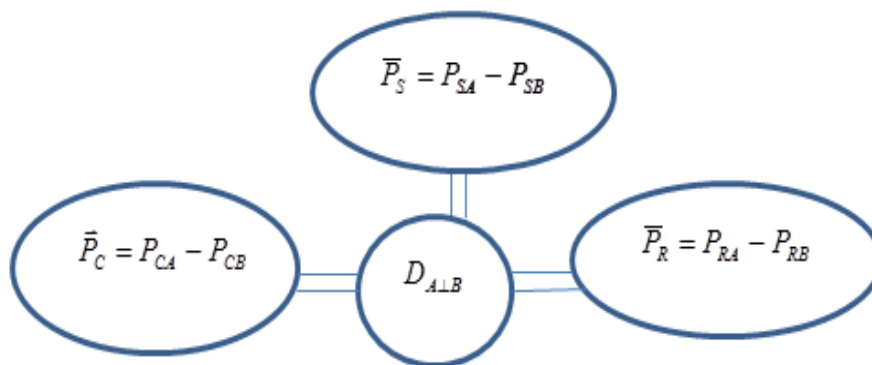


Figure 4.1. The difference of two lossy Port-Controlled-Hamiltonian systems  $A$  and  $B$ , defined with respect to power energies of the ports.

In Figure 4.1, the power energies corresponding to the storage ( $\bar{P}_S$ ), control ( $\bar{P}_C$ ) and dissipation ports ( $\bar{P}_R$ ), for the new proposed difference PCH system are equal to the difference between the power energies corresponding to the ports. It will be proved in the next

two subsections (in the continuous-time case) that this difference system is indeed a PCH system. For the **discretization error Hamiltonian**, that represents the difference between the continuous-time (system  $A$ ) and discretized (system  $B$ ) systems, we will assume that the discrete-time PCH system is defined as a perturbed continuous-time PCH system (system  $B$ ) at discretization times for the continuous-time one ( $A$ ).

#### 4.4.1 Lossless PCH systems

In this section a new PCH system is defined, for the difference between two lossless PCH systems ( $A$  and  $B$ ), with respect to the power energies of the ports. We consider two lossless PCH systems ( $A$  and  $B$ ), represented generally as follows in the explicit form:

$$\left\{ \begin{array}{l} \frac{dx_A}{dt} = J_A(x_A) \frac{\partial H_A}{\partial x_A}(x_A) + B_A(x_A) \cdot u_A \\ y_A = B_A^T(x_A) \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{dx_B}{dt} = J_B(x_B) \frac{\partial H_B}{\partial x_B}(x_B) \\ y_B = B_B^T(x_B) \frac{\partial H_B}{\partial x_B}(x_B) \end{array} \right. \quad (4.109)$$

The storage port power energies, corresponding to systems  $A$  and  $B$ , can be written as:

$$\left\{ \begin{array}{l} P_{SA} = e_{SA}^T f_{SA} = - \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) u_A \\ P_{SB} = e_{SB}^T f_{SB} = - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) u_B \end{array} \right. \quad (4.110)$$

**Remark:** The terms  $\frac{\partial H_A}{\partial x_A}(x_A) J_A(x_A) \frac{\partial H_A}{\partial x_A}(x_A)$  and  $\frac{\partial H_B}{\partial x_B}(x_B) J_B(x_B) \frac{\partial H_B}{\partial x_B}(x_B)$  in (4.110) are zero, due to the skew-symmetry of  $J_A(x_A)$  and  $J_B(x_B)$ .

The power energies corresponding to the control port for this two systems, are the following:

$$\left\{ \begin{array}{l} P_{CA} = e_{CA}^T f_{CA} = y_A^T u_A = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \cdot u_A \\ P_{CB} = e_{CB}^T f_{CB} = y_B^T u_B = \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \cdot u_B \end{array} \right. \quad (4.111)$$

Summing up the power energies corresponding to the storage and control ports for systems  $A$  and  $B$ , the total power is conserved, as follows:

$$\left\{ \begin{array}{l} P_{SA} + P_{CA} = 0 \\ P_{SB} + P_{CB} = 0 \end{array} \right. \quad (4.112)$$

For the next step, we wish to derive a lossless PCH system, which has as power energies associated to the ports, the difference between the power energies of systems  $A$  and  $B$ . The power energy difference, corresponding to the storage and control ports of systems  $A$  and  $B$ , can be defined as follows:

$$\left\{ \begin{array}{l} \tilde{P}_S \stackrel{def}{=} \tilde{e}_S^T \tilde{f}_S = e_{SA}^T f_{SA} - e_{SB}^T f_{SB} = \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \cdot u_B - \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \cdot u_A \\ \tilde{P}_C \stackrel{def}{=} \tilde{e}_C^T \tilde{f}_C = e_{CA}^T f_{CA} - e_{CB}^T f_{CB} = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \cdot u_A - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \cdot u_B \end{array} \right. \quad (4.113)$$

introducing the notations:

$$\left\{ \begin{array}{l} \tilde{e}_S^T \stackrel{def}{=} [e_{SA} \quad e_{SB}]^T, \tilde{f}_S \stackrel{def}{=} [f_{SA} \quad -f_{SB}]^T \\ \tilde{e}_C^T \stackrel{def}{=} [e_{CA} \quad e_{CB}]^T, \tilde{f}_C \stackrel{def}{=} [f_{CA} \quad -f_{CB}]^T \end{array} \right. \quad (4.114)$$

Summing both relations in (4.113), we observe that the power conservation law is satisfied and we can write the relation:

$$\tilde{P}_S + \tilde{P}_C = 0 \quad (4.115)$$

We propose then the following explicit form for the difference PCH system:

$$\left\{ \begin{array}{l} \dot{\tilde{x}} \stackrel{def}{=} \begin{bmatrix} \frac{dx_A}{dt} \\ \frac{dx_B}{dt} \end{bmatrix} = \begin{bmatrix} J_A(x_A) & 0 \\ 0 & -J_B(x_B) \end{bmatrix} \begin{bmatrix} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{bmatrix} + \begin{bmatrix} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix} \\ \tilde{y} \stackrel{def}{=} \begin{bmatrix} y_A \\ y_B \end{bmatrix} = \begin{bmatrix} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{bmatrix}^T \begin{bmatrix} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{bmatrix} \end{array} \right. \quad (4.116)$$

For this difference PCH system it can be easily proved, that the following relations hold:

$$\left\{ \begin{array}{l} \tilde{e}_S^T \tilde{f}_S = \bar{e}_S^T \bar{f}_S = - \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \cdot u_A + \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \cdot u_B \\ \tilde{e}_C^T \tilde{f}_S = \bar{e}_S^T \bar{f}_S = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \cdot u_A - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \cdot u_B \end{array} \right. \quad (4.117)$$

between the difference of the power energies (  $\tilde{P}_S$  and  $\tilde{P}_C$  ) for the storage and control ports from (4.113), with the power energies of the proposed difference PCH system from (4.116).

**Remark:** The energy (Hamiltonian) for the proposed difference PCH system can be easily derived in the lossless or lossy case from the results, based on the power conservation law.

A complete proof of the basic properties that describes a PCH system, for the difference PCH system proposed in (4.116), can be found in Appendix D.1. For the lossless difference Hamiltonian proposed in (4.116), we introduce the following definition.

#### Definition 4.1

The difference between two lossless PCH systems  $(f_{SA}, e_{SA}, f_{CA}, e_{CA}) \in D_A$  and  $(f_{SB}, e_{SB}, f_{CB}, e_{CB}) \in D_B$  defined in (4.109), with respect to the power energies corresponding to the ports, is represented by another lossless PCH system  $(\bar{f}_S, \bar{e}_S, \bar{f}_C, \bar{e}_C) \in D_{A \perp B}$ , defined as follows in explicit form:

$$\begin{cases} \dot{\bar{x}}_{A \perp B} \stackrel{\text{def}}{=} \bar{J}_{A \perp B}(x_{A \perp B}) \frac{\partial H_{A \perp B}}{\partial \bar{x}_{A \perp B}}(\bar{x}_{A \perp B}) + \bar{B}_{A \perp B}(\bar{x}_{A \perp B}) \bar{u}_{A \perp B} \\ \bar{y}_{A \perp B} \stackrel{\text{def}}{=} \bar{B}_{A \perp B}^T \frac{\partial H_{A \perp B}}{\partial \bar{x}_{A \perp B}}(\bar{x}_{A \perp B}) \end{cases} \quad (4.118)$$

where:

$$\begin{cases} \bar{x}_{A \perp B} \stackrel{\text{def}}{=} \begin{bmatrix} x_A \\ x_B \end{bmatrix}, \bar{J}_{A \perp B}(\bar{x}_{A \perp B}) \stackrel{\text{def}}{=} \begin{bmatrix} J_A(x_A) & 0 \\ 0 & -J_B(x_B) \end{bmatrix}, \frac{\partial H_{A \perp B}}{\partial \bar{x}_{A \perp B}} \stackrel{\text{def}}{=} \begin{bmatrix} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{bmatrix} \\ \bar{B}_{A \perp B}(\bar{x}_{A \perp B}) \stackrel{\text{def}}{=} \begin{bmatrix} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{bmatrix}, \bar{y}_{A \perp B} \stackrel{\text{def}}{=} \begin{bmatrix} \bar{y}_A \\ \bar{y}_B \end{bmatrix}, \bar{x}_{A \perp B} \stackrel{\text{def}}{=} \begin{bmatrix} x_A \\ x_B \end{bmatrix} \end{cases} \quad (4.119)$$

**Remark:** This definition may be used for the time-discretization process, to define a **discretization error Hamiltonian system** (in the lossless or lossy case) as the difference between two PCH systems, the original continuous-time one and a continuous-time perturbed PCH system, whose values at discrete-times are the ones of the discrete-time PCH system (approximation). This is possible only when symplectic time-discretization schemes are used to get a discrete-time PCH approximation for the original system as presented in the previous sections.

When the error between the continuous-time and discrete-time system PCH systems ( $A$  and  $B$ ) is identically zero, the following relations hold:

$$\begin{cases} P_{SA} = P_{SB}, P_{CA} = P_{CB}, J_A(x_A) = J_B(x_B), B_A(x_A) = B_B(x_B) \\ x_A = x_B, y_A = y_B, u_A = u_B \end{cases} \quad (4.120)$$

between both systems .

#### 4.4.2 Lossy PCH systems

As for the lossless case, a difference PCH system is defined in this section between two lossy PCH systems ( $A$  and  $B$ ), with respect to the power energies corresponding to the ports. The explicit form of the two lossy PCH systems  $A$  and  $B$ , can be written as:

$$\left\{ \begin{array}{l} \frac{dx_A}{dt} = (J_A(x_A) - R_A(x_A)) \frac{\partial H_A}{\partial x_A}(x_A) + B_A(x_A) \mu_A \\ y_A = B_A^T(x_A) \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{dx_B}{dt} = (J_B(x_B) - R_B(x_B)) \frac{\partial H_B}{\partial x_B}(x_B) + B_B(x_B) \mu_B \\ y_B = B_B^T(x_B) \frac{\partial H_B}{\partial x_B}(x_B) \end{array} \right. \quad (4.121)$$

The power energies corresponding to the storage port for this two systems, can be computed using (4.121) as follows:

$$\left\{ \begin{array}{l} P_{SA} = e_{SA}^T f_{SA} = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T R_A(x_A) \frac{\partial H_A}{\partial x_A}(x_A) - \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A \\ P_{SB} = e_{SB}^T f_{SB} = \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T R_B(x_B) \frac{\partial H_B}{\partial x_B}(x_B) - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \end{array} \right. \quad (4.122)$$

**Remark:** The terms  $\left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T J_A(x_A) \frac{\partial H_A}{\partial x_A}(x_A)$  and  $\left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T J_B(x_B) \frac{\partial H_B}{\partial x_B}(x_B)$  in (4.122) are zero due to the skew-symmetry of  $J_A(x_A)$  and  $J_B(x_B)$ .

For the control port power energy, we can write :

$$\left\{ \begin{array}{l} P_{CA} = e_{CA}^T f_{CA} = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A \\ P_{CB} = e_{CB}^T f_{CB} = \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \end{array} \right. \quad (4.123)$$

For the dissipation port power energy, it results ;

$$\left\{ \begin{array}{l} P_{RA} = e_{RA}^T f_{RA} = - \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T R_A(x_A) \left( \frac{\partial H_A}{\partial x_A}(x_A) \right) \\ P_{RB} = e_{RB}^T f_{RB} = - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T R_B(x_B) \left( \frac{\partial H_B}{\partial x_B}(x_B) \right) \end{array} \right. \quad (4.124)$$

Summing up, the power energies of systems A and B , from (4.122), (4.123) and (4.124), the total energy is conserved as follows:

$$\left\{ \begin{array}{l} P_{SA} + P_{CA} + P_{RA} = 0 \\ P_{SB} + P_{CB} + P_{RB} = 0 \end{array} \right. \quad (4.125)$$

where  $P_{SA}, P_{CA}, P_{RA}$  and  $P_{SB}, P_{CB}, P_{RB}$  are the power energies corresponding to the storage, control and dissipation ports of systems A and B . For the next step, we propose a lossy PCH system, which has as power energies corresponding to the ports, the difference of

the power energies for systems  $A$  and  $B$ . The difference of the power energies, corresponding to the storage port for systems  $A$  and  $B$  can be expressed as follows using (4.122):

$$\begin{aligned} \tilde{P}_S = \tilde{e}_S^T \tilde{f}_S = e_{SA}^T f_{SA} - e_{SB}^T f_{SB} = & \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T R_A(x_A) \left( \frac{\partial H_A}{\partial x_A}(x_A) \right) - \\ & - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T R_B(x_B) \left( \frac{\partial H_B}{\partial x_B}(x_B) \right) - \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A + \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \end{aligned} \quad (4.126)$$

where the notations:

$$\begin{cases} \tilde{e}_S^T \stackrel{def}{=} [e_{SA} & e_{SB}]^T \\ \tilde{f}_S \stackrel{def}{=} [f_{SA} & -f_{SB}]^T \end{cases} \quad (4.127)$$

have been adopted. The difference between the power energies corresponding to the control port, for systems  $A$  and  $B$ , using (4.123) is:

$$\begin{aligned} \tilde{P}_C = \tilde{e}_C^T \tilde{f}_C = e_{CA}^T f_{CA} - e_{CB}^T f_{CB} = & \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A - \\ & - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \end{aligned} \quad (4.128)$$

where the following notations:

$$\begin{cases} \tilde{e}_C^T \stackrel{def}{=} [e_{CA} & e_{CB}]^T \\ \tilde{f}_C \stackrel{def}{=} [f_{CA} & -f_{CB}]^T \end{cases} \quad (4.129)$$

were used. The difference between the power energies corresponding to the dissipation port, for systems  $A$  and  $B$ , can be calculated using (4.124) as follows:

$$\begin{aligned} \tilde{P}_R = \tilde{e}_R^T \tilde{f}_R = e_{RA}^T f_{RA} - e_{RB}^T f_{RB} = & - \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T R_A(x_A) \left( \frac{\partial H_A}{\partial x_A}(x_A) \right) + \\ & + \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T R_B(x_B) \left( \frac{\partial H_B}{\partial x_B}(x_B) \right) \end{aligned} \quad (4.130)$$

where the following notations were introduced:

$$\begin{cases} \tilde{e}_R^T \stackrel{def}{=} [e_{RA} & e_{RB}]^T \\ \tilde{f}_R \stackrel{def}{=} [f_{RA} & -f_{RB}]^T \end{cases} \quad (4.131)$$

Summing up the difference between the power energies of the storage, control and dissipation ports, from (4.126), (4.128) and (4.130) of systems  $A$  and  $B$ , it results:

$$\tilde{e}_S^T \tilde{f}_S + \tilde{e}_C^T \tilde{f}_C + \tilde{e}_R^T \tilde{f}_R = 0 \quad (4.132)$$

or:

$$\tilde{P}_S + \tilde{P}_C + \tilde{P}_R = 0 \quad (4.133)$$

From (4.132) or (4.133), it results that the total power energy for the difference PCH system of systems  $A$  and  $B$ , is also conserved. An explicit form of the difference PCH system of systems  $A$  and  $B$ , can be defined as:

$$\left\{ \begin{array}{l} \tilde{x} \stackrel{def}{=} \begin{bmatrix} \frac{dx_A}{dt} \\ \frac{dx_B}{dt} \end{bmatrix} = \begin{pmatrix} (J_A(x_A) - R_A(x_A)) & 0 \\ 0 & -(J_B(x_B) - R_B(x_B)) \end{pmatrix} \begin{bmatrix} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{bmatrix} + \begin{pmatrix} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{pmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix} \\ \tilde{y} \stackrel{def}{=} \begin{bmatrix} y_A \\ y_B \end{bmatrix} = \begin{pmatrix} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{pmatrix} \begin{bmatrix} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{bmatrix} \end{array} \right. \quad (4.134)$$

For the above difference lossy PCH system, it can be easily proved (see Apendix D.2), that the following relations hold:

$$\left\{ \begin{array}{l} \tilde{e}_S^T \tilde{f}_S = \bar{e}_S^T \bar{f}_S = \frac{\partial H_A}{\partial x_A}(x_A) R_A(x_A) \frac{\partial H_A}{\partial x_A}(x_A) - \frac{\partial H_B}{\partial x_B}(x_B) R_B(x_B) \frac{\partial H_B}{\partial x_B}(x_B) \\ - \frac{\partial H_A}{\partial x_A}(x_A) B_A(x_A) \mu_A + \frac{\partial H_B}{\partial x_B}(x_B) B_B(x_B) \mu_B \\ \tilde{e}_C^T \tilde{f}_C = \bar{e}_C^T \bar{f}_C = \frac{\partial H_A}{\partial x_A}(x_A) B_A(x_A) \mu_A - \frac{\partial H_B}{\partial x_B}(x_B) B_B(x_B) \mu_B \\ \tilde{e}_R^T \tilde{f}_R = \bar{e}_R^T \bar{f}_R = - \frac{\partial H_A}{\partial x_A}(x_A) R_A(x_A) \frac{\partial H_A}{\partial x_A}(x_A) + \frac{\partial H_B}{\partial x_B}(x_B) R_B(x_B) \frac{\partial H_B}{\partial x_B}(x_B) \end{array} \right. \quad (4.135)$$

between the difference of the power energies of two lossy PCH systems ( $A$  and  $B$ ) and the power energies of the difference PCH system proposed in (4.124).

**Remark:** A complete proof of the basic properties which describe a PCH system, for the proposed difference PCH system from (4.134), can be found in Appendix D.2.

Using the results presented above, we can formulate the following general definition, for the difference lossy PCH system, of two systems ( $A$  and  $B$ ), in rapport with the power energies of the ports.

#### Definition 4.2

The difference between two lossy PCH systems  $(f_{SA}, e_{SA}, f_{CA}, e_{CA}, f_{RA}, e_{RA}) \in D_A$  and  $(f_{SB}, e_{SB}, f_{CB}, e_{CB}, f_{RB}, e_{RB}) \in D_B$  defined in (4.121), with respect to the power energies

corresponding to the ports, is another lossy Port-Controlled-Hamiltonian system  $(\bar{f}_S, \bar{e}_S, \bar{f}_C, \bar{e}_C, \bar{f}_R, \bar{e}_R) \in \bar{D}_{A \perp B}$ , defined as follows in explicit form:

$$\begin{cases} \bar{x}_{A \perp B} \stackrel{\text{def}}{=} (\bar{J}_{A \perp B}(x_{A \perp B}) - \bar{R}_{A \perp B}(\bar{x}_{A \perp B})) \frac{\partial H_{A \perp B}}{\partial x_{A \perp B}}(\bar{x}_{A \perp B}) + \bar{B}_{A \perp B}(\bar{x}_{A \perp B}) \bar{u}_{A \perp B} \\ \bar{y}_{A \perp B} \stackrel{\text{def}}{=} \bar{B}_{A \perp B}^T(x_{A \perp B}) \frac{\partial H_{A \perp B}}{\partial x_{A \perp B}} \end{cases} \quad (4.136)$$

where:

$$\begin{cases} \bar{x}_{A \perp B} \stackrel{\text{def}}{=} \begin{pmatrix} x_A \\ x_B \end{pmatrix}, \bar{J}_{A \perp B}(x_{A \perp B}) \stackrel{\text{def}}{=} \begin{pmatrix} J_A(x_A) & 0 \\ 0 & -J_B(x_B) \end{pmatrix}, \bar{R}_{A \perp B} \stackrel{\text{def}}{=} \begin{pmatrix} R_A(x_A) & 0 \\ 0 & -R_B(x_B) \end{pmatrix}, \\ \frac{\partial H_{A \perp B}}{\partial x_{A \perp B}}(\bar{x}_{A \perp B}) \stackrel{\text{def}}{=} \begin{pmatrix} \frac{\partial H_A}{\partial x_A} \\ \frac{\partial H_B}{\partial x_B} \end{pmatrix}, \bar{B}_{A \perp B} \stackrel{\text{def}}{=} \begin{pmatrix} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{pmatrix}, \bar{y}_{A \perp B} \stackrel{\text{def}}{=} \begin{pmatrix} y_A \\ y_B \end{pmatrix} \end{cases} \quad (4.137)$$

When the error between the continuous-time and discretized trajectory of the PCH systems ( $A$  and  $B$ ) is identically zero, then the following relations hold, between the parameters:

$$\begin{cases} P_{SA} = P_{SB}, P_{CA} = P_{CB}, P_{RA} = P_{RB} \\ J_A(x_A) = J_B(x_B), R_A(x_A) = R_B(x_B), \\ B_A(x_A) = B_B(x_B), x_A = x_B, y_A = y_B, u_A = u_B \end{cases} \quad (4.138)$$

between the parameters that describe both systems.

## 4.5 Simulation examples

This section presents a set of simulations to test the symplectic time-discretization schemes proposed in section 4.3 and also to analyze the **discretization error Hamiltonian** concept proposed in section 4.4. As PCH systems examples, were considered two linear systems, given by an LC circuit and a capacitor microphone circuit, respectively for the lossless and lossy cases.

### 4.5.1 Lossless PCH systems

For the first symplectic time-discretization scheme proposed in section 4.3.1, that uses the explicit Euler rule for the flows and the implicit Euler rule for the efforts, it was selected a sampling time  $\Delta t = 0.005$ , a horizon length  $N = 40$  and a sinusoidal type of the input. The output response of the continuous-time/discrete-time LC circuit, can be seen in the following figure, together with the time-discretization error.



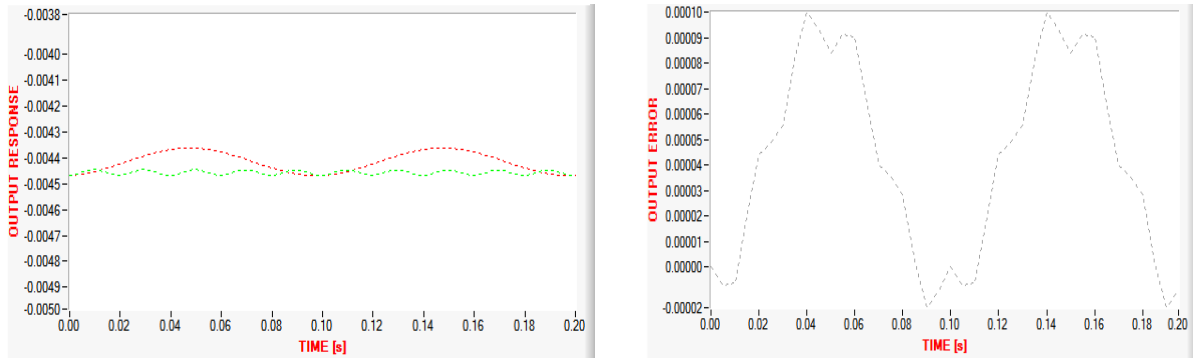


Figure 4.2. a) Output response of the continuous-time (green) /discrete-time (red) LC circuit  
 b) Time-discretization error (gray)

From Figure 4.2.b, it can be observed that the time-discretization error between the continuous-time and discrete-time system outputs, has an amplitude of evolution smaller than 0.0001 . In the next figure, the power energies of the storage and control ports will be represented for the continuous-time/discrete-time systems and also for the *discretization error Hamiltonian* system.

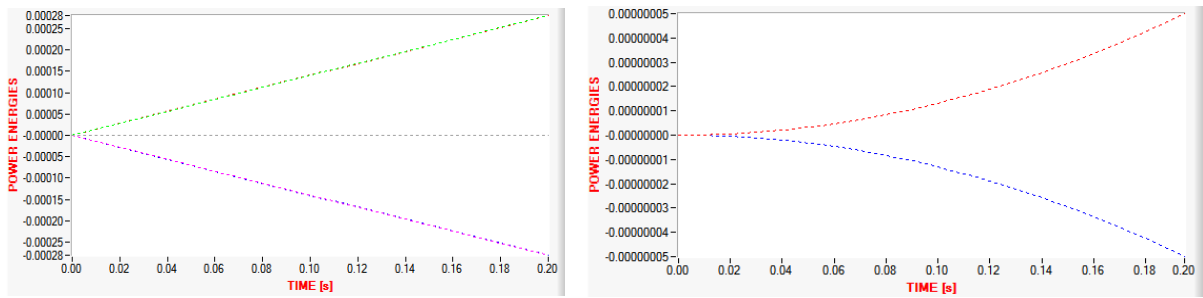


Figure 4.3. a) Power energies representation for the continuous-time (storage (green) and control (magenta) ports) and discrete-time (storage (red) and control (blue) ports) systems. b) Power energies representation for the storage (red) and control (blue) ports of the *discretization error Hamiltonian* system.

From Figure 4.3.a, and Figure 4.3.b, it can be observed that the power conservation law of PCH systems is satisfied for the continuous-time/discrete-time systems (green line) and also for the *discretization error Hamiltonian system*, as the storage and control ports have opposite power energies values at each time-step in rapport with zero.

**Remark:** *The discretization error Hamiltonian for the selected time-discretization scheme, has power port energies with maximum magnitute of order  $5 \cdot 10^{-8}$ , which is very small as compared to the continuous-time/discrete-time power port energies.*

For the second symplectic time-discretization scheme, the system output representation of the continuous-time/discrete-time systems and time-discretization error, is given in Figure 4.4.

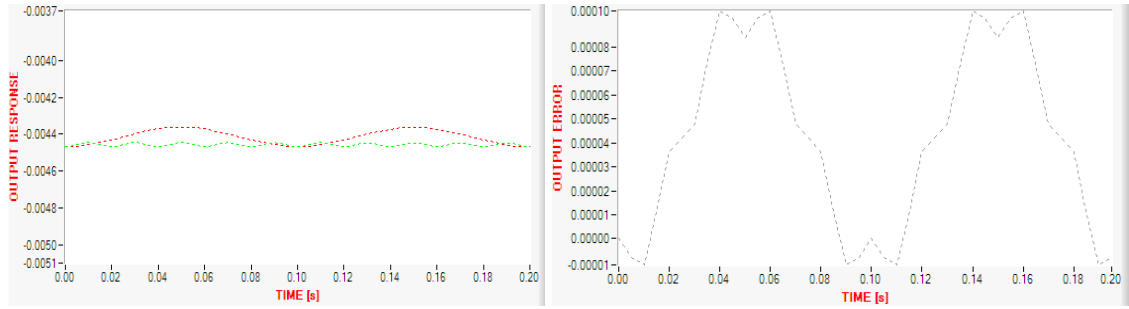


Figure 4.4. a) Output response of the continuous-time/discrete-time LC circuit (green/red) b) Discretization error (gray).

From Figure 4.4.b) we observe that the time-discretization error amplitude is smaller than 0.0001 and has a similar form to the first symplectic time-discretization scheme. The following figure represents the power energies for the continuous-time/discrete-time systems and *discretization error Hamiltonian*.

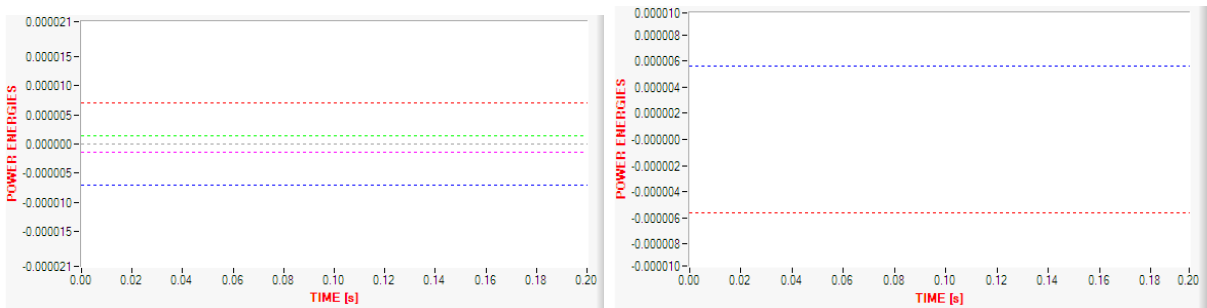


Figure 4.5 a) Power energies representation for the continuous-time (storage (green) and control (magenta) ports) and discrete-time (storage (red) and control (blue) ports) systems. b) Power energies representation for the storage (red) and control (blue) ports of the *discretization error Hamiltonian* system.

Similarly to the previous symplectic time-discretization scheme, the discrete-time Hamiltonian system and *discretization error Hamiltonian* satisfy the power conservation law of PCH systems. For the third symplectic time-discretization scheme, the system outputs and time-discretization error representation is given in Figure 4.6.

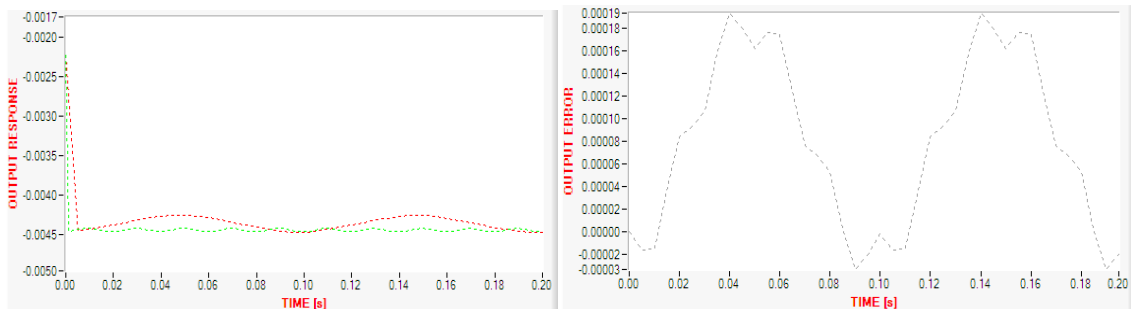


Figure 4.6. a) Output response of the continuous-time/discrete-time LC circuit (green/red). b) Discretization error (gray).

From Figure 4.6.b) the time-discretization error maximum amplitude of the system output is 0.00019, which is bigger as compared to the previous time-discretization schemes.

**Remark:** The big time-discretization error amplitudes obtained with this scheme, are due to the high errors in the first steps, as the outputs are expressed by means of the states, which propagate the errors over time, using recursive relations.

The power energy representation of the continuous-time/discrete-time systems and *discretization error Hamiltonian* are given in Figure 4.7.

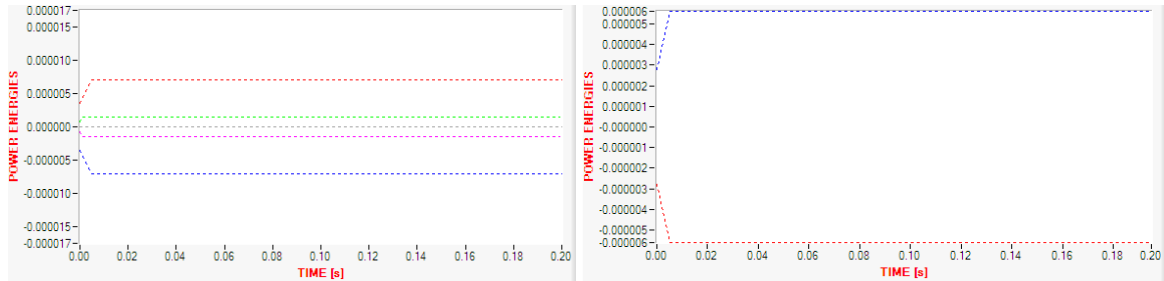


Figure 4.7 a) Power energies representation for the continuous-time (storage (green) and control (magenta) ports) and discrete-time (storage (red) and control (blue)) systems b) Power energies representation for the storage (red) and control (blue) ports of the *discretization error Hamiltonian* system.

As for the previous cases, the discrete-time Hamiltonian system and *discretization error Hamiltonian* satisfy the power conservation law, which defines PCH systems.

**Remark:** In order to perform a good time-discretization of a PCH system, the input selection is important. When selecting a non-constant input over a sampling-time interval, the discretized power energies and *discretization error Hamiltonian* might not give correct results (mainly for the storage and control ports), as the power energies are mathematically expressed as products between discrete-time flows and efforts, while for the output response the results might still be appropriate, with convergence satisfaction.

In order to make a good distinction between the time-discretization schemes, the logarithm representation is performed in Figure 4.8, between the discretization error and number of steps ( $N$ ) for a time-step  $\Delta t = 0.0005$  and a time-interval  $t_S = 5s$ .

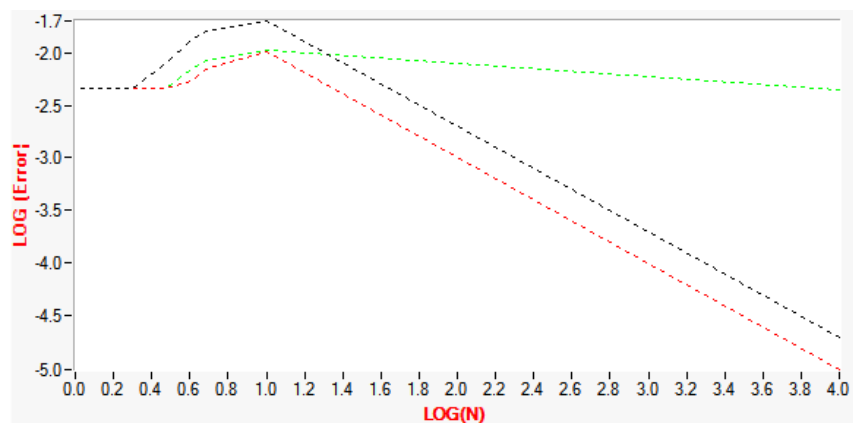


Figure 4.8. Logarithm of the maximum error and number of steps ( $N$ ) for the system output:  
: green – symplectic scheme I, red- symplectic scheme II,  
black- symplectic scheme III.

From Figure 4.8, it is obvious that the least performant symplectic time-discretization scheme for an increasing number of steps ( $N$ ), is the first one (green), which is a first order scheme ( $O(\Delta t)$ ) with a logarithm of the error between  $-2$  and  $-2.6$ . For the second and third time-discretization schemes, which are in fact of second order type ( $O(\Delta t^2)$ ), we observe improved results with an increasing number of steps ( $N$ ), with a logarithm of the error that tends to  $-5$  for the second scheme. For a small number of steps ( $N \leq 10$ ), the second time-discretization method still performs better as compared to the other two, the third one being the least performant.

**Remark:** From Figure 4.6 and 4.8, we observe that the third time-discretization scheme is not recommended to be used for a small number of steps ( $N$ ), due to big discretization errors.

In the next Figure, we also represent the logarithm of the error in rapport with the number of steps ( $N$ ) for the power energies.

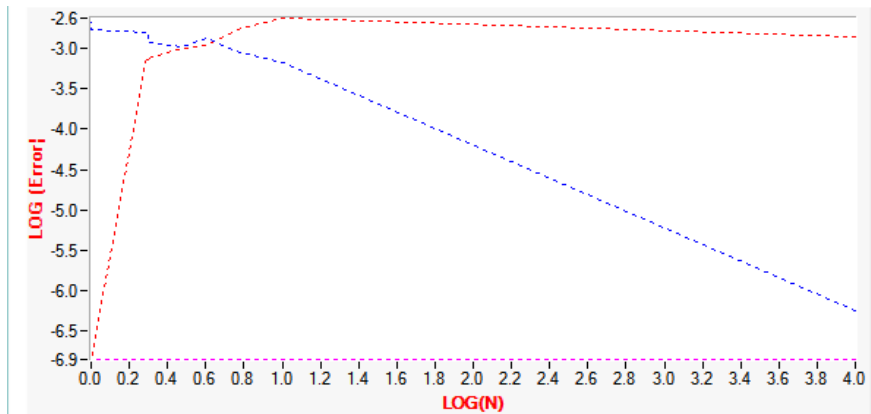


Figure 4.9. Logarithm of the maximum error and number of steps ( $N$ ) for the power energies: red – symplectic scheme I, magenta – symplectic scheme II, blue –symplectic scheme III.

Similarly to the output response case, the second order scheme obtains improved performances as compared to the other two, while the first one which is a first order type ( $O(\Delta t)$ ), is less performant. In Figure 4.10 the second scheme is represented separately, to have a better view of the result.

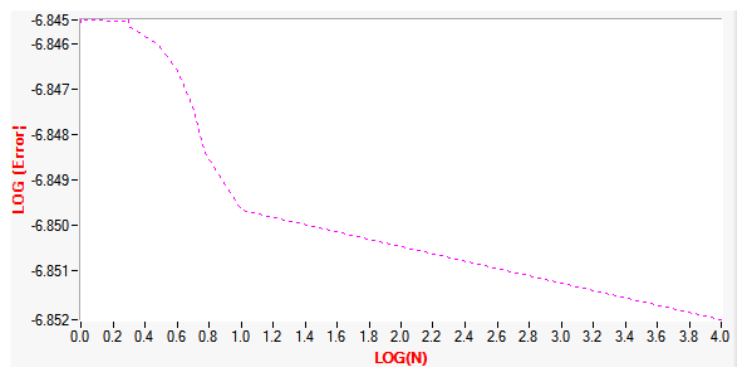


Figure 4.10. Logarithm of the maximum error and number of steps ( $N$ ) for the power energies of symplectic scheme II.

**Remark:** From the selected time-discretization schemes, the second one which is of order two type ( $O(\Delta t^2)$ ) is recommended to be used for time-discretization as compared to the other two, due to its good performances on the output and power energies time-discretization (see Figure 4.8 and 4.9).

#### 4.5.2 Lossy PCH systems

For the lossy case, the time-discretization of the capacitor microphone circuit is performed using  $N = 3000$  steps, a sampling time  $\Delta t = 0.005$  and a sinusoidal type of the input signal. For the first symplectic time-discretization scheme, the system outputs response in continuous-time/discrete-time and time-discretization error, are represented in Figure 4.11.

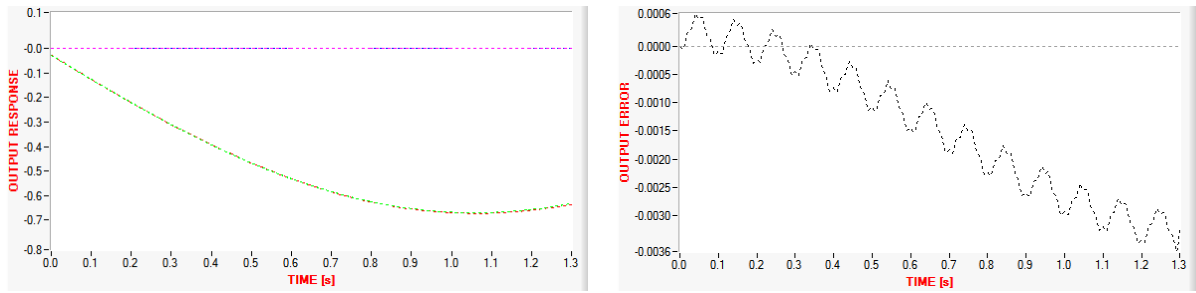


Figure 4.11. a) Output responses of the continuous-time/discrete-time capacitor microphone circuit, represented with green/red ( $y_1$ ) and magenta/blue ( $y_2$ ). b) Discretization error represented with black ( $y_1$ ) and gray ( $y_2$ ).

Figure 4.11.b shows that the time-discretization error evolution of the first system output is in the range  $-0.0036$  and  $0.0006$ , having a similar sinusoidal behavior as the first output does. In the following Figure, the power energies of the continuous-time/discrete-time systems and *discretization error Hamiltonian* are represented.

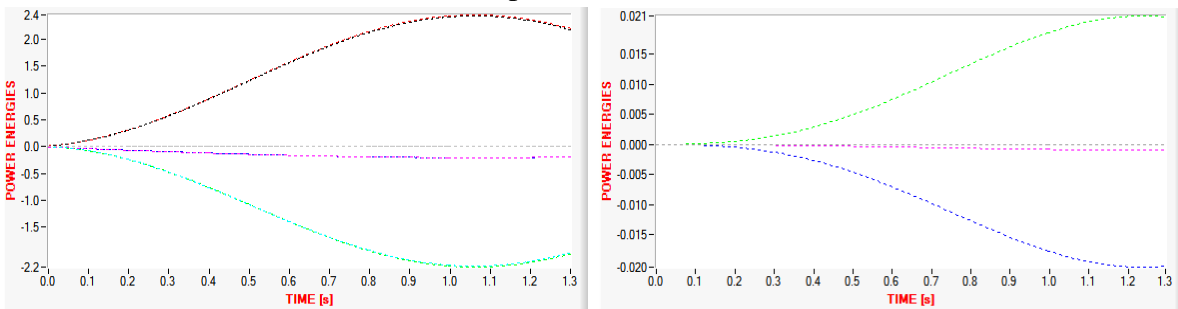


Figure 4.12. a) Power energy representation for the storage/control/dissipation ports of the continuous-time (black – storage port power energy, magenta- control port power energy, cyan – dissipation port power energy, gray-sum of power energies)/discrete-time systems (red-storage port power energy, blue – control port power energy, green – dissipation port power energy) b) Power energy representation of the *discretization error Hamiltonian* (green- storage port, blue – control port, magenta – dissipation port, gray – sum of power energies).

As this figure reflects, the power conservation law is satisfied for the continuous-time/discrete-time Hamiltonian systems and *discretization error Hamiltonian* for the selected

discretization scheme. As Figure 4.12.b shows, we observe that the *discretization error Hamiltonian* has the same form for the power energies as the continuous-time system, with bounded errors in the range  $-0.02, 0.021$ . For the second symplectic time-discretization scheme, the system outputs representation of the continuous-time/discrete-time systems, can be seen in the next figure.

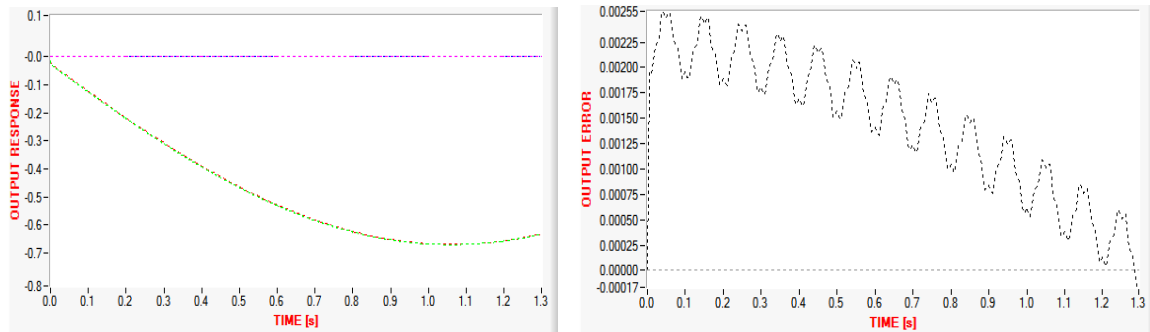


Figure 4.13. a) Output responses of the continuous-time/discrete-time capacitor microphone circuit, represented with green/red ( $y_1$ ) and magenta/blue ( $y_2$ ). b) Discretization error represented with black ( $y_1$ ) and gray ( $y_2$ ).

From Figure 4.13.b, we observe that the time-discretization error between the continuous-time/discrete-time systems of  $y_1$ , is smaller in modulus as compared to the first symplectic scheme, having a domain of variation between  $-0.0017$  and  $0.00255$ . The power energy representation of the continuous-time/discrete-time systems and *discretization error Hamiltonian* can be seen in Figure 4.14.

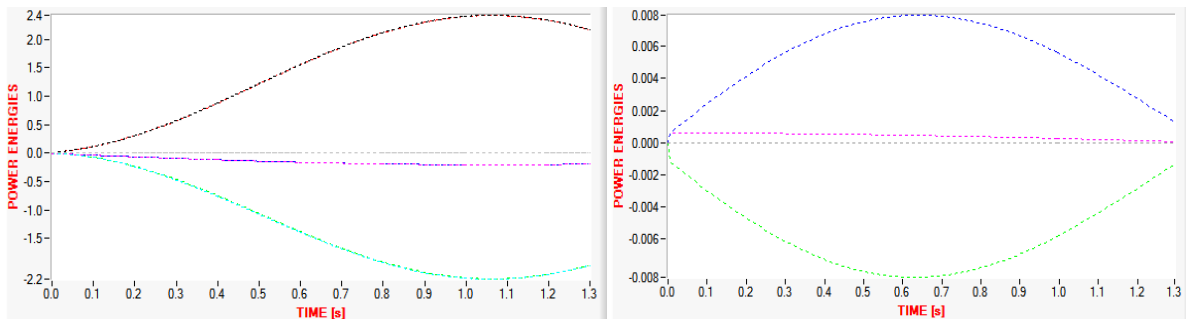


Figure 4.14. a) Power energy representation for the storage/control/dissipation ports of the continuous-time (black – storage port power energy, magenta- control port power energy, cyan – dissipation port power energy, gray-sum of power energies)/discrete-time systems (red-storage port power energy, blue – control port power energy, green – dissipation port power energy) b) Power energy representation of the *discretization error Hamiltonian* (green- storage port, blue – control port, magenta – dissipation port, gray – sum of power energies).

Similarly to the previous symplectic discretization scheme, the power conservation law is satisfied for both the continuous-time/discrete-time Hamiltonian systems and *discretization error Hamiltonian*.

**Remark:** As compared to the first symplectic time-discretization scheme, the *discretization error Hamiltonian* of this scheme, has much smaller power energies bounded between

– 0.008 ,0.008 due to the second order rule type ( $O(\Delta t^2)$ ) method selected for time-discretization.

For the third symplectic time-discretization scheme, the outputs response for the continuous-time/discrete-time systems and time-discretization error are given in Figure 4.15.

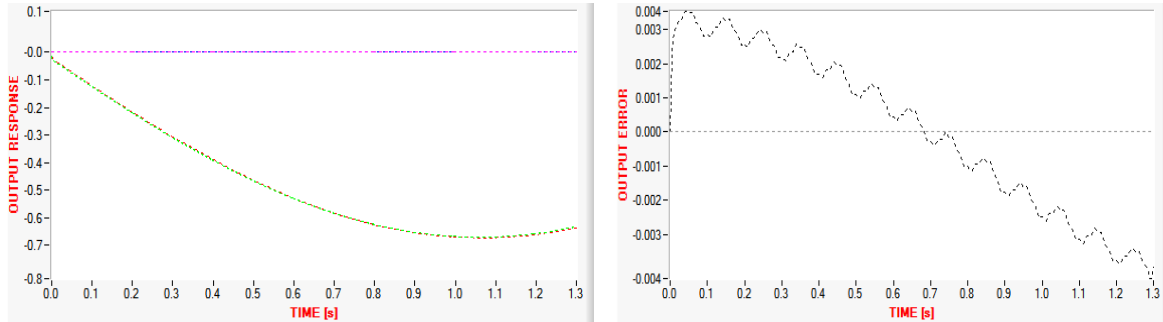


Figure 4.15. a) Output responses of the continuous-time/discrete-time capacitor microphone circuit, represented with green/red ( $y_1$ ) and magenta/blue ( $y_2$ ) b) Discretization error represented with black ( $y_1$ ) and gray ( $y_2$ ).

From Figure 4.15.b, we observe that the domain of variation for the time-discretization error is between  $-0.004$  and  $0.004$ , which makes it less performant as compared to the other two schemes, for the selected parameters. The following figure represents the power energies of the continuous-time and discrete-time systems.

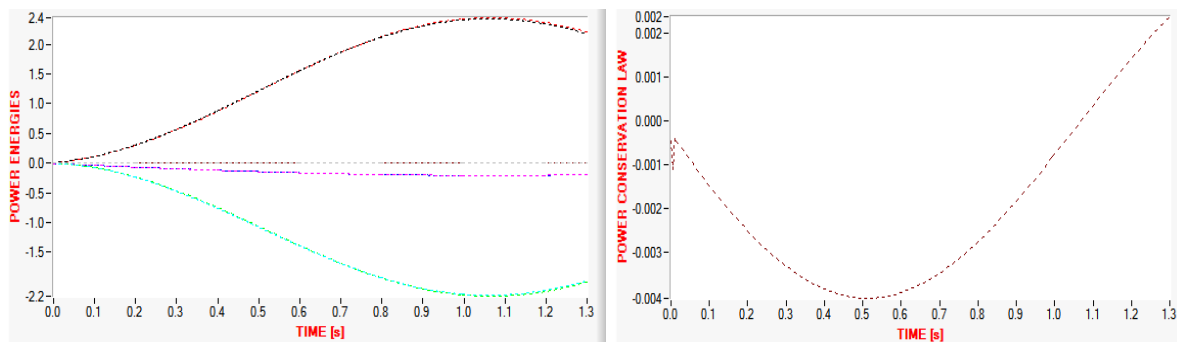


Figure 4.16. a) Power energy representation for the storage/control/dissipation ports of the continuous-time (black – storage port power energy, magenta- control port power energy, cyan – dissipation port power energy)/discrete-time system (red-storage port power energy, blue – control port power energy, green – dissipation port power energy) b) Power conservation law of the discrete-time system.

In Figure 4.16.b we can observe that in this case, the power conservation law is not satisfied, but also on Figure 4.16.a, given by the dark regions on the power conservation law (gray), which are non-zero for the discrete-time system.

**Remark:** *The power conservation law is not satisfied with this time-discretization scheme as (4.30) from the general framework, doesn't hold for this scheme. Thus the selection of this time-discretization scheme is not recommended for lossy PCH systems time-discretization.*

In order to make a comparison on the accuracy of the three symplectic time-discretization schemes for the capacitor microphone circuit, in the following figure is represented the logarithm between the error and number of steps ( $N$ ), for the system output ( $y_1$ ) for a time-step  $\Delta t = 0.0005$  and time-interval  $t_S = 5 s$ .

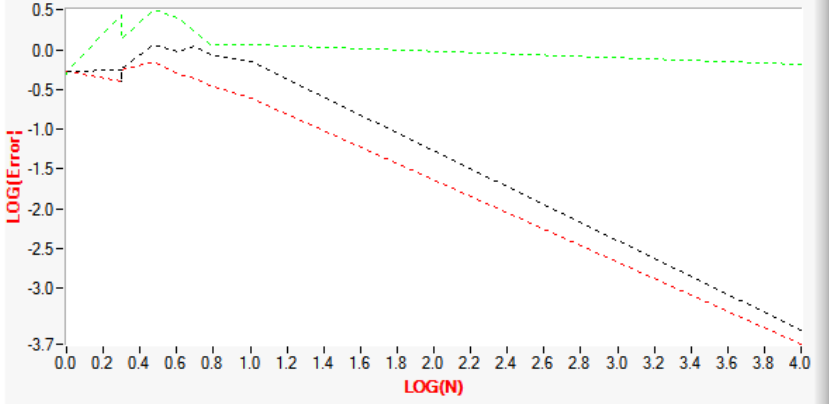


Figure 4.17. Logarithm of the maximum error and number of steps ( $N$ ) for the system output ( $y_1$ ): green – symplectic scheme I, red- symplectic scheme II, black- symplectic scheme III.

As for the lossless case, we observe that the first symplectic time-discretization scheme is less performant as compared to the other two, as it is of first order ( $O(\Delta t)$ ) and the others of second order ( $O(\Delta t)^2$ ), shown in the graphic by the slopes. Also for a small number of steps ( $N \leq 10$ ), the second symplectic time-discretization scheme performs better when compared to the other two. In the following figure, is checked the accuracy of the symplectic time-discretization methods for the power energies, by logarithm representation of the error versus number of steps ( $N$ ) for a time-step ( $\Delta t = 0.0005$ ) and time interval  $t_S = 5 s$ .

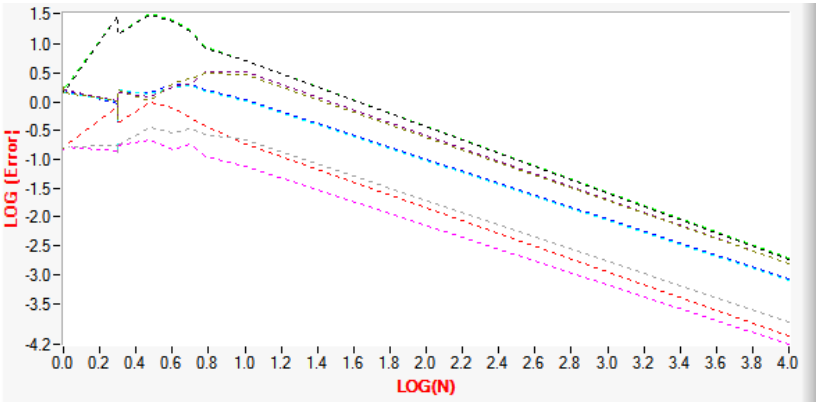


Figure 4.18. Logarithm of the maximum error and number of steps ( $N$ ) for the storage, control and dissipation port power energies: scheme I - green, red,black;scheme II – blue, magenta,cyan; scheme III –dark magenta, gray, dark yellow.



As Figure 4.18 shows, it is obvious that the second time-discretization scheme performs better as compared to the other two, with bigger accuracy (slopes) for the storage, control or dissipation ports. Also for a small number of steps ( $N \leq 10$ ) the second time-discretization scheme is still better compared to the other two on each port.

**Remark:** *For each symplectic time-discretization scheme, the storage and dissipation port power energies slopes almost coincide, as for each time-discretization scheme, the control port power energy has a small (negligible) influence on the result (see the general relations from section 4.3.2). For the selected symplectic time-discretization schemes examples, the second one is recommended to be used with linear PCH systems in the lossless or lossy case, due to its high accuracy.*

## Conclusions

This chapter treated an actual and necessary issue in control engineering, that of PCH systems structural preserving time-discretization. A general time-discretization framework was first proposed for the nonlinear PCH systems using a combined time-discretization concept for the flows and efforts. General conditions were derived to satisfy the power conservation law in discrete-time and for the rest of PCH systems structure (energy, bilinear form, implicit form). The necessary conditions proposed in the nonlinear case, were derived also in the linear one, to be used later for symplectic time-discretization examples. A set of symplectic time-discretization schemes were proposed, using distinct time-discretization rules for the flows and efforts. For the combined symplectic time-discretization concept proposed in this chapter, four representative time-discretization rules were selected to be used as examples: explicit Euler rule, implicit Euler rule, implicit mid-point rule and implicit trapezoidal rule. For each symplectic time-discretization scheme, the basic properties of PCH systems were checked and proved, writing the power conservation law, the bilinear power form together with the implicit and explicit realizations of the discrete-time PCH system. Performing all of these steps, it was proved that for the linear lossless case, all the selected schemes are symplectic (i.e. structure preserving), as the only relation to be satisfied always holds. For the linear lossy case, the power conservation law necessary conditions does not always hold and thus, the PCH structure might lose its symplecticity by time-discretization. Section 4.4 introduced a new concept to differentiate two PCH systems, with respect to the power energies corresponding to the ports. Particularly we can associate this formulation, with the difference between the continuous-time and discrete-time PCH systems. Thus a ***discretization error Hamiltonian*** system was introduced as the difference between the continuous-time and discrete-time PCH systems. Some simulation examples are presented for the lossless and lossy cases, using two linear PCH systems given by an LC circuit and a capacitor microphone circuit in section 4.5, to underline the performances of the symplectic time-discretization schemes or ***discretization error Hamiltonian*** system. The symplectic discretization techniques proposed in this chapter, can be further extended for several control strategies of PCH systems, where high performances are necessary.

## 5. State-space identification of PCH systems

*In this chapter an analysis of Port Controlled Hamiltonian systems identification is proposed using classic state-space techniques known in the literature in the deterministic case. The analysis is performed using the most simple symplectic time-discretization scheme which makes use of the Euler explicit and implicit rules for the flows and efforts and results in a discrete state space realization, similar to the “classic” one.*

*For the deterministic case, a power energy approach is proposed, which makes use of the power conservation law. Using orthogonal/oblique projections gives the possibility to determine three different but equivalent realizations (corresponding to the control, storage and dissipation ports) for the system. As an example for this analysis a DC motor model is selected, to perform the identification simulations.*

### 5.1 Introduction

Subspace identification algorithms are based on concepts from system theory, linear algebra or statistics [107]. The main conceptual novelties in *subspace identification algorithms* are: the state of the system is central when speaking of system identification as compared to the classical approaches, which uses an input/output context; the subspace system identification approach is based on concepts and algorithms from numerical linear algebra (QR decomposition, SVD decomposition and its generalizations, together with angles between subspaces); the subspace identification approach offers a *geometric framework*, where different models are treated in a unified manner; the theoretical concepts translates into user-friendly software implementations (there is no explicit need for parameterizations and thus no highly technical and theoretical issues for the user, like canonical parameterizations). As compared to the classical approaches, some of the main points introduced by subspace identification algorithms are:

- **Parameterizations:** The subspace identification approach makes use only of the system order. With classical approaches, a part of the research was concentrated on canonical models (Ex: uncontrollable or observable canonical forms with minimal number of parameters) [107,164-170]. The main drawbacks of this approach are: they can lead to numerically ill-conditioned problems, with results sensitive to small perturbations; there is a need to overlap the parametrizations, since none of the existing ones will cover all dynamic systems; in practice minimal state-space models are desirable. In case of uncontrollable but observable modes, special parametrizations are necessary. All of these problems are eliminated in case of the subspace identification approach.
- **Convergence:** Subspace identification algorithms are faster than the classical approach, as they are not iterative and no convergence issues exist. Thus the lack of convergence, slow convergence or numerical instability are eliminated.
- **Model reduction:** In subspace identification, the reduced model of the dynamic system results directly from input/output data, without having to compute in advance the high-order model.

Subspace identification algorithms represent the input-state-output generalizations of the classical theory of the seventies, which used impulse responses for state-space

identification [107,171-179]. Some research results of influence can be found in [107,180-185] and for the combined deterministic-stochastic problem in [107,118,186,187,188].

## 5.2 Deterministic identification

In this section we recall some theoretical notions useful in the deterministic state-space identification analysis of PCH systems, using time-discretization schemes developed in chapter 4. The subspace identification of purely deterministic systems, with no measurement nor process noise ( $v_k = w_k = 0$ ) represents the ideal state-space identification case with no perturbations. This form of identification was introduced only as an academic issue, to be used for the identification of the system parameters [107]. Deterministic subspace identification algorithms, represent a class of algorithms used to compute the state-space models on the basis of input/output data. Further we will formulate the deterministic identification problem, which will be used also for PCH systems identification.

### *Deterministic identification problem [107]:*

**Given:**  $s$  measurements of the input  $u_k \in R^m$  and output  $y_k \in R^l$  generated by the unknown deterministic system of order  $n$  :

$$\begin{aligned} x_{k+1}^d &= Ax_k^d + Bu_k \\ y_k &= Cx_k^d + Du_k \end{aligned} \quad (5.1)$$

### **Determine:**

- The order  $n$  of the unknown system;
- The system matrices  $A \in R^{n \times n}$ ,  $B \in R^{n \times m}$ ,  $C \in R^{l \times n}$  (up to within a similarity transformation);

**Remark:** The above classic state-space system identification problem will be adapted and used for PCH system formulation using a proper time-discretization scheme from chapter 4.

In the literature [107-118], a series of techniques regarding deterministic state-space identification have been developed. In the following lines, some useful notations will be introduced for subspace identification algorithms of classic state-space systems. A central element in subspace identification algorithms are the *Hankel matrices*, constructed from input/output data. The input *Hankel matrix* has the following form:

$$U_{0/2i-1} \stackrel{def}{=} \begin{pmatrix} u_0 & u_1 & u_2 & \dots & u_{j-1} \\ u_1 & u_2 & u_3 & \dots & u_j \\ \dots & \dots & \dots & \dots & \dots \\ u_{i-1} & u_i & u_{i+1} & \dots & u_{i+j-2} \\ u_i & u_{i+1} & u_{i+2} & \dots & u_{i+j-1} \\ u_{i+1} & u_{i+2} & u_{i+3} & \dots & u_{i+j} \\ \dots & \dots & \dots & \dots & \dots \\ u_{2i-1} & u_{2i} & u_{2i+1} & \dots & u_{2i+j-2} \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} U_{0/i-1} \\ U_{i/2i-1} \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} U_p \\ U_f \end{pmatrix} \quad (5.2)$$

where  $U_p$  stands for the past inputs and  $U_f$  for the future inputs. The number of block rows  $i$ , is a user-defined index, which should be at least as large as the system order, we want to identify. As the number of inputs is  $m$ , this matrix consists of  $2mi$  rows, while the number of columns  $j$  represents the number of data samples used, equal to  $s - 2i + 1$  [107]. Shifting one row the past inputs  $U_p$ , we can define the matrices  $U_p^+$  and  $U_f^-$ , for the past and future inputs [107]. For instance we can write the input block Hankel matrix as:

$$U_{0/2i-1} = \begin{pmatrix} u_0 & u_1 & u_2 & \dots & u_{j-1} \\ u_1 & u_2 & u_3 & \dots & u_j \\ \dots & \dots & \dots & \dots & \dots \\ u_{i-1} & u_i & u_{i+1} & \dots & u_{i+j-2} \\ u_i & u_{i+1} & u_{i+2} & \dots & u_{i+j-1} \\ - & - & - & - & - \\ u_{i+1} & u_{i+2} & u_{i+3} & \dots & u_{i+j} \\ \dots & \dots & \dots & \dots & \dots \\ u_{2i-1} & u_{2i} & u_{2i+1} & \dots & u_{2i+j-2} \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} U_p^+ \\ U_f^- \end{pmatrix} \quad (5.3)$$

which is useful in the computation of the oblique projections  $O_{i-1}$ , future states  $x_{i+1}^d$ , etc.

Similarly with the inputs, are defined the output block Hankel matrices  $Y_{0/2i-1}, Y_p, Y_f, Y_p^+, Y_f^-$ . Using the Willems notation, it was introduced the Hankel matrix consisting of inputs and outputs as  $w_{0/i-1}$ :

$$W_{0/i-1} \stackrel{\text{def}}{=} \begin{pmatrix} U_{0/i-1} \\ Y_{0/i-1} \end{pmatrix} = \begin{pmatrix} U_p \\ Y_p \end{pmatrix} = W_p \quad (5.4)$$

When augmenting one row to the past inputs/outputs, we can define:

$$W_p^+ = \begin{pmatrix} U_p^+ \\ Y_p^+ \end{pmatrix} \quad (5.5)$$

**Remark:** The matrices presented above can be determined in different forms for each symplectic discretization scheme developed in chapter 4. The explicit/implicit Euler rules were selected for time-discretization of the flows and efforts throughout the chapter, for simplicity in computation and similarity with the classic approach [107].

The state sequences for subspace identification algorithms are defined as follows:

$$X_i^d \stackrel{\text{def}}{=} \begin{pmatrix} x_i^d & x_{i+1}^d & \dots & x_{i+j-2}^d & x_{i+j-1}^d \end{pmatrix} \in R^{n \times j} \quad (5.6)$$

with  $i$  representing the subscript of the first element of the state sequence. We also define the past states sequence by  $X_p^d$  and future states sequence by  $X_f^d$ . As the subspace identification algorithms make use of observability and controllability concepts, we introduce the extended ( $i > n$ ) observability matrix  $\Gamma_i$  defined as follows in the literature [107]:

$$\Gamma_i \stackrel{\text{def}}{=} \begin{pmatrix} C & CA & \dots & CA^{i-1} \end{pmatrix} \in R^{l \times n} \quad (5.7)$$

where the pair  $\{A, C\}$  is assumed observable. The reversed extended controllability matrix denoted as  $\Delta_i^d$  is defined by:

$$\Delta_i^d \stackrel{def}{=} \begin{pmatrix} A^{i-1}B & A^{i-2}B & \dots & B \end{pmatrix} \quad (5.8)$$

where the pair  $\{A, B\}$  is assumed controllable. We also introduce the lower block triangular matrix  $H_i^d$  as [107]:

$$H_i^d \stackrel{def}{=} \begin{pmatrix} D & 0 & 0 & \dots & 0 \\ CB & D & 0 & \dots & 0 \\ CAB & CB & D & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ CA^{i-2} & CA^{i-3}B & CA^{i-4}B & \dots & D \end{pmatrix} \quad (5.9)$$

In the following lines we will introduce the *matrix input-output equations* for classic state-space systems, which is useful also for Port Controlled Hamiltonian systems identification.

### Matrix Input-Output equations

$$\begin{aligned} Y_p &= \Gamma_i X_p^d + H_i^d U_p \\ Y_f &= \Gamma_i X_f^d + H_i^d U_p \\ X_f^d &= A^i X_p^d + \Delta_i^d U_p \end{aligned} \quad (5.10)$$

**Remark:** *The proof of this theorem can be easily determined from matrix relations using recursive relations. For the symplectic discretization schemes developed in chapter 4, this theorem can be adapted and proved for further analysis with state-space identification algorithms.*

Further we introduce a definition for the *persistence of excitation* of the input signal applied for state-space identification [107]. This notion is strongly connected with the identifiability analysis presented in the previous chapters, as good excitation signals give proper identified models.

### Definition (Persistence of excitation)

*The input sequence  $u_k \in R^m$  is persistently exciting of order  $2i$ , if the input covariance matrix*

$$R^{uu} \stackrel{def}{=} \Phi[U_{0/2i-1}, U_{0/2i-1}] \quad (5.11)$$

*has full rank, which is  $2mi$ .*

**Remark:** *In the statistical theory, the covariance matrix  $\Phi[G, L]$  of two matrices  $G \in R^{p \times j}$  and  $L \in R^{q \times j}$  is defined as:*

$$\Phi[G, L] \stackrel{def}{=} \mathbf{E}_j \left[ G \cdot L^T \right] \quad (5.12)$$

where  $\mathbf{E}$  represents the *expectation operator* and  $j$  the length of the data series available. Further we introduce the *projector operator*  $\Pi_L$  by [107]:

$$\Pi_L \stackrel{def}{=} L^T \cdot (L \cdot L^T)^+ \cdot L \quad (5.13)$$

which projects the row space of a matrix onto the row space of the matrix  $L \in R^{q \times j}$ , where  $\bullet^+$  represents the *Moore-Penrose pseudo-inverse* of the matrix  $\bullet$ . The *orthogonal projection* is defined in the literature [107] by:

$$G / \mathbf{L} \stackrel{def}{=} G \cdot \Pi_{\mathbf{L}} = G \cdot L^T (L \cdot L^T)^+ \cdot L \quad (5.14)$$

and represents the projection of the row space of the matrix  $G \in R^{p \times j}$  on the row space of the matrix  $L$ . The matrix  $L$  is bolded, as the result of the orthogonal projection lies in the row space of  $L$ . We also introduce the operator:

$$G / \mathbf{L}^\perp \stackrel{def}{=} G \cdot \Pi_{\mathbf{L}^\perp} \quad (5.15)$$

where  $\Pi_{\mathbf{L}^\perp}$  is the geometric operator that projects the row space of a matrix onto the orthogonal complement of the row space of the matrix  $L$ .  $\Pi_{\mathbf{L}^\perp}$  is defined by [107]:

$$\Pi_{\mathbf{L}^\perp} = I_j - \Pi_{\mathbf{L}} \quad (5.16)$$

Further we introduce the concept of *oblique projection*, by which the matrix  $G$  can be decomposed as linear combinations of two non-orthogonal matrices ( $L$  and  $F$ ) and the orthogonal complement of  $L$  and  $F$ . We can think of the oblique projection as follows [110]: *project the row space of  $G$  orthogonally on the joint row space of  $L$  and  $F$ ; decompose the result along the row space of  $F$ .*

The oblique projection is further introduced by the following corollary [107]:

### Corollary 1 Oblique projections

*The oblique projection of the row space of  $G \in R^{p \times j}$  along the row space of  $L \in R^{q \times j}$  on the row space of  $F \in R^{r \times j}$  can also be defined as:*

$$G / \mathbf{L} / \mathbf{F} = \left[ G / \mathbf{L}^\perp \right] \left[ F / \mathbf{L}^\perp \right]^+ \cdot F \quad (5.17)$$

The proof of this corollary follows by straight computations using the *matrix inversion lemma*:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix} \quad (5.18)$$

starting from the definition of oblique projections:

### Definition 1 Oblique projections

*The oblique projection of the row space of  $G \in R^{p \times j}$  along the row space of  $L \in R^{q \times j}$  on the row space of  $F \in R^{r \times j}$  is defined as:*

$$G / \mathbf{L} \mathbf{F} \stackrel{def}{=} G \begin{pmatrix} \mathbf{F}^T & \mathbf{L}^T \end{pmatrix} \cdot \left[ \begin{array}{c} \left( \begin{array}{cc} \mathbf{F}\mathbf{F}^T & \mathbf{F}\mathbf{L}^T \\ \mathbf{L}\mathbf{F}^T & \mathbf{L}\mathbf{L}^T \end{array} \right)^+ \\ \text{first } / r / \text{ columns} \end{array} \right] \cdot \mathbf{F} \quad (5.19)$$

The concepts of orthogonal and oblique projections presented above, are necessary in the development of subspace identification algorithms and estimation of system parameters. Further we introduce the deterministic identification theorem for the classic state-space approach, useful for the state-space identification of PCH systems. Two important advantages of the deterministic identification theorem are: the state-sequence  $X_f^d$  can be calculated using input-output data, without knowing the system matrices  $A, B, C, D$ ; the extended observability matrix ( $\Gamma_i$ ) can be determined directly from the input-output data.

