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Université de Paris 1 Panthéon-Sorbonne

Ecole d'économie de Paris-PSE

THESE

pour obtenir le grade de Docteur

de l'Université de Paris 1 Panthéon-Sorbonne

et de l'Ecole d'économie de Paris-PSE

Discipline: Mathématiques Appliquées

présentée par

Mina MOSTOUFI

le 17 Décembre 2015

**ELEMENTS DE THÉORIE DU RISQUE EN
FINANCE ET ASSURANCE**

Jury

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To my beloved parents & Reza

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Résumé

Cette thèse traite de la théorie du risque en finance et en assurance. La mise en pratique du concept de comonotonie, la dépendance du risque au sens fort, est décrite pour identifier l'optimum de Pareto et les allocations individuellement rationnelles Pareto optimales, la tarification des options et la quantification des risques. De plus, il est démontré que l'aversion au risque monotone à gauche, un raffinement pertinent de l'aversion forte au risque, caractérise tout décideur à la Yaari, pour qui, l'assurance avec franchise est optimale.

Le concept de comonotonie est introduit et discuté dans le chapitre 1. Dans le cas de risques multiples, on adopte l'idée qu'une forme naturelle pour les compagnies d'assurance de partager les risques est la Pareto optimalité risque par risque. De plus, l'optimum de Pareto et les allocations individuelles Pareto optimales sont caractérisées.

Le chapitre 2 étudie l'application du concept de comonotonie dans la tarification des options et la quantification des risques. Une nouvelle variable de contrôle de la méthode de Monte Carlo est introduite et appliquée aux "basket options", aux options asiatiques et à la TVaR. Finalement dans le chapitre 3, l'aversion au risque au sens fort est raffiné par l'introduction de l'aversion au risque monotone à gauche qui caractérise l'optimalité de l'assurance avec franchise dans le modèle de Yaari. De plus, il est montré que le calcul de la franchise s'effectue aisément.

Summary

This thesis deals with the risk theory in Finance and Insurance. Application of the Comonotonicity concept, the strongest risk dependence, is described for identifying the Pareto optima and Individually Rational Pareto optima allocations, option pricing and quantification of risk. Furthermore it is shown that the left monotone risk aversion, a meaningful refinement of strong risk aversion, characterizes Yaari's decision makers for whom deductible insurance is optimal.

The concept of Comonotonicity is introduced and discussed in Chapter 1. In case of multiple risks, the idea that a natural way for insurance companies to optimally share risks is risk by risk Pareto-optimality is adopted. Moreover, the Pareto optimal and individually Pareto optimal allocations are characterized.

The Chapter 2 investigates the application of the Comonotonicity concept in option pricing and quantification of risk. A novel control variate Monte Carlo method is introduced and its application is explained for basket options, Asian options and TVaR.

Finally in Chapter 3 the strong risk aversion is refined by introducing the left-monotone risk aversion which characterizes the optimality of deductible insurance within the Yaari's model. More importantly, it is shown that the computation of the deductible is tractable.

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Introduction

0.1 General Introduction

This thesis deals with the risk theory in Finance and Insurance. Application of the Comonotonicity concept, the strongest risk dependence, is described for identifying the Pareto optima and Individually Rational Pareto optima allocations, option pricing and quantification of risk. Furthermore it is shown that the left monotone risk aversion, a meaningful refinement of strong risk aversion, characterizes Yaari's decision makers for whom deductible insurance is optimal.

The concept of comonotonicity is introduced and discussed in Chapter 1. Moreover, the Pareto optimal and individually Pareto optimal allocations are characterized. Chapter 2 investigates the application of the Comonotonicity concept in option pricing and quantification of risk. In Chapter 3 the strong risk aversion is refined by introducing the left-monotone risk aversion which characterizes the optimality of deductible insurance within the Yaari's model.

0.2 Chapter 1: Multivariate risk sharing and the derivation of individually rational Pareto optima

In a seminal paper, in case of strict strong risk averters assumed to be expected utility decision makers, Borch (1962) characterized Pareto optimal risk sharing. The optimal sharing rule which depends on the specification of the utilities is based on a Mutuality Principle for risks which are fully diversifiable, furthermore Borch (1962) derived the precise conditions of the optimal allocations, which allow to compute the sharing of the Macroeconomic Risk (See for instance chapter 10 of Eeckhoudt et al. (2005) for more details).

It turns out that for expected utility decision makers with strictly increasing and strictly concave utility functions, Pareto optima are necessarily strictly comonotone i.e. strictly increasing functions of the aggregate endowments, but the converse is false.

As noticed by Landsberger and Meilijson (1994), the specific utilities of agents are hardly even known in practice, moreover let us add that the model which is used by an agent is

hardly even known as well. Consequently, Landsberger and Meilijson (1994) only assumed that agents are strictly strong risk averters in the sense of strict second order dominance.

They obtained the nice result, that for such agents Pareto optimal allocations coincide exactly with the set of comonotone allocations i.e. the set of allocations which are non decreasing functions of the aggregate endowments. Landsberger and Meilijson (1994) gave a proof of the previous result and an algorithm allowing to reach at least one Pareto optimum, while they did not offer a method for computing all Pareto optima.

The main novelty provided by this work is to offer a complete characterization of Pareto optima, by extensively taking advantage of the polytope structure of these Pareto optima. Furthermore, it is shown that this strategy also allows to easily describe the entire convex set of individually rational Pareto optima—those for which every individual is better off when comparing with the initial situations—which clearly are those of practical interest in real life.

0.3 Chapter 2: Comonotonic Monte Carlo and its applications in option pricing and quantification of risk

Monte Carlo (MC) simulation is a well known technique in different domains of mathematics such as mathematical finance, see Glasserman (2003); Benninga (2014). A typical application of the Monte Carlo method in finance is the estimation of the no-arbitrage price of a specific derivative security (e.g. a call option), which can be expressed as the expected value of its discounted payoff under the risk neutral measure. Another application of the Monte Carlo method in finance is estimating risk measures, such as Tail Value-at-Risk.

The main shortcoming of the Monte Carlo method is its high computational cost. The standard error of the crude Monte Carlo estimate is of order $O(\frac{1}{\sqrt{n}})$ and thus, to double the precision, one must run four times the number of simulations. Alternatively, strategies for reducing σ should be considered.

Several variance reduction techniques can be used in companion with the Monte Carlo method, such as antithetic variables, control variates and importance sampling. A detailed survey of these techniques is given in Ripley (1987). In chapter three we focus on the well-known control variate method for variance reduction which is based on the comonotonicity concept.

0.4 Chapter 3: Optimality of deductible for Yaari's model: a reappraisal

In the framework of EU model, Arrow (1965) proved that for a given premium, the optimal insurance contract for a EU risk averse decision maker is a contract with deductible. Gollier and Schlesinger (1996) obtained a nice generalization of this result by proving that this result holds also under strong aversion, whatever be the decision maker's decision model under risk.

Vergnaud (1997) refined this result by proving that for any left monotone risk averse decision maker (not necessarily strongly risk averse), whatever be the decision model under risk, the optimal contract for a given premium is a deductible policy.

This last result is important since strong risk aversion is disputable in some situations, while Jewitt (1989)'s refinement i.e. left monotone risk aversion appears to be better adapted to insurance. This adds further justification to RDEU (rank-dependent expected utility) models and in particular to Yaari (1987)'s model that allow the decision maker to be left monotone risk averse without being strongly risk averse, which is impossible in the EU model, see Chateauneuf et al. (2004). In this chapter the optimality of deductible in the framework of Yaari's model is revisited.

0.5 Summary of Results

0.5.1 Multivariate risk sharing and the derivation of individually rational Pareto optima

In case of multiple risks, we did adopt the idea that a natural way for insurance companies to optimally share risks is risk by risk Pareto-optimality. Our framework is based upon the well-known results in the one dimensional case characterizing Pareto-optimality as comonotonicity in case of strong risk aversion. A simple computable method is offered for deriving all Pareto-optima and deriving all Individually Rational Pareto-optima.

Definitions and Preliminary Results

The definitions, lemmata and theorems which exploited to obtain the results of chapter 1 are given as follows.

