Separation of variables and new quantum integrable systems with boundaries

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Séparation des variables et nouveaux systèmes intégrables quantiques avec bords

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Introduction

As physicists we are interested in discovering and understanding the laws of nature. This is a rather complex process that goes along observing the multitude of phenomena around us, performing related experiments to test various contexts and parameters and finally building new models or theories that can explain and predict the corresponding phenomena. Going back and forth between theory and experiments, we try to bring out the elementary principles governing the laws of nature, and to gather them in different classes. All along such a path, the interactions between Physics and Mathematics play a fundamental role. While Mathematics is providing a natural framework for the formulation of new theories, Physics is bringing to light new questions that could have great influence in the creation of new concepts and domains in Mathematics. This fruitful interaction goes back certainly to the early history of sciences.

One of the paradigmatic examples of such a tight interaction between Physics and Mathematics is given by the modelling of the celestial bodies motion. This problem, which goes back to the Antiquity, has first been "solved" in modern phenomenological terms by Kepler [1, 2], formulating his famous three laws building on Copernic heliocentric system [3] and on the careful interpretation of the first observations of Galilee and of the impressive amount of datas collected by and together with Tycho-Brahé [4]. But if these laws could effectively describe the motion of planets of the solar system, they do not explain the deep universal physical principle behind such behaviour. The true conceptual breakthrough came with Newton in his famous *Principia* [5], with the formulation of the fundamental laws of mechanics and gravity and their universal application from the falling bodies on earth to the celestial movements of planets around the sun. These progresses were made possible on the one hand by the formulation of the laws of mechanics with Descartes, Galilee and then Newton, and by the identification of the inverse square law for the gravity forces[1], and on the other hand by the developments of differential calculus with Newton and Leibniz [5, 6]. These advances enabled Newton to give a description of the motion of celestial bodies in terms of dynamical equations that eventually can be solved explicitly. In particular he was able to give a proof of the three Kepler phenomenological laws from the few universal principles he stated for mechanics and gravity. Hence the great success of Newton was not only to build up a new theory from a few general principles but also to develop the mathematical tools to make it predictive, namely by providing solutions of the dynamical equations determining the motion. Although the solutions of the equations of motion were formulated in more conventional geometrical terms at the time of first publication of the *Principia* [5], it became clear soon after that the differential calculus of Newton and Leibniz was a powerful tool in effectively getting their analytical solutions (see e.g. the translation of *Principia* to french by Emilie du Châtelet [7], and especially her own appendix using differential calculus to derive Newton's results).

This feature of solvability in Newton's theory, enabling its full comparison to observations, was fundamental in convincing other scientists that this theory was indeed describing the fundamental laws of nature governing gravity. It certainly had a great influence on many further developments and in particular in the reformulations of Newton's theory by Lagrange and by Hamilton [8, 9]. Rapidly however, it turned out that most of the dynamical equations of mechanics cannot be solved exactly. For instance, the two bodies interaction with an $1/r$ potential admits an exact solution, but this is generically not the case for more than two bodies. Even worse, in most of the situations of classical mechanics there is no hope

[1]There is some intricate history of the inverse square law for gravity, involving several scientists as more particularly Bullialdus, Hooke, Borelli, Wren and Halley, but it was Newton that finally found the proper formulation, still acknowledging their contributions in *Principia* [5].

to get an exact solution, showing that the explicitly solvable models, or so-called integrable models, are exceptions [10].

Nevertheless, the knowledge of an exact solution, even if restricted to a very particular situation, is of prime interest. Indeed, although the motion of planets in the solar system cannot be reduced to two body motion involving the sun and each individual planet, this first approximation is an interesting starting point to perform perturbations around such a simplified picture. And if one is interested in time interval not too large[2], the scheme is good enough to get rather high accuracy in the determination of the trajectories of planets in the solar systems. Hence, in many situations, very specific integrable models can be used as a starting point for the description of many actual realistic models, mainly by a perturbation process. There are numerous examples of the impact of exact solutions on the understanding of more general (non integrable) systems. The resolution of Schrödinger equation for the hydrogen atom is one of them, revealing the layered and discrete structure in its energy levels. It led to a theory of atoms with several electrons which was able, among others, to explain the Mendeleev classification. Moreover, there is many situations where integrable models, and well defined perturbation theory around them [13] (and references therein), give access to a whole class of non perturbative phenomena, allowing to get an understanding of complex and non-linear behaviours (strong couplings, collective effects,...).

Historically, the field of integrable systems appeared just as the search for exact solutions of the dynamical equations of classical mechanics. Because not all systems are integrable, the first question was to determine what were the necessary and sufficient conditions for integrability. In this quest for constructing and solving integrable systems, important developments emerged with the works of Lagrange, Hamilton, Jacobi, and Liouville [8, 9, 14, 15]. There, the notion of conserved quantities became central, leading in particular to the first rigorous definition of integrability by Liouville [15] (see for its modern formulation Arnold in [16]). The systematic link to symmetries was elucidated later on in the works of Emmy Noether [17]. A system with n degrees of freedom is said to be Liouville integrable if it possesses n commuting (under Poisson brackets) independent (in the sense of differentials) integrals of motion; then the equations of motion can be solved (at least locally) by quadratures and, with compactness and connectivity hypothesis of the level manifold, action-angle variables can be determined through multi-variables curvilinear integrals (see e.g. [16, 18]). It should be noted however that, from a practical point of view, their effective construction in a given model has usually to rely on additional (e.g., algebraic) constructions. An important step towards this goal is played by the notion of separation of variables in Hamilton-Jacobi theory. In short, the key idea is that to solve a system with n degrees of freedom effectively we use the n independent integrals of the motion in such a way that in the adequate (separate) variables, the system of n coupled differential equations giving the dynamics ultimately separates into n independent differential equations each involving only one variable together with the fixed values of the conserved quantities. As we shall see later, this notion of separation of variables admits an interesting quantum counterpart which will be a central tool in this thesis.

From these early developments in classical mechanics, the notion of solvability has been explored and generalised within several domains of physics: continuous models (e.g. in hydrodynamics), statistical mechanics, quantum mechanics and classical and quantum field theories.

One of the most famous example of continuous model suspected to be solvable in the end of the nineteenth century was certainly the Korteweg-de Vries (KdV) equation and the discovery of its soliton solutions (after investigations of Boussinesq [19] and Rayleigh [20] in the 1870's, which lead to the formulation of Korteweg and de Vries [21] in 1895) that were first observed by Scott-Russell in 1834 [22]. Although the existence of such solutions was pointing towards the existence of (an infinite number of) conserved quantities explaining the shape conservation of the solitons, the effective construction of those has been long awaited. The main difficulty there is that contrary to the two-body gravity problem, there is no obvious geometrical symmetry leading through Noether theorem to the corresponding conserved quantities. Rather, the conservation laws are direct consequences of the special dynamics of the system, i.e. of

[2] For large time, asymptotically, the N-body problem is generically chaotic and perturbation theory does not apply. However, sophisticated perturbation and numerical techniques have been developed to give high accuracy in the determination of trajectories even for rather large scales of time [11]. Nevertheless, for scales of the order of 100 millions years or above, the behaviour of the solar system is unpredictable for uncertainties of the order of a few meters on the initial positions of the planets [12].

the specific interacting potential. These symmetries and conserved quantities are of dynamical type, and are very similar to the Runge-Lenz vector, associated to the Kepler problem [23, 24]. The possibility to construct and to be able to solve such systems, owning a dynamical symmetry, originated from the work of Gardner, Green, Kruskal, and Miura [25] in 1967 and then Lax [26] for the KdV equation, then generalised by Zakharov and Shabat [27, 28] in the form of zero curvature equations. The main idea is to equivalently rewrite the equation of motion as a particular iso-spectral evolution equation for a (Lax) matrix, allowing to write the conserved quantities from its spectral invariants. Combining these settings with the inverse scattering transform of Gel'fand, Levitan and Marchenko [29, 30], several solvable continuum models like the KdV equation can be constructed and their (multi) soliton solutions derived [25, 31–33]. The Hamiltonian interpretation of the scheme was first given by Zakharov and Faddeev [34]. The solitons, their semi-classical quantisation as particles and their scattering were intensively studied in two-dimensional relativistic quantum field theories, with the hope to describe their bound-states in strongly interacting regimes [35]. Further progress was to come with the merging of this line of research with another one stemming from solvable models in classical and quantum statistical mechanics originating from the search for a description of magnetic properties of solids.

Motivated notably by the study of phase transitions in magnetic materials, Lenz proposed a lattice model [36], solved by Ising in the one-dimensional case [37] and that became famous as the Ising model. The two-dimensional case was much more tedious to solve, and was successfully considered by Onsager and then Onsager and Kaufman twenty years later [38, 39]. They succeeded to compute the partition function, the critical exponents and the magnetisation [40, 41]. Long after, the spontaneous magnetisation (the model original raison d'être) could be achieved by Yang [42]. Their method relies on the so-called Onsager and Clifford algebras, which marked the early history of exactly solvable statistical systems. This example of the Ising model is indeed an archetype of the current philosophy behind integrability, aiming to embed the problem in algebraic structures giving the tools towards their resolution. The one-dimensional Ising solution not showing a phase transition, and Ising (wrongly) conjecturing that it should hold for higher dimensions, physicists started to study quantum models. This gave birth to the XXX spin-1/2 chain, due to Heisenberg [43], a lattice quantum model where the interactions between nearest neighbours are isotropic. In 1931 Bethe could express the spectrum of this chain thanks to his famous coordinate ansatz [44], now called after his name. Then the works of Orbach [45] and Walker [46] allowed to apply this ansatz to different models, like the XXZ Heisenberg spin chain, which introduces an anisotropy in the z direction compared to the XXX chain. For this anisotropic chain, Yang and Yang could determine the energy of the ground state for finite [47] and infinite chains [48]. Since then, XXZ spin chain has been at the heart of many studies, as it represents a simple model which allowed to lay the foundations of several fruitful techniques. Let us mention the considerable work of Baxter on the lattice systems, in particular his studies of the 6-vertex and 8-vertex models. His approach, using the notion of Q-operator and the so-called Baxter equation [49], has shown to be applicable in a large range of models and to allow a more universal study of lattice models [50–58]. It is in particular worth noticing that the 8-vertex model contains, as particular cases, the 6-vertex and Ising models, see e.g. [59–61]. Another important model was shown to belong to this class, the 1D Bose gas and associated non-linear Schrödinger equation [62, 63], which was solved by Lieb and Liniger using a technique [64, 65] similar to the Bethe ansatz. On the other hand, the classical version of this model was solvable by mean of the Inverse Scattering Method, immediately rising the question of the connection between these two approaches and of the existence of a quantised version of the Classical Inverse Problem Method.

The conceptual efforts to understand this problem led to the discovery of the Quantum Inverse Scattering Method, or Algebraic Bethe Ansatz [66–70], realising an impressive breakthrough for the field of integrable systems. This is both a quantum version of the Classical Inverse Scattering Method and an algebraic version of the coordinate Bethe ansatz. It was very soon applied on discretised (lattice) version of the sine-Gordon relativistic field theory [67], and then became a quite universal tool applicable to many integrable lattice models [71, 72]. In the first chapter of this thesis we will review in more details the working of this method, but let us emphasise on the structure. The main point is the construction of the so-called monodromy matrix, a matrix defined on an auxiliary space which gathers operators on the quantum space acting on the Hilbert space of states of the chain. This monodromy matrix satisfies commutation relations whose structure constants are given by an R-matrix, solution of the Yang-Baxter equation (it is the R-matrix of Baxter appearing as Boltzmann weights of the 2D-lattice models, e.g. the 6-vertex model if we consider the XXZ Heisenberg spin chain). From this so-called Yang-Baxter algebra, it turns out that the transfer matrix, defined as the trace of the monodromy matrix on the auxiliary space, defines a commuting family of operators. This is the mark of the integrability, the Hamiltonian being then expressed as a function (trace identity) of the transfer matrix. Thereafter, the Algebraic Bethe Ansatz allows one to find eigenvectors of the transfer matrix, by the repeated action of creation operators (some entries of the monodromy matrix) on a so-called reference state. This latter is the cornerstone of the method, it is in fact a first eigenvector of the transfer matrix, from which one can construct new eigenvectors. This framework finally proved to be able to tackle a much more difficult problem for Heisenberg chains: the computation of correlation functions. To achieve this goal, it was necessary to actually solve the Quantum Inverse Scattering Problem, namely to reconstruct the local quantum operators (like spins at a given lattice site) from the scattering data, i.e. from the entries of the quantum monodromy matrix [73, 74]. In the classical case this procedure leads to the computation of soliton solutions of the non-linear integrable differential equations. Here it allows to compute the action of local fields on eigenstates of the transfer matrix and then to access to their form factors and correlation functions via scalar products determinant formula [75–80] as we shall describe later on.

One very interesting feature of this breakthrough for the quantum case is its impact on the study of the classical integrable systems. Indeed, it was almost immediately realised by Sklyanin [81] that the concept of R-matrix, Yang-Baxter equations and algebras have classical counterparts, namely the classical r-matrix and the classical Yang-Baxter equations and algebras, shading completely new light on the algebraic structure of classical integrable models. This emergence of Yang-Baxter algebraic structures in classical integrable systems paved the way for their classification using Lie algebra representations, thanks to the works of Belavin and Drinfel'd [82–84]. What is more, the interpretation of this classification states the importance of the study of Lie algebras, which reached an height with the use of Lie-Poisson groups and Lie bi-algebras [85–88], as it allows to construct and solve classical integrable models. We can cite in this context the fundamental works of Adler [89, 90], Kostant [91], Symes [92] and of Reymann and Semenov-Tian-Shansky [93–95].

In parallel and somehow motivated by these progresses in the classical context, a considerable endeavour has been pursued to find general solutions of the Yang-Baxter equation and representations of Yang-Baxter algebras, in order to construct and to classify quantum integrable systems. The key idea was to try to promote to the quantum case the relation between Lie algebras and classical integrable models. Thanks to the pioneering works of Kulish and Reshetikhin [96, 97], Jimbo [98] and Drinfel'd [99] discovered the notion of quantum groups, and showed how it was replacing the role played by Lie algebras in the classical case. Quantum group structures written in terms of Hopf algebras [100] realise a quantification of Lie-Poisson groups. Previously known R-matrices for specific models appeared then as special representations of a universal R-matrix, an object which is at the heart of the intertwining properties of coproduct (and hence of tensor products of representations) in quantum groups. It opened a completely new field of Algebra with numerous applications ranging from Combinatorics, Algebra and Topology, see e.g. [101].

In this thesis, we especially focus on one dimensional quantum systems. Even if they are simplified models, they very often contain the essence of the physics of fully 3D systems, and can describe their relevant properties. Baxter, in the preface of his book [59], even writes: "Basically, I suppose the justification for studying these lattice models is very simple: they are relevant and they can be solved, so why not do so and see what they tell us?". In this context of lattice quantum integrable systems, spin chains are among the most studied examples with applications which range from condensed matter to high energy physics. However, to successfully apply these models to physical situations of interest, we have to be able to compare their theoretical predictions to experiments, for example performed on magnetic materials. Linear response theory give a precise link between the response of a system submitted to an external perturbation and its correlation functions. For instance, dynamical structure factors, which are given by the Fourier transform of dynamical correlation functions, can describe the response of a magnetic system scattered with neutrons [102–104]. Moreover the critical properties of such systems are also encoded in the asymptotic behaviour of their correlation functions. As they ultimately contain all the information on their dynamics, there have been strong motivations to compute these objects. Indeed, they are also at the heart of statistical mechanics, as in principle any physical observable can be expressed through these functions. It turned out however that the computation of the correlation functions is a very involved problem. The first results were obtained for the Ising model, heavily using its free fermionic structure. Even there, although the free fermion algebra is quite simple (the interaction vanishes), a considerable amount of work was necessary to obtain satisfactory results since the pioneering work of Lieb, Shultz and Mattis [105]. Several outstanding groups, including Barouch, McCoy, Tracy and Wu [106], Jimbo, Miwa, Môri and Sato [107], and also Creamer, Thacker and Wilkinson [108] leading to the Painlevé description of correlation functions in the thermodynamic limit, worked in this direction. The similar question for non-free fermionic models, like the Heisenberg spin chains, was for a long time impossible to tackle in reasonable terms [109–111]. This was mainly due to the intricate structure of Bethe eigenstates with respect to the action of local (spin) operators. In a nutshell, Bethe states are highly non-local states and the action of local operators is very complicated to characterise in handleable terms. As already anticipated above, the full solution to this problem was obtained by solving the Quantum Inverse Scattering Problem, namely by obtaining the construction of local spin operators in an arbitrary lattice site in terms of the entries of the quantum monodromy matrix. The fact that Bethe eigenstates are obtained within the Algebraic Bethe Ansatz as repetitive actions of one of the entries of the monodromy matrix on a reference state, together with the knowledge of the Yang-Baxter algebra, enabled then to compute the action of local operators on Bethe states. This action being again written in terms of (off shell) Bethe states, the computation of form factors (matrix elements of local operators in the eigenstates basis of the transfer matrix) and correlation functions reduces to the computation of scalar products of a Bethe eigenstate with an arbitrary (off shell) Bethe state. Fortunately such formulas exists for XXX and XXZ model, leading in [73, 112] to the famous determinant formula for the form factors and to the multiple integral formula for the elementary blocks for the correlation functions of the XXZ spin-1/2 chain. Thereby it gave a full derivation of the expressions found earlier by Jimbo, Miwa and collaborators using q-vertex algebra approach to the infinite volume chains [113–116]. Moreover, because the solution to this problem within Algebraic Bethe Ansatz was obtained for finite volume, numerical techniques can be applied to compute dynamical structure factors essentially by summing up the form factor series. Those being written in terms of explicit determinants involving Bethe roots, it has been possible to use efficient (and fully controlled) numerical algorithms to obtain magnetic dynamical structure factors of the XXZ Heisenberg spin chain and to successfully compare the results to actual neutron scattering experiments [117, 118]. Moreover, subsequent works led to the full determination of the asymptotic behaviour of correlation functions and explicit contact with conformal field theory [75–80]. One should also mention the works towards temperature dependent case, with a great contribution of Goehmann et al. since 2004 [119–121] (see [122] for a pedagogical review and [123, 124] for more recent works) and other groups [125, 126], and the approach using hidden fermionic structures by Jimbo, Miwa, Smirnov et al. [127–132].

Although the situation for Heisenberg spin chains with periodic boundary conditions is rather satisfactory, the questions concerning correlation functions for more general models remains largely open. It concerns in particular integrable systems associated to higher rank algebras (the Hubbard model is one of the prominent representative), systems with non periodic boundary conditions and most of the discretised quantum field theories. While in general the solution of the Quantum Inverse Scattering Problem can be obtained in rather general terms for finite dimensional cases [74], the other ingredients are not always available : lack of a reference state preventing the use of Algebraic Bethe Ansatz, lack of determinant expressions for the scalar product of states or even worse, lack of a proper algebraic structure to handle the transfer matrix spectral problem. Hence, in some of these generalised situations, even the first step of characterising the transfer matrix spectrum along some algebraic scheme is not available yet. This is a strong motivation for developing a more general method than Algebraic Bethe Ansatz that would be applicable to these general cases. Such a method has been first proposed in the framework of the Quantum Inverse Scattering Method by Sklyanin [133, 134] (see [135] for a review). It is a quantum version of the separation of variables method of Hamilton-Jacobi in classical mechanics. The key idea of the separation of variables method for quantum integrable models having N degrees of freedom is to map the (in general) highly coupled N-variables spectral problem to a set of N one-variable (hence decoupled) soluble spectral problems. In that case the map takes the form of a change of basis to the so-called separate basis. This procedure generalises to the quantum case the well-known Hamilton-Jacobi method of classical mechanics [18]. In Sklyanin approach [133] to quantum integrable models, the separate basis is identified with the eigenstate basis of some particular operator of the Yang-Baxter algebra having simple spectrum; for lattice integrable models this amounts in general to introduce inhomogeneity parameters on each site of the lattice. In the separate basis, the transfer matrix wave functions take a simple factorised form over the spectrum of the separate variables. This simplification in the spectral problem represents thus a natural motivation to develop quantum separation of variables in the analysis of integrable quantum models. This is the point of view taken in this thesis. Further motivations to work in this framework come naturally as this separation of variables approach has proven to be an efficient tool to solve the spectral problem of a large class of integrable models [136–140], even not analysable by other methods, leading to the characterisation of both the eigenvalues and the eigenstates [141–144]. Contrary to other methods, the proof that the spectrum characterisation is complete is rather straightforward and, in this approach, no reference state is needed unlike the Algebraic Bethe Ansatz. Moreover, this approach proved already to be very efficient for the study of the dynamics of quantum integrable models, as universal determinant formula has emerged (first shown in [145]) [141, 146–152] for the scalar products of the so-called separate states and for the matrix elements of local operators on transfer matrix eigenstates. It has also been shown recently that the question of homogeneous limit (the limit in which all inhomogeneity parameters, necessary for the separation of variables to apply, are set to the same value) can be tackled by rewriting the determinants found in this approach in terms of Izergin type and then Slavnov type determinants, where homogeneous limit can be taken [147].

This thesis is willing to belong to the research effort taken in the development of the separation of variables method for integrable systems having a potential impact for physical applications in condensed matter, statistical physics or quantum field theory. We will be considering quantum one-dimensional systems associated to the 6-vertex R-matrix and to general cyclic representations of the Yang-Baxter algebra (and of the reflection algebra as we aim to consider general integrable boundary conditions). Namely, they are associated to representations of the quantum group $U_q(\hat{sl}_2)$ for q root of unity. Their study is mathematically interesting but also physically attractive since they contain, as a particular case, the chiral Potts model[3]. This model was first considered by Potts as a N-states extension of the Ising model, has then been generalised by Wu and Wang in 1976 [153] with a dependence of the interaction energy depending on the direction, and then by Ästlund and Huse [154, 155], who considered a chiral version of the two-dimensional Potts model, in which the interaction differs depending on the x or y axis of the lattice. The study of this system has been carried for the superintegrable Z_N-symmetrical chiral Potts quantum chain [156, 157], which was shown to be integrable [157] because it forms representations of the Onsager algebra. Moreover, few years later, Bazhanov and Stroganov have show the integrability of the chiral Potts model [158] within the framework of the Yang-Baxter algebra, embedding it in a 6-vertex model (see also the works of Tarasov [159–161]). This is the setting that we will be using in this thesis. The important feature with the chiral Potts model is that it is associated to high genus curves, making much more involved its resolution since the dependence on spectral parameter becomes more involved (it does not have the difference property). Despite this difficulty, Baxter tackled this problem and could express several years later, inspired by a technique from Jimbo, Miwa, and Nakayashiki [162], the order parameter of the model [163]. The question of correlation functions however is still open for this model and motivated the application of separation of variables to compute the form factors [145,151]. Eventually, let us mention that the study of these cyclic models in their generality is also motivated by the fact that the lattice sine Gordon model, in particular at its q-root of unity reduction points, can be obtained, with a special set of parameter, from the Bazhanov-Stroganov cyclic solution of Yang-Baxter algebras. One related interesting question here concerns the asymptotic behaviour of the correlation functions and the link with minimal models of Conformal Field Theory [164].

In this thesis we will be considering an additional feature for such models, namely their resolution

[3]This model can describe the melting and freezing of atomic monolayers on crystalline surfaces. This liquid layer orders itself into domains that either line up with the substrate structure or follow the layer own ordering, leading to the so-called commensurate and incommensurate phases, describing for instance the adsorbtion phenomenon.

in the presence of general boundary conditions preserving integrability for the finite chain and the construction of the corresponding integrable local Hamiltonians. The description of integrable boundaries has attracted a large research enthusiasm as they can describe both equilibrium and out of equilibrium physics. Some interesting applications concern the description of classical stochastic relaxation processes, like the asymmetric exclusion process (ASEP) [165–172] and quantum transport properties in spin systems [173, 174].

Since the paper of Gaudin [175], the coordinate Bethe ansatz [176–179] has been applied to tackle the question of boundaries, as well as a development concerning the algebraic point of view [180–185]. In the framework of the Quantum Inverse Scattering Method, Sklyanin has shown how to construct classes of quantum integrable models with non-trivial boundaries [177]. His method relies on the so-called reflection algebras, generated by the elements of a monodromy matrix satisfying the so-called reflection equations, introduced by Cherednik [178]. They have been derived in the context of the study of the scattering matrix factorisation in the presence of reflecting walls, i.e. on the interval instead of on the infinite line. This approach belongs to the "bootstrap method" that was developed to study relativistic quantum field theories, where the scattering of n particles factorises in terms of products of 2 particles scattering [186–189], related to the Yang-Baxter equation.

The pioneering work of Sklyanin on integrable models with boundaries states that when one knows a representation of the reflection algebra in the quantum space and a scalar representation of its dual, one can compute a commuting family of operators, the so-called boundary transfer matrices, and extract from them local integrable Hamiltonians with general integrable boundaries (for fundamental models). Moreover, he gives a way to use solutions of the standard Yang-Baxter equation to generate solutions of the reflection equation, once a scalar solution to the latter is known. The main point for the integrability, as well as for the computation of the Hamiltonians, is the introduction of the boundary transfer matrix, which generalises the standard transfer matrix introduced for Yang-Baxter algebras. Similarly to the Yang-Baxter case, the Hamiltonians are obtained *via* some derivatives of the boundary transfer matrix; this technique being based on the reduction of the quantum Lax operator to the permutation operator (in a specific point), it holds for fundamental models only, i.e. for models where the auxiliary and quantum spaces are isomorphic. Let us comment that the boundaries of these integrable Hamiltonians are encoded into two boundary matrices, scalar solutions of the reflection equation (and its dual). As it is the case for the Yang-Baxter algebra, the most representative model to begin to understand this class of systems with boundaries is probably the XXZ spin-1/2 quantum chain with general integrable boundary conditions. To fully characterise the spectrum of this Hamiltonian, the Algebraic Bethe Ansatz [66, 67] has been extensively used, leading for diagonal boundary matrices, to a characterisation from the spectrum [177] up to the correlation functions [190,191]. This method faces however some technical difficulties when non-diagonal boundaries are considered, like the non existence of a reference state or the completeness of the spectrum description, resulting to its characterisation only under special constraints. More specifically, the method can be applied to non-diagonal boundary matrices which satisfy certain constraints which allow the definition of reference states by a gauge transformation [192–194]. The completeness is then a complicated task to prove, involving notably two sets of Bethe ansatz equations to evidence some numerical results [195]. In fact, whereas the first steps of the Quantum Inverse Scattering Method (leading to the boundary transfer matrix) are independent on the nature of the boundaries, the effective construction of the eigenstates is more difficult, if not impossible. Numerous models were however solved in this context, for example the spin s chains [196, 197] or also higher rank models with integrable boundaries [198–201], in particular concerning the Hubbard model [167, 202–207].

Several methods were introduced to overcome the problems encountered in the description of the spectrum of quantum models with general integrable boundary conditions, examples are generalised Bethe ansatz [193, 194, 208, 209], the fusion procedure [195, 210–217] (for spin chains) or the q-Onsager algebra [218, 219], which led to the same constraint on the boundaries to be able to express the eigenvalues. The separation of variables method has been shown in the recent years to be able to overcome these difficulties and to characterise completely the spectrum of such models in the cases of XXX, XXZ and XYZ Heisenberg chains with general integrable boundaries [141, 143, 144]. Hence, for lattice integrable models associated to cyclic representations, which are at the heart of this thesis, the lack of a reference state and the willing to address the general integrable boundary conditions also require the use of the quantum separation of variables.

We will first solve the spectral problem for models associated to Bazhanov-Stroganov Lax operators with the most general integrable boundary conditions. These models are associated to cyclic representations of the 6-vertex Yang-Baxter algebra, hence there is no obvious reference state, even for the periodic boundary condition. To solve this problem we develop the separation of variables method adapted to these representations; both eigenstates and eigenvalues are characterised. Furthermore, writing the reflection equation in mixed representations, involving both spin 1/2 and cyclic ones, we are able to determine new integrable local Hamiltonians with boundaries, acting in tensor products of any cyclic representations. The current manuscript is divided into five chapters.

The first chapter briefly reminds the key classical integrable structures and presents then the Quantum Inverse Scattering Method, introducing the Yang-Baxter algebra and the transfer matrices. It explicitly focuses on the 6-vertex Yang-Baxter algebra, and presents the Algebraic Bethe Ansatz method on the example of the XXZ spin 1/2 chain. The chapter ends introducing cyclic representations of the 6-vertex Yang-Baxter algebra and thus the Lax operator at the heart of our study, the so-called Bazahnov-Stroganov Lax operator.

In the second chapter, the implementation of general integrable boundaries is addressed. On the example of the XXZ spin 1/2 chain, a twist matrix is introduced, which allows to deal with quasi-periodic boundary conditions. Then, the true generalisation due to Sklyanin [177] is considered, namely the reflection algebra and in particular the 6-vertex reflection algebra. *Via* two boundary matrices, one can compute integrable local Hamiltonians with general boundaries for fundamental models, i.e. for which the auxiliary and quantum spaces are isomorphic. Both the general construction and the explicit example of the XXZ spin 1/2 chain are given. At the end, the fusion procedure is briefly introduced in order to present a method allowing the computation of integrable local Hamiltonians (with general integrable boundaries) for some non fundamental models.

The third chapter is dedicated to the quantum separation of variables. In a first part, the classical notion of separation of variables is briefly recalled. In particular, it is shown how to construct separate variables when the integrability is characterised thanks to a two dimensional Lax matrix and a Yang-Baxter classical r-matrix. Then the quantum separation of variables method is defined. An analogy with the classical method is emphasised, particularly regarding the use of the elements of the monodromy matrix. Following [148], an explicit construction of the separate basis for the XXZ spin 1/2 antiperiodic chain is done, allowing us to bring to light the promising features of this method.

The new results brought by this thesis are exposed in chapters four and five. Chapter four focuses on the characterisation of the transfer matrix spectrum associated to cyclic representations of the 6-vertex reflection algebra, with general integrable boundary conditions. It is mainly decomposed in two parts. In the first one, we consider the system with a constraint on the right boundary, explicitly one of the associated boundary matrix is taken triangular. It allows us to present the separation of variables for this reflection algebra, and to state a characterisation of the spectrum. The other part is dedicated to the general integrable boundary conditions. To this aim, we introduce a generalisation of Baxter's gauge transformations and construct a separate basis, choosing an appropriate value for the gauge parameter. We can then state a similar characterisation of the spectrum, valid for any integrable boundaries. Moreover, an equivalent functional characterisation *via* a functional Baxter T-Q type equation is given, establishing a link with the Algebraic Bethe Ansatz technique. The chapter ends with the computation of the scalar products of so-called separate states, which is a first step toward the description of the dynamics.

The aim of the fifth and last chapter is the computation of integrable local Hamiltonians associated to the non fundamental models described by the Bazhanov-Stroganov Lax operator. Starting from the known fundamental R-matrix of the chiral Potts model [158], model which is obtained in our study as a particular case, we extend the knowledge of such a matrix beyond chiral Potts. Then, using (ABCD)-type quantum algebras [220] we define reflection equations and their duals. Considering mixed equations, namely involving representation spaces of different dimensions, spin 1/2 and arbitrary cyclic ones, we are able to define (quantum) boundary matrices and thus a fundamental transfer matrix. We also find explicit solutions for these boundaries (the one diagonalising the $\hat{\mathbf{v}}$-basis, where $\hat{\mathbf{v}}$ is one of the elements of the dynamical Weyl pair $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$). At the end of this chapter, using the commutation of these multi spectral parameter fundamental transfer matrices between themselves and with the 6-vertex transfer matrix, we can define multi parameter families of integrable local Hamiltonians, with integrable boundaries. The chapter ends with some explicit models of interest, as the XXZ spin 1 chain at root of unity, the sine Gordon model or the (superintegrable) chiral Potts model, stating a symmetry on the boundaries which has to be investigated.

We conclude this thesis by discussing several open questions that could be investigated now building on our results.

The results presented in this thesis have led to three publications, referenced thereafter thanks to Roman numbers:

Art. I. **Transfer matrix spectrum for cyclic representations of the 6-vertex reflection algebra I**
J. M. Maillet, N. Niccoli and B. Pezelier. SciPost Phys. 2, 009 (2017)

Art. II. **Transfer matrix spectrum for cyclic representations of the 6-vertex reflection algebra II**
J. M. Maillet, N. Niccoli and B. Pezelier. *arXiv:1802.08853*

Art. III. **Integrable local Hamiltonians associated to cyclic representations of the 6-vertex reflection algebra,** J. M. Maillet, N. Niccoli and B. Pezelier. *To appear*

Eventually, in the main text we sometimes refer to particular formulas of these articles, specifying the number of the article and then the number of the formula, e.g. (**I**, 5.41).

Chapter 1

Quantum integrable models

As we briefly described in the Introduction, for finite number of degrees of freedom, there is a rigorous definition of the notion of integrability in classical mechanics due to Liouville. However, already for classical continuum models, there is no universal analogue of the Liouville theorem as the notion of "an infinite number of independent conserved charges" has to be examined case by case carefully. The situation for quantum models is not better, even for a finite number of degrees of freedom, as there is no analogue of Liouville theorem either. Nevertheless, since the pioneering work of Bethe [44], a lot has been achieved to design powerful methods to solve the spectrum and even the dynamics of many quantum models of interest. The most successful framework in this respect is provided by the so-called Quantum Inverse Scattering Method (QISM). It can be considered as a quantised version of the Classical Inverse Scattering Method, that helps solving classical continuum theories with soliton solutions, while providing also an algebraic version of the Bethe ansatz. It also inherits the strategy developed in 2D solvable lattice models using the notion of commuting transfer matrices. In fact, as we will describe in Chapter 3, it also contains the necessary ingredients for defining a quantum analogue of the Separation of Variables (SoV) method, that, in some sense, could be considered as a practical definition of quantum integrability.

As in the classical case, the first question to be able to diagonalise a quantum Hamiltonian is to find a complete set of commuting conserved charges, having common simple spectrum, hence leading to a complete description of eigenstates using their corresponding quantum numbers. Of course this is a priori not enough as one has still to construct the corresponding eigenvectors and compute their energy levels. In fact one would also be interested in computing dynamical correlation functions. The QISM provides the algebraic framework to tackle such problems for a large class of systems, mainly defined on 1D lattices. We will present this method in the particular example of the 6-vertex Yang-Baxter algebra, which can describe a large class of models of physical interest, such as Heisenberg spin chains, the lattice sine-Gordon model or the chiral Potts model. Then we will focus on the Algebraic Bethe Ansatz on the example of the XXZ spin-1/2 chain to give a concrete example. A short section of the chapter is then dedicated to the computation of the form factors of local operators and correlation functions of that model. Then we will define the class of models under study in this thesis, namely, associated to the cyclic representations of the 6-vertex Yang-Baxter algebra. Before doing that, let us very briefly describe the corresponding classical settings in a way that will make clear the classical-quantum correspondence.

1.1 Classical integrable structures

The modern starting point in classical integrable models is the existence of an auxiliary linear system (Lax pair) the compatibility condition of which (Lax equation or zero-curvature equation) is equivalent to the initial equations of motion. The original Lax equation takes the form [26],

$$\frac{dL}{dt} = [M, L] \tag{1.1}$$

where L and M are size n matrices, their entries being functions of the dynamical variables of the system at hand. The very nice feature of this equation is that one can immediately extract conserved quantities under time evolution from the invariants of the L-matrix. Furthermore, the fact that these invariants, as functions on the phase space of the system, are in involution under some Poisson brackets structure is equivalent to the existence of an r-matrix [221, 222] such that [223]:

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2] \tag{1.2}$$

where we have introduced standard tensor product notations, with an n-dimensional vector space V such that $L_1 \equiv L(\lambda) \otimes \mathbf{1}$, $L_2 \equiv \mathbf{1} \otimes L(\mu)$ and $r_{12} \equiv r_{12}(\lambda, \mu)$ are endomorphisms of $V \otimes V$ and λ and μ stands for the spectral parameters (they are in many cases just free complex numbers but could eventually be more sophisticated objects). In the above equation r_{21} stands for the same object as r_{12} in which the two vector spaces copies in the tensor product $V \otimes V$ have been exchanged together with the corresponding spectral parameters. While in general the matrix r_{12} is a function on the phase space, it so happens that in many classical integrable systems it is just a constant matrix function only of the spectral parameters, moreover having the following antisymmetry property $r_{21} = -r_{12}$. The Jacobi identity of the Poisson brackets then turns into the following (sufficient) [1] quadratic relations for the r-matrix known as the classical Yang-Baxter equation [221, 224]:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{1.3}$$

as in fact the quantum structure, i.e. the Yang-Baxter equation, was discovered before the classical one's, at least in these settings. The above structures are intimately connected to Lie algebras and to the algebraic resolution of the corresponding integrable models. It was also realised soon after the KdV being solved by means of the Inverse Scattering Problem that (1.1) admits a nice extension to continuum models in two-dimensions due to Zakharov and Shabat [27, 28] (we denote by x and t the space and time variables) in the form of a zero-curvature equation (here L and M are generically depending on x, t and λ):

$$[L - \partial_x, M - \partial_t] = 0 \tag{1.4}$$

associated to the linear system :

$$\begin{aligned} (L - \partial_x)\Psi(x,t) &= 0 \\ (M - \partial_t)\Psi(x,t) &= 0 \end{aligned} \tag{1.5}$$

Originally designed by Gardner,Greene, Kruskal and Miura [25] to solve the Korteweg-de-Vries equation [21], the Classical Inverse Scattering Method consists in constructing the so-called scattering data, which turns out to contain the analogue of the action-angle variables for the continuum model. Those are given by the monodromy matrix $T(x, y; t)$ of this linear system. It is defined at fixed time t to transport the solution Ψ from the point y to the point x as $\Psi(x,t) = T(x, y; t)\Psi(y, t)$ and reduces to the identity if $x = y$. It contains, e.g. for periodic boundary conditions on the interval $[a, b]$, i.e. such that $M(a, t) = M(b, t)$ for any t, conserved quantities under time evolution given by its trace $tr[T(b, a; t)]$. If one goes to the real line instead of the interval, the off-diagonal elements of T lead to the determination of angle type variables first achieved in these settings by Faddeev and Zakharov [34] for KdV. It needs to compute the Poisson brackets structure of the matrix elements of T. They are given in terms of the linear Poisson bracket for the L's which for a large class of models reads [225]:

$$\{L_1(x, t; \lambda), L_2(y, t; \mu)\} = \delta(x - y)[r_{12}(\lambda, \mu), L_1(x, t; \lambda) + L_2(y, t; \mu)] \tag{1.6}$$

leading to the quadratic Poisson bracket for T [225]:

$$\{T_1(b, a; t; \lambda), T_2(b, a; t; \mu)\} = [r_{12}(\lambda, \mu), T_1(b, a; t; \lambda)T_2(b, a; t; \mu)] \tag{1.7}$$

[1] In principle one can add a left hand side term to this equation that commutes with $L_1 + L_2 + L_3$, giving rise to the modified Yang-Baxter equation.

from which the involution of the conserved quantities follows. If there exists sufficiently many independent conserved quantities, then there exists a canonical transformation mapping the original variables to a new set of coordinates, the so-called action-angle variables. The latter are defined such that their evolution in time is linear, so easily computable. Thereafter, thanks to the inverse transformation (this is the Inverse Scattering Problem part), one can deduce the time evolved original variables (see e.g. the books [18,226]), hence solving the problem. Let us finally remark that the above structures admit an interesting discretisation preserving the integrability properties. The discrete analogue of the Lax matrix L, namely $L(n,t;\lambda)$ depends now on the dynamical variables at site n of a lattice having N sites. In fact it is useful to see such an $L(n,t;\lambda)$ as the monodromy matrix of the continuum theory $T(x_n, x_n-\delta;t;\lambda)$, with δ the lattice spacing. As a consequence, the proper Poisson brackets becomes quadratic also for the L's at a given site n while two L's on different sites have vanishing Poisson brackets. We get :

$$\{L_1(n,t;\lambda), L_2(n,t;\mu)\} = [r_{12}(\lambda,\mu), L_1(n,t;\lambda)L_2(n,t;\mu)] \tag{1.8}$$

The full monodromy matrix is then just the ordered product all along the chain of such discrete Lax matrices :

$$T(N,1;t;\lambda) = L(N,t;\lambda)\dots L(1,t;\lambda) \tag{1.9}$$

and satisfies the Poisson brackets :

$$\{T_1(N,1;t;\lambda), T_2(N,1,t;\mu)\} = [r_{12}(\lambda,\mu), T_1(N,1;t;\lambda)T_2(N,1;t;\mu)] \tag{1.10}$$

As we will see now, the structure for quantum integrable lattice models is very closely related to the classical one.

1.2 Quantum Inverse Scattering Method

The QISM discovery originates from the study of the non-linear Schrödinger model that on the one hand was solvable through the Classical Inverse Scattering Method and on the other hand admits a quantisation (the 1D quantum Bose gas) solvable by coordinate Bethe ansatz. Adapting transfer matrix techniques from Baxter's corner transfer matrix [56,57], Sklyanin was able to understand how to quantise the classical inverse scattering scheme. The main feature of QISM is to embed the quantum Hamiltonian at hand into an algebra of operators (the entries of the quantised version of the monodromy matrix) that on the one hand provides a large abelian sub-algebra of conserved charges (containing the Hamiltonian) generated by the transfer matrix (the trace of the monodromy matrix), while on the other hand it gives access through the off diagonal elements of the same monodromy matrix to creation and annihilation operators of the common eigenstates of the transfer matrix (and hence of the Hamiltonian). The whole scheme is made possible thanks to the knowledge of the quadratic Yang-Baxter algebra of the monodromy matrices entries governed by an R-matrix solving the Yang-Baxter cubic equation that appeared previously in 2D solvable models [50,52–54,227].

1.2.1 Yang-Baxter algebra and transfer matrix

Let us consider a quantum system on a one dimensional lattice having N sites. To each site n of the chain, we associate a local Hilbert space of states $\mathcal{H}_n$ and an algebra of local operators acting on $\mathcal{H}_n$. The total Hilbert space $\mathcal{H}$ of the chain is the tensor product of the local spaces:

$$\mathcal{H} = \bigotimes_{n=1}^{N} \mathcal{H}_n \tag{1.11}$$

The quantised version of the Lax matrix is the so-called quantum Lax operator. It is a $m \times m$ matrix, its entries being operators on the local Hilbert space $\mathcal{H}_n$. We can introduce the auxiliary space V_a of dimension m, such that the quantum Lax operator is an element of $End(V_a \otimes \mathcal{H}_n)$ denoted as $L_{an}(\lambda, \xi_n)$, with λ standing generically for the spectral parameter and ξ_n for the inhomogeneity parameter[2] associated to the site n. In that way, the entries of two Lax operators defined in two different sites commute. At the same site they obey quadratic commutation relations with structure constants given by an R-matrix:

$$R_{ab}(\lambda,\mu)L_{an}(\lambda,\xi_n)L_{bn}(\mu,\xi_n)=L_{bn}(\mu,\xi_n)L_{an}(\lambda,\xi_n)R_{ab}(\lambda,\mu) \tag{1.12}$$

while the R-matrix satisfies the Yang-Baxter cubic equation:

$$R_{ab}(\lambda,\mu)R_{ac}(\lambda,\nu)R_{bc}(\mu,\nu)=R_{bc}(\mu,\nu)R_{ac}(\lambda,\nu)R_{ab}(\lambda,\mu) \tag{1.13}$$

These equations hold respectively in the spaces $V_a\otimes V_b\otimes\mathcal{H}_n$ and $V_a\otimes V_b\otimes V_c$ and we use the standard tensor product notation with lower indices to specify in which spaces the various objects act. For example, the notation R_{ab} means that the matrix R acts non trivially only on the space $V_a\otimes V_b$ while it acts as the identity in all other spaces. It means in particular that it is a pure number matrix (it acts on $\mathcal{H}_n$ as the identity).

We now define the quantum monodromy matrix of the chain, denoted $M_{aQ}(\lambda)$, as the ordered product of the Lax operators all along the chain. It is an $m\times m$ matrix with entries that are quantum operators acting on the space $\mathcal{H}$ symbolised by the subindex Q, i.e. it is an element of $End(V_a\otimes\mathcal{H})$:

$$M_{aQ}(\lambda;\xi_1,\dots\xi_N)=L_{aN}(\lambda,\xi_N)...L_{a1}(\lambda,\xi_1) \tag{1.14}$$

For simplicity, in the following we will omit writing explicitly the dependency of M_{aQ} on the inhomogeneity parameters ξ_j, namely $M_{aQ}(\lambda;\xi_1,\dots\xi_N)\equiv M_{aQ}(\lambda)$. One can show that the elements of the monodromy matrix satisfy quadratic commutation relations straightforwardly coming from (1.12) and named Yang-Baxter algebra:

$$R_{ab}(\lambda,\mu)M_{aQ}(\lambda)M_{bQ}(\mu)=M_{bQ}(\mu)M_{aQ}(\lambda)R_{ab}(\lambda,\mu) \tag{1.15}$$

It is the associative algebra generated by the elements of the monodromy matrix thanks to the Yang-Baxter equation for the R-matrix (1.13). The above relations degenerate in the classical limit in which $R=\mathbf{1}+i\hbar r+o(\hbar)$ to the above classical Poisson bracket relations.

A very important object at the heart of the Inverse Scattering Method is the transfer matrix $\mathcal{T}(\lambda)$, which is defined as the trace over the auxiliary space of the monodromy matrix:

$$\mathcal{T}(\lambda)=tr_a\left\{M_{aQ}(\lambda)\right\} \tag{1.16}$$

Thus $\mathcal{T}(\lambda)$ is an operator acting on the total Hilbert space $\mathcal{H}$ of the chain. As we shall see in particular examples, the Hamiltonian can be computed through trace identities in terms of the transfer matrix at the homogeneous limit. For an invertible R-matrix, the equation (1.15) gives, taking the trace over the two auxiliary spaces V_a and V_b:

$$\forall(\lambda,\mu)\in\mathbb{C}^2,\quad[\mathcal{T}(\lambda),\mathcal{T}(\mu)]=0 \tag{1.17}$$

Hence the transfer matrix generates via an expansion in the spectral parameter λ a full series of commuting operators that also commute with the Hamiltonian. Therefore it leads to a full series of commuting conserved operators generating the symmetries responsible for the integrability properties of the system.

The main question is then to find a common basis of eigenvectors for the transfer matrix $\mathcal{T}(\lambda)$, namely a basis independent of the spectral parameter λ, and to compute the corresponding eigenvalues of $\mathcal{T}(\lambda)$, leading to the spectrum of the Hamiltonian. To effectively construct the eigenvectors and eigenvalues, one has to go one step further. The two main approaches we will consider in this thesis are the Algebraic Bethe Ansatz (ABA), developed in paragraph 1.4 and the quantum separation of variables, developed in Chapter 3. These two methods make use the above Yang-Baxter algebra. The main difference between these two methods is, as we will see later on, that the quantum separation of variables, for all the studied models for which this technique can be applied, has so far led to the full characterisation (and proof of the sim-

[2] Notice that in general these parameters could be sets of complex parameters and not only complex parameters. See for instance the chiral Potts Lax operator (1.72).

plicity) of spectrum of the transfer matrix, while such a completeness is in general out of the reach of ABA.

A quantum integrable chain is thus completely characterised by the data of an R-matrix solution of the Yang-Baxter equation and a Lax operator giving a representation in $\mathcal{H}_n$ of the Yang-Baxter algebra associated to R. In general, starting from a given system, there is no standard procedure to show its integrability finding such an R-matrix and a corresponding Lax operator. However, there are quite general techniques to construct R-matrices and Lax operators satisfying (1.12) from representation theory of quantum groups, and a considerable literature has been established classifying the solutions and the physical models which can be described [101, 228]. The equation (1.12) appears this way as a pivotal relation: it is the starting point to find Lax operators, i.e. integrable models, and it also provides the key to their solving.

1.3 The 6-vertex Yang-Baxter algebra

In this section, we focus on the 6-vertex R-matrix, solution of the Yang-Baxter equation (1.13), and on its associated algebra, the 6-vertex Yang-Baxter algebra. Very important models are described thanks to this matrix: the XXX and XXZ spin 1/2 chains, the discretised sine-Gordon model or the chiral Potts model. The main model studied in this thesis, namely the τ_2 model (with general integrable boundary conditions), is also based on the 6-vertex R-matrix.

This matrix is a solution of the Yang-Baxter equation (1.13) with an auxiliary space of dimension 2, it is explicitly written:

$$R(\lambda,\mu) = \begin{pmatrix} a(\lambda,\mu) & 0 & 0 & 0 \\ 0 & b(\lambda,\mu) & c & 0 \\ 0 & c & b(\lambda,\mu) & 0 \\ 0 & 0 & 0 & a(\lambda,\mu) \end{pmatrix} \tag{1.18}$$

$$\text{with} \quad a(\lambda,\mu) \equiv \frac{\lambda q}{\mu} - \frac{\mu}{\lambda q} \quad , \quad b(\lambda,\mu) \equiv \frac{\lambda}{\mu} - \frac{\mu}{\lambda} \quad \text{and} \quad c \equiv q - 1/q \tag{1.19}$$

The parameters λ and μ are the spectral parameters, while q is a free parameter. One of the properties of this matrix is its dependence on the ratio of the spectral parameters. In the next, the 6-vertex R-matrix will be often denoted $R(\lambda)$, keeping the same notation R:

$$R(\lambda,\mu) \equiv R(\lambda/\mu) \tag{1.20}$$

It is standard to denote such a monodromy matrix, with an auxiliary space V_0 of dimension 2, by:

$$M_{0Q}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{1.21}$$

Note that according to (1.14) the operators $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ are not only depending on the spectral parameter λ but also on the N inhomogeneity parameters ξ_j. Then, the 6-vertex Yang-Baxter algebra is generated by $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ satisfying the relations:

$$[A(\lambda), A(\mu)] = 0 \quad [B(\lambda), B(\mu)] = 0 \tag{1.22}$$
$$[D(\lambda), D(\mu)] = 0 \quad [C(\lambda), C(\mu)] = 0 \tag{1.23}$$
$$A(\lambda)B(\mu) = \tilde{c}(\lambda,\mu)A(\mu)B(\lambda) + \tilde{b}(\lambda,\mu)B(\mu)A(\lambda) \tag{1.24}$$
$$B(\lambda)A(\mu) = \tilde{b}(\lambda,\mu)A(\mu)B(\lambda) + \tilde{c}(\lambda,\mu)B(\mu)A(\lambda) \tag{1.25}$$
$$A(\mu)C(\lambda) = \tilde{c}(\lambda,\mu)A(\lambda)C(\mu) + \tilde{b}(\lambda,\mu)C(\lambda)A(\mu) \tag{1.26}$$
$$C(\mu)A(\lambda) = \tilde{b}(\lambda,\mu)A(\lambda)C(\mu) + \tilde{c}(\lambda,\mu)C(\lambda)A(\mu) \tag{1.27}$$
$$D(\mu)B(\lambda) = \tilde{b}(\lambda,\mu)B(\lambda)D(\mu) + \tilde{c}(\lambda,\mu)D(\lambda)B(\mu) \tag{1.28}$$
$$B(\mu)D(\lambda) = \tilde{c}(\lambda,\mu)B(\lambda)D(\mu) + \tilde{b}(\lambda,\mu)D(\lambda)B(\mu) \tag{1.29}$$

$$C(\lambda)D(\mu) = \tilde{c}(\lambda,\mu)C(\mu)D(\lambda) + \tilde{b}(\lambda,\mu)D(\mu)C(\lambda) \tag{1.30}$$
$$D(\lambda)C(\mu) = \tilde{b}(\lambda,\mu)C(\mu)D(\lambda) + \tilde{c}(\lambda,\mu)D(\mu)C(\lambda) \tag{1.31}$$
$$[B(\lambda), C(\mu)] = -c/b(\lambda,\mu)\,(D(\lambda)A(\mu) - D(\mu)A(\lambda)) \tag{1.32}$$
$$[C(\lambda), B(\mu)] = -c/b(\lambda,\mu)\,(A(\lambda)D(\mu) - A(\mu)D(\lambda)) \tag{1.33}$$
$$[D(\lambda), A(\mu)] = -c/b(\lambda,\mu)\,(B(\lambda)C(\mu) - B(\mu)C(\lambda)) \tag{1.34}$$
$$[A(\lambda), D(\mu)] = -c/b(\lambda,\mu)\,(C(\lambda)B(\mu) - C(\mu)B(\lambda)) \tag{1.35}$$

where we introduced the structure constants:

$$\tilde{b}(\lambda,\mu) \equiv b(\lambda,\mu)/a(\lambda,\mu) \quad \text{and} \quad \tilde{c}(\lambda,\mu) \equiv c/a(\lambda,\mu) \tag{1.36}$$

In particular, let us comment that the four families $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ are each commuting families of operators. Let us also notice here that the inhomogeneity parameters ξ_j being attached to the lattice sites, they are the same in $A(\lambda)$ and in $A(\mu)$.

Quantum determinant Let us complete the description of the 6-vertex Yang-Baxter algebra by introducing the quantum determinant $q\text{-}det\ M_{aQ}(\lambda)$. It is a central element of the algebra:

$$\forall(\lambda,\mu) \in \mathbb{C}^2\ , \quad [q\text{-}det\ M_{aQ}(\lambda), M_{bQ}(\mu)] = 0 \tag{1.37}$$

which can be defined for any representation (not necessarily associated to the monodromy matrix, but also for the Lax matrices, representations of the Yang-Baxter algebra in the local Hilbert space $\mathcal{H}_n$). Due to the particular form of the 6-vertex R-matrix, it can be written in terms of the generators as:

$$q\text{-}det\ M_{0Q}(\lambda) = A(\lambda)D(\lambda/q) - B(\lambda)C(\lambda/q) \tag{1.38}$$

Let us mention that in order to find quantum determinants, Kulish and Sklyanin [229] make use of the degeneracy of the R-matrix (1.20) at the points $\lambda = 1/q$. This one reduces indeed to the projector P^-:

$$R(1/q) = -2(q - 1/q)\ P^- \quad \text{with} \quad P^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.39}$$

Then the 6-vertex R-matrix satisfies moreover the identity (P^- is rank 1):

$$R_{12}(1/q)R_{13}(\lambda/q)R_{23}(\lambda) = c(\lambda)R_{12}(1/q) \tag{1.40}$$

with $c(\lambda)$ a scalar function, leading to the centrality of the element

$$q\text{-}det\ M_{0Q}(\lambda) \equiv tr_{00'}\left\{P^-_{00'}M_{0Q}(\lambda/q)M_{0'Q}(\lambda)\right\} \tag{1.41}$$

The scalar $c(\lambda)$ can thus be understood as the quantum determinant of the R-matrix itself. What is more, using the Yang-Baxter equation and the commutation relations (1.33) and (1.32), one can easily check the different expressions:

$$q\text{-}det\ M_{0Q}(\lambda) \equiv D(\lambda)A(\lambda/q) - C(\lambda)B(\lambda/q) \tag{1.42}$$
$$q\text{-}det\ M_{0Q}(\lambda) \equiv D(\lambda/q)A(\lambda) - B(\lambda/q)C(\lambda) \tag{1.43}$$
$$q\text{-}det\ M_{0Q}(\lambda) \equiv A(\lambda/q)D(\lambda) - C(\lambda/q)B(\lambda) \tag{1.44}$$

Lastly, let us mention that the quantum determinant of the monodromy matrix factorises in the product of the quantum determinant of the Lax operators, i.e. it holds:

$$q\text{-}det\ M_{0Q}(\lambda) = q\text{-}det\ L_{0N}(\lambda)\ ...\ q\text{-}det\ L_{01}(\lambda) \tag{1.45}$$

Inversion formula Among other properties that will be used later, the quantum determinant is useful to express the inverse of the monodromy matrix. Indeed, it holds the following inversion formula:

$$\sigma_0^y M_{0Q}^{t_0}(\lambda/q)\sigma_0^y M_{0Q}(\lambda) = M_{0Q}(\lambda)\sigma_0^y M_{0Q}^{t_0}(\lambda/q)\sigma_0^y = q\text{-}det\ M_{0Q}(\lambda) \tag{1.46}$$

where t_0 denotes the transposition in the auxiliary space V_0 and σ^y is the Pauli matrix.

The use of different representations of the 6-vertex Yang-Baxter algebra leads to different quantum systems. Let us just cite for example the XXZ spin-s chains, the discretised sine-Gordon model or the chiral Potts model for cyclic representations (the basic operators on $\mathcal{H}_n$ are the pairs $(\hat{\mathbf{u}}_n, \hat{\mathbf{v}}_n)$; they satisfy a Weyl algebra $\hat{\mathbf{u}}_n\hat{\mathbf{v}}_n = q\hat{\mathbf{v}}_n\hat{\mathbf{u}}_n$ with q is a root of unity. $\hat{\mathbf{u}}_n$ and $\hat{\mathbf{v}}_n$ are exponentials of conjugated variables, i.e. a field ϕ and its conjugate momentum π). We will focus in this thesis on the τ_2 model and its generalisations, associated to cyclic representations.

1.4 Spectrum characterisation by the Algebraic Bethe Ansatz: example of the XXZ spin 1/2 chain

As mentioned previously, the integrability can be characterised in the framework of Quantum Inverse Scattering Method. The next step is thus to solve the system, i.e. to characterise the spectrum. Several techniques have been introduced, as Baxter Q-operators, Algebraic Bethe Ansatz or quantum separation of variables to achieve this goal. We present here the Algebraic Bethe Ansatz, whereas the chapter 3 is dedicated to the quantum separation of variables.

The Algebraic Bethe Ansatz has been developed in the late 70's. The idea is to use the Yang-Baxter algebra commutation relations and the existence of a so-called reference state $|0\rangle$, which is an eigenstate of the operators $A(\lambda)$ and $D(\lambda)$ and such that the action of $C(\lambda)$ gives 0. Then the action of the operator $B(\lambda)$ on $|0\rangle$ allows to construct eigenvectors of the transfer matrix. This method led to the spectrum of a large class of models, here we present it on the example of the XXZ spin 1/2 chain.

1.4.1 The XXZ 1/2 Heisenberg spin chain

The XXZ 1/2 Heisenberg spin chain is composed of N sites, to each one is associated a local Hilbert space of dimension 2. This model is characterised by the 6-vertex R-matrix, and the following Lax operators:

$$L_{0n}(\lambda) = \begin{pmatrix} x_+(\lambda) + x_-(\lambda)\sigma_n^z & (q-1/q)\sigma_n^- \\ (q-1/q)\sigma_n^+ & x_-(\lambda) + x_+(\lambda)\sigma_n^z \end{pmatrix} \tag{1.47}$$

$$\text{with the scalar function} \quad x_\pm(\lambda) = \frac{1}{2}\left(\lambda\ q - 1/(\lambda\ q) \pm (\lambda - 1/\lambda)\right) \tag{1.48}$$

Let us comment that this Lax operator satisfies (1.12) as it is just proportional to the R-matrix itself in $V_0 \otimes \mathcal{H}_n$, and then this relation is simply the Yang-Baxter equation (1.13). We introduce thereafter the monodromy matrix, with the inhomogeneities $(\xi_n)_{1\leqslant n\leqslant N}$:

$$M_{0Q}(\lambda) = L_{0N}(\lambda/\xi_N)...L_{01}(\lambda/\xi_1) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{1.49}$$

According to the standard procedure described in the first section, the transfer matrix is given by:

$$\mathcal{T}(\lambda) = tr_0\{M_{0Q}(\lambda)\} = A(\lambda) + D(\lambda) \tag{1.50}$$

Eventually, at the homogeneous limit (for which it holds $\xi_i = 1$ for all the inhomogeneities), one can show that:

$$H = (q - q^{-1}) \left. \frac{d\, ln\,(\mathcal{T}(\lambda))}{d\lambda} \right|_{\lambda=1} - N \frac{q + q^{-1}}{2} \tag{1.51}$$

is indeed the Hamiltonian of the model under consideration, namely:

$$H = \sum_{n=1}^{N} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \frac{q + q^{-1}}{2}\ \sigma_n^z \sigma_{n+1}^z \right) \tag{1.52}$$

with periodic boundary conditions:

$$\sigma_{N+1}^s = \sigma_1^s \quad \text{for } s = x, y \text{ or } z \tag{1.53}$$

1.4.2 The Algebraic Bethe Ansatz

To construct eigenvectors of the transfer matrix $\mathcal{T}(\lambda)$, the Algebraic Bethe Ansatz needs a *reference state* $|0\rangle$ satisfying:

$$A(\lambda)\,|0\rangle = \text{A}(\lambda)\,|0\rangle\ , \qquad D(\lambda)\,|0\rangle = \text{D}(\lambda)\,|0\rangle \tag{1.54}$$

$$C(\lambda)\,|0\rangle = 0\ , \qquad B(\lambda)\,|0\rangle \neq 0 \tag{1.55}$$

For the XXZ spin 1/2 chain, the ferromagnetic state with all spins up is a reference state:

$$|0\rangle = \bigotimes_{n=1}^{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{[n]} \tag{1.56}$$

with the eigenvalues

$$\text{A}(\lambda) = \prod_{n=1}^{N} \left(\lambda\ q/\xi_n - \xi_n/(\lambda\ q)\right) \quad \text{and} \quad \text{D}(\lambda) = \prod_{n=1}^{N} \left(\lambda/\xi_n - \xi_n/\lambda\right) \tag{1.57}$$

Then, starting from the reference state, eigenvectors of the transfer matrix can be computed using the operators $B(\lambda)$ as creator operators. Explicitly, let us define the following "n-states" vectors:

$$|\lambda_1, ..., \lambda_n\rangle \equiv B(\lambda_1)...B(\lambda_n)\,|0\rangle \tag{1.58}$$

where the $(\lambda_i)_{1 \leqslant i \leqslant n}$ is a family of complex number.

The method consists in searching the eigenvectors of $\mathcal{T}(\lambda)$ among these n-states vectors. Let us remark that from the 6-vertex Yang-Baxter algebra, $B(\lambda)$ is a commuting family of operators, so the order of the $(\lambda_i)_{1 \leqslant i \leqslant n}$ in the n-states vectors is not important.

Then, thanks to the 6-vertex Yang-Baxter commutation relations, one can show the following:

$$A(\lambda)\,|\lambda_1, ..., \lambda_n\rangle\ = \Lambda\,|\lambda_1, ..., \lambda_n\rangle + \sum_{k=1}^{n} \Lambda_k B(\mu) \left|\lambda_1, ..., \widehat{\lambda_k}, ..., \lambda_n\right\rangle \tag{1.59}$$

$$D(\lambda)\,|\lambda_1, ..., \lambda_n\rangle\ = \tilde{\Lambda}\,|\lambda_1, ..., \lambda_n\rangle + \sum_{k=1}^{n} \tilde{\Lambda}_k B(\mu) \left|\lambda_1, ..., \widehat{\lambda_k}, ..., \lambda_n\right\rangle \tag{1.60}$$

where the notation $\left|\lambda_1, ..., \widehat{\lambda_k}, ..., \lambda_n\right\rangle$ means that the term $B(\lambda_k)$ is missing in (1.58). Moreover, the different scalar functions have the expressions:

$$\Lambda = \text{A}(\mu) \prod_{j=1}^{n} b^{-1}(\lambda_j, \mu) \ , \quad \Lambda_k = \text{A}(\lambda_k) \frac{c}{b(\mu, \lambda_k)} \prod_{\substack{j=1 \\ j \neq k}}^{n} b^{-1}(\lambda_j, \lambda_k) \tag{1.61}$$

$$\tilde{\Lambda} = \text{D}(\mu) \prod_{j=1}^{n} b^{-1}(\mu, \lambda_j) \ , \quad \tilde{\Lambda}_k = \text{D}(\lambda_k) \frac{c}{b(\lambda_k, \mu)} \prod_{\substack{j=1 \\ j \neq k}}^{n} b^{-1}(\lambda_k, \lambda_j) \tag{1.62}$$

This way, a sufficient condition for $|\lambda_1, ..., \lambda_n\rangle$ to be an eigenstate of the transfer matrix (1.50) is:

$$\forall k \in [\![1, n]\!] \ , \quad \Lambda_k + \tilde{\Lambda}_k = 0 \tag{1.63}$$

Due to the skew symmetry property $b(\lambda_k, \mu) = -b(\mu, \lambda_k)$, this condition is independent of λ, and can be put under the form:

$$\forall k \in [\![1, n]\!] \ , \quad \frac{\text{A}(\lambda_k)}{\text{D}(\lambda_k)} = \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{b(\lambda_j, \lambda_k)}{b(\lambda_k, \lambda_j)} \tag{1.64}$$

It means that if such a condition is satisfied, the corresponding n-state vector $|\lambda_1, ..., \lambda_n\rangle$ is a common eigenvector of $\mathcal{T}(\lambda)$, $\forall \lambda \in \mathbb{C}$ (provided of course it is not zero). These are the famous *Bethe equations*, first obtained by Bethe [44] for the XXX-1/2 spin chain (using coordinate ansatz), obtained from XXZ in the limit $q \to 1$, while rescaling all objects. Similar forms for this system of equations occur for other integrable models.

One can see that, up to the existence of the reference state $|0\rangle$, the system (1.64) is valid for all the models described thanks to the 6-vertex R-matrix. This method has been generalised to other R-matrix.

Eventually, let us mention that to construct eigenvectors on the left, one can use the same procedure as the one describe above but using the following "n-states", constructed thanks to the C operators:

$$\langle \lambda_1, ..., \lambda_n| \equiv \langle 0| \, C(\lambda_1)...C(\lambda_n) \tag{1.65}$$

The reference state $\langle 0|$ satisfies:

$$\langle 0| \, A(\lambda) = \text{A}(\lambda) \, |0\rangle \ , \quad \langle 0| \, D(\lambda) = \text{D}(\lambda) \tag{1.66}$$

$$\langle 0| \, C(\lambda) \neq 0 \ , \quad \langle 0| \, B(\lambda) = 0 \tag{1.67}$$

Limitations and advantages The ABA method is quite a powerful tool. However it suffers from two main limitations, the first one being notably the need of a reference state, that can be problematical. Indeed, for some models like the antiperiodic XXZ spin-1/2 chain, there is no obvious reference state. And then, even if a reference state has been identified, it doesn't ensure the completeness of the description of the eigenvectors. Moreover, not all solutions of Bethe equations provide a non zero state $|\lambda_1, ..., \lambda_n\rangle$.

Let us simply mention here that the Bethe equations have been extensively studied. For example, there is the possibility to describe the ground state and the excited states in the thermodynamic limit (for which $N \to +\infty$) in terms of linear integral equations. [47, 48]

1.5 Form factors and correlation functions

To solve completely a quantum integrable model it is not sufficient to get its spectrum. We also need to compute its dynamical correlation functions. Moreover, if we would like to consider perturbations around such an integrable model, we would need the matrix elements of the perturbation potential in the basis of eigenvectors of the transfer matrix. These quantities are at the heart of statistical physics, as in principle any physical and experimentally observable can be expressed through these functions.

An n-point correlation function is given by the following normalised trace:

$$\frac{tr\left\{\phi_1...\phi_n e^{-\beta H}\right\}}{tr\left\{e^{-\beta H}\right\}} \tag{1.68}$$

where H is the Hamiltonian of the system, $\beta = (k_B T)^{-1}$ with T the absolute temperature and k_B the Boltzmann constant, and the ϕ_i are a set of local operators. In the zero temperature limit, if the ground state is non degenerated and denoted $|\Psi_g\rangle$, the trace reduces to the mean value of the local operators on this ground state :

$$\frac{\langle\Psi_g|\,\phi_1...\phi_n\,|\Psi_g\rangle}{\langle\Psi_g|\Psi_g\rangle} \tag{1.69}$$

One way to compute such a quantity is to introduce the decomposition of the identity in terms of the full set of eigenvectors $\langle t|$ and $|t\rangle$ of the Hamiltonian:

$$\mathbb{1} = \sum_{t\in Sp(H)} \frac{|t\rangle\langle t|}{\langle t|t\rangle} \tag{1.70}$$

leading to the expression:

$$\frac{\langle\Psi_g|\,\phi_1...\phi_n\,|\Psi_g\rangle}{\langle\Psi_g|\Psi_g\rangle} = \sum_{t_1\in Sp(H)}\sum_{t_2\in Sp(H)}...\sum_{t_{n-1}\in Sp(H)} \frac{\langle\Psi_g|\,\phi_1\,|t_1\rangle\langle t_1|\,\phi_2\,|t_2\rangle...\langle t_{n-1}|\,\phi_n\,|\Psi_g\rangle}{\langle\Psi_g|\Psi_g\rangle\langle t_1|t_1\rangle...\langle t_{n-1}|t_{n-1}\rangle} \tag{1.71}$$

To compute such a quantity we need to solve the following problems :

- Determine the eigenstates $|t\rangle$ of the Hamiltonian using Algebraic Bethe Ansatz and solving the Bethe equations.

- Then determine the action, say to the right, of the local operators ϕ_j on the right Bethe states. This can be achieved by first solving the so-called Quantum Inverse Scattering Problem [74], namely by reconstructing any local operator ϕ_j in terms of the elements of the monodromy matrix ; then using the Yang-Baxter algebra, one can compute the action of any element of the monodromy matrix on any Bethe state. It gives again a state that is a combination of states of the form (1.58), namely the action of a product of B operators on the reference state, but which are no longer eigenstates of the transfer matrix. Those are called off shell Bethe states.

- Then we need to compute the resulting scalar products between an arbitrary left Bethe eigenstate with an off-shell right state. In the framework of the Algebraic Bethe Ansatz, determinant formulas have been obtained for such scalar products [73, 230]. For the square norm $\langle t|t\rangle$ of an eigenstate, this is given by the Gaudin determinant while for the scalar products between an eigenstate and a general state, it is the ratio of two determinants; we need both in (1.71).

- Then one would have to compute the resulting sums at least in the thermodynamic limit and in the asymptotic regime where the local operators ϕ_j are located in points n_j of the chain such that $|n_i - n_j| >> 1$ for any different i and j.

These different steps have been accomplished for the XXZ spin-1/2 chain in the massless regime in a series of works [73, 76–80, 231–234], where one can also find the expression of the form factors associated to the spin operators $\sigma_n^-, \sigma_n^+, \sigma_n^z$. These results constitute a real breakthrough as before, the only known exact results obtained from first principles were the one for the free fermion point, namely for $q = i$. Combined with powerful numerical summation techniques, it led to experimentally verifiable forecasts for scattering neutrons on magnetic materials, well described by a XXZ spin chain [117, 118].

Let us mention another way to compute quantities of the form (1.69), which consists in first computing $|\tilde{\Psi}\rangle \equiv \phi_1...\phi_n\,|\Psi_g\rangle$ and then in computing the scalar product $\langle\Psi_g|\tilde{\Psi}\rangle$. This is the way taken in [75, 112].

1.6 Cyclic representations of the 6-vertex Yang-Baxter algebra

In this thesis, we study the τ_2 model with general integrable boundary conditions. In this paragraph, we aim to describe this model for periodic boundary conditions, as the notation of this section will be used all along the manuscript.

Historically, this model is obtained from the Lax operator which is a general cyclic solution of the equation (1.12) with the 6-vertex R-matrix [158], for q a root of unity. It is defined by the following Bazhanov-Stroganov Lax operator:

$$L_{0n}(\lambda, p_n) = \begin{pmatrix} \lambda\alpha_n \hat{\mathbf{v}}_n - \lambda^{-1}\beta_n \hat{\mathbf{v}}_n^{-1} & \hat{\mathbf{u}}_n(q^{-1/2}a_n \hat{\mathbf{v}}_n + q^{1/2}b_n \hat{\mathbf{v}}_n^{-1}) \\ \hat{\mathbf{u}}_n^{-1}(q^{1/2}c_n \hat{\mathbf{v}}_n + q^{-1/2}d_n \hat{\mathbf{v}}_n^{-1}) & \lambda^{-1}\gamma_n \hat{\mathbf{v}}_n - \lambda\delta_n \hat{\mathbf{v}}_n^{-1} \end{pmatrix}_{[0]} \tag{1.72}$$

where $p_n = (\alpha_n, \beta_n, \gamma_n, \delta_n, a_n, b_n, c_n, d_n)$ denotes the 8 parameters associated to the site n. In order for this matrix to satisfy the Yang-Baxter equations (1.12), the parameters on a given site are not independent and have to satisfy :

$$\forall\ n \in [\![1, N]\!]\ , \quad \alpha_n\gamma_n = a_n c_n \ \text{ and } \ \beta_n\delta_n = b_n d_n \tag{1.73}$$

We can moreover emphasise a global normalisation, with the parameter γ_n for instance. What is more, let us highlight that the inhomogeneities ξ_n can be absorbed into the parameters p_n.

Moreover, the hermitian operators $\hat{\mathbf{u}}_n$ and $\hat{\mathbf{v}}_n$ are the generators of the following local Weyl algebra on $\mathcal{H}_n$:

$$\forall(n, m) \in [\![1, N]\!]\ ,\ \ \hat{\mathbf{u}}_n\hat{\mathbf{v}}_m = q^{\delta_{mn}}\hat{\mathbf{v}}_m\hat{\mathbf{u}}_n \tag{1.74}$$

Through all this thesis, we will study the case where q is the following particular p^{th}-root of unity:

$$q = e^{2ik\pi/p}\ \ ,\ k \in \mathbb{N}^*, \quad p \equiv 2l + 1\ ,\quad l \in \mathbb{N}^*\ ,\ \text{with k and p coprime integers,} \tag{1.75}$$

and we will assume

$$\forall n \in [\![1, N]\!]\quad ,\ \hat{\mathbf{u}}_n^p = \hat{\mathbf{v}}_n^p = \mathbb{1} \tag{1.76}$$

We consider this choice to handle irreducible representations of the local Weyl algebras, as reducible representations appear under different possible choices of q. As a consequence, we can use finite-dimensional representations of dimension p for each Weyl algebra, referred as cyclic representations. We will use mainly two different basis:

- The so-called $\hat{\mathbf{v}}$-basis, where we introduce the left states $\{\langle k, n|_{\hat{\mathbf{v}}}\}_{k\in[\![0,p-1]\!]}$, with the cyclicity $\langle k+p, n|_{\hat{\mathbf{v}}} = \langle k, n|_{\hat{\mathbf{v}}}$ and the right states $\{|k, n\rangle_{\hat{\mathbf{v}}}\}_{k\in[\![0,p-1]\!]}$ with the cyclicity $|k+p, n\rangle_{\hat{\mathbf{v}}} = |k, n\rangle_{\hat{\mathbf{v}}}$
  The action of the operators read:
  $$\langle k, n|_{\hat{\mathbf{v}}}\, \hat{\mathbf{u}}_n = \langle k+1, n|_{\hat{\mathbf{v}}} \quad \langle k, n|_{\hat{\mathbf{v}}}\, \hat{\mathbf{v}}_n = q^k \langle k, n|_{\hat{\mathbf{v}}} \quad ; \quad \hat{\mathbf{u}}_n\, |k, n\rangle_{\hat{\mathbf{v}}} = |k-1, n\rangle_{\hat{\mathbf{v}}} \quad \hat{\mathbf{v}}_n\, |k, n\rangle_{\hat{\mathbf{v}}} = q^k\, |k, n\rangle_{\hat{\mathbf{v}}} \tag{1.77}$$
- The so-called $\hat{\mathbf{u}}$-basis, where we introduce the left states $\{\langle k, n|_{\hat{\mathbf{u}}}\}_{k\in[\![0,p-1]\!]}$, with the cyclicity $\langle k+p, n|_{\hat{\mathbf{u}}} = \langle k, n|_{\hat{\mathbf{u}}}$ and the right states $\{|k, n\rangle_{\hat{\mathbf{u}}}\}_{k\in[\![0,p-1]\!]}$ with the cyclicity $|k+p, n\rangle_{\hat{\mathbf{u}}} = |k, n\rangle_{\hat{\mathbf{u}}}$
  The action of the operators read:
  $$\langle k, n|_{\hat{\mathbf{u}}}\, \hat{\mathbf{u}}_n = q^{-2k} \langle k, n|_{\hat{\mathbf{u}}} \quad \langle k, n|_{\hat{\mathbf{u}}}\, \hat{\mathbf{v}}_n = \langle k-l, n|_{\hat{\mathbf{u}}} \quad ; \quad \hat{\mathbf{u}}_n\, |k, n\rangle_{\hat{\mathbf{u}}} = q^{-2k}\, |k, n\rangle_{\hat{\mathbf{u}}} \quad \hat{\mathbf{v}}_n\, |k, n\rangle_{\hat{\mathbf{u}}} = |k+l, n\rangle_{\hat{\mathbf{u}}} \tag{1.78}$$

In a very standard way, we define the bulk monodromy matrix:

$$M_{0Q}(\lambda, p_Q) = L_{0N}(\lambda, p_N)...L_{01}(\lambda, p_1) \tag{1.79}$$

where p_Q denotes the parameters in the whole quantum space $p_Q = (p_1, ..., p_N)$. The Yang-Baxter equation implies that the transfer matrix:

$$\tau_2(\lambda) \equiv tr_0\{M_{0Q}(\lambda, p_Q)\ \} = A(\lambda) + D(\lambda) \tag{1.80}$$

defines a one parameter family of commuting operators. Moreover, for this model it is possible to define an operator:

$$\Theta \equiv \prod_{n=1}^{N} \hat{\mathbf{v}}_n \tag{1.81}$$

which commutes with the transfer matrix:

$$[\tau_2(\lambda), \Theta] = 0 \tag{1.82}$$

The eigenvalues of the operator Θ being p^{th} roots of unity, it is possible to associate to each eigenvector of the transfer matrix its eigenvalue by this operator. It notably allows a foliation of the transfer matrix spectrum Σ_{τ_2} in terms of $\Sigma^k_{\tau_2}$ (with $k \in [\![0, p-1]\!]$), the spectrum of the eigenvectors whose Θ-eigenvalue is q^k:

$$\Sigma_{\tau_2} = \bigcup_{k=0}^{p-1} \Sigma^k_{\tau_2} \tag{1.83}$$

Let us complete the description of cyclic representations of the 6-vertex Yang-Baxter algebra by making explicit the quantum determinant. Using the factorisation property (1.45), one can write:

$$q\text{-}det\ M_{0Q}(\lambda, p_Q) = \prod_{n=1}^{\mathsf{N}} k_n\left(\frac{\lambda}{\mu_{n,+}} - \frac{\mu_{n,+}}{\lambda}\right)\left(\frac{\lambda}{\mu_{n,-}} - \frac{\mu_{n,-}}{\lambda}\right) \equiv a(\lambda)d(\lambda/q) \tag{1.84}$$

where the constants k_n and the zeros $\mu_{n,\pm}$ of the quantum determinant are given by:

$$k_n \equiv (a_n b_n c_n d_n)^{1/2} \quad \text{and} \quad \mu_{n,h} \equiv \begin{cases} iq^{1/2}\,(a_n\beta_n/\alpha_n b_n)^{1/2} & h = + \\ iq^{1/2}\,(c_n\beta_n/\alpha_n d_n)^{1/2} & h = - \end{cases} \tag{1.85}$$

As for the functions $a(\lambda)$ and $d(\lambda)$, they are defined by:

$$a(\lambda) \equiv a_0 \prod_{n=1}^{\mathsf{N}} \left(\frac{\beta_n}{\lambda} + q^{-1}\frac{b_n\alpha_n}{a_n}\lambda\right) \quad \text{and} \quad d(\lambda) \equiv \frac{(-1)^{\mathsf{N}}}{a_0} \prod_{n=1}^{\mathsf{N}} \frac{a_n c_n}{\alpha_n}\left(\frac{1}{\lambda} + q\frac{d_n\alpha_n}{c_n\beta_n}\lambda\right) \tag{1.86}$$

Let us mention the close link existing between this τ_2 model and the chiral Potts model [158]. Indeed, the transfer matrix of the chiral Potts model is a Baxter Q-operator for the τ_2 model, a property that allows to obtain results concerning the chiral Potts model. For example, the spectrum Σ_{τ_2} is completely characterised (*via* the construction of a Baxter Q-operator by separation of variables [235]).

Another interesting feature with the Lax operator (1.72) is its possibility to describe different models, like the Fateev-Zamolodchikov model (linked to the discretised sine Gordon model at root of unity) or the XXZ spin $s = (p-1)/2$ at root of unity, simply by tuning the different parameters p_n.

Eventually, let us emphasise that important results are known for this model, from the characterisation of the spectrum (using separation of variables) up to the computation of some form factors [145, 151, 235] for the periodic model. Now we aim to start to describe the model with general integrable boundaries.

Chapter 2

Quantum integrable models with boundaries

As briefly stated in the introduction, the study of models with boundaries has attracted a large research enthusiasm as they can describe both equilibrium and out of equilibrium physics. Some interesting applications concern the description of classical stochastic relaxation processes, like ASEP [165, 166], [167–172] and quantum transport properties in spin systems [173, 174].

In the previous chapter, we recalled that Hamiltonians of quantum integrable systems can be obtained from the transfer matrix. However, with the previous definition for this transfer matrix as a trace of the monodromy matrix, it turns out that for all the considered models the reconstructed Hamiltonians have periodic boundary conditions. The aim of this chapter is to introduce more general transfer matrices, in order to describe more general boundaries.

A first generalisation is presented considering twisted transfer matrices. In paragraph 2.1, we show that one can indeed use scalar representations of the Yang-Baxter algebra to generate twisted transfer matrices, leading to quasi-periodic integrable Hamiltonians.

However, the main goal of this chapter is to outline the reflection algebras introduced by Sklyanin [177]. This one allows in particular to define a boundary transfer matrix (the trace of the boundary monodromy), which can be used to generate quantum integrable Hamiltonians with general integrable boundaries.

An important feature that emerges to effectively define local Hamiltonians is the notion of fundamental and non fundamental models, regarding to the relative dimensions of the auxiliary and quantum spaces. Thus, the last paragraph is dedicated to the recall of the fusion procedure, which allows to define spin chain Hamiltonians for non fundamental models. Let us mention that all along this chapter, the XXZ spin chain is used as the main example to explain the various concepts and tools.

2.1 Twisted boundary conditions

As mentioned earlier, a first generalisation and a first step to take into account boundaries in the framework of the Quantum Inverse Scattering Method, is given by the twisted transfer matrix. In the general setting of section 1.2, instead of considering the transfer matrix (1.16), one can define a twisted transfer matrix:

$$\mathcal{T}^{(\Sigma)}(\lambda) = tr_a\{\Sigma_a(\lambda) M_{aQ}(\lambda)\} \tag{2.1}$$

with $\Sigma(\lambda)$ a scalar matrix, i.e. an element of $\mathcal{M}_m(\mathbb{C})$ (We recall that m denotes the dimension of the auxiliary space V_a). Then it holds the following proposition:

If $\Sigma(\lambda)$ is a scalar representation of the Yang-Baxter algebra, namely if there is the equality:

$$R_{ab}(\lambda,\mu)\Sigma_a(\lambda)\Sigma_b(\mu) = \Sigma_b(\mu)\Sigma_a(\lambda)R_{ab}(\lambda,\mu) \tag{2.2}$$

then the twisted transfer matrices form a commuting family of operators:

$$\forall(\lambda,\mu)\in\mathbb{C}^2\ ,\quad [\mathcal{T}^{(\Sigma)}(\lambda),\mathcal{T}^{(\Sigma)}(\mu)]=0 \tag{2.3}$$

One of the most interesting feature in considering the twisted transfer matrix is the possibility to describe local Hamiltonians with twisted boundary conditions. Thanks to the trace identities, the quasi-periodic boundary conditions are linked to the choice of $\Sigma(\lambda)$. An example is given by the XXZ- spin 1/2 chain in the next paragraph.

2.1.1 The XXZ-1/2 quasi-periodic chain

Let us consider the XXZ-1/2 spin chain, whose periodic case has been studied in section 1.4.1.

One can find the following solutions for the scalar representations of the 6-vertex Yang-Baxter algebra:

$$\forall(\alpha,a)\in\mathbb{C}\times\{0,1\}\ ,\quad \Sigma(\lambda)\equiv\Sigma^{(\alpha,a)}=(\sigma^x)^a\begin{pmatrix}e^{\alpha} & 0\\ 0 & e^{-\alpha}\end{pmatrix} \tag{2.4}$$

where σ^x is the Pauli matrix. Let us denote then $\mathcal{T}^{(\alpha,a)}(\lambda)$ the associated twisted transfer matrix, given by (2.1). By taking the logarithmic derivative of this matrix, one can show that similarly to equation (1.51), the quantity

$$H^{(\alpha,a)}\equiv(q-q^{-1})\left.\frac{d\ ln\left(\mathcal{T}^{(\alpha,a)}(\lambda)\right)}{d\lambda}\right|_{\lambda=1}-(q+q^{-1})\frac{N}{2} \tag{2.5}$$

is the Hamiltonian of the XXZ spin 1/2 chain with twisted boundary conditions. Explicitly,

$$H^{(\alpha,a)}=\sum_{n=1}^{N}\left[\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y+\frac{1}{2}(q+q^{-1})\sigma_n^z\sigma_{n+1}^z\right] \tag{2.6}$$

with the following quasi-periodic boundary conditions:

$$\sigma_{N+1}^s=\Sigma_1^{(\alpha,a)}\ \sigma_1^s\ \Sigma_1^{(\alpha,a)}\quad \text{for } s=x,y \text{ or } z \tag{2.7}$$

In particular, to consider the twisted transfer matrix $\mathcal{T}^{(0,1)}(\lambda)=B(\lambda)+C(\lambda)$ leads to the anti-periodic chain. Let us simply mention here that at the thermodynamic limit, i.e. for an infinite number of sites, the results concerning the form factors and the correlation functions should not depend on the twist function Σ.

2.2 The 6-vertex reflection algebra and integrability

The question we address now is to describe chains with general integrable boundary conditions. For example, we aim to describe the action of two magnetic fields at the boundaries of the spin XXZ 1/2 chain. To answer this question, we report here the work of Sklyanin [177], who introduced the notion of reflection algebra for a boundary version of the monodromy matrix. We present this algebra on the example of the 6-vertex R-matrix, and give the definition of the boundary transfer matrix, which will be used to derive integrable local Hamiltonians with integrable boundaries.

2.2.1 The 6-vertex reflection algebra

Let us consider $R(\lambda)$ the 6-vertex R-matrix (1.20). Then following [177] let us introduce the algebra $\mathcal{U}_-$, the associative algebra defined by the generators $\mathcal{U}(\lambda)_{\alpha,\beta}$, with $(\alpha,\beta)\in\{1,2\}$, considered as the elements of a square matrix $\mathcal{U}(\lambda)$, and by the relation:

$$R_{12}(\lambda/\mu)\mathcal{U}_1(\lambda)R_{12}(\lambda\mu/q)\mathcal{U}_2(\mu)=\mathcal{U}_2(\mu)R_{12}(\lambda\mu/q)\mathcal{U}_1(\lambda)R_{12}(\lambda/\mu) \tag{2.8}$$

The algebra $\mathcal{U}_+$ is defined in the same way with a matrix $\mathcal{U}_+(\lambda)$ and the relation:

$$R_{12}(\mu/\lambda)\mathcal{U}_{+1}^{t_1}(\lambda)R_{12}(1/(\lambda\mu q))\mathcal{U}_{+2}^{t_2}(\mu) = \mathcal{U}_{+2}^{t_2}(\mu)R_{12}(1/(\lambda\mu q))\mathcal{U}_{+1}^{t_1}(\lambda)R_{12}(\mu/\lambda) \tag{2.9}$$

The equations (2.8) and (2.9) are called the reflection equations.

Moreover, Sklyanin gives a way to find representations of these algebras. In fact, his construction allows to use representations of the Yang-Baxter algebra to generate representations of the reflection algebra thanks to scalar solutions of the reflection equations. Explicitly, a solution of the reflection equation (2.8) in the total quantum space $\mathcal{H}$ can be obtained by the following boundary monodromy matrix:

$$\mathcal{U}_{0Q}(\lambda) = M_{0Q}(\lambda)K_0(\lambda)\hat{M}_{0Q}(\lambda) \quad \text{with} \quad \hat{M}_{0Q}(\lambda) = \sigma_0^y M_{0Q}^{t_0}(1/\lambda)\sigma_0^y \tag{2.10}$$

where $M_{0Q}(\lambda)$ is the monodromy matrix introduced for the Yang-Baxter algebra, and $K(\lambda)$ is a scalar solution of the reflection equation (2.8), i.e.:

$$R_{12}(\lambda/\mu)K_1(\lambda)R_{12}(\lambda\mu/q)K_2(\mu) = K_2(\mu)R_{12}(\lambda\mu/q)K_1(\lambda)R_{12}(\lambda/\mu) \tag{2.11}$$

in the space $V_1 \times V_2$ (we recall that the auxiliary spaces are of dimension 2, $V_i \simeq \mathbb{C}^2$).

The expression for $\hat{M}_{0Q}(\lambda)$ is reminiscent of the inversion formula for the bulk monodromy matrix (1.46), and indeed it is linked to the inverse of the bulk monodromy thanks to the quantum determinant:

$$\hat{M}_{0Q}(\lambda) = q\text{-}det\ M_{0Q}(q/\lambda) \cdot M_{0Q}^{-1}(q/\lambda) \tag{2.12}$$

As for the Yang-Baxter algebra, we will use a standard notation:

$$\mathcal{U}_{0Q}(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix} \tag{2.13}$$

to denote the operators of the boundary monodromy matrix. These ones are thus the generators of the reflection algebra, let us write explicitly some of the relations they satisfy:

$$[\mathcal{B}(\lambda), \mathcal{B}(\mu)] = 0 \tag{2.14}$$
$$[\mathcal{C}(\lambda), \mathcal{C}(\mu)] = 0 \tag{2.15}$$
$$[\mathcal{A}(\lambda), \mathcal{A}(\mu)] = c_5(\lambda,\mu)\left(\mathcal{B}(\lambda)\mathcal{C}(\mu) - \mathcal{B}(\mu)\mathcal{C}(\lambda)\right) \tag{2.16}$$
$$[\mathcal{D}(\lambda), \mathcal{D}(\mu)] = c_5(\lambda,\mu)\left(\mathcal{C}(\mu)\mathcal{B}(\lambda) - \mathcal{C}(\lambda)\mathcal{B}(\mu)\right) \tag{2.17}$$
$$\mathcal{A}(\mu)\mathcal{B}(\lambda) = c_1(\lambda,\mu)\ \mathcal{B}(\lambda)\mathcal{A}(\mu) + c_2(\lambda,\mu)\ \mathcal{B}(\mu)\mathcal{A}(\lambda) + c_3(\lambda,\mu)\ \mathcal{B}(\mu)\tilde{\mathcal{D}}(\lambda) \tag{2.18}$$
$$\mathcal{B}(\lambda)\mathcal{D}(\mu) = c_1(\lambda,\mu)\ \mathcal{D}(\mu)\mathcal{B}(\lambda) + c_4(\lambda,\mu)\ \mathcal{D}(\lambda)\mathcal{B}(\mu) + c_5(\lambda,\mu)\ \mathcal{A}(\lambda)\mathcal{B}(\mu) \tag{2.19}$$

where the different constant structures $c_i(\lambda,\mu)$ are given by:

$$c_1(\lambda,\mu) = \frac{(q\lambda/\mu - \mu/(q\lambda))\,(\lambda\mu/q - q/(\lambda\mu))}{(\lambda/\mu - \mu/\lambda)\,(\lambda\mu - 1/(\lambda\mu))} \qquad c_2(\lambda,\mu) = \frac{\left(\lambda^2/q - q/\lambda^2\right)(q - 1/q)}{(\mu/\lambda - \lambda/\mu)\,(\lambda^2 - 1/\lambda^2)} \tag{2.20}$$
$$c_3(\lambda,\mu) = -\frac{q - 1/q}{(\lambda\mu - 1/(\lambda\mu))\,(\lambda^2 - 1/\lambda^2)} \qquad c_4(\lambda,\mu) = -\frac{(\lambda\mu/q - q/(\lambda\mu))\,(q - 1/q)}{(\lambda/\mu - \mu/\lambda)\,(\lambda\mu - 1/(\lambda\mu))} \tag{2.21}$$
$$c_5(\lambda,\mu) = -\frac{q - 1/q}{\lambda\mu - 1/(\lambda\mu)} \tag{2.22}$$

and we have defined the operator $\tilde{\mathcal{D}}(\lambda)$ by:

$$\tilde{\mathcal{D}}(\lambda) \equiv (\lambda^2 - 1/\lambda^2)\mathcal{A}(\lambda) - (q - q^{-1})\mathcal{D}(\lambda) \tag{2.23}$$

We emphasise on these relations as they will be used in the next; let us in particular note the commutativity of the families $\mathcal{B}(\lambda)$ and $\mathcal{C}(\lambda)$.

The same way as $K(\lambda)$ is defined, let $K_+(\lambda)$ be a scalar solution of the reflection equation (2.9):

$$R_{12}(\mu/\lambda)K_{+1}^{t_1}(\lambda)R_{12}(1/(\lambda\mu q))K_{+2}^{t_2}(\mu) = K_{+2}^{t_2}(\mu)R_{12}(1/(\lambda\mu q))K_{+1}^{t_1}(\lambda)R_{12}(\mu/\lambda) \tag{2.24}$$

The most general expressions for the boundary matrices $K(\lambda)$ and $K_+(\lambda)$ are known [178, 183]. Let us recall here:

$$K_0(\lambda) \equiv K_0(\lambda|\xi_-,\tau_-,\kappa_-) = \frac{1}{\xi_- - \frac{1}{\xi_-}}\begin{pmatrix} \lambda\xi_- q^{-1/2} - 1/(\lambda\xi_- q^{-1/2}) & \kappa_- e^{\tau_-}\left(\lambda^2/q - q/\lambda^2\right) \\ \kappa_- e^{-\tau_-}\left(\lambda^2/q - q/\lambda^2\right) & q^{1/2}\xi_-/\lambda - \lambda/(q^{1/2}\xi_-) \end{pmatrix}_{[0]} \tag{2.25}$$

It is easy to remark that the general scalar solution of the reflection equation (2.24) admits the following form in terms of the solution of the reflection equation (2.11):

$$K_{+0}^{t_0}(\lambda) = K_0(q\lambda) \tag{2.26}$$

Hence the following expression:

$$K_{+0}(\lambda) \equiv K_{+0}(\lambda|\xi_+,\tau_+,\kappa_+) = \frac{1}{\xi_+ - \frac{1}{\xi_+}}\begin{pmatrix} \lambda\xi_+ q^{1/2} - 1/(\lambda\xi_+ q^{1/2}) & \kappa_+ e^{\tau_+}\left(q\lambda^2 - 1/(q\lambda^2)\right) \\ \kappa_+ e^{-\tau_+}\left(q\lambda^2 - 1/(q\lambda^2)\right) & q^{-1/2}\xi_+/\lambda - \lambda/(q^{-1/2}\xi_+) \end{pmatrix}_{[0]} \tag{2.27}$$

The parameters $\xi_\pm, \tau_\pm$ and $\kappa_\pm$ are generic, and associated to the boundaries. A standard notation to refer to these matrices is given by:

$$K_0(\lambda|\xi_-,\tau_-,\kappa_-) \equiv \begin{pmatrix} a_-(\lambda) & b_-(\lambda) \\ c_-(\lambda) & d_-(\lambda) \end{pmatrix} \quad \text{and} \quad K_{+0}(\lambda|\xi_+,\tau_+,\kappa_+) \equiv \begin{pmatrix} a_+(\lambda) & b_+(\lambda) \\ c_+(\lambda) & d_+(\lambda) \end{pmatrix} \tag{2.28}$$

As for the 6-vertex Yang-Baxter algebra, there is the notion of quantum determinant. Indeed, using once again the degeneracy of the 6-vertex R-matrix at the point $\lambda = 1/q$, Sklyanin [177] shows that the element

$$q\text{-}det\, \mathcal{U}_{0Q}(\lambda) \equiv \mathrm{tr}_{12}\left\{P_{12}^-\, \mathcal{U}_{1Q}(q^{-1/2}\lambda)R_{12}(\lambda^2/q)\mathcal{U}_{2Q}(q^{1/2}\lambda)\right\} \tag{2.29}$$

is central for any representation of the reflection algebra (2.8):

$$\forall(\lambda,\mu) \in \mathbb{C}^2\ , \quad [q\text{-}det\, \mathcal{U}_{0Q}(\lambda), \mathcal{U}_{0'Q}(\mu)] = 0 \tag{2.30}$$

The proof of this centrality is very similar to the one for the Yang-Baxter algebra.

2.2.2 Boundary transfer matrix

In a way, the reflection algebras generalise the Yang-Baxter algebra as one can define a one parameter family of commuting matrices, the so-called boundary transfer matrices, which are used to describe Hamiltonians with general integrable boundaries. The procedure is the following.

Let $K_+(\lambda)$ be a solution of the reflection equation (2.24) and let us consider the boundary monodromy matrix given by (2.10). The boundary transfer matrix is then defined as:

$$\mathcal{T}(\lambda) = tr_0\left\{K_{+0}(\lambda)\mathcal{U}_{0Q}(\lambda)\right\} \tag{2.31}$$

Thanks to the reflection equations, this is a one parameter family of commuting operators [177]:

$$\forall(\lambda,\mu) \in \mathbb{C}^2\ , \quad [\mathcal{T}(\lambda), \mathcal{T}(\mu)] = 0 \tag{2.32}$$

This relation is, as for the Yang-Baxter algebra, at the heart of the integrability of a system. Indeed, using trace identities, Hamiltonians can be expressed *via* the boundary transfer matrix (the example of the XXZ spin chain is given right after, cf (2.38)), so one exhibits a one-parameter family of commuting operators commuting also with the Hamiltonian. What is more, if the spectrum of the boundary transfer matrix is simple, i.e. if there is only one eigenvector associated to an eigenvalue, the set of conserved operators is complete as the eigenvectors are univocally labelled by the eigenvalues.

We end this section highlighting a symmetry of the boundary transfer matrix, and making explicit the generators of the reflection algebra in terms of the generators of the Yang-Baxter algebra. Following again [177], it is indeed more convenient to work with the next shifted definition of the monodromy matrix for the Yang-Baxter algebra, compared with (1.49):

$$M_{0Q}(\lambda) = L_{0N}(\lambda/(\xi_N q^{1/2}))...L_{01}(\lambda/(\xi_1 q^{1/2})) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{2.33}$$

This shift on λ ensures the symmetry:

$$\mathcal{T}(\lambda) = \mathcal{T}(1/\lambda) \tag{2.34}$$

We shall come back to that issue later, which concerns both the symmetry of the eigenvalues of the boundary transfer matrix and the symmetry of the boundary terms of the Hamiltonian.

Eventually, let us write explicitly the boundary transfer matrix, which is the object at the heart of the integrability of a system, in terms of the generators of the reflection algebra:

$$\mathcal{T}(\lambda) = a_+(\lambda)\mathcal{A}(\lambda) + d_+(\lambda)\mathcal{D}(\lambda) + c_+(\lambda)\mathcal{B}(\lambda) + b_+(\lambda)\mathcal{C}(\lambda) \tag{2.35}$$

Moreover, we recall that the generators of the reflection algebra are quadratic in terms of the generators of the Yang-Baxter algebra (2.10), for instance:

$$\mathcal{B}(\lambda) = -a_-(\lambda)A(\lambda)B(1/\lambda) + b_-(\lambda)A(\lambda)A(1/\lambda) - c_-(\lambda)B(\lambda)B(1/\lambda) + d_-(\lambda)B(\lambda)A(1/\lambda) \tag{2.36}$$

This expression for the boundary transfer matrix has to be compared with $\mathcal{T}(\lambda) = A(\lambda) + D(\lambda)$ for the periodic case. The study of general boundaries appears thus a priori much more involved than for periodic systems.

Remark Originally, these reflection algebras were introduced in [177] for general R-matrices $R(\lambda)$ verifying the following assumptions:

(i) The symmetries $R_{21}(\lambda) = R_{12}(\lambda)$ and $R_{12}^{t_1}(\lambda) = R_{12}^{t_2}(\lambda)$

(ii) The unitarity $R_{12}(\lambda)R_{12}(1/\lambda) = \rho(\lambda)$ for $\rho(\lambda)$ a scalar function

(iii) The crossing unitarity $R_{12}^{t_1}(\lambda)R_{12}^{t_1}(1/(q^2\lambda)) = \tilde{\rho}(\lambda)$ for $\tilde{\rho}(\lambda)$ a scalar function and q a characteristic constant of the R-matrix.

The 6-vertex R-matrix is indeed satisfying these properties, with

$$\rho(\lambda) = -\left(\lambda q - 1/(\lambda q)\right)\left(\lambda/q - q/\lambda\right) \quad \text{and} \quad \tilde{\rho}(\lambda) = \rho(\lambda q) \tag{2.37}$$

We aim to generalise these algebras in chapter 5. To this purpose, we will turn back to the original Cherednik picture of reflection equations [178], namely the description of the reflection of two particles on a wall, but for different types (i. e. representations) of particles. But for now, we recall in the next paragraph how to compute integrable Hamiltonians from the boundary transfer matrix.

2.3 Local Hamiltonians associated to representations of the reflection algebra

As this is the case for the Yang-Baxter algebra, one has to show that the boundary transfer matrix allows to construct the local Hamiltonians one is interested in. In this section we start by giving the representative example of the XXZ spin chain, and then we derive a more general expression, valid for more general fundamental models.

2.3.1 The XXZ 1/2 spin chain with general integrable boundary conditions

In paragraph 1.4.1, we have seen that the periodic XXZ-1/2 spin chain can be described in the framework of the Yang-Baxter algebra by the Lax operator (1.47) and the 6-vertex R-matrix. Thus, to study the XXZ-1/2 spin chain with general integrable boundaries, one can use the 6-vertex reflection algebras. Following Sklyanin, we use the monodromy matrix of the Yang Baxter algebra to find a representation of the reflection algebra. More specifically, to ensure the symmetry (2.34) we will use the shifted definition (2.33) for the monodromy matrix, where $L_{0n}(\lambda/\xi_n)$ is the Lax operator (1.47). Consequently, let us consider the boundary monodromy matrix $\mathcal{U}_{0Q}(\lambda)$ defined by (2.10), and the general scalar solutions $K(\lambda)$ and $K_+(\lambda)$ of the reflection equations, given respectively by (2.25) and (2.27). We can thus define the boundary transfer matrix (2.31).

Then, up to an additive constant, the following derivative of the boundary transfer matrix in the homogeneous limit ($\xi_n = 1$ on every site) reproduces the Hamiltonian of the XXZ 1/2 spin chain with integrable boundaries [148]:

$$\frac{2(q-q^{-1})^{1-2N}}{tr_0\{K_{+0}(q^{1/2})\}\ tr_0\{K_0(q^{1/2})\}}\ \frac{d\mathcal{T}(\lambda)}{d\lambda}\bigg|_{\lambda=q^{1/2}} + cste = H^{open}_{XXZ} \tag{2.38}$$

where H^{open}_{XXZ} denotes the Hamiltonian:

$$H^{open}_{XXZ} = \sum_{n=1}^{N-1}\left[\sigma^x_n\sigma^x_{n+1} + \sigma^y_n\sigma^y_{n+1} + \frac{1}{2}(q+q^{-1})\sigma^z_n\sigma^z_{n+1}\right] + B_1 + B_N \tag{2.39}$$

for wich the boundary terms on sites 1 and N are given by:

$$B_1 = \frac{q-1/q}{\xi_- - 1/\xi_-}\left(\frac{1}{2}(\xi_- + 1/\xi_-)\sigma^z_1 + \kappa_-(\tau_- + 1/\tau_-)\sigma^x_1 + i\ \kappa_-(\tau_- - 1/\tau_-)\sigma^y_1\right) \tag{2.40}$$

$$\text{and} \quad B_N = \frac{q-1/q}{\xi_+ - 1/\xi_+}\left(\frac{1}{2}(\xi_+ + 1/\xi_+)\sigma^z_N + \kappa_+(\tau_+ + 1/\tau_+)\sigma^x_N + i\ \kappa_+(\tau_+ - 1/\tau_+)\sigma^y_N\right) \tag{2.41}$$

The main idea of the proof is given in a slightly more general case in the next paragraph. Let us just comment that contrary to the quasi-periodic Hamiltonian, the boundary Hamiltonians are obtained thanks to the derivative of the boundary transfer matrix with respect to the spectral parameter, and not as the logarithmic derivative.

Moreover, with the expressions (2.39), (2.40) and (2.41) we can note the symmetry between the boundaries on sites 1 and N. They have indeed the same algebraic form, only the boundary parameters $\xi_\pm, \tau_\pm$ and $\kappa_\pm$ are different.