### Theorem: Deterministic identification

Under the assumptions that:

1. The input  $u_k$  is persistently exciting of order  $2i$
2. The intersection of the row space of  $U_f$  (future inputs) and the row space of  $X_p^d$  (past states) is empty.
3. The user-defined weighting matrices  $W_1 \in R^{lixli}$  and  $W_2 \in R^{jxj}$  are such that  $W_1$  is of full rank and  $W_2$  obeys:  $\text{rank}(W_p) = \text{rank}(W_p W_2)$ , where  $W_p$  is the block Hankel matrix containing the past inputs and outputs and with  $o_i$  defined as the oblique projection:

$$O_i \stackrel{def}{=} Y_f / U_f \mathbf{W}_p \quad (1)$$

and the SVD decomposition:

$$W_1 O_i W_2 = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} \quad (2)$$

we have:

1. The matrix  $o_i$  is equal to the product of the extended observability matrix and the states:

$$O_i = \Gamma_i X_f^d \quad (3)$$

2. The order of the system is equal to the number of singular values in (2) different from zero.
3. The extended observability matrix  $\Gamma_i$  is equal to:

$$\Gamma_i = W_1^{-1} U_1 S_1^{1/2} T \quad (4)$$

4. The part of the state sequence  $X_f^d$  that lies in the column space of  $W_2$  can be recovered from:

$$X_f^d W_2 = T_1^{-1} S_1^{1/2} V_1^T \quad (5)$$

5. The state sequence  $X_f^d$  is equal to:

$$X_f^d = \Gamma_i^+ \cdot O_i \quad (6)$$

Specific choices of the weighting matrices ( $w_1$  and  $w_2$ ) lead to different identification algorithms [107] and the choice of this weights determines the state-space basis in which the final model is obtained. The similarity transformation ( $T$ ) is introduced to make the recovered observability matrix ( $\Gamma_i$ ) and state sequence ( $X_f^d$ ) exactly equal to the original ones. Using different weighting matrices result in different observability and state space sequences matrices. Each set of results will lead to a set of system parameters ( $A, B, C$ ), equivalent up to a similarity transformation to the original (real) system. In (3)  $O_i$  represents the oblique projection of the row space of the future outputs ( $Y_f$ ) along the row space of the future inputs ( $U_f$ ) on the row space of  $W_p$  (past inputs  $U_p$  and outputs  $Y_p$ ) and can be determined to be equal to the product between the observability matrix ( $\Gamma_i$ ) and the state-sequence ( $X_f^d$ ). Using the SVD decomposition it is computed the system order from the non-zero singular values. Further are determined the state-sequences ( $X_f^d$ ) and the extended observability matrix ( $\Gamma_i$ ) as in (3), (4) and (5). A summary of this theorem, can be given by the following relations:

$$\begin{cases} \text{rank} \left( Y_f / U_f \cdot W_p \right) = n \\ \text{rowspace} \left( Y_f / U_f \cdot W_p \right) = \text{rowspace} \left( X_f^d \right) \\ \text{columnspac} \left( Y_f / U_f \cdot W_p \right) = \text{columnspac} \left( \Gamma_i \right) \end{cases} \quad (5.20)$$

From this summary, the resulting algorithms are called *subspace algorithms*, as they compute the system parameters ( $A, B, C$ ) as subspaces of projected data matrices.

**Remark:** *The proof of this theorem is straightforward from simple computations and can be found in extended form in [107]. The deterministic identification theorem formulated above can be developed also for the symplectic discretization schemes of chapter 4 (see Appendix F). In the deterministic identification algorithms, we will make use only of the most simple time-discretization scheme, which combines two first-order Euler rules for the flows and efforts.*

## 5.2.1 Deterministic identification algorithms

In this section we will briefly present two deterministic identification algorithms known from the literature and applied for classic state-space identification systems [107]. We have considered this two algorithms from the following reasons: they have the same state-space structure with PCH systems; they both make use of the deterministic identification theorem presented previously; they are simple to understand and implement for the user. A first class of algorithms are *the intersection algorithms* [107, 109-116, 192], where the row space of the



state sequence  $x_f^d$  is computed as the intersection between the row space of the past inputs and outputs and the row space of the future inputs and outputs:

$$\text{row space } X_f^d = \text{row space } \begin{pmatrix} U_p \\ Y_p \end{pmatrix} \cap \text{row space } \begin{pmatrix} U_f \\ Y_f \end{pmatrix} \quad (5.21)$$

Different ways to compute the intersection have been developed, from which a first way makes use of a SVD decomposition [114,115, 116] of the Hankel matrix  $\begin{pmatrix} U_{0,2i-1} & Y_{0,2i-1} \end{pmatrix}^T$ , while a second way [109, 112] is by using as basis of the intersection, the principal directions between the row space of the past inputs-outputs and the row space of the future inputs-outputs. In the literature [108,109, 117, 118,189] *projection algorithms* have also been developed, for which, we remark that the system order and extended observability matrix can be determined from a SVD decomposition. Further we introduce two deterministic identification algorithms, which will be applied and tested for PCH system state-space identification, using a simple time-discretization scheme.

### Deterministic algorithm I [107]

1. Calculate the oblique projections:

$$O_i = Y_f^- / U_f^- \mathbf{W}_p \quad (5.22)$$

$$O_{i-1} = Y_f^+ / U_f^+ \mathbf{W}_p^+ \quad (5.23)$$

2. Calculate the SVD of the weighted oblique projection:

$$W_1 O_i W_2 = USV^T \quad (5.24)$$

3. Determine the order by inspecting the singular values in  $s$  and partition the SVD accordingly to obtain  $U_1$  and  $s_1$ .
4. Determine  $\Gamma_i$  and  $\Gamma_{i-1}$  as:

$$\Gamma_i = W_1^{-1} U_1, \Gamma_{i-1} = \underline{\Gamma}_i \quad (5.25)$$

5. Determine  $x_i^d$  and  $x_{i+1}^d$  as:

$$x_i^d = \Gamma_i^+ O_i, x_{i+1}^d = \Gamma_{i-1}^+ O_{i-1} \quad (5.26)$$

6. Solve the set of linear equations for  $A, B, C$  and  $D$ :

$$\begin{pmatrix} x_{i+1}^d \\ Y_{i|i} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_i^d \\ U_{i|i} \end{pmatrix} \quad (5.27)$$

The current algorithm makes use of the following steps from the deterministic identification theorem: the order of the system from inspection of the singular values; the extended observability matrix  $\Gamma_i$ ; the state sequence  $x_i^d = x_f^d$ . This algorithm determines the system

parameters  $(A, B, C, D)$  by using the oblique projections  $O_{i-1}, O_i$  and determining the extended observability matrices  $\Gamma_{i-1}, \Gamma_i$  and system-states  $X_i^d, X_{i+1}^d$ .

### Deterministic algorithm II [107]

1. Calculate the oblique projection:

$$O_i = Y_f / U_f \mathbf{W} \mathbf{P} \quad (5.28)$$

2. Calculate the SVD of the weighted projection:

$$W_1 O_i W_2 = U S V^T \quad (5.29)$$

3. Determine the order by inspecting the singular values in  $s$  and partition the SVD accordingly to obtain  $U_1, U_2$  and  $S_1$ .

4. Determine  $\Gamma_i$  and  $\Gamma_i^\perp$  as:

$$\Gamma_i = W_1 U_1 S_1^{1/2}, \Gamma_i^\perp = U_2^T W_1 \quad (5.30)$$

5. Determine  $A$  from  $\Gamma_i$  as  $A = \Gamma_i^+ \overline{\Gamma_i}$ .

6. Determine  $C$  as the first  $l$  rows of  $\Gamma_i$ .

with:

$$\begin{aligned} (M_1 \quad M_2 \quad \dots \quad M_i) &= \Gamma_i^\perp Y_f U_f^+ \\ (L_1 \quad L_2 \quad \dots \quad L_i) &= \Gamma_i^\perp \end{aligned} \quad (5.31)$$

7. Solve  $B$  and  $D$  from:

$$\begin{pmatrix} M_1 \\ M_2 \\ \dots \\ M_i \end{pmatrix} = \begin{pmatrix} L_1 & L_2 & \dots & L_{i-1} & L_i \\ L_2 & L_3 & \dots & L_i & 0 \\ L_3 & L_4 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ L_i & 0 & \dots & 0 & 0 \end{pmatrix} \begin{pmatrix} I_l & 0 \\ 0 & \underline{\Gamma_i} \end{pmatrix} \begin{pmatrix} D \\ B \end{pmatrix} \quad (5.32)$$

The current algorithm makes use of the deterministic identification theorem to determine: the system order from singular value inspection; the extended observability matrix  $\Gamma_i$ . As compared to the first algorithm, the second one determines separately the matrices  $A, C$  from the matrices  $B, D$ .

## 5.2.2 Matrix I/O representation of symplectic discretization schemes

This section will prove, that discrete-time PCH systems can be written by means of a *matrix input/output representation* [107], for a simple time-discretization scheme, that makes use of Euler discretization rules for the flows and efforts.

**Remark:** A complete analysis for the proposed time-discretization schemes of chapter 4, can be found in Appendix F using matrix I/O representations, orthogonal or oblique projections.

Further we consider a simple time-discretization scheme that assumes an explicit Euler rule for the flows and an implicit Euler rule for the efforts. Applying this time-discretization scheme, starting from the general relations formulated in chapter 4, we can express the discrete-time form of the state-space system as follows:

$$\begin{cases} x_{k+1} = \bar{A} x_k + \bar{B} u_k \\ y_k = C x_k \end{cases} \quad (5.33)$$

where the following notations have been adopted:

$$\begin{cases} \text{---} \text{def} \\ A = I + \Delta T \cdot A \\ \text{---} \text{def} \\ B = \Delta T \cdot B \\ \text{---} \text{def} \\ C = B^T Q \end{cases} \quad (5.34)$$

**Remark:** Even if the state-space structure from (5.33) is the same with the classic state-space approach [107], the performed results are different. These differences come from the discretization methodology used, which gives different state-space matrices (see 5.34).

In the above relation  $A \in R^{n \times n}$ ,  $B \in R^{n \times m}$ ,  $C \in R^{l \times n}$  represent the matrices of the initial continuous-time PCH system. Computing the states and outputs for  $k = 1$ , we obtain:

$$\begin{cases} x_1 = \bar{A} x_0 + \bar{B} u_0 \\ y_0 = C x_0 \end{cases} \quad (5.35)$$

For  $k = 2$ , it results:

$$\begin{cases} x_2 = \bar{A}^2 x_0 + \bar{A} \bar{B} u_0 + \bar{B} u_1 \\ y_1 = C x_1 = C \bar{A} x_0 + C \bar{B} u_0 \end{cases} \quad (5.36)$$

For  $k = 3$ , it results:

$$\begin{cases} x_3 = \bar{A}^3 x_0 + \bar{A}^2 \bar{B} u_0 + \bar{A} \bar{B} u_1 + \bar{B} u_2 \\ y_2 = C x_2 = C \bar{A}^2 x_0 + C \bar{A} \bar{B} u_0 + C \bar{B} u_1 \end{cases} \quad (5.37)$$

Adopting the notations:

$$\begin{cases} \Delta_i^d \text{---} \text{def} = \begin{pmatrix} \bar{A}^{i-1} \bar{B} & \bar{A}^{i-2} \bar{B} & \dots & \bar{A} \bar{B} & \bar{B} \end{pmatrix} \\ U_p \text{---} \text{def} = (u_0 \quad u_1 \quad \dots \quad u_{i-1})^T \end{cases} \quad (5.38)$$

we can deduce the following general form for the future states, using also (5.35), (5.36), (5.37) and (5.38):

$$X_f^d = X_i^d = \bar{A}^i X_p^d + \Delta_i^d \cdot U_p \quad (5.39)$$

where  $X_p^d = X_0$  represents the initial state (*past state*). Adopting the notations:

$$\left\{ \begin{array}{l} \Gamma_i \stackrel{def}{=} \left( C \quad \overline{CA} \quad \overline{CA}^2 \quad \dots \quad \overline{CA}^{i-1} \right)^T \\ H_i^d \stackrel{def}{=} \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \overline{CB} & 0 & 0 & \dots & 0 \\ \overline{CA} \overline{B} & \overline{CB} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \overline{CA}^{i-2} & \overline{CA}^{i-3} \overline{B} & \overline{CA}^{i-4} \overline{B} & \dots & 0 \end{pmatrix} \end{array} \right. \quad (5.40)$$

where  $\Gamma_i$  is the extended observability matrix and  $H_i^d$  the lower block triangular Toeplitz matrix, we can write the following general relations:

$$\begin{cases} Y_p = \Gamma_i \cdot X_p^d + H_i^d U_p \\ Y_f = \Gamma_i X_f^d + H_i^d U_f \end{cases} \quad (5.41)$$

for the past and future output vectors. The vector of the past and future inputs and outputs are denoted as follows:

$$\left\{ \begin{array}{l} Y_p \stackrel{def}{=} (Y_0 \quad Y_1 \quad Y_2 \quad \dots \quad Y_{i-1})^T \\ U_p \stackrel{def}{=} (U_0 \quad U_1 \quad U_2 \quad \dots \quad U_{i-1})^T \\ Y_f \stackrel{def}{=} (Y_i \quad Y_{i+1} \quad Y_{i+2} \quad \dots \quad Y_{2i-1})^T \\ U_f \stackrel{def}{=} (U_i \quad U_{i+1} \quad U_{i+2} \quad \dots \quad U_{2i-1})^T \end{array} \right. \quad (5.42)$$

Using (5.39) and (5.41) we can write the following *matrix input-output equations*:

$$\begin{cases} X_f^d = \overline{A}^i X_p^d + \Delta_i^d U_p \\ Y_p = \Gamma_i X_p^d + H_i^d U_p \\ Y_f = \Gamma_i X_f^d + H_i^d U_f \end{cases} \quad (5.43)$$

which has the same structure as for the classic state-space systems [107].

**Remark:**

*The **matrix input-output equations** (5.43), of this symplectic time-discretization scheme, are similar as form to the classic state-space systems used with subspace identification algorithms (see [107]). Following the same steps of the deterministic identification theorem formulated for classic state-space systems, we can also prove the results for the current symplectic time-discretization scheme.*

### 5.2.3 Simulations and discussion

In this section, we will perform a set of simulations using the deterministic identification algorithms described in section 5.2.1 using the time-discretization scheme presented in 5.2.2, which has a similar form to the classic one [107]. The simulations are performed on a lossy PCH system represented by a DC motor, presented in chapter 1 (see section 1.3.7.2). For the

first example we will consider the DC motor with a white noise sequence as input signal, generated by software, a horizon length ( $j = 50$ ) and a time interval  $t_s = 10 s$ . In the first figure is represented the output response of the real and estimated systems for the selected parameters.

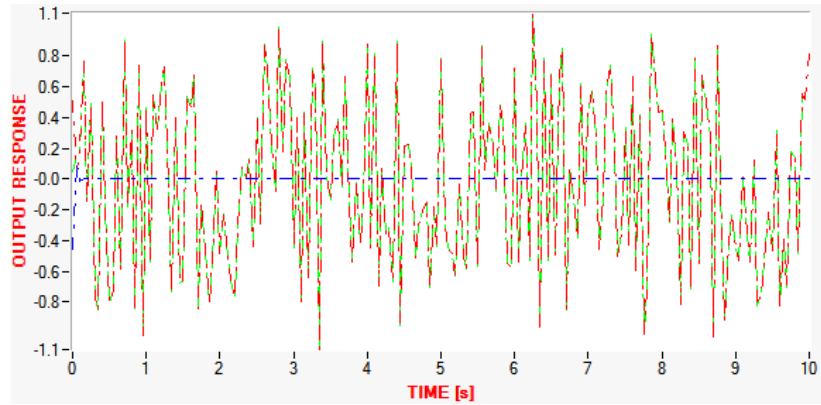


Figure 5.1. Output response of the real (green), estimated (red) and error (blue) for the deterministic algorithm I.

In Figure 5.1 the output responses of the real and estimated systems are illustrated, and as was expected for the case with no perturbations, the estimation errors are almost null, excepting the first identification steps. In the following figures, we will make a representation also for the power port energies to prove the symplecticity of the PCH system by identification.

**Remark:** *The power port energies representation for the identification algorithms of this chapter, is introduced as a consequence of the structural identifiability concepts proposed in chapter 2, but also to prove the power conservation law, when performing the system identification. Thus by system identification are obtained equivalent state-space models given by  $(A_e, B_e, C_e)$ , which approximate the real state-space system given by  $(A, B, C)$ .*

For the control port, the power energy representation of the real and estimated systems, is given in the following figure.

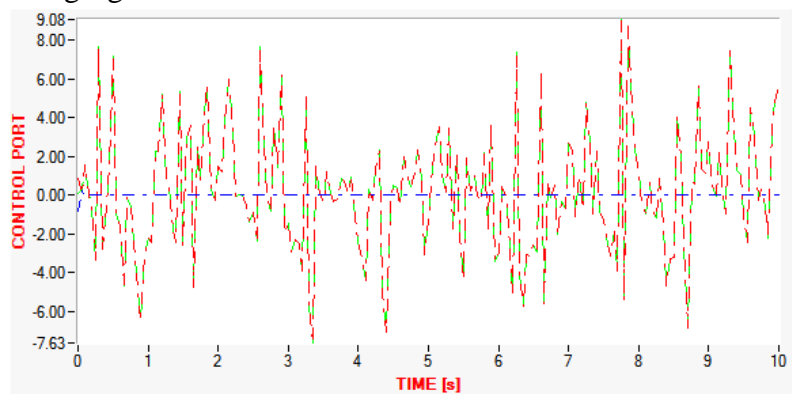


Figure 5.2. Control port power energy representation of the real (green), estimated (red) and error (blue) for the deterministic algorithm I.

From this figure we observe that the estimation of the control port power energy is almost perfect for this deterministic algorithm, with an error almost zero on the time-interval.

For the dissipation port, the power energy representation of the real and estimated systems is drawn in Figure 5.3.

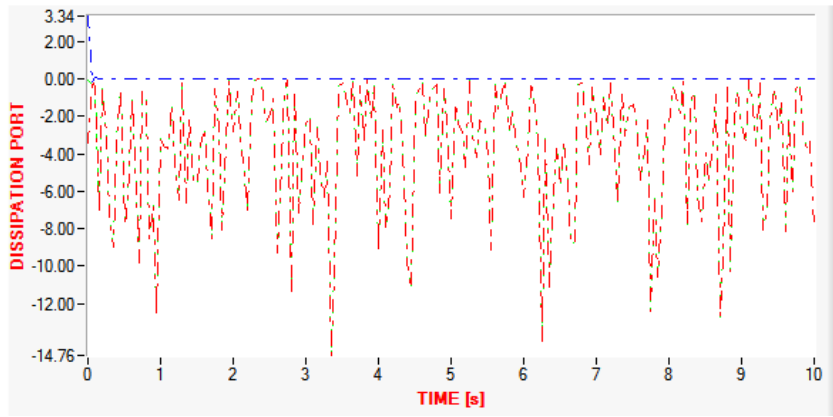


Figure 5.3. Dissipation port power energy representation of the real (green), estimated (red) and error (blue) for the deterministic algorithm I.

As Figure 5.3 shows, the estimated error on the time-interval is almost zero, excepting the first steps of estimation. For the storage port power energy, the real and estimated responses, can be seen in Figure 5.4.

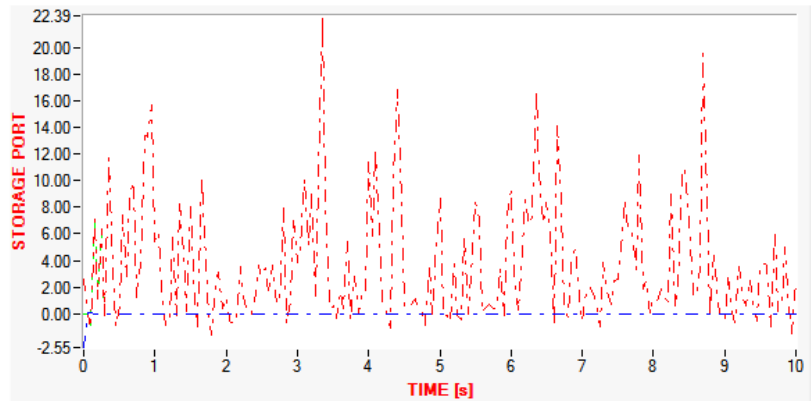


Figure 5.4. Storage port power energy representation of the real (green), estimated (red) and error (blue) for the deterministic algorithm I.

As expected, the power energy responses of the real and estimated systems, have a very good fit on the time-interval, excepting the initial estimation steps. In the following figure, is analysed the preservation of the power conservation law for the estimated system.

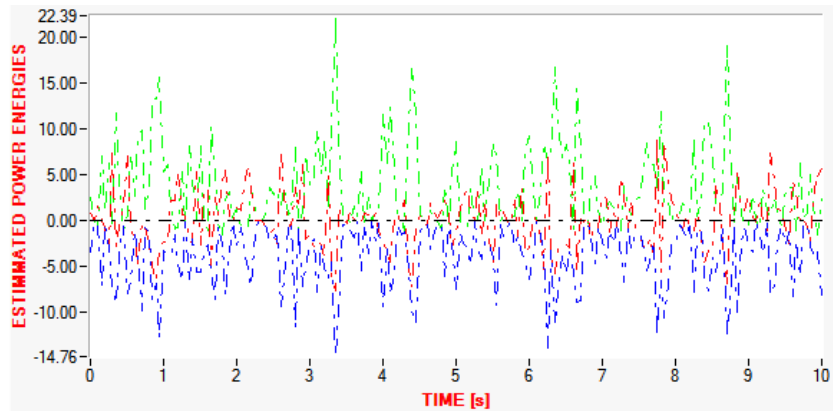


Figure 5.5. Estimated system port energies (green – storage port, red – control port, blue-dissipation port) and power conservation law (black) for deterministic algorithm I.

From Figure 5.5, the power conservation law of the estimated system is thus satisfied, which makes it an equivalent PCH system of the real system, that satisfies the basic properties.

**Remark:** *The estimated system represents an equivalent PCH system, obtained by applying a non-singular transformation matrix ( $T \in R^{n \times n}$ ) to the real one and was proved in chapter 2 to preserve Hamiltonian properties.*

For the second deterministic algorithm, we will consider the same scenario as for the first deterministic algorithm. Thus we consider a scenario with a horizon length ( $j = 50$ ) and a time-interval  $t_S = 10 s$ . In the following figure, we represent the fitness between the output response of the real and estimated systems.

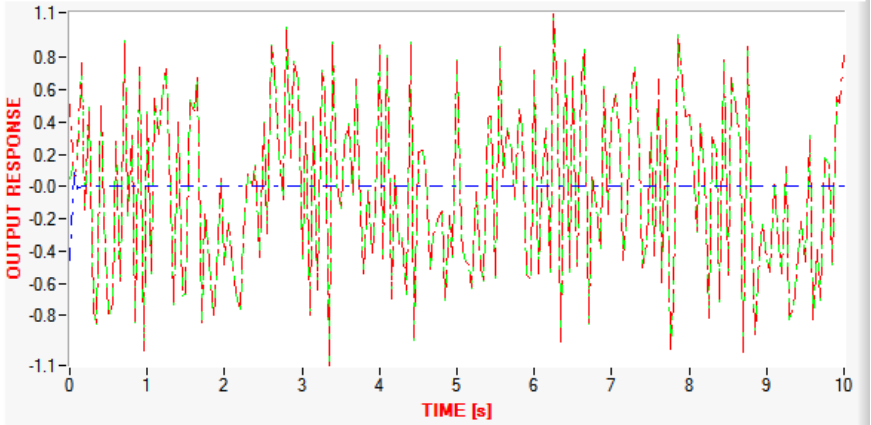


Figure 5.6. Output response of the real (green), estimated (red) and error (blue) for the deterministic algorithm II.

As expected, the fitness between the real and estimated output responses of the DC motor is almost perfect in the deterministic case, with errors close to zero as the blue line shows. For the control port, the fitness analysis of the real and estimated systems, is performed in the following figure.

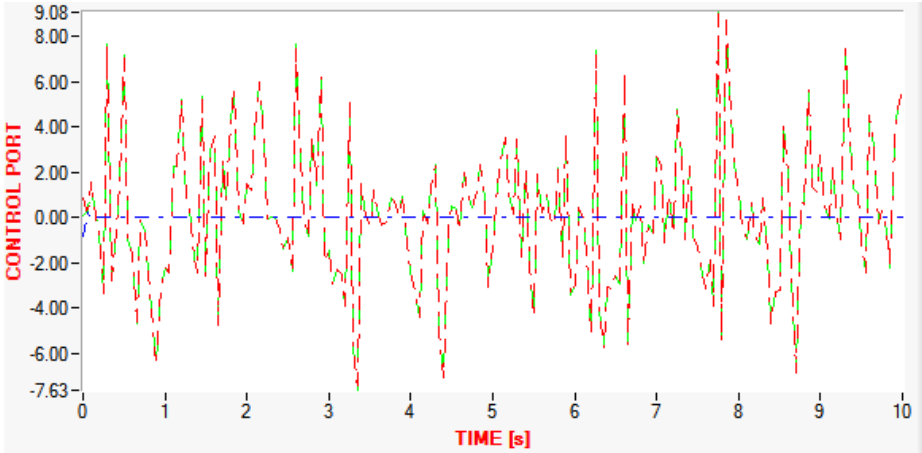


Figure 5.7. Control port power energy representation of the real (green), estimated (red) and error (blue) for the deterministic algorithm II.

From Figure 5.7 we observe a very good estimation of the real system control port power energy, with errors that are close to zero on the time-interval. For the dissipation port, the fitness analysis performed for this algorithm, can be found in the following figure.

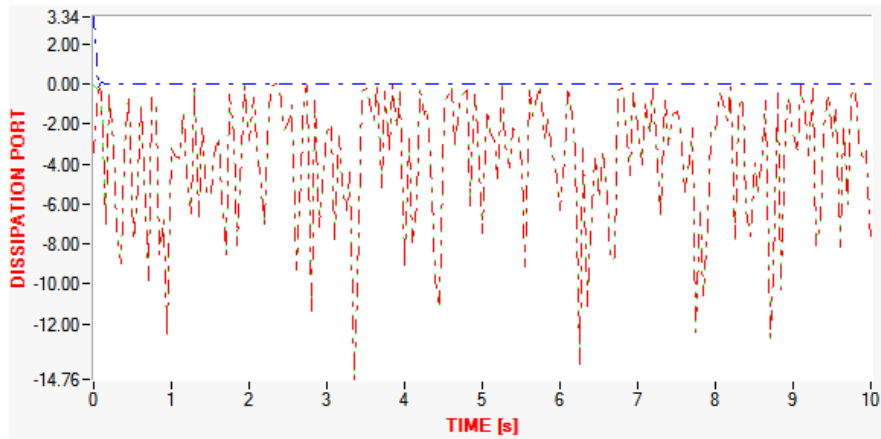


Figure 5.8. Dissipation port power energy representation of the real (green), estimated (red) and error (blue) for the deterministic algorithm II.

According to Figure 5.8, the estimated error between the real and estimated dissipation port power energy is almost zero, with higher errors on the first steps of the algorithm. For the storage port power energy, the fitness test analysis is illustrated in the following figure.

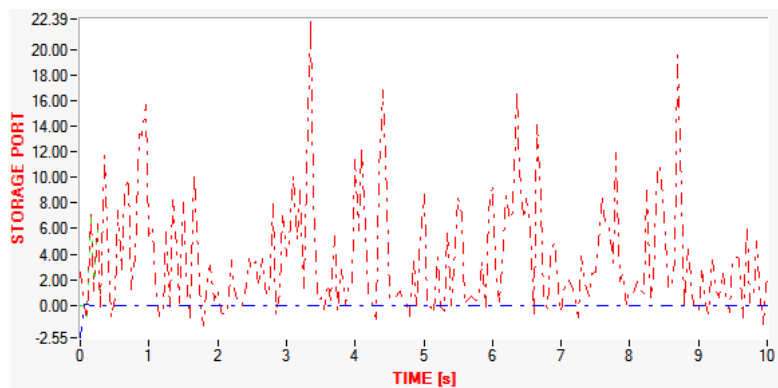


Figure 5.9. Storage port power energy representation of the real (green), estimated (red) and error (blue) for the deterministic algorithm II.

As this graphic shows, the fitness of the real and estimated storage port power energy is almost perfect, with the exception of the first estimation steps. In the next figure, the preservation of the power conservation law is performed for the estimated model.



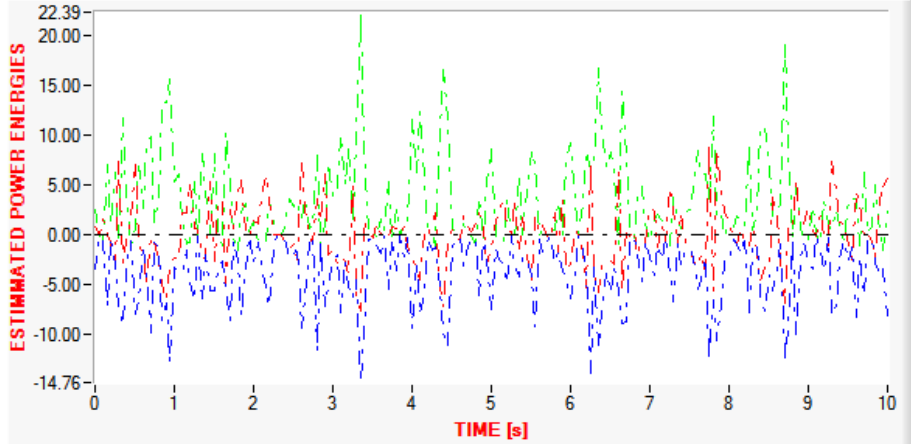


Figure 5.10. Estimated system power port energies (green – storage port, red –control port, blue-dissipation port) and power conservation law (black) for deterministic algorithm II.

As Figure 5.10 shows, the power conservation law is satisfied, preserving thus the basic properties of PCH systems for the selected scenario.

**Remark:** *The results obtained with the deterministic identification algorithms for the DC motor model are similar as expected, with an almost perfect fit between the real and estimated parameters due to the absence of the perturbations.*

### 5.3 Power energy approach

In this section it is proposed a power energy context for the deterministic identification of PCH systems in the general lossy case. The lossless case can be determined easily from the results. This section is in direct relation with the results on structural identifiability proposed in chapter 2 section 2.5, where we have defined the *global/local identifiability* or *global/local port identifiability* by means of the port energies. The current context makes use of the subspace identification approach to estimate the real system parameters from known inputs and power port energies. Thus we will formulate and prove a specific power energy deterministic identification theorem and propose an algorithm to demonstrate by simulation the results.

#### 5.3.1 Deterministic identification

This section formulates the power energy deterministic identification context, together with a specific theorem completely proved.

**Given:**  $s$  measurements of the input  $u_k \in R^m$  and power port energies  $P_C(k), P_S(k), P_R(k)$  generated by the unknown deterministic system of order  $n$  :

$$\begin{aligned} x_{k+1}^d &= \bar{A} x_k^d + \bar{B} u_k \\ y_k &= \bar{C} x_k^d \end{aligned} \quad (5.44)$$

**Determine:**

- The order  $n$  of the unknown system;

- The system matrices  $\bar{A} \in R^{n \times n}$ ,  $\bar{B} \in R^{n \times m}$ ,  $\bar{C} \in R^{l \times n}$ , up to within a similarity transformation ( $T \in R^{n \times n}$ );

**Remark:** At least two power port energies must be known, in order to perform this deterministic identification approach in the lossy case, as we make use of the power conservation law.

On this deterministic identification approach, we make use of the power conservation law, in which case we can consider the control port states as the sum between the storage and dissipation port states, as follows:

$$X_{fC}^d = L_P W_{PS} + L_P W_{PR} \quad (5.45)$$

where we define the storage and dissipation states by:

$$\begin{cases} X_{fS}^d \stackrel{def}{=} L_P W_{PS} \\ X_{fR}^d \stackrel{def}{=} L_P W_{PR} \end{cases} \quad (5.46)$$

**Remark:** The states  $X_f^d$  in the classic approach can be written as  $X_f^d = L_P \cdot W_P$ , where

$$L_P = \left( \Delta_i^d - A^i \Gamma_i^+ H_i^d \middle| A^i \Gamma_i^+ \right) \quad (\text{see 107]).$$

and thus, we can also define:

$$X_{fC}^d \stackrel{def}{=} X_{fS}^d + X_{fR}^d \quad (5.47)$$

Further we write the future outputs on the control port as:

$$Y_{fC} = \Gamma_i L_P (W_{PS} + W_{PR}) + H_i^d U_f \quad (5.48)$$

Making use of the control port power energy relation ( $P_C = y^T u$ ), we can write:

$$Y_{fC} = \left( P_{Cf} U_f^+ \right)^T \quad (5.49)$$

**Remark:**  $U_f^+$  is the Moore-Penrose pseudo-inverse of the future inputs matrix applied to the dynamic system.

(5.49) is also equivalent to:

$$Y_{fC} = - \left( (P_{SF} + P_{RF}) U_f^+ \right)^T \quad (5.50)$$

using the power conservation law. We define further the future outputs corresponding to the storage and dissipation ports, as:

$$\begin{cases} Y_{fS} \stackrel{def}{=} - \left( P_{SF} U_f^+ \right)^T \\ Y_{fR} \stackrel{def}{=} - \left( P_{RF} U_f^+ \right)^T \end{cases} \quad (5.51)$$

**Remark:** Similarly to the classic case, we can define the past/future power energies or system outputs corresponding to each port.

Multiplying in (5.48) with the orthogonal projection  $(\Pi_{U_f}^\perp)$  of the future inputs, we get:

$$Y_{fS} / U_f^\perp + Y_{fR} / U_f^\perp = \Gamma_i L_P (W_{PS} + W_{PR}) / U_f^\perp \quad (5.52)$$

Multiplying to the right with the pseudo-inverse matrix  $\left( (W_{PS} + W_{PR}) / U_f^\perp \right)^+$ , we get:

$$\left( Y_{fS} / U_f^\perp \right) \left( (W_{PS} + W_{PR}) / U_f^\perp \right)^+ + \left( Y_{fR} / U_f^\perp \right) \left( (W_{PS} + W_{PR}) / U_f^\perp \right)^+ = \Gamma_i L_P \quad (5.53)$$

Again we can multiply to the right with  $W_{PS} + W_{PR}$  :

$$\begin{aligned} & \left( Y_{fS} / U_f^\perp \right) \left( (W_{PS} + W_{PR}) / U_f^\perp \right)^+ (W_{PS} + W_{PR}) + \\ & + \left( Y_{fR} / U_f^\perp \right) \left( (W_{PS} + W_{PR}) / U_f^\perp \right)^+ (W_{PS} + W_{PR}) = \Gamma_i L_P (W_{PS} + W_{PR}) \end{aligned} \quad (5.54)$$

We observe that to the left part we can split in two oblique projections corresponding to the storage and dissipation ports, defined as:

$$\begin{cases} \overset{def}{O_{iS}} = Y_{fS} / U_f (W_{PS} + W_{PR}) \\ \overset{def}{O_{iR}} = Y_{fR} / U_f (W_{PS} + W_{PR}) \end{cases} \quad (5.55)$$

Then it results:

$$O_{iS} + O_{iR} = \Gamma_i L_P (W_{PS} + W_{PR}) \quad (5.56)$$

(5.55) can be also written as:

$$\begin{cases} O_{iS} = - \left( P_{Sf} U_f^+ \right)^T / U_f (W_{PS} + W_{PR}) \\ O_{iR} = - \left( P_{Rf} U_f^+ \right)^T / U_f (W_{PS} + W_{PR}) \end{cases} \quad (5.57)$$

using the future inputs and power energies. Further we can define the oblique projections corresponding to the storage and dissipation ports as:

$$\begin{cases} \overset{def}{O_{iS}} = \Gamma_{iC} L_P W_{PS} \\ \overset{def}{O_{iR}} = \Gamma_{iC} L_P W_{PR} \end{cases} \quad (5.58)$$

Using (5.46), we can write further:

$$\begin{cases} O_{iS} = \Gamma_{iC} X_{fS}^d \\ O_{iR} = \Gamma_{iC} X_{fR}^d \end{cases} \quad (5.59)$$

**Remark:** The oblique projection corresponding to the sum  $O_{iS} + O_{iR}$ , is defined as the oblique projection corresponding to the control port:

$$O_{iC} \stackrel{def}{=} O_{iS} + O_{iR} \quad (5.60)$$

Performing the SVD decomposition of  $O_{iC}$  as in the classic case, we can write:

$$W_1 O_{iC} W_2 = W_1 (O_{iS} + O_{iR}) W_2 \quad (5.61)$$

This can be split as:

$$W_1 O_{iC} W_2 = W_1 \cdot O_{iS} \cdot W_2 + W_1 \cdot O_{iR} \cdot W_2 \quad (5.62)$$

Then it results:

$$W_1 O_{iC} W_2 = U_S S_S V_S^T + U_R S_R V_R^T = U_C S_C V_C^T \quad (5.63)$$

by SVD decomposition on each port oblique projection.  $w_1, w_2$  are weighting matrices such that  $w_1$  is of full rank and  $\text{rank}(w_p) = \text{rank}(w_p \cdot w_2)$  [107].

**Remark:** We have denoted  $U_S, S_S, V_S$ ,  $U_R, S_R, V_R$  and  $U_C, S_C, V_C$  as the matrix elements corresponding to the SVD decomposition of the storage, dissipation and control ports oblique projections ( $O_{iS}, O_{iR}, O_{iC}$ ). We can thus inspect the singular values contribution for the storage, control or dissipation ports, to determine the dynamic system degree and recover the system parameters.

For the control port we can write the SVD decomposition as:

$$W_1 \Gamma_{iC} \begin{pmatrix} X_{fS}^d + X_{fR}^d \end{pmatrix} W_2 = U_C S_C V_C^T \quad (5.64)$$

We can split this relation as follows:

$$\begin{cases} W_1 \cdot \Gamma_{iC} = U_C \cdot S_C^{1/2} \cdot T_C \\ X_{fC}^d \cdot W_2 = T_C^{-1} \cdot S_C^{1/2} \cdot V_C^T \end{cases} \quad (5.65)$$

**Remark:** We have denoted the extended observability matrix  $\Gamma_i$  as  $\Gamma_{iC}$  corresponding to the control port power energy. Similarly we define the storage and dissipation port observability matrices as  $\Gamma_{iS}, \Gamma_{iR}$ .

For the storage port oblique projection  $O_{iS}$ , we can write:

$$W_1 \cdot \Gamma_{iS} \cdot X_{fS}^d \cdot W_2 = U_S \cdot S_S \cdot V_S^T \quad (5.66)$$

which can be also split in:

$$\begin{cases} W_1 \cdot \Gamma_{iS} = U_S \cdot S_S^{1/2} \cdot T_S \\ X_{fS}^d \cdot W_2 = T_S^{-1} \cdot S_S^{1/2} \cdot V_S^T \end{cases} \quad (5.67)$$

Similarly we can write for the dissipation port oblique projection, the SVD decomposition:

$$W_1 \cdot \Gamma_{iR} \cdot X_{fR}^d \cdot W_2 = U_R \cdot S_R \cdot V_R^T \quad (5.68)$$

This can be split in:

$$\begin{cases} W_1 \cdot \Gamma_{iR} = U_R \cdot S_R^{1/2} \cdot T_R \\ X_{fR}^d \cdot W_2 = T_R^{-1} \cdot S_R^{1/2} \cdot V_R^T \end{cases} \quad (5.69)$$

**Remark:** The similarity transformation matrices corresponding to the control, storage or dissipation ports are denoted by  $T_C, T_S, T_R$ , as they can be in different space dimension, as will be shown also by simulation.

Further we define the state vectors on each port, corresponding to the control port SVD decomposition as:

$$\begin{cases} X_{fCC}^d \stackrel{def}{=} \Gamma_{iC}^+ \cdot O_{iC} \\ X_{fCS}^d \stackrel{def}{=} \Gamma_{iC}^+ \cdot O_{iS} \\ X_{fCR}^d \stackrel{def}{=} \Gamma_{iC}^+ \cdot O_{iR} \end{cases} \quad (5.70)$$

For the storage port power energy, the state vector sequences can be defined as:

$$\begin{cases} X_{fSC}^d \stackrel{def}{=} \Gamma_{iS}^+ \cdot O_{iC} \\ X_{fSS}^d \stackrel{def}{=} \Gamma_{iS}^+ \cdot O_{iS} \\ X_{fSR}^d \stackrel{def}{=} \Gamma_{iS}^+ \cdot O_{iR} \end{cases} \quad (5.71)$$

For the dissipation port power energy, the state vector sequences are defined as:

$$\begin{cases} X_{fRC}^d \stackrel{def}{=} \Gamma_{iR}^+ \cdot O_{iC} \\ X_{fRS}^d \stackrel{def}{=} \Gamma_{iR}^+ \cdot O_{iS} \\ X_{fRR}^d \stackrel{def}{=} \Gamma_{iR}^+ \cdot O_{iR} \end{cases} \quad (5.72)$$

We have denoted the state vector sequences corresponding to each port differently for each SVD decomposition. Further we define a specific deterministic identification theorem for the power energy approach.

**Theorem: Deterministic identification (power energy approach)**

We consider the initial assumptions:

1. The input is persistently exciting of order  $2i$
2. The intersection of the row space of  $U_f$  (future inputs) and the row space of  $X_p^d$  (past states) is empty.

3. The user defined weighting matrices  $W_1 \in R^{l \times l}$  and  $W_2 \in R^{j \times j}$  are such that  $W_1$  is of full rank and  $W_2$  obeys:

$$\text{rank}(W_P) = \text{rank}(W_P \cdot W_2) \quad (5.73)$$

where  $W_P$  represents the Hankel matrices containing the past inputs and outputs corresponding to the ports.

We define the oblique projections corresponding to the storage, dissipation and control ports, as:

$$\begin{cases} O_{iS} \stackrel{\text{def}}{=} Y_{fS} / U_f W_P \\ O_{iR} \stackrel{\text{def}}{=} Y_{fR} / U_f W_P \\ O_{iC} \stackrel{\text{def}}{=} O_{iS} + O_{iR} \end{cases} \quad (5.74)$$

**Remark:** The control port oblique projection  $O_{iC}$  can be expressed as a sum between the storage port ( $O_{iS}$ ) and dissipation port oblique projections ( $O_{iR}$ ), by means of the power conservation law (see 5.60).

We consider the singular value decomposition (SVD) corresponding to the power ports:

$$\begin{aligned} W_1 O_{iC} W_2 &= W_1 (O_{iS} + O_{iR}) W_2 = \begin{pmatrix} U_{C1} & U_{C2} \end{pmatrix} \begin{pmatrix} S_{C1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_{C1}^T \\ V_{C2}^T \end{pmatrix} \\ &= U_{C1} S_{C1} V_{C1}^T = U_S S_S V_S^T + U_R S_R V_R^T \end{aligned} \quad (5.75)$$

1. The oblique projections corresponding to the storage, dissipation and control ports, equal to:

$$\begin{cases} O_{iS} \stackrel{\text{def}}{=} \Gamma_{iC} \cdot X_{fCS}^d ; O_{iR} \stackrel{\text{def}}{=} \Gamma_{iC} \cdot X_{fCR}^d ; O_{iC} \stackrel{\text{def}}{=} \Gamma_{iC} \cdot X_{fCC}^d \end{cases} \quad (5.76)$$

$$\begin{cases} O_{iS} \stackrel{\text{def}}{=} \Gamma_{iS} \cdot X_{fSS}^d ; O_{iR} \stackrel{\text{def}}{=} \Gamma_{iS} \cdot X_{fSR}^d ; O_{iC} \stackrel{\text{def}}{=} \Gamma_{iS} \cdot X_{fSC}^d \end{cases} \quad (5.77)$$

$$\begin{cases} O_{iS} \stackrel{\text{def}}{=} \Gamma_{iR} \cdot X_{fRS}^d ; O_{iR} \stackrel{\text{def}}{=} \Gamma_{iR} \cdot X_{fRR}^d ; O_{iC} \stackrel{\text{def}}{=} \Gamma_{iR} \cdot X_{fRC}^d \end{cases} \quad (5.78)$$

by SVD decomposition.

2. The order of the dynamic system corresponds to the number of singular values of the storage, dissipation and control port oblique projections in (5.75).  
3. The extended observability matrix  $\Gamma_i$  can be determined for either port as:

$$\begin{cases} \Gamma_{iS} = W_1^{-1} \cdot U_S \cdot S_S^{1/2} \cdot T_S \\ \Gamma_{iR} = W_1^{-1} \cdot U_R \cdot S_R^{1/2} \cdot T_R \\ \Gamma_{iC} = W_1^{-1} \cdot U_C \cdot S_C^{1/2} \cdot T_C \end{cases} \quad (5.79)$$

**Remark:**  $T$  represents a non-singular transformation matrix applied for each port.

4. The part of the state sequences  $X_{fS}^d, X_{fR}^d, X_{fC}^d$  corresponding to the storage, dissipation or control ports, that lies in the column space of  $w_2$ , can be recovered from:

$$\begin{cases} X_{fSS}^d \cdot w_2 = T_S^{-1} \cdot S_S^{1/2} \cdot v_S^T; \\ X_{fRR}^d \cdot w_2 = T_R^{-1} \cdot S_R^{1/2} \cdot v_R^T; \\ X_{fCC}^d \cdot w_2 = T_C^{-1} \cdot S_C^{1/2} \cdot v_C^T \end{cases} \quad (5.80)$$

5. The state sequences  $X_{fS}^d, X_{fR}^d, X_{fC}^d$  corresponding to the storage, dissipation and control ports equal to:

$$\begin{cases} X_{fCS}^d \stackrel{def}{=} \Gamma_{iC}^+ \cdot O_{iS}; X_{fCR}^d \stackrel{def}{=} \Gamma_{iC}^+ \cdot O_{iR}; X_{fCC}^d \stackrel{def}{=} \Gamma_{iC}^+ \cdot O_{iC} \end{cases} \quad (5.81)$$

$$\begin{cases} X_{fSS}^d \stackrel{def}{=} \Gamma_{iS}^+ \cdot O_{iS}; X_{fSR}^d \stackrel{def}{=} \Gamma_{iS}^+ \cdot O_{iR}; X_{fSC}^d \stackrel{def}{=} \Gamma_{iS}^+ \cdot O_{iC} \end{cases} \quad (5.82)$$

$$\begin{cases} X_{fRS}^d \stackrel{def}{=} \Gamma_{iR}^+ \cdot O_{iS}; X_{fRR}^d \stackrel{def}{=} \Gamma_{iR}^+ \cdot O_{iR}; X_{fRC}^d \stackrel{def}{=} \Gamma_{iR}^+ \cdot O_{iC} \end{cases} \quad (5.83)$$

**Proof:**

We will start the proof with the control port oblique projection, for which we prove that  $\text{rank}(W_P) = \text{rank}\left(W_P / U \frac{\perp}{f}\right)$ . But  $w_P$  can be written equivalently as:

$$W_P = \begin{pmatrix} I_{mi} & 0 \\ H_i^d & \Gamma_{iC} \end{pmatrix} \begin{pmatrix} U_P \\ X_P^d \end{pmatrix} \quad (5.84)$$

Further we can write:

$$W_P / U \frac{\perp}{f} = \begin{pmatrix} I_{mi} & 0 \\ H_i^d & \Gamma_{iC} \end{pmatrix} \begin{pmatrix} U_P / U \frac{\perp}{f} \\ X_P^d / U \frac{\perp}{f} \end{pmatrix} \quad (5.85)$$

According to the first two assumptions of the theorem, the following holds:

$$\text{rank} \begin{pmatrix} U_P \\ X_P^d \end{pmatrix} = \text{rank} \begin{pmatrix} U_P / U \frac{\perp}{f} \\ X_P^d / U \frac{\perp}{f} \end{pmatrix} \quad (5.86)$$

which proves that  $\text{rank}(W_P) = \text{rank}(W_P / U_f^\perp)$ . Further we denote the SVD decomposition of  $W_P / U_f^\perp$  as:

$$W_P / U_f^\perp = \begin{pmatrix} U_{C1} & U_{C2} \end{pmatrix} \begin{pmatrix} S_{C1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_{C1}^T \\ V_{C2}^T \end{pmatrix} = U_{C1} S_{C1} V_{C1}^T \quad (5.87)$$

Since  $W_P / U_f^\perp$  is a linear combination of the columns of  $W_P$  and since the rank of  $W_P$  and  $W_P / U_f^\perp$  are equal, we find that the column spaces of  $W_P$  and  $W_P / U_f^\perp$  are also equal. This implies that  $W_P$  can be written as:

$$W_P = U_{C1} R \quad (5.88)$$

Then using (5.87), we can write:

$$\left( W_P / U_f^\perp \right) \left( W_P / U_f^\perp \right)^+ W_P = \left( U_{C1} S_{C1} V_{C1}^T \right) \left( V_{C1} S_{C1}^{-1} U_{C1}^T \right) \left( U_{C1} R \right) \quad (5.89)$$

which proves that the following holds:

$$\left( W_P / U_f^\perp \right) \left( W_P / U_f^\perp \right)^+ W_P = W_P \quad (5.90)$$

Thus it results the oblique projection of the control port as:

$$O_{iC} = \Gamma_{iC} \cdot X_{fCC}^d \quad (5.91)$$

Using (5.59) and (5.74), we can determine the oblique projections corresponding to the storage and dissipation ports as:

$$\begin{cases} O_{iS} = \Gamma_{iC} X_{fCS}^d \\ O_{iR} = \Gamma_{iC} X_{fCR}^d \end{cases} \quad (5.92)$$

and thus (5.76) is proved. For the storage port oblique projection, it is necessary to prove that:

$$\left( W_P / U_f^\perp \right) \left( W_P / U_f^\perp \right)^+ W_{PS} = W_{PS} \quad (5.93)$$

In this case, the property  $\text{rank} W_{PS} = \text{rank} W_P / U_f^\perp$  generally might not hold and thus the column spaces of  $W_{PS}$  and  $W_P / U_f^\perp$  are not always equal. In this case, we can write  $W_{PS}$  as:

$$W_{PS} = U_2 R \quad (5.94)$$

Using (5.87), we can write then:

$$\left( W_P / U_f^\perp \right) \left( W_P / U_f^\perp \right)^+ W_{PS} = \left( U_{C1} S_{C1} V_{C1}^T \right) \left( V_{C1} S_{C1}^{-1} U_{C1}^T \right) \left( U_2 R \right) \quad (5.95)$$

Performing the possible simplifications, it results:

$$\left( W_P / U_f^\perp \right) \left( W_P / U_f^\perp \right)^+ W_{PS} = U_2 R \quad (5.96)$$



Thus the SVD decomposition of the storage port oblique projection ( $o_{iS}$ ) can be in a different space dimension from the control port oblique projection ( $o_{iC}$ ) as will be seen by simulation examples. Then the oblique projections corresponding to the storage port results as in (5.77). For the dissipation port oblique projections it is necessary to prove that:

$$\left(W_P / U \frac{\perp}{f}\right) \left(W_P / U \frac{\perp}{f}\right)^+ W_{PR} = W_{PR} \quad (5.97)$$

Similarly to the storage port  $rank W_{PR} = rank W_P / U \frac{\perp}{f}$  doesn't hold generally and the column spaces of  $W_{PR}$  and  $W_P / U \frac{\perp}{f}$  might be different. In this case, we can write:

$$\left(W_P / U \frac{\perp}{f}\right) \left(W_P / U \frac{\perp}{f}\right)^+ W_{PR} = \left(U_{C1} S_{C1} V_{C1}^T\right) \left(V_{C1} S_{C1}^{-1} U_{C1}^T\right) \left(U_3 R\right) \quad (5.98)$$

where:

$$W_{PR} = U_3 R \quad (5.99)$$

Performing all simplifications in (5.98), it results:

$$\left(W_P / U \frac{\perp}{f}\right) \left(W_P / U \frac{\perp}{f}\right)^+ W_{PR} = U_3 R \quad (5.100)$$

As for the storage port, the oblique projection corresponding to the dissipation port  $o_{iR}$ , may be in a different space dimension, but still representing an equivalent system to the real one. Thus the oblique projections corresponding to the dissipation port can be determined as in (5.78). The second claim of the theorem follows from the fact that  $w_1 o_{iC} w_2$  is equal to the product of two matrices  $w_1 \Gamma_{iC}$  ( $n_C$  columns) and  $X_{fCC}^d w_2$  ( $n_C$  rows). Since both matrices have rank  $n_C$  due to assumption 3 of the theorem, their product is also rank  $n_C$ . Thus the second claim is proved for the control port. For the storage port oblique projection we can write that  $w_1 o_{iS} w_2$  is equal to the product of  $w_1 \Gamma_{iS}$  ( $n_S$  columns) and  $X_{fSC}^d w_2$  ( $n_S$  rows). Using assumption 3 of the theorem, their product is also rank  $n_S$  and thus statement 2 is proved for the storage port. Similarly for the dissipation port oblique projection, we can write that  $w_1 o_{iR} w_2$  is equal to the product between  $w_1 \Gamma_{iR}$  ( $n_R$  columns) and  $X_{fRC}^d w_2$  ( $n_R$  rows). Using assumption 3 of the theorem, their product is rank  $n_R$  and thus statement 2 is proved for the dissipation port.

**Remark:** We have denoted with  $n_C, n_S, n_R$  the system degrees obtained for the control, storage or dissipation ports by SVD decomposition. Thus it is possible to obtain three different and equivalent systems for the real PCH model.

Statement 3 and 4 of the theorem can be easily obtained by splitting the SVD decomposition on each port as in (5.65), (5.67) and (5.69). The last statement of the theorem can be easily obtained from the first one.

**Remark:** Similarly to the classic case it is easy to prove that  $o_{i-1} = \Gamma_{i-1} X_{f+1}^d$  holds for each port.

Further we propose a specific deterministic identification algorithm for the power energy approach, which is in fact an extension of *Deterministic algorithm I* from section 5.2.1 to the power energy approach.

**Remark:** We observe that this theorem proves also the definitions and propositions proposed for the power based structural identifiability of chapter 2, as each power energy can be corresponds to a transfer function from the input to the port output and also is a function of the unknown parameters. Thus, three equivalent systems can be determined for the dynamic system by system identification as will be shown later.

### 5.3.2 Deterministic identification algorithm

1. Calculate the oblique projections corresponding to the storage, dissipation and control ports as:

$$\begin{aligned}
 (\mathbf{S}) \left\{ \begin{array}{l} O_{iS} \stackrel{\text{def}}{=} \left( Y_{fS} / U_f \right) (\mathbf{W}_P) \\ O_{(i-1)S} \stackrel{\text{def}}{=} \left( Y_{fS}^- / U_f^- \right) (\mathbf{W}_P) \end{array} \right. \\
 (\mathbf{R}) \left\{ \begin{array}{l} O_{iR} \stackrel{\text{def}}{=} \left( Y_{fR} / U_f \right) (\mathbf{W}_P) \\ O_{(i-1)R} \stackrel{\text{def}}{=} \left( Y_{fR}^- / U_f^- \right) (\mathbf{W}_P) \end{array} \right. \\
 (\mathbf{C}) \left\{ \begin{array}{l} O_{iC} \stackrel{\text{def}}{=} O_{iS} + O_{iR} \\ O_{(i-1)C} \stackrel{\text{def}}{=} O_{(i-1)S} + O_{(i-1)R} \end{array} \right.
 \end{aligned}$$

**Remark:** We have denoted with  $Y_{fS}, Y_{fR}$  the future outputs corresponding to the storage and dissipation ports (see 5.51).

2. Calculate the SVD decomposition of the weighted oblique projections corresponding to the ports:

$$\begin{cases} W_1 O_{iS} W_2 = U_S S_S V_S^T \\ W_1 O_{iR} W_2 = U_R S_R V_R^T \\ W_1 O_{iC} W_2 = U_C S_C V_C^T \end{cases}$$

3. Determine the system order, by inspecting the singular values of either the storage, dissipation or control ports and partition the SVD accordingly.

4. Determine  $\Gamma_i$  and  $\Gamma_{(i-1)}$  on each port as:

$$\begin{cases} \Gamma_{iS} = W_1^{-1} U_S S_S^{1/2} T_S; \Gamma_{(i-1)S} = \underline{\Gamma_{iS}} \\ \Gamma_{iR} = W_1^{-1} U_R S_R^{1/2} T_R; \Gamma_{(i-1)R} = \underline{\Gamma_{iR}} \\ \Gamma_{iC} = W_1^{-1} U_C S_C^{1/2} T_C; \Gamma_{(i-1)C} = \underline{\Gamma_{iC}} \end{cases}$$

**Remark:** The extended observability matrix of the system  $\Gamma_i$ , can be determined using one of the power energies corresponding to the ports.

5. Determine the states  $X_i^d$  and  $X_{i+1}^d$  corresponding to each port:

$$(S) \begin{cases} X_{iSS}^d = \Gamma_{iS}^+ O_{iS}; X_{iSR}^d = \Gamma_{iS}^+ O_{iR}; X_{iSC}^d = \Gamma_{iS}^+ O_{iC} \\ X_{(i+1)SS}^d = \Gamma_{(i-1)S}^+ O_{(i-1)S}; X_{(i+1)SR}^d = \Gamma_{(i-1)S}^+ O_{(i-1)R}; X_{(i+1)SC}^d = \Gamma_{(i-1)S}^+ O_{(i-1)C} \end{cases}$$

$$(R) \begin{cases} X_{iRS}^d = \Gamma_{iR}^+ O_{iS}; X_{iRR}^d = \Gamma_{iR}^+ O_{iR}; X_{iRC}^d = \Gamma_{iR}^+ O_{iC} \\ X_{(i+1)RS}^d = \Gamma_{(i-1)R}^+ O_{(i-1)S}; X_{(i+1)RR}^d = \Gamma_{(i-1)R}^+ O_{(i-1)R}; X_{(i+1)RC}^d = \Gamma_{(i-1)R}^+ O_{(i-1)C} \end{cases}$$

$$(C) \begin{cases} X_{iCS}^d = \Gamma_{iC}^+ O_{iS}; X_{iCR}^d = \Gamma_{iC}^+ O_{iR}; X_{iCC}^d = \Gamma_{iC}^+ O_{iC} \\ X_{(i+1)CS}^d = \Gamma_{(i-1)C}^+ O_{(i-1)S}; X_{(i+1)CR}^d = \Gamma_{(i-1)C}^+ O_{(i-1)R}; X_{(i+1)CC}^d = \Gamma_{(i-1)C}^+ O_{(i-1)C} \end{cases}$$

6. Solve the linear set of equations for  $A, B, C, D$ , using either the storage, dissipation or control ports as:

$$\begin{pmatrix} X_{(i+1)SC}^d \\ \left( \begin{matrix} A_S & B_S \\ C_S & D_S \end{matrix} \right)^T \begin{pmatrix} X_{iSC}^d \\ U(i/i) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} A_S & B_S \\ C_S & D_S \end{pmatrix} \begin{pmatrix} X_{iSC}^d \\ U(i/i) \end{pmatrix}$$

$$\begin{pmatrix} X_{(i+1)RC}^d \\ \left( \begin{matrix} A_R & B_R \\ C_R & D_R \end{matrix} \right)^T \begin{pmatrix} X_{iRC}^d \\ U(i/i) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} A_R & B_R \\ C_R & D_R \end{pmatrix} \begin{pmatrix} X_{iRC}^d \\ U(i/i) \end{pmatrix}$$

$$\begin{pmatrix} X_{(i+1)CC}^d \\ \left( \begin{matrix} A_C & B_C \\ C_C & D_C \end{matrix} \right)^T \begin{pmatrix} X_{iCC}^d \\ U(i/i) \end{pmatrix} \end{pmatrix} = \begin{pmatrix} A_C & B_C \\ C_C & D_C \end{pmatrix} \begin{pmatrix} X_{iCC}^d \\ U(i/i) \end{pmatrix}$$

**Remark:** This algorithm finds three equivalent models to the real one given by  $A_S, B_S, C_S, D_S, A_R, B_R, C_R, D_R$  and  $A_C, B_C, C_C, D_C$  corresponding to the system ports.

Further we will define and prove a proposition useful to determine the observability matrix ( $\Gamma_{iH}$ ) of an equivalent PCH system for the selected time-discretization scheme, starting from the observability matrix  $\Gamma_i$  of (5.33).

**Proposition 5.1** *The extended observability matrix  $\Gamma_{iH}$  of a discrete-time PCH system (5.33) in its equivalent form, when using a symplectic discretization scheme with an explicit Euler rule for the flows and implicit Euler rule for the efforts, can be determined recursively using the following formula:*

$$\Gamma_{iH} = \Gamma_i + \Gamma'_H \quad (5.101)$$

where  $\Gamma'_H$  is defined as:

$$\Gamma'_H \stackrel{def}{=} - \sum_{j=1}^{i-1} \mathbf{C}_B \binom{j}{i-1} \begin{pmatrix} 0_{l \times n} \\ \dots \\ 0_{lj \times n} \\ \Gamma_{(i-j)H} \end{pmatrix} + \sum_{p=0}^{i-2-p} \mathbf{C}_B \binom{j}{i-1-p} - \mathbf{C}_B \binom{j}{i-2-p} \begin{pmatrix} 0_{l \times n} \\ \dots \\ 0_{lj \times n} \\ \Gamma_{(i-3-p)H} \\ 0_{l \times n} \\ \dots \\ 0_{l(p+1) \times n} \end{pmatrix} \quad (5.102)$$

**Remark:** We have adopted the notation  $\mathbf{C}_B \binom{k}{n}$  for the binomial coefficients or combinatorial numbers in (5.101).

Further we will present a proof of the proposition which makes use of the binomial theorem.

**Proof:**

Using (5.33) and (5.34), the observability matrix of an equivalent state-space system can be described by the following formula:

$$\Gamma_i = \begin{pmatrix} C & C\bar{A} & \dots & C\bar{A}^{i-1} \end{pmatrix}^T T^{-1} \quad (5.103)$$

where  $T$  represents a non-singular similarity transformation of the dynamic system.

**Remark:** For simplicity in all computations we will consider the time-step  $\Delta t = 1$  for the selected time-discretization scheme.

Using (5.34), we can also write the observability matrix as:

$$\Gamma_i = \begin{pmatrix} C & C(A+I) & \dots & C(A+I)^{i-1} \end{pmatrix}^T T^{-1} \quad (5.104)$$

We will consider first the case with  $i = 2$ :

$$\Gamma_i = \begin{pmatrix} C \\ C(A+I) \end{pmatrix}^* T^{-1} \quad (5.105)$$

which can be split in:

$$\Gamma_i = \Gamma_{iH} + \begin{pmatrix} 0_{l \times n} \\ \Gamma_{(i-1)H} \end{pmatrix} \quad (5.106)$$

In (5.106), we have introduced the notation:

$$\Gamma_{iH} \stackrel{def}{=} \begin{pmatrix} C & CA & \dots & CA^{i-1} \end{pmatrix}^T T^{-1} \quad (5.107)$$

for the observability matrix of an equivalent PCH system.

**Remark:**  $\Gamma_{iH}$  represents the observability matrix of an equivalent PCH system for the continuous-time model, as defined in the previous chapters.