Definition 0.5.1. $X = (X_1, \dots, X_i, \dots, X_n) \in (\mathbb{R}^m)^{p \times n}$ is Pareto optimal if $\forall k \in \llbracket 1, p \rrbracket$ $(X_1^k, \dots, X_i^k, \dots, X_n^k)$ is Pareto optimal in the usual sense for the univariate case with respect to the second order stochastic dominance i.e. for k given, $(X_1^k, \dots, X_i^k, \dots, X_n^k)$ is a

feasible allocation:

$X_i^k \in \mathbb{R}_+^m \quad \forall i, \quad \sum_{i=1}^n X_i^k = w^k$ and there does not exist $Y = (Y_1^k, \dots, Y_i^k, \dots, Y_n^k)$,
 $Y_i^k \in \mathbb{R}_+^m \quad \forall i, \quad \sum_{i=1}^n Y_i^k = w^k$, such that $Y_i^k \succeq_{SSD} X_i^k \quad \forall i$ and $Y_{i_0}^k \succ_{SSD} X_{i_0}^k$ for some i_0 .

Definition 0.5.2. $X = (X_1, \dots, X_i, \dots, X_n) \in (\mathbb{R}^m)^{p \times n}$ is an individually rational Pareto optimum if X is Pareto optimal and individually rational i.e. $\forall i, k \quad X_i^k \succeq_{SSD} w_i^k$.

Definition 0.5.3. An allocation $X = (X_1, \dots, X_i, \dots, X_n)$ is comonotone if,
 $\forall (i, i') \in \llbracket 1, n \rrbracket^2 \quad (X_i(s) - X_i(t)) (X_{i'}(s) - X_{i'}(t)) \geq 0 \quad \forall (s, t) \in S^2$.

Theorem 0.5.4. The set of Pareto optimal allocations coincide with the set of comonotone allocations.

Lemma 0.5.5. Any Pareto optimal allocation is comonotone.

Lemma 0.5.6. Any comonotone allocation is Pareto optimal.

Lemma 0.5.7. Let $w(s_j) = w_j$. Then after possibly relabeling, if needed, the indices in such a way that $w_1 \leq \dots \leq w_j \leq \dots \leq w_m$, one gets: If $(X_i)_{i=1, \dots, n}$ is a feasible allocation, then the two following properties are equivalent;

- (i) $(X_i)_{i=1, \dots, n}$ is comonotone.
- (ii) $X_i(1) \leq \dots \leq X_i(j) \leq \dots \leq X_i(m) \quad \forall i = 1, \dots, n$.

Remark 0.5.8. It is worth noticing that building upon Lemma 0.5.7, Theorem 0.5.4 can be restated, in accordance with some economic terminology (For instance see Eeckhoudt et al. (2005)), see Theorem 0.5.9 below.

Theorem 0.5.9. For an allocation $X = (X_1, \dots, X_i, \dots, X_n) \in \mathbb{R}^{m \times n}$ the two following assertions are equivalent:

- (i) X is Pareto optimal
- (ii) 1. X is feasible

2. *Mutuality Principle:*

$$\forall s, t \in S \quad w(s) = w(t) \implies X_i(s) = X_i(t) \quad \forall i = 1, \dots, n$$

3. *Weak comonotonicity:*

$$\forall s, t \in S \quad w(s) < w(t) \implies X_i(s) \leq X_i(t) \quad \forall i = 1, \dots, n$$

Remark 0.5.10. *Let us recall again that for strict strong risk averters who are EU (expected utility decision makers) Pareto optima satisfy (ii).1, (ii).2 but (ii).3 should be replaced by strict comonotonicity (see Borch (1962) or also Eeckhoudt et al. (2005)) i.e. $\forall s, t \in S$ $w(s) < w(t) \implies X_i(s) < X_i(t) \quad \forall i = 1, \dots, n$.*

Theorem 0.5.11. *The set of Pareto optima is a polytope, hence it is the convex hull of its finitely many extreme points.*

Lemma 0.5.12. *Any individually rational Pareto optimum (IRPO) X_i is such that $E(X_i) = E(w_i)$.*

Lemma 0.5.13. *The set \mathcal{P}_{IR} of individually rational Pareto optima is nonempty.*

Remark 0.5.14. *Note that in the present paper, we intend to systematically derive all IRPO's at least for rational probabilities (which apparently in "real life" is not a severe limitation). Our result contrasts from the algorithms which can be found in the literature. Actually these algorithms propose a method to obtain only one IRPO (see e.g. Landsberger and Meilijson (1994) or Ludkovski and Rüschendorf (2008)), but not all IRPO's.*

Remark 0.5.15. *Note that even for a finite state space S , it is not easy to express the individually rational conditions $X_i \succeq_{SSD} w_i, i = 1, \dots, n$. Actually $X_i \succeq_{SSD} w_i$ is equivalent to*

$$\int_0^p F_{X_i}^{-1}(t) dt \geq \int_0^p F_{w_i}^{-1}(t) dt \quad \forall p \in (0, 1) \tag{0.1}$$

with equality if $p=1$, as noticed in Lemma 0.5.12, but even if (0.1) has to be checked only for a finite number $p_\ell \in (0, 1)$, in practice finding which p_ℓ must be chosen is a delicate task. In contrast, if each p_j is a rational probability, let us say of the type $p_j = \frac{k_j}{q}$ where $k_j, q \in \mathbb{N}_+^$, it is immediate that $X_i \succeq_{SSD} w_i$ iff $\int_0^{\frac{k}{q}} F_{X_i}^{-1}(t) dt \geq \int_0^{\frac{k}{q}} F_{w_i}^{-1}(t) dt \quad \forall k \in \llbracket 1, q \rrbracket$.*

Theorem 0.5.16. *The set \mathcal{P}_{IR} of individually rational Pareto optima is a polytope, hence the convex hull of its finitely many extreme points.*

Theorem 0.5.17. *Assume that the initial endowment of agent i is $w_i \in \mathbb{R}^m$ and define the IRPO's as the allocations $X = (X_1, \dots, X_i, \dots, X_n) \in \mathbb{R}^{m \times n}$ such that $\sum_i X_i = \sum_i w_i$ ($:= w$) and $X_i \succeq_{SSD} w_i \quad \forall i$, then the set \mathcal{P}_{IR} is the polytope of the feasible allocations which are comonotone and satisfy the individually rational constraints.*

0.5.2 Comonotonic Monte Carlo and its applications in option pricing and quantification of risk

A novel control variate Monte Carlo method (CoMC) is presented based on the concept of comonotonicity. This method is explained for basket options, Asian options and TVaR. We evaluated the performance of the method in realistic cases by illustrative numerical examples. The realistic benchmark examples show that the precision of estimating the price of Asian and Basket options is drastically increased by employing the CoMC method while the computation time is not increased considerably compared to the crude Monte Carlo method.

Definitions and Preliminary Results

The definitions, lemmata and theorems which exploited to obtain the results of chapter 2 are given as follows.

Definition 0.5.18. *A random vector $\underline{X} = (X_1, \dots, X_n)$ is comonotonic if and only if it has a comonotonic copula i.e. for all $\underline{x} = (x_1, \dots, x_n)$, we have*

$$F_{\underline{X}}(\underline{x}) = \min \{F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_n}(x_n)\}. \quad (0.2)$$

Proposition 0.5.19. *If \underline{X} has a comonotonic copula then for $U \sim \text{Uniform}(0, 1)$, we have*

$$\underline{X} \stackrel{d}{=} (F_{X_1}^{-1}(U), (F_{X_2}^{-1}(U), \dots, (F_{X_n}^{-1}(U)). \quad (0.3)$$

Proposition 0.5.20. *The quantile function $F_{\mathbb{S}^c}^{-1}$ of a sum \mathbb{S}^c of comonotonic random variables with distribution functions F_{X_1}, \dots, F_{X_n} is additive*

$$F_{\mathbb{S}^c}^{-1}(p) = \sum_{i=1}^n F_{X_i}^{-1}(p), \quad 0 < p < 1. \quad (0.4)$$

Definition 0.5.21. *The distorted expectation of a random variable X is defined by*

$$\rho_g[X] = \int_{-\infty}^0 (g(\bar{F}_X(x)) - 1) dx + \int_0^{\infty} g(\bar{F}_X(x)) dx, \quad (0.5)$$

where $\bar{F}_X(x) = 1 - F_X(x)$ denotes the tail function of $F_X(x)$ and the function $g(\cdot)$ is a so-called distortion function, i.e. a non-decreasing function $g : [0, 1] \rightarrow [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$.