2.3.2 Local Hamiltonians with boundaries for fundamental models

Actually, in his original paper [177], Sklyanin could give the expression of local Hamiltonians with integrable boundaries for fundamental models (for which the quantum spaces and the auxiliary space are isomorphic) associated to R-matrices satisfying the properties (i)-(iii) of paragraph 2.2.2, and for models whose Lax operators coincide with the R-matrix. The XXZ 1/2 spin chain is a fundamental model of this kind.

Sklyanin used a variant of Baxter argument (trace identities) to derive local Hamiltonians. Explicitly, if it holds the following properties:

- The auxiliary space V_a and the local quantum spaces $\mathcal{H}_n$ are isomorphic (fundamental models)
- There exists a point λ_0 such that $K(\lambda_0) = \mathbb{1}$

- The Lax operator (which is an R-matrix) reduces to the permutation between the spaces V_a and $\mathcal{H}_n$ in the same point λ_0: $R_{an}(\lambda_0) = \mathcal{P}_{an}$

then one can show that the first derivative of the boundary transfer matrix, at the homogeneous limit ($\xi_n = 1$ on every site), can be put under the form:

$$\left.\frac{d\mathcal{T}(\lambda)}{d\lambda}\right|_{\lambda=\lambda_0} = 2\ \mathcal{H}\ tr_a\{K_{+a}(\lambda_0)\} + tr_a \left\{ \left.\frac{dK_{+a}(\lambda)}{d\lambda}\right|_{\lambda=\lambda_0} \right\} \tag{2.42}$$

with the following Hamiltonian with boundaries:

$$\mathcal{H} = \sum_{n=1}^{N-1} H_{n,n+1} + \frac{1}{2}\left.\frac{d\ K_1(\lambda)}{d\ \lambda}\right|_{\lambda=\lambda_0} + \frac{tr_a\{K_{+a}(\lambda_0)H_{Na}\}}{tr_a\{K_{+a}(\lambda_0)\}} \tag{2.43}$$

where the bulk interactions are given by:

$$H_{n,n+1} \equiv \mathcal{P}_{n,n+1}\left.\frac{dR_{n,n+1}(\lambda)}{d\lambda}\right|_{\lambda=\lambda_0} \quad \text{and} \quad H_{Na} \equiv \mathcal{P}_{Na}\left.\frac{dR_{Na}(\lambda)}{d\lambda}\right|_{\lambda=\lambda_0} \tag{2.44}$$

Referring to the commutativity of the family $\mathcal{T}(\lambda)$, it follows:

$$\forall\lambda \in \mathbb{C}\ , \quad [\mathcal{T}(\lambda), \mathcal{H}] = 0 \tag{2.45}$$

This way, one finds a commuting family of operators commuting with the Hamiltonian $\mathcal{H}$ for a large class of models. Let us comment that the boundary terms on sites 1 and N in (2.43) are apparently different. Their symmetry, which was highlighted by (2.40)-(2.41) for the spin 1/2 spin chain, can in fact be proven thanks to (2.34).

Let us eventually emphasise that the symmetry $R_{n,n+1}(\lambda) = R_{n+1,n}(\lambda)$ is sufficient to have the same bulk interactions $H_{n,n+1}$ for general integrable boundaries as for the bulk interactions of the quasi-periodic systems obtained from Yang-Baxter algebra[1]. Indeed, for fundamental models with the Lax operators coinciding with the R-matrix, the transfer matrix is given by:

$$\mathcal{T}(\lambda) = tr_a\left\{R_{aN}(\lambda/\xi_N)...R_{a1}(\lambda/\xi_1)\right\} \tag{2.46}$$

and assuming that there exists a point λ_0 such that $R_{an}(\lambda_0) = \mathcal{P}_{an}$, the first derivative leads to:

$$\left.\frac{d\mathcal{T}(\lambda)}{d\lambda}\right|_{\lambda=\lambda_0} = \sum_{n=1}^{N} \mathcal{P}_{n,n+1}\left.\frac{dR_{n,n+1}(\lambda)}{d\lambda}\right|_{\lambda=\lambda_0} \tag{2.47}$$

We shall come back to that issue later, when we will derive the Hamiltonians for non fundamental models (in chapter 5).

In a nutshell, the reflection algebras mainly result in the possibility to associate to any closed integrable quantum model, characterised by a solution of the Yang-Baxter equation, its integrable quantum model with boundaries, characterised by the associated Sklyanin's solutions of the reflection equation.

2.3.3 The fusion procedure

As we have just seen, we know a way to define local integrable Hamiltonians (quasi-periodic or with general boundaries) for fundamental systems. In this paragraph, we briefly present the fusion procedure, which allows to consider the Hamiltonians for higher spin representations while keeping for example the 6-vertex

[1] We shall come back to that issue in section 5.5.4. Let us comment that more generally, it seems only necessary to have $\left.\frac{dR_{n,n+1}(\lambda)}{d\lambda}\right|_{\lambda=\lambda_0} \propto \left.\frac{dR_{n+1,n}(\lambda)}{d\lambda}\right|_{\lambda=\lambda_0}$, with a proportionality coefficient different from -1, in order to have the same bulk interactions between the chain with quasi-periodic boundary conditions and the chain with general integrable boundaries.

R-matrix. Indeed, the XXZ 1/2 spin chain is described by the 6-vertex R-matrix, and is a fundamental model having isomorphic auxiliary and quantum spaces (of dimension 2). If one wants to consider the XXZ 1 spin chain, the quantum spaces become of dimension 3 (and thus the Lax operators cannot be reduced to the permutation, as the involved spaces cannot be isomorphic).

Let us consider a fundamental R-matrix satisfying the Yang-Baxter equation (1.13). The fusion procedure, introduced in [236], is a method to construct solutions of (1.13) in higher dimensional spaces.

Here we present the fusion procedure on the example of the XXZ spin chain, which allows the construction of a solution of the Yang Baxter algebra with an auxiliary space of dimension 3, starting from the 6-vertex R-matrix (solution of the Yang-Baxter algebra with an auxiliary space of dimension 2). Let us recall the degeneracy (1.39) of the 6-vertex R-matrix in the point $\lambda = 1/q$, which reduces (up to a scalar) to the rank one projector P^-, and let us consider

$$P^+ = 1 - P^- \tag{2.48}$$

which is also a projector. Using the Yang-Baxter algebra, and the property $P^-P^+ = 0$, one can show that the so-called fused R-matrices:

$$R_{<12>3}(\lambda) \equiv P^+_{12} R_{13}(\lambda) R_{23}(q\lambda) P^+_{12} \tag{2.49}$$
$$R_{3<12>}(\lambda) \equiv P^+_{12} R_{32}(\lambda/q) R_{31}(\lambda) P^+_{12} \tag{2.50}$$

satisfy the Yang-Baxter equations:

$$R_{<12>3}(\lambda/\mu) R_{<12>4}(\lambda) R_{34}(\mu) = R_{34}(\mu) R_{<12>4}(\lambda) R_{<12>3}(\lambda/\mu) \tag{2.51}$$
$$R_{12}(\lambda/\mu) R_{1<34>}(\lambda) R_{2<34>}(\mu) = R_{2<34>}(\mu) R_{1<34>}(\lambda) R_{12}(\lambda/\mu) \tag{2.52}$$

in the spaces $V_{<12>} \times V_3 \times V_4$ and $V_1 \times V_2 \times V_{<34>}$ respectively, where as usual the spaces V_i are 2 dimensional. Regarding the definition of the spaces $V_{<12>}$ and $V_{<34>}$, obtained as the projections of $V_1 \times V_2$ and $V_3 \times V_4$ respectively, they are of dimension 3.

Similarly, in [237] the authors tackle the fusion procedure for open chains. Using the degeneracy of the R-matrix, the projectors P^- and P^+ and the reflection equation (2.11), they show that:

$$K_{<12>}(\lambda) \equiv P^+_{12} K_1(\lambda) R_{12}(\lambda^2) K_2(\lambda q) P^+_{12} \tag{2.53}$$

satisfies the following reflection equation:

$$R_{0<12>}(\lambda/\mu) K_0(\lambda) R_{<12>0}(\lambda\mu/q) K_{<12>}(\mu) = K_{<12>}(\mu) R_{<12>0}(\lambda\mu/q) K_0(\lambda) R_{0<12>}(\lambda/\mu) \tag{2.54}$$

This equation is a mixed reflection equation, as it involves different representations. This kind of equation will be the starting point to our study of more general reflection equations in chapter 5. Lastly, by considering the fusion on both the auxiliary and quantum spaces [238, 239], one can construct:

$$R_{<12><34>}(\lambda) \equiv P^+_{12} R_{1<34>}(\lambda) R_{2<34>}(q\lambda) P^+_{12} = P^+_{34} R_{<12>4}(\lambda/q) R_{<12>3}(\lambda) P^+_{34} \tag{2.55}$$

which satisfies the Yang-Baxter equation:

$$R_{<12><34>}(\lambda/\mu) R_{<12><56>}(\lambda) R_{<34><56>}(\mu) = R_{<34><56>}(\mu) R_{<12><56>}(\lambda) R_{<12><34>}(\lambda/\mu) \tag{2.56}$$

and which leads to the reflection equation:

$$R_{<12><34>}(\lambda/\mu) K_{<12>}(\lambda) R_{<34><12>}(\lambda\mu/q) K_{<34>}(\mu) = \tag{2.57}$$
$$K_{<34>}(\mu) R_{<34><12>}(\lambda\mu/q) K_{<12>}(\lambda) R_{<12><34>}(\lambda/\mu) \tag{2.58}$$

Thus, from this last equation one can apply Sklyanin's procedure to compute local Hamiltonians.
We emphasise that from the 6-vertex R-matrix, which has an auxiliary space of dimension 2, we can get

Hamiltonians for representations with arbitrary spin s, so we are able to give the Hamiltonians associated to XXZ spin s chains[2]. As already mentioned, we aim to generalise these results, in particular to be able to compute the Hamiltonians for non fundamental models. It will be done in the chapter 5, for models associated to cyclic representations of the 6-vertex reflection algebra.

We henceforth have an algebraic framework to deal with general integrable boundaries and moreover, thanks to Sklyanin's construction, we can associate to any representation of the Yang-Baxter algebra, and a boundary matrix, a representation of the reflection algebra. However, the generators of the latter are quadratic in terms of the generators of the Yang-Baxter algebra, and the boundary transfer matrix appears much more cumbersome. In particular, the Algebraic Bethe Ansatz seems hardly applicable for general boundaries. To nevertheless study this boundary transfer matrix, the next chapter introduces the quantum separation of variables, which is another approach to the characterisation of the transfer matrix spectrum in the Quantum Inverse Scattering Method framework.

[2]Indeed, in that case the Lax operators have an auxiliary space of dimension 2 and quantum spaces of dimension $2s + 1$. We can fuse the auxiliary space up to the obtention of an auxiliary space of dimension $2s + 1$.

Chapter 3

Quantum separation of variables

In this chapter, we introduce the important notion of separation of variables for integrable systems. Starting from the classical point of view, we recall the usual definition of separate variables. Then we show the implementation of this concept in quantum models, emphasising this way on the analogy between the classical and quantum descriptions. The quantum separation of variables (SoV) allows for the characterisation of the transfer matrix spectrum in the framework of the Quantum Inverse Scattering Method. Notably, one can describe models for which the reference state $|0\rangle$ used in Algebraic Bethe Ansatz cannot be found directly. As an example, we construct the explicit separate basis for the anti-periodic XXZ 1/2 spin chain and highlight on this example, the main motivations to work with this tool.

3.1 Classical separation of variables

In order to understand where the concept of quantum separation of variables comes from, we start recalling briefly the classical point of view. Thus, let us consider a classical system described by 2N canonical variables $(q_i)_{1 \leqslant i \leqslant N}$ and $(p_i)_{1 \leqslant i \leqslant N}$, and let us moreover suppose that it is Liouville integrable [16], thanks to N independent conserved quantities $(H_i)_{1 \leqslant i \leqslant N}$ in involution in the sense of the Poisson bracket $\{.,.\}$. This way we have:

$$\forall (i,j) \in [\![1,N]\!]^2, \quad \{q_i, q_j\} = \{p_i, p_j\} = 0 \quad \text{and} \quad \{q_i, p_j\} = \delta_{ij} \tag{3.1}$$

as well as

$$\forall (i,j) \in [\![1,N]\!]^2, \quad \{H_i, H_j\} = 0 \tag{3.2}$$

Then a set of 2N canonical variables $(x_i)_{1 \leqslant i \leqslant N}$ and $(v_i)_{1 \leqslant i \leqslant N}$ is called separate if there exists N separate relations of the form:

$$\forall i \in [\![1,N]\!], \quad f_i \left(x_i, v_i, H_1, ..., H_N\right) = 0 \tag{3.3}$$

linking, through some functions f_i, any couple (x_i, v_i) with the conserved charges.

The separation of variables is one of the most powerful method to deal with integrable systems. Its prime motivation is to transform the original problem in 2N variables (q_i's and p_i's) into a set of N independent problems involving only one variable and its conjugate (x_i and v_i). Indeed, the H_i's commuting with the Hamiltonian (in the sense of the Poisson bracket), they are constant and their value can be fixed. Equations (3.3) give then N relations, each one involving only one couple (x_i, v_i).

The standard procedure follows in general three steps to actually solve a system. First, one has to get a set of action-angle variables, defined such that their evolution equation is linear in time. Thus one can easily perform the time evolution of this set, and lastly one has to go back to the original variables at the time t, *via* an inverse transformation. Such a resolution by quadrature defines the Liouville integrability [16, 18].

The first part of this procedure can be solved by the Liouville-Arnold theorem [16]. In particular, if the phase space of dimension $2N$ is compact and connected, it is diffeormophic to a N-torus. The action variables are the conserved quantities $(H_i)_{1\leqslant i\leqslant N}$, and it is possible to define angle variables thanks to a generating function $S[(H_i)_{1\leqslant i\leqslant N}, (q_i)_{1\leqslant i\leqslant N}]$. The latter is defined as an integral on the level manifold where the $(H_i)_{1\leqslant i\leqslant N}$ are fixed, but hardly handleable in general as for a given $i \in [\![1, N]\!]$, p_i depends on the whole set $(q_i)_{1\leqslant i\leqslant N}$. However, if the set of variables $(q_i)_{1\leqslant i\leqslant N}$ and $(p_i)_{1\leqslant i\leqslant N}$ is separate, then one can use the separate relations (3.3) to simplify its expression. In particular, if it is possible to express p_i as a function of q_i (the quadrature), the integral splits into N one variable integrals, leading to:

$$S[(H_i)_{1\leqslant i\leqslant N}, (q_i)_{1\leqslant i\leqslant N}] = \sum_{k=1}^{N} S_k[(H_i)_{1\leqslant i\leqslant N}, q_k] \tag{3.4}$$

Each S_k, $k \in [\![1, N]\!]$, is now computable or at least handleable, and the angle variables can be computed.

A simple example of separate variables is given by the classical Hamiltonian describing the motion of a particle in a central potential. The use of the spherical coordinates r,θ, ϕ is indeed rather natural, given the symmetry of the system under rotations. But there is more in fact, as it decouples the original problem in the three variables x,y and z (orthonormal basis) in three independent problem in r,θ, ϕ. The generating function and the effective quadrature of this problem can be read for example in [18], in particular in paragraph 2.7.

3.1.1 Construction of separate variables

For a given integrable system there is in general no universal method to construct a set of canonical separate variables. However, when there exists a Lax pair ensuring the integrability of the system, a general construction allows to define such a set. As we will highlight here, the key relation to the construction of separate variables is the quadratic Poisson bracket for the Lax operators, involving the r-matrix (cf paragraph 1.1).

To make a clear statement, let $L(\lambda)$ be a Lax matrix of dimension 2×2, associated to a given system with phase space of dimension 2N. It depends on a spectral parameter λ, and of course on the canonical dynamical variables $\{q_i, p_i\}_{1\leqslant i\leqslant N}$ (we will omit writing them for simplicity):

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{3.5}$$

By definition, the conserved quantities H_i are obtained from the spectral invariants of $L(\lambda)$. Let us assume that the coefficients of the Lax matrix are polynomials in the spectral parameter, and that they satisfy the Sklyanin quadratic Poisson bracket:

$$\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda, \mu), L_1(\lambda)L_2(\mu)] \tag{3.6}$$

with the following r-matrix:

$$r_{12}(\lambda, \mu) = \frac{\mathcal{P}_{12}}{\lambda - \mu} \tag{3.7}$$

where $\mathcal{P}_{12}$ is the permutation operator between the spaces V_1 and V_2. This represents already a large class of models, as for instance the Toda chain [18, 240] satisfies these assumptions.

The main line of the method is to consider the eigenvalues of the Lax matrix. Let us denote this way $z(\lambda)$ an eigenvalue of this matrix and $\Omega(\lambda)$ the associated eigenvector, such that it holds:

$$L(\lambda)\Omega(\lambda) = z(\lambda)\Omega(\lambda) \tag{3.8}$$

The characteristic equation for the eigenvalue problem (3.8) is thus:

$$det\,(L(\lambda) - z(\lambda)\mathbb{1}) = 0 \tag{3.9}$$

It defines an algebraic curve Γ, the so-called spectral curve, which is the locus of the points λ and $z(\lambda)$ such that it holds (3.9). We shall come back to that notion later, but we can as of now claim that this object is at the heart of the separation of variables methods: it indeed provides separate relations. The cornerstone to understand this property is the following decomposition of the determinant entering in (3.9):

$$det\,(L(\lambda) - z(\lambda)) = z^2(\lambda) - z(\lambda)\; trL(\lambda) + det\; L(\lambda) \tag{3.10}$$

The trace and the determinant of the Lax matrix being spectral invariants, they can be expressed *via* the conserved quantities H_i. This way, the equation (3.9) only involves 2 variables, λ and $z(\lambda)$, and the conserved quantities.

In this context of 2-dimensional Lax matrices, it is easy to find eigenvalues. Indeed, let x_i be a zero of $B(\lambda)$. This variable, which non trivially depends on the dynamical variables of the system, is of prime interest as in such a point the Lax matrix is lower triangular:

$$L(x_i) = \begin{pmatrix} A(x_i) & 0 \\ C(x_i) & D(x_i) \end{pmatrix} \tag{3.11}$$

Thus the eigenvalues are immediately known, and in particular the couple $(x_i, A(x_i))$ belongs to the spectral curve (3.9). Let us denote $\gamma_i \equiv A(x_i)$.

The main feature with these variables is that we can moreover show that the couples (x_i, γ_i) can be easily used to construct separate variables, namely canonical variables. The quadratic Poisson bracket (3.6)-(3.7), i. e. the integrability in some sense, is the cornerstone of the method. Indeed, from these commutation relations it follows [18, 135]:

$$\{x_i, x_j\} = 0 \quad , \quad \{\gamma_i, \gamma_j\} = 0 \;\; \text{and} \quad \{\gamma_i, x_j\} = \gamma_i \delta_{ij} \tag{3.12}$$

We can mention that the first Poisson commutation relation comes from the commutativity of the polynomial family $B(\lambda)$, and that the two others can be computed using for instance from the Poisson bracket $\{B(\lambda), A(\mu)\}$.

Thus, the couples $(x_i, v_i = log(\gamma_i))$ form a set of canonical coordinates, and thus of separate variables with respect to the spectral curves (3.9).

Let us mention that a more general framework has been developed to deal with Lax matrices of arbitrary size $m \times m$. This makes use of the Baker-Akhiezer function, a properly normalised eigenvector of the Lax matrix (see [135] and references therein). This way the separate relations are still given by spectral curves. The variables x_i are obtained as the zeros of a function defined by a combination of the entries of the Lax matrix (which reduces to $B(\lambda)$ in dimension 2), while the v_i variables are obtained from the value in x_i of a function defined by another combination of the entries of the Lax matrix (which reduces to $A(\lambda)$ in dimension 2).

As we will see in the next paragraph, a very similar structure can be used to compute separate variables in quantum systems. In particular, the use of the operators $B(\lambda)$ and $A(\lambda)$ for 2 dimensional Lax operators is an efficient way to construct separate variables.

3.2 Quantum separation of variables

Let us now move to the quantum version of separation of variables, and let us consider a quantum integrable model having N degrees of freedom, described by an (operatorial) Hamiltonian H.

The main idea remains the same as for the classical case [133, 135, 241–247], namely to map the coupled N-variables spectral problem to a set of N one-variable (thus decoupled) soluble spectral problems, involving one operator and its canonical conjugate. Here, the notion of canonical operators has to be understood in the sense of the usual commutator, that is a set of 2N canonical operators $(X_i)_{1\leqslant i\leqslant N}$ and $(P_i)_{1\leqslant i\leqslant N}$ satisfies by definition the commutation relations:

$$\forall (i,j) \in [\![1,N]\!]^2, \quad [X_i, X_j] = [P_i, P_j] = 0 \quad \text{and} \quad [X_i, P_j] = i\hbar\delta_{ij} \tag{3.13}$$

Let us suppose that there exists a complete set of N commuting observables $(H_i)_{1\leqslant i\leqslant N}$, ensuring the integrability of the system:

$$\forall (i,j) \in [\![1,N]\!]^2, \quad [H_i, H_j] = [H_i, H] = 0 \tag{3.14}$$

The aim of the method is then to find a set of canonical operators $(X_i, P_i)_{1\leqslant i\leqslant N}$ satisfying N separate relations of the form:

$$\forall i \in [\![1,N]\!], \quad F_i\left(X_i, P_i, H_1, ..., H_N\right)|\Psi\rangle = 0 \tag{3.15}$$

where $|\Psi\rangle$ is the wave function.

Moreover, when the operators X_i are diagonalisable and with simple spectrum, the common eigenbasis is called the separate basis. In this basis $\langle \mathbf{x}| \equiv \langle x_1, ..., x_N|$, the wave function $\Psi(\mathbf{x}) \equiv \langle \mathbf{x}|\Psi\rangle = \Psi(x_1, ..., x_N)$ factorises in a product of $\Psi_i(x_i)$.

3.2.1 The example of the Hydrogen atom

Let us briefly illustrate the notion of quantum separation of variables with the quantum model of Hydrogen atom. In spherical coordinates, and in position representation, the Hamiltonian is given by:

$$\langle r,\theta,\phi|\, H = \langle r,\theta,\phi| \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial^2 r} + \frac{1}{2mr^2}L^2 - \frac{e^2}{r} \right] \tag{3.16}$$

where L is the angular momentum, depending only on θ and ϕ:

$$\langle r,\theta,\phi|\, L^2 = \langle r,\theta,\phi| \left[\frac{1}{tan(\theta)}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial^2\theta} + \frac{1}{sin^2(\theta)}\frac{\partial^2}{\partial^2\phi} \right] \tag{3.17}$$

A complete set of commuting observables is obtained with:

$$H_1 \equiv L_z \quad , \quad H_2 \equiv L^2 \quad \text{and} \quad H_3 \equiv H \quad \text{where } L_z = -i\hbar\frac{\partial}{\partial\phi} \tag{3.18}$$

as these operators satisfy:

$$[H, L^2] = [H, L_z] = [L^2, L_z] = 0 \tag{3.19}$$

Similarly to the classical case, the operators position X_i and impulsion P_i satisfy, in these coordinates, separate relations of the form (3.15) with:

$$F_1\left(\phi, -i\hbar\frac{\partial}{\partial\phi}, h_1(m)\right) = -i\hbar\frac{\partial}{\partial\phi} - h_1(m) \tag{3.20}$$

$$F_2\left(\theta, -i\hbar\frac{\partial}{\partial\theta}, h_1(m), h_2(l)\right) = \frac{1}{tan(\theta)}\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial^2\theta} - \frac{h_1^2(m)}{\hbar^2}\frac{1}{sin^2(\theta)} - h_2(l) \tag{3.21}$$

$$F_3\left(r, -i\hbar\frac{\partial}{\partial r}, h_2(k), h_3(k,l)\right) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial^2 r} + \frac{1}{2mr^2}h_2(l) - \frac{e^2}{r} - h_3(k,l) \tag{3.22}$$

One can solve these three equations, and the wave functions are separate and expressed as the product of the solutions:

$$\Psi(r,\theta,\phi) = \langle r,\theta,\phi|\Psi\rangle = R_{k,l}(r)F_{l,m}(\theta)e^{im\phi} \tag{3.23}$$

where $R_{k,l}(r)$ and $Y_{l,m}(\theta,\phi) \equiv F_{l,m}(\theta)e^{im\phi}$ are respectively the radial part, solution of (3.22) (expressed *via* Laguerre polynomials), and the spherical harmonics. $F_{l,m}(\theta)$ and $e^{im\phi}$ are the respective solutions of (3.21) and (3.20). We emphasise that the wave functions are univocally described by the triplet (k,l,m) of quantum numbers which describe all the eigenvalues of the Hamiltonian, which is the essence of the integrability.

To generalise this notion to integrable systems with N degrees of freedom is the aim of the next paragraph. In particular, we show how to define the separate operators in the framework of the 6-vertex Yang-Baxter algebra.

3.2.2 Construction of separate variables from the 6-vertex Yang-Baxter algebra

As for classical systems, there is in general no standard method to get quantum separation of variables for a given quantum system. However, as already mentioned, the Lax operators in the framework of Quantum Inverse Scattering Method can be wisely used, and provide the necessary ingredients.

The first one to obtain results this way has been Sklyanin for the Toda chain [133], relying on the works of Gutzwiller [246] and Komarov [247]. It led to the so called functional Bethe ansatz. The method has thereafter been well developed and many models were characterised, one can cite among others the XXX [137, 138] and XXZ quasi-periodic spin chain [148, 248], the sinh Gordon model [249], the sine Gordon model [250] or even the more general τ_2 chain [235]. See also the works [136, 139, 140].

In the framework of the 6-vertex Yang-Baxter algebra, Sklyanin's approach to the separation of variables is the following. The aim being to find a separate basis for the transfer matrix spectral problem, i.e. a basis where this problem separates in independent one variable problems, let us consider the standard monodromy matrix (1.21). If the operator $B(\lambda)$ is diagonalisable and with simple spectrum, the commutativity (1.22) of this family shows the existence of a common eigenbasis. Then we can consider the zero operators $\hat{\eta}_i$ of $B(\lambda)$, diagonal operators in the eigenbasis, such that (for example for $B(\lambda)$ polynomial of degree N in λ):

$$B(\lambda) = B_0 \prod_{i=1}^{N} (\lambda - \hat{\eta}_i) \tag{3.24}$$

The $\hat{\eta}_i$ being diagonals in the same basis, they commute between themselves.

Then, as in the classical case, we use the integrability to construct the canonical operator associated to $\hat{\eta}_i$. And very similarly to the classical case, using the 6-vertex Yang-Baxter algebra one can show that the operators $A(\eta_i)$, where η_i are the eigenvalues of $\hat{\eta}_i$, are of prime interest. Indeed, let us parametrise the eigenbasis of $B(\lambda)$ by the eigenvalues η_i, i.e. let us consider the covectors $\langle \eta_1, ..., \eta_N|$ such that it holds:

$$\langle \eta_1, ..., \eta_N| \, B(\lambda) = b_\eta(\lambda) \, \langle \eta_1, ..., \eta_N| \quad \text{with} \quad b_\eta(\lambda) \equiv b_0 \prod_{i=1}^{N} (\lambda - \eta_i) \tag{3.25}$$

Using the simplicity of spectrum of the operator $B(\lambda)$, and the Yang-Baxter algebra, one can show that the action of $A(\eta_i)$ and $D(\eta_i)$ on the elements $\langle \eta_1, ..., \eta_N|$ are simply shifts on the zero η_i solely:

$$\langle \eta_1, ..., \eta_i, ..., \eta_N| \, A(\eta_i) \propto \langle \eta_1, ..., q\,\eta_i, ..., \eta_N| \tag{3.26}$$

$$\langle \eta_1, ..., \eta_i, ..., \eta_N| \, D(\eta_i) \propto \langle \eta_1, ..., \eta_i/q, ..., \eta_N| \tag{3.27}$$

The action of the operators $A(\eta_i)$ is known only on the basis elements containing η_i, namely $\langle \eta_1, ..., \eta_i, ..., \eta_N|$. However, up to some assumptions (e. g. the polynomiality of $A(\lambda)$), the action on $\mathcal{H}$ of $A(\lambda)$ can in general be obtained (e. g. thanks to an interpolation formula).

Thus in this basis, the spectral problem for $\mathcal{T}(\lambda) = A(\lambda) + D(\lambda)$ can be written in terms of one variable equations only, the so-called Baxter equations. And the diagonalisation basis of $B(\lambda)$ is a separate basis.

Let us now see an explicit example to deal with concrete operators, in the case of the antiperiodic

XXZ 1/2 spin chain.

3.2.3 The example of the XXZ 1/2 antiperiodic spin chain

Here we present the method on the simple and representative example of the XXZ 1/2 antiperiodic spin chain, the generalisation to other representations of the 6-vertex Yang-Baxter algebra being discussed just after. This way we consider the chain described in paragraph 2.1.1, in particular the one associated to the twist $\Sigma_0 = \Sigma_0^{(0,1)} = \sigma_0^x$. The corresponding monodromy will be denoted:

$$\bar{M}_{0Q}(\lambda) \equiv M_{0Q}^{(0,1)}(\lambda) \equiv \begin{pmatrix} \bar{A}(\lambda) & \bar{B}(\lambda) \\ \bar{C}(\lambda) & \bar{D}(\lambda) \end{pmatrix} \tag{3.28}$$

and we will consider the associated antiperiodic transfer matrix:

$$\mathcal{T}^{(0,1)}(\lambda) \equiv \bar{\mathcal{T}}(\lambda) = tr_0 \left\{ \bar{M}_{0Q}(\lambda) \right\} \tag{3.29}$$

The distinction with a bar notation emphasises on the difference between these operators and the operators associated to the periodic transfer matrix, which will be denoted here by:

$$M_{0Q}(\lambda) \equiv M_{0Q}^{(0,0)}(\lambda) \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \quad \text{leading to} \quad \begin{pmatrix} \bar{A}(\lambda) & \bar{B}(\lambda) \\ \bar{C}(\lambda) & \bar{D}(\lambda) \end{pmatrix} = \begin{pmatrix} C(\lambda) & D(\lambda) \\ A(\lambda) & B(\lambda) \end{pmatrix} \tag{3.30}$$

In particular, the quantum determinant reads, using the re-writings of paragraph 1.3:

$$\begin{aligned} q\text{-}det\ \bar{M}_{0Q}(\lambda) &= \bar{A}(\lambda)\bar{D}(\lambda/q) - \bar{B}(\lambda)\bar{C}(\lambda/q) && (3.31) \\ &= C(\lambda)B(\lambda/q) - D(\lambda)A(\lambda/q) && (3.32) \\ &= -\ q\text{-}det\ M_{0Q}(\lambda) && (3.33) \end{aligned}$$

Moreover, for instance using the reference state $\langle 0|$ given by (1.66)-(1.67), the quantum determinant can be written:

$$q\text{-}det\ \bar{M}_{0Q}(\lambda) = \bar{\mathrm{A}}(\lambda)\bar{\mathrm{D}}(\lambda/q) = -\mathrm{A}(\lambda)\mathrm{D}(\lambda/q) \tag{3.34}$$

where the functions $\mathrm{A}(\lambda)$ and $\mathrm{D}(\lambda)$ are the same as in (1.57) and we can choose:

$$\bar{\mathrm{A}}(\lambda) \equiv -\mathrm{A}(\lambda) = -\prod_{n=1}^{N} (\lambda\ q/\xi_n - \xi_n/(\lambda\ q)) \quad \text{and} \quad \bar{\mathrm{D}}(\lambda) \equiv \mathrm{D}(\lambda) = \prod_{n=1}^{N} (\lambda/\xi_n - \xi_n/\lambda) \tag{3.35}$$

Let us now sketch the proof that the operator $\bar{B}(\lambda)$ is diagonalisable and with simple spectrum, and that its diagonalisation basis is a separate basis for the transfer matrix spectral problem. Here we reproduce the results given in [148], where the proof follows from the explicit construction of the diagonalisation basis. Let us also remark that $\bar{B}(\lambda) = D(\lambda)$, and that the eigenbasis of $D(\lambda)$ is the so-called F-basis of [251, 252].

The key observation is that the operators $\bar{A}(\lambda)$ and $\bar{D}(\lambda)$ are Laurent polynomials of degree $N-1$ in λ, respectively even for N odd and odd for N even. Thus, we only need to know their action in N different points to know their action on the whole space, by a Lagrange interpolation formula. The same holds thus for the antiperiodic transfer matrix $\bar{\mathcal{T}}(\lambda) = \bar{A}(\lambda) + \bar{D}(\lambda)$.

Diagonalisation Let $\langle 0|$ be the reference state with all spins up. By its very definition, it satisfies:

$$\langle 0|\, \bar{B}(\lambda) = \mathrm{D}(\lambda)\, \langle 0| \tag{3.36}$$

The key point to remark is that, due to the Yang-Baxter algebra (1.24), we have:

$$\forall (\lambda, \mu) \in \mathbb{C}^2, \quad \bar{A}(\mu)\bar{B}(\lambda) = f(\lambda, \mu)\bar{B}(\lambda)\bar{A}(\mu) + g(\lambda, \mu)\bar{B}(\mu)\bar{A}(\lambda) \tag{3.37}$$

with the scalar functions:

$$f(\lambda,\mu)=\frac{q\lambda/\mu-\mu/(\lambda q)}{\lambda/\mu-\mu/\lambda}\quad\text{and}\quad g(\lambda,\mu)=\frac{q-1/q}{\lambda/\mu-\mu/\lambda}\tag{3.38}$$

Thus, by acting on the state $\langle 0|$ with $\bar{A}(\mu)$, for μ a root of $\mathrm{D}(\lambda)$ we have that $\langle 0|\,\bar{A}(\mu)$ is another eigenstate of $\bar{B}(\lambda)$, with the eigenvalue $f(\lambda,\mu)\mathrm{D}(\lambda)$. Repeating this procedure for the different roots ξ_n of $\mathrm{D}(\lambda)$, we can consider $\langle h_1,...,h_N|$ the following states:

$$\langle h_1,...,h_N|\equiv\langle\mathbf{h}|\equiv\frac{1}{\mathcal{N}}\langle 0|\prod_{n=1}^{N}\left(\frac{\bar{A}(\xi_n)}{\bar{\mathrm{A}}(\xi_n)}\right)^{h_n}\quad\text{with}\quad\forall n\in[\![1,N]\!]\ ,\ h_n\in\{0,1\}\tag{3.39}$$

where $\mathcal{N}$ is a global normalisation (it is moreover convenient to make explicit the term $\bar{\mathrm{A}}(\xi_n)$ of the quantum determinant). Then one easily gets that the states $\langle\mathbf{h}|$ are eigenvectors of $\bar{B}(\lambda)$:

$$\langle\mathbf{h}|\,\bar{B}(\lambda)=b_{h_1,...,h_N}(\lambda)\,\langle\mathbf{h}|\quad\text{with}\quad b_{h_1,...,h_N}(\lambda)=\prod_{a=1}^{N}\left(\frac{\lambda q^{h_a}}{\xi_a}-\frac{\xi_a}{\lambda q^{h_a}}\right)\tag{3.40}$$

Let us sketch the proof of (3.40). First, using the commutativity of the $A(\lambda)$ family, one can write:

$$\forall a\in[\![1,N]\!]\ ,\quad\langle\mathbf{h}|=\frac{1}{\mathcal{N}}\langle 0|\left(\frac{\bar{A}(\xi_a)}{\bar{\mathrm{A}}(\xi_a)}\right)^{h_a}\prod_{\substack{n=1\\n\neq a}}^{N}\left(\frac{\bar{A}(\xi_n)}{\bar{\mathrm{A}}(\xi_n)}\right)^{h_n}\tag{3.41}$$

Applying thus the commutation (3.37) with $\mu=\xi_a$, it holds:

$$\begin{aligned}\langle\mathbf{h}|\,\bar{B}(\lambda)=\mathrm{D}(\lambda)\prod_{a=1}^{N}f(\lambda,\xi_a)^{h_a}\,\langle\mathbf{h}|+&\\ \sum_{a=1}^{N}\frac{1}{\mathcal{N}}\mathrm{D}(\xi_a)g(\lambda,\xi_a)\delta_{h_a,1}\prod_{\substack{n=1\\n\neq a}}^{N}f(\lambda,\xi_n)^{h_n}\,\langle 0|\,\frac{\bar{A}(\lambda)}{\bar{\mathrm{A}}(\xi_a)}\prod_{\substack{n=1\\n\neq a}}^{N}\left(\frac{\bar{A}(\xi_n)}{\bar{\mathrm{A}}(\xi_n)}\right)^{h_n}&\end{aligned}\tag{3.42}$$

Let us comment that the first term of (3.42) is obtained using the first term of the right hand side of equation (3.37) to commute all the $\bar{A}(\xi_n)$ with $\bar{B}(\lambda)$, while the generic term (a) in the sum comes from the use of the first term of the right hand side of equation (3.37) to commute all the $\bar{A}(\xi_n)$, $n\neq a$, with $\bar{B}(\lambda)$ but the second one to commute the $\bar{A}(\xi_a)$ with $\bar{B}(\lambda)$. Using the fact that $\mathrm{D}(\xi_a)=0$, one eventually gets (3.40).

Thus, for generic inhomogeneities satisfying:

$$\forall(n,m)\in[\![1,N]\!]\ ,\forall j\in\{-1,0,1\}\ ,\quad\xi_n\neq q^j\xi_m\quad\text{for }m\neq n\tag{3.43}$$

the states $\langle\mathbf{h}|$ form a set of 2^N independent vectors and so they are the eigenbasis of $\bar{B}(\lambda)$ which is therefore diagonalisable and with simple spectrum (to each eigenvalue is associated a unique eigenvector).

Let us comment that the eigenvalues η_n of the zeros of $\bar{B}(\lambda)$ are given by:

$$\eta_n=\xi_n/q^{h_n}\quad\text{for}\quad h_n\in\{0,1\}\tag{3.44}$$

In the parametrisation (3.39) of the eigenbasis, we use the constants h_n to distinguish the different states, but the basis is genuinely parametrised by the eigenvalues of the zeros of $\bar{B}(\lambda)$, as h_n and η_n are in one-to-one correspondence.

Separate relations Following the general procedure introduced in 3.2.2, let us now compute the action of the operator $\bar{A}(\eta_i)$ on the eigenbasis elements containing η_i, namely $\langle\eta_1,...,\eta_i,...,\eta_N|$. As we explain here, this allows to show that the eigenbasis of $\bar{B}(\lambda)$ is indeed a separate basis.

From the definition of the basis elements $\langle\mathbf{h}|$, it follows:

$$\langle\mathbf{h}|\,\bar{A}(\xi_i/q^{h_i}) = \bar{\mathrm{A}}(\xi_i/q^{h_i})\,\langle h_1,...,h_i+1,...,h_N| \tag{3.45}$$

Using then the quantum determinant (3.31) and (3.34), it holds:

$$\langle h_1,...,h_i+1,...,h_N|\,\bar{D}(\xi_i/q^{h_i+1}) = \bar{\mathrm{D}}(\xi_i/q^{h_i+1})\,\langle\mathbf{h}| \tag{3.46}$$

As we highlighted in 3.2.2, the operators $\bar{A}(\xi_i/q^{h_i})$ and $\bar{D}(\xi_i/q^{h_i+1})$ simply shift the eigenvalue η_i when they act on the states $\langle h_1,...,h_i,...,h_N|$ and $\langle h_1,...,h_i+1,...,h_N|$ respectively. With these known actions, a simple Lagrange interpolation formula gives the action of $\bar{A}(\lambda)$ and $\bar{D}(\lambda)$.

We can now consider the spectral problem of the transfer matrix $\bar{\mathcal{T}}(\lambda) = \bar{A}(\lambda) + \bar{D}(\lambda)$. So let $|\bar{t}\rangle$ be an eigenvector of the transfer matrix, associated to the eigenvalue $\bar{t}(\lambda)$:

$$\bar{\mathcal{T}}(\lambda)\,|\bar{t}\rangle = \bar{t}(\lambda)\,|\bar{t}\rangle \tag{3.47}$$

and let us denote by $\Psi_{\bar{t}}(\mathbf{h})$ the wave functions in this basis:

$$\Psi_{\bar{t}}(\mathbf{h}) \equiv \langle\mathbf{h}|\bar{t}\rangle \tag{3.48}$$

Computing $\langle\mathbf{h}|\,\bar{\mathcal{T}}(\lambda)\,|\bar{t}\rangle$ thanks to the action of $\bar{\mathcal{T}}(\lambda)$ on the right and on the left, one easily obtains the so called Baxter equations:

$$\begin{aligned}&\forall n \in [\![1,N]\!]\ ,\\ &\quad \bar{t}(\xi_n/q^{h_n})\Psi_{\bar{t}}(\mathbf{h}) = \bar{\mathrm{A}}(\xi_n/q^{h_n})\Psi_{\bar{t}}(h_1,...,h_n+1,...h_N) + \bar{\mathrm{D}}(\xi_n/q^{h_n})\Psi_{\bar{t}}(h_1,...,h_n-1,...h_N)\end{aligned} \tag{3.49}$$

These are the separate relations (3.15) in the eigenbasis. Indeed, each of these N equations only involves the parameter η_n (or h_n equivalently). Thus, let $Q_{\bar{t}}(h_n)$ be a solution of the Baxter equation (3.49). It holds the following separate form of the wave functions:

$$\Psi_{\bar{t}}(\mathbf{h}) = \prod_{n=1}^{N} Q_{\bar{t}}(h_n) \tag{3.50}$$

It is worth emphasising once again on the analogy with the classical case, notably as the Baxter equations can be seen as the analogue of the spectral curve (3.9).

We would like to end this paragraph using the example of the XXZ antiperiodic spin chain to highlight the main features of the separation of variables method.

In particular, the spectrum of $\bar{\mathcal{T}}(\lambda)$ is simple (i.e. to any eigenvalue is associated a unique eigenvector), and the eigenvalues are characterised, in a class of function, by the solutions to a discrete system of equations.

Indeed, the eigenvalues of the transfer matrix inherit the symmetries of the transfer matrix itself. Here, this operator is a Laurent polynomial in λ of degree $N-1$, with the opposite parity with respect to the parity of N. And so are the eigenvalues.

Then, it is worth rewriting the Baxter equations in a matrix form:

$$\forall n \in [\![1,N]\!]\ , \quad \begin{pmatrix} \bar{t}(\xi_n) & -\bar{\mathrm{A}}(\xi_n) \\ -\bar{\mathrm{D}}(\xi_n/q) & \bar{t}(\xi_n/q) \end{pmatrix} \begin{pmatrix} \Psi_{\bar{t}}(h_1,...,h_n=0,...h_N) \\ \Psi_{\bar{t}}(h_1,...,h_n=1,...h_N) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.51}$$

Thus, if $\bar{t}(\lambda)$ is an eigenvalue of $\bar{\mathcal{T}}(\lambda)$, the previous system has to have a non trivial solution, and so the eigenvalues are characterised by functions satisfying the previous symmetries, and such that

$$det\ D_{\bar{t}}(\xi_n) = 0 \tag{3.52}$$

where $D_{\bar{t}}(\xi_n)$ are the 2×2 matrices entering in (3.51). On top of that, the rank of this matrix being one, the solution is unique (up to normalisation), and $\bar{\mathcal{T}}$-spectrum is simple.

Let us comment that the condition (3.52) is written:

$$\bar{t}(\xi_n)\bar{t}(\xi_n/q) - q\text{-}det\ \bar{M}_{0Q}(\xi_n) = 0 \tag{3.53}$$

It is interesting to note that the characterisation (3.52) appears thus as the fusion relation [210, 237] for the transfer matrix $\bar{\mathcal{T}}(\lambda)$ in the point $\lambda = \xi_n$.

Moreover, the provided description of the spectrum is complete in the following sense: not only any eigenvector of the transfer matrix is characterised by the Baxter equations (3.49), but also any solution of the Baxter equations characterises an eigenvector.

Indeed, let $\bar{t}(\lambda)$ be a function in the class previously defined (i.e. a Laurent polynomial of degree $N-1$ in λ, with the opposite parity with respect to the parity of N), let $Q_{\bar{t}}(h_n)$ denote solutions to the associated Baxter system (3.49), and let us consider:

$$\left|\bar{t}\right\rangle = \sum_{h_1,\ldots,h_N=0}^{1} \prod_{n=1}^{N} Q_{\bar{t}}(h_n) \left(\prod_{1\leqslant b<a\leqslant N} \eta_a/\eta_b - \eta_b/\eta_a \right)^{-1} |h_1, \ldots, h_N\rangle \tag{3.54}$$

This is equivalent to the factorisation property (3.50) when one uses the following decomposition of the identity:

$$\mathbb{1} = \sum_{h_1,\ldots,h_N=0}^{1} \left(\prod_{1\leqslant b<a\leqslant N} \eta_a/\eta_b - \eta_b/\eta_a \right)^{-1} |h_1, \ldots, h_N\rangle\langle h_1, \ldots, h_N| \tag{3.55}$$

Then, it holds:

$$\forall n \in [\![1, N]\!] \ , \quad \langle h_1, \ldots, h_N|\, \bar{\mathcal{T}}(\eta_n) \left|\bar{t}\right\rangle = \bar{t}(\eta_n) \left\langle h_1, \ldots, h_N \middle| \bar{t}\right\rangle \tag{3.56}$$

Thus, being $\bar{\mathcal{T}}(\lambda)$ a polynomial of degree $N-1$ in λ, it holds:

$$\langle h_1, \ldots, h_N|\, \bar{\mathcal{T}}(\lambda) \left|\bar{t}\right\rangle = \bar{t}(\lambda) \left\langle h_1, \ldots, h_N \middle| \bar{t}\right\rangle \tag{3.57}$$

that is $\bar{t}(\lambda)$ is an eigenvalue and $\left|\bar{t}\right\rangle$ given by (3.54) the associated eigenvector.

3.2.4 A powerful and promising tool

The quantum separation of variables exposed here can be generalised to different representations. This method is *a priori* more generally applicable than the Algebraic Bethe Ansatz, as no reference state is needed. In the example of the XXZ 1/2 anti-periodic spin chain, such a state is used to give an explicit basis of diagonalisation of the operator $\bar{B}(\lambda)$. And here lies maybe the only limitation: in order for the method to be applicable, the operator $B(\lambda)$ of a generic transfer matrix must be diagonalisable and with simple spectrum. For instance, the $B(\lambda)$ operator associated to the periodic transfer matrix of the XXZ 1/2 spin chain is not diagonalisable (it is nilpotent) whereas the ones associated to all the other quasi-periodic transfer matrices are. Still there are some conditions for the separation of variables method to be applicable, but simplicity of spectrum for $B(\lambda)$ is related to the inhomogeneity parameters being in generic position, see (3.43).

Eventually, let us emphasise two points that emerged in all the models studied so far with the quantum separation of variables [148, 235, 253]. First, one can show, almost by construction, the completeness of the description of the eigenvectors, i.e. all the eigenvectors of the transfer matrix are described this way. Second, this technique leads to the simplicity of spectrum of the transfer matrix, hence characterising the integrability (up to the fact that the Hamiltonian commutes with this family).

In a nutshell, the quantum separation of variables seems very promising as compared to other methods to analyse the spectral problem (Algebraic Bethe Ansatz, Baxter Q-operator, analytic Bethe ansatz) as it works in a large class of models for which the others do not apply. It characterises both the spectrum and the eigenvectors, the completeness of the description is almost straightforward and the same way it is not a hard task to prove the simplicity of spectrum of the transfer matrix.

3.2.5 Within the 6-vertex reflection algebra

The question of quantum separation of variables henceforth naturally arises for models with general integrable boundary conditions. For some models associated to the 6-vertex reflection algebra for example, with triangular boundary matrices, the operator $\mathcal{B}(\lambda)$ can be used to construct a separate basis.

But to deal with the general case is more involved, this problem is at the heart of the next chapter, considering cyclic representations of the 6-vertex reflection algebra.

Chapter 4

Boundary transfer matrix spectrum by quantum separation of variables

In this chapter we study the boundary transfer matrix spectral problem for cyclic representations of the 6-vertex reflection algebra associated to the Bazahnaov-Stroganov Lax operator. As already mentioned in the introduction, particular cases include the spin s chains with the anisotropy coupling at root of unity, the chiral Potts model or the discretised sine Gordon model at root of unity.

Implementing the quantum separation of variables for these representations with the most general integrable boundaries, we will characterise in two different ways the spectrum of the transfer matrix: a discrete characterisation, as the set of solutions to a discrete system of polynomial equations in a given class of functions, and an equivalent characterisation as the set of solutions to a functional Baxter like TQ equation.

In order to be more understandable, we present our method in two steps. The first one is to consider one general $K(\lambda)$ matrix (left boundary) and one triangular $K_+(\lambda)$ matrix (right boundary), such that in this case a separate basis is obtained as the diagonalisation basis of $\mathcal{B}(\lambda)$. Then, to allow the description of completely general integrable boundaries, we will introduce a gauge transformation. Wisely choosing the gauge parameters, we will be able to deal with a gauged 6-vertex reflection algebra and we will put the gauge right boundary matrix triangular, leading to the framework of the case treated before.

The results described in this chapter have been published in the two articles **I** and **II**.

4.1 The model in the framework of the Quantum Inverse Scattering Method

To begin with, we make explicit some key properties and symmetries of the cyclic representations of the 6-vertex reflection algebra. Thanks to Sklyanin's construction of the boundary transfer matrix [177], we consider cyclic representations of the 6-vertex Yang-Baxter algebra, i.e. the Bazhanov-Stroganov Lax operators (1.72), and define the monodromy matrix with a shift (cf paragraph 2.2.2):

$$M_{0Q}(\lambda, p_Q) = L_{0N}(\lambda/q^{1/2}, p_N)...L_{01}(\lambda/q^{1/2}, p_1) \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{4.1}$$

We recall that $p_Q = (p_1, ..., p_N)$ denotes the parameters in the whole quantum space. The expression for the quantum determinant is thus the following shifted one:

$$q\text{-}det\ M_{0Q}(\lambda, p_Q) = A(\lambda q^{1/2})D(\lambda q^{-1/2}) - B(\lambda q^{1/2})C(\lambda q^{-1/2}) = a(\lambda)d(\lambda/q) \tag{4.2}$$

where the functions $a(\lambda)$ and $d(\lambda)$ have been introduced in (1.86).

Let $K_0(\lambda)$ and $K_{+0}(\lambda)$ be the general scalar solutions of the reflection equations (2.25) and (2.27).

The boundary monodromy matrix is defined in the standard way:

$$\mathcal{U}_{0Q}(\lambda, p_Q) = M_{0Q}(\lambda, p_Q)K_0(\lambda)\hat{M}_{0Q}(\lambda, p_Q) \quad \text{with} \quad \hat{M}_{0Q}(\lambda, p_Q) = (-1)^N \sigma_0^y M_{0Q}^{t_0}(1/\lambda, p_Q)\sigma_0^y \tag{4.3}$$

as well as the boundary transfer matrix:

$$\mathcal{T}(\lambda) = tr_0 \left\{K_{+0}(\lambda)\mathcal{U}_{0Q}(\lambda, p_Q)\right\} \tag{4.4}$$

Following the same proof given in Sklyanin's paper [177], one shows that this is a one parameter family of commuting operators and our aim here is to characterise its spectrum. Let us recall here some notations and properties of these representations.

First, we can compute the quantum determinant (2.29) associated to the boundary matrices, it holds:

$$q\text{-}det\ K_0(\lambda) = \left(\lambda^2/q^2 - q^2/\lambda^2\right) g_-(\lambda q^{1/2})g_-(q^{1/2}/\lambda) \tag{4.5}$$

$$\text{and} \quad q\text{-}det\ K_{+0}(\lambda) = -\left(\lambda^2 q^2 - 1/(\lambda^2 q^2)\right) g_+(\lambda q^{1/2})g_+(q^{1/2}/\lambda) \tag{4.6}$$

with the functions:

$$g_\pm(\lambda) \equiv \frac{(\lambda\alpha_\pm/q^{1/2} - q^{1/2}/(\lambda\alpha_\pm))(\lambda\beta_\pm/q^{1/2} + q^{1/2}/(\lambda\beta_\pm))}{(\alpha_\pm - 1/\alpha_\pm)\left(\beta_\pm + 1/\beta_\pm\right)} \tag{4.7}$$

where we used a more convenient parametrisation for the boundary parameters:

$$(\alpha_\pm - 1/\alpha_\pm)\left(\beta_\pm + 1/\beta_\pm\right) \equiv \frac{\xi_\pm - 1/\xi_\pm}{\kappa_\pm} \quad \text{and} \quad (\alpha_\pm + 1/\alpha_\pm)\left(\beta_\pm - 1/\beta_\pm\right) \equiv \frac{\xi_\pm + 1/\xi_\pm}{\kappa_\pm} \tag{4.8}$$

Let $\mathcal{A}(\lambda), \mathcal{B}(\lambda), \mathcal{C}(\lambda)$ and $\mathcal{D}(\lambda)$ be the elements of the boundary monodromy matrix, namely the generators of the 6-vertex reflection algebra. Then, following Sklyanin's paper [177], one can show the symmetries:

$$\mathcal{D}(\lambda) = \frac{(\lambda^2/q - q/\lambda^2)}{(\lambda^2 - 1/\lambda^2)}\mathcal{A}(\lambda^{-1}) + \frac{(q - 1/q)}{(\lambda^2 - 1/\lambda^2)}\mathcal{A}(\lambda) \tag{4.9}$$

as well as

$$\mathcal{B}(\lambda^{-1}) = -\frac{(\lambda^2 q - 1/\left(q\lambda^2\right))}{(\lambda^2/q - q/\lambda^2)}\mathcal{B}(\lambda) \quad \text{and} \quad \mathcal{C}(\lambda^{-1}) = -\frac{(\lambda^2 q - 1/\left(q\lambda^2\right))}{(\lambda^2/q - q/\lambda^2)}\mathcal{C}(\lambda) \tag{4.10}$$

Moreover, by using the previous symmetries, and the definition (2.29), the boundary quantum determinant can be expressed:

$$q\text{-}det\ \mathcal{U}_{0Q}(\lambda, p_Q) = \left(\lambda^2/q^2 - q^2/\lambda^2\right)\left[\mathcal{A}(\lambda q^{1/2})\mathcal{A}(q^{1/2}/\lambda) + \mathcal{B}(\lambda q^{1/2})\mathcal{C}(q^{1/2}/\lambda)\right] \tag{4.11}$$

or equivalently:

$$q\text{-}det\ \mathcal{U}_{0Q}(\lambda, p_Q) = \left(\lambda^2/q^2 - q^2/\lambda^2\right)\left[\mathcal{D}(\lambda q^{1/2})\mathcal{D}(q^{1/2}/\lambda) + \mathcal{C}(\lambda q^{1/2})\mathcal{B}(q^{1/2}/\lambda)\right] \tag{4.12}$$

Moreover, we have the factorisation:

$$q\text{-}det\ \mathcal{U}_{0Q}(\lambda, p_Q) = q\text{-}det\ K_0(\lambda)\ q\text{-}det\ M_{0Q}(\lambda, p_Q)\ q\text{-}det\ M_{0Q}(1/\lambda, p_Q) \tag{4.13}$$

leading to the expression:

$$q\text{-}det\ \mathcal{U}_{0Q}(\lambda, p_Q) = (\lambda^2/q^2 - q^2/\lambda^2)\mathsf{A}_-(\lambda q^{1/2})\mathsf{A}_-(q^{1/2}/\lambda) \tag{4.14}$$

with the function:

$$\mathsf{A}_-(\lambda) \equiv g_-(\lambda)a(\lambda q^{-1/2})d(1/(q^{1/2}\lambda)) \tag{4.15}$$

Similarly, we introduce (for further use) the function:

$$\mathsf{D}_-(\lambda) \equiv k(\lambda)\mathsf{A}_-(q/\lambda) \quad \text{where} \quad k(\lambda) = \left(\lambda^2 - 1/\lambda^2\right)/(\lambda^2/q^2 - q^2/\lambda^2) \tag{4.16}$$

which is such that the boundary quantum determinant can also be written:

$$q\text{-}det\, \mathcal{U}_{0Q}(\lambda, p_Q) = (\lambda^2/q^2 - q^2/\lambda^2)\mathsf{D}_-(\lambda q^{1/2})\mathsf{D}_-(q^{1/2}/\lambda) \tag{4.17}$$

Furthermore, let us introduce the coefficients:

$$\mathsf{a}_+(\lambda) \equiv \frac{(\lambda^2 q - 1/\left(q\lambda^2\right))(\lambda\xi_+/q^{1/2} - q^{1/2}/(\lambda\xi_+))}{(\lambda^2 - 1/\lambda^2)(\xi_+ - 1/\xi_+)} \tag{4.18}$$

$$\text{and} \quad \mathsf{d}_+(\lambda) \equiv \frac{(\lambda^2 q - 1/\left(q\lambda^2\right))(\xi_+ q^{1/2}/\lambda - \lambda/(q^{1/2}\xi_+))}{(\lambda^2 - 1/\lambda^2)(\xi_+ - 1/\xi_+)} \tag{4.19}$$

It follows two important rewritings of the boundary transfer matrix (2.35):

$$\mathcal{T}(\lambda) = \mathsf{a}_+(\lambda)\mathcal{A}(\lambda) + \mathsf{a}_+(1/\lambda)\mathcal{A}(1/\lambda) + c_+(\lambda)\mathcal{B}(\lambda) + b_+(\lambda)\mathcal{C}(\lambda) \tag{4.20}$$

$$= \mathsf{d}_+(\lambda)\mathcal{D}(\lambda) + \mathsf{d}_+(1/\lambda)\mathcal{D}(1/\lambda) + c_+(\lambda)\mathcal{B}(\lambda) + b_+(\lambda)\mathcal{C}(\lambda) \tag{4.21}$$

Lastly, using the symmetries (4.10) and the coefficients of the boundary transfer matrix, we can easily show the following proposition.

Proposition 4.1.1. *The most general boundary transfer matrix admits the following symmetries:*

$$\mathcal{T}(\lambda) = \mathcal{T}(1/\lambda) \quad \textit{and} \quad \mathcal{T}(-\lambda) = \mathcal{T}(\lambda) \tag{4.22}$$

This symmetry is thus satisfied for all the parameters and it will be used in the spectrum characterisation.

Now that the main notations have been introduced, and some symmetries characterised, we can tackle the spectral problem of this operator family.

4.2 The case of a triangular right boundary matrix

As stated previously, we start the characterisation of the spectrum for models with the special boundary condition associated to a general $K(\lambda)$ (left boundary) and a lower triangular $K_+(\lambda)$ (right boundary) to give a clear example. It lays the foundations of the quantum separation of variables method which will be then generalised to the case of general left and right boundary matrices.

The idea developed here is a straightforward generalisation of the concept of separation of variables developed for the Yang-Baxter algebra: we use the operator $\mathcal{B}(\lambda)$ of the boundary monodromy matrix to construct a separate basis. We first show that this operator is indeed diagonalisable and with simple spectrum, then we show that the diagonalisation basis is indeed a separate basis for this particular transfer matrix spectral problem, and we finally give a discrete characterisation of the spectrum.

The procedure to show that $\mathcal{B}(\lambda)$ is diagonalisable is as follows. First, we consider a subclass of parameters for which we can show the proposition thanks to an explicit construction, *via* an explicit reference state. This is done in the proposition 4.2.1. Then for the most general parameters, we use a general argument to state the existence of a reference state, and from the explicit case treated before we use a continuity argument to show that the proposition holds for almost any value of the parameters.

4.2.1 Prelude: an explicit construction of $\mathcal{B}(\lambda)$ diagonalisation basis for a subclass of parameters

In this paragraph, we show the diagonalisability of the operator $\mathcal{B}(\lambda)$ for a general boundary matrix $K(\lambda)$ but with the following extra constraint on the quantum parameters:

$$\forall n \in [\![1, \mathsf{N}]\!] \,, \quad b_n^p + a_n^p = 0 \tag{4.23}$$

This constraint allows us to explicitly construct a diagonalisation basis in this subclass of parameters, thanks to an explicit reference state. Indeed, from (4.23) we can construct, by a tensor product of simple local states[1], a covector $\langle\Omega|$ and a vector $|\bar{\Omega}\rangle$ which are eigenstates of the operators $A(\lambda)$ and $D(\lambda)$, and such that the action of $B(\lambda)$ is zero:

$$\langle\Omega| A(\lambda q^{1/2}) \;=\; a(\lambda)\langle\Omega| \,, \quad \langle\Omega| D(\lambda q^{1/2}) = d(\lambda)\langle\Omega| \,, \quad \langle\Omega| B(\lambda) = \underline{0}, \quad \langle\Omega| C(\lambda) \neq \underline{0} \tag{4.24}$$

$$A(\lambda q^{1/2}) |\bar{\Omega}\rangle \;=\; |\bar{\Omega}\rangle a(\lambda q), \; D(\lambda q^{1/2}) |\bar{\Omega}\rangle = |\bar{\Omega}\rangle d(\lambda/q), \; B(\lambda) |\bar{\Omega}\rangle = \underline{0}, \; C(\lambda) |\bar{\Omega}\rangle \neq \underline{0} \tag{4.25}$$

Thereafter, we use the states $\langle\Omega|$ and $|\bar{\Omega}\rangle$ to construct explicitly the elements of the basis in which the operator $\mathcal{B}(\lambda)$ is diagonal. Using the explicit form of $\mathcal{B}(\lambda)$:

$$\mathcal{B}(\lambda) = (-1)^N(-a_-(\lambda)A(\lambda)B(1/\lambda) + b_-(\lambda)A(\lambda)A(1/\lambda) - c_-(\lambda)B(\lambda)B(1/\lambda) + d_-(\lambda)B(\lambda)A(1/\lambda)) \tag{4.26}$$

and using the identities (4.24)-(4.25), one easily obtains that $\langle\Omega|$ and $|\bar{\Omega}\rangle$ are $\mathcal{B}(\lambda)$-eigenstates:

$$\langle\Omega| \mathcal{B}(\lambda) = \mathrm{B}_{\mathbf{0}}(\lambda)\langle\Omega| \quad \text{and} \quad \mathcal{B}(\lambda) |\bar{\Omega}\rangle = |\bar{\Omega}\rangle \mathrm{B}_{\mathbf{p-1}}(\lambda) \tag{4.27}$$

with non-zero eigenvalues:

$$\mathrm{B}_{\mathbf{0}}(\lambda) = b_-(\lambda)a(\lambda/q^{1/2})a(1/(\lambda q^{1/2})) \quad \text{and} \quad \mathrm{B}_{\mathbf{p-1}}(\lambda) = b_-(\lambda)a(q^{1/2}\lambda)a(q^{1/2}/\lambda) \tag{4.28}$$

From these first eigenstates, we can in fact construct new eigenvectors of $\mathcal{B}(\lambda)$ thanks to the repeated action of the operator $\mathcal{A}(\lambda)$ in some particular points, up to the determination of the $\mathcal{B}(\lambda)$-eigenstates basis of the Hilbert space (and of its dual). These points are zeros of the quantum determinant shifted by powers of the parameter q. We introduce them here, in order to state the proposition 4.2.1.

Let $(\mu_{n,+})_{1\leqslant n\leqslant N}$ and $(\mu_{n,-})_{1\leqslant n\leqslant N}$ be respectively the zeros of the functions $a(\lambda)$ and $d(\lambda/q)$, they read:

$$\mu_{n,+} = iq^{1/2}(a_n\beta_n/\alpha_n b_n)^{1/2} \quad \text{and} \quad \mu_{n,-} = iq^{1/2}(c_n\beta_n/\alpha_n d_n)^{1/2} \tag{4.29}$$

We define the following functions parametrised thanks to N-tuples $\mathbf{h} = (h_1, ..., h_N) \in [\![0, ..., p-1]\!]^N$:

$$\mathrm{B}_{\mathbf{h}}(\lambda) \equiv b_-(\lambda)a_{\mathbf{h}}(\lambda)a_{\mathbf{h}}(1/\lambda) \tag{4.30}$$

with

$$a_{\mathbf{h}}(\lambda) \equiv (-1)^{\mathsf{N}} \prod_{n=1}^{\mathsf{N}} (\alpha_n\beta_n)^{1/2} \Big(\frac{\lambda}{\xi_n^{(h_n)}} - \frac{\xi_n^{(h_n)}}{\lambda}\Big) \tag{4.31}$$

The variables $\xi_n^{(h)}$ are simply the zeros $\mu_{n,+}$, shifted by a power $q^{h+1/2}$:

$$\forall n \in [\![1, \mathsf{N}]\!] \,, \quad \xi_n^{(h)} = \mu_{n,+}q^{h+1/2} \tag{4.32}$$

Proposition 4.2.1. *For generic boundary and bulk parameters satisfying* (4.23) *and:*

$$b_-(\lambda) \neq 0, \; \mu_{n,+}^p \neq \mu_{m,+}^p \quad \forall n \neq m \in [\![1, \mathsf{N}]\!] \tag{4.33}$$

[1] The states $\langle j_n - 1, n|_{\hat{\mathbf{v}}}$ and $|j_n, n\rangle_{\hat{\mathbf{v}}}$, where j_n is such that (4.23) is satisfied by imposing $b_n = -q^{2j_n - 1}a_n$.

as well as $\forall h \in [\![1, p-1]\!]$, $\forall (n,m) \in [\![1, \mathsf{N}]\!]^2$, $\forall \epsilon \in \{-1, 1\}$,

$$\mu_{n,+}^{2p} \neq \pm 1,\ \mu_{n,+}^2 \neq q^{-2h}\alpha_-^{2\epsilon},\ \mu_{n,+}^2 \neq -q^{-2h}\beta_-^{2\epsilon},\ \mu_{n,+}^2 \neq q^{-2h-2\epsilon}\mu_{m,-}^{2\epsilon} \tag{4.34}$$

the states:

$$\langle h_1, ..., h_{\mathsf{N}}| \equiv \langle \boldsymbol{h}| \equiv \frac{1}{\mathrm{N}}\langle\Omega| \prod_{n=1}^{\mathsf{N}} \prod_{k_n=1}^{h_n} \frac{\mathcal{A}(1/\xi_n^{(k_n-1)})}{\mathsf{A}_-(1/\xi_n^{(k_n-1)})} \tag{4.35}$$

$$|h_1, ..., h_{\mathsf{N}}\rangle \equiv |\boldsymbol{h}\rangle \equiv \frac{1}{\mathrm{N}} \prod_{n=1}^{\mathsf{N}} \prod_{k_n=h_n}^{p-2} \frac{\mathcal{D}(\xi_n^{(k_n+1)})}{\mathsf{D}_-(\xi_n^{(k_n+1)})} |\bar{\Omega}\rangle \tag{4.36}$$

where $h_n \in [\![0, p-1]\!]$ *and* N *is a free normalisation, define a* $\mathcal{B}(\lambda)$*-eigenstates basis of* $\mathcal{H}^*$ *and* $\mathcal{H}$ *respectively:*

$$\langle \boldsymbol{h}|\mathcal{B}(\lambda) = \mathrm{B}_{\boldsymbol{h}}(\lambda)\langle \boldsymbol{h}| \tag{4.37}$$

$$\mathcal{B}(\lambda)|\boldsymbol{h}\rangle = |\boldsymbol{h}\rangle \mathrm{B}_{\boldsymbol{h}}(\lambda) \tag{4.38}$$

In particular, this result states that for almost all the values of the parameters, when the constraints (4.23) are satisfied, the operator $\mathcal{B}(\lambda)$ is diagonalisable and with simple spectrum.

In order to understand the role of the constraints (4.33)-(4.34), let us highlight the main steps of the proof. The idea is to efficiently use the reflection algebra to construct, as mentioned before, new eigenvectors from $\langle\Omega|$ and $|\bar{\Omega}\rangle$, the key elements being the commutation relations (2.14)-(2.19) of the 6-vertex reflection algebra generators. In more details, for the left case it holds (2.16):

$$c_1(\lambda,\mu)\ \mathcal{B}(\lambda)\mathcal{A}(\mu) = \mathcal{A}(\mu)\mathcal{B}(\lambda) - \mathcal{B}(\mu)\left(c_2(\lambda,\mu)\ \mathcal{A}(\lambda) + c_3(\lambda,\mu)\tilde{\mathcal{D}}(\lambda)\right) \tag{4.39}$$

This way, one can repeat the procedure given for the diagonalisation of the operator $\bar{B}(\lambda)$ in the paragraph 3.2.3: starting from (4.27), the action of $\mathcal{A}(\mu)$ for μ a root of $\mathrm{B}_{\mathbf{0}}(\lambda)$ gives that the covector $\langle\Omega|\,\mathcal{A}(\mu)$ is another eigenvector of $\mathcal{B}(\lambda)$, and so on for repeated actions, up to (4.37). The same is true for (4.38), using the commutation relation (2.17).

As by condition (4.33) each state $\langle\mathbf{h}|$ (respectively $|\mathbf{h}\rangle$) is associated to a different eigenvalue of $\mathcal{B}(\lambda)$, the only thing that we need to prove to get their linear independence, and thus the fact that they form a $\mathcal{B}(\lambda)$-eigenstate basis of $\mathcal{H}^*$ (respectively $\mathcal{H}$), is that each such state is non-zero. This can be proven using the action of the operator $\mathcal{A}(\lambda)$ (respectively $\mathcal{D}(\lambda)$) in some points on the states $\langle\mathbf{h}|$ (respectively $|\mathbf{h}\rangle$). Then this non-zero condition becomes equivalent to the non annihilation of the quantum determinant, which hold thanks to conditions (4.34).

In this framework where the bulk parameters satisfy the extra constraints (4.23), we can explicitly construct the eigenbasis of $\mathcal{B}(\lambda)$. In the next, we extend this procedure to unconstrained parameters, leading to a non explicit basis. However, as we will emphasise, the exact knowledge of the basis elements is not necessary for the computation of the dynamics of the model.

4.2.2 For generic parameters $\mathcal{B}(\lambda)$ is diagonalisable and with simple spectrum: the general proof

In this paragraph, we give a procedure leading to the existence, for general parameters, of at least one eigenvectors of $\mathcal{B}(\lambda)$ on the right $|\Omega_R\rangle$ and on the left $\langle\Omega_L|$. These vectors play the same role as the vectors $|\bar{\Omega}\rangle$ and $\langle\Omega|$ introduced in the previous proposition, thus from these states one can prove the diagonalisability of $\mathcal{B}(\lambda)$, using the procedure previously described.

Proposition 4.2.2. *There exists at least one eigenstate on the right* $|\Omega_R\rangle$ *and one eigenstate on the left*

$\langle\Omega_L|$ of the one parameter family of commuting operators $\mathcal{B}(\lambda)$:

$$\mathcal{B}(\lambda)|\Omega_R\rangle = |\Omega_R\rangle \mathrm{B}_-(\frac{\lambda^2}{q} - \frac{q}{\lambda^2}) \prod_{a=1}^{\mathsf{N}} (\frac{\lambda}{\bar{b}_{-,a}} - \frac{\bar{b}_{-,a}}{\lambda})(\lambda \bar{b}_{-,a} - \frac{1}{\lambda \bar{b}_{-,a}}) \tag{4.40}$$

$$\langle\Omega_L|\mathcal{B}(\lambda) = \mathrm{B}_-(\frac{\lambda^2}{q} - \frac{q}{\lambda^2}) \prod_{a=1}^{\mathsf{N}} (\frac{\lambda}{\hat{b}_{-,a}} - \frac{\hat{b}_{-,a}}{\lambda})(\lambda \hat{b}_{-,a} - \frac{1}{\lambda \hat{b}_{-,a}})\langle\Omega_L| \tag{4.41}$$

where B_-, $\bar{b}_{-,a}$ and $\hat{b}_{-,a}$ for $a \in [\![1, N]\!]$ are non-zero complex numbers.

To understand these formulas, let us sketch the proof of the existence of $|\Omega_R\rangle$. First, from the expression (4.26) of the generator $\mathcal{B}(\lambda)$ in terms of the generators of the Yang-Baxter algebra, $\mathcal{B}(\lambda)$ is a Laurent polynomial in the spectral parameter λ (of degree $2N + 2$) if the left boundary matrix is non-diagonal ($b_-(\lambda) \neq 0$) and the bulk parameters are generals, i.e. if the leading coefficient is non-zero:

$$\mathrm{B}_- \equiv (-1)^{\mathsf{N}} \kappa_- e^{\tau_-} \prod_{a=1}^{\mathsf{N}} \alpha_n \beta_n \ \neq 0 \tag{4.42}$$

Then, we can deduce thanks to the symmetry (4.10), the following functional form:

$$\mathcal{B}(\lambda) = \mathrm{B}_-(\frac{\lambda^2}{q} - \frac{q}{\lambda^2}) \prod_{a=1}^{\mathsf{N}} (\frac{\lambda}{\mathcal{B}_{-,a}} - \frac{\mathcal{B}_{-,a}}{\lambda})(\lambda \mathcal{B}_{-,a} - \frac{1}{\lambda \mathcal{B}_{-,a}}) \tag{4.43}$$

where the $\mathcal{B}_{-,a}$ are invertible commuting operators.

After, reasoning with the Jordan normal form of the operators $\mathcal{B}_{-,1}$, $\mathcal{B}_{-,2}$ and so on, we show that there exists at least one simultaneous right eigenstate of all the operators $\mathcal{B}_{-,a}$, which is thus an eigenvector of $\mathcal{B}(\lambda)$.

At this point, we can state the following proposition 4.2.3, which is the generalisation of 4.2.1 to almost any value of the parameters.

Proposition 4.2.3. *Under the condition* (4.42)*, for almost all the values of the bulk and boundary parameters, the operator family $\mathcal{B}(\lambda)$ is diagonalisable and it has simple spectrum.*

In particular it holds

$$\forall n \neq m \in [\![1, N]\!] \ , \quad \bar{b}^p_{-,m} \neq \bar{b}^p_{-,n} \tag{4.44}$$

as well as $\forall h \in [\![1, p-1]\!]$, $\forall (n, m) \in [\![1, \mathsf{N}]\!]^2$, $\forall \varepsilon \in \{-1, 1\}$,

$$\bar{b}^{2p}_{-,n} \neq \pm 1, \ \bar{b}^2_{-,n} \neq q^{-2h+1}\alpha^{2\varepsilon}_-, \ \bar{b}^2_{-,n} \neq -q^{-2h+1}\beta^{2\varepsilon}_-, \ \bar{b}^2_{-,n} \neq q^{-2\varepsilon+1-2h}\mu^{2\varepsilon}_{m,-}, \ \bar{b}^2_{-,n} \neq q^{-2\varepsilon+1-2h}\mu^{2\varepsilon}_{m,+} \tag{4.45}$$

and the states:

$$\langle h_1, ..., h_{\mathsf{N}}| \equiv \langle \boldsymbol{h}| \equiv \frac{1}{\mathrm{N}}\langle\Omega_L| \prod_{n=1}^{\mathsf{N}} \prod_{k_n=1}^{h_n} \frac{\mathcal{A}(1/\xi_n^{(k_n-1)})}{\mathsf{A}_-(1/\xi_n^{(k_n-1)})} \tag{4.46}$$

$$|h_1, ..., h_{\mathsf{N}}\rangle \equiv |\boldsymbol{h}\rangle \equiv \frac{1}{\mathrm{N}} \prod_{n=1}^{\mathsf{N}} \prod_{k_n=h_n}^{p-2} \frac{\mathcal{D}(\xi_n^{(k_n+1)})}{\mathsf{D}_-(\xi_n^{(k_n+1)})}|\Omega_R\rangle \tag{4.47}$$

where $h_n \in [\![0, p-1]\!]$, $\xi_n^{(h)} \equiv \bar{b}_{-,n}q^h$ and N is a free normalisation, define a $\mathcal{B}(\lambda)$-eigenstates basis of $\mathcal{H}^*$ and $\mathcal{H}$ respectively:

$$\langle \boldsymbol{h}|\mathcal{B}(\lambda) = \mathrm{B}_{\boldsymbol{h}}(\lambda)\langle \boldsymbol{h}| \tag{4.48}$$

$$\mathcal{B}(\lambda)|\boldsymbol{h}\rangle = |\boldsymbol{h}\rangle \mathrm{B}_{\boldsymbol{h}}(\lambda) \tag{4.49}$$

The eigenvalues $\mathrm{B}_{\boldsymbol{h}}(\lambda)$ have the same expressions (4.30)-(4.31)*, but now with the more general roots $\xi_n^{(h)}$.*

As already mentioned, the proof of this proposition uses a continuity argument: we know that it holds for one particular subclass of parameters, we can extend it to almost every value of the parameters. Let us give some more details.

First, we observe that by definition the operator family $\mathcal{B}_-(\lambda)$ is a polynomial in the bulk and boundary parameters, so the same must be true for its spectrum. This implies in particular that the quantities $\bar{b}^p_{-,n} - \bar{b}^p_{-,m}$, for any $n \neq m \in [\![1, N]\!]$ are either identically zero or they can be zero only over subspaces of non-zero co-dimension in the space of the bulk parameters. So here we have just to prove that it is not identically zero to derive that (4.44) holds for almost all the values of the parameters. To do so we can just recall that from the results derived in the previous paragraph it holds:

$$\forall m \in [\![1, N]\!] \ , \quad \bar{b}^p_{-,m} = q^{p/2} \mu^p_{m,+} \tag{4.50}$$

under the condition $B(\lambda)$ nilpotent, i.e. (4.23). Then, the condition (4.44) holds as soon as we impose:

$$\forall n \neq m \in [\![1, N]\!] \ , \quad \beta^p_n / \alpha^p_n \neq \beta^p_m / \alpha^p_m \tag{4.51}$$

Thereafter, one can follow step by step the proof of the proposition 4.2.1 to show that the states (4.46) and (4.47) indeed lead to the relations (4.48) and (4.49). The simplicity of spectrum comes from the condition (4.44).

Eventually, let us note that the repeated action of the operator $\mathcal{A}(\lambda)$ in the zeros of $\mathcal{B}(\lambda)$ can generate a null vector only if some of the zeros of $\mathcal{B}(\lambda)$ coincides with the zeros of the quantum determinant. Anyhow, as discussed in the previous paragraph, under the condition (4.34) we are always able to chose an appropriate set of p^N zeros of $\mathcal{B}(\lambda)$ which do not coincide with those of the quantum determinant, so that we generate exactly p^N independent states for almost all the parameters.

Remark. *Propositions similar to the previous two hold also for $\mathcal{C}(\lambda)$ so that we can say that for general values of the boundary and bulk parameters the standard Algebraic Bethe Ansatz does not applies, being $\mathcal{C}(\lambda)$ and $\mathcal{B}(\lambda)$ not nilpotent operators.*

4.2.3 The separate basis for a lower triangular right boundary

In this paragraph, we show that the eigenbasis of $\mathcal{B}(\lambda)$ defined previously in proposition 4.2.3 are separate basis for the boundary transfer matrix spectral problem associated to a lower triangular $K_+(\lambda)$ matrix. Here and in the next of this section, we consider the general $\xi^{(h_n)}_n$ defined in the proposition 4.2.3, i.e. we do not suppose the extra condition (4.23).

We start by giving the needed expressions of the actions of the generators of the reflection algebra on the elements of the basis, and then we show thanks to these expressions, that the spectral problem indeed separates for this special boundary.

But first, let us introduce some more notation. Let $h_n \in [\![0, p-1]\!]$ be associated to the site n of the chain, $n \in [\![1, N]\!]$. We define:

$$\zeta^{(h_n)}_n \equiv \begin{cases} \xi^{(h_n)}_n & \forall n \in [\![1, \mathsf{N}]\!] \\ \left(\xi^{(h_{n-N})}_{n-N}\right)^{-1} & \forall n \in [\![N+1, 2\mathsf{N}]\!] \end{cases} \tag{4.52}$$

which is just a convenient way to deal with $\xi^{(h_n)}_n$ and its inverse for $n > N$. Moreover, in the next the following notations will give more compact expressions:

$$\Lambda \equiv (\lambda^2 + 1/\lambda^2) \quad \text{and} \quad X \equiv q + 1/q \tag{4.53}$$

$$\forall b \in [\![1, 2\mathsf{N}]\!] \ , \quad X^{(h_b)}_b \equiv (\zeta^{(h_b)}_b)^2 + 1/(\zeta^{(h_b)}_b)^2 \tag{4.54}$$

Proposition 4.2.4. *The action on the left of the operator $\mathcal{A}(\lambda)$ on a generic state $\langle \boldsymbol{h}|$ reads:*

$$\begin{aligned}\langle \boldsymbol{h}|\mathcal{A}(\lambda) = &\sum_{a=1}^{2\mathsf{N}} \frac{(\lambda^2/q - q/\lambda^2)(\lambda\zeta_a^{(h_a)} - 1/\zeta_a^{(h_a)}\lambda)}{((\zeta_a^{(h_a)})^2/q - q/(\zeta_a^{(h_a)})^2)((\zeta_a^{(h_a)})^2 - 1/(\zeta_a^{(h_a)})^2)} \prod_{\substack{b=1 \\ b\neq a \ mod \mathsf{N}}}^{\mathsf{N}} \frac{\Lambda - X_b^{(h_b)}}{X_a^{(h_a)} - X_b^{(h_b)}} \mathsf{A}_-(\zeta_a^{(h_a)}) \\ &\times \langle \boldsymbol{h}| T_a^{-\varphi_a} + (-1)^{\mathsf{N}} q\text{-}det\ M_{0Q}(1) \frac{(\lambda/q^{1/2} + q^{1/2}/\lambda)}{2} \prod_{b=1}^{\mathsf{N}} \frac{\Lambda - X_b^{(h_b)}}{X - X_b^{(h_b)}} \langle \boldsymbol{h}| \\ &+ (-1)^{\mathsf{N}} \frac{(\xi_- + 1/\xi_-)}{(\xi_- - 1/\xi_-)} q\text{-}det\ M_{0Q}(i) \frac{(\lambda/q^{1/2} - q^{1/2}/\lambda)}{2} \prod_{b=1}^{\mathsf{N}} \frac{\Lambda - X_b^{(h_b)}}{X + X_b^{(h_b)}} \langle \boldsymbol{h}| \end{aligned} \tag{4.55}$$

whereas the action on the right of the operator $\mathcal{D}(\lambda)$ on a generic state $|\boldsymbol{h}\rangle$ reads:

$$\begin{aligned}\mathcal{D}(\lambda)|\boldsymbol{h}\rangle = &\sum_{a=1}^{2\mathsf{N}} T_a^{-\varphi_a}|\boldsymbol{h}\rangle \frac{(\lambda^2/q - q/\lambda^2)(\lambda\zeta_a^{(h_a)} - 1/\zeta_a^{(h_a)}\lambda)}{((\zeta_a^{(h_a)})^2/q - q/(\zeta_a^{(h_a)})^2)((\zeta_a^{(h_a)})^2 - 1/(\zeta_a^{(h_a)})^2)} \prod_{\substack{b=1 \\ b\neq a \ mod \mathsf{N}}}^{\mathsf{N}} \frac{\Lambda - X_b^{(h_b)}}{X_a^{(h_a)} - X_b^{(h_b)}} \\ &\times \mathsf{D}_-(\zeta_a^{(h_a)}) + |\boldsymbol{h}\rangle (-1)^{\mathsf{N}} q\text{-}det\ M_{0Q}(1) \frac{(\lambda/q^{1/2} + q^{1/2}/\lambda)}{2} \prod_{b=1}^{\mathsf{N}} \frac{\Lambda - X_b^{(h_b)}}{X - X_b^{(h_b)}} \\ &+ (-1)^{\mathsf{N}+1} |\boldsymbol{h}\rangle \frac{(\xi_- + 1/\xi_-)}{(\xi_- - 1/\xi_-)} q\text{-}det\ M_{0Q}(i) \frac{(\lambda/q^{1/2} - q^{1/2}/\lambda)}{2} \prod_{b=1}^{\mathsf{N}} \frac{\Lambda - X_b^{(h_b)}}{X + X_b^{(h_b)}} \end{aligned} \tag{4.56}$$

In these expressions, the operators $T_a^{\pm}$ are simply shifts:

$$\langle h_1, ..., h_a, ..., h_{\mathsf{N}}| T_a^{\pm} = \langle h_1, ..., h_a \pm 1, ..., h_{\mathsf{N}}| \tag{4.57}$$

$$T_a^{\pm}|h_1, ..., h_a, ..., h_{\mathsf{N}}\rangle = |h_1, ..., h_a \pm 1, ..., h_{\mathsf{N}}\rangle \tag{4.58}$$

while φ_a is defined by:

$$\forall a \in [\![1, 2N]\!]\ , \quad \varphi_a \equiv \begin{cases} 1 & \forall a \in [\![1, \mathsf{N}]\!] \\ -1 & \forall a \in [\![N+1, 2\mathsf{N}]\!] \end{cases} \tag{4.59}$$

Proof. This proposition can be derived without difficulty from the known form of the vectors $\langle \mathbf{h}|$ and $|\mathbf{h}\rangle$. Indeed, one easily computes the action of $\mathcal{A}(\zeta_b^{(h_b)})$ for $b \in [\![1, 2N]\!]$ thanks to the commutation (4.39) for the 6-vertex reflection algebra generators and the simplicity of spectrum of $\mathcal{B}(\lambda)$. This is the standard computation, we presented it in paragraph 3.2.3 for the Yang-Baxter algebra. Up to simple modifications to deal with the reflection algebra, and using the expression for the boundary quantum determinant (4.11), we have:

$$\langle \mathbf{h}|\mathcal{A}(\zeta_b^{(h_b)}) = \mathsf{A}_-(\zeta_b^{(h_b)})\langle \mathbf{h}|\mathrm{T}_b^{-\varphi_b} \tag{4.60}$$

Moreover, $\mathcal{A}(\lambda)$ is a Laurent polynomial of degree 2N+1 in λ. Thus, remarking that we can compute the action of the operators in the points $q^{1/2}$ and $iq^{1/2}$:

$$\mathcal{U}(q^{1/2}) = (-1)^{\mathsf{N}}\, q\text{-}det\ M_{0Q}(1)\ \mathbb{1} \quad \text{and} \quad \mathcal{U}(iq^{1/2}) = i(-1)^{\mathsf{N}+1} \frac{\xi_- + 1/\xi_-}{\xi_- - 1/\xi_-} q\text{-}det\ M_{0Q}(i)\ \sigma_0^z \tag{4.61}$$

we get the interpolation formula for its action on $\langle \mathbf{h}|$. We proceed using the same steps for representations on the right. □

The previous proposition gives in fact the action of the four generators of the reflection algebra on the left and on the right. Indeed, the representation of $\mathcal{D}(\lambda)$ on the left and of $\mathcal{A}(\lambda)$ on the right follow from the symmetry (4.9), while the ones of $\mathcal{C}(\lambda)$ follow from by the quantum determinant.

As mentioned before, we restrict our attention in this paragraph to the special boundary condition $b_+(\lambda) = 0$ (see the notation (2.28)). Indeed, we aim now to prove that a separate basis is given by the $\mathcal{B}(\lambda)$-eigenstates basis. That is we consider the spectral problem of the boundary transfer matrix under the following conditions on the boundary parameters:

$$b_+ (\lambda) = 0 \text{ and } b_- (\lambda) \neq 0. \tag{4.62}$$

Note that the condition $b_+ (\lambda) = 0$, keeping instead if desired a $c_+ (\lambda) \neq 0$, can be simply realised by the following renormalisation of the boundary parameters $\kappa_+ = e^{-\gamma}\kappa'_+$ and $e^{\tau_+} = e^{\tau'_+ - \gamma}$ by sending $\gamma \to +\infty$. So, by using the symmetry (4.20), we are left with:

$$\mathcal{T}(\lambda) = \mathsf{a}_+(\lambda)\mathcal{A}_-(\lambda) + \mathsf{a}_+(1/\lambda)\mathcal{A}_-(1/\lambda) + c_+ (\lambda)\, \mathcal{B}(\lambda) \tag{4.63}$$

Separate basis Let $|\tau\rangle$ be a right eigenvector of the boundary transfer matrix associated to the eigenvalue $\tau(\lambda)$:

$$\mathcal{T}(\lambda)\, |\tau\rangle = \tau(\lambda)\, |\tau\rangle \tag{4.64}$$

By acting with $\mathcal{T}(\xi_n^{(h_n)})$ on the left and on the right, the computation of the element $\langle \mathbf{h}|\, \mathcal{T}(\xi_n^{(h_n)})\, |\tau\rangle$ leads to:

$$\tau(\xi_n^{(h_n)})\Psi_\tau(\mathbf{h}) \;= \textsc{a}(\xi_n^{(h_n)})\Psi_\tau(\mathsf{T}_n^-(\mathbf{h})) + \textsc{a}(1/\xi_n^{(h_n)})\Psi_\tau(\mathsf{T}_n^+(\mathbf{h})) \tag{4.65}$$

where we introduced the coefficients and the wave functions:

$$\textsc{a}(\lambda) \equiv \mathsf{a}_+(\lambda)\mathsf{A}_-(\lambda) \quad \text{and} \quad \Psi_\tau(\mathbf{h}) \equiv \langle \mathbf{h}|\tau\rangle \tag{4.66}$$

as well as the shifted states:

$$\mathsf{T}_n^\pm(\mathbf{h}) \equiv (h_1, \dots, h_n \pm 1, \dots, h_\mathsf{N}) \tag{4.67}$$

The spectrum, both the eigenvalues and eigenstates, of the transfer matrix $\mathcal{T}(\lambda)$ is thus characterised by the discrete system of equations (4.65) for any $n \in [\![1, N]\!]$ and $\mathbf{h} \in [\![0, p-1]\!]^\mathsf{N}$, i.e. a system of p^N Baxter-like equations in the wave functions. These are separate equations as for any fixed n the coefficients are only function of the spectrum of one separate variable as well as the shifts on the wave functions are only along the same variable.

The same kind of Baxter equations are obtained for the left eigenvectors of $\mathcal{T}(\lambda)$, using this time the expression (4.21) for the boundary transfer matrix, and the right eigenbasis of $\mathcal{B}(\lambda)$.

So the eigenbasis of $\mathcal{B}(\lambda)$ are indeed separate basis for this special boundary. In the next section, we refer to it as the left separate basis (for $\langle \mathbf{h}|$) and right separate basis (for $|\mathbf{h}\rangle$), and use it to completely characterise both the eigenvalues and eigenstates of the boundary transfer matrix.

4.2.4 Discrete characterisation of the spectrum

In this paragraph we present the complete characterisation of the spectrum of the boundary transfer matrix $\mathcal{T}(\lambda)$ associated to cyclic representations of the 6-vertex reflection algebra with the constraint $K_+(\lambda)$ lower triangular. In the next, we will denote by Σ_τ the transfer matrix spectrum, namely the set of all the eigenvalues of $\mathcal{T}(\lambda)$.

But first, let us present a preliminary property satisfied by all the eigenvalue functions of all the boundary transfer matrices $\mathcal{T}(\lambda)$ associated to cyclic representations of the 6-vertex reflection. This one, given in the following proposition, states the functional form of the eigenvalues, which is the same for all the parameters, even if the boundary matrix $K_+(\lambda)$ is not triangular.

Proposition 4.2.5. *Any function $\tau(\lambda) \in \Sigma_\tau$ is an even function of λ invariant under the transformation*

$\lambda \to 1/\lambda$ *which admits the following interpolation formula:*

$$\tau(\lambda) = \sum_{a=1}^{\mathsf{N}} \frac{\Lambda^2 - X^2}{(X_a^{(0)})^2 - X^2} \prod_{\substack{b=1\\ b\neq a}}^{\mathsf{N}} \frac{\Lambda - X_b^{(0)}}{X_a^{(0)} - X_b^{(0)}} \tau(\zeta_a^{(0)}) + (-1)^{\mathsf{N}} \frac{(\Lambda + X)}{2} \prod_{b=1}^{\mathsf{N}} \frac{\Lambda - X_b^{(0)}}{X - X_b^{(0)}} q\text{-}det\ M_{0Q}(1)$$
$$+ (-1)^{\mathsf{N}} \frac{(\Lambda - X)}{2} \prod_{b=1}^{\mathsf{N}} \frac{\Lambda - X_b^{(0)}}{X + X_b^{(0)}} \frac{(\xi_+ + 1/\xi_+)}{(\xi_+ - 1/\xi_+)} \frac{(\xi_- + 1/\xi_-)}{(\xi_- - 1/\xi_-)} q\text{-}det\ M_{0Q}(i)$$
$$+ (\Lambda^2 - X^2)\tau_\infty \prod_{b=1}^{\mathsf{N}} (\Lambda - X_b^{(0)}) \tag{4.68}$$

where:

$$\tau_\infty \equiv \frac{\kappa_+\kappa_-(e^{\tau_+ - \tau_-} \prod_{b=1}^{\mathsf{N}} \delta_b\gamma_b + e^{\tau_- - \tau_+} \prod_{b=1}^{\mathsf{N}} \alpha_b\beta_b)}{(\xi_+ - 1/\xi_+)(\xi_- - 1/\xi_-)} \tag{4.69}$$

This proposition is simply an interpolation formula. Indeed, we know the symmetries of $\mathcal{T}(\lambda)$, and thus the ones of $\tau(\lambda) \in \Sigma_\tau$, and we use the polynomiality in the spectral parameter. The asymptotic behaviour

$$\tau_\infty = \lim_{\log\lambda \to \pm\infty} \lambda^{\mp 2(\mathsf{N}+2)} \mathcal{T}(\lambda) \tag{4.70}$$

is also known after some simple computations.

Let us now go back to a lower triangular right boundary. Under the reparametrisation $\kappa_+ = e^{-\gamma}\kappa'_+$ and $e^{\tau_+} = e^{\tau'_+ - \gamma}$, with $\gamma \to +\infty$, the asymptotic of the transfer matrix reads:

$$\tau_\infty = \frac{(-1)^{\mathsf{N}} \kappa'_+ \kappa_- e^{\tau_- - \tau'_+} \prod_{b=1}^{\mathsf{N}} \alpha_b\beta_b}{(\xi_+ - 1/\xi_+)(\xi_- - 1/\xi_-)} \tag{4.71}$$

In the following we will suppress the unnecessary prime in κ_+ and τ_+ when dealing with the constraint $b_+(\lambda) \equiv 0$ but $c_+(\lambda) \neq 0$.

The previous proposition defines the set of polynomials to which belong the transfer matrix eigenvalues. In order to completely characterise the eigenvalues, we introduce now the following one-parameter family $D_t(\lambda)$ of $p \times p$ matrices, depending on a generic function $t(\lambda)$:

$$D_t(\lambda) \equiv \begin{pmatrix} t(\lambda) & -\mathrm{A}(1/\lambda) & 0 & \cdots & 0 & -\mathrm{A}(\lambda) \\ -\mathrm{A}(q\lambda) & t(q\lambda) & -\mathrm{A}(1/(q\lambda)) & 0 & \cdots & 0 \\ 0 & \ddots & & & & \vdots \\ \vdots & & \cdots & & & \vdots \\ \vdots & & & \cdots & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \ldots & 0 & -\mathrm{A}(q^{2l-1}\lambda) & t(q^{2l-1}\lambda) & -\mathrm{A}(1/(q^{2l-1}\lambda)) \\ -\mathrm{A}(1/(q^{2l}\lambda)) & 0 & \ldots & 0 & -\mathrm{A}(q^{2l}\lambda) & t(q^{2l}\lambda) \end{pmatrix} \tag{4.72}$$

It is clear from the equation (4.65) that the matrix $D_\tau(\lambda)$ will allow to write the Baxter system, for $\tau(\lambda) \in \Sigma_{\mathcal{T}}$, in a matrix form. Moreover, with some simple computations on the rows and columns, one can prove that if $t(\lambda)$ is a function of λ invariant under the transformation $\lambda \to 1/\lambda$ then det $D_t(\lambda)$ is a function of λ^p invariant under the transformation $\lambda^p \to 1/\lambda^p$. This property will be used in the following.

The last point we need before to state the result of the characterisation is to compute the scalar product between states of the left and right separate basis:

Proposition 4.2.6. *Using an appropriate choice for the norm* N *in the basis elements* (4.46) *and* (4.47), *it holds:*

$$\forall\ \boldsymbol{h},\boldsymbol{k} \in [\![0,p-1]\!]^N\ ,\quad \langle \boldsymbol{h}|\boldsymbol{k}\rangle = \delta_{\boldsymbol{hk}} \prod_{1\leqslant b<a\leqslant \mathsf{N}} \frac{1}{X_a^{(h_a)} - X_b^{(h_b)}} \tag{4.73}$$

In particular, this implies the spectral decomposition of the identity in the separate basis:

$$\sum_{h_1,\dots,h_{\mathsf{N}}=0}^{p-1} \prod_{1\leqslant b<a\leqslant \mathsf{N}} (X_a^{(h_a)} - X_a^{(h_a)})|h_1,\dots,h_{\mathsf{N}}\rangle\langle h_1,\dots,h_{\mathsf{N}}| = \mathbb{1} \tag{4.74}$$

The fact that the states are orthogonal comes from the fact that all the eigenvalues of $\mathcal{B}(\lambda)$ are different. Then, to compute the value of the scalar product, we compute the matrix element $\theta_a \equiv \langle h_1,\dots,h_a,\dots,h_{\mathsf{N}}|\mathcal{A}(\xi_a^{(h_a+1)})|h_1,\dots,h_a+1,\dots,h_{\mathsf{N}}\rangle$, for $a \in [\![1,N]\!]$, using the left action and the right action of $\mathcal{A}(\lambda)$ from the proposition 4.2.4.

We are now able to give the complete characterisation of the spectrum in the separate basis, it is stated in the next proposition:

Proposition 4.2.7 (Characterisation of the spectrum)**.** *If the conditions:*

$$\forall h \in [\![1,p-1]\!],\ \forall n \in [\![1,N]\!]\ ,\quad \bar{b}_{-,n}^2 \neq q^{-2h+1}\xi_+^{\pm 2} \tag{4.75}$$

and (4.44)-(4.45) *are satisfied, then $\mathcal{T}(\lambda)$ has simple spectrum and $\Sigma_{\mathcal{T}}$ coincides with the set of polynomials $\tau(\lambda)$ of the form* (4.68) *with* (4.71) *which satisfy the following discrete system of equations:*

$$\forall a \in [\![1,N]\!]\ ,\quad det D_\tau(\zeta_a^{(0)}) = 0 \tag{4.76}$$

- *The right $\mathcal{T}$-eigenstate corresponding to $\tau(\lambda) \in \Sigma_\tau$ is defined by the following decomposition in the right separate basis:*

$$|\tau\rangle = \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{p-1} \prod_{a=1}^{\mathsf{N}} Q_{\tau,a}^{(h_a)} \prod_{1\leqslant b<a\leqslant \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)})|h_1,\dots,h_{\mathsf{N}}\rangle \tag{4.77}$$

  where the $Q_{\tau,a}^{(h_a)}$ are the unique nontrivial solution up to normalisation of the linear homogeneous system:

$$D_\tau(\zeta_a^{(0)}) \begin{pmatrix} Q_{\tau,a}^{(0)} \\ \vdots \\ Q_{\tau,a}^{(p-1)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.78}$$

- *The left $\mathcal{T}$-eigenstate corresponding to $\tau(\lambda) \in \Sigma_\tau$ is defined by the following decomposition in the left separate basis:*

$$\langle\tau| = \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{p-1} \prod_{a=1}^{\mathsf{N}} \hat{Q}_{\tau,a}^{(h_a)} \prod_{1\leqslant b<a\leqslant \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)})\langle h_1,\dots,h_{\mathsf{N}}| \tag{4.79}$$

  where the $\hat{Q}_{\tau,a}^{(h_a)}$ are the unique nontrivial solution up to normalisation of the linear homogeneous system:

$$\begin{pmatrix} \hat{Q}_{\tau,a}^{(0)} & \dots & \hat{Q}_{\tau,a}^{(p-1)} \end{pmatrix} \left(\hat{D}_\tau(\zeta_a^{(0)})\right)^{t_0} = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix} \tag{4.80}$$

  and $\hat{D}_\tau(\lambda)$ is the family of $p \times p$ matrices defined substituting in $D_\tau(\lambda)$ the coefficient $\mathrm{A}(\lambda)$ with

$$\mathrm{D}(\lambda) = \mathsf{d}_+(\lambda)\mathsf{D}_-(\lambda) \tag{4.81}$$

In the followig, we just aim to highlight the main points of the proof. The key is the use of the matrix $D_\tau(\lambda)$ to rewrite the Baxter equations. Details are given in article **I**.

Key elements of the proof

The Baxter system admits the following equivalent representation as a p-linear system of homogeneous equations:

$$D_\tau(\xi_n^{(0)}) \begin{pmatrix} \Psi_\tau(h_1, ..., h_n = 0, ..., h_{\mathsf{N}}) \\ \Psi_\tau(h_1, ..., h_n = 1, ..., h_{\mathsf{N}}) \\ \vdots \\ \Psi_\tau(h_1, ..., h_n = p-1, ..., h_{\mathsf{N}}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.82}$$

for any $n \in [\![1, N]\!]$ and for any $h_{m\neq n}$ in $[\![0, p-1]\!]$.

Then the condition $\tau(\lambda) \in \Sigma_\tau$ implies the compatibility equations for these linear systems, i.e. it must hold:

$$\forall a \in [\![1, N]\!] , \quad \det D_\tau(\xi_a^{(0)}) = 0 \tag{4.83}$$

The rank of the matrices $D_\tau(\xi_n^{(0)})$ in (4.82) is $p-1$, being:

$$\forall h \in [\![1, p-1]\!] , \quad \forall n \in [\![1, N]\!] , \quad \mathrm{A}_-(\xi_n^{(h)}) \neq 0 \quad \text{and} \quad \mathrm{A}_-(1/\xi_n^{(h-1)}) \neq 0 \tag{4.84}$$

from the conditions (4.44)-(4.45) and (4.75). Then, up to an overall normalisation, the solution is unique and independent from the $h_{m\neq n} \in [\![0, p-1]\!]$ for any $n \in [\![1, N]\!]$.

So fixing $\tau(\lambda) \in \Sigma_\tau$ there exists, up to normalisation, one and only one corresponding $\mathcal{T}(\lambda)$ eigenstate $|\tau\rangle$, with coefficients of the factorised form given by (4.77) and (4.78). Hence $\mathcal{T}(\lambda)$ has simple spectrum.

Vice versa, if $\tau(\lambda)$ is in the set of functions (4.68) with (4.71) and satisfies (4.76), then the state $|\tau\rangle$ defined by (4.77)-(4.78) satisfies:

$$\forall n \in [\![1, N]\!] , \quad \langle \mathbf{h}| \mathcal{T}(\zeta_n^{(h_n)})|\tau\rangle = \tau(\zeta_n^{(h_n)})\langle \mathbf{h}|\tau\rangle \tag{4.85}$$

for any eigenstate $\langle \mathbf{h}|$ of $\mathcal{B}(\lambda)$ and this implies:

$$\forall \lambda \in \mathbb{C} , \quad \langle \mathbf{h}| \mathcal{T}(\lambda)|\tau\rangle = \tau(\lambda)\langle \mathbf{h}|\tau\rangle \tag{4.86}$$

which is equivalent to $\tau(\lambda) \in \Sigma_\tau$ and $|\tau\rangle$ is the corresponding eigenstate.

For the left $\mathcal{T}(\lambda)$ eigenstates the proof follows as above, we just remark in this case that the matrix elements $\langle \tau|\mathcal{T}(\zeta_n^{(h_n)})|\mathbf{h}\rangle$ are computed in the right separate basis, so it holds:

$$\forall n \in [\![1, N]\!] , \quad \tau(\zeta_n^{(h_n)})\hat{\Psi}_\tau(\mathbf{h}) = \mathrm{D}(\zeta_n^{(h_n)})\hat{\Psi}_\tau(\mathsf{T}_n^-(\mathbf{h})) + \mathrm{D}(1/\zeta_n^{(h_n)})\hat{\Psi}_\tau(\mathsf{T}_n^+(\mathbf{h})) \tag{4.87}$$

where we denote:

$$\hat{\Psi}_\tau(\mathbf{h}) \equiv \langle \tau|\mathbf{h}\rangle \quad \text{and} \quad \mathrm{D}(\lambda) \equiv \mathsf{d}_+(\lambda)\mathsf{D}_-(\lambda) \tag{4.88}$$

After some simple computations, one can prove that it holds $\det \hat{D}_\tau(\lambda) = \det D_\tau(\lambda)$.