From (5.106) we can determine recursively  $\Gamma_{iH}$  for  $i = 2$  as:

$$\Gamma_{iH} = \Gamma_i - \begin{pmatrix} 0_{l \times n} \\ \Gamma_{(i-1)H} \end{pmatrix} \quad (5.108)$$

For  $i = 3$  we can write:

$$\Gamma_i = \begin{pmatrix} C \\ C(A+I) \\ C(A+I)^2 \end{pmatrix} T^{-1} \quad (5.109)$$

This relation can be further split as:

$$\Gamma_i = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} T^{-1} + \begin{pmatrix} 0 \\ C \\ 2CA \end{pmatrix} T^{-1} + \begin{pmatrix} 0 \\ 0 \\ C \end{pmatrix} T^{-1} \quad (5.110)$$

Using (5.107) we can write equivalently:

$$\Gamma_i = \Gamma_{iH} + \begin{pmatrix} 0_{l \times n} \\ 2\Gamma_{(i-1)H} \end{pmatrix} + \begin{pmatrix} 0_{l \times n} \\ 0_{l \times n} \\ \Gamma_{(i-2)H} \end{pmatrix} - \begin{pmatrix} 0_{l \times n} \\ \Gamma_{(i-2)H} \\ 0_{l \times n} \end{pmatrix} \quad (5.111)$$

For  $i = 4$  we can write the observability matrix as:

$$\Gamma_i = \begin{pmatrix} C \\ C(A+I) \\ C(A+I)^2 \\ C(A+I)^3 \end{pmatrix} T^{-1} \quad (5.112)$$

Using (5.107) we can write this relation as:

$$\Gamma_i = \Gamma_{iH} + \begin{pmatrix} 0 \\ 3\Gamma_{(i-1)H} \end{pmatrix} - \begin{pmatrix} 0 \\ 2C \\ CA \\ 0 \end{pmatrix} T^{-1} + \begin{pmatrix} 0 \\ 0 \\ 3\Gamma_{(i-2)H} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2C \\ 0 \end{pmatrix} T^{-1} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma_{(i-3)H} \end{pmatrix} \quad (5.113)$$

This relation can be equivalently written as:

$$\Gamma_i = \Gamma_{iH} + \begin{pmatrix} 0 \\ 3\Gamma_{(i-1)H} \end{pmatrix} - \begin{pmatrix} 0 \\ \Gamma_{(i-2)H} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3\Gamma_{(i-2)H} \end{pmatrix} - \begin{pmatrix} 0 \\ 2\Gamma_{(i-3)H} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Gamma_{(i-3)H} \end{pmatrix} - \begin{pmatrix} 0 \\ \Gamma_{(i-3)H} \\ 0 \\ 0 \end{pmatrix}$$

(5.114)

Further we will use the *binomial theorem* (binomial expansion) and write  $\Gamma_i$  as:

$$\Gamma_i = \left[ \begin{array}{l} C \cdot \mathbf{C}_B \binom{0}{0} A^0 \\ C \cdot \left( \mathbf{C}_B \binom{0}{1} A^1 + \mathbf{C}_B \binom{1}{1} A^0 \right) \\ C \cdot \left( \mathbf{C}_B \binom{0}{2} A^2 + \mathbf{C}_B \binom{1}{2} A^1 + \mathbf{C}_B \binom{2}{2} A^0 \right) \\ \dots \\ C \cdot \left( \mathbf{C}_B \binom{0}{i-2} A^{i-2} + \dots + \mathbf{C}_B \binom{i-2}{i-2} A^0 \right) \\ C \cdot \left( \mathbf{C}_B \binom{0}{i-1} A^{i-1} + \dots + \mathbf{C}_B \binom{i-1}{i-1} A^0 \right) \end{array} \right] T^{-1} \quad (5.115)$$

We observe that we can split this relation as:

$$\begin{aligned} \Gamma_i = & \Gamma_{iH} + \mathbf{C}_B \binom{1}{i-1} \left( \begin{array}{l} 0 \\ \Gamma_{(i-1)H} \end{array} \right) - \left[ \begin{array}{l} 0 \\ C \cdot \left( \mathbf{C}_B \binom{1}{i-1} - \mathbf{C}_B \binom{1}{1} \right) A^0 \\ C \cdot \left( \mathbf{C}_B \binom{1}{i-1} - \mathbf{C}_B \binom{1}{2} \right) A^1 \\ \dots \\ C \cdot \left( \mathbf{C}_B \binom{1}{i-1} - \mathbf{C}_B \binom{1}{i-2} \right) A^{i-3} \\ 0 \end{array} \right] + \mathbf{C}_B \binom{2}{i-1} \left( \begin{array}{l} 0 \\ 0 \\ \Gamma_{(i-2)H} \end{array} \right) \\ & - \left[ \begin{array}{l} 0 \\ 0 \\ C \cdot \left( \mathbf{C}_B \binom{2}{i-1} - \mathbf{C}_B \binom{2}{2} \right) A^0 \\ \dots \\ C \cdot \left( \mathbf{C}_B \binom{2}{i-1} - \mathbf{C}_B \binom{2}{i-2} \right) A^{i-4} \\ 0 \end{array} \right] + \dots + \mathbf{C}_B \binom{i-2}{i-1} \left( \begin{array}{l} 0 \\ 0 \\ \dots \\ \Gamma_{2H} \end{array} \right) - \left[ \begin{array}{l} 0 \\ 0 \\ \dots \\ C \cdot \left( \mathbf{C}_B \binom{i-2}{i-1} - \mathbf{C}_B \binom{i-2}{i-2} \right) A^0 \end{array} \right] + \\ & \mathbf{C}_B \binom{i-1}{i-1} \left( \begin{array}{l} 0 \\ \dots \\ 0 \\ \Gamma_{1H} \end{array} \right) - \left( \begin{array}{l} 0 \\ 0 \\ \dots \\ 0 \end{array} \right) \end{aligned} \quad (5.116)$$

**Remark:** In order to write the above relation in a simplified form, we have added and subtracted  $\mathbf{C}_B \binom{j}{i-1}$  ( $j = 1, \dots, i-2$ ) on each line as in (5.116).

Further we can develop as follows:

$$\Gamma_i = \Gamma_{iH} + \sum_{j=1}^{i-1} \mathbf{C}_B \binom{j}{i-1} \begin{pmatrix} 0_{l1xn} \\ \dots \\ 0_{ljxn} \\ \Gamma_{(i-j)H} \end{pmatrix} - \begin{pmatrix} 0_{l1xn} \\ 0_{l2xn} \\ \dots \\ 0_{ljxn} \\ C \left( \mathbf{C}_B \binom{j}{i-1} - \mathbf{C}_B \binom{j}{j} \right) A^0 \\ \dots \\ C \left( \mathbf{C}_B \binom{j}{i-1} - \mathbf{C}_B \binom{j}{i-2} \right) A^{i-2-j} \\ 0 \end{pmatrix} \quad (5.117)$$

From (5.117) we can determine  $\Gamma_{iH}$  as:

$$\Gamma_{iH} = \Gamma_i - \sum_{j=1}^{i-1} \mathbf{C}_B \binom{j}{i-1} \begin{pmatrix} 0_{l1xn} \\ \dots \\ 0_{ljxn} \\ \Gamma_{(i-j)H} \end{pmatrix} - \begin{pmatrix} 0_{l1xn} \\ \dots \\ 0_{ljxn} \\ C \left( \mathbf{C}_B \binom{j}{i-1} - \mathbf{C}_B \binom{j}{j} \right) \\ \dots \\ C \left( \mathbf{C}_B \binom{j}{i-1} - \mathbf{C}_B \binom{j}{i-2} \right) \\ 0 \end{pmatrix} \quad (5.118)$$

The last term of (5.118) can be written recursively as:

$$\begin{pmatrix} 0_{l1xn} \\ \dots \\ 0_{ljxn} \\ \sum_{j=1}^{i-1} C \left( \mathbf{C}_B \binom{j}{i-1} - \mathbf{C}_B \binom{j}{j} \right) A^0 \\ \dots \\ C \left( \mathbf{C}_B \binom{j}{i-1} - \mathbf{C}_B \binom{j}{i-2} \right) A^{i-2-j} \\ 0 \end{pmatrix}$$

$$= \sum_{j=1}^{i-1} \left( \mathbf{C}_B \binom{j}{i-1} - \mathbf{C}_B \binom{j}{i-2} \right) \begin{pmatrix} 0_{l1xn} \\ \dots \\ 0_{ljxn} \\ \Gamma^{(i-3-j)H} \\ 0 \end{pmatrix} - \begin{pmatrix} 0_{l1xj} \\ \dots \\ 0_{ljxn} \\ C \left( \mathbf{C}_B \binom{j}{i-1} - \mathbf{C}_B \binom{j}{i-2} - \left( \mathbf{C}_B \binom{j}{i-1} - \mathbf{C}_B \binom{j}{j} \right) \right) A^0 \\ \dots \\ C \left( \mathbf{C}_B \binom{j}{i-1} - \mathbf{C}_B \binom{j}{i-2} - \left( \mathbf{C}_B \binom{j}{i-1} - \mathbf{C}_B \binom{j}{i-3} \right) \right) A^{i-3-j} \\ 0 \\ 0 \end{pmatrix} \quad (5.119)$$

Performing the simplifications in the last term of (5.119) we obtain:

$$\begin{pmatrix} 0_{l1xn} \\ \dots \\ 0_{ljxn} \\ \dots \\ C \left( \mathbf{C}_B \binom{j}{i-1} - \mathbf{C}_B \binom{j}{j} \right) A^0 \\ \dots \\ C \left( \mathbf{C}_B \binom{j}{i-1} - \mathbf{C}_B \binom{j}{i-2} \right) A^{i-2-j} \\ 0 \end{pmatrix} = \sum_{j=1}^{i-1} \begin{pmatrix} 0_{l1xn} \\ \dots \\ 0_{ljxn} \\ C \left( \mathbf{C}_B \binom{j}{i-2} - \mathbf{C}_B \binom{j}{j} \right) A^0 \\ \dots \\ C \left( \mathbf{C}_B \binom{j}{i-2} - \mathbf{C}_B \binom{j}{i-3} \right) A^{i-3-j} \\ 0_{l(i-1)xn} \\ 0_{lixn} \end{pmatrix} + \begin{pmatrix} 0_{l1xn} \\ \dots \\ 0_{ljxn} \\ \Gamma^{(i-3-j)H} \\ 0 \end{pmatrix} \quad (5.120)$$

Repeating the development of the 2<sup>nd</sup> term in (5.120), we arrive by recursive computation at (5.102) and the proof ends. Similarly to **Proposition 5.1** we can introduce the following proposition for the computation of an extended controllability matrix of a PCH system.

**Proposition 5.2** *The extended controllability matrix  $c_{iH}$  of a discrete-time PCH system (5.33) in its equivalent form, when using a symplectic discretization scheme with an explicit*



Euler rule for the flows and implicit Euler rule for the efforts, can be determined recursively using the following formula:

$$C_{iH} = C_i + C'_H \quad (5.121)$$

where  $C'_H$  is defined as:

$$C'_H = - \sum_{j=1}^{i-1} C_B \binom{j}{i-1} \begin{pmatrix} 0 & l_{1 \times n} \\ \dots \\ 0 & l_{j \times n} \\ C_{(i-j)H} \end{pmatrix} + \sum_{p=0}^{i-3} C_B \binom{j}{i-1-p} - C_B \binom{j}{i-2-p} \begin{pmatrix} 0 & l_{1 \times n} \\ \dots \\ 0 & l_{j \times n} \\ C_{(i-3-p)H} \\ 0 & l_{1 \times n} \\ \dots \\ 0 & l_{(p+1) \times n} \end{pmatrix} \quad (5.122)$$

**Proof:**

The proof follows the same steps as **Proposition 5.1** to determine the recursive formula 5.121.

**Remark:** The controllability matrix of a classic state space system is defined by

$C_i = \begin{bmatrix} B & AB & \dots & A^{i-1}B \end{bmatrix}^T$ , and it can also be determined for PCH systems as for the observability matrix (see chapter 2). The previous two propositions are useful in determining the equivalent state-space system matrices  $(A_H, B_H, C_H)$  of a PCH system given by:

$$\begin{cases} A_H = T \cdot A \cdot T^{-1} \\ B_H = T \cdot B \\ C_H = C \cdot T^{-1} \end{cases} \quad (5.123)$$

using  $\Gamma_i$  and  $C_i$  of (5.33). Using (5.123) we can determine also the observability and controllability Grammians as:

$$\begin{cases} \tilde{W}_O^2 = T^{-T} W_O^2 T^{-1} \\ \tilde{W}_C^2 = T W_C^2 T^T \end{cases} \quad (5.124)$$

where:

$$\begin{cases} W_O^2 = \int_0^\infty e^{A^T t} C^T C e^{A t} dt \\ 0 \\ W_C^2 = \int_0^\infty e^{A t} B B^T e^{A^T t} dt \\ 0 \end{cases} \quad (5.125)$$

Further we will prove and determine that between the inputs and power port energies, we can determine separate transfer functions, which are in fact associated to the oblique projections

defined in the power energy deterministic identification approach. The power energies associated to the ports in continuous-time form, can be determined as:

$$\begin{cases} P_C = (Qx)^T Bu \\ P_S = (Qx)^T R(Qx) - (Qx)^T Bu \\ P_R = -(Qx)^T R(Qx) \end{cases} \quad (5.126)$$

Changing to Laplace transform representation, we can write equivalently:

$$\begin{cases} P_C(s) = \left( Q(sI - A)^{-1} B \cdot U(s) \right)^T \cdot U(s) \\ P_S(s) = \left( Q(sI - A)^{-1} BU(s) \right)^T R \left( Q(sI - A)^{-1} B \cdot U(s) \right) - \left( Q(sI - A)^{-1} BU(s) \right)^T BU(s) \\ P_R(s) = - \left( Q(sI - A)^{-1} BU(s) \right)^T R \left( Q(sI - A)^{-1} B \cdot U(s) \right) \end{cases} \quad (5.127)$$

where  $X(s) = (sI - A)^{-1} BU(s)$  is determined from the continuous-time state-space representation. But the control port power energy can be also written as:

$$P_C(s) = Y^T(s) \cdot U(s) \quad (5.128)$$

Equivalently we can write then:

$$P_C(s) = \left( C(sI - A)^{-1} BU(s) \right)^T U(s) \quad (5.129)$$

Expressed using the transfer function, we can write:

$$P_C(s) = \left( H_C(s) \cdot U(s) \right)^T U(s) \quad (5.130)$$

Then (5.127) can be equivalently written as:

$$\begin{cases} P_C(s) = U^T(s) \cdot H_C^T(s) \cdot U(s) \\ P_S(s) = U^T(s) \cdot H_S^T(s) \cdot U(s) \\ P_R(s) = U^T(s) \cdot H_R^T(s) \cdot U(s) \end{cases} \quad (5.131)$$

where  $H_C(s), H_S(s), H_R(s)$  can be deduced from straight computations:

$$\begin{cases} H_C(s) = C(sI - A)^{-1} B \\ H_S(s) = B^T (sI - A)^{-T} Q \cdot R \cdot Q(sI - A)^{-1} B - C(sI - A)^{-1} B \\ H_R(s) = -B^T (sI - A)^{-T} Q \cdot R \cdot Q(sI - A)^{-1} B \end{cases} \quad (5.132)$$

Thus we can write the following relation between the power port transfer functions:

$$H_C(s) = -H_R(s) - H_S(s) \quad (5.133)$$

Equivalently we can write:

$$Y_C(s) = -Y_S(s) - Y_R(s) \quad (5.134)$$

where:

$$\begin{cases} Y_C(s) \stackrel{def}{=} H_C(s) \cdot U(s) \\ Y_S(s) \stackrel{def}{=} H_S(s) \cdot U(s) \\ Y_R(s) \stackrel{def}{=} H_R(s) \cdot U(s) \end{cases} \quad (5.135)$$

**Remark:** *This results show that between the input and power port energies, we can define separate transfer functions, which satisfy also the conservation law (5.133) and are in correspondence with the oblique projections defined in the deterministic identification context.*

### 5.3.3 Simulations and discussion

This section is reserved for a set of simulations using the algorithm presented above and the deterministic identification context, formulated for the power energy approach, using a DC motor model. For simulations, we consider a case scenario with a horizon length ( $j = 100$ ) over a time interval  $t_S = 10$  s. For the control port, we obtain an equivalent 2<sup>nd</sup> order system represented with red in the following figure.

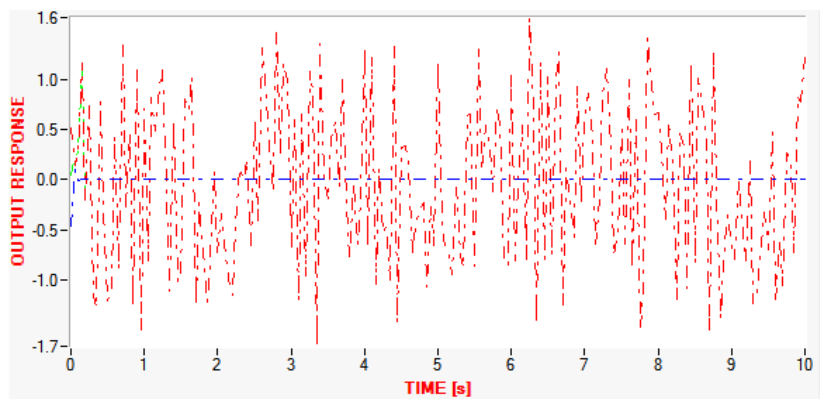


Figure 5.11. Output response of the real (green), estimated (red) and error (blue) for the control port.

As the graphic shows, the fitness between the real (green) and estimated (red) systems is almost perfect (error almost zero), excepting the beginning of the time interval. For the storage port power energy, we obtain an equivalent 3<sup>rd</sup> order model, with the following graphic representation.

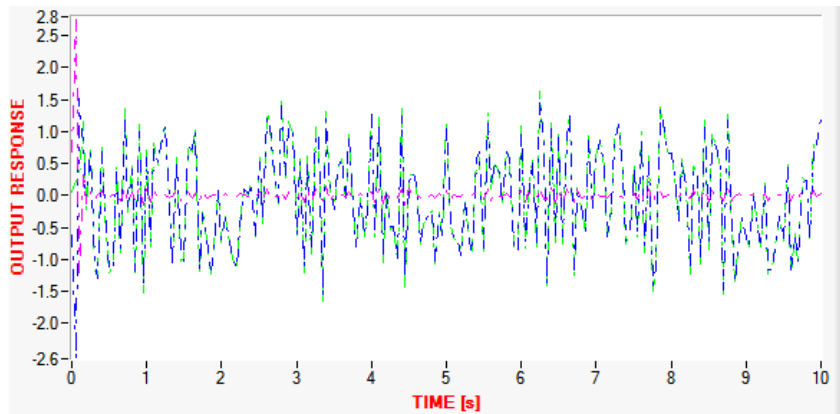


Figure 5.12. Output response of the real (green), estimated (blue) and error (magenta) for the storage port.

From this graphic, the estimated 3<sup>rd</sup> order model cannot approximate perfectly the real system response, which is a 2<sup>nd</sup> order system and thus, the errors (magenta) have increased amplitudes over time. It should be remarked that for the first part of the time-interval, the errors have higher amplitudes and they decrease with increasing time. Similarly to the control and storage ports, we represent the output response obtained for the equivalent system on the dissipation port, which is also a 3<sup>rd</sup> order model.

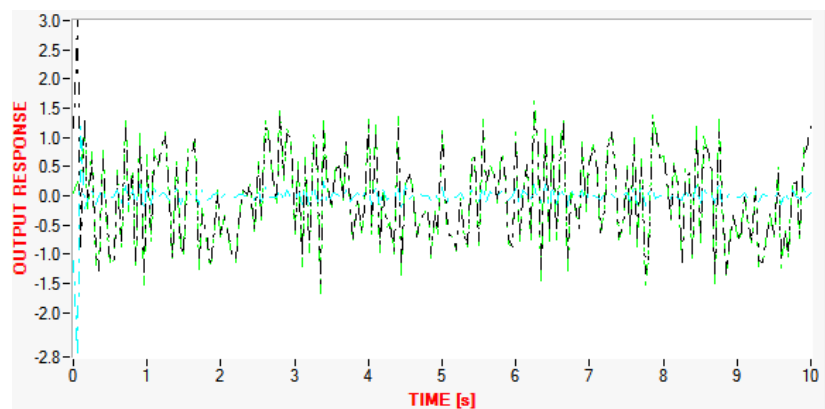


Figure 5.13. Output response of the real (green), estimated (black) and error (cyan) for the dissipation port.

As expected the 3<sup>rd</sup> order model estimation obtained for the dissipation port, doesn't provide a perfect approximation of the real dynamic system (2<sup>nd</sup> order model), with errors (cyan) of higher amplitudes. As for the storage port, the estimation errors have higher amplitudes in the beginning of the time-interval as the model approximation is weaker compared to the control port model. In the following figure, we represent the system responses for the real and estimated models for all three ports.

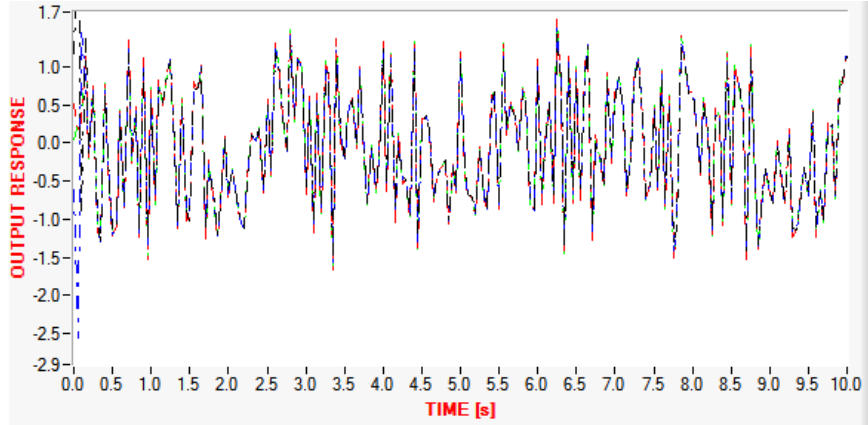


Figure 5.14. Output response of the real (green), control port (red), storage port (blue), dissipation port (black) estimated models.

As this figure shows, we can observe a good estimation of the DC motor model on each port, excepting the first part of the time-interval, where big estimation errors can be found mainly for the storage and dissipation port models. In the following figure, we also represent the output errors obtained for each port model.

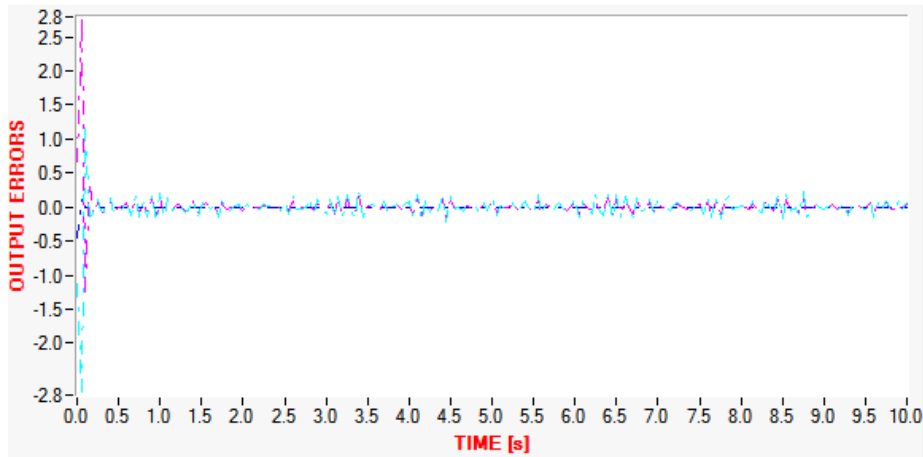


Figure 5.15. Output errors for the control port (blue), storage port (magenta) and dissipation port (cyan) estimated models.

Further we will perform a set of simulations for the perturbed PCH model proposed in chapter 3, using the power energy identification algorithm proposed in this section.

**Remark:** *The perturbed PCH system model proposed in chapter 3, is more a deterministic one, as the outputs are perturbed using a fixed model that preserve the PCH system structure, while the states are perturbed using a white noise sequence.*

In order to choose a model that better approximates the real one, we will introduce for this analysis, the following validation index known in the literature [195]:

$$v\_index = \sqrt{\frac{\sum_{i=1}^N (y_i - \bar{y})^2}{\sum_{i=1}^N (\hat{y}_i - \bar{y})^2}} \quad (5.136)$$

where  $y_i$  represents the real values of the system outputs,  $\bar{y}$  represents the medium value of the system outputs and  $\hat{y}_i$  the estimated values of the system outputs. We will consider three models for the output perturbation model of the  $\kappa$  matrix (see eq. (3.36)), to illustrate the results by simulation. As a first model of the perturbation matrix  $\kappa$ , we will consider the following model  $\kappa = [0.13 \quad -0.24]^T$  on a horizon length  $j = 200$ . In the following figure, we represent with different colours the real and estimated models for each port.

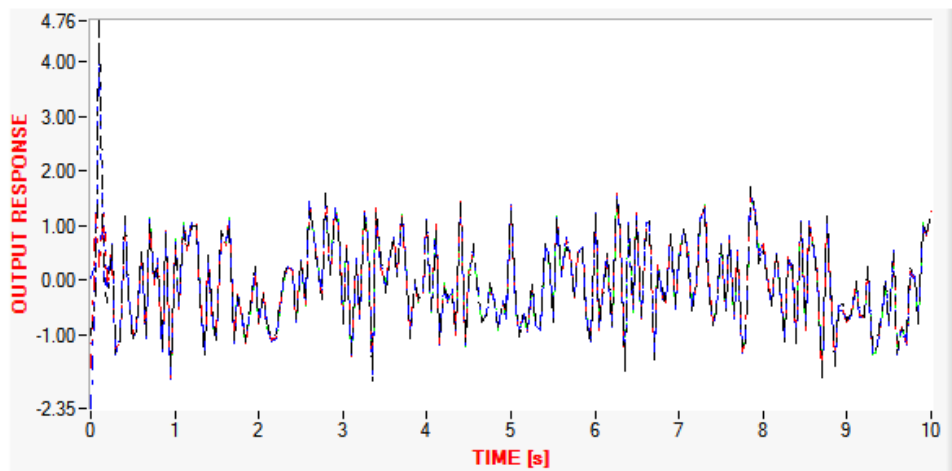


Figure 5.16. Output response of the real (green), control port (red), storage port (blue), dissipation port (black) estimated models.

For the first simulation, with the control port system outputs we have obtained the following validation index  $v\_index = 0.9946$ , with a third order model, being also the best approximation for the real model of the system. For the storage port model, it was determined a validation index  $v\_index = 0.9394$  for a third order model, being also the model with less performances as compared to the other two port models. For the dissipation port model, the validation index has the following value  $v\_index = 0.9404$ , which makes it a good approximation of the real model. In the second case, we have considered the following model  $\kappa = [0.5 \quad -0.6]^T$  for the output perturbation matrix. Applying the power energy deterministic identification algorithm of this section, we obtain the following responses of the system outputs.

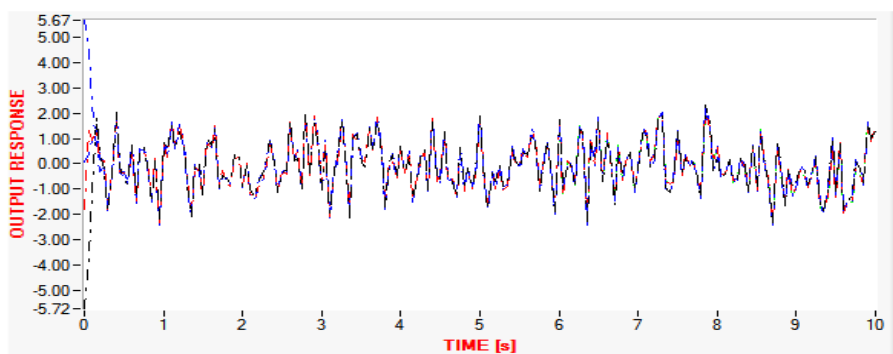


Figure 5.17. Output response of the real (green), control port (red), storage port (blue), dissipation port (black) estimated models.

As for the first case, the best approximation of the real model is obtained for the control port estimation with a third order model and validation index  $v\_index = 0.9997$ . The less performant estimation is realized for the storage port, with a 3<sup>rd</sup> order model and validation  $v\_index = 0.8939$ , which makes it the worse approximation of the real model. For the dissipation port model, the estimation gets a 3<sup>rd</sup> order model with a validation index  $v\_index = 0.9170$  which makes it a better approximation compared to the storage port case. For the 3<sup>rd</sup> case we will consider the following perturbation matrix  $K = [0.75 \quad -0.8]^T$  to check the models performance on each port. We thus obtain the following responses of the system outputs for the real and estimated models.

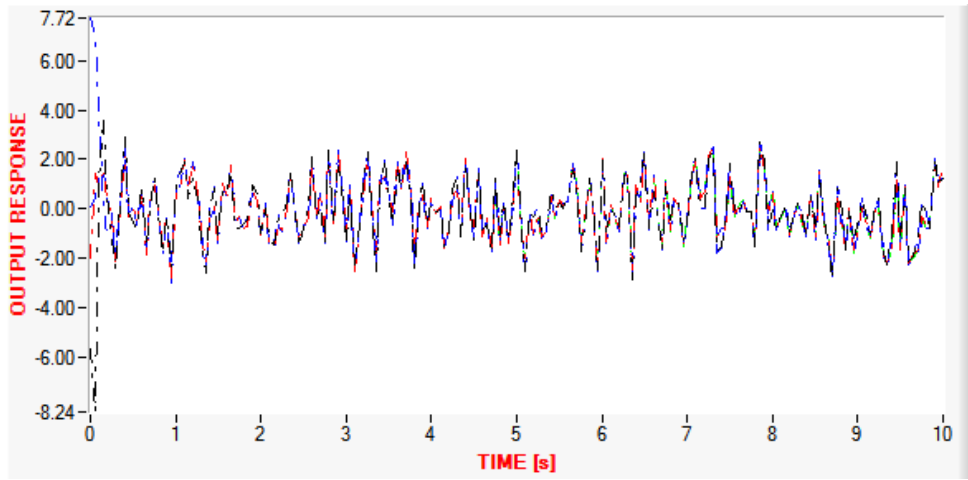


Figure 5.18. Output response of the real (green), control port (red), storage port (blue), dissipation port (black) estimated models.

As for the previous two cases, the control port obtains the best approximation of the real model, with a 3<sup>rd</sup> order model and a validation index  $v\_index = 0.9998$ , which is very close to one. As expected, for the storage port we obtain the less performant model of the real one, with a 3<sup>rd</sup> order model and a validation index  $v\_index = 0.8340$ , showing higher errors of estimation. Also for the dissipation port model, we obtain poor results with a 3<sup>rd</sup> order model and a validation index of  $v\_index = 0.8492$ .

**Remark:** When determining a higher order (equivalent) state-space system from system identification, we can write it equivalently as follows:

$$\begin{cases} \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} T_1 \bar{A} T_1^{-1} & T_1 \bar{A}_{12} T_2^{-1} \\ T_2 \bar{A}_{21} T_1^{-1} & T_2 \bar{A}_{22} T_2^{-1} \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} T_1 \bar{B} \\ T_2 B_2 \end{pmatrix} u \\ y_k = \begin{pmatrix} \bar{C} T_1^{-1} & \bar{C}_2 T_2^{-1} \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \end{cases} \quad (5.137)$$

where  $(\bar{A}, \bar{B}, \bar{C})$  represents the internal dominant system as defined in the literature [176], given generally by the analytical model. Thus we can determine an equivalent PCH system  $(T_1 A T_1^{-1}, T_1 B, C T_1^{-1})$  for the dynamic system, expressed in analytic form. From (5.134) we can derive the case of an equivalent PCH system of the same order as the analytic model, for

$\bar{A}_{12} = 0, \bar{A}_{21} = 0, \bar{A}_{22} = 0, \bar{B}_2 = 0, \bar{C}_2 = 0$ . By system identification, it is never possible to determine the exact system matrices  $(A, B, C)$  of the real system, only an equivalent form  $(T_1 A T_1^{-1}, T_1 B, C T_1^{-1})$ . From (5.137), it is not really necessary to determine an equivalent high order PCH system to the initial one.

## Conclusions

This chapter proposed an analysis of subspace identification algorithms known from the literature for PCH systems, using a simple symplectic time-discretization scheme of chapter 4, that gives a similar (PCH state-space structure) to the classic one. It can be proved, that for the selected time-discretization scheme, a matrix I/O representation, orthogonal or oblique projections can be determined, which are used with the subspace identification algorithms. We should remark that the subspace identification analysis of this chapter, can be extended also for other symplectic time-discretization schemes presented in chapter 4, increasing the complexity of computations and time duration. Using two classic deterministic identification algorithms, several simulations were realized, that prove to give equivalent PCH system models for the real system (DC motor) considered. Further it was proposed a new (power energy approach) in the lossy case, starting from the classic one, using the identifiability concepts introduced in chapter 2, that makes use of the inputs and power port energies knowledge to determine the system parameters. Thus the new identification framework, can estimate equivalent PCH models corresponding to each port. Several simulations were performed to illustrate the deterministic power energy approach for the DC motor model. The perturbation model proposed in chapter 3, which preserve the PCH structure was also analysed. This model structure was tested on the power energy deterministic identification context, for several perturbation models using a validation index to check the performances of the estimated models when compared to the real one.



## 6. Conclusions, personal contributions and perspectives

The final chapter presents the main conclusions, personal contributions and perspectives, that result at the end of the proposed thesis on the *identification of Port-Controlled-Hamiltonian systems*. The *main objectives* proposed for the thesis, were: the selection of the main reasons and advantages to develop a *system identification theory* for Port-Controlled-Hamiltonian systems, as presented in chapter 1; the structural identifiability analysis of Port-Controlled-Hamiltonian systems using different techniques: observability/controllability concepts, direct test, power series expansion or a new (specific) power energy approach; the practical identifiability analysis of Port-Controlled-Hamiltonian systems using the observability/controllability concepts; the introduction of a *perturbed PCH model* by means of the interaction port, which is also proved to conserve PCH systems properties; the selection of a set of symplectic discretization schemes, which combine different first or second order discretization rules for the flows and efforts, conserving the basic PCH systems structure; the introduction of a *discretization error Hamiltonian* as a difference between the continuous and discrete-time PCH systems, which is proved to conserve the main properties; the analysis of known state-space identification algorithms from the literature to be used for PCH systems; the introduction of matrix I/O representation, orthogonal or oblique projections for the selected time-discretization schemes in view of a subspace identification approach; the selection of the most simple and reliable symplectic discretization scheme to be used with the identification algorithms for analysis; the introduction of a new context for subspace identification of PCH systems using the knowledge of the inputs/power energies to determine equivalent PCH systems on each port; the analysis of the proposed subspace identification algorithms or new (power based) for a proposed PCH system model; the introduction of a criterion for performance validation of the estimated models for a selected perturbation model.

*Chapter 1*, made a general presentation of the system identification control theory, from the modelling of dynamic systems, types of models, modeling procedures with a general schema known in practice or theory. Further in section 1.3, the PCH modeling and control approach was introduced, as a new approach, with their explicit/implicit/bilinear forms, Dirac interconnection structure or dynamics. For the identifiability analysis throughout the thesis, three representative examples were proposed for the lossless or lossy case, as follows: LC circuit, capacitor microphone circuit or DC motor. Section 1.4, introduced the main reasons for the development of an *identification theory* of PCH systems starting from their remarkable properties. The last two sections of chapter 1, 1.5 and 1.6, introduced the main objectives of the thesis and a summary on chapters.

*Chapter 2*, proposed an analysis of the structural identifiability of PCH systems using different known or proposed techniques. As classic structural identifiability techniques, the following ones were selected: the observability/controllability based test; the direct test or the power series expansion test. For each approach, general expressions have been proved and found and proper definitions have been formulated specific for PCH systems. Beside this classic identifiability techniques, a new (power based approach), was proposed using the power energies corresponding to each port. Thus, new propositions and definitions were

introduced for the global and local identifiability of a PCH system and also, a new concept of *port-identifiability*, was proposed specific to one of the systems ports. For each structural identifiability technique, examples have been considered for the lossless or lossy case, using an LC circuit, capacitor microphone or a DC motor.

*Chapter 3*, made a practical identifiability analysis of PCH systems using the observability/controllability concepts known from classic state-space identification theory.

A perturbation model of PCH systems was proposed in this chapter, by means of an input/output pair corresponding to the interaction port, such as the preservation of the main properties holds. An equivalent observable canonical form known from the literature was proposed for PCH systems, on the proposed model. For a known perturbation model, examples were considered for the analysis of the practical identifiability using the observability/controllability concepts in the lossless or lossy case, using the model of an LC circuit, capacitor microphone or a DC motor.

*Chapter 4*, proposed a new framework for symplectic time-discretization of PCH systems, using different discretization rules for the flows and efforts. As discretization rules, were selected: implicit/explicit Euler rule, implicit mid-point rule or implicit trapezoidal rule. For the combined symplectic discretization framework, general expressions were defined for the explicit/implicit forms, power conservation law or bilinear form of discrete-time PCH systems in the lossless or lossy case. For the selected first or second order discretization rules, it was checked the conservation of the main properties in discrete-time in the lossless or lossy case. It was proved that some symplectic time-discretization schemes, doesn't preserve the power conservation law in discrete-time, when introducing dissipation elements in the system. Section 4.4, introduced a methodology to differentiate two different PCH systems in the lossless or lossy case, by creating a new PCH system, which conserves the main characteristic properties. The new system, can be particularly used for the discretization analysis, by introducing a *discretization error Hamiltonian*, as the difference between the continuous and discrete-time models. The last part of the chapter, made several simulations for the lossless or lossy case, using a capacitor microphone circuit, to test the performances of the selected symplectic discretization schemes and *discretization error Hamiltonian*. A complete set of symplectic discretization schemes can be found in Appendix E for the lossless or lossy case, for which the main properties of PCH systems are checked using the general relations introduced in section 4.2.

*Chapter 5*, proposed an analysis of classic subspace identification algorithms for PCH systems using one time-discretization scheme developed in chapter 4. Thus it was determined a matrix I/O representation, together with orthogonal and oblique projections for the selected time-discretization schemes. As time-discretization scheme for the analysis using the identification algorithms a simple scheme was selected, which uses Euler time-discretization rules for the flows and efforts and has a similar state-space structure to the classic one. A new (power energy) context was also proposed, to determine the system parameters using the inputs and power port energies knowledge, which is proved to determine equivalent PCH systems corresponding to each port. Simulations using the proposed or new (energy based) deterministic identification algorithms are performed for a DC motor model.

## Personal contributions

- The analysis of the usefulness, necessity and possibilities of development of a *system identification theory* for PCH systems, in view of an improved real-time modelling and control;
- The analysis of the PCH systems model structure, which is necessary in process identification;
- The selection of appropriate structural identifiability techniques for PCH systems from the classic identification theory;
- The analysis and proof of the selected structural identifiability techniques, that can be used for PCH systems in the lossless or lossy case;
- The introduction of a new (specific) structural identifiability approach for PCH systems, by means of the power energies corresponding to the ports, by proposing specific propositions for the global or local identifiability;
- The introduction of a new concept of *port identifiability*, with specific definitions for the global or local identifiability for a closer analysis;
- The introduction of a *perturbed PCH model* by means of an input/output pair corresponding to the interaction port of the system with the environment;
- The analysis and proof that PCH systems conserve their main characteristic properties, when converted to the observable canonical form in the presence of a perturbation model;
- The selection of symplectic time-discretization schemes to be used for PCH systems;
- The introduction of general expressions for the explicit/implicit/bilinear forms or power conservation law of discrete-time PCH systems in the lossless or lossy case, for the combined time-discretization framework;
- The analysis and proof of the main characteristic properties of PCH systems in the lossless or lossy case, for the selected symplectic discretization schemes;
- The introduction of a difference PCH model between two PCH systems, which is proved to conserve the main properties;
- The adoption of a *discretization error Hamiltonian* model, as a difference between the continuous-time and discrete-time PCH systems, for a description using the power energies corresponding to the ports;
- The development of a simulation software (Labwindows environment) for the analysis and simulation of the selected symplectic time-discretization schemes;
- The general formulation of *matrix I/O equations* for the selected symplectic time-discretization schemes;
- The introduction of *orthogonal or oblique projections* for the selected symplectic time-discretization schemes;
- The selection of appropriate classic or new subspace identification algorithms, for PCH systems;
- The simulation of the selected identification algorithms, using one symplectic time-discretization scheme, which is similar to the classic approach.

## **Main perspectives**

- The development of improved real-time identification and control applications using the new PCH modelling and control approach for complex dynamic systems;
- The development of new (specific) system identification theory and algorithms using the power energy approach, introduced in chapter 2;
- The development of real-time test platforms in the laboratory, to test the proposed identification and control theory and algorithms;
- The possibility to analyse for identification and control purpose, the PCH systems not only in continuous-time but also in discrete-time, using the proposed symplectic time-discretization schemes;
- The development of other appropriate symplectic time-discretization schemes for PCH systems using more general and complex discretization rules, like Runge-Kutta or collocation methods;
- The possibility to develop specific identification algorithms, corresponding to each symplectic time-discretization scheme proposed in the thesis;
- The development of high-quality software modules for time-discretization of PCH systems, using symplectic discretization schemes;
- The development of performant software modules for the real-time identification and control of PCH systems using the proposed algorithms;
- The introduction and use of the proposed identification and control techniques or algorithms in an industrial environment.

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## Appendix A: OBSERVABLE CANONICAL FORM REPRESENTATION OF PCH SYSTEMS

### A.1. LOSSLESS PCH SYSTEMS

For the lossless PCH systems from (2.3), it is known, that  $C = B^T Q$  and in this case, (2.9) becomes:

$$\begin{cases} T_{obs} J Q T_{obs}^{-1} = J_{obs} Q_{obs} \\ B^T Q T_{obs}^{-1} = C_{obs} \\ C_{obs} = B_{obs}^T Q_{obs} \end{cases} \quad (A.1)$$

Replacing  $B_{obs} = T_{obs} B$  from (2.7) in (A.1), we get:

$$\begin{cases} T_{obs} J Q T_{obs}^{-1} = J_{obs} Q_{obs} \\ B^T Q T_{obs}^{-1} = (T_{obs} B)^T Q_{obs} \\ C_{obs} = (T_{obs} B)^T Q_{obs} \end{cases} \quad (A.2)$$

Developing further the 2<sup>nd</sup> relation in (A.2), we get:

$$B^T Q T_{obs}^{-1} = B^T T_{obs}^T Q_{obs} \quad (A.3)$$

Multiplying with the right pseudo-inverse  $T_B^T T_B^T$  of  $B$  in (A.3), the following relation is obtained:

$$Q T_{obs}^{-1} = T_{obs}^T Q_{obs} \quad (A.4)$$

In the above relation,  $T_B$  is a pseudo-inverse matrix  $T_B \in R^{m \times n}$ , which satisfies:

$$B T_B = I \quad (A.5)$$

Multiplying to the left with the pseudo-inverse  $T_{obs}^{-T}$  of  $T_{obs}^T$ , in (A.4), the following form is obtained for the  $Q_{obs}$  matrix:

$$Q_{obs} = T_{obs}^{-T} Q T_{obs}^{-1} \quad (A.6)$$

Replacing the value of  $Q_{obs}$  from (A.6) in the 1<sup>st</sup> relation of (A.2), it results:

$$T_{obs} J Q T_{obs}^{-1} = J_{obs} T_{obs}^{-T} Q T_{obs}^{-1} \quad (A.7)$$

It can be observed, that we can multiply to the right with  $T_{obs}$ ,  $Q^{-1} Q^{-1}$  and  $T_{obs}^T$ , and  $J_{obs}$  becomes then:

$$J_{obs} = T_{obs} J T_{obs}^T \quad (A.8)$$

Using matrix transpose properties, we compute  $Q_{obs}^T$  as follows:

$$Q_{obs}^T = T_{obs}^{-T} Q^T T_{obs}^{-1} \quad (A.9)$$

Using the property  $Q = Q^T$  of a PCH system, (A.9) becomes:

$$Q_{obs}^T = T_{obs}^{-T} Q_{obs}^{-1} \quad (A.10)$$

From (A.6) and (A.10), it can be observed that  $Q_{obs} = Q_{obs}^T$ . Because  $Q$  is positive semi-definite  $Q = Q^T \geq 0$  for a PCH system, it results that for all real vectors  $x \in R^n$ , the following relation  $x^T Q x \geq 0$  is also satisfied. From (A.10), we observe that the matrix  $Q_{obs}$  is also a positive semi-definite matrix, as it is composed of positive semi-definite elements, given by the column elements of  $T_{obs}^{-1}$ . We conclude, that for the  $Q_{obs}$  matrix from (A.6), the following relation holds:

$$Q_{obs} = Q_{obs}^T \geq 0 \quad (A.11)$$

Using (A.8) together with matrix transpose properties, the matrix transpose of  $J_{obs}$  is:

$$J_{obs}^T = \left( T_{obs} \quad J_{obs}^T \right)^T = \left( T_{obs}^T \right)^T \left( T_{obs} \quad J_{obs} \right)^T \quad (A.12)$$

The above relation, can be equivalently written as:

$$J_{obs}^T = T_{obs} J_{obs}^T T_{obs}^T \quad (A.13)$$

Using the property  $J = -J^T$  of a PCH system, (A.13) becomes:

$$J_{obs}^T = -T_{obs} J_{obs}^T T_{obs}^T \quad (A.14)$$

From (A.8) and (A.14), it results that  $J_{obs} = -J_{obs}^T$ . Using (A.1) and (A.6), we prove the following relation:

$$C_{obs} = B_{obs}^T Q_{obs} = C_{obs}^{-1} \quad (A.15)$$

Using (2.6) and (A.6), the above relation can be also expressed as:

$$C_{obs} = \left( T_{obs} \quad B \right)^T T_{obs}^{-T} Q_{obs}^{-1} \quad (A.16)$$

(A.16) can be written further as:

$$C_{obs} = B^T T_{obs}^T T_{obs}^{-T} Q_{obs}^{-1} \quad (A.17)$$

After possible simplifications, the final result of  $C_{obs}$  is:

$$C_{obs} = B^T Q_{obs}^{-1} \quad (A.18)$$

From (A.18) and (A.1), it results:

$$C_{obs} = C_{obs}^{-1} = B_{obs}^T Q_{obs} \quad (A.19)$$

Using (A.6), (A.8), (A.11), (A.14) and (A.19), the observable canonical-form from (2.8):

$$\begin{cases} \dot{x}_{obs} = J_{obs} Q_{obs} x_{obs} + B_{obs} u \\ y = C_{obs} x_{obs} \end{cases} \quad (A.20)$$

is also a PCH system, where the PCH systems parameters satisfy:

$$\begin{cases} J_{obs} = -J_{obs}^T = T_{obs} J_{obs}^T \\ Q_{obs} = Q_{obs}^T = T_{obs}^{-T} Q_{obs}^{-1} \\ C_{obs} = B_{obs}^T Q_{obs} = B_{obs}^T Q_{obs}^{-1} \end{cases} \quad (A.21)$$

This ends the proof of (2.10).

## A.2 LOSSY PCH SYSTEMS

For the lossy case, replacing  $B_{obs} = T_{obs} B$  from (2.12) in the 2<sup>nd</sup> relation from (2.14), the following relation can be written:

$$B^T Q_{obs}^{-1} = (T_{obs} B)^T Q_{obs} \quad (A.22)$$

(A.22) can be further expressed as:

$$B^T Q_{obs}^{-1} = B^T T_{obs}^T Q_{obs} \quad (A.23)$$

In order to find the pseudo-inverse of  $B$ , a matrix  $T_B \in R^{m \times n}$  exists, for which the following relation holds:

$$B^T T_B = I \quad (A.24)$$

where  $I \in R^{m \times m}$  represents the identity matrix. Multiplying to the left with  $T_B^T$  in (A.23), we get:

$$Q_{obs}^{-1} = T_{obs}^T Q_{obs} \quad (A.25)$$

Multiplying further to the left with the inverse  $T_{obs}^{-T}$  of  $T_{obs}$  in (A.25), we obtain the following relation:

$$T_{obs}^{-T} Q_{obs}^{-1} = Q_{obs} \quad (A.26)$$

In order to prove that  $Q_{obs} = Q_{obs}^T \geq 0$ ,  $Q_{obs}^T$  is calculated as follows:

$$Q_{obs}^T = \left( T_{obs}^{-T} Q_{obs}^{-1} \right)^T \quad (A.27)$$

Using the matrix transpose properties, (A.27) can be expressed also as:

$$Q_{obs}^T = T_{obs}^{-T} \left( T_{obs}^{-T} Q_{obs}^{-1} \right)^T \quad (A.28)$$

(A.28) becomes further:

$$Q_{obs}^T = T_{obs}^{-T} Q_{obs}^T T_{obs}^{-1} \quad (A.29)$$

Using the property  $Q = Q^T$  of a PCH system, (A.29) can be expressed as follows:

$$Q_{obs}^T = T_{obs}^{-T} Q_{obs}^{-1} \quad (A.30)$$

From (A.26) and (A.30), it can be observed that:

$$Q_{obs} = Q_{obs}^T = T_{obs}^{-T} Q_{obs}^{-1} \quad (A.31)$$

It is known, that  $Q$  is a positive semi-definite matrix, which means also, that for every vector  $x \in \mathbb{R}^n$ , the following relation holds:

$$x^T Q x \geq 0 \quad (\text{A.32})$$

Using (A.32), it can be observed that  $Q_{obs}$  in (A.31) is also formed by positive semi-definite elements, generated with the column elements of  $T_{obs}^{-1}$ , together with the  $Q$  matrix. The following relation results then:

$$Q_{obs} = Q_{obs}^T \geq 0 \quad (\text{A.33})$$

The matrices  $J_{obs}$  and  $R_{obs}$  can be calculated, using the 1<sup>st</sup> relation from (2.14), where  $Q_{obs}$  is replaced from (A.31), as follows:

$$T_{obs} (J - R) Q_{obs}^{-1} = \left( J_{obs} - R_{obs} \right) T_{obs}^{-T} Q_{obs}^{-1} \quad (\text{A.34})$$

In (A.34), it can be seen that a multiplication can be realized to the right with  $T_{obs}$ ,  $Q^{-1}$  and  $T_{obs}^T$  and the relation becomes:

$$T_{obs} (J - R) T_{obs}^T = \left( J_{obs} - R_{obs} \right) \quad (\text{A.35})$$

(A.35) can be also expressed as follows, on the left part:

$$T_{obs} J T_{obs}^T - T_{obs} R T_{obs}^T = J_{obs} - R_{obs} \quad (\text{A.36})$$

From (A.36), it can be seen that the relation holds for:

$$\begin{cases} T_{obs} J T_{obs}^T = J_{obs} \\ R_{obs} = T_{obs} R T_{obs}^T \end{cases} \quad (\text{A.37})$$

Using (A.37),  $J_{obs}^T$  is computed as follows:

$$J_{obs}^T = \left( T_{obs} J T_{obs}^T \right)^T \quad (\text{A.38})$$

From matrix transpose properties, (A.38) becomes:

$$J_{obs}^T = T_{obs} \left( T_{obs} J \right)^T \quad (\text{A.39})$$

(A.39) can be equivalently written as:

$$J_{obs}^T = T_{obs} J^T T_{obs}^T \quad (\text{A.40})$$

Using the property  $J = -J^T$  of a PCH system from (2.11), (A.40) can be expressed as:

$$J_{obs}^T = T_{obs} (-J) T_{obs}^T \quad (\text{A.41})$$

From (A.37) and (A.41), it results that:

$$J_{obs} = -J_{obs}^T = T_{obs} J T_{obs}^T \quad (\text{A.42})$$

Using (A.37), the value of  $R_{obs}^T$ , becomes:

$$R_{obs}^T = \left( T_{obs} R T_{obs}^T \right)^T \quad (\text{A.43})$$

From matrix transpose properties, the following relation is obtained from (A.43):

$$R_{obs}^T = T_{obs} (T_{obs} R)^T \quad (A.44)$$

which can be expressed also as follows:

$$R_{obs}^T = T_{obs} R^T T_{obs}^T \quad (A.45)$$

Using the property  $R = R^T$  of a PCH system, (A.45) becomes:

$$R_{obs}^T = T_{obs} R T_{obs}^T \quad (A.46)$$

From (A.37) and (A.46), we observe that:

$$R_{obs} = R_{obs}^T = T_{obs} R T_{obs}^T \quad (A.47)$$

It is known that  $R$  is positive semi-definite, which means that for every vector  $x \in R^n$ , the following relation is satisfied:

$$x^T R x \geq 0 \quad (A.48)$$

In (A.47) it can be observed, that  $R_{obs}$  is formed by semi-positive elements, given by the columns of  $T_{obs}^T$ . Using (A.47) and (A.48), the matrix  $R_{obs}$  is also positive semi-definite and it can be concluded that:

$$R_{obs} = R_{obs}^T \geq 0 \quad (A.49)$$

In order to prove that  $C_{obs} = B_{obs}^T Q_{obs} = C T_{obs}^{-1}$  for the observable canonical form of the PCH system from (2.11), we replace  $B_{obs} = T_{obs} B$  from (2.12) and  $Q_{obs} = T_{obs}^{-T} Q T_{obs}^{-1}$  from (A.31) in  $C_{obs}$  and it results:

$$C_{obs} = (T_{obs} B)^T T_{obs}^{-T} Q T_{obs}^{-1} \quad (A.50)$$

(A.50) becomes further:

$$C_{obs} = B^T T_{obs}^T T_{obs}^{-T} Q T_{obs}^{-1} \quad (A.51)$$

After performing all the simplifications in (A.51), the relation becomes:

$$C_{obs} = B^T Q T_{obs}^{-1} \quad (A.52)$$

Using (2.12), (2.14) and (A.52), it can be observed that:

$$C_{obs} = C T_{obs}^{-1} = B^T Q T_{obs}^{-1} = B_{obs}^T Q_{obs} \quad (A.53)$$

From (2.12), (2.13), (2.14), (A.31), (A.33), (A.37), (A.42), (A.49) and (A.53), it results that the observable canonical form representation from (2.12) of the initial PCH system from (2.10), is also a PCH system which conserves the basic properties. The general form of this state-space system is:

$$\begin{cases} \dot{x}_{obs} = (J_{obs} - R_{obs}) Q_{obs} x_{obs} + B_{obs} u \\ y = C_{obs} x_{obs} \end{cases} \quad (A.54)$$

For the lossy PCH system from (A.54), the following relations hold then:

$$\begin{cases} Q_{obs} = Q_{obs}^T = T_{obs}^{-T} Q_{obs}^{-1} \geq 0 \\ J_{obs} = -J_{obs}^T = T_{obs} J_{obs}^T \\ R_{obs} = R_{obs}^T = T_{obs} R_{obs}^T \geq 0 \end{cases} \quad (A.55)$$

which represent also the result from (2.15).

## **Appendix B: STRUCTURAL IDENTIFIABILITY OF PCH SYSTEMS**

### **B.1.OBSERVABILITY/ CONTROLABILITY ANALYSIS**

#### **B.1.1 LOSSLESS PCH SYSTEMS**

For the observability/controllability based structural identifiability test, replacing  $B_{obs} = T_{obs} B$ , and  $A_{obs} = T_{obs} A T_{obs}^{-1}$  from (2.7), in the lossless case, (2.16) becomes:

$$R_{ID} = \begin{bmatrix} T_{obs} B & T_{obs} A T_{obs}^{-1} T_{obs} B & \dots & \left( T_{obs} A T_{obs}^{-1} \right)^n T_{obs} B \end{bmatrix} \quad (B.1)$$

In the following lines, a general form will be found for the term  $A_{obs}^k$  from (B.1). For the term  $A_{obs}^2$ , the following relation can be written:

$$A_{obs}^2 = T_{obs} A T_{obs}^{-1} T_{obs} A T_{obs}^{-1} \quad (B.2)$$

which becomes after simplification:

$$A_{obs}^2 = T_{obs} A^2 T_{obs}^{-1} \quad (B.3)$$

For the term  $A_{obs}^3$ , we obtain the following form using (B.3):

$$A_{obs}^3 = T_{obs} A^2 T_{obs}^{-1} T_{obs} A T_{obs}^{-1} \quad (B.4)$$

which becomes after simplification:

$$A_{obs}^3 = T_{obs} A^3 T_{obs}^{-1} \quad (B.5)$$

From (B.1), (B.3) and (B.5), the following general relation:

$$A_{obs}^k = T_{obs} A^k T_{obs}^{-1} \quad (B.6)$$

can be deduced for a  $k$  power of  $A_{obs}$ . Replacing the result from (B.6) in (B.1), the identifiability matrix  $R_{ID}$  becomes:

$$R_{ID} = \begin{bmatrix} T_{obs} B & T_{obs} A T_{obs}^{-1} T_{obs} B & \dots & T_{obs} A^{n-1} T_{obs}^{-1} T_{obs} B \end{bmatrix} \quad (B.7)$$

Equivalently, (B.7) becomes:

$$R_{ID} = \begin{bmatrix} T_{obs} B & T_{obs} AB & \dots & T_{obs} A^{n-1} B \end{bmatrix} \quad (B.8)$$

The above relation, can be also expressed as:

$$R_{ID} = T_{obs} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \quad (\text{B.9})$$

The transformation matrix  $T_{obs}$  from (2.5), can be also expressed as follows:

$$T_{obs} = \begin{bmatrix} B^T Q \\ B^T QA \\ B^T QA^2 \\ \dots \\ B^T QA^{n-1} \end{bmatrix} \quad (\text{B.10})$$

The first 4 elements of the transformation matrix  $T_{obs}$ , can be also expressed as follows:

$$\begin{cases} C = B^T Q \\ CA = -(AB)^T Q \\ CA^2 = (-1)^2 (A^2 B)^T Q \\ CA^3 = (-1)^3 (A^3 B)^T Q \end{cases} \quad (\text{B.11})$$

using the PCH systems and matrix transpose properties.

From (B.11), the following relation can be deduced:

$$CA^{k-1} = (-1)^{k-1} (A^{k-1} B)^T Q \quad (\text{B.12})$$

for the elements of the transformation matrix  $T_{obs}$  with  $k \geq 1$ . Using (B.12), the expression of the identifiability matrix from (B.9) becomes:

$$R_{ID} = \begin{bmatrix} B^T Q \\ -(AB)^T Q \\ \dots \\ (-1)^{n-1} (A^{n-1} B)^T Q \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{n-1} B \end{bmatrix} \quad (\text{B.13})$$

Using the property  $Q = Q^T$  of PCH systems, (B.13) can be written as follows:

$$R_{ID} = \begin{bmatrix} B^T Q^T \\ -(AB)^T Q^T \\ \dots \\ (-1)^{n-1} (A^{n-1} B)^T Q^T \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{n-1} B \end{bmatrix} \quad (\text{B.14})$$

The above relation becomes:

$$R_{ID} = \begin{bmatrix} (QB)^T \\ -(QAB)^T \\ \dots \\ (-1)^{n-1} (QA^{n-1}B)^T \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \quad (\text{B.15})$$

which represent also the relation (2.17). Using **Proposition 2.1**, if the lossless PCH system from (2.3) is observable, the following two relations:

$$\text{rank} \begin{bmatrix} B & -AB & \dots & (-1)^{n-1} A^{n-1}B \end{bmatrix} = n \quad (\text{B.16})$$

$$\text{rank} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \quad (\text{B.17})$$

hold for the elements of the identifiability matrix in (2.17). Using the properties of the matrix determinant, the following relation:

$$\det R_{ID} = \det (QR_1R_2) \quad (\text{B.18})$$

can be written, for the determinant of the identifiability matrix  $R_{ID}$  from (B.15), where the following notations:

$$R_1 = \begin{bmatrix} B & -AB & \dots & (-1)^{n-1} A^{n-1}B \end{bmatrix} \quad (\text{B.19})$$

$$R_2 = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \quad (\text{B.20})$$

were introduced. Using the following property of the determinant  $\det (AB) = \det A \det B$ , (B.18) is equivalent to:

$$\det R_{ID} = \det Q \det R_1 \det R_2 \quad (\text{B.21})$$

The above determinant is different from 0, which means the identifiability matrix  $R_{ID}$  is full rank and the system controllable, for  $\det Q \neq 0$ ,  $\det R_1 \neq 0$  and  $\det R_2 \neq 0$ .

### B.1.2 LOSSY PCH-SYSTEMS

For the lossy case of PCH systems, an equivalent form of (2.18), can be obtained by searching for a general form of the term  $A_{obs}^k$ . Computing the term  $A_{obs}^2$  using (2.14), the following relation is obtained:

$$A_{obs}^2 = T_{obs} (J - R) Q T_{obs}^{-1} T_{obs} (J - R) Q T_{obs}^{-1} \quad (\text{B.22})$$

After the possible simplifications in (B.22), the relation becomes:

$$A_{obs}^2 = T_{obs} ((J - R) Q)^2 T_{obs}^{-1} \quad (\text{B.23})$$

The next term  $A_{obs}^3$  can be written as:

$$A_{obs}^3 = T_{obs} (J - R) Q T_{obs}^{-1} T_{obs} ((J - R) Q)^2 T_{obs}^{-1} \quad (\text{B.24})$$

After the possible simplifications in (B.24), it results:



$$A_{obs}^3 = T_{obs} ((J - R)Q)^3 T_{obs}^{-1} \quad (B.25)$$

Using (B.23) and (B.25), the following general relation:

$$A_{obs}^k = T_{obs} ((J - R)Q)^k T_{obs}^{-1} \quad (B.26)$$

can be deduced for the term  $A_{obs}^k$  with  $k \geq 1$ . Replacing the result of  $A_{obs}^k$  from (B.26) in (2.18), we get:

$$R_{ID} = T_{obs} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \quad (B.27)$$

where  $A = (J - R)Q$  from (2.11). In the next lines an equivalent form will be found, for the observable transformation matrix  $T_{obs}$  from (2.5) in the lossy case. The 1<sup>st</sup> two elements of the transformation matrix  $T_{obs}$ , can be written as follows:

$$C = B^T Q = (QB)^T \quad (B.28)$$

$$C = CA = B^T Q^T \left( \begin{bmatrix} -J^T & -R^T \end{bmatrix} Q \right) \quad (B.29)$$

using the PCH systems properties. Introducing the notations  $A_{loss} = JQ$  and  $A_{res} = RQ$  with the matrix transpose properties, (B.29) becomes:

$$CA = B^T \begin{bmatrix} -A_{loss} & -A_{res} \end{bmatrix}^T Q \quad (B.30)$$

Using the property  $Q = Q^T$  of a PCH system together with matrix transpose properties, (B.30) can be also expressed as follows:

$$CA = (-1)^1 \left( Q \begin{bmatrix} A_{loss} & A_{res} \end{bmatrix} B \right)^T \quad (B.31)$$

The next term  $CA^2$  from the transformation matrix  $T_{obs}$ , has the following form:

$$CA^2 = (-1)^1 B^T \left( \begin{bmatrix} A_{loss} & A_{res} \end{bmatrix}^T Q \right) (J - R)Q \quad (B.32)$$

which becomes further:

$$CA^2 = (-1)^2 B^T \left( \left( \begin{bmatrix} A_{loss} & A_{res} \end{bmatrix}^T \right)^2 Q \right) \quad (B.33)$$

using the properties of PCH systems and matrix transpose. Using (B.33),  $CA^3$  becomes further:

$$CA^3 = (-1)^2 B^T \left( \left( \begin{bmatrix} A_{loss} & A_{res} \end{bmatrix}^T \right)^2 Q \right) (J - R)Q \quad (B.34)$$

From PCH systems and matrix transpose properties, the above relation can be equivalently in the following form:

$$CA^3 = (-1)^3 B^T \left( \left( \begin{bmatrix} A_{loss} & A_{res} \end{bmatrix}^T \right)^3 Q \right) \quad (B.35)$$

From (B.30), (B.33) and (B.35), the following general form can be deduced for the term  $CA^k$ , with  $k \geq 1$ :

$$CA^k = (-1)^k B^T \left( \left( \begin{bmatrix} A_{loss} & A_{res} \end{bmatrix}^T \right)^k Q \right) \quad (B.36)$$

Using matrix transpose properties, the following relation can be easily deduced:

$$\left( (A_{loss} + A_{res})^T \right)^k = \left( (A_{loss} + A_{res})^k \right)^T \quad (\text{B.37})$$

In this case, (B.36) becomes:

$$CA^k = (-1)^k B^T \left( (A_{loss} + A_{res})^k \right)^T \quad (\text{B.38})$$

The corresponding transformation matrix  $T_{obs}$ , can be written then:

$$T_{obs} = \begin{bmatrix} B^T Q \\ B^T (-1)^1 (A_{loss} + A_{res})^T Q \\ \dots \\ B^T (-1)^{n-1} \left( (A_{loss} + A_{res})^k \right)^T Q \end{bmatrix} \quad (\text{B.39})$$

Using PCH systems properties, (B.39) can be also expressed as follows:

$$T_{obs} = \begin{bmatrix} B^T Q^T \\ B^T (-1)^1 (A_{loss} + A_{res})^T Q^T \\ \dots \\ B^T (-1)^{n-1} \left( (A_{loss} + A_{res})^k \right)^T Q^T \end{bmatrix} \quad (\text{B.40})$$

Another expression for (B.40) is:

$$T_{obs} = \begin{bmatrix} (-1)^0 (QB)^T \\ (-1)^1 (Q(A''B))^T \\ \dots \\ B^T (-1)^{n-1} \left( (A_{loss} + A_{res})^k \right)^T Q^T \end{bmatrix} \quad (\text{B.41})$$

This represents the result from (2.19), where the following notations were introduced:

$$\begin{cases} A_{loss} = JQ \\ A_{res} = RQ \\ A'' = (A_{loss} + A_{res}) \end{cases} \quad (\text{B.42})$$

For the identifiability matrix in (2.21) to be full rank and the pair  $(A_{obs}, B_{obs})$  controllable for lossy PCH systems, the following matrix notations are introduced:

$$R_1 = Q \quad (\text{B.43})$$

$$R_2 = \begin{bmatrix} (-1)^0 B & (-1)^1 (A''B) & \dots & (-1)^{n-1} (A''^{n-1} B) \end{bmatrix} \quad (\text{B.44})$$

$$R_3 = \begin{bmatrix} B & AB & \dots & A^{n-1} B \end{bmatrix} \quad (\text{B.45})$$

The identifiability matrix  $R_{ID}$  from (2.21), becomes then:

$$R_{ID} = R_1 R_2 R_3 \quad (\text{B.46})$$

Using the determinant properties, we can write the following relation:

$$\det(R_{ID}) = \det(R_1) \det(R_2) \det(R_3) \quad (\text{B.47})$$

Then the identifiability matrix  $R_{ID}$  is full rank and the system controllable, when  $\det R_1 \neq 0$ ,  $\det R_2 \neq 0$  and  $\det R_3 \neq 0$ . Using the expression of  $A$ ,  $A_{loss}$  and  $A''$  matrices from (2.11) and (B.42), the following relation can be written:

$$A = 2A_{loss} - A'' \quad (B.48)$$

The identifiability matrix  $R_{ID}$  from (2.21), becomes also:

$$\begin{aligned} R'_{ID} &= Q \begin{bmatrix} (-1)^0 B & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \\ R'_{ID} &= Q \begin{bmatrix} (-1)^0 B & (-1)^1 (A''B) & \dots & (-1)^{n-1} (A''^{n-1} B) \end{bmatrix} \\ & \begin{bmatrix} B & (2A_{loss} - A'')B & \dots & (2A_{loss} - A'')^{n-1} B \end{bmatrix} \end{aligned} \quad (B.49)$$

## **B.2 DIRECT IDENTIFIABILITY**

### **B.2.1 LOSSLESS PCH SYSTEMS**

For the direct structural identifiability test applied to a lossless PCH system, the following relations can be written using (2.24) and (2.25):

$$\begin{cases} (JQ)(\theta_{HA1})x + B(\theta_{HB1})u = (JQ)(\theta_{HA2})x + B(\theta_{HB2})u \\ \left[ \begin{matrix} B^T & Q \end{matrix} \right] (\theta_{HC1})x = \left[ \begin{matrix} B^T & Q \end{matrix} \right] (\theta_{HC2})x \end{cases} \quad (B.50)$$

(B.50) can be further expressed as:

$$\begin{cases} \left[ \begin{matrix} (JQ)(\theta_{HA1}) - (JQ)(\theta_{HA2}) \end{matrix} \right] x = \left[ \begin{matrix} B(\theta_{HB2}) - B(\theta_{HB1}) \end{matrix} \right] u \\ \left[ \left[ \begin{matrix} B^T & Q \end{matrix} \right] (\theta_{HC1}) - \left[ \begin{matrix} B^T & Q \end{matrix} \right] (\theta_{HC2}) \right] x = 0 \end{cases} \quad (B.51)$$

From (B.51), we observe that the relations hold (for every admissible input  $u$  and generated states  $x$ ), only if:

$$\begin{cases} (JQ)(\theta_{HA1}) = (JQ)(\theta_{HA2}) \\ B(\theta_{HB1}) = B(\theta_{HB2}) \\ \left[ \begin{matrix} B^T & Q \end{matrix} \right] (\theta_{HC1}) = \left[ \begin{matrix} B^T & Q \end{matrix} \right] (\theta_{HC2}) \end{cases} \quad (B.52)$$

Solving the above system of equations, the following relations:

$$\begin{cases} \theta_{HA1} = \theta_{HA2} \\ \theta_{HB1} = \theta_{HB2} \\ \theta_{HC1} = \theta_{HC2} \end{cases} \quad (B.53)$$

need to be satisfied between the unknown parameters  $\theta_{HA1}, \theta_{HB1}, \theta_{HC1}$  and  $\theta_{HA2}, \theta_{HB2}, \theta_{HC2}$ , in order to have identifiable parameters  $\theta_{HA}, \theta_{HB}, \theta_{HC}$ . This represents also the result from (2.26).