Proposition 0.5.22. *The distortion risk measure for a sum of comonotonic variables is additive i.e. for any distortion function g and all random variables X_i we have*

$$\rho_g[\mathbb{S}^c] = \sum_{i=1}^n \rho_g[X_i]. \quad (0.6)$$

Corollary 0.5.23. *The Tail Value-at-Risk, $TVaR_X(p)$, at level $p \in (0, 1)$ given by*

$$TVaR_X(p) = \frac{1}{1-p} \int_p^1 F_X^{-1}(q) dq \quad (0.7)$$

is a distortion risk measure with distortion function

$$g(x) = \min\left(\frac{x}{1-p}, 1\right), 0 \leq x \leq 1,$$

hence it is additive for comonotonic random variables.

Corollary 0.5.24. *The ESF (Expected shortfall) can be written as a linear combination of distortion risk measures given by*

$$TVaR_X(p) = F_X^{-1}(p) + \frac{1}{1-p} ESF_X(p),$$

see Dhaene et al. (2006), thus it follows

$$\begin{aligned} ESF_{\mathbb{S}^c}(p) &= (1-p)(TVaR_{\mathbb{S}^c}(p) - F_{\mathbb{S}^c}^{-1}(p)) \\ &= (1-p) \left(\sum_{i=1}^n TVaR_{X_i}(p) - \sum_{i=1}^n F_{X_i}^{-1}(p) \right) \\ &= \sum_{i=1}^n ESF_{X_i}(p), \quad 0 < p < 1. \end{aligned}$$

Corollary 0.5.25. *By choosing $p = F_{\mathbb{S}^c}(K)$ in Corollary 0.5.24, it follows that the stop-loss premium $E[(\mathbb{S}^c - K)_+]$ of a sum \mathbb{S}^c of comonotonic random variables with strictly increasing distribution functions F_{X_1}, \dots, F_{X_n} can be written as*

$$E[(\mathbb{S}^c - K)_+] = \sum_{i=1}^n [(X_i - F_{X_i}^{-1}(F_{\mathbb{S}^c}(K)))_+], \quad \forall K \in \mathbb{R}. \quad (0.8)$$

0.5.3 Optimality of deductible for Yaari's model: a reappraisal

The main purpose of this chapter is to show that, within the Yaari's model, left-monotone risk aversion does characterize the optimality of deductible insurance. Moreover, it is shown that for such left-monotone Yaari's risk averters, the computation of the deductible is very tractable.

Definitions and Preliminary Results

The definitions, Lemmata and theorems which exploited to obtain the results of chapter 3 are given as follows.

Definition 0.5.26. *For random variables X and Y with the same mean, Y is a left monotone increase in risk of X if $\int_{-\infty}^{F_Y^{-1}(p)} F_Y(p) \geq \int_{-\infty}^{F_X^{-1}(p)} F_X(p)$, $\forall p \in [0, 1]$. Let us recall that for any distribution F i.e. any mapping $F : \mathbb{R} \rightarrow \mathbb{R}$ non-decreasing, right-continuous such that $\lim_{t \rightarrow -\infty} F(t) = 0$, $\lim_{t \rightarrow +\infty} F(t) = 1$, $F^{-1} : [0, 1] \rightarrow \mathbb{R}$ is defined $\forall p \in [0, 1]$ by $F^{-1}(p) = \inf \{t \in \bar{\mathbb{R}}, F(t) \geq p\}$. Note that $F^{-1}(0) = -\infty$.*

Lemma 0.5.27. *For every pair (X, Y) of discrete random variables with $E(X) = E(Y)$ such that Y is a left monotone increase in risk of X , Y can be reached from X by a finite sequence of transfers as in (3.3).*

Definition 0.5.28. *Distribution G is a left-monotone simple spread of F if*

1. $E(G) = E(F)$

2. $\exists p_0 \in (0, 1)$ such that:

$$p \leq p_0 \implies (2.1) \quad G^{-1}(p) \leq F^{-1}(p)$$

$$(2.2) \quad d(p) = F^{-1}(p) - G^{-1}(p) \text{ is non-increasing on } (0, p_0]$$

$$p > p_0 \implies (2.3) \quad G^{-1}(p) \geq F^{-1}(p).$$

Lemma 0.5.29. *If G is a left-monotone simple spread of F then F is left-monotone less risky than G .*

Lemma 0.5.30. *Any Yaari decision maker is a left monotone increase in risk if and only if the probability transformation function is star shaped¹ at 1 i.e. $\frac{1-f(p)}{1-p}$ is an increasing function of p on $[0, 1)$.*

Theorem 0.5.31. *Any Yaari's decision maker who has preference for deductibles with any given premium is a left-monotone risk averse.*

Lemma 0.5.32. *Any decision maker who exhibits preference for deductible will prefer $\mathcal{L}(X) = (x_1, p_1; x_2, p_2; x_3, p_3; x_4, p_4)$ to $\mathcal{L}(Y) = (x_1 - \epsilon p_3, p_1; x_2, p_2; x_3 + \epsilon p_1, p_3; x_4, p_4)$, [Recall that through the definitions of the “ \mathcal{L} ”, one has $p_i \geq 0$, $\sum_{i=1}^4 p_i = 1$ and $x_1 < x_2 < x_3 < x_4$ and $x_1 - \epsilon p_3 < x_2 < x_3 + \epsilon p_1 < x_4$].*

Remark 0.5.33. *Note that if we had required that indemnities should satisfy the Moral Hazard requirement i.e. that what remains to be paid by the decision maker namely $D - I(D)$ should increase with the amount of the damage our Lemma 0.5.32 would remain valid. Actually: $d_4 - I(d_4) = 0 < d_3 - I(d_3) = x_4 - x_3 - \epsilon p_1 < d_2 - I(d_2) = x_4 - x_2 < d_1 - I(d_1) = x_4 - x_1 + \epsilon p_3$.*

Remark 0.5.34. *The proof of Theorem 0.5.31 shows that it is enough that a Yaari's decision maker has preference for deductible only in case of finite discrete random losses, in order to be a left-monotone risk averter.*

Theorem 0.5.35. *(Vergnaud (1997)) Any left-monotone risk-averse decision maker has preference for deductible.*

Theorem 0.5.36. *A strict left monotone risk averse Yaari decision maker will purchase full insurance if*

$$(1 + m)(1 - F(0)) - (1 - f(F(0))) < 0 \tag{0.9}$$

¹A function $f \in F$ is star-shaped at m , if:
 $\frac{f(m) - f(p)}{m - p}$
 is an increasing function of p on $[0, m) \cup (m, 1]$.

Otherwise, \bar{d} is an optimal level of deductible if and only if it satisfies

$$(1+m)(1-F(\bar{d}_-)) - (1-f(F(\bar{d}_-))) \geq 0 \geq (1+m)(1-F(\bar{d})) - (1-f(F(\bar{d}))). \quad (0.10)$$

Remark 0.5.37. If F is continuous, indeed the inequality 0.10 in theorem 0.5.36 reduces to the following simple equation:

$$(1+m)(1-F(\bar{d})) - (1-f(F(\bar{d}))) = 0$$

Lemma 0.5.38. Let $u : [a, b] \rightarrow \mathbb{R}$ be continuous and such that $u'_+(\cdot)$ exists on (a, b) with $u'_+(x) \leq 0 \forall x \in (a, b)$ then u is non-increasing on $[a, b]$.

Lemma 0.5.39. Let $u : [a, b] \rightarrow \mathbb{R}$ be continuous and such that $u'_+(\cdot)$ exists and strictly negative on $J = (a, b)$ where $J \neq \emptyset$ then u is strictly decreasing on $[a, b]$.

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Chapter 1

Multivariate risk sharing and the derivation of individually rational Pareto optima¹

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Abstract

Considering that a natural way of sharing risks in insurance companies is to require risk by risk Pareto optimality, we offer in case of strong risk aversion, a simple computable method for deriving all Pareto optima. More importantly all Individually Rational Pareto optima can be computed according to our method.

Keywords : Multivariate risk sharing, Comonotonicity, Individually rational Pareto optima.

JEL classification: D70, D81.

1.1 Introduction

In a seminal paper, in case of strict strong risk averters assumed to be expected utility decision makers, Borch (1962) characterized Pareto optimal risk sharing. The optimal sharing rule which depends on the specification of the utilities is based on a Mutuality Principle for risks which are fully diversifiable, furthermore Borch (1962) derived the precise conditions of the optimal allocations, which allow to compute the

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sharing of the Macroeconomic Risk (See for instance chapter 10 of Eeckhoudt et al. (2005) for more details).

It turns out that for expected utility decision makers with strictly increasing and strictly concave utility functions, Pareto optima are necessarily strictly comonotone i.e. strictly increasing functions of the aggregate endowments, but the converse is false.