A link between the right and left eigenvectors

For future applications it is interesting to show that we can obtain the coefficients of a left eigenstate of the boundary transfer matrix in terms of those of the right one. In particular, we can show the next proposition:

Proposition 4.2.8. *Let $\tau(\lambda) \in \Sigma_\tau$ then we have:*

$$\frac{\hat{Q}_{\tau,a}^{(h)}}{\hat{Q}_{\tau,a}^{(h-1)}} = \frac{\mathrm{A}(1/\zeta_a^{(h-1)})}{\mathrm{D}(1/\zeta_a^{(h-1)})} \frac{Q_{\tau,a}^{(h)}}{Q_{\tau,a}^{(h-1)}} \tag{4.89}$$

This property will be used later in the functional characterisation of the spectrum.

4.3 The most general integrable boundaries

The aim of this section is now to tackle the spectral problem of the boundary transfer matrix associated to cyclic representations of the 6-vertex reflection algebra, for the most general integrable boundaries. To characterise the eigenfunctions, we aim to find a separate basis, in which the spectral problem separates as in the previous section.

The main idea is to use a gauge transformation on the generators of the 6-vertex reflection algebra and on the boundary matrices, and to use this freedom to put lower triangular the gauged boundary matrix $K_+(\lambda)$, to be able to do the same steps of computation that we did in the previous section. As we will see, the separate basis are given by the pseudo-diagonalisation basis of the gauged $\mathcal{B}(\lambda)$ operator, in which this operator is pseudo-diagonal (in a precise sense that we are going to define).

We first introduce the gauge transformation in question, and then give the sketch of the proof that the gauged $\mathcal{B}(\lambda)$ is indeed pseudo-diagonalisable. Then we give the discrete characterisation of the spectrum, as it was done in the previous section, and eventually provide another characterisation *via* Baxter's type functional equation, which leads to a Bethe ansatz like form for the eigenvectors.

4.3.1 The gauge transformation

Keeping in mind that the boundary monodromy matrix is expressed in terms of the Yang-Baxter monodromy matrix, we first use a gauge transformation on the generators of the Yang-Baxter algebra. We follow the same approach used in the spin 1/2 case, namely the use of the gauge transformation introduced by Baxter's series of paper [52–54] (and its use to simplify the boundary matrix [192,193]), but with different gauge matrices, adapted to the cyclic case. This results in a gauge modification of the 6-vertex Yang-Baxter algebra. Then we define a gauge transformation on the generators of the 6-vertex reflection algebra, which satisfy a gauge modification of the reflection algebra commutation relations. The modified generators define a modified algebra which remains closed and allow for the introduction of the separation of variables in a way similar to the ungauged case.

Gauge transformation of the Yang-Baxter algebra

For arbitrary non zero complex parameters α, β and γ, let us introduce the following two matrices:

$$G(\lambda|\alpha,\beta) \equiv \begin{pmatrix} 1/(\alpha\beta\lambda) & \beta/(\alpha\lambda) \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \bar{G}(\lambda|\gamma) \equiv \begin{pmatrix} 1/(\lambda\gamma) & 0 \\ 1 & 1 \end{pmatrix} \tag{4.90}$$

as well as their inverses:

$$G^{-1}(\lambda|\alpha,\beta) = \frac{\alpha\lambda}{\beta - 1/\beta} \begin{pmatrix} -1 & \beta/(\alpha\lambda) \\ 1 & -1/(\alpha\beta\lambda) \end{pmatrix} \quad \text{and} \quad \bar{G}^{-1}(\lambda|\gamma) = \begin{pmatrix} \lambda\gamma & 0 \\ -\lambda\gamma & 1 \end{pmatrix} \tag{4.91}$$

The gauge transformed bulk monodromy matrix is then defined by:

$$M_{0Q}(\lambda|\alpha,\beta,\gamma) \equiv G^{-1}(\lambda q^{1/2}|\alpha,\beta)\, M_{0Q}(\lambda)\bar{G}(\lambda q^{1/2}|\gamma) \equiv \begin{pmatrix} A(\lambda|\alpha,\beta,\gamma) & B(\lambda|\alpha,\beta) \\ C(\lambda|\alpha,\beta,\gamma) & D(\lambda|\alpha,\beta) \end{pmatrix} \tag{4.92}$$

and, in a similar way:

$$\hat{M}_{0Q}(\lambda|\alpha,\beta,\gamma) \equiv \bar{G}^{-1}(q^{1/2}/\lambda|\gamma)\, \hat{M}_{0Q}(\lambda) G(q^{1/2}/\lambda|\alpha,\beta) \equiv \begin{pmatrix} \bar{A}(\lambda|\alpha,\beta,\gamma) & \bar{B}(\lambda|\alpha,\beta,\gamma) \\ \bar{C}(\lambda|\alpha,\beta,\gamma) & \bar{D}(\lambda|\alpha,\beta,\gamma) \end{pmatrix} \tag{4.93}$$

Let us recall that $\hat{M}_{0Q}(\lambda)$, defined in (4.3), is a modification of the monodromy matrix used to express the boundary monodromy matrix.

The definition here chosen of these gauge transformations differs with respect to that used previously in the literature [192, 193]. On the one hand for the particular choice of the gauge matrix $\bar{G}(\lambda|\gamma)$, which can be seen as $\lim_{a\to+\infty} G(\lambda|a\gamma^{1/2},\gamma^{1/2}/a)$ and, on the other hand, as the parameters on the left and the right transformations are a priori independent.

These gauge transformed Yang-Baxter generators are of special interest as they define a closed set of commutation relations. Here we will not write all of them, let us simply highlight how the commutation between the generators are modified (with respect to (1.22)-(1.31)) for some relevant couples of algebra generators:

$$A(\lambda|\alpha,\beta,\gamma)A(\mu|\alpha,\beta/q,\gamma/q) = A(\mu|\alpha,\beta,\gamma)A(\lambda|\alpha,\beta/q,\gamma/q) \tag{4.94}$$
$$B(\lambda|\alpha,\beta)B(\mu|\alpha,\beta/q) = B(\mu|\alpha,\beta)B(\lambda|\alpha,\beta/q) \tag{4.95}$$
$$C(\lambda|\alpha,\beta,\gamma)C(\mu|\alpha,\beta q,\gamma/q) = C(\mu|\alpha,\beta,\gamma)C(\lambda|\alpha,\beta q,\gamma/q) \tag{4.96}$$
$$D(\lambda|\alpha,\beta)D(\mu|\alpha,\beta/q) = D(\mu|\alpha,\beta)D(\lambda|\alpha,\beta/q) \tag{4.97}$$

$$\begin{aligned} A(\lambda|\alpha,\beta,\gamma)B(\mu|\alpha,\beta/q) &= \frac{q(\lambda/\mu-\mu/\lambda)}{\lambda q/\mu-\mu/q\lambda}B(\mu|\alpha,\beta)A(\lambda|\alpha,\beta/q,\gamma q) \\ &+ \frac{(q-1/q)\mu/\lambda}{\lambda q/\mu-\mu/q\lambda}A(\mu|\alpha,\beta,\gamma)B(\lambda|\alpha,\beta/q) \end{aligned} \tag{4.98}$$
$$\begin{aligned} B(\lambda|\alpha,\beta)A(\mu|\alpha,\beta/q,\gamma q) &= \frac{\lambda/\mu-\mu/\lambda}{q(\lambda q/\mu-\mu/q\lambda)}A(\mu|\alpha,\beta,\gamma)B(\lambda|\alpha,\beta/q) \\ &+ \frac{(q-1/q)\lambda/\mu}{\lambda q/\mu-\mu/q\lambda}B(\mu|\alpha,\beta)A(\lambda|\alpha,\beta/q,\gamma q) \end{aligned} \tag{4.99}$$

We can prove by direct computation these relations, using the action of the gauge matrices on the Yang-Baxter equation.

As far as the quantum determinant is concerned, it is simple to prove that it holds:

$$\hat{M}_{0Q}(\lambda|\alpha,\beta,\gamma) = f(\alpha,\beta,\gamma)(-1)^N\sigma_0^y M_{0Q}^{t_0}(\lambda|\alpha,\beta,\gamma)\sigma_0^y \tag{4.100}$$

where we define the following function of the gauge parameters:

$$f(\alpha,\beta,\gamma) \equiv \frac{\gamma(1-\beta^2)}{\alpha\beta} \tag{4.101}$$

This way, it implies that we can express the quantum determinant thanks to the gauge transformed generators. The same way as in paragraph 1.3, it holds four different re-writings, let us here mention two of them:

$$q\text{-}det\ M_{0Q}(\lambda) = f(\alpha,q\beta,q\gamma)\left[A(\lambda q^{1/2}|\alpha,\beta,\gamma)D(\lambda/q^{1/2}|\alpha,\beta q) - B(\lambda q^{1/2}|\alpha,\beta)C(\lambda/q^{1/2}|\alpha,\beta q,\gamma q)\right] \tag{4.102}$$

and

$$q\text{-}det\ M_{0Q}(\lambda) = f(\alpha,\beta/q,q\gamma)\left[D(\lambda q^{1/2}|\alpha,\beta)A(\lambda/q^{1/2}|\alpha,\beta/q,\gamma q) - C(\lambda q^{1/2}|\alpha,\beta,\gamma)B(\lambda/q^{1/2}|\alpha,\beta/q)\right] \tag{4.103}$$

The results here presented highlight that this modified Yang-Baxter algebra is very similar to the original one allowing for the extension of the quantum separation of variables to this framework.

Gauge transformation of the reflection algebra

In this paragraph, we now construct the gauge boundary monodromy matrix. Using the previous gauge matrices, we define:

$$\mathcal{U}(\lambda|\alpha,\beta) \equiv \frac{q^{1/2}}{\lambda} G^{-1}(\lambda q^{1/2}|\alpha,\beta)\,\mathcal{U}(\lambda)\,G(q^{3/2}/\lambda|\alpha,\beta) \equiv \begin{pmatrix} \mathcal{A}(\lambda|\alpha,\beta q^2) & \mathcal{B}(\lambda|\alpha,\beta) \\ \mathcal{C}(\lambda|\alpha,\beta q^2) & \mathcal{D}(\lambda|\alpha,\beta) \end{pmatrix} \tag{4.104}$$

Note the definition of the gauge generators, with a shift of q^2 on β for further convenience. Thus we consider the following linear combination of operators:

$$\mathcal{A}(\lambda|\alpha,\beta) = \Big[-\lambda q^{3/2}/\beta\ \mathcal{A}(\lambda) - \alpha q\ \mathcal{B}(\lambda) + 1/(\alpha q)\ \mathcal{C}_{-}(\lambda) + \beta/\left(\lambda q^{3/2}\right)\mathcal{D}(\lambda)\Big]/(\beta/q^2 - q^2/\beta) \tag{4.105}$$

$$\mathcal{B}(\lambda|\alpha,\beta) = \Big[-\lambda\beta/q^{1/2}\mathcal{A}(\lambda) - \alpha q\ \mathcal{B}(\lambda) + \beta^2/\left(\alpha q\right)\mathcal{C}(\lambda) + \beta q^{1/2}/\lambda\mathcal{D}(\lambda)\Big]/(\beta - 1/\beta) \tag{4.106}$$

$$\mathcal{C}(\lambda|\alpha,\beta) = \Big[\lambda q^{3/2}/\beta\ \mathcal{A}(\lambda) + \alpha q\ \mathcal{B}(\lambda) - q^3/\left(\alpha\beta^2\right)\mathcal{C}(\lambda) - q^{5/2}/\left(\lambda\beta\right)\mathcal{D}(\lambda)\Big]/(\beta/q^2 - q^2/\beta) \tag{4.107}$$

$$\mathcal{D}(\lambda|\alpha,\beta) = \Big[\lambda\beta/q^{1/2}\mathcal{A}(\lambda) + \alpha q\ \mathcal{B}(\lambda) - 1/(\alpha q)\mathcal{C}(\lambda) - q^{1/2}/\left(\lambda\beta\right)\mathcal{D}(\lambda)\Big]/(\beta - 1/\beta) \tag{4.108}$$

Moreover, one can expand this gauged boundary monodromy matrix in terms of the gauged generators of the Yang-Baxter algebra. In particular, it holds:

$$\begin{pmatrix} \mathcal{A}(\lambda|\alpha,\beta q^2) \\ \mathcal{C}(\lambda|\alpha,\beta q^2) \end{pmatrix} = M(\lambda|\alpha,\beta,\gamma)\bar{K}(\lambda|\gamma)\begin{pmatrix} \bar{A}(\lambda|\alpha,\beta q,\gamma q) \\ \bar{C}(\lambda|\alpha,\beta q,\gamma q) \end{pmatrix} \tag{4.109}$$

$$\begin{pmatrix} \mathcal{B}(\lambda|\alpha,\beta) \\ \mathcal{D}(\lambda|\alpha,\beta) \end{pmatrix} = M(\lambda|\alpha,\beta,\gamma)\bar{K}(\lambda|\gamma)\begin{pmatrix} \bar{B}(\lambda|\alpha,\beta/q,\gamma q) \\ \bar{D}(\lambda|\alpha,\beta/q,\gamma q) \end{pmatrix} \tag{4.110}$$

where the gauged left boundary matrix reads:

$$\bar{K}(\lambda|\gamma) \equiv \frac{q^{1/2}}{\lambda}\bar{G}^{-1}(\lambda q^{1/2}|\gamma)\ K(\lambda)\,\bar{G}(q^{1/2}/\lambda|\gamma q) \tag{4.111}$$

In these gauge transformed boundary matrix extra degrees of freedom are added that can be used wisely to eventually characterise the boundary transfer matrix spectrum.

Here, we can remark that $\mathcal{U}(\lambda|\alpha,\beta)$ does not depend on the internal gauge parameter γ, so we are free to choose it at will. In our separation of variables construction, we will exclude the following values of γ:

$$\gamma = \gamma_{\pm} \equiv \frac{-\xi_{-} \pm \sqrt{\xi_{-}^2 + 4\kappa_{-}^2}}{2qe^{\tau_{-}}\kappa_{-}} \tag{4.112}$$

which leads to the vanishing of the coefficient (2,1) of this matrix:

$$\forall\lambda\in\mathbb{C}\quad,\quad \bar{K}(\lambda|\gamma_{\pm})_{21} \equiv 0 \tag{4.113}$$

(An explicit expression of the coefficients can be read in article **II**, formulas (B.6),(B.7),(B.8) and (B.9)). In fact, one can argue that the function $\bar{K}(\lambda|\gamma)_{21}$ plays the same role as the function $b_{-}(\lambda)$ for the ungauged case and, as we have already seen, we had to suppose $b_{-}(\lambda) \neq 0$ in order to implement the separation of variables.

The two next propositions state that the gauged generators satisfy similar commutation relations to the 6-vertex reflection algebra, and similar symmetry properties compared to the ungauged ones.

Proposition 4.3.1. *The gauge transformed boundary operators satisfy, among others, the following com-*

mutation relations:

$$\forall(\lambda_1,\lambda_2)\in\mathbb{C}^2\ ,\quad \mathcal{B}(\lambda_2|\alpha,\beta)\mathcal{B}(\lambda_1|\alpha,\beta/q^2)=\mathcal{B}(\lambda_1|\alpha,\beta)\mathcal{B}(\lambda_2|\alpha,\beta/q^2) \tag{4.114}$$

$$\begin{aligned}\mathcal{A}(\lambda_2|\alpha,\beta q^2)\mathcal{B}(\lambda_1|\alpha,\beta)=&\frac{(\lambda_1 q/\lambda_2-\lambda_2/q\lambda_1)(\lambda_1\lambda_2/q-q/\lambda_1\lambda_2)}{(\lambda_1/\lambda_2-\lambda_2/\lambda_1)(\lambda_1\lambda_2-1/\lambda_1\lambda_2)}\mathcal{B}(\lambda_1|\alpha,\beta)\mathcal{A}_-(\lambda_2|\alpha,\beta)\\&+\frac{(\lambda_1\lambda_2/q-q/\lambda_1\lambda_2)(\lambda_1\beta/q\lambda_2-\lambda_2 q/\beta\lambda_1)(q-1/q)}{(\lambda_1/\lambda_2-\lambda_2/\lambda_1)(\lambda_1\lambda_2-1/\lambda_1\lambda_2)(\beta/q-q/\beta)}\mathcal{B}(\lambda_2|\alpha,\beta)\mathcal{A}(\lambda_1|\alpha,\beta)\\&+\frac{(\lambda_1\lambda_2/\beta-\beta/\lambda_1\lambda_2)(q-1/q)}{(\lambda_1\lambda_2-1/\lambda_1\lambda_2)(\beta/q-q/\beta)}\mathcal{B}(\lambda_2|\alpha,\beta)\mathcal{D}(\lambda_1|\alpha,\beta)\end{aligned} \tag{4.115}$$

$$\begin{aligned}\mathcal{B}(\lambda_1|\alpha,\beta)\mathcal{D}(\lambda_2|\alpha,\beta)=&\frac{(\lambda_1 q/\lambda_2-\lambda_2/q\lambda_1)(\lambda_1\lambda_2/q-q/\lambda_1\lambda_2)}{(\lambda_1/\lambda_2-\lambda_2/\lambda_1)(\lambda_1\lambda_2-1/\lambda_1\lambda_2)}\mathcal{D}(\lambda_2|\alpha,\beta q^2)\mathcal{B}_-(\lambda_1|\alpha,\beta)\\&-\frac{(\lambda_1\lambda_2/q-q/\lambda_1\lambda_2)(\lambda_2\beta q/\lambda_1-\lambda_1/\lambda_2\beta q)(q-1/q)}{(\lambda_1/\lambda_2-\lambda_2/\lambda_1)(\lambda_1\lambda_2-1/\lambda_1\lambda_2)(\beta q-1/\beta q)}\mathcal{D}(\lambda_1|\alpha,\beta q^2)\mathcal{B}(\lambda_2|\alpha,\beta)\\&-\frac{(\lambda_1\lambda_2\beta-1/\lambda_1\lambda_2\beta)(q-1/q)}{(\lambda_1\lambda_2-1/\lambda_1\lambda_2)(\beta q-1/q\beta)}\mathcal{A}(\lambda_1|\alpha,\beta q^2)\mathcal{B}_-(\lambda_2|\alpha,\beta)\end{aligned} \tag{4.116}$$

and

$$\begin{aligned}&\mathcal{A}(\lambda_1|\alpha,\beta q^2)\mathcal{A}(\lambda_2|\alpha,\beta q^2)-\frac{(\lambda_1\lambda_2/\beta-1/\lambda_1\lambda_2)(q-1/q)}{(\lambda_1\lambda_2-1/\lambda_1\lambda_2)(\beta/q-q/\beta)}\mathcal{B}(\lambda_1|\alpha,\beta)\mathcal{C}_-(\lambda_2|\alpha,\beta q^2)=\\&\qquad\mathcal{A}(\lambda_2|\alpha,\beta q^2)\mathcal{A}(\lambda_1|\alpha,\beta q^2)-\frac{(\lambda_1\lambda_2/\beta-1/\lambda_1\lambda_2)(q-1/q)}{(\lambda_1\lambda_2-1/\lambda_1\lambda_2)(\beta/q-q/\beta)}\mathcal{B}(\lambda_2|\alpha,\beta)\mathcal{C}_-(\lambda_1|\alpha,\beta q^2)\end{aligned} \tag{4.117}$$

In this proposition, we emphasise only some of these commutation relations. They are similar to the ones used in the ungauged case and they allow to extend to this gauged case the quantum separation of variables.

Proposition 4.3.2 (Symmetries)**.** *Similar commutation relations of proposition 4.3.1 involving $\mathcal{C}(\lambda|\alpha,\beta)$ can be written by using the following β-symmetries:*

$$\mathcal{B}(\lambda|\alpha,\beta)=\mathcal{C}(\lambda|\alpha,q^2/\beta)\quad \textit{and}\quad \mathcal{A}(\lambda|\alpha,\beta)=\mathcal{D}(\lambda|\alpha,q^2/\beta) \tag{4.118}$$

Moreover, these gauge transformed operators satisfy the following parity properties:

$$\mathcal{D}(\lambda|\alpha,\beta)=\frac{(q-1/q)(\lambda^2\beta/q-q/\lambda^2\beta)}{(\beta/q^2-q^2/\beta)(\lambda^2-1/\lambda^2)}\mathcal{A}(\lambda|\alpha,\beta)+\frac{(\beta/q-q/\beta)(\lambda^2/q-q/\lambda^2)}{(\beta-1/\beta)(\lambda^2-1/\lambda^2)}\mathcal{A}(1/\lambda|\alpha,\beta) \tag{4.119}$$

$$\mathcal{B}(1/\lambda|\alpha,\beta)=-\frac{(\lambda^2 q-1/q\lambda^2)}{(\lambda^2/q-q/\lambda^2)}\mathcal{B}(\lambda|\alpha,\beta)\quad \textit{and}\quad \mathcal{C}(1/\lambda|\alpha,\beta)=-\frac{(\lambda^2 q-1/q\lambda^2)}{(\lambda^2/q-q/\lambda^2)}\mathcal{C}_-(\lambda|\alpha,\beta) \tag{4.120}$$

Lastly, the boundary quantum determinant can be written in terms of the gauge transformed boundary generators as it follows:

$$\frac{det_q\mathcal{U}(\lambda)}{(\lambda^2/q^2-q^2/\lambda^2)}=\mathcal{A}(q^{1/2}\lambda^\epsilon|\alpha,\beta q^2)\mathcal{A}(q^{1/2}/\lambda^\epsilon|\alpha,\beta q^2)+\mathcal{B}_-(q^{1/2}\lambda^\epsilon|\alpha,\beta)\mathcal{C}(q^{1/2}/\lambda^\epsilon|\alpha,\beta q^2) \tag{4.121}$$

$$=\mathcal{D}(q^{1/2}\lambda^\epsilon|\alpha,\beta)\mathcal{D}(q^{1/2}/\lambda^\epsilon|\alpha,\beta)+\mathcal{C}(q^{1/2}\lambda^\epsilon|\alpha,\beta q^2)\mathcal{B}(q^{1/2}/\lambda^\epsilon|\alpha,\beta) \tag{4.122}$$

for any $\epsilon\in\{-1,1\}$.

Once again, one can remark very similar relations compared to the ungauged case.

4.3.2 Pseudo-diagonalisability of the gauged operator $\mathcal{B}(\lambda|\alpha,\beta)$

In this paragraph, we show that the gauged operator $\mathcal{B}(\lambda|\alpha,\beta)$ is pseudo-diagonalisable with simple pseudo-spectrum. Here, the term *pseudo* has to be understood as the fact that we can exhibit states which satisfy a similar relation to the one defining an eigenstate, only the value of the gauge parameter β is modified by the action of $\mathcal{B}(\lambda|\alpha,\beta)$.

However, this pseudo-diagonalisation basis of $\mathcal{B}(\lambda|\alpha,\beta)$ is extremely important as, for a wise choice of the gauge parameter α, it defines a separate basis for $\mathcal{T}(\lambda)$.

Statement of result

As usual, we start with some notation. Let $\text{B}_{-,a}(\beta)$ be a set of complex numbers, for $1 \leqslant a \leqslant N$, and let $\text{B}_{-}(\beta) \in \mathbb{C}$. We introduce the following function of the N-tuple $\mathbf{h} = (h_1, ..., h_n) \in [\![0, p-1]\!]^N$:

$$\text{B}_{\mathbf{h}}(\lambda|\beta) \equiv \text{B}_{-}(\beta)(\frac{\lambda^2}{q} - \frac{q}{\lambda^2}) \prod_{a=1}^{\mathsf{N}} (\frac{\lambda}{\text{B}_{-,a}(\beta)q^{h_a}} - \frac{\text{B}_{-,a}(\beta)q^{h_a}}{\lambda})(\lambda q^{h_a}\text{B}_{-,a}(\beta) - \frac{1}{\lambda q^{h_a}\text{B}_{-,a}(\beta)}) \tag{4.123}$$

This function is a straightforward generalisation of the eigenvalues (4.30)-(4.31), it will of course be used to express the pseudo-eigenvalues. Then we can state the following:

Theorem 4.3.3. *For almost all the values of the boundary and bulk and gauge parameters, there exist states $\langle\Omega_\beta|$ and $|\Omega_\beta\rangle$, as well as complex numbers $\text{B}_{-}(\beta) \in \mathbb{C}$ and $(\text{B}_{-,a}(\beta))_{1\leqslant a\leqslant N}$, such that for any $\epsilon = \pm1$, $n, m \in [\![1, N]\!]$ and $h \in [\![1, p-1]\!]$ it holds:*

$$\text{B}^{2p}_{-,n}(\beta) \neq \pm 1, \;\; \text{B}^{p}_{-,m}(\beta) \neq \text{B}^{p}_{-,n}(\beta), \quad \forall n \neq m \in [\![1, N]\!] \tag{4.124}$$

$$\text{B}^{2}_{-,n}(\beta) \neq q^{1-2h}\alpha_{-}^{2\epsilon}, \;\; \text{B}^{2}_{-,n}(\beta) \neq -q^{1-2h}\beta_{-}^{2\epsilon} \tag{4.125}$$

$$\text{and} \;\; \text{B}^{2}_{-,n}(\beta) \neq q^{1-2h-2\epsilon}\mu_{m,+}^{2\epsilon}, \;\; \text{B}^{2}_{-,n}(\beta) \neq q^{1-2h-2\epsilon}\mu_{m,-}^{2\epsilon} \tag{4.126}$$

and such that the following set of states:

$$\langle\beta, \boldsymbol{h}| \equiv \langle\beta, h_1, ..., h_{\mathsf{N}}| \quad \equiv \quad \frac{1}{\text{N}_\beta}\langle\Omega_\beta| \prod_{n=1}^{\mathsf{N}} \prod_{k_n=1}^{h_n} \frac{\mathcal{A}(q^{1-k_n}/\text{B}_{-,n}(\beta)|\alpha, \beta q^2)}{\mathsf{A}_{-}(q^{1-k_n}/\text{B}_{-,n}(\beta))} \tag{4.127}$$

$$|\beta, \boldsymbol{h}\rangle \equiv |\beta, h_1, ..., h_{\mathsf{N}}\rangle \quad \equiv \quad \frac{1}{\text{N}_{\beta/q^2}} \prod_{n=1}^{\mathsf{N}} \prod_{k_n=1}^{h_n} \frac{\mathcal{D}(q^{1-k_n}/\text{B}_{-,n}(\beta)|\alpha, \beta)}{\mathsf{D}_{-}(q^{1-k_n}/\text{B}_{-,n}(\beta))}|\Omega_\beta\rangle \tag{4.128}$$

form a left and a right basis of the representation space. Moreover, in this basis the operator family $\mathcal{B}(\lambda|\alpha,\beta)$ is pseudo-diagonalised in the following sense:

$$\langle\beta, \boldsymbol{h}|\mathcal{B}(\lambda|\alpha,\beta) \quad = \quad \text{B}_{\boldsymbol{h}}(\lambda|\beta)\langle\beta/q^2, \boldsymbol{h}| \tag{4.129}$$

$$\mathcal{B}(\lambda|\alpha,\beta)|\beta, \boldsymbol{h}\rangle \quad = \quad |q^2\beta, \boldsymbol{h}\rangle\text{B}_{\boldsymbol{h}}(\lambda|\beta) \tag{4.130}$$

and with simple pseudo-spectrum, namely there is only one pseudo-eigenvector associated to a pseudo-eigenvalue.

Sketch of proof

The proof of this theorem, which paves the way to one of the main result of this thesis, follows the same line as the one of the proof of diagonalisability of the operator $\mathcal{B}(\lambda)$. However, the proof is much more technical, we simply highlight in the following the main ideas.

As for the ungauged case, the existence of the states $\langle\Omega_\beta|$ and $|\Omega_\beta\rangle$ can be proven by an explicit construction for a subclass of parameters. The same holds for the coefficients $(\text{B}_{-,a}(\beta))_{1\leqslant a\leqslant N}$ and $\text{B}_{-}(\beta)$, and so we explicitly show the Theorem 4.3.3 for this subclass of parameters.

Then, a general argument states the existence of the vectors $\langle\Omega_\beta|$ and $|\Omega_\beta\rangle$ for any parameters, i.e. the existence of at least one pseudo-eigenvector on the left and on the right. Thereafter, the theorem is proven using a continuity argument based on the fact that the quantities in question are polynomials in the bulk, boundary and gauge parameters, and the fact that an explicit construction under special values of these parameters is known. Let us give some more details:

- **Special representation** In the following, we want to study the conditions for which a nonzero state identically annihilated by the action of the operator family $A(\lambda|\alpha,\beta,\gamma)$ exists:

$$\langle\Omega,\alpha,\beta,\gamma|A(\lambda|\alpha,\beta,\gamma) = 0 \tag{4.131}$$

It is an easy consequence of the gauge transformed Yang-Baxter commutation relations that under the condition that this state exists and is unique then it is a pseudo-reference state for the gauge transformed Yang-Baxter algebra:

$$\langle\Omega,\alpha,\beta,\gamma|A(\lambda|\alpha,\beta,\gamma) = 0,\ \ \langle\Omega,\alpha,\beta,\gamma|B(\lambda|\alpha,\beta) = b(\lambda|\alpha,\beta)\langle\Omega,\alpha,\beta/q,\gamma q| \tag{4.132}$$

$$\langle\Omega,\alpha,\beta/q,\gamma q|C(\lambda|\alpha,\beta q,\gamma q) = c(\lambda|\alpha,\beta q)\langle\Omega,\alpha,\beta,\gamma|,\ \ \langle\Omega,\alpha,\beta,\gamma|D(\lambda|\alpha,\beta) \neq 0 \tag{4.133}$$

In paper **II**, we show that we can construct such a pseudo-reference state of tensor product form if we impose at least $N+1$ constraints on the set of bulk and gauge parameters. The technical proof consists first in constructing local pseudo-reference states, and to use it to explicitly construct states $\langle\Omega_\beta|$ and $|\Omega_\beta\rangle$.

At a local level, we thus find an explicit expression for the states $\langle\Omega_{n,\alpha,\beta,\gamma}|$ such that:

$$\langle\Omega_{n,\alpha,\beta,\gamma}|A_n(\lambda|\alpha,\beta) = 0 \ \text{ and }\ \langle\Omega_{n,\alpha,\beta,\gamma}|B_n(\lambda|\alpha,\beta) = b_n(\lambda|\alpha,\beta)\langle\Omega_{n,\alpha,\beta/q,\gamma q}| \tag{4.134}$$

where $b_n(\lambda|\alpha,\beta)$ is the local pseudo-eigenvalue, non identically zero, depending on the bulk, the boundary and gauge parameters (see (**II**, A.34) for its exact value). Then, considering their tensor product along the chain, we can define:

$$\langle\Omega,\alpha,\beta,\gamma| = \bigotimes_{n=1}^{N}\langle\Omega_{n,\alpha,\beta,\gamma}| \tag{4.135}$$

which satisfies:

$$\langle\Omega,\alpha,\beta,\gamma|A(\lambda|\alpha,\beta,\gamma) = 0 \ \text{ and }\ \langle\Omega,\alpha,\beta,\gamma|B(\lambda|\alpha,\beta) = b(\lambda|\alpha,\beta)\langle\Omega,\alpha,\beta/q,\gamma q| \tag{4.136}$$

with the pseudo-eigenvalue:

$$b(\lambda|\alpha,\beta) = \prod_{n=1}^{\mathsf{N}} b_n(\lambda|\alpha,\beta) \tag{4.137}$$

We can now give the following theorem, which states the pseudo-diagonalisability and simplicity of pseudo-spectrum of the operator $\mathcal{B}(\lambda|\alpha,\beta)$ for this special representation.

Proposition 4.3.4 (pseudo-diagonalisability and simplicity of spectrum of the operator $\mathcal{B}(\lambda|\alpha,\beta)$ for a subclass of parameters)**.** *Let us assume that the bulk and gauge parameters are such that there exists a vector $\langle\Omega,\alpha,\beta,\gamma|$ satisfying (4.136)-(4.137) and that $\gamma \neq \gamma_\pm$ (where $\gamma_\pm$ have been introduced in (4.112)). Moreover, let us take parameters such that for any $n,m \in [\![1,N]\!]$ and $\epsilon \in \{-1,1\}$, it holds:*

$$\mu_{n,\epsilon_n}^{2p} \neq \pm 1,\ \mu_{n,\epsilon_n}^{2p} \neq \alpha_-^{2p\epsilon},\ \mu_{n,\epsilon_n}^{2p} \neq -\beta_-^{2p\epsilon},\ \mu_{n,+}^{2p} \neq \mu_{m,-}^{2p\epsilon},\ \mu_{n,\epsilon_n}^{p} \neq \mu_{m,\epsilon_n}^{p} \tag{4.138}$$

Then the operator family $\mathcal{B}(\lambda|\alpha,\beta)$ is pseudo-diagonalised:

$$\langle\beta,h_1,...,h_{\mathsf{N}}|\mathcal{B}(\lambda|\alpha,\beta) = \mathrm{B}_{\boldsymbol{h}}(\lambda|\beta)\langle\beta/q^2,h_1,...,h_{\mathsf{N}}| \tag{4.139}$$

and has simple pseudo-spectrum characterised by fixing:

$$\forall n \in [\![1, N]\!]\ , \quad \mathrm{B}_{-,n}(\beta) = \mu_{n,\epsilon_n} q^{1/2}\ ,\ \textit{(i.e. independent of } \beta) \tag{4.140}$$

$$\textit{and} \quad \mathrm{B}_{-}(\beta) = \mathcal{F}\left[\alpha, \beta, \{\epsilon_n, k_n\}, \kappa_-, \tau_-, \xi_-\right] \tag{4.141}$$

*The explicit value of the function $\mathcal{F}$ is not of prime interest in this presentation, we just emphasise that it is a rational fraction in all the bulk, boundary and gauge parameters. The exact expression can be found in (**II**, B.67).*

The left pseudo-eigenbasis characterised by the formulas (4.127) *is obtained by fixing:*

$$\langle \Omega_\beta| = \langle \Omega, \alpha, \beta, \gamma| \tag{4.142}$$

Proof. The proof of this proposition follows quite straightforwardly the steps of the proof of diagonalisability of the operator $\mathcal{B}(\lambda)$ given in proposition 4.2.1. More explicitly, for $\mathcal{B}(\lambda|\alpha,\beta)$ it holds the next expression from the definition (4.110) of the gauged generators:

$$\begin{aligned}\frac{\mathcal{B}(\lambda|\alpha,\beta)}{f(\alpha,\beta/q,\gamma q)} &= \bar{K}_-(\lambda|\gamma)_{12} A(\lambda|\alpha,\beta,\gamma) A(1/\lambda|\alpha,\beta/q,\gamma q) - \bar{K}_-(\lambda|\gamma)_{11} A(\lambda|\alpha,\beta,\gamma) B(1/\lambda|\alpha,\beta/q) \\ &+ \bar{K}_-(\lambda|\gamma)_{21} B(\lambda|\alpha,\beta) B(1/\lambda|\alpha,\beta/q) - \bar{K}_-(\lambda|\gamma)_{22} B(\lambda|\alpha,\beta) A(1/\lambda|\alpha,\beta/q,\gamma q)\end{aligned} \tag{4.143}$$

This way we easily find a first pseudo-eigenvector of $\mathcal{B}(\lambda|\alpha,\beta)$:

$$\langle \Omega, \alpha, \beta, \gamma|\mathcal{B}(\lambda|\alpha,\beta) = \mathrm{B}_{\mathbf{0}}(\lambda|\beta)\ \langle \Omega, \alpha, \beta/q^2, \gamma| \tag{4.144}$$

with

$$\mathrm{B}_{\mathbf{0}}(\lambda|\beta) = f(\alpha,\beta/q,\gamma q)\bar{K}_-(\lambda|\gamma)_{21} b(\lambda|\alpha,\beta) b(1/\lambda|\alpha,\beta/q) \tag{4.145}$$

This pseudo-eigenvalue is not identically zero as soon as $\gamma \neq \gamma_\pm$. This fixes the values of the $\mathrm{B}_-(\beta)$ and $\mathrm{B}_{-,a}(\beta)$ to those stated in this proposition, recalling that an explicit expression for the coefficients $\bar{K}_-(\lambda|\gamma)_{ij}$ can be found in paper **II**. Thereafter, one can repeat the standard procedure already developed using now the modified commutation relations (4.116)-(4.117) to show that the relations (4.139) of the proposition is satisfied.

As for the previous propositions, the conditions (4.138) implies the simplicity. Indeed, each state $\langle \beta, \mathbf{h}|$ is associated to a different pseudo-eigenvalue of $\mathcal{B}(\lambda|\alpha,\beta)$, then the only thing that we need to prove to get their linear independence is that each such state is nonzero. This is done as for the ungauged case, namely we know by construction that $\langle \Omega_\beta|$ is nonzero and we prove by induction, thanks to the action of $\mathcal{A}(\lambda|\alpha,\beta q^2)$ on the left in some points, that all the states are non zero as it holds (4.138). Thus, they define a left basis of linear space of the representation. □

One can repeat the same procedure for the elements on the right, and it holds a very similar proposition for the right pseudo-eigenbasis. Thus, we have a subclass of parameters for which we have an explicit construction of the elements of theorem 4.3.3.

- **General parameters** As pointed out in the beginning, the proof that the operator $\mathcal{B}(\lambda|\alpha,\beta)$ is pseudo-diagonalisable and with simple pseudo-spectrum for general parameters follows from two steps. The first one is the existence of the states $|\Omega_\beta\rangle$ and $\langle \Omega_\beta|$ in general, which can be done using the same kind of reasoning as in the ungauged case, and the second one relies on the special case of proposition 4.3.4, using a continuity argument. Let us give some more details.

  To begin with, we can show that there exists at least one left and right pseudo-eigenstate $|\Omega_\beta\rangle$ and $\langle \Omega_\beta|$ of the one parameter family of pseudo-commuting operators $\mathcal{B}(\lambda|\alpha,\beta)$, with pseudo-eigenvalue $\mathrm{B}_{\mathbf{0}}(\lambda|\beta)$ satisfying the conditions (4.124), (4.125) and (4.126).

Indeed, the operator family $\mathcal{B}(\lambda|\alpha,\beta)$ admits the following representation:

$$\mathcal{B}(\lambda|\alpha,\beta) = (\frac{\lambda^2}{q} - \frac{q}{\lambda^2}) \sum_{a=0}^{\mathsf{N}} \Lambda^a \bar{\mathcal{B}}_a(\beta) T_\beta^{-2} \tag{4.146}$$

where the following commutation relations hold:

$$\forall a,b \in [\![1,N]\!] \ , \quad \bar{\mathcal{B}}_a(\beta)T_\beta = T_\beta \bar{\mathcal{B}}_a(\beta q) \quad \text{and} \quad \left[\bar{\mathcal{B}}_a(\beta), \bar{\mathcal{B}}_b(\beta)\right] = 0 \tag{4.147}$$

as a consequence of the commutation relations (4.114). The result of the proposition 4.3.4 implies that for the special choice of the boundary, bulk and gauge parameters introduced, all the operators $\bar{\mathcal{B}}_{-,a,\beta}$ are invertible and $\mathcal{B}(\lambda|\alpha,\beta)$ admits the following representation:

$$\mathcal{B}(\lambda|\alpha,\beta) = \mathrm{B}_-(\beta)(\frac{\lambda^2}{q} - \frac{q}{\lambda^2}) \prod_{a=1}^{\mathsf{N}} (\frac{\lambda}{\mathcal{B}_a(\beta)} - \frac{\mathcal{B}_a(\beta)}{\lambda})(\lambda \mathcal{B}_a(\beta) - \frac{1}{\lambda \mathcal{B}_a(\beta)}) T_\beta^{-2} \tag{4.148}$$

where the $\mathcal{B}_a(\beta)$ are commuting and invertible operators. Then the fact that these operators depend continuously on the parameters implies that this statement is true for almost any values of these parameters. This also implies that for almost all the values of the boundary, bulk and gauge parameters we can use the above representation for $\mathcal{B}(\lambda|\alpha,\beta)$.

We can now recall that, repeating the reasoning of proposition 4.2.2, we can always find a nonzero simultaneous eigenstate of commuting operators such as the $\mathcal{B}_a(\beta)$ for any $a \in [\![1,N]\!]$. This is a pseudo-eigenstate of the operator family $\mathcal{B}(\lambda|\alpha,\beta)$.

Now, for the same set of representations considered in the proposition 4.3.4 we know that the pseudo-eigenvalues of $\mathcal{B}(\lambda|\alpha,\beta)$ satisfy the conditions (4.125) and (4.126). Then, we can use once again the continuity argument to argue that the eigenvalues on the common eigenstate still satisfy (4.125) and (4.126).

We can thus prove the theorem 4.3.3, namely the pseudo-diagonalisability and simplicity of pseudo-spectrum of $\mathcal{B}(\lambda|\alpha,\beta)$. Indeed, under the conditions (4.125) and (4.126) we can prove that all the left and right states are well defined and nonzero states which are pseudo-eigenstates of $\mathcal{B}(\lambda|\alpha,\beta)$ associated to different pseudo-eigenvalues as a consequence of the gauge transformed commutation relations. The proof of the fact that the states (4.127) and (4.128) are all nonzero is done reproducing the standard argument.

4.3.3 The separate basis

In this paragraph lies the heart of the quantum separation of variables. In a first part, we show a rewriting of the boundary transfer matrix for general parameters in terms of a gauged $K_+(\lambda)$ right boundary matrix. Then, for an appropriate choice of the gauge parameter, we show that its spectral problem separates in the pseudo-eigenbasis of $\mathcal{B}(\lambda|\alpha,\beta)$. To this end, we give the action of the generators of the gauged reflection algebra on the elements of the pseudo-eigenbasis.

A gauge transformation on the right boundary matrix $K_+(\lambda)$

Let us denote by $X(\lambda|\alpha,\beta)$ and $Y(\lambda|\alpha,\beta)$ the respectively first and second columns of the gauge matrix $G(\lambda|\alpha,\beta)$, and similarly let us define $\tilde{Y}(\lambda|\alpha,\beta)$ and $\tilde{X}(\lambda|\alpha,\beta)$ the respectively first and second lines of the gauge matrix $G^{-1}(\lambda|\alpha,\beta)$.

Moreover, we define some modifications of these quantities, namely:

$$\hat{X}(\lambda|\alpha,q^2\beta) \equiv q\frac{\beta/q - q/\beta}{\beta - 1/\beta} X(\lambda|\alpha,q^2\beta) \quad \text{and} \quad \hat{Y}(\lambda|\alpha,\beta/q^2) \equiv q\frac{q\beta - 1/(q\beta)}{\beta - 1/\beta} Y(\lambda|\alpha,\beta/q^2) \tag{4.149}$$

$$\overline{X}(\lambda|\alpha,\beta) \equiv q\frac{\beta - 1/\beta}{\beta/q - q/\beta} \tilde{X}(\lambda|\alpha,\beta) \quad \text{and} \quad \overline{Y}(\lambda|\alpha,\beta) \equiv q\frac{\beta - 1/\beta}{q\beta - 1/(q\beta)} \tilde{Y}(\lambda|\alpha,\beta) \tag{4.150}$$

Then the gauged boundary $K_+(\lambda)$, denoted $\bar{K}_+(\lambda,\alpha;\beta)$, is defined by its elements:

$$\bar{K}_+(\lambda|\alpha,\beta) \equiv \frac{\lambda}{q^{1/2}} \begin{pmatrix} \overline{Y}(q^{1/2}/\lambda|\alpha,\beta/q^2)K_+(\lambda)\hat{X}(\lambda/q^{1/2}|\alpha,q^2\beta) & \overline{Y}(q^{1/2}/\lambda|\alpha,\beta)K_+(\lambda)\hat{Y}(\lambda/q^{1/2}|\alpha,\beta/q^2) \\ \overline{X}(q^{1/2}/\lambda|\alpha,\beta)K_+(\lambda)\hat{X}(\lambda/q^{1/2}|\alpha,q^2\beta) & \overline{X}(q^{1/2}/\lambda|\alpha,q^2\beta)K_+(\lambda)\hat{Y}(\lambda/q^{1/2}|\alpha,\beta/q^2) \end{pmatrix} \tag{4.151}$$

One can thus compute the explicit expressions for the elements of the matrix $\bar{K}_+(\lambda|\alpha,\beta)$. For example, it holds:

$$\bar{K}_+(\lambda|\alpha,\beta/q)_{12} = 1/(\xi_+ - 1/\xi_+)1/(\beta-1/\beta)\alpha/q\left(q\lambda^2 - 1/(q\lambda^2)\right)\left(q\kappa_+e^{-\tau_+}(\beta/\alpha)^2 - q^2/\xi_+\beta/\alpha - \kappa_+e^{\tau_+}\right) \tag{4.152}$$

Proposition 4.3.5. *The boundary transfer matrix admits the following expression in terms of the gauged boundary matrix $\bar{K}_+(\lambda|\alpha,\beta)$ and of the gauged generators:*

$$\begin{aligned}\mathcal{T}(\lambda) = \bar{K}_+(\lambda|\alpha,\beta/q)_{11}\mathcal{A}(\lambda|\alpha,\beta) + \bar{K}_+(\lambda|\alpha,\beta/q)_{22}\mathcal{D}(\lambda|\alpha,\beta)+\\ \bar{K}_+(\lambda|\alpha,\beta/q)_{21}\mathcal{B}(\lambda|\alpha,\beta/q^2) + \bar{K}_+(\lambda|\alpha,\beta/q)_{12}\mathcal{C}(\lambda|\alpha,q^2\beta)\end{aligned} \tag{4.153}$$

The interest in these gauge transformation is that for a judicious choice of the gauge, one gets that $\bar{K}_+(\lambda|\alpha,\beta)$ is triangular. Then, we are in the same framework as what has been done for the constraint boundary. More precisely, we can state:

Proposition 4.3.6. *If we fix the gauge parameter α as it follows:*

$$\alpha = -\beta\beta_+/q^2\alpha_+e^{\tau_+} \tag{4.154}$$

then $\bar{K}_+(\lambda|\alpha,\beta)_{12} \equiv 0$ and the boundary transfer matrix can be equivalently written:

$$\mathcal{T}(\lambda) = \mathsf{a}_+(\lambda)\mathcal{A}(\lambda|\alpha,\beta) + \mathsf{a}_+(1/\lambda)\mathcal{A}(1/\lambda|\alpha,\beta) + q\ \mathsf{c}_+(\lambda|\beta)\mathcal{B}(\lambda|\alpha,\beta/q^2) \tag{4.155}$$

$$\textit{or}\ \ \mathcal{T}(\lambda) = \mathsf{d}_+(\lambda)\mathcal{D}(\lambda|\alpha,\beta) + \mathsf{d}_+(1/\lambda)\mathcal{D}(1/\lambda|\alpha,\beta) + q^{-1}\mathsf{c}_+(\lambda|\beta)\mathcal{B}(\lambda|\alpha,\beta) \tag{4.156}$$

where we have defined:

$$\mathsf{a}_+(\lambda) = -\frac{\lambda^2q - 1/q\lambda^2}{\lambda^2 - 1/\lambda^2}g_+(\lambda)\ ,\quad \mathsf{d}_+(\lambda) = \frac{\lambda^2q - 1/q\lambda^2}{\lambda^2 - 1/\lambda^2}g_+(q/\lambda) \tag{4.157}$$

$$\mathsf{c}_+(\lambda|\beta) = \frac{-q(\lambda^2q - 1/q\lambda^2)\left(\beta\beta_+/q\alpha_+ - q\alpha_+/\beta\beta_+\right)}{\beta\left(\alpha_+ - 1/\alpha_+\right)\left(\beta_+ + 1/\beta_+\right)} \tag{4.158}$$

(we recall that the function $g_+(\lambda)$ has been introduced in (4.7)*).*

Note that (4.155) and (4.156) are direct consequences of the symmetry (4.120). Let us emphasise on the fact that these are the same kind of formulas than the symmetries (4.20) and (4.21) proven in paragraph 4.1 for the ungauged case.

The spectral problem separates

As it was expected, the pseudo-eigenbasis of the operator $\mathcal{B}(\lambda|\alpha,\beta)$ is a separate basis for the boundary transfer matrix with general parameters. Indeed, we show here that the action of the generators $\mathcal{A}(\lambda|\alpha,\beta)$ and $\mathcal{D}(\lambda|\alpha,\beta)$ are simply shifts on the elements of this basis, leading to a system of Baxter equations.

First, we introduce the notation:

$$\zeta_n^{(h_n)} \equiv \begin{cases} \mathrm{B}_{-,n}(\beta)q^{h_n} & \forall n \in [\![1,\mathsf{N}]\!] \\ \left(\mathrm{B}_{-,n-N}(\beta)q^{h_{n-N}}\right)^{-1} & \forall n \in [\![N+1,2\mathsf{N}]\!] \end{cases} \tag{4.159}$$

Then, thanks to the boundary quantum determinant (4.121)-(4.122) and the simplicity of pseudo-spectrum of $\mathcal{B}(\lambda|\alpha,\beta)$, one can check *via* the same procedure already seen for the ungauged case that the operator families $\mathcal{A}(\lambda|\alpha,\beta)$ and $\mathcal{D}(\lambda|\alpha,\beta)$ in the zeros of $\mathcal{B}(\lambda|\alpha,\beta)$ act as simple shift operators:

$$\langle\beta,\mathbf{h}|\mathcal{A}(\zeta_a^{(h_a)}|\alpha,\beta q^2) = \mathsf{A}_-(\zeta_a^{(h_a)})\langle\beta,\mathbf{h}|T_a^{-\varphi_a} \tag{4.160}$$

$$\text{and}\quad \mathcal{D}(\zeta_a^{(h_a)}|\alpha,\beta)|\beta,\mathbf{h}\rangle = T_a^{-\varphi_a}|\beta,\mathbf{h}\rangle\mathsf{D}_-(\zeta_a^{(h_a)}) \tag{4.161}$$

where

$$\langle\beta,h_1,...,h_a,...,h_{\mathsf{N}}|T_a^{\pm} \;=\; \langle\beta,h_1,...,h_a\pm1,...,h_{\mathsf{N}}| \tag{4.162}$$

$$\text{and}\quad T_a^{\pm}|\beta,h_1,...,h_a,...,h_{\mathsf{N}}\rangle \;=\; |\beta,h_1,...,h_a\pm1,...,h_{\mathsf{N}}\rangle \tag{4.163}$$

and φ_a is simply defined in (4.59).

The simple representations (4.155)-(4.156) of the transfer matrix in terms of the gauge transformed boundary generators and the known actions (4.160)-(4.161) of these operators imply that the transfer matrix spectral problem is separate in the pseudo-eigenbasis of $\mathcal{B}(\lambda|\alpha,\beta)$. Explicitly, it follows the Baxter equations:

$$\tau(\xi_n^{(h_n)})\Psi_\tau(\beta/q^2,\mathbf{h}) \;=\; \mathrm{A}(\xi_n^{(h_n)})\Psi_\tau(\beta/q^2,\mathsf{T}_n^-(\mathbf{h})) + \mathrm{A}(1/\xi_n^{(h_n)})\Psi_\tau(\beta/q^2,\mathsf{T}_n^+(\mathbf{h})) \tag{4.164}$$

where we introduced:

$$\mathrm{A}(\lambda)\equiv\mathsf{a}_+(\lambda)\mathsf{A}_-(\lambda)\quad\text{and}\quad\Psi_\tau(\beta/q^2,\mathbf{h})\equiv\langle\beta/q^2,\mathbf{h}|\tau\rangle \tag{4.165}$$

for the characterisation of the eigenvalue $\tau(\lambda)$ and $|\tau\rangle$ the associated eigenvector. While for the eigencovector $\langle\tau|$ it holds:

$$\forall n\in[\![1,N]\!]\ ,\quad \tau(\xi_n^{(h_n)})\hat{\Psi}_\tau(\beta,\mathbf{h}) \;=\; \mathrm{D}(\xi_n^{(h_n)})\hat{\Psi}_\tau(\beta,\mathsf{T}_n^-(\mathbf{h})) + \mathrm{D}(1/\xi_n^{(h_n)})\hat{\Psi}_\tau(\beta,\mathsf{T}_n^+(\mathbf{h})) \tag{4.166}$$

with:

$$\mathrm{D}(\lambda)\equiv\mathsf{d}_+(\lambda)\mathsf{D}_-(\lambda)\quad\text{and}\quad\hat{\Psi}_\tau(\beta,\mathbf{h})\equiv\langle\tau|\beta,h_1,...,h_{\mathsf{N}}\rangle \tag{4.167}$$

4.3.4 Discrete characterisation of the spectrum of the boundary transfer matrix

In this paragraph we characterise the spectrum of the boundary transfer matrix associated to general integrable boundaries. In view of what has been done previously, the results have a very similar form as for the ungauged case, proposition 4.2.7. Simply here the coefficients $\zeta_a^{(h_a)}$ are given by (4.159), and the separate basis are different.

As before, we can state a proposition on the scalar products of the elements of the left and right separate basis:

Proposition 4.3.7. *Using an appropriate choice for the norm* N_β *in the basis elements* (4.127) *and* (4.128)*, it holds:*

$$\forall\ \boldsymbol{h},\boldsymbol{k}\in[\![0,p-1]\!]^N\ ,\quad \langle\beta,\boldsymbol{h}|\beta q^2,\boldsymbol{k}\rangle = \delta_{\boldsymbol{hk}}\prod_{1\leqslant b<a\leqslant\mathsf{N}}\frac{1}{X_a^{(h_a)}-X_b^{(h_b)}} \tag{4.168}$$

In particular, this implies the spectral decomposition of the identity in the separate basis:

$$\sum_{h_1,...,h_{\mathsf{N}}=0}^{p-1}\prod_{1\leqslant b<a\leqslant\mathsf{N}}(X_a^{(h_a)}-X_a^{(h_a)})|\beta q^2,h_1,...,h_{\mathsf{N}}\rangle\langle\beta,h_1,...,h_{\mathsf{N}}| = \mathbb{1} \tag{4.169}$$

where we recall the definition

$$\forall b \in [\![1, N]\!], \quad X_b^{(h_b)} \equiv (\text{B}_{-,b}(\beta)q^{h_b})^2 + 1/(\text{B}_{-,b}(\beta)q^{h_b})^2 \equiv (\zeta_b^{(h_b)})^2 + 1/(\zeta_b^{(h_b)})^2 \tag{4.170}$$

Therefore, we have the following characterisation of the boundary transfer matrix:

Proposition 4.3.8. *For almost all the values of the boundary and bulk parameters, $\mathcal{T}(\lambda)$ has simple spectrum and Σ_τ, the set of all the eigenvalues, coincides with the set of polynomials $\tau(\lambda)$ of the form* (4.68) *with* (4.69) *which satisfy the following discrete system of equations:*

$$\forall a \in [\![1, N]\!] \ , \quad detD_\tau(\zeta_a^{(0)}) = 0 \tag{4.171}$$

- *The right $\mathcal{T}(\lambda)$ eigenstate corresponding to $\tau(\lambda) \in \Sigma_\tau$ is defined by the following decomposition in the right separate basis:*

$$|\tau\rangle = \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{p-1} \prod_{a=1}^{\mathsf{N}} q_{\tau,a}^{(h_a)} \prod_{1\leqslant b<a\leqslant \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)})|\beta, h_1, \dots, h_{\mathsf{N}}\rangle \tag{4.172}$$

  where the gauge parameters α and β satisfy the condition (4.154) *and the $q_{\tau,a}^{(h_a)}$ are the unique nontrivial solutions up to normalisation of the linear homogeneous system:*

$$D_\tau(\zeta_a^{(0)}) \begin{pmatrix} q_{\tau,a}^{(0)} \\ \vdots \\ q_{\tau,a}^{(p-1)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.173}$$

- *The left $\mathcal{T}(\lambda)$ eigenstate corresponding to $\tau(\lambda) \in \Sigma_\tau$ is defined by the following decomposition in the left separate basis:*

$$\langle\tau| = \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{p-1} \prod_{a=1}^{\mathsf{N}} \hat{q}_{\tau,a}^{(h_a)} \prod_{1\leqslant b<a\leqslant \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)})\langle\beta/q^2, h_1, \dots, h_{\mathsf{N}}| \tag{4.174}$$

  where the gauge parameters α and β satisfy the condition (4.154) *and the $\hat{q}_{\tau,a}^{(h_a)}$ are the unique nontrivial solutions up to normalisation of the linear homogeneous system:*

$$\begin{pmatrix} \hat{q}_{\tau,a}^{(0)} & \dots & \hat{q}_{\tau,a}^{(p-1)} \end{pmatrix} \left(\hat{D}_\tau(\zeta_a^{(0)})\right)^{t_0} = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix} \tag{4.175}$$

  with $\hat{D}_\tau(\lambda)$ is the family of $p \times p$ matrices defined substituting in $D_\tau(\lambda)$ the coefficient $\text{A}(\lambda)$ with $\text{D}(\lambda)$.

Lastly, there is the following link: let $\tau(\lambda) \in \Sigma_\tau$ then it holds:

$$\frac{\hat{q}_{\tau,a}^{(h)}}{\hat{q}_{\tau,a}^{(h-1)}} = \frac{\text{A}(1/\zeta_a^{(h-1)})}{\text{D}(1/\zeta_a^{(h-1)})} \frac{q_{\tau,a}^{(h)}}{q_{\tau,a}^{(h-1)}} \tag{4.176}$$

The proof of this proposition follows the same lines as the one dedicated to the characterisation of the spectrum under the constraint that the right boundary is triangular, cf proposition 4.2.7.

This proposition is one of the main result of this thesis. Before to move to an equivalent characterisation in terms of functional equations, we would like to state the following proposition:

Proposition 4.3.9. *For almost any values of the bulk and boundary parameters, the set $\{|\tau\rangle\}$ of all the eigenvectors of the boundary transfer matrix forms a basis of the Hilbert space $\mathcal{H}$.*

Proof. Let us consider the following special representation, where the bulk parameters satisfy:

$$c_n = -b_n^* \quad ; \quad d_n = -a_n^* \quad \text{and} \quad \alpha_n^*\beta_n = a_n^* b_n \tag{4.177}$$

and where the boundary matrices are diagonal, $K_0(\lambda|\xi_-, 0, 0)$ and $K_{+0}(\lambda|\xi_+, 0, 0)$, with the associated boundary parameters satisfying moreover $|\xi_-| = |\xi_+| = 1$. The $*$ operation is the complex conjugation. A simple direct calculation made for example in [235] leads to the following Hermitian conjugate of the monodromy matrix:

$$M_{0Q}^\dagger(\lambda|P_Q|P_C) = \sigma_0^y M_{0Q}(\lambda^*|P_Q|P_C)\sigma_0^y \tag{4.178}$$

where σ_0^y denotes the Pauli matrix.

From this relation, and using the specific left boundary matrix introduced, one can compute the Hermitian conjugate of the boundary monodromy matrix:

$$\mathcal{U}_{0Q}^\dagger(\lambda) = \mathcal{U}_{0Q}^{t_0}(1/\lambda^*) \tag{4.179}$$

Then, from the definition of the boundary transfer matrix, and for the special choice of representation here chosen, we can show:

$$\mathcal{T}^\dagger(\lambda) = \mathcal{T}(1/\lambda^*) \tag{4.180}$$

Thus for this special representation the boundary transfer matrix is normal. Then it follows that the determinant of the matrix $p^{\mathsf{N}} \times p^{\mathsf{N}}$ of elements $\langle e_i|\tau_j\rangle$, where $\langle e_i|$ is the generic element of a left orthogonal basis and $|\tau_j\rangle$ is the generic transfer matrix eigenvector, is non zero.

Noting that this determinant is a fractional function of the bulk and boundary parameters, non zero for the special choice of the parameters above defined, it follows that it is non zero for almost every choice of the parameters. Which concludes the proof. □

This property will be used later, notably in chapter 5.

4.3.5 Functional characterisation of $\mathcal{T}(\lambda)$ spectrum

In this section, we give an equivalent characterisation of $\mathcal{T}(\lambda)$ spectrum as the set of solutions to a Baxter like T-Q functional equation, and rewrite the eigenstates in an Algebraic Bethe Ansatz form. This makes an interesting link with the Bethe ansatz characterisation, but as we are going to see, for generic parameters the equation has an inhomogeneous term which has to be tackled. The literature on the analysis of the inhomogeneous Baxter equations is still at a preliminary stage. Let us mention that they were first introduced in [217] for the XXZ spin 1/2 chain and the work of Nepomechie [254] has given, for the XXX case, some first numerical evidence that they define a complete characterisation of the transfer matrix spectrum while a first rigorous proof of the completeness is appeared in [144].

In order to get a shorter presentation, we here introduce the functional characterisation for the most general integrable boundaries. The case of a triangular right boundary matrix can be explicitly read in paper **I**, or can be traced back from the following.

Here we mainly state the result, the fairly technical proofs can be found in paper **II**. The propositions 4.3.10 and 4.3.11 are preliminary results in order to understand the main result, the theorems 4.3.12 and 4.3.13.

Proposition 4.3.10. *Let $t(\lambda)$ be a function of λ invariant under the transformation $\lambda \to 1/\lambda$ and $\lambda \to -\lambda$ then det $D_t(\lambda)$ is a function of*

$$Z = \lambda^{2p} + \frac{1}{\lambda^{2p}} \tag{4.181}$$

i.e. it is invariant under the transformations $\lambda^p \to 1/\lambda^p$ and $\lambda \to -\lambda$. Moreover, if $t(\lambda)$ is a Laurent polynomial of degree $\mathsf{N}+2$ in Λ then det $D_t(\lambda)$ is a Laurent polynomial of degree $\mathsf{N}+2$ in Z.

The first symmetry is the same one as the one already seen for the ungauged case. The proof of the second statement consists in showing that *det* $D_t(q^{1/2})$ and *det* $D_t(i\, q^{1/2})$ are finite when $t(\lambda)$ is a Laurent polynomial of degree $\mathsf{N}+2$ in Λ, even if the elements $\textsc{a}(\pm 1)$ and $\textsc{a}(\pm i)$ diverge.

Let us thus introduce the following asymptotic behaviours:

$$\textsc{a}_\infty \equiv \lim_{\lambda\to+\infty} \lambda^{-2(\mathsf{N}+2)}\textsc{a}(\lambda) \quad \text{and} \quad \textsc{a}_0 \equiv \lim_{\lambda\to 0} \lambda^{2(\mathsf{N}+2)}\textsc{a}(\lambda) \tag{4.182}$$

as well as the function:

$$F(\lambda) = \prod_{b=1}^{2\mathsf{N}} \left(\frac{\lambda^p}{\left(\zeta_b^{(0)}\right)^p} - \frac{\left(\zeta_b^{(0)}\right)^p}{\lambda^p} \right) \tag{4.183}$$

Then we have a functional expression for the determinant of the matrix $D_\tau(\lambda)$:

Proposition 4.3.11. *For almost all the values of the boundary and bulk parameters, $\mathcal{T}(\lambda)$ has simple spectrum and $\tau(\lambda)$ of the form* (4.68) *is an element of Σ_τ if and only if det $D_\tau(\lambda)$ is a Laurent polynomial of degree $\mathsf{N}+2$ in the variable Z which satisfies the following functional equation:*

$$det\ D_\tau(\lambda) = F(\lambda) \left(\lambda^{2p} - \frac{1}{\lambda^{2p}}\right)^2 \prod_{k=0}^{p-1} (\tau_\infty - (q^k \textsc{a}_\infty + q^{-k}\textsc{a}_0)) \tag{4.184}$$

The proof consists in the following. From the previous proposition, we know that *det* $D_\tau(\lambda)$ is a polynomial of degree N+2 in Z. We can explicitly compute the asymptotic behaviour, and N zeros are known when $\tau(\lambda)$ is an eigenvalue, thanks to the discrete characterisation (4.171). Moreover, we show that in fact *det* $D_\tau(q^{1/2})$ and *det* $D_\tau(i\ q^{1/2})$ are vanishing (for $\tau(\lambda)$ an eigenvalue).

Let us now turn to the main results of this section, Baxter's TQ-functional equations. In the following, we denote with $Q(\lambda)$ a polynomial in $\Lambda = \lambda^2 + 1/\lambda^2$ of degree N_Q of the form:

$$Q(\lambda) = \prod_{b=1}^{\mathsf{N}_Q} (\Lambda - \Lambda_b) \tag{4.185}$$

Theorem 4.3.12. *For almost all the values of the boundary and bulk parameters such that:*

$$\forall k \in [\![0, p-1]\!]\ , \quad \tau_\infty \neq q^{-k} a_\infty + q^k a_0 \tag{4.186}$$

it holds $\tau(\lambda) \in \Sigma_\tau$ if and only if $\tau(\lambda)$ is an entire function and there exists and is unique a polynomial $Q(\lambda)$ of the form (4.185) *with $\mathsf{N}_Q = (p-1)\,\mathsf{N}$, satisfying the following functional equation:*

$$\tau(\lambda)Q(\lambda) = \textsc{a}(\lambda)Q(\lambda/q) + \textsc{a}(1/\lambda)Q(\lambda q) + \left[\tau_\infty - (q^{-\mathsf{N}_Q}\textsc{a}_\infty + q^{\mathsf{N}_Q}\textsc{a}_0)\right]\left(\Lambda^2 - X^2\right) F(\lambda) \tag{4.187}$$

and the conditions:

$$\forall a \in [\![1, N]\!]\ , \quad (Q(\zeta_a^{(0)}), ..., Q(\zeta_a^{(p-1)})) \neq (0, ..., 0) \tag{4.188}$$

The difficult task is to prove the existence of such a polynomial $Q(\lambda)$, the other implication being quite straightforward thanks to the transfer matrix discrete characterisation (4.171). Thus, the trick is to consider a more general equation than (4.187) (namely (**II**, 5.30)), and to show that the solution $X_0(\lambda)$ which can be computed thanks to the Cramer's rule, is in fact the searched $Q(\lambda)$. Using similar techniques as for the propositions 4.3.10 and 4.3.11 to show that apparently diverging quantities are in fact finite, we show each statements: the equation (4.187) is satisfied, $Q(\lambda)$ has the form (4.185), it is unique and satisfies (4.188). For this last point, we use the polynomiality in the parameters, and show that the statement holds for a particular set of parameters, so it holds almost everywhere.

In the previous theorem we have excluded the constraint on the boundary and bulk parameters leading to an identically zero $det\ D_\tau(\lambda)$ for any $\tau(\lambda) \in \Sigma_\tau$. These specific cases are considered in the next theorem.

Theorem 4.3.13. *Let us assume that there exists a $k \in [\![0, p-1]\!]$ such that it holds:*

$$\tau_\infty = q^{-k}\mathrm{A}_\infty + q^k \mathrm{A}_0 \tag{4.189}$$

Then for almost all the values of the boundary and bulk parameters, $\tau(\lambda) \in \Sigma_\tau$ if and only if $\tau(\lambda)$ is an entire function and there exists and is unique a polynomial $Q(\lambda)$ of the form (4.185) *with $\mathsf{N}_Q \leqslant (p-1)(\mathsf{N}+1)$ and $\mathsf{N}_Q = k\ mod p$, satisfying the following homogeneous Baxter equation:*

$$\tau(\lambda)Q(\lambda) = \mathrm{A}(\lambda)Q(\lambda/q) + \mathrm{A}(1/\lambda)Q(\lambda q) \tag{4.190}$$

and the conditions:

$$\forall a \in [\![1, N]\!]\ , \quad (Q(\zeta_a^{(0)}), ..., Q(\zeta_a^{(p-1)})) \neq (0, ..., 0) \tag{4.191}$$

Let us simply comment that in this case, the polynomial $Q(\lambda)$ can be explicitly constructed *via* the cofactor of the element $(1,1)$ of the matrix $D_\tau(\lambda)$, the proof being based on some symmetries of the cofactors.

We end this section by giving a Bethe ansatz type formulation for the eigenvectors. In more details, let us introduce the states:

$$\langle \beta, \omega| \equiv \sum_{h_1,\dots,h_\mathsf{N}=0}^{p-1} \prod_{a=1}^{\mathsf{N}} \prod_{k_a=0}^{h_a-1} \frac{\mathrm{A}(1/\zeta_a^{(k_a)})}{\mathrm{D}(1/\zeta_a^{(k_a)})} \prod_{1\leqslant b<a\leqslant \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)}) \langle \beta, h_1, ..., h_\mathsf{N}| \tag{4.192}$$

$$|\beta, \bar{\omega}\rangle \equiv \sum_{h_1,\dots,h_\mathsf{N}=0}^{p-1} \prod_{1\leqslant b<a\leqslant \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)}) |\beta, h_1, ..., h_\mathsf{N}\rangle \tag{4.193}$$

and a renormalisation of the operator

$$\hat{\mathcal{B}}(\lambda|\alpha,\beta) \equiv \frac{\mathcal{B}(\lambda|\alpha,\beta)T_\beta^2}{(\lambda^2/q - q/\lambda^2)\mathrm{B}_-(\beta)} \tag{4.194}$$

which is a degree N polynomial in Λ. As first remarked in paper [136], from the functional (polynomial) characterisation of the Q-function and from the discrete characterisation of the boundary transfer matrix eigenvectors, it follows the Bethe-like rewriting of these eigenstates:

Corollary 4.3.14. *The left and right transfer matrix eigenstates associated to $\tau(\lambda) \in \Sigma_\tau$ admit the following Bethe ansatz like representations:*

$$\langle \tau| = \langle \beta, \omega| \prod_{b=1}^{\mathsf{N}_Q} \hat{\mathcal{B}}(\lambda_b|\alpha,\beta) \quad \textit{and} \quad |\tau\rangle = \prod_{b=1}^{\mathsf{N}_Q} \hat{\mathcal{B}}(\lambda_b|\alpha,\beta)|\beta,\bar{\omega}\rangle \tag{4.195}$$

where the λ_b (fixed up to the symmetries $\lambda_b \to -\lambda_b$ and $\lambda_b \to 1/\lambda_b$), for $b \in [\![1, N_Q]\!]$, are the zeros of $Q(\lambda)$ and we have imposed the condition (4.154) *on the gauge parameters.*

4.4 A first step toward the dynamic

As already mention, the characterisation of the spectrum is the very first step in the computation of physical quantities, aiming to characterise the dynamic of the system. In this short section, we give a little step toward the computation of the form factors and correlation functions for the class of models under consideration, namely we compute the scalar products of the so-called separate states.

4.4.1 Separate states and their scalar products

Let us introduce a class of left and right states, the so-called separate states, characterised by the following type of decompositions in the left and right separate basis:

$$\langle\alpha| \equiv \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{p-1} \prod_{a=1}^{\mathsf{N}} \alpha_a^{(h_a)} \prod_{1\leqslant b<a\leqslant \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)}) \langle \beta/q^2, h_1, ..., h_{\mathsf{N}}| \tag{4.196}$$

$$|\beta\rangle = \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{p-1} \prod_{a=1}^{\mathsf{N}} \beta_a^{(h_a)} \prod_{1\leqslant b<a\leqslant \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)}) |\beta, h_1, ..., h_{\mathsf{N}}\rangle \tag{4.197}$$

where the coefficients $\alpha_a^{(h_a)}$ and $\beta_a^{(h_a)}$ are arbitrary complex numbers.

These separate states are interesting at least for two reasons: the eigenstates of the boundary transfer matrix are special separate states; they admit a simple determinant scalar product, as it is stated in the next proposition:

Proposition 4.4.1. *Let us take an arbitrary separate left state $\langle\alpha|$ and an arbitrary separate right state $|\beta\rangle$. Then it holds:*

$$\langle\alpha|\beta\rangle = det\ \mathcal{M}^{(\alpha,\beta)} \tag{4.198}$$

where the elements of the size N matrix $\mathcal{M}^{(\alpha,\beta)}$ are given by:

$$\forall (a,b) \in [\![1,N]\!]^2, \quad \mathcal{M}_{a,b}^{(\alpha,\beta)} \equiv \sum_{h=0}^{p-1} \alpha_a^{(h)} \beta_a^{(h)} (X_a^{(h)})^{(b-1)} \tag{4.199}$$

The proof is quite straightforward, it is based on the fact that Vandermonde determinants appear when computing the scalar product. One of the main corollary is the orthogonality of two eigenstates $\langle\tau|$ and $|\tau'\rangle$ of the boundary transfer matrix associated to two different eigenvalues $\tau(\lambda)$ and $\tau'(\lambda)$:

$$\langle\tau|\tau'\rangle = 0 \tag{4.200}$$

The computation of such scalar products is the very first step toward the dynamic, several further steps are required to reach this characterisation for the models associated to cyclic representations of the 6-vertex reflection algebra (see paragraph 1.5): the reconstruction of the local operators in separate variables, the identification of the ground state, the homogeneous and the thermodynamic limit. For example a rewriting of the determinant representations for the form factors obtained from separation of variables will be necessary to overcome the standard problems related to the homogeneous limit. This problem has been addressed and solved for the XXX spin 1/2 chain, linking the separation of variables type determinants with Izergin's, Slavnov's and Gaudin's type determinants [146, 147].

Let us comment that we have chosen, for both the triangular and general K_+ boundary matrices, to present the existence of the separate basis avoiding a recursive construction, which was instead done for example in the sine Gordon model [250]. Here, we first considered some special representations in which we could construct explicitly the separate basis (in both triangular and general cases), while the existence of the separate basis for general representations is proven by some general arguments. Let us argue that this does not lead to severe limitations per se. For the spectrum indeed, we are still able to prove that the functional equation characterises the eigenvalues for general parameters. Moreover, for the future computations of the dynamics, the relevant point is that we can still characterise Sklyanin's measure (4.73) and (4.168) by a Vandermonde formula, leading to formula of the kind (4.198)-(4.199) for the scalar products of separate states. Then, the rewriting of the scalar products and form factors by determinant formulas adapted to the homogeneous limit should allow to absorbe the separate variable eigenvalue spectrum in Laurent polynomials whose homogeneous limit is finite and known; as it happens in the spin 1/2 quantum chains [146, 147].

Chapter 5

Local Hamiltonians associated to cyclic representations of the reflection algebra

In this chapter, we solve the problem to define local Hamiltonians with integrable boundaries, commuting with the boundary transfer matrix associated to cyclic representations of the 6-vertex reflection algebra. The search of an Hamiltonian is an interesting problem, as this results in the modelling of the interactions at a microscopic level, emphasising on the physics of the system at hand.

To that end, we follow the subsequent steps. First we consider the fundamental R-matrix, solution of the so-called mixed Yang-Baxter equation intertwining two Bazhanov-Stroganov Lax operators acting in two different quantum spaces. We define then a mixed reflection equation intertwining these Lax operators and two scalar boundary matrices, respectively defined on the auxiliary space (of dimension 2) and quantum space (of dimension p). By explicitly solving this mixed reflection equation, we find diagonal scalar solutions for the boundary matrices on the quantum space, associated to the known diagonal scalar matrices on the auxiliary space (the matrices $K(\lambda)$ and $K_+(\lambda)$ of the previous chapter, but taken here diagonal). Then we can define a multi spectral parameter family of fundamental transfer matrices, commuting between themselves and with the transfer matrix associated to the original cyclic reflection algebra. Moreover, we prove that these fundamental transfer matrices generate multi parameter families of commuting local Hamiltonians with integrable boundaries.

We end the chapter making totally explicit the Hamiltonians for some models of interest, for 3-dimensional local quantum spaces (i.e. p=3): the XXZ spin 1 chain at root of unity, the super integrable chiral Potts model, the sine-Gordon model at root of unity and a more general model. They all can be obtained from the Bazhanov-Stroganov Lax operator, by an appropriate choice of the parameters.

The results of this chapter have been published in article **III**.

5.1 Fundamental R-matrix

For fundamental quantum integrable models, e.g. the XXZ spin 1/2 chain, the construction of the local Hamiltonians is given by standard techniques, see paragraph 2.3.2, relying on the fact that the associated Lax operator reduces to the permutation operator in some special value of the spectral parameter. However, the Bazhanov-Stroganov models are not of this type, as the Lax operator (1.72) does not reduce in general to the permutation operator (its bi-dimensional auxiliary space and its p-dimensional quantum space are non isomorphic).

Hence, in order to define local Hamiltonians for Bazhanov-Stroganov models, one has to consider the fundamental R-matrix [158, 161], denoted S, solution of the so-called mixed Yang-Baxter equation which intertwines two Lax operators acting in two different quantum spaces V_a and V_b, while acting on the same auxiliary space V_0:

$$\forall\ \lambda \in \mathbb{C} \quad , \ S_{ba}(p_b, p_a) L_{0a}(\lambda, p_a) L_{0b}(\lambda, p_b) = L_{0b}(\lambda, p_b) L_{0a}(\lambda, p_a) S_{ba}(p_b, p_a) \tag{5.1}$$

Such a fundamental R-matrix acts in the tensor product of two cyclic representations.

In [158], this Yang-Baxter equation has first appeared and it has been solved for the subclass of cyclic representations associated to the chiral-Potts model. In [161] the fundamental R-matrix has been determined for the most general cyclic representations. Here, we reproduce these results and fundamental steps in their derivations, following our convention making use of the untwisted 6-vertex R-matrix unlike what it is done in the literature [158, 161]. Thus in this section we first use the chiral Potts model as a starting point to present the fundamental R-matrix. Then, we extend to more general representations the computation of this fundamental R-matrix and we eventually give two important properties, namely its unitarity and crossing-unitarity.

Parametrisation by points in $\mathbb{C}^4$

Let us first give a different parametrisation for the Lax operator $L_{0n}(\lambda, p_n)$ (1.72). Following [158], let us thus consider a point $P \in \mathbb{C}^4$. Generally, we will refer to its canonical coordinates by the notation $P = (a_P, b_P, c_P, d_P)$ or equivalently we will use the parameters:

$$x_P \equiv \frac{a_P}{d_P} \qquad y_P \equiv \frac{b_P}{c_P} \qquad s_P \equiv \frac{d_P}{c_P} \qquad t_P \equiv x_P y_P \tag{5.2}$$

Then a more convenient parametrisation of the Lax operator (1.72), for future computations, can be given by the triplet (P, Q_n, R_n) thanks to the correspondence:

$$\lambda\alpha_n = -t_P^{-1/2} b_{Q_n} b_{R_n} \qquad q^{-1/2} b_n = \frac{1}{q^2} a_{Q_n} d_{R_n} \tag{5.3}$$

$$\frac{\beta_n}{\lambda} = -t_P^{1/2} d_{Q_n} d_{R_n} \qquad q^{1/2} a_n = -c_{Q_n} b_{R_n} \tag{5.4}$$

$$\lambda\delta_n = \frac{1}{q^2} t_P^{-1/2} a_{Q_n} a_{R_n} \qquad q^{1/2} d_n = -d_{Q_n} a_{R_n} \tag{5.5}$$

$$\frac{\gamma_n}{\lambda} = t_P^{1/2} c_{Q_n} c_{R_n} \qquad q^{-1/2} c_n = b_{Q_n} c_{R_n}. \tag{5.6}$$

We recall here that q is a parameter entering in the R-matrix (1.18), and that it is assumed to be a p^{th}-root of unity: $q = e^{2ik\pi/p}$ for some $k \in \mathbb{N}^*$, with k and p coprime integers. We also recall that p is odd: $p \equiv 2l + 1$ with $l \in \mathbb{N}^*$.

With this parametrisation, the dependence in the point P is related to the spectral parameter λ by choosing $\lambda \propto t_P^{-1/2}$, and we can use the notation

$$L_{0n}(\lambda|Q_n, R_n) = \gamma_n \begin{pmatrix} -\lambda y_{Q_n} y_{R_n} \hat{\mathbf{v}}_n + \frac{1}{\lambda}\sigma_n \hat{\mathbf{v}}_n^{-1} & \hat{\mathbf{u}}_n(-\frac{1}{q} y_{R_n} \hat{\mathbf{v}}_n + \frac{1}{q} x_{Q_n} \sigma_n \hat{\mathbf{v}}_n^{-1}) \\ \hat{\mathbf{u}}_n^{-1}(q y_{Q_n} \hat{\mathbf{v}}_n - \frac{1}{q}\sigma_n x_{R_n} \hat{\mathbf{v}}_n^{-1}) & \frac{1}{\lambda}\hat{\mathbf{v}}_n - \frac{1}{q^2}\lambda\sigma_n x_{Q_n} x_{R_n} \hat{\mathbf{v}}_n^{-1} \end{pmatrix} \tag{5.7}$$

where we introduced the parameter $\sigma_n = s_{Q_n} s_{R_n}$ and where we used the parameter γ_n as a global normalisation.

Let us emphasise that each Bazhanov-Stroganov Lax operator depends on the spectral parameter λ and on a bunch of 5 independent parameters $x_{Q_n}, x_{R_n}, y_{Q_n}, y_{R_n}$ and σ_n associated to the site n. The total number of independent parameters in the monodromy matrix $M_{0Q}(\lambda)$ is thus $5N$ (indeed, one can always absorb for example σ_1 in λ).

In the following, this parametrisation with points in $\mathbb{C}^4$ is almost exclusively used to deal with the fundamental R-matrix and the computation of the local Hamiltonians.

5.1.1 The chiral Potts representation

In [158], the authors have shown that the chiral Potts model can be described and studied through the Yang-Baxter algebra associated to the 6-vertex R-matrix. Indeed, the Bazhanov-Stroganov Lax operators are general solutions of the Yang-Baxter equation, and to particularise the points Q_n and R_n of (5.7) to a specific curve leads to the description of the chiral Potts model.

In this short section, we recall the expression of the so-called chiral Potts curves and of the intertwiner $S_{ba}(p_b, p_a)$ (5.1) in this context.

Definition 5.1.1. *The algebraic chiral Potts curve $\mathcal{C}^k_{chP}$ of modulus k is defined by*[1]*:*

$$\mathcal{C}^k_{chP} \equiv \left\{ S \in \mathbb{C}^4 \mid x_S^p + y_S^p = k(1 + x_S^p y_S^p) \;\; ; \;\; k x_S^p = 1 - \frac{k'}{s_S^p} \;\; \textit{and} \;\; k y_S^p = 1 - k' s_S^p \;,\; \textit{with} \;\; k^2 + k'^2 = 1 \right\} \tag{5.8}$$

If for each site $n \in [\![1, N]\!]$ we have $(Q_n, R_n) \in \mathcal{C}^k_{chP} \times \mathcal{C}^k_{chP}$, then the set of the Lax operators $L_{0n}(\lambda|Q_n, R_n)$ describes the chiral Potts model.

The main tool to express the intertwiners is the dilogarithm function [255–257], which for the chiral Potts representations reduces to the Boltzmann weights. The next two propositions 5.1.1 and 5.1.2 recall respectively its definition and how the fundamental R-matrix is computed for the chiral Potts representations:

Proposition 5.1.1 (Dilogarithm functions)**.** *Let us denote by W_{QP} and $\overline{W}_{QP}$ the dilogarithm functions associated to a couple $(Q, P) \in \mathbb{C}^4 \times \mathbb{C}^4$. By definition, these functions are given, up to normalisation, by:*

$$\forall n \in \mathbb{N} \;, \quad \frac{W_{QP}(n)}{W_{QP}(0)} = \left(\frac{s_Q}{s_P}\right)^n \prod_{k=1}^{n} \frac{y_P - q^{-2k} x_Q}{y_Q - q^{-2k} x_P} \qquad \textit{and} \qquad \frac{\overline{W}_{QP}(n)}{\overline{W}_{QP}(0)} = (s_Q s_P)^n \prod_{k=1}^{n} \frac{q^{-2} x_Q - q^{-2k} x_P}{y_P - q^{-2k} y_Q} \tag{5.9}$$

The dilogarithms are geometric quasi-periodic functions of quasi-period p:

$$\frac{W_{QP}(n+p)}{W_{QP}(n)} = k_{QP}^{-p} \qquad \frac{\overline{W}_{QP}(n+p)}{\overline{W}_{QP}(n)} = \bar{k}_{QP}^{-p} \tag{5.10}$$

where we have defined k_{QP} and $\bar{k}_{QP}$ the constants, defined up to a p^{th}-root of unity, such that:

$$k_{QP}^p = \frac{s_P^p}{s_Q^p} \frac{y_Q^p - x_P^p}{y_P^p - x_Q^p} \quad \textit{and} \quad \bar{k}_{QP}^p = \frac{1}{s_P^p s_Q^p} \frac{y_P^p - y_Q^p}{x_Q^p - x_P^p} \tag{5.11}$$

The key point with these functions is the fact that they are solutions of the following recursion relations:

$$\forall n \in \mathbb{N} \;, \quad \frac{W_{QP}(n)}{W_{QP}(n+1)} = -\frac{s_P}{s_Q} \frac{x_P}{y_P} q^{-2(n+1)} \frac{1 - \frac{y_Q}{x_P} q^{2(n+1)}}{1 - \frac{x_Q}{y_P} q^{-2(n+1)}} \tag{5.12}$$

$$\textit{and} \quad \frac{\overline{W}_{QP}(n)}{\overline{W}_{QP}(n+1)} = -q^{2(n+1)} \frac{1}{s_P s_Q} \frac{y_P}{x_P} \frac{1 - \frac{y_Q}{y_P} q^{-2(n+1)}}{1 - \frac{x_Q}{x_P} q^{2n}} \tag{5.13}$$

Moreover, if the points P and Q belong to a given chiral Potts curve, then these dilogarithm functions are periodic of period p, i.e. it holds $k_{QP}^p = \bar{k}_{QP}^p = 1$

Then we can recall the following proposition, from the work of Bazhanov and Stroganov [158]:

Proposition 5.1.2 ([158])**.** *Let $(Q_a, R_a, Q_b, R_b) \in \left(\mathcal{C}^k_{chP}\right)^4$, and let us consider two Lax operators $L_{0a}(\lambda|Q_a, R_a)$ and $L_{0b}(\lambda|Q_b, R_b)$ on two different quantum spaces V_a and V_b.*

[1] Let us comment that the three relations in this definition are not independent, as for example the first one can be obtained from the others.

Let us consider the following basis for the tensor product of the quantum spaces:

$$\forall(h_1,h_2)\in[\![0,p-1]\!]^2\ ,\quad \langle h_1,h_2|\equiv\langle h_1,a|_{\hat{\mathbf{u}}}\otimes\langle h_2,b|_{\hat{\mathbf{u}}}\quad \textit{and}\quad |h_1,h_2\rangle\equiv|h_1,a\rangle_{\hat{\mathbf{u}}}\otimes|h_2,b\rangle_{\hat{\mathbf{u}}} \tag{5.14}$$

and let $\mathbb{S}_{ba}(Q_b,R_b|Q_a,R_a)$ be the operator defined, on the tensor product of the two p-dimensional spaces, by its elements:

$$\forall(h_1,h_2,h_1'',h_2'')\in[\![0,p-1]\!]^2\ ,\quad \langle h_1,h_2|\,\mathbb{S}_{ba}(Q_b,R_b|Q_a,R_a)\,|h_1'',h_2''\rangle\equiv \overline{W}_{Q_bQ_a}(h_1-h_2'')W_{R_bQ_a}(h_1''-h_2'')\overline{W}_{R_bR_a}(h_2-h_1'')W_{Q_bR_a}(h_2-h_1) \tag{5.15}$$

Then $\mathbb{S}_{ba}(Q_b,R_b|Q_a,R_a)$ is well defined, in the sense that $\langle h_1,h_2|\,\mathbb{S}_{ba}(Q_b,R_b|Q_a,R_a)\,|h_1'',h_2''\rangle$ is p-periodic for each of the variables h_1,h_2,h_1'' and h_2'', and is the following intertwiner in the quantum spaces:

$$\forall\lambda\in\mathbb{C}\ ,\quad \mathbb{S}_{ba}(Q_b,R_b|Q_a,R_a)L_{0a}(\lambda|Q_a,R_a)L_{0b}(\lambda|Q_b,R_b)= L_{0b}(\lambda|Q_b,R_b)L_{0a}(\lambda|Q_a,R_a)\mathbb{S}_{ba}(Q_b,R_b|Q_a,R_a) \tag{5.16}$$

Proof. The fact that $\mathbb{S}_{ba}(Q_b,R_b|Q_a,R_a)$ is p-periodic is clear since each of the Boltzmann weight is p-periodic for points on the chiral Potts curve. The proof of the fact that the relation (5.16) holds follows from the recursion relations satisfied by the dilogarithm functions. This equation can be seen as an equality of matrices in the auxiliary space, with elements in $End(\mathcal{H}_a\otimes\mathcal{H}_b)$, and we show this equality for any matrix element $(i_1,i_2)\in\{1,2\}^2$ by direct computation. □

The key observation, at the basis of our work, is that this proof only requires points on the chiral Potts curves to get a well defined intertwiner, i.e. p-periodic in the four variables. The relation (5.16) holds independently of the curves (5.8), it is only based on the recursion relations satisfied by the dilogarithm functions.

5.1.2 More general representations

Here we present an expression of the fundamental R-matrix for Bazhanov-Stroganov Lax operators beyond the chiral Potts model (giving the expression of each of its components), i.e. we compute an intertwiner for the Lax operators (5.7) when the points Q_n and R_n are not necessarily on a chiral Potts curve.

To this end, thanks to the observation of the previous paragraph, rather than imposing the points Q_n and R_n to live on the chiral Potts curves to have p periodic dilogarithm functions, we choose here to use a bunch of four scalars to make p periodic the whole product of the four terms in (5.15).

Explicitly, let $S_{ba}(Q_b,R_b|Q_a,R_a)$ be the matrix, acting on the tensor product of the two p-dimensional spaces V_a and V_b, whose matrix elements read:

$$\forall(h_1,h_2,h_1'',h_2'')\in[\![0,p-1]\!]^2\ ,\quad \langle h_1,h_2|\,S_{ba}(Q_b,R_b|Q_a,R_a)\,|h_1'',h_2''\rangle= \mathbb{c}_1^{h_1}\mathbb{c}_2^{h_2}\mathbb{c}_3^{h"_1}\mathbb{c}_4^{h"_2}\overline{W}_{Q_bQ_a}(h_1-h_2'')W_{R_bQ_a}(h_1''-h_2'')\overline{W}_{R_bR_a}(h_2-h_1'')W_{Q_bR_a}(h_2-h_1) \tag{5.17}$$

with the following scalars:

$$\mathbb{c}_1=\frac{\bar{k}_{Q_bQ_a}}{k_{Q_bR_a}}\quad;\quad \mathbb{c}_2=\bar{k}_{R_bR_a}k_{Q_bR_a}\quad;\quad \mathbb{c}_3=\frac{k_{R_bQ_a}}{\bar{k}_{R_bR_a}}\quad\text{and}\quad \mathbb{c}_4=\frac{1}{\bar{k}_{Q_bQ_a}k_{R_bQ_a}} \tag{5.18}$$

By construction, these matrix elements are p periodic in each of the variables h_1,h_2,h_1'' and h_2''. From now on, we can emphasise the fact that:

$$\mathbb{c}_1\mathbb{c}_2\mathbb{c}_3\mathbb{c}_4=1 \tag{5.19}$$

Remark. *In our study, we do not consider degenerate cases for which one of the parameters x_{Q_n}, x_{R_n}, y_{Q_n}, y_{R_n} or σ_n is zero. Moreover, we take parameters for which the different scalars $\mathbb{c}_i$ are well defined, i.e. we suppose:*

$$y^p_{Q_b} \neq y^p_{Q_a} \ ; \ y^p_{R_b} \neq y^p_{R_a} \ ; \ x^p_{Q_b} \neq x^p_{Q_a} \ ; \ x^p_{R_b} \neq x^p_{R_a} \tag{5.20}$$
$$\text{and} \ \ y^p_{Q_b} \neq x^p_{R_a} \ ; \ x^p_{Q_b} \neq y^p_{R_a} \ ; \ y^p_{R_b} \neq x^p_{Q_a} \ ; \ x^p_{R_b} \neq y^p_{Q_a} \tag{5.21}$$

Thus, the operator family $S_{ba}(Q_b, R_b|Q_a, R_a) \in End(\mathcal{H}_a \otimes \mathcal{H}_b)$ is well defined for any choice of (Q_a, R_a, Q_b, R_b) in this set of parameters.

Existence condition

The use of the expression (5.17) seems promising as the matrix elements are p-periodic without any requirement. However the next proposition defines some restriction on the parameters:

Proposition 5.1.3. *Let $S_{ba}(Q_b, R_b|Q_a, R_a)$ be defined by (5.17). It satisfies the mixed Yang-Baxter equation for any $\lambda \in \mathbb{C}$:*

$$S_{ba}(Q_b, R_b|Q_a, R_a)L_{0a}(\lambda|Q_a, R_a)L_{0b}(\lambda|Q_b, R_b) = L_{0b}(\lambda|Q_b, R_b)L_{0a}(\lambda|Q_a, R_a)S_{ba}(Q_b, R_b|Q_a, R_a) \tag{5.22}$$

if and only if the points $(Q_a, R_a, Q_b, R_b) \in \left(\mathbb{C}^4\right)^4$ are such that the following three independent conditions are satisfied:

$$\forall i \in \{1, 2, 3\} \quad , \quad \mathbb{c}_i = 1. \tag{5.23}$$

Proof. To show the equation (5.22) from (5.23) is trivial thanks to proposition 5.1.2 and the remark on the fact that points on chiral Potts curves are only necessary to have p periodic functions. The other way of the proposition is done using the same lines as for the proof of proposition 5.1.2, namely to consider (5.22) as four equations in the quantum space $V_a \times V_b$, but now the quantities are dressed with the coefficients $\mathbb{c}_i$. At the end, to require (5.22) leads to the condition (5.23). A more explicit proof can be read in appendix of paper **III**. □

Thus, the existence of an intertwiner of the form (5.17) is guaranteed if the pre factors satisfy the condition (5.23), which can be put under the form:

$$\begin{cases} \bar{k}_{Q_bQ_a} = k_{Q_bR_a} \\ k_{R_bQ_a} = \bar{k}_{R_bR_a} \\ \bar{k}_{R_bR_a}k_{Q_bR_a} = 1 \end{cases} \tag{5.24}$$

The pre factors being defined up to a p^{th}-root of unity, one can always choose these roots such that (5.24) is equivalent to:

$$\begin{cases} \bar{k}^p_{Q_bQ_a} = k^p_{Q_bR_a} \\ k^p_{R_bQ_a} = \bar{k}^p_{R_bR_a} \\ \bar{k}^p_{R_bR_a}k^p_{Q_bR_a} = 1 \end{cases} \tag{5.25}$$

Let us introduce the following notation:

$$\forall i \in \{a, b\} \quad , \ X_{Q_i} = x^p_{Q_i} \ ; \ X_{R_i} = x^p_{R_i} \ ; \ Y_{Q_i} = y^p_{Q_i} \ ; \ Y_{R_i} = y^p_{R_i} \ ; \ S_i = \sigma^p_i = s^p_{Q_i}s^p_{R_i} \tag{5.26}$$

Then, if the quantum parameters on sites a and b satisfy the following system (5.27), the intertwinner

$S_{ba}(Q_b, R_b|Q_a, R_a)$ exists and is given by (5.17).

$$\left\{\begin{array}{l}(Y_{Q_a}-Y_{Q_b})/(X_{Q_b}-X_{Q_a})=S_a(Y_{Q_b}-X_{R_a})/(Y_{R_a}-X_{Q_b})\\(Y_{R_a}-Y_{R_b})/(X_{R_b}-X_{R_a})=S_b(Y_{R_a}-X_{Q_b})/(Y_{Q_b}-X_{R_a})\\(Y_{R_a}-Y_{R_b})/(X_{R_b}-X_{R_a})=S_a(Y_{R_b}-X_{Q_a})/(Y_{Q_a}-X_{R_b})\end{array}\right. \tag{5.27}$$

Explicit parametrisations

In this paragraph, we find explicit solutions of the system (5.27), i.e. we find explicit conditions on the parameters for the fundamental R-matrix to exist.

Let us first introduce the following notation:

$$L_i=\frac{S_iX_{Q_i}-Y_{R_i}}{1-S_i}\qquad M_i=\frac{S_iX_{Q_i}-Y_{R_i}}{S_iX_{Q_i}X_{R_i}-Y_{R_i}Y_{Q_i}}\qquad N_i=\frac{Y_{Q_i}-S_iX_{R_i}}{1-S_i} \tag{5.28}$$

and

$$R_i=\frac{X_{Q_i}X_{R_i}-Y_{Q_i}Y_{R_i}}{X_{Q_i}-Y_{R_i}}\qquad K_i=\frac{Y_{Q_i}-X_{R_i}}{X_{Q_i}-Y_{R_i}}\qquad T_i=\frac{Y_{Q_i}Y_{R_i}}{S_i} \tag{5.29}$$

or equivalently in the original parametrisation of the Bazhanov-Stroganov Lax operator:

$$L_i = q^{p/2}\,\frac{a_i^p+b_i^p}{\gamma_i^p+\beta_i^p}\qquad M_i=q^{p/2}\,\frac{a_i^p+b_i^p}{\alpha_i^p+\delta_i^p}\qquad N_i=q^{p/2}\,\frac{c_i^p+d_i^p}{\gamma_i^p+\beta_i^p} \tag{5.30}$$

$$\text{and}\quad R_i = q^{p/2}\,\frac{\alpha_i^p+\delta_i^p}{a_i^p+b_i^p}\qquad K_i=\frac{d_i^p+c_i^p}{a_i^p+b_i^p}\qquad T_i=\frac{\alpha_i^p}{\beta_i^p} \tag{5.31}$$

Then the following proposition holds:

Proposition 5.1.4. *Let us assume that the conditions* (5.23) *are satisfied. The solutions to* (5.23) *can be parametrised by the following four sets of conditions:*

A) If the points $(Q_a, R_a, Q_b, R_b) \in (\mathbb{C}^4)^4$ satisfy the conditions:

$$\forall i\ \in\{a,b\},\quad S_i\neq 1 \text{ and } Y_{Q_i}Y_{R_i}-S_iX_{Q_i}X_{R_i}\neq 0 \tag{5.32}$$

$$\text{as well as } Y_{Q_i}-S_iX_{R_i}\neq 0 \text{ and } Y_{R_i}-S_iX_{Q_i}\neq 0 \tag{5.33}$$

then (5.23) *is equivalent to the following three conditions of homogeneities between the sites a and b:*

$$L_a=L_b\equiv L,\quad M_a=M_b\equiv M,\quad N_a=N_b\equiv N \tag{5.34}$$

where L, M and N are three free parameters.

B) If the points $(Q_a, R_a, Q_b, R_b) \in (\mathbb{C}^4)^4$ satisfy the conditions:

$$S_i=1,\quad X_{Q_i}X_{R_i}-Y_{Q_i}Y_{R_i}\neq 0,\quad \forall i\in\{a,b\}$$

then (5.23) *is equivalent to the following two conditions of homogeneities between the sites a and b:*

$$R_a=R_b\equiv R,\quad K_a=K_b\equiv K \tag{5.35}$$

where R and K are two free parameters.

C) If the points $(Q_a, R_a, Q_b, R_b) \in (\mathbb{C}^4)^4$ satisfy the conditions:

$$(S_a,S_b)\neq(1,1)\ \text{ and }\ \forall i\in\{a,b\}\ ,\quad Y_{R_i}=S_iX_{Q_i}\quad\text{and}\quad Y_{Q_i}=S_iX_{R_i} \tag{5.36}$$

then (5.23) *is equivalent to the following condition of homogeneity between the sites a and b:*

$$T_a=T_b=T \tag{5.37}$$

where T is a free parameter.

D) If the points $(Q_a, R_a, Q_b, R_b) \in \left(\mathbb{C}^4\right)^4$ satisfy the conditions:

$$\forall i \in \{a, b\} \ , \quad S_i = 1 \quad ; \quad Y_{R_i} = X_{Q_i} \quad \textit{and} \quad Y_{Q_i} = X_{R_i} \tag{5.38}$$

then (5.23) *is satisfied without any more requirement.*

Proof. The proof of this proposition is based on the following equivalent formulation for (5.27):

$$\left\{ \begin{array}{c} Y_{Q_a}Y_{R_a} - S_aX_{Q_a}X_{R_a} = (Y_{Q_a} - S_aX_{R_a})X_{Q_b} - (S_aX_{Q_a} - Y_{R_a})Y_{Q_b} + (S_a - 1)Y_{Q_b}X_{Q_b} \\ Y_{Q_b}Y_{R_b} - S_bX_{Q_b}X_{R_b} = (Y_{R_b} - S_bX_{Q_b})X_{R_a} - (S_bX_{R_b} - Y_{Q_b})Y_{R_a} + (S_b - 1)Y_{R_a}X_{R_a} \\ (S_aX_{Q_a} - Y_{R_a})(Y_{Q_b} - X_{R_b}) = (Y_{Q_a} - S_aX_{R_a})(X_{Q_b} - Y_{R_b}) + (S_a - 1)(X_{Q_b}Y_{Q_b} - X_{R_b}Y_{R_b}) \end{array} \right. \tag{5.39}$$

□

Remark. *Points C) and D) of the proposition correspond to the special representations with nilpotent off-diagonal elements of the Bazhanov-Stroganov Lax operators.*

Remark. *Following the idea given by [161], one of the simplest necessary condition for the intertwiner S_{ba} to exist is probably the equality of the mean values[2], i.e. it must hold:*

$$\forall \lambda \in \mathbb{C} \quad , \ < L_{0a}(\lambda|Q_a, R_a) >< L_{0b}(\lambda|Q_b, R_b) >=< L_{0b}(\lambda|Q_b, R_b) >< L_{0a}(\lambda|Q_a, R_a) > \tag{5.40}$$

We can show that the generic solution of this equation is the generic solution given by point A) of proposition 5.1.4, thus this proposition states that for generic parameters, the equality of mean values is in fact equivalent to the existence of S_{ba}.

However, this result does not hold for example for the point C), where the system (5.40) *is trivially satisfied whereas the extra condition* (5.37) *is needed for the existence of the intertwiner.*

Let us resume the results we have obtained so far. In each of the four points of proposition 5.1.4, we state the existence of the intertwiner $S_{ba}(Q_b, R_b|Q_a, R_a)$ of the form (5.17) when imposing that three parameters have to be the same on the both spaces V_a and V_b:

- (L, M, N) for point A)
- $(S = 1, R, K)$ for point B)
- $(Y_R - SX_Q = 0, Y_Q - SX_R = 0, T)$ for point C)
- and $(S = 1, Y_R - X_Q = 0, Y_Q - X_R = 0)$ for point D)

As a consequence, these homogeneous constant parameters along the chain can be seen as characterising the model. We will denote by P_C this set of three parameters in a general way.

Then, as on each site n of the chain are associated 5 independent parameters, let us denote P_n the two remaining quantum parameters on each site. They can be seen for example, as the remaining 6-vertex inhomogeneity plus a quantum inhomogeneity. Eventually, we will denote the set of all the quantum parameters of the chain (2N parameters) by:

$$P_Q \equiv \{P_n\}_{1 \leqslant n \leqslant N} \tag{5.41}$$

Accordingly with this notation, to emphasise the fact that the parameters belong to certain sets for which the intertwiner exists, in all the next we will denote the fundamental R-matrix by $S_{ba}(P_b|P_a|P_C)$ and we will denote the Bazhanov-Stroganov Lax operators by $L_{0n}(\lambda|P_n|P_C)$.

[2] We recall that the mean value of an operator $O(\lambda)$, belonging to a commutative family of operators, is by definition $< O(\lambda) >\equiv \prod_{k=0}^{p-1} O(q^k\lambda) = \mathcal{O}(\lambda^p)$

Let us comment here that the fundamental R-matrix (5.17) has formally the same expression as for the chiral Potts representations (5.15). The difference lies in the fact that each Boltzmann weight is now quasi-periodic, keeping periodic however the product of the four weights with respect to the variables $h_1, h_2, h"_1$ and $h"_2$.

Remark. *The chiral Potts model is obtained as a particular case of our study, as it holds for it for this model:*

$$\bar{k}^p_{Q_bQ_a} = 1 \quad ; \quad k^p_{R_bQ_a} = 1 \quad ; \quad \bar{k}^p_{R_bR_a} = 1 \quad \textit{and} \quad k^p_{Q_bR_a} = 1 \tag{5.42}$$

so the system (5.27) *is trivially satisfied.*

Moreover, it is included for a generic parameter $k \neq 0$ of $\mathcal{C}^k_{chP}$, in the point A) of proposition 5.1.4. The constants of the model read:

$$L = -\frac{1}{k} \qquad M = \frac{1}{k} \quad \textit{and} \quad N = \frac{1}{k} \tag{5.43}$$

For a chiral Potts model on a curve with $k = 0$, the point B) applies and the constants of the models read:

$$R = 0 \quad \textit{and} \quad K = 1 \tag{5.44}$$

5.1.3 Unitarity and crossing-unitarity

In this paragraph, we give some properties satisfied by the fundamental R-matrix (5.17). In particular, we state its unitarity and crossing-unitarity, two properties that will be of interest when dealing with reflection equations.