### B.2.2 LOSSY PCH SYSTEMS

For the lossy Port-Controlled-Hamiltonian systems, the following relations can be written using the direct identifiability test:

$$\begin{cases} ((J - R)Q)(\theta_{HA1})x + B(\theta_{HB1})\mu = ((J - R)Q)(\theta_{HA2})x + B(\theta_{HB2})\mu \\ \left[ \begin{matrix} B^T & Q \end{matrix} \right] (\theta_{HC1})x = \left[ \begin{matrix} B^T & Q \end{matrix} \right] (\theta_{HC2})x \end{cases} \quad (B.54)$$

The above relation, can be also written as:

$$\begin{cases} ((J - R)Q)(\theta_{HA1}) - ((J - R)Q)(\theta_{HA2})x + (B(\theta_{HB1}) - B(\theta_{HB2}))\mu = 0 \\ \left[ \begin{matrix} B^T & Q \end{matrix} \right] (\theta_{HC1}) - \left[ \begin{matrix} B^T & Q \end{matrix} \right] (\theta_{HC2})x = 0 \end{cases} \quad (B.55)$$

by moving the right elements to the left. From (B.55), it can be seen that the relations hold for:

$$\begin{cases} ((J - R)Q)(\theta_{HA1}) = ((J - R)Q)(\theta_{HA2}) \\ B(\theta_{HB1}) = B(\theta_{HB2}) \\ \left[ \begin{matrix} B^T & Q \end{matrix} \right] (\theta_{HC1}) = \left[ \begin{matrix} B^T & Q \end{matrix} \right] (\theta_{HC2}) \end{cases} \quad (B.56)$$

Solving the system of equations from (B.56), the following relations:

$$\begin{cases} \theta_{HA1} = \theta_{HA2} \\ \theta_{HB1} = \theta_{HB2} \\ \theta_{HC1} = \theta_{HC2} \end{cases} \quad (B.57)$$

need to be satisfied, between the unknown parameters  $\theta_{HA1}, \theta_{HB1}, \theta_{HC1}$  and  $\theta_{HA2}, \theta_{HB2}, \theta_{HC2}$  in order to have identifiable parameters  $\theta_{HA}, \theta_{HB}, \theta_{HC} \in \Theta$ . This represents the result from (2.28).

## **B3.POWER SERIES EXPANSION IDENTIFIABILITY**

### B.3.1 LOSSLESS PCH SYSTEMS

For the power series expansion identifiability test, the first term in (2.29) is:

$$a_0(t_0) = C(\theta_{HC})x(t_0) \quad (B.58)$$

The 2<sup>nd</sup> term in (2.29) is:

$$a_1(t_0) = C(\theta_{HC})\dot{x}(t_0) \quad (B.59)$$

Replacing  $\dot{x}(t_0) = A(\theta_{HA})x(t_0) + B(\theta_{HB})\mu$  from (2.23) in (B.59), we get:

$$a_1(t_0) = C(\theta_{HC})\left(A(\theta_{HA})x(t_0) + B(\theta_{HB})\mu\right) \quad (B.60)$$

Replacing  $C(\theta_{HC}) = B^T Q$  in (B.60) using the PCH systems properties, it results:

$$a_1(t_0) = \left[ \begin{matrix} B^T & Q \end{matrix} \right] (\theta_{HC}) \left( A(\theta_{HA})x + B(\theta_{HB})\mu \right) \quad (B.61)$$

Using (B.61), the 3<sup>rd</sup> element of the power series from (2.23),  $a_2(t_0)$  is:

$$a_2(t_0) = \dot{a}_1(t_0) = C(\theta_{HC}) \left( A(\theta_{HA}) \dot{x} + B(\theta_{HB}) \dot{u} \right) \quad (\text{B.62})$$

Developing further (B.62), the relation becomes:

$$a_2(t_0) = C(\theta_{HC}) \left( A(\theta_{HA}) \left( A(\theta_{HA}) x + B(\theta_{HB}) u \right) + B(\theta_{HB}) \dot{u} \right) \quad (\text{B.63})$$

The 4<sup>th</sup> element of the power series,  $a_3(t_0)$  from (2.29), has the following form:

$$a_3(t_0) = C(\theta_{HC}) \left\{ x^{(3)}(t_0) \right\} \quad (\text{B.64})$$

Using (B.63),(B.64) becomes:

$$a_3(t_0) = \left( B^T Q \right) (\theta_{HC}) A(\theta_{HA}) \left( A(\theta_{HA}) \left( A(\theta_{HA}) x + B(\theta_{HB}) u \right) + B(\theta_{HB}) \dot{u} \right) \quad (\text{B.65})$$

Computing further the 4<sup>th</sup> element of the power-series expansion from (B.65), it becomes:

$$a_3(t_0) = \left( B^T Q \right) (\theta_{HC}) \left( A^2(\theta_{HA}) \dot{x} + A(\theta_{HA}) B(\theta_{HB}) \dot{u} + B(\theta_{HB}) \ddot{u} \right) \quad (\text{B.66})$$

Using (2.23) in (B.66), it results:

$$a_3(t_0) = \left( B^T Q \right) (\theta_{HC}) \left\{ \begin{aligned} & A^3(\theta_{HA}) x + A^2(\theta_{HA}) B(\theta_{HB}) u + \\ & + A(\theta_{HA}) B(\theta_{HB}) \dot{u} + B(\theta_{HB}) \ddot{u} \end{aligned} \right\} \quad (\text{B.67})$$

The 5<sup>th</sup> element of the power-series expansion, has the following form:

$$a_4(t_0) = C(\theta_{HC}) \left\{ x^{(4)}(t_0) \right\} \quad (\text{B.68})$$

Developing (B.68) using (B.67), it results:

$$a_4(t_0) = \left( B^T Q \right) (\theta_{HC}) \left\{ \begin{aligned} & A^3(\theta_{HA}) \dot{x} + A^2(\theta_{HA}) B(\theta_{HB}) \dot{u} + \\ & + A(\theta_{HA}) B(\theta_{HB}) \ddot{u} + B(\theta_{HB}) \ddot{u} \end{aligned} \right\} \quad (\text{B.69})$$

Replacing the 1<sup>st</sup> derivate of the states,  $\dot{x}$  from (2.23), (B.69) becomes:

$$a_4(t_0) = \left( B^T Q \right) (\theta_{HC}) \left\{ \begin{aligned} & A^4(\theta_{HA}) x + A^3(\theta_{HA}) B(\theta_{HB}) u + A^2(\theta_{HA}) B(\theta_{HB}) \dot{u} + \\ & + A(\theta_{HA}) B(\theta_{HB}) \ddot{u} + B(\theta_{HB}) \ddot{u} \end{aligned} \right\} \quad (\text{B.70})$$

Using (B.58),(B.61),(B.63), (B.67) and (B.70), a general form of the power series from (2.29) can be deduced, in the lossless case of PCH systems:

$$a_k(t_0) = \left( B^T Q \right) (\theta_{HC}) \left\{ A^k(\theta_{HA}) x + A^{k-1}(\theta_{HA}) B(\theta_{HB}) u + \dots + A^0(\theta_{HA}) B(\theta_{HB}) u^{(k-1)} \right\} \quad (\text{B.71})$$

Developing further the right part of (B.71), the following relation results:

$$a_k(t_0) = \left( B^T Q \right) (\theta_{HC}) \left\{ A^k(\theta_{HA}) x + \sum_{i=0}^{k-1} A^i(\theta_{HA}) B(\theta_{HB}) u^{(k-1-i)} \right\} \quad (\text{B.72})$$

In the above equation, the term  $u^{(k-1-i)}$  represents the the  $(k-1-i)^{th}$  order derivate of the input signal  $u$ . When  $(k-1-i) = 0$ , the term  $u^{(k-1-i)}$  from (B.72) is  $u$ . Replacing the matrix  $A(\theta_{HA})$  with  $JQ$  in the lossless case, it results:

$$a_k(t_0) = \left( B^T Q \right) \left( \theta_{HC} \right) \left\{ (JQ)^k (\theta_{HA})_x + \sum_{i=0}^{k-1} (JQ)^i (\theta_{HA})_B (\theta_{HB})_u^{(k-1-i)} \right\} \quad (B.73)$$

which is equivalent to (2.30). In order to express in a different way (B.73), the term  $Q_A^k$ , will be written in a general form for  $k \geq 1$ . For  $k = 1$ , the following relation can be written for a lossless PCH system:

$$Q_A = -A^T Q \quad (B.74)$$

using the PCH systems properties. For  $k = 2$ , the following relation results:

$$Q_A^2 = \left( \left( Q^T \left( -J^T \right) \right) \left( Q^T \left( -J^T \right) \right) Q \right) \quad (B.75)$$

(B.75) can be equivalently written as follows:

$$Q_A^2 = (-1)^2 (JQ)^T (JQ)^T Q \quad (B.76)$$

Replacing  $A = JQ$ , in the lossless case, it results:

$$Q_A^2 = (-1)^2 \left( A^T \right)^2 Q \quad (B.77)$$

For  $k = 3$ , the term  $Q_A^k$  can be written in the following form:

$$Q_A^3 = Q(JQ)(JQ)(JQ) \quad (B.78)$$

(B.78) can be also expressed as follows:

$$Q_A^3 = (QJ)(QJ)(QJ)Q \quad (B.79)$$

Using PCH systems properties, (B.79) becomes:

$$Q_A^3 = \left( Q^T \left( -J^T \right) \right) \left( Q^T \left( -J^T \right) \right) \left( Q^T \left( -J^T \right) Q \right) \quad (B.80)$$

which is equivalent to:

$$Q_A^3 = (-1)^3 \left( A^T \right)^3 Q \quad (B.81)$$

Using the results from (B.74), (B.77) and (B.81), a general form can be deduced for  $Q_A^k$ :

$$Q_A^k = (-1)^k \left( A^T \right)^k Q \quad (B.82)$$

Using (B.82), (B.72) becomes:

$$a_k(t_0) = B^T \left( \theta_{HB} \right) \left\{ (-1)^k A^T \left( \theta_{HA} \right) Q_x + \sum_{i=0}^{k-1} (-1)^i \left( A^T \left( \theta_{HA} \right) \right)^i Q_B \left( \theta_{HB} \right)_u^{(k-1-i)} \right\} \quad (B.83)$$

The unknown parameters of a lossless PCH system, are contained in the matrices  $Q$  and  $B$ , and the dimensions of the unknown parameter space  $\theta_H$ , for  $Q$  and  $B$  matrices, are the following:

$$\begin{cases} n_{\theta Q} = \frac{n^2 + n}{2} \\ n_{\theta B} = nm \end{cases} \quad (B.84)$$

From (B.84), the total number of unknown parameters of a lossless PCH system results directly:

$${}^n\theta_H = {}^n\theta_Q + {}^n\theta_B \quad (\text{B.85})$$

### B.3.2 LOSSY PCH SYSTEMS

In the lossy case of PCH systems, the expression from (2.32), can be further rewritten in an equivalent form, by computing the terms  $QA^k$  in a general form. For  $k = 1$ ,  $QA^k$  becomes as follows:

$$QA = -\left(Q^T J^T - Q^T R^T\right)Q \quad (\text{B.86})$$

using the PCH systems properties. (B.86) can be also written as:

$$QA = -\left((JQ)^T - (RQ)^T\right)Q \quad (\text{B.87})$$

For an easier representation of (B.87), the following notations can be used:

$$\begin{cases} \text{def} \\ A_{loss} = JQ \\ \text{def} \\ A_{res} = RQ \end{cases} \quad (\text{B.88})$$

where  $A_{loss} \in R^{n \times n}$  and  $A_{res} \in R^{n \times n}$  correspond to the lossless respectively resistive matrices, that compose the matrix  $A \in R^{n \times n}$ , which defines a lossy PCH system. Using the notations from (B.88), (B.87) becomes:

$$QA = -\left(A_{loss}^T + A_{res}^T\right)Q \quad (\text{B.89})$$

(B.89) can be further written as:

$$QA = (-1)^1 \left(A_{loss} + A_{res}\right)^T Q \quad (\text{B.90})$$

The next term of  $QA^k$  for  $k = 2$ , has the form:

$$QA^2 = Q(J - R)Q(J - R)Q \quad (\text{B.91})$$

Using the PCH systems properties, (B.91) can be expressed as follows:

$$QA^2 = Q^T \left(-J^T - R^T\right)Q^T \left(-J^T - R^T\right)Q \quad (\text{B.92})$$

Using (B.88), (B.92) can be written as:

$$QA^2 = (-1)^2 \left(A_{loss}^T + A_{res}^T\right)^2 Q \quad (\text{B.93})$$

For  $k = 3$ , the term  $QA^k$  has the following form:

$$QA^3 = Q(J - R)Q(J - R)Q(J - R)Q \quad (\text{B.94})$$

(B.94) can be expressed as:

$$QA^3 = Q^T \left(-J^T - R^T\right)Q^T \left(-J^T - R^T\right)Q^T \left(-J^T - R^T\right)Q \quad (\text{B.95})$$

using the PCH systems properties. Using (B.88), (B.95) becomes:

$$QA^3 = (-1)^3 \left(A_{loss}^T + A_{res}^T\right)^3 Q \quad (\text{B.96})$$

Using (B.90), (B.93), (B.96), we can deduce the general term for  $QA^k$ :

$$QA^k = (-1)^k \left( A_{loss}^T + A_{res}^T \right)^k Q \quad (\text{B.97})$$

for the lossy case of PCH systems. Using (B.97), (2.32) becomes:

$$a_k(t_0) = B^T (\theta_{HB}) \left\{ \begin{array}{l} \left( (-1)^k \left( A_{loss}^T + A_{res}^T \right) (\theta_{HA}) \right)^k Q_x + \\ + \sum_{i=0}^{k-1} (-1)^i \left( A_{loss}^T + A_{res}^T \right) (\theta_{HA})^i Q_B (\theta_{HB}) \mu^{(k-1-i)} \end{array} \right\} \quad (\text{B.98})$$

In this case, the unknown parameters, are included in the matrices  $R \in R^{n \times n}$ ,  $Q \in R^{n \times n}$  and  $B \in R^{n \times m}$ . The dimensions of the unknown parameters  $\theta_H$ , for  $R$ ,  $Q$  and  $B$  matrices are the following:

$$\left\{ \begin{array}{l} n_{\theta R} = \frac{n^2 + n}{2} \\ n_{\theta Q} = \frac{n^2 + n}{2} \\ n_{\theta B} = nm \end{array} \right. \quad (\text{B.99})$$

From (B.99), the total number of unknown parameters of a lossy PCH system is:

$$n_{\theta H} = n_{\theta R} + n_{\theta Q} + n_{\theta B} = \left( n^2 + n + nm \right) \quad (\text{B.100})$$

For the lossy case of the linear PCH systems, the necessary system of equations that need to be solved, for the power-series expansion identifiability test, is the following:

$$a_k(t_0) = B^T (\theta_{HB}) \left\{ \begin{array}{l} (-1)^k \left( A_{loss} + A_{res} \right)^T (\theta_{HA})^k Q_x + \\ + \sum_{i=0}^{k-1} (-1)^i \left( A_{loss} + A_{res} \right)^T (\theta_{HA})^i Q_B (\theta_{HB}) \mu^{(k-1-i)} \end{array} \right\} \quad (\text{B.101})$$

which represents also (2.33).



## Appendix C: STRUCTURAL IDENTIFIABILITY EXAMPLES

### C.1 OBSERVABILITY AND CONTROLABILITY ANALYSIS

#### C.1.1 LOSSLESS PCH SYSTEMS

In the case of the lossless PCH systems, represented by an LC circuit, the matrix terms of the state-space system, represented in the observable canonical form from (2.6), are the following:

$$A_{obs} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\left(\frac{1}{CL_1} + \frac{1}{CL_2}\right) & 0 \end{bmatrix}, B_{obs} = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ -\frac{1}{CL_1^2} \end{bmatrix}, C_{obs} = [1 \quad 0 \quad 0] \quad (C.1)$$

where  $A_{obs}$ ,  $B_{obs}$  and  $C_{obs}$ , were computed using (2.10). The transformation matrix  $T_{obs}$  from (2.5) has the following form, for the LC circuit:

$$T_{obs} = \begin{pmatrix} 0 & \frac{1}{L_1} & 0 \\ -\frac{1}{L_1 C} & 0 & 0 \\ 0 & -\frac{1}{L_1^2 C} & \frac{1}{L_1 L_2 C} \end{pmatrix} \quad (C.2)$$

The structural identifiability matrix from (2.16), computed for the LC circuit, becomes:

$$R_{ID} = \begin{bmatrix} -\frac{1}{CL_1} & 0 & \frac{1}{(CL_1)^2} \\ 0 & \frac{1}{L_1^2} & -\frac{1}{C^2 L_1^3} \\ 0 & 0 & \frac{1}{C^2 L_1^2 L_2} \end{bmatrix} \quad (C.3)$$

The determinant of this identifiability matrix is:

$$\det R_{ID} = -\frac{1}{C^3 L_1^5 L_2} \quad (C.4)$$

This determinant is different from 0, when the following relations are satisfied:

$$\begin{cases} C \neq \pm\infty \\ L_1 \neq \pm\infty \\ L_2 \neq \pm\infty \end{cases} \quad (C.5)$$

which represents also (2.67).

### C.1.2 LOSSY PCH SYSTEMS

In the lossy case of PCH systems, represented by the capacitor microphone circuit described in section 1.3.7, the observable transformation matrix  $T_{obs}$  has the following form:

$$T_{obs} = [T_{obs\ 1} \quad T_{obs\ 2}] \quad (C.6)$$

where:

$$T_{obs\ 1} = \begin{bmatrix} 0 & \frac{1}{m} & 0 \\ -\frac{k}{m} & -\frac{c}{m^2} & 0 \\ \frac{ck}{m^2} & -\frac{k}{m^2} + \frac{c^2}{m^3} & 0 \end{bmatrix} \quad (C.7)$$

represents the transformation matrix corresponding to the first input  $B_1 = [0 \quad 1 \quad 0]^T$  and:

$$T_{obs\ 2} = \begin{bmatrix} 0 & 0 & \frac{1}{RC} \\ 0 & 0 & -\frac{1}{(RC)^2} \\ 0 & 0 & \frac{1}{(RC)^3} \end{bmatrix} \quad (C.8)$$

represents the transformation matrix corresponding to the 2<sup>nd</sup> input  $B_2 = \left[0 \quad 0 \quad \frac{1}{R}\right]^T$ .

From (C.7) and (C.8), it can be observed that the observable transformation matrices  $T_{obs\ 1}, T_{obs\ 2}$  corresponding to the inputs  $B_1, B_2$  are rank deficient and in conclusion, the system is not observable. In the case of lossy PCH systems, represented by the DC motor, the matrix elements of the observable canonical form from (2.13) are:

$$A_{obs} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2J_E} - \frac{Rb}{LJ_E} & -\frac{R}{L} - \frac{b}{J_E} \end{bmatrix}, \quad B_{obs} = \begin{bmatrix} \frac{1}{L} \\ -\frac{R}{L^2} \end{bmatrix}, \quad C_{obs} = [1 \quad 0], \quad (C.9)$$

The observable canonical form transformation matrix is:

$$T_{obs} = \begin{bmatrix} \frac{1}{L} & \frac{R}{L^2} \\ -\frac{R}{L^2} & -\frac{1}{LJ_E} \end{bmatrix} \quad (C.10)$$

The structural identifiability matrix from (2.21), applied to the DC motor is:

$$R_{ID} = \begin{bmatrix} \frac{1}{L} & -\frac{R}{L^2} \\ -\frac{R}{L^2} & \frac{1}{L^2 J_E} + \frac{R^2}{L^3} \end{bmatrix} \quad (C.11)$$

The determinant of the above identifiability matrix, is the following:

$$\det R_{ID} = -\frac{1}{L^3 J_E} \quad (\text{C.12})$$

This determinant is different from 0 , when the following relations are satisfied:

$$\begin{cases} L \neq \pm\infty \\ J_E \neq \pm\infty \end{cases} \quad (\text{C.13})$$

which represents also the result from (2.69).

## **C.2 DIRECT IDENTIFIABILITY**

### **C.2.1 LOSSLESS PCH SYSTEMS**

For the direct identifiability test applied to the LC circuit, (2.70) becomes further:

$$\left\{ \begin{array}{l} \left[ \begin{array}{ccc} 0 & \frac{1}{L_{11}} - \frac{1}{L_{12}} & -\frac{1}{L_{21}} + \frac{1}{L_{22}} \\ -\frac{1}{C_{11}} + \frac{1}{C_{12}} & 0 & 0 \\ \frac{1}{C_{11}} - \frac{1}{C_{12}} & 0 & 0 \end{array} \right] x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \left[ \begin{array}{ccc} 0 & \frac{1}{L_{11}} - \frac{1}{L_{12}} & 0 \end{array} \right] x = 0 \end{array} \right. \quad (\text{C.14})$$

by moving the right elements, to the left. From (C.14), we observe that the relations hold when:

$$\begin{cases} \frac{1}{L_{11}} - \frac{1}{L_{12}} = 0 \\ \frac{1}{C_{11}} - \frac{1}{C_{12}} = 0 \\ \frac{1}{L_{21}} - \frac{1}{L_{22}} = 0 \end{cases} \quad (\text{C.15})$$

Using (C.15), it results:

$$\begin{cases} L_{11} = L_{12} \\ C_{11} = C_{12} \\ L_{21} = L_{22} \end{cases} \quad (\text{C.16})$$

which represents the result from (2.71).

### C.2.2 LOSSY PCH SYSTEMS

For the lossy case represented by a DC motor, from section 1.3.7, the application of the direct identifiability test, results in the following relations:

$$\left\{ \begin{array}{l} \left[ \begin{array}{cc} -\frac{R_{11}}{L_{11}} + \frac{R_{12}}{L_{12}} & -\frac{1}{J_{E11}} + \frac{1}{J_{E12}} \\ \frac{1}{L_{11}} - \frac{1}{L_{12}} & -\frac{b_{11}}{J_{E11}} + \frac{b_{12}}{J_{E12}} \end{array} \right] x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left[ \begin{array}{cc} \frac{1}{L_{11}} - \frac{1}{L} & 0 \end{array} \right] x \end{array} \right. \quad (C.17)$$

by moving the right elements to the left in (2.72). We can write then:

$$\left\{ \begin{array}{l} \frac{R_{11}}{L_{11}} = \frac{R_{12}}{L_{12}}, \frac{1}{J_{E11}} = \frac{1}{J_{E12}} \\ \frac{1}{L_{11}} = \frac{1}{L_{12}}, \frac{b_{11}}{J_{E11}} = \frac{b_{12}}{J_{E12}} \end{array} \right. \quad (C.18)$$

From the 2<sup>nd</sup> and 3<sup>rd</sup> relation of (C.18), it results that:

$$\left\{ \begin{array}{l} J_{E11} = J_{E12} \\ L_{11} = L_{12} \end{array} \right. \quad (C.19)$$

Using (C.19) in (C.18), we can write then:

$$\left\{ \begin{array}{l} R_{11} = R_{12} \\ b_{11} = b_{12} \end{array} \right. \quad (C.20)$$

From (C.19) and (C.20), the following relations result:

$$\left\{ \begin{array}{l} J_{E11} = J_{E12}, L_{11} = L_{12} \\ R_{11} = R_{12}, b_{11} = b_{12} \end{array} \right. \quad (C.21)$$

which represents also (2.73). This result concludes that the unknown parameters of the DC motor  $J_E, L, R, b \in \Theta$ , are structurally identifiable for this test.

## **C.3 POWER SERIES EXPANSION IDENTIFIABILITY**

### C.3.1 LOSSLESS PCH SYSTEMS

In the lossless case of PCH systems, represented by the LC circuit, the 1<sup>st</sup> element of the power series expansion identifiability test, is computed using (2.29) as follows:

$$a_0(t_0) = \frac{1}{L_1} \quad (C.22)$$

and using (2.74), we conclude that  $L_1 = 1$ . Using (2.29), the 2<sup>nd</sup> element of the power series expansion, can be written as follows:

$$a_1(t_0) = B^T Q(Ax(t_0) + Bu(t_0)) \quad (C.23)$$

Replacing all the known elements in (C.23), the following relation can be written:

$$a_1(t_0) = -\frac{1}{C} + 2 \quad (\text{C.24})$$

for the 2<sup>nd</sup> element of the power series expansion. Using (2.29), the 1<sup>st</sup> derivate of  $y(t)$  at  $t_0 = 1$  is  $y'(t_0) = 1$  and using also (C.24) and (2.30), the value of the capacity results:

$$C = 1 \quad (\text{C.25})$$

Using (2.30), the 3<sup>rd</sup> element of the power series expansion results as:

$$a_2(t_0) = B^T Q \left( A^2 x(t_0) + ABu(t_0) \right) \quad (\text{C.26})$$

Replacing all the known elements in (C.26), the relation becomes after computations:

$$a_2(t_0) = \left[ -1 + \frac{1}{L_2} \right] = 0 \quad (\text{C.27})$$

Using (2.29), the 2<sup>nd</sup> derivate of  $y(t)$  is 0, and from (2.30) and (C.27), the following result:

$$L_2 = 1 \quad (\text{C.28})$$

is obtained for  $L_2$ . Using (2.30), the 4<sup>th</sup> element of the power-series expansion has the following form:

$$a_3(t_0) = B^T Q \left( A^3 x(t_0) + A^2 Bu(t_0) \right) \quad (\text{C.29})$$

Replacing all the known elements in (C.29), the following relation is obtained:

$$a_3(t_0) = 0 \quad (\text{C.30})$$

Using (2.29), (2.74) and (C.30), the following result:

$$a_3(t_0) = y'''(t_0) = 0 \quad (\text{C.31})$$

can be calculated. Using (2.30), the 5<sup>th</sup> element of the power series expansion, has the following form:

$$a_4(t_0) = B^T Q \left( A^4 x(t_0) + A^3 Bu(t_0) \right) \quad (\text{C.32})$$

Replacing all the known elements in (C.32), the following relation results:

$$a_4(t_0) = 0 \quad (\text{C.33})$$

Using (2.29), (2.74) and (C.33), the following relation:

$$a_4(t_0) = y^{(4)}(t_0) = 0 \quad (\text{C.34})$$

can be written. From (C.27), (C.31), (C.34), the relation (2.76) is deduced for the power series expansion identifiability test, applied to the LC circuit.

## Appendix D: Difference between two PCH systems

### Appendix D.1 Lossless PCH systems

This appendix proves in detail the basic properties of the difference lossless PCH system defined between two nonlinear systems (  $A$  and  $B$  ) in rapport with the power energies. The total energy for two lossless PCH systems  $A$  and  $B$  is conserved and thus, we can write the following relations:

$$\begin{cases} P_{SA} + P_{CA} = 0 \\ P_{SB} + P_{CB} = 0 \end{cases} \quad (D.1)$$

where  $P_{SA}, P_{CA}$  and  $P_{SB}, P_{CB}$  represent the power energies corresponding to the storage and control ports. Using (1.24), we can write:

$$\begin{cases} \frac{dH_A}{dt} = -e_{SA}^T f_{SA} = e_{CA}^T f_{CA} \\ \frac{dH_B}{dt} = -e_{SB}^T f_{SB} = e_{CB}^T f_{CB} \end{cases} \quad (D.2)$$

for two lossless PCH systems (  $A$  and  $B$  ). The power energies corresponding to the storage and control ports, for systems  $A$  and  $B$  , can be written as:

$$\begin{cases} P_{SA} = e_{SA}^T f_{SA}, P_{CA} = e_{CA}^T f_{CA} \\ P_{SB} = e_{SB}^T f_{SB}, P_{CB} = e_{CB}^T f_{CB} \end{cases} \quad (D.3)$$

using flows and efforts. Subtracting both relations in (D.2), it results:

$$\frac{d(H_A - H_B)}{dt} = -e_{SA}^T f_{SA} + e_{SB}^T f_{SB} = e_{CA}^T f_{CA} - e_{CB}^T f_{CB} \quad (D.4)$$

(D.2) can be also written as:

$$\begin{cases} \frac{dH_A}{dt} = \left( \frac{\partial H_A}{\partial x_A} (x_A) \right)^T \frac{dx_A}{dt} = -e_{SA}^T f_{SA} \\ \frac{dH_B}{dt} = \left( \frac{\partial H_B}{\partial x_B} (x_B) \right)^T \frac{dx_B}{dt} = -e_{SB}^T f_{SB} \end{cases} \quad (D.5)$$

using the chain rule differentiation. The explicit form of the two nonlinear lossless PCH systems (  $A$  and  $B$  ), can be written as:

$$\left\{ \begin{array}{l} \frac{dx_A}{dt} = J_A(x_A) \frac{\partial H_A}{\partial x_A}(x_A) + B_A(x_A) \mu_A \\ y_A = B_A^T(x_A) \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{dx_B}{dt} = J_B(x_B) \frac{\partial H_B}{\partial x_B}(x_B) + B_B(x_B) \mu_B \\ y_B = B_B^T(x_B) \frac{\partial H_B}{\partial x_B}(x_B) \end{array} \right. \quad (D.6)$$

The energy difference for the storage and control ports of systems  $A$  and  $B$ , can be written as follows:

$$\left\{ \begin{array}{l} e_{SA}^T f_{SA} - e_{SB}^T f_{SB} = [e_{SA} \quad e_{SB}]^T \begin{bmatrix} f_{SA} \\ -f_{SB} \end{bmatrix} \\ e_{CA}^T f_{CA} - e_{CB}^T f_{CB} = [e_{CA} \quad e_{CB}]^T \begin{bmatrix} f_{CA} \\ -f_{CB} \end{bmatrix} \end{array} \right. \quad (D.7)$$

The flows  $f_{SA}$ ,  $f_{SB}$  of systems  $A$  and  $B$ , can be expressed also as:

$$\left\{ f_{SA} = -\frac{dx_A}{dt}, f_{SB} = -\frac{dx_B}{dt} \right. \quad (D.8)$$

The power energies corresponding to systems  $A$  and  $B$ , can be written as follows using (D.6) and (D.8) for the storage port:

$$\left\{ \begin{array}{l} e_{SA}^T f_{SA} = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T \left( -J_A(x_A) \frac{\partial H_A}{\partial x_A}(x_A) - B_A(x_A) \mu_A \right) \\ e_{SB}^T f_{SB} = \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T \left( -J_B(x_B) \frac{\partial H_B}{\partial x_B}(x_B) - B_B(x_B) \mu_B \right) \end{array} \right. \quad (D.9)$$

Equivalently we can write:

$$\left\{ \begin{array}{l} e_{SA}^T f_{SA} = -\left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T J_A(x_A) \frac{\partial H_A}{\partial x_A}(x_A) - \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A \\ e_{SB}^T f_{SB} = -\left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T J_B(x_B) \frac{\partial H_B}{\partial x_B}(x_B) - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \end{array} \right. \quad (D.10)$$

The 1<sup>st</sup> terms in both relations of (D.10) are reduced by skew-symmetry of  $J_A(x_A)$ ,  $J_B(x_B)$ .

Then it is straightforward:

$$\left\{ \begin{array}{l} e_{SA}^T f_{SA} = -\left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A \\ e_{SB}^T f_{SB} = -\left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \end{array} \right. \quad (D.11)$$

For the control port power energies of systems  $A$  and  $B$ , we can write :

$$\begin{cases} e_{CA}^T f_{CA} = y_A^T u_A = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) u_A \\ e_{CB}^T f_{CB} = y_B^T u_B = \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) u_B \end{cases} \quad (D.12)$$

Summing up the power energies from (D.11) and (D.12), the power conservation law is satisfied. At the next step, a lossless PCH system is defined, which has as energy corresponding to the ports, the difference between the energies of systems  $A$  and  $B$ . Then the following relations results for the power energies of the new defined lossless difference PCH system using (D.7), (D.11) and (D.12):

$$\begin{cases} \tilde{P}_S = \tilde{e}_S^T \tilde{f}_S = e_{SA}^T f_{SA} - e_{SB}^T f_{SB} = \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) u_B - \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) u_A \\ \tilde{P}_C = \tilde{e}_C^T \tilde{f}_C = e_{CA}^T f_{CA} - e_{CB}^T f_{CB} = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) u_A - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) u_B \end{cases} \quad (D.13)$$

with the following notations:

$$\begin{cases} \tilde{e}_S^T \stackrel{def}{=} \begin{bmatrix} e_{SA} & e_{SB} \end{bmatrix}^T, \tilde{f}_S^T \stackrel{def}{=} \begin{bmatrix} f_{SA} & -f_{SB} \end{bmatrix}^T \\ \tilde{e}_C^T \stackrel{def}{=} \begin{bmatrix} e_{CA} & e_{CB} \end{bmatrix}^T, \tilde{f}_C^T \stackrel{def}{=} \begin{bmatrix} f_{CA} & -f_{CB} \end{bmatrix}^T \end{cases} \quad (D.14)$$

Summing both relations in (D.13), the total power conservation law for the new defined *difference PCH system*, is conserved:

$$\tilde{P}_S + \tilde{P}_C = 0 \quad (D.15)$$

For the next step, an explicit form is proposed for the difference lossless PCH system, as follows:

$$\begin{cases} \dot{\tilde{x}} \stackrel{def}{=} \begin{bmatrix} \frac{dx_A}{dt} \\ \frac{dx_B}{dt} \end{bmatrix} = \begin{bmatrix} J_A(x_A) & 0 \\ 0 & -J_B(x_B) \end{bmatrix} \begin{bmatrix} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{bmatrix} + \begin{bmatrix} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix} \\ \tilde{y} \stackrel{def}{=} \begin{bmatrix} y_A \\ y_B \end{bmatrix} = \begin{bmatrix} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{bmatrix}^T \begin{bmatrix} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{bmatrix} \end{cases} \quad (D.16)$$

for which, (D.13) must be satisfied. Using (D.16), the power energies corresponding to the storage and control ports, can be defined as:



$$\left\{ \begin{array}{l} \overline{P}_S \stackrel{\text{def}}{=} \overline{e}_S^T \overline{f}_S = \begin{bmatrix} \frac{\partial H_A}{\partial x_A}^T(x_A) & \frac{\partial H_B}{\partial x_B}^T(x_B) \end{bmatrix} \begin{bmatrix} -\frac{dx_A}{dt} \\ \frac{dx_B}{dt} \end{bmatrix} \\ \overline{P}_C \stackrel{\text{def}}{=} \overline{e}_C^T \overline{f}_C = \begin{bmatrix} \frac{\partial H_A}{\partial x_A}^T(x_A) & \frac{\partial H_B}{\partial x_B}^T(x_B) \end{bmatrix} \begin{bmatrix} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix} \end{array} \right. \quad (\text{D.17})$$

Using (D.16), (D.17) becomes:

$$\left\{ \begin{array}{l} \overline{e}_S^T \overline{f}_S = \begin{bmatrix} \frac{\partial H_A}{\partial x_A}^T(x_A) & \frac{\partial H_B}{\partial x_B}^T(x_B) \end{bmatrix} \begin{bmatrix} -J_A(x_A) & 0 \\ 0 & J_B(x_B) \end{bmatrix} \begin{bmatrix} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{bmatrix} + \begin{bmatrix} -B_A(x_A) & 0 \\ 0 & B_B(x_B) \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix} \\ \overline{e}_C^T \overline{f}_C = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \end{array} \right. \quad (\text{D.18})$$

Making the possible simplifications in (D.18) due to the skew-symmetry of the  $J_A(x_A), J_B(x_B)$  matrices, it results:

$$\left\{ \begin{array}{l} \overline{P}_S = \overline{e}_S^T \overline{f}_S = -\left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A + \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \\ \overline{P}_C = \overline{e}_C^T \overline{f}_C = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \end{array} \right. \quad (\text{D.19})$$

Comparing (D.19) with (D.13), we get the equalities:

$$\left\{ \begin{array}{l} \tilde{e}_S^T \tilde{f}_S = \overline{e}_S^T \overline{f}_S = -\left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A + \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \\ \tilde{e}_C^T \tilde{f}_C = \overline{e}_C^T \overline{f}_C = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \end{array} \right. \quad (\text{D.20})$$

between the difference of the power energies of the ports, for systems  $A$  and  $B$  and the power energies for the new defined difference lossless PCH system from (D.16). Using (D.20), we can also write the power balance relation of the proposed difference PCH system, as follows:

$$\overline{e}_S^T \overline{f}_S + \overline{e}_C^T \overline{f}_C = 0 \quad (\text{D.21})$$

or:

$$\overline{P}_S + \overline{P}_C = 0 \quad (\text{D.22})$$

For the next step, we wish to prove the following definition of a Dirac structure, for the proposed difference lossless PCH system from (D.16).

**Definition D.1**

A Dirac structure on  $\overline{F} \times \overline{F}^*$  is a subspace  $\overline{D}_{A \perp B} \subset \overline{F} \times \overline{F}^*$ , such that:

$$\text{i) } \quad \langle \overline{e} | \overline{f} \rangle = 0$$

$$\text{ii) } \quad \dim \overline{D}_{A \perp B} = \dim \overline{F}$$

The first property, is satisfied due to the power-conservation law of the energies given in (D.21) or (D.22). This property can be also written as follows:

$$\langle (\overline{e}_S, \overline{e}_C), (\overline{f}_S, \overline{f}_C) \rangle = 0 \quad (\text{D.23})$$

where the following notations:

$$\overline{e} \stackrel{\text{def}}{=} (\overline{e}_S, \overline{e}_C) \quad (\text{D.24})$$

$$\overline{f} \stackrel{\text{def}}{=} (\overline{f}_S, \overline{f}_C) \quad (\text{D.25})$$

are introduced for the lossless case. For the second statement of the above definition, an equivalent characterization of a Dirac structure is given from the total space of flows and efforts variables  $\overline{F} \times \overline{F}^*$ , by introducing a bilinear form  $\ll, \gg$  on the space  $\overline{F} \times \overline{F}^*$ , as:

$$\overline{B}_F \stackrel{\text{def}}{=} \ll (\overline{f}^a, \overline{e}^a), (\overline{f}^b, \overline{e}^b) \gg := \langle \overline{e}^a | \overline{f}^b \rangle + \langle \overline{e}^b | \overline{f}^a \rangle \quad (\text{D.26})$$

with  $(\overline{f}^a, \overline{e}^a), (\overline{f}^b, \overline{e}^b) \in \overline{F} \times \overline{F}^*$ . The space of flows is defined by  $\overline{F} = \overline{F}_S \times \overline{F}_C$ , while the space of efforts  $\overline{F}^*$  is defined by  $\overline{E}_S \times \overline{E}_C$ . For the next step, we wish to prove the following proposition for the proposed difference lossless PCH system from (D.16).

**Proposition D.1**

A constant Dirac structure on  $\overline{F} \times \overline{F}^*$  is a subspace  $\overline{D}_{A \perp B} \subset \overline{F} \times \overline{F}^*$  such that:

$$\overline{D}_{A \perp B} = \overline{D}^\perp_{A \perp B} \quad (\text{D.27})$$

where  $\perp$  denotes the orthogonal complement with respect to the bilinear form:

$$\ll, \gg .$$

**Proof:**

Let  $\overline{D}_{A \perp B}$  satisfy (D.27). Then for every  $(\overline{f}, \overline{e}) \in \overline{D}_{A \perp B}$ ,

$$\ll (\overline{f}, \overline{e}), (\overline{f}, \overline{e}) \gg = \langle \overline{e} | \overline{f} \rangle + \langle \overline{e} | \overline{f} \rangle \quad (\text{D.28})$$

In the above relation, the flow  $\overline{f}$  and effort  $\overline{e}$  are defined for the proposed difference lossless PCH system in (D.24) and (D.25). Using (D.28), we can write further:

$$\langle (\overline{e}_S, \overline{e}_C) | (\overline{f}_S, \overline{f}_C) \rangle \quad (\text{D.29})$$

expressing the flows and efforts of each port. By the non-degeneracy of the bilinear form  $\langle\langle, \rangle\rangle$  :

$$\dim \bar{D}^\perp_{A \perp B} = \dim \left( \bar{F} x \bar{F}^* \right) - \dim \bar{D}_{A \perp B} = 2 \dim \bar{F} - \dim \bar{D}_{A \perp B} \quad (\text{D.30})$$

Using (D.27), it results that  $\dim \bar{D}_{A \perp B} = \dim \bar{F}$ . We also assume that  $\bar{D}_{A \perp B}$  is a Dirac structure, satisfying the properties i) and ii) of **Definition D.1**. Let  $(\bar{f}^a, \bar{e}^a), (\bar{f}^b, \bar{e}^b)$  be any flow and effort vectors contained in  $\bar{D}_{A \perp B}$ . Using the linearity property, we can write:

$(\bar{f}^a + \bar{f}^b, \bar{e}^a + \bar{e}^b) \in \bar{D}_{A \perp B}$ . Using property i) of **Definition D.1**:

$$\begin{aligned} 0 &= \left\langle \bar{e}^a + \bar{e}^b \middle| \bar{f}^a + \bar{f}^b \right\rangle = \left\langle \bar{e}^a \middle| \bar{f}^b \right\rangle + \left\langle \bar{e}^b \middle| \bar{f}^a \right\rangle + \\ &+ \left\langle \bar{e}^a \middle| \bar{f}^a \right\rangle + \left\langle \bar{e}^b \middle| \bar{f}^b \right\rangle = \left\langle \bar{e}^a \middle| \bar{f}^b \right\rangle + \left\langle \bar{e}^b \middle| \bar{f}^a \right\rangle \end{aligned} \quad (\text{D.31})$$

Due to the property i) of the **Definition D.1**:

$$\left\langle \bar{e}^a \middle| \bar{f}^a \right\rangle = \left\langle \bar{e}^b \middle| \bar{f}^b \right\rangle = 0 \quad (\text{D.32})$$

Using (D.24) and (D.25), (D.32) can be written as:

$$\left\langle \left( \bar{e}_S^a, \bar{e}_C^a \right) \middle| \left( \bar{f}_S^a, \bar{f}_C^a \right) \right\rangle = \left\langle \left( \bar{e}_S^b, \bar{e}_C^b \right) \middle| \left( \bar{f}_S^b, \bar{f}_C^b \right) \right\rangle = 0 \quad (\text{D.33})$$

This implies also that  $\bar{D}_{A \perp B} \subset \bar{D}^\perp_{A \perp B}$ . Using the 2<sup>nd</sup> property of **Definition D.1** and  $\dim \bar{D}^\perp_{A \perp B} = 2 \dim \bar{F} - \dim \bar{D}_{A \perp B}$ , it follows that:

$$\dim \bar{D}_{A \perp B} = \dim \bar{D}^\perp_{A \perp B} \quad (\text{D.34})$$

yielding that  $\bar{D}_{A \perp B} = \bar{D}^\perp_{A \perp B}$ . For the difference lossless PCH system proposed in (D.16), between two systems ( $A$  and  $B$ ) in rapport with the power energies, the following notation is introduced for the skew-symmetric matrix of the new defined system:

$$\bar{J}_{A \perp B}(\bar{x}_{A \perp B}) \stackrel{\text{def}}{=} \begin{pmatrix} J_A(x_A) & 0 \\ 0 & -J_B(x_B) \end{pmatrix} \quad (\text{D.35})$$

Next we check the skew-symmetry property of  $\bar{J}_{A \perp B}(\bar{x}_{A \perp B})$  matrix and we can write:

$$\bar{J}_{A \perp B}^T(\bar{x}_{A \perp B}) = \begin{pmatrix} J_A^T(x_A) & 0 \\ 0 & -J_B^T(x_B) \end{pmatrix} \quad (\text{D.36})$$

Using the skew-symmetry of the  $J_A(x_A)$  and  $J_B(x_B)$  matrices, (D.36) becomes:

$$\bar{J}_{A \perp B}^T(\bar{x}_{A \perp B}) = \begin{pmatrix} -J_A(x_A) & 0 \\ 0 & J_B(x_B) \end{pmatrix} \quad (\text{D.37})$$

From (D.35) and (D.37), it results also the skew-symmetry of  $\bar{J}_{A \perp B}(\bar{x}_{A \perp B})$  :

$$\bar{J}_{A \perp B}(\bar{x}_{A \perp B}) = -\bar{J}_{A \perp B}^T(x_{A \perp B}) = \begin{pmatrix} J_A(x_A) & 0 \\ 0 & -J_B(x_B) \end{pmatrix} \quad (\text{D.38})$$

Further we introduce the notation:

$$\bar{Q}_{A \perp B}(\bar{x}_{A \perp B}) \stackrel{\text{def}}{=} \begin{pmatrix} Q_A(x_A) & 0 \\ 0 & Q_B(x_B) \end{pmatrix} \quad (\text{D.39})$$

for which we compute the matrix transpose and we obtain:

$$\bar{Q}_{A \perp B}^T(\bar{x}_{A \perp B}) = \begin{pmatrix} Q_A^T(x_B) & 0 \\ 0 & Q_B^T(x_B) \end{pmatrix} \quad (\text{D.40})$$

From the positive definiteness of the  $Q_A(x_A), Q_B(x_B)$  matrices, we can write also:

$$\bar{Q}_{A \perp B}(\bar{x}_{A \perp B}) = \bar{Q}_{A \perp B}^T(\bar{x}_{A \perp B}) = \begin{pmatrix} Q_A(x_A) & 0 \\ 0 & Q_B(x_B) \end{pmatrix} \geq 0 \quad (\text{D.41})$$

In the above relations, the following notation has been adopted for the states:

$$\bar{x}_{A \perp B} \stackrel{\text{def}}{=} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \quad (\text{D.42})$$

inputs:

$$\bar{u}_{A \perp B} \stackrel{\text{def}}{=} \begin{pmatrix} u_A \\ u_B \end{pmatrix} \quad (\text{D.43})$$

matrix inputs:

$$\bar{B}_{A \perp B}(\bar{x}_{A \perp B}) \stackrel{\text{def}}{=} \begin{pmatrix} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{pmatrix} \quad (\text{D.44})$$

and outputs:

$$\bar{y}_{A \perp B} \stackrel{\text{def}}{=} \begin{pmatrix} y_A \\ y_B \end{pmatrix} \quad (\text{D.45})$$

for the difference lossless PCH system, from (D.16). The difference lossless PCH system of systems  $A$  and  $B$ , can be defined in explicit form as follows:

$$\begin{cases} \dot{\bar{x}}_{A \perp B} \stackrel{\text{def}}{=} \bar{J}_{A \perp B}(\bar{x}_{A \perp B}) \frac{\partial H_{A \perp B}}{\partial x_{A \perp B}}(\bar{x}_{A \perp B}) + \bar{B}_{A \perp B}(\bar{x}_{A \perp B}) \bar{u}_{A \perp B} \\ \bar{y}_{A \perp B} \stackrel{\text{def}}{=} \bar{B}_{A \perp B}^T(\bar{x}_{A \perp B}) \frac{\partial H_{A \perp B}}{\partial x_{A \perp B}}(\bar{x}_{A \perp B}) \end{cases} \quad (\text{D.46})$$

The implicit form of the proposed difference lossless PCH system, is defined as:

$$(\bar{f}_S, \bar{e}_S, \bar{f}_C, \bar{e}_C) \in \bar{D}_{A \perp B} \quad (\text{D.47})$$

Equivalently (D.47) can be written as:

$$\left( -\dot{\bar{x}}_{A \perp B}, \frac{\partial H_{A \perp B}}{\partial x_{A \perp B}}(\bar{x}_{A \perp B}), \bar{u}_{A \perp B}, \bar{y}_{A \perp B} \right) \in \bar{D}_{A \perp B} \quad (\text{D.48})$$

As the basic properties of lossless PCH systems, are validated also for the proposed lossless difference PCH system, of system  $A$  and  $B$ , the proof of **Definition 4.1** ends.

## Appendix D.2 Lossy PCH systems

Similar to the lossless case, this appendix is reserved to the proof of the basic properties which describe a lossy PCH system for the proposed difference PCH system of two systems (  $A$  and  $B$  ). Due to the power energy conservation law of systems  $A$  and  $B$ , we can write:

$$\begin{cases} P_{SA} + P_{CA} + P_{RA} = 0 \\ P_{SB} + P_{CB} + P_{RB} = 0 \end{cases} \quad (\text{D.49})$$

where  $P_{SA}, P_{CA}, P_{RA}$  and  $P_{SB}, P_{CB}, P_{RB}$  represent the power energies corresponding to the storage, control and dissipation ports of systems  $A$  and  $B$ . Using (1.24) for systems  $A$  and  $B$ , we can write for the energy (Hamiltonian):

$$\begin{cases} \frac{dH_A}{dt} = -e_{SA}^T f_{SA} = e_{CA}^T f_{CA} + e_{RA}^T f_{RA} \\ \frac{dH_B}{dt} = -e_{SB}^T f_{SB} = e_{CB}^T f_{CB} + e_{RB}^T f_{RB} \end{cases} \quad (\text{D.50})$$

Using flows and efforts, the power port energies from (D.49), can be defined as follows:

$$\begin{cases} P_{SA} \stackrel{\text{def}}{=} e_{SA}^T f_{SA}, P_{CA} \stackrel{\text{def}}{=} e_{CA}^T f_{CA}, P_{RA} \stackrel{\text{def}}{=} e_{RA}^T f_{RA} \\ P_{SB} \stackrel{\text{def}}{=} e_{SB}^T f_{SB}, P_{CB} \stackrel{\text{def}}{=} e_{CB}^T f_{CB}, P_{RB} \stackrel{\text{def}}{=} e_{RB}^T f_{RB} \end{cases} \quad (\text{D.51})$$

Subtracting both relations from (D.50), we can write:

$$\frac{d(H_A - H_B)}{dt} = -e_{SA}^T f_{SA} + e_{SB}^T f_{SB} = e_{CA}^T f_{CA} - e_{CB}^T f_{CB} + e_{RA}^T f_{RA} - e_{RB}^T f_{RB} \quad (\text{D.52})$$

(D.50), can also be written as:

$$\begin{cases} \frac{dH_A}{dt} = \left( \frac{\partial H_A}{\partial x_A} (x_A) \right)^T \frac{dx_A}{dt} = -e_{SA}^T f_{SA} \\ \frac{dH_B}{dt} = \left( \frac{\partial H_B}{\partial x_B} (x_B) \right)^T \frac{dx_B}{dt} = -e_{SB}^T f_{SB} \end{cases} \quad (\text{D.53})$$

using the chain rule differentiation. The explicit form of the two lossy PCH systems (  $A$  and  $B$  ), can be written as follows:

$$\begin{cases} \frac{dx_A}{dt} = (J_A(x_A) - R_A(x_A)) \frac{\partial H_A}{\partial x_A} (x_A) + B_A(x_A) \mu_A \\ y_A = B_A^T(x_A) \frac{\partial H_A}{\partial x_A} (x_A) \\ \frac{dx_B}{dt} = (J_B(x_B) - R_B(x_B)) \frac{\partial H_B}{\partial x_B} (x_B) + B_B(x_B) \mu_B \\ y_B = B_B^T(x_B) \frac{\partial H_B}{\partial x_B} (x_B) \end{cases} \quad (\text{D.54})$$

The difference between the power energies of the storage, control and dissipation ports, can be expressed using the relations:

$$\begin{cases} e_{SA}^T f_{SA} - e_{SB}^T f_{SB} = [e_{SA} & e_{SB}]^T \begin{bmatrix} f_{SA} \\ -f_{SB} \end{bmatrix} \\ e_{CA}^T f_{CA} - e_{CB}^T f_{CB} = [e_{CA} & e_{CB}]^T \begin{bmatrix} f_{CA} \\ -f_{CB} \end{bmatrix} \\ e_{RA}^T f_{RA} - e_{RB}^T f_{RB} = [e_{RA} & e_{RB}]^T \begin{bmatrix} f_{RA} \\ -f_{RB} \end{bmatrix} \end{cases} \quad (D.55)$$

The flows  $f_{SA}, f_{SB}$ , corresponding to the systems  $A$  and  $B$ , can be written as:

$$\begin{cases} f_{SA} = -\frac{dx_A}{dt}, f_{SB} = -\frac{dx_B}{dt} \end{cases} \quad (D.56)$$

Using (D.54) and (D.56), the power energies corresponding to the storage port of systems  $A$  and  $B$ , result as:

$$\begin{cases} P_{SA} = e_{SA}^T f_{SA} = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T \left( (-J_A(x_A) - R_A(x_A)) \frac{\partial H_A}{\partial x_A}(x_A) - B_A(x_A) \mu_A \right) \\ P_{SB} = e_{SB}^T f_{SB} = \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T \left( (-J_B(x_B) - R_B(x_B)) \frac{\partial H_B}{\partial x_B}(x_B) - B_B(x_B) \mu_B \right) \end{cases} \quad (D.57)$$

(D.57) can be written further as:

$$\begin{cases} P_{SA} = e_{SA}^T f_{SA} = -\left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T J_A(x_A) \left( \frac{\partial H_A}{\partial x_A}(x_A) \right) + \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T R_A(x_A) \left( \frac{\partial H_A}{\partial x_A}(x_A) \right) - \\ - \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A \\ P_{SB} = e_{SB}^T f_{SB} = -\left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T J_B(x_B) \left( \frac{\partial H_B}{\partial x_B}(x_B) \right) + \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T R_B(x_B) \left( \frac{\partial H_B}{\partial x_B}(x_B) \right) - \\ - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \end{cases} \quad (D.58)$$

**Remark:** From the skew-symmetry of the  $J_A(x_A)$  and  $J_B(x_B)$  matrices, the terms

$$\left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T J_A(x_A) \left( \frac{\partial H_A}{\partial x_A}(x_A) \right) \text{ and } \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T J_B(x_B) \left( \frac{\partial H_B}{\partial x_B}(x_B) \right) \text{ are zero.}$$

(D.58) becomes then:

$$\left\{ \begin{array}{l} P_{SA} = e_{SA}^T f_{SA} = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T R_A(x_A) \left( \frac{\partial H_A}{\partial x_A}(x_A) \right) - \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A \\ P_{SB} = e_{SB}^T f_{SB} = \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T R_B(x_B) \left( \frac{\partial H_B}{\partial x_B}(x_B) \right) - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \end{array} \right. \quad (D.59)$$

The difference between the storage port power energies of systems  $A$  and  $B$ , is given as follows:

$$\begin{aligned} \tilde{e}_S^T \tilde{f}_S &= e_{SA}^T f_{SA} - e_{SB}^T f_{SB} = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T R_A(x_A) \left( \frac{\partial H_A}{\partial x_A}(x_A) \right) - \\ &- \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T R_B(x_B) \left( \frac{\partial H_B}{\partial x_B}(x_B) \right) - \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A + \\ &+ \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \end{aligned} \quad (D.60)$$

In the above relation, the notations:

$$\tilde{e}_S^T \stackrel{def}{=} \begin{bmatrix} e_{SA} & e_{SB} \end{bmatrix}^T \quad (D.61)$$

$$\tilde{f}_S \stackrel{def}{=} \begin{bmatrix} f_{SA} \\ -f_{SB} \end{bmatrix} \quad (D.62)$$

have been adopted for the effort and flow of the storage port difference of systems  $A$  and  $B$ . For the control port, the power energies of systems  $A$  and  $B$ , can be written as:

$$\left\{ \begin{array}{l} P_{CA} = e_{CA}^T f_{CA} = \left( B_A^T(x_A) \frac{\partial H_A}{\partial x_A}(x_A) \right)^T \mu_A \\ P_{CB} = e_{CB}^T f_{CB} = \left( B_B^T(x_B) \frac{\partial H_B}{\partial x_B}(x_B) \right) \mu_B \end{array} \right. \quad (D.63)$$

Using matrix transpose properties, (D.63) becomes further:

$$\left\{ \begin{array}{l} P_{CA} = e_{CA}^T f_{CA} = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A \\ P_{CB} = e_{CB}^T f_{CB} = \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \end{array} \right. \quad (D.64)$$

The difference between the power energies of the control port, of systems  $A$  and  $B$  from (D.64), can be written as:

$$\tilde{e}_C^T \tilde{f}_C = e_{CA}^T f_{CA} - e_{CB}^T f_{CB} = \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T B_A(x_A) \mu_A - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T B_B(x_B) \mu_B \quad (D.65)$$

with the following notations adopted:

$$\tilde{e}_C^T \stackrel{def}{=} \begin{bmatrix} e_{CA} & e_{CB} \end{bmatrix}^T \quad (D.66)$$

$$\tilde{f}_C \stackrel{def}{=} \begin{bmatrix} f_{CA} \\ -f_{CB} \end{bmatrix} \quad (D.67)$$

For the dissipation port, the power energies corresponding to systems  $A$  and  $B$  are:

$$\begin{cases} P_{RA} = e_{RA}^T f_{RA} = - \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T R_A(x_A) \left( \frac{\partial H_A}{\partial x_A}(x_A) \right) \\ P_{RB} = e_{RB}^T f_{RB} = - \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T R_B(x_B) \left( \frac{\partial H_B}{\partial x_B}(x_B) \right) \end{cases} \quad (D.68)$$

The difference between the dissipation port power energies of systems  $A$  and  $B$ , is then:

$$\tilde{e}_R^T \tilde{f}_R = e_{RA}^T f_{RA} - e_{RB}^T f_{RB} = - \left( \frac{\partial H_A}{\partial x_A}(x_A) \right)^T R_A(x_A) \left( \frac{\partial H_A}{\partial x_A}(x_A) \right) + \left( \frac{\partial H_B}{\partial x_B}(x_B) \right)^T R_B(x_B) \left( \frac{\partial H_B}{\partial x_B}(x_B) \right) \quad (D.69)$$

with the notations:

$$\tilde{e}_R^T \stackrel{def}{=} \begin{bmatrix} e_{RA} & e_{RB} \end{bmatrix}^T \quad (D.70)$$

$$\tilde{f}_R \stackrel{def}{=} \begin{bmatrix} f_{RA} \\ -f_{RB} \end{bmatrix} \quad (D.71)$$

for the efforts and flows. Summing up the power energies for the ports, of systems  $A$  and  $B$  in (D.59), (D.64) and (D.68), it results:

$$\begin{cases} e_{SA}^T f_{SA} + e_{CA}^T f_{CA} + e_{RA}^T f_{RA} = 0 \\ e_{SB}^T f_{SB} + e_{CB}^T f_{CB} + e_{RB}^T f_{RB} = 0 \end{cases} \quad (D.72)$$

or:

$$\begin{cases} P_{SA} + P_{CA} + P_{RA} = 0 \\ P_{SB} + P_{CB} + P_{RB} = 0 \end{cases} \quad (D.73)$$

For the next step, a difference lossy PCH system is defined, which has as power energies of the ports, the difference between the power energies of systems  $A$  and  $B$ . Summing up the power energies associated to the *difference lossy PCH system* from (D.60), (D.65) and (D.69), it results:

$$\tilde{e}_S^T \tilde{f}_S + \tilde{e}_C^T \tilde{f}_C + \tilde{e}_R^T \tilde{f}_R = 0 \quad (D.74)$$

It results that the difference between the power energies of the ports of systems  $A$  and  $B$ , also satisfies the power conservation law. An explicit form of the *difference lossy PCH system*, is defined as:



$$\left\{ \begin{array}{l} \dot{x} \stackrel{\text{def}}{=} \begin{bmatrix} \frac{dx_A}{dt} \\ \frac{dx_B}{dt} \end{bmatrix} = \begin{pmatrix} (J_A(x_A) - R_A(x_A)) & 0 \\ 0 & -(J_B(x_B) - R_B(x_B)) \end{pmatrix} \begin{bmatrix} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{bmatrix} \\ + \begin{pmatrix} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} \\ \bar{y} \stackrel{\text{def}}{=} \begin{bmatrix} y_A \\ y_B \end{bmatrix} = \begin{pmatrix} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{pmatrix}^T \begin{bmatrix} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{bmatrix} \end{array} \right. \quad (\text{D.75})$$

Further we check if the relations from (D.60), (D.65), (D.69) and (D.74), are also satisfied for the new defined difference lossy PCH system from (D.75). Computing the storage port power energy for the new defined system, we can write:

$$\begin{aligned} \bar{e}_S^T \bar{f}_S &= \begin{bmatrix} \frac{\partial H_A}{\partial x_A} & \frac{\partial H_B}{\partial x_B} \end{bmatrix}^T \begin{bmatrix} -\frac{dx_A}{dt} \\ \frac{dx_B}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial H_A}{\partial x_A} & \frac{\partial H_B}{\partial x_B} \end{bmatrix}^T \begin{bmatrix} (J_A(x_A) - R_A(x_A)) \frac{\partial H_A}{\partial x_A}(x_A) - B_A(x_A) u_A \\ (J_B(x_B) - R_B(x_B)) \frac{\partial H_B}{\partial x_B}(x_B) + B_B(x_B) u_B \end{bmatrix} \end{aligned} \quad (\text{D.76})$$

(D.76) becomes further:

$$\begin{aligned} \bar{e}_S^T \bar{f}_S &= -\frac{\partial H_A}{\partial x_A} &^T (x_A) \cdot J_A(x_A) \frac{\partial H_A}{\partial x_A}(x_A) + \frac{\partial H_A}{\partial x_A} &^T (x_A) \cdot R_A(x_A) \frac{\partial H_A}{\partial x_A}(x_A) \\ &- \frac{\partial H_A}{\partial x_A} &^T (x_A) B_A(x_A) u_A + \frac{\partial H_B}{\partial x_B} &^T (x_B) \cdot J_B(x_B) \frac{\partial H_B}{\partial x_B}(x_B) - \frac{\partial H_B}{\partial x_B} &^T (x_B) R_B(x_B) \frac{\partial H_B}{\partial x_B}(x_B) \\ &+ \frac{\partial H_B}{\partial x_B} &^T (x_B) \cdot B_B(x_B) u_B \end{aligned} \quad (\text{D.77})$$

**Remark:** Due to the skew-symmetry property of  $J_A(x_A)$  and  $J_B(x_B)$  matrices, the terms

$$\frac{\partial H_A}{\partial x_A} &^T (x_A) J_A(x_A) \frac{\partial H_A}{\partial x_A}(x_A) \text{ and } \frac{\partial H_B}{\partial x_B} &^T (x_B) J_B(x_B) \frac{\partial H_B}{\partial x_B}(x_B) \text{ equal zero.}$$

(D.77) becomes then:

$$\begin{aligned} \bar{e}_S^T \bar{f}_S &= \frac{\partial H_A}{\partial x_A} (x_A) R_A (x_A) \frac{\partial H_A}{\partial x_A} (x_A) - \frac{\partial H_A}{\partial x_A} (x_A) B_A (x_A) \mu_A \\ &- \frac{\partial H_B}{\partial x_B} (x_B) R_B (x_B) \frac{\partial H_B}{\partial x_B} (x_B) + \frac{\partial H_B}{\partial x_B} (x_B) B_B (x_B) \mu_B \end{aligned} \quad (D.78)$$

which is equivalent to:

$$\begin{aligned} \bar{e}_S^T \bar{f}_S &= \frac{\partial H_A}{\partial x_A} (x_A) R_A (x_A) \frac{\partial H_A}{\partial x_A} (x_A) - \frac{\partial H_B}{\partial x_B} (x_B) R_B (x_B) \frac{\partial H_B}{\partial x_B} (x_B) \\ &- \frac{\partial H_A}{\partial x_A} (x_A) B_A (x_A) \mu_A + \frac{\partial H_B}{\partial x_B} (x_B) B_B (x_B) \mu_B \end{aligned} \quad (D.79)$$

Comparing the result from (D.60) and (D.79), we can write:

$$\begin{aligned} \tilde{e}_S^T \tilde{f}_S &= \bar{e}_S^T \bar{f}_S = \frac{\partial H_A}{\partial x_A} (x_A) R_A (x_A) \frac{\partial H_A}{\partial x_A} (x_A) - \frac{\partial H_B}{\partial x_B} (x_B) R_B (x_B) \frac{\partial H_B}{\partial x_B} (x_B) \\ &- \frac{\partial H_A}{\partial x_A} (x_A) B_A (x_A) \mu_A + \frac{\partial H_B}{\partial x_B} (x_B) B_B (x_B) \mu_B \end{aligned} \quad (D.80)$$

which proves the power energy equality, between the difference of the storage port power energies of systems  $A$  and  $B$  and the storage port power energy of the new defined difference lossy PCH system from (D.75). Next we determine if the control port power energy of the proposed *difference PCH system* from (D.75), is equal to the difference between the control port power energies of systems  $A$  and  $B$ . Using (D.75), the control port power energy, can be defined as:

$$\bar{e}_C^T \bar{f}_C \stackrel{def}{=} \bar{y}^T \bar{u} = \left[ \begin{array}{cc} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{array} \right]^T \left[ \begin{array}{c} \frac{\partial H_A}{\partial x_A} (x_A) \\ \frac{\partial H_B}{\partial x_B} (x_B) \end{array} \right] \begin{array}{c} u_A \\ u_B \end{array} \quad (D.81)$$

Equivalently we can write:

$$\bar{e}_C^T \bar{f}_C = \left[ \frac{\partial H_A}{\partial x_A} (x_A) \quad \frac{\partial H_B}{\partial x_B} (x_B) \right] \left[ \begin{array}{cc} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{array} \right] \begin{array}{c} u_A \\ u_B \end{array} \quad (D.82)$$

Expressed further (D.82) results as:

$$\bar{e}_C^T \bar{f}_C = \frac{\partial H_A}{\partial x_A} (x_A) B_A (x_A) \mu_A - \frac{\partial H_B}{\partial x_B} (x_B) B_B (x_B) \mu_B \quad (D.83)$$

Using (D.65) and (D.83), the following equality results:

$$\bar{e}_C^T \bar{f}_C = \tilde{e}_C^T \tilde{f}_C = \frac{\partial H_A}{\partial x_A} (x_A) B_A (x_A) \mu_A - \frac{\partial H_B}{\partial x_B} (x_B) B_B (x_B) \mu_B \quad (D.84)$$

and thus the difference of the control port power energies, of systems  $A$  and  $B$  is equal to the control port power energy of the new defined *difference lossy PCH system* from (D.75).

For the next step, we determine, if the the difference between the dissipation port power energies of two systems (  $A$  and  $B$  ) from (D.69), is equal to the dissipation port power energy of the lossy PCH system defined in (D.75). The dissipation port power energy of the difference lossy PCH system proposed in (D.75), can be computed using the relation:

$$\bar{e}_R^T \bar{f}_R \stackrel{def}{=} -\bar{e}_R^T \bar{R}(\bar{x}_{A \perp B}) \bar{e}_R \quad (D.85)$$

where we have used the linear relation between the flows and efforts:

$$\bar{f}_R = -\bar{R}(\bar{x}_{A \perp B}) \bar{e}_R \quad (D.86)$$

for the flows and efforts. The dissipation matrix  $\bar{R}$  in (D.86), is defined as:

$$\bar{R}(\bar{x}_{A \perp B}) \stackrel{def}{=} \begin{pmatrix} R_A(x_A) & 0 \\ 0 & -R_B(x_B) \end{pmatrix} \quad (D.87)$$

For the dissipation port effort variable, we can adopt the notation:

$$\bar{e}_R \stackrel{def}{=} \begin{pmatrix} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{pmatrix} \quad (D.88)$$

Using (D.87) and (D.88), (D.85) becomes :

$$\bar{e}_R^T \bar{f}_R \stackrel{def}{=} - \begin{pmatrix} \frac{\partial H_A}{\partial x_A} & \frac{\partial H_B}{\partial x_B} \end{pmatrix}^T \begin{pmatrix} R_A(x_A) & 0 \\ 0 & -R_B(x_B) \end{pmatrix} \begin{pmatrix} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{pmatrix} \quad (D.89)$$

After basic computations, (D.89) becomes:

$$\bar{e}_R^T \bar{f}_R = - \frac{\partial H_A}{\partial x_A} &^T (x_A) R_A(x_A) \frac{\partial H_A}{\partial x_A}(x_A) + \frac{\partial H_B}{\partial x_B} &^T (x_B) R_B(x_B) \frac{\partial H_B}{\partial x_B}(x_B) \quad (D.90)$$

Comparing the results from (D.69) and (D.90), it results:

$$\tilde{e}_R^T \tilde{f}_R = \bar{e}_R^T \bar{f}_R = - \frac{\partial H_A}{\partial x_A} &^T (x_A) R_A(x_A) \frac{\partial H_A}{\partial x_A}(x_A) + \frac{\partial H_B}{\partial x_B} &^T (x_B) R_B(x_B) \frac{\partial H_B}{\partial x_B}(x_B) \quad (D.91)$$

which proves the equality, between the difference of the dissipation port power energies of systems  $A$  and  $B$  and the dissipation port power energy, of the new defined difference lossy PCH system in (D.75). Using (D.80), (D.84) and (D.91), the power conservation law is satisfied for the new defined *difference PCH system* in (D.75):

$$\bar{e}_S^T \bar{f}_S + \bar{e}_C^T \bar{f}_C + \bar{e}_R^T \bar{f}_R = 0 \quad (D.92)$$

or:

$$\bar{P}_S + \bar{P}_C + \bar{P}_R = 0 \quad (D.93)$$

In (D.93),  $\bar{P}_S, \bar{P}_C, \bar{P}_R$  represent the power energies associated to the storage, control and dissipation ports for the new defined difference lossy PCH systems of systems  $A$  and  $B$ . For the next step, it is introduced and proved a definition of a Dirac structure, corresponding to the difference lossy PCH system defined in (D.75).