As noticed by Landsberger and Meilijson (1994), the specific utilities of agents are hardly even known in practice, moreover let us add that the model which is used by an agent is hardly even known as well. Consequently, Landsberger and Meilijson (1994) only assumed that agents are strictly strong risk averters in the sense of strict second order dominance.

They obtained the nice result, that for such agents Pareto optimal allocations coincide exactly with the set of comonotone allocations i.e. the set of allocations which are non decreasing functions of the aggregate endowments. Landsberger and Meilijson (1994) gave a proof of the previous result and an algorithm allowing to reach at least one Pareto optimum, while they did not offer a method for computing all Pareto optima. The main novelty provided by this work is to offer a complete characterization of Pareto optima, by extensively taking advantage of the polytope structure of these Pareto optima. Furthermore, it is shown that this strategy also allows to easily describe the entire convex set of individually rational Pareto optima—those for which every individual is better off when comparing with the initial situations—which clearly are those of practical interest in real life.

This is performed under the mild assumption that the underlying probability information (we just consider a finite set of states of nature) consists of rational probabilities. This is not a too restrictive assumption since any probabilistic information can indeed be approximated as far as needed by such rational probabilities.

As a dividend in case of multidimensional risk sharing if multidimensional risk aversion, is defined as strict strong risk aversion component by component, which would prove to be meaningful in case of extreme caution, then multidimensional Pareto optima, reduce to one dimensional Pareto optima component by component and therefore can be easily computed through our proposed method.

Indeed, in this way we avoid using a generalized comonotone dominance principle, which is in accordance with the multidimensional second order stochastic dominance, as this is developed by Carlier et al. (2012) in order to obtain other types of Pareto optima, which apparently might be difficult to derive analytically.

The paper is organized as follows. Section 1.2 presents the general framework of multidimensional risk sharing, recalls some definitions and shows how the problem reduces to one dimensional Pareto optima. Section 1.3 deals with the characterization of Pareto optimal risk sharing, while section 1.4 offers a description of individually rational Pareto optima.

As an application, we derive all individually rational Pareto optima linked with the insurance problem examined by Landsberger and Meilijson (1994). This example illustrates how *IRPO*'s (individually Rational Pareto optimal risk sharings) allow reducing risks which are not initially covered by the insurance policy. Finally, section 1.5 discusses the obtained results and concludes the paper.

1.2 Framework and Definitions

Consider, for the purpose of illustration, n insurance companies, $i = 1, \dots, n$, each holds at date zero, p portfolios of insurance of type $k = 1, \dots, p$ leading at date one to future stochastic wealth $X_i^k : (S, 2^S, P) \rightarrow \mathbb{R}_+$, where $S = (s_1, \dots, s_j, \dots, s_m)$ is the finite space of the sets of nature, and P the probability on 2^S is given and satisfies $P(s_j) = p_j > 0 \quad \forall j$.

Let $w_i = (w_i^1, \dots, w_i^k, \dots, w_i^p)$ be the initial endowment of insurance i with respect to each portfolio of type k , i.e. each future wealth in each state with respect to premia and reimbursements related to type k . Denote $w^k = \sum_{i=1}^n w_i^k$.

By definition, X is a feasible allocation if $X = (X_1, \dots, X_i, \dots, X_n)$ with $X_i \in (\mathbb{R}_+^m)^p \forall i = 1, \dots, n$ and $\sum_{i=1}^n X_i^k = w^k \quad \forall k \in \llbracket 1, p \rrbracket$.

Let us now recall that if X and Y are bounded real random variables, X dominates Y by the second order stochastic dominance i.e. X is considered as less risky than Y denoted by $X \succeq_{SSD} Y$ if $\int_0^p F_X^{-1}(t) dt \geq \int_0^p F_Y^{-1}(t) dt \quad \forall p \in \llbracket 0, 1 \rrbracket$ where F^{-1} is the usual quantile function.

Moreover $X \succ_{SSD} Y$ i.e. X strictly dominates Y for the second order stochastic dominance if furthermore $\int_0^{p_0} F_X^{-1}(t) dt > \int_0^{p_0} F_Y^{-1}(t) dt$ for some $p_0 \in (0, 1]$.

We assume that each agent i has preferences \succeq_i associated with the component by component second order stochastic dominance that is for $X_i = (X_i^1, \dots, X_i^k, \dots, X_i^p) \in (\mathbb{R}_+^m)^p$ and $Y_i = (Y_i^1, \dots, Y_i^k, \dots, Y_i^p) \in (\mathbb{R}_+^m)^p$ then if $X_i^k \succeq_{SSD} Y_i^k \quad \forall k \in \llbracket 1, p \rrbracket$ one has X_i weakly preferred to Y_i i.e. $X_i \succeq_i Y_i$ and if furthermore there exist $k_0 \in \llbracket 1, p \rrbracket$ such

that $X_i^{k_0} \succ_{SSD} Y_i^{k_0}$ then X_i is strictly preferred to Y_i i.e. $X_i \succ_i Y_i$.

From the above assumptions it turns out that:

Definition 1.2.1. $X = (X_1, \dots, X_i, \dots, X_n) \in (\mathbb{R}^m)^{p \times n}$ is Pareto optimal if $\forall k \in \llbracket 1, p \rrbracket$ $(X_1^k, \dots, X_i^k, \dots, X_n^k)$ is Pareto optimal in the usual sense for the univariate case with respect to the second order stochastic dominance i.e. for k given, $(X_1^k, \dots, X_i^k, \dots, X_n^k)$ is a feasible allocation:

$X_i^k \in \mathbb{R}_+^m \forall i, \sum_{i=1}^n X_i^k = w^k$ and there does not exist $Y = (Y_1^k, \dots, Y_i^k, \dots, Y_n^k), Y_i^k \in \mathbb{R}_+^m \forall i, \sum_{i=1}^n Y_i^k = w^k$, such that $Y_i^k \succeq_{SSD} X_i^k \forall i$ and $Y_{i_0}^k \succ_{SSD} X_{i_0}^k$ for some i_0 .

Definition 1.2.2. $X = (X_1, \dots, X_i, \dots, X_n) \in (\mathbb{R}^m)^{p \times n}$ is an individually rational Pareto optimum if X is Pareto optimal and individually rational i.e. $\forall i, k X_i^k \succeq_{SSD} w_i^k$.

1.3 Deriving all Pareto optima

From Definition 1.2.1 it turns out that the p -dimensional case reduces to p one dimensional situations. So we just have to deal with the following situation:

$X = (X_1, \dots, X_i, \dots, X_n) X_i : (S, 2^S, P) \rightarrow \mathbb{R}_+, w \in \mathbb{R}_+^m$ given.

In subsection 2.3.1 for the sake of completeness we just propose what we hope to be a very simple, direct and complete proof of the well-known characterization of Pareto optimal allocations in terms of comonotonicity.

1.3.1 Pareto optima in the one-dimensional case

Definition 1.3.1. An allocation $X = (X_1, \dots, X_i, \dots, X_n)$ is comonotone if, $\forall (i, i') \in \llbracket 1, n \rrbracket^2 (X_i(s) - X_i(t))(X_{i'}(s) - X_{i'}(t)) \geq 0 \forall (s, t) \in S^2$.

We intend to retrieve, in a simple way, the well-known following theorem, which is implicit in Landsberger and Meilijson, see Landsberger and Meilijson (1994).

Theorem 1.3.2. The set of Pareto optimal allocations coincide with the set of comonotone allocations.

The proof will result from the following two lemmas.

Lemma 1.3.3. Any Pareto optimal allocation is comonotone.

Proof: We just sketch the proof given in Chateauneuf et al. (2000). It is enough to show that any non-comonotone allocation $X = (X_1, \dots, X_i, \dots, X_n)$ can be improved

to a new allocation $X' = (X'_1, \dots, X'_i, \dots, X'_n)$ which is mutually beneficial for all agents and strictly beneficial for at least one.

Let us assume, without loss of generality, that comonotonicity is not satisfied for X_1 , X_2 and for s_1 , s_2 . Let $X_1(s_1) = x_1$, $X_1(s_2) = x_2$, $X_2(s_1) = y_1$, $X_2(s_2) = y_2$ and assume without loss of generality that $x_1 + y_1 \leq x_2 + y_2$, $x_1 > x_2$ and $y_1 < y_2$.

Let us modify (x_1, x_2) to (x'_1, x'_2) and (y_1, y_2) to (y'_1, y'_2) where $x'_1 = x'_2 = \frac{p_1 x_1 + p_2 x_2}{p_1 + p_2}$, $y'_1 = x_1 + y_1 - x'_1$ and $y'_2 = x_2 + y_2 - x'_2$.