Let us consider two quantum spaces V_a and V_b isomorphic to $\mathbb{C}^p$, and parameters such that it exists a set P_C entering in one of the points of proposition 5.1.4, leading to the existence of $S_{ba}(P_b|P_a|P_C)$. We recall that the parameters are given by sets $(x_{Q_a}, x_{R_a}, y_{Q_a}, y_{R_a}, \sigma_a)$ and $(x_{Q_b}, x_{R_b}, y_{Q_b}, y_{R_b}, \sigma_b)$ with $\sigma_i = s_{Q_i}s_{R_i}$, associated to couples of points Q_a, R_a and Q_b, R_b.

We first show by direct computation the next lemma, which is at the basis of many of the properties:

Lemma 5.1.5. *Let $(h_2, h"_2) \in [\![0, p-1]\!]^2$. It holds the following identities:*

$$\sum_{h=0}^{p-1} \overline{W}_{X_bX_a}(h - h_2)\overline{W}_{X_aX_b}(h"_2 - h) = \delta_{h_2h"_2}\Sigma^{(X)}_{ab} \tag{5.45}$$

where the functions $\Sigma^{(X)}_{ab}$, for $X = R$ or Q, are independent of h_2 and given by:

$$\Sigma^{(X)}_{ab} = \sum_{h=0}^{p-1} \chi^{(X)}_{ab,h} \quad \textit{with} \quad \chi^{(X)}_{ab,h} = \frac{(x_{X_a} - x_{X_b})(y_{X_b} - y_{X_a})}{(x_{X_a}q^{h-1} - x_{X_b}q^{1-h})(y_{X_b}q^{h-1} - y_{X_a}q^{1-h})} \tag{5.46}$$

Unitarity

The next proposition states the unitarity of the fundamental R-matrix, a property which is known for the 6-vertex R-matrix, cf [177].

Proposition 5.1.6 (Unitarity of the fundamental R-matrix)**.** *When the fundamental R-matrix exits, it satisfies:*

$$S_{ab}(P_a|P_b|P_C)S_{ba}(P_b|P_a|P_C) = \rho(P_a|P_b|P_C)\ \mathbb{1} \tag{5.47}$$

with the scalar function

$$\rho(P_a|P_b|P_C) = \Sigma^{(Q)}_{ab}\Sigma^{(R)}_{ab} \tag{5.48}$$

Proof. By the mixed Yang-Baxter equation (5.1), it follows that $S_{ab}(P_a|P_b|P_C)S_{ba}(P_b|P_a|P_C) \propto \mathbb{1}$. Using the next relation on the Boltzmann weights, clear from the definition of proposition 5.1.1:

$$\forall h \in [\![0, p-1]\!] \ , \quad W_{PQ}(h)W_{QP}(h) = 1 \quad \text{for any points } (P, Q) \in \left(\mathbb{C}^4\right)^2 \tag{5.49}$$

the expression (5.17) for the matrix elements, and the lemma 5.1.5 we can in fact compute the value of the proportionality coefficient. The symmetry $\Sigma_{ab}^{(X)} = \Sigma_{ba}^{(X)}$ concludes the proof. □

Crossing-unitarity

In order to make explicit another symmetry of the fundamental R-matrix, we introduce an operation Υ on the parameters. On the original parameters, it reads:

$$\Upsilon : (\alpha, \beta, \gamma, \delta, a, b, c, d) \to (-\delta/q, -q\gamma, -q\beta, -\alpha/q, -b, -a, -d, -c) \tag{5.50}$$

while, using the parametrisation with points in $\mathbb{C}^4$, we have:

$$\Upsilon : (x_Q, x_R, y_Q, y_R, \sigma) \to (y_R, y_Q, x_R/q^2, x_Q/q^2, 1/\sigma) \tag{5.51}$$

This non involutive operation is useful to state the crossing-unitarity of the fundamental transfer matrix. But first, we give this important proposition:

Proposition 5.1.7. *The Υ operation leaves invariant the different sets P_C of proposition 5.1.4.*

It means that one can apply the Υ operation on S_{ba}, the existence of $S_{ba}^{\Upsilon_a}$, $S_{ba}^{\Upsilon_b}$ and $S_{ba}^{\Upsilon_b\Upsilon_a}$ is guaranteed without any other requirements.

Proposition 5.1.8 (Crossing-unitarity of the fundamental R-matrix)**.** *When the fundamental R-matrix S_{ba} exists, it holds:*

$$S_{ba}^{t_a}(P_b|P_a|P_C)S_{ab}^{t_a}(P_a|P_b^{\Upsilon_b\Upsilon_b}|P_C) = \tilde{\rho}(P_a|P_b|P_C)\mathbb{1} \tag{5.52}$$

with the scalar function

$$\tilde{\rho}(P_a|P_b|P_C) = \rho(P_a^{\Upsilon_a^{-1}}|P_b|P_C) \tag{5.53}$$

The transpositions are given in the $\hat{\mathbf{u}}$*-basis.*

Proof. The first relation that we use is given thanks to the operation Υ, which acts as the following on Bazhanov-Stroganov Lax operators:

$$L_{0b}^{t_b}(\lambda|P_b^{\Upsilon_b}) \propto L_{0b}^{-1}(\lambda|P_b) \tag{5.54}$$

Then, one can show the proposition thanks to simple computations, similar to the proof of the unitarity, and using the symmetries:

$$\forall h \in [\![0, p-1]\!] \ , \quad W_{Q_aR_b}^{\Upsilon_b}(h) = \overline{W}_{Q_bQ_a}(h) \quad \text{and} \quad \overline{W}_{Q_aQ_b}^{\Upsilon_b}(h) = W_{R_bQ_a}(-h) \tag{5.55}$$

□

Remark. *Moreover, one easily shows by direct computation:*

$$S_{ab}(P_a^{\Upsilon_a\Upsilon_a}|P_b^{\Upsilon_b\Upsilon_b}|P_C) = S_{ab}(P_a|P_b|P_C) \quad \textit{and} \quad L_{0b}(\lambda|P_b^{\Upsilon_b\Upsilon_b}) = qL_{0b}(q\lambda|P_b) \tag{5.56}$$

So far, we have obtained the existence and the expression of the fundamental R-matrix for models beyond the chiral Potts case, as well as the unitarity and crossing-unitarity for this matrix. As introduced previously, one of the role of this matrix is to overcome the fact that the auxiliary and quantum spaces of the Bazhanov-Stroganov Lax operators are not isomorphic. In the following, we use this fundamental operator to compute local Hamiltonians. Moreover, we aim to describe integrable boundaries, for systems associated to cyclic representations of the 6-vertex reflection algebra. In order to implement this idea, we now turn to the definition of the reflection equation involving cyclic representations.

5.2 Reflection equations

In this short paragraph, we give a general point of view on the reflection equations, introducing the fact that an automorphism can describe the reflection process at the boundaries. We emphasise the role of these equations in integrability, as they can be used to construct a family of commuting operators, and we make a link with (ABCD)-type quantum algebras [220]. This point of view will be useful to tackle reflection equations involving one or more cyclic representations. As an example, we particularise this general frame to the 6-vertex case and find back the Sklyanin's description of the reflection algebras.

5.2.1 General considerations

Introduced by Cherednik [178], and at the heart of Sklyanin's construction of the one parameter family of boundary transfer matrices in the 6-vertex case [177], the reflection equations can be used to describe the behaviour of particles reflecting on a wall while they have factorised scattering in the bulk. Considering this idea of two particles labelled 1 and 2, we can give this general expression for the reflection equation:

$$\mathbb{R}_{21}^{\phi_2\phi_1}(P_2,P_1)K_1(P_1)\mathbb{R}_{12}^{\phi_2}(P_1,P_2)K_2(P_2) = K_2(P_2)\mathbb{R}_{21}^{\phi_1}(P_2,P_1)K_1(P_1)\mathbb{R}_{12}(P_1,P_2) \tag{5.57}$$

Here, the matrix $\mathbb{R}_{12}(P_1,P_2)$ encodes the collision of the two particles, P_i stands for the spectral parameters associated to space V_i describing the "colour" of the particle, K_i describes the reflection of the particle i on the boundary and the automorphism ϕ_i describes the reflected particle i. (cf. figure 5.1)

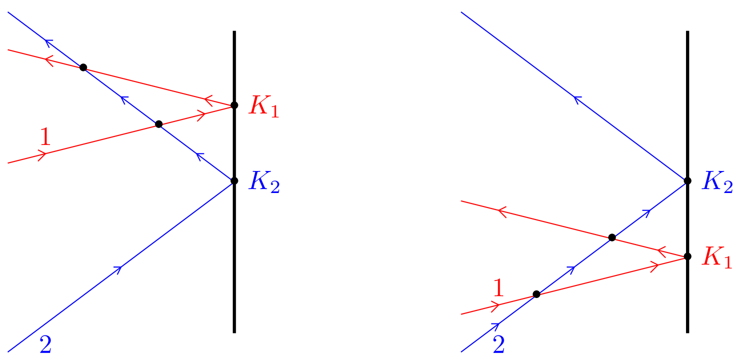

Figure 5.1: On the left, particle 1 and particle 2 are reflecting on a wall. On the right, the same situation but with particle 1 translated. By conservation laws, the equation (5.57) holds. The operations ϕ_i caracterise the reflection. The order $\mathbb{R}_{12}$ or $\mathbb{R}_{21}$ is determined thanks to the relative position of the particles 1 and 2 at the asymptotic times.

In parallel, let us recall that an (ABCD)-type quantum quadratic algebra [220] is defined as the associative algebra generated by the elements of a matrix T_-, which satisfy the relations

$$A_{12}T_{-,1}B_{12}T_{-,2} = T_{-,2}C_{12}T_{-,1}D_{12} \tag{5.58}$$

for four given operators A_{12}, B_{12}, C_{12} and D_{12}. To be a consistent algebra, the latter have to satisfy eight compatibility conditions, explicitly given by the relations (15) of [220]. These ones are involving a third space V_3, labelled 3. As an example, it must hold:

$$A_{12}A_{13}A_{23} = A_{23}A_{13}A_{12} \tag{5.59}$$

and similar conditions involving the other operators, for instance $A_{12}B_{31}B_{32} = B_{32}B_{31}A_{12}$.

The point is that if the matrix $\mathbb{R}$, describing the reflection, satisfies the Yang-Baxter equation:

$$\mathbb{R}_{12}(P_1,P_2)\mathbb{R}_{13}(P_1,P_3)\mathbb{R}_{23}(P_2,P_3) = \mathbb{R}_{23}(P_2,P_3)\mathbb{R}_{13}(P_1,P_3)\mathbb{R}_{12}(P_1,P_2) \tag{5.60}$$

then the boundary matrix K is a scalar representation of an (ABCD)-type quantum quadratic algebra, associated to the generators

$$A_{12} = \mathbb{R}_{21}^{\phi_2\phi_1}(P_2,P_1),\ B_{12} = \mathbb{R}_{12}^{\phi_2}(P_1,P_2),\ C_{12} = \mathbb{R}_{21}^{\phi_1}(P_2,P_1) \text{ and } D_{12} = \mathbb{R}_{12}(P_1,P_2) \tag{5.61}$$

There are two main interests in the (ABCD)-quantum type algebra.

- First, the compatibility conditions ensure the following property [220]:

  Proposition 5.2.1. *Let V_a and V_b be two vector spaces. If one knows a representation $T_{-,a}$ in the space V_a of an (ABCD)-type quantum quadratic algebra $\mathcal{A}$, then one can dress the solution thanks to the operators A_{ab} and B_{ab} or thanks to the operators C_{ab} and D_{ab} to get a representation in the space $V_a \otimes V_b$.*

  For example, $A_{ab}T_{-,a}B_{ab}$ and $\left(C_{ab}^{t_b}\right)^{-1} T_{-,a} \left(D_{ab}^{-1}\right)^{t_b}$ are representations of the (ABCD)-quantum quadratic algebra $\mathcal{A}$ in the vector space $V_a \otimes V_b$.

  In particular, starting from a scalar representation of the algebra (5.58) (so a solution to the reflection equation (5.57)), one can dress it up to a representation in the total quantum space $\mathcal{H}$.

- Second, this algebra is useful to define a commuting family of operators, making thus a link with integrable systems. Indeed, following [220] we can consider the dual algebra of (5.58), defined as the associative algebra generated by the elements of a matrix T_+ which satisfy the relations:

  $$\left(A_{12}^{t_1t_2}\right)^{-1} T_{+,1} \left(\left(B_{12}^{t_1}\right)^{-1}\right)^{t_2} T_{+,2} = T_{+,2} \left(\left(C_{12}^{t_2}\right)^{-1}\right)^{t_1} T_{+,1} \left(D_{12}^{t_1t_2}\right)^{-1} \tag{5.62}$$

  This algebra has been introduced in order to define the following boundary transfer matrices:

  $$\mathcal{T}(P_1) = tr_1\left\{K_{+,1}(P_1)T_{-,1Q}(P_1)\right\} \tag{5.63}$$

  with $T_{-,1Q}$ a representation of the reflection algebra in the total quantum space, and $K_{+,1}^{t_1}$ a scalar representation of the dual algebra. Indeed, this defines a family of commuting operators:

  $$[\mathcal{T}(P_1), \mathcal{T}(P_1^*)] = 0 \tag{5.64}$$

  for P_1 and P_1^* two different sets of spectral parameters associated to the (auxiliary) space V_1.

  To understand where this commutativity and this dual equation come from, let us give some details of the procedure explained in [220]. The very efficient way used by the authors is a rewriting of the reflection algebra (5.58) using bi-vector indices: the first step is to consider the matrices $T_{-,i}$ as elements of $V_i \otimes V_i^*$ with the notation $T_{-,i} = T_{-,\mathbf{ii'}}^{t_{i'}}$. The convention is that any bold index is a vector index. It holds thus:

  $$A_{12}T_{-,\mathbf{11'}}^{t_{1'}}B_{1'2}T_{-,\mathbf{22'}}^{t_{2'}} = T_{-,\mathbf{22'}}^{t_{2'}}C_{12'}T_{-,\mathbf{11'}}^{t_{1'}}D_{1'2'} \tag{5.65}$$

  This can be rewritten:

  $$\mathcal{R}_{11',22'}\ T_{-,\mathbf{11'}}\ T_{-,\mathbf{22'}} = T_{-,\mathbf{22'}}\ T_{-,\mathbf{11'}} \tag{5.66}$$

  with the following matrix (assuming the invertibility of the involved operators):

  $$\mathcal{R}_{11',22'} = \left(C_{12'}^{t_{2'}}\right)^{-1} \left(D_{1'2'}^{t_{1'}t_{2'}}\right)^{-1} A_{12}\ B_{1'2}^{t_{1'}} \tag{5.67}$$

In view of the application of this structure to the obtention of commuting families, it is defined the following algebra, that can be considered as the dual of (5.65):

$$T^{t_{11'}}_{+,\mathbf{11'}}\, T^{t_{22'}}_{+,\mathbf{22'}} = T^{t_{22'}}_{+,\mathbf{22'}}\, T^{t_{11'}}_{+,\mathbf{11'}}\, \mathcal{R}_{11',22'} \tag{5.68}$$

Indeed, one directly can check that for K_+ a scalar representation of this dual algebra, it holds (by simply evaluating (5.66) on the bi-covector $K^{t_{22'}}_{+,\mathbf{22'}} K^{t_{11'}}_{+,\mathbf{11'}}$):

$$K^{t_{11'}}_{+,\mathbf{11'}} T_{-,\mathbf{11'}} K^{t_{22'}}_{+,\mathbf{22'}} T_{-,\mathbf{22'}} = K^{t_{22'}}_{+,\mathbf{22'}} T_{-,\mathbf{22'}} K^{t_{11'}}_{+,\mathbf{11'}} T_{-,\mathbf{11'}} \tag{5.69}$$

As by definition of the bi-vector index notations it holds $K^{t_{11'}}_{+,\mathbf{11'}} T_{-,\mathbf{11'}} = tr_1 \left\{ K^{t_1}_{+,1} T_{-,1} \right\}$, it is exactly the commutation (5.64) of (5.63).

The origin of the dual algebra (5.62) is thus clear, as in matrix notation the equation (5.68) simply reads (5.62).

Let us comment that Sklyanin was the first to introduce and prove the commutativity of the boundary transfer matrices using two algebras, in [177], for the (fundamental) 6-vertex representation of the Yang-Baxter algebra. His proof is equivalent as the one in [220], but more technical as it remains in matrix notations. Note that also in the case of different spaces V_1 and V_2 a proof à-*la* Sklyanin of the transfer matrix commutativity can be still applied.

This set up lays the foundation of the next of the chapter, namely the use of this framework for mixed representations, i.e. spaces V_1 and V_2 non isomorphic. But before, let us end this section considering again the 6-vertex reflection algebra (2.8) of Sklyanin, under the light of this general reflection equation framework.

5.2.2 Reflection algebra associated to the 6-vertex R-matrix

In this paragraph we have a fresh look at the reflection algebra associated to the 6-vertex R-matrix, cf paragraph 2.2.1. In particular, we show that the reflection equations (2.8) and (2.9) define indeed (ABCD)-type quantum quadratic algebras for an appropriate choice of the automorphism ϕ describing the reflection.

Here, let 1 and 2 denote two quantum spaces of dimension 2. Following the work of Sklyanin [177], we know an automorphism ϕ for which the reflection equation (5.57), associated to the 6-vertex R-matrix, admits a non-trivial solution (and which leads, at the end of the day, to the fact that the algebra (5.58)-(5.61) and its dual (5.62) effectively describe the quantum XXZ spin chain with open boundaries conditions.). Denoting σ this automorphism, it consists in the following action on the associated spectral parameter λ :

$$\sigma : \lambda \to \frac{q}{\lambda} \tag{5.70}$$

The action will be indifferently denoted by its action on the operator or on the parameters:

$$L^{\sigma_0}_{0a}(\lambda|P_a|P_C) \equiv L_{0a}(\lambda^{\sigma_0}|P_a|P_C) \tag{5.71}$$

This operation leaves unchanged P_a and P_C and it is consistent with the Yang-Baxter equations. That is we can apply σ_0 to the Yang-Baxter equation (1.12),(1.13) and (5.1), leaving all them satisfied.

The reflection equation (5.57) is thus:

$$R^{\sigma_2\sigma_1}_{21}(\mu,\lambda) K_1(\lambda) R^{\sigma_2}_{12}(\lambda,\mu) K_2(\mu) = K_2(\mu) R^{\sigma_1}_{21}(\mu,\lambda) K_1(\lambda) R_{12}(\lambda,\mu) \tag{5.72}$$

Using the following properties of the 6-vertex R matrix:

$$R_{12}(\lambda,\mu) = R_{12}(\lambda/\mu) \tag{5.73}$$

$$R_{21}(\lambda/\mu) = R_{12}(\lambda/\mu) \tag{5.74}$$

the 6-vertex reflection equation is then written in the usual form [177]:

$$R_{12}(\lambda/\mu)K_1(\lambda)R_{12}(\frac{\lambda\mu}{q})K_2(\mu) = K_2(\mu)R_{12}(\frac{\lambda\mu}{q})K_1(\lambda)R_{12}(\lambda/\mu) \tag{5.75}$$

This R-matrix satisfying the Yang-Baxter equation (1.13), the associated algebra:

$$R_{12}(\lambda/\mu)T_{-,1}(\lambda)R_{12}(\frac{\lambda\mu}{q})T_{-,2}(\mu) = T_{-,2}(\mu)R_{12}(\frac{\lambda\mu}{q})T_{-,1}(\lambda)R_{12}(\lambda/\mu) \tag{5.76}$$

is an (ABCD)-quantum type algebra.

We can also use the (ABCD)-quantum type algebra to understand the second algebra introduced by Sklyanin in [177]. Indeed, taking the dual algebra T_+ from (5.62), let us consider:

$$\begin{aligned}&\left(R_{12}^{-1}(\lambda/\mu)\right)^{t_1t_2} T_{+,1}(\lambda) \left[\left(R_{12}^{t_1}(\lambda\mu/q)\right)^{-1}\right]^{t_2} T_{+,2}(\mu)\\ &\qquad = T_{+,2}(\mu) \left[\left(R_{12}^{t_1}(\lambda\mu/q)\right)^{-1}\right]^{t_2} T_{+,1}(\lambda) \left(R_{12}^{-1}(\lambda/\mu)\right)^{t_1t_2}\end{aligned} \tag{5.77}$$

By using the crossing unitarity of the 6-vertex R-matrix:

$$R_{12}^{t_1}(\lambda)R_{12}^{t_1}(1/\lambda q^2) = \left(q - \frac{1}{q}\right)^2 - \left(\lambda q - \frac{1}{\lambda q}\right)^2 \tag{5.78}$$

as well as the unitarity property:

$$R_{12}(\lambda)R_{12}(1/\lambda) = \left(\frac{q}{\lambda} - \frac{\lambda}{q}\right)\left(\lambda q - \frac{1}{\lambda q}\right) \tag{5.79}$$

and the fact that for this R-matrix, it also holds:

$$R_{12}^{t_1}(\lambda) = R_{12}^{t_2}(\lambda) \tag{5.80}$$

this dual reflection algebra reads:

$$R_{12}(\mu/\lambda)\; T_{+,1}(\lambda)\; R_{12}(1/\lambda\mu q)\; T_{+,2}(\mu) = T_{+,2}(\mu)\; R_{12}(1/\lambda\mu q)\; T_{+,1}(\lambda)\; R_{12}(\mu/\lambda) \tag{5.81}$$

Thus it is the second reflection algebra introduced by Sklyanin, cf (2.24) (up to the definition of T_+ with a transposition). Let $K_{+0}^{t_0}(\lambda)$ be a scalar representation of this dual reflection algebra (5.81). Then we immediately know that the boundary transfer matrix studied in chapter 4, given by (5.63), is a commuting family. Here, we add the upper-script $(6V)$ to emphasise that this matrix is constructed from the 6-vertex R-matrix:

$$\mathcal{T}^{(6V)}(\lambda|P_Q|P_C) = tr_0\left\{K_{+0}(\lambda)\mathcal{U}_{0Q}(\lambda|P_Q|P_C)\right\} \tag{5.82}$$

Let us recall that in chapter 4 we have shown that the boundary transfer matrices associated to the Bazhanov-Stroganov Lax operators (1.72) have simple spectrum. Let us see what happens when considering two non isomorphic spaces.

5.3 Mixed reflection equation

In this section, we start the study of mixed representations, i.e. the reflection equation (5.57) for two non isomorphic spaces 1 and 2.

We consider thus the reflection equation (5.57) for mixed representations, with a space $V_a \simeq \mathbb{C}^p$ (quantum space) and a space $V_0 \simeq \mathbb{C}^2$ (auxiliary space). The Bazhanov-Stroganov Lax operator mixing both the representations, our starting point is thus the following mixed reflection equation, associated to the known scalar matrix $K_0(\lambda)$ (2.25) in the auxiliary space:

$$\begin{aligned} L_{a0}^{\theta_a\sigma_0}(P_a|\lambda|P_C)K_0(\lambda)L_{0a}^{\theta_a}(\lambda|P_a|P_C)K_a(P_a|P_C) \\ = K_a(P_a|P_C)L_{a0}^{\sigma_0}(P_a|\lambda|P_C)K_0(\lambda)L_{0a}(\lambda|P_a|P_C) \end{aligned} \tag{5.83}$$

Similarly to σ_0, the operation θ_a is a transformation on the parameters associated to the quantum space:

$$L_{0a}^{\theta_a}(\lambda|P_a|P_C) \equiv L_{0a}(\lambda|P_a^{\theta_a}|P_C), \quad S_{ab}^{\theta_a}(P_a|P_b|P_C) \equiv S_{ab}(P_a^{\theta_a}|P_b|P_C) \tag{5.84}$$

which leaves λ and P_C unchanged, and which is consistent with the Yang-Baxter equations. That is we can apply θ_a to the Yang-Baxter equation (1.12) and (5.1) leaving them satisfied.

A solution to this equation, i.e. both an automorphism θ_a and a matrix $K_a(P_a|P_C)$ in the quantum space, is a first step toward the computation of the fundamental boundary transfer matrix. This one will serve to derive the local Hamiltonian, and the matrix $K_a(P_a|P_C)$ will take part in the description of the boundaries of the system.

An interesting point of view is to consider that the matrix $K_a(P_a|P_C)$ can be seen as a scalar representation of the following algebra, generated by the elements of a matrix T_- satisfying:

$$L_{a0}^{\theta_a\sigma_0}(P_a|\lambda|P_C)T_{-0}(\lambda)L_{0a}^{\theta_a}(\lambda|P_a|P_C)T_{-a}(P_a) = T_{-a}(P_a)L_{a0}^{\sigma_0}(P_a|\lambda|P_C)T_{-0}(\lambda)L_{0a}(\lambda|P_a|P_C) \tag{5.85}$$

In order to interpret (5.85) as an (ABCD)-quantum type algebra, the operators L_{0a} and L_{a0} must satisfy the Freidel-Maillet compatibility, cf relations (15) of [220].

Proposition 5.3.1. *Let us take parameters such that a set P_C ensures the existence of an intertwiner* (5.1)*. Then, once we do the identification:*

$$L_{a0}(P_a|\lambda|P_C) = L_{0a}^{-1}(\lambda|P_a|P_C) \tag{5.86}$$

the algebra (5.85) *is an (ABCD)-quantum type algebra.*

Proof. To show the proposition, only the compatibility conditions have to be checked. So one needs to verify the eight relations (15) of [220], as (5.59), when the third space (labelled 3) is of dimension 2 and also when the third space is of dimension p.

Let 0 and 0′ be two indexes denoting two spaces of dimension 2, V_0 and $V_{0'}$, and let a and b be two indexes denoting two spaces of dimension p, V_a and V_b.

The proof relies on the following identifications:

$$A_{00'} = R_{0'0}^{\sigma_{0'}\sigma_0} \quad ; \quad A_{0a} = L_{a0}^{\sigma_0\theta_a} \quad \text{and} \quad A_{ab} = S_{ba}^{\theta_b\theta_a} \tag{5.87}$$

$$B_{00'} = R_{00'}^{\sigma_{0'}} \quad ; \quad B_{0a} = L_{0a}^{\theta_a} \quad \text{and} \quad B_{ab} = S_{ab}^{\theta_b} \tag{5.88}$$

$$C_{00'} = R_{0'0}^{\sigma_0} \quad ; \quad C_{0a} = L_{a0}^{\sigma_0} \quad \text{and} \quad C_{ab} = S_{ba}^{\theta_a} \tag{5.89}$$

$$D_{00'} = R_{00'} \quad ; \quad D_{0a} = L_{0a} \quad \text{and} \quad D_{ab} = S_{ab} \tag{5.90}$$

where R, L and S are of course respectively the 6-vertex R-matrix, the Bazhanov-Stroganov Lax operator and the fundamental R-matrix. Accordingly to the spaces, the automorphisms are $\phi_0 = \sigma_0$ on two-dimensional spaces and $\phi_a = \theta_a$ on p-dimensional spaces.

With this identification, when the third space is taken bi-dimensional (and denoted 0′), the equation (5.59) can be written:

$$L_{a0}^{\theta_a\sigma_0}R_{0'0}^{\sigma_{0'}\sigma_0}L_{0'a}^{\sigma_{0'}\theta_a} = L_{0'a}^{\sigma_{0'}\theta_a}R_{0'0}^{\sigma_{0'}\sigma_0}L_{a0}^{\theta_a\sigma_0} \tag{5.91}$$

Using the fact that σ and θ are automorphisms, it must hold:

$$L_{a0}R_{0'0}L_{0'a} = L_{0'a}R_{0'0}L_{a0} \tag{5.92}$$

so that the identification of the proposition is well-posed, using the mixed Yang-Baxter equation (1.12).

The same holds for all the other relations: when the third space is taken bi-dimensional, the equation (1.12) ensures the compatibility.

When the third space is taken p-dimensional (and denoted b), the equation (5.59) is written:

$$L_{a0}^{\theta_a\sigma_0}L_{b0}^{\theta_b\sigma_0}S_{ba}^{\theta_b\theta_a} = L_{b0}^{\theta_a\sigma_0}L_{b0}^{\theta_a\sigma_0}S_{ba}^{\theta_b\theta_a} \tag{5.93}$$

Then using the identifications introduced, this equation is equivalent to the intertwining relation (5.1). The same holds for all the other conditions: when the third space is taken p-dimensional, the equation (5.1) ensures the compatibility. □

Let us note that this compatibility condition is also imposed to be able to dress scalar solutions of (5.83) by Yang-Baxter solutions, in order to get new solutions of (5.83), i.e. non scalar representations of (5.85).

5.3.1 Choice of the automorphism θ and symmetries

In this paragraph, we define an automorphism θ describing the reflection in the cyclic space.

According to the fact that P_C must be invariant under the action of θ, we restrict ourself to look for solutions to the above problem once we impose one condition on the parameters P_C.

In particular, we impose:

$$\mathrm{L} = -\mathrm{M} \quad \text{for parameters of point A) of proposition 5.1.4} \tag{5.94}$$

$$\mathrm{R} = 0 \quad \text{for parameters of point B) of proposition 5.1.4} \tag{5.95}$$

$$\text{and} \qquad T^2 = 1 \quad \text{for parameters of point C) of proposition 5.1.4} \tag{5.96}$$

In this context we can define the local transformation θ_a, acting in space V_a, as it follows:

$$\theta_a : (\alpha_a, \beta_a, \gamma_a, \delta_a, a_a, b_a, c_a, d_a) \to (-\gamma_a, -\delta_a, -\alpha_a, -\beta_a, a_a, b_a, c_a, d_a) \tag{5.97}$$

or equivalently, using the parametrisation with points in $\mathbb{C}^4$:

$$\theta_a : (x_{Q_a}, x_{R_a}, y_{Q_a}, y_{R_a}, \sigma_a) \to \left(\frac{q^2}{x_{R_a}}, \frac{q^2}{x_{Q_a}}, \frac{1}{y_{R_a}}, \frac{1}{y_{Q_a}}, \sigma_a \frac{x_{Q_a}x_{R_a}}{q^2 y_{Q_a}y_{R_a}}\right) \tag{5.98}$$

This acts only on the local parameters P_a of the space V_a while leaving the parameters λ and P_C unchanged. In particular $S_{ba}^{\theta_a}$, $S_{ba}^{\theta_b}$ and $S_{ba}^{\theta_b\theta_a}$ exist when S_{ba} exists. Moreover, the 6-vertex consistency conditions:

$$\alpha_a\gamma_a = a_a c_a \quad \text{and} \quad \beta_a\delta_a = b_a d_a \tag{5.99}$$

are invariant under θ_a so that the action of θ_a on the Yang-Baxter equations leaves all them satisfied.

Remark. *There is a point that could be investigated further: the existence of an automorphism θ which does not require* (5.94)-(5.96) *and which reduces to our definition of θ when these conditions are satisfied. If there exists such an automorphism, the results we will obtain should remain true in a wider class of parameters. It is worth to mention anyhow that the conditions* (5.94)-(5.96) *correspond to still quite general 6-vertex cyclic representations which contains as specific representations those associated to the chiral Potts, the sine-Gordon model and the XXZ spin chain at the root of units.*

Symmetries and rewritings

Following from the definitions of σ_0 and θ_a, by direct computation one can show:

$$L_{a0}^{\theta_a\sigma_0}(P_a|\lambda|P_C) = -L_{0a}(\lambda|P_a|P_C) \tag{5.100}$$

Let us comment that, using the identifications (5.87) and (5.88), the dressing of a scalar representation $K_0(\lambda)$ of the reflection algebra (5.58) by the operators $A_{0n} = -L_{0n}(\lambda|P_n|P_C)$ and $B_{0n} = L_{0n}^{-1}(q/\lambda|P_n|P_C)$

(for V_0 a bi-dimensional space and V_n a local quantum space of dimension p) leads to the construction of the boundary transfer matrix $\mathcal{U}_{0Q}(\lambda|P_Q|P_C)$.

Moreover, we have the following identity:

Proposition 5.3.2. *When it exists, the fundamental R-matrix satisfies:*

$$S_{ba}^{\theta_b\theta_a}(P_b|P_a|P_C) = S_{ab}(P_a|P_b|P_C) \tag{5.101}$$

Proof. Looking to the mixed Yang-Baxter equation (5.1), and using (5.100), we can prove that it holds:

$$S_{ba}^{\theta_b\theta_a}(P_b|P_a|P_C) \propto S_{ab}(P_a|P_b|P_C) \tag{5.102}$$

In fact, looking to the definition (5.17) for the intertwiner S_{ba}, one can prove by calculations that the proportionality coefficient is one. □

Let us use now the symmetry outlined above to rewrite the mixed reflection equation (5.83). This can be put under the form:

$$L_{0a}(\lambda|P_a|P_C)K_0(\lambda)L_{a0}^{\sigma_0}(P_a|\lambda|P_C)K_a(P_a|P_C) = K_a(P_a|P_C)L_{a0}^{\sigma_0}(P_a|\lambda|P_C)K_0(\lambda)L_{0a}(\lambda|P_a|P_C) \tag{5.103}$$

or equivalently:

$$L_{0a}(\lambda|P_a|P_C)K_0(\lambda)L_{0a}^{\theta_a}(\lambda|P_a|P_C)K_a(P_a|P_C) = K_a(P_a|P_C)L_{0a}^{\theta_a}(\lambda|P_a|P_C)K_0(\lambda)L_{0a}(\lambda|P_a|P_C) \tag{5.104}$$

These symmetries are of prime interest, as we are now able to show that these equations contain the equations obtained by the fusion procedure (cf paragraph 2.3.3):

The fusion procedure as a particular case

Let us rewrite the mixed equation proposed by Nepomechie et al for the fusion case [237]. Using the symmetry of the 6-vertex R-matrix, it can be rewritten in the form ($\langle 12\rangle$ denotes the fused space, of dimension 3):

$$\begin{aligned} &R_{0\langle 12\rangle}(\lambda|P_{\langle 12\rangle})K_0(\lambda)R^{\tilde{\sigma}_0}_{\langle 12\rangle 0}(P_{\langle 12\rangle}|\lambda)K^-_{\langle 12\rangle}(P_{\langle 12\rangle}) \\ &\qquad = K^-_{\langle 12\rangle}(P_{\langle 12\rangle})R^{\tilde{\sigma}_0}_{\langle 12\rangle 0}(P_{\langle 12\rangle}|\lambda)K_0(\lambda|\xi)R_{0\langle 12\rangle}(\lambda|P_{\langle 12\rangle}) \end{aligned} \tag{5.105}$$

Here we have done the identifications:

$$P_{\langle 12\rangle} \equiv \mu \quad \text{and} \quad R_{0\langle 12\rangle}(\lambda|P_{\langle 12\rangle}) \equiv R_{0\langle 12\rangle}(\lambda/\mu) \tag{5.106}$$

and $\tilde{\sigma}_0$ is defined, on the space 0, by:

$$\tilde{\sigma}_0 \;:\; \lambda \to 1/\lambda \tag{5.107}$$

Let us observe that the 6-vertex cyclic representations corresponding to these fusion procedure are contained in point D) of proposition 5.1.4. It is now easy to remark that a part the shifted definition of the $\tilde{\sigma}_0$ with respect to σ_0, the equation proposed by Nepomechie et al [237] is a particular case of the equation (5.103) and so it is equivalent to the reflection equation (5.83).

Let us comment that the shift in question is due to a different choice of convention. Indeed, in [237] they base on the reflection algebra $\mathcal{T}$ of Sklyanin (equation (12) of [177]) while in this thesis we consider the shifted version, the algebra $\mathcal{U}$ (equation (35) of [177]).

5.3.2 Solutions of the mixed reflection equation

The aim of this paragraph is to compute solutions of the mixed reflection equations. In a first part, we show how to dress a known scalar solution $K_a(P_a|P_C)$ to get a solution in the whole quantum space.

Then, we explicitly find a scalar solution $K_a(P_a|P_C)$, associated to a diagonal boundary matrix in the space V_0. As we have already seen, the knowledge of such matrices are of prime interest, as they will allow to describe integrable boundaries for the models associated to the 6-vertex cyclic representations.

Dressing of solutions

Let us suppose that one knows a matrix $K_a(P_a|P_C)$ solution of the mixed reflection equation. In this short part, we dress this boundary matrix to get a solution of the mixed reflection equation in the whole quantum space. Later on, subscripts $n \in [\![1, N]\!]$ will refer to the local quantum spaces $V_n \simeq \mathbb{C}^p$.

Proposition 5.3.3. *Let us define the fundamental boundary monodromy matrix:*

$$\mathcal{V}_{aQ}(P_a|P_Q|P_C) \equiv S_{Qa}^{\theta_Q\theta_a}(P_Q|P_a|P_C)K_a(P_a|P_C)S_{aQ}^{\theta_Q}(P_a|P_Q|P_C) \tag{5.108}$$

where we have defined:

$$S_{Qa}^{\theta_Q\theta_a}(P_Q|P_a|P_C) = S_{Na}^{\theta_N\theta_a}(P_N|P_a|P_C)\cdots S_{1a}^{\theta_1\theta_a}(P_1|P_a|P_C) \tag{5.109}$$

$$\text{and} \quad S_{aQ}^{\theta_Q}(P_a|P_Q|P_C) = S_{a1}^{\theta_1}(P_a|P_1|P_C)\cdots S_{aN}^{\theta_N}(P_a|P_N|P_C) \tag{5.110}$$

Then, it satisfies the following mixed reflection equation:

$$\begin{aligned} &L_{a0}^{\theta_a\sigma_0}(P_a|\lambda|P_C)\mathcal{U}_{0Q}(\lambda|P_Q|P_C)L_{0a}^{\theta_a}(\lambda|P_a|P_C)\mathcal{V}_{aQ}(P_a|P_Q|P_C) \\ &\qquad = \mathcal{V}_{aQ}(P_a|P_Q|P_C)L_{a0}^{\sigma_0}(P_a|\lambda|P_C)\mathcal{U}_{0Q}(\lambda|P_Q|P_C)L_{0a}(\lambda|P_a|P_C) \end{aligned} \tag{5.111}$$

where $\mathcal{U}_{0Q}(\lambda|P_Q|P_C)$ is the boundary monodromy matrix (studied in chapter 4).

Proof. The proof is a direct consequence of proposition 5.2.1, using the identifications (5.87) and (5.88). □

The next proposition states the unicity of the solutions:

Proposition 5.3.4. *Up to a normalisation the following statements are true: i) there exists at most one $K_a(P_a|P_C)$ solution to the corresponding mixed reflection equation* (5.83) *once $K_0(\lambda)$ is fixed. ii) if the $K_a(P_a|P_C)$ associated to $K_0(\lambda)$ exists, then $\mathcal{V}_{aQ}(P_a|P_Q|P_C)$ is the only solution of* (5.111) *once $\mathcal{U}_{0Q}(\lambda|P_Q|P_C)$ is fixed.*

Moreover, if the normalisation can be fixed such that:

$$K_a(P_a^{\theta_a}|P_C) = K_{-a}^{-1}(P_a|P_C) \tag{5.112}$$

then it follows :

$$\mathcal{V}_{aQ}^{\theta_a}(P_a|P_Q|P_C) = \rho_{aQ}(P_a|P_Q|P_C)\rho_{aQ}(P_a|P_Q^{\theta_Q}|P_C)\mathcal{V}_{aQ}^{-1}(P_a|P_Q|P_C) \tag{5.113}$$

with the scalar function:

$$\rho_{aQ}(P_a|P_Q|P_C) = \prod_{k=1}^{N}\rho_{ak}(P_a|P_k|P_C) \tag{5.114}$$

We recall that the function $\rho_{ak}(P_a|P_k|P_C)$ have been introduced to deal with the unitarity and crossing-unitarity of the fundamental R-matrix.

Proof. The proof of the proposition is based on the following lemma:

Lemma 5.3.5 (Center of the Reflection Algebra)**.** *The centre of the reflection algebra is proportional to the identity.*

This one is easily proven using the basis (4.127) for the Hilbert space $\mathcal{H}$ and the simplicity of pseudo-spectrum of the operator $\mathcal{B}(\lambda|\alpha, \beta)$. Then, from the reflection equation (5.83) (respectively (5.111)), one can show that $K_a(P_a|P_C)K_a(P_a^{\theta_a}|P_C)$ (respectively $\mathcal{V}_{aQ}(P_a|P_Q|P_C)\mathcal{V}_{aQ}^{\theta_a}(P_a|P_Q|P_C)$) is central with respect to the reflection algebra associated to $\mathcal{U}_{0\{aQ\}}(\lambda|P_a, P_Q|P_C)$, the dressed boundary monodromy matrix on a chain with $N+1$ quantum sites, the last site coinciding with V_a. The relation (5.113) follow from the appropriate choice of normalisation. □

Scalar solutions to the mixed reflection equations

Let us compute now an explicit scalar solution of the mixed reflection equation. In this paragraph, we restrict to the case of a diagonal $K_0(\lambda)$ solution of the original 6-vertex reflection equation, i.e. a matrix $K_0(\lambda|\xi,0,0)$ given by (2.25). For this type of boundaries, we could construct the unique associated scalar solution in the quantum space of the mixed reflection equation. The following proposition holds:

Proposition 5.3.6. *Let P_C be a set of constant parameters such that the fundamental R-matrix $S_{aQ}(P_a|P_Q|P_C)$ exists and let us denote P_a the arbitrary parameters of the cyclic representation in the space V_a. Then the mixed reflection equation* (5.83) *admits the following non-trivial solution:*

- *In the framework of points A) and B) of proposition 5.1.4, the boundary matrix $K_a(\xi|P_a|P_C)$ is diagonal in the $\hat{\mathbf{v}}$-basis (cf.* (1.77)*), and its elements satisfy:*

$$\frac{\langle h+1|_{\hat{\mathbf{v}}} K_a(\xi|P_a|P_C)|h+1\rangle_{\hat{\mathbf{v}}}}{\langle h|_{\hat{\mathbf{v}}} K_a(\xi|P_a|P_C)|h\rangle_{\hat{\mathbf{v}}}} = -\frac{\gamma_a q^{h+1/2} + \delta_a q^{k-(h+1/2)}}{\alpha_a q^{h+1/2} + \beta_a q^{k-(h+1/2)}} \quad \forall h \in \{0, ..., p-2\} \tag{5.115}$$

This solution exists if and only if the boundary parameter ξ satisfies:

$$\frac{\delta_a^p/\xi^{2p} - \gamma_a^p}{\alpha_a^p - \beta_a^p/\xi^{2p}} = 1 \tag{5.116}$$

- *In the framework of points C) and D) of proposition 5.1.4, satisfying moreover the identities:*

$$b_a = -q^{2j-1}a_a, \quad d_a = -q^{2j-1}c_a \tag{5.117}$$

for a given $j \in \{0, ..., p-1\}$, it holds for any value of ξ that the boundary matrix $K_a(\xi|P_a|P_C)$ is diagonal in the $\hat{\mathbf{v}}$-basis and its element satisfy:

$$\frac{\langle h+1|_{\hat{\mathbf{v}}} K_a(\xi|P_a|P_C)|h+1\rangle_{\hat{\mathbf{v}}}}{\langle h|_{\hat{\mathbf{v}}} K_a(\xi|P_a|P_C)|h\rangle_{\hat{\mathbf{v}}}} = -\frac{\gamma_a q^{h+1/2} - \delta_a q^{-(h+1/2)}/\xi^2}{\alpha_a q^{h+1/2} - \beta_a q^{-(h+1/2)}/\xi^2} \quad \forall h \in \{j, ..., p+j-2\} \tag{5.118}$$

Proof. The idea is to consider the reflection equation (5.83) as four equations in the auxiliary space of dimension 2, with elements in the quantum space V_a. The fact that $K_a(P_a|P_C)$ is diagonal (in the $\hat{\mathbf{v}}$-basis) follows from the fact that $K_0(\lambda)$ is diagonal, and from the equations associated to the elements $(1,1)$ and $(2,2)$ of the auxiliary space. Then, looking for diagonal solutions, we find the result of the proposition. Let us comment that the condition (5.116) is a compatibility condition, needed to ensure the periodicity of the solution. For nilpotent models, no condition is required. □

Remark. *Let us mention that the scalar solutions of the reflection equation hold when the set P_C is general, namely the extra constraints* (5.94)-(5.96) *are not necessary. For further applications, we will nevertheless have to impose these constraints, leading to the following simplification of the compatibility condition* (5.116)*: the boundary parameter ξ must satisfy $\xi^2 = -q^{-k}$, for some $k \in \{0, ..., p-1\}$.*

Thanks to the previous proposition, we can compute diagonal boundary matrices in the $\hat{\mathbf{v}}$-basis. Let us mention that the existence and the derivation of non diagonal solutions is currently under examination as it can lead to describe more general boundaries for these classes of representations.

5.3.3 The dual mixed reflection equation

In this paragraph, we consider the dual mixed reflection equation. As we have already seen, it allows to define a family of commuting operators. First we rewrite this equation in a more handleable way, and then we find diagonal scalar solutions similarly to the original equation, i.e. associated to a diagonal scalar boundary matrix $K_{+0}(\lambda)$.

For a general cyclic representation, the mixed dual reflection equation (5.62) associated to the most general scalar solution $K_{+0}(\lambda)$ (2.27) of the 6-vertex dual reflection equation reads:

$$(L_{0a}^{-1}(\lambda|P_a|P_C))^{t_0 t_a} K_{+0}(\lambda) \left[\left(L_{0a}^{t_0}(\lambda|P_a^{\theta_a}|P_C) \right)^{-1} \right]^{t_a} K_{+a}(P_a|P_C) = \\ K_{+a}(P_a|P_C) \left[\left(L_{0a}^{t_0}(\lambda|P_a^{\theta_a}|P_C) \right)^{-1} \right]^{t_a} K_{+0}(\lambda)(L_{0a}^{-1}(\lambda|P_a|P_C))^{t_0 t_a} \tag{5.119}$$

In order to solve the problem to compute the scalar solution $K_{+a}(P_a|P_C)$, we show the following rewriting:

Proposition 5.3.7. *The dual mixed reflection equation* (5.119) *is equivalent to the equation:*

$$K_{+a}^{t_a}(P_a|P_C) L_{0a}(1/q\lambda|P_a|P_C) K_{+0}^{t_0}(\lambda) L_{0a}^{-1}(\lambda|P_a|P_C) = \\ L_{0a}^{-1}(\lambda|P_a|P_C) K_{+0}^{t_0}(\lambda) L_{0a}(1/q\lambda|P_a|P_C) K_{+a}^{t_a}(P_a|P_C) \tag{5.120}$$

Proof. The equivalence is shown with some simple computations, using the transposition on the different spaces. The key is that we can show

$$\left[\left(L_{0a}^{t_0}(\lambda|P_a^{\theta_a}|P_C) \right)^{-1} \right]^{t_0} \propto L_{0a}(1/q\lambda|P_a|P_C) \tag{5.121}$$

which is proven also using the definition of the quantum determinant (1.46). ☐

Scalar solutions to the dual mixed reflection equations

As before, in this part we restrict to the case of a diagonal $K_{+0}(\lambda)$ solution of the dual 6-vertex reflection equation, i.e. a matrix $K_{+0}(\lambda|\xi, 0, 0)$ given by (2.27). For this type of boundaries, we construct the associated scalar solution in the quantum space of the dual mixed reflection equation. The following proposition holds:

Proposition 5.3.8. *Let P_C be a set of constant parameters such that the fundamental R-matrix $S_{aQ}(P_a|P_Q|P_C)$ exists and let us denote P_a the arbitrary parameters of the cyclic representation in the space V_a. Then the dual mixed reflection equation* (5.119) *admits the following non-trivial solution:*

- *In the framework of points A) and B) of proposition 5.1.4, the matrix $K_{+a}(\xi|P_a|P_C)$ is diagonal in the $\hat{\mathbf{v}}$-basis and its elements satisfy:*

$$\frac{\langle h+1|_{\hat{\mathbf{v}}} K_{+a}(\xi|P_a|P_C)|h+1\rangle_{\hat{\mathbf{v}}}}{\langle h|_{\hat{\mathbf{v}}} K_{+a}(\xi|P_a|P_C)|h\rangle_{\hat{\mathbf{v}}}} = -\frac{\alpha_a q^{h-1/2} + \beta_a q^{k-(h-1/2)}}{\gamma_a q^{h+3/2} + \delta_a q^{k-(h+3/2)}} \quad \forall h \in \{0, ..., p-2\} \tag{5.122}$$

The solution exists if and only if the boundary parameter ξ satisfies:

$$\frac{\beta_a^p \xi^{2p} - \alpha_a^p}{\gamma_a^p - \delta_a^p \xi^{2p}} = 1 \tag{5.123}$$

- *In the framework of points C) and D) of proposition 5.1.4, satisfying moreover the identities:*

$$b = -q^{2j-1} a, \quad d = -q^{2j-1} c \tag{5.124}$$

for a given $j \in \{0, ..., p-1\}$, it holds for any value of ξ that the boundary matrix $K_{+a}(\xi|P_a|P_C)$ is diagonal in the $\hat{\mathbf{v}}$-basis and its element satisfy:

$$\frac{\langle h+1|_{\hat{\mathbf{v}}} K_{+a}(\xi|P_a|P_C)|h+1\rangle_{\hat{\mathbf{v}}}}{\langle h|_{\hat{\mathbf{v}}} K_{+a}(\xi|P_a|P_C)|h\rangle_{\hat{\mathbf{v}}}} = -\frac{\alpha_a q^{h-1/2} - \beta_a q^{-(h-1/2)} \xi^2}{\gamma_a q^{h+3/2} - \delta_a q^{-(h+3/2)} \xi^2} \quad \forall h \in \{j, ..., p+j-2\} \tag{5.125}$$

Proof. The proof follows the same line as the one dedicated to the proof of the mixed reflection equation, using the form (5.121) for the equation. A similar condition on the boundary parameter ξ is needed to ensure the periodicity. □

Remark. *As for the reflection equation, let us mention that the scalar solutions of the dual reflection equation hold when the set P_C is general, namely the extra constraints* (5.94)-(5.96) *are not necessary. However, it is important to note that when these ones are imposed, the condition* (5.123) *reduces to the same condition as for the reflection equation, namely ξ has to satisfy: $\xi^2 = -q^{-k}$, for some $k \in \{0, ..., p-1\}$.*

Remark. *In the same way as for the original mixed reflection equation, let us comment that the question of the existence and expression of more general (non-diagonal in the $\hat{\mathbf{v}}$-basis) boundaries is still under examination. Nevertheless in the next, $K_a(P_a|P_C)$ and $K_{+a}(P_a|P_C)$ will denote as before, general scalar solutions of the mixed reflection equation and its dual, while $K_a(\xi|P_a|P_C)$ and $K_{+a}(\xi|P_a|P_C)$ will stand for the explicit diagonal scalar solutions given in propositions 5.3.6 and 5.3.8. We keep these general matrices just to emphasise that the results that we will present in the next for the fundamental transfer matrix are not related to the specific form of the boundary matrices but just to the fact that they are solutions of the reflection equations.*

5.3.4 The fundamental transfer matrix

We can now define the fundamental transfer matrix (5.63) for cyclic representations of the 6-vertex Yang-Baxter algebra associated to Bazhanov-Stroganov Lax operators:

$$\mathcal{T}^{(fund)}(P_a|P_Q|P_C) = tr_a \left\{ K_{+a}(P_a|P_C) \mathcal{V}_{aQ}(P_a|P_Q|P_C) \right\} \tag{5.126}$$

Here $K_{+a}^{t_a}(P_a|P_C)$ is a scalar solution of the mixed dual equation, and $\mathcal{V}_{aQ}(P_a|P_Q|P_C)$ is the fundamental boundary monodromy matrix defined by (5.108). In this definition, the space V_a can be considered as an auxiliary space, but this time of dimension p. Thus the set P_a of two parameters can be considered as a set of spectral parameters for the fundamental transfer matrix.

Using the fact that the mixed reflection algebra and its dual are (ABCD)-quantum type algebras, it immediately holds the following proposition which is at the heart of the computation of local Hamiltonians commuting with the 6-vertex boundary transfer matrix:

Proposition 5.3.9. *Let P_C be a set of constant parameters ensuring the existence of the fundamental R-matrix $S_{aQ}(P_a|P_Q|P_C)$ and satisfying the associated constraints* (5.94), (5.95) *or* (5.96), *and let P_a be the set of two spectral parameters on the space V_a. Then it holds:*

$$[\mathcal{T}^{(fund)}(P_a|P_Q|P_C), \mathcal{T}^{(6V)}(\lambda|P_Q|P_C)] = 0 \tag{5.127}$$

Let us comment that this commutativity can also be proved by a direct computation, using an approach à-la Sklyanin. Moreover, to compute local Hamiltonians which commute between themselves, we will use the essential next proposition:

Proposition 5.3.10. *Let P_C be a set of constant parameters ensuring the existence of the fundamental R-matrix $S_{aQ}(P_a|P_Q|P_C)$ and satisfying the associated constraints* (5.94), (5.95) *or* (5.96), *and let P_a and P'_a be two sets of spectral parameters on the space V_a. Then it holds:*

$$[\mathcal{T}^{(fund)}(P_a|P_Q|P_C), \mathcal{T}^{(fund)}(P'_a|P_Q|P_C)] = 0 \tag{5.128}$$

Proof. The proof is based on the simplicity of the spectrum of $\mathcal{T}^{(6V)}(\lambda|P_Q|P_C)$, which has been derived in chapter 4. Indeed, let $|\tau\rangle$ be an eigenvector of $\mathcal{T}^{(6V)}(\lambda|P_Q|P_C)$:

$$\mathcal{T}^{(6V)}(\lambda|P_Q|P_C)\,|\tau\rangle = \tau(\lambda|P_Q|P_C)\,|\tau\rangle \tag{5.129}$$

with $\tau(\lambda|P_Q|P_C)$ the associated eigenvalue. Thanks to proposition 5.3.9 and the mentioned simplicity, we can set

$$\mathcal{T}^{(fund)}(P_a|P_Q|P_C)\,|\tau\rangle = \tilde{\tau}(P_a|P_Q|P_C)\,|\tau\rangle \tag{5.130}$$

with $\tilde{\tau}$ a scalar function. Then, using the proposition 4.3.9, we know that for almost all the parameters the set $\{|\tau\rangle\}$ of all the eigenvectors forms a basis of the Hilbert space.

Thus the equation (5.128) holds for almost all the values of the parameters. Observing that the fundamental transfer matrix $\mathcal{T}^{(fund)}(P_a|P_Q|P_C)$ is a rational fraction in all the parameters, then the left hand side of (5.128) is a continuous functions of the parameters zero almost for every values of the parameters and so it is identically zero and (5.128) is proven. □

This fundamental transfer matrix family plays a crucial role in the determination of local Hamiltonians for models associated to cyclic representations, it is the purpose of paragraph 5.4 to explain how. Let us early mention that the main point to understand this is the fact that we could compute a fundamental transfer matrix which is similar to the standard one $\mathcal{T}^{(6V)}$, but which has an auxiliary space of dimension p. Thus the auxiliary and quantum spaces are isomorphic for this matrix and the hope, to compute local Hamiltonians, is to be able to use a similar standard computation than the one dedicated to fundamental models (cf paragraph 2.3.2).

Before to end this section, we emphasise on the link between the mixed reflection equation and its dual (for the particular diagonal boundary matrices), and we eventually give a reflection equation at a cyclic-cyclic level. As we will see, this latter point has to be investigated more (this work is under consideration) to be able to deal with an (ABCD)-quantum type algebra at a cyclic-cyclic level.

5.3.5 The dual mixed reflection equation from the mixed reflection equation

In this paragraph, we make explicit the link between the mixed dual reflection equation and its dual. First, from the expression for the $K_a(\xi|P_a|P_C)$ and $K_{+a}(\xi|P_a|P_C)$ boundary matrices, cf. propositions 5.3.6 and 5.3.8, it follows the corollary:

Corollary 5.3.11. *The scalar solution $K_{+a}(\xi|P_a|P_C)$ of the dual reflection equation can be written in terms of the original $K_a(\xi|P_a|P_C)$ as it follows:*

$$K_{+a}(\xi|P_a|P_C) = K_a(1/\xi|P_a^{\bar{\theta}_a}|P_C) \tag{5.131}$$

where we make use of the automorphism $\bar{\theta}_a$ acting on the parameters as:

$$\bar{\theta}_a : (\alpha_a, \beta_a, \gamma_a, \delta_a, a_a, b_a, c_a, d_a) \to (-q\gamma_a, -\delta_a/q, -\alpha_a/q, -q\beta_a, a_a, b_a, c_a, d_a) \tag{5.132}$$

Using the parametrisation with points in $\mathbb{C}^4$, we have:

$$\bar{\theta}_a : (x_{Q_a}, x_{R_a}, y_{Q_a}, y_{R_a}, \sigma_a) \to \left(\frac{q^3}{x_{R_a}}, \frac{q^3}{x_{Q_a}}, \frac{q}{y_{R_a}}, \frac{q}{y_{Q_a}}, \sigma_a\frac{x_{Q_a}x_{R_a}}{q^2 y_{Q_a}y_{R_a}}\right) \tag{5.133}$$

This property is the equivalent of the link (2.26) between the boundary matrices on the auxiliary space V_0 of dimension 2. As a consequence, we can show the next proposition:

Proposition 5.3.12. *The original mixed reflection equation* (5.83) *for the diagonal scalar boundary matrix $K_a(\xi|P_a|P_C)$ reduces to the dual one* (5.119) *for the diagonal scalar boundary matrix $K_{+a}(1/\xi|P_a|P_C)$ under the transformation $\bar{\sigma}_0\bar{\theta}_a$, where $\bar{\sigma}_0$ is the following involutive automorphism on the auxiliary space:*

$$\bar{\sigma}_0 \ : \ \lambda \to \frac{1}{\lambda} \tag{5.134}$$

The proof directly follows from the action of $\bar{\sigma}_0$ on $K_0(\lambda|\xi)$:

$$K_0^{\bar{\sigma}_0}(\lambda|\xi) = K_{+0}(\lambda|1/\xi) \tag{5.135}$$

and the following two identities:

$$L_{0a}(\lambda|P_a^{\bar{\theta}_a}|P_C) = L_{0a}(\lambda q|P_a^{\theta_a}|P_C) = -L_{0a}^{-1}(1/\lambda|P_a|P_C) \tag{5.136}$$

$$\text{and} \quad L_{0a}(\lambda|\left(P_a^{\theta_a}\right)^{\bar{\theta}_a}|P_C) = L_{0a}(\lambda/q|P_a|P_C) \tag{5.137}$$

We end this section by giving some properties related to the operation $\bar{\theta}$, allowing to write differently the equations. By a proof similar to the one of proposition 5.3.2, one easily shows:

$$S_{ab}^{\bar{\theta}_a\bar{\theta}_b}(P_a|P_b|P_C) = S_{ba}(P_b|P_a|P_C) \tag{5.138}$$

Moreover, the three operations θ, $\bar{\theta}$ and Υ (introduced in (5.50)) are linked via the obvious identity:

$$\theta\bar{\theta} = \bar{\theta}\theta\Upsilon^{-1}\Upsilon^{-1} \tag{5.139}$$

Then, in order to write the dual mixed reflection equation in a form similar to the original one, we can introduce the operations:

$$\phi_0 \equiv \bar{\sigma}_0\sigma_0\bar{\sigma}_0 \quad \text{and} \quad \Psi_a = \bar{\theta}_a\theta_a\bar{\theta}_a \tag{5.140}$$

The dual mixed reflection equation can be written:

$$\begin{aligned} &L_{0a}^{\phi_0\Psi_a}(\lambda|P_a|P_C)K_{+0}(\lambda|\xi)L_{a0}^{\Psi_a}(\lambda|P_a|P_C)K_{+a}(\xi|P_a|P_C) = \\ &\qquad K_{+a}(\xi|P_a|P_C)L_{0a}^{\phi_0}(\lambda|P_a|P_C)K_{+0}(\lambda|\xi)L_{a0}(\lambda|P_a|P_C) \end{aligned} \tag{5.141}$$

Thanks to this form, we can state the following:

Proposition 5.3.13 (Equations at a cyclic-cyclic level)**.** *Let $K_a(\xi|P_a|P_C)$ and $K_b(\xi|P_b|P_C)$ be the two diagonal solutions of the mixed reflection equation* (5.83)*. Then they satisfy the following cyclic-cyclic reflection equation:*

$$\begin{aligned} c(P_a, P_b, P_C)\ &S_{ba}^{\theta_b\theta_a}(P_b|P_a|P_C)K_a(\xi|P_a|P_C)S_{ab}^{\theta_b}(P_a|P_b|P_C)K_b(\xi|P_b|P_C) \\ &= K_b(\xi|P_b|P_C)S_{ba}^{\theta_a}(P_b|P_a|P_C)K_a(\xi|P_a|P_C)S_{ab}(P_a|P_b|P_C) \end{aligned} \tag{5.142}$$

The dual matrices $K_{+a}(\xi|P_a|P_C)$ and $K_{+b}(\xi|P_b|P_C)$, solutions of the dual mixed reflection equation (5.141)*, satisfy:*

$$\begin{aligned} &c(P_a^{\bar{\theta}_a}, P_b^{\bar{\theta}_b}, P_C)S_{ab}^{\Psi_a\Psi_b}(P_a|P_b|P_C)K_{+a}(\xi|P_a|P_C)S_{ba}^{\Psi_b}(P_b|P_a|P_C)K_{+b}(\xi|P_b|P_C) = \\ &\qquad K_{+b}(\xi|P_b|P_C)S_{ab}^{\Psi_a}(P_a|P_b|P_C)K_{+a}(\xi|P_a|P_C)S_{ba}(P_b|P_a|P_C) \end{aligned} \tag{5.143}$$

The scalar $c(P_a, P_b, P_C)$ satisfies, for any sets of spectral parameters P_a and P_b, the following properties:

$$c(P_a, P_b, P_C) = \frac{1}{c(P_b, P_a, P_C)} = \frac{1}{c(P_a^{\theta_a}, P_b, P_C)} = \frac{1}{c(P_a, P_b^{\theta_b}, P_C)} \quad \textit{and} \quad c(P_a, P_a, P_C) = 1 \tag{5.144}$$

The equation (5.143) *can be equivalently written:*

$$\begin{aligned} &\left(S_{ab}^{-1}(P_a|P_b|P_C)\right)^{t_at_b} K_{+a}(\xi|P_a|P_C)\left[\left(S_{ab}^{t_a}(P_a^{\theta_a}|P_b|P_C)\right)^{-1}\right]^{t_b} K_{+b}(\xi|P_b|P_C) \\ &= c(P_a^{\bar{\theta}_a}, P_b^{\bar{\theta}_b}, P_C)K_{+b}(\xi|P_b|P_C)\left[\left(S_{ab}^{t_a}(P_a^{\theta_a}|P_b|P_C)\right)^{-1}\right]^{t_b} K_{+a}(\xi|P_a|P_C)\left(S_{ab}^{-1}(P_a|P_b|P_C)\right)^{t_at_b} \end{aligned} \tag{5.145}$$

where t_a and t_b are the transposition on spaces a and b respectively, given in the $\hat{\mathbf{u}}$-basis.

As it will be shown in paragraph 5.5, we can explicitly compute this scalar for $p = 3$, and it indeed reduces to one. The proof that this is the case for any value of p odd is currently under consideration.

Remark. *The interest in this proposition is that, with dressed solutions of these cyclic-cyclic reflection equations, one could provide a direct proof of the commutativity of the fundamental transfer matrices between themselves, with the standard procedure induced by the (ABCD)-quantum type algebras (see* (5.69)*). In this way it would avoid the indirect argument that we have used, based on the simplicity of 6-vertex fundamental transfer matrix.*

5.4 Expression of local Hamiltonians

Now that the reflection equations have been studied, and that we explicitly know some boundary matrices, we can use them and tackle the problem to define integrable local Hamiltonians associated to cyclic representations of the 6-vertex reflection equation.

Let us consider the fundamental transfer matrix associated to the diagonal boundary scalar matrices, with boundary parameters denoted ξ_- and ξ_+:

$$\mathcal{T}^{(fund)}(P_a|P_Q|P_C) = tr_a \left\{ K_{+a}(\xi_+|P_a|P_C)\mathcal{V}_{aQ}(P_a|P_Q|P_C) \right\} \tag{5.146}$$

with

$$\mathcal{V}_{aQ}(P_a|P_Q|P_C) = S^{\theta_Q \theta_a}_{Qa}(P_Q|P_a|P_C)K_a(\xi_-|P_a|P_C)S^{\theta_Q}_{aQ}(P_a|P_Q|P_C) \tag{5.147}$$

Making use of (5.146), the standard procedure in order to define local Hamiltonians consists in taking its (first order) derivative with respect to the spectral parameters, in a particular value (see paragraph 2.3.2). Indeed, for the fundamental transfer matrix, the auxiliary space V_a and the local quantum spaces V_n are isomorphic so that one can try to derive local Hamiltonians following the procedure introduced by Sklyanin in [177].

In this section we show that for cyclic representations, the first order derivative does not produce non-trivial local Hamiltonians. However, one can compute the second order derivative and obtain from it the local Hamiltonians. At the end, we get a model of interactions where the two boundary matrices $K_a(\xi_-|P_a|P_C)$ and $K_{+a}(\xi_+|P_a|P_C)$ effectively describe the left and right boundaries of a chain, and where the nearest neighbour interactions are given thanks to derivatives of the fundamental R-matrix.

5.4.1 First order derivative of the fundamental transfer matrix

Following Sklyanin's procedure [177], an important property is needed to construct local Hamiltonians from the fundamental transfer matrix: there must exist at least one set of spectral parameters such that the boundary matrix $K_a(P_a|P_C)$ reduces to the identity and such that the fundamental R-matrix reduces to the permutation operator. This property is satisfied for the models under consideration:

- The matrices $K(\xi|P_a|P_C)$ and $K_+(\xi|P_a|P_C)$ reduce to the identity when the spectral parameters P_a belong to certain hypersurfaces. Indeed, let us introduce the sets:

$$\mathcal{P}^- = \left\{ P_a = (x_{Q_a}, x_{R_a}, y_{Q_a}, y_{R_a}, \sigma_a) \mid x_{Q_a}x_{R_a} = q^2 \text{ and } y_{Q_a}y_{R_a} = 1 \right\} \tag{5.148}$$

$$\text{and} \quad \mathcal{P}^+ = \left\{ P_a = (x_{Q_a}, x_{R_a}, y_{Q_a}, y_{R_a}, \sigma_a) \mid x_{Q_a}x_{R_a} = 1/q^2 \text{ and } y_{Q_a}y_{R_a} = q^2 \right\} \tag{5.149}$$

  Then we have, by choosing the appropriate normalisation for the boundary transfer matrices, that for any complex parameter ξ it holds:

$$\forall P_a^- \in \mathcal{P}^-, \quad K(\xi|P_a^-|P_C) = \mathbb{1} \qquad \text{and} \qquad \forall P_a^+ \in \mathcal{P}^+, \quad K_+(\xi|P_a^+|P_C) = \mathbb{1} \tag{5.150}$$

- From the Yang-Baxter equation, we directly have that for any set P of spectral parameters:

$$S_{ab}(P|P|P_C) = \mathcal{P}_{ab} \tag{5.151}$$

where $\mathcal{P}_{ab}$ denotes the permutation operator between the spaces V_a and V_b.

Moreover, from the definition of the action of θ on the parameters, it is clear that:

$$\forall P^- \in \mathcal{P}^-, \quad S_{ab}^{\theta_a}(P^-|P^-|P_C) = \mathcal{P}_{ab} \tag{5.152}$$

Definition 5.4.1 (Homogeneous chain)**.** *Let P be a set of spectral parameters. A chain is said homogeneous when, on every site n, the quantum parameters P_n are given by $P_n = P$. For brevity, we will simply denote this by $P_Q = P$, where the meaning is $\forall n \in [\![1, N]\!]$, $P_n = P$.*

Let $P^- \in \mathcal{P}^-$, and let us denote $(x_Q, x_R, y_Q, y_R, \sigma)$ its (non-independent) components. The standard computation to get local Hamiltonians is to calculate the first derivative of the fundamental transfer matrix in the homogeneous limit, i.e. where the chain is homogeneous ($P_Q = P^-$) and where the spectral parameters are evaluated in the same values ($P_a = P^-$). Indeed, one can then extract from the obtained formula the expression of Hamiltonians which commute with the fundamental transfer matrix (see paragraph 2.3.2).

So let x_a be a component of the set of spectral parameters P_a. We aim to compute:

$$\left.\frac{d\,\mathcal{T}^{(fund)}(P_a|P_Q|P_C)}{dx_a}\right|_{\substack{P_a=P^- \\ P_Q=P^-}} \tag{5.153}$$

Proposition 5.4.1. *The first derivative in an homogeneous limit $P^- \in \mathcal{P}^-$ of the fundamental transfer matrix has the following expression:*

$$\begin{aligned}\left.\frac{d\,\mathcal{T}^{(fund)}(P_a|P_Q|P_C)}{dx_a}\right|_{\substack{P_a=P^- \\ P_Q=P^-}} &= tr_a\left\{\left.\frac{dK_{+a}(\xi_+|P_a|P_C)}{dx_a}\right|_{P_a=P^-}\right\} + tr_a\left\{K_{+a}(\xi_+|P^-|P_C)\right\}\sum_{k=1}^{N-1} H_{k,k+1}^{(x)} \\ &+ tr_a\left\{K_{+a}(\xi_+|P^-|P_C)\right\}\left.\frac{dK_1(\xi_-|P_1|P_C)}{dx_1}\right|_{P_1=P^-} + tr_a\left\{K_{+a}(\xi_+|P^-|P_C)H_{Na}^{(x)}\right\}\end{aligned} \tag{5.154}$$

The local interactions are given by:

$$H_{k,k+1}^{(x)} = \left.\frac{dS_{k+1,k}(P_{k+1}|P_k|P_C)}{dx_{k+1}}\right|_{\substack{P_{k+1}=P^- \\ P_k=P^-}} \mathcal{P}_{k,k+1} + \mathcal{P}_{k,k+1}\left.\frac{dS_{k+1,k}(P_{k+1}|P_k|P_C)}{dx_{k+1}}\right|_{\substack{P_{k+1}=P^- \\ P_k=P^-}} \tag{5.155}$$

and

$$H_{Na}^{(x)} = \left.\frac{dS_{aN}(P_a|P_N|P_C)}{dx_a}\right|_{\substack{P_a=P^- \\ P_N=P^-}} \mathcal{P}_{Na} + \mathcal{P}_{Na}\left.\frac{dS_{aN}(P_a|P_N|P_C)}{dx_a}\right|_{\substack{P_a=P^- \\ P_N=P^-}} \tag{5.156}$$

The upper-script (x) emphasises on the fact that we derive in the direction x_a.

Proof. The proof is a direct computation. We have:

$$\begin{aligned}\left.\frac{d\,\mathcal{T}^{(fund)}(P_a|P_Q|P_C)}{dx_a}\right|_{\substack{P_a=P^- \\ P_Q=P^-}} &= tr_a\left\{\left.\frac{dK_{+a}(\xi_+|P_a|P_C)}{dx_a}\right|_{P_a=P^-}\mathcal{V}_{aQ}(P^-|P^-|P_C)\right\} \\ &+ tr_a\left\{K_{+a}(\xi_+|P^-|P_C)\left.\frac{d\mathcal{V}_{aQ}(P_a|P_Q|P_C)}{dx_a}\right|_{\substack{P_a=P^- \\ P_Q=P^-}}\right\}\end{aligned} \tag{5.157}$$

Then using the properties (5.151) and (5.152), we have $\mathcal{V}_{aQ}(P^-|P^-|P_C) = \mathbb{1}$. Indeed, the unitarity in the point P^- gives $\rho(P^-|P^-|P_C) = 1$.

For the other term, it holds:

$$\left.\frac{d\mathcal{V}_{aQ}(P_a|P_Q|P_C)}{dx_a}\right|_{\substack{P_a=P^- \\ P_Q=P^-}} = \left.\frac{dS_{Qa}^{\theta_a\theta_Q}(P_Q|P_a|P_C)}{dx_a}\right|_{\substack{P_a=P^- \\ P_Q=P^-}} K_a(\xi_-|P^-|P_C)\; S_{aQ}^{\theta_Q}(P^-|P^-|P_C) \tag{5.158}$$

$$+S_{Qa}^{\theta_a\theta_Q}(P^-|P^-|P_C)\; \left.\frac{dK_a(\xi_-|P^-|P_C)}{dx_a}\right|_{P_a=P^-} S_{aQ}^{\theta_Q}(P^-|P^-|P_C) \tag{5.159}$$

$$+S_{Qa}^{\theta_a\theta_Q}(P^-|P^-|P_C)\; K_a(\xi_-|P^-|P_C)\; \left.\frac{dS_{aQ}^{\theta_Q}(P_a|P_Q|P_C)}{dx_a}\right|_{\substack{P_a=P^- \\ P_Q=P^-}} \tag{5.160}$$

Each one is now calculated using the reduction to the permutation operator in the special value P^- considered here. □

Two comments are required. The first one concerns the interactions $H_{k,k+1}^{(x)}$. Indeed, let us introduce the following notation :

$$h_{k,k+1}^{(x)} = \left.\frac{dS_{k+1,k}(P_{k+1}|P_k|P_C)}{dx_{k+1}}\right|_{\substack{P_{k+1}=P^- \\ P_k=P^-}} \mathcal{P}_{k,k+1} \quad \text{and} \quad h_{Na}^{(x)} = \left.\frac{dS_{aN}(P_a|P_N|P_C)}{dx_a}\right|_{\substack{P_a=P^- \\ P_N=P^-}} \mathcal{P}_{Na} \tag{5.161}$$

Then the local interactions can be written:

$$H_{k,k+1}^{(x)} = h_{k,k+1}^{(x)} + h_{k+1,k}^{(x)} \quad \text{and} \quad H_{Na}^{(x)} = h_{Na}^{(x)} + h_{aN}^{(x)} \tag{5.162}$$

Let us early mention that within this formalism, the bulk Hamiltonian for a chain described by the reflection algebra (so with integrable boundaries) is a symmetrised version of the bulk Hamiltonian obtained for the same chain described by the Yang-Baxter algebra (with quasi-periodic boundary conditions):

$$\left(H_{bulk}^{reflection}\right)_{k,k+1}^{(x)} = h_{k,k+1}^{(x)} + h_{k+1,k}^{(x)} \quad \text{while} \quad \left(H_{bulk}^{Yang-Baxter}\right)_{k,k+1}^{(x)} = h_{k,k+1}^{(x)} \tag{5.163}$$

We shall come back later to the impact of the boundaries on the bulk interactions, for some specific models in paragraph 5.5.

The second comment concerns the Hamiltonian itself. Indeed, one is tempted to divide by the quantity $tr_a\left\{K_{+a}(\xi_+|P^-|P_C)\right\}$ and thus to obtain an Hamiltonian which expression is extremely similar to the one derived in (2.43), paragraph 2.3.2. However, as it is explained in the following paragraph, this cannot be done when one considers cyclic representations.

5.4.2 Peculiarities for cyclic representations

As already mentioned, for cyclic representations some technical difficulties emerge due to the fact that q is a p^{th} root of unity. In particular, the following proposition holds:

Proposition 5.4.2. *For cyclic representations (associated to diagonal boundary matrices), the first order derivative of the fundamental transfer matrix reduces to a scalar multiple of the identity:*

$$\forall P^- \in \mathcal{P}^-, \qquad \left.\frac{d\,\mathcal{T}^{(fund)}(P_a|P_Q|P_C)}{dx_a}\right|_{\substack{P_a=P^- \\ P_Q=P^-}} \propto \mathbb{1} \tag{5.164}$$

As a consequence, the proof that the local Hamiltonians commute with the boundary transfer matrix cannot be done using this derivative.

The aim of this paragraph is to give the main steps of the proof of the proposition. It relies on the propositions 5.4.3 and 5.4.5. One important property is given by the corollary 5.4.4, stating that the trace of the boundary matrix $tr_a\left\{K_{+a}(\xi_+|P^-|P_C)\right\}$ is zero. In particular, it explains that we cannot define local Hamiltonians of the form (2.43). The proofs of the propositions 5.4.3 and 5.4.5 are based on the fact that in the special homogeneous limit P^- considered, we can give an explicit and handleable formula of the right and left boundary matrices, in terms of the operators $\hat{\mathbf{v}}$. Indeed, in this point, the recursion formulas of the propositions 5.3.6 and 5.3.8 can be solved explicitly.

Proposition 5.4.3. *In the $\hat{\mathbf{u}}$-basis, the diagonal elements of the matrices $K_{+a}(\xi|P^-|P_C)$ and $K_a(\xi|P^+|P_C)$ are vanishing:*

$$\forall (P^-,\xi)\in\mathcal{P}^-\otimes\mathbb{C}\ ,\ \forall h\in[\![0,p-1]\!]\ ,\quad \langle h|_{\hat{\mathbf{u}}}\, K_{+a}(\xi|P^-|P_C)\,|h\rangle_{\hat{\mathbf{u}}}=0 \tag{5.165}$$

$$\forall (P^+,\xi)\in\mathcal{P}^+\otimes\mathbb{C}\ ,\ \forall h\in[\![0,p-1]\!]\ ,\quad \langle h|_{\hat{\mathbf{u}}}\, K_{a}(\xi|P^+|P_C)\,|h\rangle_{\hat{\mathbf{u}}}=0 \tag{5.166}$$

We chose to give here the explicit proof, as it introduces an important reconstruction of the boundaries matrices that will be used later.

Proof. We show the first relation (5.165). From proposition 5.3.8, we know the ratios defining the diagonal coefficients of the matrix $K_{+a}(\xi|P_a|P_C)$ in the $\hat{\mathbf{v}}$-basis. At the special point $P_a = P^-$, this ratio $R_h^{(+)}$ (5.122) or (5.125) can be written:

$$R_h^{(+)}=q^2\frac{f(h)}{f(h+2)} \tag{5.167}$$

with $f(h)=q^{2h}-s$, where $s=\sigma q\xi^2$ is a scalar depending on the parameters at the homogeneous limit P^- and on the boundary parameter ξ.

Thanks to the specific form (5.167) of these ratios, we can compute:

$$\langle h|_{\hat{\mathbf{v}}}\, K_{+a}(\xi|P^-|P_C)\,|h\rangle_{\hat{\mathbf{v}}}=\mathcal{N}\left(\frac{1}{q^{2(h+1)}-s}-\frac{1}{q^{2h}-s}\right)$$

$$\text{where the pre-factor } \mathcal{N} \text{ is explicitly given by}\quad \mathcal{N}\equiv\langle 0|_{\hat{\mathbf{v}}}\, K_{+a}(\xi|P^-|P_C)\,|0\rangle_{\hat{\mathbf{v}}}\,\frac{f(0)f(1)}{1-q^2}. \tag{5.168}$$

The key observation to note is that we can then give an expression of the matrix $K_{+a}(\xi|P^-|P_C)$ in terms of the $\hat{\mathbf{v}}$ operators:

$$K_{+a}(\xi|P^-|P_C)=\frac{\mathcal{N}}{q^2\hat{\mathbf{v}}^2-s}-\frac{\mathcal{N}}{\hat{\mathbf{v}}^2-s} \tag{5.169}$$

Now, the cornerstone of the proof is the following inversion formula (see (5.12) of [145]):

$$\forall y\in\mathbb{C}\ ,\quad \frac{1+y^p}{y\ \hat{\mathbf{v}}^2+1}=\sum_{j=0}^{p-1}(-1)^j y^j\,\hat{\mathbf{v}}^{2j} \tag{5.170}$$

which leads, with a simple computation, to:

$$K_{+a}(\xi|P^-|P_C)=\sum_{j=0}^{p-1}\mathcal{C}_j\ \hat{\mathbf{v}}^{2j}\qquad\text{with}\quad \mathcal{C}_j=\mathcal{N}\frac{s^{p-1}}{1-s^p}s^{-j}\left(q^{2j}-1\right) \tag{5.171}$$

It follows the proposition as:

$$\langle h|_{\hat{\mathbf{u}}}\, K_{+a}(\xi|P^-|P_C)\,|h\rangle_{\hat{\mathbf{u}}}=\sum_{j=0}^{p-1}\mathcal{C}_j\ \langle h|_{\hat{\mathbf{u}}}\,|h-j\rangle_{\hat{\mathbf{u}}}=\mathcal{C}_0=0 \tag{5.172}$$

Using a very similar method, one can prove (5.166). Let us simply comment that the reconstruction of the matrix is different, as it holds:

$$K_a(\xi|P^+|P_C) = \bar{\mathcal{N}}\left(\hat{\mathbf{v}}^{p-2} - t\ \hat{\mathbf{v}}^{p-4}\right) \tag{5.173}$$

with $\bar{\mathcal{N}}$ and t are scalars depending on the parameters at the homogeneous limit P^- and on the boundary parameter ξ. □

Corollary 5.4.4. *In the points where $K_{+a}(\xi|P_a|P_C)$ (respectively $K_a(\xi|P_a|P_C)$) reduces to a scalar, the trace of $K_a(\xi|P_a|P_C)$ (respectively $K_{+a}(\xi|P_a|P_C)$) is vanishing:*

$$\forall(P^-,\xi) \in \mathcal{P}^- \otimes \mathbb{C}\ , \quad tr_a\left\{K_{+a}(\xi|P^-|P_C)\right\} = 0 \tag{5.174}$$

$$\forall(P^+,\xi) \in \mathcal{P}^+ \otimes \mathbb{C}\ , \quad tr_a\left\{K_a(\xi|P^+|P_C)\right\} = 0 \tag{5.175}$$

As already mentioned, this is an important obstacle in our way to the computation of local Hamiltonians. Moreover, using this corollary, it remains only the term $tr_a\left\{K_{+a}(\xi_+|P^-|P_C)H_{Na}\right\}$ in (5.154). The following lemma is then at the heart of the proof of proposition 5.4.2:

Proposition 5.4.5. *In any direction x for the derivative, it holds:*

$$\forall(P_a^-,\xi) \in \mathcal{P}^-\otimes\mathbb{C} \quad tr_a\left\{K_{+a}(\xi|P^-|P_C)h_{Na}^{(x)}\right\} = \kappa^{(x)}\ \mathbb{1} \ \ \textit{and} \ \ tr_a\left\{K_{+a}(\xi|P^-|P_C)h_{aN}^{(x)}\right\} = \tilde{\kappa}^{(x)}\ \mathbb{1} \tag{5.176}$$

The two scalars are explicitly known, and given by:

$$\kappa^{(x)} = \sum_{\Delta_a=0}^{p-1} \mathcal{C}_{\Delta_a}\ \frac{d\ \bar{W}_{R_aR_N}(\Delta_a)}{d\ x_a}\bigg|_{\substack{P_N=P^-\\P_a=P^-}} \sum_{h_a=0}^{p-1} W_{RQ}(h_a)W_{QR}(h_a+\Delta_a) \tag{5.177}$$

$$\textit{and}\ \ \tilde{\kappa}^{(x)} = \sum_{\Delta_a=0}^{p-1} \mathcal{C}_{\Delta_a}\ \frac{d\ \bar{W}_{Q_aQ_N}(\Delta_a)}{d\ x_a}\bigg|_{\substack{P_N=P^-\\P_a=P^-}} \sum_{h_a=0}^{p-1} W_{RQ}(h_a+\Delta_a)W_{QR}(h_a) \tag{5.178}$$

In these formulas, the coefficient $\mathcal{C}_j$ is the one introduced in (5.171) *for the decomposition of the boundary matrix in terms of the generators of the Weyl algebra, W and $\bar{W}$ denote the dilogarithm functions entering in the expression of the fundamental R-matrix, and the points denoted R and Q are the homogenous limits of the points R_n and Q_n of the chain.*

The proof of these equalities is rather technical (see paper **III**). We simply highlight that the heart of the proof lies in some particular relations satisfied by the dilogarithm functions, for example $\bar{W}_{QQ}(h-l) = \delta_{h,l}$ (the Kronecker symbol). Computing the trace in the $\hat{\mathbf{u}}$-basis allows to use the proposition 5.4.3, which simplifies the computation. Lastly, we use the reconstruction (5.171) of $K_{+a}(\xi|P^-|P_C)$ in terms of the generators of the Weyl algebra, which permits to complete the computations.

This last proposition ends the proof of proposition 5.4.2. Indeed, in the expression (5.154) the bulk interactions $\sum_{k=1}^{N-1} H_{k,k+1}^{(x)}$ are eliminated by the zero factor $tr_a\left\{K_{+a}(\xi|P^-|P_C)\right\}$ (proposition 5.4.4), and the remaining terms are scalars (proposition 5.4.5).

Thus the first order derivative reduces to a scalar in the considered limit, and the standard procedure to obtain local Hamiltonians cannot be used.

5.4.3 Expression of local Hamiltonians

Here, we prove that the second order derivative with respect to the spectral parameters can be used to express local Hamiltonians and to show their commutativity with the fundamental transfer matrix. The key point of the computation relies on the fact that the non-trivial expression for the second order derivative simplifies thanks to proposition 5.4.2. In particular, we show that the highly non local interactions are in fact cancelled.

Second derivative of the fundamental transfer matrix

Thus we tackle the computation of the second order derivative with respect to the spectral parameters, in the homogeneous limit, of the fundamental transfer matrix:

$$\left.\frac{d^2\ \mathcal{T}^{(fund)}(P_a|P_Q|P_C)}{dx_a^2}\right|_{\substack{P_a=P^-\\P_Q=P^-}} \tag{5.179}$$

After a rather long computation, with more involved steps than for the first order derivative but of the same kind (explicit computation of the different terms in the derivative, reduction of the fundamental R-matrix to the permutation operator, reduction of the boundary matrix to the identity and so on), we find:

$$\begin{aligned}
\left.\frac{d^2\ \mathcal{T}^{(fund)}(P_a|P_Q|P_C)}{dx_a^2}\right|_{\substack{P_a=P^-\\P_Q=P^-}} &= tr_a\left\{\left.\frac{d^2K_{+a}(\xi_+|P_a|P_C)}{dx_a^2}\right|_{P_a=P^-}\right\} + tr_a\left\{K_{+a}(\xi_+|P^-|P_C)\right\}\mathcal{H}^{(x)}_{nonlocal}\\
&+\ 2\left(tr_a\left\{\left.\frac{dK_{+a}(\xi_+|P_a|P_C)}{dx_a}\right|_{P_a=P^-}\right\} + tr_a\left\{K_{+a}(\xi_+|P^-|P_C)H^{(x)}_{Na}\right\}\right)\sum_{k=1}^{N-1}H^{(x)}_{k,k+1}\\
&+\ 2\left(tr_a\left\{\left.\frac{dK_{+a}(\xi_+|P_a|P_C)}{dx_a}\right|_{P_a=P^-}\right\} + tr_a\left\{K_{+a}(\xi_+|P^-|P_C)H^{(x)}_{Na}\right\}\right)\left.\frac{dK_1(\xi_-|P_1|P_C)}{dx_1}\right|_{P_1=P^-}\\
&+2\ tr_a\left\{\left.\frac{dK_{+a}(\xi_+|P_a|P_C)}{dx_a}\right|_{P_a=P^-}H^{(x)}_{Na}\right\} + tr_a\left\{K_{+a}(\xi_+|P^-|P_C)H^{(2)(x)}_{Na}\right\} + 2\ tr_a\left\{K_{+a}(P_a^-|\xi_+)h^{(x)}_{Na}h^{(x)}_{aN}\right\}\\
&+2\ tr_a\left\{K_{+a}(\xi_+|P^-|P_C)\left[H^{(x)}_{N-1,N}, h^{(x)}_{aN}\right]\right\}
\end{aligned} \tag{5.180}$$

In this formula, $\mathcal{H}^{(x)}_{nonlocal}$ denotes a highly non local and non trivial term, which is however irrelevant in our calculations as it is multiplied by 0 (proposition 5.4.4). We also introduced:

$$H^{(2)(x)}_{Na} \equiv \left.\frac{d^2S_{aN}(P_a|P_N|P_C)}{dx_a^2}\right|_{\substack{P_a=P^-\\P_N=P^-}}\mathcal{P}_{aN} + \mathcal{P}_{aN}\ \left.\frac{d^2S^{\theta_N}_{aN}(P_a|P_N|P_C)}{dx_a^2}\right|_{\substack{P_a=P^-\\P_N=P^-}} \tag{5.181}$$

Moreover, we can rewrite the previous equality (5.180) as:

$$\begin{aligned}
\left.\frac{d^2\ \mathcal{T}^{(fund)}(P_a|P_Q|P_C)}{dx_a^2}\right|_{\substack{P_a=P^-\\P_Q=P^-}} &= tr_a\left\{\left.\frac{d^2K_{+a}(\xi_+|P_a|P_C)}{dx_a^2}\right|_{P_a=P^-}\right\}\ +\ \alpha^{(x)}\sum_{k=1}^{N-1}H^{(x)}_{k,k+1}\\
&+\ \alpha^{(x)}\left.\frac{dK_1(\xi_-|P_1|P_C)}{dx_1}\right|_{P_1=P^-} + 2\ tr_a\left\{\left.\frac{dK_{+a}(\xi_+|P_a|P_C)}{dx_a}\right|_{P_a=P^-}H^{(x)}_{Na}\right\}\\
&+ tr_a\left\{K_{+a}(\xi_+|P^-|P_C)H^{(2)(x)}_{Na}\right\} + 2\ tr_a\left\{K_{+a}(\xi_+|P^-|P_C)h^{(x)}_{Na}h^{(x)}_{aN}\right\}
\end{aligned} \tag{5.182}$$

with the scalar (proposition 5.4.5) $\alpha^{(x)}$ given by:

$$\alpha^{(x)} = 2\left(tr_a\left\{\left.\frac{dK_{+a}(\xi_+|P_a|P_C)}{dx_a}\right|_{P_a=P^-}\right\} + tr_a\left\{K_{+a}(\xi_+|P^-|P_C)H^{(x)}_{Na}\right\}\right) \tag{5.183}$$

This form (5.182) for the second order derivative is very interesting as it is similar to the one obtained for the first order derivative: we can see the interactions $\sum_{k=1}^{N-1}H^{(x)}_{k,k+1}$ and two terms only acting on site 1 and site N, the boundaries. What is more, contrary to what happens with the first order derivative, the next proposition makes usable the second order derivative both to compute local Hamiltonians and to show their commutation with the boundary transfer matrix:

Proposition 5.4.6. *For generic parameters, and for any direction* (x) *of derivation, it holds:*

$$\alpha^{(x)} \neq 0 \tag{5.184}$$

Proof. The proof uses a continuity argument, similar to the one used previously many times in chapter 4. From the expression (5.183), one can see that $\alpha^{(x)}$ is a continuous function of the parameters. Then, it is sufficient to show that there is at least one set of parameters for which $\alpha^{(x)} \neq 0$, and the result will hold for almost any value of the parameters.

So let us consider the following set of parameters:

$$P_a = (x_{Q_a}, x_{R_a}, y_{Q_a}, y_{R_a}, \sigma_a) = (q\ \chi_a, q\ \chi_a, \chi_a, \chi_a, \Sigma) \tag{5.185}$$

$$\text{and} \quad P_N = (x_{Q_N}, x_{R_N}, y_{Q_N}, y_{R_N}, \sigma_N) = (q\ \chi_N, q\ \chi_N, \chi_N, \chi_N, \Sigma) \tag{5.186}$$

with χ_a, χ_N and Σ free parameters. The homogeneous limit is given by $\chi_a = \chi_N = \chi = 1$. The idea is to check the behaviour of $\alpha^{(\chi)}$ under the limit $\Sigma \to +\infty$. After some computations, using for instance the reconstruction (5.171), we show that $\alpha^{(\chi)} \neq 0$ for this model, which concludes the proof. ☐

This proposition leads to another main result of this thesis, exposed in the following conclusion.

Expression of local Hamiltonians associated to cyclic representations of the 6-vertex reflection algebra

For $\alpha^{(x)} \neq 0$, we can define the following integrable local Hamiltonians with boundaries:

$$\mathcal{H}^{(x)} \equiv \sum_{k=1}^{N-1} H^{(x)}_{k,k+1} + \left.\frac{dK_1(\xi_-|P_1|P_C)}{dx_1}\right|_{P_1=P^-} + \mathcal{B}^{(x)}_N(\xi_+|P_C) \tag{5.187}$$

where the right boundary (on site N) is given by:

$$\begin{aligned}\mathcal{B}^{(x)}_N(\xi_+|P_C) = \frac{1}{\alpha^{(x)}}\Bigg(& 2\ tr_a\left\{\left.\frac{dK_{+a}(\xi_+|P_a|P_C)}{dx_a}\right|_{P_a^-} H^{(x)}_{Na}\right\} \\ & + tr_a\left\{K_{+a}(\xi_+|P_a^-|P_C) H^{(2)(x)}_{Na}\right\} + 2tr_a\left\{K_{+a}(\xi_+|P_a^-|P_C) h^{(x)}_{Na} h^{(x)}_{aN}\right\}\Bigg)\end{aligned} \tag{5.188}$$

Thanks to the commutativity exposed in propositions 5.3.10 and 5.3.9, these Hamiltonians commute with the fundamental transfer matrix and with the 6-vertex transfer matrix for any direction x of derivation:

$$\forall (P_a, P^-) \in \mathcal{P} \otimes \mathcal{P}^- \ , \qquad \left[\mathcal{T}^{(fund)}(P_a|P^-|P_C), \mathcal{H}^{(x)}\right] = 0 \tag{5.189}$$

$$\forall (\lambda, P^-) \in \mathbb{C} \otimes \mathcal{P}^- \ , \qquad \left[\mathcal{T}^{(6V)}(\lambda|P^-|P_C), \mathcal{H}^{(x)}\right] = 0 \tag{5.190}$$

The left and right boundaries are encoded via the parameters ξ_- and ξ_+ respectively. Let us comment that similarly to Sklyanin's procedure [177] for the expression of the local Hamiltonian, cf (2.43), the boundaries have different expressions, with the one for the right boundary seeming much more involved. We shall come back to that issue very soon, in section 5.5.

Remark. *Let us comment that in the more general case, two Hamiltonians* $\mathcal{H}^{(x)}$ *and* $\mathcal{H}^{(y)}$ *of the form* (5.187) *can be obtained thanks to the derivative in two different direction* x *and* y*. Then, one can combine these two expressions with a free parameter* z *to obtain* $\mathcal{H} = \mathcal{H}^{(x)} + z\ \mathcal{H}^{(y)}$*, another commuting family which commutes with the fundamental transfer matrix and with the 6-vertex transfer matrix. Indeed, from the commutativity of proposition 5.3.10, it holds* $\left[\mathcal{H}^{(x)}, \mathcal{H}^{(y)}\right] = 0$*. Thus the Hamiltonian* $\mathcal{H}$ *depends on*

2 constants of the model (L and N for example, cf point A. of proposition 5.1.4) and on another free parameter z.

5.5 Some explicit (new) models for 3-dimensional local quantum spaces

In the last section of this thesis, we explicitly study four models of interest with finite cyclic representation of dimension 3, i.e. at the value $p = 3$. The fact to take a small numerical value for p allows to get explicit expressions and thus to develop an intuition for what happens with generic representations. In particular a symmetry on the boundaries of the Hamiltonians appears, as well as some properties for the bulk interactions. Moreover, we have results concerning the impact of the boundaries on the bulk Hamiltonians, and we check the reflection equations at a cyclic-cyclic level.

The models we are considering are the followings. For each of them, we give the parameters $(x_{Q_n}, x_{R_n}, y_{Q_n}, y_{R_n}, \sigma_n)$ in the quantuml spaces V_n (of dimension 3) of the chain. The same sets of parameters hold in the auxiliary space V_a (of dimension 3), simply replacing the subscript n by a. Moreover, we will give the homogeneous limit of these models, i.e. for which the parameters $P^- = (x_Q, x_R, y_Q, y_R, \sigma) \in \mathcal{P}^-$. In addition to this, we give an explicit expression for the generators of the Weyl algebra. For example in the so-called $\hat{\mathbf{v}}$-basis, it holds:

$$\hat{\mathbf{u}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{v}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^2 \end{pmatrix}, \quad \text{with} \quad q^3 = 1. \tag{5.191}$$

- **The XXZ spin chain at root of unity**. In our study, the XXZ spin $s = 1$ chain can be obtained with the following set of parameters:

$$\forall n \in [\![1, N]\!], \;\; \sigma_n = q\,, \quad x_{Q_n} = i\epsilon q^2 \frac{1}{\xi_n}\,, \quad x_{R_n} = -qi\epsilon \frac{1}{\xi_n}\,, \quad y_{Q_n} = -i\epsilon \frac{1}{\xi_n} \;\; \text{and} \;\; y_{R_n} = qi\epsilon \frac{1}{\xi_n} \tag{5.192}$$

where $\epsilon = \pm 1$ and ξ_n are free parameters. Indeed, the Lax operator (5.7) reduces to the R-matrix obtained by fusion [197, 237] from the 6-vertex R-matrix under the following similarity transformation $P.\hat{\mathbf{u}}.P^{-1}$ and $P.\hat{\mathbf{v}}.P^{-1}$ on the generators of the Weyl algebra represented by (5.191). The matrix P is explicitly given by:

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{5.193}$$

The anisotropy Δ of the model is related to q by $\Delta = (q + 1/q)/2$. For this model, which corresponds to the point D) of proposition 5.1.4, the homogeneous limit in a point $P^- \in \mathcal{P}^-$ is obtain when the parameters satisfy $\xi_n = \xi$ with $\xi^2 = q$.

This model is used, in this section, mainly as a verification tool. Indeed, as we shall see our method allows to find back the known Hamiltonian of the spin 1 XXZ chain.

- **The super integrable chiral Potts chain**. With the following set of parameters:

$$\forall n \in [\![1, N]\!], \;\; \sigma_n = 1\,, \quad x_{Q_n} = q^{2-i}\, y_{R_n} \quad \text{and} \quad x_{R_n} = q^i\, y_{Q_n} \tag{5.194}$$

where the parameters y_{Q_n} and y_{R_n} are free and i is a free integer, one can describe the super integrable chiral Potts model. This one corresponds also to the point D) of proposition 5.1.4. The homogeneous limit in a point $P^- \in \mathcal{P}^-$ is obtain when the parameters y_Q and y_R are satisfying $y_Q y_R = 1$.

- **The Fateev-Zamolodchikov model**. With the set of parameters:

$$\forall n \in [\![1, N]\!], \quad \sigma_n = 1 \ , \quad x_{Q_n} = q^2 y_{Q_n} \ , \quad x_{R_n} = y_{R_n} \ , \quad y_{Q_n} = \frac{1}{iq^{1/2}\kappa_n \xi_n} \quad \text{and} \quad y_{R_n} = \frac{iq^{1/2}\kappa_n}{\xi_n} \tag{5.195}$$

where ξ_n and κ_n are free, one can describe the so-called Fateev-Zamolodchikov model [258]. This model is interesting *per se*, but also for its link with the discretised sine-Gordon model at q root of unity. Indeed, the Lax operator (5.7) satisfies:

$$L_{0n}(\lambda|Q_n, R_n) = L_{0n}^{(sG)}(\lambda).\sigma_0^x \tag{5.196}$$

where $L_{0n}^{(sG)}(\lambda)$ is the Lax operator of the sine-Gordon model, see [250] for instance, and σ_0^x denotes the Pauli matrix in the space V_0. It has been shown that, for an even number of sites (N=2M), there is a mapping between the transfer matrices of these two models. As far as an odd N is concerned, the two transfer matrices have different eigenvalues (see appendix D of [250] for an explicit comparison between these two models).

This model enters in the framework of point B) of proposition 5.1.4 with the constant parameters $R = 0$ and $K = 1$. Let us mention that the homogeneous limit in a point $P^- \in \mathcal{P}^-$ is obtain when the parameter ξ_n and κ_n satisfy $\xi_n = \xi$ and $\kappa_n = \kappa$, with $\xi^2 = 1$ and κ free.

- **A general model**. We will also consider a model which generalises the Fateev-Zamolodchikov model, parametrised as follow:

$$\forall n \in [\![1, N]\!], \quad \sigma_n = q^k \ , \quad x_{Q_n} = q^i \frac{y_{Q_n}}{K} \quad \text{and} \quad x_{R_n} = q^{2-i} \ K \ y_{R_n} \tag{5.197}$$

with y_{Q_n}, y_{R_n} and K free, and free integers k and i. It enters in the framework of point B) of proposition 5.1.4, where the constants parameters read $R = 0$ and K. The homogeneous limit in a point $P^- \in \mathcal{P}^-$ can be easily realised, the parameters y_Q and y_R have to satisfy $y_Q y_R = 1$.

Let us simply note that for the XXZ model, there is only one direction of derivation, ξ_a. For the other models, (x) can take two different values: ξ_a or κ_a for the sine Gordon model, and y_{Q_a} or y_{R_a} for the two others.

These models being now characterised, let us compute their local Hamiltonians and analyse some of their properties.

5.5.1 Symmetry of the boundary Hamiltonians

For all the four considered models, we check that when the coefficient $\alpha^{(x)}$ given by (5.183) is non zero, the following symmetry holds:

$$\mathcal{B}_N^{(x)}(\xi_+|P_C) = \left.\frac{dK_N(\xi_+|P_N|P_C)}{dx_N}\right|_{P_N=P^-} + cst \tag{5.198}$$

where *cst* is a scalar whose exact value is irrelevant, as the Hamiltonians are anyhow defined up to an additive constant.

Thus, the left and right boundaries have a symmetric expression on the two boundary sites 1 and N with just different parameters ξ_- and ξ_+.

The Hamiltonians are given (up to an additive constant) by:

$$\mathcal{H}^{(x)} = \sum_{k=1}^{N-1} H_{k,k+1}^{(x)} + \left.\frac{dK_1(\xi_-|P_1|P_C)}{dx_1}\right|_{P_1=P^-} + \left.\frac{dK_N(\xi_+|P_N|P_C)}{dx_N}\right|_{P_N=P^-} \tag{5.199}$$

This is a nice compact form of the boundaries terms, which should probably hold for generic values of p. For the boundary transfer matrix associated to the spin 1/2 XXZ spin chain, the symmetry with respect to λ going in $1/\lambda$ allows to show the same kind of symmetrical rewriting of the boundaries for the Hamiltonians (2.43).

Here, we have to investigate the symmetry of the fundamental transfer matrix with respect to the set of parameters P_a. A symmetry of $\mathcal{T}^{(fund)}(P_a|P_Q|P_C)$ may be responsible for the symmetry of the boundaries.

5.5.2 Reflection equations at a cyclic-cyclic level

For these particular models with a cyclic representation of dimension 3, we can check by direct computation that for all the considered models, the cyclic-cyclic reflection equation (5.142) and its dual (5.145) are satisfied, with the scalar:

$$c(P_a|P_b|P_C) = 1 \tag{5.200}$$

Thus for the considered models, it holds:

$$\begin{aligned}&S_{ba}^{\theta_b\theta_a}(P_b|P_a|P_C)K_a(\xi_-|P_a|P_C)S_{ab}^{\theta_b}(P_a|P_b|P_C)K_b(\xi_-|P_b|P_C)\\ &\quad= K_b(\xi_-|P_b|P_C)S_{ba}^{\theta_a}(P_b|P_a|P_C)K_a(\xi_-|P_a|P_C)S_{ab}(P_a|P_b|P_C)\end{aligned} \tag{5.201}$$

as well as

$$\begin{aligned}&(S_{ba}(P_b|P_a|P_C))^{t_at_b}\, K_{+a}(\xi_+|P_a|P_C)\left[\left(S_{ab}^{t_a}(P_a^{\theta_a}|P_b|P_C)\right)^{-1}\right]^{t_b} K_{+b}(\xi_+|P_b|P_C)\\ &\quad= K_{+b}(\xi_+|P_b|P_C)\left[\left(S_{ab}^{t_a}(P_a^{\theta_a}|P_b|P_C)\right)^{-1}\right]^{t_b} K_{+a}(\xi_+|P_a|P_C)\,(S_{ba}(P_b|P_a|P_C))^{t_at_b}\end{aligned} \tag{5.202}$$

As already mentioned, these equations allow to show the commutativity of the fundamental transfer matrix between themselves by an elegant and simple algebraic proof, using the framework of the (ABCD)-quantum type algebras. The needed compatibility conditions are satisfied as we can check the Yang-Baxter equation for the fundamental R-matrix:

$$S_{ab}(P_a|P_b|P_C)S_{ac}(P_a|P_c|P_C)S_{bc}(P_b|P_c|P_C) = S_{bc}(P_b|P_c|P_C)S_{ac}(P_a|P_b|P_C)S_{ab}(P_a|P_b|P_C) \tag{5.203}$$

We expect that these results hold for every dimension p of the representation, and for every scalar solution (not only diagonal) for the boundaries.

In the two last paragraphs of this chapter, we explicitly study the Hamiltonians of the four considered models. To that end, we first give the explicit expressions of the bulk Hamiltonians for the quasi-periodic models, i.e. associated to the Yang-Baxter algebra, in 5.5.3. Then in 5.5.4 we give the expressions of the boundary matrices, and see how the bulk interactions in the reflection algebra case are modified compared with the Yang-Baxter algebra framework.