## Definition D.2

A Dirac structure on  $\overline{F \times F^*}$ , is a subspace  $\overline{D}_{A \perp B} \subset \overline{F \times F^*}$ , such that:

$$\text{i) } \langle \overline{e} | \overline{f} \rangle = 0$$

$$\text{ii) } \dim \overline{D}_{A \perp B} = \dim \overline{F}$$

The first property of the **Definition D.2** corresponds to the *power-conservation law*, which is equivalent to (D.92) for the difference PCH system, that expresses the fact that the total power entering (or leaving) a Dirac structure is always zero. For the second statement of **Definition D.2**, an equivalent characterization of a Dirac structure is introduced related to the geometric structure of the total space of flows and efforts variables  $\overline{F \times F^*}$ , by defining a bilinear-form. The bilinear form corresponding to the difference lossy PCH system, can be defined as follows:

$$\overline{B}_F \stackrel{\text{def}}{=} \langle\langle \overline{f}^a, \overline{e}^a \rangle, \langle \overline{f}^b, \overline{e}^b \rangle \rangle := \langle \overline{e}^a, \overline{f}^b \rangle + \langle \overline{e}^b, \overline{f}^a \rangle \quad (\text{D.94})$$

with  $(\overline{f}^a, \overline{e}^a), (\overline{f}^b, \overline{e}^b) \in \overline{F \times F^*}$ , where  $\overline{F} \stackrel{\text{def}}{=} \overline{F}_S \times \overline{F}_C \times \overline{F}_R$  and  $\overline{F}^* \stackrel{\text{def}}{=} \overline{E}_S \times \overline{E}_C \times \overline{E}_R$

represent the total space of the flows and efforts corresponding to the new defined difference lossy PCH system for systems  $A$  and  $B$ , in (D.75). Developing further (D.94), we can write:

$$\overline{B}_F \stackrel{\text{def}}{=} \left\langle \left( \overline{e}_S, \overline{e}_C, \overline{e}_R \right)^a, \left( \overline{f}_S, \overline{f}_C, \overline{f}_R \right)^b \right\rangle + \left\langle \left( \overline{e}_S, \overline{e}_C, \overline{e}_R \right)^b, \left( \overline{f}_S, \overline{f}_C, \overline{f}_R \right)^a \right\rangle \quad (\text{D.95})$$

where:

$$\left\langle \left( \overline{e}_S, \overline{e}_C, \overline{e}_R \right), \left( \overline{f}_S, \overline{f}_C, \overline{f}_R \right) \right\rangle = 0 \quad (\text{D.96})$$

as it has been proven from (D.92). For the next step, we introduce and prove the following proposition for the new defined difference lossy PCH system from (D.75).

## Proposition D.2

A constant Dirac structure on  $\overline{F \times F^*}$  is a subspace  $\overline{D}_{A \perp B} \subset \overline{F \times F^*}$  such that:

$$\overline{D}_{A \perp B} = \overline{D}^\perp_{A \perp B} \quad (\text{D.97})$$

where  $\perp$  denotes the orthogonal complement with respect to the bilinear form  $\langle\langle, \rangle\rangle$ .

In the above definition  $\overline{D}_{A \perp B}$  represents the Dirac structure corresponding to the proposed difference lossy PCH system, of systems  $A$  and  $B$ .

## Proof:

We first consider  $\overline{D}_{A \perp B}$  to satisfy (D.97). Then for every  $(\overline{f}, \overline{e}) \in \overline{D}_{A \perp B}$ , we can write:

$$\langle\langle (\overline{f}, \overline{e}), (\overline{f}, \overline{e}) \rangle\rangle = \langle \overline{e} | \overline{f} \rangle + \langle \overline{e} | \overline{f} \rangle = 2 \langle \overline{e} | \overline{f} \rangle \quad (\text{D.98})$$

Decomposing the flows and efforts from (D.98) on each port, it results:

$$2\langle(\bar{e}_S, \bar{e}_C, \bar{e}_R), (\bar{f}_S, \bar{f}_C, \bar{f}_R)\rangle = 0 \quad (\text{D.99})$$

By the non-degeneracy of the bilinear form  $\langle\langle, \rangle\rangle$  :

$$\dim \bar{D}^\perp_{A\perp B} = \dim(\bar{F}x\bar{F}^*) - \dim \bar{D}_{A\perp B} = 2 \dim \bar{F} - \dim \bar{D}_{A\perp B} \quad (\text{D.100})$$

Using (D.97) it results that  $\dim \bar{D}_{A\perp B} = \dim \bar{F}$ . We also assume that  $\bar{D}_{A\perp B}$  is a Dirac structure, satisfying the properties i) and ii) of **Definition D.2**. Then we consider  $(\bar{f}^a, \bar{e}^a), (\bar{f}^b, \bar{e}^b)$  to be any vectors contained in  $\bar{D}_{A\perp B}$ . Using the linearity of the flows and efforts, we can write  $(\bar{f}^a + \bar{f}^b, \bar{e}^a + \bar{e}^b) \in \bar{D}_{A\perp B}$ . Using property i) of **Definition D.2**, we can write:

$$\begin{aligned} 0 &= \langle \bar{e}^a + \bar{e}^b | \bar{f}^a + \bar{f}^b \rangle = \langle \bar{e}^a | \bar{f}^b \rangle + \langle \bar{e}^b | \bar{f}^a \rangle + \langle \bar{e}^a | \bar{f}^a \rangle + \langle \bar{e}^b | \bar{f}^b \rangle = \langle \bar{e}^a | \bar{f}^b \rangle + \langle \bar{e}^b | \bar{f}^a \rangle \\ &= \langle\langle (\bar{f}^a, \bar{e}^a), (\bar{f}^b, \bar{e}^b) \rangle\rangle \end{aligned} \quad (\text{D.101})$$

Due to the property i) of **Definition D.2**:

$$\langle \bar{e}^a | \bar{f}^a \rangle = \langle \bar{e}^b | \bar{f}^b \rangle = 0 \quad (\text{D.102})$$

Expressing the flows and efforts in (D.102) for each port, it results:

$$\langle\langle (\bar{e}_S^a, \bar{e}_C^a, \bar{e}_R^a), (\bar{f}_S^a, \bar{f}_C^a, \bar{f}_R^a) \rangle\rangle = \langle\langle (\bar{e}_S^b, \bar{e}_C^b, \bar{e}_R^b), (\bar{f}_S^b, \bar{f}_C^b, \bar{f}_R^b) \rangle\rangle \quad (\text{D.103})$$

This implies that  $\bar{D}_{A\perp B} \subset \bar{D}^\perp_{A\perp B}$ . Using the 2<sup>nd</sup> point of **Definition D.2** and  $\dim \bar{D}^\perp_{A\perp B} = 2 \dim \bar{F} - \dim \bar{D}_{A\perp B}$ , it follows that:

$$\dim \bar{D}_{A\perp B} = \dim \bar{D}^\perp_{A\perp B} \quad (\text{D.104})$$

yielding that  $\bar{D}_{A\perp B} = \bar{D}^\perp_{A\perp B}$ , which ends the proof. For the difference lossy PCH system defined in (D.75), we adopt the following notation:

$$\bar{J}_{A\perp B}(\bar{x}_{A\perp B}) \stackrel{\text{def}}{=} \begin{pmatrix} J_A(x_A) & 0 \\ 0 & -J_B(x_B) \end{pmatrix} \quad (\text{D.105})$$

for the skew-symmetric matrix. Computing  $\bar{J}_{A\perp B}^T(\bar{x}_{A\perp B})$ , we get:

$$\bar{J}_{A\perp B}^T(\bar{x}_{A\perp B}) = \begin{pmatrix} J_A^T(x_A) & 0 \\ 0 & -J_B^T(x_B) \end{pmatrix} \quad (\text{D.106})$$

Due to the skew-symmetry of  $J_A(x_A)$  and  $J_B(x_B)$  matrices, the above relation can be also written as:

$$\bar{J}_{A\perp B}^T(\bar{x}_{A\perp B}) = \begin{pmatrix} -J_A(x_A) & 0 \\ 0 & J_B(x_B) \end{pmatrix} \quad (\text{D.107})$$

From (D.105) and (D.107), it results that we can write the following relation:

$$\bar{J}_{A \perp B}(\bar{x}_{A \perp B}) = -\bar{J}_{A \perp B}^T(\bar{x}_{A \perp B}) = \begin{pmatrix} J_A(x_A) & 0 \\ 0 & -J_B(x_B) \end{pmatrix} \quad (\text{D.108})$$

which proves that the matrix  $\bar{J}_{A \perp B}(\bar{x}_{A \perp B})$ , corresponding to the new defined difference lossy PCH system from (D.75), preserves the skew-symmetry property. For the dissipation matrix corresponding to the difference lossy PCH, we adopt the notation:

$$\bar{R}_{A \perp B}(\bar{x}_{A \perp B}) \stackrel{\text{def}}{=} \begin{pmatrix} R_A(x_A) & 0 \\ 0 & -R_B(x_B) \end{pmatrix} \quad (\text{D.109})$$

Computing the matrix transpose of  $\bar{R}_{A \perp B}(\bar{x}_{A \perp B})$ , we can write:

$$\bar{R}_{A \perp B}^T(\bar{x}_{A \perp B}) = \begin{pmatrix} R_A^T(x_A) & 0 \\ 0 & -R_B^T(x_B) \end{pmatrix} \quad (\text{D.110})$$

Due to the positive definiteness of  $R_A(x_A)$  and  $R_B(x_B)$  matrices, it results that:

$$\bar{R}_{A \perp B}^T(\bar{x}_{A \perp B}) = \begin{pmatrix} R_A(x_A) & 0 \\ 0 & -R_B(x_B) \end{pmatrix} \quad (\text{D.111})$$

and then we can write:

$$\bar{R}_{A \perp B}(\bar{x}_{A \perp B}) = \bar{R}_{A \perp B}^T(\bar{x}_{A \perp B}) = \begin{pmatrix} R_A(x_A) & 0 \\ 0 & -R_B(x_B) \end{pmatrix} \quad (\text{D.112})$$

**Remark:** The presence of the negative term ( $R_B(x_B)$ ) in (D.112) for the **difference PCH system** (D.75) dissipation matrix, is due to the **power port energy difference concept** proposed for systems A and B. Thus, the **difference PCH system** (D.75) will augment internally to the dissipation port, the energy corresponding to system B and not dissipate it.

For the input matrix corresponding to the difference lossy PCH system proposed in (D.75), the following notation is adopted:

$$\bar{B}_{A \perp B} \stackrel{\text{def}}{=} \begin{pmatrix} B_A(x_A) & 0 \\ 0 & -B_B(x_B) \end{pmatrix} \quad (\text{D.113})$$

Using the notations introduced in (D.105), (D.109) and (D.113), we can equivalently write the explicit form of the difference lossy PCH system from (D.75), as:

$$\left\{ \begin{array}{l} \bar{\dot{x}} \stackrel{\text{def}}{=} \left( \bar{J}_{A \perp B}(\bar{x}_{A \perp B}) - \bar{R}_{A \perp B}(\bar{x}_{A \perp B}) \right) \begin{pmatrix} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{pmatrix} + \bar{B}_{A \perp B}(\bar{x}_{A \perp B}) \begin{pmatrix} u_A \\ u_B \end{pmatrix} \\ \bar{y}_{A \perp B} \stackrel{\text{def}}{=} \bar{B}_{A \perp B}^T(\bar{x}_{A \perp B}) \begin{pmatrix} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{pmatrix} \end{array} \right. \quad (\text{D.114})$$

(D.114) can be equivalently written as:

$$\left\{ \begin{array}{l} \overset{def}{\dot{\bar{x}}} = (\bar{J}_{A \perp B}(x \perp B) - \bar{R}_{A \perp B}(\bar{x}_{A \perp B})) \left( \frac{\partial H_{A \perp B}}{\partial \bar{x}_{A \perp B}}(\bar{x}_{A \perp B}) \right) + \bar{B}_{A \perp B}(\bar{x}_{A \perp B}) \mu_{A \perp B} \\ \bar{y}_{A \perp B} \overset{def}{=} \bar{B}_{A \perp B}^T(\bar{x}_{A \perp B}) \frac{\partial H_{A \perp B}}{\partial x_{A \perp B}}(\bar{x}_{A \perp B}) \end{array} \right. \quad (D.115)$$

using the notations:

$$\frac{\partial H_{A \perp B}}{\partial x_{A \perp B}}(\bar{x}_{A \perp B}) \overset{def}{=} \left[ \begin{array}{c} \frac{\partial H_A}{\partial x_A}(x_A) \\ \frac{\partial H_B}{\partial x_B}(x_B) \end{array} \right] \quad (D.116)$$

$$\bar{x}_{A \perp B} \overset{def}{=} \begin{pmatrix} x_A \\ x_B \end{pmatrix} \quad (D.117)$$

$$\bar{u}_{A \perp B} \overset{def}{=} \begin{pmatrix} u_A \\ u_B \end{pmatrix} \quad (D.118)$$

This represents the explicit form of the new defined difference lossy PCH system for systems  $A$  and  $B$ . The implicit form of the difference lossy PCH system proposed for systems  $A$  and  $B$ , can be defined as:

$$(\bar{f}_S, \bar{e}_S, \bar{f}_C, \bar{e}_C, \bar{f}_R, \bar{e}_R) \in \bar{D}_{A \perp B} \quad (D.119)$$

Using the explicit form from (D.115), (D.119) becomes:

$$\left( -\overset{def}{\dot{\bar{x}}}_{A \perp B}, \frac{\partial H_{A \perp B}}{\partial x_{A \perp B}}(\bar{x}_{A \perp B}), \bar{u}_{A \perp B}, \bar{y}_{A \perp B}, -\bar{R}_{A \perp B}(\bar{x}_{A \perp B}) \frac{\partial H_{A \perp B}}{\partial x_{A \perp B}}(\bar{x}_{A \perp B}), \frac{\partial H_{A \perp B}}{\partial \bar{x}_{A \perp B}}(\bar{x}_{A \perp B}) \right) \quad (D.120)$$

Due to the satisfaction of the general propositions and definitions which define a PCH system, the difference lossy PCH system proposed in section 4.4.2 by the **Definition 4.2** is also a PCH system.

## Appendix E Symplectic discretization schemes

### E.1 Lossless PCH systems

This part of the appendix presents a set of symplectic discretization schemes for the lossless and lossy case, which make use of the explicit/implicit Euler, implicit mid-point and implicit trapezoidal rules for the flows and efforts.

#### E.1.1 Symplectic discretization scheme I

This symplectic discretization scheme assumes an *implicit Euler* rule for the flows and an *explicit Euler* rule for the efforts. The initial continuous-time form of the explicit lossless PCH system is:

$$\begin{cases} \dot{x} = J \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.1})$$

Applying the general discretization framework from (4.1) and (4.14), we can write in discrete-time as follows:

$$\begin{cases} x_{k+1} = x_k + \Delta t \cdot J \frac{\Delta H}{\Delta x}(k+1) + \Delta t \cdot Bu_{k+1} \\ H_{k+1} = H_k + \Delta x \cdot Qx_k \\ y_{k+1} = B^T \frac{\Delta H}{\Delta x}(k+1) \end{cases} \quad (\text{E.2})$$

**Remark:** The composed time-discretization  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$  of the effort becomes  $Qx_{k+1}$ ,

by applying the effort time-discretization  $\frac{\Delta H}{\Delta x}$  at step  $k+1$ .

Equivalently (E.2) can be written as follows:

$$\begin{cases} \frac{\Delta H}{\Delta t}(k+1) = JQx_{k+1} + Bu_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \\ y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.3})$$

**Remark:** The step  $k+1$  in the parenthesis of flows, efforts and power energies, is due to the implicit form of the scheme, given by the flow time-discretization rule.

The discrete-time power energy associated to the storage port  $\hat{P}_S(k+1)$ , can be defined as:

$$\hat{P}_S(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) \hat{f}_S(k+1) = (Qx_{k+1})^T \left( -\frac{\Delta x}{\Delta t}(k+1) \right) \quad (\text{E.4})$$

using (E.3). From simple mathematical relations, it is straightforward that we can write:

$$\hat{P}_S(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.5})$$



For the control port, the following relation can be defined as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_C^T(k+1)\hat{f}_C(k+1) \quad (\text{E.6})$$

Using (E.3) and simple mathematical relations, (E.6) can be equivalently written:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (\text{E.7})$$

From (E.5) and (E.7), the power conservation law for the discrete-time PCH system is satisfied, as follows:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) = -\hat{e}_S^T(k+1)B\hat{f}_C(k+1) + \hat{e}_S^T(k+1)B\hat{f}_C(k+1) = 0 \quad (\text{E.8})$$

Using (4.18), we can define the discrete-time energy (Hamiltonian) as:

$$\frac{\Delta H}{\Delta x}(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1)\hat{f}_S(k+1) = y_{k+1}^T u_{k+1} \quad (\text{E.9})$$

which is equivalent to:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (\text{E.10})$$

when using (E.5) and (E.7). Using the general relation from (4.19) for the discrete-time bilinear form, we can define it as follows:

$$\begin{aligned} \hat{B}_F(k+1) &\stackrel{def}{=} \left\langle \left( Q_{k+1}, B^T Q_{k+1} \right)^a, \left( -\frac{\Delta x}{dt}(k+1), u_{k+1} \right)^b \right\rangle + \\ &+ \left\langle \left( Q_{k+1}, B^T Q_{k+1} \right)^b, \left( -\frac{\Delta x}{\Delta t}(k+1), u_{k+1} \right)^a \right\rangle \end{aligned} \quad (\text{E.11})$$

Using (4.21), the implicit form of the discrete-time lossless PCH system, for the selected symplectic discretization scheme is defined as:

$$\left( \hat{f}_S(k+1), \hat{e}_S(k+1), \hat{f}_C(k+1), \hat{e}_C(k+1) \right) \in D \quad (\text{E.12})$$

Replacing the known values of the flow and energy from (E.3), we can write further:

$$\left( -\frac{\Delta x}{\Delta t}(k+1), \frac{\Delta H}{\Delta x}(k+1), u_{k+1}, y_{k+1} \right) \in D \quad (\text{E.13})$$

**Remark:** *The Dirac interconnection structure (D) and rules, are conserved by time-discretization as they only serve for power port interconnection.*

### E.1.2. Symplectic discretization scheme II

This symplectic time-discretization scheme assumes an explicit Euler rule for the flows and an implicit mid-point rule for the efforts. The initial continuous-time explicit form of the lossless PCH system, for which we apply this time-discretization scheme, has the following form:

$$\begin{cases} \dot{x} = J \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.14})$$

Considering the explicit Euler rule time-discretization for the flows and the implicit mid-point rule for the efforts, using the general framework from (4.1) and (4.14), it results:

$$\begin{cases} x_{k+1} = x_k + \Delta t \cdot J \cdot \frac{\Delta H}{\Delta x}(k) + Bu_k \\ H_{k+1} = H_k + \Delta x \cdot Q \cdot \left( \frac{x_k + x_{k+1}}{2} \right) \\ y_k = B^T \cdot \frac{\Delta H}{\Delta x}(k) \end{cases} \quad (\text{E.15})$$

**Remark:** The composed time-discretization  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$  of the effort, becomes

$$Q \left( \frac{x_{k-1} + x_k}{2} \right) \text{ by applying the flow discretization rule for the discrete-time effort } \frac{\Delta H}{\Delta x}.$$

Equivalently we can write (E.15) as:

$$\begin{cases} \frac{\partial H}{\Delta t}(k) = JQ \left( \frac{x_{k-1} + x_k}{2} \right) + Bu_k \\ \frac{\Delta H}{\Delta x}(k+1) = Q \left( \frac{x_k + x_{k+1}}{2} \right), y_k = B^T Q \left( \frac{x_{k-1} + x_k}{2} \right) \end{cases} \quad (\text{E.16})$$

**Remark:** The step  $k$  in the parenthesis of the flows, efforts and power energies is due to the explicit form of the scheme, given by the flow time-discretization rule.

The discrete-time power energy of the storage port, can be defined as:

$$\hat{P}_S(k) \stackrel{def}{=} \hat{e}_S^T(k) \hat{f}_S(k) = \left( \frac{\Delta H}{\Delta x}(k) \right)^T \left( -\frac{\Delta x}{\Delta t}(k) \right) \quad (\text{E.17})$$

Using simple computations, we can derive further the following relation:

$$\hat{P}_S(k) = -\hat{e}_S^T(k) B \hat{f}_C(k) \quad (\text{E.18})$$

For the control port, we define the following discrete-time power energy:

$$\hat{P}_C(k) \stackrel{def}{=} \hat{e}_C^T(k) \hat{f}_C(k) \quad (\text{E.19})$$

which is equivalent to:

$$\hat{P}_C(k) \stackrel{def}{=} \hat{e}_S^T(k) \cdot B \cdot \hat{f}_C(k) \quad (\text{E.20})$$

using simple mathematical relations. By making the sum between the storage power energy and control power energy from (E.18) and (E.20), we get:

$$\hat{P}_S(k) + \hat{P}_C(k) = -\hat{e}_S^T(k) B \hat{f}_C(k) + \hat{e}_S^T(k) B \hat{f}_C(k) \quad (\text{E.21})$$

from which it results that the discrete-time power conservation law holds for this scheme.

Using (4.18), we can define the discrete-time energy (Hamiltonian) as:

$$\frac{\Delta H}{\Delta t}(k) \stackrel{def}{=} -\hat{e}_S^T(k) \hat{f}_S(k) = y_k^T u_k \quad (\text{E.22})$$

Using (E.18) and (E.20), E(22) is equivalent to:

$$\frac{\Delta H}{\Delta t}(k) \stackrel{def}{=} \hat{e}_S^T(k) \cdot B \cdot \hat{f}_C(k) \quad (\text{E.23})$$

Using (4.19), the bilinear form of the lossless PCH system, can be defined as:

$$\hat{B}_F(k) \stackrel{def}{=} \left\langle \left( Q\bar{x}_k, B^T Q\bar{x}_k \right)^a, \left( -\frac{\Delta x}{\Delta t}(k), u_k \right)^b \right\rangle + \left\langle \left( Q\bar{x}_k, B^T Q\bar{x}_k \right)^b, \left( -\frac{\Delta x}{\Delta t}(k), u_k \right)^a \right\rangle \quad (\text{E.24})$$

where we have adopted the notation:

$$\bar{x}_k \stackrel{def}{=} \left( \frac{x_{k-1} + x_k}{2} \right) \quad (\text{E.25})$$

The implicit form of the discrete PCH system is defined using (4.21) as:

$$\left( \hat{f}_S(k), \hat{e}_S(k), \hat{f}_C(k), \hat{e}_C(k) \right) \in D \quad (\text{E.26})$$

which is equivalent to:

$$\left( -\frac{\Delta x}{\Delta t}(k), Q\bar{x}_k, u_k, B^T Q\bar{x}_k \right) \in D \quad (\text{E.27})$$

### E.1.3.Symplectic discretization scheme III

This symplectic time-discretization scheme, makes use of the implicit mid-point rule for the flows and explicit Euler rule for the efforts. The initial continuous-time lossless PCH system, for which we apply the time-discretization scheme, has the following form:

$$\begin{cases} \dot{x} = J \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.28})$$

Using (4.1) and (4.14), for the selected time-discretization rules, it results:

$$\begin{cases} x_{k+1} = x_k + \Delta t \cdot J \cdot \frac{\Delta H}{\Delta x} \left( \frac{x_k + x_{k+1}}{2} \right) + \Delta t \cdot B \cdot u_{k+1} \\ H_{k+1} = H_k + \Delta x \cdot Qx_k \\ y_{k+1} = B^T \frac{\Delta H}{\Delta x}(k+1) \end{cases} \quad (\text{E.29})$$

Equivalently we can write (E.29) as:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = J \cdot \frac{\Delta H}{\Delta x} \left( \frac{x_k + x_{k+1}}{2} \right) + Bu_{k+1} \\ \frac{\Delta H}{\Delta x}(k) = Qx_k \\ y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.30})$$

**Remark:** The composed time-discretization  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$  of the effort, becomes

$Q \left( \frac{x_k + x_{k+1}}{2} \right)$  by applying the flow discretization rule for  $\frac{\Delta H}{\Delta x}$ , by replacing  $x_{k+1}$  with  $\frac{x_k + x_{k+1}}{2}$ .

Equivalently we can write:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = JQ \left( \frac{x_k + x_{k+1}}{2} \right) + Bu_{k+1} \\ \frac{\Delta H}{\Delta x}(k) = Qx_k \\ y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.31})$$

**Remark:** The step  $k+1$ , in the parenthesis of the discrete-time flows, efforts and power energies, is due to the implicit discretization rule selected for the flows and thus for the scheme.

The discrete power energy associated to the storage port, can be defined as:

$$\hat{P}_S(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) \hat{f}_S(k+1) = (Qx_{k+1})^T \left( -\frac{\Delta x}{\Delta t}(k+1) \right) \quad (\text{E.32})$$

From simple relations, we can write equivalently:

$$\hat{P}_S(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.33})$$

For the control port, we can define the power energy as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_C^T(k+1) \hat{f}_C(k+1) \quad (\text{E.34})$$

From (E.34) we can further derive the relation:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.35})$$

From (E.33) and (E.35) it results the power conservation law of the discrete-time PCH system at step  $k+1$ :

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) = -\hat{e}_S^T(k+1) B \hat{f}_C(k+1) + \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.36})$$

Thus the power conservation law is satisfied in discrete-time. Using (4.18), we define the discrete-time energy (Hamiltonian) as:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1) \hat{f}_S(k+1) = y_{k+1}^T u_{k+1} \quad (\text{E.37})$$

which is equivalent to:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.38})$$

when using (E.33) and (E.35). Using (4.19), the discrete-time bilinear form for the selected symplectic discretization scheme is defined as:

$$\begin{aligned} \hat{B}_F(k+1) &\stackrel{def}{=} \left\langle \left( Q_{k+1}^x, B^T Q_{k+1}^x \right)^a, \left( -\frac{\Delta x}{\Delta t}(k+1), u_{k+1} \right)^b \right\rangle + \\ &+ \left\langle \left( Q_{k+1}^x, B^T Q_{k+1}^x \right)^b, \left( -\frac{\Delta x}{\Delta t}(k+1), u_{k+1} \right)^a \right\rangle \end{aligned} \quad (\text{E.39})$$

The implicit form of the discrete-time lossless PCH system is defined as:

$$\left( \hat{f}_S(k+1), \hat{e}_S(k+1), \hat{f}_C(k+1), \hat{e}_C(k+1) \right) \quad (\text{E.40})$$

where the interconnection Dirac structure conserves its structure by time-discretization.

Replacing the discrete-time flows and efforts in (E.40), it results:

$$\left( -\frac{\Delta x}{\Delta t}(k+1), Q_{k+1}^x, u_{k+1}, B^T Q_{k+1}^x \right) \in D \quad (\text{E.41})$$

### E.1.4 Symplectic discretization scheme IV

This symplectic time-discretization scheme assumes an implicit Euler rule for the flows and an implicit mid-point rule for the efforts. The initial continuous-time lossless PCH system for which we apply the time-discretization scheme, has the following form:

$$\begin{cases} \dot{x} = J \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.42})$$

Applying the selected time-discretization rules for the flows and efforts using (4.1) and (4.14), it results:

$$\begin{cases} x_{k+1} = x_k + \Delta t \cdot J \cdot \frac{\Delta H}{\Delta x}(k+1) + Bu_{k+1} \\ H_{k+1} = H_k + \Delta x \cdot Q \cdot \left( \frac{x_k + x_{k+1}}{2} \right) \\ y_{k+1} = B^T \cdot \frac{\Delta H}{\Delta x}(k+1) \end{cases} \quad (\text{E.43})$$

**Remark:** The composed time-discretization of the effort  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$ , becomes

$Q \left( \frac{x_k + x_{k+1}}{2} \right)$  by applying the implicit Euler rule over  $\frac{\Delta H}{\Delta x}$  at step  $k+1$ .

Equivalently we can write:

$$\begin{cases} \frac{\Delta H}{\Delta x}(k+1) = JQ \cdot \left( \frac{x_k + x_{k+1}}{2} \right) + Bu_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Q \left( \frac{x_k + x_{k+1}}{2} \right) \\ y_{k+1} = B^T \cdot Q \cdot \left( \frac{x_k + x_{k+1}}{2} \right) \end{cases} \quad (\text{E.44})$$

**Remark:** The step  $k+1$ , in the parenthesis of the discrete-time flows, efforts and power energies is due to the implicit discretization rule selected for the flows and thus for the scheme.

The discrete-time power energy associated to the storage port is defined as:

$$\hat{P}_S(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1)\hat{f}_S(k+1) = \left\langle Q \left( \frac{x_k + x_{k+1}}{2} \right) \right\rangle^T \left( -\frac{\Delta x}{\Delta t}(k+1) \right) \quad (E.45)$$

Using simple computations, we can write:

$$\hat{P}_S(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (E.46)$$

For the control port, we define the discrete-time power energy as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_C^T(k+1)\hat{f}_C(k+1) = \left\langle B^T Q \left( \frac{x_k + x_{k+1}}{2} \right) \right\rangle^T u_{k+1} \quad (E.47)$$

which results in:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (E.48)$$

from simple computations. Using (E.46) and (E.48), the power energy conservation law in discrete-time is satisfied:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) = -\hat{e}_S^T(k+1)B\hat{f}_C(k+1) + \hat{e}_S^T(k+1)B\hat{f}_C(k+1) = 0 \quad (E.49)$$

Using (4.18), the discrete-time energy (Hamiltonian) is defined as:

$$\frac{\Delta H}{\Delta x}(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1)\hat{f}_S(k+1) = y_{k+1}^T u_{k+1} \quad (E.50)$$

(E.50) can be equivalently written as:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (E.51)$$

using (E.46) and (E.48).

Using (4.19), the discrete-time bilinear form is defined as:

$$\begin{aligned} \hat{B}_F(k+1) &\stackrel{def}{=} \left\langle \left( Q\bar{x}_{k+1}, B^T Q\bar{x}_{k+1} \right)^a, \left( -\frac{\Delta x}{\Delta t}(k+1), u_{k+1} \right)^b \right\rangle + \\ &+ \left\langle \left( Q\bar{x}_{k+1}, B^T Q\bar{x}_{k+1} \right)^b, \left( -\frac{\Delta x}{\Delta t}(k+1), u_{k+1} \right)^a \right\rangle \end{aligned} \quad (E.52)$$

where  $\bar{x}_{k+1}$  is defined as:

$$\bar{x}_{k+1} \stackrel{def}{=} \frac{x_k + x_{k+1}}{2} \quad (E.53)$$

The implicit form of the discrete-time PCH system for the selected symplectic discretization scheme is defined as:

$$\left( \hat{f}_S(k+1), \hat{e}_S(k+1), \hat{f}_C(k+1), \hat{e}_C(k+1) \right) \in D \quad (E.54)$$

Replacing the discrete-time flows and efforts using (E.44), we can write equivalently:

$$\left( -\frac{\Delta x}{\Delta t}(k+1), Q\bar{x}_{k+1}, u_{k+1}, y_{k+1} \right) \in D \quad (\text{E.55})$$

### E.1.5 Symplectic discretization scheme V

For this symplectic time-discretization scheme it is assumed an implicit mid-point rule for the flows and an implicit Euler rule for the efforts. The initial continuous-time lossless PCH system for which we consider the selected symplectic discretization scheme, is:

$$\begin{cases} \dot{x} = J \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.56})$$

Applying this symplectic time-discretization scheme using (4.1) and (4.14), it results:

$$\begin{cases} x_{k+1} = x_k + J \cdot \frac{\Delta H}{\Delta x} \left( \frac{x_k + x_{k+1}}{2} \right) + Bu_{k+1} \\ H_{k+1} = H_k + \Delta x \cdot Qx_{k+1} \\ y_{k+1} = B^T \cdot \frac{\Delta H}{\Delta x} (k+1) \end{cases} \quad (\text{E.57})$$

Using the first two relations, we can write:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = J \cdot Q \cdot \left( \frac{x_k + x_{k+1}}{2} \right) + Bu_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \\ y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.58})$$

**Remark:** In this case the combined operator  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$  becomes  $Q \left( \frac{x_k + x_{k+1}}{2} \right)$ , as

we apply the implicit mid-point rule for  $\frac{\Delta H}{\Delta x}$ , which replaces  $x_{k+1}$  with  $\frac{x_k + x_{k+1}}{2}$  for the effort discretization. The step  $k+1$  in the parenthesis of the flows, efforts or power energies, corresponds to the implicit form of the discretization scheme, given by the flows time-discretization rule.

The discrete-time power energy associated to the storage port, can be defined as:

$$\hat{P}_S(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) \hat{f}_S(k+1) = (Qx_{k+1})^T \left( -\frac{\Delta x}{\Delta t}(k+1) \right) \quad (\text{E.59})$$

Equivalently we can write:

$$\hat{P}_S(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.60})$$

For the control port, the discrete-time power energy is defined as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_C^T(k+1) \hat{f}_C(k+1) \quad (\text{E.61})$$

We can write equivalently (E.61) as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (\text{E.62})$$

Using (E.60) and (E.62), the total power energy of the system is:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) = -\hat{e}_S^T(k+1)B\hat{f}_C(k+1) + \hat{e}_S^T(k+1)B\hat{f}_C(k+1) = 0 \quad (\text{E.63})$$

and thus the discrete-time power conservation law of the PCH system is satisfied.

Using (4.18), we define the discrete-time energy (Hamiltonian) as:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1)\hat{f}_S(k+1) = y_{k+1}^T u_{k+1} \quad (\text{E.64})$$

Using (E.60) and (E.62), it results further:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (\text{E.65})$$

Using (4.19), we define the discrete-time bilinear form as:

$$\begin{aligned} \hat{B}_F(k+1) &\stackrel{def}{=} \left\langle \left( Qx_{k+1}, B^T Qx_{k+1} \right)^a, \left( -\frac{\Delta x}{\Delta t}(k+1), u_{k+1} \right)^b \right\rangle + \\ &+ \left\langle \left( Qx_{k+1}, B^T Qx_{k+1} \right)^b, \left( -\frac{\Delta x}{\Delta t}(k+1), u_{k+1} \right)^a \right\rangle \end{aligned} \quad (\text{E.66})$$

The implicit form of the discrete-time PCH system is defined as:

$$\left( \hat{f}_S(k+1), \hat{e}_S(k+1), \hat{f}_C(k+1), \hat{e}_C(k+1) \right) \in D \quad (\text{E.67})$$

which results as:

$$\left( -\frac{\Delta x}{\Delta t}(k+1), Qx_{k+1}, u_{k+1}, y_{k+1} \right) \in D \quad (\text{E.68})$$

by replacing the discrete-time flows and efforts. As for the previous cases, the Dirac interconnection structure (  $D$  ) conserves its properties by time-discretization.

### E.1.6 Symplectic discretization scheme VI

For the current symplectic discretization scheme, it is assumed an implicit trapezoidal rule for the flows and an explicit Euler for the efforts. The initial continuous-time lossless PCH system for which we apply this time-discretization scheme is:

$$\begin{cases} \dot{x} = J \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.69})$$

Applying the general framework from (4.14) for the selected symplectic time-discretization scheme, we can write:

$$\begin{cases} x_{k+1} = x_k + \frac{\Delta t}{2} J \left( \frac{\Delta H}{\Delta x}(k) + \frac{\Delta H}{\Delta x}(k+1) \right) + \frac{\Delta t}{2} (Bu_k + Bu_{k+1}) \\ H_{k+1} = H_k + Qx_k, H_{k+2} = H_{k+1} + Qx_{k+1} \\ y_{k+1} = B^T \frac{\Delta H}{\Delta x}(k+1) \end{cases} \quad (\text{E.70})$$



or equivalently:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = \frac{1}{2}(JQx_k + JQx_{k+1}) + \frac{1}{2}B(u_k + u_{k+1}) \\ \frac{\Delta H}{\Delta x}(k) = Qx_k, \quad \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \\ y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.71})$$

**Remark:** By applying the composed time-discretization rule  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$  for the effort, it

results in  $Q \left( \frac{x_k + x_{k+1}}{2} \right)$ , by introducing  $\frac{\Delta H}{\Delta x}$  in the flow time-discretization rule, after

applying the Euler rule discretization. The step  $k+1$  in the parenthesis of the discrete-time flows, efforts or power energies corresponds to the implicit form of the discretization scheme, given by the flows time-discretization rule.

Next we define the power energy associated to the storage port at step  $k+1$ , as follows:

$$\hat{P}_S(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) \hat{f}_S(k+1) = (Qx_{k+1})^T \left( -\frac{\Delta x}{\Delta t}(k+1) \right) \quad (\text{E.72})$$

Using simple mathematical relations, (E.72) can be written equivalently as:

$$\hat{P}_S(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.73})$$

where we have adopted the notation:

$$\hat{f}_C(k+1) \stackrel{def}{=} \left( \frac{u_k + u_{k+1}}{2} \right) \quad (\text{E.74})$$

The control port power energy, can be defined as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_C^T(k+1) \hat{f}_C(k+1) = y_{k+1}^T \bar{u}_{k+1} \quad (\text{E.75})$$

which can be equivalently written as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.76})$$

from straightforward computations. From (E.73) and (E.76), the total power energy of the discrete-time system is:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) = -\hat{e}_S^T(k+1) B \hat{f}_C(k+1) + \hat{e}_S^T(k+1) B \hat{f}_C(k+1) = 0 \quad (\text{E.77})$$

and thus, the power conservation law is satisfied in discrete-time.

Using (4.18), the discrete-time energy (Hamiltonian) is defined as:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1) \hat{f}_S(k+1) = y_{k+1}^T u_{k+1} \quad (\text{E.78})$$

Using (E.73) and (E.76), it results as:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.79})$$

Using (4.19), the discrete bilinear form of this symplectic discretization scheme is defined as:

$$\begin{aligned} \hat{B}_F(k+1) &\stackrel{def}{=} \left\langle \left( Qx_{k+1}, B^T Qx_{k+1} \right)^a, \left( -\frac{\Delta x}{\Delta t}(k+1), \bar{u}_{k+1} \right)^b \right\rangle + \\ &+ \left\langle \left( Qx_{k+1}, B^T Qx_{k+1} \right)^b, \left( -\frac{\Delta x}{\Delta t}(k+1), \bar{u}_{k+1} \right)^a \right\rangle \end{aligned} \quad (\text{E.80})$$

The implicit form of the discrete-time PCH system is defined as:

$$\left( \hat{f}_S(k+1), \hat{e}_S(k+1), \hat{f}_C(k+1), \hat{e}_C(k+1) \right) \in D \quad (\text{E.81})$$

Replacing the discrete-time flows and efforts in (E.81), it results:

$$\left( -\frac{\Delta x}{\Delta t}(k+1), Qx_{k+1}, \bar{u}_{k+1}, y_{k+1} \right) \in D \quad (\text{E.82})$$

### E.1.7. Symplectic discretization scheme VII

This symplectic discretization scheme assumes an explicit Euler rule for the flows and an implicit trapezoidal rule for the efforts. The continuous-time form of the lossless PCH system, for which we apply the time-discretization scheme is:

$$\begin{cases} \dot{x} = J \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.83})$$

Applying the general discretization rules from (4.1) and (4.14), for the current symplectic time-discretization scheme it results:

$$\begin{cases} x_{k+1} = x_k + \Delta t \cdot J \cdot \frac{\Delta H}{\Delta x}(k) + \Delta t \cdot B \cdot u_k \\ H_k = H_{k-1} + \Delta x \cdot Q \left( \frac{x_{k-1} + x_k}{2} \right) \\ y_k = B^T \cdot \frac{\Delta H}{\Delta x}(k) \end{cases} \quad (\text{E.84})$$

**Remark:** The composed time-discretization  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$  of the effort, results in

$Q \left( \frac{x_{k-1} + x_k}{2} \right)$  by applying the flow discretization rule over the effort  $\frac{\Delta H}{\Delta x}$ .

Equivalently we can write (E.84) as:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k) = JQ \left( \frac{x_{k-1} + x_k}{2} \right) + Bu_k \\ \frac{\Delta H}{\Delta x}(k) = Q \left( \frac{x_{k-1} + x_k}{2} \right) \\ y_k = B^T Q \left( \frac{x_{k-1} + x_k}{2} \right) \end{cases} \quad (\text{E.85})$$

**Remark:** The step  $k$  in the parenthesis of the flows, efforts or power energies is due to the explicit time-discretization rule selected for the flows, which applies for the whole discretization scheme.

The discrete-time power energy associated to the storage port can be defined as:

$$\hat{P}_S(k) \stackrel{def}{=} \hat{e}_S^T(k) \hat{f}_S(k) = \left( Q \left( \frac{x_{k-1} + x_k}{2} \right) \right)^T \left( -JQ \left( \frac{x_{k-1} + x_k}{2} \right) - Bu_k \right) \quad (E.86)$$

which results in:

$$\hat{P}_S(k) \stackrel{def}{=} -\hat{e}_S^T(k) \hat{f}_C(k) \quad (E.87)$$

from straightforward computations. For the control port, we define the following discrete-time power energy:

$$\hat{P}_C(k) \stackrel{def}{=} \hat{e}_C^T(k) \hat{f}_C(k) \quad (E.88)$$

which can be equivalently written as:

$$\hat{P}_C(k) \stackrel{def}{=} \hat{e}_S^T(k) B \hat{f}_C(k) \quad (E.89)$$

using straightforward relations. From (E.87) and (E.89) the power conservation law of PCH systems is satisfied in discrete-time:

$$\hat{P}_S(k) + \hat{P}_C(k) = -\hat{e}_S^T(k) B \hat{f}_C(k) + \hat{e}_S^T(k) B \hat{f}_C(k) = 0 \quad (E.90)$$

Using (4.18), the discrete-time energy (Hamiltonian) can be defined as:

$$\frac{\Delta H}{\Delta t}(k) \stackrel{def}{=} -\hat{e}_S^T(k) \hat{f}_S(k) = y_k^T u_k \quad (E.91)$$

which is equivalent to:

$$\frac{\Delta H}{\Delta t}(k) \stackrel{def}{=} -\hat{e}_S^T(k) B \hat{f}_C(k) \quad (E.92)$$

using (E.87) and (E.89). Using (4.19), the discrete-time bilinear form is defined as:

$$\hat{B}_F(k) \stackrel{def}{=} \left\langle \left( Q \bar{x}_k, B^T Q \bar{x}_k \right)^a, \left( -\frac{\Delta x}{\Delta t}(k), u_k \right)^b \right\rangle + \left\langle \left( Q \bar{x}_k, B^T Q \bar{x}_k \right)^b, \left( -\frac{\Delta x}{\Delta t}(k), u_k \right)^a \right\rangle \quad (E.93)$$

where the notation:

$$\bar{x}_k \stackrel{def}{=} \frac{x_{k-1} + x_k}{2} \quad (E.94)$$

has been adopted. The implicit form of the discrete-time PCH system is defined as:

$$\left( \hat{f}_S(k), \hat{e}_S(k), \hat{f}_C(k), \hat{e}_C(k) \right) \in D \quad (E.95)$$

Replacing the discrete-time flows and efforts in (E.95) it results:

$$\left( -\frac{\Delta x}{\Delta t}(k), Q \bar{x}_k, u_k, y_k \right) \in D \quad (E.96)$$

### E.1.8 Symplectic discretization scheme VIII

This symplectic time-discretization scheme assumes an implicit trapezoidal rule for the flows and an implicit Euler rule for the efforts. The initial continuous-time lossless PCH system for which we apply this symplectic time-discretization scheme, is:

$$\begin{cases} \dot{x} = J \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.97})$$

Applying the general framework from (4.1) and (4.14) for the selected scheme, results in:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = \frac{1}{2} \left( \frac{\Delta H}{\Delta x}(k) + \frac{\Delta H}{\Delta x}(k+1) \right) + \frac{1}{2} (Bu_k + u_{k+1}) \\ \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \\ y_{k+1} = B^T \frac{\Delta H}{\Delta x}(k+1) \end{cases} \quad (\text{E.98})$$

Adopting the following notations:

$$\begin{cases} \overline{\frac{\Delta H}{\Delta x}}(k+1) \stackrel{\text{def}}{=} \frac{1}{2} (Qx_k + Qx_{k+1}) \\ \overline{u}_{k+1} \stackrel{\text{def}}{=} \frac{u_k + u_{k+1}}{2} \end{cases} \quad (\text{E.99})$$

it results:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = J \overline{\frac{\Delta H}{\Delta x}}(k+1) + B\overline{u}_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1}, y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.100})$$

**Remark:** The composed time-discretization of the effort  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$ , results in  $Q \left( \frac{x_k + x_{k+1}}{2} \right)$ , by applying the flow discretization rule for  $\frac{\Delta H}{\Delta x}$ . The step  $k+1$  in the parenthesis of the flows, efforts or power energies, are due to the implicit discretization rule selected for the flows, which apply for the whole scheme.

For the storage port, the discrete-time power energy can be defined as:

$$\hat{P}_S(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1) \hat{f}_S(k+1) = (Qx_{k+1})^T \left( -\frac{\Delta x}{\Delta t}(k+1) \right) \quad (\text{E.101})$$

Equivalently we can write (E.101) as:

$$\hat{P}_S(k+1) = -\hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.102})$$

The discrete-time power energy associated to the control port, can be defined as:

$$\hat{P}_C(k+1) \stackrel{\text{def}}{=} \hat{e}_C^T(k+1) \hat{f}_C(k+1) \quad (\text{E.103})$$

From straightforward computations, this relation is equivalent to:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.104})$$

From (E.102) and (E.104), the discrete-time power conservation law is satisfied:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) = -\hat{e}_S^T(k+1) B \hat{f}_C(k+1) + \hat{e}_S^T(k+1) B \hat{f}_C(k+1) = 0 \quad (\text{E.105})$$

Using (4.18), we can define the discrete-time energy (Hamiltonian) as:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1) \hat{f}_S(k+1) = y_{k+1}^T \bar{u}_{k+1} \quad (\text{E.106})$$

which can be equivalently written as:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.107})$$

using (E.102) and (E.104). Using (4.19), the discrete-time bilinear form can be defined as:

$$\begin{aligned} \hat{B}_F(k+1) &\stackrel{def}{=} \left\langle \left( Q_{x_{k+1}}, B^T Q_{x_{k+1}} \right)^a, \left( -\frac{\Delta x}{\Delta t}(k+1), \bar{u}_{k+1} \right)^b \right\rangle + \\ &+ \left\langle \left( Q_{x_{k+1}}, B^T Q_{x_{k+1}} \right)^b, \left( -\frac{\Delta x}{\Delta t}(k+1), \bar{u}_{k+1} \right)^a \right\rangle \end{aligned}$$

The implicit form of the discrete-time PCH system is defined as:

$$\left( \hat{f}_S(k+1), \hat{e}_S(k+1), \hat{f}_C(k+1), \hat{e}_C(k+1) \right) \in D \quad (\text{E.108})$$

where the Dirac interconnection structure ( $D$ ) is conserved by time-discretization.

Replacing the discrete-time flows and efforts, we can write equivalently:

$$\left( -\frac{\Delta x}{\Delta t}(k+1), Q_{x_{k+1}}, y_{k+1}, \bar{u}_{k+1} \right) \in D \quad (\text{E.109})$$

### E.1.9 Symplectic discretization scheme IX

This symplectic time-discretization scheme makes use of an implicit mid-point rule for the flows and an implicit trapezoidal rule for the efforts. The continuous-time lossless PCH system, for which we apply this symplectic time-discretization scheme is:

$$\begin{cases} \dot{x} = J \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.110})$$

Using (4.1) and (4.14) for the selected symplectic time-discretization rule, it results:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = \frac{1}{2} J Q \left( \frac{x_{k-1} + x_k}{2} + \frac{x_k + x_{k+1}}{2} \right) + Bu_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Q \left( \frac{x_k + x_{k+1}}{2} \right) \\ y_{k+1} = B^T Q \left( \frac{x_k + x_{k+1}}{2} \right) \end{cases} \quad (\text{E.111})$$

**Remark:** The composed time-discretization of the effort  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$  results in

$\frac{1}{2} Q \left( \frac{x_{k-1} + x_k}{2} + \frac{x_k + x_{k+1}}{2} \right)$ , by applying the flow discretization rule on  $\frac{\Delta H}{\Delta x}$ , which

replaces  $x_k$  and  $x_{k+1}$  with  $\frac{x_{k-1} + x_k}{2}$  and  $\frac{x_k + x_{k+1}}{2}$ .

Adopting the following notations:

$$\left\{ \begin{array}{l} \bar{x}_k \stackrel{def}{=} \left( \frac{x_{k-1} + x_k}{2} \right) \\ \bar{x}_{k+1} \stackrel{def}{=} \frac{1}{2} (\bar{x}_k + \bar{x}_{k+1}) \end{array} \right. \quad (\text{E.112})$$

(E.111) becomes:

$$\left\{ \begin{array}{l} \frac{\Delta x}{\Delta t}(k+1) = JQ \bar{x}_{k+1} + Bu_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Q \bar{x}_{k+1} \\ y_{k+1} = B^T Q \bar{x}_{k+1} \end{array} \right. \quad (\text{E.113})$$

**Remark:** The step  $k+1$ , in the parenthesis of the flows, efforts or power port energies corresponds to the implicit form of the discrete-time scheme, given by the flow discretization rule.

The discrete-time power energy corresponding to the storage port, can be defined as:

$$\hat{P}_S(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) \hat{f}_S(k+1) = (Q \bar{x}_{k+1})^T (-JQ \bar{x}_{k+1} - Bu_{k+1}) \quad (\text{E.114})$$

which is equivalent to:

$$\hat{P}_S(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.115})$$

For the control port power energy, we define the discrete-time energy as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_C^T(k+1) \hat{f}_C(k+1) = (B^T Q \bar{x}_{k+1})^T u_{k+1} \quad (\text{E.116})$$

which can be also equivalently written as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.117})$$

From (E.115) and (E.117), the the discrete-time power conservation law is satisfied:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) = -\hat{e}_S^T(k+1) B \hat{f}_C(k+1) + \hat{e}_S^T(k+1) B \hat{f}_C(k+1) = 0 \quad (\text{E.118})$$

Using (4.18), the discrete-time energy (Hamiltonian) is:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1) \hat{f}_S(k+1) = \hat{e}_C^T(k+1) \hat{f}_C(k+1) \quad (\text{E.119})$$

which is equivalent to:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.120})$$

using (E.115) and (E.117). Using (4.19), the discrete-time bilinear form for the selected discretization scheme is defined as:

$$\begin{aligned} \hat{B}_F(k+1) &\stackrel{def}{=} \left\langle \left( \mathcal{Q}\bar{x}_{k+1}, B^T \mathcal{Q}\bar{x}_{k+1} \right)^a, \left( -\frac{\Delta x}{\Delta t}(k+1), u_{k+1} \right)^b \right\rangle + \\ &+ \left\langle \left( \mathcal{Q}\bar{x}_{k+1}, B^T \mathcal{Q}\bar{x}_{k+1} \right)^b, \left( -\frac{\Delta x}{\Delta t}(k+1), u_{k+1} \right)^a \right\rangle \end{aligned} \quad (\text{E.121})$$

The implicit discrete-time form of the PCH system is:

$$\left( \hat{f}_S(k+1), \hat{e}_S(k+1), \hat{f}_C(k+1), \hat{e}_C(k+1) \right) \quad (\text{E.122})$$

which is equivalent to:

$$\left( -\frac{\Delta x}{\Delta t}(k+1), \mathcal{Q}\bar{x}_{k+1}, u_{k+1}, y_{k+1} \right) \in D \quad (\text{E.123})$$

when replacing the discrete-time flows and efforts.

## E.2 Symplectic discretization schemes for lossy PCH systems

### E.2.1 Symplectic discretization scheme I

This symplectic discretization rule assumes an implicit Euler rule for the flows and an explicit Euler rule for the efforts. The initial continuous-time lossy PCH system, for which we apply the current symplectic time-discretization scheme is:

$$\begin{cases} \dot{x} = (J - R) \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.124})$$

By applying the implicit Euler for the flows and explicit Euler for the efforts, we get the following relations:

$$\begin{cases} x_{k+1} = x_k + \Delta t \cdot (J - R) \frac{\Delta H}{\Delta x}(k+1) + \Delta t \cdot B \cdot u_{k+1} \\ H_{k+1} = H_k + \Delta x \cdot \mathcal{Q}x_k \\ y_{k+1} = B^T \frac{\Delta H}{\Delta x}(k+1) \end{cases} \quad (\text{E.125})$$

**Remark:** The composed time-discretization of the effort  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$  becomes  $\mathcal{Q}x_{k+1}$ , by

applying the the implicit Euler rule for  $\frac{\Delta H}{\Delta x}$ .

Equivalently we can write:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = (J-R)Qx_{k+1} + Bu_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \\ y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.126})$$

**Remark:** The step  $k+1$ , in the parenthesis of the flows, effort or power energies, correspond to the implicit form of the scheme, given by the implicit time-discretization rule selected for the flows.

The discrete-time storage port power energy is defined as:

$$\hat{P}_S(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1)\hat{f}_S(k+1) = (Qx_{k+1})^T \left( -(J-R)Qx_{k+1} - Bu_{k+1} \right) \quad (\text{E.127})$$

which is equivalent to:

$$\hat{P}_S(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1)R\hat{e}_S^T(k+1) - \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (\text{E.128})$$

For the control port, we can define the following discrete-time power energy:

$$\hat{P}_C(k+1) \stackrel{\text{def}}{=} \hat{e}_C^T(k+1)\hat{f}_C(k+1) = \left( B^T Qx_{k+1} \right)^T u_{k+1} \quad (\text{E.129})$$

which is equivalent to:

$$\hat{P}_C(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (\text{E.130})$$

For the dissipation port, we define the power energy as:

$$\hat{P}_R(k+1) \stackrel{\text{def}}{=} \hat{e}_R^T(k+1)\hat{f}_R(k+1) \quad (\text{E.131})$$

Using the general relation:

$$f_R = -\text{Re}_R \quad (\text{E.132})$$

we can write further the discrete-time power energy as:

$$\hat{P}_R(k+1) \stackrel{\text{def}}{=} -\hat{e}_R^T(k+1)R\hat{e}_R(k+1) \quad (\text{E.133})$$

The discrete-time dissipation port effort variable, can be written as:

$$\hat{e}_R(k+1) \stackrel{\text{def}}{=} \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \quad (\text{E.134})$$

Then the discrete-time power energy of (E.133) becomes:

$$\hat{P}_R(k+1) \stackrel{\text{def}}{=} -\left( Qx_{k+1} \right)^T R \left( Qx_{k+1} \right) \quad (\text{E.135})$$

which is equivalent to:

$$\hat{P}_R(k+1) \stackrel{\text{def}}{=} -\hat{e}_S^T(k+1)R\hat{e}_S(k+1) \quad (\text{E.136})$$

Using (E.128), (E.130) and (E.136), the power conservation law is satisfied in discrete-time:

$$\begin{aligned} \hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) &= \hat{e}_S^T(k+1)R\hat{e}_S(k+1) - \hat{e}_S^T(k+1)B\hat{f}_C(k+1) + \\ &+ \hat{e}_S^T(k+1)B\hat{f}_C(k+1) - \hat{e}_S^T(k+1)R\hat{e}_S(k+1) = 0 \end{aligned} \quad (\text{E.137})$$

Using (4.31), the discrete-time energy (Hamiltonian) can be defined as:



$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1)\hat{f}_S(k+1) = y_{k+1}^T u_{k+1} + \hat{e}_R^T(k+1)\hat{f}_R(k+1) \quad (\text{E.138})$$

which can be written as:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1)B\hat{f}_C(k+1) - \hat{e}_S^T(k+1)R\hat{e}_S(k+1) \quad (\text{E.139})$$

Using (E.128), (E.130) and (E.136). Further we define the discrete-time bilinear form as:

$$\hat{B}_F(k+1) \stackrel{def}{=} \left\langle \left( Qx_{k+1}, B^T Qx_{k+1}, Qx_{k+1} \right)^a, \left( -\frac{\Delta x}{\Delta t}(k+1), u_{k+1}, -RQx_{k+1} \right)^b \right\rangle + \left\langle \left( Qx_{k+1}, B^T Qx_{k+1}, Qx_{k+1} \right)^b, \left( -\frac{dx}{dt}(k+1), u_{k+1}, -RQx_{k+1} \right)^a \right\rangle$$

The implicit form of the discrete-time PCH system is defined as:

$$\left( \hat{f}_S(k+1), \hat{e}_S(k+1), \hat{f}_C(k+1), \hat{e}_C(k+1), \hat{f}_R(k+1), \hat{e}_R(k+1) \right) \in D \quad (\text{E.140})$$

which is equivalent to:

$$\left( -\frac{dx}{dt}(k+1), Qx_{k+1}, u_{k+1}, y_{k+1}, -RQx_{k+1}, Qx_{k+1} \right) \in D \quad (\text{E.141})$$

when replacing the discrete-time flows and efforts.

## E.2.2. Symplectic discretization scheme II

For this scheme it is assumed an explicit Euler rule for the flows and an implicit mid-point rule for the efforts. The initial continuous-time lossy PCH system, for which we apply the current time-discretization scheme is:

$$\begin{cases} \dot{x} = (J - R) \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.142})$$

Using (4.1) and (4.24) for the current symplectic time-discretization scheme, it results:

$$\begin{cases} x_{k+1} = x_k + \Delta t \cdot (J - R) \frac{\Delta H}{\Delta x}(k) + \Delta t \cdot Bu_k \\ H_k = H_{k-1} + \Delta x \cdot Q \left( \frac{x_{k-1} + x_k}{2} \right) \\ y_k = B^T \frac{\Delta H}{\Delta x}(k) \end{cases} \quad (\text{E.143})$$

**Remark:** By applying the composed time-discretization for the effort  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$ , it

results  $Q \left( \frac{x_{k-1} + x_k}{2} \right)$ , when using the explicit Euler rule for  $\frac{\Delta H}{\Delta x}$ .

We can write equivalently:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k) = (J - R)Q\bar{x}_k + Bu_k \\ \frac{\Delta x}{\Delta t}(k) = Q\bar{x}_k \\ y_k = B^T Q\bar{x}_k \end{cases} \quad (\text{E.144})$$

when adopting the notation:

$$\bar{x}_k \stackrel{\text{def}}{=} \left( \frac{x_{k-1} + x_k}{2} \right) \quad (\text{E.145})$$

**Remark:** The step  $k$  in the parenthesis of the flows, efforts or power energies, correspond to the explicit form of the discrete-time scheme, given by the flow discretization rule.

For the discrete-time storage port power energy, we can write:

$$\hat{P}_S(k) \stackrel{\text{def}}{=} \hat{e}_S^T(k) \hat{f}_S(k) = (Q\bar{x}_k)^T (- (J - R)Q\bar{x}_k - Bu_k) \quad (\text{E.146})$$

More compact we can express this relation as:

$$\hat{P}_S(k) \stackrel{\text{def}}{=} \hat{e}_S^T(k) R \hat{e}_S(k) - \hat{e}_S^T(k) B \hat{f}_C(k) \quad (\text{E.147})$$

For the control port, the power energy can be defined as:

$$\hat{P}_C(k) \stackrel{\text{def}}{=} \hat{e}_C^T(k) \hat{f}_C(k) \quad (\text{E.148})$$

which becomes:

$$\hat{P}_C(k) \stackrel{\text{def}}{=} \hat{e}_S^T(k) B \hat{f}_C(k) \quad (\text{E.149})$$

from straightforward computations.

For the dissipation port, the discrete-time power energy can be defined as:

$$\hat{P}_R(k) \stackrel{\text{def}}{=} \hat{e}_R^T(k) \hat{f}_R(k) \quad (\text{E.150})$$

which is equivalent to:

$$\hat{P}_R(k) \stackrel{\text{def}}{=} -\hat{e}_R^T(k) R \hat{e}_R(k) \quad (\text{E.151})$$

by using the general relation:

$$f_R = -\text{Re}_R \quad (\text{E.152})$$

But the discrete-time effort variable can be defined as:

$$\hat{e}_R(k) \stackrel{\text{def}}{=} \frac{\Delta H}{\Delta x}(k) \quad (\text{E.153})$$

and (E.151) becomes:

$$\hat{P}_R(k) \stackrel{\text{def}}{=} -\hat{e}_S^T(k) R \hat{e}_S(k) \quad (\text{E.154})$$

Using (E.147), (E.149) and (E.154), the discrete-time power conservation law holds:

$$\hat{P}_S(k) + \hat{P}_C(k) + \hat{P}_R(k) = \hat{e}_S^T(k) R \hat{e}_S(k) - \hat{e}_S^T(k) B \hat{f}_C(k) + \hat{e}_S^T(k) B \hat{f}_C(k) - \hat{e}_S^T(k) R \hat{e}_S(k) \quad (\text{E.155})$$

Using (4.31), the discrete-time energy (Hamiltonian) can be defined as:

$$\frac{\Delta H}{\Delta t}(k) \stackrel{def}{=} -\hat{e}_S^T(k)\hat{f}_S(k) = y_k^T u_k + \hat{e}_R^T \hat{f}_R \quad (\text{E.156})$$

Using (E.147), (E.149) and (E.154), it results:

$$\frac{\Delta H}{\Delta t}(k) \stackrel{def}{=} \hat{e}_S^T(k)B\hat{f}_C(k) - \hat{e}_S^T(k)R\hat{e}_S(k) \quad (\text{E.157})$$

Using (4.32), the discrete-time bilinear form is defined as:

$$\begin{aligned} \hat{B}_F(k) \stackrel{def}{=} & \left\langle \left( Q\bar{x}_k, B^T Q\bar{x}_k, Q\bar{x}_k \right)^a, \left( -\frac{dx}{dt}(k), u_k, -RQ\bar{x}_k \right)^b \right\rangle + \\ & + \left\langle \left( Q\bar{x}_k, B^T Q\bar{x}_k, Q\bar{x}_k \right)^b, \left( -\frac{dx}{dt}(k), u_k, -RQ\bar{x}_k \right)^a \right\rangle \end{aligned} \quad (\text{E.158})$$

The implicit form of the discrete-time PCH system is defined as:

$$\left( \hat{f}_S(k), \hat{e}_S(k), \hat{f}_C(k), \hat{e}_C(k), \hat{f}_R(k), \hat{e}_R(k) \right) \in D \quad (\text{E.159})$$

using (4.33) and (4.34). Replacing the discrete-time flows and efforts we can write:

$$\left( -\frac{dx}{dt}(k), Q\bar{x}_k, u_k, y_k, -RQ\bar{x}_k, Q\bar{x}_k \right) \in D \quad (\text{E.160})$$

### E.2.3 Symplectic discretization scheme III

For this symplectic discretization scheme, we assume an implicit mid-point rule for the flows and an explicit Euler rule for the efforts. The initial continuous-time lossy PCH system, for which we apply this time-discretization scheme, is:

$$\begin{cases} \dot{x} = (J - R) \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.161})$$

Applying the current symplectic time-discretization scheme using (4.1) and (4.24), it results:

$$\begin{cases} x_{k+1} = x_k + \Delta t \cdot (J - R) \frac{\Delta H}{\Delta x} \left( k+1, \frac{x_k + x_{k+1}}{2} \right) + \Delta t \cdot Bu_{k+1} \\ H_{k+1} = H_k + \Delta x \cdot Qx_k \\ y_{k+1} = B^T \frac{\Delta H}{\Delta x} (k+1) \end{cases} \quad (\text{E.162})$$

**Remark:** The composed time-discretization of the effort  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$ , results in

$Q \left( \frac{x_k + x_{k+1}}{2} \right)$ , by applying the implicit mid-point rule for  $\frac{\Delta H}{\Delta x}$ .

Adopting the following notation:

$$\bar{x}_{k+1} \stackrel{def}{=} \left( \frac{x_k + x_{k+1}}{2} \right) \quad (\text{E.163})$$

It results:

$$\begin{cases} \frac{\Delta x}{\Delta t} = (J - R)Q\bar{x}_{k+1} + Bu_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \\ y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.164})$$

**Remark:** The step  $k+1$  in the parenthesis of the discrete-time flows, efforts or power energies, correspond to the implicit form of the discretization scheme, given by the flow time-discretization rule.

The discrete-time power energy associated to the storage port can be defined as:

$$\hat{P}_S(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1)\hat{f}_S(k+1) = (Qx_{k+1})^T \left( -(J - R)Q\bar{x}_{k+1} - Bu_{k+1} \right) \quad (\text{E.165})$$

Using simple mathematical relations, we can write further:

$$\hat{P}_S(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1)R\hat{e}_S(k+1) - \hat{e}_S^T(k+1)Bf_C(k+1) \quad (\text{E.166})$$

where:

$$\hat{e}_S(k+1) \stackrel{\text{def}}{=} \left( \frac{\hat{e}_S(k) + \hat{e}_S(k+1)}{2} \right) = Q \left( \frac{x_k + x_{k+1}}{2} \right) \quad (\text{E.167})$$

The discrete-time power energy associated to the control port can be defined as:

$$\hat{P}_C(k+1) \stackrel{\text{def}}{=} \hat{e}_C^T(k+1)\hat{f}_C(k+1) = \left( B^T Qx_{k+1} \right)^T u_{k+1} \quad (\text{E.168})$$

Equivalently we can write:

$$\hat{P}_C(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1)Bf_C(k+1) \quad (\text{E.169})$$

For the dissipation port, the discrete-time power energy is defined as:

$$\hat{P}_R(k+1) \stackrel{\text{def}}{=} \hat{e}_R^T(k+1)\hat{f}_R(k+1) \quad (\text{E.170})$$

Using the general relation:

$$f_R = -\text{Re}_R \quad (\text{E.171})$$

between flows and efforts, we can write (E.170) as:

$$\hat{P}_R(k+1) \stackrel{\text{def}}{=} -\hat{e}_R^T(k+1)R\hat{e}_R(k+1) \quad (\text{E.172})$$

Using the discrete-time effort relation:

$$\hat{e}_R(k+1) \stackrel{\text{def}}{=} \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \quad (\text{E.173})$$

the discrete-time power energy becomes:

$$\hat{P}_R(k+1) \stackrel{\text{def}}{=} -\hat{e}_S^T(k+1)R\hat{e}_S(k+1) \quad (\text{E.174})$$

Using (E.166),(E.169) and (E.174), the discrete-time power conservation law is not satisfied:

$$\begin{aligned} \hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) &= \hat{e}_S^T(k+1)R\bar{e}_S(k+1) - \hat{e}_S^T(k+1)Bf_C(k+1) + \\ &+ \hat{e}_S^T(k+1)Bf_C(k+1) - \hat{e}_S^T(k+1)R\hat{e}_S(k+1) \end{aligned}$$

(E.175)

Equivalently, we can write (E.175) as:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) = \hat{e}_S^T(k+1)R \left\{ \frac{Q^x_k - Q^x_{k+1}}{2} \right\} \quad (\text{E.176})$$

and thus the power conservation law doesn't hold for this discretization scheme, as (4.30) is not satisfied.