Therefore $X = (X_1, \dots, X_i, \dots, X_n)$ has been modified to, $X' = (X'_1, X'_2, X'_3, \dots, X'_n)$ where $X'_i = X_i \quad \forall i = 3, \dots, n$.

It is then straightforward to see that we obtain a new allocation X' and that X'_i is strictly less risky than X_i for $i = 1, 2$ since $E(u(X'_i)) > E(u(X_i))$ for any strictly concave utility function u , which completes the proof. \square

Lemma 1.3.4. *Any comonotone allocation is Pareto optimal.*

Proof: Let $X = (X_1, \dots, X_i, \dots, X_n)$ be a comonotone allocation. We just intend to show that it is impossible that a feasible allocation $Y = (Y_1, \dots, Y_i, \dots, Y_n)$ strictly dominates X . Without loss of generality, we assume that $Y_1 \succ_{SSD} X_1$ i.e. there exists $p_0 \in (0, 1]$ such that:

$$\int_0^{p_0} F_{Y_1}^{-1}(t) dt > \int_0^{p_0} F_{X_1}^{-1}(t) dt$$

and,

$$\int_0^p F_{Y_1}^{-1}(t) dt \geq \int_0^p F_{X_1}^{-1}(t) dt \text{ and } \forall p \in [0, 1].$$

Moreover $\int_0^p F_{Y_i}^{-1}(t) dt \geq \int_0^p F_{X_i}^{-1}(t) dt \quad \forall i \quad \forall p \in [0, 1]$.

Hence we get,

$$\sum_{i=1}^n \int_0^{p_0} F_{Y_i}^{-1}(t) dt > \sum_{i=1}^n \int_0^{p_0} F_{X_i}^{-1}(t) dt \quad (1.1)$$

Let us now show that,

$$\sum_{i=1}^n \int_0^{p_0} F_{Y_i}^{-1}(t) dt \leq \int_0^{p_0} F_{\sum_{i=1}^n Y_i}^{-1}(t) dt \quad (1.2)$$

Recall that $TVAR$ is sub-additive see Denuit and Dhaene (2012), i.e. for any random variable Z , $TVAR(Z, p) = \frac{1}{1-p} \int_p^1 F_Z^{-1}(t) dt$, where $p \in [0, 1)$ is such that for any

random variables T and Z one gets:

$$TVAR(Z+T,p) \leq TVAR(Z,p) + TVAR(T,p).$$

From $E(Z) = \int_0^1 F_Z^{-1}(t)dt$, $E(T) = \int_0^1 F_T^{-1}(t)dt$ and indeed $E(Z+T) = E(Z) + E(T)$ it is then straightforward to obtain:

$$\int_0^{p_0} F_{Z+T}^{-1}(t)dt \geq \int_0^{p_0} F_Z^{-1}(t)dt + \int_0^{p_0} F_T^{-1}(t)dt.$$

And therefore by induction one gets (1.2). Combining (1.1) and (1.2) we obtain:

$$\sum_{i=1}^n \int_0^{p_0} F_{X_i}^{-1}(t)dt < \int_0^{p_0} F_{\sum_{i=1}^n Y_i}^{-1}(t)dt \quad (1.3)$$

But $\sum_{i=1}^n X_i = w = \sum_{i=1}^n Y_i$ hence $F_w^{-1} = F_{\sum_{i=1}^n X_i}^{-1} = F_{\sum_{i=1}^n Y_i}^{-1}$, moreover since X is comonotone $F_{\sum_{i=1}^n X_i}^{-1} = \sum_{i=1}^n F_{X_i}^{-1}$ a.e. (almost everywhere) thus (1.3) implies: $\int_0^{p_0} F_w^{-1}(t)dt < \int_0^{p_0} F_w^{-1}(t)dt$ a contradiction, which completes the proof of lemma 1.3.4 and henceforth of Theorem 1.3.2. \square

1.3.2 Deriving all Pareto optima

We intend now to show that the set of Pareto optima is a polytope. Therefore by implementing the vertex identification algorithm as can be found in MATLAB (2010), one can easily obtain all Pareto optima. Let us start by a preliminary lemma.

Lemma 1.3.5. *Let $w(s_j) = w_j$. Then after possibly relabeling, if needed, the indices in such a way that $w_1 \leq \dots \leq w_j \leq \dots \leq w_m$, one gets: If $(X_i)_{i=1,\dots,n}$ is a feasible allocation, then the two following properties are equivalent;*

- (i) $(X_i)_{i=1,\dots,n}$ is comonotone.
- (ii) $X_i(1) \leq \dots \leq X_i(j) \leq \dots \leq X_i(m) \forall i = 1, \dots, n$.

Proof:

(i) \implies (ii): Let $w_j \leq w_{j+1}$ and assume there exists i_0 such that $X_{i_0}(j) > X_{i_0}(j+1)$. Since $(X_i)_{i=1,\dots,n}$ is comonotone, we have $X_i(j) \geq X_i(j+1) \forall i \neq i_0$. Summing up both sides of the inequality over $i = 1, \dots, n$ gives $\sum_{i=1}^n X_i(j) > \sum_{i=1}^n X_i(j+1)$.

Since $(X_i)_{i=1,\dots,n}$ is a feasible allocation, the following relation $w_j > w_{j+1}$ is obtained, which is a contradiction.

(ii) \implies (i): Take $j \neq j'$. From (ii) we have:

$X_i(1) \leq \dots \leq X_i(j) \leq \dots \leq X_i(m) \forall i = 1, \dots, n$, therefore it is immediate that $\forall i \neq i' (X_i(j) - X_i(j'))(X_{i'}(j) - X_{i'}(j')) \geq 0$, which means that the X_i 's are pairwise comonotone. \square

Remark 1.3.6. *It is worth noticing that building upon Lemma 1.3.5, Theorem 1.3.2 can be restated, in accordance with some economic terminology (For instance see Eeckhoudt et al. (2005)), see Theorem 1.3.7 below.*

Theorem 1.3.7. *For an allocation $X = (X_1, \dots, X_i, \dots, X_n) \in \mathbb{R}^{m \times n}$ the two following assertions are equivalent:*

(i) X is Pareto optimal

(ii) 1. X is feasible

2. Mutuality Principle:

$$\forall s, t \in S \quad w(s) = w(t) \implies X_i(s) = X_i(t) \quad \forall i = 1, \dots, n$$

3. Weak comonotonicity:

$$\forall s, t \in S \quad w(s) < w(t) \implies X_i(s) \leq X_i(t) \quad \forall i = 1, \dots, n$$

Proof:

(i) \implies (ii) is immediate since by definition, Pareto optima are feasible. It can be concluded from Theorem 1.3.2 that X is comonotone, therefore from lemma 1.3.5 we have:

$$w(s) \leq w(t) \text{ and } w(t) \leq w(s) \text{ implies } X_i(s) = X_i(t) \forall i.$$

$$\text{Also we have, } w(s) < w(t) \implies X_i(s) \leq X_i(t) \forall i.$$

(ii) \implies (i) Let $i, j \in \{1, \dots, n\}$ and $s, t \in S$ indeed if $X_i(s) = X_i(t)$ then

$$(X_i(s) - X_i(t))(X_j(s) - X_j(t)) \geq 0, \text{ and by symmetry if } X_i(s) < X_i(t) \text{ then } w(s) < w(t), \text{ therefore it can be concluded from (ii).3 that } X_j(s) \leq X_j(t) \text{ hence, } (X_i(s) - X_i(t))(X_j(s) - X_j(t)) \geq 0. \quad \square$$

Remark 1.3.8. *Let us recall again that for strict strong risk averters who are EU (expected utility decision makers) Pareto optima satisfy (ii).1, (ii).2 but (ii).3 should be replaced by strict comonotonicity (see Borch (1962) or also Eeckhoudt et al. (2005)) i.e. $\forall s, t \in S \quad w(s) < w(t) \implies X_i(s) < X_i(t) \quad \forall i = 1, \dots, n$.*

As mentioned in the introduction, for EU decision makers the converse is false, only some particular feasible allocations satisfying the Mutuality Principle, and strict comonotonicity are Pareto optima. Their derivation requires the use of first order conditions intimately linked with the *Von Neumann* utilities of the considered decision makers.