A part the XXZ model which is considered as a verification of our approach, we are generating new local Hamiltonians, both for the quasi-periodic conditions and for integrable boundary conditions.

5.5.3 On the bulk Hamiltonians associated to the Yang-Baxter algebra

Following the standard procedure presented in chapter 1 (trace identity), we easily show that the Hamiltonians:

$$H^{(x)} \equiv \sum_{k=1}^{N} h_{k,k+1}^{(x)} \tag{5.204}$$

where we recall that $h_{k,k+1}^{(x)}$ is defined in (5.161), are commuting with the bulk transfer matrix:

$$\mathcal{T}(P_a|P_Q|P_C) = tr_a\left\{S_{aN}(P_a|P_N|P_C)...S_{a1}(P_a|P_1|P_C)\right\} \tag{5.205}$$

Then, for each of the four models previously described, we give the explicit bulk Hamiltonians $h^{(x)}_{k,k+1}$ acting on $V_k \otimes V_{k+1}$ of this quasi-periodic chain, in terms of the generators of the Weyl algebra.

The XXZ spin 1 model at root of unity

Computing the derivative in (5.161) for the set of parameters (5.192), the bulk Hamiltonians have the following expression in terms of the generators $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ of the Weyl algebra:

$$h^{(\xi)}_{k,k+1} = \Gamma_0 \sum_{\alpha_k=0}^{2} \sum_{\beta_k=0}^{2} \sum_{\alpha_{k+1}=0}^{2} \sum_{\beta_{k+1}=0}^{2} h(\alpha_k, \beta_k, \alpha_{k+1}, \beta_{k+1})\ \hat{\mathbf{u}}_k^{\alpha_k}\ \hat{\mathbf{v}}_k^{\beta_k}\ \hat{\mathbf{u}}_{k+1}^{\alpha_{k+1}}\ \hat{\mathbf{v}}_{k+1}^{\beta_{k+1}} \tag{5.206}$$

In this formula, the parameter Γ_0 denotes a global normalisation and $h(0,0,0,0)$ is free, the Hamiltonian being defined up to an additive constant. All the coefficients $h(\alpha_k, \beta_k, \alpha_{k+1}, \beta_{k+1})$ are zero, except:

$$\begin{aligned} h(0,0,0,1) = -1/2 \ \ ; \ \ h(0,0,0,2) = -q/2 \ \ ; \ \ h(0,1,0,0) = -1/2 \ \ ; \ \ h(0,2,0,0) = -q/2 \\ h(1,0,2,0) = -q^2 \ \ ; \ \ h(1,0,2,1) = q^2 \ \ ; \ \ h(1,0,2,2) = q^2 \ \ ; \ \ h(1,1,2,0) = q \\ h(1,2,2,0) = 1 \ \ ; \ \ h(2,0,1,0) = -q^2 \ \ ; \ \ h(2,0,1,1) = q \ \ ; \ \ h(2,0,1,2) = 1 \\ h(2,1,1,0) = q^2 \ \ ; \ \ h(2,2,1,0) = q^2 \end{aligned} \tag{5.207}$$

As mentioned before, we can check that this Hamiltonian is indeed the bulk one expected of the XXZ spin 1 chain. This spin 1 Hamiltonian appeared with the work of Fateev and Zamolodchikov [259], it has the following expression:

$$\begin{aligned} h^{XXZ}_{k,k+1} = \tau_{k,k+1} - \tau^2_{k,k+1} + 2sh(\eta)^2 \left(\tau^z_{k,k+1} - (\tau^z_{k,k+1})^2 + (S^z_k)^2 + (S^z_{k+1})^2\right) \\ - 4sh^2(\eta/2)\left(\tau^\perp_{k,k+1}\tau^z_{k,k+1} + \tau^z_{k,k+1}\tau^\perp_{k,k+1}\right) \end{aligned} \tag{5.208}$$

where:

$$\tau_{k,k+1} = \overrightarrow{S}_k.\overrightarrow{S}_{k+1} \ \ ; \ \ \tau^\perp_{k,k+1} = S^x_k S^x_{k+1} + S^y_k S^y_{k+1} \ \text{ and } \ \tau^z_{k,k+1} = S^z_k S^z_{k+1} \tag{5.209}$$

and $\overrightarrow{S}$ are the standard spin 1 generators of $su(2)$. Here, the parameter η is related to the anisotropy of the chain *via* $\Delta = ch(\eta)$, and linked to our notation thanks to $q = e^\eta$. For the particular choice $\Gamma_0 = -q/3$ for the normalisation, and for q a 3^{rd}-root of unity, it holds:

$$h^{XXZ}_{k,k+1} = P_k P_{k+1}.h^{(\xi)}_{k,k+1}.P_k^{-1} P_{k+1}^{-1} + cst \tag{5.210}$$

for the representation (5.191) of the generators $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$.

The super integrable chiral Potts model

When the parameters satisfy (5.194), the Hamiltonians, for example in the case $i = 1$, have the following expressions:

$$h^{(y_Q)}_{k,k+1} = \Gamma_0^{(Q)} \sum_{\alpha_k=0}^{2} \sum_{\beta_k=0}^{2} \sum_{\alpha_{k+1}=0}^{2} \sum_{\beta_{k+1}=0}^{2} h^{(Q)}(\alpha_k, \beta_k, \alpha_{k+1}, \beta_{k+1})\ \hat{\mathbf{u}}_k^{\alpha_k}\ \hat{\mathbf{v}}_k^{\beta_k}\ \hat{\mathbf{u}}_{k+1}^{\alpha_{k+1}}\ \hat{\mathbf{v}}_{k+1}^{\beta_{k+1}} \tag{5.211}$$

$$h^{(y_R)}_{k,k+1} = \Gamma_0^{(R)} \sum_{\alpha_k=0}^{2} \sum_{\beta_k=0}^{2} \sum_{\alpha_{k+1}=0}^{2} \sum_{\beta_{k+1}=0}^{2} h^{(R)}(\alpha_k, \beta_k, \alpha_{k+1}, \beta_{k+1})\ \hat{\mathbf{u}}_k^{\alpha_k}\ \hat{\mathbf{v}}_k^{\beta_k}\ \hat{\mathbf{u}}_{k+1}^{\alpha_{k+1}}\ \hat{\mathbf{v}}_{k+1}^{\beta_{k+1}} \tag{5.212}$$

The parameters $\Gamma_0^{(Q)}$ and $\Gamma_0^{(R)}$ denote a global normalisation, $h^{(Q)}(0,0,0,0)$ and $h^{(R)}(0,0,0,0)$ are free, and all the $h^{(Q)}(\alpha_k,\beta_k,\alpha_{k+1},\beta_{k+1})$ and $h^{(R)}(\alpha_k,\beta_k,\alpha_{k+1},\beta_{k+1})$ are zero, except:

$$\begin{aligned} h^{(Q)}(0,0,0,1) = -2 \ ; \ h^{(Q)}(0,0,0,2) = 1 \ ; \ h^{(Q)}(0,1,0,0) = 1 \ ; \ h^{(Q)}(0,2,0,0) = -2 \\ h^{(Q)}(1,0,2,0) = -1 \ ; \ h^{(Q)}(1,0,2,1) = q^2 \ ; \ h^{(Q)}(1,0,2,2) = q \ ; \ h^{(Q)}(1,1,2,0) = q \\ h^{(Q)}(1,2,2,0) = q^2 \ ; \ h^{(Q)}(2,0,1,0) = -1 \ ; \ h^{(Q)}(2,0,1,1) = q \ ; \ h^{(Q)}(2,0,1,2) = q^2 \\ h^{(Q)}(2,1,1,0) = q^2 \ ; \ h^{(Q)}(2,2,1,0) = q \end{aligned} \tag{5.213}$$

and

$$\begin{aligned} h^{(R)}(0,0,0,1) = 1 \ ; \ h^{(R)}(0,0,0,2) = -2 \ ; \ h^{(R)}(0,1,0,0) = -2 \ ; \ h^{(R)}(0,2,0,0) = 1 \\ h^{(R)}(1,0,2,0) = -1 \ ; \ h^{(R)}(1,0,2,1) = q^2 \ ; \ h^{(R)}(1,0,2,2) = q \ ; \ h^{(R)}(1,1,2,0) = q \\ h^{(R)}(1,2,2,0) = q^2 \ ; \ h^{(R)}(2,0,1,0) = -1 \ ; \ h^{(R)}(2,0,1,1) = q \ ; \ h^{(R)}(2,0,1,2) = q^2 \\ h^{(R)}(2,1,1,0) = q^2 \ ; \ h^{(R)}(2,2,1,0) = q \end{aligned} \tag{5.214}$$

Let us mention that these Hamiltonians are different from the Hamiltonians introduced by Baxter [260] and then generalised by Tarasov [160, 161], for the so-called chiral Potts model with fixed spin boundary conditions. Indeed, they could give an expression for the Hamiltonian of the chiral Potts model under a special boundary condition, for which only the operator $A(\lambda)$ of the Lax matrix is used to construct the Hamiltonian. Moreover, this Hamiltonian is obtained from a completely different method than the one presented here, as it is defined as the coefficient in λ^{N-1} of the operator $A(\lambda)$ (for a chain composed of N sites). Our Hamiltonian is obtained from the full transfer matrix.

The Fateev-Zamolodchikov model

For the parameters describing the Fateev-Zamolodchikov model, i.e. when they satisfy (5.195), we can also have two different directions of derivation.

- In the direction ξ_a, the Hamiltonian is given by:

$$h^{(\xi)}_{k,k+1} = \Gamma_0 \sum_{\alpha_k=0}^{2}\sum_{\beta_k=0}^{2}\sum_{\alpha_{k+1}=0}^{2}\sum_{\beta_{k+1}=0}^{2} h(\alpha_k,\beta_k,\alpha_{k+1},\beta_{k+1})\ \hat{\mathbf{u}}_k^{\alpha_k}\ \hat{\mathbf{v}}_k^{\beta_k}\ \hat{\mathbf{u}}_{k+1}^{\alpha_{k+1}}\ \hat{\mathbf{v}}_{k+1}^{\beta_{k+1}} \tag{5.215}$$

  The parameter Γ_0 denotes a global normalisation, $h(0,0,0,0)$ is once again free and all the $h(\alpha_k,\beta_k,\alpha_{k+1},\beta_{k+1})$ are zero, except:

$$\begin{aligned} h(0,0,0,1) = -q^2\kappa^2 \ ; \ h(0,0,0,2) = -q^2\kappa^2 \ ; \ h(0,1,0,0) = -q^2\kappa^2 \\ h(0,2,0,0) = -q^2\kappa^2 \ ; \ h(1,0,2,0) = -2q^2\kappa^2 \ ; \ h(1,0,2,1) = -q\kappa^2(\kappa^2-1) \\ h(1,0,2,2) = \kappa^2-1 \ ; \ h(1,1,2,0) = \kappa^2-1 \ ; \ h(1,2,2,0) = -q\kappa^2(\kappa^2-1) \\ h(2,0,1,0) = -2q^2\kappa^2 \ ; \ h(2,0,1,1) = \kappa^2-1 \ ; \ h(2,0,1,2) = -q\kappa^2(\kappa^2-1) \\ h(2,1,1,0) = -q\kappa^2(\kappa^2-1) \ ; \ h(2,2,1,0) = \kappa^2-1 \end{aligned} \tag{5.216}$$

- In the direction κ_a, the Hamiltonian is given by:

$$h^{(\kappa)}_{k,k+1} = \tilde{\Gamma}_0 \sum_{\alpha_k=0}^{2}\sum_{\beta_k=0}^{2}\sum_{\alpha_{k+1}=0}^{2}\sum_{\beta_{k+1}=0}^{2} h^{(\kappa)}(\alpha_k,\beta_k,\alpha_{k+1},\beta_{k+1})\ \hat{\mathbf{u}}_k^{\alpha_k}\ \hat{\mathbf{v}}_k^{\beta_k}\ \hat{\mathbf{u}}_{k+1}^{\alpha_{k+1}}\ \hat{\mathbf{v}}_{k+1}^{\beta_{k+1}} \tag{5.217}$$

  The parameter $\tilde{\Gamma}_0$ stands for a global normalisation, $h^{(\kappa)}(0,0,0,0)$ is free and all the

$h^{(\kappa)}(\alpha_k, \beta_k, \alpha_{k+1}, \beta_{k+1})$ are zero, except:

$$\begin{aligned} h^{(\kappa)}(0,0,0,1) = -q\kappa \;\; ; \;\; h^{(\kappa)}(0,0,0,2) = -q\kappa \;\; ; \;\; h^{(\kappa)}(0,1,0,0) = q\kappa \\ h^{(\kappa)}(0,2,0,0) = q\kappa \;\; ; \;\; h^{(\kappa)}(1,0,2,1) = -\kappa(\kappa^2-1) \;\; ; \;\; h^{(\kappa)}(1,0,2,2) = q^2(\kappa^2-1)/\kappa \\ h^{(\kappa)}(1,1,2,0) = -q^2(\kappa^2-1)/\kappa \;\; ; \;\; h^{(\kappa)}(1,2,2,0) = \kappa(\kappa^2-1) \;\; ; \;\; h^{(\kappa)}(2,0,1,1) = q^2(\kappa^2-1)/\kappa \\ h^{(\kappa)}(2,0,1,2) = -\kappa(\kappa^2-1) \;\; ; \;\; h^{(\kappa)}(2,1,1,0) = \kappa(\kappa^2-1) \;\; ; \;\; h^{(\kappa)}(2,2,1,0) = -q^2(\kappa^2-1)/\kappa \end{aligned} \tag{5.218}$$

The general model

Lastly, for the parameters satisfying (5.197) with $k=1$ and $i=2$ for instance, the bulk Hamiltonians have the following expressions:

$$h^{(y_Q)}_{k,k+1} = \Gamma_0^{(Q)} \sum_{\alpha_k=0}^{2} \sum_{\beta_k=0}^{2} \sum_{\alpha_{k+1}=0}^{2} \sum_{\beta_{k+1}=0}^{2} h^{(Q)}(\alpha_k, \beta_k, \alpha_{k+1}, \beta_{k+1}) \; \hat{\mathbf{u}}_k^{\alpha_k} \; \hat{\mathbf{v}}_k^{\beta_k} \; \hat{\mathbf{u}}_{k+1}^{\alpha_{k+1}} \; \hat{\mathbf{v}}_{k+1}^{\beta_{k+1}} \tag{5.219}$$

$$h^{(y_R)}_{k,k+1} = \Gamma_0^{(R)} \sum_{\alpha_k=0}^{2} \sum_{\beta_k=0}^{2} \sum_{\alpha_{k+1}=0}^{2} \sum_{\beta_{k+1}=0}^{2} h^{(R)}(\alpha_k, \beta_k, \alpha_{k+1}, \beta_{k+1}) \; \hat{\mathbf{u}}_k^{\alpha_k} \; \hat{\mathbf{v}}_k^{\beta_k} \; \hat{\mathbf{u}}_{k+1}^{\alpha_{k+1}} \; \hat{\mathbf{v}}_{k+1}^{\beta_{k+1}} \tag{5.220}$$

The parameters $\Gamma_0^{(Q)}$ and $\Gamma_0^{(R)}$ denote a global normalisation, $h^{(Q)}(0,0,0,0)$ and $h^{(R)}(0,0,0,0)$ are free and all the $h^{(Q)}(\alpha_k, \beta_k, \alpha_{k+1}, \beta_{k+1})$ and $h^{(R)}(\alpha_k, \beta_k, \alpha_{k+1}, \beta_{k+1})$ are zero, except:

$$\begin{aligned} h^{(Q)}(0,1,0,0) = q^2 \;\; ; \;\; h^{(Q)}(0,2,0,0) = 1 \;\; ; \;\; h^{(Q)}(1,0,2,0) = q \;\; ; \;\; h^{(Q)}(1,1,2,0) = -(1+q/G) \\ h^{(Q)}(1,2,2,0) = -q(G+q) \;\; ; \;\; h^{(Q)}(2,0,1,0) = q \;\; ; \;\; h^{(Q)}(2,1,1,0) = -(G+q) \\ h^{(Q)}(2,2,1,0) = -q(1+q/G) \end{aligned} \tag{5.221}$$

and

$$\begin{aligned} h^{(R)}(0,0,0,1) = q^2 \;\; ; \;\; h^{(R)}(0,0,0,2) = 1 \;\; ; \;\; h^{(R)}(1,0,2,0) = q \;\; ; \;\; h^{(R)}(1,0,2,1) = -\left(q+G\right) \\ h^{(R)}(1,0,2,2) = -\frac{q}{G}\left(q+G\right) \;\; ; \;\; h^{(R)}(2,0,1,0) = q \;\; ; \;\; h^{(R)}(2,0,1,1) = -\frac{1}{G}\left(q+G\right) \\ h^{(R)}(2,0,1,2) = -q\left(q+G\right) \end{aligned} \tag{5.222}$$

We denoted $G \equiv K/y_Q^2$ the re-scaled free parameter K.

5.5.4 On the bulk Hamiltonians associated to the reflection algebra

In this final paragraph we analyse the impact of the presence of integrable boundaries on the bulk Hamiltonians previously derived, i.e. the local interactions $H^{(x)}_{k,k+1}$ of the four models (5.192)-(5.197). We recall that it holds $H^{(x)}_{k,k+1} = h^{(x)}_{k,k+1} + h^{(x)}_{k+1,k}$ for the bulk interactions of the chain with boundaries. In a way we are thus considering a symmetrised version of the chain.

Using symmetries specific to each of the models, and to each direction x_a of derivation, we show that the behaviour of the bulk of the chain with integrable boundaries really depends on the considered model.

The XXZ spin 1 model at root of unity

When the parameters are tuned to describe the XXZ spin 1 chain, the fundamental R-matrix only depends on the ratio of the spectral parameters. It is of the form:

$$S_{k+1,k}\left(P_{k+1}|P_k|P_C\right) = S_{k+1,k}\left(\frac{\xi_{k+1}}{\xi_k}\right) \tag{5.223}$$

Moreover, we can check the symmetry:

$$S_{k+1,k}\left(\frac{\xi_{k+1}}{\xi_k}\right) = S_{k,k+1}\left(\frac{\xi_{k+1}}{\xi_k}\right) \tag{5.224}$$

This way, it holds:

$$h^{(\xi)}_{k,k+1} = h^{(\xi)}_{k+1,k} \quad \text{and thus} \quad \sum_{k=1}^{N-1} H^{(\xi)}_{k,k+1} = 2\sum_{k=1}^{N-1} h^{(\xi)}_{k,k+1} \tag{5.225}$$

We find thus that up to a factor 2, the bulk interactions of the chain with general integrable boundary conditions are the same as the bulk interactions of the periodic chain.

We can also explicitly see the relation $h^{(\xi)}_{k,k+1} = h^{(\xi)}_{k+1,k}$ as the coefficients (5.207) satisfy:

$$\forall(\alpha_k,\beta_k,\alpha_{k+1},\beta_{k+1}) \in [\![0,2]\!]^4\ , \quad h(\alpha_k,\beta_k,\alpha_{k+1},\beta_{k+1}) = h(\alpha_{k+1},\beta_{k+1},\alpha_k,\beta_k) \tag{5.226}$$

Let us write the diagonal boundary matrices for this model. Thanks to propositions 5.3.6 and 5.3.8, we know the coefficients of these matrices in the $\hat{\mathbf{v}}$-basis:

$$K_a(\xi_-|\xi_a|P_C) = \begin{pmatrix} \frac{1-q\xi_a^2\xi_-^2}{\xi_a^2-\xi_-^2}\frac{1-\xi_a^2\xi_-^2}{q\xi_a^2-\xi_-^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q\frac{1-q\xi_a^2\xi_-^2}{\xi_a^2-\xi_-^2} \end{pmatrix} \tag{5.227}$$

$$\text{and} \quad K_{+a}(\xi_+|\xi_a|P_C) = \begin{pmatrix} q^2\frac{q-\xi_a^2\xi_+^2}{\xi_a^2-\xi_+^2}\frac{1-\xi_a^2\xi_+^2}{q^2\xi_a^2-\xi_+^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q\frac{q-\xi_a^2\xi_+^2}{\xi_a^2-\xi_+^2} \end{pmatrix} \tag{5.228}$$

Thus, we can equivalently write them in terms of the generators. It holds:

$$K_a(\xi_-|\xi_a|P_C) = 1/N(\xi_a,\xi_-)\left[\alpha_0(\xi_a,\xi_-)\mathbb{1} + \alpha_1(\xi_a,\xi_-)\hat{\mathbf{v}}_a + \alpha_2(\xi_a,\xi_-)\hat{\mathbf{v}}_a^2\right] \tag{5.229}$$

The factor $N(\xi_a,\xi_-)$ is simply a normalisation, and the three coefficients $\alpha_i(\xi_a,\xi_-)$ are polynomials in the variables ξ_a and ξ_-. Explicitly, for the choice of normalisation (5.227), it holds:

$$N(\xi_a,\xi_-) = 3(\xi_a^2-\xi_-^2)(q\xi_a^2-\xi_-^2) \tag{5.230}$$

and the coefficients $\alpha_i(\xi_a,\xi_-)$ read:

$$\alpha_i(\xi_a,\xi_-) = \alpha_{i0}(\xi_-) + \alpha_{i2}(\xi_-)\xi_a^2 + \alpha_{i4}(\xi_-)\xi_a^4 \tag{5.231}$$

The polynomials $\alpha_{ij}(\xi_-)$ are given by:

$$\alpha_{00}(\xi_-) = 1 - q\xi_-^2 + \xi_-^4 \ \ ; \ \ \alpha_{02}(\xi_-) = q^2 + 2q^2\xi_-^2 + q^2\xi_-^4 \ \ ; \ \ \alpha_{04}(\xi_-) = q - \xi_-^2 + q\xi_-^4 \tag{5.232}$$
$$\alpha_{10}(\xi_-) = 1 - q^2\xi_-^2 + q^2\xi_-^4 \ \ ; \ \ \alpha_{12}(\xi_-) = 1 - \xi_-^2 + \xi_-^4 \ \ ; \ \ \alpha_{14}(\xi_-) = 1 - q\xi_-^2 + q\xi_-^4 \tag{5.233}$$
$$\alpha_{20}(\xi_-) = 1 - \xi_-^2 + q\xi_-^4 \ \ ; \ \ \alpha_{22}(\xi_-) = q - q\xi_-^2 + q\xi_-^4 \ \ ; \ \ \alpha_{24}(\xi_-) = q^2 - q^2\xi_-^2 + q\xi_-^4 \tag{5.234}$$

Exploiting the symmetry (5.131), we can state:

$$K_{+a}(\xi_+|\xi_a|P_C) = K_a(1/\xi_+|1/\xi_a|P_C) \tag{5.235}$$

Thus, the integrable boundaries on the sites $j = 1$ and $j = N$ explicitly read:

$$\mathcal{B}^{(\xi)}_j(\xi_\pm|P_C) = \left.\frac{dK_j(\xi_\pm|\xi_j|P_C)}{d\xi_j}\right|_{\xi_j=q^{1/2}} = 1/\bar{N}(\xi_\pm)\left[\gamma_0(\xi_\pm)\mathbb{1} + \gamma_1(\xi_\pm)\hat{\mathbf{v}}_a + \gamma_2(\xi_\pm)\hat{\mathbf{v}}_a^2\right] \tag{5.236}$$

where the coefficients $\gamma_i(\xi)$ read:

$$\gamma_0(\xi) = 3q^2 + (q-1)\xi^2 - 3q^2\xi^4 \;\; ; \;\; \gamma_1(\xi) = -q(q-1)(1+\xi^2-\xi^4) \;\; ; \;\; \gamma_2(\xi) = q^2(q-1)(1-\xi^2-\xi^4) \quad (5.237)$$

and the normalisation $\bar{N}(\xi)$ is given by:

$$\bar{N}(\xi) = -3/2\; q(1+\xi^2+\xi^4) \quad (5.238)$$

As mentioned before, this model serves as a tool for the verification of the good working of our approach. Indeed, we find back the known results for the integrable boundaries of the XXZ spin 1 chain at root of unity. For example, the boundary matrices $K_a(\xi_-|\xi_a|P_C)$ and $K_{+a}(\xi_+|\xi_a|P_C)$ are effectively the diagonal matrices reduction of the more general ones derived in [197]: up to an additive constant, the diagonal boundary term H_b of [197] on site 1, that we will note $H_{bound,1}(\xi_-)$ is indeed recovered *via*:

$$P_1.\mathcal{B}_1^{(\xi)}(\xi_-|P_C).P_1^{-1} = \gamma\; H_{bound,1}(\xi_-) \quad (5.239)$$

when we use the representation (5.191) for the generators $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$. (γ is a numerical factor)

The super integrable chiral Potts model (at $p=3$)

For the super integrable chiral Potts model, the fundamental R-matrix is of the form

$$S_{k+1,k}\,(P_{k+1}|P_k|P_C) = S_{k+1,k}(\frac{y_{Q_{k+1}}}{y_{Q_k}}, \frac{y_{R_{k+1}}}{y_{R_k}}) \quad (5.240)$$

When taking the derivatives with respect to y_{Q_k} or y_{R_k}, the Hamiltonians satisfy:

$$h_{k,k+1}^{(y_Q)} = h_{k+1,k}^{(y_Q)} + \gamma_{k+1}^{(y_Q)} - \gamma_k^{(y_Q)} \quad (5.241)$$

$$\text{and} \quad h_{k,k+1}^{(y_R)} = h_{k+1,k}^{(y_R)} + \gamma_{k+1}^{(y_R)} - \gamma_k^{(y_R)} \quad (5.242)$$

where $\gamma_i^{(y_Q)}$ and $\gamma_i^{(y_R)}$ are matrices acting only in space i.

These symmetries come from symmetries on the fundamental R-matrix:

$$\forall x \in \mathbb{C}\;, \quad S_{k+1,k}(x,1) = S_{k,k+1}(x,1) + \mathcal{P}_{k,k+1}\left(\Gamma_{k+1}^{(y_Q)}(x) - \Gamma_k^{(y_Q)}(x) + \mathcal{M}_{k+1,k}^{(y_Q)}(x)\right) \quad (5.243)$$

$$\forall x \in \mathbb{C}\;, \quad S_{k+1,k}(1,x) = S_{k,k+1}(1,x) + \mathcal{P}_{k,k+1}\left(\Gamma_{k+1}^{(y_R)}(x) - \Gamma_k^{(y_R)}(x) + \mathcal{M}_{k+1,k}^{(y_R)}(x)\right) \quad (5.244)$$

where $\Gamma_i^{(y_Q)}$ and $\Gamma_i^{(y_R)}$ are matrices acting only in space V_i, and where matrices $\mathcal{M}_{k+1,k}^{(y_Q)}$ and $\mathcal{M}_{k+1,}^{(y_R)}$ satisfy:

$$\left.\frac{d\mathcal{M}_{k+1,k}^{(y_Q)}(x)}{dx}\right|_{x=1} = 0 \quad \text{and} \quad \left.\frac{d\mathcal{M}_{k+1,k}^{(y_R)}(x)}{dx}\right|_{x=1} = 0 \quad (5.245)$$

Then it holds:

$$\sum_{k=1}^{N-1} H_{k,k+1}^{(y_Q)} = 2\sum_{k=1}^{N-1} h_{k,k+1}^{(y_Q)} + \gamma_1^{(y_Q)} - \gamma_N^{(y_Q)} \quad \text{and} \quad \sum_{k=1}^{N-1} H_{k,k+1}^{(y_R)} = 2\sum_{k=1}^{N-1} h_{k,k+1}^{(y_R)} + \gamma_1^{(y_R)} - \gamma_N^{(y_R)} \quad (5.246)$$

This way, as previously, the bulk interactions for the chain with general integrable boundary conditions are the same as for the quasi-periodic chain up to a factor 2, but moreover here there is an extra modification of the boundaries via the matrices γ_1 and γ_N.

One can read the matrices γ from the coefficients (5.213)-(5.214):

$$\gamma^{(y_Q)} = \Gamma_0^{(Q)}\left(\hat{\mathbf{v}} - \hat{\mathbf{v}}^2\right) + cst \quad \text{and} \quad \gamma^{(y_R)} = \Gamma_0^{(R)}\left(\hat{\mathbf{v}}^2 - \hat{\mathbf{v}}\right) + cst' \quad (5.247)$$

(they are defined up to additive constants cst and cst')

In that case, the boundary matrices (in the $\hat{\mathbf{v}}$-basis) are given by:

$$K_a(\xi_-|y_{Q_a},y_{R_a}|P_C) = \begin{pmatrix} \frac{y_{Q_a}y_{R_a}-q^2\xi_-^2}{1-q^2y_{Q_a}y_{R_a}\xi_-^2} & 0 & 0 \\ 0 & \frac{(q^2y_{Q_a}y_{R_a}-\xi_-^2)(y_{Q_a}y_{R_a}-q^2\xi_-^2)}{(q^2-y_{Q_a}y_{R_a}\xi_-^2)(1-q^2y_{Q_a}y_{R_a}\xi_-^2)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.248}$$

$$\text{and} \quad K_{+a}(\xi_+|y_{Q_a},y_{R_a}|P_C) = \begin{pmatrix} \frac{y_{Q_a}y_{R_a}-\xi_+^2}{q^2-qy_{Q_a}y_{R_a}\xi_+^2} & 0 & 0 \\ 0 & \frac{(y_{Q_a}y_{R_a}-\xi_+^2)(y_{Q_a}y_{R_a}-q\xi_+^2)}{(1-y_{Q_a}y_{R_a}\xi_+^2)(q-y_{Q_a}y_{R_a}\xi_+^2)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.249}$$

We can equivalently write them in terms of the generators. It holds:

$$K_a(\xi_-|y_{Q_a},y_{R_a}|P_C) = 1/N(y_{Q_a},y_{R_a},\xi_-)\left[\alpha_0(y_{Q_a},y_{R_a},\xi_-)\mathbb{1} + \alpha_1(y_{Q_a},y_{R_a},\xi_-)\hat{\mathbf{v}}_a + \alpha_2(y_{Q_a},y_{R_a},\xi_-)\hat{\mathbf{v}}_a^2\right] \tag{5.250}$$

The factor $N(y_{Q_a},y_{R_a},\xi_-)$ is simply a normalisation, and the three coefficients $\alpha_i(y_{Q_a},y_{R_a},\xi_-)$ are polynomials in the variables $y_{Q_a}y_{R_a}$ and ξ_-. Explicitly, for the choice of normalisation (5.248) , it holds:

$$N(y_{Q_a},y_{R_a},\xi_-) = 3(1 + y_{Q_a}y_{R_a}\xi_-^2 + y_{Q_a}y_{R_a}\xi_-^4) \tag{5.251}$$

and the coefficients $\alpha_i(y_{Q_a},y_{R_a},\xi_-)$ read:

$$\alpha_i(y_{Q_a},y_{R_a},\xi_-) = \alpha_{i0}(\xi_-) + \alpha_{i1}(\xi_-)y_{Q_a}y_{R_a} + \alpha_{i2}(\xi_-)(y_{Q_a}y_{R_a})^2 \tag{5.252}$$

The polynomials $\alpha_{ij}(\xi_-)$ are given by:

$$\alpha_{00}(\xi_-) = 1 - q^2\xi_-^2 + \xi_-^4 \quad ; \quad \alpha_{01}(\xi_-) = 1 + 2\xi_-^2 + \xi_-^4 \quad ; \quad \alpha_{02}(\xi_-) = 1 - q\xi_-^2 + \xi_-^4 \tag{5.253}$$

$$\alpha_{10}(\xi_-) = q - q^2\xi_-^2 + q^2\xi_-^4 \quad ; \quad \alpha_{11}(\xi_-) = 1 - \xi_-^2 + \xi_-^4 \quad ; \quad \alpha_{12}(\xi_-) = q^2 - q\xi_-^2 + q\xi_-^4 \tag{5.254}$$

$$\alpha_{20}(\xi_-) = q^2 - q^2\xi_-^2 + q\xi_-^4 \quad ; \quad \alpha_{21}(\xi_-) = 1 - \xi_-^2 + \xi_-^4 \quad ; \quad \alpha_{22}(\xi_-) = q - q\xi_-^2 + q^2\xi_-^4 \tag{5.255}$$

Exploiting the symmetry (5.131), we can state:

$$K_{+a}(\xi_+|y_{Q_a},y_{R_a}|P_C) = K_a(1/\xi_+|q/y_{R_a},q/y_{Q_a}|P_C) \tag{5.256}$$

Thus, the integrable boundaries on the sites $j=1$ and $j=N$ explicitly read:

$$\begin{aligned} \mathcal{B}_j^{(y_Q)}(\xi_\pm|P_C) &= \left.\frac{dK_j(\xi_\pm|y_{Q_j},y_{R_j}|P_C)}{dy_{Q_j}}\right|_{y_{Q_j}y_{R_j}=y_Qy_R=1} \qquad (5.257) \\ &= y_R/\bar{N}(y_{Q_a},y_{R_a},\xi_\pm)\left[\gamma_0(y_{Q_a},y_{R_a},\xi_\pm)\mathbb{1} + \gamma_1(y_{Q_a},y_{R_a},\xi_\pm)\hat{\mathbf{v}}_a + \gamma_2(y_{Q_a},y_{R_a},\xi_\pm)\hat{\mathbf{v}}_a^2\right] \end{aligned}$$

The normalisation $\bar{N}(y_{Q_a},y_{R_a},\xi)$ is given by:

$$\bar{N}(y_{Q_a},y_{R_a},\xi) = 3(1+\xi^2+\xi^4) \tag{5.258}$$

while the three other coefficients are written:

$$\gamma_0(y_{Q_a},y_{R_a},\xi) = 3 + q(q-1)\xi^2 - 3\xi^4 \quad ; \quad \gamma_1(y_{Q_a},y_{R_a},\xi) = q(q-1)(1+\xi^2-\xi^4) \tag{5.259}$$

$$\gamma_2(y_{Q_a},y_{R_a},\xi) = -q(q-1)(1-\xi^2-\xi^4) \tag{5.260}$$

As the boundary matrices depend on the product $y_{Q_a}y_{R_a}$, then it follows:

$$\mathcal{B}_j^{(y_R)}(\xi_\pm|P_C) = \left.\frac{dK_j(\xi_\pm|y_{Q_j},y_{R_j}|P_C)}{dy_{R_j}}\right|_{y_{Q_j}y_{R_j}=y_Qy_R=1} = 1/y_R^2\ \mathcal{B}_j^{(y_Q)}(\xi_\pm|P_C) \tag{5.261}$$

The Fateev-Zamolodchikov model

As far as the Fateev-Zamolodchikov model is concerned, the fundamental R-matrix is written:

$$S_{k+1,k}\left(P_{k+1}|P_k|P_C\right) = S_{k+1,k}\left(\frac{\xi_{k+1}}{\xi_k}, \kappa_{k+1}, \kappa_k\right) \tag{5.262}$$

The boundary matrices are:

$$K_a(\xi_-|\xi_a,\kappa_a|P_C) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{q^2-\xi_a^2\xi_-^2}{q^2\xi_a^2-\xi_-^2} & 0 \\ 0 & 0 & \frac{(-1+\xi_a^2\xi_-^2)(-q^2+\xi_a^2\xi_-^2)}{(\xi_a^2-\xi_-^2)(q^2\xi_a^2-\xi_-^2)} \end{pmatrix} \tag{5.263}$$

$$\text{and} \quad K_{+a}(\xi_+|\xi_a,\kappa_a|P_C) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1-q\xi_a^2\xi_+^2}{q^2(\xi_a^2-\xi_+^2)} & 0 \\ 0 & 0 & \frac{(-q+\xi_a^2\xi_+^2)(-1+q\xi_a^2\xi_+^2)}{(\xi_a^2-\xi_+^2)(q^2\xi_a^2-\xi_+^2)} \end{pmatrix} \tag{5.264}$$

In order for the mixed reflection equation and its dual to be satisfied, it must hold $\xi_-^{2p} = 1$ and $\xi_+^{2p} = 1$. We can equivalently write them in terms of the generators. It holds:

$$K_a(\xi_-|\xi_a,\kappa_a|P_C) = 1/N(\xi_a,\xi_-)\left[\alpha_0(\xi_a,\xi_-)\mathbb{1} + \alpha_1(\xi_a,\xi_-)\hat{\mathbf{v}}_a + \alpha_2(\xi_a,\xi_-)\hat{\mathbf{v}}_a^2\right] \tag{5.265}$$

The factor $N(\xi_a,\xi_-)$ is simply a normalisation, and the three coefficients $\alpha_i(\xi_a,\xi_-)$ are polynomials in the variables ξ_a and ξ_-. Explicitly, for the choice of normalisation (5.263) , it holds:

$$N(\xi_a,\xi_-) = 3(\xi_a^2-\xi_-^2)(q\xi_a^2-\xi_-^2) \tag{5.266}$$

and the coefficients $\alpha_i(\xi_a,\xi_-)$ read:

$$\alpha_i(\xi_a,\xi_-) = \alpha_{i0}(\xi_-) + \alpha_{i2}(\xi_-)\xi_a^2 + \alpha_{i4}(\xi_-)\xi_a^4 \tag{5.267}$$

The polynomials $\alpha_{ij}(\xi_-)$ are given by:

$$\alpha_{00}(\xi_-) = 1-\xi_-^2+q\xi_-^4 \quad ; \quad \alpha_{02}(\xi_-) = 1+2q^2\xi_-^2+q\xi_-^4 \quad ; \quad \alpha_{04}(\xi_-) = 1-q\xi_-^2+q\xi_-^4 \tag{5.268}$$

$$\alpha_{10}(\xi_-) = q-q^2\xi_-^2+q\xi_-^4 \quad ; \quad \alpha_{12}(\xi_-) = q^2-q\xi_-^2+\xi_-^4 \quad ; \quad \alpha_{14}(\xi_-) = 1-\xi_-^2+q^2\xi_-^4 \tag{5.269}$$

$$\alpha_{20}(\xi_-) = q^2-q\xi_-^2+q\xi_-^4 \quad ; \quad \alpha_{22}(\xi_-) = q-\xi_-^2+q^2\xi_-^4 \quad ; \quad \alpha_{24}(\xi_-) = 1-q^2\xi_-^2+\xi_-^4 \tag{5.270}$$

Exploiting the symmetry (5.131), we can state:

$$K_{+a}(\xi_+|\xi_a,\kappa_a|P_C) = K_a(1/\xi_+|q^2/\xi_a,\kappa_a|P_C) \tag{5.271}$$

For this model, the two directions of derivation lead to different behaviours.

- In the direction ξ_a, we use the next symmetry which is similar to the one of the XXZ model:

$$S_{k+1,k}\left(\frac{\xi_{k+1}}{\xi_k},\kappa,\kappa\right) = S_{k,k+1}\left(\frac{\xi_{k+1}}{\xi_k},\kappa,\kappa\right) \tag{5.272}$$

  Then, it holds

$$h^{(\xi)}_{k,k+1} = h^{(\xi)}_{k+1,k} \quad \text{leading to} \quad \sum_{k=1}^{N-1} H^{(\xi)}_{k,k+1} = 2\sum_{k=1}^{N-1} h^{(\xi)}_{k,k+1} \tag{5.273}$$

We explicitly see the relation $h^{(\xi)}_{k,k+1} = h^{(\xi)}_{k+1,k}$ as the coefficients satisfy:

$$\forall(\alpha_k, \beta_k, \alpha_{k+1}, \beta_{k+1}) \in [\![0, 2]\!]^4 \ , \quad h(\alpha_k, \beta_k, \alpha_{k+1}, \beta_{k+1}) = h(\alpha_{k+1}, \beta_{k+1}, \alpha_k, \beta_k) \tag{5.274}$$

As far as the integrable boundaries are concerned, it holds:

$$\mathcal{B}^{(\xi)}_j(\xi_\pm|P_C) = \left.\frac{dK_j(\xi_\pm|\xi_j, \kappa_j|P_C)}{d\xi_j}\right|_{\xi_j=1;\ \kappa_j=\kappa} = 1/\bar{N}(\xi_\pm)\left[\gamma_0(\xi_\pm)\mathbb{1} + \gamma_1(\xi_\pm)\hat{\mathbf{v}}_a + \gamma_2(\xi_\pm)\hat{\mathbf{v}}_a^2\right] \tag{5.275}$$

where the coefficients $\gamma_i(\xi)$ read:

$$\gamma_0(\xi) = 3q^2+q^2(q-1)\xi^2-3\xi^4 \ ; \ \gamma_1(\xi) = (q-1)(q^2+q\xi^2-\xi^4) \ ; \ \gamma_2(\xi) = (q-1)(q-\xi^2-q^2\xi^4) \tag{5.276}$$

and the normalisation $\bar{N}(\xi)$ is given by:

$$\bar{N}(\xi) = 3/2(\xi^2 - 1)(q^2 - \xi^2) \tag{5.277}$$

- In the direction κ_a however, we have $\alpha^{(\kappa)} = 0$. So the formula (5.187) does not apply.

Thus, one may try to compute for example a higher derivative of the fundamental transfer matrix to get an expression. However, let us just remark here that the symmetry:

$$S_{k+1,k}(1, \kappa_{k+1}, \kappa_k) = S_{k,k+1}(1, \kappa_k, \kappa_{k+1}) \tag{5.278}$$

and the expression:

$$S_{k+1,k}(1, \kappa_{k+1}, \kappa_k) = \mathcal{P}_{k,k+1} + \tilde{S}_{k+1,k}(\frac{\kappa_{k+1}}{\kappa_k}) + (\kappa_{k+1} - \kappa_k)\check{S}_{k+1,k}(\kappa_{k+1}, \kappa_k) \tag{5.279}$$

where $\tilde{S}$ and $\check{S}$ are regular functions, allow us to write:

$$h^{(\kappa)}_{k,k+1} = -h^{(\kappa)}_{k+1,k} \tag{5.280}$$

This way, even if in the case of periodic boundary conditions the κ_a direction of derivation generates a non trivial local hamiltonian, the previous symmetry implies that the integrable boundary conditions lead to a trivial local hamiltonian in this direction:

$$\sum_{k=1}^{N-1} H^{(\kappa)}_{k,k+1} = 0 \tag{5.281}$$

One can moreover check that the $K_j(\xi|P_a|P_C)$ matrix does not depend on κ:

$$\left.\frac{dK_j(\xi_\pm|\xi_j, \kappa_j|P_C)}{d\kappa_j}\right|_{\xi_j=1;\ \kappa_j=\kappa} = 0 \tag{5.282}$$

In the direction κ_a, the reflection makes to vanish all the interactions.

The general model

For the general model, let us consider the parameters (5.197) with $k = 1$ and $i = 2$. The boundary matrices have the following form:

$$K_a(\xi_-|y_{Q_a}, y_{R_a}|P_C) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{y_{Q_a}y_{R_a}-\xi_-^2}{1-y_{Q_a}y_{R_a}\xi_-^2} & 0 \\ 0 & 0 & \frac{(y_{Q_a}y_{R_a}-\xi_-^2)(qy_{Q_a}y_{R_a}-\xi_-^2)}{(1-y_{Q_a}y_{R_a}\xi_-^2)(q-y_{Q_a}y_{R_a}\xi_-^2)} \end{pmatrix} \tag{5.283}$$

$$\text{and} \quad K_{+a}(\xi_+|y_{Q_a}, y_{R_a}|P_C) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{y_{Q_a}y_{R_a}-q^2\xi_+^2}{q^2-y_{Q_a}y_{R_a}\xi_+^2} & 0 \\ 0 & 0 & \frac{(y_{Q_a}y_{R_a}-\xi_+^2)(y_{Q_a}y_{R_a}-q^2\xi_+^2)}{(q-y_{Q_a}y_{R_a}\xi_+^2)(1-qy_{Q_a}y_{R_a}\xi_+^2)} \end{pmatrix} \tag{5.284}$$

In order for the mixed reflection equation and its dual to be satisfied, it must hold $\xi_-^{2p} = 1$ and $\xi_+^{2p} = 1$ for the boundary parameters.

We can equivalently write them in terms of the generators. It holds:

$$K_a(\xi_-|y_{Q_a}, y_{R_a}|P_C) = 1/N(y_{Q_a}, y_{R_a}, \xi_-)\left[\alpha_0(y_{Q_a}, y_{R_a}, \xi_-)\mathbb{1} + \alpha_1(y_{Q_a}, y_{R_a}, \xi_-)\hat{\mathbf{v}}_a + \alpha_2(y_{Q_a}, y_{R_a}, \xi_-)\hat{\mathbf{v}}_a^2\right] \tag{5.285}$$

The factor $N(y_{Q_a}, y_{R_a}, \xi_-)$ is simply a normalisation, and the three coefficients $\alpha_i(y_{Q_a}, y_{R_a}, \xi_-)$ are polynomials in the variables $y_{Q_a}y_{R_a}$ and ξ_-. Explicitly, for the choice of normalisation (5.248), it holds:

$$N(y_{Q_a}, y_{R_a}, \xi_-) = 3(1 - y_{Q_a}y_{R_a}\xi_-^2)(1 - q^2y_{Q_a}y_{R_a}\xi_-^2) \tag{5.286}$$

and the coefficients $\alpha_i(y_{Q_a}, y_{R_a}, \xi_-)$ read:

$$\alpha_i(y_{Q_a}, y_{R_a}, \xi_-) = \alpha_{i0}(\xi_-) + \alpha_{i1}(\xi_-)y_{Q_a}y_{R_a} + \alpha_{i2}(\xi_-)(y_{Q_a}y_{R_a})^2 \tag{5.287}$$

The polynomials $\alpha_{ij}(\xi_-)$ are given by:

$$\alpha_{00}(\xi_-) = 1 - \xi_-^2 + q^2\xi_-^4 \;\;;\;\; \alpha_{01}(\xi_-) = 1 + 2q\xi_-^2 + q^2\xi_-^4 \;\;;\;\; \alpha_{02}(\xi_-) = 1 - q^2\xi_-^2 + q^2\xi_-^4 \tag{5.288}$$

$$\alpha_{10}(\xi_-) = 1 - q^2\xi_-^2 + \xi_-^4 \;\;;\;\; \alpha_{11}(\xi_-) = q^2 - \xi_-^2 + q\xi_-^4 \;\;;\;\; \alpha_{12}(\xi_-) = q - q\xi_-^2 + q^2\xi_-^4 \tag{5.289}$$

$$\alpha_{20}(\xi_-) = 1 - q\xi_-^2 + q\xi_-^4 \;\;;\;\; \alpha_{21}(\xi_-) = q - q^2\xi_-^2 + \xi_-^4 \;\;;\;\; \alpha_{22}(\xi_-) = q^2 - \xi_-^2 + q^2\xi_-^4 \tag{5.290}$$

Exploiting the symmetry (5.131), we can state:

$$K_{+a}(\xi_+|y_{Q_a}, y_{R_a}|P_C) = K_a(1/\xi_+|q/y_{R_a}, q/y_{Q_a}|P_C) \tag{5.291}$$

Thus, the integrable boundaries on the sites $j = 1$ and $j = N$ explicitly read:

$$\begin{aligned} \mathcal{B}_j^{(y_Q)}(\xi_\pm|P_C) &= \left.\frac{dK_j(\xi_\pm|y_{Q_j}, y_{R_j}|P_C)}{dy_{Q_j}}\right|_{y_{Q_j}y_{R_j}=y_Qy_R} \\ &= y_R/\bar{N}(y_{Q_a}, y_{R_a}, \xi_\pm)\left[\gamma_0(y_{Q_a}, y_{R_a}, \xi_\pm)\mathbb{1} + \gamma_1(y_{Q_a}, y_{R_a}, \xi_\pm)\hat{\mathbf{v}}_a + \gamma_2(y_{Q_a}, y_{R_a}, \xi_\pm)\hat{\mathbf{v}}_a^2\right] \end{aligned} \tag{5.292}$$

The normalisation $\bar{N}(y_{Q_a}, y_{R_a}, \xi)$ is given by:

$$\bar{N}(y_{Q_a}, y_{R_a}, \xi) = 3q\;(\xi^2 - 1)(q^2\xi^2 - 1) \tag{5.293}$$

while the three other coefficients are written:

$$\gamma_0(y_{Q_a}, y_{R_a}, \xi) = 3q + (q-1)\xi^2 - 3q^2\xi^4 \;\;;\;\; \gamma_1(y_{Q_a}, y_{R_a}, \xi) = q(q-1)(1 + q\xi^2 - q^2\xi^4) \tag{5.294}$$

$$\gamma_2(y_{Q_a}, y_{R_a}, \xi) = -(q-1)(1 - q\xi^2 - q^2\xi^4) \tag{5.295}$$

As the boundary matrices depend on the product $y_{Q_a} y_{R_a}$, then it follows:

$$\mathcal{B}_j^{(y_R)}(\xi_\pm|P_C) = \left.\frac{dK_j(\xi_\pm|y_{Q_j}, y_{R_j}|P_C)}{dy_{R_j}}\right|_{y_{Q_j} y_{R_j} = y_Q y_R = 1} = 1/y_R^2 \ \mathcal{B}_j^{(y_Q)}(\xi_\pm|P_C) \tag{5.296}$$

For this model, the bulk Hamiltonians of the chain with integrable boundary conditions seem not to satisfy any particular symmetry with the bulk Hamiltonians for the Yang-Baxter chain. The boundaries completely change the behaviour of the bulk of the chain.

With these four models, we have presented a representative sample of the different behaviours of a chain with integrable boundaries. Depending on the symmetries of the fundamental transfer matrix, the fact to consider integrable boundaries can lead to the same bulk interactions, or to completely different bulk interactions, even possibly leading to the vanishing of the interactions.

Conclusion

The main theoretical steps to understand the macroscopic behaviour of quantum systems are certainly the determination of their Hamiltonian spectrum and the computation of their correlation functions, where all the information is encoded. This thesis takes place in the development of such a research program to study quantum integrable models with general integrable boundary conditions, the long-range goal being to be able to exactly describe both equilibrium and out of equilibrium physics.

Precisely, we started the analysis of the class of integrable quantum models associated to cyclic representations of the 6-vertex reflection algebra, including as particular cases the XXZ spin chain at root of unity, the lattice sine-Gordon model at root of unity and the chiral Potts model.

A large part of the work has been devoted to the development of the quantum separation of variables method to solve the transfer matrix spectral problem for the models associated to general integrable boundary conditions, described by boundary K-matrices scalar solutions of the 6-vertex reflection equation, and to the general Bazhanov-Stroganov cyclic Lax operator, local solution of the 6-vertex Yang-Baxter algebra. In a first time, we have proven that the quantum separation of variables was applicable for the general cyclic representation under one constraint on the boundary parameters. In particular, we have considered one completely general and one triangular boundary matrix. Thereafter, we have extended this method to the most general integrable boundaries. This has been done solving the problem to generalise Baxter's gauge transformations to these cyclic reflection algebras. Indeed, without any loss of generality, we have used these gauge transformations to transform one general boundary matrix in a triangular form, leading to the separate basis construction in a framework similar to the triangular case.

The boundary transfer matrix spectrum, both eigenvalues and eigenstates, is completely characterised in terms of the set of solutions to a discrete system of polynomial equations. Moreover, we have proven the equivalence of this characterisation with the set of solutions, in a given class of functions, to a Baxter like functional equation, showing also the rewriting of the boundary transfer matrix eigenstates in an Algebraic Bethe Ansatz form. It is worth remarking that this functional equation has in general an inhomogeneous term, leading to a new type of equation for which new methods have to be developed. However, let us mention that this inhomogeneous term vanishes when just one constraint is imposed, either on the boundary or on the bulk parameters, resulting this way in a homogeneous Baxter like equation. In that particular case, the boundary transfer matrix spectral problem is reduced to the standard analysis of an equivalent system of Bethe equations.

In the continuity of this work, we have solved the problem to define cyclic local Hamiltonians with integrable boundaries commuting with the above boundary transfer matrix, i.e. associated to general cyclic representations of the reflection algebra. The main difficulty has been to overcome the fact that the auxiliary and quantum spaces of the Bazahanov-Stroganov Lax operators are not isomorphic. Hence, we have considered first the fundamental cyclic R-matrix, solution of the so-called mixed Yang-Baxter equation intertwining two Lax operators acting in two different quantum spaces. Such an R-matrix acts in the tensor product of two cyclic representations. Then, we defined a mixed reflection equation intertwining Lax operators and two scalar boundary matrices, one defined on the auxiliary space and one defined on the quantum space. We found diagonal scalar solutions (boundary matrices) on the quantum space, associated to known scalar diagonal solutions on the auxiliary space. This achievement allowed us to define a multi spectral parameter family of fundamental transfer matrices commuting between themselves and with the boundary transfer matrix associated to the original cyclic reflection algebra.

In addition to that, we have proven that these fundamental transfer matrices generate multi parameter families of commuting local Hamiltonians with boundaries. It is worth mentioning that these cyclic local Hamiltonians are obtained as second derivatives with respect to the spectral parameters of the fundamental transfer matrices, as the first order derivatives reduce to the identity. It is worth mentioning also that the Hamiltonians we have constructed are new. In particular, they differ from the one considered by Baxter and Tarasov

The natural continuation of this work, already partly under consideration, is to clarify the role of the fundamental R-matrix for building up reflection equations at a cyclic-cyclic level and their compatibility conditions. This should also lead to new type of boundary conditions and new integrable local Hamiltonians, generalising the ones we have obtained to non diagonal boundary matrices.

Moreover, it should be interesting to investigate the Baxter T-Q equations in the context of general integrable boundaries and in particular the link between the Q-operator [51] and the fundamental transfer matrix. This last connection is done, for periodic boundaries and non compact representations of general quantum affine algebras, in [261]. What is more, it must be investigated wether the fundamental transfer matrix introduced in this thesis is related to the fused transfer matrix obtained in [262]. In both cases, this would allow to compute (or at least to characterise) the eigenvalues of the fundamental transfer matrix, and thus the ones of the corresponding local Hamiltonians.

Another continuation of this work is to tackle the study of the dynamics. The first step is to consider the matrix elements, in the eigenstate basis, of local operators for these models. These form factors are an essential ingredient for computing correlation functions. At this point, the approach being through the quantum separation of variables, the determinants one will obtain should suffer from the standard problem regarding their non-trivial homogeneous limits, i.e. the limit in which all inhomogeneity parameters are put to the same value. It should be interesting to investigate, in this cyclic representation framework, the equivalence between these types of determinants for scalar products, essentially related to dressings of Vandermonde determinants, with Izergin's and Slavnov's type determinants, as was demonstrated recently for the XXX and XXZ boundary integrable models [146, 147]. This should make possible to take their homogeneous limit and then their thermodynamic limit (where the size N of the chain becomes large). Let us comment that the impact of boundaries in the thermodynamic limit is non trivial for a half-infinite chain only, where solely one boundary is sent away. In addition to that, it is worth emphasising that due to the boundary, the 1-point correlation functions are already non trivial.

Even if there is still a lot to be done, for example concerning the characterisation of the ground state, if successful these techniques should also allow for the computation of the order parameters of the chiral Potts model in a rather direct way along the preliminary results described in [151]. Another application would be the computation of form factors for the lattice sine-Gordon model, in particular at its q-root of unity reduction points, linked to minimal models of Conformal Field Theory.

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Article I

Transfer matrix spectrum for cyclic representations of the 6-vertex reflection algebra I

Transfer matrix spectrum for cyclic representations of the 6-vertex reflection algebra I

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Abstract

We study the transfer matrix spectral problem for the cyclic representations of the trigonometric 6-vertex reflection algebra associated to the Bazanov-Stroganov Lax operator. The results apply as well to the spectral analysis of the lattice sine-Gordon model with integrable open boundary conditions. This spectral analysis is developed by implementing the method of separation of variables (SoV). The transfer matrix spectrum (both eigenvalues and eigenstates) is completely characterized in terms of the set of solutions to a discrete system of polynomial equations in a given class of functions. Moreover, we prove an equivalent characterization as the set of solutions to a Baxter's like T-Q functional equation and rewrite the transfer matrix eigenstates in an algebraic Bethe ansatz form. In order to explain our method in a simple case, the present paper is restricted to representations containing one constraint on the boundary parameters and on the parameters of the Bazanov-Stroganov Lax operator. In a next article, some more technical tools (like Baxter's gauge transformations) will be introduced to extend our approach to general integrable boundary conditions.

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1 Introduction

The study of quantum models with integrable open boundary conditions has attracted a large research interest, e.g. see [1–52] and references therein. These models are of physical interest as they can describe both equilibrium and out of equilibrium physics; e.g. some interesting applications concern the description of classical stochastic relaxation processes, like ASEP [53, 54], [48–51], and quantum transport properties in spin systems [55, 56].

In this paper we start the analysis of the class of open integrable quantum models associated to cyclic representations [57–61] of the 6-vertex reflection algebra. The literature of these models is so far rather sparse with the exception of some rather special representations and boundary conditions, like the open XXZ chains at the roots of unity, that can be traced back to these representations under some special constraints, and for which some results are known in the framework of algebraic Bethe ansatz (ABA) [52, 62, 63], fusion of transfer matrices and truncation identities [6, 7, 64–67].

In order to study the general representations and boundary conditions we have to go beyond traditional methods [62–83] which do not apply for these general settings. This is done by developing the Sklyanin's SoV method [84–87] for this class of models, a method that has the advantage to lead (mainly by construction) to the complete characterization of the spectrum (eigenvalues and eigenvectors) and has proven to be applicable for a large variety of integrable quantum models [28–36, 88–102], where traditional methods fail. Moreover, the SoV approach has the advantage to allow also for the study of the dynamics of the models as it leads to universal determinant formulae for matrix elements of local operators on transfer matrix eigenstates as shown for different classes of models, first in [97], and then in many other cases in [33, 88, 89, 99, 100, 103]. Moreover, the analysis developed in [104, 105] makes it possible to compute homogeneous and then thermodynamic limit of these matrix elements opening the way to the computation of the corresponding correlation functions.

Let us recall that in [3], Sklyanin has shown how to construct classes of quantum models with integrable boundaries in the framework of the so-called Quantum Inverse Scattering Method [62–79], constructing in particular associated families of commuting transfer matrices (conserved charges of the model). In fact, Sklyanin's construction allows to use solutions of the Yang-Baxter equation to generate new solutions of the reflection equation [4] once a scalar solution of this last equation is known. Then as a consequence of the reflection equation these new solutions generate commuting transfer matrices [3].

Sklyanin has used the 6-vertex case and the associated XXZ spin 1/2 quantum chains [80, 106–108] to develop an explicit example of this construction. However as pointed out in [3] similar constructions applies also to other 6-vertex cases like the non-linear Schrödinger and the Toda chains as well as for models associated to the 8-vertex case like the XYZ spin 1/2 quantum chains. Further integrable quantum models with open boundary conditions have been presented following the Sklyanin's construction. Interesting examples are the higher spin open quantum chains [17, 18], the higher rank open quantum spin chains [40–42] and the Hubbard model [109–112] with integrable open boundaries [43–48]. In fact, this mainly results in the possibility to associate to any closed integrable quantum model (characterized by a solution of the Yang-Baxter equation) new open integrable quantum models (characterized by the associated Sklyanin's solutions of the reflection equation).

Here, we characterize the spectral problem and we start the analysis of the dynamical problem (by determining scalar products of separate states) for the class of models in the Sklyanin's construction associated to general scalar solutions of the 6-vertex reflection equation [11, 13] and the general Bazhanov-Stroganov cyclic solution of the 6-vertex Yang-Baxter algebra [113].

It may be instructive to recall some main literature on open integrable quantum chains, even for different representations with regards to those studied here. Indeed, this allows us to point out the difficulties that arise in their analysis and that we have also encountered for the models studied in this paper and to give the motivations for the approach that we have followed to overcome them.

The spectrum of the open XXZ spin 1/2 quantum chain with parallel z-oriented magnetic fields on the boundaries has been characterized in [3], in the framework of the algebraic Bethe ansatz. While its dynamics has been studied by the exact computation of correlation functions first in [14, 15] and then in [16], there generalizing in the ABA framework the method established in [114, 115] for the periodic chains. These open quantum spin chains with z-oriented boundary magnetic fields correspond in the Sklyanin's construction to the diagonal scalar solution of the reflection equation. However, the most general scalar solution of the 6-vertex/8-vertex reflection equation is non-diagonal [11, 13] producing unparalleled and not z-oriented boundaries magnetic fields. Under this general setting the analysis of the spectrum and dynamics has shown to be much more involved. The ABA method cannot be directly applied to these open chains with general boundary. In fact, it was first understood in [37], for the XYZ spin 1/2 open chain, that the use of the Baxter's gauge transformations [116] allows to generalize the ABA method limitedly to non-diagonal boundary matrices which satisfy one special constraint. After that, the same approach, based now on the trigonometric version of the Baxter's gauge transformations, was used in [38, 39] to describe the XXZ spectrum by ABA under similar constraints. Let us comment that for the XXZ spectrum the same constraint was derived independently by a pure functional method based on the use of the fusion of transfer matrices and truncations identities for the roots of unit case in [22]- [24]. This has given access to the study of the spectrum leaving however the study of the dynamics for these open models still unsolved.

Results on the spectrum for the most general unconstrained boundary conditions have been achieved only more recently and they have required the introduction of methods different from the ABA. Pure eigenvalue analysis has been implemented in [25] by a functional method leading to nested Bethe ansatz type equations similar to those previously introduced in [26]. Moreover, an ansatz for polynomial T-Q functional equations with an inhomogeneous term has been recently argued in [27]. Eigenstate construction has been first considered under these general boundary in [21] by the q-Onsager algebra formalism. A different approach, based on the generalization of the Sklyanin's separation of variables (SoV) method to the reflection algebra framework, has then lead to the complete eigenvalues and eigenstates characterization [29, 30, 33–36], proving its equivalence to an inhomogeneous TQ functional equation [36],

and also giving access to first computations of matrix elements of local operators [33] in the eigenstates basis.

The aim of the paper is to generalize this type of results for the 6-vertex cyclic representations. Here we solve this problem in the case of one triangular and one general boundary matrix, so that our current results define also the setup for the solution of the most general boundary case as well as the paper [33] has introduced the tools to solve the case with the most general boundary in [35].

The paper is organized as it follows. In section 2, we recall the cyclic representations of the 6-vertex Yang-Baxter algebra associated to the Bazhanov-Stroganov Lax-operator. In section 3, we define the associated representations of the cyclic 6-vertex reflection algebra. In section 4, we prove the diagonalizability of the generator $\mathcal{B}_-(\lambda)$ of the reflection algebra generated by $\mathcal{U}_-(\lambda)$ for the most general $K_-(\lambda)$ boundary matrix while we impose one constraint on the parameters of the Bazhanov-Stroganov Lax-operator for any quantum site to make easier the explicit construction of the $\mathcal{B}_-$-eigenstates basis. Moreover, we compute the scalar product for the so-called separate states in the $\mathcal{B}_-$-eigenstates basis. In section 5, we show that the $\mathcal{B}_-$-eigenstates basis is the SoV-basis for the transfer matrix spectral problem associated to the most general K_--boundary matrix and upper triangular K_+-boundary matrix and we solve in this SoV basis this spectral problem. In section 6, we show that the SoV characterization of the transfer matrix spectrum is equivalent to inhomogeneous Baxter's like TQ-functional equation with polynomial Q-functions. We present four appendices, in the first one we extend the proof of section 4 for the diagonalizability of the operator $\mathcal{B}_-(\lambda)$ to the case of general values of the boundary and bulk parameters. The remaining three appendices deal with the reduction of our representations to those associated to the chiral-Potts, the sine-Gordon and the XXZ spin s-chains at the 2s+1 roots of unit.

2 Cyclic representations of the 6-vertex Yang-Baxter algebra

In this section we recall the basics of the cyclic representations of the 6-vertex Yang-Baxter algebra associated to the Bazhanov-Stroganov Lax-operator. We consider the representations defined by the tensor product of N local representations of the 6-vertex Yang-Baxter algebra on the local Hilbert spaces $\mathcal{R}_n$. Each local representation is defined as the representation of a local Weyl algebra

$$u_n v_m = q^{\delta_{n,m}} v_m u_n \quad \forall n, m \in \{1, ..., \mathsf{N}\}, \tag{2.1}$$

associated to a root of unit q, where u_n and v_n are the Weyl algebra generators on the Hilbert spaces $\mathcal{R}_n$. Here, we assume that u_n and v_n are unitary operators and that it holds:

$$u_n^p = v_n^p = 1 \quad \text{for} \quad q = e^{-i\pi\beta^2}, \tag{2.2}$$

with $\beta^2 = p'/p$, p' even and $p = 2l + 1$ odd. This type of representation can be defined on a p-dimensional linear space $\mathcal{R}_n$, imposing that the v_n spectrum coincides with the p-roots of the unit:

$$v_n |k, n\rangle = q^k |k, n\rangle \;\; \forall (n, k) \in \{1, ..., \mathsf{N}\} \times \{-l, ..., l\}. \tag{2.3}$$

On $\mathcal{R}_n$ is defined a p-dimensional representation of the Weyl algebra by setting:

$$u_n |k, n\rangle = |k + 1, n\rangle \quad \forall k \in \{-l, ..., l\}, \tag{2.4}$$

with the cyclicity condition:

$$|k + p, n\rangle = |k, n\rangle. \tag{2.5}$$

$\mathscr{R}_n$ is also called the right local quantum space at the site n of the chain. Let $\mathscr{L}_n$ be the dual space of $\mathscr{R}_n$ then we can define the following scalar products:

$$\langle k,n|k',n\rangle = ((\langle k,n|)^{\dagger},|k',n\rangle) \equiv \delta_{k,k'}, \tag{2.6}$$

for any $k,k' \in \{-l,...,l\}$.

The local generators of the cyclic 6-vertex Yang-Baxter algebra can be now defined as the elements of the following Bazhanov-Stroganov Lax operator:

$$L_{a,n}(\lambda) \equiv \begin{pmatrix} \lambda\alpha_n v_n - \beta_n\lambda^{-1}v_n^{-1} & u_n\left(q^{-1/2}a_n v_n + q^{1/2}b_n v_n^{-1}\right) \\ u_n^{-1}\left(q^{1/2}c_n v_n + q^{-1/2}d_n v_n^{-1}\right) & \gamma_n v_n/\lambda - \delta_n\lambda/v_n \end{pmatrix}_a \in \mathrm{End}(\mathbb{C}^2 \otimes \mathscr{R}_n), \tag{2.7}$$

where a denote the so-called auxiliary space $V_a \simeq \mathbb{C}^2$. Indeed, under the condition

$$\gamma_n = a_n c_n/\alpha_n, \qquad \delta_n = b_n d_n/\beta_n, \tag{2.8}$$

$L_{a,n}(\lambda)$ is a solution of the 6-vertex Yang-Baxter equation:

$$R_{12}(\lambda/\mu)L_{1,n}(\lambda)L_{2,n}(\mu) = L_{2,n}(\mu)L_{1,n}(\lambda)R_{12}(\lambda/\mu), \tag{2.9}$$

with regards to the standard 6-vertex R-matrix:

$$R_{ab}(\lambda) = \begin{pmatrix} q\lambda - q^{-1}\lambda^{-1} & & & \\ & \lambda-\lambda^{-1} & q-q^{-1} & \\ & q-q^{-1} & \lambda-\lambda^{-1} & \\ & & & q\lambda - q^{-1}\lambda^{-1} \end{pmatrix}, \tag{2.10}$$

where a and b denote two bidimensional spaces $V_a, V_b \equiv \mathbb{C}^2$ and $R_{ab}(\lambda)$ is an endomorphism on their tensor product, i.e. $R_{ab}(\lambda) \in\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Then, the following monodromy matrix:

$$M_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_a \equiv L_{a,\mathsf{N}}(\lambda q^{-1/2})\cdots L_{a,1}(\lambda q^{-1/2}) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathscr{H}), \tag{2.11}$$

is also a solution of the Yang-Baxter equation:

$$R_{12}(\lambda/\mu)M_1(\lambda)M_2(\mu) = M_2(\mu)M_1(\lambda)R_{12}(\lambda/\mu), \tag{2.12}$$

and its elements define a representation of the Yang-Baxter algebra on the tensor product of the local representation spaces, i.e. $\mathscr{H} = \otimes_{n=1}^{\mathsf{N}}\mathscr{R}_n$. Note that one can also consider cyclic representations of the 6-vertex Yang-Baxter algebra associated to q, an even root of unit, these have been studied in [117].

2.1 Bulk transfer matrix and quantum determinant

The Yang-Baxter equation implies that the bulk transfer matrix $\tau_2(\lambda) \equiv \mathrm{tr}_a M_a(\lambda)$ defines a one parameter family of commuting operators. Note that we have:

$$[\tau_2(\lambda),\Theta] = 0, \quad \text{where} \quad \Theta \equiv \prod_{n=1}^{\mathsf{N}} v_n. \tag{2.13}$$

In [113, 118] it was related to the analysis of the chP-model [119–126] and characterized by SoV in [92–97]. The Yang-Baxter equation also implies that the so-called quantum determinant is a central element and it has the following factorized form:

$$\begin{aligned} \mathrm{det}_{\mathrm{q}}M_a(\lambda) &\equiv A(\lambda q^{1/2})D(\lambda q^{-1/2}) - B(\lambda q^{1/2})C(\lambda q^{-1/2}) && (2.14)\\ &= D(\lambda q^{1/2})A(\lambda q^{-1/2}) - C(\lambda q^{1/2})B(\lambda q^{-1/2}) && (2.15)\\ &= \prod_{n=1}^{\mathsf{N}} \mathrm{det}_{\mathrm{q}}L_{a,n}(\lambda), && (2.16) \end{aligned}$$

where the local quantum determinants read:

$$\begin{aligned} \mathrm{det}_{\mathrm{q}}L_{a,n}(\lambda) &\equiv \left(L_{a,n}(\lambda)\right)_{11}\left(L_{a,n}(\lambda q^{-1})\right)_{22} - \left(L_{a,n}(\lambda)\right)_{12}\left(L_{a,n}(\lambda q^{-1})\right)_{21} && (2.17)\\ &= \left(L_{a,n}(\lambda)\right)_{22}\left(L_{a,n}(\lambda q^{-1})\right)_{11} - \left(L_{a,n}(\lambda)\right)_{21}\left(L_{a,n}(\lambda q^{-1})\right)_{12}. && (2.18) \end{aligned}$$

They admit the following explicit form:

$$\begin{aligned} \mathrm{det}_{\mathrm{q}}M_a(\lambda) &= \prod_{n=1}^{\mathsf{N}} k_n\left(\frac{\lambda}{\mu_{n,+}} - \frac{\mu_{n,+}}{\lambda}\right)\left(\frac{\lambda}{\mu_{n,-}} - \frac{\mu_{n,-}}{\lambda}\right) && (2.19)\\ &= (-q)^{\mathsf{N}}\prod_{n=1}^{\mathsf{N}}\frac{\beta_n a_n c_n}{\alpha_n}\left(\frac{1}{\lambda} + q^{-1}\frac{b_n\alpha_n}{a_n\beta_n}\lambda\right)\left(\frac{1}{\lambda} + q^{-1}\frac{d_n\alpha_n}{c_n\beta_n}\lambda\right) && (2.20)\\ &= a(\lambda)d(\lambda/q), && (2.21) \end{aligned}$$

where:

$$k_n \equiv (a_n b_n c_n d_n)^{1/2}, \quad \mu_{n,h} \equiv \begin{cases} iq^{1/2}(a_n\beta_n/\alpha_n b_n)^{1/2} & h = +, \\ iq^{1/2}(c_n\beta_n/\alpha_n d_n)^{1/2} & h = -. \end{cases} \tag{2.22}$$

$$a(\lambda) \equiv a_0\prod_{n=1}^{\mathsf{N}}\left(\frac{\beta_n}{\lambda} + q^{-1}\frac{b_n\alpha_n}{a_n}\lambda\right), \quad d(\lambda) \equiv \frac{(-1)^{\mathsf{N}}}{a_0}\prod_{n=1}^{\mathsf{N}}\frac{a_n c_n}{\alpha_n}\left(\frac{1}{\lambda} + q\frac{d_n\alpha_n}{c_n\beta_n}\lambda\right), \tag{2.23}$$

and a_0 is a free non zero parameter.

3 Cyclic representations of the 6-vertex reflection algebra

In this section we define the most general cyclic representations of the 6-vertex reflection algebra associated to the Bazhanov-Stroganov Lax-operator. This is done by following the general procedure introduced by Sklyanin [3] which us allows to associate to any solution $M_a(\lambda) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathscr{H})$ of the 6-vertex Yang-Baxter equation a solution $\mathscr{U}_a(\lambda) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathscr{H})$ of the 6-vertex reflection equation:

$$R_{12}(\lambda/\mu)\,\mathscr{U}_1(\lambda)R_{12}(\lambda\mu/q)\,\mathscr{U}_2(\mu) = \mathscr{U}_2(\mu)R_{12}(\lambda\mu/q)\,\mathscr{U}_1(\lambda)R_{12}(\lambda/\mu). \tag{3.1}$$

Here, we have defined:

$$\mathscr{U}_a(\lambda) = M_a(\lambda)K_a(\lambda)\hat{M}_a(\lambda), \tag{3.2}$$

where $K_a(\lambda;\zeta,\kappa,\tau)$ is the most general scalar (boundary matrix) solution of the 6-vertex reflection equation [11, 12]:

$$K_a(\lambda;\zeta,\kappa,\tau) = \frac{1}{\zeta - \frac{1}{\zeta}}\begin{pmatrix} \frac{\lambda\zeta}{q^{1/2}} - \frac{q^{1/2}}{\lambda\zeta} & \kappa e^{\tau}\left(\frac{\lambda^2}{q} - \frac{q}{\lambda^2}\right) \\ \kappa e^{-\tau}\left(\frac{\lambda^2}{q} - \frac{q}{\lambda^2}\right) & \frac{q^{1/2}\zeta}{\lambda} - \frac{\lambda}{\zeta q^{1/2}} \end{pmatrix}_a, \tag{3.3}$$

and we have defined:

$$\hat{M}_a(\lambda) = (-1)^{\mathsf{N}}\,\sigma_a^y\,M_a^{t_a}(1/\lambda)\,\sigma_a^y. \tag{3.4}$$

Using this correspondence, the most general cyclic representations of the 6-vertex Yang-Baxter algebra, associated to the bulk monodromy matrix (2.11), define the most general cyclic representations of the 6-vertex reflection algebra, corresponding to the boundary monodromy matrices:

$$\mathscr{U}_{a,-}(\lambda) = M_a(\lambda)K_{a,-}(\lambda)\hat{M}_a(\lambda) = \begin{pmatrix} \mathscr{A}_-(\lambda) & \mathscr{B}_-(\lambda) \\ \mathscr{C}_-(\lambda) & \mathscr{D}_-(\lambda) \end{pmatrix}_a, \tag{3.5}$$

$$\mathscr{U}_{a,+}^{t_a}(\lambda) = M_a^{t_a}(\lambda)K_{a,+}^{t_a}(\lambda)\hat{M}_a^{t_a}(\lambda) = \begin{pmatrix} \mathscr{A}_+(\lambda) & \mathscr{C}_+(\lambda) \\ \mathscr{B}_+(\lambda) & \mathscr{D}_+(\lambda) \end{pmatrix}_a, \tag{3.6}$$

with $\mathcal{U}_{a,-}(\lambda)$ and $\mathcal{V}_{a,+}(\lambda) \equiv \mathcal{U}_{a,+}^{t_a}(-\lambda)$ both solution of the reflection equation (3.1), where:

$$K_{a,\pm}(\lambda) = K_a(\lambda q^{(1\pm 1)/2}; \zeta_\pm, \kappa_\pm, \tau_\pm) = \begin{pmatrix} a_\pm(\lambda) & b_\pm(\lambda) \\ c_\pm(\lambda) & d_\pm(\lambda) \end{pmatrix}_a, \tag{3.7}$$

and $\zeta_\pm, \delta_\pm, \tau_\pm$ are arbitrary complex parameters.

3.1 Boundary transfer matrix and quantum determinant

These boundary monodromy matrices define a one parameter family of commuting transfer matrices:

$$\begin{aligned} \mathcal{T}(\lambda) &\equiv \operatorname{tr}_a\{K_{a,+}(\lambda)\, M_a(\lambda)\, K_{a,-}(\lambda)\hat{M}_a(\lambda)\} && (3.8)\\ &= \operatorname{tr}_a\{K_{a,-}(\lambda)\mathcal{U}_{a,+}(\lambda)\} = \operatorname{tr}_a\{K_{a,+}(\lambda)\mathcal{U}_{a,-}(\lambda)\} && (3.9)\\ &= a_+(\lambda)\mathcal{A}_-(\lambda) + d_+(\lambda)\mathcal{D}_-(\lambda) + b_+(\lambda)\mathcal{C}_-(\lambda) + c_+(\lambda)\mathcal{B}_-(\lambda). && (3.10) \end{aligned}$$

This statement follows by using the reflection equation as Sklyanin has proven in [3]. The characterization of the spectrum (eigenvalues and eigenstates) of this class of transfer matrices is the main subject of this paper. In particular, we will restrict our attention to the special boundary condition $b_+(\lambda) = 0$, which can be analyzed by implementing the SoV approach once is proven the diagonalizability of the $\mathcal{B}_-(\lambda)$ family of commuting operators. In order to introduce this spectral analysis we start pointing out some important properties satisfied by the generators of the reflection algebra $\mathcal{A}_-(\lambda)$, $\mathcal{B}_-(\lambda)$, $\mathcal{C}_-(\lambda)$ and $\mathcal{D}_-(\lambda)$.

Let us start with the following re-parametrization of the boundary parameters [22]:

$$(\alpha_- - 1/\alpha_-)(\beta_- + 1/\beta_-) \equiv \frac{\zeta_- - 1/\zeta_-}{\kappa_-}, \quad (\alpha_- + 1/\alpha_-)(\beta_- - 1/\beta_-) \equiv \frac{\zeta_- + 1/\zeta_-}{\kappa_-}. \tag{3.11}$$

Then we define the following functions:

$$\mathsf{A}_-(\lambda) \equiv g_-(\lambda) a(\lambda q^{-1/2}) d(1/(q^{1/2}\lambda)), \tag{3.12}$$

where:

$$g_-(\lambda) \equiv \frac{(\lambda\alpha_-/q^{1/2} - q^{1/2}/(\lambda\alpha_-))(\lambda\beta_-/q^{1/2} + q^{1/2}/(\lambda\beta_-))}{(\alpha_- - 1/\alpha_-)(\beta_- + 1/\beta_-)}. \tag{3.13}$$

Proposition 3.1. *The following boundary quantum determinant:*

$$\begin{aligned} det_q\,\mathcal{U}_{a,-}(\lambda) &\equiv ((\lambda/q)^2 - (q/\lambda)^2)[\mathcal{A}_-(\lambda q^{1/2})\mathcal{A}_-(q^{1/2}/\lambda) + \mathcal{B}_-(\lambda q^{1/2})\mathcal{C}_-(q^{1/2}/\lambda)] && (3.14)\\ &= ((\lambda/q)^2 - (q/\lambda)^2)[\mathcal{D}_-(\lambda q^{1/2})\mathcal{D}_-(q^{1/2}/\lambda) + \mathcal{C}_-(\lambda q^{1/2})\mathcal{B}_-(q^{1/2}/\lambda)], && (3.15) \end{aligned}$$

is a central element in the reflection algebra, i.e.

$$[det_q\,\mathcal{U}_{a,-}(\lambda), \mathcal{U}_{a,-}(\mu)] = 0, \tag{3.16}$$

and its explicit expression reads:

$$det_q\,\mathcal{U}_{a,-}(\lambda) = (\lambda^2/q^2 - q^2/\lambda^2)\mathsf{A}_-(\lambda q^{1/2})\mathsf{A}_-(q^{1/2}/\lambda). \tag{3.17}$$

Moreover, the generators of the reflection algebra satisfy the following properties:

$$\mathcal{D}_-(\lambda) = \frac{(\lambda^2/q - q/\lambda^2)}{(\lambda^2 - 1/\lambda^2)}\mathcal{A}_-(\lambda^{-1}) + \frac{(q - 1/q)}{(\lambda^2 - 1/\lambda^2)}\mathcal{A}_-(\lambda), \tag{3.18}$$

and

$$\mathcal{B}_-(\lambda^{-1}) = -\frac{(\lambda^2 q - 1/(q\lambda^2))}{(\lambda^2/q - q/\lambda^2)}\mathcal{B}_-(\lambda), \quad \mathcal{C}_-(\lambda^{-1}) = -\frac{(\lambda^2 q - 1/(q\lambda^2))}{(\lambda^2/q - q/\lambda^2)}\mathcal{C}_-(\lambda). \tag{3.19}$$

We omit the proof of this proposition as it can be derived repeating the main steps of the original Sklyanin's paper, where similar statements were proven for the case of spin 1/2 representations of the 6-vertex reflection algebra. Let us introduce now the following notation:

$$\mathcal{T}_\setminus(\lambda) \equiv a_+(\lambda)\,\mathcal{A}_-(\lambda) + d_+(\lambda)\,\mathcal{D}_-(\lambda), \tag{3.20}$$

for the diagonal part of the transfer matrix, i.e. the one associated to the diagonal elements of the matrix $\mathcal{U}_{a,-}(\lambda)$, and the coefficients:

$$\mathsf{a}_+(\lambda) \equiv \frac{(\lambda^2 q - 1/(q\lambda^2))(\lambda\zeta_+/q^{1/2} - q^{1/2}/(\lambda\zeta_+))}{(\lambda^2 - 1/\lambda^2)(\zeta_+ - 1/\zeta_+)}, \tag{3.21}$$

$$\mathsf{d}_+(\lambda) \equiv \frac{(\lambda^2 q - 1/(q\lambda^2))(\zeta_+ q^{1/2}/\lambda - \lambda/(q^{1/2}\zeta_+))}{(\lambda^2 - 1/\lambda^2)(\zeta_+ - 1/\zeta_+)}. \tag{3.22}$$

then we can prove the following:

Corollary 3.1. *The most general transfer matrix admits the following symmetries:*

$$\mathcal{T}(\lambda) = \mathcal{T}(1/\lambda), \quad \mathcal{T}(-\lambda) = \mathcal{T}(\lambda), \tag{3.23}$$

and the diagonal part $\mathcal{T}_\setminus(\lambda)$ has the following explicitly symmetric forms:

$$\begin{aligned} \mathcal{T}_\setminus(\lambda) &\equiv \mathsf{a}_+(\lambda)\mathcal{A}_-(\lambda) + \mathsf{a}_+(1/\lambda)\mathcal{A}_-(1/\lambda) && (3.24)\\ &= \mathsf{d}_+(\lambda)\mathcal{D}_-(\lambda) + \mathsf{d}_+(1/\lambda)\mathcal{D}_-(1/\lambda). && (3.25) \end{aligned}$$

4 SoV representation of cyclic 6-vertex reflection algebra

In this section we construct the left and right basis which diagonalize the one-parameter family of commuting operators $\mathcal{B}_-(\lambda)$ associated to the most general $K_-(\lambda)$ matrix. Here we impose one constraint on the parameters of the representation at any quantum site:

$$b_n^p + a_n^p = 0, \quad \forall n \in \{1, ..., \mathsf{N}\}. \tag{4.1}$$

This is done in order to make completely explicit the construction of this basis; however, the proof of the diagonalizability of $\mathcal{B}_-(\lambda)$ can be done without these constraints and under completly general values of the inner boundary matrix and of the bulk parameters and it will be presented in appendix.

4.1 Pseudo-vacuum states

We implement the above constraints by imposing:

$$b_n = -q^{2j_n - 1} a_n, \tag{4.2}$$

where for any $n \in \{1, ..., \mathsf{N}\}$ we have fixed $j_n \in \{0, ..., p-1\}$, then we have:

$$\langle j_n - 1, n | \left(L_{a,n}\right)_{12} = \underline{0}, \quad \left(L_{a,n}\right)_{12} | j_n, n\rangle = \underline{0}, \tag{4.3}$$

as well as:

$$\langle j_n - 1, n | \left(L_{a,n}(\lambda)\right)_{11} = \mathsf{a}_n(\lambda q^{j_n - 1}) \langle j_n - 1, n| \tag{4.4}$$

$$\langle j_n - 1, n | \left(L_{a,n}(\lambda)\right)_{22} = \mathsf{d}_n(\lambda q^{1 - j_n}) \langle j_n - 1, n|, \tag{4.5}$$

and

$$\left(L_{a,n}(\lambda)\right)_{11}|j_n,n\rangle = |j_n,n\rangle\,\mathrm{a}_n(\lambda q^{j_n}), \qquad \left(L_{a,n}(\lambda)\right)_{22}|j_n,n\rangle = |j_n,n\rangle\,\mathrm{d}_n(\lambda q^{-j_n}), \tag{4.6}$$

where:

$$\mathrm{a}_n(\lambda) = \lambda\alpha_n - \beta_n/\lambda, \quad \mathrm{d}_n(\lambda) = \gamma_n/\lambda - \lambda\delta_n, \tag{4.7}$$

which is of course compatible with the local quantum determinant at site n:

$$\langle j_n - 1, n|\det_{\mathrm{q}} L_{a,n}(\lambda) = \langle j_n - 1, n|\left[\left(L_{a,n}(\lambda)\right)_{11}\left(L_{a,n}(\lambda/q)\right)_{22} - \left(L_{a,n}\right)_{12}\left(L_{a,n}\right)_{21}\right] \tag{4.8}$$
$$= \mathrm{a}_n(\lambda q^{j_n-1})\mathrm{d}_n(\lambda q^{-j_n})\langle j_n - 1, n| \tag{4.9}$$
$$\det_{\mathrm{q}} L_{a,n}(\lambda)|j_n,n\rangle = \left[\left(L_{a,n}(\lambda)\right)_{22}\left(L_{a,n}(\lambda/q)\right)_{11} - \left(L_{a,n}\right)_{21}\left(L_{a,n}\right)_{12}\right]|j_n,n\rangle \tag{4.10}$$
$$= |j_n,n\rangle\,\mathrm{a}_n(\lambda q^{j_n-1})\mathrm{d}_n(\lambda q^{-j_n}), \tag{4.11}$$

being:

$$\mathrm{a}_n(\lambda q^{j_n-1})\mathrm{d}_n(\lambda q^{-j_n}) = -q\frac{\beta_n a_n c_n}{\alpha_n}\left(\frac{1}{\lambda} - q^{2(j_n-1)}\frac{\alpha_n}{\beta_n}\lambda\right)\left(\frac{1}{\lambda} + q^{-1}\frac{d_n\alpha_n}{c_n\beta_n}\lambda\right). \tag{4.12}$$

Then we can define the following left and right "reference states":

$$\langle\Omega| = \otimes_{n=1}^{\mathrm{N}}\langle j_n - 1, n|, \quad \left|\bar{\Omega}\right\rangle = \otimes_{n=1}^{\mathrm{N}}|j_n,n\rangle. \tag{4.13}$$

The following properties are satisfied:

$$\langle\Omega|A(\lambda q^{1/2}) = a(\lambda)\langle\Omega|, \quad \langle\Omega|D(\lambda q^{1/2}) = d(\lambda)\langle\Omega|, \quad \langle\Omega|B(\lambda) = \underline{0}, \quad \langle\Omega|C(\lambda) \neq \underline{0}, \tag{4.14}$$
$$A(\lambda q^{1/2})\left|\bar{\Omega}\right\rangle = \left|\bar{\Omega}\right\rangle a(\lambda q), \; D(\lambda q^{1/2})\left|\bar{\Omega}\right\rangle = \left|\bar{\Omega}\right\rangle d(\lambda/q), \; B(\lambda)\left|\bar{\Omega}\right\rangle = \underline{0}, \; C(\lambda)\left|\bar{\Omega}\right\rangle \neq \underline{0}, \tag{4.15}$$

where it is simple to verify that as it should:

$$a(\lambda) = \prod_{n=1}^{\mathrm{N}}\mathrm{a}_n(\lambda q^{j_n-1}), \quad d(\lambda) = \prod_{n=1}^{\mathrm{N}}\mathrm{d}_n(\lambda q^{1-j_n}), \tag{4.16}$$

once we have fixed the free parameter:

$$a_0 = (-q)^{\mathrm{N}}\prod_{n=1}^{\mathrm{N}} q^{-j_n}.$$

Of course, the coefficients $a(\lambda)$ and $d(\lambda)$ as well as the reference states depend on the choice of the N-tuple $\{j_1,\ldots,j_{\mathrm{N}}\}$ but for simplicity we do not write it explicitly.

4.2 Representation of the reflection algebra in $\mathscr{B}_-(\lambda)$-eigenstates basis

The left and right SoV-representations of the cyclic 6-vertex reflection algebra are now defined by constructing the left and right $\mathscr{B}_-(\lambda)$-eigenstates basis and by determining in this new basis the representation of the other generators of the algebra. In order to present our results we need to introduce some notations. We define the following functions parametrized by the N-tuples $\boldsymbol{h} \equiv (h_1,\ldots,h_{\mathrm{N}}) \in \{0,\ldots,p-1\}^{\mathrm{N}}$:

$$\mathrm{B}_{\boldsymbol{h}}(\lambda) \equiv \kappa_- e^{\tau_-}\frac{(\lambda^2/q - q/\lambda^2)}{(\zeta_- - 1/\zeta_-)}a_{\boldsymbol{h}}(\lambda)a_{\boldsymbol{h}}(1/\lambda), \tag{4.17}$$

with

$$a_{\boldsymbol{h}}(\lambda) \equiv (-1)^{\mathrm{N}}\prod_{n=1}^{\mathrm{N}}(\alpha_n\beta_n)^{1/2}\left(\frac{\lambda}{\xi_n^{(h_n)}} - \frac{\xi_n^{(h_n)}}{\lambda}\right), \tag{4.18}$$

where:

$$\xi_n^{(h)} = \mu_{n,+}q^{h+1/2}, \; \xi_{n+\mathrm{N}}^{(h)} \equiv \xi_n^{(h)} \;\; \forall n \in \{1,\ldots,\mathrm{N}\}, \; a_{\mathbf{0}}(\lambda) = a(\lambda/q^{1/2}). \tag{4.19}$$

Moreover, next, we will need also the following notations:

$$\Lambda = (\lambda^2 + 1/\lambda^2), \quad X_b^{(h_b)} = (\zeta_b^{(h_b)})^2 + 1/(\zeta_b^{(h_b)})^2, \; X = q + 1/q \tag{4.20}$$
$$\zeta_n^{(h)} = \left(\xi_n^{(h)}\right)^{\varphi_n} \;\text{ for }\; h \in \{0,\ldots,p-1\} \;\text{ and }\; \forall n \in \{1,\ldots,2\mathrm{N}\}, \tag{4.21}$$
$$\varphi_a = 1 - 2\theta(a - \mathrm{N}) \quad\text{with}\quad \theta(x) = \{0 \text{ for } x \leq 0, \; 1 \text{ for } x > 0\}. \tag{4.22}$$

Theorem 4.1 (Left $\mathscr{B}_-(\lambda)$ SOV-representations). *If $b_-(\lambda) \neq 0$ and it holds:*

$$\mu_{n,+}^p \neq \mu_{m,+}^p \quad \forall n \neq m \in \{1,\ldots,\mathrm{N}\}, \tag{4.23}$$

and

$$\mu_{n,+}^{2p} \neq \pm 1, \; \mu_{n,+}^2 \neq q^{-2h}\alpha_-^{2\epsilon}, \; \mu_{n,+}^2 \neq -q^{-2h}\beta_-^{2\epsilon}, \; \mu_{n,+}^2 \neq q^{-2\epsilon-2h}\mu_{m,-}^{2\epsilon} \tag{4.24}$$

for any $\epsilon = \pm 1$, $h \in \{1,\ldots,p-1\}$ and $n,m \in \{1,\ldots,\mathrm{N}\}$, then the states:

$$\langle h_1,\ldots,h_{\mathrm{N}}| \equiv \frac{1}{\mathrm{N}}\langle\Omega|\prod_{n=1}^{\mathrm{N}}\prod_{k_n=1}^{h_n}\frac{\mathscr{A}_-(1/\xi_n^{(k_n-1)})}{\mathrm{A}_-(1/\xi_n^{(k_n-1)})}, \tag{4.25}$$

where $h_n \in \{0,\ldots,p-1\}$ and N is a free normalization, define a $\mathscr{B}_-(\lambda)$-eigenstates basis of $\mathscr{H}^*$:

$$\langle\boldsymbol{h}|\mathscr{B}_-(\lambda) = \mathrm{B}_{\boldsymbol{h}}(\lambda)\langle\boldsymbol{h}|. \tag{4.26}$$

Here we have denoted $\langle\boldsymbol{h}| \equiv \langle h_1,\ldots,h_{\mathrm{N}}|$. The quantum determinant and the following left action on the generic state $\langle\boldsymbol{h}|$:

$$\langle\boldsymbol{h}|\mathscr{A}_-(\lambda) = \sum_{a=1}^{2\mathrm{N}}\frac{(\lambda^2/q - q/\lambda^2)(\lambda\zeta_a^{(h_a)} - 1/\zeta_a^{(h_a)}\lambda)}{((\zeta_a^{(h_a)})^2/q - q/(\zeta_a^{(h_a)})^2)((\zeta_a^{(h_a)})^2 - 1/(\zeta_a^{(h_a)})^2)}\prod_{\substack{b=1 \\ b\neq a \bmod \mathrm{N}}}^{\mathrm{N}}\frac{\Lambda - X_b^{(h_b)}}{X_a^{(h_a)} - X_b^{(h_b)}}\mathrm{A}_-(\zeta_a^{(h_a)})$$
$$\times\langle\boldsymbol{h}|T_a^{-\varphi_a} + (-1)^{\mathrm{N}}det_q M(1)\frac{(\lambda/q^{1/2} + q^{1/2}/\lambda)}{2}\prod_{b=1}^{\mathrm{N}}\frac{\Lambda - X_b^{(h_b)}}{X - X_b^{(h_b)}}\langle\boldsymbol{h}|$$
$$+ (-1)^{\mathrm{N}}\frac{(\zeta_- + 1/\zeta_-)}{(\zeta_- - 1/\zeta_-)}det_q M(i)\frac{(\lambda/q^{1/2} - q^{1/2}/\lambda)}{2}\prod_{b=1}^{\mathrm{N}}\frac{\Lambda - X_b^{(h_b)}}{X + X_b^{(h_b)}}\langle\boldsymbol{h}|, \tag{4.27}$$

where:

$$\langle h_1,\ldots,h_a,\ldots,h_{\mathrm{N}}|T_a^{\pm} = \langle h_1,\ldots,h_a \pm 1,\ldots,h_{\mathrm{N}}|, \tag{4.28}$$

completely determine the representation of the other generators of the reflection algebra in the $\mathscr{B}_-(\lambda)$-eigenstates basis. Indeed, the representation of $\mathscr{D}_-(\lambda)$ follows from the identity (3.18) while $\mathscr{C}_-(\lambda)$ by the quantum determinant.

Proof. Let us write explicitly the decomposition of the reflection algebra generator:

$$\mathscr{B}_-(\lambda) = (-1)^{\mathrm{N}}\left[-a_-(\lambda)A(\lambda)B(1/\lambda) + b_-(\lambda)A(\lambda)A(1/\lambda)\right.$$
$$\left.-c_-(\lambda)B(\lambda)B(1/\lambda) + d_-(\lambda)B(\lambda)A(1/\lambda)\right], \tag{4.29}$$

in terms of the generators of the Yang-Baxter algebra. Then, by using the identities (4.14) it follows that $\langle\Omega|$ is a $\mathscr{B}_-(\lambda)$-eigenstate with non-zero eigenvalue:

$$\langle\Omega|\mathscr{B}_-(\lambda) \equiv \mathrm{B}_{\mathbf{0}}(\lambda)\langle\Omega|. \tag{4.30}$$

Now by using the reflection algebra commutation relations:

$$\begin{aligned}\mathscr{A}_{-}(\lambda_2)\mathscr{B}_{-}(\lambda_1) =& \frac{(\lambda_1 q/\lambda_2-\lambda_2/(\lambda_1 q))(\lambda_1\lambda_2/q-q/(\lambda_1\lambda_2))}{(\lambda_1/\lambda_2-\lambda_2/\lambda_1)(\lambda_1\lambda_2-1/(\lambda_1\lambda_2))}\mathscr{B}_{-}(\lambda_1)\mathscr{A}_{-}(\lambda_2)\\ &+\frac{(\lambda_1^2/q-q/\lambda_1^2)(q-1/q)}{(\lambda_2/\lambda_1-\lambda_1/\lambda_2)(\lambda_1^2-1/\lambda_1^2)}\mathscr{B}_{-}(\lambda_2)\mathscr{A}_{-}(\lambda_1)\\ &-\frac{q-1/q}{(\lambda_1^2-1/\lambda_1^2)(\lambda_1\lambda_2-1/(\lambda_1\lambda_2))}\mathscr{B}_{-}(\lambda_2)\tilde{\mathscr{D}}_{-}(\lambda_1)\end{aligned} \tag{4.31}$$

we can follow step by step the proof given in [99] to prove the validity of (4.26). The action of $\mathscr{A}_{-}(\zeta_b^{(h_b)})$ for $b\in\{1,...,2\mathsf{N}\}$ follows from the definition of the states $\langle\mathbf{h}|$, the reflection algebra commutation relations (4.31) and the quantum determinant relations. Let us show now that the conditions (4.23) and (4.24) imply that the set of p^{N} states $\langle\boldsymbol{h}|$ is a $\mathscr{B}_{-}(\lambda)$-eigenstates basis of $\mathscr{H}^*$. As by condition (4.23) each such state is associated to a different eigenvalue of $\mathscr{B}_{-}(\lambda)$ the only thing that we need to prove to get their linear independence is that each such state is nonzero. We know by construction that the state $\langle\Omega|$ is nonzero so let us assume by induction that the same is true for the state $\langle\boldsymbol{h}^{(0)}|=\langle h_1^{(0)},...,h_{\mathsf{N}}^{(0)}|$ with $h_j^{(0)}\in\{0,...,p-2\}$ and let us show that $\langle\boldsymbol{h}_j^{(0)}|=\langle h_1^{(0)},...,h_j^{(0)}+1,...,h_{\mathsf{N}}^{(0)}|$ is nonzero. We have that:

$$\langle\boldsymbol{h}_j^{(0)}|\mathscr{A}_{-}(\xi_j^{(h_j^{(0)}+1)})=\mathsf{A}_{-}(\xi_j^{(h_j^{(0)}+1)})\langle\boldsymbol{h}^{(0)}|\neq\underline{0}\qquad j\in\{1,...,\mathsf{N}\} \tag{4.32}$$

so that $\langle\boldsymbol{h}_j^{(0)}|$ is nonzero. Using this we can prove that all states $\langle\boldsymbol{h}^{(1)}|=\langle h_1^{(0)}+x_1,...,h_{\mathsf{N}}^{(0)}+x_{\mathsf{N}}|$ with $x_j\in\{0,1\}$ for any $j\in\{1,...,\mathsf{N}\}$ are nonzero, which just prove the validity of the induction. Finally, by using the identities:

$$\mathscr{U}_{-}(q^{1/2})=(-1)^{\mathsf{N}}\det{}_qM(1)\,I_0,\quad \mathscr{U}_{-}(iq^{1/2})=i(-1)^{\mathsf{N}+1}\frac{\zeta_{-}+1/\zeta_{-}}{\zeta_{-}-1/\zeta_{-}}\det{}_qM(i)\,\sigma_0^z, \tag{4.33}$$

and remarking that $\mathscr{A}_{-}(\lambda)$ has the following functional dependence with regards to λ:

$$\mathscr{A}_{-}(\lambda)=\sum_{a=0}^{2\mathsf{N}+1}\lambda^{(2a-2\mathsf{N}+1)}\mathscr{A}_{-,a}, \tag{4.34}$$

where $\mathscr{A}_{-,a}\in\mathrm{End}(\mathscr{H})$ are some fixed operators, we get our interpolation formula for its action on $\langle\boldsymbol{h}|$. □

Similarly, defining:

$$\kappa_a^{(h)}=k(\zeta_a^{(h)}),\quad \text{for } h\in\{0,...,p-1\},\ a\in\{1,...,2\mathsf{N}\}, \tag{4.35}$$

and the function:

$$k(\lambda)=\left(\lambda^2-1/\lambda^2\right)/(\lambda^2/q^2-q^2/\lambda^2), \tag{4.36}$$

we have similar properties for the right representations:

Theorem 4.2 (Right $\mathscr{B}_{-}(\lambda)$ SOV-representations). *If $b_{-}(\lambda)\neq 0$ and (4.23)-(4.24) are satisfied, the states:*

$$|h_1,...,h_{\mathsf{N}}\rangle\equiv\frac{1}{\mathrm{N}}\prod_{n=1}^{\mathsf{N}}\prod_{k_n=h_n}^{p-2}\frac{\mathscr{D}_{-}(\xi_n^{(k_n+1)})}{\kappa_n^{(k_n+1)}\mathsf{A}_{-}(1/\xi_n^{(k_n)})}|\bar{\Omega}\rangle, \tag{4.37}$$

with N *the same coefficient as in (4.25), define a $\mathscr{B}_{-}(\lambda)$-eigenstates basis of $\mathscr{H}$:*

$$\mathscr{B}_{-}(\lambda)|\boldsymbol{h}\rangle=|\boldsymbol{h}\rangle\mathrm{B}_{\boldsymbol{h}}(\lambda). \tag{4.38}$$

On the generic state $|\boldsymbol{h}\rangle$, the action of the remaining reflection algebra generators follows by:

$$\begin{aligned}\mathscr{D}_{-}(\lambda)|\boldsymbol{h}\rangle=&\sum_{a=1}^{2\mathsf{N}}T_a^{-\varphi_a}|\boldsymbol{h}\rangle\frac{(\lambda^2/q-q/\lambda^2)(\lambda\zeta_a^{(h_a)}-1/\zeta_a^{(h_a)}\lambda)}{((\zeta_a^{(h_a)})^2/q-q/(\zeta_a^{(h_a)})^2)((\zeta_a^{(h_a)})^2-1/(\zeta_a^{(h_a)})^2)}\prod_{\substack{b=1\\ b\neq a\bmod \mathsf{N}}}^{\mathsf{N}}\frac{\Lambda-X_b^{(h_b)}}{X_a^{(h_a)}-X_b^{(h_b)}}\\ &\times\mathsf{D}_{-}(\zeta_a^{(h_a)})+|\boldsymbol{h}\rangle(-1)^{\mathsf{N}}det_qM(1)\frac{(\lambda/q^{1/2}+q^{1/2}/\lambda)}{2}\prod_{b=1}^{\mathsf{N}}\frac{\Lambda-X_b^{(h_b)}}{X-X_b^{(h_b)}}\\ &+(-1)^{\mathsf{N}+1}|\boldsymbol{h}\rangle\frac{(\zeta_{-}+1/\zeta_{-})}{(\zeta_{-}-1/\zeta_{-})}det_qM(i)\frac{(\lambda/q^{1/2}-q^{1/2}/\lambda)}{2}\prod_{b=1}^{\mathsf{N}}\frac{\Lambda-X_b^{(h_b)}}{X+X_b^{(h_b)}},\end{aligned} \tag{4.39}$$

where:

$$\mathsf{D}_{-}(\lambda)=k(\lambda)\mathsf{A}_{-}(q/\lambda),\quad T_a^{\pm}|h_1,...,h_a,...,h_{\mathsf{N}}\rangle=|h_1,...,h_a\pm1,...,h_{\mathsf{N}}\rangle. \tag{4.40}$$

Indeed, the representation of $\mathscr{A}_{-}(\lambda)$ follows from the identity (3.18) while $\mathscr{C}_{-}(\lambda)$ is given by the quantum determinant.