**Remark:** *As the discrete-time power conservation law doesn't hold for this scheme, the rest of discrete-time PCH systems properties will not be defined.*

#### E.2.4 Symplectic discretization scheme IV

In this case, an implicit Euler rule for the flows and an implicit mid-point rule for the efforts are selected for time-discretization. The continuous-time PCH system for which we apply the symplectic discretization scheme is:

$$\begin{cases} \dot{x} = (J - R) \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.177})$$

Using the general framework relations from (4.1) and (4.24), for the current symplectic time-discretization scheme, it results:

$$\begin{cases} x_{k+1} = x_k + \Delta t \cdot (J - R) \frac{\Delta H}{\Delta x} \left( k+1, \frac{x_k + x_{k+1}}{2} \right) + \Delta t \cdot Bu_{k+1} \\ H_{k+1} = H_k + \Delta x \cdot Q \left( \frac{x_k + x_{k+1}}{2} \right) \\ y_{k+1} = B^T \frac{\Delta H}{\Delta x} (k+1) \end{cases} \quad (\text{E.178})$$

**Remark:** *The composed time discretization of the effort  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$ , results in  $Q \left( \frac{x_k + x_{k+1}}{2} \right)$ , as we apply the flow discretization rule over  $\frac{\Delta H}{\Delta x}$ .*

Adopting the following notation:

$$\bar{x}_{k+1} \stackrel{def}{=} \left( \frac{x_k + x_{k+1}}{2} \right) \quad (\text{E.179})$$

(E.178) becomes:

$$\begin{cases} \frac{\Delta x}{\Delta t} (k+1) = (J - R) Q \bar{x}_{k+1} + Bu_{k+1} \\ \frac{\Delta H}{\Delta x} (k+1) = Q \bar{x}_{k+1} \\ y_{k+1} = Q \bar{x}_{k+1} \end{cases} \quad (\text{E.180})$$

**Remark:** The step  $k+1$  in the parenthesis of the discrete-time flows, efforts or power energies correspond to the implicit form of the discretization scheme, given by the flows time-discretization rule.

The discrete-time power energy associated to the storage port, can be defined as:

$$\hat{P}_S(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1)\hat{f}_S(k+1) = (\mathcal{Q}\bar{x}_{k+1})^T \left( -(J-R)\mathcal{Q}\bar{x}_{k+1} - Bu_{k+1} \right) \quad (\text{E.181})$$

which is equivalent to:

$$\hat{P}_S(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1)R\hat{e}_S(k+1) - \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (\text{E.182})$$

from simple calculus. For the control port, the discrete-time power energy can be defined as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_C^T(k+1)\hat{f}_C(k+1) = \left( B^T \mathcal{Q}\bar{x}_{k+1} \right)^T \hat{f}_C(k+1) \quad (\text{E.183})$$

which is equivalent to:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (\text{E.184})$$

For the dissipation port, the discrete-time power energy is defined as:

$$\hat{P}_R(k+1) \stackrel{def}{=} \hat{e}_R^T(k+1)\hat{f}_R(k+1) \quad (\text{E.185})$$

which transforms to:

$$\hat{P}_R(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1)R\hat{e}_R(k+1) \quad (\text{E.186})$$

using the general relation:

$$f_R = -\text{Re}_R \quad (\text{E.187})$$

Using the discrete-time effort variable:

$$\hat{e}_R(k+1) \stackrel{def}{=} \frac{\Delta H}{\Delta x}(k+1) = \mathcal{Q}\bar{x}_{k+1} \quad (\text{E.188})$$

(E.186) becomes:

$$\hat{P}_R(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1)R\hat{e}_S(k+1) \quad (\text{E.189})$$

From (E.182), (E.184) and (E.186), the discrete-time power conservation law is satisfied:

$$\begin{aligned} \hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) &= \hat{e}_S^T(k+1)R\hat{e}_S(k+1) - \hat{e}_S^T(k+1)B\hat{f}_C(k+1) + \\ &+ \hat{e}_S^T(k+1)B\hat{f}_C(k+1) - \hat{e}_S^T(k+1)R\hat{e}_S(k+1) = 0 \end{aligned} \quad (\text{E.190})$$

Using (4.31), the discrete-time energy (Hamiltonian) is defined as:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1)\hat{f}_S(k+1) = y_{k+1}^T u_{k+1} + \hat{e}_R^T(k+1)\hat{f}_R(k+1) \quad (\text{E.191})$$

Using (E.182), (E.184) and (E.189), we can write equivalently:

$$\frac{\Delta H}{\Delta t}(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1)B\hat{f}_C(k+1) - \hat{e}_S^T(k+1)R\hat{e}_S(k+1) \quad (\text{E.192})$$

Using (4.32), the discrete-time bilinear form can be defined as:

$$\hat{B}_F(k+1) \stackrel{def}{=} \left\langle \left( Q\bar{x}_{k+1}, B^T Q\bar{x}_{k+1}, Q\bar{x}_{k+1} \right)^a ; \left( -\frac{\Delta x}{\Delta t}(k+1), u_{k+1}, -RQ\bar{x}_{k+1} \right)^b \right\rangle + \left\langle \left( Q\bar{x}_{k+1}, B^T Q\bar{x}_{k+1}, Q\bar{x}_{k+1} \right)^b ; \left( -\frac{\Delta x}{\Delta t}(k+1), u_{k+1}, -RQ\bar{x}_{k+1} \right)^a \right\rangle \quad (\text{E.193})$$

The implicit discrete-time form is defined as:

$$\left( \hat{f}_S(k+1), \hat{e}_S(k+1), \hat{f}_C(k+1), \hat{e}_C(k+1), \hat{f}_R(k+1), \hat{e}_R(k+1) \right) \in D \quad (\text{E.194})$$

Replacing the discrete-time flows and efforts we can write equivalently:

$$\left( -\frac{\Delta x}{\Delta t}(k+1), Q\hat{x}_{k+1}, u_{k+1}, y_{k+1}, -RQ\bar{x}_{k+1}, Q\bar{x}_{k+1} \right) \in D \quad (\text{E.195})$$

### E.2.5 Symplectic discretization scheme V

On this scheme, we assume an implicit mid-point rule for the flows and an implicit Euler for the efforts. The initial continuous-time lossy PCH system, for which we perform the time-discretization is:

$$\begin{cases} \dot{x} = (J - R) \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.196})$$

Applying the time-discretization rules for the flows and efforts using the general framework of (4.1) and (4.24), it results:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = (J - R) Q \left( \frac{x_k + x_{k+1}}{2} \right) + Bu_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \\ y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.197})$$

**Remark:** The composed time-discretization of the effort  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$ , results in

$$Q \left( \frac{x_k + x_{k+1}}{2} \right), \text{ by applying the implicit mid-point rule on } \frac{\Delta H}{\Delta x}.$$

Introducing the notation:

$$\bar{x}_{k+1} \stackrel{def}{=} \left( \frac{x_k + x_{k+1}}{2} \right) \quad (\text{E.198})$$

we can write equivalently:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = (J-R)Q\bar{x}_{k+1} + Bu_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \\ y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.199})$$

**Remark:** The step  $k+1$  in the parenthesis of the discrete-time flows, efforts or power energies correspond to the implicit form of the discretization scheme, given by the flows time-discretization rule.

For the storage port, the discrete-time power energy can be defined as:

$$\hat{P}_S(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1)\hat{f}_S(k+1) = (Qx_{k+1})^T \left[ - (J-R)Q \left( \frac{x_k + x_{k+1}}{2} \right) - Bu_{k+1} \right] \quad (\text{E.200})$$

Equivalently we can develop as:

$$\hat{P}_S(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1)R\hat{e}_S(k+1) - \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (\text{E.201})$$

For the control port power energy, we define the discrete-time power energy as:

$$\hat{P}_C(k+1) \stackrel{\text{def}}{=} \hat{e}_C^T(k+1)\hat{f}_C(k+1) = \left( B^T Qx_{k+1} \right)^T u_{k+1} \quad (\text{E.202})$$

which is equivalent to:

$$\hat{P}_C(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (\text{E.203})$$

For the dissipation port, the discrete-time power energy is defined as:

$$\hat{P}_R(k+1) \stackrel{\text{def}}{=} \hat{e}_R^T(k+1)\hat{f}_R(k+1) \quad (\text{E.204})$$

which is equivalent to:

$$\hat{P}_R(k+1) \stackrel{\text{def}}{=} -\hat{e}_R^T(k+1)R\hat{e}_R(k+1) \quad (\text{E.205})$$

when using the general relation:

$$f_R = -Re_R \quad (\text{E.206})$$

The discrete-time effort variable is defined as:

$$\hat{e}_R(k+1) \stackrel{\text{def}}{=} \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \quad (\text{E.207})$$

Then the discrete-time power energy becomes:

$$\hat{P}_R(k+1) \stackrel{\text{def}}{=} -\hat{e}_S^T(k+1)R\hat{e}_S(k+1) \quad (\text{E.208})$$

Using (E.201), (E.203) and (E.208), the power conservation law doesn't hold in this case:

$$\begin{aligned} \hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) &= \hat{e}_S^T(k+1)R\hat{e}_S(k+1) - \hat{e}_S^T(k+1)B\hat{f}_C(k+1) + \\ &+ \hat{e}_S^T(k+1)B\hat{f}_C(k+1) - \hat{e}_S^T(k+1)R\hat{e}_S(k+1) \end{aligned} \quad (\text{E.209})$$

More compact, we can write this relation as:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) = \hat{e}_S^T(k+1)R \left( \hat{e}_S^T(k+1) - \hat{e}_S(k+1) \right) \quad (\text{E.210})$$



or replacing the discrete-time flows and efforts:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) = (Qx_{k+1})^T RQ \left( \frac{x_k - x_{k+1}}{2} \right) \quad (\text{E.211})$$

**Remark:** The current time-discretization scheme is not a symplectic one, as (4.30) doesn't hold and thus, the rest of PCH systems properties are not defined.

### E.2.6 Symplectic discretization scheme VI

In this case, an implicit trapezoidal rule for the flows and an explicit Euler rule are selected for time-discretization. The initial continuous-time lossy PCH system, for which we apply the discretization scheme is:

$$\begin{cases} \dot{x} = (J - R) \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.212})$$

Applying this symplectic time-discretization scheme using the general framework of (4.1) and (4.24), it results:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = (J - R)Q \left( \frac{x_k + x_{k+1}}{2} \right) + B \left( \frac{u_k + u_{k+1}}{2} \right) \\ H_{k+1} = H_k + Qx_k \\ y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.213})$$

**Remark:** The composed time-discretization of the effort  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$ , results in

$$Q \left( \frac{x_k + x_{k+1}}{2} \right), \text{ by applying the implicit trapezoidal rule on } \frac{\Delta H}{\Delta x}.$$

Equivalently (E.213) can be written as:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = (J - R)Q \left( \frac{x_k + x_{k+1}}{2} \right) + B \left( \frac{u_k + u_{k+1}}{2} \right) \\ \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \\ y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.214})$$

Introducing the following notations:

$$\bar{x}_{k+1} \stackrel{\text{def}}{=} \frac{x_k + x_{k+1}}{2} \quad (\text{E.215})$$

$$\bar{u}_{k+1} \stackrel{\text{def}}{=} \frac{u_k + u_{k+1}}{2} \quad (\text{E.216})$$

we can write equivalently:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = (J-R)Q\bar{x}_{k+1} + B\bar{u}_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \\ y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.217})$$

**Remark:** The step  $k+1$  in the parenthesis of the discrete-time flows, efforts or power energies correspond to the implicit form of the discrete-time scheme, given by the flows discretization rule.

The discrete-time power energy associated to the storage port can be defined as:

$$\hat{P}_S(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1)\hat{f}_S(k+1) = (Qx_{k+1})^T \left( -(J-R)Q\bar{x}_{k+1} - B\bar{u}_{k+1} \right) \quad (\text{E.218})$$

which is equivalent to:

$$\hat{P}_S(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1)R\hat{e}_S(k+1) - \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (\text{E.219})$$

where:

$$\hat{e}_S(k+1) \stackrel{\text{def}}{=} \left\{ \frac{\hat{e}_S(k) + \hat{e}_S(k+1)}{2} \right\} \quad (\text{E.220})$$

For the control port, the discrete-time energy is defined as:

$$\hat{P}_C(k+1) \stackrel{\text{def}}{=} \hat{e}_C^T(k+1)\hat{f}_C(k+1) = \left( B^T Qx_{k+1} \right)^T \bar{u}_{k+1} \quad (\text{E.221})$$

which is equivalent to:

$$\hat{P}_C(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (\text{E.222})$$

For the dissipation port, the discrete-time power energy can be defined as:

$$\hat{P}_R(k+1) \stackrel{\text{def}}{=} \hat{e}_R^T(k+1)\hat{f}_R(k+1) \quad (\text{E.223})$$

which is equivalent to:

$$\hat{P}_R(k+1) \stackrel{\text{def}}{=} -\hat{e}_S^T(k+1)R\hat{e}_S(k+1) \quad (\text{E.224})$$

when using the following general relation:

$$f_R = -\text{Re}_R \quad (\text{E.225})$$

The dissipation port effort variable can be defined as:

$$\hat{e}_R(k+1) = \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \quad (\text{E.226})$$

From (E.219), (E.222) and (E.224), the discrete-time power conservation law results as:

$$\begin{aligned} \hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) &= \hat{e}_S^T(k+1)R\hat{e}_S(k+1) - \hat{e}_S^T(k+1)B\hat{f}_C(k+1) + \\ &+ \hat{e}_S^T(k+1)B\hat{f}_C(k+1) - \hat{e}_S^T(k+1)R\hat{e}_S(k+1) \end{aligned} \quad (\text{E.227})$$

which is equivalent to:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) = (Qx_{k+1})^T RQ \left( \frac{x_k - x_{k+1}}{2} \right) \quad (\text{E.228})$$

and thus, the discrete-time power conservation law doesn't hold.

**Remark:** *The power conservation law doesn't hold for this scheme as (4.30) is not satisfied.*

### E.2.7 Symplectic discretization scheme VII

This symplectic time-discretization scheme assumes an explicit Euler rule for the flows and an implicit trapezoidal rule for the efforts. The initial continuous-time lossy PCH system for which we perform the symplectic discretization is:

$$\begin{cases} \dot{x} = (J - R) \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.229})$$

Using the general time-discretization framework from (4.1) and (4.24) for the selected discretization scheme, it results:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k) = (J - R)Q \left( \frac{x_{k-1} + x_k}{2} \right) + Bu_k \\ \frac{\Delta H}{\Delta x}(k) = Q \left( \frac{x_{k-1} + x_k}{2} \right) \\ y_k = B^T Q \left( \frac{x_{k-1} + x_k}{2} \right) \end{cases} \quad (\text{E.230})$$

**Remark:** *The composed time-discretization of the effort  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$ , results in  $Q \left( \frac{x_{k-1} + x_k}{2} \right)$  by applying the flow discretization rule on  $\frac{\Delta H}{\Delta x}$ .*

Introducing the notation:

$$\bar{x}_k \stackrel{\text{def}}{=} \left( \frac{x_{k-1} + x_k}{2} \right) \quad (\text{E.231})$$

it results:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k) = (J - R)Q \bar{x}_k + Bu_k \\ \frac{\Delta H}{\Delta x}(k) = Q \bar{x}_k \\ y_k = B^T Q \bar{x}_k \end{cases} \quad (\text{E.232})$$

**Remark:** *The step  $k$  in the parenthesis of the flows, efforts or power energies correspond to the explicit form of the discrete-time scheme, given by the flows discretization rule.*

The discrete-time power energy associated to the storage port can be defined as:

$$\hat{P}_S(k) \stackrel{\text{def}}{=} \hat{e}_S^T(k) \hat{f}_S(k) = (Q \bar{x}_k)^T \left( -(J - R)Q \bar{x}_k - Bu_k \right) \quad (\text{E.233})$$

Equivalently we can write:

$$\hat{P}_S(k) \stackrel{\text{def}}{=} \hat{e}_S^T(k) R \hat{e}_S(k) - \hat{e}_S^T(k) B \hat{f}_C \quad (\text{E.234})$$

from straightforward computations. The discrete-time control port power energy, can be defined as:

$$\hat{P}_C(k) \stackrel{def}{=} \hat{e}_C^T(k) \hat{f}_C(k) = \left( B^T Q \bar{x}_k \right)^T u_k \quad (\text{E.235})$$

which is equivalent to:

$$\hat{P}_C(k) \stackrel{def}{=} \hat{e}_S^T(k) B \hat{f}_C(k) \quad (\text{E.236})$$

The dissipation port power energy, can be defined as:

$$\hat{P}_R(k) \stackrel{def}{=} \hat{e}_R^T(k) \hat{f}_R(k) \quad (\text{E.237})$$

which is equivalent to:

$$\hat{P}_R(k) = \hat{e}_S^T(k) R \hat{e}_S(k) \quad (\text{E.238})$$

When using the following general relation:

$$f_R = -\text{Re}_R \quad (\text{E.239})$$

The discrete-time effort variable is defined as:

$$\hat{e}_R(k) \stackrel{def}{=} \frac{\Delta H}{\Delta x}(k) = Q \bar{x}_k \quad (\text{E.240})$$

Using (E.234), (E.236) and (E.238), the discrete-time power conservation law is satisfied:

$$\hat{P}_S(k) + \hat{P}_C(k) + \hat{P}_R(k) = \hat{e}_S^T(k) R \hat{e}_S(k) - \hat{e}_S^T(k) B \hat{f}_C(k) + \hat{e}_S^T(k) B \hat{f}_C(k) - \hat{e}_S^T(k) R \hat{e}_S(k) = 0 \quad (\text{E.241})$$

Using (4.31), the discrete-time energy (Hamiltonian) is defined as:

$$\frac{\Delta H}{\Delta t}(k) \stackrel{def}{=} -\hat{e}_S^T(k) \hat{f}_S(k) = y_k^T u_k + \hat{e}_R^T(k) \hat{f}_R(k) \quad (\text{E.242})$$

Using (2.234), (2.236) and (2.238), (2.242) results as:

$$\frac{\Delta H}{\Delta t}(k) \stackrel{def}{=} \hat{e}_S^T(k) B \hat{f}_C(k) - \hat{e}_S^T(k) R \hat{e}_S(k) \quad (\text{E.243})$$

Using (4.32), the discrete-time bilinear form can be defined as:

$$\begin{aligned} \hat{B}_F(k) \stackrel{def}{=} & \left\langle \left( Q \bar{x}_k, B^T Q \bar{x}_k, Q \bar{x}_k \right)^a, \left( -\frac{\Delta x}{\Delta t}(k), u_k, -R Q \bar{x}_k \right)^b \right\rangle + \\ & + \left\langle \left( Q \bar{x}_k, B^T Q \bar{x}_k, Q \bar{x}_k \right)^b, \left( -\frac{\Delta x}{\Delta t}(k), u_k, -R Q \bar{x}_k \right)^a \right\rangle \end{aligned} \quad (\text{E.244})$$

The implicit form of the discrete-time PCH system can be defined as:

$$\left( \hat{f}_S(k), \hat{e}_S(k), \hat{f}_C(k), \hat{e}_C(k), \hat{f}_R(k), \hat{e}_R(k) \right) \in D \quad (\text{E.245})$$

Replacing the discrete-time flows and efforts of each port, it results:

$$\left( -\frac{\Delta x}{\Delta t}(k), Q \bar{x}_k, u_k, y_k, -R Q \bar{x}_k, Q \bar{x}_k \right) \in D \quad (\text{E.246})$$

## E.2.8 Symplectic discretization scheme VIII

In this case, an implicit trapezoidal rule for the flows and an implicit Euler rule for the efforts were selected for time-discretization. The initial continuous-time lossy PCH system, for which we apply this symplectic discretization scheme is:

$$\begin{cases} \dot{x} = (J - R) \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.247})$$

Applying this time-discretization rules for the flows and efforts, it results:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = (J - R)Q \left( \frac{x_k + x_{k+1}}{2} \right) + B \left( \frac{u_k + u_{k+1}}{2} \right) \\ \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \\ y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.248})$$

**Remark:** The composed time-discretization of the effort  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$ , results in

$Q \left( \frac{x_k + x_{k+1}}{2} \right)$  by applying the implicit trapezoidal rule on  $\frac{\Delta H}{\Delta x}$ .

Adopting the notations:

$$\bar{x}_{k+1} \stackrel{\text{def}}{=} \left( \frac{x_k + x_{k+1}}{2} \right) \quad (\text{E.249})$$

$$\bar{u}_{k+1} \stackrel{\text{def}}{=} \left( \frac{u_k + u_{k+1}}{2} \right) \quad (\text{E.250})$$

(E.248) becomes:

$$\begin{cases} \frac{\Delta x}{\Delta t}(k+1) = (J - R)Q\bar{x}_{k+1} + B\bar{u}_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \\ y_{k+1} = B^T Qx_{k+1} \end{cases} \quad (\text{E.251})$$

**Remark:** The step  $k+1$  in the parenthesis of the discrete-time flows, efforts or power energies correspond to the implicit form of the discretization scheme, given by the flows time-discretization rule.

The discrete-time power energy corresponding to the storage port can be defined as:

$$\hat{P}_S(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1) \hat{f}_S(k+1) = (Qx_{k+1})^T \left( -(J - R)Q\bar{x}_{k+1} - B\bar{u}_{k+1} \right) \quad (\text{E.252})$$

which is equivalent to:

$$\hat{P}_S(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1) R \hat{e}_S(k+1) - \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.253})$$

from straight computations. The discrete-time power energy associated to the control port can be defined as:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_C^T(k+1)\hat{f}_C(k+1) = \left( B^T Q x_{k+1} \right)^T \bar{u}_{k+1} \quad (\text{E.254})$$

which is equivalent to:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (\text{E.255})$$

from straightforward computations. For the dissipation port, the discrete-time power energy can be defined as:

$$\hat{P}_R(k+1) \stackrel{def}{=} \hat{e}_R^T(k+1)\hat{f}_R(k+1) \quad (\text{E.256})$$

which results in:

$$\hat{P}_R(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1)R\hat{e}_S(k+1) \quad (\text{E.257})$$

Using the general relation:

$$f_R = -\text{Re}_R \quad (\text{E.258})$$

the discrete-time effort variable is defined as:

$$\hat{e}_R(k+1) \stackrel{def}{=} \frac{\Delta H}{\Delta x}(k+1) = Qx_{k+1} \quad (\text{E.259})$$

From (E.253), (E.255) and (E.257), the discrete-time power conservation law is:

$$\begin{aligned} \hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) &= \hat{e}_S^T(k+1)R\bar{e}_S(k+1) - \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \\ &+ \hat{e}_S^T(k+1)B\hat{f}_C(k+1) - \hat{e}_S^T(k+1)R\hat{e}_S(k+1) \end{aligned} \quad (\text{E.260})$$

(E.260) is equivalent to:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) = \hat{e}_S^T(k+1)R\left(\bar{e}_S^T(k+1) - \hat{e}_S^T(k+1)\right) \quad (\text{E.261})$$

or:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) = \left( Qx_{k+1} \right)^T RQ \left[ \frac{x_k - x_{k+1}}{2} \right] \quad (\text{E.262})$$

and thus the discrete-time power conservation law is not satisfied.

**Remark:** *This time-discretization scheme is not symplectic as (4.30) doesn't hold.*

## E.2.9 Symplectic discretization scheme IX

For this time-discretization scheme, it was selected an implicit mid-point rule for the flows and an implicit trapezoidal rule for the efforts. The initial continuous-time lossy PCH system for which we perform the time-discretization is:

$$\begin{cases} \dot{x} = (J - R) \frac{\partial H}{\partial x} + Bu \\ y = B^T \frac{\partial H}{\partial x} \end{cases} \quad (\text{E.263})$$

Applying this time-discretization scheme, using (4.1) and (4.24) it results:

$$\left\{ \begin{array}{l} \frac{\Delta x}{\Delta t}(k+1) = (J-R)Q \frac{1}{2} \left( \frac{x_{k-1} + x_k}{2} + \frac{x_k + x_{k+1}}{2} \right) \\ \frac{\Delta H}{\Delta x}(k+1) = Q \left( \frac{x_k + x_{k+1}}{2} \right) \\ y_{k+1} = B^T Q \left( \frac{x_k + x_{k+1}}{2} \right) \end{array} \right. \quad (\text{E.264})$$

**Remark:** The composed time-discretization of the effort  $F_{di} * E_{di} \left( \frac{\partial H}{\partial x} \right)$ , results in

$$Q \frac{1}{2} \left( \frac{x_{k-1} + x_k}{2} + \frac{x_k + x_{k+1}}{2} \right), \text{ by applying the flow discretization rule on } \frac{\Delta H}{\Delta x}.$$

Adopting the notations:

$$\bar{x}_{k+1} \stackrel{\text{def}}{=} \frac{x_k + x_{k+1}}{2} \quad (\text{E.265})$$

and:

$$\bar{\bar{x}}_{k+1} \stackrel{\text{def}}{=} \left( \frac{\bar{x}_k + \bar{x}_{k+1}}{2} \right) \quad (\text{E.266})$$

(E.264) becomes:

$$\left\{ \begin{array}{l} \frac{\Delta x}{\Delta t}(k+1) = (J-R)Q\bar{\bar{x}}_{k+1} + Bu_{k+1} \\ \frac{\Delta H}{\Delta x}(k+1) = Q\bar{x}_{k+1} \\ y_{k+1} = B^T Q\bar{x}_{k+1} \end{array} \right. \quad (\text{E.267})$$

**Remark:** The step  $k+1$ , in the parenthesis of the flows, efforts or power energies correspond to the implicit form of the time-discretization scheme, given by the time-discretization rule selected for the flows.

The discrete-time power energy corresponding to the storage port, can be defined as:

$$\hat{P}_S(k+1) \stackrel{\text{def}}{=} \hat{e}_S^T(k+1) \hat{f}_S(k+1) = (Q\bar{x}_{k+1})^T \left( -(J-R)Q\bar{\bar{x}}_{k+1} - Bu_{k+1} \right) \quad (\text{E.268})$$

Equivalently we can write:

$$\hat{P}_S(k+1) = \hat{e}_S^T(k+1) \mathbf{R} \hat{e}_S(k+1) - \hat{e}_S^T(k+1) B \hat{f}_C(k+1) \quad (\text{E.269})$$

from straightforward computations. In (E.269),  $\hat{e}_S(k+1)$  is defined as:

$$\hat{e}_S(k+1) \stackrel{\text{def}}{=} \left( \frac{\hat{e}_S(k) + \hat{e}_S(k+1)}{2} \right) \quad (\text{E.270})$$

The discrete-time power energy corresponding to the control port can be defined as:

$$\hat{P}_C(k+1) \stackrel{\text{def}}{=} \hat{e}_C^T(k+1) \hat{f}_C(k+1) = \left( B^T Q\bar{x}_{k+1} \right)^T u_{k+1} \quad (\text{E.271})$$

which is equivalent to:

$$\hat{P}_C(k+1) \stackrel{def}{=} \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \quad (\text{E.272})$$

The dissipation port power energy, can be defined as:

$$\hat{P}_R(k+1) \stackrel{def}{=} \hat{e}_R^T(k+1)\hat{f}_R(k+1) \quad (\text{E.273})$$

which results in:

$$\hat{P}_R(k+1) \stackrel{def}{=} -\hat{e}_S^T(k+1)R\hat{e}_S(k+1) \quad (\text{E.274})$$

Using the general relation:

$$f_R = -\text{Re}_R \quad (\text{E.275})$$

the discrete-time effort variable is defined as:

$$\hat{e}_R(k+1) \stackrel{def}{=} \frac{\Delta H}{\Delta x}(k+1) = Q\bar{x}_{k+1} \quad (\text{E.276})$$

Using (E.269), (E.272) and (E.274), the discrete-time power conservation law is:

$$\begin{aligned} \hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) &= \hat{e}_S^T(k+1)R\hat{e}_S(k+1) - \hat{e}_S^T(k+1)B\hat{f}_C(k+1) \\ &+ \hat{e}_S^T(k+1)B\hat{f}_C(k+1) - \hat{e}_S^T(k+1)R\hat{e}_S(k+1) \end{aligned} \quad (\text{E.277})$$

Equivalently we can write:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) = \hat{e}_S^T(k+1)R\left(\hat{e}_S(k+1) - \hat{e}_S(k+1)\right) \quad (\text{E.278})$$

or:

$$\hat{P}_S(k+1) + \hat{P}_C(k+1) + \hat{P}_R(k+1) = (Q\bar{x}_{k+1})^T R Q \left( \frac{\bar{x}_k - \bar{x}_{k+1}}{2} \right) \quad (\text{E.279})$$

Thus the discrete-time power conservation law is not satisfied with this discretization scheme.

**Remark:** *This time-discretization scheme is not symplectic, as (4.30) from the general framework is not satisfied.*



## Appendix F Matrix I/O representation of symplectic time-discretization schemes and deterministic identification

**General remark:** *This Appendix comes to prove that the symplectic time-discretization schemes proposed in chapter 4, can be used for deterministic identification of PCH systems using orthogonal or oblique projections. The main differences between the discretization schemes are given by different structures of the past/future states and output vectors, maintaining in the same time the classic approach of state-space form. Thus all symplectic time-discretization schemes can be used as well for determining the system parameters.*

### F.1 Symplectic discretization scheme II

For the 2<sup>nd</sup> symplectic discretization scheme of chapter 5, it is proposed an implicit Euler rule for the flows and an explicit Euler rule for the efforts. Applying the general discretization schemes introduced in chapter 4, we can write the following state-space form:

$$\begin{cases} x_{k+1} = \bar{A} x_k + \bar{B} u_{k+1} \\ y_{k+1} = C x_k \end{cases} \quad (\text{F.1})$$

where the  $\bar{A}, \bar{B}, \bar{C}$  matrices represent the state-space matrices of the discrete-time PCH system. For  $k = 0$ , the above equation is:

$$\begin{cases} x_1 = \bar{A} x_0 + \bar{B} u_1 \\ y_1 = C x_0 \end{cases} \quad (\text{F.2})$$

For  $k = 1$ , (F.1) becomes:

$$\begin{cases} x_2 = \bar{A} x_1 + \bar{B} u_2 = \bar{A}^2 x_0 + A B u_1 + B u_2 \\ y_2 = C x_1 = C \bar{A} x_0 + C \bar{B} u_1 \end{cases} \quad (\text{F.3})$$

For  $k = 2$ , we can write:

$$\begin{cases} x_3 = \bar{A}^3 x_0 + \bar{A}^2 \bar{B} u_1 + \bar{A} \bar{B} u_2 + B u_3 \\ y_3 = C \bar{A}^2 x_0 + C \bar{A} \bar{B} u_1 + C \bar{B} u_2 \end{cases} \quad (\text{F.4})$$

Using (F.2), (F.3) and (F.4), the following relations can be deduced:

$$\begin{cases} X_f^d = X_i^d = \bar{A}^i X_p^d + \Delta_i^d U_p^+ \\ Y_p = \begin{pmatrix} 0 \\ \Gamma_{i-1} \end{pmatrix} X_p^d + H_{i-1}^d U_p^+ \\ Y_f = \Gamma_i X_{f-1}^d + H_{i-1}^d U_f^+ \end{cases} \quad (\text{F.5})$$

In the above relation,  $U_p^+$  represents the vector:

$$U_p^+ = (u_1 \quad u_2 \quad u_3 \quad \dots \quad u_i) \quad (\text{F.6})$$

from which the 1<sup>st</sup> input  $u_0$  is missing, while  $u_i$  is augmented.

**Remark:**

The matrix input-output equations of this symplectic discretization scheme are similar with the classic state-space form, with the 1<sup>st</sup> output always zero. In this case, as for the previous symplectic discretization scheme, the proof of the deterministic identification theorem follows the same steps as for the classic case and it won't be performed further.

**F.2 Symplectic discretization scheme III**

This symplectic discretization scheme, assumes an explicit Euler rule for the flows and an implicit mid-point rule for the efforts. Applying the general discretization rules introduced in chapter 4, we can write the following state-space representation:

$$\begin{cases} x_{k+1} = x_k + (\bar{A} - I) \left( \frac{x_{k-1} + x_k}{2} \right) + \bar{B} u_k \\ y_k = C \left( \frac{x_{k-1} + x_k}{2} \right) \end{cases} \quad (\text{F.7})$$

The implicit form of this time-discretization scheme is:

$$\left( \hat{f}_S(k), \hat{e}_S(k), \hat{f}_C(k), \hat{e}_C(k), \hat{f}_R(k), \hat{e}_R(k) \right) \in D \quad (\text{F.8})$$

**Remark:** Using the same rules as for the 1<sup>st</sup> two symplectic discretization schemes, the results are laborious and no general (recursive) formulas can be found for a vectorial representation of the states and outputs. By combining this symplectic time-discretization scheme with the 1<sup>st</sup> one, a vectorial representation of the states and outputs may be obtained.

Using (5.24) and (F.7), it can be observed that the following relations can be written:

$$x_{k+1} = \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} + \left( \frac{\bar{A} - I}{2} \right) \left( \bar{A}^{k-1} x_0 + \Delta_{k-1}^d U_{0/k-2} + \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} \right) + \bar{B} u_k \quad (\text{F.9})$$

More compact this relation can be written as:

$$x_{k+1} = \left( \frac{\bar{A}^2 + 2\bar{A} - I}{2} \right) \bar{A}^{k-1} x_0 + \left( \frac{\bar{A} + I}{2} \right) \Delta_k^d U_{0/k-1} + \left( \frac{\bar{A} - I}{2} \right) \Delta_{k-1}^d U_{0/k-2} + \left( \frac{2\bar{B} u_k}{2} \right) \quad (\text{F.10})$$

Next we adopt the notations for simplicity:

$$\begin{cases} \Delta_{k+1}^{-d} \stackrel{\text{def}}{=} \bar{A} \cdot \Delta_k^d = \left( \bar{A}^k \bar{B} \quad \bar{A}^{k-1} \bar{B} \quad \dots \quad \bar{A} \bar{B} \right) \\ \Delta_k^{-d} \stackrel{\text{def}}{=} \bar{A} \Delta_{k-1}^d = \left( \bar{A}^{k-1} \bar{B} \quad \bar{A}^{k-2} \bar{B} \quad \dots \quad \bar{A} \bar{B} \right) \\ \bar{A}^{k+1} \stackrel{\text{def}}{=} \left( \frac{\bar{A}^2 + 2\bar{A} - I}{2} \right) \bar{A}^{k-1} \end{cases} \quad (\text{F.11})$$

**Remark:** The minus sign was introduced for the reversed extended controllability matrix in (F.11) as the last term is missing.

Using (F.11), (F.10) becomes:

$$x_{k+1} = \overline{\overline{A}}^{k+1} x_0 + \frac{\Delta_{k+1}^{-d} U_{0/k-1} + \Delta_k^d U_{0/k-1} + \Delta_k^{-d} U_{0/k-2} - \Delta_{k-1}^d U_{0/k-2} + 2\overline{B} u_k}{2} \quad (\text{F.12})$$

Further we can write:

$$x_{k+1} = \overline{\overline{A}}^{k+1} x_0 + \frac{\Delta_{k+1}^d U_{0/k} + \left( \Delta_k^d \quad \overline{B} \right) U_{0/k} + \left( \Delta_k^{-d} - \Delta_{k-1}^d \right) U_{0/k-2}}{2} \quad (\text{F.13})$$

More compact the above relation can be written as:

$$x_{k+1} = \overline{\overline{A}}^{k+1} x_0 + \frac{1}{2} \left[ \begin{array}{cc} \Delta_k^{-d} - \Delta_{k-1}^d & \Delta_{k+1}^d + \left( \Delta_k^d \quad \overline{B} \right) \end{array} \right] \begin{bmatrix} U_{0/k-2} \\ U_{0/k} \end{bmatrix} \quad (\text{F.14})$$

Adopting the notations:

$$\left\{ \begin{array}{l} \Delta_{k+1}^{*d} \stackrel{\text{def}}{=} \frac{1}{2} \left[ \Delta_k^{-d} - \Delta_{k-1}^d \quad \Delta_{k+1}^d + \left( \Delta_k^d \quad \overline{B} \right) \right] \\ U_{0/k}^* \stackrel{\text{def}}{=} \left( U_{0/k-2} \quad U_{0/k} \right)^T \end{array} \right. \quad (\text{F.15})$$

(F.14) becomes:

$$x_{k+1} = \overline{\overline{A}}^{k+1} x_0 + \Delta_{k+1}^{*d} U_{0/k}^* \quad (\text{F.16})$$

**Remark:** The state-space form of (F.16) is similar as form to the classic state-space representation.

Then we define the future states, as follows:

$$X_f^d = X_i^d = \overline{\overline{A}}^i X_P^d + \Delta_i^{*d} U_P^* \quad (\text{F.17})$$

For the system outputs, using the 2<sup>nd</sup> relation in (F.7), we can write:

$$y_k = C \left( \frac{x_{k-1} + x_k}{2} \right) = \frac{C}{2} \left( \overline{\overline{A}}^{k-1} X_P^d + \Delta_{k-1}^{*d} U_{0/k-2}^* + \overline{\overline{A}}^k X_P^d + \Delta_k^{*d} U_{0/k-1}^* \right) \quad (\text{F.18})$$

This relation can be further written as:

$$y_k = \frac{C}{2} \left( \overline{\overline{A}}^{k-1} + \overline{\overline{A}}^k \right) X_P^d + \frac{C}{2} \left( \Delta_{k-1}^{*d} \quad \Delta_k^{*d} \right) \begin{bmatrix} U_{0/k-2}^* \\ U_{0/k-1}^* \end{bmatrix} \quad (\text{F.19})$$

Then we can represent the past and future outputs vectorially as:

$$\left\{ \begin{array}{l} Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**} \\ Y_f = \Gamma_i^* \overline{X}_f^d + H_i^{**d} U_f^{**} \end{array} \right. \quad (\text{F.20})$$

where:

$$\Gamma_i^{**} \stackrel{\text{def}}{=} \frac{1}{2} \left( \begin{bmatrix} 0 \\ \Gamma_{i-1}^* \end{bmatrix} + \Gamma_i^* \right), \Gamma_i^* \stackrel{\text{def}}{=} \left( C \quad C\overline{\overline{A}} \quad \dots \quad C\overline{\overline{A}}^{i-1} \right)^T \quad (\text{F.21})$$

$$X_P^d \stackrel{\text{def}}{=} x_0, H_i^{**d} \stackrel{\text{def}}{=} \begin{bmatrix} \Delta_{-1}^{*d} & 0 & 0 & \Delta_0^{*d} & 0 & 0 \\ \Delta_0^{*d} & 0 & 0 & \Delta_1^{*d} & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \Delta_{i-1}^{*d} & \dots & 0 & \Delta_i^{*d} & 0 & 0 \end{bmatrix}, \quad (\text{F.22})$$

$$\bar{X}_f^d = \frac{X_{f-1}^d + X_f^d}{2} \quad (\text{F.23})$$

Then the final form of the *matrix input-output equations* is:

$$\begin{cases} X_f^d = X_i^d = \bar{A}^i X_P^d + \Delta_i^{*d} U_P^* \\ \bar{X}_f^d = \frac{X_{f-1}^d + X_f^d}{2} \\ Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^* \\ Y_f = \Gamma_i^* \bar{X}_f^d + H_i^{**d} U_f^{**} \end{cases} \quad (\text{F.24})$$

Thus we introduce the following theorem for the *matrix I/O representation* of this discretization scheme.

### Theorem F.1

The *matrix I/O equations* of a discrete-time PCH system represented in explicit form in (F.7) and in implicit form in (F.8), for a symplectic time-discretization scheme with an explicit Euler rule for the flows and implicit mid-point rule for the efforts are:

$$\begin{cases} X_f^d = X_i^d = \bar{A}^i X_P^d + \Delta_i^{*d} U_P^* \\ \bar{X}_f^d = \frac{X_{f-1}^d + X_f^d}{2} \\ Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^* \\ Y_f = \Gamma_i^* \bar{X}_f^d + H_i^{**d} U_f^{**} \end{cases} \quad (\text{F.25})$$

At the next step, we search for an equivalent state-space representation using (5.33) and (F.7). Replacing  $x_k$  from (5.33) in (F.7), it results:

$$x_{k+1} = Ax_{k-1} + \bar{B}u_{k-1} + \left( \frac{\bar{A} - I}{2} \right) (x_{k-1} + \bar{A}x_{k-1} + \bar{B}u_{k-1}) + \bar{B}u_k \quad (\text{F.26})$$

Equivalently we can write:

$$x_{k+1} = \frac{2\bar{A}x_{k-1} + 2\bar{B}u_{k-1} + \bar{A}x_{k-1} + \bar{A}^2x_{k-1} + \bar{A}\bar{B}u_{k-1} + 2\bar{B}u_k - x_{k-1} - \bar{A}x_{k-1} - \bar{B}u_{k-1}}{2} \quad (\text{F.27})$$

Performing the possible simplifications, this relation becomes:

$$x_{k+1} = \left( \frac{\bar{A}^2 + 2\bar{A} - I}{2} \right) x_{k-1} + \frac{1}{2} [\bar{A}\bar{B} + \bar{B}] \begin{pmatrix} u_{k-1} \\ u_k \end{pmatrix} \quad (\text{F.28})$$

Using the 2<sup>nd</sup> relation from (F.7), we can write for the outputs:

$$y_k = C \left( \frac{x_{k-1} + \bar{A}x_{k-1} + \bar{B}u_{k-1}}{2} \right) \quad (\text{F.29})$$

(F.29) can be further written as:

$$y_k = \left( \frac{C + \bar{A}}{2} \right) x_{k-1} + \frac{C\bar{B}}{2} u_{k-1} \quad (\text{F.30})$$

Using (F.28) and (F.30), we can write the following two relations:

$$\begin{pmatrix} x_{k+1} \\ y_k \end{pmatrix} = \begin{pmatrix} \frac{2\bar{A} + \bar{A}^2 - I}{2} & \frac{1}{2}(\bar{A}\bar{B} + \bar{B} \quad 2\bar{B}) \\ \frac{C + \bar{A}}{2} & \frac{1}{2}(C\bar{B} \quad 0) \end{pmatrix} \begin{pmatrix} x_{k-1} \\ u_{k-1} \\ u_k \end{pmatrix} \quad (\text{F.31})$$

A simpler representation of the above system is:

$$\begin{pmatrix} x_{k+1} \\ y_k \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \begin{pmatrix} x_{k-1} \\ \tilde{u}_{k-1} \end{pmatrix} \quad (\text{F.32})$$

where the following notations have been adopted:

$$\left\{ \begin{array}{l} \tilde{A} \stackrel{\text{def}}{=} \left( \frac{\bar{A}^2 + 2\bar{A} - I}{2} \right), \tilde{B} \stackrel{\text{def}}{=} \frac{1}{2}(\bar{A}\bar{B} + \bar{B} \quad 2\bar{B}) \\ \tilde{C} \stackrel{\text{def}}{=} \frac{C + \bar{A}}{2}, \tilde{D} \stackrel{\text{def}}{=} \frac{1}{2}(C\bar{B} \quad 0) \end{array} \right. \quad (\text{F.33})$$

**Remark:** Similar to the classic state-space approach, it is possible to recover the initial state-space matrices  $(A, B, C)$  by solving (F.32) when having knowledge of the input/output vectors.

In the following lines, we will prove the deterministic identification theorem for this symplectic time-discretization scheme. We propose thus a deterministic identification theorem as in the classic state-space case for the selected symplectic discretization scheme, which is proved further.

## Theorem F.2

Under the assumptions that:

- I. The input  $u_k$  is persistently exciting of order  $2i$ .
- II. The intersection of the row space of  $U_f^{**}$  (future inputs) and the row space of  $X_P^d$  (past states) is empty.
- III. The user-defined weighting matrices  $W_1 \in R^{lixli}$  and  $W_2 \in R^{jxj}$  are such that  $W_1$  is of full rank and  $W_2$  obeys:  $\text{rank} \left( W_P^* \right) = \text{rank} \left( W_P^* W_2 \right)$ , where  $W_P^*$  is the block Hankel matrix containing the past inputs and outputs. and with  $O_i$  defined as the oblique projection:

$$O_i = \left( Y_f / U_f^{**} \right) W_P^* \quad (1)$$

and the singular value decomposition:

$$W_1 O_i W_2 = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1 S_1 V_1^T \quad (2)$$

we have:

1. The matrix  $O_i$  is equal to the product of the extended observability matrix and the states:

$$O_i = \Gamma_i^* \bar{X}_f^d \quad (3)$$

2. The order of the dynamic system is equal to the number of singular values in (2) different from 0.
3. The extended observability matrix  $\Gamma_i^*$  is equal to:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (4)$$

4. The part of the state sequence  $\bar{X}_f^d$  that lies in the column space of  $w_2$  can be recovered from:

$$\bar{X}_f^d w_2 = T^{-1} S_1^{1/2} V_1^T \quad (5)$$

5. The state sequence  $\bar{X}_f^d$  is equal to:

$$\bar{X}_f^d = \Gamma_i^{*+} O_i \quad (6)$$

**Remark:** The proof follows the same steps as for the classic state-space approach [107] in the deterministic identification.

**Proof:**

Using (F.24), we can write the average value of the future states as:

$$\bar{X}_f^d = \left\{ \frac{X_{f-1}^d + X_f^d}{2} \right\} = \frac{1}{2} \left( \bar{A}^{i-1} X_P^d + \Delta_{i-1}^{*d} U_{P-1}^* + \bar{A}^i X_P^d + \Delta_i^{*d} U_P^* \right) \quad (F.34)$$

More compact we can write this relation as:

$$\bar{X}_f^d = \left\{ \frac{\bar{A}^{i-1} + \bar{A}^i}{2} \right\} X_P^d + \frac{1}{2} \begin{pmatrix} \Delta_{i-1}^{*d} & \Delta_i^{*d} \end{pmatrix} \begin{pmatrix} U_{P-1}^* \\ U_P^* \end{pmatrix} \quad (F.35)$$

Adopting the notations:

$$\begin{cases} A_M^i & \stackrel{\text{def}}{=} \left\{ \frac{\bar{A}^{i-1} + \bar{A}^i}{2} \right\} \\ \Delta_M & \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} \Delta_{i-1}^{*d} & \Delta_i^{*d} \end{pmatrix} \end{cases} \quad (F.36)$$

(F.35) becomes:

$$\bar{X}_f^d = A_M^i X_P^d + \Delta_M U_P^{**} \quad (F.37)$$

where:

$$U_P^{**} \stackrel{\text{def}}{=} \begin{pmatrix} U_{P-1}^* & U_P^* \end{pmatrix}^T \quad (F.38)$$

Using (F.24), we can write (F.37) equivalently:

$$\bar{X}_f^d = A_M^i \left( \Gamma_i^{**+} Y_P - \Gamma_i^{**+} H_i^{**d} U_P^{**} \right) + \Delta_M U_P^{**} \quad (F.39)$$

More compact, we can write:

$$\bar{X}_f^d = \begin{pmatrix} \Delta_M - A_M^i \Gamma_i^{**+} H_i^{**d} & A_M^i \Gamma_i^{**+} \\ \hline U_P^{**} \\ Y_P \end{pmatrix} \quad (\text{F.40})$$

or equivalently:

$$\bar{X}_f^d = L_P^* W_P^* \quad (\text{F.41})$$

by adopting the notations:

$$\begin{cases} L_P^* \stackrel{\text{def}}{=} \begin{pmatrix} \Delta_M - A_M^i \Gamma_i^{**+} H_i^{**d} & A_M^i \Gamma_i^{**+} \\ \hline U_P^{**} & Y_P \end{pmatrix} \\ W_P^* \stackrel{\text{def}}{=} \begin{pmatrix} U_P^{**} & Y_P \end{pmatrix}^T \end{cases} \quad (\text{F.42})$$

Using (F.24), we can write for the future states as:

$$Y_f = \Gamma_i^* L_P^* W_P^* + H_i^{**d} U_f^{**} \quad (\text{F.43})$$

As for the classic case, we multiply this relation with the orthogonal projection of  $U_f^{**}$  (future inputs) and we get:

$$Y_f \Pi_{U_f^{**\perp}} = \Gamma_i^* L_P^* W_P^* \Pi_{U_f^{**\perp}} + H_i^{**d} U_f^{**} \Pi_{U_f^{**\perp}} \quad (\text{F.44})$$

We remark that the term  $U_f^{**} \Pi_{U_f^{**\perp}}$  is zero and thus, it results:

$$Y_f \Pi_{U_f^{**\perp}} = \Gamma_i^* L_P^* W_P^* \Pi_{U_f^{**\perp}} \quad (\text{F.45})$$

Using the definition of the orthogonal projection, (F.45) becomes:

$$Y_f / U_f^{**\perp} = \Gamma_i^* L_P^* W_P^* / U_f^{**\perp} \quad (\text{F.46})$$

Multiplying to the right with the pseudo-inverse  $(W_P^* / U_f^{**\perp})^+$  and  $W_P^*$ , we can further write:

$$\left( Y_f / U_f^{**\perp} \right) \left( W_P^* / U_f^{**\perp} \right)^+ W_P^* = \Gamma_i^* L_P^* W_P^* \quad (\text{F.47})$$

Similarly to the definition of classic state-space systems, we can define an oblique projection in this case:

$$O_i = \Gamma_i^* L_P^* W_P^* \quad (\text{F.48})$$

where:

$$O_i = \left( Y_f / U_f^{**} \right) W_P^* \quad (\text{F.49})$$

Using (F.41), (F.48) becomes:

$$O_i = \Gamma_i^* \bar{X}_f^d \quad (\text{F.50})$$

where:

$$\left( W_P^* / U_f^{**\perp} \right) \left( W_P^* / U_f^{**\perp} \right)^+ W_P^* = W_P^* \quad (\text{F.51})$$

holds.

Further the weighting matrices  $w_1 \in R^{lixli}$  and  $w_2 \in R^{jxj}$  are introduced, such that  $w_1$  is of full rank and  $w_2$  obeys  $rank \left( W_P^* \right) = rank \left( W_P^* w_2 \right)$ . It is necessary to prove that:

$$rank \left( W_P^* \right) = rank \left( W_P^* / U_f^{**\perp} \right) \quad (F.52)$$

Using (F.22) and (F.24), we can write  $w_P^*$  as:

$$W_P^* = \begin{pmatrix} I_{(4i-6)m} & 0 \\ H_i^{**d} & \Gamma_i^{**} \end{pmatrix} \begin{pmatrix} U_P^{**} \\ X_P^d \end{pmatrix} \quad (F.53)$$

where:

$$U_P^{**} = U_{0/i-1}^{**} = \left( U_{0/i-2}^* \quad U_{0/i-1}^* \right)^T \quad (F.54)$$

Multiplying to the right with the orthogonal complement of  $U_f^{**\perp}$ , we can write:

$$W_P^* \Pi_{U_f^{**\perp}} = \begin{pmatrix} I_{(4i-6)m} & 0 \\ H_i^{**d} & \Gamma_i^{**} \end{pmatrix} \begin{pmatrix} U_P^{**} \Pi_{U_f^{**\perp}} \\ X_P^d \Pi_{U_f^{**\perp}} \end{pmatrix} \quad (F.55)$$

which becomes further:

$$W_P^* / U_f^{**\perp} = \begin{pmatrix} I_{(4i-6)m} & 0 \\ H_i^{**d} & \Gamma_i^{**} \end{pmatrix} \begin{pmatrix} U_P^{**} / U_f^{**\perp} \\ X_P^d / U_f^{**\perp} \end{pmatrix} \quad (F.56)$$

Using also the initial conditions assumed for **Theorem F.2**, it results:

$$rank \begin{pmatrix} U_P^{**} \\ X_P^d \end{pmatrix} = rank \begin{pmatrix} U_P^{**} / U_f^{**\perp} \\ X_P^d / U_f^{**\perp} \end{pmatrix} \quad (F.57)$$

which proves then (F.52).

**Remark:**

*The initial conditions are maintained as for the classic case, the only difference is that  $w_P^*$  is changing his structure according to the symplectic time-discretization scheme.*

As for the classic case, we can develop the (SVD) decomposition of  $w_P^* / U_f^{**\perp}$  as:

$$W_P^* / U_f^{**\perp} = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1 S_1 V_1^T \quad (F.58)$$

**Remark:** *Since  $w_P^* / U_f^{**\perp}$  is a linear combination of the columns of  $w_P^*$  and since the rank of  $w_P^*$  and  $w_P^* / U_f^{**\perp}$  are equal, the column spaces of  $w_P^*$  and  $w_P^* / U_f^{**\perp}$  must be also equal.*

We can write then  $w_P^*$  as:



$$W_P^* = U_1 R \quad (\text{F.59})$$

Using the SVD decomposition (F.58), we can write:

$$\left( W_P^* / U_f^{**\perp} \right) \left( W_P^* / U_f^{**\perp} \right)^+ W_P^* = \left( U_1 S_1 V_1^T \right) \left( V_1 S_1^{-1} U_1^T \right) \left( U_1 R \right) \quad (\text{F.60})$$

Using the properties:

$$U_1^T U_1 = V_1^T V_1 = I \quad (\text{F.61})$$

assumed for the SVD decomposition, it results:

$$\left( W_P^* / U_f^{**\perp} \right) \left( W_P^* / U_f^{**\perp} \right)^+ W_P^* = W_P^* = U_1 R \quad (\text{F.62})$$

Using (F.59) and (F.62) it is obvious that:

$$\left( W_P^* / U_f^{**\perp} \right) \left( W_P^* / U_f^{**\perp} \right)^+ W_P^* = W_P^* \quad (\text{F.63})$$

which proves also (F.51). For the 2<sup>nd</sup> claim of the theorem, we can write:

$$W_1 O_i W_2 = \left( W_1 \Gamma_i^* \right) \left( \bar{X}_f^d W_2 \right) \quad (\text{F.64})$$

where  $\Gamma_i^*$  has  $n$  columns and  $\bar{X}_f^d$   $n$  rows. The rank of this product is also  $n$ , which proves the statement of the theorem. Introducing a non-singular similarity transformation ( $T \in \mathbb{R}^{n \times n}$ ), we can write  $W_1 \Gamma_i^*$  and  $\bar{X}_f^d W_2$  as follows:

$$W_1 \Gamma_i^* = U_1 S_1^{1/2} T \quad (\text{F.65})$$

$$\bar{X}_f^d W_2 = T^{-1} S_1^{1/2} V_1^T \quad (\text{F.66})$$

Multiplying to the left with  $W_1^{-1}$  in (F.65), we get:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (\text{F.67})$$

which proves statement 4 of the theorem. From (F.66) also the 4<sup>th</sup> statement of the theorem results. Using (F.50), the future states can be written as:

$$\bar{X}_f^d = \Gamma_i^* O_i \quad (\text{F.68})$$

which proves the last statement of the theorem. From (F.50), (F.64), (F.66), (F.67) and (F.68), it results that the deterministic identification theorem proposed for the current discretization scheme is proved.

### F.3 Symplectic discretization scheme IV

In this case, it is assumed an implicit mid-point rule for the flows and an explicit Euler rule for the efforts. The discrete-time state-space representation of this symplectic time-discretization scheme, starting from the general relations of chapter 4, is the following:

$$\begin{cases} x_{k+1} = x_k + (\bar{A} - I) \left( \frac{x_{k-1} + x_k}{2} \right) + \bar{B} u_{k+1} \\ y_{k+1} = C x_k \end{cases} \quad (\text{F.69})$$

The general implicit form of the PCH system for this symplectic discretization scheme, is the following:

$$(f_S(k+1), e_S(k+1), f_C(k+1), e_C(k+1), f_R(k+1), e_R(k+1)) \in D \quad (\text{F.70})$$

Using the 1<sup>st</sup> relation from (5.24) in (F.69), we can write the following equivalent relation:

$$x_{k+1} = \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} + \left( \frac{\bar{A} - I}{2} \right) \left( \bar{A}^{k-1} x_0 + \Delta_{k-1}^d U_{0/k-2} + \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} \right) + \bar{B} u_{k+1} \quad (\text{F.71})$$

Equivalently we can write:

$$x_{k+1} = \frac{\bar{A}^{k+1} + 2\bar{A}^k - \bar{A}^{k-1}}{2} x_0 + \left( \frac{\bar{A} + I}{2} \right) \Delta_k^d U_{0/k-1} + \left( \frac{\bar{A} - I}{2} \right) \Delta_{k-1}^d U_{0/k-2} + \frac{2\bar{B} u_{k+1}}{2} \quad (\text{F.72})$$

Adopting further the notations:

$$\left\{ \begin{array}{l} \bar{A}_{k+1} \stackrel{\text{def}}{=} \frac{\bar{A}^{k+1} + 2\bar{A}^k - \bar{A}^{k-1}}{2} \\ \Delta_{k+1}^{-d} \stackrel{\text{def}}{=} \bar{A} \Delta_k^d = \begin{pmatrix} \bar{A}^k \bar{B} & \bar{A}^{k-1} \bar{B} & \dots & \bar{A} \bar{B} \end{pmatrix} \end{array} \right. \quad (\text{F.73})$$

(F.72) becomes:

$$x_{k+1} = \bar{A}^{k+1} x_0 + \frac{1}{2} \begin{pmatrix} \Delta_k^{-d} - \Delta_{k-1}^d & \Delta_{k+1}^{-d} + \Delta_k^d & 2\bar{B} \end{pmatrix} \begin{pmatrix} U_{0/k-2} \\ U_{0/k-1} \\ U_{k+1} \end{pmatrix} \quad (\text{F.74})$$

Further we adopt the notations:

$$\left\{ \begin{array}{l} \Delta_{k+1}^{*d} \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} \Delta_k^{-d} - \Delta_{k-1}^d & \Delta_{k+1}^{-d} + \Delta_k^d & 2\bar{B} \end{pmatrix} \\ U_{0/k}^* \stackrel{\text{def}}{=} \begin{pmatrix} U_{0/k-2} & U_{0/k-1} & U_{k+1} \end{pmatrix}^T \\ X_P^d \stackrel{\text{def}}{=} x_0 \end{array} \right. \quad (\text{F.75})$$

and (F.74) is equivalent to:

$$x_{k+1} = \bar{A}^{k+1} X_P^d + \Delta_{k+1}^{*d} U_{0/k}^* \quad (\text{F.76})$$

Then we can write then the following relation for the future states as for the classic case:

$$X_f^d = X_i^d = \bar{A}^i X_P^d + \Delta_i^{*d} U_P^* \quad (\text{F.77})$$

where:

$$U_P^* = \begin{pmatrix} U_{0/i-3} & U_{0/i-2} & U_i \end{pmatrix}^T \quad (\text{F.78})$$

For the system outputs we can write:

$$y_{k+1} = Cx_k = C \left( \bar{A}^k x_0 + \Delta_k^{*d} U_{0/k-1}^* \right) \quad (\text{F.79})$$

Vectorially we can write then the past outputs  $Y_P$  of the state-space system, as follows:

$$Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**} \quad (\text{F.80})$$

with the following notations:

$$\left\{ \begin{array}{l} \Gamma_i^{**} \stackrel{def}{=} \begin{pmatrix} 0 & \Gamma_{i-1}^* \end{pmatrix}^T, U_P^{**} \stackrel{def}{=} \begin{pmatrix} 0 & 0 & u_0^* & \dots & u_{i-2}^* \end{pmatrix}^T \\ H_i^{**d} \stackrel{def}{=} C \begin{pmatrix} 0 & 0 & 0 & 0 \\ \Delta_0^{*d} & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ \Delta_{i-2}^{*d} & 0 & \dots & 0 \end{pmatrix}^T \end{array} \right. \quad (\text{F.81})$$

The first zero appears in (F.81), because the initial output is zero. Similarly with (F.80), we can write for the future outputs the following relation:

$$Y_f = \Gamma_i^* X_{f-1}^d + H_i^{*d} U_f^* \quad (\text{F.82})$$

The future states term  $X_{f-1}^d$  stands for the future states  $X_i^d$  vector, with one step behind.

The future inputs  $U_f^*$  and  $H_i^{*d}$  are defined by:

$$U_f^* \stackrel{def}{=} \begin{pmatrix} u_{i-1}^* & u_i^* & \dots & u_{2i-2}^* \end{pmatrix}^T, H_i^{*d} = C \begin{pmatrix} \Delta_0^{*d} & 0 & \dots & 0 \\ \Delta_1^{*d} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \Delta_{i-1}^{*d} & 0 & \dots & 0 \end{pmatrix} \quad (\text{F.83})$$

From (F.77), (F.80) and (F.82), we can write *the matrix input-output equations* for this symplectic time-discretization scheme:

$$\left\{ \begin{array}{l} X_f^d = \overline{A}^i X_P^d + \Delta_i^{*d} U_P^* \\ Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**} \\ Y_f = \Gamma_i^* X_{f-1}^d + H_i^{*d} U_f^* \end{array} \right. \quad (\text{F.84})$$

As for the classic state-space approach, we formulate the following theorem:

### Theorem F.3

*The matrix input-output equations of a discrete-time PCH system, represented in explicit form in (F.69) and in implicit form in (F.70), for a symplectic time-discretization scheme with an implicit mid-point rule for the flows and explicit Euler for the efforts are:*

$$\left\{ \begin{array}{l} X_f^d = \overline{A}^i X_P^d + \Delta_i^{*d} U_P^* \\ Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**} \\ Y_f = \Gamma_i^* X_{f-1}^d + H_i^{*d} U_f^* \end{array} \right. \quad (\text{F.85})$$

Next an equivalent state-space representation will be written using (F.69) and (5.24).

Replacing the states from (5.33) in (F.69), we can write:

$$x_{k+1} = \overline{A} x_{k-1} + \overline{B} u_{k-1} + \left( \frac{\overline{A} - I}{2} \right) (x_{k-1} + \overline{A} x_{k-1} + \overline{B} u_{k-1}) + \overline{B} u_{k+1} \quad (\text{F.86})$$

Equivalently we can write:

$$x_{k+1} = \frac{2\bar{A}x_{k-1} + 2\bar{B}u_{k-1} + \bar{A}x_{k-1} + \bar{A}^2x_{k-1} + \bar{A}\bar{B}u_{k-1} - x_{k-1} - \bar{A}x_{k-1} - \bar{B}u_{k-1} + 2\bar{B}u_{k+1}}{2} \quad (\text{F.87})$$

More compact this relation can be written as:

$$x_{k+1} = \left\{ \frac{2\bar{A} + \bar{A}^2 - I}{2} \right\} x_{k-1} + \frac{1}{2} \begin{bmatrix} \bar{B} + \bar{A}\bar{B} & 2\bar{B} \end{bmatrix} \begin{bmatrix} u_{k-1} \\ u_{k+1} \end{bmatrix} \quad (\text{F.88})$$

The system outputs can be computed as follows:

$$y_{k+1} = Cx_k = C(\bar{A}x_{k-1} + \bar{B}u_{k-1}) \quad (\text{F.89})$$

Equivalently we can write:

$$y_{k+1} = C\bar{A}x_{k-1} + \begin{pmatrix} C\bar{B} & 0 \end{pmatrix} \begin{pmatrix} u_{k-1} \\ u_{k+1} \end{pmatrix} \quad (\text{F.90})$$

Using (F.88) and (F.90), the following relation results:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{2\bar{A} + \bar{A}^2 - I}{2} & \frac{1}{2}(\bar{B} + \bar{A}\bar{B} \quad 2\bar{B}) \\ C\bar{A} & \begin{pmatrix} C\bar{B} & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} x_{k-1} \\ u_{k-1} \\ u_k \end{pmatrix} \quad (\text{F.91})$$

Adopting the following notations for a compact representation:

$$\begin{cases} \tilde{A} \stackrel{\text{def}}{=} \frac{2\bar{A} + \bar{A}^2 - I}{2}, \tilde{B} \stackrel{\text{def}}{=} \frac{1}{2}(\bar{B} + \bar{A}\bar{B} \quad 2\bar{B}) \\ \tilde{C} \stackrel{\text{def}}{=} C\bar{A}, \tilde{D} \stackrel{\text{def}}{=} \begin{pmatrix} C\bar{B} & 0 \end{pmatrix}, \tilde{u}_{k-1} \stackrel{\text{def}}{=} \begin{pmatrix} u_{k-1} & u_{k+1} \end{pmatrix}^T \end{cases} \quad (\text{F.92})$$

we can simplify then (F.91) as follows:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \begin{pmatrix} x_{k-1} \\ \tilde{u}_{k-1} \end{pmatrix} \quad (\text{F.93})$$

Further a deterministic identification theorem, will be proved for the selected symplectic time-discretization scheme.

**Remark:** Similar to the classic state-space case, it is possible to recover the state-space matrices  $(A, B, C)$  by solving (F.93), knowing the input/output vectors.

Thus we propose the following theorem:

#### Theorem F.4

Under the assumptions that:

- I. The input  $u_k$  is persistently exciting of order  $2i$
- II. The intersection of the row space of  $U_f^*$  (the future inputs) and the row space of  $X_p^d$  (the past states) is empty.

III. The user-defined weighting matrices  $W_1 \in R^{l \times l}$  and  $W_2 \in R^{(j-1) \times (j-1)}$  are such that  $W_1$  is of full rank and  $W_2$  obeys:  $\text{rank} \left( W_{p-1}^* \right) = \text{rank} \left( W_{p-1}^* W_2 \right)$ , where  $W_p^*$  is the block Hankel matrix containing the past inputs and outputs and with  $O_i$  defined as the oblique projection:

$$O_i = \left( Y_f / U_{f-1}^* \right) W_{p-1}^* \quad (1)$$

and the singular value decomposition:

$$W_1 O_i W_2 = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1 S_1 V_1^T \quad (2)$$

we have:

1. The matrix  $O_i$  is equal to the product of the extended observability matrix and the states:

$$O_i = \Gamma_i^* X_{f-1}^d \quad (3)$$

2. The order of the system is equal to the number of singular values in (2) different from 0.

3. The extended observability matrix  $\Gamma_i^*$  is equal to:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (4)$$

4. The part of the state sequence  $X_{f-1}^d$  that lies in the column space of  $W_2$  can be recovered from:

$$X_{f-1}^d W_2 = T^{-1} S_1^{1/2} V_1^T \quad (5)$$

5. The state sequence  $X_{f-1}^d$  is equal to:

$$X_{f-1}^d = \Gamma_i^{*+} O_i \quad (6)$$

**Proof:**

Using (F.85), we can write:

$$X_f^d = \overline{A}^i \left( \Gamma_i^{**+} Y_P - \Gamma_i^{**+} H_i^{**d} U_P^{**} \right) + \Delta_i^{*d} U_P \quad (F.94)$$

where:

$$\Gamma_i^{**} = \begin{pmatrix} 0 & \Gamma_{i-1}^* \end{pmatrix}^T \quad (F.95)$$

$$H_i^{**d} = C \begin{pmatrix} 0 & \Delta_0^{*d} & \dots & \Delta_{i-2}^{*d} \end{pmatrix}^T \quad (F.96)$$

$$U_P^{**} = \begin{pmatrix} 0 & 0 & u_0^* & \dots & u_{i-2}^* \end{pmatrix} \quad (F.97)$$

**Remark:** The first zero terms in (F.95), (F.96) and (F.97) corresponds to the first output of the system response, which is zero.

(F.94) can be also written as:

$$X_f^d = \begin{pmatrix} -\Gamma_i^{**+} & H_i^{**d} & \Delta_i^{*d} \end{pmatrix} \begin{pmatrix} U_P^{**} \\ U_P^* \end{pmatrix} + \overline{\overline{A}}^i \Gamma_i^{**+} Y_P \quad (\text{F.98})$$

Adopting the following notations:

$$\begin{cases} L_P^* \stackrel{\text{def}}{=} \begin{pmatrix} -\Gamma_i^{**+} & H_i^{**d} & \Delta_i^{*d} & \overline{\overline{A}}^i \Gamma_i^{**+} \end{pmatrix} \\ W_P^* \stackrel{\text{def}}{=} \begin{pmatrix} U_P^{**} & U_P^* & Y_P \end{pmatrix}^T \end{cases} \quad (\text{F.99})$$

we can write (F.98) as:

$$X_f^d = L_P^* W_P^* \quad (\text{F.100})$$

Using (F.85) and (F.100), we can write:

$$Y_f = \Gamma_i^* L_{P-1}^* W_{P-1}^* + H_i^{*d} U_f^* \quad (\text{F.101})$$

Multiplying to the right with the orthogonal projection of  $U_f^*$ , it results:

$$Y_f \Pi_{U_f^{*\perp}} = \Gamma_i^* L_{P-1}^* W_{P-1}^* \Pi_{U_f^{*\perp}} + H_i^{*d} U_f^* \Pi_{U_f^{*\perp}} \quad (\text{F.102})$$

**Remark:** The terms  $L_{P-1}^*$  and  $W_{P-1}^*$  come from the fact that the future outputs, are expressed in rapport with the future states, with one step behind  $X_{f-1}^d$ .