Theorem 1.3.9. *The set of Pareto optima is a polytope, hence it is the convex hull of its finitely many extreme points.*

Proof: Since from Theorem 1.3.2, Pareto optima are comonotone allocations, it is straightforward from lemma 1.3.5 to see that the set \mathcal{P} of Pareto optima is a nonempty compact convex subset K of \mathbb{R}^n , defined as the intersection of some finite collection of closed half-spaces (that is the set of solutions of finitely many linear inequalities). Hence from proposition 3.2.1 in Florenzano et al. (2001) we conclude that \mathcal{P} is a polytope, and from the well-known Krein-Milman theorem (see for instance, proposition 3.1.4 in Florenzano et al. (2001)) that \mathcal{P} is the convex hull of its extreme points. Since \mathcal{P} is a polytope, the number of these extreme points is finite. \square

1.3.3 Two illustrating examples

Here we present two examples to illustrate the procedure of the calculation of the extreme points; also we study the properties of the obtained vertices through these examples. In the first example, we chose the following values for the parameters introduced in the problem framework. We take $m = n = 2$ and the initial vector of endowment $w = (2, 4)$, which leads to the following constraints:

$$\mathcal{P} = \{(x_1, x_2), (y_1, y_2)\} \text{ such that: } \begin{cases} -x_1 \leq 0, & -x_2 \leq 0 \\ -y_1 \leq 0, & -y_2 \leq 0 \\ x_1 - x_2 \leq 0, & y_1 - y_2 \leq 0 \\ -x_1 - y_1 \leq -2, & -x_2 - y_2 \leq -4 \\ x_1 + y_1 \leq 2, & x_2 + y_2 \leq 4 \end{cases}$$

As it was discussed before, \mathcal{P} is a polytope. Furthermore, \mathcal{P} is the convex hull of the finite set of extreme points. For the sake of illustration, we obtain the extreme points of \mathcal{P} . First we construct the related system \mathcal{S} of linear equations with the four unknown quantities x_1 , x_2 , y_1 and y_2 :

$$\begin{aligned}
 \textcircled{1} \quad & x_1 + y_1 = 2 \\
 \textcircled{2} \quad & x_2 + y_2 = 4 \\
 \textcircled{3} \quad & x_1 = 0 \\
 \textcircled{4} \quad & x_2 = 0 \\
 \textcircled{5} \quad & y_1 = 0 \\
 \textcircled{6} \quad & y_2 = 0 \\
 \textcircled{7} \quad & x_1 - x_2 = 0 \\
 \textcircled{8} \quad & y_1 - y_2 = 0.
 \end{aligned}$$

Clearly any extreme point satisfies the endowment constraints (1) and (2). So building for instance upon Proposition 3.3.1 in Florenzano et al. (2001), we know that any extreme point is a solution of a subsystem of \mathcal{S} including equalities (1) and (2), which is of rank 4, and which satisfies, the corresponding remaining inequalities of \mathcal{P} . First we pick equations (1), (2), (3) and (4), which leads to $(x_1 = 0, y_1 = 2)$ and $(x_2 = 0, y_2 = 4)$. Now we check the obtained result with the remaining inequality constraints, which proves that $((x_1, x_2), (y_1, y_2))$ is situated inside the polytope, therefore it is a feasible solution and can be regarded as one of the vertices of the polytope. We add $(x_1 = 0, y_1 = 2)$ and $(x_2 = 0, y_2 = 4)$ to the set of vertices and continue the algorithm for the remaining set of equations to identify all the extreme points. In this way four extreme points are identified as below:

$$\{((x_1, x_2), (y_1, y_2))\} = \{((0, 0), (2, 4)), ((2, 4), (0, 0)), ((0, 2), (2, 2)), ((2, 2), (0, 2))\}.$$

As already mentioned in theorem 1.3.9, the set of Pareto optima can be defined as the convex hull of the identified extreme points. Hence, considering the extreme points obtained for this example, the set of Pareto optimal can be presented as below:

$$\begin{aligned}
 \{((x_1, x_2), (y_1, y_2))\} = \{ & (2\alpha_2 + 2\alpha_4, 4\alpha_2 + 2\alpha_3 + 2\alpha_4), \\
 & (2\alpha_1 + 2\alpha_3, 4\alpha_1 + 2\alpha_3 + 2\alpha_4)\}. \quad (1.4)
 \end{aligned}$$

In the second example we consider the parameters, $m = 3, n = 2$ and $w = (3, 5, 3)$.

The comonotonicity assumption requires reordering the initial endowment in an increasing order, which gives $w = (3, 3, 5)$. Now we can construct the system of equations and inequalities as described in the problem framework. The obtained feasible extreme points are as following:

$$\begin{aligned} \{(x_1, x_2, x_3), (y_1, y_2, y_3)\} = \{((0, 0, 0), (3, 3, 5)), ((3, 3, 5), (0, 0, 0)), \\ ((0, 0, 2), (3, 3, 3)), ((3, 3, 3), (0, 0, 2))\}. \end{aligned} \quad (1.5)$$

One important property of the obtained extreme points in this example and the previous one is the symmetry of results with respect to the replacing of x_i by y_i . This property is due to the structure of equalities, imposed by endowment constraints. Hence if (x_i, y_i) is an extreme point, (y_i, x_i) is also an extreme point of the polytope².

1.4 Deriving all individually rational Pareto optima

As for Pareto optima, it turns out from Definition 1.2.2 that the p -dimensional case reduces to p one dimensional cases. Let $w_i : (S, 2^S, P) \rightarrow \mathbb{R}_+$, $i=1, \dots, n$ be the initial endowments of the agents. We are looking for a Pareto optimum $X = (X_1, \dots, X_i, \dots, X_n)$ such that $X_i \succeq_{SSD} w_i$, $i = 1, \dots, n$.

Some preliminary lemmata:

Lemma 1.4.1. *Any individually rational Pareto optimum (IRPO) X_i is such that $E(X_i) = E(w_i)$.*

Proof: One has $\sum_{i=1}^n w_i = \sum_{i=1}^n X_i$ hence $\sum_{i=1}^m E(w_i) = \sum_{i=1}^m E(X_i)$ but $X_i \succeq_{SSD} w_i$ so $\int_0^1 F_{X_i}^{-1}(t)dt \geq \int_0^1 F_{w_i}^{-1}(t)dt$ i.e. $E(X_i) \geq E(w_i)$ which gives the result. \square

Lemma 1.4.2. *The set \mathcal{P}_{IR} of individually rational Pareto optima is nonempty.*

Proof: Landsberger and Meilijson (1994) nicely proved in their Proposition 1 (page 100) that every allocation is dominated by some comonotone allocation, therefore from Theorem 1.3.2 one can conclude that there exist at least one individually rational Pareto optimum. \square

²Indeed implementing the vertex identification algorithm in MATLAB(2010) would give the set of Pareto optima in any case.

Remark 1.4.3. *Note that in the present paper, we intend to systematically derive all IRPO's at least for rational probabilities (which apparently in "real life" is not a severe limitation). Our result contrasts from the algorithms which can be found in the literature. Actually these algorithms propose a method to obtain only one IRPO (see e.g. Landsberger and Meilijson (1994) or Ludkovski and Rüschenendorf (2008)), but not all IRPO's.*

Remark 1.4.4. *Note that even for a finite state space S , it is not easy to express the individually rational conditions $X_i \succeq_{SSD} w_i, i = 1, \dots, n$. Actually $X_i \succeq_{SSD} w_i$ is equivalent to*

$$\int_0^p F_{X_i}^{-1}(t)dt \geq \int_0^p F_{w_i}^{-1}(t)dt \quad \forall p \in (0, 1) \quad (1.6)$$

with equality if $p=1$, as noticed in Lemma 1.4.1, but even if (1.6) has to be checked only for a finite number $p_\ell \in (0, 1)$, in practice finding which p_ℓ must be chosen is a delicate task. In contrast, if each p_j is a rational probability, let us say of the type $p_j = \frac{k_j}{q}$ where $k_j, q \in \mathbb{N}_+^$, it is immediate that $X_i \succeq_{SSD} w_i$ iff*

$$\int_0^{\frac{k}{q}} F_{X_i}^{-1}(t)dt \geq \int_0^{\frac{k}{q}} F_{w_i}^{-1}(t)dt \quad \forall k \in \llbracket 1, q \rrbracket.$$

We then state Theorem 1.4.5, which is the main result of this paper.