Proof. The proof is given along the same steps used in the previous theorem, we just need to make the following remarks. First of all by using the identities (4.15) it follows that $\left|\bar{\Omega}\right\rangle$ is a $\mathscr{B}_{-}(\lambda)$-eigenstate with non-zero eigenvalue:

$$\mathscr{B}_{-}(\lambda)\left|\bar{\Omega}\right\rangle\equiv\left|\bar{\Omega}\right\rangle\mathrm{B}_{\mathbf{p-1}}(\lambda). \tag{4.41}$$

Now all we need are the following reflection algebra commutation relations:

$$\begin{aligned}\mathscr{B}_{-}(\lambda_1)\mathscr{D}_{-}(\lambda_2)=&\frac{(\lambda_1q/\lambda_2-\lambda_2/(\lambda_1q))(\lambda_1\lambda_2/q-q/(\lambda_1\lambda_2))}{(\lambda_1/\lambda_2-\lambda_2/\lambda_1)(\lambda_1\lambda_2-1/(\lambda_1\lambda_2))}\mathscr{D}_{-}(\lambda_2)\mathscr{B}_{-}(\lambda_1)\\ &-\frac{(q-1/q)(\lambda_1\lambda_2/q-q/(\lambda_1\lambda_2))}{(\lambda_1/\lambda_2-\lambda_2/\lambda_1)(\lambda_1\lambda_2-1/(\lambda_1\lambda_2))}\mathscr{D}_{-}(\lambda_1)\mathscr{B}_{-}(\lambda_2)\\ &-\frac{q-1/q}{(\lambda_1\lambda_2-1/(\lambda_1\lambda_2))}\mathscr{A}_{-}(\lambda_1)\mathscr{B}_{-}(\lambda_2).\end{aligned} \tag{4.42}$$

By using them, the definition of the states $|\boldsymbol{h}\rangle$ and the quantum determinant, we get our interpolation formula for the right action of $\mathscr{D}_{-}(\lambda)$ on $|\boldsymbol{h}\rangle$. Let us remark that in fact, the chosen gauge for the coefficients of $\mathscr{D}_{-}(\lambda)$ is consistent with the quantum determinant condition as we have:

$$\mathsf{D}_{-}(\zeta_a^{(h)})=\kappa_a^{(h)}\mathsf{A}_{-}(q/\zeta_a^{(h)}) \tag{4.43}$$

for any $h\in\{0,...,p-1\}$ and $a\in\{1,...,2\mathsf{N}\}$ and

$$\frac{\det_q\mathscr{U}_{-}(\xi_a^{(h+1/2)})}{((\xi_a^{(h+3/2)})^2-1/(\xi_a^{(h+3/2)})^2)}=\mathsf{D}_{-}(\xi_a^{(h+1)})\mathsf{D}_{-}(1/\xi_a^{(h)})=\mathsf{A}_{-}(\xi_a^{(h+1)})\mathsf{A}_{-}(1/\xi_a^{(h)}), \tag{4.44}$$

since,

$$\kappa_a^{(h+1)}\kappa_{a+\mathsf{N}}^{(h)}=1 \tag{4.45}$$

for any $h\in\{0,...,p-1\}$ and $a\in\{1,...,\mathsf{N}\}$. □

4.3 Change of basis and SoV spectral decomposition of the identity

In this section we present the main properties of the $p^{\mathsf{N}} \times p^{\mathsf{N}}$ matrices $U^{(L)}$ and $U^{(R)}$ defining respectively the change of basis from the original left and right basis, formed by the v_n-eigenstates basis:

$$\underline{\langle \boldsymbol{h}|} \equiv \otimes_{n=1}^{\mathsf{N}} \langle h_n, n| \quad \text{and} \quad \underline{|\boldsymbol{h}\rangle} \equiv \otimes_{n=1}^{\mathsf{N}} |h_n, n\rangle, \tag{4.46}$$

to the left and right $\mathscr{B}_-$-eigenstates basis:

$$\langle \boldsymbol{h}| = \underline{\langle \boldsymbol{h}|} U^{(L)} = \sum_{i=1}^{p^{\mathsf{N}}} U^{(L)}_{\varkappa(\boldsymbol{h}),i} \langle \varkappa^{-1}(i)| \quad \text{and} \quad |\boldsymbol{h}\rangle = U^{(R)} \underline{|\boldsymbol{h}\rangle} = \sum_{i=1}^{p^{\mathsf{N}}} U^{(R)}_{i,\varkappa(\boldsymbol{h})} |\varkappa^{-1}(i)\rangle, \tag{4.47}$$

where $\varkappa$ is the isomorphism between the sets $\{0,\dots,p-1\}^{\mathsf{N}}$ and $\{1,\dots,p^{\mathsf{N}}\}$ defined by:

$$\varkappa : \boldsymbol{h} \in \{0,\dots,p-1\}^{\mathsf{N}} \to \varkappa(\boldsymbol{h}) \equiv 1 + \sum_{a=1}^{\mathsf{N}} p^{(a-1)} h_a \in \{1,\dots,p^{\mathsf{N}}\}. \tag{4.48}$$

From the diagonalizability of $\mathscr{B}_-(\lambda)$ it follows that $U^{(L)}$ and $U^{(R)}$ are invertible matrices for which it holds:

$$U^{(L)} \mathscr{B}_-(\lambda) = \Delta_{\mathscr{B}_-}(\lambda) U^{(L)}, \quad \mathscr{B}_-(\lambda) U^{(R)} = U^{(R)} \Delta_{\mathscr{B}_-}(\lambda), \tag{4.49}$$

where $\Delta_{\mathscr{B}_-}(\lambda)$ is the $p^{\mathsf{N}} \times p^{\mathsf{N}}$ diagonal matrix defined by:

$$\left(\Delta_{\mathscr{B}_-}(\lambda)\right)_{i,j} \equiv \delta_{i,j} \mathtt{B}_{\varkappa^{-1}(i)}(\lambda) \quad \forall i,j \in \{1,\dots,p^{\mathsf{N}}\}. \tag{4.50}$$

We can prove that it holds:

Proposition 4.1. *The $p^{\mathsf{N}} \times p^{\mathsf{N}}$ matrix $M \equiv U^{(L)} U^{(R)}$ consisting of scalar products of left and right $\mathscr{B}_-$-eigenstates is diagonal and it is characterized by the following diagonal entries:*

$$M_{\varkappa(\boldsymbol{h})\varkappa(\boldsymbol{h})} = \langle \boldsymbol{h}|\boldsymbol{h}\rangle = \prod_{1\le b<a\le \mathsf{N}} \frac{1}{X_a^{(h_a)} - X_b^{(h_b)}}. \tag{4.51}$$

Proof. Note that the action of a left $\mathscr{B}_-$-eigenstate on a right $\mathscr{B}_-$-eigenstate is zero, for two different $\mathscr{B}_-$-eigenvalues. This implies that the matrix M is diagonal; then to compute its diagonal elements we compute the matrix elements

$$\theta_a \equiv \langle h_1,\dots,h_a,\dots,h_{\mathsf{N}}| \mathscr{A}_-(\xi_a^{(h_a+1)}) |h_1,\dots,h_a+1,\dots,h_{\mathsf{N}}\rangle \quad \text{where} \quad a \in \{1,\dots,\mathsf{N}\}. \tag{4.52}$$

Using the left action of the operator $\mathscr{A}_-(\xi_a^{(h_a+1)})$ we get:

$$\begin{aligned}\theta_a =& \frac{(q-1/q)\mathsf{A}_-(1/\xi_a^{(h_a)})}{((\xi_a^{(h_a)})^2 - 1/(\xi_a^{(h_a)})^2)} \prod_{\substack{b=1\\ b\neq a}}^{\mathsf{N}} \frac{X_a^{(h_a+1)} - X_b^{(h_b)}}{X_a^{(h_a)} - X_b^{(h_b)}} \\ & \times \langle h_1,\dots,h_a+1,\dots,h_{\mathsf{N}}|h_1,\dots,h_a+1,\dots,h_{\mathsf{N}}\rangle\end{aligned} \tag{4.53}$$

while using the decomposition (3.18) and the fact that:

$$\langle h_1,\dots,h_a,\dots,h_{\mathsf{N}}| \mathscr{D}_-(1/\xi_a^{(h_a+1)}) |h_1,\dots,h_a+1,\dots,h_{\mathsf{N}}\rangle = 0 \tag{4.54}$$

it holds:

$$\theta_a = \frac{k_a^{(h_a+1)}(q-1/q)\mathsf{A}_-(1/\xi_a^{(h_a)})}{((\xi_a^{(h_a+1)})^2 - 1/(\xi_a^{(h_a+1)})^2)} \langle h_1,\dots,h_a,\dots,h_{\mathsf{N}}|h_1,\dots,h_a,\dots,h_{\mathsf{N}}\rangle, \tag{4.55}$$

and so:

$$\theta_a = \frac{(q-1/q)\mathsf{A}_-(1/\xi_a^{(h_a)})}{((\xi_a^{(h_a)})^2 - 1/(\xi_a^{(h_a)})^2)} \langle h_1,\dots,h_a,\dots,h_{\mathsf{N}}|h_1,\dots,h_a,\dots,h_{\mathsf{N}}\rangle. \tag{4.56}$$

These results lead to the identity:

$$\frac{\langle h_1,\dots,h_a+1,\dots,h_{\mathsf{N}}|h_1,\dots,h_a+1,\dots,h_{\mathsf{N}}\rangle}{\langle h_1,\dots,h_a,\dots,h_{\mathsf{N}}|h_1,\dots,h_a,\dots,h_{\mathsf{N}}\rangle} = \prod_{\substack{b=1\\ b\neq a}}^{\mathsf{N}} \frac{X_a^{(h_a)} - X_b^{(h_b)}}{X_a^{(h_a+1)} - X_b^{(h_b)}}, \tag{4.57}$$

from which one can prove:

$$\frac{\langle h_1,\dots,h_{\mathsf{N}}|h_1,\dots,h_{\mathsf{N}}\rangle}{\langle p-1,\dots,p-1|p-1,\dots,p-1\rangle} = \prod_{1\le b<a\le \mathsf{N}} \frac{X_a^{(p-1)} - X_b^{(p-1)}}{X_a^{(h_a)} - X_b^{(h_b)}}. \tag{4.58}$$

This proves the proposition being

$$\langle p-1,\dots,p-1|p-1,\dots,p-1\rangle = \prod_{1\le b<a\le \mathsf{N}} \frac{1}{X_a^{(p-1)} - X_b^{(p-1)}}, \tag{4.59}$$

using the following choice of the normalization:

$$\mathsf{N} = \left(\prod_{1\le b<a\le \mathsf{N}} \left(X_a^{(p-1)} - X_b^{(p-1)}\right) \langle \Omega| \prod_{n=1}^{\mathsf{N}} \prod_{k_n=0}^{p-2} \frac{\mathscr{A}_-(1/\xi_n^{(k_n)})}{\mathsf{A}_-(1/\xi_n^{(k_n)})} |\bar{\Omega}\rangle \right)^{1/2}. \tag{4.60}$$

□

The previous theorem implies the following spectral decomposition of the identity $\mathbb{I}$ in the SoV basis:

$$\mathbb{I} \equiv \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{p-1} \prod_{1\le b<a\le \mathsf{N}} (X_a^{(h_a)} - X_a^{(h_a)}) |h_1,\dots,h_{\mathsf{N}}\rangle\langle h_1,\dots,h_{\mathsf{N}}|. \tag{4.61}$$

4.4 Separate states and their scalar products

Let us introduce a class of left and right states, the so-called separate states, characterized by the following type of decompositions in the left and right SoV-basis:

$$\langle \alpha| = \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{p-1} \prod_{a=1}^{\mathsf{N}} \alpha_a^{(h_a)} \prod_{1\le b<a\le \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)}) \langle h_1,\dots,h_{\mathsf{N}}|, \tag{4.62}$$

$$|\beta\rangle = \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{p-1} \prod_{a=1}^{\mathsf{N}} \beta_a^{(h_a)} \prod_{1\le b<a\le \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)}) |h_1,\dots,h_{\mathsf{N}}\rangle, \tag{4.63}$$

where the coefficients $\alpha_a^{(h_a)}$ and $\beta_a^{(h_a)}$ are arbitrary complex numbers, meaning that the coefficients of these separate states have a factorized form in this basis. These separate states are interesting at least for two reasons : they admit simple determinant scalar products, as it will be shown in the next proposition, and the eigenstates of the transfer matrix are special separate states, as we will show in the next section.

Proposition 4.2. *Let us take an arbitrary separate left state $\langle \alpha|$ (separate covector) and an arbitrary separate right state $|\beta\rangle$ (separate vector) then it holds:*

$$\langle \alpha|\beta\rangle = det_{\mathsf{N}} ||\mathscr{M}_{a,b}^{(\alpha,\beta)}|| \quad with \quad \mathscr{M}_{a,b}^{(\alpha,\beta)} \equiv \sum_{h=0}^{p-1} \alpha_a^{(h)} \beta_a^{(h)} (X_a^{(h)})^{(b-1)}. \tag{4.64}$$

Proof. The proof follows the same method as in [97]. The formula (4.51) implies, using the representation of the states $\langle\alpha|$ and $|\beta\rangle$, the following:

$$\langle\alpha|\beta\rangle = \sum_{h_1,\dots,h_{\mathsf{N}}=0}^{p-1} V(X_1^{(h_1)},\dots,X_{\mathsf{N}}^{(h_{\mathsf{N}})})\prod_{a=1}^{\mathsf{N}}\alpha_a^{(h_a)}\beta_a^{(h_a)}, \tag{4.65}$$

where we have denoted by $V(x_1,\dots,x_{\mathsf{N}})\equiv\prod_{1\le b<a\le \mathsf{N}}(x_a-x_b)$ the Vandermonde determinant. Finally, using the multilinearity of the determinant we get our result. □

5 $\mathcal{T}$-spectrum characterization in the SoV basis

In this section we present the complete characterization of the spectrum of the transfer matrix $\mathcal{T}(\lambda)$ associated to the cyclic representations of the 6-vertex reflection algebra. We first present some preliminary properties satisfied by all the eigenvalue functions of the transfer matrix $\mathcal{T}(\lambda)$:

Lemma 5.1. *Denote by $\Sigma_{\mathcal{T}}$ the transfer matrix spectrum, then any $\tau(\lambda)\in\Sigma_{\mathcal{T}}$ is an even function of λ invariant under the transformation $\lambda\to 1/\lambda$ which admits the following interpolation formula:*

$$\begin{aligned}\tau(\lambda) ={}& \sum_{a=1}^{\mathsf{N}}\frac{\Lambda^2-X^2}{(X_a^{(0)})^2-X^2}\prod_{\substack{b=1\\ b\neq a}}^{\mathsf{N}}\frac{\Lambda-X_b^{(0)}}{X_a^{(0)}-X_b^{(0)}}\tau(\zeta_a^{(0)})+(-1)^{\mathsf{N}}\frac{(\Lambda+X)}{2}\prod_{b=1}^{\mathsf{N}}\frac{\Lambda-X_b^{(0)}}{X-X_b^{(0)}}det_qM(1)\\ &-(-1)^{\mathsf{N}}\frac{(\Lambda-X)}{2}\prod_{b=1}^{\mathsf{N}}\frac{\Lambda-X_b^{(0)}}{X+X_b^{(0)}}\frac{(\zeta_++1/\zeta_+)}{(\zeta_+-1/\zeta_+)}\frac{(\zeta_-+1/\zeta_-)}{(\zeta_--1/\zeta_-)}det_qM(i)\\ &+(\Lambda^2-X^2)\tau_\infty\prod_{b=1}^{\mathsf{N}}(\Lambda-X_b^{(0)}),\end{aligned} \tag{5.1}$$

where:

$$\tau_\infty\equiv\frac{\kappa_+\kappa_-(e^{\tau_+-\tau_-}\prod_{b=1}^{\mathsf{N}}\delta_b\gamma_b+e^{\tau_--\tau_+}\prod_{b=1}^{\mathsf{N}}\alpha_b\beta_b)}{(\zeta_+-1/\zeta_+)(\zeta_--1/\zeta_-)}. \tag{5.2}$$

Proof. In the previous section, we have shown that the transformations $\lambda\to-\lambda$ and $\lambda\to 1/\lambda$ are symmetries of the transfer matrix $\mathcal{T}(\lambda)$ so if $\tau(\lambda)\in\Sigma_{\mathcal{T}}$ then $\tau(\lambda)$ is left unchanged under these transformations. Moreover, the asymptotic of the transfer matrix can be easily derived by direct computations, it is central and it holds:

$$\tau_\infty=\lim_{\log\lambda\to\pm\infty}\lambda^{\mp 2(\mathsf{N}+2)}\mathcal{T}(\lambda). \tag{5.3}$$

The identities (4.33) imply after some simple computation that the transfer matrix is central in $q^{\pm 1/2}$ and $iq^{\pm1/2}$ and that it holds:

$$\mathcal{T}(q^{\pm1/2})=(-1)^{\mathsf{N}}X\det_qM(1),\quad \mathcal{T}(iq^{\pm1/2})=(-1)^{\mathsf{N}}X\frac{(\zeta_++1/\zeta_+)}{(\zeta_+-1/\zeta_+)}\frac{(\zeta_-+1/\zeta_-)}{(\zeta_--1/\zeta_-)}\det_qM(i). \tag{5.4}$$

The known functional form of $\mathcal{T}(\lambda)$ with regards to λ together with this identities imply the interpolation formula in the lemma. □

The previous lemma defines the set of polynomials to which belong the transfer matrix eigenvalues; in order to completely characterize the eigenvalues we introduce now the following one-parameter family $D_\tau(\lambda)$ of $p\times p$ matrices:

$$D_\tau(\lambda)\equiv\begin{pmatrix}\tau(\lambda) & -\mathrm{A}(1/\lambda) & 0 & \cdots & 0 & -\mathrm{A}(\lambda)\\ -\mathrm{A}(q\lambda) & \tau(q\lambda) & -\mathrm{A}(1/(q\lambda)) & 0 & \cdots & 0\\ 0 & \ddots & & & & \vdots\\ \vdots & & \cdots & & & \vdots\\ \vdots & & & \cdots & & \vdots\\ \vdots & & & & \ddots & 0\\ 0 & \cdots & 0 & -\mathrm{A}(q^{2l-1}\lambda) & \tau(q^{2l-1}\lambda) & -\mathrm{A}(1/(q^{2l-1}\lambda))\\ -\mathrm{A}(1/(q^{2l}\lambda)) & 0 & \cdots & 0 & -\mathrm{A}(q^{2l}\lambda) & \tau(q^{2l}\lambda)\end{pmatrix}, \tag{5.5}$$

where for now $\tau(\lambda)$ is a generic function and we have defined:

$$\mathrm{A}(\lambda)=\mathsf{a}_+(\lambda)\mathsf{A}_-(\lambda). \tag{5.6}$$

Note that the coefficient $\mathrm{A}(\lambda)$ satisfies the quantum determinant condition:

$$\mathrm{A}(\lambda q^{1/2})\mathrm{A}(q^{1/2}/\lambda)=\frac{\mathsf{a}_+(\lambda q^{1/2})\mathsf{a}_+(q^{1/2}/\lambda)\det_q\mathcal{U}_-(\lambda)}{(\lambda/q)^2-(q/\lambda)^2}. \tag{5.7}$$

Lemma 5.2. *Let $\tau(\lambda)$ be a function of λ invariant under the transformation $\lambda\to1/\lambda$ then $det_pD_\tau(\lambda)$ is a function of λ^p invariant under the transformation $\lambda^p\to1/\lambda^p$.*

Proof. Let us observe that for the invariance of the function $\tau(\lambda)$ under $\lambda\to1/\lambda$, we have that:

$$D_\tau(1/\lambda)=O_CO_R(D_\tau(\lambda)), \tag{5.8}$$

where we have denoted by O_R the operation on a $p\times p$ matrix which exchanges the couple of rows $p-i$ with $i+2$ for any $i\in\{0,\dots,(p-3)/2\}$, similarly O_C is the operation on a $p\times p$ matrix which exchanges the couple of columns $p-i$ with $i+2$ for any $i\in\{0,\dots,(p-3)/2\}$. It is then trivial to see that:

$$\det_pD_\tau(1/\lambda)=\det_pD_\tau(\lambda). \tag{5.9}$$

Let us now observe that:

$$D_\tau(\lambda q)=C_{p\to1}R_{p\to1}(D_\tau(\lambda)), \tag{5.10}$$

where $R_{p\to1}$ is the operation on a $p\times p$ matrix which move the last row in the first row leaving the order of the others unchanged and similarly $C_{p\to1}$ is the operation on a $p\times p$ matrix which move the last column in the first column leaving the order of the others unchanged. This clearly implies that:

$$\det_pD_\tau(q\lambda)=\det_pD_\tau(\lambda), \tag{5.11}$$

which completes the proof of the lemma. □

As mentioned before, we will restrict our attention to the special boundary condition $b_+(\lambda)=0$, for which the SoV-basis coincides with the $\mathcal{B}_-$-eigenstates basis. That is we consider the spectral problem of the transfer matrix:

$$\mathcal{T}(\lambda)\equiv\mathcal{T}_\backslash(\lambda)+c_+(\lambda)\,\mathcal{B}_-(\lambda), \tag{5.12}$$

under the following conditions on the boundary parameters:

$$b_+(\lambda)=0\quad\text{and}\quad b_-(\lambda)\neq0. \tag{5.13}$$

Note that the condition $b_+(\lambda)=0$, keeping instead if desired a $c_+(\lambda)\neq 0$, can be simply realized by the following renormalization of the boundary parameters $\kappa_+=e^{-\gamma}\kappa'_+$ and $e^{\tau_+}=e^{\tau'_+-\gamma}$ by sending $\gamma\rightarrow+\infty$. Under this limit the asymptotic of the transfer matrix reads:

$$\tau_\infty=\frac{(-1)^{\mathsf{N}}\kappa'_+\kappa_- e^{\tau_--\tau'_+}\prod_{b=1}^{\mathsf{N}}\alpha_b\beta_b}{(\zeta_+-1/\zeta_+)(\zeta_--1/\zeta_-)}. \tag{5.14}$$

In the following we will suppress the unnecessary prime in κ_+ and τ_+.

Theorem 5.1. *If the conditions:*

$$\mu_{n,+}^2\neq q^{-2h}\zeta_+^{\pm 2},\quad \forall h\in\{1,...,p-1\},\ n\in\{1,...,\mathsf{N}\} \tag{5.15}$$

and (4.23)-(4.24) *are satisfied, then* $\mathcal{T}(\lambda)$ *has simple spectrum and* $\Sigma_{\mathcal{T}}$ *coincides with the set of polynomials* $\tau(\lambda)$ *of the form* (5.1) *with* (5.14) *which satisfy the following discrete system of equations:*

$$detD_\tau(\zeta_a^{(0)})=0,\ \forall a\in\{1,...,\mathsf{N}\}. \tag{5.16}$$

I) The right $\mathcal{T}$*-eigenstate corresponding to* $\tau(\lambda)\in\Sigma_{\mathcal{T}}$ *is defined by the following decomposition in the right SoV-basis:*

$$|\tau\rangle=\sum_{h_1,...,h_{\mathsf{N}}=0}^{p-1}\prod_{a=1}^{\mathsf{N}}\mathrm{Q}_{\tau,a}^{(h_a)}\prod_{1\leq b<a\leq\mathsf{N}}(X_a^{(h_a)}-X_b^{(h_b)})|h_1,...,h_{\mathsf{N}}\rangle, \tag{5.17}$$

where the $\mathrm{Q}_{\tau,a}^{(h_a)}$ *are the unique nontrivial solution up to normalization of the linear homogeneous system:*

$$D_\tau(\zeta_a^{(0)})\begin{pmatrix}\mathrm{Q}_{\tau,a}^{(0)}\\ \vdots\\ \mathrm{Q}_{\tau,a}^{(p-1)}\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}. \tag{5.18}$$

II) The left $\mathcal{T}$*-eigenstate corresponding to* $\tau(\lambda)\in\Sigma_{\mathcal{T}}$ *is defined by the following decomposition in the left SoV-basis:*

$$\langle\tau|=\sum_{h_1,...,h_{\mathsf{N}}=0}^{p-1}\prod_{a=1}^{\mathsf{N}}\hat{\mathrm{Q}}_{\tau,a}^{(h_a)}\prod_{1\leq b<a\leq\mathsf{N}}(X_a^{(h_a)}-X_b^{(h_b)})\langle h_1,...,h_{\mathsf{N}}|, \tag{5.19}$$

where the $\hat{\mathrm{Q}}_{\tau,a}^{(h_a)}$ *are the unique nontrivial solution up to normalization of the linear homogeneous system:*

$$\begin{pmatrix}\hat{\mathrm{Q}}_{\tau,a}^{(0)} & \ldots & \hat{\mathrm{Q}}_{\tau,a}^{(p-1)}\end{pmatrix}\left(\hat{D}_\tau(\zeta_a^{(0)})\right)^{t_0}=\begin{pmatrix}0 & \ldots & 0\end{pmatrix}, \tag{5.20}$$

and $\hat{D}_\tau(\lambda)$ *is the family of* $p\times p$ *matrices defined substituting in* $D_\tau(\lambda)$ *the coefficient* $\mathrm{A}(\lambda)$ *with*

$$\mathrm{D}(\lambda)=\mathsf{d}_+(\lambda)\mathsf{D}_-(\lambda). \tag{5.21}$$

Finally, using ideas from [97], let us note that if $\tau(\lambda)\neq\tau'(\lambda)\in\Sigma_{\mathcal{T}}$*:*

$$\sum_{b=1}^{\mathsf{N}}\mathcal{M}_{a,b}^{(\tau,\tau')}x_b^{(\tau,\tau')}=0\qquad\forall a\in\{1,...,\mathsf{N}\}, \tag{5.22}$$

where the $x_b^{(\tau,\tau')}$ *are defined by:*

$$\tau(\lambda)-\tau'(\lambda)\equiv\left(\Lambda^2-X^2\right)\sum_{b=1}^{\mathsf{N}}x_b^{(\tau,\tau')}\Lambda^{b-1}, \tag{5.23}$$

which in particular implies that the action of $\langle\tau|$ *on* $|\tau'\rangle$ *is zero.*

Proof. The spectrum (eigenvalues and eigenstates) of the transfer matrix $\mathcal{T}(\lambda)$ in the SoV-basis is characterized by the following discrete system of equations:

$$\tau(\xi_n^{(h_n)})\Psi_\tau(\boldsymbol{h})=\mathrm{A}(\xi_n^{(h_n)})\Psi_\tau(\mathsf{T}_n^-(\boldsymbol{h}))+\mathrm{A}(1/\xi_n^{(h_n)})\Psi_\tau(\mathsf{T}_n^+(\boldsymbol{h})), \tag{5.24}$$

for any $n\in\{1,...,\mathsf{N}\}$ and $\boldsymbol{h}\in\{0,...,p-1\}^{\mathsf{N}}$, i.e. a system of p^{N} Baxter-like equations in the *wave-functions*:

$$\Psi_\tau(\boldsymbol{h})\equiv\langle h_1,...,h_{\mathsf{N}}|\tau\rangle, \tag{5.25}$$

of the $\mathcal{T}$-eigenstate $|\tau\rangle$ associated to $\tau(\lambda)\in\Sigma_{\mathcal{T}}$, where:

$$\mathsf{T}_n^\pm(\boldsymbol{h})\equiv(h_1,\ldots,h_n\pm 1,\ldots,h_{\mathsf{N}}). \tag{5.26}$$

This system admits the following equivalent representation as N linear systems of homogeneous equations:

$$D_\tau(\xi_n^{(0)})\begin{pmatrix}\Psi_\tau(h_1,...,h_n=0,...,h_{\mathsf{N}})\\ \Psi_\tau(h_1,...,h_n=1,...,h_{\mathsf{N}})\\ \vdots\\ \Psi_\tau(h_1,...,h_n=p-1,...,h_{\mathsf{N}})\end{pmatrix}=\begin{pmatrix}0\\0\\ \vdots\\0\end{pmatrix}, \tag{5.27}$$

for any $n\in\{1,...,\mathsf{N}\}$ and for any $h_{m\neq n}$ in $\{0,...,p-1\}$. Then the condition $\tau(\lambda)\in\Sigma_{\mathcal{T}}$ implies the compatibility equations for these linear systems, i.e. it must hold:

$$\det D_\tau(\xi_a^{(0)})=0,\ \forall a\in\{1,...,\mathsf{N}\} \tag{5.28}$$

Note that for the previous lemma this condition is verified also in the points $\zeta_a^{(0)}=1/\xi_{a-\mathsf{N}}^{(0)}$ for any $a\in\{\mathsf{N}+1,...,2\mathsf{N}\}$. The rank of the matrices in (5.27) is $p-1$ being

$$\mathrm{A}(\xi_n^{(h)})\neq 0,\ \mathrm{A}(1/\xi_n^{(h-1)})\neq 0\ \ \forall h\in\{1,...,p-1\},\ n\in\{1,...,\mathsf{N}\}, \tag{5.29}$$

for the conditions (4.23), (4.24) and (5.15). Then (up to an overall normalization) the solution is unique and independent from the $h_{m\neq n}\in\{0,...,p-1\}$ for any $n\in\{1,...,\mathsf{N}\}$. So fixing $\tau(\lambda)\in\Sigma_{\mathcal{T}}$ there exists (up to normalization) one and only one corresponding $\mathcal{T}$-eigenstate $|\tau\rangle$ with coefficients of the factorized form given in (5.17)-(5.18); i.e. the $\mathcal{T}$-spectrum is simple.

Vice versa, if $\tau(\lambda)$ is in the set of functions (5.1) and satisfies (5.16), then the state $|\tau\rangle$ defined by (5.17)-(5.18) satisfies:

$$\langle h_1,...,h_{\mathsf{N}}|\mathcal{T}(\zeta_n^{(h_n)})|\tau\rangle=\tau(\zeta_n^{(h_n)})\langle h_1,...,h_{\mathsf{N}}|\tau\rangle\ \ \forall n\in\{1,...,\mathsf{N}\} \tag{5.30}$$

for any $\mathcal{B}_-$-eigenstate $\langle h_1,...,h_{\mathsf{N}}|$ and this implies:

$$\langle h_1,...,h_{\mathsf{N}}|\mathcal{T}(\lambda)|\tau\rangle=\tau(\lambda)\langle h_1,...,h_{\mathsf{N}}|\tau\rangle\quad\forall\lambda\in\mathbb{C}, \tag{5.31}$$

i.e. $\tau(\lambda)\in\Sigma_{\mathcal{T}}$ and $|\tau\rangle$ is the corresponding $\mathcal{T}$-eigenstate.

For the left $\mathcal{T}$-eigenstates the proof follows as above, we just remark in this case that the matrix elements:

$$\langle\tau|\mathcal{T}(\zeta_n^{(h_n)})|h_1,...,h_{\mathsf{N}}\rangle, \tag{5.32}$$

are computed in the right $\mathcal{B}_-$-representation:

$$\tau(\zeta_n^{(h_n)})\hat{\Psi}_\tau(\boldsymbol{h})=\mathrm{D}(\zeta_n^{(h_n)})\hat{\Psi}_\tau(\mathsf{T}_n^-(\boldsymbol{h}))+\mathrm{D}(1/\zeta_n^{(h_n)})\hat{\Psi}_\tau(\mathsf{T}_n^+(\boldsymbol{h})),\quad\forall n\in\{1,...,\mathsf{N}\} \tag{5.33}$$

where:

$$\hat{\Psi}_\tau(\boldsymbol{h})\equiv\langle\tau|h_1,...,h_{\mathsf{N}}\rangle,\quad\mathrm{D}(\lambda)\equiv\mathsf{d}_+(\lambda)\mathsf{D}_-(\lambda). \tag{5.34}$$

It is simple to observe that it holds:

$$\det D_\tau(\lambda) = \det \hat{D}_\tau(\lambda), \tag{5.35}$$

as a consequence of the following identities:

$$\mathrm{D}(\lambda) = \alpha(\lambda)\mathrm{A}(q/\lambda) \tag{5.36}$$

and:

$$\alpha(1/\lambda)\alpha(q\lambda) = 1, \quad \prod_{a=0}^{p-1} \alpha(\lambda q^a) = 1 \tag{5.37}$$

where:

$$\alpha(\lambda) = \frac{s(\lambda)}{s(q/\lambda)} k(\lambda), \quad s(\lambda) = \frac{\lambda^2 q - 1/\left(q\lambda^2\right)}{\lambda^2 - 1/\lambda^2}. \tag{5.38}$$

Finally, the identity (5.22) is quite general in the SoV framework and can be proven also in our case of cyclic representations of the 6-vertex reflection algebra by following the same proof first given in the case of a periodic lattice [97]. □

For next applications it is interesting to show that we can obtain the coefficients of a left transfer matrix eigenstates in terms of those of the right one by introducing a recursion formula that produces both coefficients in terms of the transfer matrix eigenvalues. The following lemma holds:

Lemma 5.3. *Let $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ then we have:*

$$\frac{\hat{Q}^{(h)}_{\tau,a}}{\hat{Q}^{(h-1)}_{\tau,a}} = \frac{\mathrm{A}(1/\zeta_a^{(h-1)})}{\mathrm{D}(1/\zeta_a^{(h-1)})} \frac{Q^{(h)}_{\tau,a}}{Q^{(h-1)}_{\tau,a}}, \tag{5.39}$$

being:

$$\frac{\hat{Q}^{(h)}_{\tau,a}}{\hat{Q}^{(0)}_{\tau,a}} = \frac{t^{(h)}_{\tau,a}}{\prod_{b=0}^{h-1} \mathrm{D}(1/\zeta_a^{(b)})}, \quad \frac{Q^{(h)}_{\tau,a}}{Q^{(0)}_{\tau,a}} = \frac{t^{(h)}_{\tau,a}}{\prod_{b=0}^{h-1} \mathrm{A}(1/\zeta_a^{(b)})}, \tag{5.40}$$

where the $t^{(h)}_{\tau,a}$ are defined by the following recursion formula:

$$t^{(h)}_{\tau,a} = \tau(\zeta_a^{(h-1)}) t^{(h-1)}_{\tau,a} - \frac{\det_q K_+(\xi_a^{(h-3/2)}) \det_q \mathcal{U}_-(\xi_a^{(h-3/2)})}{(\xi_a^{(h-1/2)})^2 - 1/(\xi_a^{(h-1/2)})^2} t^{(h-2)}_{\tau,a} \text{ for } h \in \{1, ..., p-1\} \tag{5.41}$$

with the following initial conditions $t^{(-1)}_{\tau,a} = 0$, $t^{(0)}_{\tau,a} = 1$.

Proof. We just need to prove the last two identities in this lemma as the first ones are simple consequences of them. Let us prove the second identity in (5.40). For $h = 1$ we have:

$$\frac{Q^{(1)}_{\tau,a}}{Q^{(0)}_{\tau,a}} = \frac{\tau(\zeta_a^{(0)})}{\mathrm{A}(1/\zeta_a^{(0)})}, \tag{5.42}$$

by the SoV Baxter's like equation and the condition:

$$\mathrm{A}(\zeta_a^{(0)}) = 0, \quad \mathrm{A}(1/\zeta_a^{(0)}) \neq 0. \tag{5.43}$$

This means that the second formula in (5.40) and (5.41) are both satisfied for $h = 1$ once the initial conditions $t^{(-1)}_{\tau,a} = 0$, $t^{(0)}_{\tau,a} = 1$ are imposed. So let us assume that these two identities

are satisfied for any $h \in \{1, ..., x\}$ and let us prove it for $h = x + 1 \leq p - 1$. We have that by the SoV equations it holds:

$$\frac{Q^{(x+1)}_{\tau,a}}{Q^{(x-1)}_{\tau,a}} = \frac{\tau(\zeta_a^{(x)})}{\mathrm{A}(1/\zeta_a^{(x)})} \frac{Q^{(x)}_{\tau,a}}{Q^{(x-1)}_{\tau,a}} - \frac{\mathrm{A}(\zeta_a^{(x)})}{\mathrm{A}(1/\zeta_a^{(x)})} \tag{5.44}$$

and so:

$$\frac{Q^{(x+1)}_{\tau,a}}{Q^{(0)}_{\tau,a}} = \frac{\tau(\zeta_a^{(x)})}{\mathrm{A}(1/\zeta_a^{(x)})} \frac{Q^{(x)}_{\tau,a}}{Q^{(0)}_{\tau,a}} - \frac{\mathrm{A}(\zeta_a^{(x)})}{\mathrm{A}(1/\zeta_a^{(x)})} \frac{Q^{(x-1)}_{\tau,a}}{Q^{(0)}_{\tau,a}} \tag{5.45}$$

which by using the formula (5.40) for $h = x - 1$ and $h = x$ reads:

$$\frac{Q^{(x+1)}_{\tau,a}}{Q^{(0)}_{\tau,a}} = \frac{\tau(\zeta_a^{(x)}) t^{(x)}_{\tau,a} - \mathrm{A}(\zeta_a^{(x)}) \mathrm{A}(1/\zeta_a^{(x-1)}) t^{(x-1)}_{\tau,a}}{\prod_{b=0}^{x} \mathrm{A}(1/\zeta_a^{(b)})}, \tag{5.46}$$

which just proves the formulae (5.40) and (5.41) for $h = x + 1$ once we recall that:

$$\mathrm{A}(\zeta_a^{(x)}) \mathrm{A}(1/\zeta_a^{(x-1)}) = \frac{\det_q K_{a,+}(\xi_a^{(x-1/2)}) \det_q \mathcal{U}_{a,-}(\xi_a^{(x-1/2)})}{(\xi_a^{(h+1/2)})^2 - 1/(\xi_a^{(h+1/2)})^2}. \tag{5.47}$$

The proof of the first identity in (5.40) can be done in the same way, we have just to use now that:

$$\mathrm{D}(\zeta_a^{(x)}) \mathrm{D}(1/\zeta_a^{(x-1)}) = \frac{\det_q K_{a,+}(\xi_a^{(x-1/2)}) \det_q \mathcal{U}_{a,-}(\xi_a^{(x-1/2)})}{(\xi_a^{(h+1/2)})^2 - 1/(\xi_a^{(h+1/2)})^2}. \tag{5.48}$$

□

6 Functional equation characterizing the $\mathcal{T}$-spectrum

In this section we assume that at any quantum site the following constraints are satisfied:

$$b_n^p + a_n^p = 0, \quad \forall n \in \{1, ..., \mathrm{N}\}, \tag{6.1}$$

$$c_n^p + d_n^p = 0, \quad \forall n \in \{1, ..., \mathrm{N}\}. \tag{6.2}$$

Under these conditions we can explicitly construct the left and right basis which diagonalize the one-parameter family of commuting operators $\mathcal{B}_-(\lambda)$ and $\mathcal{C}_-(\lambda)$ associated to the most general $K_-(\lambda)$ matrix. In the previous section we have done this construction for the $\mathcal{B}_-(\lambda)$ eigenstates basis, clearly we can do a similar construction also for $\mathcal{C}_-(\lambda)$. In the previous sections we have explained how the spectral problem of the transfer matrix $\mathcal{T}(\lambda)$ can be characterized by SoV in the $\mathcal{B}_-(\lambda)$ eigenstates basis when $b_+(\lambda) = 0$ keeping instead if desired a $c_+(\lambda) \neq 0$, by the same approach we can characterize the $\mathcal{T}(\lambda)$-spectrum by SoV in the $\mathcal{C}_-(\lambda)$ eigenstates basis when $c_+(\lambda) = 0$ keeping instead if desired a $b_+(\lambda) \neq 0$.

Here, we show that the SoV characterization of the $\mathcal{T}(\lambda)$-spectrum can be reformulated by functional equations. Before this, let us just observe that the central value of the asymptotic of $\mathcal{T}(\lambda)$ is given by:

$$\tau_\infty = \frac{(-1)^{\mathrm{N}} \kappa_+ \kappa_- e^{\epsilon(\tau_- - \tau_+)} \prod_{b=1}^{\mathrm{N}} \alpha_b \beta_b}{(\zeta_+ - 1/\zeta_+)(\zeta_- - 1/\zeta_-)}, \tag{6.3}$$

where $\epsilon = +1$ for $b_+(\lambda) = 0$ (keeping instead a $c_+(\lambda) \neq 0$) and $\epsilon = -1$ for $c_+(\lambda) = 0$ (keeping instead a $b_+(\lambda) \neq 0$). Let us moreover introduce the following notations:

$$\bar{\mathrm{A}}(\lambda) = \lambda q^{-1/2} \mathrm{A}(\lambda), \tag{6.4}$$

$$\bar{\mathrm{A}}_\infty = \lim_{\lambda \to +\infty} \lambda^{-2(\mathrm{N}+2)} \bar{\mathrm{A}}(\lambda) = \frac{(-1)^{\mathrm{N}} \alpha_- \beta_- \zeta_+ \kappa_- \prod_{n=1}^{\mathrm{N}} b_n c_n}{q^{1+\mathrm{N}} (\zeta_+ - 1/\zeta_+)(\zeta_- - 1/\zeta_-)}, \tag{6.5}$$

and:

$$F(\lambda)=\prod_{b=1}^{2\mathsf{N}}\left(\frac{\lambda^p}{\left(\zeta_b^{(0)}\right)^p}-\frac{\left(\zeta_b^{(0)}\right)^p}{\lambda^p}\right), \tag{6.6}$$

then the following results hold:

Proposition 6.1. *If the conditions (4.23), (4.24) and (5.15) are satisfied and:*

$$b_n=-q^{2j_n-1}a_n,\quad d_n=-q^{2j_n-1}c_n, \tag{6.7}$$

then $\mathcal{T}(\lambda)$ has simple spectrum and $\tau(\lambda)$ of the form (5.1) with (5.14) is an element of $\Sigma_{\mathcal{T}}$ if and only if $det_p\bar{D}_\tau(\lambda)$ is a Laurent polynomial of degree $\mathsf{N}+2$ in the variable:

$$Z=\lambda^{2p}+\frac{1}{\lambda^{2p}} \tag{6.8}$$

which satisfies the following functional equation:

$$det_p\bar{D}_\tau(\lambda)-F(\lambda)\sum_{a=0}^{1}det_p\bar{D}_\tau(i^a q^{1/2})\frac{(\lambda^p+(-1)^a/\lambda^p)^2}{4(-1)^a F(i^a q^{1/2})}=\left(\tau_\infty^p-\bar{\mathrm{A}}_\infty^p\right)F(\lambda)\left(\lambda^{2p}-\frac{1}{\lambda^{2p}}\right)^2. \tag{6.9}$$

Here $\bar{D}_\tau(\lambda)$ is obtained from $D_\tau(\lambda)$ by substituting in it $\mathrm{A}(\lambda)$ with $\bar{\mathrm{A}}(\lambda)$.

Proof. Let us observe that $\det_p\bar{D}_\tau(\lambda)$ is an even function of λ as a consequence of the parity of $\tau(\lambda)$ and $\bar{\mathrm{A}}(\lambda)$ moreover following the same steps of Lemma 5.2 we can prove that $\det_p\bar{D}_\tau(\lambda)$ is invariant under the transformations $\lambda\to 1/\lambda$ and $\lambda\to q\lambda$, so that $\det_p\bar{D}_\tau(\lambda)$ is indeed a function of $Z=\lambda^{2p}+1/\lambda^{2p}$. Let us now observe that in the points $\lambda=q^{1/2}$ and $\lambda=iq^{1/2}$, the central row of the matrix $\bar{D}_\tau(\lambda)$ has two elements which are divergent as proportional to $\bar{\mathrm{A}}(\pm q^{p/2})$ and $\bar{\mathrm{A}}(\pm iq^{p/2})$, respectively. In the following we prove that:

$$\det_p\bar{D}_\tau(q^{1/2})\text{ and }\det_p\bar{D}_\tau(iq^{1/2})\text{ are finite,} \tag{6.10}$$

if we ask that the function $\tau(\lambda)$ has the functional form (5.1). Let us first introduce the notations:

$$\bar{\mathrm{X}}(\lambda)=\left(\lambda^2-\frac{1}{\lambda^2}\right)\bar{\mathrm{A}}(\lambda), \tag{6.11}$$

and using it let us expand the determinant:

$$\begin{aligned}\det_p\bar{D}_\tau(\lambda q^{1/2})=&\tau(\lambda)\det_{p-1}\bar{D}_{\tau,(p+1)/2,(p+1)/2}(\lambda q^{1/2})+\frac{\bar{\mathrm{X}}(\lambda)\det_{p-1}\bar{D}_{\tau,(p+1)/2,(p+1)/2-1}(\lambda q^{1/2})}{\lambda^2-1/\lambda^2}\\ &-\frac{\bar{\mathrm{X}}(1/\lambda)\det_{p-1}\bar{D}_{\tau,(p+1)/2,(p+1)/2+1}(\lambda q^{1/2})}{\lambda^2-1/\lambda^2},\end{aligned} \tag{6.12}$$

with regards to the central row. Here, we have denoted with $\bar{D}_{\tau,i,j}(\lambda)$ the $(p-1)\times(p-1)$ matrix defined by removing the row i and the column j to the matrix $\bar{D}_\tau(\lambda)$. The following identity holds:

$$\det_{p-1}\bar{D}_{\tau,(p+1)/2,(p+1)/2+1}(\lambda q^{1/2})=\det_{p-1}\bar{D}_{\tau,(p+1)/2,(p+1)/2-1}(q^{1/2}/\lambda), \tag{6.13}$$

and it follows that just exchanging the row j with the row $p-j$, for any $j\in\{1,..,(p+1)/2\}$, and then the column j with the column $p-j$, for any $j\in\{1,..,(p+1)/2\}$, in the matrix $\bar{D}_{\tau,(p+1)/2,(p+1)/2+1}(\lambda q^{1/2})$. Note that the determinants $\det_{p-1}\bar{D}_{\tau,(p+1)/2,(p+1)/2+1}(\lambda q^{1/2})$ and $\det_{p-1}\bar{D}_{\tau,(p+1)/2,(p+1)/2-1}(\lambda q^{1/2})$ are Laurent's rational functions both finite for $\lambda\to\pm 1$ and

$\lambda\to\pm i$. This implies that in both the limits $\lambda\to\pm1$ and $\lambda\to\pm i$ the function $\det_p\bar{D}_\tau(\lambda q^{1/2})$ is finite. As $Z=2$ and $Z=-2$ are the only points for which $\det_p\bar{D}_\tau(\lambda)$ may have divergencies, then, from $\tau(\lambda)$ and $\bar{\mathrm{X}}(\lambda)$ Laurent's polynomial in λ of degree $2\mathsf{N}+4$, it follows that $\det_p\bar{D}_\tau(\lambda)$ is a polynomial of degree $\mathsf{N}+2$ in the variable Z. Let us now remark that by the SoV characterization of the spectrum we have that $\tau(\lambda)\in\Sigma_{\mathcal{T}}$ if and only if it has the form (5.1) and satisfies:

$$\det_p\bar{D}_\tau(\xi_a^{(0)})=0,\ \forall a\in\{1,...,\mathsf{N}\}, \tag{6.14}$$

as we are assuming that (6.7) is satisfied. Finally, it is simple to verify that the following asymptotic hold:

$$\det_p\left[\lim_{\lambda\to+\infty}\lambda^{-2(\mathsf{N}+2)}\bar{D}_\tau(\lambda)\right]=\det_p\left[\lim_{\lambda\to0}\lambda^{2(\mathsf{N}+2)}\bar{D}_\tau(\lambda)\right]^t \tag{6.15}$$

$$=\det_p\begin{pmatrix}\tau_\infty & 0 & 0 & \cdots & 0 & -\bar{\mathrm{A}}_\infty\\ -x\bar{\mathrm{A}}_\infty & x\tau_\infty & 0 & 0 & \cdots & 0\\ 0 & -x^2\bar{\mathrm{A}}_\infty & x^2\tau_\infty & \ddots & & \vdots\\ \vdots & & \ddots & \ddots & 0 & 0\\ 0 & \cdots & 0 & -x^{2l-1}\bar{\mathrm{A}}_\infty & x^{2l-1}\tau_\infty & 0\\ 0 & 0 & \cdots & 0 & -x^{2l}\bar{\mathrm{A}}_\infty & x^{2l}\tau_\infty\end{pmatrix}, \tag{6.16}$$

where we have denoted with ^{t} the transpose of the matrix and $x=q^{2(\mathsf{N}+2)}$, so that it holds:

$$\lim_{\log\lambda\to\pm\infty}\lambda^{\mp2p(\mathsf{N}+2)}\det_p\bar{D}_\tau(\lambda)=\tau_\infty^p-\bar{\mathrm{A}}_\infty^p. \tag{6.17}$$

These results fix completely the Laurent's polynomial $\det_p\bar{D}_\tau(\lambda)$ to satisfy (6.9). □

Let us introduce now the following function:

$$\begin{aligned}G(\lambda|x,y)=&F(\lambda)\Bigg[\left(\tau_\infty-q^{-2(p-1)\mathsf{N}}\bar{\mathrm{A}}_\infty\right)x\prod_{a=0}^{1}\left(\frac{\lambda}{i^a q^{1/2}}-\frac{i^a q^{1/2}}{\lambda}\right)\left(i^a q^{1/2}\lambda-\frac{1}{i^a q^{1/2}\lambda}\right)\\ &+(i-1)\frac{y\mathrm{A}(iq^{1/2})}{4F(iq^{1/2})}\prod_{a=0}^{1}\left(\lambda q^{(1-2a)/2}-\frac{1}{\lambda q^{(1-2a)/2}}\right)\Bigg],\end{aligned} \tag{6.18}$$

and the states:

$$\langle\omega|=\sum_{h_1,\dots,h_\mathsf{N}=0}^{p-1}\prod_{a=1}^{\mathsf{N}}\prod_{k_a=0}^{h_a-1}\frac{\mathrm{A}(1/\zeta_a^{(k_a)})}{\mathrm{D}(1/\zeta_a^{(k_a)})}\prod_{1\leq b<a\leq\mathsf{N}}(X_a^{(h_a)}-X_b^{(h_b)})\,\langle h_1,\dots,h_\mathsf{N}|, \tag{6.19}$$

$$|\bar{\omega}\rangle=\sum_{h_1,\dots,h_\mathsf{N}=0}^{p-1}\prod_{1\leq b<a\leq\mathsf{N}}(X_a^{(h_a)}-X_b^{(h_b)})|h_1,\dots,h_\mathsf{N}\rangle, \tag{6.20}$$

and a renormalization of the operator

$$\hat{\mathscr{B}}_-(\lambda)=\frac{(\zeta_--1/\zeta_-)}{\kappa_-e^{\tau_-}(\lambda^2/q-q/\lambda^2)\prod_{n=1}^{\mathsf{N}}\alpha_n\beta_n}\mathscr{B}_-(\lambda), \tag{6.21}$$

which is a degree N polynomial in Λ. In the following we denote with $Q(\lambda)$ a polynomial in $\Lambda=\lambda^2+1/\lambda^2$ of degree $\mathsf{N}_Q\leq(p-1)\mathsf{N}$ which admits the following interpolation formula:

$$Q(\lambda)=\sum_{a=1}^{(p-1)\mathsf{N}+1}\prod_{b=1,b\neq a}^{(p-1)\mathsf{N}+1}\frac{\Lambda-w_b}{w_a-w_b}Q(\xi_a), \tag{6.22}$$

where $w_b = \xi_b^2 + 1/\xi_b^2$ for any $b \in \{1, ..., (p-1)\mathsf{N}+1\}$ and where we have defined:

$$\xi_{s(n,h_n)} \equiv \xi_n^{(h_n-1)},\ s(n,h_n) \equiv (n-1)(p-1)+h_n,\ \forall n \in \{1,...,\mathsf{N}\}, h_n \in \{1,...,p-1\}, \quad (6.23)$$

while $\xi_{(p-1)\mathsf{N}+1}$ is arbitrary. Moreover, we use the notation:

$$q_\infty \equiv \sum_{a=1}^{(p-1)\mathsf{N}+1} \prod_{b=1,b\neq a}^{(p-1)\mathsf{N}+1} \frac{Q(\xi_a)}{w_a - w_b},\ \ q_0 \equiv Q(iq^{1/2}). \quad (6.24)$$

Here, q_∞ is the coefficient in $\Lambda^{(p-1)\mathsf{N}}$ of the power expansion of the polynomial $Q(\lambda)$. Once this notation are introduced, the previous characterization of the spectrum can be reformulated in terms of Baxter's type TQ-functional equations and the eigenstates admit an algebraic Bethe ansatz like reformulation, as we show in the next theorem.

Theorem 6.1. *Let the conditions (4.23), (4.24), (5.15) and (6.7) be satisfied and let $\tau(\lambda)$ be an entire function for which there exists a polynomial $Q(\lambda)$ of the form (6.22) satisfying the conditions:*

$$Q(\xi_a^{(0)}) \neq 0,\ \forall a \in \{1,...,\mathsf{N}\}, \quad (6.25)$$

and the following functional equation:

$$\tau(\lambda)Q(\lambda) = \bar{\mathrm{A}}(\lambda)Q(\lambda/q) + \bar{\mathrm{A}}(1/\lambda)Q(\lambda q) + G(\lambda|q_\infty, q_0), \quad (6.26)$$

with $(p-1)\mathsf{N}-1 \leq \mathsf{N}_Q$, then $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ and (up to normalization) the left and right transfer matrix eigenstates associated to it admit the following Bethe ansatz like representations:

$$\langle\tau| = \langle\omega|\prod_{b=1}^{\mathsf{N}_Q}\hat{\mathscr{B}}_-(\lambda_b),\quad |\tau\rangle = \prod_{b=1}^{\mathsf{N}_Q}\hat{\mathscr{B}}_-(\lambda_b)|\bar{\omega}\rangle, \quad (6.27)$$

where the λ_b (fixed up the symmetry $\lambda_b \to -\lambda_b$, $\lambda_b \to 1/\lambda_b$) for $b \in \{1,...,\mathsf{N}_Q\}$ are the zeros of $Q(\lambda)$. Vice versa, if $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ then there exists a polynomial $Q(\lambda)$ of the form (6.22) with $(p-1)\mathsf{N}-1 \leq \mathsf{N}_Q$ satisfying with $\tau(\lambda)$ the Baxter's equation (6.26).

Proof. The proof of this type of reformulation of the spectrum is by now quite standard and it has been proven for several models once they admit SoV description, see for example [36, 94–96, 101–103]. So we will try to point out just some main features of the proof. Let us start proving the first part of the statement. It is simple to remark that the r.h.s. of the equation (6.26) is a Laurent polynomial in λ, indeed we can write:

$$\bar{\mathrm{A}}(\lambda)Q(\lambda/q) + \bar{\mathrm{A}}(1/\lambda)Q(\lambda q) = \frac{\bar{\mathrm{X}}(\lambda)Q(q/\lambda) - \bar{\mathrm{X}}(1/\lambda)Q(\lambda q)}{\lambda^2 - 1/\lambda^2} \quad (6.28)$$

so that the limits $\lambda \to \pm 1$, $\lambda \to \pm i$ are all finite. Moreover, it is simple to observe that the r.h.s. of (6.26) is invariant under the transformations $\lambda \to -\lambda$ and $\lambda \to 1/\lambda$, so that the r.h.s. of (6.26) is in fact a polynomial of degree $\mathsf{N}_Q + \mathsf{N} + 2 \leq p\mathsf{N} + 2$ in Λ. Then, the fact that $\tau(\lambda)$ is entire in λ implies by the equation (6.26) that $\tau(\lambda)$ is a polynomial in Λ of the form (5.1) with (5.14). This together with the equations:

$$\det_p \bar{D}_\tau(\xi_a^{(0)}) = 0,\ \forall a \in \{1,...,\mathsf{N}\}, \quad (6.29)$$

which are trivial consequences of (6.26) and of (6.25), imply by the SoV characterization that $\tau(\lambda) \in \Sigma_{\mathcal{T}}$. Let us show now that the eigenstates associated to this $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ can be written

in the form (6.27). In order to do so let us observe that the $Q(\lambda)$ which is solution of (6.26) is defined up to a constant factor so that we are free to fix it by writing:

$$Q(\lambda) = \prod_{b=1}^{\mathsf{N}_Q}(\Lambda - \Lambda_b) \quad (6.30)$$

where $\Lambda_b = \lambda_b^2 + 1/\lambda_b^2$, then we have that it holds:

$$\begin{aligned}\prod_{b=1}^{\mathsf{N}_Q}\hat{\mathscr{B}}_-(\lambda_b)|\bar{\omega}\rangle &= \sum_{h_1,...,h_\mathsf{N}=0}^{p-1}\prod_{b=1}^{\mathsf{N}_Q}\prod_{a=1}^{\mathsf{N}}(X_a^{(h_a)} - \Lambda_b)\prod_{1\leq b<a\leq \mathsf{N}}(X_a^{(h_a)} - X_b^{(h_b)})|h_1,...,h_\mathsf{N}\rangle \\ &= \sum_{h_1,...,h_\mathsf{N}=0}^{p-1}\prod_{a=1}^{\mathsf{N}}Q(X_a^{(h_a)})\prod_{1\leq b<a\leq \mathsf{N}}(X_a^{(h_a)} - X_b^{(h_b)})|h_1,...,h_\mathsf{N}\rangle, \end{aligned} \quad (6.31)$$

where in the last line appears the SoV characterization of the right eigenstate associated to $\tau(\lambda) \in \Sigma_{\mathcal{T}}$. The same steps are used to prove (6.27) for the left eigenstate. Let us comment that the polynomiality of $Q(\lambda)$ is the central property that we have used to prove these rewriting of the SoV characterizations of the transfer matrix eigenstates. This type of rewriting was first observed in the case of the noncompact XXX chains [90]. It is quite general and it has been used already for different models in the SoV framework, see for example [101–103].

Let us now assume that $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ and let us prove that it exists $Q(\lambda)$ of the form (6.22). In order to do so it is enough to show that the Baxter's equation holds in $p\mathsf{N}+2$ different values of Λ, that we chose to be $\Lambda = \pm(q+1/q)$ and the $\Lambda = X_a^{(h_a)}$ for any $a \in \{1,...,\mathsf{N}\}$ and $h_a \in \{0,...,p-1\}$. This set of equations can be organized in the following form:

$$\bar{D}_\tau(\xi_a^{(0)})\begin{pmatrix} Q(\xi_a^{(h=0)}) \\ Q(\xi_a^{(h=1)}) \\ \vdots \\ \vdots \\ Q(\xi_a^{(h=p-1)}) \end{pmatrix}_{p\times 1} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}_{p\times 1} \qquad \forall a \in \{1,...,\mathsf{N}\}. \quad (6.32)$$

They are equivalent to the following system of equations:

$$Q(\xi_a^{(0)})\tau(\xi_a^{(0)})/\bar{\mathrm{A}}(1/\xi_a^{(0)}) = Q(\xi_a^{(1)}) \quad (6.33)$$

$$\frac{\tau(\xi_a^{(h)})}{\bar{\mathrm{A}}(1/\xi_a^{(h)})}Q(\xi_a^{(h)}) - \frac{\bar{\mathrm{A}}(\xi_a^{(h)})}{\bar{\mathrm{A}}(1/\xi_a^{(h)})}Q(\xi_a^{(h-1)}) = Q(\xi_a^{(h+1)}),\ \forall h \in \{1,...,p-3\}, \quad (6.34)$$

$$\begin{aligned}\frac{\tau(\xi_a^{(p-2)})}{\bar{\mathrm{A}}(1/\xi_a^{(p-2)})}Q(\xi_a^{(p-2)}) - \frac{\bar{\mathrm{A}}(\xi_a^{(p-2)})}{\bar{\mathrm{A}}(1/\xi_a^{(p-2)})}Q(\xi_a^{(p-3)}) &= \prod_{b=1}^{(p-1)\mathsf{N}}\frac{X_a^{(p-1)} - w_b}{w_{(p-1)\mathsf{N}+1} - w_b}Q(\xi_{(p-1)\mathsf{N}+1}) \\ &+ \sum_{n=1}^{\mathsf{N}_Q}\prod_{\substack{b=1\\ b\neq n}}^{(p-1)\mathsf{N}+1}\frac{X_a^{(p-1)} - w_b}{w_n - w_b}Q(\xi_n), \end{aligned} \quad (6.35)$$

as $\tau(\lambda) \in \Sigma_{\mathcal{T}}$. So we are left with (6.33)-(6.35) a linear system of $(p-1)\mathsf{N}$ inhomogeneous equations in the $(p-1)\mathsf{N}$ unknowns $Q(\xi_a)$ for all $a \in \{1,...,(p-1)\mathsf{N}\}$. Let us define by induction the following coefficients:

$$\mathbb{C}_{a,h+1} = x_{a,0,h}\mathbb{C}_{a,h} + x_{a,-h}\mathbb{C}_{a,h-1},\ \ \forall h \in \{1,...,p-2\},\ \mathbb{C}_{a,1} = \tau(\xi_a^{(0)})/\bar{\mathrm{A}}(1/\xi_a^{(0)}),\ \ \mathbb{C}_{a,0} = 1, \quad (6.36)$$

where we have defined:

$$x_{a,0,h}=\frac{\tau(\xi_a^{(h)})}{\bar{\text{A}}(1/\xi_a^{(h)})},\quad x_{a,-,h}=-\frac{\bar{\text{A}}(\xi_a^{(h)})}{\bar{\text{A}}(1/\xi_a^{(h)})},\ \ \forall h\in\{1,...,p-2\},\ \forall a\in\{1,...,\mathsf{N}\}. \tag{6.37}$$

One can rewrite the previous system as it follows:

$$Q(\xi_a^{(h)})=\mathbb{C}_{a,h}Q(\xi_a^{(0)}),\ \ \forall h\in\{1,...,p-1\},\ \forall a\in\{1,...,\mathsf{N}\}, \tag{6.38}$$

and

$$\mathbb{C}_{a,p-1}Q(\xi_a^{(0)})=\prod_{b=1}^{(p-1)\mathsf{N}}\frac{X_a^{(p-1)}-w_b}{w_{(p-1)\mathsf{N}+1}-w_b}Q(\xi_{(p-1)\mathsf{N}+1})+\sum_{n=1}^{\mathsf{N}}\mathbb{D}_{a,n}Q(\xi_n^{(0)}), \tag{6.39}$$

where

$$\mathbb{D}_{b,n}=\sum_{h=0}^{p-2}\prod_{c=1,c\neq a(n,h)}^{(p-1)\mathsf{N}+1}\frac{X_b^{(p-1)}-w_c}{w_n-w_c}\mathbb{C}_{n,h}. \tag{6.40}$$

So finally we are left with a system of N inhomogeneous equations in the N unknown $Q(\xi_a^{(0)})$ for all $a\in\{1,...,\mathsf{N}\}$, which always admits a nontrivial solution, i.e.

$$\{Q(\xi_1^{(0)}),...,Q(\xi_{\mathsf{N}}^{(0)})\}\neq\{0,...,0\}. \tag{6.41}$$

Finally, let us point out that the degree of the polynomial $Q(\lambda)$ is constrained by the Baxter's equation (6.26). Indeed, it is trivial to remark that if $\mathsf{N}_Q\leq(p-1)\mathsf{N}-2$, then the equation (6.26) admits only the trivial solution $Q(\lambda)=0$ which is not compatible with equation (6.41). Instead, the equation (6.26) may still be satisfied with a nontrivial $Q(\lambda)$ if $\mathsf{N}_Q=(p-1)\mathsf{N}-1$ but only if the following condition:

$$\lim_{\Lambda\to\infty}\Lambda^{(p-1)\mathsf{N}-1}Q(\lambda)=\frac{(i-1)\text{A}(iq^{1/2})q_0}{4\left(\tau_\infty-q^{2-2(p-1)\mathsf{N}}\bar{\text{A}}_\infty\right)F(iq^{1/2})}, \tag{6.42}$$

is satisfied. It is clear that this represents one supplementary condition and one can expect that up to some exceptional case, related to the values of the parameters of the representation and to some special choice of the $\tau(\lambda)\in\Sigma_{\mathcal{T}}$, it is not satisfied and so that $\mathsf{N}_Q=(p-1)\mathsf{N}$. □

It is then simple to prove the following corollary which provides under some further constraints a complete characterization of the transfer matrix eigenvalues and eigenstates in terms of solution of ordinary Bethe equations.

Corollary 6.1. *If the conditions (4.23), (4.24), (5.15) and (6.7) are satisfied and if we fix the boundary parameter $\zeta_+=iz_+$ with $z_+\in\{-1,+1\}$ and the following global condition:*

$$\frac{\kappa_+e^{\epsilon(\tau_--\tau_+)}}{iz_+\alpha_-\beta_-}=q^{1+\mathsf{N}}\prod_{n=1}^{\mathsf{N}}\frac{b_nc_n}{\alpha_n\beta_n}, \tag{6.43}$$

where $\epsilon=+1$ for $b_+(\lambda)=0$, $c_+(\lambda)\neq0$ and $\epsilon=-1$ for $c_+(\lambda)=0$, $b_+(\lambda)\neq0$, then $\mathcal{T}(\lambda)$ has simple spectrum and $\tau(\lambda)\in\Sigma_{\mathcal{T}}$ if and only if $\tau(\lambda)$ is entire and there exists a polynomial $Q(\lambda)$ of the form (6.22), with $\mathsf{N}_Q\leq(p-1)\mathsf{N}$, which satisfies the following homogeneous Baxter equation:

$$\tau(\lambda)Q(\lambda)=\bar{\text{A}}(\lambda)Q(\lambda/q)+\bar{\text{A}}(1/\lambda)Q(\lambda q). \tag{6.44}$$

The $(\lambda_1,...,\lambda_{\mathsf{N}_Q})$ entering in the Bethe ansatz like representations of the eigenstates (6.27) are solutions of the associated ordinary Bethe equations.

Proof. The condition $\zeta_+=iz_+$ with $z_+\in\{-1,+1\}$ implies:

$$\tau(iq^{1/2})=\text{A}(iq^{1/2})=0, \tag{6.45}$$

so that the condition (6.43) implies that $G(\lambda|q_\infty,q_1)=0$, for any choice of q_∞, q_1. Then, the previous theorem directly implies our corollary. Finally, let us remark that in this case we do not have any restriction on the degree of the polynomial $Q(\lambda)$ imposed by the Baxter's equation (6.26), so that we are only left with $\mathsf{N}_Q\leq(p-1)\mathsf{N}$. □

7 Conclusions

In this paper we have studied the transfer matrix spectrum of the class of cyclic 6-vertex representations of the reflection algebra in the case of one completely general and one triangular boundary matrices and for bulk parameters satisfying some specific constraints. Our result is the complete characterization of the spectrum (eigenvalues and eigenstates) of this class of models both by a discrete system of Baxter's like second order difference equations and by a single inhomogeneous TQ-functional equation within a class of polynomial Q-functions.

The present paper represents a natural starting point to solve the spectral problem in the most general setting. In order to do so we need to generalize to the 6-vertex cyclic representations the Baxter's gauge transformations used in the 6-vertex spin 1/2 highest weight representations and prove then the pseudo-diagonalizability of the associated family of gauge transformed $\mathcal{B}_-$-operators. These points are currently under analysis.

An interesting point to which we would like to come back in future investigations is the explicit construction of the Q-operator for the cyclic representations of the 6-vertex reflection algebra. This can lead to new connections with transfer matrices of exactly solvable class of models of non 6-vertex type. Indeed, let us recall that for the special class of 6-vertex cyclic representations of the Yang-Baxter algebra, parametrized by points on the algebraic curve (A.37), the associated integrable models have some remarkable connections with the inhomogeneous p-state chiral Potts models. Indeed, the chiral Potts transfer matrices play the role of the Q-operators for the transfer matrix associated to the cyclic representations of the 6-vertex Yang-Baxter algebra [113].

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A Appendix

A.1 General proof of diagonalizability of $\mathcal{B}_-(\lambda)$

The generator $\mathcal{B}_-(\lambda)$ admits the following boundary bulk decomposition:

$$\mathcal{B}_-(\lambda)=-a_-(\lambda)A(\lambda)B(1/\lambda)+b_-(\lambda)A(\lambda)A(1/\lambda)-c_-(\lambda)B(\lambda)B(1/\lambda)+d_-(\lambda)B(\lambda)A(1/\lambda), \tag{A.1}$$

if the inner boundary matrix is non-diagonal and the bulk parameters are generals, i.e. if it holds:

$$\text{B}_-=(-1)^{\mathsf{N}}\kappa_-e^{\tau_-}\prod_{a=1}^{\mathsf{N}}\alpha_n\beta_n\neq0, \tag{A.2}$$

then it trivially follows that $\mathscr{B}_-(\lambda)$ admits the following functional form:

$$\mathscr{B}_-(\lambda)=\mathrm{B}_-\left(\frac{\lambda^2}{q}-\frac{q}{\lambda^2}\right)\prod_{a=1}^{\mathrm{N}}\left(\frac{\lambda}{\mathscr{B}_{-,a}}-\frac{\mathscr{B}_{-,a}}{\lambda}\right)\left(\lambda\mathscr{B}_{-,a}-\frac{1}{\lambda\mathscr{B}_{-,a}}\right), \tag{A.3}$$

where the $\mathscr{B}_{-,a}$ are invertible commuting operators. The above functional form implies that under the condition (A.2) the operator family is not nilpotent, in particular does not exist any state annihilated by $\mathscr{B}_-(\lambda)$ for any $\lambda\in\mathbb{C}$. That is does not exist a reference state and we cannot use ABA to analyze the spectral problem of the transfer matrix. Here, we show that under the condition (A.2) for general values of the bulk parameters the $\mathscr{B}_-(\lambda)$ is indeed diagonalizable and with simple spectrum. Let us first prove the following lemma:

Lemma A.1. *There exists at least one simultaneous eigenstate:*

$$|\Omega_R\rangle,\ \ \langle\Omega_L| \tag{A.4}$$

of the one parameter family of commuting operators $\mathscr{B}_-(\lambda)$:

$$\mathscr{B}_-(\lambda)|\Omega_R\rangle=|\Omega_R\rangle\mathrm{B}_-\left(\frac{\lambda^2}{q}-\frac{q}{\lambda^2}\right)\prod_{a=1}^{\mathrm{N}}\left(\frac{\lambda}{\bar{b}_{-,a}}-\frac{\bar{b}_{-,a}}{\lambda}\right)\left(\lambda\bar{b}_{-,a}-\frac{1}{\lambda\bar{b}_{-,a}}\right) \tag{A.5}$$

$$\langle\Omega_L|\mathscr{B}_-(\lambda)=\mathrm{B}_-\left(\frac{\lambda^2}{q}-\frac{q}{\lambda^2}\right)\prod_{a=1}^{\mathrm{N}}\left(\frac{\lambda}{\hat{b}_{-,a}}-\frac{\hat{b}_{-,a}}{\lambda}\right)\left(\lambda\hat{b}_{-,a}-\frac{1}{\lambda\hat{b}_{-,a}}\right)\langle\Omega_L|. \tag{A.6}$$

Proof. We can always put $\mathscr{B}_{-,1}$ in a Jordan normal form, let us denote with B_1 the right eigenspace associated to a given eigenvalue $\bar{b}_{-,1}\neq 0$ of $\mathscr{B}_{-,1}$, as $\mathscr{B}_{-,1}$ is invertible. If this eigenspace is one dimensional we have found our simultaneous eigenstate of $\mathscr{B}_-(\lambda)$. If this is not the case then by the commutativity we have that B_1 is an invariant space with regards to $\mathscr{B}_{-,n}$ for any $n\in\{1,...,N\}$. So we can always put $\mathscr{B}_{-,2}$ in a Jordan normal form in B_1, let us denote with $\mathrm{B}_{1,2}$ the eigenspace associated to a given eigenvalue $\bar{b}_{-,2}\neq 0$ of $\mathscr{B}_{-,2}$. If this eigenspace is one dimensional we have found our simultaneous eigenstate of $\mathscr{B}_-(\lambda)$ otherwise we can reiterate this procedure. We can have two possibilities both at the step $n\leq N$ we find that the eigenspace $\mathrm{B}_{1,...,n}$ is one-dimensional or we arrive up to the eigenspace $\mathrm{B}_{1,...,N}$ in both the cases we have (at least) one simultaneous eigenstate of the $\mathscr{B}_{-,n}$ for any $n\in\{1,...,N\}$ and so one eigenstate of $\mathscr{B}_-(\lambda)$. Similarly, we can prove the existence of $\langle\Omega_L|$. □

From the previous Lemma and the reflection algebra equation we can prove the following proposition.

Proposition A.1. *Under the condition (A.2) for almost all the values of the bulk parameters, the operator family $\mathscr{B}_-(\lambda)$ is diagonalizable and it has simple spectrum and its average value is central and it holds:*

$$\mathbb{B}_-(\lambda)=\prod_{a=1}^{p}\mathscr{B}_-(\lambda q^a)=\mathrm{B}_-^p\left(\frac{\lambda^{2p}}{1}-\frac{1}{\lambda^{2p}}\right)\prod_{a=1}^{\mathrm{N}}\left(\frac{\lambda^p}{\bar{b}_{-,a}^p}-\frac{\bar{b}_{-,a}^p}{\lambda^p}\right)\left(\lambda^p\bar{b}_{-,a}^p-\frac{1}{\lambda^p\bar{b}_{-,a}^p}\right), \tag{A.7}$$

with:

$$\bar{b}_{-,m}^p\neq\bar{b}_{-,n}^p,\ \forall n\neq m\in\{1,...,N\}\,. \tag{A.8}$$

Proof. Let us observe that by definition the operator family $\mathscr{B}_-(\lambda)$ is a polynomial in the bulk parameters so the same must be true for its spectrum. This implies in particular that defined

$$\hat{b}_{n,m}=\bar{b}_{-,n}^p-\bar{b}_{-,m}^p,\ \forall n\neq m\in\{1,...,N\}\,, \tag{A.9}$$

or they are identically zero or they can be zero only over subspaces of nonzero codimension in the space of the bulk parameters. So here we have just to prove that the $\hat{b}_{n,m}$ are not identically zero to derive that (A.8) holds for almost all the values of the parameters. To do so we can just recall that from the results derived in the previous section it holds:

$$\bar{b}_{-,m}^p=q^{p/2}\mu_{m,+}^p,\ \ \forall m\in\{1,...,N\}\,, \tag{A.10}$$

under the condition $B(\lambda)$ nilpotent, i.e. $a_n^p=-b_n^p$ for any $n\in\{1,...,N\}$. From which the condition (A.8) holds as soon as we impose:

$$\beta_n^p/\alpha_n^p\neq\beta_m^p/\alpha_m^p,\ \ \forall n\neq m\in\{1,...,N\}\,. \tag{A.11}$$

Once we have proven this statement, then we have just to use the reflection algebra to construct an eigenbasis of $\mathscr{B}_-(\lambda)$ and this is done just by a repeated action of the generators $\mathscr{A}_-(\lambda)$ computed in the zeros of $\mathscr{B}_-(\lambda)$ on the eigenstates $|\Omega_R\rangle$ and $\langle\Omega_L|$. That is we repeat the construction of the eigenbasis presented in Section 4.2 by substituting to the reference states defined in (4.1) with the $\mathscr{B}_-(\lambda)$ on the eigenstates $|\Omega_R\rangle$ the state $\langle\Omega_L|$. Note that such an action can generate a null vector only if some of the zeros of $\mathscr{B}_-(\lambda)$ coincides with the zeros of the quantum determinant anyhow as discussed in Section 4.2 under the condition (4.24) we are always able to chose an appropriate set of p^N zeros of $\mathscr{B}_-(\lambda)$ which do not coincides with those of the quantum determinant so that we generate exactly p^N independent states. □

A.2 The lattice sine-Gordon model with integrable boundaries

In this appendix we point out that the results on the transfer matrix spectrum for general cyclic representations developed in this paper apply also to characterize the spectrum of the transfer matrix of the lattice sine-Gordon model with integrable open boundary conditions. Let us recall that the Lax operator defining the lattice sine-Gordon model has the following form:

$$L_{a,n}^{sG}(\lambda|\mathrm{u}_n,\mathrm{v}_n,\kappa_n,r_n,s_n)\equiv\begin{pmatrix}\mathrm{u}_n\left(q^{-1/2}\kappa_n^2 r_n s_n\mathrm{v}_n+\frac{q^{1/2}s_n}{\mathrm{v}_n r_n}\right) & \frac{\kappa_n}{i}\left(\lambda r_n\mathrm{v}_n-\frac{1}{\lambda r_n\mathrm{v}_n}\right)\\ \frac{\kappa_n}{i}\left(\frac{\lambda}{r_n\mathrm{v}_n}-\frac{r_n\mathrm{v}_n}{\lambda}\right) & \mathrm{u}_n^{-1}\left(\frac{q^{1/2}r_n\mathrm{v}_n}{s_n}+\frac{q^{-1/2}\kappa_n^2}{s_n r_n\mathrm{v}_n}\right)\end{pmatrix}_a, \tag{A.12}$$

each local representation being defined as the representation of a local Weyl algebra

$$\mathrm{u}_n\mathrm{v}_m=q^{\delta_{n,m}}\mathrm{v}_m\mathrm{u}_n\quad\forall n,m\in\{1,...,\mathrm{N}\}, \tag{A.13}$$

associated to a root of unit q, where u_n and v_n are the Weyl algebra generators on the Hilbert space $\mathscr{R}_n$. Let us introduce the monodromy matrices for the Yang-Baxter and reflection equations of the lattice sine-Gordon model, they read:

$$M_a^{sG}(\lambda)\ =\ L_{a,\mathrm{N}}(\lambda q^{-1/2}/\xi_{\mathrm{N}})\cdots L_{a,1}(\lambda q^{-1/2}/\xi_1)\in\mathrm{End}(\mathbb{C}^2\otimes\mathscr{H}), \tag{A.14}$$

$$\mathscr{U}_{a,-}^{sG}(\lambda)\ =\ M_a^{sG}(\lambda)K_{a,-}(\lambda)\hat{M}_a^{sG}(\lambda)\in\mathrm{End}(\mathbb{C}^2\otimes\mathscr{H}), \tag{A.15}$$

and also the boundary transfer matrix of the sine-Gordon model, which reads:

$$\mathscr{T}^{sG}(\lambda)\equiv tr_a\{K_{a,+}(\lambda)\mathscr{U}_{a,-}^{sG}(\lambda)\}. \tag{A.16}$$

Let us remark that we have used the upper index sG in the above two monodromy matrices and transfer matrix to point out that they are related to the sine-Gordon model while we will continue to denote with $M_a(\lambda)$, $\mathscr{U}_{a,-}(\lambda)$ and $\mathscr{T}(\lambda)$ those associated to the original τ_2-model. The following lemma establishes the connection between the monodromy and transfer matrices of the sine-Gordon model and the original τ_2-model.

Lemma A.2. *Let us denote* $\mathsf{N}=2\mathsf{M}+\mathsf{x}$ *with* $\mathsf{x}\in\{0,1\}$ *and let us impose the following identification of the generators of the local Weyl algebras:*

$$\mathrm{u}_{2n+\mathsf{y}}=u_{2n+\mathsf{y}}^{(1-2\mathsf{x})(1-2\mathsf{y})},\quad \mathrm{v}_{2n+\mathsf{y}}=v_{2n+\mathsf{y}}^{(1-2\mathsf{x})(1-2\mathsf{y})},\tag{A.17}$$

with $\mathsf{y}\in\{0,1\}$ *and* $2n+\mathsf{y}\in\{1,...,\mathsf{N}\}$. *Then the following identity holds:*

$$M_a^{sG}(\lambda)=M_a(\lambda)\left(\sigma_a^x\right)^{\mathsf{x}},\tag{A.18}$$

once we define the parameters of the τ_2*-model in terms of those of the lattice sine-Gordon model as it follows:*

$$a_{2n+\mathsf{y}}=\kappa_{2n+\mathsf{y}}^2\left(r_{2n+\mathsf{y}}s_{2n+\mathsf{y}}\right)^{(1-2\mathsf{x})(1-2\mathsf{y})},\qquad b_{2n+\mathsf{y}}=\left(\frac{s_{2n+\mathsf{y}}}{r_{2n+\mathsf{y}}}\right)^{(1-2\mathsf{x})(1-2\mathsf{y})},\tag{A.19}$$

$$c_{2n+\mathsf{y}}=\left(\frac{r_{2n+\mathsf{y}}}{s_{2n+\mathsf{y}}}\right)^{(1-2\mathsf{x})(1-2\mathsf{y})},\qquad d_{2n+\mathsf{y}}=\frac{\kappa_{2n+\mathsf{y}}^2}{\left(r_{2n+\mathsf{y}}s_{2n+\mathsf{y}}\right)^{(1-2\mathsf{x})(1-2\mathsf{y})}},\tag{A.20}$$

$$\alpha_{2n+\mathsf{y}}=\frac{\kappa_{2n+\mathsf{y}}r_{2n+\mathsf{y}}^{(1-2\mathsf{x})(1-2\mathsf{y})}}{i\xi_{2n+\mathsf{y}}},\qquad \beta_{2n+\mathsf{y}}=\frac{\kappa_{2n+\mathsf{y}}\xi_{2n+\mathsf{y}}}{ir_{2n+\mathsf{y}}^{(1-2\mathsf{x})(1-2\mathsf{y})}}.\tag{A.21}$$

Moreover, under the above conditions and imposing the following identifications:

$$\tau_\epsilon=\epsilon^{\mathsf{x}}\tau_\epsilon^{sG},\ \kappa_\epsilon=\epsilon^{\mathsf{x}}\kappa_\epsilon^{sG},\ \zeta_\epsilon=\left(\zeta_\epsilon^{sG}\right)^{\epsilon^{\mathsf{x}}}\ \textit{for}\ \epsilon=\pm,\tag{A.22}$$

of the boundary parameters of the τ_2*-model and the lattice sine-Gordon model, we get:*

$$\mathscr{U}_{a,-}(\lambda)=\mathscr{U}_{a,-}^{sG}(\lambda),\ \ \mathscr{T}(\lambda)=\mathscr{T}^{sG}(\lambda).\tag{A.23}$$

Proof. It is simple to observe that the following identities hold:

$$L_{a,n}^{sG}(\lambda/\xi_n|\mathrm{u}_n,\mathrm{v}_n,\kappa_n,r_n,s_n)=L_{a,n}(\lambda)\sigma_a^x\tag{A.24}$$

if:

$$\mathrm{u}_n=u_n,\ \ \mathrm{v}_n=v_n,\tag{A.25}$$

and

$$a_n=\kappa_n^2r_ns_n,\ \ b_n=s_n/r_n,\ \ c_n=r_n/s_n,\tag{A.26}$$

$$d_n=\kappa_n^2/(r_ns_n),\ \ \alpha_n=-i\kappa_nr_n/\xi_n,\ \ \beta_n=-i\kappa_n\xi_n/r_n.\tag{A.27}$$

Similarly by direct computations it is easy to show that defined:

$$\tilde{L}_{a,n}^{sG}(\lambda)\equiv\sigma_a^xL_{a,n}^{sG}(\lambda|\mathrm{u}_n,\mathrm{v}_n,\kappa_n,r_n,s_n)\sigma_a^x\tag{A.28}$$

it holds:

$$\tilde{L}_{a,n}^{sG}(\lambda)=L_{a,n}^{sG}(\lambda|\mathrm{u}_n^{-1},\mathrm{v}_n^{-1},\kappa_n,r_n^{-1},s_n^{-1}).\tag{A.29}$$

Let us now observe that for $\mathsf{x}=0$, we can write:

$$M_a^{sG}(\lambda)=[L_{a,2\mathsf{M}}(\frac{\lambda q^{-1/2}}{\xi_{2\mathsf{M}}})\sigma_a^x][\tilde{L}_{a,2\mathsf{M}-1}(\frac{\lambda q^{-1/2}}{\xi_{2\mathsf{M}-1}})\sigma_a^x]\cdots[L_{a,2}(\frac{\lambda q^{-1/2}}{\xi_2})\sigma_a^x][\tilde{L}_{a,1}(\frac{\lambda q^{-1/2}}{\xi_1})\sigma_a^x]\tag{A.30}$$

while for $\mathsf{x}=1$, we can write:

$$M_a^{sG}(\lambda)\sigma_a^x=[L_{a,2\mathsf{M}+1}(\frac{\lambda q^{-1/2}}{\xi_{2\mathsf{M}+1}})\sigma_a^x][\tilde{L}_{a,2\mathsf{M}}(\frac{\lambda q^{-1/2}}{\xi_{2\mathsf{M}}})\sigma_a^x]\cdots[\tilde{L}_{a,2}(\frac{\lambda q^{-1/2}}{\xi_2})\sigma_a^x][L_{a,1}(\frac{\lambda q^{-1/2}}{\xi_1})\sigma_a^x],\tag{A.31}$$

then these two identities together with the identity (A.24), (A.29) and the parametrization (A.26)-(A.27) imply the identity (A.18) with the parametrization (A.19)-(A.21)and the local operator identification (A.17). Finally, let us observe that from the identity (A.18) it follows:

$$\hat{M}_a^{sG}(\lambda)\equiv(-1)^{\mathsf{N}}\sigma_a^y\left(M_a^{sG}(1/\lambda)\right)^{t_0}\sigma_a^y=\left(-\sigma_a^x\right)^{\mathsf{x}}\hat{M}_a(\lambda),\tag{A.32}$$

which together with:

$$K_{a,+}(\lambda|\tau_\epsilon,\ \kappa_\epsilon,\ \zeta_\epsilon)=\left(\sigma_a^x\right)^{\mathsf{x}}K_{a,+}^{sG}(\lambda|\tau_-^{sG},\kappa_-^{sG},\zeta_-^{sG})\left(-\sigma_a^x\right)^{\mathsf{x}},\tag{A.33}$$

holding under the parametrization (A.22), implies the identities (A.23). □

A.3 Reduction to inhomogeneous chiral Potts representations

In this appendix we want to point out that a nontrivial class of representations of the inhomogeneous chiral Potts model can be described on the closed chain in the space of the parameters of the τ_2-model considered in this paper. In order to do so let us recall that the transfer matrix $\mathsf{T}_\lambda^{\text{chP}}$ of the inhomogeneous chiral Potts model [113] is characterized by the following kernel:

$$\mathsf{T}_\lambda^{\text{chP}}(\mathrm{z},\mathrm{z}')\equiv\langle\mathrm{z}|\mathsf{T}_\lambda^{\text{chP}}|\mathrm{z}'\rangle=\prod_{n=1}^{\mathsf{N}}W_{\mathrm{q}_n\mathrm{p}}(z_n/z_n')\bar{W}_{\mathrm{r}_n\mathrm{p}}(z_n/z_{n+1}'),\tag{A.34}$$

in the basis $\langle\mathrm{z}|\equiv\langle z_1,...,z_N|$ and $|\mathrm{z}\rangle\equiv|z_1,...,z_N\rangle$ formed by the left and right u_n-eigenstates:

$$\langle\mathrm{z}|u_n=z_n\langle\mathrm{z}|,\quad u_n|\mathrm{z}\rangle=z_n|\mathrm{z}\rangle\ \text{with}\ z_n\in\mathbb{S}_p\equiv\{q^{2r},\ r=1,..,p\},\tag{A.35}$$

and where:

$$\lambda=t_{\mathrm{p}}^{-1/2}\mathrm{c}_0,\quad \mathrm{p},\mathrm{r}_n,\mathrm{q}_n\in\mathscr{C}_k,\mathrm{c}_0\in\mathbb{C}.\tag{A.36}$$

The algebraic curve $\mathscr{C}_k$ of modulus k is by definition the locus of the points in the four-dimensional complex space $p\equiv(a_{\mathrm{p}},b_{\mathrm{p}},c_{\mathrm{p}},d_{\mathrm{p}})\in\mathbb{C}^4$ which satisfy the equations:

$$x_{\mathrm{p}}^p+y_{\mathrm{p}}^p=k(1+x_{\mathrm{p}}^py_{\mathrm{p}}^p),\quad kx_{\mathrm{p}}^p=1-k's_{\mathrm{p}}^{-p},\quad ky_{\mathrm{p}}^p=1-k's_{\mathrm{p}}^p,\tag{A.37}$$

where:

$$x_{\mathrm{p}}\equiv a_{\mathrm{p}}/d_{\mathrm{p}},\quad y_{\mathrm{p}}\equiv b_{\mathrm{p}}/c_{\mathrm{p}},\quad s_{\mathrm{p}}\equiv d_{\mathrm{p}}/c_{\mathrm{p}},t_{\mathrm{p}}\equiv x_{\mathrm{p}}y_{\mathrm{p}},\quad k^2+(k')^2=1,\tag{A.38}$$

and $W_{\mathrm{qp}}(z(n))$ and $\bar{W}_{\mathrm{qp}}(z(n))$ are the Boltzmann weights of the chiral Potts model. Then, $\mathsf{T}_\lambda^{\text{chP}}$ is a Baxter Q-operator with regards to the bulk τ_2-transfer matrix in $\mathscr{H}_{\mathsf{N}}$.

Let us here directly characterize the class of the inhomogeneous chiral Potts representations once we restrict the space of the parameters to that used in the section 6; in particular, we assume that it holds:

$$b_n=-q^{-1}a_n,\ \ d_n=-q^{-1}c_n.\tag{A.39}$$

The parameters of the τ_2-Lax operators are written in terms of the coordinate of the points p, r_n, q_n by using the equations (5.3) of the paper [96]. Then we have that the points r_n, q_n are elements of $\mathscr{C}_k$ if and only if beyond (A.39) the parameters of the τ_2-Lax operators satisfy the following conditions:

$$\alpha_n\beta_n=a_nc_n\tag{A.40}$$

and

$$\left(\frac{c_0\alpha_n}{q^{1/2}a_n}\right)^p+\left(\frac{q^{1/2}c_0\alpha_n}{c_n}\right)^p=k\left(1+\left(\frac{c_0^2\alpha_n^2}{c_n a_n}\right)^p\right). \tag{A.41}$$

Under these constraints the class of the inhomogeneous chiral Potts representations is characterized by the following identity:

$$\mathrm{r}_n=\Delta(\mathrm{q}_n),\quad \forall n\in\{1,\dots,\mathrm{N}\}, \tag{A.42}$$

where the q_n are free elements of $\mathscr{C}_k$ and Δ is the following discrete automorphism of the curve:

$$\Delta:\mathrm{x}=(a_\mathrm{x},b_\mathrm{x},c_\mathrm{x},d_\mathrm{x})\in\mathscr{C}_k\to\Delta(\mathrm{x})=(b_\mathrm{x},a_\mathrm{x},d_\mathrm{x},c_\mathrm{x})\in\mathscr{C}_k, \tag{A.43}$$

which implies:

$$x_{\mathrm{p}_n}=y_{\mathrm{q}_n},\ y_{\mathrm{p}_n}=x_{\mathrm{q}_n},\ s_{\mathrm{p}_n}=s_{\mathrm{q}_n}^{-1}. \tag{A.44}$$

Finally, this class of representations reduce to the superintegrable chiral Potts model under the following special homogeneous limits:

$$x_{\mathrm{q}_n}^p\to(1+k')/k,\quad \forall n\in\{1,\dots,\mathrm{N}\}. \tag{A.45}$$

A.4 Reduction to the XXZ spin s open chains at the $p=2s+1$ roots of unit

Here we show that imposing a set of conditions on the parameters of the τ_2-Lax operator we can reduce it to the one of the spin $s=(p-1)/2$ XXZ case at the p roots of unit. This has the interesting consequence that the analysis done of the open τ_2-chain reduces for these special representations to that of an open spin chain under the same boundary conditions. In particular, we have that our functional equation characterization of the spectrum under a special homogeneous limit defines the spectrum of the following local Hamiltonian given by fusion and the Sklyanin formula:

$$\mathscr{H}_s=c_0\frac{d}{d\lambda}\mathscr{T}_p(\lambda)_{|_{\lambda=q^s}}+\text{constant}, \tag{A.46}$$

with

$$\mathscr{H}_s=\sum_{n=1}^{\mathrm{N}-1}H_{n,n+1}^{(s)}+\frac{d}{d\lambda}K_{1,-}^{(p)}(q^s)+\frac{\mathrm{tr}_0\{K_{0,+}^{(p)}(q^s)H_{0,\mathrm{N}}^{(s)}\}}{\mathrm{tr}_0\{K_{0,+}^{(p)}(q^s)\}}, \tag{A.47}$$

where $\mathscr{T}_p(\lambda)$ is the p-fused open transfer matrix, $H_{n,n+1}^{(s)}$ is the two sites local Hamiltonian of the spin s XXZ chain, $K_{0,\pm}^{(p)}(\lambda)$ are the $p\times p$ matrices p-fused scalar solutions of the reflection algebra obtained by doing the fusion $p-1$ times starting from the original 2×2 scalar solutions $K_{0,\pm}(\lambda)$, respectively.

Lemma A.3. *Let us fix the parameters of the τ_2-representations as follows:*

$$\alpha_n=\beta_n=1/2,\ a_n=q^{-1/2}/2i, \tag{A.48}$$

$$b_n=iq^{1/2}/2,\ c_n=q^{-1/2}/2i,\ d_n=iq^{1/2}/2, \tag{A.49}$$

and let

$$L_{a,n}^{XXZ}(\lambda)=\begin{pmatrix}[\lambda q^{s+S_n^z/2}-1/(\lambda q^{s+S_n^z/2})]/2 & S_n^- \\ S_n^+ & [\lambda q^{s-S_n^z/2}-1/(\lambda q^{s-S_n^z/2})]/2\end{pmatrix} \tag{A.50}$$

be the Lax operator of the spin s XXZ chain with anisotropy $\cosh\eta=(q+1/q)/2$, then it holds:

$$L_{a,n}^{XXZ}(\lambda)=L_{a,n}(\lambda/q) \tag{A.51}$$

for $s=(p-1)/2$ and $q=e^{i\pi p'/p}$ with p' even and p odd and coprime, which is equivalent to the following identities among the generators of the local algebras:

$$S_n^+=u_n^{-1}(v_n-1/v_n)/2i,\quad S_n^-=u_n(v_n/q-q/v_n)/2i, \tag{A.52}$$

and

$$S_n^z=\frac{2p}{i\pi p'}\log v_n-(p+1)\in\{-2s,-2(s-1),\dots,2s\}\ \ mod\,2p \tag{A.53}$$

Proof. Let us denote

$$\overline{|a,n\rangle}=\begin{pmatrix}0 & \cdots & 1 & \cdots & 0\end{pmatrix}^{t_0},\quad a\in\{1,\dots,2s+1\} \tag{A.54}$$

the element a of the canonical basis given by the column vector with all elements zero except that in row a which is 1. In this basis we have the following representations for the generators

$$S_n^+=\begin{pmatrix}0 & f(1) & & \\ & \ddots & & \\ & & \ddots & f(2s) \\ & & & 0\end{pmatrix},\quad S_n^-=\begin{pmatrix}0 & & & \\ f(1) & \ddots & & \\ & \ddots & \ddots & \\ & & f(2s) & 0\end{pmatrix}, \tag{A.55}$$

where:

$$f(j)=\sqrt{\sinh j\eta\sinh(p-j)\eta}=i(q^j-q^{-j})/2, \tag{A.56}$$

and the second identity holds for $q^p=1$, and:

$$S_n^z=\begin{pmatrix}2s & 0 & & \\ 0 & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ & & 0 & -2s\end{pmatrix}. \tag{A.57}$$

Let us now impose that in our representation the v_n-eigenstates coincide with the elements of the canonical basis:

$$|p-a,n\rangle=\overline{|a+1,n\rangle}\quad\forall a\in\{0,\dots,p-1\}, \tag{A.58}$$

we can verify now the formulae (A.52)-(A.53). The formula (A.53) is equivalent to:

$$v_n=q^{(S_n^z+p+1)/2} \tag{A.59}$$

which holds for the following identities:

$$q^{(S_n^z+p+1)/2}\overline{|a+1,n\rangle}=\overline{|a+1,n\rangle}q^{(2(s-a)+p+1)/2}=|p-a,n\rangle q^{p-a}=v_n|p-a,n\rangle. \tag{A.60}$$

Similarly we have:

$$S_n^+\overline{|a,n\rangle}=\overline{|a+1,n\rangle}f(a)=\overline{|a+1,n\rangle}(q^{-a}-q^a)/2i \tag{A.61}$$

$$=\left[u_n^{-1}(v_n-1/v_n)/2i\right]|p-a,n\rangle\quad\forall a\in\{1,\dots,p\} \tag{A.62}$$

and

$$S_n^-\overline{|a,n\rangle}=\overline{|a-1,n\rangle}f(a-1)=\overline{|a-1,n\rangle}(q^{1-a}-q^{a-1})/2i \tag{A.63}$$

$$=[u_n(v_n/q-q/v_n)/2i]\,|p+2-a,n\rangle\quad\forall a\in\{1,\dots,p\}. \tag{A.64}$$

□

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Article II

Transfer matrix spectrum for cyclic representations of the 6-vertex reflection algebra II

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Transfer matrix spectrum for cyclic representations of the 6-vertex reflection algebra II

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Abstract. This article is a direct continuation of [1] where we begun the study of the transfer matrix spectral problem for the cyclic representations of the trigonometric 6-vertex reflection algebra associated to the Bazhanov-Stroganov Lax operator. There we addressed this problem for the case where one of the K-matrices describing the boundary conditions is triangular. In the present article we consider the most general integrable boundary conditions, namely the most general boundary K-matrices satisfying the reflection equation. The spectral analysis is developed by implementing the method of Separation of Variables (SoV). We first design a suitable gauge transformation that enable us to put into correspondence the spectral problem for the most general boundary conditions with another one having one boundary K-matrix in a triangular form. In these settings the SoV resolution can be obtained along an extension of the method described in [1]. The transfer matrix spectrum is then completely characterized in terms of the set of solutions to a discrete system of polynomial equations in a given class of functions and equivalently as the set of solutions to an analogue of Baxter's T-Q functional equation. We further describe scalar product properties of the separate states including eigenstates of the transfer matrix.

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1 Introduction

In the recent years, the out-of-equilibrium behavior of close and open physical systems has attracted a lot of interest motivated in particular by new experimental results, see e.g. [2–10]. Microscopic models able to describe such situations are thus in general not only described through their bulk Hamiltonian but also by specifying appropriate boundary conditions. It leads eventually to rather complicated dynamical properties with possible deformations of the bulk symmetries. In the context of strongly coupled systems, integrable models in low dimension, with boundary conditions preserving integrability properties, can be used to gain insights into the non-perturbative behavior of such out-of-equilibrium dynamics, see e.g. [11] and references therein. In particular, they can also describe classical stochastic relaxation processes, like ASEP [12–16] or transport properties in one dimensional quantum systems, see e.g. [17, 18].

The algebraic description of quantum integrable models with non-trivial boundary conditions (namely going beyond periodic boundary conditions) goes back to Cherednik [19] and Sklyanin [20]. Such models have already a long history, that started with spin chains and Bethe ansatz [21–29], and continued using its modern developments, see e.g. [20, 30–81]. The key point of the algebraic approaches is an extension of the standard Quantum Inverse Scattering method , see e.g. [82–97], and its associated Yang-Baxter algebra ; it takes the form of the so-called reflection equations [19,20] satisfied by the boundary version of the quantum monodromy matrix. The integrable structure of the model with boundaries can be described in terms of the corresponding bulk quantities supplemented with boundary conditions encoded in some K-matrices. To preserve integrability properties, these K-matrices should satisfy reflection equations driven by the R-matrix of the model in the bulk which solves the usual Yang-Baxter equation. As shown by Cherednik in [19] these reflection equations are just consequences of the factorization property of the scattering of particles on a segment having reflecting ends described by the boundary K-matrices. It leads to compatibility properties between the scattering in the bulk described by the R-matrix and the reflection properties of the ends encoded in the K-matrices. These are such that there still exists full series of commuting conserved quantities for the model with boundaries generated by the boundary transfer matrix [20]. Its expression is quadratic in the bulk monodromy matrix entries and depends on the right and left boundary K-matrices. Then, as for the periodic case, the local Hamiltonian for the boundary model can be obtained from this boundary transfer matrix. This is the standard framework to then address the resolution of the common spectral problem for the transfer matrix and its associated local Hamiltonian. There have been quite a number of works devoted to boundary integrable models using, as in the periodic situation, various versions of the Bethe ansatz [30–81]. It appeared however, that while for special models and boundary K-matrices a method very similar to the standard algebraic Bethe ansatz (ABA), here based on the reflection equations, can be applied, the case of the most general boundary conditions (and associated K-matrices) preserving integrability turns out of be out of the reach of these methods. This motivated the use of different approaches like in particular the use of q-Onsager algebras, see e.g. [47, 48], modifications of the Bethe ansatz [45, 46, 49–53, 59] and the implementation for this case of the separation of variable (SoV) method [98–119]. For an extensive discussion and comparison of these various methods in the case of boundary integrable models, we refer to the general discussion given in the introduction of our first article [1] and to the references therein.

In our article [1] we started the implementation of the SoV method to cyclic representation of the 6-vertex reflection equation associated to the most general Bazhanov-Stroganov quantum Lax operator [120–125]. Let us recall that the periodic boundary conditions case (spectrum and

form factors) was considered in previous works [106–111], generalizing in particular [126, 127]. The interest in such a problem is due to the fact that special cases include the Sine-Gordon lattice model at roots of unity and the Chiral Potts model [128–137]. In [1] we started the analysis considering the special case where one of the boundary K-matrices has triangular form (which is equivalent to one constraint on the boundary parameters). For that situation we have been able to apply successfully the SoV method by identifying the separate basis as the eigenstate basis of a special diagonalizable B-operator with simple spectrum which can be constructed from the boundary monodromy matrix entries. Then using this separate basis, the spectrum (eigenvalues and eigenstates) for the boundary transfer matrix was completely characterized in terms of the set of solutions to a discrete system of polynomial equations in a given class of functions.

The purpose of the present article is to address this spectral problem for the most general boundary conditions preserving integrability, namely for the most general K-matrices solution of the reflection equation. The method to reach this goal is to design a gauge transformation that enable us to put this general situation into correspondence with the previous one, namely with a model having one triangular K-matrix. For that purpose, the standard idea of Baxter's gauge transformations, see e.g. [83, 95] and references therein, has to be adapted in a way similar to [68, 70] and generalized to these cyclic representations of the 6-vertex Yang-Baxter algebra. Then using this correspondence, the method and tools obtained in our first paper [1] can be used, leading to the complete characterization of the spectrum (again eigenvalues and eigenstates) of the general boundary transfer matrix. We also give determinant formula for the scalar products of the separate states. Further, we show that the spectrum characterization admits a representation in terms of functional equations of Baxter TQ-equation type. Let us remark that an analogous inhomogeneous Baxter's like equation has been already proposed in [60] on the basis of pure functional arguments on the fusion of transfer matrices. Thanks to our SoV construction, we prove in the present article that our inhomogeneous Baxter's like equation does characterize the full transfer matrix spectrum. A direct comparison with [60] is complicated as the Q-function proposed there has 4p more zeros than the Q-function we consider here (see section 5).

This article is organized as follows. In section 2 we just recall the basics of the cyclic representations associated to the Bazhanov-Stroganov quantum Lax operator. In section 3 we define the gauged transformed reflection algebra that put into correspondence the most general boundary condition K-matrix with a triangular one. It enables us to adapt the SoV method that we already described in our first article [1] to this more general context, leading in section 4 to the transfer matrix spectrum characterization in this SoV basis. There we also present the scalar product formulae for the so-called separate states containing the transfer matrix eigenstates. In section 5 we show that the spectrum characterization admit a representation in terms of functional equations of Baxter TQ-equation type. Details about the construction of the gauge transformation are given in Appendices A and B together with determinant identities used in the spectrum characterization in Appendix C.