We can write then:

$$Y_f / U_f^{*\perp} = \Gamma_i^* L_{P-1}^* W_{P-1}^* / U_f^{*\perp} \quad (\text{F.103})$$

using the orthogonal projection operator and the property:

$$U_f^* \Pi_{U_f^{*\perp}} \quad (\text{F.104})$$

$$\left( Y_f / U_f^{*\perp} \right) \left( W_{P-1}^* / U_f^{*\perp} \right)^+ W_{P-1}^* = \Gamma_i^* L_{P-1}^* W_{P-1}^*$$

(F.105)

We introduce then the oblique projection for the left term:

$$O_i = \left( Y_f / U_f^{*\perp} \right) \left( W_{P-1}^* / U_f^{*\perp} \right)^+ W_{P-1}^* \quad (\text{F.106})$$

Using (F.100) and (F.105), we can write:

$$O_i = \Gamma_i^* X_{f-1}^d \quad (\text{F.107})$$

In (F.105), the following relation has been used:

$$\left( W_{P-1}^* / U_f^{*\perp} \right) \left( W_{P-1}^* / U_f^{*\perp} \right)^+ W_{P-1}^* = W_{P-1}^* \quad (\text{F.108})$$

For the next step it is necessary to prove that:

$$\text{rank} W_{P-1}^* = \text{rank} \left( W_{P-1}^* / U_f^{*\perp} \right) \quad (\text{F.109})$$

We can then write the following relation:

$$W_{P-1}^* = \begin{pmatrix} I_{(2i-4)m} & 0 & 0 \\ 0 & I_{(2i-4)m} & 0 \\ H_{i-1}^{**d} & 0 & \Gamma_{i-1}^{**} \end{pmatrix} \begin{pmatrix} U_P \\ U_P^* \\ X_P^d \end{pmatrix} \quad (\text{F.110})$$

where:

$$W_P^* = L_P^* W_P^* \quad (\text{F.111})$$

Further we multiply to the right with the orthogonal projector of  $U_f^*$ , which satisfies:

$$U_f^{*\Pi} U_{f^* \perp} = 0 \quad (\text{F.112})$$

and we can write:

$$W_{P-1}^* \Pi_{U_{f^* \perp}} = \begin{pmatrix} I_{(2i-4)m} & 0 & 0 \\ 0 & I_{(2i-4)m} & 0 \\ H_{i-1}^{**d} & 0 & \Gamma_{i-1}^{**} \end{pmatrix} \begin{pmatrix} U_P^{**\Pi} U_{f^* \perp} \\ U_P^* \Pi_{U_{f^* \perp}} \\ X_P^d \Pi_{U_{f^* \perp}} \end{pmatrix} \quad (\text{F.113})$$

Equivalently we can write:

$$W_{P-1}^* / U_{f^* \perp} = \begin{pmatrix} I_{(2i-4)m} & 0 & 0 \\ 0 & I_{(2i-4)m} & 0 \\ H_{i-1}^{**d} & 0 & \Gamma_{i-1}^{**} \end{pmatrix} \begin{pmatrix} U_P^{**\Pi} U_{f^* \perp} \\ U_P^* \Pi_{U_{f^* \perp}} \\ X_P^d \Pi_{U_{f^* \perp}} \end{pmatrix} \quad (\text{F.114})$$

or:

$$W_{P-1}^* / U_{f^* \perp} = \begin{pmatrix} I_{(2i-4)m} & 0 & 0 \\ 0 & I_{(2i-4)m} & 0 \\ H_{i-1}^{**d} & 0 & \Gamma_{i-1}^{**} \end{pmatrix} \begin{pmatrix} U_P^{**} / U_{f^* \perp} \\ U_P^* / U_{f^* \perp} \\ X_P^d / U_{f^* \perp} \end{pmatrix} \quad (\text{F.115})$$

Using the initial conditions assumed on this theorem, it results:

$$\text{rank} \begin{pmatrix} U_P^{**} \\ U_P^* \\ X_P^d \end{pmatrix} = \text{rank} \begin{pmatrix} U_P^{**} / U_{f^* \perp} \\ U_P^* / U_{f^* \perp} \\ X_P^d / U_{f^* \perp} \end{pmatrix} \quad (\text{F.116})$$

which proves (F.109).

**Remark:** It is necessary to use the same initial conditions, as for the classic state-space case, in order to prove (F.109), the only difference is given by the structure of  $W_P^*$ , which is changing according with the symplectic time-discretization scheme.

We then introduce the SVD decomposition:

$$W_{P-1}^* / U_f^{*\perp} = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1 S_1 V_1^T \quad (\text{F.117})$$

where  $W_{P-1}^* / U_f^{*\perp}$  is a linear combination of the columns of  $W_{P-1}^*$  and using (F.109), the column spaces of  $W_{P-1}^*$  and  $W_{P-1}^* / U_f^{*\perp}$  are also equal. We can write then the following relation:

$$W_{P-1}^* = U_1 R \quad (\text{F.118})$$

Using the decomposition from (F.117), we can write:

$$\left( W_{P-1}^* / U_f^{*\perp} \right) \left( W_{P-1}^* / U_f^{*\perp} \right)^+ W_{P-1}^* = \begin{pmatrix} U_1 & S_1 & V_1^T \end{pmatrix} \begin{pmatrix} V_1 & S_1^{-1} & U_1^T \end{pmatrix} \begin{pmatrix} U_1 & R \end{pmatrix} \quad (\text{F.119})$$

Grouping the terms in (F.119), it results:

$$\left( W_{P-1}^* / U_f^{*\perp} \right) \left( W_{P-1}^* / U_f^{*\perp} \right)^+ W_{P-1}^* = U_1 S_1 \left( V_1^T V_1 \right) S_1^{-1} \left( U_1^T U_1 \right) R \quad (\text{F.120})$$

Using the properties:

$$U_1^T U_1 = V_1^T V_1 = I \quad (\text{F.121})$$

related to SVD decomposition, it results:

$$\left( W_{P-1}^* / U_f^{*\perp} \right) \left( W_{P-1}^* / U_f^{*\perp} \right)^+ W_{P-1}^* = U_1 R \quad (\text{F.122})$$

which is also equal to (F.118). We then compute:

$$W_1 O_i W_2 = W_1 \Gamma_i^* X_{f-1}^d W_2 \quad (\text{F.123})$$

where we consider  $W_1 \in R^{l \times l_i}$  and  $W_2 \in R^{(j-1)x(j-1)}$ . The future states vector  $X_{f-1}^d$  in this case is:

$$X_{f-1}^d = \begin{pmatrix} x_{i-1}^d & x_i^d & \dots & x_{i+j-3}^d & x_{i+j-2}^d \end{pmatrix} \in R^{n \times (j-1)} \quad (\text{F.124})$$

The rank of  $W_1 \Gamma_i^*$  and  $X_{f-1}^d W_2$  is  $n$  and also is their product. Thus statement 2 of the theorem is proved. Introducing a non-singular transformation matrix ( $T \in R^{n \times n}$ ), we can write:

$$\begin{cases} W_1 \Gamma_i^* = U_1 S_1^{1/2} T \\ X_{f-1}^d W_2 = T^{-1} S_1^{1/2} V_1^T \end{cases} \quad (\text{F.125})$$

from which the observability matrix  $\Gamma_i^*$  is:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (\text{F.126})$$

This proves statement 4 of the theorem. Statement 5 of the theorem is given by the 2<sup>nd</sup> relation of (F.125). From (F.107), we can recover the states  $X_{f-1}^d$ :

$$X_{f-1}^d = \Gamma_i^{*+} O_i \quad (\text{F.127})$$



which proves the last statement of the theorem.

#### F.4 Symplectic discretization scheme V

This symplectic time-discretization scheme assumes an implicit Euler rule for the flows and an implicit mid-point rule for the efforts. The general form of this symplectic discretization scheme is:

$$\begin{cases} x_{k+1} = x_k + \left( \frac{\bar{A} - I}{2} \right) (x_k + x_{k+1}) + \bar{B} u_{k+1} \\ y_{k+1} = C \left( \frac{x_k + x_{k+1}}{2} \right) \end{cases} \quad (\text{F.128})$$

The implicit form of the PCH system for this symplectic discretization scheme is:

$$(f_S(k+1), e_S(k+1), f_C(k+1), e_C(k+1), f_R(k+1), e_R(k+1)) \in D \quad (\text{F.129})$$

Using (5.33) and (F.128), we can write equivalently:

$$x_{k+1} = \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} + \left( \frac{\bar{A} - I}{2} \right) \left( \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} + \bar{A}^{k+1} x_0 + \Delta_{k+1}^d U_{0/k} \right) + \bar{B} u_{k+1} \quad (\text{F.130})$$

This relation can be further written as:

$$x_{k+1} = \frac{2\bar{A}^k x_0 + 2\Delta_k^d U_{0/k-1} + \bar{A}^{k+1} x_0 + \bar{A} \Delta_k^d U_{0/k-1} + \bar{A}^{k+2} x_0 + \bar{A} \Delta_{k+1}^d U_{0/k}}{2} - \frac{\bar{A}^k x_0 - \Delta_k^d U_{0/k-1} - \bar{A}^{k+1} x_0 - \Delta_{k+1}^d U_{0/k} + 2\bar{B} u_{k+1}}{2} \quad (\text{F.131})$$

This relation can be further written as:

$$x_{k+1} = \left( \frac{\bar{A}^{k+2} + \bar{A}^k}{2} \right) x_0 + \left( \frac{\bar{A} + I}{2} \right) \Delta_k^d U_{0/k-1} + \left( \frac{\bar{A} - I}{2} \right) \Delta_{k+1}^d U_{0/k} + \frac{2\bar{B} u_{k+1}}{2} \quad (\text{F.132})$$

Adopting the notations:

$$\begin{cases} \Delta_{k+1}^{-d} \stackrel{\text{def}}{=} \begin{pmatrix} \bar{A}^k \bar{B} & \bar{A}^{k-1} \bar{B} & \dots & \bar{A}^2 \bar{B} & \bar{A} \bar{B} \end{pmatrix} \\ \bar{\bar{A}}_{k+1} \stackrel{\text{def}}{=} \left( \frac{\bar{A}^{k+2} + \bar{A}^k}{2} \right), X_P^d \stackrel{\text{def}}{=} x_0 \end{cases} \quad (\text{F.133})$$

(F.132) becomes:

$$x_{k+1} = \bar{\bar{A}}_{k+1} X_P^d + \frac{\Delta_{k+1}^{-d} U_{0/k-1} + \Delta_k^d U_{0/k-1} + \Delta_{k+2}^d U_{0/k+1} + \begin{pmatrix} -\Delta_{k+1}^d & \bar{B} \end{pmatrix} U_{0/k+1}}{2} \quad (\text{F.134})$$

More compact we can write (F.134) as:

$$x_{k+1} = \bar{\bar{A}}_{k+1} X_P^d + \frac{1}{2} \begin{pmatrix} \Delta_{k+1}^{-d} + \Delta_k^d & \Delta_{k+2}^d + \begin{pmatrix} -\Delta_{k+1}^d & \bar{B} \end{pmatrix} \end{pmatrix} \begin{pmatrix} U_{0/k-1} \\ U_{0/k+1} \end{pmatrix} \quad (\text{F.135})$$

Adopting further the notations:

$$\begin{cases} \Delta_{k+1}^{*d} \stackrel{def}{=} \frac{1}{2} \left( \Delta_{k+1}^{-d} + \Delta_k^d \quad \Delta_{k+2}^d + \left( -\Delta_{k+1}^d \quad \overline{B} \right) \right) \\ U_{0/k}^* \stackrel{def}{=} \left( U_{0/k-1} \quad U_{0/k+1} \right) \end{cases} \quad (\text{F.136})$$

we can express the states as:

$$x_{k+1} = \overline{A}^{k+1} X_P^d + \Delta_{k+1}^{*d} U_{0/k}^* \quad (\text{F.137})$$

The future states become then:

$$X_f^d = X_i^d = \overline{A}^i X_P^d + \Delta_i^{*d} U_P^* \quad (\text{F.138})$$

where:

$$U_P^* = \left( u_0^* \quad u_1^* \quad \dots \quad u_{i-2}^* \quad u_{i-1}^* \right)^T \quad (\text{F.139})$$

stands for the past outputs. We compute then the system outputs, as follows:

$$y_{k+1} = \frac{C}{2} (x_k + x_{k+1}) = \frac{C}{2} \left( \overline{A}^k X_P^d + \Delta_k^{*d} U_{0/k-1}^* + \overline{A}^{k+1} X_P^d + \Delta_{k+1}^{*d} U_{0/k}^* \right) \quad (\text{F.140})$$

At state  $k$ , the output can be written as:

$$y_k = \frac{C}{2} \overline{A}^{k-1} X_P^d + \frac{C}{2} \overline{A}^k X_P^d + \frac{C}{2} \begin{bmatrix} \Delta_{k-1}^{*d} & \Delta_k^{*d} \\ U_{0/k-2}^* \\ U_{0/k-1}^* \end{bmatrix} \quad (\text{F.141})$$

A vectorial representation of the past outputs is:

$$Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**} \quad (\text{F.142})$$

Using the notations:

$$\begin{aligned} \Gamma_i^{**} &\stackrel{def}{=} \frac{1}{2} \left( \begin{pmatrix} 0 \\ \Gamma_i^* \end{pmatrix} + \Gamma_i^* \right), \Gamma_i^* \stackrel{def}{=} \left( C \quad C\overline{A} \quad \dots \quad C\overline{A}^{i-1} \right)^T, \\ H_i^{**d} &\stackrel{def}{=} \frac{1}{2} C \begin{pmatrix} \Delta_{-1}^{*d} & \dots & 0 & \Delta_0^{*d} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \Delta_{i-2}^{*d} & \dots & 0 & \Delta_{i-1}^{*d} & \dots & 0 \end{pmatrix}, U_P^{**} \stackrel{def}{=} \begin{bmatrix} U_{P-1}^* & U_P^* \end{bmatrix}^T \end{aligned} \quad (\text{F.143})$$

similar to the classic case. For the vector of future outputs, we can write:

$$Y_f = \Gamma_i^* \overline{X}_f^d + H_i^{**d} U_f^{**} \quad (\text{F.144})$$

where:

$$\overline{X}_f^d \stackrel{def}{=} \frac{X_{f-1}^d + X_f^d}{2} \quad (\text{F.145})$$

Using (F.138), (F.143), (F.144) and (F.145), the matrix input-output equations result as:

$$\begin{cases} X_f^d = \bar{A}^i X_P^d + \Delta_i^{*d} U_P^* \\ \bar{X}_f^d = \frac{X_{f-1}^d + X_f^d}{2} \\ Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**} \\ Y_f = \Gamma_i^* \bar{X}_f^d + H_i^{**d} U_f^{**} \end{cases} \quad (\text{F.146})$$

### Theorem F.5

The matrix input-output equations of a discrete PCH system, represented in explicit form in (F.128) and in implicit form in (F.129), for a symplectic discretization scheme with an implicit Euler rule for the flows and implicit mid-point rule for the efforts are:

$$\begin{cases} X_f^d = \bar{A}^i X_P^d + \Delta_i^{*d} U_P^* \\ \bar{X}_f^d = \frac{X_{f-1}^d + X_f^d}{2} \\ Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**} \\ Y_f = \Gamma_i^* \bar{X}_f^d + H_i^{**d} U_f^{**} \end{cases} \quad (\text{F.147})$$

Next we write a more compact form of (F.128), using (5.24) as:

$$x_{k+1} = x_k + \left( \frac{\bar{A} - I}{2} \right) (x_k + \bar{A} x_k + \bar{B} u_k) + \bar{B} u_{k+1} \quad (\text{F.148})$$

Developing this relation, we can write:

$$x_{k+1} = \frac{2x_k + \bar{A}x_k + \bar{A}^2x_k + \bar{A}\bar{B}u_k - x_k - \bar{A}x_k - \bar{B}u_k + 2\bar{B}u_{k+1}}{2} \quad (\text{F.149})$$

$$x_{k+1} = \left( \frac{\bar{A}^2 + I}{2} \right) x_k + \frac{\bar{A}\bar{B}u_k - \bar{B}u_k + 2\bar{B}u_{k+1}}{2} \quad (\text{F.150})$$

Equivalently we can write:

$$x_{k+1} = \left( \frac{\bar{A}^2 + I}{2} \right) x_k + \frac{1}{2} \begin{pmatrix} \bar{A}\bar{B} - \bar{B} & 2\bar{B} \end{pmatrix} \begin{pmatrix} u_k \\ u_{k+1} \end{pmatrix} \quad (\text{F.151})$$

For the system outputs, we can write:

$$y_{k+1} = \frac{C}{2} (x_k + x_{k+1}) = \frac{C}{2} (x_k + \bar{A}x_k + \bar{B}u_k) \quad (\text{F.152})$$

Equivalently we can write:

$$y_{k+1} = \left( \frac{C + \bar{A}C}{2} \right) x_k + \frac{1}{2} \begin{pmatrix} C\bar{B} & 0 \end{pmatrix} \begin{pmatrix} u_k \\ u_{k+1} \end{pmatrix} \quad (\text{F.153})$$

Using (F.151) and (F.153) it results:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{\bar{A}^2 + I}{2} & \frac{1}{2}(\bar{A}\bar{B} - \bar{B} \ 2\bar{B}) \\ \frac{C + \bar{A}}{2} & \frac{1}{2}(C\bar{B} \ 0) \end{pmatrix} \quad (\text{F.154})$$

Further we can adopt the notations:

$$\begin{cases} \tilde{A} \stackrel{\text{def}}{=} \begin{pmatrix} \bar{A}^2 + I \\ 2 \end{pmatrix}; \tilde{B} \stackrel{\text{def}}{=} (\bar{A}\bar{B} - \bar{B} \ 2\bar{B}) \\ \tilde{C} \stackrel{\text{def}}{=} \frac{C + \bar{A}}{2}; \tilde{D} \stackrel{\text{def}}{=} \frac{1}{2}(C\bar{B} \ 0); \tilde{u}_k \stackrel{\text{def}}{=} (u_k \ u_{k+1})^T \end{cases} \quad (\text{F.155})$$

and (F.154) becomes:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \begin{pmatrix} x_k \\ \tilde{u}_k \end{pmatrix} \quad (\text{F.156})$$

**Remark:** As for the classic state-space case, it is possible to recover the state-space matrices  $(A, B, C)$  by solving (F.156), when knowing the input/output vectors.

Next we propose a deterministic identification theorem corresponding to this symplectic time-discretization scheme and perform the proof as in the classic state-space case.

### Theorem F.6

Under the assumptions that:

I. The input  $u_k$  is persistently exciting of order  $2i$

I. The intersection of the row space of  $U_f^{**}$  (the future inputs) and the row space of  $X_P^d$  (the past states) is empty.

II. The user-defined weighting matrices  $W_1 \in R^{lixli}$  and  $W_2 \in R^{jxj}$  are such that  $w_1$  is of

full rank and  $w_2$  obeys:  $\text{rank}(W_P^*) = \text{rank}(W_P^* W_2)$ , where  $W_P^*$  is the block Hankel

matrix containing the past inputs and outputs and with  $o_i$  defined as the oblique

projection:

$$O_i = \left( Y_f / U_f^{**} \right) \cdot W_P^* \quad (1)$$

and with the singular value decomposition:

$$W_1 O_i W_2 = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} \quad (2)$$

we have:

I. The matrix  $o_i$  is equal to the product of the extended observability matrix and the states:

$$O_i = \Gamma_i^* \bar{X}_f^d \quad (3)$$

2. The order of the system is equal to the number of singular values in (2) different from 0.

3. The extended observability matrix  $\Gamma_i^*$  is equal to:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (4)$$

4. The part of the state sequence  $\bar{X}_f^d$  that lies in the column space of  $w_2$  can be recovered from:

$$\bar{X}_f^d W_2 = T^{-1} S_1^{1/2} V_1^T \quad (5)$$

5. The state sequence  $\bar{X}_f^d$  is equal to:

$$\bar{X}_f^d = \Gamma_i^{*+} O_i \quad (6)$$

**Proof:**

Using the first three relations of (F.147) we can write:

$$\bar{X}_f^d = \left\{ \frac{X_i^d + X_{i-1}^d}{2} \right\} = A_M^i \left( \Gamma_i^{**+} Y_P - \Gamma_i^{**+} H_i^{**d} U_P^{**} \right) + \Delta_M U_P^{**}$$

(F.157)

where we define:

$$\left\{ \begin{aligned} A_M^i & \stackrel{\text{def}}{=} \left( \frac{\bar{A}^{i-1} + \bar{A}^i}{2} \right) \\ \Delta_M & \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} \Delta_{i-1}^{*d} & \Delta_i^{*d} \end{pmatrix} \end{aligned} \right. \quad (F.158)$$

Equivalently we can write:

$$\bar{X}_f^d = \left( \Delta_M - A_M^i \Gamma_i^{**+} H_i^{**d} \quad A_M^i \Gamma_i^{**+} \right) \begin{pmatrix} U_P^{**} \\ Y_P \end{pmatrix} \quad (F.159)$$

Adopting the notations:

$$\left\{ \begin{aligned} L_P^* & = \left( \Delta_M - A_M^i \Gamma_i^{**+} H_i^{**d} \quad A_M^i \Gamma_i^{**+} \right) \\ W_P^* & = \begin{pmatrix} U_P^{**} \\ Y_P \end{pmatrix}^T \end{aligned} \right. \quad (F.160)$$

(F.159) becomes:

$$\bar{X}_f^d = L_P^* W_P^* \quad (F.161)$$

Using (F.161), the future outputs from (F.147) become:

$$Y_f = \Gamma_i^* L_P^* W_P^* + H_i^{**d} U_f^{**} \quad (F.162)$$

Next we apply the orthogonal operator  $\Pi_{uf}^{**+}$ , to the right:

$$Y_f \cdot \Pi_{U_f^{**\perp}} = \Gamma_i^* L_P^* W_P^* \Pi_{U_f^{**\perp}} + H_i^{**d} U_f^{**} \Pi_{U_f^{**\perp}} \quad (\text{F.163})$$

Using the orthogonal projection operator, we can write equivalently:

$$Y_f / U_f^{**\perp} = \Gamma_i^* L_P^* W_P^* / U_f^{**\perp} \quad (\text{F.164})$$

Multiplying to the right with the pseudo-inverse  $\left(W_P^* / U_f^{**\perp}\right)^+$  and  $W_P^*$ , we can write:

$$\left(Y_f / U_f^{**\perp}\right) \left(W_P^* / U_f^{**\perp}\right)^+ W_P^* = \Gamma_i^* L_P^* W_P^* \quad (\text{F.165})$$

Using (F.161) and the oblique projection operator, we can write:

$$O_i = \Gamma_i^* \bar{X}_f^d \quad (\text{F.166})$$

For the next step, it is necessary to prove that:

$$\text{rank} \left(W_P^*\right) = \text{rank} \left(W_P^* / U_f^{**\perp}\right) \quad (\text{F.167})$$

Using the expression of  $W_P^*$  from (F.160), we can write equivalently:

$$W_P^* = \begin{pmatrix} I_{(4i-2)m} & 0 \\ H_i^{**d} & \Gamma_i^{**} \end{pmatrix} \begin{pmatrix} U_P^{**} \\ X_P^d \end{pmatrix} \quad (\text{F.168})$$

where  $U_P^{**}$  has the following form:

$$U_P^{**} = \begin{pmatrix} U_{0:/i-2}^* & U_{0:/i-1}^* \end{pmatrix}^T \quad (\text{F.169})$$

We then multiply in (F.168) with the orthogonal complement of  $U_f^{**\perp}$  and we get:

$$W_P^* \Pi_{U_f^{**\perp}} = \begin{pmatrix} I_{(4i-2)m} & 0 \\ H_i^{**d} & \Gamma_i^{**} \end{pmatrix} \begin{pmatrix} U_P^{**} \Pi_{U_f^{**\perp}} \\ X_P^d \Pi_{U_f^{**\perp}} \end{pmatrix} \quad (\text{F.170})$$

Using the two initial conditions assumed for the theorem, it results that:

$$\text{rank} \begin{pmatrix} U_P^{**} \\ X_P^d \end{pmatrix} = \text{rank} \begin{pmatrix} U_P^{**} / U_f^{**\perp} \\ X_P^d / U_f^{**\perp} \end{pmatrix} \quad (\text{F.171})$$

Then we get:

$$\text{rank} \left(W_P^*\right) = \text{rank} \left(W_P^* / U_f^{**\perp}\right) \quad (\text{F.172})$$

**Remark:** The initial conditions assumed on the deterministic identification theorem for this symplectic discretization scheme are similar to the classic case, the difference being given by the block Hankel matrix  $W_P^*$  which changes the structure.

We denote then the SVD decomposition of  $W_P^* / U_f^{**\perp}$  as:

$$W_P^* / U_f^{**\perp} = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1 S_1 V_1^T \quad (\text{F.173})$$

As  $W_P^* / U_f^{**\perp}$  is a linear combination of the columns of  $W_P^*$  and (F.172) is satisfied, then the column spaces of  $W_P^*$  and  $W_P^* / U_f^{**\perp}$  are also equal. We can write then:

$$W_P^* = U_1 R \quad (\text{F.174})$$

Using the SVD decomposition from (F.173), we can write:

$$\left( W_P^* / U_f^{**\perp} \right) \left( W_P^* / U_f^{**\perp} \right)^+ W_P^* = \left( U_1 S_1 V_1^T \right) \left( V_1 S_1^{-1} U_1^T \right) \left( U_1 R \right) \quad (\text{F.175})$$

Using the properties:

$$V_1^T V_1 = U_1^T U_1 = I \quad (\text{F.176})$$

associated to the  $U, V$  matrices from the SVD decomposition, it results:

$$\left( W_P^* / U_f^{**\perp} \right) \left( W_P^* / U_f^{**\perp} \right)^+ W_P^* = U_1 R \quad (\text{F.177})$$

Using (F.174) and (F.177), we can write the following relation:

$$\left( W_P^* / U_f^{**\perp} \right) \left( W_P^* / U_f^{**\perp} \right)^+ W_P^* = W_P^* \quad (\text{F.178})$$

Using (F.166), we can write the product:

$$W_1 O_i W_2 = W_1 \Gamma_i^* \bar{X}_f^d W_2 \quad (\text{F.179})$$

We observe that the products  $W_1 \Gamma_i^*$  and  $\bar{X}_f^d W_2$  contain  $n$  columns and respectively  $n$  rows, from which the product rank is also  $n$ . This proves statement 2 of the theorem.

Introducing a non-singular similarity transformation ( $T \in R^{n \times n}$ ) in (F.179), we can split (F.179) in two pieces, as follows:

$$W_1 \Gamma_i^* = U_1 S_1^{1/2} T \quad (\text{F.180})$$

$$\bar{X}_f^d W_2 = T^{-1} S_1^{1/2} V_1^T \quad (\text{F.181})$$

Multiplying (F.180) to the left with  $W_1^{-1}$ , we can write further:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (\text{F.182})$$

From (F.181) and (F.182), statements 4 and 5 of the theorem are proved. Using (F.166) it results also that the future states sequence is equal to:

$$\bar{X}_f^d = \Gamma_i^{*+} O_i \quad (\text{F.183})$$

which proves the last statement of the theorem.

## F.5 Symplectic discretization scheme VI

For this symplectic time-discretization scheme, it is assumed an implicit mid-point rule for the flows and implicit Euler rule for the efforts. The general explicit form of this symplectic discretization scheme is:

$$\begin{cases} x_{k+1} = x_k + \Delta T \cdot A \left( \frac{x_k + x_{k+1}}{2} \right) + \Delta T \cdot B u_{k+1} \\ y_{k+1} = C x_{k+1} \end{cases} \quad (\text{F.184})$$

The implicit form of the PCH system for this symplectic discretization scheme is:

$$(f_S(k+1), e_S(k+1), f_C(k+1), e_C(k+1), f_R(k+1), e_R(k+1)) \in D \quad (\text{F.185})$$

Combining (5.33) with (F.184), it results:

$$x_{k+1} = \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} + \left( \frac{\bar{A} - I}{2} \right) \left( \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} + \bar{A}^{k+1} x_0 + \Delta_{k+1}^d U_{0/k} \right) + \bar{B} u_{k+1} \quad (\text{F.186})$$

This relation can be further written as:

$$x_{k+1} = \left( \frac{\bar{A}^{k+2} + \bar{A}^k}{2} \right) x_0 + \left( \frac{\bar{A} + I}{2} \right) \Delta_k^d U_{0/k-1} + \left( \frac{\bar{A} - I}{2} \right) \Delta_{k+1}^d U_{0/k} + \frac{2 B u_{k+1}}{2} \quad (\text{F.187})$$

Adopting the notation:

$$\bar{\bar{A}}^{k+1} \stackrel{def}{=} \left( \frac{\bar{A}^{k+2} + \bar{A}^k}{2} \right) \quad (\text{F.188})$$

we can write further:

$$x_{k+1} = \bar{\bar{A}}^{k+1} x_0 + \left( \frac{\bar{A} + I}{2} \right) \Delta_k^d U_{0/k-1} + \left( \frac{\bar{A} - I}{2} \right) \Delta_{k+1}^d U_{0/k} + \frac{2 B u_{k+1}}{2} \quad (\text{F.189})$$

As for the previous cases, we adopt the notation:

$$\Delta_{k+1}^{-d} \stackrel{def}{=} \bar{A} \cdot \Delta_k^d = \begin{pmatrix} \bar{A}^k \bar{B} & \bar{A}^{k-1} \bar{B} & \dots & \bar{A} \bar{B} \end{pmatrix} \quad (\text{F.190})$$

and (F.189) becomes:

$$x_{k+1} = \bar{\bar{A}}^{k+1} x_0 + \frac{\Delta_{k+1}^{-d} U_{0/k-1} + \Delta_{k+2}^d U_{0/k+1} - \Delta_{k+1}^d U_{0/k} + \bar{B} u_{k+1}}{2} \quad (\text{F.191})$$

Further we can write more compact as:

$$x_{k+1} = \bar{\bar{A}}^{k+1} x_0 + \frac{1}{2} \left[ \Delta_{k+1}^{-d} + \Delta_k^d \quad \Delta_{k+2}^d + \begin{pmatrix} -\Delta_{k+1}^d & \bar{B} \end{pmatrix} \right] \begin{bmatrix} U_{0/k-1} \\ U_{0/k+1} \end{bmatrix} \quad (\text{F.192})$$

Adopting the notations:

$$\begin{cases} \Delta_{k+1}^{*d} \stackrel{def}{=} \frac{1}{2} \left( \Delta_{k+1}^{-d} + \Delta_k^d \quad \Delta_{k+2}^d + \begin{pmatrix} -\Delta_{k+1}^d & \bar{B} \end{pmatrix} \right)^T \\ U_k^* \stackrel{def}{=} (U_{0/k-1} \quad U_{0/k+1})^T \\ X_P^d \stackrel{def}{=} x_0 \end{cases} \quad (\text{F.193})$$

we can write the states as:

$$x_{k+1} = \bar{\bar{A}}^{k+1} X_P^d + \Delta_{k+1}^{*d} U_k^* \quad (\text{F.194})$$

For the future states we can write then:



$$X_f^d = X_i^d = \overline{\overline{A}}^i X_P^d + \Delta_i^{*d} U_P^* \quad (\text{F.195})$$

where:

$$U_P^* = \begin{pmatrix} u_0^* & u_1^* & \dots & u_{i-1}^* \end{pmatrix}^T \quad (\text{F.196})$$

represents the vector of past outputs. We can write the output vector as:

$$y_{k+1} = C \left( \overline{\overline{A}}^{k+1} X_P^d + \Delta_{k+1}^{*d} U_{0/k}^* \right) \quad (\text{F.197})$$

We can write then the following relations for the past outputs:

$$Y_P = \Gamma_i^* X_P^d + H_i^{*d} U_P^* \quad (\text{F.198})$$

where:

$$\begin{cases} \Gamma_i^* \stackrel{\text{def}}{=} \begin{pmatrix} C & C\overline{\overline{A}} & \dots & C\overline{\overline{A}}^{i-1} \end{pmatrix}^T \\ H_i^{*d} \stackrel{\text{def}}{=} C \begin{pmatrix} \Delta_0^{*d} & \dots & 0 \\ \dots & \dots & \dots \\ \Delta_{i-1}^{*d} & \dots & 0 \end{pmatrix}, U_P^* \stackrel{\text{def}}{=} \begin{pmatrix} u_0^* & u_1^* & \dots & u_{i-1}^* \end{pmatrix}^T \end{cases} \quad (\text{F.199})$$

Similarly we can write the vector of the future outputs as follows:

$$Y_f = \Gamma_i^* X_f^d + H_i^{*d} U_f^* \quad (\text{F.200})$$

From (F.195), (F.198) and (F.200), we can write the following relations for the *matrix input-output equations*:

$$\begin{cases} Y_P = \Gamma_i^* X_P^d + H_i^{*d} U_P^* \\ Y_f = \Gamma_i^* X_f^d + H_i^{*d} U_f^* \\ X_f^d = X_i^d = \overline{\overline{A}}^i X_P^d + \Delta_i^{*d} U_P^* \end{cases} \quad (\text{F.201})$$

### Theorem F.7

*The matrix input-output equations of a discrete-time PCH system, represented in explicit form in (F.184) and in implicit form in (F.185), for a symplectic discretization scheme with an implicit mid-point rule for the flows and implicit Euler for the efforts are:*

$$\begin{cases} Y_P = \Gamma_i^* X_P^d + H_i^{*d} U_P^* \\ Y_f = \Gamma_i^* X_f^d + H_i^{*d} U_f^* \\ X_f^d = X_i^d = \overline{\overline{A}}^i X_P^d + \Delta_i^{*d} U_P^* \end{cases}$$

Next we write an equivalent representation of (F.184) using (5.33) as for the classic case. Thus we can write:

$$x_{k+1} = x_k + \left( \frac{\overline{\overline{A}} - I}{2} \right) \left( x_k + \overline{\overline{A}} x_k + \overline{\overline{B}} u_k \right) + \overline{\overline{B}} u_{k+1} \quad (\text{F.202})$$

Further this relation becomes:

$$x_{k+1} = \frac{2x_k + \bar{A}x_k + \bar{A}^2x_k + \bar{A}\bar{B}u_k - x_k - \bar{A}x_k - \bar{B}u_k + 2\bar{B}u_{k+1}}{2} \quad (\text{F.203})$$

After making possible simplifications, we can write:

$$x_{k+1} = \left( \frac{\bar{A}^2 + I}{2} \right) x_k + \frac{\bar{A}\bar{B}u_k - \bar{B}u_k + 2\bar{B}u_{k+1}}{2} \quad (\text{F.204})$$

An equivalent representation of (F.204) is:

$$x_{k+1} = \left( \frac{\bar{A}^2 + I}{2} \right) x_k + \frac{1}{2} \begin{pmatrix} \bar{A}\bar{B} - \bar{B} & 2\bar{B} \end{pmatrix} \begin{pmatrix} u_k \\ u_{k+1} \end{pmatrix} \quad (\text{F.205})$$

For the system outputs, we can write:

$$y_{k+1} = Cx_{k+1} = C(\bar{A}x_k + \bar{B}u_k) \quad (\text{F.206})$$

Equivalently we can write:

$$y_{k+1} = C\bar{A}x_k + \begin{pmatrix} C\bar{B} & 0 \end{pmatrix} \begin{pmatrix} u_k \\ u_{k+1} \end{pmatrix} \quad (\text{F.207})$$

Using (F.205) and (F.207), it results:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{\bar{A}^2 + I}{2} & \frac{1}{2} \begin{pmatrix} \bar{A}\bar{B} - \bar{B} & 2\bar{B} \end{pmatrix} \\ C\bar{A} & \begin{pmatrix} C\bar{B} & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} x_k \\ u_k \\ u_{k+1} \end{pmatrix} \quad (\text{F.208})$$

Adopting the notations:

$$\begin{cases} \tilde{A} \stackrel{\text{def}}{=} \frac{\bar{A}^2 + I}{2}; \tilde{B} \stackrel{\text{def}}{=} \begin{pmatrix} \bar{A}\bar{B} - \bar{B} & 2\bar{B} \end{pmatrix} \\ C \stackrel{\text{def}}{=} C\bar{A}; \tilde{D} \stackrel{\text{def}}{=} \begin{pmatrix} C\bar{B} & 0 \end{pmatrix}; \tilde{u}_k = \begin{pmatrix} u_k & u_{k+1} \end{pmatrix}^T \end{cases} \quad (\text{F.209})$$

it results:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \begin{pmatrix} x_k \\ \tilde{u}_k \end{pmatrix} \quad (\text{F.210})$$

### Remark:

As for the classic state-space case, the state-space matrices  $(A, B, C)$  can be recovered by solving (F.210) when knowing the input/output vectors.

Further we will propose and prove a deterministic identification theorem associated to the current symplectic discretization scheme.

### Theorem F.8

Under the assumptions that:

- I. The input  $u_k$  is persistently exciting of order  $2i$
- II. The intersection of the row space of  $U_f^*$  (the future inputs) and the row space of  $X_P^d$  (the past states) is empty.

III. The user-defined weighting matrices  $W_1 \in R^{l \times l}$  and  $W_2 \in R^{j \times j}$  are such that  $W_1$  is of full rank and  $W_2$  obeys:  $\text{rank} \left( W_P^* \right) = \text{rank} \left( W_P^* W_2 \right)$ , where  $W_P^*$  is the block Hankel matrix containing the past inputs and outputs and with  $O_i$  defined as the oblique projection:

$$O_i = \left( Y_f / U_f^* \right) W_P^* \quad (1)$$

and the singular value decomposition:

$$W_1 O_i W_2 = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} \quad (2)$$

we have:

1. The matrix  $O_i$  is equal to the product of the extended observability matrix and the states:

$$O_i = \Gamma_i^* X_f^d \quad (3)$$

2. The order of the system is equal to the number of singular values in (2) different from 0.
3. The extended observability matrix  $\Gamma_i^*$  is equal to:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (4)$$

4. The part of the state sequence  $X_f^d$  that lies in the column space of  $W_2$  can be recovered from:

$$X_f^d W_2 = T^{-1} S_1^{1/2} V_1^T \quad (5)$$

5. The state sequence  $X_f^d$  is equal to:

$$X_f^d = \Gamma_i^{*+} O_i \quad (6)$$

**Proof:**

Using (F.201), the state vector can be written equivalently:

$$X_f^d = X_i^d = \overline{\overline{A}}^i \left( \Gamma_i^{*+} Y_P - \Gamma_i^{*+} H_i^{*d} U_P^* \right) + \Delta_i^{*d} U_P^* \quad (F.211)$$

Equivalently this relation can be written as:

$$X_f^d = \left( \Delta_i^{*d} - \Gamma_i^{*+} H_i^{*d} \right) U_P^* + \overline{\overline{A}}^i \Gamma_i^{*+} Y_P \quad (F.212)$$

Adopting the notations:

$$\begin{cases} L_P^* \stackrel{\text{def}}{=} \left( \Delta_i^{*d} - \Gamma_i^{*+} H_i^{*d} & \overline{\overline{A}}^i \Gamma_i^{*+} \right) \\ W_P^* \stackrel{\text{def}}{=} \begin{pmatrix} U_P^* & Y_P \end{pmatrix}^T \end{cases} \quad (F.213)$$

(F.212) becomes:

$$X_f^d = L_P^* W_P^* \quad (F.214)$$

Using (F.214), the future outputs have the following form:

$$Y_f = \Gamma_i^* L_P^* W_P^* + H_i^* d U_f^* \quad (\text{F.215})$$

Multiplying to the right with the orthogonal projector of  $U_f^*$ , we get:

$$Y_f \Pi_{U_f^* \perp} = \Gamma_i^* L_P^* W_P^* \Pi_{U_f^* \perp} \quad (\text{F.216})$$

On the left and right hand, we can apply the orthogonal projection operator and it results:

$$Y_f / U_{f^* \perp} = \Gamma_i^* L_P^* W_P^* / U_{f^* \perp} \quad (\text{F.217})$$

Multiplying to the right with the pseudo-inverse  $\left( W_P^* / U_{f^* \perp} \right)^+$  it results:

$$\left( Y_f / U_{f^* \perp} \right) \left( W_P^* / U_{f^* \perp} \right)^+ W_P^* = \Gamma_i^* L_P^* W_P^* \quad (\text{F.218})$$

We observe that the left term is the expression of an oblique projection and then we can write:

$$O_i = \Gamma_i^* L_P^* W_P^* \quad (\text{F.219})$$

or equivalently:

$$O_i = \Gamma_i^* X_f^d \quad (\text{F.220})$$

At the next step, it is necessary to prove:

$$\text{rank} \left( W_P^* \right) = \text{rank} \left( W_P^* / U_{f^* \perp} \right) \quad (\text{F.221})$$

$W_P^*$  can be equivalently written as:

$$W_P^* = \begin{pmatrix} I_{2mi} & 0 \\ H_i^* d & \Gamma_i^* \end{pmatrix} \begin{pmatrix} U_P^* \\ X_P^d \end{pmatrix} \quad (\text{F.222})$$

where:

$$U_P^* = U_{0/i-1}^* = \left( U_{0/i-2} \quad U_{0/i} \right)^T \quad (\text{F.223})$$

Multiplying to the right with the orthogonal complement of  $U_{f^* \perp}$  it results:

$$W_P^* \Pi_{U_{f^* \perp}} = \begin{pmatrix} I_{2mi} & 0 \\ H_i^* d & \Gamma_i^* \end{pmatrix} \begin{pmatrix} U_P^* \Pi_{U_{f^* \perp}} \\ X_P^d \Pi_{U_{f^* \perp}} \end{pmatrix} \quad (\text{F.224})$$

Using the orthogonal projection operator, we can write further:

$$W_P^* / U_{f^* \perp} = \begin{pmatrix} I_{2mi} & 0 \\ H_i^* d & \Gamma_i^* \end{pmatrix} \begin{pmatrix} U_P^* / U_{f^* \perp} \\ X_P^d / U_{f^* \perp} \end{pmatrix} \quad (\text{F.225})$$

Using the initial conditions on the theorem, it results:

$$\text{rank} \begin{pmatrix} U_P^* \\ X_P^d \end{pmatrix} = \text{rank} \begin{pmatrix} U_P^* / U_{f^{*\perp}} \\ X_P^d / U_{f^{*\perp}} \end{pmatrix} \quad (\text{F.226})$$

which proves F.221.

**Remark:** As for the classic case, the initial conditions assumed on this symplectic discretization scheme maintain the same form, changing in accordance only the block Hankel matrix ( $W_P^*$ ) structure and inputs ( $U_P^*, U_f^*$ ).

Then we write the SVD decomposition of  $W_P^* / U_{f^{*\perp}}$  as:

$$W_P^* / U_{f^{*\perp}} = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} \quad (\text{F.227})$$

where  $W_P^* / U_{f^{*\perp}}$  is known to be a linear combination of the columns of  $W_P^*$  and since the rank of  $W_P^*$  and  $W_P^* / U_{f^{*\perp}}$  are equal, then the column spaces of  $W_P^*$  and  $W_P^* / U_{f^{*\perp}}$  are also equal. We can introduce then the relation:

$$W_P^* = U_1 R \quad (\text{F.228})$$

from which we can write:

$$\begin{pmatrix} W_P^* / U_{f^{*\perp}} \end{pmatrix} \begin{pmatrix} W_P^* / U_{f^{*\perp}} \end{pmatrix}^+ W_P^* = \begin{pmatrix} U_1 S_1 V_1^T \end{pmatrix} \begin{pmatrix} V_1 S_1^{-1} U_1^T \end{pmatrix} \begin{pmatrix} U_1 R \end{pmatrix} \quad (\text{F.229})$$

Using the properties:

$$V_1^T V_1 = U_1^T U_1 = I \quad (\text{F.230})$$

associated to the SVD decomposition, it results:

$$\begin{pmatrix} W_P^* / U_{f^{*\perp}} \end{pmatrix} \begin{pmatrix} W_P^* / U_{f^{*\perp}} \end{pmatrix}^+ W_P^* = W_P^* \quad (\text{F.231})$$

Using (F.228) it results:

$$\begin{pmatrix} W_P^* / U_{f^{*\perp}} \end{pmatrix} \begin{pmatrix} W_P^* / U_{f^{*\perp}} \end{pmatrix}^+ W_P^* = W_P^* \quad (\text{F.232})$$

For the 2<sup>nd</sup> claim of the theorem, we observe that:

$$W_1 \cdot O_i \cdot W_2 = W_1 \cdot \Gamma_i \cdot X_f^d \cdot W_2 \quad (\text{F.233})$$

We know that the product  $W_1 \cdot \Gamma_i^*$  has  $n$  columns and  $X_f^d W_2$   $n$  rows, from which we conclude that:

$$\text{rank} (W_1 \cdot O_i \cdot W_2) = n \quad (\text{F.234})$$

This proves statement 2 of the theorem.

Introducing a non-singular transformation matrix ( $T \in R^{n \times n}$ ), we can write:

$$\begin{cases} W_1 \Gamma_i^* = U_1 S_1^{1/2} T \\ X_f^d \cdot W_2 = T^{-1} S_1^{1/2} V_1^T \end{cases} \quad (\text{F.235})$$

Multiplying to the left with  $w_1^{-1}$  in the 1<sup>st</sup> relation, it results:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (\text{F.236})$$

Thus statement 3 of the theorem is proved. The 2<sup>nd</sup> relation from (F.235) proves statement 4 of the theorem. Using (F.220) we can compute the state sequence  $x_f^d$  as follows:

$$X_f^d = \Gamma_i^* O_i \quad (\text{F.237})$$

This proves statement 5 of the theorem.

## F.6 Symplectic discretization scheme VII

For this symplectic discretization scheme, it is assumed an implicit trapezoidal rule for the flows and an explicit Euler rule for the efforts. The general explicit form of the PCH system for this symplectic discretization scheme, is the following:

$$\begin{cases} x_{k+1} = x_k + \left( \frac{\bar{A} - I}{2} \right) (x_{k-1} + x_k) + \frac{1}{2} \bar{B} (u_k + u_{k+1}) \\ y_{k+1} = C x_k \end{cases} \quad (\text{F.238})$$

while the implicit form of the PCH system for this symplectic discretization scheme is:

$$(f_S(k+1), e_S(k+1), f_C(k+1), e_C(k+1), f_R(k+1), e_R(k+1)) \in D \quad (\text{F.239})$$

Using (5.24) and (F.238), we can write equivalently:

$$\begin{aligned} x_{k+1} &= \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} + \frac{1}{2} (\bar{A} - I) \left( \bar{A}^{k-1} x_0 + \Delta_{k-1}^d U_{0/k-2} + \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} \right) + \\ &+ \frac{\bar{B}}{2} (u_k + u_{k+1}) \end{aligned} \quad (\text{F.240})$$

Equivalently we can write:

$$x_{k+1} = \frac{\bar{A}^{k+1} x_0 + 2\bar{A}^k x_0 - \bar{A}^{k-1} x_0}{2} + \left( \frac{\bar{A} + I}{2} \right) \Delta_k^d U_{0/k-1} + \left( \frac{\bar{A} - I}{2} \right) \Delta_{k-1}^d U_{0/k-2} + \frac{\bar{B}}{2} (u_k + u_{k+1}) \quad (\text{F.241})$$

Further we adopt the notation:

$$\Delta_{k+1}^{-d} \stackrel{def}{=} \bar{A} \Delta_k^d = \begin{pmatrix} \bar{A}^k \bar{B} & \bar{A}^{k-1} \bar{B} & \dots & \bar{A} \bar{B} \end{pmatrix} \quad (\text{F.242})$$

Then we can write:

$$x_{k+1} = \left( \frac{\bar{A}^2 + 2\bar{A} - I}{2} \right) \bar{A}^{k-1} x_0 + \frac{\Delta_{k+1}^d U_{0/k} + \Delta_k^d U_{0/k-1}}{2} + \frac{\Delta_k^{-d} U_{0/k-2} - \Delta_{k-1}^d U_{0/k-2}}{2} + \frac{\bar{B} u_{k+1}}{2} \quad (\text{F.243})$$

We can write this relation further:

$$x_{k+1} = \left( \frac{\bar{A}^2 + 2\bar{A} - I}{2} \right) \bar{A}^{k-1} x_0 + \frac{1}{2} \left( \Delta_k^{-d} \quad -\Delta_{k-1}^d \quad \Delta_k^d \quad \left( \Delta_{k+1}^d \quad \bar{B} \right) \right) \begin{pmatrix} U_{0/k-2} \\ U_{0/k-1} \\ U_{0/k+1} \end{pmatrix} \quad (\text{F.244})$$

Then we adopt the notations:

$$\begin{cases} \bar{A}^{k+1} \stackrel{\text{def}}{=} \frac{\bar{A}^{k+1} + 2\bar{A}^k - \bar{A}^{k-1}}{2} \\ \Delta_{k+1}^{*d} \stackrel{\text{def}}{=} \frac{1}{2} \left( \Delta_k^{-d} \quad -\Delta_{k-1}^d \quad \Delta_k^d \quad \left( \Delta_{k+1}^d \quad \bar{B} \right) \right) \\ U_{0/k}^* \stackrel{\text{def}}{=} \left( U_{0/k-2} \quad U_{0/k-1} \quad U_{0/k+1} \right)^T \end{cases} \quad (\text{F.245})$$

and (F.244) becomes:

$$x_{k+1} = \bar{A}^{k+1} X_P^d + \Delta_{k+1}^{*d} U_k^* \quad (\text{F.246})$$

We can write the future states as:

$$X_f^d = X_i^d = \bar{A}^i X_P^d + \Delta_i^{*d} U_P^* \quad (\text{F.247})$$

where:

$$U_P^* = \left( u_0^* \quad u_1^* \quad \dots \quad u_{i-1}^* \right)^T \quad (\text{F.248})$$

is defined in (F.245). Similarly we compute the system outputs using the formula:

$$y_{k+1} = Cx_k = C \left( \bar{A}^k X_P^d + \Delta_k^{*d} U_{0/k-1}^* \right) \quad (\text{F.249})$$

We can generalize then a vectorial representation for the past outputs:

$$Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^* \quad (\text{F.250})$$

where:

$$\begin{cases} \Gamma_i^{**} \stackrel{\text{def}}{=} \left( 0 \quad \Gamma_{i-1}^* \right)^T, U_P^{**} \stackrel{\text{def}}{=} \left( 0 \quad 0 \quad u_0^* \quad \dots \quad u_{i-2}^* \right)^T \\ H_i^{**d} \stackrel{\text{def}}{=} C \begin{pmatrix} 0 & 0 & \dots & 0 \\ \Delta_0^{*d} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \Delta_{i-2}^{*d} & 0 & \dots & 0 \end{pmatrix} \end{cases} \quad (\text{F.251})$$

**Remark:** In (F.251), we have zero terms for  $\Gamma_i^{**}$ ,  $U_P^{**}$  and  $H_i^{**d}$ , due to the fact that the 1<sup>st</sup> output is zero, on this symplectic scheme.

For the future outputs we can write:

$$Y_f = \Gamma_i^{*d} X_{f-1}^d + H_i^{*d} U_{f-1}^* \quad (\text{F.252})$$

where  $H_i^{*d}, U_f^*$  are defined as follows:

$$\left\{ \begin{array}{l} H_i^{*d} \stackrel{def}{=} C \begin{pmatrix} \Delta_0^{*d} & 0 & \dots & 0 \\ \Delta_1^{*d} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \Delta_{i-1}^{*d} & 0 & \dots & 0 \end{pmatrix} \\ U_f^* \stackrel{def}{=} \left( u_{i-1}^* \quad u_i^* \quad \dots \quad u_{2i-2}^* \right)^T \end{array} \right.$$

Using (F.247), (F.250) and (F.252), the *matrix input-output equations* of this scheme, can be written as:

$$\left\{ \begin{array}{l} Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**} \\ Y_f = \Gamma_i^* X_{f-1}^d + H_i^{*d} U_f^* \\ X_f^d = X_i^d = \bar{A}^i X_P^d + \Delta_i^{*d} U_P^* \end{array} \right. \quad (\text{F.253})$$

Then we can introduce the following theorem:

### Theorem F.9

*The matrix input-output equations of a discrete PCH system, represented in explicit form in (F.238) and in implicit form in (F.239), for a symplectic discretization scheme with an implicit trapezoidal rule for the flows and explicit Euler for the efforts are:*

$$\left\{ \begin{array}{l} Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**} \\ Y_f = \Gamma_i^* X_{f-1}^d + H_i^{*d} U_f^* \\ X_f^d = X_i^d = \bar{A}^i X_P^d + \Delta_i^{*d} U_P^* \end{array} \right. \quad (\text{F.254})$$

An equivalent state-space representation of (F.238) can be obtained by combining it with (5.24). Then we can write:

$$x_{k+1} = \bar{A} x_{k-1} + \bar{B} u_{k-1} + \left( \frac{\bar{A} - I}{2} \right) (x_{k-1} + \bar{A} x_{k-1} + \bar{B} u_{k-1}) + \frac{1}{2} \bar{B} (u_k + u_{k+1}) \quad (\text{F.255})$$

This relation becomes further:

$$x_{k+1} = \frac{2\bar{A} x_{k-1} + 2\bar{B} u_{k-1} + \bar{A} x_{k-1} + \bar{A}^2 x_{k-1} + \bar{A} \bar{B} u_{k-1} - x_{k-1}}{2} + \frac{-\bar{A} x_{k-1} - \bar{B} u_{k-1} + \bar{B} u_k + \bar{B} u_{k+1}}{2} \quad (\text{F.256})$$

A simplified form of the above relation is:

$$x_{k+1} = \left( \frac{2\bar{A} + \bar{A}^2 - I}{2} \right) x_{k-1} + \frac{1}{2} (\bar{B} + \bar{A} \bar{B}) u_{k-1} + \frac{1}{2} \bar{B} u_k + \frac{1}{2} \bar{B} u_{k+1}$$

(F.257)

Equivalently, we can write:



$$x_{k+1} = \left\{ \frac{2\bar{A} + \bar{A}^2 - I}{2} \right\} x_{k-1} + \frac{1}{2} \begin{pmatrix} \bar{B} + \bar{A}\bar{B} & \bar{B} & \bar{B} \end{pmatrix} \begin{pmatrix} u_{k-1} \\ u_k \\ u_{k+1} \end{pmatrix} \quad (\text{F.258})$$

For the system outputs, we can write the following relation:

$$y_{k+1} = Cx_k = C(\bar{A}x_{k-1} + \bar{B}u_{k-1}) \quad (\text{F.259})$$

This relation can be also written as:

$$y_{k+1} = C\bar{A}x_{k-1} + \begin{pmatrix} C\bar{B} & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{k-1} \\ u_k \\ u_{k+1} \end{pmatrix} \quad (\text{F.260})$$

Using (F.258) and (F.260), we can write the following state-space representation:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{2\bar{A} + \bar{A}^2 - I}{2} & \frac{1}{2} \begin{pmatrix} \bar{B} + \bar{A}\bar{B} & \bar{B} & \bar{B} \end{pmatrix} \\ C\bar{A} & \begin{pmatrix} C\bar{B} & 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} x_{k-1} \\ u_{k-1} \\ u_k \\ u_{k+1} \end{pmatrix} \quad (\text{F.261})$$

Adopting the following notations, we can write:

$$\left\{ \begin{array}{l} \tilde{A} \stackrel{\text{def}}{=} \left\{ \frac{2\bar{A} + \bar{A}^2 - I}{2} \right\}, \tilde{B} \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} \bar{B} + \bar{A}\bar{B} & \bar{B} & \bar{B} \end{pmatrix} \\ \tilde{C} \stackrel{\text{def}}{=} C\bar{A}; \tilde{D} \stackrel{\text{def}}{=} \begin{pmatrix} C\bar{B} & 0 & 0 \end{pmatrix}; \tilde{u}_{k-1} \stackrel{\text{def}}{=} \begin{pmatrix} u_{k-1} & u_k & u_{k+1} \end{pmatrix}^T \end{array} \right. \quad (\text{F.262})$$

and it results:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \begin{pmatrix} x_{k-1} \\ \tilde{u}_{k-1} \end{pmatrix} \quad (\text{F.263})$$

**Remark:**

*As for the classic case, the original state-space matrices (A, B, C) can be recovered by solving (F.263) when knowing the input/output vectors.*

Next we will introduce and prove a deterministic identification theorem for the selected symplectic time-discretization scheme.

**Theorem F.10**

*Under the assumptions that:*

1. *The input  $u_k$  is persistently exciting of order  $2i$*
2. *The intersection of the row space of  $U_f^*$  (the future inputs) and the row space of  $X_P^d$  (the past states) is empty.*
3. *The user-defined weighting matrices  $W_1 \in R^{lixli}$  and  $W_2 \in R^{(j-1)x(j-1)}$  are such that  $W_1$  is of full rank and  $W_2$  obeys:  $\text{rank} \left( W_{P-1}^* \right) = \text{rank} \left( W_{P-1}^* W_2 \right)$ , where  $W_P^*$  is the block*

Hankel matrix containing the past inputs and outputs and with  $o_i$  defined as the oblique projection:

$$O_i = \begin{pmatrix} Y_f \\ U_f \end{pmatrix} W_{P-1}^* \quad (1)$$

and the singular value decomposition:

$$W_1 O_i W_2 = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1 S_1 V_1^T \quad (2)$$

we have:

1. The matrix  $o_i$  is equal to the product of the extended observability matrix and the states:

$$O_i = \Gamma_i^* X_{f-1}^d \quad (3)$$

2. The order of the system is equal to the number of singular values in (2) different from 0.
3. The extended observability matrix  $\Gamma_i^*$  is equal to:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (4)$$

4. The part of the state sequence  $X_{f-1}^d$  that lies in the column space of  $w_2$  can be recovered from:

$$X_{f-1}^d W_2 = T^{-1} S_1^{1/2} V_1^T \quad (5)$$

5. The state sequence  $X_{f-1}^d$  is equal to:

$$X_{f-1}^d = \Gamma_i^{*+} O_i \quad (6)$$

### Proof:

Using (F.254), the future outputs can be expressed as:

$$X_f^d = \overline{\overline{A}}^i \left( \Gamma_i^{**+} Y_P - \Gamma_i^{**+} H_i^{**d} U_P^{**} \right) + \Delta_i^{*d} U_P^* \quad (F.264)$$

We can write this relation as:

$$X_f^d = \left( \Delta_i^{*d} - \overline{\overline{A}}^i \Gamma_i^{**+} H_i^{**d} \right) U_P^* + \overline{\overline{A}}^i \Gamma_i^{**+} Y_P \quad (F.265)$$

Adopting the notations:

$$\begin{cases} L_P^* \stackrel{def}{=} \left( \Delta_i^{*d} - \overline{\overline{A}}^i \Gamma_i^{**+} H_i^{**d} \quad \overline{\overline{A}}^i \Gamma_i^{**+} \right) \\ W_P^* \stackrel{def}{=} \begin{pmatrix} U_P^* & Y_P \end{pmatrix}^T \end{cases} \quad (F.266)$$

Then (F.265) simplifies to:

$$X_f^d = L_P^* W_P^* \quad (F.267)$$

Using (F.254) we can write for the future outputs:

$$Y_f = \Gamma_i^* L_{P-1}^* W_{P-1}^* + H_i^{*d} U_f^* \quad (\text{F.268})$$

By multiplying to the right with the orthogonal projector of  $U_f^*$  we get:

$$Y_f \Pi_{U_f^{*\perp}} = \Gamma_i^* L_{P-1}^* W_{P-1}^* \Pi_{U_f^{*\perp}} + H_i^{*d} U_f^* \Pi_{U_f^{*\perp}} \quad (\text{F.269})$$

which is equivalent to:

$$Y_f / U_f^{*\perp} = \Gamma_i^* L_{P-1}^* W_{P-1}^* / U_f^{*\perp} \quad (\text{F.270})$$

Next we multiply to the right with  $\left( W_{P-1}^* / U_f^{*\perp} \right)^+$  and it results:

$$\left( Y_f / U_f^{*\perp} \right) \left( W_{P-1}^* / U_f^{*\perp} \right)^+ W_{P-1}^* = \Gamma_i^* L_{P-1}^* W_{P-1}^* \quad (\text{F.271})$$

We observe that to the left we have an oblique projection ( $O_i = Y_f / U_f^{*\perp} W_{P-1}^*$ ) and using (F.267) it results:

$$O_i = \Gamma_i^* X_{f-1}^d \quad (\text{F.272})$$

which proves the first statement of the theorem. In (F.271) we have used the relation:

$$\left( W_{P-1}^* / U_f^{*\perp} \right) \left( W_{P-1}^* / U_f^{*\perp} \right)^+ W_{P-1}^* = W_{P-1}^* \quad (\text{F.273})$$

We next prove the relation:

$$\text{rank} \left( W_{P-1}^* \right) = \text{rank} \left( W_{P-1}^* / U_f^{*\perp} \right) \quad (\text{F.274})$$

We write next  $W_{P-1}^*$  equivalently:

$$W_{P-1}^* = \begin{pmatrix} I_{(3i-5)} & 0 & 0 \\ 0 & H_i^{**d} & \Gamma_i^{**} \end{pmatrix} \begin{pmatrix} U_P^* \\ U_P^{**} \\ X_P^d \end{pmatrix} \quad (\text{F.275})$$

Next we can multiply to the right with the orthogonal projector of  $U_f^{*\perp}$  and it results:

$$W_{P-1}^* \Pi_{U_f^{*\perp}} = \begin{pmatrix} I_{(3i-5)} & 0 & 0 \\ 0 & H_i^{**d} & \Gamma_i^{**} \end{pmatrix} \begin{pmatrix} U_P^* \Pi_{U_f^{*\perp}} \\ U_P^{**} \Pi_{U_f^{*\perp}} \\ X_P^d \Pi_{U_f^{*\perp}} \end{pmatrix} \quad (\text{F.276})$$

Using the orthogonal projection operator, we can write equivalently:

$$W_{P-1}^* / U_{f^* \perp} = \begin{pmatrix} I_{(3i-5)m} & 0 & 0 \\ 0 & H_i^{**d} & \Gamma_i^{**} \end{pmatrix} \begin{pmatrix} U_P^* \Pi_{U_{f^* \perp}} \\ U_P^{**} \Pi_{U_{f^* \perp}} \\ X_P^d \Pi_{U_{f^* \perp}} \end{pmatrix} \quad (\text{F.277})$$

Assuming the initial conditions on the theorem, we can write:

$$\text{rank} \begin{pmatrix} U_P^* \\ U_P^{**} \\ X_P^d \end{pmatrix} = \text{rank} \begin{pmatrix} U_P^* / U_{f^* \perp} \\ U_P^{**} / U_{f^* \perp} \\ X_P^d / U_{f^* \perp} \end{pmatrix} \quad (\text{F.278})$$

and thus (F.274) is proved.

**Remark:** The initial conditions assumed for this symplectic time-discretization scheme, follow the same form with the classic state-space one, where the block Hankel matrix  $W_P^*$  and past/future inputs  $(U_P^*, U_f^*)$  change their structure.

Then we introduce the SVD decomposition of  $W_{P-1}^* / U_{f^* \perp}$  as:

$$W_{P-1}^* / U_{f^* \perp} = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} \quad (\text{F.279})$$

Since  $W_{P-1}^* / U_{f^* \perp}$  is a linear combination of the columns of  $W_{P-1}^*$  and since the rank of  $W_{P-1}^*$  and  $W_{P-1}^* / U_{f^* \perp}$  are equal, the column spaces of  $W_{P-1}^*$  and  $W_{P-1}^* / U_{f^* \perp}$  are the same.

We can write the relation:

$$W_{P-1}^* = U_1 R \quad (\text{F.280})$$

Also we can write:

$$\left( W_{P-1}^* / U_{f^* \perp} \right) \left( W_{P-1}^* / U_{f^* \perp} \right)^+ W_{P-1}^* = \begin{pmatrix} U_1 S_1 V_1^T \end{pmatrix} \begin{pmatrix} V_1 S_1^{-1} U_1^T \end{pmatrix} U_1 R \quad (\text{F.281})$$

Using the SVD decomposition properties:

$$V_1^T V_1 = U_1^T U_1 = I \quad (\text{F.282})$$

corresponding to the SVD decomposition from (F.279). Grouping the terms in (F.281), we can write:

$$\left( W_{P-1}^* / U_{f^* \perp} \right) \left( W_{P-1}^* / U_{f^* \perp} \right)^+ W_{P-1}^* = U_1 S_1 \left( V_1^T V_1 \right) S_1^{-1} \left( U_1^T U_1 \right) R \quad (\text{F.283})$$

Performing the simplifications, it results:

$$\left( W_{P-1}^* / U_{f^* \perp} \right) \left( W_{P-1}^* / U_{f^* \perp} \right)^+ W_{P-1}^* = U_1 R \quad (\text{F.284})$$

Using (F.280) and (F.284), it results that:

$$\left( W_{P-1}^* / U_{f^* \perp} \right) \left( W_{P-1}^* / U_{f^* \perp} \right)^+ W_{P-1}^* = W_{P-1}^* \quad (\text{F.285})$$

For the 2<sup>nd</sup> claim of the theorem, we observe that:

$$W_1 O_i W_2 = W_1 \Gamma_i^* X_{f-1}^d W_2 \quad (\text{F.286})$$

where  $W_1 \Gamma_i^*$  has  $n$  columns and  $X_{f-1}^d W_2$   $n$  rows. It results that the product  $W_1 O_i W_2$  has rank  $n$ . This proves statement 2 of the theorem. Next we introduce a non-singular similarity transformation matrix ( $T \in R^{n \times n}$ ) and we can write the next two relations:

$$\begin{cases} W_1 \Gamma_i^* = U_1 S_1^{1/2} T \\ X_{f-1}^d W_2 = T^{-1} S_1^{1/2} V_1^T \end{cases} \quad (\text{F.287})$$

From the 1<sup>st</sup> relation of (F.287) we can recover  $\Gamma_i^*$  by multiplying to the left with  $W_1^{-1}$  as follows:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (\text{F.288})$$

(F.287) and (F.288) proves statements 3 and 4 of the theorem. Using (F.272) we compute the future states as follows:

$$X_{f-1}^d = \Gamma_i^{*+} O_i \quad (\text{F.289})$$

This proves statement 5 of the theorem for this particular symplectic time-discretization scheme and ends the proof.