Theorem 1.4.5. *The set \mathcal{P}_{IR} of individually rational Pareto optima is a polytope, hence the convex hull of its finitely many extreme points.*

Proof: The proof is similar as the one of theorem 1.3.9 except that we have now to take into account the new "closed half-spaces" constraints $E(X_i) = E(w_i) \forall i$ and $\int_0^{p_\ell} F_{X_i}^{-1}(t)dt \geq \int_0^{p_\ell} F_{w_i}^{-1}(t)dt \quad \forall i \forall p_\ell$. Note that IRPO's depend on the probability P while PO's are independent of probability P . \square

1.4.1 Two illustrating examples

Example 1

One can imagine that 1 and 2 are agricultural producers and that w_1, w_2 represents the possible production of tomatoes during one year depending on the climate conditions s_1 and s_2 . Note by the way that one could imagine that these agricultural producers produce also potatoes, so our definition of individually rational Pareto optima as

State	s_1	s_2
p	$\frac{1}{2}$	$\frac{1}{2}$
w_1	2	0
w_2	0	4
w	2	4

Table 1.1: Probability of the states for example 1.

Pareto optima separately with respect to tomatoes productions and potatoes productions would apparently make sense in such a situation.

Here we are looking for individually rational Pareto optima $X = (x_1, x_2)$ and $Y = (y_1, y_2)$. Clearly the individually rational Pareto optima (X, Y) are characterized by:

Comonotonicity condition: $x_1 \leq x_2$ and $y_1 \leq y_2$.

Dominance: $X \succeq_{SSD} w_1$ so $x_1 \geq 0$ and $E(X) = E(w_1)$ i.e. $x_1 + x_2 = 2$

$Y \succeq_{SSD} w_2$ so $y_1 \geq 0$ and $E(Y) = E(w_2)$ i.e. $y_1 + y_2 = 4$.

Feasibility: $x_1 \geq 0$ and $x_2 \geq 0$, $y_1 \geq 0$ and $y_2 \geq 0$

$$x_1 + y_1 = 2, \quad x_2 + y_2 = 4.$$

Hence direct computations give that the extreme *IRPO*'s are $((0, 2), (2, 2))$ and $((1, 1), (1, 3))$ so:

$$\mathcal{P}_{IR} = \{(\alpha_2, 2\alpha_1 + \alpha_2), (2\alpha_1 + \alpha_2, 2\alpha_1 + 3\alpha_2), \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1\}.$$

Example 2

Note that we can write the initial situation as in table 1.3 by taking into account that the true states that will occur are s_1 and s_2 , and not s_1, s_{21}, s_{22} and s_{23} . So Pareto optima will be $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ or fictitious $\hat{X} = (x_1, x_2, x_2, x_2)$ and $\hat{Y} = (y_1, y_2, y_2, y_2)$, so the set \mathcal{P}_{IR} will now satisfy the polytope property:

Comonotonicity condition: $x_1 \leq x_2$ and $y_1 \leq y_2$.

State	s_1	s_2
p	$\frac{1}{4}$	$\frac{3}{4}$
w_1	2	0
w_2	0	4
w	2	4

Table 1.2: Initial probability of the states for example 2.

State	s_1	s_{21}	s_{22}	s_{23}
p	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
\hat{w}_1	2	0	0	0
\hat{w}_2	0	4	4	4
\hat{w}	2	4	4	4

Table 1.3: Converted to the uniform probability

Dominance: $X \succeq_{SSD} w_1$, $x_1 \geq 0$ and $x_1 + x_2 \geq 0$, $x_1 + 2x_2 \geq 0$,

$$E(X) = E(w_1) \text{ i.e. } x_1 + 3x_2 = 2$$

$Y \succeq_{SSD} w_2$, $y_1 \geq 0$ and $y_1 + y_2 \geq 4$, $y_1 + 2y_2 \geq 8$,

$$E(Y) = E(w_2) \text{ i.e. } y_1 + 3y_2 = 12.$$

Feasibility: $x_1 \geq 0$ and $x_2 \geq 0$, $y_1 \geq 0$ and $y_2 \geq 0$

$$x_1 + y_1 = 2, x_2 + y_2 = 4.$$

Hence direct computation gives that the extreme *IRPO*'s are $((0, \frac{2}{3}), (2, \frac{10}{3}))$ and $((\frac{1}{2}, \frac{1}{2}), (\frac{3}{2}, \frac{7}{2}))$ so:

$$\mathcal{P}_{IR} = \left\{ \left(\frac{1}{2}\alpha_2, \frac{2}{3}\alpha_1 + \frac{1}{2}\alpha_2 \right), \left(2\alpha_1 + \frac{3}{2}\alpha_2, \frac{10}{3}\alpha_1 + \frac{7}{2}\alpha_2 \right), \alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1 \right\}.$$

1.4.2 Revisiting the insurance example of Landsberger and Meilijson (1994)

Landsberger and Meilijson (1994) considered the following risky situation. In this example the following risky situation is considered. A risk averse agent owns a house worth 10\$. The house is susceptible to a total loss with probability 1/10. The agent also owns 1\$ in cash and a source of random income (stock) which pays 2\$ or nothing with probability 1/2 each.

Fair insurance of the house is available but uncertainty associated with capital (stock) income is retained. Under fair insurance, the wealth positions of the insured and the insurer are given in table 1.4:

State	s_1	s_2	s_3	s_4
Situation	Stock pays zero	Stock pays 2\$	Stock pays zero	Stock pays 2\$
	House-total loss	House-total loss	House intact	House intact
Probability	0.05	0.05	0.45	0.45
Insured's wealth				
position X	10	12	10	12
Insurer's wealth				
position Z	-9	-9	1	1

Table 1.4: Insurance example of Landsberger and Meilijson (1994).

We assume as Landsberger and Meilijson (1994) that both agents are strict strong risk averters. We intend to derive all the *IRPO*'s, while Landsberger and Meilijson (1994) just proposed one sharing rule through their specific algorithm.

Note that theorem 1.4.5 assumes that all the considered allocations are non-negative, so we need to prove that the results of theorem 1.4.5 remains valid if no boundedness constraints are imposed on the initial allocations and the Pareto optima. To this end, theorem 1.4.5 can be improved in the following way:

Theorem 1.4.6. *Assume that the initial endowment of agent i is $w_i \in \mathbb{R}^m$ and define the *IRPO*'s as the allocations $X = (X_1, \dots, X_i, \dots, X_n) \in \mathbb{R}^{m \times n}$ such that $\sum_i X_i = \sum_i w_i$ ($:= w$) and $X_i \succeq_{SSD} w_i \quad \forall i$, then the set \mathcal{P}_{IR} is the polytope of the feasible allocations which are comonotone and satisfy the individually rational constraints.*

Proof: A simple examination of the proof of theorem 1.4.5 shows that it is just required to check that the set \mathcal{P}_{IR} is bounded.

We may assume since the probabilities are rational, that in fact we are considering the situation where all the states $j = 1, \dots, m$ are with probability $1/m$.

So translating our problem in this setting, we may assume:

$$w(1) \leq \dots \leq w(j) \leq \dots \leq w(m).$$

Thus we are looking for X_i , $i = 1, \dots, n$ such that from comonotonicity:

$$X_i(1) \leq \dots \leq X_i(j) \leq \dots \leq X_i(m) \quad \forall i$$

We know that for *IRPO* X_i , we have $E(X_i) = E(w_i)$ so the dominance condition 1 gives: $\sum_{j=1}^m X_i(j) = \sum_{j=1}^m w_i(j)$.

Let $\sigma_i : \{1, \dots, m\} \rightarrow \{1, \dots, m\}$ be the permutation such that,
 $w_i(\sigma_i(1)) \leq \dots \leq w_i(\sigma_i(j)) \leq \dots \leq w_i(\sigma_i(m))$.

Hence from dominance condition 2: $X_i \succeq_{SSD} w_i$, we get:

$$\sum_{j=1}^k X_i(j) \geq \sum_{j=1}^k w_i(\sigma_i(j)) \quad \forall k \in \{1, \dots, m\}.$$

Therefore for a given i , $X_i(1) \geq \text{Min}_j w_i(j)$, and then comonotonicity implies that
 $X_i(1) \geq \text{Min}_j w_i(j) \quad \forall i$, hence the X_i 's are bounded from below. Condition 1 immediately leads to the fact that each X_i is bounded from above.

Considering that for a given i we have:

$$X_i(m) = \sum_{j=1}^m w_i(j) - \sum_{j=1}^{m-1} X_i(j) \leq \sum_{j=1}^m w_i(j) - (m-1)\text{Min}_j w_i(j).$$

Therefore from comonotonicity $X_i \leq \sum_{j=1}^m w_i(j) - (m-1)\text{Min}_j w_i(j)$ which completes the proof. \square

Having proved the validity of our *IRPO* computation algorithm for the cases like Landsberger and Meilijson (1994)'s example, we apply our algorithm and we obtain the following two extreme points for the set of *IRPO* which are depicted in tables 1.5 and 1.6.