2 Cyclic representations of 6-vertex reflection algebra.

Following Sklyanin's paper [20], we consider the most general cyclic solutions of the 6-vertex reflection equation associated to the Bazhanov-Stroganov Lax operator [121]:

$$R_{12}(\lambda/\mu)\,\mathcal{U}_{1,-}(\lambda)\,R_{21}(\lambda\mu/q)\,\mathcal{U}_{2,-}(\mu) = \mathcal{U}_{2,-}(\mu)\,R_{21}(\lambda\mu/q)\,\mathcal{U}_{1,-}(\lambda)\,R_{12}(\lambda/\mu) \qquad (2.1)$$

where the two sides of the equation belong to $\mathrm{End}(V_1 \otimes V_2 \otimes \mathcal{H})$ and are defined by the following boundary monodromy matrices:

$$\mathcal{U}_{a,-}(\lambda) = M_a(\lambda) K_{a,-}(\lambda) \hat{M}_a(\lambda) = \begin{pmatrix} \mathcal{A}_-(\lambda) & \mathcal{B}_-(\lambda) \\ \mathcal{C}_-(\lambda) & \mathcal{D}_-(\lambda) \end{pmatrix}_a \in \mathrm{End}(V_a \otimes \mathcal{H}), \tag{2.2}$$

where:

$$\hat{M}_a(\lambda) = (-1)^{\mathsf{N}} \sigma_a^y M_a^{t_a}(1/\lambda)\, \sigma_a^y, \tag{2.3}$$

and $V_a \simeq \mathbb{C}^2$ is the so-called auxiliary space. Here,

$$M_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_a \equiv L_{a,\mathsf{N}}(\lambda q^{-1/2}) \cdots L_{a,1}(\lambda q^{-1/2}) \in \mathrm{End}(V_a \otimes \mathcal{H}), \tag{2.4}$$

is the cyclic solution of the 6-vertex Yang-Baxter equation:

$$R_{12}(\lambda/\mu) M_1(\lambda) M_2(\mu) = M_2(\mu) M_1(\lambda) R_{12}(\lambda/\mu) \in \mathrm{End}(V_1 \otimes V_2 \otimes \mathcal{H}), \tag{2.5}$$

associated to the R-matrix

$$R_{ab}(\lambda) = \begin{pmatrix} q\lambda - q^{-1}\lambda^{-1} & 0 & 0 & 0 \\ 0 & \lambda - \lambda^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & \lambda - \lambda^{-1} & 0 \\ 0 & 0 & 0 & q\lambda - q^{-1}\lambda^{-1} \end{pmatrix} \in \mathrm{End}(V_a \otimes V_b), \tag{2.6}$$

and defined in terms of the Bazhanov-Stroganov's Lax operators [121]:

$$L_{a,n}(\lambda) \equiv \begin{pmatrix} \lambda \alpha_n v_n - \beta_n \lambda^{-1} v_n^{-1} & u_n \left(q^{-1/2} a_n v_n + q^{1/2} b_n v_n^{-1}\right) \\ u_n^{-1}\left(q^{1/2} c_n v_n + q^{-1/2} d_n v_n^{-1}\right) & \gamma_n v_n/\lambda - \delta_n \lambda / v_n \end{pmatrix}_a \in \mathrm{End}(V_a \otimes \mathcal{R}_n), \tag{2.7}$$

where:

$$\gamma_n = a_n c_n/\alpha_n, \qquad \delta_n = b_n d_n/\beta_n. \tag{2.8}$$

The $u_n \in \mathrm{End}(\mathcal{R}_n)$ and $v_m \in \mathrm{End}(\mathcal{R}_m)$ are unitary Weyl algebra generators:

$$u_n v_m = q^{\delta_{n,m}} v_m u_n \quad \text{with} \quad u_n^p = v_m^p = 1 \ \forall n, m \in \{1, ..., \mathsf{N}\}, \tag{2.9}$$

and:

$$q = e^{-i\pi\beta^2},\ \beta^2 = p'/p \text{ with } p' \text{ even and } p = 2l+1 \text{ odd}. \tag{2.10}$$

The local quantum spaces $\mathcal{R}_n$ are p-dimensional Hilbert spaces and the full representation space of the cyclic Yang-Baxter and reflection algebra is defined by the tensor product of the local quantum spaces, i.e. $\mathcal{H} = \otimes_{n=1}^{\mathsf{N}} \mathcal{R}_n$. Moreover, we consider here the most general boundary matrices defined as:

$$K_{a,\pm}(\lambda) = \begin{pmatrix} a_\pm(\lambda) & b_\pm(\lambda) \\ c_\pm(\lambda) & d_\pm(\lambda) \end{pmatrix}_a \equiv K_a(\lambda q^{(1\pm1)/2}; \zeta_\pm, \kappa_\pm, \tau_\pm), \tag{2.11}$$

where:

$$K_a(\lambda; \zeta, \kappa, \tau) = \frac{1}{\zeta - \frac{1}{\zeta}} \begin{pmatrix} \frac{\lambda\zeta}{q^{1/2}} - \frac{q^{1/2}}{\lambda\zeta} & \kappa e^{\tau}\left(\frac{\lambda^2}{q} - \frac{q}{\lambda^2}\right) \\ \kappa e^{-\tau}\left(\frac{\lambda^2}{q} - \frac{q}{\lambda^2}\right) & \frac{q^{1/2}\zeta}{\lambda} - \frac{\lambda}{\zeta q^{1/2}} \end{pmatrix}_a \in \mathrm{End}(V_a). \tag{2.12}$$

We introduce the functions:

$$\mathsf{A}_-(\lambda) \equiv g_-(\lambda) a(\lambda q^{-1/2}) d(1/(q^{1/2}\lambda)), \quad \mathsf{D}_-(\lambda) = k(\lambda) \mathsf{A}_-(q/\lambda), \tag{2.13}$$

where:

$$a(\lambda) \equiv a_0 \prod_{n=1}^{\mathsf{N}} \left(\frac{\beta_n}{\lambda} + q^{-1}\frac{b_n \alpha_n}{a_n}\lambda\right), \quad k(\lambda) = \frac{(\lambda^2 - 1/\lambda^2)}{(\lambda^2/q^2 - q^2/\lambda^2)}, \tag{2.14}$$

$$d(\lambda) \equiv \frac{(-1)^{\mathsf{N}}}{a_0} \prod_{n=1}^{\mathsf{N}} \frac{a_n c_n}{\alpha_n}\left(\frac{1}{\lambda} + q\frac{d_n \alpha_n}{c_n \beta_n}\lambda\right), \tag{2.15}$$

a_0 is a free nonzero parameter and

$$g_\epsilon(\lambda) \equiv \frac{(\lambda\alpha_\epsilon/q^{1/2} - q^{1/2}/(\lambda\alpha_\epsilon))(\lambda\beta_\epsilon^{-\epsilon}/q^{1/2} + q^{1/2}/(\lambda\beta_\epsilon^{-\epsilon}))}{(\alpha_\epsilon - 1/\alpha_\epsilon)(\beta_\epsilon + 1/\beta_\epsilon)}, \tag{2.16}$$

where $\epsilon = \pm 1$ and we have defined:

$$(\alpha_\epsilon - 1/\alpha_\epsilon)(\beta_\epsilon + 1/\beta_\epsilon) \equiv \frac{\zeta_\epsilon - 1/\zeta_\epsilon}{\kappa_\epsilon}, \quad (\alpha_\epsilon + 1/\alpha_\epsilon)(\beta_\epsilon - 1/\beta_\epsilon) \equiv \frac{\zeta_\epsilon + 1/\zeta_\epsilon}{\kappa_\epsilon}, \tag{2.17}$$

Moreover, later we will use:

$$\mu_{n,h} \equiv \begin{cases} iq^{1/2}(a_n\beta_n/\alpha_n b_n)^{1/2} & h = +, \\ iq^{1/2}(c_n\beta_n/\alpha_n d_n)^{1/2} & h = -. \end{cases} \tag{2.18}$$

Following the Sklyanin's paper [20] the next proposition holds:

Proposition 2.1. *The most general boundary transfer matrix associated to the Bazhanov-Stroganov Lax operator in the cyclic representations of the reflection algebra is defined by:*

$$\mathcal{T}(\lambda) \equiv tr_a\{K_{a,+}(\lambda)\mathcal{U}_{a,-}(\lambda)\} \tag{2.19}$$

$$= a_+(\lambda)\mathcal{A}_-(\lambda) + d_+(\lambda)\mathcal{D}_-(\lambda) + b_+(\lambda)\mathcal{C}_-(\lambda) + c_+(\lambda)\mathcal{B}_-(\lambda), \tag{2.20}$$

It is a one parameter family of commuting operators satisfying the following symmetries proprieties:

$$\mathcal{T}(\lambda) = \mathcal{T}(1/\lambda), \quad \mathcal{T}(-\lambda) = \mathcal{T}(\lambda). \tag{2.21}$$

The boundary quantum determinant:

$$det_q\, \mathcal{U}_{a,-}(\lambda) \equiv ((\lambda/q)^2 - (q/\lambda)^2)[\mathcal{A}_-(\lambda q^{1/2})\mathcal{A}_-(q^{1/2}/\lambda) + \mathcal{B}_-(\lambda q^{1/2})\mathcal{C}_-(q^{1/2}/\lambda)] \tag{2.22}$$

$$= ((\lambda/q)^2 - (q/\lambda)^2)[\mathcal{D}_-(\lambda q^{1/2})\mathcal{D}_-(q^{1/2}/\lambda) + \mathcal{C}_-(\lambda q^{1/2})\mathcal{B}_-(q^{1/2}/\lambda)], \tag{2.23}$$

is a central element in the reflection algebra, i.e.

$$[det_q\, \mathcal{U}_{a,-}(\lambda), \mathcal{U}_{a,-}(\mu)] = 0, \tag{2.24}$$

and its explicit expression reads:

$$det_q\, \mathcal{U}_{a,-}(\lambda) = (\lambda^2/q^2 - q^2/\lambda^2)\mathsf{A}_-(\lambda q^{1/2})\mathsf{A}_-(q^{1/2}/\lambda). \tag{2.25}$$

3 Gauged cyclic reflection algebra and SoV representations

In our previous paper we solved the spectral problem associated to the transfer matrix of the cyclic representations under the requirement that one of the boundary matrices is triangular, i.e. $b_+(\lambda) \equiv 0$. In this paper we want to solve the same type of spectral problem but for the most general boundary conditions. In order to do so we can follow the same approach used in the case of the transfer matrix associated to the spin-1/2 reflection algebra [68]. That is, we introduce the following linear combinations of the original reflection algebra generators:

$$\mathcal{A}_-(\lambda|\beta) = \left[-\left(\lambda q^{3/2}/\beta\right)\mathcal{A}_-(\lambda) - \alpha q\mathcal{B}_-(\lambda) + \mathcal{C}_-(\lambda)/(\alpha q) + \beta\mathcal{D}_-(\lambda)/\left(\lambda q^{3/2}\right)\right]/(\beta/q^2 - q^2/\beta) \tag{3.1}$$

$$\mathcal{B}_-(\lambda|\beta) = \left[-\left(\lambda\beta/q^{1/2}\right)\mathcal{A}_-(\lambda) - \alpha q\mathcal{B}_-(\lambda) + (\beta^2/\alpha q)\,\mathcal{C}_-(\lambda) + \left(\beta q^{1/2}/\lambda\right)\mathcal{D}_-(\lambda)\right]/(\beta - 1/\beta) \tag{3.2}$$

$$\mathcal{C}_-(\lambda|\beta) = \left[\left(\lambda q^{3/2}/\beta\right)\mathcal{A}_-(\lambda) + \alpha q\mathcal{B}_-(\lambda) - (q^3/\alpha\beta^2)\,\mathcal{C}_-(\lambda) - \left(q^{5/2}/\lambda\beta\right)\mathcal{D}_-(\lambda)\right]/(\beta/q^2 - q^2/\beta) \tag{3.3}$$

$$\mathcal{D}_-(\lambda|\beta) = \left[\left(\lambda\beta/q^{1/2}\right)\mathcal{A}_-(\lambda) + \alpha q\mathcal{B}_-(\lambda) - \mathcal{C}_-(\lambda)/\alpha q - \left(q^{1/2}/\lambda\beta\right)\mathcal{D}_-(\lambda)\right]/(\beta - 1/\beta), \tag{3.4}$$

where $\beta \neq \pm 1, \pm q^2$ and α are arbitrary complex values; to simplify the notation, we won't explicit the dependance in α. As it is discussed in the appendices A and B, these operators families still satisfy a set of commutation relations which are gauged versions of the reflection algebra commutation relations. In the following we will refer to these families as the gauge transformed reflection algebra generators. In the same appendices, we prove the following theorem, characterizing the representation of these generators:

Theorem 3.1. *For almost all the values of the boundary-bulk-gauge parameters there exit a left $\langle\Omega_\beta|$ and a right $|\Omega_\beta\rangle$ pseudo-eigenstate of $\mathcal{B}_-(\lambda|\beta)$:*

$$\langle\Omega_\beta|B_-(\lambda|\beta) = \mathrm{B}_{\boldsymbol{0}}(\lambda|\beta)\langle\Omega_{\beta/q^2}|, \quad B_-(\lambda|\beta)|\Omega_\beta\rangle = |\Omega_{q^2\beta}\rangle\mathrm{B}_{\boldsymbol{0}}(\lambda|\beta), \tag{3.5}$$

where:

$$\mathrm{B}_{\boldsymbol{h}}(\lambda|\beta) = \mathrm{B}_-(\beta)\left(\frac{\lambda^2}{q} - \frac{q}{\lambda^2}\right)\prod_{a=1}^{\mathsf{N}}\left(\frac{\lambda}{\mathrm{B}_{-,a}(\beta)q^{h_a}} - \frac{\mathrm{B}_{-,a}(\beta)q^{h_a}}{\lambda}\right)\left(\lambda q^{h_a}\mathrm{B}_{-,a}(\beta) - \frac{1}{\lambda q^{h_a}\mathrm{B}_{-,a}(\beta)}\right), \tag{3.6}$$

for $\boldsymbol{h} = (h_1, ..., h_{\mathsf{N}}) \in \{0, ..., p-1\}^{\mathsf{N}}$ with

$$\mathrm{B}^2_{-,n}(\beta) \neq q^{1-2h}\alpha_-^{2\epsilon}, \ \mathrm{B}^2_{-,n}(\beta) \neq -q^{1-2h}\beta_-^{2\epsilon}, \tag{3.7}$$
$$\mathrm{B}^2_{-,n}(\beta) \neq q^{1-2h}\mu_{m,+}^{2\epsilon}, \ \mathrm{B}^2_{-,n}(\beta) \neq q^{1-2h}\mu_{m,-}^{2\epsilon}, \tag{3.8}$$

for any $\epsilon = \pm 1$, $n, m \in \{1, ..., \mathsf{N}\}$ and $h \in \{1, ..., p-1\}$. Then, the following set of states:

$$\langle\beta, h_1, ..., h_{\mathsf{N}}| = \frac{1}{\mathrm{N}_\beta}\langle\Omega_\beta|\prod_{n=1}^{\mathsf{N}}\prod_{k_n=1}^{h_n}\frac{\mathcal{A}_-(q^{1-k_n}/\mathrm{B}_{-,n}(\beta)|\beta q^2)}{\mathsf{A}_-(q^{1-k_n}/\mathrm{B}_{-,n}(\beta))}, \tag{3.9}$$

$$|\beta, h_1, ..., h_{\mathsf{N}}\rangle \equiv \frac{1}{\mathrm{N}_{\beta/q^2}}\prod_{n=1}^{\mathsf{N}}\prod_{k_n=1}^{h_n}\frac{\mathcal{D}_-(q^{1-k_n}/\mathrm{B}_{-,n}(\beta)|\beta)}{\mathsf{D}_-(q^{1-k_n}/\mathrm{B}_{-,n}(\beta))}|\Omega_\beta\rangle, \tag{3.10}$$

form a left and a right basis of the representation space defining the following decomposition of the identity:

$$\mathbb{I} \equiv \sum_{h_1,...,h_{\mathsf{N}}=0}^{p-1}\prod_{1\leq b<a\leq\mathsf{N}}(X_a^{(h_a)} - X_a^{(h_a)})|\beta q^2, h_1, ..., h_{\mathsf{N}}\rangle\langle\beta, h_1, ..., h_{\mathsf{N}}|, \tag{3.11}$$

with

$$\langle\beta, h_1, ..., h_{\mathsf{N}}|\beta q^2, k_1, ..., k_{\mathsf{N}}\rangle = \prod_{1\leq a\leq\mathsf{N}}\delta_{h_a,k_a}\prod_{1\leq b<a\leq\mathsf{N}}\frac{1}{X_a^{(h_a)} - X_b^{(h_b)}}, \tag{3.12}$$

where

$$X_b^{(h_b)} = (\mathrm{B}_{-,b}(\beta)q^{h_b})^2 + 1/(\mathrm{B}_{-,b}(\beta)q^{h_b})^2 \tag{3.13}$$

for the non-zero normalization fixed by

$$\mathrm{N}_\beta = \left(\prod_{1\leq b<a\leq\mathsf{N}}\left(X_a^{(p-1)} - X_b^{(p-1)}\right)\langle\Omega_\beta|\Omega_{\beta q^2}\rangle\right)^{1/2}. \tag{3.14}$$

In this basis the operator family $\mathcal{B}_-(\lambda|\beta)$ is pseudo-diagonalized:

$$\langle\beta, h_1, ..., h_{\mathsf{N}}|\mathcal{B}_-(\lambda|\beta) = \mathrm{B}_{\boldsymbol{h}}(\lambda|\beta)\langle\beta/q^2, h_1, ..., h_{\mathsf{N}}|, \tag{3.15}$$
$$\mathcal{B}_-(\lambda|\beta)|\beta, h_1, ..., h_{\mathsf{N}}\rangle = |q^2\beta, h_1, ..., h_{\mathsf{N}}\rangle\mathrm{B}_{\boldsymbol{h}}(\lambda|\beta), \tag{3.16}$$

with simple pseudo-spectrum

$$\mathrm{B}^{2p}_{-,n}(\beta) \neq \pm 1, \ \mathrm{B}^{p}_{-,m}(\beta) \neq \mathrm{B}^{p}_{-,n}(\beta), \quad \forall n \neq m \in \{1, ..., \mathsf{N}\}, \tag{3.17}$$

and the operator families $\mathcal{A}_-(\lambda|\beta)$ and $\mathcal{D}_-(\lambda|\beta)$ in the zeros of $\mathcal{B}_-(\lambda|\beta)$ act as simple shift operators:

$$\langle\beta, h_1, ..., h_{\mathsf{N}}|\mathcal{A}_-(\zeta_a^{(h_a)}|\beta q^2) = \mathsf{A}_-(\zeta_a^{(h_a)})\langle\beta, h_1, ..., h_{\mathsf{N}}|T_a^{-\varphi_a}, \tag{3.18}$$
$$\mathcal{D}_-(\zeta_a^{(h_a)}|\beta)|\beta, h_1, ..., h_{\mathsf{N}}\rangle = T_a^{-\varphi_a}|\beta, h_1, ..., h_{\mathsf{N}}\rangle\mathsf{D}_-(\zeta_a^{(h_a)}), \tag{3.19}$$

where

$$\langle\beta, h_1, ..., h_a, ..., h_{\mathsf{N}}|T_a^{\pm} = \langle\beta, h_1, ..., h_a \pm 1, ..., h_{\mathsf{N}}|, \tag{3.20}$$
$$T_a^{\pm}|\beta, h_1, ..., h_a, ..., h_{\mathsf{N}}\rangle = |\beta, h_1, ..., h_a \pm 1, ..., h_{\mathsf{N}}\rangle, \tag{3.21}$$

and:

$$\zeta_n^{(h)} = \left(\mathrm{B}_{-,n}(\beta)q^h\right)^{\varphi_n} \quad \text{for} \ \ h \in \{0, ..., p-1\} \ \ \text{and} \ \ \forall n \in \{1, ..., 2\mathsf{N}\}, \tag{3.22}$$
$$\varphi_a = 1 - 2\theta(a - \mathsf{N}) \quad \text{with} \ \ \theta(x) = \{0 \text{ for } x \leq 0, \ 1 \text{ for } x > 0\}. \tag{3.23}$$

Let us comment that the existence of the states $\langle\Omega_\beta|$ and $|\Omega_\beta\rangle$ can be proven by a general argument which we present in Appendix B. For general representations, the pseudo-spectrum of $\mathcal{B}_-(\lambda|\beta)$, i.e. the values of $\mathrm{B}_{-,n}(\beta)$ and $\mathrm{B}_-(\beta)$, must be computed by recursion on the number of sites. However, in Appendix B we present the explicit expression for $\mathrm{B}_{-,n}(\beta)$ and $\mathrm{B}_-(\beta)$ in some particular representations.

The interest in these gauge transformed boundary generators is due to the possibility to use them to rewrite the transfer matrix associated to the most general cyclic 6-vertex reflection algebra representations in a simple form, as presented in the following proposition:

Proposition 3.1. *The quantum determinant can be written in terms of the gauge transformed boundary generators as:*

$$\frac{det_q\,\mathcal{U}_-(\lambda)}{(\lambda^2/q^2-q^2/\lambda^2)} = \mathcal{A}_-(q^{1/2}\lambda^\epsilon|\beta q^2)\mathcal{A}_-(q^{1/2}/\lambda^\epsilon|\beta q^2)+\mathcal{B}_-(q^{1/2}\lambda^\epsilon|\beta)\mathcal{C}_-(q^{1/2}/\lambda^\epsilon|\beta q^2) \tag{3.24}$$

$$= \mathcal{D}_-(q^{1/2}\lambda^\epsilon|\beta)\mathcal{D}_-(q^{1/2}/\lambda^\epsilon|\beta)+\mathcal{C}_-(q^{1/2}\lambda^\epsilon|\beta q^2)\mathcal{B}_-(q^{1/2}/\lambda^\epsilon|\beta), \tag{3.25}$$

$\epsilon=\pm 1$. *Moreover, if we set the gauge parameter α to:*

$$\alpha=-\beta\beta_+/q^2\alpha_+e^{\tau_+}, \tag{3.26}$$

the transfer matrix can be written as

$$\mathcal{T}(\lambda)=\mathsf{a}_+(\lambda)\mathcal{A}_-(\lambda|\beta)+\mathsf{a}_+(1/\lambda)\mathcal{A}_-(1/\lambda|\beta)+q\mathsf{c}_+(\lambda|\beta)\mathcal{B}_-(\lambda|\beta/q^2) \tag{3.27}$$

$$\mathcal{T}(\lambda)=\mathsf{d}_+(\lambda)\mathcal{D}_-(\lambda|\beta)+\mathsf{d}_+(1/\lambda)\mathcal{D}_-(1/\lambda|\beta)+\mathsf{c}_+(\lambda|\beta)\mathcal{B}_-(\lambda|\beta)/q, \tag{3.28}$$

where we have defined:

$$\mathsf{a}_+(\lambda)=-\frac{\lambda^2q-1/q\lambda^2}{\lambda^2-1/\lambda^2}g_+(\lambda),\quad \mathsf{d}_+(\lambda)=\frac{\lambda^2q-1/q\lambda^2}{\lambda^2-1/\lambda^2}g_+(q/\lambda), \tag{3.29}$$

$$\mathsf{c}_+(\lambda|\beta)=\frac{-q(\lambda^2q-1/q\lambda^2)\,(\beta\beta_+/q\alpha_+-q\alpha_+/\beta\beta_+)}{\beta\,(\alpha_+-1/\alpha_+)\,(\beta_++1/\beta_+)}. \tag{3.30}$$

Proof. The proof of this statement coincides with the one given in [68] for the XXZ spin 1/2 quantum chain with general integrable boundaries; in fact, this statement is representation independent. The only difference is that here we have used a Laurent polynomial form while in the XXZ case it was a trigonometric form. □

The simple representations (3.27)-(3.28) of the transfer matrix in terms of the gauge transformed boundary generators and the known actions (3.18)-(3.19) of these operators imply that the transfer matrix spectral problem is separated in the pseudo-eigenbasis of $\mathcal{B}_-(\lambda|\beta)$.

4 $\mathcal{T}$-spectrum characterization in SoV basis and scalar products

In this section we present the complete characterization of the spectrum of the transfer matrix $\mathcal{T}(\lambda)$ associated to the cyclic representations of the 6-vertex reflection algebra. We first present some preliminary properties satisfied by all the eigenvalue functions of the transfer matrix $\mathcal{T}(\lambda)$:

Lemma 4.1. *Denote by $\Sigma_{\mathcal{T}}$ the transfer matrix spectrum, then any $\tau(\lambda)\in\Sigma_{\mathcal{T}}$ is an even function of λ symmetrical under the transformation $\lambda\to 1/\lambda$ which admits the following interpolation formula:*

$$\begin{aligned}\tau(\lambda)=&\sum_{a=1}^{\mathsf{N}}\frac{\Lambda^2-X^2}{(X_a^{(0)})^2-X^2}\prod_{\substack{b=1\\b\neq a}}^{\mathsf{N}}\frac{\Lambda-X_b^{(0)}}{X_a^{(0)}-X_b^{(0)}}\tau(\zeta_a^{(0)})+(-1)^{\mathsf{N}}\frac{(\Lambda+X)}{2}\prod_{b=1}^{\mathsf{N}}\frac{\Lambda-X_b^{(0)}}{X-X_b^{(0)}}det_qM(1)\\&-(-1)^{\mathsf{N}}\frac{(\Lambda-X)}{2}\prod_{b=1}^{\mathsf{N}}\frac{\Lambda-X_b^{(0)}}{X+X_b^{(0)}}\frac{(\zeta_++1/\zeta_+)}{(\zeta_+-1/\zeta_+)}\frac{(\zeta_-+1/\zeta_-)}{(\zeta_--1/\zeta_-)}det_qM(i)\\&+(\Lambda^2-X^2)\tau_\infty\prod_{b=1}^{\mathsf{N}}(\Lambda-X_b^{(0)}),\end{aligned} \tag{4.1}$$

where:

$$\Lambda\equiv(\lambda^2+1/\lambda^2)\quad\text{and}\quad X\equiv q+1/q \tag{4.2}$$

and

$$\tau_\infty\equiv\frac{\kappa_+\kappa_-(e^{\tau_+-\tau_-}\prod_{b=1}^{\mathsf{N}}\delta_b\gamma_b+e^{\tau_--\tau_+}\prod_{b=1}^{\mathsf{N}}\alpha_b\beta_b)}{(\zeta_+-1/\zeta_+)\,(\zeta_--1/\zeta_-)}. \tag{4.3}$$

Proof. This lemma coincides with Lemma 5.1 of our previous paper. □

We introduce the following one-parameter family $D_\tau(\lambda)$ of $p\times p$ matrices:

$$D_\tau(\lambda)\equiv\begin{pmatrix}\tau(\lambda)&-\text{A}(1/\lambda)&0&\cdots&0&-\text{A}(\lambda)\\-\text{A}(q\lambda)&\tau(q\lambda)&-\text{A}(1/(q\lambda))&0&\cdots&0\\0&\ddots&&&&\vdots\\\vdots&&\cdots&&&\vdots\\\vdots&&&\cdots&&\vdots\\\vdots&&&&\ddots&0\\0&\ldots&0&-\text{A}(q^{2l-1}\lambda)&\tau(q^{2l-1}\lambda)&-\text{A}(1/(q^{2l-1}\lambda))\\-\text{A}(1/(q^{2l}\lambda))&0&\ldots&0&-\text{A}(q^{2l}\lambda)&\tau(q^{2l}\lambda)\end{pmatrix}, \tag{4.4}$$

where for now $\tau(\lambda)$ is a generic function and we have defined:

$$\text{A}(\lambda)=\mathsf{a}_+(\lambda)\mathsf{A}_-(\lambda), \tag{4.5}$$

where the coefficient $\text{A}(\lambda)$ satisfies the quantum determinant condition:

$$\text{A}(\lambda q^{1/2})\text{A}(q^{1/2}/\lambda)=\frac{\mathsf{a}_+(\lambda q^{1/2})\mathsf{a}_+(q^{1/2}/\lambda)\text{det}_q\mathcal{U}_-(\lambda)}{(\lambda/q)^2-(q/\lambda)^2}. \tag{4.6}$$

The separation of variables lead to the following discrete characterization of the transfer matrix spectrum.

Theorem 4.1. *For almost all the values of the boundary-bulk parameters $\mathcal{T}(\lambda)$ is diagonalizable and it has simple spectrum and $\Sigma_{\mathcal{T}}$ coincides with the set of polynomials $\tau(\lambda)$ of the form (4.1) which satisfy the following discrete system of equations:*

$$det\ D_\tau(\zeta_a^{(0)})=0,\ \forall a\in\{1,...,\mathsf{N}\}. \tag{4.7}$$

I) The right $\mathcal{T}$-eigenstate corresponding to $\tau(\lambda)\in\Sigma_{\mathcal{T}}$ is defined by the following decomposition in the right SoV-basis:

$$|\tau\rangle=\sum_{h_1,\ldots,h_{\mathsf{N}}=0}^{p-1}\prod_{a=1}^{\mathsf{N}}q_{\tau,a}^{(h_a)}\prod_{1\leq b<a\leq\mathsf{N}}(X_a^{(h_a)}-X_b^{(h_b)})|\beta,h_1,...,h_{\mathsf{N}}\rangle, \tag{4.8}$$

where the gauge parameters α and β satisfy the condition (3.26) and the $q_{\tau,a}^{(h_a)}$ are the unique nontrivial solutions up to normalization of the linear homogeneous system:

$$D_\tau(\zeta_a^{(0)})\begin{pmatrix}q_{\tau,a}^{(0)}\\\vdots\\q_{\tau,a}^{(p-1)}\end{pmatrix}=\begin{pmatrix}0\\\vdots\\0\end{pmatrix}. \tag{4.9}$$

II) The left $\mathcal{T}$-eigenstate corresponding to $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ is defined by the following decomposition in the left SoV-basis:

$$\langle \tau | = \sum_{h_1,...,h_{\mathsf{N}}=0}^{p-1} \prod_{a=1}^{\mathsf{N}} \hat{q}_{\tau,a}^{(h_a)} \prod_{1\leq b<a\leq \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)}) \langle h_1, ..., h_{\mathsf{N}}, \beta/q^2 |, \tag{4.10}$$

where the gauge parameters α and β satisfy the condition (3.26) and the $\hat{q}_{\tau,a}^{(h_a)}$ are the unique nontrivial solutions up to normalization of the linear homogeneous system:

$$\left(\begin{array}{ccc} \hat{q}_{\tau,a}^{(0)} & \dots & \hat{q}_{\tau,a}^{(p-1)} \end{array} \right) \left(\hat{D}_\tau(\zeta_a^{(0)})\right)^{t_0} = \left(\begin{array}{ccc} 0 & \dots & 0 \end{array} \right), \tag{4.11}$$

and $\hat{D}_\tau(\lambda)$ is the family of $p \times p$ matrices defined substituting in $D_\tau(\lambda)$ the coefficient $\mathrm{A}(\lambda)$ with

$$\mathrm{D}(\lambda) = \mathsf{d}_+(\lambda)\mathsf{D}_-(\lambda). \tag{4.12}$$

Proof. The Theorem 3.1 implies that for almost all the values of the gauge-boundary-bulk parameters the conditions (3.7)-(3.8) hold. Here, we need to prove also that for almost all the values of the boundary-bulk parameters we have,

$$\mathrm{B}^2_{-,n}(\beta) \neq q^{1-2h}\alpha_+^{\pm 2},\ \mathrm{B}^2_{-,n}(\beta) \neq -q^{1-2h}\beta_+^{\pm 2},\quad \forall h \in \{1,...,p-1\}, n \in \{1,...,\mathsf{N}\}, \tag{4.13}$$

once we set the ratio α/β as in (3.26). Let us first observe that $\mathcal{B}_-(\lambda|\beta)$ is a Laurent polynomial in α, β, the inner boundary parameters and the bulk parameters. So that by (3.26), the one parameter family $\mathcal{B}_-(\lambda|\beta)$ becomes Laurent polynomial in the outer boundary parameters too. Consequently, to prove that (4.13) is satisfied for almost all the values of the boundary-bulk parameters it is enough to prove that we can find some values of these parameters for which (4.13) is satisfied. Indeed, we can chose arbitrary boundary-bulk parameters satisfying the following inequalities:

$$\mu^p_{+,n} \neq \alpha_+^{\pm p} \text{ and } \mu^p_{+,n} \neq -\beta_+^{\pm p},\quad \forall n \in \{1,...,\mathsf{N}\}, \tag{4.14}$$

together with those in $(B.64)$ and $(B.65)$ and impose the N conditions $(B.63)$. Under these conditions, Theorem 3.1 implies the pseudo-diagonalizability of $\mathcal{B}_-(\lambda|\beta)$ and fixes the spectrum of its zeros $\mathrm{B}_{-,n}(\beta)$ by $(B.66)$; so that the inequality (4.13) is satisfied.

As we have proven that for almost all the values of the boundary-bulk parameters the inequalities (4.13), $(B.64)$ and $(B.65)$ hold, to prove this theorem we have just to follow the same proof given in the non-gauged case, i.e. the proof of Theorem 5.1 of our previous paper.

Let us comment that with respect to this last theorem here we are stating also the diagonalizability of the transfer matrix for almost any value of the parameters of the representation. This last statement can be proven as it follows. Let us consider the following special representation, where the bulk parameters satisfy:

$$c_n = -b_n^* \quad ; \quad d_n = -a_n^* \quad \text{and} \quad \alpha_n^*\beta_n = a_n^* b_n \tag{4.15}$$

and where the boundary matrices are diagonal, $K_0(\lambda|\xi_-,0,0)$ and $K_{+0}(\lambda|\xi_+,0,0)$, with the associated boundary parameters satisfying moreover $|\xi_-| = |\xi_+| = 1$. The $*$ operation is the complex conjugation. A simple direct calculation made for example in [110] leads to the following Hermitian conjugate of the monodromy matrix:

$$M^\dagger_{0Q}(\lambda|P_Q|P_C) = \sigma_0^y M_{0Q}(\lambda^*|P_Q|P_C)\sigma_0^y \tag{4.16}$$

where σ_0^y denotes the Pauli matrix.

From this relation, and using the specific inner boundary matrix introduced, one can compute the Hermitian conjugate of the boundary monodromy matrix:

$$\mathcal{U}^\dagger_{0Q}(\lambda) = \mathcal{U}^{t_0}_{0Q}(1/\lambda^*) \tag{4.17}$$

Then, from the definition of the boundary transfer matrix, and for the special choice of representation here chosen, we can show:

$$\mathcal{T}^\dagger(\lambda) = \mathcal{T}(1/\lambda^*) \tag{4.18}$$

Thus for this special representation the boundary transfer matrix is normal. Then it follows that the determinant of the $p^{\mathsf{N}} \times p^{\mathsf{N}}$ matrix of elements $\langle e_i|\tau_j\rangle$, where $\langle e_i|$ is the generic element of a given basis of covectors and $|\tau_j\rangle$ is the generic transfer matrix eigenvector, is non zero.

Noticing that this determinant is a fractional function of the bulk and boundary parameters, non zero for the special choice of the parameters above defined, it follows that it is non zero for almost every choice of the parameters. Which concludes the proof. □

It is also interesting to remark that we can obtain the coefficients of a left transfer matrix eigenstates in terms of those of the right one. The following lemma defines this characterization and can be proven as in the standard case [1]:

Lemma 4.2. *Let $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ then it holds:*

$$\frac{\hat{q}^{(h)}_{\tau,a}}{\hat{q}^{(h-1)}_{\tau,a}} = \frac{\mathrm{A}(1/\zeta_a^{(h-1)})}{\mathrm{D}(1/\zeta_a^{(h-1)})} \frac{q^{(h)}_{\tau,a}}{q^{(h-1)}_{\tau,a}}. \tag{4.19}$$

Let us introduce a class of left and right states, the so-called separate states, characterised by the following type of decompositions in the left and right separate basis:

$$\langle \alpha | \equiv \sum_{h_1,...,h_{\mathsf{N}}=0}^{p-1} \prod_{a=1}^{\mathsf{N}} \alpha_a^{(h_a)} \prod_{1\leq b<a\leq \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)}) \langle \beta/q^2, h_1, ..., h_{\mathsf{N}} | \tag{4.20}$$

$$| \beta \rangle = \sum_{h_1,...,h_{\mathsf{N}}=0}^{p-1} \prod_{a=1}^{\mathsf{N}} \beta_a^{(h_a)} \prod_{1\leq b<a\leq \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)}) | \beta, h_1, ..., h_{\mathsf{N}} \rangle \tag{4.21}$$

where the coefficients $\alpha_a^{(h_a)}$ and $\beta_a^{(h_a)}$ are arbitrary complex numbers, meaning that the coefficients of these separate states have a factorised form in these basis.

These separate states are interesting at least for two reasons: the eigenstates of the boundary transfer matrix are special separate states, and they admit a simple determinant scalar product, as it is stated in the next proposition:

Proposition 4.1. *Let us take an arbitrary separate left state $\langle \alpha |$ and an arbitrary separate right state $|\beta\rangle$. Then it holds:*

$$\langle \alpha | \beta \rangle = det\ \mathcal{M}^{(\alpha,\beta)} \tag{4.22}$$

where the elements of the size N matrix $\mathcal{M}^{(\alpha,\beta)}$ are given by:

$$\forall (a,b) \in [1,N]^2,\quad \mathcal{M}^{(\alpha,\beta)}_{a,b} \equiv \sum_{h=0}^{p-1} \alpha_a^{(h)} \beta_a^{(h)} (X_a^{(h)})^{(b-1)} \tag{4.23}$$

The proof is quite straightforward, it is based on the fact that one can see a Vandermonde determinant when computing the scalar product. One of the main corollary is the orthogonality of two eigenstates $\langle\tau|$ and $|\tau'\rangle$ of the boundary transfer matrix associated to two different eigenvalues $\tau(\lambda)$ and $\tau'(\lambda)$:

$$\langle\tau|\tau'\rangle = 0 \tag{4.24}$$

The computation of such scalar products is the very first step towards the dynamics, several further steps being required to reach this characterization for the models associated to cyclic representations of the 6-vertex reflection algebra: the reconstruction of the local operators in separate variables, the identification of the ground state, the homogeneous and the thermodynamic limit. For example a rewriting of the determinant representations for the form factors obtained from separation of variable will be necessary to overcome the standard problems related to the homogeneous limit. This problem has been addressed and solved for the XXX spin 1/2 chain, linking the separation of variable type determinants with Izergin's, Slavnov's and Gaudin's type determinants [118, 119].

5 Functional equation characterizing the $\mathcal{T}$-spectrum

The purpose of this section is to characterize the spectrum by functional relations analogous to Baxter's TQ-equation. To begin with, we first need the following property.

Lemma 5.1. *Let $\tau(\lambda)$ be a function of λ invariant under the transformation $\lambda \to 1/\lambda$ and $\lambda \to -\lambda$ then $det_p D_\tau(\lambda)$ is a function of*

$$Z = \lambda^{2p} + \frac{1}{\lambda^{2p}}, \tag{5.1}$$

i.e. it is a function of λ^p invariant under the transformations $\lambda^p \to 1/\lambda^p$ and $\lambda \to -\lambda$. Moreover, if $\tau(\lambda)$ is a Laurent polynomial of degree $\mathsf{N}+2$ in Λ then $det_p D_\tau(\lambda)$ is a Laurent polynomial of degree $\mathsf{N}+2$ in Z.

Proof. The first part of this lemma about the dependence w.r.t. Z of $\det_p D_\tau(\lambda)$ has been already proven in Lemma 5.2 of our previous paper [1] while the second part of this lemma can be proven following the proof given in Proposition 6.1 of the same paper. To adapt this proof here, let us observe that the matrix $D_\tau(i^a q^{h+1/2})$ for $a \in \{0,1\}$ and $h \in \{0,...,p-1\}$ contains one row with two divergent elements, i.e. $-\text{A}(\pm 1)$ and $-\text{A}(\pm i)$, respectively for $a = 0$ and $a = 1$. Nevertheless the determinants $\det_p D_\tau(i^a q^{h+1/2})$ are all finites for any $a \in \{0,1\}$ and $h \in \{0,...,p-1\}$ if $\tau(i^b q^{k+1/2})$ are finite for any $b \in \{0,1\}$ and $k \in \{0,...,p-1\}$. Indeed, by the symmetries $\lambda^p \to 1/\lambda^p$ and $\lambda \to -\lambda$ all the determinants $\det_p D_\tau(q^{h+1/2})$ coincide as well as all the determinants $\det_p D_\tau(iq^{h+1/2})$. So that we have to prove our statement for one value of $q^{h+1/2}$ and one value of $iq^{h+1/2}$. Now, we can use the expansion of the determinant w.r.t. the central row:

$$\begin{aligned}\det_p D_\tau(\lambda q^{1/2}) = {}& \tau(\lambda)\det_{p-1} D_{\tau,(p+1)/2,(p+1)/2}(\lambda q^{1/2}) + \frac{\text{X}(\lambda)\det_{p-1} D_{\tau,(p+1)/2,(p+1)/2-1}(\lambda q^{1/2})}{\lambda^2 - 1/\lambda^2} \\ & - \frac{\text{X}(1/\lambda)\det_{p-1} D_{\tau,(p+1)/2,(p+1)/2+1}(\lambda q^{1/2})}{\lambda^2 - 1/\lambda^2},\end{aligned} \tag{5.2}$$

where

$$\text{X}(\lambda) = \left(\lambda^2 - \frac{1}{\lambda^2}\right)\text{A}(\lambda), \tag{5.3}$$

and $D_{\tau,i,j}(\lambda)$ denotes the $(p-1)\times(p-1)$ matrix obtained from $D_\tau(\lambda)$ removing the row i and the column j. From the identity:

$$\det_{p-1} D_{\tau,(p+1)/2,(p+1)/2+1}(\lambda q^{1/2}) = \det_{p-1} D_{\tau,(p+1)/2,(p+1)/2-1}(q^{1/2}/\lambda), \tag{5.4}$$

and the regularity of these two determinants for $\lambda \to \pm 1$ and $\lambda \to \pm i$, it follows that $\det_p D_\tau(i^a q^{1/2})$ are finites too for $a \in \{0,1\}$. Now, our statement about the Laurent polynomiality of degree $\mathsf{N}+2$ of $\det_p D_\tau(\lambda)$ w.r.t. Z follows from the symmetries and from the fact that $\tau(\lambda)$ and $\text{X}(\lambda)$ are Laurent polynomials in λ of degree $2\mathsf{N}+4$. □

Let us introduce the following notations:

$$\text{A}_\infty = \lim_{\lambda\to+\infty} \lambda^{-2(\mathsf{N}+2)}\text{A}(\lambda) = \frac{(-1)^{\mathsf{N}+1}\kappa_+\kappa_-\alpha_-\beta_-\alpha_+\prod_{n=1}^{\mathsf{N}} b_n c_n}{q^{3+\mathsf{N}}\beta_+(\zeta_+ - 1/\zeta_+)(\zeta_- - 1/\zeta_-)}, \tag{5.5}$$

$$\text{A}_0 = \lim_{\lambda\to 0} \lambda^{2(\mathsf{N}+2)}\text{A}(\lambda) = \frac{(-1)^{\mathsf{N}+1}q^{3+\mathsf{N}}\kappa_+\kappa_-\beta_+\prod_{n=1}^{\mathsf{N}} a_n d_n}{\alpha_-\beta_-\alpha_+(\zeta_+ - 1/\zeta_+)(\zeta_- - 1/\zeta_-)}, \tag{5.6}$$

and

$$F(\lambda) = \prod_{b=1}^{2\mathsf{N}}\left(\frac{\lambda^p}{\left(\zeta_b^{(0)}\right)^p} - \frac{\left(\zeta_b^{(0)}\right)^p}{\lambda^p}\right), \tag{5.7}$$

then the following results hold:

Proposition 5.1. *For almost all the values of the boundary-bulk parameters, $\mathcal{T}(\lambda)$ has simple spectrum and $\tau(\lambda)$ of the form (4.1) is an element of $\Sigma_{\mathcal{T}}$ if and only if $det_p D_\tau(\lambda)$ is a Laurent polynomial of degree $\mathsf{N}+2$ in the variable Z which satisfies the following functional equation:*

$$det_p D_\tau(\lambda) = F(\lambda)\left(\lambda^{2p} - \frac{1}{\lambda^{2p}}\right)^2 \prod_{k=0}^{p-1}(\tau_\infty - (q^k\text{A}_\infty + q^{-k}\text{A}_0)). \tag{5.8}$$

Proof. The SoV characterization of the spectrum implies that $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ if and only if it holds:

$$\det_p D_\tau(\zeta_a^{(0)}) = 0, \ \forall a \in \{1,...,\mathsf{N}\}, \tag{5.9}$$

and $\tau(\lambda)$ has the form (4.1). In the previous lemma we have shown that $\det_p D_\tau(\lambda)$ is a Laurent polynomial of degree $\mathsf{N}+2$ in Z, here we show that from $\tau(\lambda)$ of form (4.1) it follows the identities:

$$\lim_{\lambda\to\pm1,\pm i} \det_p D_\tau(\lambda q^{1/2+h}) = 0 \quad \forall h \in \{0,...,p-1\}. \tag{5.10}$$

For the symmetry it is enough to consider the above limit in the case $h = 0$. Let us denote with $\bar{D}_\tau(\lambda q^{1/2})$ the matrix whose first row is the sum of the first and the last row of $D_\tau(\lambda q^{1/2})$ divided for $(\lambda^2 - 1/\lambda^2)$ and whose row $(p+1)/2$ is the row $(p+1)/2$ of $D_\tau(\lambda q^{1/2})$ multiplied for $(\lambda^2 - 1/\lambda^2)$ while all the others rows of $\bar{D}_\tau(\lambda q^{1/2})$ and $D_\tau(\lambda q^{1/2})$ coincide. Clearly it holds:

$$\det_p \bar{D}_\tau(\lambda q^{1/2}) = \det_p D_\tau(\lambda q^{1/2}), \tag{5.11}$$

so that we can compute the limits directly for $\det_p \bar{D}_\tau(\lambda q^{1/2})$. The interesting point is that now all the rows of the matrix $\bar{D}_\tau(\lambda q^{1/2})$ are finites in the limits $\lambda \to \pm 1, \pm i$, this is a consequence of the identities:

$$\tau(\pm i^a q^{1/2}) = \textsc{a}(\pm i^a q^{1/2}), \quad \textsc{a}(\pm i^a q^{-1/2}) = 0, \ \forall a \in \{0,1\}. \tag{5.12}$$

Explicitly, we have that the nonzero elements of the rows 1, $(p+1)/2$ and p are:

$$\left[\bar{D}_\tau(\pm i^a q^{1/2})\right]_{1,1} = r_{a,\pm}, \ \left[\bar{D}_\tau(\pm i^a q^{1/2})\right]_{1,2} = s_{a,\pm}, \tag{5.13}$$

$$\left[\bar{D}_\tau(\pm i^a q^{1/2})\right]_{1,p-1} = -\, s_{a,\pm}, \ \left[\bar{D}_\tau(\pm i^a q^{1/2})\right]_{1,p} = -\, r_{a,\pm}, \tag{5.14}$$

$$\left[\bar{D}_\tau(\pm i^a q^{1/2})\right]_{(p+1)/2,(p+1)/2-1} = -\,\textsc{x}(\pm i^a), \tag{5.15}$$

$$\left[\bar{D}_\tau(\pm i^a q^{1/2})\right]_{(p+1)/2,(p+1)/2+1} = \textsc{x}(\pm i^a), \tag{5.16}$$

$$\left[\bar{D}_\tau(\pm i^a q^{1/2})\right]_{p,1} = -\,\tau(\pm i^a q^{1/2}), \ \left[\bar{D}_\tau(\pm i^a q^{1/2})\right]_{p,p} = \tau(\pm i^a q^{1/2}), \tag{5.17}$$

where we have defined:

$$r_{a,\pm} = (-1)^a \lim_{\lambda \to \pm 1} \frac{\tau(i^a q^{1/2}\lambda) - \textsc{a}(i^a q^{1/2}/\lambda)}{\lambda^2 - 1/\lambda^2}, \quad s_{a,\pm} = (-1)^a \lim_{\lambda \to \pm 1} \frac{\textsc{a}(i^{-a} q^{-1/2}/\lambda)}{\lambda^2 - 1/\lambda^2}. \tag{5.18}$$

The remaining rows of $\bar{D}_\tau(\pm i^a q^{1/2})$ produce the tridiagonal part of this matrix. Then, it is possible to prove that this matrix has linear dependent rows; so that $\det_p \bar{D}_\tau(\pm i^a q^{1/2}) = 0$. Finally, we can compute the following asymptotic formulae:

$$\Delta_\infty \equiv \lim_{\lambda\to\infty} \lambda^{-2p(\mathsf{N}+2)} \det_p D_\tau(\lambda) = \det_p \left[\lim_{\lambda\to\infty} \lambda^{-2(\mathsf{N}+2)} D_\tau(\lambda)\right] \tag{5.19}$$

$$= \lim_{\lambda\to 0} \lambda^{2p(\mathsf{N}+2)} \det_p D_\tau(\lambda) = \det_p \left[\lim_{\lambda\to 0} \lambda^{2(\mathsf{N}+2)} D_\tau(\lambda)\right]^t \tag{5.20}$$

$$= \det_p \begin{pmatrix} \tau_\infty & -\textsc{a}_0 & 0 & \cdots & 0 & -\textsc{a}_\infty \\ -x\textsc{a}_\infty & x\tau_\infty & -x\textsc{a}_0 & 0 & \cdots & 0 \\ 0 & -x^2\textsc{a}_\infty & x^2\tau_\infty & -x^2\textsc{a}_0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 \\ 0 & \ldots & 0 & -x^{2l-1}\textsc{a}_\infty & x^{2l-1}\tau_\infty & -x^{2l-1}\textsc{a}_0 \\ -x^{2l}\textsc{a}_0 & 0 & \ldots & 0 & -x^{2l}\textsc{a}_\infty & x^{2l}\tau_\infty \end{pmatrix}, \tag{5.21}$$

where we have denoted with ^{t} the transpose of the matrix and $x = q^{2(\mathsf{N}+2)}$. We have that Δ_∞ is a degree p polynomial in τ_∞ whose zeros are known from the identities:

$$\Delta_\infty|_{\tau_\infty = q^k \textsc{a}_\infty + q^{-k}\textsc{a}_0} = 0 \ \ \forall k \in \{0,...,p-1\}, \tag{5.22}$$

so that we get:

$$\Delta_\infty = \prod_{k=0}^{p-1} (\tau_\infty - (q^k \textsc{a}_\infty + q^{-k}\textsc{a}_0)). \tag{5.23}$$

This means that we have determined $\det_p D_\tau(\lambda)$ in $\mathsf{N}+2$ different values of Z together with the asymptotic for $Z \to \infty$. From which the characterization (5.8) trivially follows. □

The discrete characterization of the spectrum given in Theorem 4.1 can be reformulated in terms of Baxter's type TQ-functional equations and the eigenstates admit an algebraic Bethe ansatz like reformulation, as we show in the next theorem. These type of reformulations of the spectrum holds for several models once they admit SoV description, see for example [69, 108–110, 115–117].

In the following we denote with $Q(\lambda)$ a polynomial in Λ of degree N_Q of the form:

$$Q(\lambda) = \prod_{b=1}^{\mathsf{N}_Q} (\Lambda - \Lambda_b). \tag{5.24}$$

Theorem 5.1. *For almost all the values of the boundary-bulk parameters such that:*

$$\tau_\infty \neq q^{-k} a_\infty + q^k a_0 \ \ \forall k \in \{0,...,p-1\}, \tag{5.25}$$

$\tau(\lambda) \in \Sigma_{\mathcal{T}}$ *if and only if* $\tau(\lambda)$ *is an entire function and there exists and is unique a polynomial* $Q(\lambda)$ *of the form (5.24) with* $\mathsf{N}_Q = (p-1)\,\mathsf{N}$, *satisfying the following functional equation:*

$$\tau(\lambda)Q(\lambda) = \textsc{a}(\lambda)Q(\lambda/q) + \textsc{a}(1/\lambda)Q(\lambda q) + \left[\tau_\infty - (q^{-\mathsf{N}_Q}\textsc{a}_\infty + q^{\mathsf{N}_Q}\textsc{a}_0)\right](\Lambda^2 - X^2)\,F(\lambda), \tag{5.26}$$

and the conditions:

$$(Q(\zeta_a^{(0)}),...,Q(\zeta_a^{(p-1)})) \neq (0,...,0) \quad \forall a \in \{1,...,\mathsf{N}\}. \tag{5.27}$$

Proof. Let us prove first that if it exists a $Q(\lambda)$ of the form (5.24) with $\mathsf{N}_Q = (p-1)\,\mathsf{N}$ satisfying (5.27) and (5.26) with $\tau(\lambda)$ an entire function, then $\tau(\lambda) \in \Sigma_{\mathcal{T}}$. The r.h.s of the equation (5.26) is a Laurent polynomial in λ as we have:

$$\textsc{a}(\lambda)Q(\lambda/q) + \textsc{a}(1/\lambda)Q(\lambda q) = \frac{\textsc{x}(\lambda)Q(q/\lambda) - \textsc{x}(1/\lambda)Q(\lambda q)}{\lambda^2 - 1/\lambda^2} \tag{5.28}$$

which is finite in the limits $\lambda \to \pm 1$, $\lambda \to \pm i$. So that the r.h.s. of (5.26) is a polynomial of degree $p\mathsf{N}+2$ in Λ, as it is invariant w.r.t. the transformations $\lambda \to -\lambda$ and $\lambda \to 1/\lambda$. Then, the assumption that $\tau(\lambda)$ is entire in λ implies by the equation (5.26) that $\tau(\lambda)$ is a polynomial in Λ of the form (4.1) and that it satisfies the equations:

$$\det_p D_\tau(\xi_a^{(0)}) = 0, \ \forall a \in \{1,...,\mathsf{N}\}, \tag{5.29}$$

thanks to (5.26) and (5.27), so that we obtain by SoV characterization $\tau(\lambda) \in \Sigma_{\mathcal{T}}$.

Let us now prove the reverse statement, i.e. we assume $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ and we prove that there exists $Q(\lambda)$ of the form (5.24) with degree $\mathsf{N}_Q = (p-1)\,\mathsf{N}$ satisfying (5.27) and (5.26). Let us consider the system of equations:

$$D_\tau(\lambda) \begin{pmatrix} X_0(\lambda) \\ X_1(\lambda) \\ \vdots \\ \vdots \\ X_{p-1}(\lambda) \end{pmatrix}_{p\times 1} = \left[\tau_\infty - (q^{-\mathsf{N}_Q}\textsc{a}_\infty + q^{\mathsf{N}_Q}\textsc{a}_0)\right] F(\lambda) \begin{pmatrix} \Lambda_0^2 - X^2 \\ \Lambda_1^2 - X^2 \\ \vdots \\ \vdots \\ \Lambda_{p-1}^2 - X^2 \end{pmatrix}_{p\times 1}, \tag{5.30}$$

where we have used the notations:

$$\Lambda_i = q^{2i}\lambda^2 + \frac{1}{q^{2i}\lambda^2}. \tag{5.31}$$

From the condition $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ and the assumption of general values of the boundary-bulk parameters (5.25), we know that $\det_p D_\tau(\lambda)$ is a non-zero polynomial, so defining:

$$Z_{det_p D_\tau} = \left\{ \pm i^a q^{h+1/2}, \pm\zeta_n^{(h)} \ \forall a \in \{0,1\}, n \in \{1,...,2\mathsf{N}\}, h \in \{0,...,p-1\} \right\}, \tag{5.32}$$

we can solve the previous system of equations for any value of $\lambda \in \mathbb{C}\backslash Z_{det_p D_\tau}$ by the Cramer's rule:

$$X_i(\lambda) = \frac{\tau_\infty - (q^{-\mathsf{N}_Q}\mathrm{A}_\infty + q^{\mathsf{N}_Q}\mathrm{A}_0)}{(Z^2-4)\prod_{k=0}^{p-1}[\tau_\infty - (q^k\mathrm{A}_\infty + q^{-k}\mathrm{A}_0)]}\det_p D_\tau^{(i+1)}(\lambda), \tag{5.33}$$

where $D_\tau^{(i)}(\lambda)$ is the $p \times p$ matrix obtained replacing the column i by the column at the r.h.s. of (5.30). Let us now rewrite the system of equation (5.30) bringing the first element in the last one for the two column vectors:

$$\tilde{D}_\tau(\lambda) \begin{pmatrix} X_1(\lambda) \\ X_2(\lambda) \\ \vdots \\ X_{p-1}(\lambda) \\ X_0(\lambda) \end{pmatrix}_{p\times 1} = \left[\tau_\infty - (q^{-\mathsf{N}_Q}\mathrm{A}_\infty + q^{\mathsf{N}_Q}\mathrm{A}_0)\right] F(\lambda) \begin{pmatrix} \Lambda_1^2 - X^2 \\ \Lambda_2^2 - X^2 \\ \vdots \\ \Lambda_{p-1}^2 - X^2 \\ \Lambda_0^2 - X^2 \end{pmatrix}_{p\times 1}, \tag{5.34}$$

where it is easy to see that $\tilde{D}_\tau(\lambda) = D_\tau(\lambda q)$. Rescaling now the argument of the functions, we can rewrite it as it follows:

$$D_\tau(\lambda) \begin{pmatrix} X_1(\lambda/q) \\ X_2(\lambda/q) \\ \vdots \\ X_{p-1}(\lambda/q) \\ X_0(\lambda/q) \end{pmatrix}_{p\times 1} = \left[\tau_\infty - (q^{-\mathsf{N}_Q}\mathrm{A}_\infty + q^{\mathsf{N}_Q}\mathrm{A}_0)\right] F(\lambda) \begin{pmatrix} \Lambda_0^2 - X^2 \\ \Lambda_1^2 - X^2 \\ \vdots \\ \Lambda_{p-2}^2 - X^2 \\ \Lambda_{p-1}^2 - X^2 \end{pmatrix}_{p\times 1}, \tag{5.35}$$

so that it must hold:

$$X_{i+1}(\lambda/q) = X_i(\lambda) \quad \forall\lambda \in \mathbb{C}\backslash Z_{det_p D_\tau},\ i \in \{0,...,p-1\} \tag{5.36}$$

where we have used the notation $X_p(\lambda) \equiv X_0(\lambda)$, or equivalently:

$$X_a(\lambda) = X_0(\lambda q^a) \quad \forall\lambda \in \mathbb{C}\backslash Z_{det_p D_\tau},\ a \in \{1,...,p-1\}. \tag{5.37}$$

Let us observe now that, from their definition, $X_a(\lambda)$ are continuous functions of λ so the above equation must be indeed satisfied for any value of $\lambda \in \mathbb{C}$. Moreover, from the identity:

$$\det_p D_\tau^{(1)}(\lambda) = \det_p D_\tau^{(1)}(1/\lambda), \tag{5.38}$$

which we can prove by some simple exchange of rows and columns, and from the fact that:

$$\forall i \in \{0...p-1\},\ \lambda \to 1/\lambda \ \Rightarrow\ \Lambda_i \to \Lambda_{p-i} \tag{5.39}$$

we get the symmetry:

$$X_0(\lambda) = X_0(1/\lambda), \tag{5.40}$$

which together with the symmetry $X_0(\lambda) = X_0(-\lambda)$ implies that $X_0(\lambda)$ is a function of Λ.

By using this last result we can rewrite the first equation of the system (5.30) as it follows $\forall\lambda \in \mathbb{C}$:

$$\tau(\lambda)X_0(\lambda) - \mathrm{A}(\lambda)X_0(\lambda/q) - \mathrm{A}(1/\lambda)X_0(\lambda q) = \left[\tau_\infty - (q^{-\mathsf{N}_Q}\mathrm{A}_\infty + q^{\mathsf{N}_Q}\mathrm{A}_0)\right]\left(\Lambda^2 - X^2\right)F(\lambda). \tag{5.41}$$

Let us now prove that $\det_p D_\tau^{(1)}(\lambda)$ is indeed a polynomial of degree $(p-1)\mathsf{N} + 2p$ in Λ. Note that in the following when we refer to a row $k \in \mathbb{Z}$ what we mean is the row $k' \in \{1,...,p\}$ with $k' = k \bmod p$. In the row $\bar{h} = (p+1)/2 + h$ of $D_\tau^{(1)}(\pm i^a q^{1/2-h}\lambda)$ at least one of the three non-zero elements is diverging under the limit $\lambda \to \pm 1, \pm i$. We can proceed as done in the previous theorem, we define the matrix $\bar{D}_{\tau,h}^{(1)}(\lambda)$ as the matrix with all the rows coinciding with those of $D_\tau^{(1)}(\lambda)$ except the row $h+1$, which is obtained by summing the row h and $h+1$ of $D_\tau^{(1)}(\lambda)$ and dividing them by $((i^a q^{h-1/2}\lambda)^2 - 1/(i^a q^{h-1/2}\lambda)^2)$, and the row $\bar{h}$, obtained multiplying the row $\bar{h}$ of $D_\tau^{(1)}(\lambda)$ by $((i^a q^{h-1/2}\lambda)^2 - 1/(i^a q^{h-1/2}\lambda)^2)$. Clearly we have:

$$\det_p \bar{D}_{\tau,h}^{(1)}(\lambda) = \det_p D_\tau^{(1)}(\lambda), \tag{5.42}$$

and the interesting point is that now all the rows of the matrix $D_{\tau,h}^{(1)}(\pm i^a q^{1/2-h}\lambda)$ are finite in the limits $\lambda \to \pm 1, \pm i$. We have that the nonzero elements of the rows h, $h+1$ and $\bar{h}$ of $\bar{D}_{\tau,h}^{(1)}(\pm i^a q^{1/2-h})$ reads:

$$\left[\bar{D}_{\tau,h}^{(1)}(\pm i^a q^{1/2-h})\right]_{h+1,h-1} = -\, s_{a,\pm}(1 - \delta_{h-1,1}) + \delta_{h-1,1}\omega_a, \tag{5.43}$$

$$\left[\bar{D}_{\tau,h}^{(1)}(\pm i^a q^{1/2-h})\right]_{h+1,h} = -\, r_{a,\pm}(1 - \delta_{h,1}) + \delta_{h,1}\omega_a, \tag{5.44}$$

$$\left[\bar{D}_{\tau,h}^{(1)}(\pm i^a q^{1/2-h})\right]_{h+1,h+1} = r_{a,\pm}(1 - \delta_{h+1,1}) + \delta_{h+1,1}\omega_a, \tag{5.45}$$

$$\left[\bar{D}_{\tau,h}^{(1)}(\pm i^a q^{1/2-h})\right]_{h+1,h+2} = s_{a,\pm}(1 - \delta_{h+2,1}) + \delta_{h+2,1}\omega_a, \tag{5.46}$$

$$\left[\bar{D}_{\tau,h}^{(1)}(\pm i^a q^{1/2-h})\right]_{\bar{h},\bar{h}-1} = -\,\mathrm{X}(\pm i^a)(-1)^a(1 - \delta_{\bar{h}-1,1}), \tag{5.47}$$

$$\left[\bar{D}_{\tau,h}^{(1)}(\pm i^a q^{1/2-h})\right]_{\bar{h},\bar{h}+1} = \mathrm{X}(\pm i^a)(-1)^a(1 - \delta_{\bar{h}+1,1}), \tag{5.48}$$

$$\left[\bar{D}_{\tau,h}^{(1)}(\pm i^a q^{1/2-h})\right]_{h,h-1} = \tau(\pm i^a q^{1/2})(1 - \delta_{h+1,1}), \tag{5.49}$$

$$\left[\bar{D}_{\tau,h}^{(1)}(\pm i^a q^{1/2-h})\right]_{h,h+1} = -\,\tau(\pm i^a q^{1/2})(1 - \delta_{h+1,1}), \tag{5.50}$$

where we have defined:

$$\omega_a = (-1)^a\left(q^2 - 1/q^2\right). \tag{5.51}$$

The remaining rows of $\bar{D}_{\tau,h}^{(1)}(\pm i^a q^{1/2-h})$ produce the tridiagonal part of this matrix. It is possible to prove than that for any $h \in \{0,...,p-1\}$ the matrix $\bar{D}_{\tau,h}^{(1)}(\pm i^a q^{1/2-h})$ has linear dependent rows; so that $\det_p D_\tau^{(1)}(\pm i^a q^{1/2-h}) = 0$ and the following factorization holds:

$$\det_p D_\tau^{(1)}(\lambda) = \left(\lambda^{2p} - \frac{1}{\lambda^{2p}}\right) P_\tau(\lambda). \tag{5.52}$$

Here $P_\tau(\lambda)$ is a Laurent polynomial of degree $2(p-1)\mathsf{N}+2p$ in λ, with the following odd parity:

$$P_\tau(1/\lambda) = -P_\tau(\lambda)\,, \tag{5.53}$$

being $\det_p D_\tau^{(1)}(\lambda)$ a polynomial of degree $(p-1)\mathsf{N}+2p$ in Λ. Here, we want to prove that in fact:

$$\det_p D_\tau^{(1)}(\lambda) = \left(\lambda^{2p} - \frac{1}{\lambda^{2p}}\right)^2 \bar{Q}_\tau(\lambda)\,, \tag{5.54}$$

where $\bar{Q}_\tau(\lambda)$ is a polynomial of degree $(p-1)\mathsf{N}$ in Λ. In order to do so we write down the equation:

$$\begin{aligned}\tau(\lambda)R_\tau(\lambda) &= \mathrm{A}(\lambda)R_\tau(\lambda/q) + \mathrm{A}(1/\lambda)R_\tau(\lambda q)\\ &+ \left(Z^2-4\right)\left(\Lambda^2 - X^2\right)\prod_{k=0}^{p-1}\left[\tau_\infty - (q^k\mathrm{A}_\infty + q^{-k}\mathrm{A}_0)\right]F(\lambda),\end{aligned} \tag{5.55}$$

where for convenience we have denoted $R_\tau(\lambda) = \det_p D_\tau^{(1)}(\lambda)$. The above equation is a direct consequence of the equation satisfied by $X_0(\lambda)$ and of the definition of this last function in terms of $\det_p D_\tau^{(1)}(\lambda)$. Now let us consider the following limit on the above equation $\lambda \to \pm i^a$ with $a \in \{0,1\}$:

$$\begin{aligned}\tau(\pm i^a)R_\tau(\pm i^a) &= \frac{1}{\pm 2i^a}\frac{d\mathrm{X}}{d\lambda}(\pm i^a)\left(R_\tau(\pm i^a/q) - R_\tau(\pm i^a q)\right)\\ &+ \mathrm{X}(\pm i^a)\lim_{\lambda\to\pm i^a}\left[\frac{R_\tau(\lambda/q)}{\lambda^2-1/\lambda^2} - \frac{R_\tau(\lambda q)}{\lambda^2-1/\lambda^2}\right],\end{aligned} \tag{5.56}$$

now by using the known identities:

$$R_\tau(\pm i^a) = R_\tau(\pm i^a/q) = R_\tau(\pm i^a q) = 0, \tag{5.57}$$

$$\frac{R_\tau(\lambda/q)}{\lambda^2-1/\lambda^2} = \frac{R_\tau(q/\lambda)}{\lambda^2-1/\lambda^2}, \tag{5.58}$$

we get:

$$\lim_{\lambda\to\pm i^a}\frac{R_\tau(\lambda/q)}{\lambda^2-1/\lambda^2} = -\lim_{\lambda\to\pm i^a}\frac{R_\tau(\lambda q)}{\lambda^2-1/\lambda^2}, \tag{5.59}$$

and so being $\mathrm{X}(\pm i^a) \neq 0$

$$\lim_{\lambda\to\pm i^a}\frac{R_\tau(\lambda/q)}{\lambda^2-1/\lambda^2} = 0. \tag{5.60}$$

These results imply the identities:

$$P_\tau(\pm i^a/q) = -P_\tau(\pm i^a q) = 0. \tag{5.61}$$

We can now write the functional equation for $P_\tau(\lambda)$:

$$\begin{aligned}\tau(\lambda)P_\tau(\lambda) &= \mathrm{A}(\lambda)P_\tau(\lambda/q) + \mathrm{A}(1/\lambda)P_\tau(\lambda q)\\ &+ \left(\lambda^{2p} - \frac{1}{\lambda^{2p}}\right)\left(\Lambda^2 - X^2\right)\prod_{k=0}^{p-1}\left[\tau_\infty - (q^k\mathrm{A}_\infty + q^{-k}\mathrm{A}_0)\right]F(\lambda).\end{aligned} \tag{5.62}$$

Taking the limit $\lambda \to \pm i^a$ with $a \in \{0,1\}$, we obtain:

$$\begin{aligned}\tau(\pm i^a)P_\tau(\pm i^a) &= \frac{1}{\pm 2i^a}\frac{d\mathrm{X}}{d\lambda}(\pm i^a)\left(P_\tau(\pm i^a/q) - P_\tau(\pm i^a q)\right)\\ &+ \mathrm{X}(\pm i^a)\lim_{\lambda\to\pm i^a}\left[\frac{P_\tau(\lambda/q)}{\lambda^2-1/\lambda^2} - \frac{P_\tau(\lambda q)}{\lambda^2-1/\lambda^2}\right],\end{aligned} \tag{5.63}$$

so that using the previous result (5.61) and the identity:

$$\lim_{\lambda\to\pm i^a}\frac{P_\tau(\lambda/q)}{\lambda^2-1/\lambda^2} = \lim_{\lambda\to\pm i^a}\frac{P_\tau(\lambda q)}{\lambda^2-1/\lambda^2} \tag{5.64}$$

we obtain

$$P_\tau(\pm i^a) = 0, \tag{5.65}$$

being $\tau(\pm i^a) \neq 0$. Let us now compute the functional equation for $P_\tau(\lambda)$ in the points $\lambda = \pm i^a q^\epsilon$ for $a \in \{0,1\}$, $\epsilon \in \{-1,1\}$, we obtain:

$$\tau(\pm i^a q)P_\tau(\pm i^a q) = \mathrm{A}(\pm i^a q)P_\tau(\pm i^a) + \mathrm{A}(\pm i^a/q)P_\tau(\pm i^a q^2), \tag{5.66}$$

$$\tau(\pm i^a/q)P_\tau(\pm i^a/q) = \mathrm{A}(\pm i^a/q)P_\tau(\pm i^a/q^2) + \mathrm{A}(\pm i^a q)P_\tau(\pm i^a), \tag{5.67}$$

implying:

$$P_\tau(\pm i^a/q^2) = -P_\tau(\pm i^a q^2) = 0, \tag{5.68}$$

being $\mathrm{A}(\pm i^a q^\epsilon) \neq 0$ for $a, \epsilon \in \{0,1\}$. We can iterate these computations for $\lambda = \pm i^a q^{b\epsilon}$ for any $a \in \{0,1\}$, $\epsilon \in \{-1,1\}$ and $b \in \{2,...,(p-3)/2\}$ obtaining that:

$$P_\tau(\pm i^a/q^{2b}) = -P_\tau(\pm i^a q^{2b}) = 0, \text{ for any } b \in \{1,...,(p-3)/2\}\,. \tag{5.69}$$

In the cases $\lambda = \pm i^a q^{\pm 1/2}$ as $\mathrm{A}(\pm i^a/q^{1/2}) = 0$ the functional equation for $P_\tau(\lambda)$ give us:

$$\tau(\pm i^a q^{\pm 1/2})P_\tau(\pm i^a q^{\pm 1/2}) = \mathrm{A}(\pm i^a q^{1/2})P_\tau(\pm i^a q^{\mp 1/2}), \tag{5.70}$$

which being $P_\tau(\pm i^a q^{1/2}) = -P_\tau(\pm i^a q^{-1/2})$ and $\tau(\pm i^a q^{\pm 1/2}) = \mathrm{A}(\pm i^a q^{1/2}) \neq 0$ implies the identity:

$$P_\tau(\pm i^a q^{1/2}) = -P_\tau(\pm i^a q^{-1/2}) = 0, \tag{5.71}$$

so that the factorization (5.54) is proven and we get that:

$$X_0(\lambda) = \frac{\tau_\infty - (q^{-\mathsf{N}_Q}\mathrm{A}_\infty + q^{\mathsf{N}_Q}\mathrm{A}_0)}{\prod_{k=0}^{p-1}\left[\tau_\infty - (q^k\mathrm{A}_\infty + q^{-k}\mathrm{A}_0)\right]}\bar{Q}_\tau(\lambda), \tag{5.72}$$

is a polynomial of degree $\mathsf{N}_Q = (p-1)\,\mathsf{N}$ in Λ which has the form (5.24). This follows by taking the asymptotic of its functional equation so that we can fix:

$$Q(\lambda) \equiv X_0(\lambda), \tag{5.73}$$

hence giving a constructive proof of the existence of the polynomial Q-function solution of the equation (5.26). The fact that it is unique is shown observing that if $\hat{Q}(\lambda)$ is another polynomial solution then:

$$D_\tau(\lambda)\begin{pmatrix} Q(\lambda) - \hat{Q}(\lambda) \\ Q(\lambda q) - \hat{Q}(\lambda q) \\ \vdots \\ \vdots \\ Q(\lambda q^{p-1}) - \hat{Q}(\lambda q^{p-1}) \end{pmatrix}_{p\times 1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}_{p\times 1}, \tag{5.74}$$

from which it follows $Q(\lambda) \equiv \hat{Q}(\lambda)$ as $D_\tau(\lambda)$ is invertible for any $\lambda \in \mathbb{C}\backslash Z_{det_p D_\tau}$.

Finally, let us show that $Q(\lambda)$ satisfies the condition (5.27). By the definition (5.33), $Q(\lambda)$ is a continuous function of the boundary-bulk parameters, then it is enough to prove this statement for some value of these parameters to show that it holds for almost all the values of these parameters.

Let us impose the condition $(B.63)$, where the ratio β/α is fixed by (3.26), then the following identities are satisfied:

$$\mathrm{A}(\zeta_a^{(0)}) = 0 \quad \forall a \in \{1, ..., \mathsf{N}\}, \tag{5.75}$$

and the SoV characterization of the transfer matrix spectrum holds for any value of the boundary-bulk parameters satisfying the inequalities $(B.64)$-$(B.65)$. So in particular if we impose:

$$\mu_{n_k,-} = 1/(q^{1+k}\mu_{a,+}) \quad \forall k \in \{1, ..., p-1\}, \tag{5.76}$$

for some $n_k \in \{1, ..., \mathsf{N}\}\backslash\{a\}$ once we have chosen any $a \in \{1, ..., \mathsf{N}\}$. Under these conditions it holds:

$$\mathrm{A}(\zeta_a^{(k)}) = 0, \ \forall k \in \{1, ..., p-1\}, \tag{5.77}$$

and the SoV representation implies the following centrality condition:

$$\prod_{k=0}^{p-1} \mathcal{T}(\zeta_a^{(k)}) = \prod_{k=0}^{p-1} \mathrm{A}(1/\zeta_a^{(k)}), \tag{5.78}$$

from which in particular follows:

$$\prod_{k=0}^{p-1} \tau(\zeta_a^{(k)}) = \prod_{k=0}^{p-1} \mathrm{A}(1/\zeta_a^{(k)}). \tag{5.79}$$

Let us remark now that the r.h.s and the l.h.s of the above equation are continuous w.r.t. the boundary-bulk parameters so that the above identity holds also if we take the special limit $\mu_{a,-} \to q^{1-p}/\mu_{a,+}$ for which it holds $\mathrm{A}(1/\zeta_a^{(p-1)}) = 0$ and so we get:

$$\exists ! \bar{h} \in \{0, ..., p-1\} : \tau(\zeta_a^{(\bar{h})}) = 0. \tag{5.80}$$

By definition of the function $Q(\lambda)$ under these conditions and limit on the bulk parameters we get:

$$Q(\zeta_a^{(\bar{h})}) \propto \det_p \begin{pmatrix} W_{a,\bar{h}} & -\mathrm{A}(1/\zeta_a^{(\bar{h})}) & 0 & \cdots & 0 & 0 \\ W_{a,\bar{h}+1} & \tau(\zeta_a^{(\bar{h}+1)}) & -\mathrm{A}(1/\zeta_a^{(\bar{h}+1)}) & 0 & \cdots & 0 \\ W_{a,\bar{h}+2} & 0 & \tau(\zeta_a^{(\bar{h}+2)}) & -\mathrm{A}(1/\zeta_a^{(\bar{h}+1)}) & & \vdots \\ \vdots & & \cdots & & & \vdots \\ W_{a,p-1} & 0 & \cdots 0 & \tau(\zeta_a^{(p-1)}) & 0 \cdots & \vdots \\ \vdots & & & & \ddots & 0 \\ W_{a,\bar{h}} & \cdots & 0 & 0 & \tau(\zeta_a^{(\bar{h}-2)}) & -\mathrm{A}(1/\zeta_a^{(\bar{h}-2)}) \\ W_{a,\bar{h}} & 0 & \cdots & 0 & 0 & \tau(\zeta_a^{(\bar{h}-1)}) \end{pmatrix}, \tag{5.81}$$

where we have defined:

$$W_{a,k} = \left((\zeta_a^{(k)})^2 + 1/(\zeta_a^{(k)})^2\right)^2 - X^2. \tag{5.82}$$

Now replacing the first row R_1 with the following linear combination of rows:

$$\bar{R}_1 = R_1 + \sum_{i=0}^{p-2-\bar{h}} \prod_{j=0}^{i} \frac{\mathrm{A}(1/\zeta_a^{(\bar{h}+j)})}{\tau(\zeta_a^{(\bar{h}+j+1)})} R_{2+i}, \tag{5.83}$$

we get

$$\bar{R}_1 = \begin{pmatrix} \bar{W}_{a,\bar{h}} & 0 & \cdots & 0 & 0 \end{pmatrix}_{1\times p} \tag{5.84}$$

where:

$$\bar{W}_{a,\bar{h}} = W_{a,\bar{h}} + \sum_{i=0}^{p-2-\bar{h}} \prod_{j=0}^{i} \frac{\mathrm{A}(1/\zeta_a^{(\bar{h}+j)})}{\tau(\zeta_a^{(\bar{h}+j+1)})} W_{a,\bar{h}+1+i} \tag{5.85}$$

and so:

$$Q(\zeta_a^{(\bar{h})}) = \bar{W}_{a,\bar{h}} \prod_{k\neq \bar{h}, k=0}^{p-1} \tau(\zeta_a^{(k)}) \neq 0, \tag{5.86}$$

for generic values of the boundary-bulk parameters. Indeed, as the $W_{a,\bar{h}+1+i}$ are functions only of the bulk parameter $\mu_{a,+}$ while the ratios $\mathrm{A}(1/\zeta_a^{(\bar{h}+j)})/\tau(\zeta_a^{(\bar{h}+j+1)})$ are functions of both the boundary and the bulk parameters then we can prove that $\bar{W}_{a,\bar{h}} \neq 0$. Explicitly we can compute the asymptotic of $\bar{W}_{a,\bar{h}}$ in the limit $\mu_{a,+} \to \infty$, by using the know asymptotic of the transfer matrix, therefore showing that it is non-zero for general values of boundary-bulk parameters. □

In the previous theorem we have excluded the boundary-bulk one-constraint cases leading to an identically zero $\det D_\tau(\lambda)$ for any $\tau(\lambda) \in \Sigma_{\mathcal{T}}$, these specific cases are considered in the next theorem.

Theorem 5.2. *Let us assume that there exists a $k \in \{0, ..., p-1\}$ such that it holds:*

$$\tau_\infty = q^{-k}\mathrm{A}_\infty + q^k \mathrm{A}_0, \tag{5.87}$$

then, for almost all the values of the boundary-bulk parameters, $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ if and only if $\tau(\lambda)$ is an entire function and there exists and is unique a polynomial $Q(\lambda)$ of the form (5.24) with $\mathsf{N}_Q \leq (p-1)(\mathsf{N}+1)$ and $\mathsf{N}_Q = k \bmod p$, satisfying the following homogeneous Baxter equation:

$$\tau(\lambda)Q(\lambda) = \mathrm{A}(\lambda)Q(\lambda/q) + \mathrm{A}(1/\lambda)Q(\lambda q), \tag{5.88}$$

and the conditions:

$$(Q(\zeta_a^{(0)}), ..., Q(\zeta_a^{(p-1)})) \neq (0, ..., 0) \quad \forall a \in \{1, ..., \mathsf{N}\}. \tag{5.89}$$

Proof. First let us assume that $\tau(\lambda)$ and $Q(\lambda)$ satisfies the homogeneous Baxter equation with $\tau(\lambda)$ entire function and $Q(\lambda)$ polynomial of the form (5.24) with $\mathsf{N}_Q \leq (p-1)(\mathsf{N}+1)$ and $\mathsf{N}_Q = k \bmod p$, then from this same equation it follows that $\tau(\lambda)$ is a polynomial of the form (4.1). Moreover, for any fixed $\lambda \in \mathbb{C}$ we can construct the following homogeneous system of equations:

$$D_\tau(\lambda) \begin{pmatrix} Q(\lambda) \\ Q(\lambda q) \\ \vdots \\ \vdots \\ Q(\lambda q^{p-1}) \end{pmatrix}_{p\times 1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}_{p\times 1}, \tag{5.90}$$

which is satisfied as a consequence of the Baxter equation. Finally, being $(Q(\lambda), ..., Q(\lambda q^{p-1}))$ non-zero for any $\lambda \in \mathbb{C}$, up to at most a finite number of values, we get:

$$\det D_\tau(\lambda) = 0 \quad \forall \lambda \in \mathbb{C} \tag{5.91}$$

so that Proposition 5.1 implies $\tau(\lambda) \in \Sigma_{\mathcal{T}}$.

To prove the reverse statement we use the results of the Lemma C.1 on the matrix $D_\tau(\lambda)$ and on its cofactors:

$$\mathrm{C}_{i,j}(\lambda) = (-1)^{i+j} \det_{p-1} D_{\tau,i,j}(\lambda). \tag{5.92}$$

We take now $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ from which it holds:

$$\det D_\tau(\lambda) = 0 \quad \forall \lambda \in \mathbb{C}, \tag{5.93}$$

and so by Lemma C.1 it follows that $\mathrm{rank} D_\tau(\lambda) = p - 1$ for any $\lambda \in \mathbb{C}\backslash K$, where K is a finite set of complex numbers if not empty. Then the matrix composed of the cofactors of the matrix $D_\tau(\lambda)$ has rank 1 for any $\lambda \in \mathbb{C}\backslash K$. This just means the proportionality:

$$\mathrm{V}_i(\lambda) = \mathrm{A}_{i,j}(\lambda)\mathrm{V}_j(\lambda)\ \forall \lambda \in \mathbb{C}\backslash K, \forall i,j \in \{1,...,p\} \tag{5.94}$$

where we have defined:

$$\mathrm{V}_i(\lambda) \equiv (\mathrm{C}_{i,1}(\lambda), \mathrm{C}_{i,2}(\lambda), ..., \mathrm{C}_{i,p}(\lambda))\ \forall \lambda \in \mathbb{C}\backslash K, \forall i \in \{1,...,p\} \tag{5.95}$$

and $\mathrm{A}_{i,j}(\lambda)$ are some functions such that:

$$\mathrm{A}_{i,j}(\lambda) \neq 0 \text{ and finite for any } \lambda \in \mathbb{C}\backslash\{K \cup K_0 \cup K_i \cup K_j\} \tag{5.96}$$

where K_0 is the set of the p-roots of unit and

$$K_a \equiv \{x \in \mathbb{C} : \mathrm{V}_a(x) \equiv (0, ..., 0)\}\ \forall a \in \{1, ..., p\} \tag{5.97}$$

such sets are finite if not empty, being the elements of the vectors $(\Lambda^p - X^p)\mathrm{V}_i(\lambda)$ Laurent polynomials. The above identities in particular imply:

$$\mathrm{A}_{1,2}(\lambda)\mathrm{C}_{1,1}(\lambda)\mathrm{C}_{2,2}(\lambda) = \mathrm{A}_{1,2}(\lambda)\mathrm{C}_{1,2}(\lambda)\mathrm{C}_{2,1}(\lambda)\ \forall \lambda \in \mathbb{C}\backslash K \tag{5.98}$$

so that for any $\lambda \in \mathbb{C}\backslash\{K \cup K_0 \cup K_i \cup K_j\}$ it holds:

$$\mathrm{C}_{1,1}(\lambda)\mathrm{C}_{2,2}(\lambda) = \mathrm{C}_{1,2}(\lambda)\mathrm{C}_{2,1}(\lambda). \tag{5.99}$$

Hence it holds for any $\lambda \in \mathbb{C}$ using continutiy properties of the cofactors, being $\{K \cup K_0 \cup K_i \cup K_j\}$ a finite set of values. Similarly, the fact that the vectorial condition $D(\lambda)\mathrm{V}_1(\lambda) = \underline{0}$ holds true for any $\lambda \in \mathbb{C}\backslash K$ implies that it is indeed satisfied for any $\lambda \in \mathbb{C}$. Here, we write explicitly the first element of this vectorial condition:

$$\tau(\lambda)\mathrm{C}_{1,1}(\lambda) = \mathrm{A}(\lambda)\mathrm{C}_{1,p}(\lambda) + \mathrm{A}(1/\lambda)\mathrm{C}_{1,2}(\lambda), \tag{5.100}$$

together with the rewriting of (5.99) by using the identity (C.1):

$$\mathrm{C}_{1,1}(\lambda)\mathrm{C}_{1,1}(\lambda q) = \mathrm{C}_{1,2}(\lambda)\mathrm{C}_{1,p}(q\lambda). \tag{5.101}$$

Once we recall that $\mathrm{C}_{1,1}(\lambda)$, $\mathrm{C}_{1,2}(\lambda)$ and $\mathrm{C}_{1,p}(\lambda)$ are Laurent polynomial in λ satisfying the factorizations ($C.4$), ($C.5$) and ($C.6$), respectively, it follows that the above two equations holds as well as if written in terms of the functions $\widehat{\mathrm{C}}_{1,1}(\lambda)$, $\widehat{\mathrm{C}}_{1,2}(\lambda)$ and $\widehat{\mathrm{C}}_{1,p}(\lambda)$.

Similarly to what has been done in the Lemma 5 of the paper [109], we can show that the two above equations for $\widehat{\mathrm{C}}_{1,1}(\lambda)$, $\widehat{\mathrm{C}}_{1,2}(\lambda)$ and $\widehat{\mathrm{C}}_{1,p}(\lambda)$ and their symmetry properties ($C.3$) imply that if $\widehat{\mathrm{C}}_{1,1}(\lambda)$ has a common zero with $\widehat{\mathrm{C}}_{1,2}(\lambda)$ then this is also a zero of $\widehat{\mathrm{C}}_{1p}(\lambda)$ and also the inverse of such zero is a common zero of these polynomials. Moreover, the same statement holds exchanging $\widehat{\mathrm{C}}_{1,2}(\lambda)$ with $\widehat{\mathrm{C}}_{1,p}(\lambda)$. So we can denote with $\mathrm{c}_{1,1}\overline{\mathrm{C}}_{1,1}(\lambda)$, $\mathrm{c}_{1,2}\overline{\mathrm{C}}_{1,2}(\lambda)$ and $\mathrm{c}_{1,p}\overline{\mathrm{C}}_{1,p}(\lambda)$ the polynomials obtained simplifying the common factors in $\widehat{\mathrm{C}}_{1,1}(\lambda)$, $\widehat{\mathrm{C}}_{1,2}(\lambda)$ and $\widehat{\mathrm{C}}_{1,p}(\lambda)$. Then, they have to satisfy the relations:

$$\overline{\mathrm{C}}_{1,p}(\lambda) = y_{1,1}\overline{\mathrm{C}}_{1,1}(\lambda q^{-1}), \quad \overline{\mathrm{C}}_{1,2}(\lambda) = y_{1,1}^{-1}\overline{\mathrm{C}}_{1,1}(\lambda q) \tag{5.102}$$

and defined $x_{1,1} \equiv \mathrm{c}_{1,1}/\mathrm{c}_{1,2} = \mathrm{c}_{1,p}/\mathrm{c}_{1,1}$, we obtain the following Baxter equation in the polynomial $\overline{\mathrm{C}}_{1,1}(\lambda)$:

$$t(\lambda)\overline{\mathrm{C}}_{1,1}(\lambda) = (x_{1,1}y_{1,1})\,\mathrm{A}(\lambda)\overline{\mathrm{C}}_{1,1}(\lambda q^{-1}) + (1/(x_{1,1}y_{1,1}))\,\mathrm{A}(1/\lambda)\overline{\mathrm{C}}_{1,1}(\lambda q), \tag{5.103}$$

and computing the above equation in $\lambda = q^{1/2}$ we get:

$$t(q^{1/2})\overline{\mathrm{C}}_{1,1}(q^{1/2}) = (x_{1,1}y_{1,1})\,\mathrm{A}(q^{1/2})\overline{\mathrm{C}}_{1,1}(q^{-1/2}), \tag{5.104}$$

from which it follows $x_{1,1}y_{1,1} = 1$ once we recall that $\mathrm{C}_{1,1}(q^{1/2}) \neq 0$ and that $\overline{\mathrm{C}}_{1,1}(\lambda)$ is even under $\lambda \to 1/\lambda$. So, we can define:

$$Q(\lambda) \equiv \overline{\mathrm{C}}_{1,1}(\lambda), \tag{5.105}$$

a polynomial in Λ of maximal degree $(p-1)(\mathsf{N}+1)$, which satisfies the homogeneous Baxter equation as required. □

Let us introduce now the following states:

$$\langle \beta, \omega| = \sum_{h_1,...,h_{\mathsf{N}}=0}^{p-1} \prod_{a=1}^{\mathsf{N}} \prod_{k_a=0}^{h_a-1} \frac{\mathrm{A}(1/\zeta_a^{(k_a)})}{\mathrm{D}(1/\zeta_a^{(k_a)})} \prod_{1\leq b<a\leq \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)})\, \langle \beta, h_1, ..., h_{\mathsf{N}}|, \tag{5.106}$$

$$|\beta, \bar{\omega}\rangle = \sum_{h_1,...,h_{\mathsf{N}}=0}^{p-1} \prod_{1\leq b<a\leq \mathsf{N}} (X_a^{(h_a)} - X_b^{(h_b)}) |\beta, h_1, ..., h_{\mathsf{N}}\rangle, \tag{5.107}$$

and the following renormalization of the $\mathcal{B}_-$-operator family

$$\hat{\mathcal{B}}_-(\lambda|\beta) = \frac{\mathcal{B}_-(\lambda|\beta)T_\beta^2}{(\lambda^2/q - q/\lambda^2)\mathrm{B}_-(\beta)}, \tag{5.108}$$

which is a degree N polynomial in Λ. As first remarked in the papers [61,104], from the polynomial characterization of the Q-function and the SoV characterization it follows the Bethe-like rewriting of the transfer matrix eigenstates stated in the following:

Corollary 5.1. *The left and right transfer matrix eigenstates associated to $\tau(\lambda) \in \Sigma_{\mathcal{T}}$ admit the following Bethe ansatz like representations:*

$$\langle \tau| = \langle \beta, \omega| \prod_{b=1}^{\mathsf{N}_Q} \hat{\mathcal{B}}_-(\lambda_b|\beta), \quad |\tau\rangle = \prod_{b=1}^{\mathsf{N}_Q} \hat{\mathcal{B}}_-(\lambda_b|\beta)|\beta, \bar{\omega}\rangle, \tag{5.109}$$

where the λ_b (fixed up the symmetry $\lambda_b \to -\lambda_b$, $\lambda_b \to 1/\lambda_b$) for $b \in \{1, ..., \mathsf{N}_Q\}$ are the zeros of $Q(\lambda)$ and we have imposed the condition (3.26) on the gauge parameters.

6 Conclusions

In this second article we have shown how to implement the SoV method to characterize the transfer matrix spectrum for integrable models associated to the Bazhanov-Stroganov quantum Lax operator and to the most general integrable boundary conditions. For that purpose it was necessary to perform a gauge transformation so as to recast the problem in a form similar to the one studied in our first article, i.e., such that one of the boundary K-matrices becomes triangular after the gauge transformation. Let us stress that the separate basis was designed again as the (pseudo)-eigenvector basis of some gauged operator of the reflection algebra having simple spectrum. What remains to be done is the construction of integrable local cyclic Hamiltonian having appropriate boundary conditions and commuting with the boundary transfer matrices considered here. This amounts to use trace identities involving the fundamental R-matrix acting in the tensor product of two cyclic representations [121,124,125] and to construct the associated K-matrices, hence also acting in these cyclic representations. The reflection equations will have to be written for arbitrary choices (and mixing) of the spin-1/2 and cyclic representations. Correspondingly, there will be compatibility conditions between the different K-matrices acting in these two different representations. We will address this question in a forthcoming article [138].

Appendices

A Gauge transformed Yang-Baxter algebra

A.1 Gauge transformed Yang-Baxter generators

For arbitrary complex parameters α and β let us introduce the following two matrices:

$$G(\lambda|\alpha,\beta) \equiv \begin{pmatrix} 1/(\alpha\beta\lambda) & \beta/(\alpha\lambda) \\ 1 & 1 \end{pmatrix}, \quad \bar{G}(\lambda|\gamma) \equiv \begin{pmatrix} 1/(\lambda\gamma) & 0 \\ 1 & 1 \end{pmatrix}, \tag{A.1}$$

and their inverses:

$$G^{-1}(\lambda|\alpha,\beta) = \frac{\alpha\lambda}{\beta - 1/\beta}\begin{pmatrix} -1 & \beta/(\alpha\lambda) \\ 1 & -1/(\alpha\beta\lambda) \end{pmatrix}, \quad \bar{G}^{-1}(\lambda|\gamma) = \begin{pmatrix} \lambda\gamma & 0 \\ -\lambda\gamma & 1 \end{pmatrix}. \tag{A.2}$$

Now we can construct the gauge transformed bulk monodromy matrix:

$$M(\lambda|\alpha,\beta,\gamma) = G^{-1}(\lambda q^{1/2}|\alpha,\beta)\, M(\lambda)\bar{G}(\lambda q^{1/2}|\gamma) = \begin{pmatrix} A(\lambda|\alpha,\beta,\gamma) & B(\lambda|\alpha,\beta) \\ C(\lambda|\alpha,\beta,\gamma) & D(\lambda|\alpha,\beta) \end{pmatrix}, \tag{A.3}$$

and, in a similar way, we can define:

$$\hat{M}(\lambda|\alpha,\beta,\gamma) = \bar{G}^{-1}(q^{1/2}/\lambda|\gamma)\,\hat{M}(\lambda)G(q^{1/2}/\lambda|\alpha,\beta) = \begin{pmatrix} \bar{A}(\lambda|\alpha,\beta,\gamma) & \bar{B}(\lambda|\alpha,\beta,\gamma) \\ \bar{C}(\lambda|\alpha,\beta,\gamma) & \bar{D}(\lambda|\alpha,\beta,\gamma) \end{pmatrix}. \tag{A.4}$$

The definition here chosen of these gauge transformations differ w.r.t. that used previously in the literature on one hand for the particular choice of the right transformation in $M(\lambda|\alpha,\beta,\gamma)$ and, on the other hand, as the parameters on the left and the right transformation are a priori independent. It is simple to prove by direct computations that:

$$\begin{aligned} \bar{D}(\lambda|\alpha,\beta,\gamma) &= f(\alpha,\beta,\gamma)A(1/\lambda|\alpha,\beta,\gamma), \quad \bar{B}(\lambda|\alpha,\beta,\gamma) = -f(\alpha,\beta,\gamma)B(1/\lambda|\alpha,\beta), && \text{(A.5)} \\ \bar{C}(\lambda|\alpha,\beta,\gamma) &= -f(\alpha,\beta,\gamma)C(1/\lambda|\alpha,\beta,\gamma), \quad \bar{A}(\lambda|\alpha,\beta,\gamma) = f(\alpha,\beta,\gamma)D(1/\lambda|\alpha,\beta), && \text{(A.6)} \end{aligned}$$

where:

$$f(\alpha,\beta,\gamma) = (-1)^N \frac{\gamma(1-\beta^2)}{\alpha\beta}. \tag{A.7}$$

Moreover, the identity:

$$\det{}_q M(\lambda) = (-1)^N M(\lambda q^{1/2})\hat{M}(q^{1/2}/\lambda), \tag{A.8}$$

and the corollaries:

$$\begin{aligned} \det{}_q M(\lambda) &= (-1)^N M(\lambda q^{1/2}|\alpha,\beta,\gamma)\hat{M}(q^{1/2}/\lambda|\alpha q,\beta,\gamma q) && \text{(A.9)} \\ &= (-1)^N \hat{M}(q^{1/2}/\lambda|\alpha q,\beta,\gamma q)M(\lambda q^{1/2}|\alpha,\beta,\gamma), && \text{(A.10)} \end{aligned}$$

imply the following two equivalent expressions of the quantum determinant by the gauge transformed generators:

$$\det{}_q M(\lambda) = \frac{\gamma(1-q^2\beta^2)\left[A(\lambda q^{1/2}|\alpha,\beta,\gamma)D(\lambda/q^{1/2}|\alpha,\beta q) - B(\lambda q^{1/2}|\alpha,\beta)C(\lambda/q^{1/2}|\alpha,\beta q,\gamma q)\right]}{\alpha\beta} \tag{A.11}$$

$$= \frac{\gamma(q^2-\beta^2)\left[D(\lambda q^{1/2}|\alpha,\beta)A(\lambda/q^{1/2}|\alpha,\beta/q,\gamma q) - C(\lambda q^{1/2}|\alpha,\beta,\gamma)B(\lambda/q^{1/2}|\alpha,\beta/q)\right]}{\alpha\beta}, \tag{A.12}$$

plus other two equivalent rewriting. The gauge transformed Yang-Baxter generators are of special interest as they define a closed set of commutation relations:

$$\begin{aligned} A(\lambda|\alpha,\beta,\gamma)A(\mu|\alpha,\beta/q,\gamma/q) &= A(\mu|\alpha,\beta,\gamma)A(\lambda|\alpha,\beta/q,\gamma/q) && \text{(A.13)} \\ B(\lambda|\alpha,\beta)B(\mu|\alpha,\beta/q) &= B(\mu|\alpha,\beta)B(\lambda|\alpha,\beta/q) && \text{(A.14)} \\ C(\lambda|\alpha,\beta,\gamma)C(\mu|\alpha,\beta q,\gamma/q) &= C(\mu|\alpha,\beta,\gamma)C(\lambda|\alpha,\beta q,\gamma/q) && \text{(A.15)} \\ D(\lambda|\alpha,\beta)D(\mu|\alpha,\beta/q) &= D(\mu|\alpha,\beta)D(\lambda|\alpha,\beta/q) && \text{(A.16)} \\ A(\lambda|\alpha,\beta,\gamma)B(\mu|\alpha,\beta/q) &= \frac{q(\lambda/\mu-\mu/\lambda)}{\lambda q/\mu-\mu/q\lambda}B(\mu|\alpha,\beta)A(\lambda|\alpha,\beta/q,\gamma q) \\ &\quad + \frac{(q-1/q)\mu/\lambda}{\lambda q/\mu-\mu/q\lambda}A(\mu|\alpha,\beta,\gamma)B(\lambda|\alpha,\beta/q) && \text{(A.17)} \\ B(\lambda|\alpha,\beta)A(\mu|\alpha,\beta/q,\gamma q) &= \frac{\lambda/\mu-\mu/\lambda}{q(\lambda q/\mu-\mu/q\lambda)}A(\mu|\alpha,\beta,\gamma)B(\lambda|\alpha,\beta/q) \\ &\quad + \frac{(q-1/q)\lambda/\mu}{\lambda q/\mu-\mu/q\lambda}B(\mu|\alpha,\beta)A(\lambda|\alpha,\beta/q,\gamma q), && \text{(A.18)} \end{aligned}$$

We can prove these commutation relations by direct computations using the properties of the gauge transformations and their action on the Yang-Baxter equation.

A.2 Pseudo-reference state for the gauge transformed Yang-Baxter algebra

In the following, we want to study the conditions for which a nonzero state identically annihilated by the action of the operator family $A(\lambda|\alpha,\beta,\gamma)$ exists:

$$\langle\Omega,\alpha,\beta,\gamma|A(\lambda|\alpha,\beta,\gamma) = 0. \tag{A.19}$$

It is an easy consequence of the gauge transformed Yang-Baxter commutation relations that under the condition that this state exists and is unique then it is a pseudo-reference state for the gauge transformed Yang-Baxter algebra, i.e. it holds:

$$\langle\Omega,\alpha,\beta,\gamma|A(\lambda|\alpha,\beta,\gamma)=0,\quad \langle\Omega,\alpha,\beta,\gamma|B(\lambda|\alpha,\beta)=b(\lambda|\alpha,\beta)\langle\Omega,\alpha,\beta/q,\gamma q|, \tag{A.20}$$

$$\langle\Omega,\alpha,\beta/q,\gamma q|C(\lambda|\alpha,\beta q,\gamma q)=c(\lambda|\alpha,\beta q)\langle\Omega,\alpha,\beta,\gamma|,\quad \langle\Omega,\alpha,\beta,\gamma|D(\lambda|\alpha,\beta)\neq 0, \tag{A.21}$$

with:

$$b(\lambda q^{1/2}|\alpha,\beta)c(\lambda q^{-1/2}|\alpha,\beta q)=-\det{}_q M(\lambda). \tag{A.22}$$

Here, we show that we can construct such a pseudo-reference state if and only if we impose at least $N+1$ constraints on the bulk and gauge parameters.

Let us start our analysis looking to the local conditions to be imposed, in order to do so let us define the local gauge transformed bulk operators:

$$\begin{pmatrix} A_n(\lambda|\alpha,\beta,\gamma) & B_n(\lambda|\alpha,\beta) \\ C_n(\lambda|\alpha,\beta,\gamma) & D_n(\lambda|\alpha,\beta) \end{pmatrix}_0 = G_0^{-1}(\lambda q^{1/2}|\alpha,\beta)\, L_{0,n}(\lambda q^{-1/2})\bar{G}_0(\lambda q^{1/2}|\gamma) \tag{A.23}$$

and let us introduce the parameters:

$$z_n^{(\epsilon_n,k_n)} = -q^{k_n+1/2}\left[\left(\frac{c_n^p+d_n^p}{a_n^p+b_n^p}\right)^{1/p}\frac{a_n}{c_n}\right]^{(1+\epsilon_n)/2}\frac{\alpha_n}{a_n}, \tag{A.24}$$

$$w_n^{(\epsilon_n,k_n)} = -q^{1/2-k_n}\left[\left(\frac{a_n^p+b_n^p}{c_n^p+d_n^p}\right)^{1/p}\frac{d_n}{b_n}\right]^{(1+\epsilon_n)/2}\frac{\beta_n}{d_n}, \tag{A.25}$$

where $n\in\{1,...,N\}$, $\epsilon_n=\pm1$, $k_n\in\{0,...,p-1\}$. Let us denote with:

$$\langle h_n,n|v_n=q^{h_n}\langle h_n,n|,\quad v_n|h_n,n\rangle=|h_n,n\rangle q^{h_n}, \tag{A.26}$$

the left and right eigenbasis of the operators v_n.

Lemma A.1. *Let us assume that:*

$$a_n^p+b_n^p\neq 0,\qquad c_n^p+d_n^p\neq 0, \tag{A.27}$$

then the non-zero left state annihilated by the local operator $A_n(\lambda|\alpha,\beta,\gamma)$ exists if and only if we impose the following two constraints on the gauge parameters:

$$\beta/\alpha=w_n^{(\epsilon_n,k_n)},\quad \gamma=z_n^{(\epsilon_n,k_n)}, \tag{A.28}$$

for some fixed $\epsilon_n=\pm1$ and $k_n\in\{0,...,p-1\}$, moreover this state is uniquely defined by:

$$\langle\Omega_{n,\alpha,\beta,\gamma}|=\sum_{h_n=0}^{p-1}q^{h_n(k_n+1)}\left[\prod_{r_n=1}^{h_n}\frac{a_nq^{r_n-1/2}+b_nq^{1/2-r_n}}{c_nq^{r_n-1/2}+d_nq^{1/2-r_n}}\left(\frac{c_n^p+d_n^p}{a_n^p+b_n^p}\right)^{1/p}\right]^{(1+\epsilon_n)/2}\langle h_n,n|. \tag{A.29}$$

Similarly, if the condition (A.27) holds the non-zero right state annihilated by the local operator $A_n(\lambda|\alpha,\beta,\gamma)$ exists if and only if we impose the following two constraints on the gauge parameters:

$$\beta/\alpha=w_n^{(\epsilon_n,k_n)},\quad \gamma=z_n^{(\epsilon_n,k_n-2)}, \tag{A.30}$$

for some fixed $\epsilon_n=\pm1$ and $k_n\in\{0,...,p-1\}$, moreover this state is uniquely defined by:

$$|\Omega_{n,\alpha,\beta,\gamma}\rangle=\sum_{h_n=0}^{p-1}|h_n,n\rangle q^{-h_nk_n}\left[\prod_{r_n=1}^{h_n}\frac{c_nq^{r_n-1/2}+d_nq^{1/2-r_n}}{a_nq^{r_n-1/2}+b_nq^{1/2-r_n}}\left(\frac{a_n^p+b_n^p}{c_n^p+d_n^p}\right)^{1/p}\right]^{(1+\epsilon_n)/2}. \tag{A.31}$$

These are pseudo-eigenstates of the operator $B_n(\lambda|\alpha,\beta)$:

$$\langle\Omega_{n,\alpha,\beta,\gamma}|B_n(\lambda|\alpha,\beta) = b_n(\lambda|\alpha,\beta)\langle\Omega_{n,\alpha,\beta/q,\gamma q}|, \tag{A.32}$$

$$B_n(\lambda|\alpha,\beta)|\Omega_{n,\alpha,\beta,\gamma}\rangle = qb_n(\lambda q|\alpha,\beta)|\Omega_{n,\alpha,\beta/q,\gamma q}\rangle, \tag{A.33}$$

where:

$$b_n(\lambda|\alpha,\beta)=\frac{\beta^2\gamma_n}{(1-\beta^2)\mu_{n,\epsilon_n}}\left(\frac{\lambda}{q^{1/2}\mu_{n,\epsilon_n}}-\frac{q^{1/2}\mu_{n,\epsilon_n}}{\lambda}\right). \tag{A.34}$$

Proof. The lemma is proven by direct construction. Let us introduce a state:

$$\langle\Omega_{n,\alpha,\beta,\gamma}|=\sum_{h=0}^{p-1}c_h(n,\alpha,\beta,\gamma)\langle h,n|, \tag{A.35}$$

and look for the conditions to be imposed on $c_h(n,\alpha,\beta,\gamma)$ in order to satisfy the equation:

$$\langle\Omega_{n,\alpha,\beta,\gamma}|A_n(\lambda|\alpha,\beta,\gamma)=0,\quad\forall\lambda\in\mathbb{C}. \tag{A.36}$$

By the definition of $A_n(\lambda|\alpha,\beta,\gamma)$ it is easy to verify that we have:

$$\langle\Omega_{n,\alpha,\beta,\gamma}|A_n(\lambda|\alpha,\beta,\gamma)=\left(-\lambda\sum_{h=0}^{p-1}C_h^+\langle h,n|+\frac{1}{\lambda}\sum_{h=0}^{p-1}C_h^-\langle h,n|\right)\frac{\beta^2}{1-\beta^2}, \tag{A.37}$$

where:

$$C_h^+ = q^{-1/2}c_h(\alpha_nq^h+(\beta\gamma/\alpha)\,\delta_nq^{-h})+c_{h-1}\gamma q^{1/2}(a_nq^{h-1/2}+b_nq^{1/2-h}), \tag{A.38}$$

$$C_h^- = q^{1/2}c_h(\beta_nq^{-h}+(\beta\gamma/\alpha)\,\gamma_nq^h)+(\beta/\alpha)\,q^{-1/2}c_{h+1}(c_nq^{h+1/2}+d_nq^{-1/2-h}), \tag{A.39}$$

and we omit to write explicitly the dependence on n,α,β,γ in c_h when it is not misleading. So that we get the following system of equations:

$$C_h^+=0,\quad C_h^-=0\quad\forall h\in\{0,...,p-1\}. \tag{A.40}$$

As we have assumed that the bulk parameters are generic and satisfy (A.27), the equations $C_h^+=0$ fix the values of the ratios $E_h\equiv c_{h-1}/c_h$ and the equations $C_h^-=0$ fix the value of ratios $F_h\equiv c_{h+1}/c_h$ for any $h\in\{0,...,p-1\}$ and one has to impose the compatibility of these values:

$$E_h=1/F_{h-1}\ \forall h\in\{0,...,p-1\}, \tag{A.41}$$

together with the cyclicity condition:

$$\prod_{h=0}^{p-1}E_h=1. \tag{A.42}$$

Then it is easy to show that the only solution of this system of equation is obtained fixing the two gauge parameters by $(A.28)$ which correspondingly fixes the form of the state $(A.29)$.