## F.7 Symplectic discretization scheme VIII

For this symplectic time-discretization it was assumed an explicit Euler rule for the flows and an implicit trapezoidal rule for the efforts. The general explicit state-space form of this symplectic discretization scheme, is the following:

$$\begin{cases} x_{k+1} = x_k + \left( \frac{\bar{A} - I}{2} \right) (x_{k-1} + x_k) + \bar{B} u_k \\ y_k = C \left( \frac{x_{k-1} + x_k}{2} \right) \end{cases} \quad (\text{F.290})$$

using the notations from (5.25). The implicit form of the PCH system for this symplectic discretization scheme is:

$$(f_S(k), e_S(k), f_C(k), e_C(k), f_R(k), e_R(k)) \in D \quad (\text{F.291})$$

Combining (5.33) with (F.290), it results:

$$\begin{aligned} x_{k+1} &= \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} + \left( \frac{\bar{A} - I}{2} \right) \left( \bar{A}^{k-1} x_0 + \Delta_{k-1}^d U_{0/k-2} + \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} \right) + \\ &+ \bar{B} u_k \end{aligned} \quad (\text{F.292})$$

Equivalently we can write this relation as:

$$x_{k+1} = \frac{2\bar{A}^k x_0 + 2\Delta_k^d U_{0/k-1} + \bar{A}^k x_0 + \bar{A}\Delta_{k-1}^d U_{0/k-2} + \bar{A}^{k+1} x_0 + \bar{A}\Delta_k^d U_{0/k-1} - \bar{A}^{k-1} x_0 - \Delta_{k-1}^d U_{0/k-2} - \bar{A}^k x_0 - \Delta_k^d U_{0/k-1} + 2\bar{B}u_k}{2} \quad (\text{F.293})$$

More compact we can write:

$$x_{k+1} = \left( \frac{\bar{A}^2 + 2\bar{A} - I}{2} \right) \bar{A}^{k-1} x_0 + \left( \frac{\bar{A} + I}{2} \right) \Delta_k^d U_{0/k-1} + \left( \frac{\bar{A} - I}{2} \right) \Delta_{k-1}^d U_{0/k-2} + \frac{2\bar{B}u_k}{2} \quad (\text{F.294})$$

Adopting the notations:

$$\left\{ \begin{array}{l} \bar{\bar{A}}^{k+1} \stackrel{\text{def}}{=} \left( \frac{\bar{A}^{k+1} + 2\bar{A}^k - \bar{A}^{k-1}}{2} \right) \\ \Delta_{k+1}^{-d} \stackrel{\text{def}}{=} \bar{A}\Delta_k^d = \left( \bar{A}^k \bar{B} \quad \bar{A}^{k-1} \bar{B} \quad \dots \quad \bar{A}\bar{B} \right), X_P^d \stackrel{\text{def}}{=} x_0 \end{array} \right. \quad (\text{F.295})$$

(F.294) becomes:

$$x_{k+1} = \bar{\bar{A}}^{k+1} X_P^d + \frac{\Delta_{k+1}^d U_{0/k} + \Delta_k^d U_{0/k-1}}{2} + \frac{1}{2} \left( \Delta_k^{-d} U_{0/k-2} - \Delta_{k-1}^d U_{0/k-2} \right) + \frac{\bar{B}u_k}{2} \quad (\text{F.296})$$

This relation can be written as follows:

$$x_{k+1} = \bar{\bar{A}}^{k+1} X_P^d + \frac{1}{2} \left( \Delta_k^{-d} - \Delta_{k-1}^d \quad \Delta_{k+1}^d + \left( \Delta_k^d \quad \bar{B} \right) \right) \begin{pmatrix} U_{0/k-2} \\ U_{0/k} \end{pmatrix} \quad (\text{F.297})$$

We introduce the notations:

$$\left\{ \begin{array}{l} \Delta_{k+1}^{*d} \stackrel{\text{def}}{=} \left( \Delta_k^{-d} - \Delta_{k+1}^d \quad \Delta_{k+1}^d + \left( \Delta_k^d \quad \bar{B} \right) \right) \\ U_{0/k}^* \stackrel{\text{def}}{=} \left( U_{0/k-2} \quad U_{0/k} \right)^T \end{array} \right. \quad (\text{F.298})$$

Then (F.297) becomes:

$$x_{k+1} = \bar{\bar{A}}^{k+1} X_P^d + \Delta_{k+1}^{*d} U_k^* \quad (\text{F.299})$$

Next we can write the future states as follows:

$$X_f^d = X_i^d = \bar{\bar{A}}^i X_P^d + \Delta_i^{*d} U_P^* \quad (\text{F.300})$$

where:

$$U_P^* = \left( u_0^* \quad u_1^* \quad \dots \quad u_{i-1}^* \right)^T \quad (\text{F.301})$$

represent the past inputs associated to the states. Using (5.33) and (F.290) the system outputs become:

$$y_k = C \left\{ \frac{\bar{\bar{A}}^{k-1} X_P^d + \Delta_{k-1}^{*d} U_{0/k-2}^* + \bar{A}^k X_P^d + \Delta_k^{*d} U_{0/k-1}^*}{2} \right\} \quad (\text{F.302})$$

This relation can be also written as:

$$y_k = C \left( \frac{\overline{A}^{k-1} + \overline{A}^k}{2} \right) X_P^d + \frac{C}{2} \begin{pmatrix} \Delta_{k-1}^{*d} & \Delta_k^{*d} \end{pmatrix} \begin{pmatrix} U_{0/k-2}^* \\ U_{0/k-1}^* \end{pmatrix} \quad (\text{F.303})$$

For the vector of past outputs we can write:

$$Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**} \quad (\text{F.304})$$

where the following notations have been adopted:

$$\left\{ \begin{array}{l} \Gamma_i^{**} \stackrel{\text{def}}{=} \frac{1}{2} \left( \begin{pmatrix} 0 \\ \Gamma_{i-1}^* \end{pmatrix} + \Gamma_i^* \right), \Gamma_i^* \stackrel{\text{def}}{=} \begin{pmatrix} C & C\overline{A} & \dots & C\overline{A}^{i-1} \end{pmatrix}^T \\ H_i^{**d} \stackrel{\text{def}}{=} \frac{C}{2} \begin{pmatrix} \Delta_{-1}^{*d} & \dots & 0 & \Delta_0^{*d} & \dots & 0 \\ \dots & \dots & \ddots & \dots & \dots & 0 \\ \Delta_{i-2}^{*d} & \dots & 0 & \Delta_{i-1}^{*d} & \dots & 0 \end{pmatrix}, U_P^{**} \stackrel{\text{def}}{=} \begin{pmatrix} u_{-2/0}^* & u_{-1/0}^* & \dots & u_{0/i-3}^* \\ u_{-1/0}^* & u_{0/0}^* & \ddots & u_{0/i-2}^* \end{pmatrix}^T \end{array} \right. \quad (\text{F.305})$$

Next we introduce the future outputs vector as:

$$Y_f = \Gamma_i^* \overline{X}_f^d + H_i^{**d} U_f^{**} \quad (\text{F.306})$$

where  $\overline{X}_f^d$  is defined as:

$$\overline{X}_f^d \stackrel{\text{def}}{=} \frac{X_{f-1}^d + X_f^d}{2} \quad (\text{F.307})$$

and  $U_f^{**}$  represents the matrix of future outputs, similar to the past system outputs ( $U_P^{**}$ ).

From (F.300), (F.304), (F.306) and (F.307) the following *matrix input-output equations* result:

$$\left\{ \begin{array}{l} X_f^d = \overline{A}^i X_P^d + \Delta_i^{*d} U_P^*; \overline{X}_f^d = \frac{X_{f-1}^d + X_f^d}{2} \\ Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**}; Y_f = \Gamma_i^* \overline{X}_f^d + H_i^{**d} U_f^{**} \end{array} \right. \quad (\text{F.308})$$

### Theorem F.11

*The matrix input-output equations of a discrete-time PCH system, represented in explicit form in (F.290) and in implicit form in (F.291), for a symplectic discretization scheme with an explicit Euler rule for the flows and implicit trapezoidal rule for the efforts are:*

$$\left\{ \begin{array}{l} X_f^d = \overline{A}^i X_P^d + \Delta_i^{*d} U_P^*; \overline{X}_f^d = \frac{X_{f-1}^d + X_f^d}{2} \\ Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**}; Y_f = \Gamma_i^* \overline{X}_f^d + H_i^{**d} U_f^{**} \end{array} \right. \quad (\text{F.309})$$

Further we will find an equivalent representation of (F.290), when combined with (5.33). Replacing the state-space values from (5.33) in (F.290) and using the notations from (5.34), it results:

$$x_{k+1} = \bar{A}x_{k-1} + \bar{B}u_{k-1} + \left(\frac{\bar{A} - I}{2}\right)(x_{k-1} + \bar{A}x_{k-1} + \bar{B}u_{k-1}) + \bar{B}u_k \quad (\text{F.310})$$

This relation can be further written as:

$$x_{k+1} = \frac{2\bar{A}x_{k-1} + 2\bar{B}u_{k-1} + \bar{A}x_{k-1} + \bar{A}^2x_{k-1} + \bar{A}\bar{B}u_{k-1} - x_{k-1} - \bar{A}x_{k-1} - \bar{B}u_{k-1} + 2\bar{B}u_k}{2} \quad (\text{F.311})$$

Equivalently we can write:

$$x_{k+1} = \left(\frac{\bar{A}^2 + 2\bar{A} - I}{2}\right)x_{k-1} + \frac{1}{2}(\bar{B} + \bar{A}\bar{B} \quad 2\bar{B}) \begin{pmatrix} u_{k-1} \\ u_k \end{pmatrix} \quad (\text{F.312})$$

Similarly we can write for the outputs:

$$y_k = C \left(\frac{x_{k-1} + x_k}{2}\right) = C \left(\frac{x_{k-1} + Ax_{k-1}}{2}\right) \quad (\text{F.313})$$

Further this relation can be written as:

$$y_k = \frac{C}{2}(\bar{A} + I)x_{k-1} + \frac{1}{2}(C\bar{B} \quad 0) \begin{pmatrix} u_{k-1} \\ u_k \end{pmatrix} \quad (\text{F.314})$$

Using (F.312) and (F.314), we can write the following relations for the state-space system of (F.290):

$$\begin{pmatrix} x_{k+1} \\ y_k \end{pmatrix} = \begin{pmatrix} \frac{\bar{A}^2 + 2\bar{A} - I}{2} & \frac{1}{2}(\bar{B} + \bar{A}\bar{B}) & 2\bar{B} \\ \frac{C + C\bar{A}}{2} & \frac{1}{2}C\bar{B} & 0 \end{pmatrix} \begin{pmatrix} x_{k-1} \\ u_{k-1} \\ u_k \end{pmatrix} \quad (\text{F.315})$$

Adopting the notations:

$$\begin{cases} \tilde{A} \stackrel{\text{def}}{=} \frac{\bar{A}^2 + 2\bar{A} - I}{2}; \tilde{B} \stackrel{\text{def}}{=} (\bar{B} + \bar{A}\bar{B} \quad 2\bar{B}) \\ \tilde{C} \stackrel{\text{def}}{=} \frac{C + C\bar{A}}{2}; \tilde{D} \stackrel{\text{def}}{=} \frac{1}{2}(C\bar{B} \quad 0); \tilde{u}_{k-1} \stackrel{\text{def}}{=} (u_{k-1} \quad u_k)^T \end{cases} \quad (\text{F.316})$$

(F.314) becomes:

$$\begin{pmatrix} x_{k+1} \\ y_k \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \begin{pmatrix} x_{k-1} \\ \tilde{u}_{k-1} \end{pmatrix} \quad (\text{F.317})$$

**Remark:**

The state-space matrices  $(A, B, C)$  can be recovered by solving (F.317) when having knowledge on the input/output vectors.

As for the classic state-space case, we propose and prove further a theorem associated to the current symplectic time-discretization scheme.

**Theorem F.12**

Under the assumptions that:

- I. The input  $u_k$  is persistently exciting of order  $2i$



II. The intersection of the row space of  $U_f^{**}$  (the future inputs) and the row space of  $X_P^d$  (the past states) is empty.

III. The user-defined weighting matrices  $W_1 \in R^{lixli}$  and  $W_2 \in R^{jxj}$  are such that  $W_1$  is of full rank and  $W_2$  obeys:  $\text{rank}(W_P^*) = \text{rank}(W_P^* \cdot W_2)$ , where  $W_P^*$  is the block Hankel matrix containing the past inputs and outputs and with  $O_i$  defined as the oblique projection:

$$O_i = (Y_f / U_f^{**}) \cdot W_P^* \quad (1)$$

and the singular value decomposition:

$$W_1 O_i W_2 = (U_1 \quad U_2) \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1 S_1 V_1^T \quad (2)$$

we have:

1. The matrix  $O_i$  is equal to the product of the extended observability matrix and the states:

$$O_i = \Gamma_i^* \bar{X}_f^d \quad (3)$$

2. The order of the system is equal to the number of singular values in (2) different from 0.

3. The extended observability matrix  $\Gamma_i^*$  is equal to:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (4)$$

4. The part of the state sequence  $X_f^d$  that lies in the column space of  $W_2$  can be recovered from:

$$\bar{X}_f^d W_2 = T^{-1} S_1^{1/2} V_1^T \quad (5)$$

5. The state sequence  $\bar{X}_f^d$  is equal to:

$$\bar{X}_f^d = \Gamma_i^{*+} O_i \quad (6)$$

**Proof:**

Using (F.309) we can write the average value of the future states, as follows:

$$\bar{X}_f^d = A_M^i \left( \Gamma_i^{**+} Y_P - \Gamma_i^{**+} H_i^{**d} U_P \right) + \Delta_M U_P^{**} \quad (F.318)$$

where the notations have been adopted:

$$\left\{ \begin{array}{l} A_M^i \stackrel{\text{def}}{=} \frac{\bar{A}^{i-1} + \bar{A}^i}{2} \\ \Delta_M \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} \Delta_{i-1}^{*d} & \Delta_i^{*d} \end{pmatrix} \end{array} \right. \quad (F.319)$$

Equivalently we can write this relation as:

$$\bar{X}_f^d = \left( \Delta_M - A_M^i \Gamma_i^{**+} H_i^{**d} \quad A_M^i \Gamma_i^{**+} \right) \begin{pmatrix} U_P^{**} \\ Y_P \end{pmatrix} \quad (F.320)$$

Adopting the notations:

$$\begin{cases} L_P^* \stackrel{def}{=} \begin{pmatrix} \Delta_M & -A_M^i \Gamma_i^{**+} H_i^{**d} & A_M^i \Gamma_i^{**+} \end{pmatrix} \\ W_P^* \stackrel{def}{=} \begin{pmatrix} U_P^{**} & Y_P \end{pmatrix} \end{cases} \quad (\text{F.321})$$

Then (F.320) becomes:

$$\bar{X}_f^d = L_P^* W_P^* \quad (\text{F.322})$$

For the future outputs, we can write then:

$$Y_f = \Gamma_i^* L_P^* W_P^* + H_i^{**d} U_f^{**} \quad (\text{F.323})$$

Multiplying to the right with the orthogonal projector of  $U_{f^{**\perp}}$  it results:

$$Y_f \Pi_{U_{f^{**\perp}}} = \Gamma_i^* L_P^* W_P^* \Pi_{U_{f^{**\perp}}} \quad (\text{F.324})$$

Using the orthogonal projection, we can write equivalently:

$$Y_f / U_{f^{**\perp}} = \Gamma_i^* L_P^* W_P^* / U_{f^{**\perp}} \quad (\text{F.325})$$

Multiplying to the right with  $\left( W_P^* / U_{f^{**\perp}} \right)^+$  we get:

$$\left( Y_f / U_{f^{**\perp}} \right) \left( W_P^* / U_{f^{**\perp}} \right)^+ W_P^* = \Gamma_i^* L_P^* W_P^* \quad (\text{F.326})$$

On the left in (F.326), we have an oblique projection:

$$O_i = \left( Y_f / U_{f^{**\perp}} \right) W_P^* \quad (\text{F.327})$$

Then we can write:

$$O_i = \Gamma_i^* L_P^* W_P^* \quad (\text{F.328})$$

Using (F.322), it results:

$$O_i = \Gamma_i^* \bar{X}_f^d \quad (\text{F.329})$$

In (F.326), the following relation was used:

$$\left( W_P^* / U_{f^{**\perp}} \right) \left( W_P^* / U_{f^{**\perp}} \right)^+ W_P^* = W_P^* \quad (\text{F.330})$$

At the next step it is necessary to prove that:

$$\text{rank} \left( W_P^* \right) = \text{rank} \left( W_P^* / U_{f^{**\perp}} \right) \quad (\text{F.331})$$

First we observe that we can write  $W_P^*$  as follows:

$$W_P^* = \begin{pmatrix} I_{(4i-6)m} & 0 \\ H_i^{**d} & \Gamma_i^{**} \end{pmatrix} \begin{pmatrix} U_P^{**} \\ X_P^d \end{pmatrix} \quad (\text{F.332})$$

Multiplying to the right with the orthogonal projector of  $U_{f^{**\perp}}$ , we get:

$$W_P^* \Pi_{Uf^{**\perp}} = \begin{pmatrix} I(4i-6)m & 0 \\ H_i^{**d} & \Gamma_i^{**} \end{pmatrix} \begin{pmatrix} U_P^{**\Pi} Uf^{**\perp} \\ X_P^d \Pi_{Uf^{**\perp}} \end{pmatrix} \quad (\text{F.333})$$

Using the initial conditions assumed on the theorem, it results:

$$\text{rank} \begin{pmatrix} U_P^{**} \\ X_P^d \end{pmatrix} = \text{rank} \begin{pmatrix} U_P^{**} / U_{f^{**\perp}} \\ X_P^d / U_{f^{**\perp}} \end{pmatrix} \quad (\text{F.334})$$

and (F.331) holds.

**Remark:** The same initial conditions, as for the classic case are used in order for (F.334) to hold, the only difference is that  $w_P^*$  is changing form according to the symplectic time-discretization scheme.

We introduce the SVD decomposition of  $w_P^* / U_{f^{**\perp}}$  as:

$$w_P^* / U_{f^{**\perp}} = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} = U_1 S_1 V_1^T \quad (\text{F.335})$$

As  $w_P^* / U_{f^{**\perp}}$  is a linear combination of the columns of  $w_P^*$  and because the rank of  $w_P^*$  and  $w_P^* / U_{f^{**\perp}}$  are equal, it results also that the column spaces of  $w_P^*$  and  $w_P^* / U_{f^{**\perp}}$  are equal. We can write then  $w_P^*$  as:

$$w_P^* = U_1 R \quad (\text{F.336})$$

Using the SVD decomposition in (F.335), we can write then:

$$\left( w_P^* / U_{f^{**\perp}} \right) \left( w_P^* / U_{f^{**\perp}} \right)^+ w_P^* = \begin{pmatrix} U_1 S_1 V_1^T \end{pmatrix} \begin{pmatrix} V_1 S_1^{-1} U_1^T \end{pmatrix} \begin{pmatrix} U_1 R \end{pmatrix} \quad (\text{F.337})$$

where:

$$U_1^T U_1 = V_1^T V_1 = I \quad (\text{F.338})$$

Performing the simplifications in (F.337), we get:

$$\left( w_P^* / U_{f^{**\perp}} \right) \left( w_P^* / U_{f^{**\perp}} \right)^+ w_P^* = U_1 R \quad (\text{F.339})$$

From (F.336) and (F.339), it results that:

$$\left( w_P^* / U_{f^{**\perp}} \right) \left( w_P^* / U_{f^{**\perp}} \right)^+ w_P^* = w_P^* \quad (\text{F.340})$$

For the 2<sup>nd</sup> claim of the theorem, we can write:

$$w_1 O_i w_2 = w_1 \Gamma_i^* \bar{X}_f^d w_2 \quad (\text{F.341})$$

In (F.340) we observe that  $w_1 \Gamma_i^*$  has  $n$  columns and  $\bar{X}_f^d w_2$   $n$  rows. From this it results that their product is of rank  $n$ . Thus the 2<sup>nd</sup> claim of the theorem is proved. For the 3<sup>rd</sup> statement of the theorem, we introduce a non-singular similarity transformation ( $T \in R^{n \times n}$ ), for which we can write:

$$\begin{cases} w_1 \Gamma_i^* = U_1 S_1^{1/2} T \\ \bar{X}_f^d w_2 = T^{-1} S_1^{1/2} V_1^T \end{cases} \quad (\text{F.342})$$

Multiplying to the left in the 1<sup>st</sup> equation with  $w_1^{-1}$ , we get:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (\text{F.343})$$

which represents statement 3 of the theorem. From (F.342) and (F.343) it results that statements 3 and 4 are satisfied for the deterministic theorem. Using (F.329), we can determine the average value of the states as:

$$\bar{X}_f^d = \Gamma_i^* + O_i \quad (\text{F.344})$$

This proves the last statement of the theorem.

## F.8 Symplectic discretization scheme IX

For this case, the symplectic discretization scheme assumed an implicit Euler method for the flows and implicit trapezoidal rule for the efforts. The explicit form of the discrete-time PCH system with this symplectic discretization scheme is:

$$\begin{cases} x_{k+1} = x_k + \left( \frac{\bar{A} - I}{2} \right) (x_k + x_{k+1}) + B u_{k+1} \\ y_{k+1} = C \left( \frac{x_k + x_{k+1}}{2} \right) \end{cases} \quad (\text{F.345})$$

while the implicit form is:

$$(f_S(k+1), e_S(k+1), f_C(k+1), e_C(k+1), f_R(k+1), e_R(k+1)) \in D \quad (\text{F.346})$$

Combining (5.33) with (F.345), we can write the following relation:

$$x_{k+1} = \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} + \left( \frac{\bar{A} - I}{2} \right) \left( \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} + \bar{A}^{k+1} x_0 + \Delta_{k+1}^d U_{0/k} \right) + \left( \frac{2\bar{B} u_{k+1}}{2} \right) \quad (\text{F.347})$$

Equivalently we can write this relation as:

$$\begin{aligned} x_{k+1} = & \frac{2\bar{A}^k x_0 + 2\Delta_k^d U_{0/k-1}}{2} + \frac{\bar{A}^{k+1} x_0 + \bar{A} \Delta_k^d U_{0/k-1} + \bar{A}^{k+2} x_0 + \bar{A} \Delta_{k+1}^d U_{0/k} - \bar{A}^k x_0}{2} + \\ & + \frac{-\Delta_k^d U_{0/k-1} - \bar{A}^{k+1} x_0 - \Delta_{k+1}^d U_{0/k} + 2\bar{B} u_{k+1}}{2} \end{aligned} \quad (\text{F.348})$$

More compact this relation can be written as:

$$x_{k+1} = \left( \frac{\bar{A}^{k+2} + \bar{A}^k}{2} \right) x_0 + \left( \frac{\bar{A} + I}{2} \right) \Delta_k^d U_{0/k-1} + \left( \frac{\bar{A} - I}{2} \right) \Delta_{k+1}^d U_{0/k} + \frac{2\bar{B}u_{k+1}}{2} \quad (\text{F.349})$$

Adopting the notation:

$$\Delta_{k+1}^{-d} \stackrel{\text{def}}{=} \bar{A} \Delta_k^d = \begin{pmatrix} \bar{A}^k \bar{B} & \bar{A}^{k-1} \bar{B} & \dots & \bar{A} \bar{B} \end{pmatrix} \quad (\text{F.350})$$

We can write:

$$x_{k+1} = \left( \frac{\bar{A}^{k+2} + \bar{A}^k}{2} \right) x_0 + \left( \frac{\Delta_{k+1}^{-d} U_{0/k-1} + \Delta_k^d U_{0/k-1}}{2} \right) + \left( \frac{\Delta_{k+2}^d U_{0/k+1} - \Delta_{k+1}^d U_{0/k}}{2} \right) + \frac{Bu_{k+1}}{2} \quad (\text{F.351})$$

or:

$$x_{k+1} = \left( \frac{\bar{A}^{k+2} + \bar{A}^k}{2} \right) x_0 + \frac{1}{2} \begin{pmatrix} \Delta_{k+1}^{-d} + \Delta_k^d & \Delta_{k+2}^d + (-\Delta_{k+1}^d \quad \bar{B}) \end{pmatrix} \begin{pmatrix} U_{0/k-1} \\ U_{0/k+1} \end{pmatrix} \quad (\text{F.352})$$

Then we can adopt the notations:

$$\begin{cases} \bar{A}^{k+1} \stackrel{\text{def}}{=} \left( \frac{\bar{A}^{k+2} + \bar{A}^k}{2} \right), X_P^d = x_0 \\ \Delta_{k+1}^{*d} \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} \Delta_{k+1}^{-d} + \Delta_k^d & \Delta_{k+2}^d + (-\Delta_{k+1}^d \quad \bar{B}) \end{pmatrix} \\ U_{0/k}^* \stackrel{\text{def}}{=} \begin{pmatrix} U_{0/k-1} & U_{0/k+1} \end{pmatrix} \end{cases} \quad (\text{F.353})$$

and it results:

$$x_{k+1} = \bar{A}^{k+1} X_P^d + \Delta_{k+1}^{*d} U_{0/k}^* \quad (\text{F.354})$$

As for the classic case, we can express the future states as:

$$X_f^d = X_i^d = \bar{A}^i X_P^d + \Delta_i^{*d} U_P^* \quad (\text{F.355})$$

where  $U_P^*$  represent the past states, given as:

$$U_P^* \stackrel{\text{def}}{=} \begin{pmatrix} U_{0/i-2} & U_{0/i} \end{pmatrix} \quad (\text{F.356})$$

Using (F.355) we can compute also the system outputs:

$$y_{k+1} = C \left( \frac{x_k + x_{k+1}}{2} \right) = \frac{C}{2} \left( \bar{A}^k X_P^d + \Delta_k^{*d} U_{0/k-1}^* + \bar{A}^{k+1} X_P^d + \Delta_{k+1}^{*d} U_{0/k}^* \right) \quad (\text{F.357})$$

Equivalently we can write:

$$y_{k+1} = \frac{C}{2} \left( \bar{A}^k + \bar{A}^{k+1} \right) X_P^d + \frac{C}{2} \begin{pmatrix} \Delta_k^{*d} & \Delta_{k+1}^{*d} \end{pmatrix} \begin{pmatrix} U_{0/k-1}^* \\ U_{0/k}^* \end{pmatrix} \quad (\text{F.358})$$

We can define the vector of past outputs as:

$$Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**} \quad (\text{F.359})$$

when adopting the notations:

$$\left\{ \begin{array}{l} \Gamma_i^{**} \stackrel{def}{=} \frac{1}{2} \left( \begin{pmatrix} 0 \\ \Gamma_{i-1}^* \end{pmatrix} + \Gamma_i^* \right); \Gamma_i^* \stackrel{def}{=} \left( C \quad \overline{CA} \quad \dots \quad \overline{CA}^{i-1} \right)^T \\ H_i^{**d} \stackrel{def}{=} \frac{C}{2} \begin{pmatrix} \Delta_{-1}^{*d} & \dots & 0 & \Delta_0^{*d} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \Delta_{i-2}^{*d} & \dots & \dots & \Delta_{i-1}^{*d} & \dots & 0 \end{pmatrix}; U_P^{**} \stackrel{def}{=} \left( U_{0/i-3}^* \quad U_{0/i-2}^* \right)^T \end{array} \right. \quad (\text{F.360})$$

For the future outputs, we can write:

$$Y_f = \Gamma_i^* \overline{X}_f^d + H_i^{**d} U_f^{**} \quad (\text{F.361})$$

where:

$$\overline{X}_f^d = \frac{X_{f-1}^d + X_f^d}{2} \quad (\text{F.362})$$

Thus we can formulate, the *matrix input-output equations* using (F.355),(F.359),(F.361) and (F.362), as follows:

$$\left\{ \begin{array}{l} X_f^d = \overline{A}^i X_P^d + \Delta_i^{*d} U_P^*, \overline{X}_f^d = \frac{X_{f-1}^d + X_f^d}{2} \\ Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**}, Y_f = \Gamma_i^* \overline{X}_f^d + H_i^{**d} U_f^{**} \end{array} \right. \quad (\text{F.363})$$

Thus we propose the following theorem as in the classic state-space case:

### Theorem F.13

*The matrix input-output equations of a discrete-time PCH system, represented in explicit form in (F.345) and in implicit form in (F.346), for a symplectic discretization scheme with an implicit Euler rule for the flows and implicit trapezoidal rule for the efforts are:*

$$\left\{ \begin{array}{l} X_f^d = \overline{A}^i X_P^d + \Delta_i^{*d} U_P^*, \overline{X}_f^d = \frac{X_{f-1}^d + X_f^d}{2} \\ Y_P = \Gamma_i^{**} X_P^d + H_i^{**d} U_P^{**}, Y_f = \Gamma_i^* \overline{X}_f^d + H_i^{**d} U_f^{**} \end{array} \right. \quad (\text{F.364})$$

Using (5.33) and (F.345), we can write:

$$x_{k+1} = \frac{2x_k + \overline{A}x_k + \overline{A}^2x_k + \overline{A}\overline{B}u_k - x_k - \overline{A}x_k - \overline{B}u_k + 2\overline{B}u_{k+1}}{2} \quad (\text{F.365})$$

Performing the possible simplifications in (F.365), it results:

$$x_{k+1} = \left( \frac{\overline{A}^2 + I}{2} \right) x_k + \frac{1}{2} \begin{pmatrix} \overline{A}\overline{B} - \overline{B} & 2\overline{B} \end{pmatrix} \begin{pmatrix} u_k \\ u_{k+1} \end{pmatrix} \quad (\text{F.366})$$

For the system outputs in (F.345), we can write then:

$$y_{k+1} = C \left( \frac{x_k + \overline{A}x_k + \overline{B}u_k}{2} \right) \quad (\text{F.367})$$

More compact (F.367) becomes:

$$y_{k+1} = \left( \frac{C + C\bar{A}}{2} \right) x_k + \frac{C\bar{B}}{2} u_k \quad (\text{F.368})$$

Using (F.365) and (F.367), we can write then:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{\bar{A}^2 + I}{2} & \frac{1}{2}(\bar{A}\bar{B} - \bar{B} \quad 2\bar{B}) \\ \frac{C + C\bar{A}}{2} & \frac{1}{2}(C\bar{B} \quad 0) \end{pmatrix} \begin{pmatrix} x_k \\ u_k \\ u_{k+1} \end{pmatrix} \quad (\text{F.369})$$

Further we adopt the notations:

$$\begin{cases} \tilde{A} \stackrel{\text{def}}{=} \frac{\bar{A}^2 + I}{2}; \tilde{B} \stackrel{\text{def}}{=} \frac{1}{2}(\bar{A}\bar{B} - \bar{B} \quad 2\bar{B}) \\ \tilde{C} \stackrel{\text{def}}{=} \frac{C + C\bar{A}}{2}; \tilde{D} \stackrel{\text{def}}{=} (C\bar{B} \quad 0); \tilde{u}_k \stackrel{\text{def}}{=} (u_k \quad u_{k+1})^T \end{cases} \quad (\text{F.370})$$

and it results:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \begin{pmatrix} x_k \\ \tilde{u}_k \end{pmatrix} \quad (\text{F.371})$$

**Remark:** As for the classic case, the state-space matrices  $A, B, C$  can be recovered when solving (F.371) for known input/output vectors.

We propose and prove further a theorem suited for this symplectic discretization scheme.

#### Theorem F.14

Under the assumptions that:

- I. The input  $u_k$  is persistently exciting of order  $2i$
- II. The intersection of the row space of  $U_f^{**}$  (the future inputs) and the row space of  $X_P^d$  (the past states) is empty.
- III. The user-defined weighting matrices  $W_1 \in R^{lixli}$  and  $W_2 \in R^{jxj}$  are such that  $w_1$  is of full rank and  $w_2$  obeys:  $\text{rank} \begin{pmatrix} W_P^* \\ W_2 \end{pmatrix} = \text{rank} \begin{pmatrix} W_P^* W_2 \end{pmatrix}$ , where  $W_P^*$  is the block Hankel matrix containing the past inputs and outputs and with  $o_i$  defined as the oblique projection:

$$O_i = \left( Y_f / U_f^{**} \right) W_P^* \quad (1)$$

and the singular value decomposition:

$$W_1 O_i W_2 = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1 S_1 V_1^T \quad (2)$$

we have:

1. The matrix  $o_i$  is equal to the product of the extended observability matrix and the states:

$$O_i = \Gamma_i^* \bar{X}_f^d \quad (3)$$

2. The order of the system is equal to the number of singular values in (2) different from 0.
3. The extended observability matrix  $\Gamma_i^*$  is equal to:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (4)$$

4. The part of the state sequence  $\bar{X}_f^d$  that lies in the column space of  $w_2$  can be recovered from:

$$\bar{X}_f^d W_2 = T^{-1} S_1^{1/2} V_1^T \quad (5)$$

5. The state sequence  $\bar{X}_f^d$  is equal to:

$$\bar{X}_f^d = \Gamma_i^{*+} O_i \quad (6)$$

**Proof :**

Using (F.364), we can write the relation:

$$\bar{X}_f^d = A_M^i \left( \Gamma_i^{**+} Y_P - \Gamma_i^{**+} H_i^{**d} U_P^{**} \right) + \Delta_M U_P^{**} \quad (F.372)$$

for the future states, where:

$$\begin{cases} A_M^i & \stackrel{\text{def}}{=} \frac{\bar{A}^{i-1} + \bar{A}^i}{2} \\ \Delta_M & \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} \Delta_{i-1}^{*d} & \Delta_i^{*d} \end{pmatrix} \end{cases} \quad (F.373)$$

Equivalently we can write:

$$\bar{X}_f^d = \begin{pmatrix} \Delta_M - A_M^i \Gamma_i^{**+} H_i^{**d} & A_M^i \Gamma_i^{**+} \end{pmatrix} \begin{pmatrix} U_P^{**} \\ Y_P \end{pmatrix} \quad (F.374)$$

Adopting the notations:

$$\begin{cases} L_P^* & \stackrel{\text{def}}{=} \begin{pmatrix} \Delta_M - A_M^i \Gamma_i^{**+} H_i^{**d} & A_M^i \Gamma_i^{**+} \end{pmatrix} \\ W_P^* & \stackrel{\text{def}}{=} \begin{pmatrix} U_P^{**} & Y_P \end{pmatrix}^T \end{cases} \quad (F.375)$$

it results:

$$\bar{X}_f^d = L_P^* W_P^* \quad (F.376)$$

Using (F.364), we can compute the future outputs as follows:

$$Y_f = \Gamma_i^* L_P^* W_P^* + H_i^{**d} U_f^{**} \quad (F.377)$$

Multiplying to the right with the orthogonal projector of  $U_{f^{**\perp}}$ , it results:

$$Y_f \Pi_{U_f^{**\perp}} = \Gamma_i^* L_P^* W_P^* \Pi_{U_f^{**\perp}} + H_i^{**d} U_f^{**} \Pi_{U_f^{**\perp}} \quad (F.378)$$

As the 2<sup>nd</sup> term to the right in (F.378) is null, it results:



$$Y_f / U_{f^{**\perp}} = \Gamma_i^* L_P^* W_P^* / U_{f^{**\perp}} \quad (\text{F.379})$$

Multiplying to the right with the pseudo-inverse  $\left( W_P^* / U_{f^{**\perp}} \right)^+$  and  $W_P^*$ , it results:

$$\left( Y_f / U_{f^{**\perp}} \right) \left( W_P^* / U_{f^{**\perp}} \right) \left( W_P^* / U_{f^{**\perp}} \right)^+ W_P^* = \Gamma_i^* L_P^* W_P^* \quad (\text{F.380})$$

Then we observe that in the left part, we have an oblique projector as follows:

$$O_i = \left( Y_f / U_{f^{**\perp}} \right) W_P^* = \left( Y_f / U_{f^{**\perp}} \right) \left( W_P^* / U_{f^{**\perp}} \right) \left( W_P^* / U_{f^{**\perp}} \right)^+ W_P^* \quad (\text{F.381})$$

Then it results:

$$O_i = \Gamma_i^* L_P^* W_P^* \quad (\text{F.382})$$

Using (F.376) it results:

$$O_i = \Gamma_i^* \bar{X}_f^d \quad (\text{F.383})$$

In (F.381) we have used the relation:

$$\left( W_P^* / U_{f^{**\perp}} \right) \left( W_P^* / U_{f^{**\perp}} \right)^+ W_P^* = W_P^* \quad (\text{F.384})$$

Then we have to prove the following relation:

$$\text{rank} \left( W_P^* \right) = \text{rank} \left( W_P^* / U_{f^{**\perp}} \right) \quad (\text{F.385})$$

In this case  $W_P^*$  can be written as follows:

$$W_P^* = \begin{pmatrix} I_{(4i-2)m} & 0 \\ H_i^{**d} & \Gamma_i^{**} \end{pmatrix} \begin{pmatrix} U_P^{**} \\ X_P^d \end{pmatrix} \quad (\text{F.386})$$

Assuming the initial conditions of the theorem, it results:

$$\text{rank} \begin{pmatrix} U_P^{**} \\ X_P^d \end{pmatrix} = \text{rank} \begin{pmatrix} U_P^{**} / U_{f^{**\perp}} \\ X_P^d / U_{f^{**\perp}} \end{pmatrix} \quad (\text{F.387})$$

which proves (F.385).

**Remark:** As in the classic state-space case, the initial conditions assumed on the theorem follow the same lines, the only difference being given by the block Hankel matrix  $w_P^*$  or past/future inputs  $(U_P^{**}, U_f^{**})$ , which change the structure according to the symplectic discretization scheme.

For  $W_P^* / U_{f^{**\perp}}$  we can perform the SVD decomposition:

$$W_P^* / U_{f^{**\perp}} = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} \quad (\text{F.388})$$

Since  $W_P^* / U_{f^{**\perp}}$  is a linear combination of the columns of  $W_P^*$  and  $\text{rank} \left( W_P^* / U_{f^{**\perp}} \right) = \text{rank} \left( W_P^* / U_{f^{**\perp}} \right)$ , the column spaces of  $W_P^*$  and  $W_P^* / U_{f^{**\perp}}$  result also as equal.

We can write in this case:

$$W_P^* = U_1 R \quad (\text{F.389})$$

Using (F.388) and (F.389), it results:

$$\left( W_P^* / U_{f^{**\perp}} \right) \left( W_P^* / U_{f^{**\perp}} \right)^+ W_P^* = \left( U_1 S_1 V_1^T \right) \left( V_1 S_1^{-1} U_1^T \right) \left( U_1 R \right) \quad (\text{F.390})$$

Using the properties:

$$V_1^T V_1 = U_1^T U_1 = I \quad (\text{F.391})$$

associated to the SVD decomposition, we get:

$$\left( W_P^* / U_{f^{**\perp}} \right) \left( W_P^* / U_{f^{**\perp}} \right)^+ W_P^* = U_1 R \quad (\text{F.392})$$

Using (F.389) it results:

$$\left( W_P^* / U_{f^{**\perp}} \right) \left( W_P^* / U_{f^{**\perp}} \right)^+ W_P^* = W_P^* \quad (\text{F.393})$$

In order to prove statement 2, we can write the following relation:

$$W_1 O_i W_2 = W_1 \Gamma_i^* \bar{X}_f^d W_2 \quad (\text{F.394})$$

We observe in (F.393) that  $W_1 \Gamma_i^*$  has  $n$  columns and  $\bar{X}_f^d W_2$  has  $n$  rows and it results that their product is also of rank  $n$ . This proves statement 2 of the theorem. For the statement 3 of the theorem, we consider a non-singular similarity transformation ( $T \in R^{n \times n}$ ), in which case we can write:

$$\begin{cases} W_1 \Gamma_i^* = U_1 S_1^{1/2} T \\ \bar{X}_f^d W_2 = T^{-1} S_1^{1/2} V_1^T \end{cases} \quad (\text{F.395})$$

We can deduce then that:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (\text{F.396})$$

which represents also the statement 3 of the theorem. From (F.395) and (F.396) statements 3 and 4 are proved. Using (F.383) we can deduce the average value of the states  $\bar{X}_f^d$  as:

$$\bar{X}_f^d = \Gamma_i^{*+} O_i \quad (\text{F.397})$$

This proves the last statement of the theorem.

## F.9 Symplectic discretization scheme X

For this symplectic discretization scheme it is assumed an implicit trapezoidal rule for the flows and an implicit Euler rule for the efforts. The general explicit form of this PCH system is:

$$\begin{cases} x_{k+1} = x_k + \left( \frac{\bar{A} - I}{2} \right) (x_k + x_{k+1}) + \frac{1}{2} \bar{B} (u_k + u_{k+1}) \\ y_{k+1} = Cx_{k+1} \end{cases} \quad (\text{F.398})$$

and the implicit form is:

$$(f_S(k+1), e_S(k+1), f_C(k+1), e_C(k+1), f_R(k+1), e_R(k+1)) \in D \quad (\text{F.399})$$

Similar with the previous schemes, we combine this scheme with the 1<sup>st</sup> time-discretization scheme and using (5.33) we can write:

$$x_{k+1} = \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} + \frac{1}{2} (\bar{A} - I) \left( \bar{A}^k x_0 + \Delta_k^d U_{0/k-1} + \bar{A}^{k+1} x_0 + \Delta_{k+1}^d U_{0/k} \right) + \frac{1}{2} \bar{B} (u_k + u_{k+1}) \quad (\text{F.400})$$

Equivalently this relation becomes:

$$x_{k+1} = \frac{2\bar{A}^k x_0 + 2\Delta_k^d U_{0/k-1} + \bar{A}^{k+1} x_0 + \bar{A} \Delta_k^d U_{0/k-1} + \bar{A}^{k+2} x_0 + \bar{A} \Delta_{k+1}^d U_{0/k} - \bar{A}^k x_0 - \Delta_k^d U_{0/k-1}}{2} + \frac{-\bar{A}^{k+1} x_0 - \Delta_{k+1}^d U_{0/k}}{2} + \frac{1}{2} \bar{B} (u_k + u_{k+1}) \quad (\text{F.401})$$

Grouping the terms in (F.401):

$$x_{k+1} = \left\{ \frac{\bar{A}^k x_0 + \bar{A}^{k+2} x_0}{2} \right\} + \left\{ \frac{\bar{A} + I}{2} \right\} \Delta_k^d U_{0/k-1} + \left\{ \frac{\bar{A} - I}{2} \right\} \Delta_{k+1}^d U_{0/k} + \frac{1}{2} \bar{B} (u_k + u_{k+1}) \quad (\text{F.402})$$

and adopting the notation:

$$\Delta_{k+1}^{-d} \stackrel{def}{=} \bar{A} \Delta_k^d = \begin{pmatrix} \bar{A}^k \bar{B} & \bar{A}^{k-1} \bar{B} & \dots & \bar{A} \bar{B} \end{pmatrix} \quad (\text{F.403})$$

it results:

$$x_{k+1} = \left( \frac{I + \bar{A}^2}{2} \right) \bar{A}^k x_0 + \frac{\Delta_{k+1}^{-d} U_{0/k-1} + \Delta_k^d U_{0/k-1} + \Delta_{k+2}^{-d} U_{0/k} - \Delta_{k+1}^d U_{0/k}}{2} + \frac{1}{2} \bar{B} (u_k + u_{k+1}) \quad (\text{F.404})$$

Introducing also the past states:

$$X_P^d \stackrel{def}{=} x_0 \quad (\text{F.405})$$

results in:

$$x_{k+1} = \left( \frac{I + \bar{A}^2}{2} \right) \bar{A}^k X_P^d + \frac{\Delta_{k+1}^d U_{0/k} + \Delta_k^d U_{0/k-1} + \Delta_{k+2}^d U_{0/k+1} - \Delta_{k+1}^d U_{0/k}}{2} \quad (\text{F.406})$$

Performing the simplifications it results:

$$x_{k+1} = \left( \frac{I + \bar{A}^2}{2} \right) \bar{A}^k X_P^d + \frac{\Delta_k^d U_{0/k-1} + \Delta_{k+2}^d U_{0/k+1}}{2} \quad (\text{F.407})$$

Equivalently we can write:

$$x_{k+1} = \left( \frac{I + \bar{A}^2}{2} \right) \bar{A}^k X_P^d + \frac{1}{2} \begin{pmatrix} \Delta_k^d & \Delta_{k+2}^d \end{pmatrix} \quad (\text{F.408})$$

Using the notations:

$$\left\{ \begin{array}{l} \bar{A}^{k+1} \stackrel{\text{def}}{=} \left( \frac{\bar{A}^k + \bar{A}^{k+2}}{2} \right) \\ \Delta_{k+1}^{*d} \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} \Delta_k^d & \Delta_{k+2}^d \end{pmatrix} \\ U_{0/k}^* \stackrel{\text{def}}{=} \left( U_{0/k-1} \quad U_{0/k+1} \right)^T \end{array} \right. \quad (\text{F.409})$$

we can express then the states as:

$$x_{k+1} = \bar{A}^{k+1} X_P^d + \Delta_{k+1}^{*d} U_{0/k}^* \quad (\text{F.410})$$

As in the classic state-space case, the future states can be written as:

$$X_f^d = X_i = \bar{A}^i X_P^d + \Delta_i^{*d} U_P^* \quad (\text{F.411})$$

where  $U_P^*$  denotes the past outputs defined as:

$$U_P^* \stackrel{\text{def}}{=} \left( U_{0/i-2} \quad U_{0/i} \right)^T \quad (\text{F.412})$$

As for the states, we can write the outputs as follows:

$$y_{k+1} = \frac{C}{2} \left( \bar{A}^{k+1} X_P^d + \Delta_{k+1}^{*d} U_{0/k}^* \right) \quad (\text{F.413})$$

or equivalently:

$$y_{k+1} = \frac{C}{2} \bar{A}^{k+1} X_P^d + \frac{C}{2} \Delta_{k+1}^{*d} U_{0/k}^* \quad (\text{F.414})$$

Adopting the notations:

$$\left\{ \begin{array}{l} \Gamma_i^* \stackrel{\text{def}}{=} \frac{C}{2} \left( I \quad \bar{A} \quad \dots \quad C \bar{A}^{i-1} \right)^T \\ H_i^{*d} \stackrel{\text{def}}{=} \frac{C}{2} \begin{pmatrix} \Delta_0^{*d} & \dots & 0 \\ \dots & \dots & \dots \\ \Delta_{i-1}^{*d} & \dots & 0 \end{pmatrix} \\ U_P^* \stackrel{\text{def}}{=} U_{0/i-1}^* \end{array} \right. \quad (\text{F.415})$$

we can write then the past outputs vector as:

$$Y_P = \Gamma_i^* X_P^d + H_i^{*d} U_P^* \quad (\text{F.416})$$

Similarly we can write the relation for the future outputs:

$$Y_f = \Gamma_i^* X_f^d + H_i^{*d} U_f^* \quad (\text{F.417})$$

Using (F.411),(F.416) and (F.417), it results the *matrix input-output equations*:

$$\begin{cases} X_f^d = X_i = \overline{\overline{A}}^i X_P^d + \Delta_i^{*d} U_P^* \\ Y_P = \Gamma_i^* X_P^d + H_i^{*d} U_P^* \\ Y_f = \Gamma_i^* X_f^d + H_i^{*d} U_f^* \end{cases} \quad (\text{F.418})$$

We propose thus the following theorem for the *matrix input-output equations*:

**Theorem F.15**

*The matrix input-output equations of a discrete-time PCH system, represented in explicit form in (F.398) and in implicit form in (F.399), for a symplectic discretization scheme with an implicit trapezoidal rule for the flows and implicit Euler rule for the efforts are:*

$$\begin{cases} X_f^d = X_i = \overline{\overline{A}}^i X_P^d + \Delta_i^{*d} U_P^* \\ Y_P = \Gamma_i^* X_P^d + H_i^{*d} U_P^* \\ Y_f = \Gamma_i^* X_f^d + H_i^{*d} U_f^* \end{cases} \quad (\text{F.419})$$

In order to get an equivalent representation of (F.398), we combine it with (5.33) and it results:

$$\begin{cases} x_{k+1} = x_k + \frac{1}{2}(\overline{A} - I)(x_k + \overline{A}x_k + \overline{B}u_k) + \overline{B}\frac{1}{2}(u_k + u_{k+1}) \\ y_{k+1} = C(\overline{A}x_k + \overline{B}u_k) \end{cases} \quad (\text{F.420})$$

Equivalently we can write:

$$\begin{cases} x_{k+1} = \frac{2x_k + \overline{A}x_k + \overline{A}^2x_k + \overline{A}\overline{B}u_k - x_k - \overline{A}x_k - \overline{B}u_k + \overline{B}u_k + \overline{B}u_{k+1}}{2} \\ y_{k+1} = C\overline{A}x_k + C\overline{B}u_k \end{cases} \quad (\text{F.421})$$

A more compact form of the above relation is:

$$\begin{cases} x_{k+1} = \left( \frac{\overline{A}^2 + I}{2} \right) x_k + \frac{1}{2} \overline{A}\overline{B}u_k + \frac{1}{2} \overline{B}u_{k+1} \\ y_{k+1} = C\overline{A}x_k + C\overline{B}u_k \end{cases} \quad (\text{F.422})$$

This relation can be also written as:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{\overline{A}^2 + I}{2} & \frac{1}{2}(\overline{A}\overline{B} & \overline{B}) \\ C\overline{A} & (C\overline{B} & 0) \end{pmatrix} \begin{pmatrix} x_k \\ u_k \\ u_{k+1} \end{pmatrix} \quad (\text{F.423})$$

Introducing the notations:

$$\begin{cases} \tilde{A} \stackrel{def}{=} \left( \frac{\overline{A}^2 + I}{2} \right); \tilde{B} \stackrel{def}{=} \frac{1}{2}(\overline{A}\overline{B} & \overline{B}) \\ \tilde{C} \stackrel{def}{=} C\overline{A}; \tilde{D} \stackrel{def}{=} (C\overline{B} & 0); \tilde{u}_k \stackrel{def}{=} (u_k \quad u_{k+1})^T \end{cases} \quad (\text{F.424})$$

it results:

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \begin{pmatrix} x_k \\ \tilde{u}_k \end{pmatrix} \quad (\text{F.425})$$

**Remark:**

The initial state-space matrices  $A, B, C$  can be recovered by solving (F.425), when having knowledge on the input/output vectors.

Similar to the classic state-space case, we propose further a theorem for the deterministic identification corresponding to this symplectic discretization scheme.

**Theorem F.16**

Under the assumptions that:

- I. The input  $u_k$  is persistently exciting of order  $2i$
- II. The intersection of the row space of  $U_f^*$  (the future inputs) and the row space of  $X_P^d$  (the past states) is empty.
- III. The user-defined weighting matrices  $W_1 \in R^{lixli}$  and  $W_2 \in R^{jxj}$  are such that  $w_1$  is of full rank and  $w_2$  obeys:  $\text{rank} \left( W_P^* \right) = \text{rank} \left( W_P^* W_2 \right)$ , where  $W_P^*$  is the block Hankel matrix containing the past inputs  $U_P^{**}$  and outputs  $Y_P$  and with  $O_i$  defined as the oblique projection:

$$O_i = \left( Y_f / U_f^* \right) W_P^* \quad (1)$$

and the singular value decomposition:

$$W_1 O_i W_2 = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1 S_1 V_1^T \quad (2)$$

we have:

1. The matrix  $O_i$  is equal to the product of the extended observability matrix and the states:

$$O_i = \Gamma_i^* X_f^d \quad (3)$$

2. The order of the system is equal to the number of singular values in (2) different from 0.
3. The extended observability matrix  $\Gamma_i^*$  is equal to:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (4)$$

4. The part of the state sequence  $X_f^d$  that lies in the column space of  $w_2$  can be recovered from:

$$X_f^d W_2 = T^{-1} S_1^{1/2} V_1^T \quad (5)$$

5. The state sequence  $X_f^d$  is equal to:

$$X_f^d = \Gamma_i^{*+} O_i \quad (6)$$

**Proof:**

Using (F.419), we can write the future states as:

$$X_f^d = \overline{\overline{A}}^i \left( \Gamma_i^{*+} Y_P - \Gamma_i^{*+} H_i^{*d} U_P^* \right) + \Delta_i^{*d} U_P^* \quad (F.426)$$

Or equivalently:

$$X_f^d = \left( \Delta_i^{*d} - \overline{\overline{A}}^i \Gamma_i^{*+} H_i^{*d} \right) U_P^* + \overline{\overline{A}}^i \Gamma_i^{*+} Y_P \quad (F.427)$$

More compact we can write:

$$X_f^d = L_P^* W_P^* \quad (F.428)$$

Adopting the notations:

$$\begin{cases} L_P^* \stackrel{def}{=} \left( \Delta_i^{*d} - \overline{\overline{A}}^i \Gamma_i^{*+} H_i^{*d} \quad \overline{\overline{A}}^i \Gamma_i^{*+} \right) \\ W_P^* \stackrel{def}{=} \begin{pmatrix} U_P^* & Y_P \end{pmatrix}^T \end{cases} \quad (F.429)$$

Using (F.419) and (F.429), the future outputs are:

$$Y_f = \Gamma_i^* L_P^* W_P^* + H_i^{*d} U_f^* \quad (F.430)$$

Multiplying to the right with the orthogonal projector of  $U_{f^* \perp}$ , it results:

$$Y_f \Pi_{U_{f^* \perp}} = \Gamma_i^* L_P^* W_P^* \Pi_{U_{f^* \perp}} + H_i^{*d} U_f \Pi_{U_{f^* \perp}} \quad (F.431)$$

The 2<sup>nd</sup> term to the right is null and then it results:

$$Y_f / U_{f^* \perp} = \Gamma_i^* L_P^* W_P^* / U_{f^* \perp} \quad (F.432)$$

Multiplying to the right with the pseudo-inverse  $\left( W_P^* / U_{f^* \perp} \right)^+$  and  $W_P^*$ , it results:

$$\left( Y_f / U_{f^* \perp} \right) \left( W_P^* / U_{f^* \perp} \right)^+ W_P^* = \Gamma_i^* L_P^* W_P^* \quad (F.433)$$

It can easily be observed to the left in (F.433) an oblique projection and we can write then:

$$O_i = \left( Y_f / U_{f^* \perp} \right) W_P^* = \Gamma_i^* L_P^* W_P^* \quad (F.434)$$

Using (F.428) it results:

$$O_i = \Gamma_i^* X_f^d \quad (F.435)$$

In (F.433) we have used the relation:

$$\left( W_P^* / U_{f^* \perp} \right) \left( W_P^* / U_{f^* \perp} \right)^+ W_P^* = W_P^* \quad (\text{F.436})$$

For the next step we will prove that:

$$\text{rank} \left( W_P^* \right) = \text{rank} \left( W_P^* / U_{f^* \perp} \right) \quad (\text{F.437})$$

For this symplectic scheme, we can write  $W_P^*$  as:

$$W_P^* = \begin{pmatrix} I_{2mi} & 0 \\ H_i^{*d} & \Gamma_i^* \end{pmatrix} \begin{pmatrix} U_P^* \\ X_P^d \end{pmatrix} \quad (\text{F.438})$$

where:

$$U_P^* = U_{0/i-1}^* = \begin{pmatrix} U_{0/i-2} & U_{0/i} \end{pmatrix}^T \quad (\text{F.439})$$

Multiplying to the right with  $\Pi_{U_{f^* \perp}}$ , we get:

$$W_P^* \Pi_{U_{f^* \perp}} = \begin{pmatrix} I_{2mi} & 0 \\ H_i^{*d} & \Gamma_i^* \end{pmatrix} \begin{pmatrix} U_P^* \Pi_{U_{f^* \perp}} \\ X_P^d \Pi_{U_{f^* \perp}} \end{pmatrix} \quad (\text{F.440})$$

Using the orthogonal projection operator, (F.440) becomes:

$$W_P^* / U_{f^* \perp} = \begin{pmatrix} I_{2mi} & 0 \\ H_i^{*d} & \Gamma_i^* \end{pmatrix} \begin{pmatrix} U_P^* / U_{f^* \perp} \\ X_P^d / U_{f^* \perp} \end{pmatrix} \quad (\text{F.441})$$

Assuming the initial conditions of the theorem, it results:

$$\text{rank} \begin{pmatrix} U_P^* \\ X_P^d \end{pmatrix} = \text{rank} \begin{pmatrix} U_P^* / U_{f^* \perp} \\ X_P^d / U_{f^* \perp} \end{pmatrix} \quad (\text{F.442})$$

**Remark:** The initial conditions assumed for the current symplectic discretization scheme, are the same with the classic state-space case approach, the only difference comes in the structure of the block Hankel matrix  $W_P^*$  and past/future inputs  $(U_P^*, U_f^*)$ .

which proves (F.437). Introducing the SVD decomposition of  $W_P^* / U_{f^* \perp}$  it results:

$$W_P^* / U_{f^* \perp} = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} \quad (\text{F.443})$$

Since  $W_P^* / U_{f^* \perp}$  is a linear combination of the columns of  $W_P^*$  and since the rank of  $W_P^*$  and  $W_P^* / U_{f^* \perp}$  are equal, then also the column spaces of  $W_P^*$  and  $W_P^* / U_{f^* \perp}$  are also equal.

Thus we can write  $W_P^*$ , as:

$$W_P^* = U_1 R \quad (\text{F.444})$$

Using the SVD decomposition of (F.443), we can write:



$$\left( W_P^* / U_{f^* \perp} \right) \left( W_P^* / U_{f^* \perp} \right)^+ W_P^* = \left( U_1 S_1 V_1^T \right) \left( V_1 S_1^{-1} U_1^T \right) \left( U_1 R \right) \quad (\text{F.445})$$

Together with the properties of the SVD decomposition:

$$V_1^T V_1 = U_1^T U_1 = I \quad (\text{F.446})$$

it results:

$$\left( W_P^* / U_{f^* \perp} \right) \left( W_P^* / U_{f^* \perp} \right)^+ W_P^* = U_1 R \quad (\text{F.447})$$

From (F.444) and (F.447), it results:

$$\left( W_P^* / U_{f^* \perp} \right) \left( W_P^* / U_{f^* \perp} \right)^+ W_P^* = W_P^* \quad (\text{F.448})$$

The 2<sup>nd</sup> claim of the theorem can be proved by writing:

$$W_1 O_i W_2 = W_1 \Gamma_i^* X_f^d W_2 \quad (\text{F.449})$$

We observe that the product  $W_1 \Gamma_i^*$  has  $n$  columns and  $X_f^d W_2$   $n$  rows, from which the rank of  $W_1 O_i W_2$  results to be also  $n$ . According to the SVD decomposition, we can write:

$$\begin{cases} W_1 \Gamma_i^* = U_1 S_1^{1/2} T \\ X_f^d W_2 = T^{-1} S_1^{1/2} V_1^T \end{cases} \quad (\text{F.450})$$

Using the first equation of (F.450), it results the extended observability matrix:

$$\Gamma_i^* = W_1^{-1} U_1 S_1^{1/2} T \quad (\text{F.451})$$

From (F.450) and (F.451), statements 3 and 4 are proved. Multiplying to the left with the pseudo-inverse  $\Gamma_i^{*+}$  in (F.435), it results the states value:

$$X_f^d = \Gamma_i^{*+} O_i \quad (\text{F.452})$$

This proves statement 5 of the theorem.

## Appendix G: Controllability and observability grammians

In this section, we determine a general relation between the controllability and observability grammians in the lossy case. In order to do that, we write first the observability grammian as follows:

$$W_O^2 = \int_0^\infty e^{A^T t} C^T C e^{-At} dt \quad (\text{F.453})$$

Replacing  $C = B^T Q$ , we can write equivalently:

$$W_O^2 = \int_0^\infty e^{A^T t} Q B B^T Q e^{-At} dt \quad (\text{F.454})$$

Further we will develop an equivalent form for  $e^{A^T t} Q$  and  $Q e^{-At}$  using the general relation of the exponential  $e^x$ . Thus for  $e^{A^T t} Q$  we can write:

$$e^{A^T t} Q = \left( I + \frac{1}{1!} (A^T t) + \frac{1}{2!} (A^T t)^2 + \dots \right) Q \quad (\text{F.455})$$

Further we will determine an equivalent relation for  $(A^T)^k Q$  using the PCH systems properties. For  $k = 1$  we can write:

$$A^T Q = ((J - R)Q)^T Q = Q^T (J - R)^T Q = Q(-J - R)Q \quad (\text{F.456})$$

Introducing the notations:

$$\begin{cases} \text{def} \\ A_{loss} = JQ \\ \text{def} \\ A_{res} = RQ \end{cases} \quad (\text{F.457})$$

It results:

$$A^T Q = (-1)^1 Q (A_{loss} + A_{res}) \quad (\text{F.458})$$

For  $k = 2$ , we can write as follows:

$$(A^T)^2 Q = ((J - R)Q)^T ((J - R)Q)^T Q \quad (\text{F.459})$$

Using the matrix transpose properties, we get:

$$(A^T)^2 Q = Q^T (J - R)^T Q^T (J - R)^T Q \quad (\text{F.460})$$

Using the PCH systems properties this is equivalent to:

$$(A^T)^2 Q = (-1)^2 Q (A_{loss} + A_{res})^2 \quad (\text{F.461})$$

Thus it is straightforward that:

$$(A^T)^k Q = (-1)^k Q (A_{loss} + A_{res})^k \quad (\text{F.462})$$

It results then that:

$$e^{A^T t} Q = Q e^{-A' t} \quad (\text{F.463})$$

where:

$$A' \stackrel{\text{def}}{=} A_{loss} + A_{res} \quad (\text{F.464})$$

Further we determine an equivalent form for  $Q e^{A t}$  using the matrix exponential form. Thus we will find a general relation for  $Q A^k$  using the PCH systems properties. For  $k = 1$ , we can write:

$$Q A = Q (J - R)Q = Q^T (J^T - R^T) Q \quad (\text{F.465})$$

Using (F.457) we can write equivalently:

$$Q A = (-1)^1 (A_{loss} + A_{res})^T Q = (-1)^1 A'^T Q \quad (\text{F.466})$$

For  $k = 2$  we can write:

$$QA^2 = Q(J - R)Q(J - R)Q = Q^T \left( J^T - R^T \right) Q^T \left( J^T - R^T \right) Q \quad (F.467)$$

Using the PCH systems properties and (F.457), we can develop (F.467) as:

$$QA^2 = (-1)^2 \left( (A_{loss} + A_{res})^T \right)^2 Q \quad (F.468)$$

From (F.466) and (F.468) we can deduce that:

$$QA^k = (-1)^k \left( (A_{loss} + A_{res})^T \right)^k Q \quad (F.469)$$

It results then that:

$$Qe^{At} = e^{-A^T t} Q \quad (F.470)$$

Using (F.463) and (F.470), we can write the observability grammian as:

$$W_O^2 = \int_0^{\infty} Qe^{-A^T t} BB^T e^{-A^T t} Q dt \quad (F.471)$$

which is equivalent to:

$$W_O^2 = Q \int_0^{\infty} e^{-A^T t} BB^T e^{-A^T t} dt Q \quad (F.472)$$

We observe that in the middle of (F.472) we have the controllability Grammian:

$$W_C^2 = \int_0^{\infty} e^{-A^T t} BB^T e^{-A^T t} dt \quad (F.473)$$

and thus it results:

$$W_O^2 = Q W_C^2 Q \quad (F.474)$$

from which we obtain the following general relation:

$$W_O = Q W_C \quad (F.475)$$

**Remark:** The lossless case of (F.475) can be easily deduced when  $A_{res} = 0$ , by straightforward computations, following the same pattern of the proof.

Further we will determine the necessary conditions for the internal balancing of (5.134) as in [176] for  $x_1, x_2$  states. For the  $x_1$  states we can write:

$$\bar{X}_1 = e^{T^T A T^{-1} t} T^T B \quad (F.476)$$

where  $T = [T_1 \ 0]$  and  $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ . By straightforward computation it results:

$$\bar{X}_1 = T^T e^{A t} B \quad (F.477)$$

Then the controllability Grammian results as:

$$W_{C1}^2 = \int_0^{\infty} T^T e^{A t} BB^T e^{A^T t} T^T dt \quad (F.478)$$

which is equivalent to:

$$W_{C1}^2 = T^T W_C^2 T^T \quad (F.479)$$

Then we determine  $\bar{Y}_1$  as:

$$\bar{Y}_1 = C T'^{-1} e^{T' A T'^{-1} t} \quad (F.480)$$

By straightforward computation we can write:

$$\bar{Y}_1 = C e^{A t T'^{-1}} \quad (F.481)$$

Then the observability Grammian of  $x_1$  states, results as:

$$W_{O1}^2 = \int_0^{\infty} \bar{Y}_1^T \bar{Y}_1 dt = \int_0^{\infty} T'^{-T} e^{A^T t} C^T C e^{A t T'^{-1}} dt \quad (F.482)$$

Using (F.474) it results:

$$W_{O1}^2 = T'^{-T} W_O^2 T'^{-1} = T'^{-T} Q W_C^2 Q T'^{-1} \quad (F.483)$$

Then we write the internal balancing relation [179]:

$$W_{C1}^2 = W_{O1}^2 = \Sigma_1^2 \quad (F.484)$$

where  $\Sigma_1$  represent the singular values from grammian decomposition. Using (F.479) we can easily deduce that:

$$\begin{cases} W_O = \Sigma_1 T' \\ W_C = T'^{-1} \Sigma_1 \end{cases} \quad (F.485)$$

Using (F.479), (F.483) and (F.484) we will determine further a relation between the Grammians  $w'_C$  and  $w_C$  by straightforward computation:

$$W_C^2 = T'^{-1} T'^{-T} Q W_C^2 Q T'^{-1} T'^{-T} \quad (F.486)$$

This relation can be equivalently written as:

$$W_C W_C^T = T'^{-1} T'^{-T} Q W_C' \left( T'^{-1} T'^{-T} Q W_C' \right)^T \quad (F.487)$$

Then it results directly:

$$W_C = T'^{-1} T'^{-T} Q W_C' \quad (F.488)$$

Using (F.483) and (F.484) we can write the following relation:

$$W_C' W_C'^T = Q^{-T} T'^T \Sigma_1 \Sigma_1^T T' Q^{-T} \quad (F.489)$$

from which it results:

$$W_C' = Q^{-1} T'^T \Sigma_1 \quad (F.490)$$

**Remark:** The results obtained for the  $x_2$  states for internal balancing [176] can be easily derived from the previous relations by straight computation.

## Abstract

The present thesis '*Identification des systèmes hamiltoniens à ports*', proposed the development of a specific theory for Port Controlled Hamiltonian (PCH) systems identification, due to their remarkable properties of energy preserving, interconnection of systems from multiple domains of physics, etc. An analysis of structural identifiability is first performed using classic or new (power energy) techniques to check the model structure identifiability. A new context of identifiability is proposed here, which makes use of the inputs and power port energies knowledge and also a '*port identifiability*' concept is proposed for the analysis of PCH systems. Then a practical identifiability analysis is performed using a classic test, by adopting a perturbation model introduced on the interaction port of the PCH system. Further a discrete-time framework is proposed, with different classic discretization rules for the flows and efforts, that prove to preserve the continuous-time structure. The last part of the thesis proposes the analysis of classic subspace identification algorithms for PCH systems using the previous time-discretization schemes. Beside this it is proposed a new (*power energy*), which makes use of the known inputs and power energies to determine the unknown system parameters. On this new context it is proved by algorithms and simulation, that equivalent PCH systems for the real model, can be identified on each port. Thus it is proved that the system parameters can be determined in a classic or new (*power energy*) manner.

**Keywords:** system identification, Port Hamiltonian systems, structural identifiability, observability, controllability, energy based identifiability, practical identifiability, discretization error, symplectic discretization, deterministic identification.

## Résumé

**L'Objectif** de cette thèse est de développer une **théorie de l'identification** spécifique pour les systèmes Hamiltonien à ports. Les raisons principales pour motiver cette théorie résident dans les propriétés remarquables de ces systèmes, notamment leur structure de Dirac et sa stabilité par interconnexion conservative de puissance (e.g. parallèle, séries ou feedback). Dans la première partie, les systèmes Hamiltoniens sont analysés en ce qui concerne leur identifiabilité structurelle, par analyse de leur observabilité/commandabilité, par tests directs, par l'analyse en série de puissance de leur fonction de transfert ou par une nouvelle approche *énergétique* d'analyse d'une identifiabilité spécifique associée à un port. Dans la partie suivante, des modèles de perturbation par port d'interaction sont introduits et permettent l'analyse de l'identifiabilité 'pratique' des systèmes hamiltoniens à ports. Le quatrième chapitre présente des schémas de discrétisation en temps qui préserve les bilans de puissance et d'énergie et leur application sur des exemples de système hamiltoniens à ports linéaires et non linéaires. L'erreur de discrétisation est analysée en introduisant la notion de représentation hamiltonienne de l'erreur de discrétisation. Dans la dernière partie de cette thèse, une approche d'identification dans l'espace d'état est développée pour les systèmes obtenus par discrétisation symplectique des systèmes hamiltoniens à ports. Les cas déterministe est analysé et une approche énergétique basée sur les résultats d'identifiabilité structurelle développé dans la première partie est proposée. Enfin, dans la dernière partie, les contributions du travail sont rappelées et quelques perspectives pour des travaux futurs sont présentées.

**Mots clés:** identification des systèmes, systèmes Hamiltonian à ports, identifiabilité structurelle, observabilité, commandabilité, identifiabilité énergétique, identifiabilité pratique, erreur de discretization, discrétisation symplectique, identification déterministe.