Let us compute now the action of the operator $B_n(\lambda|\alpha,\beta)$ on this state; by definition it holds:

$$B_n(\lambda|\alpha,\beta)=\frac{\beta^2 D_n(\lambda)-\alpha\beta\lambda q^{1/2}B_n}{\beta^2-1} \tag{A.43}$$

so that:

$$\begin{aligned}&\langle\Omega_{n,\alpha,\beta,\gamma}|B_n(\lambda|\alpha,\beta)|h_n,n\rangle=\\ &\quad\frac{\beta^2}{1-\beta^2}\left\{\lambda\left[c_{h_n}\frac{q^{-1/2}\delta_n}{q^{h_n}}+\frac{\alpha}{\beta}c_{h_n-1}(a_n q^{h_n-1/2}+b_n q^{1/2-h_n})\right]-c_{h_n}\frac{q^{1/2}\gamma_n q^{h_n}}{\lambda}\right\}\\ &=\frac{\beta^2 c_{h_n}q^{h_n}}{\beta^2-1}\left\{\frac{\lambda}{q^{1/2}}\frac{\alpha}{\beta\gamma}\alpha_n+\frac{q^{1/2}\gamma_n}{\lambda}\right\},\end{aligned} \tag{A.44}$$

where to get the third line we used the identity $C^+_{h_n}=0$. Now remarking that:

$$c_{h_n}(n,\alpha,\beta,\gamma)q^{h_n}=c_{h_n}(n,\alpha,\beta/q,\gamma q), \tag{A.45}$$

as the effect of q^{h_n} is to bring k_n to k_n+1 in the state $\langle\Omega_{n,\alpha,\beta,\gamma}|$, this, for the gauge choice $(A.28)$, being equivalent to the above redefinitions of the gauge parameters. So that we get:

$$\langle\Omega_{n,\alpha,\beta,\gamma}|B_n(\lambda|\alpha,\beta)=b_n(\lambda|\alpha,\beta)\langle\Omega_{n,\alpha,\beta/q,\gamma q}| \tag{A.46}$$

and so:

$$b_n(\lambda|\alpha,\beta)=\frac{\beta^2}{\beta^2-1}\left\{\frac{\lambda}{q^{1/2}}\frac{\alpha}{\beta\gamma}\alpha_n+\frac{q^{1/2}\gamma_n}{\lambda}\right\} \tag{A.47}$$

$$=\frac{\beta^2}{\beta^2-1}\begin{cases}\frac{q^{1/2}\gamma_n}{\lambda}+\frac{\lambda}{q^{1/2}}q^{-1}\frac{a_n d_n}{\beta_n} & \text{for } \epsilon_n=-1\\ \frac{q^{1/2}\gamma_n}{\lambda}+\frac{\lambda}{q^{1/2}}q^{-1}\frac{c_n b_n}{\beta_n} & \text{for } \epsilon_n=1\end{cases}. \tag{A.48}$$

Similarly, one can prove our statements for the right state and the action on it of $B_n(\lambda|\alpha,\beta)$. □

Let us remark that if the condition $(A.27)$ are not satisfied we can still derive the left and right local reference states imposing some case dependent condition on the gauge parameters; here for simplicity we have chosen to omit the description of these cases.

Proposition A.1. *Let us assume that for any $n\in\{1,...,\mathsf{N}\}$ the conditions $(A.27)$ is satisfied then the non-zero left state annihilated by the operator family $A(\lambda|\alpha,\beta,\gamma)$ exists if and only if we impose the following $\mathsf{N}+1$ constraints on the bulk and gauge parameters:*

$$\gamma=z_1^{(\epsilon_1,k_1)},\quad \beta/\alpha=w_N^{(\epsilon_N,k_N)},\quad w_n^{(\epsilon_n,k_n)}=1/z_{n+1}^{(\epsilon_{n+1},k_{n+1})}\quad\forall n\in\{1,...,\mathsf{N}-1\}, \tag{A.49}$$

for fixed N-tuples of $\epsilon_n=\pm1$ and $k_n\in\{0,...,p-1\}$, moreover it is uniquely defined by:

$$\langle\Omega,\alpha,\beta,\gamma|=\sum_{h_1,...,h_N=0}^{p-1}\prod_{n=1}^{N}q^{h_n(k_n+1)}\left[\prod_{r_n=1}^{h_n}\frac{a_n q^{r_n-1/2}+b_n q^{1/2-r_n}}{c_n q^{r_n-1/2}+d_n q^{1/2-r_n}}\left(\frac{a_n^p+d_n^p}{a_n^p+b_n^p}\right)^{1/p}\right]^{(1+\epsilon_n)/2}\bigotimes_{n=1}^{N}\langle h_n,n|. \tag{A.50}$$

Under the same condition the non-zero right state annihilated by the operator family $A(\lambda|\alpha,\beta,\gamma)$ exists if and only if we impose the following $\mathsf{N}+1$ constraints on the bulk and gauge parameters:

$$\gamma=z_1^{(\epsilon_1,k_1-2)},\quad \beta/\alpha=w_N^{(\epsilon_N,k_N)},\quad w_n^{(\epsilon_n,k_n)}=1/z_{n+1}^{(\epsilon_{n+1},k_{n+1}-2)}\quad\forall n\in\{1,...,N-1\}, \tag{A.51}$$

for fixed N-tuples of $\epsilon_n=\pm1$ and $k_n\in\{0,...,p-1\}$, moreover it is uniquely defined by:

$$|\Omega,\alpha,\beta,\gamma\rangle=\sum_{h_1,...,h_N=0}^{p-1}\prod_{n=1}^{\mathsf{N}}q^{-h_n k_n}\left[\prod_{r_n=1}^{h_n}\frac{c_n q^{r_n-1/2}+d_n q^{1/2-r_n}}{a_n q^{r_n-1/2}+b_n q^{1/2-r_n}}\left(\frac{a_n^p+b_n^p}{c_n^p+d_n^p}\right)^{1/p}\right]^{(1+\epsilon_n)/2}\bigotimes_{n=1}^{\mathsf{N}}|h_n,n\rangle. \tag{A.52}$$

These are pseudo-eigenstates of $B(\lambda|\alpha,\beta)$:

$$\langle\Omega,\alpha,\beta,\gamma|B(\lambda|\alpha,\beta)=b(\lambda|\alpha,\beta)\langle\Omega,\alpha,\beta/q,\gamma q|, \tag{A.53}$$

$$B(\lambda|\alpha,\beta)|\Omega,\alpha,\beta,\gamma\rangle=|\Omega,\alpha,\beta q,\gamma/q\rangle q^N b(\lambda q|\alpha,\beta), \tag{A.54}$$

with:

$$b(\lambda|\alpha,\beta)=\prod_{n=1}^{\mathsf{N}}b_n(\lambda|\alpha,\beta). \tag{A.55}$$

Proof. The operator family $A(\lambda|\alpha,\beta,\gamma)$ is a degree N Laurent polynomial of the form:

$$A(\lambda|\alpha,\beta,\gamma)=\sum_{n=0}^{\mathsf{N}}\lambda^{2n-N}A_n(\alpha,\beta,\gamma) \tag{A.56}$$

where the $A_n(\alpha,\beta,\gamma)$ are operators, for example we write explicitly:

$$A_0(\alpha,\beta,\gamma)=\frac{-\alpha\prod_{n=1}^{\mathsf{N}}\beta_n v_n^{-1}+\beta\gamma\prod_{n=1}^{\mathsf{N}}\gamma_n v_n+q^{-1/2}\beta\sum_{a=1}^{\mathsf{N}}\left(\prod_{n=a+1}^{\mathsf{N}}\beta_n v_n^{-1}\right)C_a\prod_{n=1}^{a-1}\gamma_n v_n}{(\beta-1/\beta)\gamma}, \tag{A.57}$$

$$A_N(\alpha,\beta,\gamma)=\frac{-\alpha\prod_{n=1}^{\mathsf{N}}\alpha_n v_n+\beta\gamma\prod_{n=1}^{\mathsf{N}}\delta_n v_n^{-1}-q^{1/2}\alpha\gamma\sum_{a=1}^{\mathsf{N}}\left(\prod_{n=a+1}^{\mathsf{N}}\alpha_n v_n\right)B_a\prod_{n=1}^{a-1}\delta_n v_n^{-1}}{(\beta-1/\beta)\gamma}. \tag{A.58}$$

For general values of the parameters these are invertible operators so that we have to impose at least $\mathsf{N}+1$ constraints to have that their common kernel is at least one dimensional. We can find the set of constraints by using induction and decomposing $A(\lambda|\alpha,\beta,\gamma)$ in terms of gauged operators on two subchains one of $\mathsf{N}-1$ sites and one of 1 site. The most general decomposition reads:

$$A_{\mathsf{N},...,1}(\lambda|\alpha,\beta,\gamma)=A_{\mathsf{N},...,2}(\lambda|\alpha,\beta,x_1,y_1)A_1(\lambda|x_1,y_1,\gamma)+B_{\mathsf{N},...,2}(\lambda|\alpha,\beta,x_1,y_1)C_1(\lambda|x_1,y_1,\gamma) \tag{A.59}$$

where we have defined:

$$M(\lambda|\alpha,\beta,x,y)=G^{-1}(\lambda q^{1/2}|\alpha,\beta)\,M(\lambda)G(\lambda q^{1/2}|x,y)=\begin{pmatrix}A(\lambda|\alpha,\beta,x,y) & B(\lambda|\alpha,\beta,x,y)\\ C(\lambda|\alpha,\beta,x,y) & D(\lambda|\alpha,\beta,x,y)\end{pmatrix}, \tag{A.60}$$

and we have explicitly pointed out in the subscripts the quantum sites to which the operator are referred. The following identities holds:

$$A(\lambda|\alpha,\beta,x,y) = -A(\lambda|\alpha,\beta,xy), \tag{A.61}$$
$$B(\lambda|\alpha,\beta,x,y) = A(\lambda|\alpha,\beta,x/y), \tag{A.62}$$
$$C(\lambda|x,y,\gamma) = \frac{x^2y^2-1}{1-y^2}A(\lambda|1,1/xy,\gamma), \tag{A.63}$$
$$A(\lambda|x,y,\gamma) = \frac{x^2-y^2}{1-y^2}A(\lambda|1,y/x,\gamma), \tag{A.64}$$

from which it follows:

$$\begin{aligned} A_{\mathsf{N},\dots,1}(\lambda|\alpha,\beta,\gamma) =& \frac{(y_1^2-x_1^2)\,A_{\mathsf{N},\dots,2}(\lambda|\alpha,\beta,x_1y_1)A_1(\lambda|1,y_1/x_1,\gamma)}{1-y_1^2} \\ &+\frac{(x_1^2y_1^2-1)A_{\mathsf{N},\dots,2}(\lambda|\alpha,\beta,x_1/y_1)A_1(\lambda|1,1/x_1y_1,\gamma)}{1-y_1^2}. \end{aligned} \tag{A.65}$$

Then $A_{\mathsf{N},\dots,1}(\lambda|\alpha,\beta,\gamma)$ admits a non-zero state annihilated by its action once we impose that it is true for $A_{\mathsf{N},\dots,2}(\lambda|\alpha,\beta,x_1y_1)$ and $A_1(\lambda|1,1/x_1y_1,\gamma)$ or for $A_1(\lambda|1,y_1/x_1,\gamma)$ and $A_{\mathsf{N},\dots,2}(\lambda|\alpha,\beta,x_1/y_1)$, and this state is given by the tensor product of the ones on the two subchains. As the parameters x_1 and y_1 are arbitrary in fact these two conditions are equivalents and so we can chose just one of them. So let us say we ask the second one and we repeat the same argument for $A_{\mathsf{N},\dots,2}(\lambda|\alpha,\beta,x_1/y_1)$, i.e. $A_{\mathsf{N},\dots,2}(\lambda|\alpha,\beta,x_1/y_1)$ admits such a state if $A_2(\lambda|1,y_2/x_2,x_1/y_1)$ and $A_{\mathsf{N},\dots,3}(\lambda|\alpha,\beta,x_2/y_2)$ do. So on by induction we get that the existence condition is equivalent to the existence conditions for the following N local operators:

$$A_n(\lambda|1,y_n/x_n,x_{n-1}/y_{n-1}) \text{ for any } n\in\{1,\dots,N\}, \tag{A.66}$$

where we have denoted

$$y_N/x_N=\beta/\alpha, \quad x_0/y_0=\gamma, \tag{A.67}$$

while the y_n/x_n for any $n\in\{1,\dots,\mathsf{N}-1\}$ are free parameters to be used to satisfy the existence condition for the local operators $A_n(\lambda|1,y_n/x_n,x_{n-1}/y_{n-1})$. From the previous lemma for $A_n(\lambda|1,s_n,r_n)$, the existence condition is equivalent to:

$$s_n=w_n^{(\epsilon_n)} \quad\text{and}\quad r_n=z_n^{(\epsilon_n)} \tag{A.68}$$

for any $\epsilon_n=\pm1$ and the right state annihilated by $A_n(\lambda|1,s_n,r_n)$ reads:

$$\langle\Omega_n,s_n,r_n|=\sum_{h_n=0}^{p-1}q^{h_n(k_n+1)}\left[\prod_{k_n=1}^{h_n}\frac{a_nq^{k_n-1/2}+b_nq^{1/2-k_n}}{c_nq^{k_n-1/2}+d_nq^{1/2-k_n}}\left(\frac{c_n^p+d_n^p}{a_n^p+b_n^p}\right)^{1/p}\right]^{(1+\epsilon_n)/2}\langle h_n,n|. \tag{A.69}$$

From this it is clear that the existence conditions of such a state for $A_{\mathsf{N},\dots,1}(\lambda|\alpha,\beta,\gamma)$ coincides with the simultaneous existence for the N local operators $(A.66)$ and that the state is just the tensor product of the states $(A.69)$ so that our proposition is proven. Similarly, we can prove the statement for the right state and using the previous lemma we can prove our statement on the action of the operator $B(\lambda|\alpha,\beta)$ on these states. □

B Gauge transformed Reflection algebra

B.1 Gauge transformed boundary operators

The gauged two-row monodromy matrix can be defined as it follows:

$$\mathcal{U}_-(\lambda|\alpha,\beta)\equiv\frac{q^{1/2}}{\lambda}G^{-1}(\lambda q^{1/2}|\alpha,\beta)\mathcal{U}_-(\lambda)\,G(q^{1/2}/\lambda|\alpha,\beta)=\begin{pmatrix}\mathcal{A}_-(\lambda|\alpha,\beta q^2) & \mathcal{B}_-(\lambda|\alpha,\beta)\\ \mathcal{C}_-(\lambda|\alpha,\beta q^2) & \mathcal{D}_-(\lambda|\alpha,\beta)\end{pmatrix}. \tag{B.1}$$

Note that one can expand this last gauged monodromy matrix in terms of the gauged bulk ones. Moreover, $\mathcal{U}_-(\lambda|\alpha,\beta)$ does not depend on the internal gauge parameter γ, so we are free to chose it at will. The following decompositions hold:

$$\begin{pmatrix}\mathcal{A}_-(\lambda|\alpha,\beta q^2)\\ \mathcal{C}_-(\lambda|\alpha,\beta q^2)\end{pmatrix}=M(\lambda|\alpha,\beta,\gamma)\bar{K}_-(\lambda|\gamma)\begin{pmatrix}\bar{A}(\lambda|\alpha,\beta q,\gamma q)\\ \bar{C}(\lambda|\alpha,\beta q,\gamma q)\end{pmatrix} \tag{B.2}$$
$$\begin{pmatrix}\mathcal{B}_-(\lambda|\alpha,\beta)\\ \mathcal{D}_-(\lambda|\alpha,\beta)\end{pmatrix}=M(\lambda|\alpha,\beta,\gamma)\bar{K}_-(\lambda|\gamma)\begin{pmatrix}\bar{B}(\lambda|\alpha,\beta/q,\gamma q)\\ \bar{D}(\lambda|\alpha,\beta/q,\gamma q)\end{pmatrix}, \tag{B.3}$$

where

$$\bar{K}_-(\lambda|\gamma)=\frac{q^{1/2}}{\lambda}\bar{G}^{-1}(\lambda q^{1/2}|\gamma)\;K_-(\lambda)\,\bar{G}(q^{1/2}/\lambda|\gamma q). \tag{B.4}$$

Explicitly, for $\mathcal{B}_-(\lambda|\alpha,\beta)$, it holds:

$$\begin{aligned}\frac{\mathcal{B}_-(\lambda|\alpha,\beta)}{f(\alpha,\beta/q,\gamma q)}=&\bar{K}_-(\lambda|\gamma)_{12}A(\lambda|\alpha,\beta,\gamma)A(1/\lambda|\alpha,\beta/q,\gamma q)-\bar{K}_-(\lambda|\gamma)_{11}A(\lambda|\alpha,\beta,\gamma)B(1/\lambda|\alpha,\beta/q)\\ &+\bar{K}_-(\lambda|\gamma)_{21}B(\lambda|\alpha,\beta)B(1/\lambda|\alpha,\beta/q)-\bar{K}_-(\lambda|\gamma)_{22}B(\lambda|\alpha,\beta)A(1/\lambda|\alpha,\beta/q,\gamma q).\end{aligned} \tag{B.5}$$

where:

$$\bar{K}_-(\lambda|\gamma)_{11}=\frac{\lambda^2(\zeta_-/q+q^2\gamma\kappa_-e^{\tau_-})-q^2\gamma\kappa_-e^{\tau_-}/\left(\zeta_-\lambda^2\right)-1/\zeta_-}{\zeta_--1/\zeta_-}, \tag{B.6}$$
$$\bar{K}_-(\lambda|\gamma)_{22}=\frac{(\zeta_-+q\gamma\kappa_-e^{\tau_-})/\lambda^2-\gamma\lambda^2-1/\zeta_-}{\zeta_--1/\zeta_-}, \tag{B.7}$$
$$\bar{K}_-(\lambda|\gamma)_{12}=\frac{q\gamma\kappa_-e^{\tau_-}(\lambda^2/q-q/\lambda^2)}{\zeta_--1/\zeta_-} \tag{B.8}$$
$$\bar{K}_-(\lambda|\gamma)_{21}=\frac{(q/\lambda^2-\lambda^2/q)\left[q\gamma e^{\tau_-}\kappa_-+\zeta_--\kappa_-/(q\gamma e^{\tau_-})\right]}{\zeta_--1/\zeta_-}. \tag{B.9}$$

Then it holds:

$$\bar{K}_-(\lambda|\gamma)_{21}\equiv0,\ \forall\lambda\in\mathbb{C}\quad\text{for }\gamma=\gamma_\epsilon \tag{B.10}$$

for $\epsilon=\pm$ and

$$\gamma_\epsilon=\frac{-\zeta_-+\epsilon\sqrt{\zeta_-^2+4\kappa_-^2}}{2qe^{\tau_-}\kappa_-}. \tag{B.11}$$

These gauge transformed boundary operators satisfies the following gauge deformed reflection algebra.

Proposition B.1. *The gauge transformed boundary operators satisfy the following commutation relations:*

$$\mathcal{B}_-(\lambda_2|\beta)\mathcal{B}_-(\lambda_1|\beta/q^2) = \mathcal{B}_-(\lambda_1|\beta)\mathcal{B}_-(\lambda_2|\beta/q^2), \tag{B.12}$$

$$\begin{aligned}\mathcal{A}_-(\lambda_2|\beta q^2)\mathcal{B}_-(\lambda_1|\beta) &= \frac{(\lambda_1 q/\lambda_2 - \lambda_2/q\lambda_1)(\lambda_1\lambda_2/q - q/\lambda_1\lambda_2)}{(\lambda_1/\lambda_2 - \lambda_2/\lambda_1)(\lambda_1\lambda_2 - 1/\lambda_1\lambda_2)}\mathcal{B}_-(\lambda_1|\beta)\mathcal{A}_-(\lambda_2|\beta)\\ &+ \frac{(\lambda_1\lambda_2/q - q/\lambda_1\lambda_2)(\lambda_1\beta/q\lambda_2 - \lambda_2 q/\beta\lambda_1)(q-1/q)}{(\lambda_1/\lambda_2 - \lambda_2/\lambda_1)(\lambda_1\lambda_2 - 1/\lambda_1\lambda_2)(\beta/q - q/\beta)}\mathcal{B}_-(\lambda_2|\beta)\mathcal{A}_-(\lambda_1|\beta)\\ &+ \frac{(\lambda_1\lambda_2/\beta - \beta/\lambda_1\lambda_2)(q-1/q)}{(\lambda_1\lambda_2 - 1/\lambda_1\lambda_2)(\beta/q - q/\beta)}\mathcal{B}_-(\lambda_2|\beta)\mathcal{D}_-(\lambda_1|\beta),\end{aligned} \tag{B.13}$$

$$\begin{aligned}\mathcal{B}_-(\lambda_1|\beta)\mathcal{D}_-(\lambda_2|\beta) &= \frac{(\lambda_1 q/\lambda_2 - \lambda_2/q\lambda_1)(\lambda_1\lambda_2/q - q/\lambda_1\lambda_2)}{(\lambda_1/\lambda_2 - \lambda_2/\lambda_1)(\lambda_1\lambda_2 - 1/\lambda_1\lambda_2)}\mathcal{D}_-(\lambda_2|\beta q^2)\mathcal{B}_-(\lambda_1|\beta)\\ &- \frac{(\lambda_1\lambda_2/q - q/\lambda_1\lambda_2)(\lambda_2\beta q/\lambda_1 - \lambda_1/\lambda_2\beta q)(q-1/q)}{(\lambda_1/\lambda_2 - \lambda_2/\lambda_1)(\lambda_1\lambda_2 - 1/\lambda_1\lambda_2)(\beta q - 1/\beta q)}\mathcal{D}_-(\lambda_1|\beta q^2)\mathcal{B}_-(\lambda_2|\beta)\\ &- \frac{(\lambda_1\lambda_2\beta - 1/\lambda_1\lambda_2\beta)(q-1/q)}{(\lambda_1\lambda_2 - 1/\lambda_1\lambda_2)(\beta q - 1/q\beta)}\mathcal{A}_-(\lambda_1|\beta q^2)\mathcal{B}_-(\lambda_2|\beta),\end{aligned} \tag{B.14}$$

and

$$\begin{aligned}&\mathcal{A}_-(\lambda_1|\beta q^2)\mathcal{A}_-(\lambda_2|\beta q^2) - \frac{(\lambda_1\lambda_2/\beta - 1/\lambda_1\lambda_2)(q-1/q)}{(\lambda_1\lambda_2 - 1/\lambda_1\lambda_2)(\beta/q - q/\beta)}\mathcal{B}_-(\lambda_1|\beta)\mathcal{C}_-(\lambda_2|\beta q^2) =\\ &\qquad \mathcal{A}_-(\lambda_2|\beta q^2)\mathcal{A}_-(\lambda_1|\beta q^2) - \frac{(\lambda_1\lambda_2/\beta - 1/\lambda_1\lambda_2)(q-1/q)}{(\lambda_1\lambda_2 - 1/\lambda_1\lambda_2)(\beta/q - q/\beta)}\mathcal{B}_-(\lambda_2|\beta)\mathcal{C}_-(\lambda_1|\beta q^2).\end{aligned} \tag{B.15}$$

Similar commutation relations involving $\mathcal{C}_-(\lambda|\beta)$ can be written by using the following β-symmetries:

$$\mathcal{B}_-(\lambda|\beta) = \mathcal{C}_-(\lambda|q^2/\beta), \quad \mathcal{A}_-(\lambda|\beta) = \mathcal{D}_-(\lambda|q^2/\beta). \tag{B.16}$$

Moreover, these gauge transformed operators satisfy the following parity properties:

$$\mathcal{A}_-(\lambda|\beta) = -\frac{(q-1/q)(\lambda^2 q/\beta - \beta/\lambda^2 q)}{(\beta/q^2 - q^2/\beta)(\lambda^2 - 1/\lambda^2)}\mathcal{D}_-(\lambda|\beta) + \frac{(\beta/q - q/\beta)(\lambda^2/q - q/\lambda^2)}{(\beta/q^2 - q^2/\beta)(\lambda^2 - 1/\lambda^2)}\mathcal{D}_-(1/\lambda|\beta), \tag{B.17}$$

$$\mathcal{D}_-(\lambda|\beta) = \frac{(q-1/q)(\lambda^2\beta/q - q/\lambda^2\beta)}{(\beta/q^2 - q^2/\beta)(\lambda^2 - 1/\lambda^2)}\mathcal{A}_-(\lambda|\beta) + \frac{(\beta/q - q/\beta)(\lambda^2/q - q/\lambda^2)}{(\beta - 1/\beta)(\lambda^2 - 1/\lambda^2)}\mathcal{A}_-(1/\lambda|\beta), \tag{B.18}$$

$$\mathcal{B}_-(1/\lambda|\beta) = -\frac{(\lambda^2 q - 1/q\lambda^2)}{(\lambda^2/q - q/\lambda^2)}\mathcal{B}_-(\lambda|\beta)\ , \qquad \mathcal{C}_-(1/\lambda|\beta) = -\frac{(\lambda^2 q - 1/q\lambda^2)}{(\lambda^2/q - q/\lambda^2)}\mathcal{C}_-(\lambda|\beta). \tag{B.19}$$

Proof. Both the commutation relations and the parity properties here presented coincide with those derived in [68] for the case of the XXZ spin 1/2 quantum chain with general integrable boundaries. This is the case as they are clearly representation independent. Here we are just writing them in a Laurent polynomial form instead of a trigonometric form. □

B.2 Representation of the gauge transformed Reflection algebra

In the bulk of the paper we have anticipated that for almost all the values of the boundary, bulk and gauge parameters the operator family $\mathcal{B}_-(\lambda|\beta)$ is pseudo-diagonalizable. We will show this

statement in the last subsection of this appendix, but for now we want to write explicitly the representation of the other gauge transformed boundary operator families in the left and right basis formed out of the pseudo-eigenstates of $\mathcal{B}_-(\lambda|\beta)$.

Theorem B.1. *The action of the reflection algebra generator $\mathcal{A}_-(\lambda|\beta q^2)$ on the generic state $\langle\beta,\mathbf{h}|$ is given by the following expression:*

$$\begin{aligned}\langle\beta,\mathbf{h}|\mathcal{A}_-(\lambda|\beta q^2) &= \sum_{a=1}^{2\mathsf{N}} \frac{\zeta_a^{(h_a)}\left(\lambda^2/q - q/\lambda^2\right)(\lambda\zeta_a^{(h_a)} - 1/(\lambda\zeta_a^{(h_a)}))\mathsf{A}_-(\zeta_a^{(h_a)})}{\lambda\left((\zeta_a^{(h_a)})^2/q - q/(\zeta_a^{(h_a)})^2\right)\left((\zeta_a^{(h_a)})^2 - 1/(\zeta_a^{(h_a)})^2\right)} \prod_{\substack{b=1\\ b\neq a \ mod\mathsf{N}}}^{\mathsf{N}} \frac{\Lambda - X_b^{(h_b)}}{X_a^{(h_a)} - X_b^{(h_b)}}\\ &\times \langle\beta,\mathbf{h}|T_a^{-\varphi_a} + (-1)^{\mathsf{N}}\frac{q^{1/2}}{2\lambda}det_q M(1)\left(\frac{\lambda}{q^{1/2}} + \frac{q^{1/2}}{\lambda}\right)\prod_{b=1}^{\mathsf{N}}\frac{\Lambda - X_b^{(h_b)}}{X - X_b^{(h_b)}}\langle\beta,\mathbf{h}|\\ &+ (-1)^{\mathsf{N}+1}\frac{iq^{1/2}}{2\lambda}\frac{\zeta_- + 1/\zeta_-}{\zeta_- - 1/\zeta_-}det_q M(i)\left(\frac{\lambda}{q^{1/2}} - \frac{q^{1/2}}{\lambda}\right)\prod_{b=1}^{\mathsf{N}}\frac{\Lambda - X_b^{(h_b)}}{X + X_b^{(h_b)}}\langle\beta,\mathbf{h}|\\ &+ q\left(\lambda^2/q - q/\lambda^2\right)\prod_{b=1}^{\mathsf{N}}(\Lambda - X_b^{(h_b)})\langle\beta,\mathbf{h}|\mathcal{A}_-^{\infty}(\beta q^2),\end{aligned} \tag{B.20}$$

where

$$\begin{aligned}\langle\beta,\mathbf{h}|\mathcal{A}_-^{\infty}(\beta q^2) &= \frac{1}{q(1-\beta^2)}\Bigg[\sum_{a=1}^{2\mathsf{N}}\frac{\prod_{b=1, b\neq a \ mod\mathsf{N}}^{\mathsf{N}}(X_a^{(h_a)} - X_b^{(h_b)})^{-1}\mathsf{A}_-(\zeta_a^{(h_a)})\langle\beta,\mathbf{h}|T_a^{-\varphi_a}}{\left((\zeta_a^{(h_a)})^2/q - q/(\zeta_a^{(h_a)})^2\right)\left((\zeta_a^{(h_a)})^2 - 1/(\zeta_a^{(h_a)})^2\right)}\\ &+(-1)^{\mathsf{N}}\left(\frac{1}{2}det_q M(1)\prod_{b=1}^{\mathsf{N}}\frac{1}{X - X_b^{(h_b)}} + \frac{i}{2}\frac{\zeta_- + 1/\zeta_-}{\zeta_- - 1/\zeta_-}det_q M(i)\prod_{b=1}^{\mathsf{N}}\frac{1}{X + X_b^{(h_b)}}\right)\langle\beta,\mathbf{h}|\Bigg]\\ &+ \frac{\kappa_-}{q(\zeta_- - 1/\zeta_-)}\left(\frac{\prod_{a=1}^{\mathsf{N}}\gamma_a\delta_a}{q\alpha e^{\tau_-}} - q\alpha e^{\tau_-}\prod_{a=1}^{\mathsf{N}}\alpha_a\beta_a\right)\langle\beta,\mathbf{h}|,\end{aligned} \tag{B.21}$$

and

$$\langle\beta, h_1, ..., h_a, ..., h_{\mathsf{N}}|T_a^{\pm} = \langle\beta, h_1, ..., h_a \pm 1, ..., h_{\mathsf{N}}|, \tag{B.22}$$

once the parameter α has been fixed by (3.26)*.*

Proof. The following interpolation formula:

$$\begin{aligned}\langle\beta,\mathbf{h}|\mathcal{A}_-(\lambda|\beta q^2) &= \sum_{a=1}^{2\mathsf{N}}\frac{\zeta_a^{(h_a)}\left(\lambda^2/q - q/\lambda^2\right)\mathsf{A}_-(\zeta_a^{(h_a)})}{\lambda\left((\zeta_a^{(h_a)})^2/q - q/(\zeta_a^{(h_a)})^2\right)}\prod_{\substack{b=1\\ b\neq a}}^{2\mathsf{N}}\frac{\lambda/\zeta_b^{(h_b)} - \zeta_b^{(h_b)}/\lambda}{\zeta_a^{(h_a)}/\zeta_b^{(h_b)} - \zeta_b^{(h_b)}/\zeta_a^{(h_a)}}\\ &\times\langle\beta,\mathbf{h}|T_a^{-\varphi_a} + (-1)^{\mathsf{N}}\frac{q^{1/2}}{2\lambda}\mathrm{det}_q M(1)\left(\frac{\lambda}{q^{1/2}} + \frac{q^{1/2}}{\lambda}\right)\prod_{b=1}^{\mathsf{N}}\frac{\lambda/\zeta_b^{(h_b)} - \zeta_b^{(h_b)}/\lambda}{q^{1/2}/\zeta_b^{(h_b)} - \zeta_b^{(h_b)}/q^{1/2}}\langle\beta,\mathbf{h}|\\ &+ (-1)^{\mathsf{N}+1}\frac{iq^{1/2}}{2\lambda}\frac{\zeta_- + 1/\zeta_-}{\zeta_- - 1/\zeta_-}\mathrm{det}_q M(i)\left(\frac{\lambda}{q^{1/2}} - \frac{q^{1/2}}{\lambda}\right)\prod_{b=1}^{\mathsf{N}}\frac{\lambda/\zeta_b^{(h_b)} - \zeta_b^{(h_b)}/\lambda}{q^{1/2}/\zeta_b^{(h_b)} - \zeta_b^{(h_b)}/q^{1/2}}\langle\beta,\mathbf{h}|\\ &+ q\left(\lambda^2/q - q/\lambda^2\right)\prod_{b=1}^{\mathsf{N}}(\Lambda - X_b^{(h_b)})\langle\beta,\mathbf{h}|\mathcal{A}_-^{\infty}(\beta q^2),\end{aligned} \tag{B.23}$$

where we have defined:

$$\mathcal{A}_-^{\infty,0}(\beta q^2) = \lim_{\lambda\to\infty,0} \lambda^{\mp 2(\mathsf{N}+1)}\mathcal{A}_-(\lambda|\beta q^2), \tag{B.24}$$

is a direct consequence of the functional dependence with respect to λ:

$$\mathcal{A}_-(\lambda|\beta) = \sum_{a=0}^{2\mathsf{N}+2} \lambda^{2(a-(\mathsf{N}+1))}\mathcal{A}_a(\beta) \tag{B.25}$$

and of the identities:

$$\mathcal{U}_-(q^{1/2}) = (-1)^{\mathsf{N}}\det_q M(1)\ I_0, \quad \mathcal{U}_-(iq^{1/2}) = -i\frac{\zeta_- + 1/\zeta_-}{\zeta_- - 1/\zeta_-}\det_q M(i)\ \sigma_0^z, \tag{B.26}$$

which are representation independent. Instead the asymptotic operators $\mathcal{A}_-^{\infty,0}(\beta q^2)$ depend on the representation and we can compute them observing that using the definition (3.1) of $\mathcal{A}_-(\lambda|\beta q^2)$ it holds:

$$\begin{aligned}
\mathcal{A}_-^{\infty}(\beta q^2) &= \left[-\mathcal{A}_-^{\infty}/\beta q^{1/2} - \alpha q\mathcal{B}_-^{\infty} + \mathcal{C}_-^{\infty}/\alpha q\right]/(\beta - 1/\beta) && \text{(B.27)}\\
\mathcal{A}_-^{0}(\beta q^2) &= \left[\beta q^{1/2}\mathcal{D}_-^{0} - \alpha q\mathcal{B}_-^{0} + \mathcal{C}_-^{0}/\alpha q\right]/(\beta - 1/\beta) && \text{(B.28)}
\end{aligned}$$

where the $\mathcal{A}_-^{\infty,0}$, $\mathcal{D}_-^{\infty,0}$, $\mathcal{B}_-^{\infty,0}$ and $\mathcal{C}_-^{\infty,0}$ are the asymptotic limits of the ungauged elements of $\mathcal{U}_-(\lambda)$. The identities:

$$\mathcal{A}_-^{\infty,0} = q^{\mp 1}\mathcal{D}_-^{0,\infty},\ \mathcal{B}_-^{0} = -q^2\mathcal{B}_-^{\infty},\ \mathcal{C}_-^{0} = -q^2\mathcal{C}_-^{\infty}, \tag{B.29}$$

following from (B.17)-(B.19), and

$$\mathcal{B}_-^{\infty} = \frac{\kappa_- e^{\tau_-}\prod_{a=1}^{\mathsf{N}}\alpha_a\beta_a}{q(\zeta_- - 1/\zeta_-)},\ \mathcal{C}_-^{\infty} = \frac{\kappa_- e^{-\tau_-}\prod_{a=1}^{\mathsf{N}}\gamma_a\delta_a}{q(\zeta_- - 1/\zeta_-)}, \tag{B.30}$$

imply the main identity:

$$\beta q\mathcal{A}_-^{\infty}(\beta q^2) + \mathcal{A}_-^{0}(\beta q^2)/(\beta q) = \frac{\kappa_-(\beta - 1/\beta)}{\zeta_- - 1/\zeta_-}\left[\frac{\prod_{a=1}^{\mathsf{N}}\gamma_a\delta_a}{q\alpha e^{\tau_-}} - q\alpha e^{\tau_-}\prod_{a=1}^{\mathsf{N}}\alpha_a\beta_a\right]. \tag{B.31}$$

This identity allows to compute these asymptotic operators once we use the interpolation formula to write $\mathcal{A}_-^{0}(\beta q^2)$ in terms of $\mathcal{A}_-^{\infty}(\beta q^2)$ as it follows:

$$\begin{aligned}
\langle\beta,\mathbf{h}|\mathcal{A}_-^{0}(\beta q^2) = &\sum_{a=1}^{2\mathsf{N}}\frac{q\mathsf{A}_-(\zeta_a^{(h_a)})}{\left((\zeta_a^{(h_a)})^2/q - q/(\zeta_a^{(h_a)})^2\right)\left((\zeta_a^{(h_a)})^2 - 1/(\zeta_a^{(h_a)})^2\right)}\prod_{\substack{b=1\\ b\neq a \bmod \mathsf{N}}}^{\mathsf{N}}\frac{1}{X_a^{(h_a)} - X_b^{(h_b)}}\\
&\times\langle\beta,\mathbf{h}|T_a^{-\varphi_a} + (-1)^{\mathsf{N}}\frac{q}{2}\det_q M(1)\prod_{b=1}^{\mathsf{N}}\frac{1}{X - X_b^{(h_b)}}\langle\beta,\mathbf{h}|\\
&+ (-1)^{\mathsf{N}}\frac{iq}{2}\frac{\zeta_- + 1/\zeta_-}{\zeta_- - 1/\zeta_-}\det_q M(i)\prod_{b=1}^{\mathsf{N}}\frac{1}{X + X_b^{(h_b)}}\langle\beta,\mathbf{h}|\\
&- q^2\langle\beta,\mathbf{h}|\mathcal{A}_-^{\infty}(\beta q^2).
\end{aligned}\tag{B.32}$$

□

Similarly, the following theorem characterizes the right SoV representation of the gauged cyclic reflection algebra:

Theorem B.2. *The action of the reflection algebra generators $\mathcal{D}_-(\lambda|\beta)$ on the generic state $|\beta,\mathbf{h}\rangle$, can be written as it follows:*

$$\begin{aligned}
\mathcal{D}_-(\lambda|\beta)|\beta,\mathbf{h}\rangle = &\sum_{a=1}^{2\mathsf{N}} T_a^{-\varphi_a}|\beta,\mathbf{h}\rangle\frac{\zeta_a^{(h_a)}\left(\lambda^2/q - q/\lambda^2\right)\left(\lambda\zeta_a^{(h_a)} - 1/(\lambda\zeta_a^{(h_a)})\right)\mathsf{D}_-(\zeta_a^{(h_a)})}{\lambda\left((\zeta_a^{(h_a)})^2/q - q/(\zeta_a^{(h_a)})^2\right)\left((\zeta_a^{(h_a)})^2 - 1/(\zeta_a^{(h_a)})^2\right)}\cdot\\
&\cdot\prod_{\substack{b=1\\ b\neq a \bmod \mathsf{N}}}^{\mathsf{N}}\frac{\Lambda - X_b^{(h_b)}}{X_a^{(h_a)} - X_b^{(h_b)}} + |\beta,\mathbf{h}\rangle(-1)^{\mathsf{N}}\frac{q^{1/2}}{2\lambda}det_q M(1)\left(\frac{\lambda}{q^{1/2}} + \frac{q^{1/2}}{\lambda}\right)\prod_{b=1}^{\mathsf{N}}\frac{\Lambda - X_b^{(h_b)}}{X - X_b^{(h_b)}}\\
&+ |\beta,\mathbf{h}\rangle(-1)^{\mathsf{N}}\frac{iq^{1/2}}{2\lambda}\frac{\zeta_- + 1/\zeta_-}{\zeta_- - 1/\zeta_-}det_q M(i)\left(\frac{\lambda}{q^{1/2}} - \frac{q^{1/2}}{\lambda}\right)\prod_{b=1}^{\mathsf{N}}\frac{\Lambda - X_b^{(h_b)}}{X + X_b^{(h_b)}}\\
&+ q\left(\lambda^2/q - q/\lambda^2\right)\prod_{b=1}^{\mathsf{N}}(\Lambda - X_b^{(h_b)})\mathcal{D}_-^{\infty}(\beta)|\beta,\mathbf{h}\rangle,
\end{aligned}\tag{B.33}$$

where:

$$\begin{aligned}
\mathcal{D}_-^{\infty}(\beta) = &\frac{\beta}{q(\beta - 1/\beta)}\left[\sum_{a=1}^{2\mathsf{N}}\frac{\mathsf{D}_-(\zeta_a^{(h_a)})T_a^{-\varphi_a}|\beta,\mathbf{h}\rangle}{\left((\zeta_a^{(h_a)})^2/q - q/(\zeta_a^{(h_a)})^2\right)\left((\zeta_a^{(h_a)})^2 - 1/(\zeta_a^{(h_a)})^2\right)}\prod_{\substack{b=1\\ b\neq a \bmod \mathsf{N}}}^{\mathsf{N}}\frac{1}{X_a^{(h_a)} - X_b^{(h_b)}}\right.\\
&\left.+(-1)^{\mathsf{N}}\left(\frac{1}{2}det_q M(1)\prod_{b=1}^{\mathsf{N}}\frac{1}{X - X_b^{(h_b)}} - \frac{i}{2}\frac{\zeta_- + 1/\zeta_-}{\zeta_- - 1/\zeta_-}det_q M(i)\prod_{b=1}^{\mathsf{N}}\frac{1}{X + X_b^{(h_b)}}\right)|\beta,\mathbf{h}\rangle\right]\\
&+\frac{\kappa_-[q\alpha e^{\tau_-}\prod_{a=1}^{\mathsf{N}}\alpha_a\beta_a - \frac{\prod_{a=1}^{\mathsf{N}}\gamma_a\delta_a}{q\alpha e^{\tau_-}}]}{q(\zeta_- - 1/\zeta_-)}|\beta,\mathbf{h}\rangle.
\end{aligned}\tag{B.34}$$

and

$$T_a^{\pm}|\beta,h_1,...,h_a,...,h_{\mathsf{N}}\rangle = |\beta,h_1,...,h_a \pm 1,...,h_{\mathsf{N}}\rangle. \tag{B.35}$$

Proof. The following interpolation formula is derived as in the previous theorem using the polynomiality of the operator family $\mathcal{D}_-(\lambda|\beta)$:

$$\mathcal{D}_-(\lambda|\beta) = \sum_{a=0}^{2\mathsf{N}+2}\lambda^{(2a-(2\mathsf{N}+2))}\mathcal{D}_a(\beta), \tag{B.36}$$

we have just to compute the asymptotic operator $\mathcal{D}_-^{\infty}(\beta)$. The following identities:

$$\begin{aligned}
\mathcal{D}_-^{\infty}(\beta) &= \left[\beta\mathcal{A}_-^{\infty}/q^{1/2} + \alpha q\mathcal{B}_-^{\infty} - \mathcal{C}_-^{\infty}/\alpha q\right]/(\beta - 1/\beta) && \text{(B.37)}\\
\mathcal{D}_-^{0}(\beta) &= -\left[q^{3/2}\mathcal{A}_-^{\infty}/\beta + q^2(\alpha q\mathcal{B}_-^{\infty} - \mathcal{C}_-^{\infty}/\alpha q)\right]/(\beta - 1/\beta) && \text{(B.38)}
\end{aligned}$$

trivially follows by the definition of the operator family $\mathcal{D}_-(\lambda|\beta)$, from which we get:

$$q\mathcal{D}_-^{\infty}(\beta)/\beta + \beta\mathcal{D}_-^{0}(\beta)/q = \frac{\kappa_-(\beta - 1/\beta)}{\zeta_- - 1/\zeta_-}\left[\frac{\prod_{a=1}^{\mathsf{N}}\gamma_a\delta_a}{q\alpha e^{\tau_-}} - q\alpha e^{\tau_-}\prod_{a=1}^{\mathsf{N}}\alpha_a\beta_a\right], \tag{B.39}$$

while from the interpolation formula we get:

$$\begin{aligned}\mathcal{D}_{-}^{0}(\beta) =& \sum_{a=1}^{2\mathsf{N}} \frac{q\mathsf{D}_{-}(\zeta_a^{(h_a)})T_a^{-\varphi_a}|\beta,\mathbf{h}\rangle}{\left((\zeta_a^{(h_a)})^2/q - q/(\zeta_a^{(h_a)})^2\right)\left((\zeta_a^{(h_a)})^2 - 1/(\zeta_a^{(h_a)})^2\right)} \prod_{\substack{b=1\\ b\neq a \text{ mod}\mathsf{N}}}^{\mathsf{N}} \frac{1}{X_a^{(h_a)} - X_b^{(h_b)}} \\ &+ (-1)^{\mathsf{N}}\left(\frac{q}{2}\det{}_q M(1)\prod_{b=1}^{\mathsf{N}}\frac{1}{X - X_b^{(h_b)}} - \frac{iq}{2}\frac{\zeta_- + 1/\zeta_-}{\zeta_- - 1/\zeta_-}\det{}_q M(i)\prod_{b=1}^{\mathsf{N}}\frac{1}{X + X_b^{(h_b)}}\right)|\beta,\mathbf{h}\rangle \\ &- q^2\mathcal{D}_{-}^{\infty}(\beta)|\beta,\mathbf{h}\rangle. \end{aligned} \tag{B.40}$$

from which the statement of the theorem follows easily . □

B.3 SoV spectral decomposition of the identity

The Theorem 3.1 states the pseudo-diagonalizability of $\mathcal{B}_{-}(\lambda|\beta)$ for almost all the values of the boundary-bulk-gauge parameters, so that for almost all the values of these parameters the left and right states $\langle\beta,\mathbf{h}|$ and $|\beta,\mathbf{k}\rangle$ are well defined nonzero left and right states describing a left and right basis in the space of the representation.

We can now defines the following $p^{\mathsf{N}}\times p^{\mathsf{N}}$ matrices $U^{(L,\beta)}$ and $U^{(R,\beta q^2)}$ defining the change of basis from the original left and right basis:

$$\underline{\langle\mathbf{h}|} \equiv \otimes_{n=1}^{\mathsf{N}}\langle h_n,n| \quad \text{and} \quad \underline{|\mathbf{h}\rangle} \equiv \otimes_{n=1}^{\mathsf{N}}|h_n,n\rangle, \tag{B.41}$$

composed by v_n-eigenstates, to the left and right pseudo-eigenbasis of $\mathcal{B}_{-}(\lambda|\beta)$:

$$\langle\beta,\mathbf{h}| = \underline{\langle\mathbf{h}|}U^{(L,\beta)} = \sum_{i=1}^{p^{\mathsf{N}}} U_{\varkappa(\mathbf{h}),i}^{(L,\beta)}\underline{\langle\varkappa^{-1}(i)|} \quad \text{and} \quad |\beta q^2,\mathbf{h}\rangle = U^{(R,\beta q^2)}\underline{|\mathbf{h}\rangle} = \sum_{i=1}^{p^{\mathsf{N}}} U_{i,\varkappa(\mathbf{h})}^{(R,\beta q^2)}\underline{|\varkappa^{-1}(i)\rangle}, \tag{B.42}$$

where $\varkappa$ is an isomorphism between the sets $\{0,...,p-1\}^{\mathsf{N}}$ and $\{1,...,p^{\mathsf{N}}\}$ defined by:

$$\varkappa : \mathbf{h}\in\{0,...,p-1\}^{\mathsf{N}} \to \varkappa(\mathbf{h}) \equiv 1 + \sum_{a=1}^{\mathsf{N}} p^{(a-1)}h_a \in \{1,...,p^{\mathsf{N}}\}. \tag{B.43}$$

It follows from the pseudo-diagonalizability of $\mathcal{B}_{-}(\lambda|\beta)$ that the $p^{\mathsf{N}}\times p^{\mathsf{N}}$ square matrices $U^{(L,\beta)}$ and $U^{(R,\beta q^2)}$ are invertible matrices for which it holds:

$$U^{(L,\beta)}\mathcal{B}_{-}(\lambda|\beta) = \Delta_{\mathcal{B}_{-}}(\lambda|\beta)U^{(L,\beta)}, \quad \mathcal{B}_{-}(\lambda|\beta)U^{(R,\beta q^2)} = U^{(R,\beta q^2)}\Delta_{\mathcal{B}_{-}}(\lambda|\beta), \tag{B.44}$$

where $\Delta_{\mathcal{B}_{-}}(\lambda|\beta)$ is the $p^{\mathsf{N}}\times p^{\mathsf{N}}$ diagonal matrix defined by:

$$\left(\Delta_{\mathcal{B}_{-}}(\lambda|\beta)\right)_{i,j} \equiv \delta_{i,j}\mathrm{B}_{\varkappa^{-1}(i)}(\lambda|\beta) \quad \forall i,j\in\{1,...,p^{\mathsf{N}}\}. \tag{B.45}$$

We can prove that it holds, with the same notation as in Theorem 3.1:

Proposition B.2. *For almost all the values of the boundary-bulk-gauge parameters it holds:*

$$\langle\Omega_\beta|\Omega_{\beta q^2}\rangle \neq 0, \tag{B.46}$$

so that fixed the normalization factor:

$$\mathrm{N}_\beta = \left(\prod_{1\leq b<a\leq\mathsf{N}}\left(X_a^{(p-1)} - X_b^{(p-1)}\right)\langle\Omega_\beta|\Omega_{\beta q^2}\rangle\right)^{1/2}, \tag{B.47}$$

in the left and right pseudo-eigenstates, then the $p^{\mathsf{N}}\times p^{\mathsf{N}}$ matrix $M\equiv U^{(L,\beta)}U^{(R,\beta q^2)}$ is the following invertible diagonal matrix:

$$M_{\varkappa(\boldsymbol{h})\varkappa(\boldsymbol{k})} = \langle\beta,\boldsymbol{h}|\beta q^2,\boldsymbol{k}\rangle = \prod_{1\leq a\leq\mathsf{N}}\delta_{h_a,k_a}\prod_{1\leq b<a\leq\mathsf{N}}\frac{1}{X_a^{(h_a)} - X_b^{(h_b)}}, \tag{B.48}$$

from which the following spectral decomposition of the identity $\mathbb{I}$ follows:

$$\mathbb{I} \equiv \sum_{h_1,...,h_{\mathsf{N}}=0}^{p-1}\prod_{1\leq b<a\leq\mathsf{N}}(X_a^{(h_a)} - X_a^{(h_a)})|\beta q^2,h_1,...,h_{\mathsf{N}}\rangle\langle\beta,h_1,...,h_{\mathsf{N}}|. \tag{B.49}$$

Proof. The following identity holds:

$$\mathrm{B}_{\mathbf{h}}(\lambda|\beta)\langle\beta/q^2,\mathbf{h}|\beta,\mathbf{k}\rangle = \langle\beta,\mathbf{h}|\mathcal{B}_{-}(\lambda|\beta)|\beta,\mathbf{k}\rangle = \mathrm{B}_{\mathbf{k}}(\lambda|\beta)\langle\beta,\mathbf{h}|\beta q^2,\mathbf{k}\rangle. \tag{B.50}$$

From it follows the fact that the matrix M is diagonal:

$$\langle\beta/q^2,\mathbf{h}|\beta,\mathbf{k}\rangle = \langle\beta,\mathbf{h}|\beta q^2,\mathbf{k}\rangle = 0 \quad \forall\mathbf{h}\neq\mathbf{k}\in\{0,...,p-1\}^{\mathsf{N}} \tag{B.51}$$

as there exists at least a $n\in\{1,...,\mathsf{N}\}$ such that $h_n\neq k_n$ and then:

$$\mathrm{B}_{\mathbf{h}}(\zeta_n^{(k_n)}|\beta)\neq 0, \quad \mathrm{B}_{\mathbf{k}}(\zeta_n^{(k_n)}|\beta) = 0. \tag{B.52}$$

Moreover, independently from the choice of the nonzero normalization factor N_β, the Theorem 3.1 implies that the matrices $U^{(L,\beta)}$ and $U^{(R,\beta q^2)}$ are invertible for almost all the values of the boundary-bulk-gauge parameters so that the same must be true for the diagonal matrix M, i.e. it must holds:

$$M_{\varkappa(\mathbf{h})\varkappa(\mathbf{h})} = \langle\mathbf{h},\beta|\beta q^2,\mathbf{h}\rangle \neq 0 \quad \forall\mathbf{h}\in\{0,...,p-1\}^{\mathsf{N}}, \tag{B.53}$$

which implies $(B.46)$ for $\mathbf{p-1}\equiv(p-1,...,p-1)$ being:

$$M_{\varkappa(\mathbf{p-1})\varkappa(\mathbf{p-1})} = \frac{\langle\Omega_\beta|\Omega_{\beta q^2}\rangle}{\mathrm{N}_\beta^2}, \tag{B.54}$$

and we can define the normalization factor according to $(B.47)$. The computation now of the remaining diagonal matrix elements $M_{\varkappa(\mathbf{h})\varkappa(\mathbf{h})}$ for $\mathbf{h}\neq\mathbf{p-1}$ can be done in a standard way by computing the matrix elements:

$$\theta_{a,h_a}(\beta) \equiv \langle\beta,h_1,...,h_a,...,h_{\mathsf{N}}|\mathcal{A}_{-}(\zeta_a^{(h_a+1)}|\beta q^2)|\beta q^2,h_1,...,h_a+1,...,h_{\mathsf{N}}\rangle, \tag{B.55}$$

where $h_a \in \{0,...,p-1\}$, $a \in \{1,...,\mathsf{N}\}$. Using the left action of the operator $\mathcal{A}_-(\zeta_a^{(h_a+1)}|\beta q^2)$ we get:

$$\theta_{a,h_a}(\beta) = \left[\frac{1}{q\left(\zeta_a^{(h_a)}\right)^2} - \frac{\left(\zeta_a^{(h_a)2} q - 1/q\zeta_a^{(h_a)2}\right)}{\beta(\beta - 1/\beta)}\right] \prod_{\substack{b=1\\b\neq a}}^{\mathsf{N}} \frac{X_a^{(h_a+1)} - X_b^{(h_b)}}{X_a^{(h_a)} - X_b^{(h_b)}}$$
$$\times \frac{\mathsf{A}_-(1/\zeta_a^{(h_a)})(q-1/q)}{\left((\zeta_a^{(h_a)})^2 - 1/(\zeta_a^{(h_a)})^2\right)} \langle \beta, h_1, ..., h_a+1, ..., h_{\mathsf{N}}|\beta, h_1, ..., h_a+1, ..., h_{\mathsf{N}}\rangle \tag{B.56}$$
$$= \frac{\mathsf{A}_-(1/\zeta_a^{(h_a)})(1/q-q)((\zeta_a^{(h_a)})^2 q/\beta - \beta/q(\zeta_a^{(h_a)})^2)}{\left((\zeta_a^{(h_a)})^2 - 1/(\zeta_a^{(h_a)})^2\right)(\beta - 1/\beta)} \prod_{\substack{b=1\\b\neq a}}^{\mathsf{N}} \frac{X_a^{(h_a+1)} - X_b^{(h_b)}}{X_a^{(h_a)} - X_b^{(h_b)}}$$
$$\times \langle \beta, h_1, ..., h_a+1, ..., h_{\mathsf{N}}|\beta, h_1, ..., h_a+1, ..., h_{\mathsf{N}}\rangle \tag{B.57}$$

while using the decomposition (B.17) and the fact that:

$$\langle \beta, h_1, ..., h_a, ..., h_{\mathsf{N}}|\mathcal{D}_-(1/\zeta_a^{(h_a+1)}|\beta q^2)|\beta q^2, h_1, ..., h_a+1, ..., h_{\mathsf{N}}\rangle = 0 \tag{B.58}$$

it holds:

$$\theta_{a,h_a}(\beta) = \frac{\mathsf{A}_-(1/\zeta_a^{(h_a)})(1/q-q)((\zeta_a^{(h_a)})^2 q/\beta - \beta/q(\zeta_a^{(h_a)})^2)k_a^{(h_a+1)}}{\left((\zeta_a^{(h_a+1)})^2 - 1/(\zeta_a^{(h_a+1)})^2\right)(\beta - 1/\beta)}$$
$$\times \langle \beta, h_1, ..., h_a, ..., h_{\mathsf{N}}|\beta q^2, h_1, ..., h_a, ..., h_{\mathsf{N}}\rangle. \tag{B.59}$$

These results lead to the identity:

$$\frac{\langle \beta, h_1, ..., h_a+1, ..., h_{\mathsf{N}}|\beta q^2, h_1, ..., h_a+1, ..., h_{\mathsf{N}}\rangle}{\langle \beta, h_1, ..., h_a, ..., h_{\mathsf{N}}|\beta q^2, h_1, ..., h_a, ..., h_{\mathsf{N}}\rangle} = \prod_{\substack{b=1\\b\neq a}}^{\mathsf{N}} \frac{X_a^{(h_a)} - X_b^{(h_b)}}{X_a^{(h_a+1)} - X_b^{(h_b)}}, \tag{B.60}$$

from which one can prove:

$$\frac{\langle \beta, h_1, ..., h_a, ..., h_{\mathsf{N}}|\beta q^2, h_1, ..., h_a, ..., h_{\mathsf{N}}\rangle}{\langle \beta, p-1, ..., p-1|\beta q^2, p-1, ..., p-1\rangle} = \prod_{1\leq b<a\leq \mathsf{N}} \frac{X_a^{(p-1)} - X_b^{(p-1)}}{X_a^{(h_a)} - X_b^{(h_b)}}. \tag{B.61}$$

This proves the proposition being by our choice of normalization:

$$\langle \beta, p-1, ..., p-1|\beta q^2, p-1, ..., p-1\rangle = \prod_{1\leq b<a\leq \mathsf{N}} \frac{1}{X_a^{(p-1)} - X_b^{(p-1)}}. \tag{B.62}$$

□

B.4 Proof of pseudo diagonalizability and simplicity of $\mathcal{B}_-(\lambda|\beta)$

We prove the pseudo-digonalizability and pseudo-simplicity of $\mathcal{B}_-(\lambda|\beta)$ in two steps. We first consider some special representation for which such statement is proven by direct computation then we use this result to prove our statement for general representations.

B.4.1 Pseudo diagonalizability and simplicity of $\mathcal{B}_-(\lambda|\beta)$: special representations

The following theorem holds:

Theorem B.3. *Let us assume that the conditions on the bulk-gauge parameters:*

$$\beta = \alpha w_{\mathsf{N}}^{(\epsilon_{\mathsf{N}},k_{\mathsf{N}})},\ z_{n+1}^{(\epsilon_{n+1},k_{n+1})} = 1/w_n^{(\epsilon_n,k_n)} \quad \forall n \in \{1,...,\mathsf{N}-1\}, \tag{B.63}$$

are satisfied for fixed N*-tuples of* $\epsilon_n = \pm 1$ *and* $k_n \in \{0,...,p-1\}$ *and that the conditions* ($A.27$) *hold together with the following ones:*

$$\left(z_1^{(\epsilon_1,k_1)}\right)^p \neq (-\zeta_- + \epsilon_0\sqrt{\zeta_-^2 + 4\kappa_-^2})^p/(2qe^{\tau_-}\kappa_-)^p, \tag{B.64}$$

and

$$\mu_{n,\epsilon_n}^{2p} \neq \pm 1,\ \mu_{n,\epsilon_n}^{2p} \neq \alpha_-^{2p\epsilon},\ \mu_{n,\epsilon_n}^{2p} \neq -\beta_-^{2p\epsilon},\ \mu_{n,+}^{2p} \neq \mu_{m,-}^{2p\epsilon},\ \mu_{n,\epsilon_n}^{p} \neq \mu_{m,\epsilon_n}^{p}, \tag{B.65}$$

for any $\epsilon_0 = \pm 1$ *and* $n, m \in \{1,...,\mathsf{N}\}$, *then the operator family* $\mathcal{B}_-(\lambda|\beta)$ *has simple pseudo-spectrum characterized by:*

$$\mathrm{B}_{-,n}(\beta) = \mu_{n,\epsilon_n} q^{1/2}\ \forall n \in \{1,...,\mathsf{N}\},\ \text{i.e. independent w.r.t. } \beta, \tag{B.66}$$
$$\mathrm{B}_-(\beta) = f(\alpha, \beta/q, z_1^{(\epsilon_1,k_1+1)}) \left(\frac{\beta^4/q^2}{(1-\beta^2)(1-(\beta/q)^2)}\right)^{\mathsf{N}} (-1)^{\mathsf{N}} \prod_{n=1}^{\mathsf{N}} \frac{\gamma_n^2}{\mu_{n,\epsilon_n}^2}$$
$$\times \left[z_1^{(\epsilon_1,k_1+1)} e^{\tau_-}\kappa_- + \zeta_- - \kappa_-/(z_1^{(\epsilon_1,k_1+1)} e^{\tau_-})\right]/(\zeta_- - 1/\zeta_-), \tag{B.67}$$

and the left pseudo-eigenbasis characterized by the formulae (3.9) *by fixing:*

$$\langle \Omega_\beta| = \sum_{h_1,...,h_N=0}^{p-1} \bigotimes_{n=1}^{\mathsf{N}} q^{h_n(k_n+1)} \left[\prod_{k_n=1}^{h_n} \frac{a_n q^{k_n-1/2} + b_n q^{1/2-k_n}}{c_n q^{k_n-1/2} + d_n q^{1/2-k_n}} \left(\frac{c_n^p + d_n^p}{a_n^p + b_n^p}\right)^{1/p}\right]^{(1+\epsilon_n)/2} \langle h_n, n|. \tag{B.68}$$

Similarly, let us assume that the conditions ($A.27$) *and* ($B.64$)*-*($B.65$) *are satisfied together with:*

$$\beta = \alpha w_{\mathsf{N}}^{(\epsilon_{\mathsf{N}},k_{\mathsf{N}})},\ z_{n+1}^{(\epsilon_{n+1},k_{n+1}-2)} = 1/w_n^{(\epsilon_n,k_n)} \quad \forall n \in \{1,...,\mathsf{N}-1\}, \tag{B.69}$$

for fixed N*-tuples of* $\epsilon_n = \pm 1$ *and* $k_n \in \{0,...,p-1\}$, *then the operator family* $\mathcal{B}_-(\lambda|\beta)$ *has simple pseudo-spectrum characterized by fixing :*

$$\mathrm{B}_{-,n}(\beta) = \mu_{n,\epsilon_n} q^{-1/2}\ \forall n \in \{1,...,\mathsf{N}\},\ \text{i.e. independent w.r.t. } \beta, \tag{B.70}$$
$$\mathrm{B}_-(\beta) = f(\alpha, \beta/q, z_1^{(\epsilon_1,k_1-1)}) \left(\frac{\beta^4}{(1-\beta^2)(1-(\beta/q)^2)}\right)^{\mathsf{N}} (-1)^{\mathsf{N}} \prod_{n=1}^{\mathsf{N}} \frac{\gamma_n^2}{\mu_{n,\epsilon_n}^2}$$
$$\times \left[z_1^{(\epsilon_1,k_1-1)} e^{\tau_-}\kappa_- + \zeta_- - \kappa_-/(z_1^{(\epsilon_1,k_1-1)} e^{\tau_-})\right]/(\zeta_- - 1/\zeta_-), \tag{B.71}$$

and right pseudo-eigenbasis characterized by the formulae (3.10) *by fixing:*

$$|\Omega_\beta\rangle = \prod_{n=1}^{\mathsf{N}} \prod_{r_n=1}^{p-1} D_-(\mathrm{B}_{-,n}(\beta)/q^{r_n}|\beta)|\bar{\Omega}_\beta\rangle \tag{B.72}$$

with

$$|\bar{\Omega}_\beta\rangle = \left(\frac{\beta}{q}\right)^{\mathsf{N}} \sum_{h_1,...,h_N=0}^{p-1} \prod_{n=1}^{\mathsf{N}} q^{-h_n k_n} \left[\prod_{r_n=1}^{h_n} \frac{c_n q^{r_n-1/2} + d_n q^{1/2-r_n}}{a_n q^{r_n-1/2} + b_n q^{1/2-r_n}} \left(\frac{a_n^p + b_n^p}{c_n^p + d_n^p}\right)^{1/p} \right]^{(1+\epsilon_n)/2} \bigotimes_{n=1}^{N} |h_n, n\rangle. \tag{B.73}$$

Proof. The conditions $(B.63)$ and the choice of internal gauge parameter:

$$\gamma = z_1^{(\epsilon_1,k_1)}, \tag{B.74}$$

imply that the states $(B.68)$ and $(B.73)$ are annihilated by $A(\lambda|\alpha,\beta,\gamma)$ and $A(1/\lambda|\alpha,\beta/q,\gamma q)$ respectively as the following identifications hold:

$$\langle\Omega_\beta| = \langle\Omega,\alpha,\beta,\gamma|, \quad |\bar{\Omega}_\beta\rangle = |\Omega,\alpha,\beta/q,\gamma q\rangle \left(\frac{\beta}{q}\right)^{\mathsf{N}}, \tag{B.75}$$

moreover, it holds:

$$\begin{aligned} \langle\Omega_\beta|B(\lambda|\alpha,\beta) &= b(\lambda|\alpha,\beta)\langle\Omega,\alpha,\beta/q,\gamma q|, & \text{(B.76)}\\ B(\lambda|\alpha,\beta/q)|\bar{\Omega}_\beta\rangle &= |\Omega,\alpha,\beta,\gamma\rangle\beta^{\mathsf{N}} b(\lambda q|\alpha,\beta/q), & \text{(B.77)} \end{aligned}$$

so that it holds:

$$\begin{aligned} \langle\Omega_\beta|\mathcal{B}_-(\lambda|\alpha,\beta) &= f(\alpha,\beta/q,\gamma q)\bar{K}_-(\lambda|\gamma)_{21}\langle\Omega_\beta|B(\lambda|\alpha,\beta)B(1/\lambda|\alpha,\beta/q), & \text{(B.78)}\\ \mathcal{B}_-(\lambda|\alpha,\beta)|\bar{\Omega}_\beta\rangle &= B(\lambda|\alpha,\beta)B(1/\lambda|\alpha,\beta/q)|\bar{\Omega}_\beta\rangle f(\alpha,\beta/q,\gamma q)\bar{K}_-(\lambda|\gamma)_{21}, & \text{(B.79)} \end{aligned}$$

and consequently:

$$\begin{aligned} \langle\Omega_\beta|\mathcal{B}_-(\lambda|\alpha,\beta) &= f(\alpha,\beta/q,\gamma q)\bar{K}_-(\lambda|\gamma)_{21} b(\lambda|\alpha,\beta)b(1/\lambda|\alpha,\beta/q)\langle\Omega_{\beta/q^2}|, & \text{(B.80)}\\ \mathcal{B}_-(\lambda|\alpha,\beta)|\bar{\Omega}_\beta\rangle &= |\bar{\Omega}_{\beta q^2}\rangle b(\lambda q|\alpha,\beta)b(q/\lambda|\alpha,\beta/q)f(\alpha,\beta/q,\gamma q)\bar{K}_-(\lambda|\gamma)_{21}, & \text{(B.81)} \end{aligned}$$

so that for the pseudo-eigenvalue it holds:

$$\begin{aligned} \mathrm{B}_{\mathbf{0}}(\lambda|\beta) &= f(\alpha,\beta/q,\gamma q)\bar{K}_-(\lambda|\gamma)_{21} b(\lambda|\alpha,\beta)b(1/\lambda|\alpha,\beta/q), & \text{(B.82)}\\ \mathrm{B}_{\mathbf{1}}(\lambda|\beta) &= f(\alpha,\beta/q,\gamma q)\bar{K}_-(\lambda|\gamma)_{21} b(\lambda q|\alpha,\beta)b(q/(\lambda)|\alpha,\beta/q), & \text{(B.83)} \end{aligned}$$

respectively on the left and the right. This fixes the values of the $\mathrm{B}_-(\beta)$ and $\mathrm{B}_{-,a}(\beta)$ to those stated in this theorem. Note that the condition $(B.64)$ implies that:

$$\mathrm{B}_-(\beta/q^{2a}) \neq 0 \quad \forall a \in \{0,...,p-1\}\,. \tag{B.84}$$

Let us now prove that the states (3.9) and (3.10) are all nonzero states. The reasoning is done explicitly only for the left case as for the right one we can proceed similarly. We know by construction that the state $\langle\Omega_\beta|$ is nonzero so let us assume by induction that the same is true for the state $\langle\beta,\mathbf{h}^{(0)}| = \langle\beta,h_1^{(0)},...,h_{\mathsf{N}}^{(0)}|$ with $h_j^{(0)} \in \{0,...,p-2\}$ and let us show that $\langle\beta,\mathbf{h}_j^{(0)}| = \langle\beta,h_1^{(0)},...,h_j^{(0)}+1,...,h_{\mathsf{N}}^{(0)}|$ is nonzero. We have that:

$$\langle\beta,\mathbf{h}_j^{(0)}|\mathcal{A}_-(\zeta_j^{(h_j^{(0)}+1)}|\beta q^2) = \mathsf{A}_-(\zeta_j^{(h_j^{(0)}+1)})\langle\beta,\mathbf{h}^{(0)}| \neq \underline{0} \quad \forall j\in\{1,...,\mathsf{N}\} \tag{B.85}$$

so that $\langle\beta,\mathbf{h}_j^{(0)}|$ is nonzero. Using this we can prove that all the states $\langle\beta,h_1^{(0)}+x_1,...,h_{\mathsf{N}}^{(0)}+x_{\mathsf{N}}|$ with $x_j\in\{0,1\}$ for any $j\in\{1,...,\mathsf{N}\}$ are nonzero, which just proves the validity of the induction. Note that the same statements hold if we substitute the given value of β fixed in $(B.69)$ with any value β/q^{2a} for any $a\in\{1,...,p-1\}$; i.e. we have that $\langle\Omega_{\beta/q^{2a}}|$ is nonzero and from that we prove similarly the induction.

Let us now prove that the sets of left and right states define respectively a left and a right basis of the linear space of the representation. Let us consider the linear combination to zero of the left states:

$$\underline{0} = \sum_{\mathbf{k}\in Z_p^{\mathsf{N}}} c_{\mathbf{k}}\langle\beta,\mathbf{k}|, \tag{B.86}$$

and let us act on it with the following product of operator:

$$\begin{aligned} \mathcal{B}_{-,\mathbf{h}}(\beta) \equiv\ & \mathcal{B}_-(\zeta_1^{(0)}|\beta)\mathcal{B}_-(\zeta_1^{(1)}|\beta/q^2)\cdots\mathcal{B}_-(\zeta_1^{(p-1)}|\beta/q^{2(p-2)})\\ &\times\mathcal{B}_-(\zeta_2^{(0)}|\beta/q^{2(p-1)})\mathcal{B}_-(\zeta_2^{(1)}|\beta/q^{2p})\cdots\mathcal{B}_-(\zeta_2^{(p-1)}|\beta/q^{4(p-1)-2})\\ &\cdots\times\mathcal{B}_-(\zeta_{\mathsf{N}}^{(0)}|\beta/q^{2(\mathsf{N}-1)(p-1)})\cdots\mathcal{B}_-(\zeta_2^{(p-1)}|\beta/q^{2\mathsf{N}(p-1)-2}), \end{aligned} \tag{B.87}$$

where the generic monomial in it:

$$\mathcal{B}_-(\zeta_m^{(0)}|\beta/q^{2(m-1)(p-1)})\mathcal{B}_-(\zeta_2^{(1)}|\beta/q^{2(m-1)(p-1)+2})\cdots\mathcal{B}_-(\zeta_2^{(p-1)}|\beta/q^{2m(p-1)-2}), \tag{B.88}$$

contains only the $p-1$ arguments $\zeta_m^{(k_m)}$ with $k_m\in\{0,...,p-1\}\backslash\{h_n\}$ and $\mathbf{h}\equiv\{h_1,...,h_{\mathsf{N}}\}$ is a generic element of Z_p^{N}. Then, it easily to understand that it holds:

$$\begin{aligned} \underline{0} &= \left(\sum_{\mathbf{k}\in Z_p^{\mathsf{N}}} c_{\mathbf{k}}\langle\beta,\mathbf{k}|\right)\mathcal{B}_{-,\mathbf{h}}(\beta) = c_{\mathbf{h}}\langle\beta,\mathbf{h}|\mathcal{B}_{-,\mathbf{h}}(\beta)\\ &= c_{\mathbf{h}}\prod_{n=\mathbf{1}}^{\mathsf{N}}\prod_{k_n\in\{0,...,p-1\}\backslash\{h_n\}} \mathrm{B}_{\mathbf{h}}(\zeta_n^{(k_n)}|\beta/q^{2(n-1)(p-1)+k_n'})\langle\beta/q^{2\mathsf{N}(p-1)},\mathbf{h}|, \end{aligned} \tag{B.89}$$

where $k_n'=k_n$ if $k_n'<h_n$ and $k_n'=k_n-1$ if $h_n<k_n$. Now the simplicity of the pseudo-spectrum of $\mathcal{B}_-(\lambda|\beta)$ implies that:

$$\prod_{n=\mathbf{1}}^{\mathsf{N}}\prod_{k_n\in\{0,...,p-1\}\backslash\{h_n\}} \mathrm{B}_{\mathbf{h}}(\zeta_n^{(k_n)}|\beta/q^{2(n-1)(p-1)+k_n'}) \neq 0, \tag{B.90}$$

from which we derive:

$$c_{\mathbf{h}} = 0 \tag{B.91}$$

having already proven:

$$\langle\beta/q^{2\mathsf{N}(p-1)},\mathbf{h}| \neq \underline{0}. \tag{B.92}$$

The generality of the chosen $\mathbf{h}\in Z_p^{\mathsf{N}}$ implies that the linear combination to zero $(B.86)$ is satisfied if and only if $(B.91)$ holds for any $\mathbf{h}\in Z_p^{\mathsf{N}}$, that is the left pseudo-eigenstates $\langle\beta,\mathbf{k}|$ are a left basis. $\square$

Note that in the bulk of the paper we have chosen to present the construction of the SoV-basis starting from a state $|\Omega_\beta\rangle$ associated to the pseudo-eigenvalue $\text{B}_{\mathbf{0}}(\lambda|\beta)$ just to simplify the simultaneous presentation of the left and right basis; in fact, we can construct the right basis also starting from the state $|\bar{\Omega}_\beta\rangle$ associated to $\text{B}_{\mathbf{1}}(\lambda|\beta)$, which is the state constructed directly here for the considered special representations.

B.4.2 Pseudo diagonalizability and simplicity of $\mathcal{B}_-(\lambda|\beta)$: general representations

In this section we prove the Theorem 3.1 stating the pseudo diagonalizability and simplicity of the operator family $\mathcal{B}_-(\lambda|\beta)$ for almost all the values of the boundary-bulk-gauge parameters. Let us first prove the following lemma:

Lemma B.1. *There exists at least one left and one right pseudo-eigenstate $|\Omega_\beta\rangle$ and $\langle\Omega_\beta|$ of the one parameter family of pseudo-commuting operators $\mathcal{B}_-(\lambda|\beta)$ satisfying the condition* (3.5) *with pseudo-eigenvalue $\text{B}_{\mathbf{0}}(\lambda|\beta)$ satisfying the conditions* (3.7) *and* (3.8).

Proof. The operator family $\mathcal{B}_-(\lambda|\beta)$ admits the following representation:

$$\mathcal{B}_-(\lambda|\beta) = (\frac{\lambda^2}{q} - \frac{q}{\lambda^2})\sum_{a=0}^{\mathsf{N}} \Lambda^a \bar{\mathcal{B}}_{-,a}(\beta) T_\beta^{-2}, \tag{B.93}$$

where the following commutation relations holds:

$$\bar{\mathcal{B}}_{-,a}(\beta) T_\beta = T_\beta \bar{\mathcal{B}}_{-,a}(\beta q), \quad \left[\bar{\mathcal{B}}_{-,a}(\beta), \bar{\mathcal{B}}_{-,b}(\beta)\right] = 0 \ \ \forall a, b \in \{1, ..., \mathsf{N}\}, \tag{B.94}$$

as a consequence of the commutation relations $(B.12)$. The result of the previous section implies that for some special choice of the boundary-bulk-gauge parameters all the operators $\bar{\mathcal{B}}_{-,a,\beta}$ are invertible as $\mathcal{B}_-(\lambda|\beta)$ is pseudo-diagonalizable and it admits the following representation:

$$\mathcal{B}_-(\lambda|\beta) = \text{B}_-(\beta)(\frac{\lambda^2}{q} - \frac{q}{\lambda^2})\prod_{a=1}^{\mathsf{N}}(\frac{\lambda}{\mathcal{B}_{-,a}(\beta)} - \frac{\mathcal{B}_{-,a}(\beta)}{\lambda})(\lambda B_{-,a}(\beta) - \frac{1}{\lambda \mathcal{B}_{-,a}(\beta)})T_\beta^{-2}, \tag{B.95}$$

where the $\mathcal{B}_{-,a}(\beta)$ are commuting and invertible operators. Then the fact that this operators depend continuously on these parameters implies that this statement is true for almost any values of these parameters. This also implies that for almost all the value of the boundary-bulk-gauge parameters we can use the above representation for $\mathcal{B}_-(\lambda|\beta)$.

We can now recall that, thanks to the result of the Lemma A.1 of our previous paper, we can always find a nonzero simultaneous eigenstate of commuting operators such as the $\mathcal{B}_{-,a}(\beta)$ for any $a \in \{1, ..., \mathsf{N}\}$. This is a pseudo-eigenstate of the operator family $\mathcal{B}_-(\lambda|\beta)$.

Now, for the same set of representations considered in the previous section we know that the pseudo-eigenvalues of $\mathcal{B}_-(\lambda|\beta)$ satisfy the conditions (3.7) and (3.8). Then, we can use once again the continuity argument to argue that the eigenvalues on the common eigenstate still satisfy (3.7) and (3.8). □

We can now prove the Theorem 3.1, by using the results of the previous sections.

Proof of Theorem 3.1. The proof of the pseudo-diagonalizability of $\mathcal{B}_-(\lambda|\beta)$ is a direct consequence of the previous lemma. Indeed, under the conditions (3.7) and (3.8) we can prove that all the

left and right states are well defined and nonzero states which are pseudo-eigenstates of $\mathcal{B}_-(\lambda|\beta)$ associated to different pseudo-eigenvalues as a consequence of the gauge transformed commutation relations. The proof of the fact that the states (3.9) and (3.10) are all nonzero is done reproducing the argument presented in the proof of Theorem $B.3$.

The statements about the spectral decomposition of the identity of the theorem have been already given in Proposition B.2. □

C Properties of cofactor

In this appendix we prove a lemma giving the main properties of the *cofactors* of the matrix $D_\tau(\lambda)$.

Lemma C.1. *The matrix $D_\tau(\lambda)$ has at least rank $p-1$ for any $\lambda \in \mathbb{C}$, up to at most a finite number of values. The following symmetries:*

$$\begin{aligned} \text{C}_{i+h,j+h}(\lambda) &= \text{C}_{i,j}(\lambda q^h) \quad \forall i,j,h \in \{1,...,p\}, &\text{(C.1)}\\ \text{C}_{i,j}(-\lambda) &= \text{C}_{i,j}(\lambda) \quad \forall i,j \in \{1,...,p\}, &\text{(C.2)}\\ \text{C}_{1,1}(1/\lambda) &= \text{C}_{1,1}(\lambda), \quad \text{C}_{1,2}(1/\lambda) = \text{C}_{1,p}(\lambda), &\text{(C.3)} \end{aligned}$$

hold. Moreover, the cofactors $\text{C}_{1,1}(\lambda)$, $\text{C}_{1,2}(\lambda)$ and $\text{C}_{1,p}(\lambda)$ are polynomials in λ of maximal degree $(p-1)(2\mathsf{N}+4)$ which admit the following decomposition:

$$\text{C}_{1,1}(\lambda) = \widehat{\text{C}}_{1,1}(\lambda)\left(\lambda^2 - \frac{1}{\lambda^2}\right)^2 \prod_{\substack{k=1\\ k\neq(p-1)/2}}^{p-2}\left(\lambda^2 q^{1+2k} - \frac{1}{\lambda^2 q^{1+2k}}\right), \tag{C.4}$$

where $\widehat{\text{C}}_{1,1}(\lambda)$ is a polynomial in Λ of degree $(p-1)(\mathsf{N}+1)$, and:

$$\begin{aligned} \text{C}_{1,2}(\lambda) &= \widehat{\text{C}}_{1,2}(\lambda)\left(\lambda^2 - \frac{1}{\lambda^2}\right)^2 \prod_{\substack{k=1\\ k\neq(p-1)/2}}^{p-2}\left(\lambda^2 q^{1+2k} - \frac{1}{\lambda^2 q^{1+2k}}\right), &\text{(C.5)}\\ \text{C}_{1,p}(\lambda) &= \widehat{\text{C}}_{1,p}(\lambda)\left(\lambda^2 - \frac{1}{\lambda^2}\right)^2 \prod_{\substack{k=1\\ k\neq(p-1)/2}}^{p-2}\left(\lambda^2 q^{1+2k} - \frac{1}{\lambda^2 q^{1+2k}}\right), &\text{(C.6)} \end{aligned}$$

where $\widehat{\text{C}}_{1,2}(\lambda)$ and $\widehat{\text{C}}_{1,p}(\lambda)$ are polynomials of maximal degree $2(p-1)(\mathsf{N}+1)$ in λ.

Proof. Let us remark that independently from the explicit form of $\tau(\lambda)$ the following identities hold:

$$\text{C}_{1,p}(q^{1/2-p}/\mu_{a,+}) = \prod_{j=1}^{p-1} \text{A}(\mu_{a,+} q^j) \neq 0 \quad \forall a \in \{1, ..., \mathsf{N}\}, \tag{C.7}$$

so that $\overline{\text{C}}_{1,p}(\lambda)$ is a non-zero polynomial in λ which implies the statement on the rank of $D_\tau(\lambda)$. The proof of the above symmetry properties is standard we just need to make some exchange of rows and columns to bring the matrix in the determinant defining the cofactor in the l.h.s into the matrix defining the cofactor in the r.h.s..

Let us show our statement on the form of $\mathrm{C}_{1,1}(\lambda)$. In order to do so we have to prove that $\mathrm{C}_{1,1}(\lambda)$ is finite in the points[1] $\lambda = \pm i^a q^h$ for any $h \in \{1, ..., p-1\}$. More precisely, in the line $p-h$ there is at least one element of the matrix $\mathrm{M}_{1,1}(\lambda)$ associated to $\mathrm{C}_{1,1}(\lambda)$ which is diverging in the limit $\lambda \to \pm i^a q^h$. Here, we have to distinguish three cases. For the case $h \neq (p \pm 1)/2$, we can proceed as done in the bulk of the paper. We can define the matrix $\mathrm{M}_{1,1}^{(h)}(\lambda)$ as the matrix with all the rows coinciding with those of $\mathrm{M}_{1,1}(\lambda)$ except the row $(p+1)/2 - h$, which is obtained by summing the row $(p-1)/2-h$ and $(p+1)/2-h$ of $\mathrm{M}_{1,1}(\lambda)$ and dividing them by $((\lambda/q^h)^2 - (q^h/\lambda)^2)$, and the row $p-h$, obtained multiplying the row $p-h$ of $\mathrm{M}_{1,1}^{(h)}(\lambda)$ by $((\lambda/q^h)^2 - (q^h/\lambda)^2)$. Clearly it holds $\det_{p-1}\mathrm{M}_{1,1}^{(h)}(\lambda) = (-1)^{i+j}\mathrm{C}_{1,1}(\lambda)$ and all the rows of the matrix $\mathrm{M}_{1,1}^{(h)}(\pm i^a q^h \lambda)$ are finite in the limits $\lambda \to 1$ and so the same is true for their determinants. In fact, it is possible to show that these lines are linear dependents in each one of the matrices $\mathrm{M}_{1,1}^{(h)}(\pm i^a q^h)$, so that:

$$\det_{p-1}\mathrm{M}_{1,1}^{(h)}(\pm i^a q^h) = 0 \quad \forall h \in \{1, ..., p\} \setminus \{0, (p \pm 1)/2\}. \tag{C.8}$$

In the remaining cases, if $h = (p \pm 1)/2$ then the row $(p \pm 1)/2p - h = p \bmod(p)$ is not contained in $\mathrm{M}_{1,1}(\pm i^a q^h)$ so that we cannot remove here the divergence as we have done before. However, we can proceed differently, let us explain it in the case $h = (p+1)/2$ as in the other case we can proceed similarly. In the last row of $\mathrm{M}_{1,1}(\pm i^a q^{(p+1)/2}\lambda)$ under the limit $\lambda \to 1$ the last element tend to $\tau(i^a q^{1/2})$, finite nonzero value, and the next to last tend to $\mathrm{A}(\pm i^a q^{-1/2}) = 0$, all the others on this row are zero. So that $\mathrm{C}_{1,1}(\pm i^a q^{(p-1)/2})$ is finite iff $\det_{p-2} D_{(1,p),(1,p)}(\pm i^a q^{(p+1)/2})$ is finite. This is shown using the following expansion of the determinant:

$$\det_{p-2} D_{(1,p),(1,p)}(\pm i^a q^{(p+1)/2}\lambda) = \tau(\lambda)\det_{p-3} D_{\tau,(1,(p+1)/2,p),(1,(p+1)/2,p)}(\pm i^a q^{(p+1)/2}\lambda) \tag{C.9}$$

$$+ \frac{\mathrm{X}(\lambda)\det_{p-3} D_{\tau,(1,(p+1)/2,p),(1,(p+1)/2-1,p)}(\pm i^a q^{(p+1)/2}\lambda)}{\lambda^2 - 1/\lambda^2} \tag{C.10}$$

$$- \frac{\mathrm{X}(1/\lambda)\det_{p-3} D_{\tau,(1,(p+1)/2,p),(1,(p+1)/2+1,p)}(\pm i^a q^{(p+1)/2}\lambda)}{\lambda^2 - 1/\lambda^2}, \tag{C.11}$$

and the identity:

$$\det_{p-3} D_{\tau,(1,(p+1)/2,p),(1,(p+1)/2-1,p)}(\pm i^a q^{(p+1)/2}) = \det_{p-3} D_{\tau,(1,(p+1)/2,p),(1,(p+1)/2-1,p)}(\pm i^a q^{(p+1)/2}). \tag{C.12}$$

Finally, let us remark that in the case $h = 0$ the lines $(p-1)/2$ and $(p+1)/2$ of $\mathrm{M}_{1,1}(\pm i^a)$ are one the opposite of the other so that $\det_{p-1}\mathrm{M}_{1,1}(\pm i^a) = 0$. We can so define the matrix $\mathrm{M}_{1,1}^{(0)}(\lambda)$ as the matrix with all the rows coinciding with those of $\mathrm{M}_{1,1}(\lambda)$ except the row $(p+1)/2$, which is obtained by summing the row $(p-1)/2$ and $(p+1)/2$ of $\mathrm{M}_{1,1}(\lambda)$ and dividing them by $(\lambda^2 - 1/\lambda^2)$, this matrix has finite elements on the row $(p+1)/2$ also in the limit $\lambda \to \pm i^a$. Similarly to the previous cases one can show that the rows of $\mathrm{M}_{1,1}^{(0)}(\pm i^a)$ are linear dependent so that it holds also:

$$\lim_{\lambda \to \pm i^a} \frac{\det_{p-1}\mathrm{M}_{1,1}(\pm i^a \lambda)}{\lambda^2 - 1/\lambda^2} = (-1)^a \det_{p-1}\mathrm{M}_{1,1}^{(0)}(\pm i^a) = 0, \tag{C.13}$$

from which our statement on the form of $\mathrm{C}_{1,1}(\lambda)$ follows. Similarly, we can prove our statement on $\mathrm{C}_{1,p}(\lambda)$. □

[1] As for $h = 0$ the matrix $\mathrm{M}_{1,1}(\pm i^a)$ does not contain any singular elements.

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Résumé de thèse

Les principaux outils pour la compréhension du comportement macroscopique de systèmes quantiques à partir de leur description microscopique sont la détermination du spectre du Hamiltonien associé et le calcul des fonctions de corrélation. Cette thèse se place dans le cadre du développement d'un tel programme de recherche afin d'étudier des systèmes intégrables quantiques avec des conditions aux bords intégrables générales, le but à long terme étant la description exacte d'une physique quantique hors équilibre.

Chapitre un : modèles intégrables quantiques

Pour un nombre fini de degrés de liberté, il existe une définition rigoureuse de l'intégrabilité en mécanique classique due à Liouville. Cependant, déjà pour les modèles classiques continus, il n'y a pas d'analogue universel du théorème de Liouville, car la notion "d'un nombre infini de charges conservées indépendantes" doit être minutieusement examinée au cas par cas. La situation pour les systèmes quantiques n'est pas meilleure, et ce même pour un nombre fini de degrés de liberté, car il n'existe pas non plus un analogue du théorème de Liouville. Cependant, depuis les travaux pionniers de Bethe, de grands efforts ont été menés pour inventer des méthodes puissantes afin de résoudre le spectre et même la dynamique de nombreux systèmes quantiques d'intérêt. Le cadre de travail le plus fructueux dans cette logique est fourni par la méthode de diffusion inverse quantique (QISM) introduite par Fadeev, Sklyanin et Taktadjan. Celle-ci peut-être considérée comme une version quantifiée de la méthode de diffusion inverse classique, qui permet de résoudre des théories continues avec des solutions de type solitons, tout en fournissant également une version algébrique de la méthode de l'ansatz de Bethe. Il hérite également de la stratégie développée pour les modèles sur réseau à 2 dimensions en utilisant la notion de matrices de transfert, qui commutent. En fait, ce cadre de travail contient également les ingrédients nécessaires à la définition d'un analogue quantique à la méthode de séparation des variables (SoV), qui, dans un certain sens, pourrait être considéré comme une définition pratique de l'intégrabilité quantique. Comme pour le cas classique, la première étape pour être capable de diagonaliser un Hamiltonien quantique est de trouver un ensemble complet de charges conservées qui commutent, ayant un spectre commun simple, menant ainsi à une caractérisation complète des états propres en utilisant les nombres quantiques correspondants. Bien sûr ceci n'est *a priori* pas suffisant car il faut encore construire les vecteurs propres correspondants et calculer leurs niveaux d'énergie.

En fait, nous voulons également déterminer les fonctions de corrélations dynamiques. La méthode QISM fournit le cadre algébrique pour aborder un tel problème pour une large classe de systèmes, principalement définis sur des réseaux 1D. Le premier chapitre présente cette méthode pour l'exemple particulier de l'algèbre de Yang-Baxter à 6-vertex, qui peut décrire une large classe de modèles d'intérêts physiques, tels la chaine de spins de Heisenberg, le modèle de sine-Gordon sur réseau ou bien le modèle de chiral Potts. Ensuite, nous rappelons l'ansatz de Bethe algébrique en nous focalisant sur l'exemple de la chaine de spins XXZ-1/2. Une petite partie du chapitre est dédiée au calcul des facteurs de forme des opérateurs locaux et aux fonctions de corrélations de ce modèle. Ensuite, nous définissons rigoureusement la classe de modèle à l'étude dans cette thèse, c'est-à-dire ceux associés à des représentations cycliques de l'algèbre de Yang-Baxter à 6-vertex.

Chapitre deux : modèles intégrables quantiques avec bords

L'étude des modèles avec bords a soulevé un grand enthousiasme de recherche, car ils peuvent décrire à la fois une physique à l'équilibre et hors équilibre. Quelques applications intéressantes concernent la description de processus de relaxation stochastique classique, comme le modèle ASEP et les propriétés de transport dans les chaines de spin.

Dans le premier chapitre, nous avons rappelé que le Hamiltonien d'un système intégrable quantique peut-être obtenu à partir de la matrice de transfert. Cependant, avec la définition précédente comme la trace de la matrice de monodromie, il se trouve que les modèles que l'on peut reconstruire possèdent

des conditions aux bords périodiques. Dans le but de décrire des conditions aux bords intégrables plus générales, le chapitre deux rappelle une première généralisation, en considérant une matrice de transfert twistée. En utilisant une représentation scalaire de l'algèbre de Yang-Baxter, on peut en effet déformer la matrice de monodromie, ce qui *in fine* mène à la description de conditions aux bords quasi-périodiques.

Cependant, le principal but de ce chapitre est de souligner le rôle majeur joué par l'algèbre de réflexion introduite par Sklyanin. En particulier, celle-ci permet de définir une matrice de transfert avec bords (définie comme la trace d'une matrice de monodromie avec bords), qui peut être utilisée pour générer des Hamiltoniens quantiques intégrables avec des bords intégrables généraux.

Le dernier paragraphe est dédié au rappel de la procédure de fusion, qui permet de définir des Hamiltonians de chaines de spins s quelconque. Dans tout ce chapitre, la chaine de spin XXZ est utilisée comme l'exemple principal pour illustrer et expliquer les différents outils et concepts.

Chapitre trois : séparation quantique des variables

Nous introduisons ici l'importante notion de separation des variables pour les systèmes intégrables. En commençant par le point de vue classique, nous rappelons la définition usuelle de séparation des variables. Ensuite, nous montrons l'implémentation de ce concept aux modèles quantiques (Sklyanin), en insistant sur l'analogie entre les descriptions classiques et quantiques. La séparation quantique des variables permet la caractérisation du spectre de la matrice de transfert dans le cadre de la méthode de diffusion inverse quantique (QISM). De façon remarquable, elle permet la description de modèles pour lesquels un état de référence $|0\rangle$, utilisé dans la méthode de l'ansatz de Bethe algébrique, ne peut pas être obtenu directement. Une fois encore, afin d'illustrer nos propos nous utilisons la chaine de spin XXZ-1/2 (antiperiodique) et construisons de façon explicite pour ce modèle la base séparée (le principal objet associé à la méthode de séparation quantique des variables). Nous profitons également de cet exemple pour souligner les principales motivations à travailler avec cette méthode.

Chapitre quatre : le spectre de la matrice de transfert avec bords par la séparation quantique des variables

Dans ce chapitre nous étudions le problème spectral de la matrice de transfert avec bords pour des représentations cycliques de l'algèbre de réflexion à 6-vertex associée aux opérateurs de Lax de Bazhanov-Stroganov. Comme déjà mentionné, des cas particuliers importants sont contenus dans ce modèle, tels les chaines de spin s avec anisotropie à la racine de l'unité, le modèle de sine-Gordon sur réseau à la racine de l'unité ou encore le modèle de Potts chiral.

En implémentant la séparation quantique des variables pour ces représentations avec les bords intégrables les plus généraux, nous caractérisons de deux façons différentes le spectre de la matrice de transfert. La première caractérisation fait intervenir les solutions d'un système discret d'équations polynomiales dans une certaine classe de fonctions. La seconde caractérisation, équivalente, fait quant à elle intervenir les solutions d'une équation de Baxter fonctionnelle de type TQ. Cela permet de faire le lien dans certains cas particuliers avec la méthode de l'anstaz de Bethe algébrique, qui ne permet pas d'étudier ces modèles en toute généralité.

Les conditions aux bords sont décrites par une matrice $K_-(\lambda)$ à gauche et $K_+(\lambda)$ à droite. Afin d'être plus compréhensible, nous présentons notre méthode en deux étapes. Dans un premier temps, nous considérons une matrice $K_-(\lambda)$ générale et une matrice $K_+(\lambda)$ triangulaire. Dans ce cas, la base séparée est obtenue comme la base de diagonalisation de l'opérateur $\mathcal{B}(\lambda)$ (une des entrées de la matrice de monodromie avec bords). Ensuite, afin de décrire des bords intégrables complètement généraux, nous avons introduit une transformation de jauge, principalement en adaptant pour notre situation, une méthode introduite par Baxter. En choisissant astucieusement les paramètres de jauge, nous pouvons ramener ce problème général à l'étude d'une matrice de bords (jaugée) triangulaire, nous ramenant ainsi au cas traité précédemment. Dans ce cas, la base séparée est obtenue comme la base de (pseudo-)diagonalisation de l'opérateur $\mathcal{B}(\lambda)$ jaugé, qui est une combinaison linéaire de toutes les entrées de la matrice de monodromie avec bords.

Les résultats décrit dans ce chapitre ont donnés lieu aux deux publications **I** et **II**.

Chapitre cinq : Hamiltoniens locaux associés aux représentations cycliques de l'algèbre de réflexion

Nous résolvons le problème de définir des Hamiltoniens locaux avec des bords intégrables qui commutent avec la matrice de transfert avec bords associée à des représentations cycliques de l'algèbre de réflexion à 6-vertex. Ce résultat permet de modéliser les interactions à un niveau microscopique, et d'ainsi comprendre la physique du système étudié.

Nous suivons les étapes suivantes afin de déterminer une expression pour les Hamiltoniens. Tout d'abord, nous considérons la matrice R fondamentale, solution de l'équation de Yang-Baxter qui entremêle deux opérateurs de Lax de Bazhanov-Stroganov agissant sur deux espaces quantiques différents. Nous définissons ensuite une équation de réflexion mélangée, entremêlant ces opérateurs de Lax et deux matrices scalaires de bords, définies respectivement sur l'espace auxiliaire (de dimension 2) et l'espace quantique (de dimension p). En résolvant explicitement cette équation mélangée, nous trouvons des solutions scalaires diagonales pour les matrices de bords sur l'espace quantique, associées aux matrices scalaires diagonales connues sur l'espace auxiliaire (les matrices $K_{-}(\lambda)$ et $K_{+}(\lambda)$ du chapitre quatre, mais choisies ici diagonales).

Nous pouvons alors définir une famille à plusieurs paramètres (spectraux) de matrices de transfert fondamentales, qui commutent entre elles et avec la matrice de transfert avec bords associée à l'algèbre de réflexion cyclique originelle. De plus, nous prouvons que ces matrices de transfert fondamentales génèrent des familles à plusieurs paramètres de Hamiltoniens locaux avec bords intégrables. Il est utile de mentionner que ces Hamiltoniens locaux sont obtenus comme des dérivées secondes, par rapport aux paramètres spectraux, des matrices de transfert fondamentales.

Le chapitre finit en explicitant les Hamiltoniens locaux pour quelques modèles d'intérêt, pour des espaces quantiques locaux de dimension 3 : la chaine XXZ de spin 1 à la racine de l'unité, le modèle de Potts chiral super intégrable, le modèle de sine-Gordon à la racine de l'unité et une généralisation de celui-ci. Ils sont tous obtenus à partir de l'opérateur de Lax de Bazhanov-Stroganov, avec un choix approprié des paramètres.

Les résultats décrit dans ce chapitre ont donnés lieu à la publication **III**.

La conclusion de la thèse énonce des directions dans lesquelles ces travaux peuvent être poursuivis et utilisés. Par exemple, le rôle de la matrice R fondamentale dans les équations de réflexions pour des représentations cycliques doit être clarifié. Une autre direction concerne l'étude des équations de Baxter de type TQ dans le contexte de bords intégrables généraux, et en particulier la compréhension du lien entre la matrice de transfert fondamentale et l'opérateur Q. Une autre façon encore de continuer ces travaux est de s'attaquer au problème de la dynamique en utilisant la séparation quantique des variables.

Même s'il reste encore beaucoup à faire, par exemple en ce qui concerne la caractérisation de l'état fondamental, ces techniques devraient permettre de calculer le paramètre d'ordre du modèle de Potts chiral d'une façon assez directe.

Thesis abstract

The main theoretical tools to understand the macroscopic behaviour of quantum systems from their microscopic description are the determination of their Hamiltonian spectrum and the computation of their correlation functions. This thesis takes place in the development of such a research program to study quantum integrable models with general integrable boundary conditions, the long-range goal being to be able to exactly describe out of equilibrium physics.

More specifically, we have analysed the class of integrable quantum models on the lattice associated to cyclic representations of the 6-vertex reflection algebra, including as particular cases the lattice sine-Gordon model at root of unity and the chiral Potts model with general integrable boundaries.

A large part of the work has been devoted to the development of the quantum separation of variables method to solve the spectral problem for these models with general integrable boundary conditions, by generalising the Baxter's gauge transformations to these cyclic reflection algebras.

We have completely characterised the transfer matrix spectrum (both eigenvalues and eigenstates) in terms of the set of solutions to a discrete system of polynomial equations and equivalently as the set of solutions, in a given class of functions, to a Baxter like functional equation. This last point allows in particular cases to make a link with the Algebraic Bethe Ansatz approach, which in general, cannot be used for the study of these models.

We have then constructed families of new local Hamiltonians with integrable boundaries commuting with the above transfer matrix. To that end, we have defined a hierarchy of new mixed reflection equations, involving different representations of the 6-vertex algebra and using, among others, the fundamental R-matrix.

keywords: Yang-Baxter algebra, 6-vertex reflection algebra, cyclic representations, quantum separation of variables, local Hamiltonians

Résumé de thèse

Les principaux outils pour la compréhension du comportement macroscopique de systèmes quantiques à partir de leur description microscopique sont la détermination du spectre du Hamiltonien associé et le calcul des fonctions de corrélation. Cette thèse se place dans le cadre du développement d'un tel programme de recherche afin d'étudier des systèmes intégrables quantiques avec des conditions aux bords intégrables générales, le but à long terme étant la description exacte d'une physique quantique hors équilibre.

Plus spécifiquement, nous avons analysé la classe des systèmes intégrables quantiques sur réseau associés aux représentations cycliques de l'algèbre de réflexion à 6-vertex, avec comme exemples les modèles de sine Gordon et de Potts chiral avec conditions aux bords intégrables.

Une large partie du travail a été consacrée au développement de la méthode de séparation quantique des variables pour résoudre le problème spectral de la matrice de transfert de ces modèles avec conditions de bords intégrables les plus générales, en étendant l'idée des transformations de jauge de Baxter à ces algèbres de réflexion.

Nous avons caractérisé complètement le spectre de la matrice de transfert (valeurs propres et vecteurs propres) en termes des solutions d'un système discret d'équations polynomiales et d'une façon équivalente en termes des solutions, dans une certaine classe de fonctions, d'une équation de type Baxter fonctionnelle. Cela permet de faire le lien dans certains cas particuliers avec la méthode de l'anstaz de Bethe algébrique qui ne permet pas d'étudier ces modèles en toute généralité.

Nous avons ensuite construit des familles de nouveaux Hamiltoniens locaux avec conditions aux bords intégrables qui commutent avec la matrice de transfert. Pour ce faire nous avons défini une hiérarchie de nouvelles équations de réflexion mélangeant différentes représentations de l'algèbre quantique à 6-vertex et utilisant, entre autres, la matrice R fondamentale cyclique.

mots-clés : algèbre de Yang-Baxter, algèbre de réflexion à 6-vertex, représentations cycliques, séparation quantique des variables, Hamiltoniens locaux