State	s_1	s_2	s_3	s_4
X_1	10	10.2	10.2	12
Z_1	-9	-7.2	0.8	1
w	1	3	11	13

Table 1.5: First extreme point

State	s_1	s_2	s_3	s_4
X_2	10	10	10.22	12
Z_2	-9	-7	0.78	1
w	1	3	11	13

Table 1.6: Second extreme point

In fact the first extreme point (X_1, Z_1) turns out to be the *IRPO* found by Landsberger and Meilijson (1994). Clearly any of the *IRPO* (i.e. any convex combination of (X_1, Z_1) and (X_2, Z_2)) allows both agents to reduce their risk compared to the initial situation, but indeed while only the insurable risk was diversified using the insurance market, risk sharing allowed also social risks (stock risk) to be reduced.

1.5 Conclusion

In this paper, in case of multiple risks, we did adopt the idea that a natural way for insurance companies to optimally share risks is risk by risk Pareto-optimality. Our framework is based upon the well-known results in the one dimensional case characterizing Pareto-optimality as comonotonicity in case of strong risk aversion. Two main results are obtained in this work.

Due to the polytope structure of Pareto-optima and also of Individually Rational Pareto-optima, we offer a simple computable method. First for deriving all Pareto-optima and second—in the not severely restrictive case of rational probabilities—for deriving all Individually Rational Pareto-optima. The method merely consists in systematically obtaining the finitely many extreme points of the respective polytopes. The method is illustrated using the numerical examples. Moreover the application of this approach in insurance industry is examined by computing all the IRPO's of the Landsberger and Meilijson (1994)'s example.

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Chapter 2

Comonotonic Monte Carlo and its applications in option pricing and quantification of risk¹

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Abstract

Monte Carlo (MC) simulation is a technique that provides approximate solutions to a broad range of mathematical problems. A drawback of the method is its high computational cost, especially in a high-dimensional setting, such as estimating the Tail Value-at-Risk for large portfolios or pricing basket options and Asian options. For these types of problems, one can construct an upper bound in the convex order by replacing the copula by the comonotonic copula. This comonotonic upper bound can be computed very quickly, but it gives only a rough approximation. In this paper we introduce the Comonotonic Monte Carlo (CoMC) simulation, by using the comonotonic approximation as a control variate. The CoMC is of broad applicability and numerical results show a remarkable speed improvement. We illustrate the method for estimating Tail Value-at-Risk and pricing basket options and Asian options when the logreturns follow a Black-Scholes model or a variance gamma model.

Keywords : Control Variate Monte Carlo, Comonotonicity, Option pricing.

JEL classification: C02, C13, C15, G17.

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2.1 Introduction

Monte Carlo (MC) simulation is a well known technique in different domains of mathematics such as mathematical finance, see Glasserman (2003); Benninga (2014). The method is based on the estimation of the expectation of a real-valued random variable X by generating many independent and identically distributed samples of X , denoted X_1, \dots, X_n . The natural unbiased estimator for $E(X)$ is then the sample mean $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$.

A typical application of the Monte Carlo method in finance is the estimation of the no-arbitrage price of a specific derivative security (e.g. a call option), which can be expressed as the expected value of its discounted payoff under the risk neutral measure. For instance the price at time t of a European call option with strike price K and maturity date T on an underlying with price process S_t can be obtained as the expectation of its discounted payoff $e^{-r(T-t)}(S_T - K)_+$ under the risk-neutral probability Q ,

$$EC(K, T, t) = E^Q[e^{-r(T-t)}(S_T - K)_+].$$

For the computation of this price by Monte Carlo simulation, we generate a large number of price paths S_T and compute the discounted payoffs and their sample mean. The obtained result is an unbiased estimate of the option price.

Another application of the Monte Carlo method in finance is estimating risk measures, such as Tail Value-at-Risk. The Tail Value-at-Risk of a portfolio at the probability level p is the arithmetic average of its quantiles from the threshold p to 1. The Monte Carlo method estimates these quantiles by generating a huge number of portfolio values for which the exceedance probabilities $Pr[X \geq x] = E[I(X \geq x)]$ are computed, where $I(\cdot)$ denotes the indicator function. A classical interpolation and inversion then gives an estimate for the quantile.

The main shortcoming of the Monte Carlo method is its high computational cost. By the Central Limit Theorem, if X_1, \dots, X_n have finite variance σ^2 , then \bar{X}_n is approximately Gaussian and $\text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$. Consequently, the standard error of the crude Monte Carlo estimate is of order $O(\frac{1}{\sqrt{n}})$ and thus, to double the precision, one must run four times the number of simulations. Alternatively, strategies for reducing σ should be considered.

Several variance reduction techniques can be used in companion with the Monte Carlo method, such as antithetic variables, control variates and importance sampling. A

detailed survey of these techniques is given in Ripley (1987). In this paper we focus on the well-known control variate method for variance reduction.

The applications considered in this paper are simulation problems based on multivariate random variables, such as basket options where the price depends on several underlying securities. In these problems the closed form expressions are often available for the univariate cases. For instance, in a lognormal world the price of a European call option (which only depends on S_T) can be calculated with the Black-Scholes pricing formula. As comonotonicity essentially reduces a multivariate problem to univariate ones, leaving the marginal distributions intact, we propose to use the comonotonic approximation as a control variate in a so-called Comonotonic Monte Carlo (CoMC) framework. One further step that can be considered is utilizing the CoMC method in addition to other existing control variates in the framework of a multi-variable control variate method.

The Comonotonic Monte Carlo method is particularly useful to estimate distortion risk measures for sums of random variables, such as Tail Value-at-Risk (TVaR). The application domain of this method can also be extended to the risk measures which can be written as a linear combination of distortion risk measures, such as the Expected Shortfall (ESF). As the ESF basically consists of a stop-loss transform, its mathematical concept is very similar to option pricing, so the technique is useful in this domain as well.

The structure of this paper is as follows. First we discuss the control variate method for reducing the variance. Next, we describe the application of the comonotonicity concept to construct the comonotonic control variate. In the fourth section, we illustrate the CoMC framework for Asian options, Basket options and TVaR. In the final section we conclude the results.

2.2 Control Variate Monte Carlo Method

The control variate method is a classical approach for reducing the variance, and hence improving the efficiency, in Monte Carlo simulation, see e.g. Kemna and Vorst (1990) for the pricing of arithmetic Asian options.

In the control variate Monte Carlo method, when we generate the sample values to estimate $E[X]$, we use the same values to estimate the expectation of a different random variable Y which resembles X and for which the analytical calculation of its expectation, $E[Y]$, is straightforward. Assuming that $E[Y]$ is known, we can then

determine the error of estimating $E[Y]$ and use it to correct the estimate of $E[X]$. As an example, in the case of Asian option pricing, we can calculate the value of a geometric Asian call option using both the (analytical) Black-Scholes formula and Monte Carlo simulation. If the simulation turns out to underestimate the real option price, one could argue that the corresponding estimate for the arithmetic Asian option will also be too low and adjust the Monte Carlo estimate accordingly.

In general, the control variate method can be formulated as follows. Suppose that there exists a random variable Y , related to X , for which $E[Y]$ is known. Considering that the sample means \bar{X}_n and \bar{Y}_n are unbiased estimators for $E(X)$ and $E[Y]$ respectively, the adjusted estimator

$$\bar{X}_n(\lambda) = \bar{X}_n - \lambda(\bar{Y}_n - E[Y]), \quad \lambda \in \mathbb{R}$$

is also an unbiased estimator of $E[X]$, i.e. $E[\bar{X}_n(\lambda)] = E[\bar{X}_n] = E[X]$. The control parameter λ is an arbitrary scalar, but in order to minimize the variance of $\bar{X}_n(\lambda)$ we should set it to

$$\lambda^* = \frac{\text{Cov}(X, Y)}{\text{Var}[Y]} = \rho \sqrt{\frac{\text{Var}[X]}{\text{Var}[Y]}}$$

with ρ denoting the correlation between X and Y . This choice yields a minimum variance $(1 - \rho^2)\text{Var}[\bar{X}_n]$, which is obviously smaller than $\text{Var}[\bar{X}_n]$ as $-1 \leq \rho \leq 1$. Therefore the control variate unbiased estimator $\bar{X}_n(\lambda)$ leads to a smaller variance compared to the obtained variance from the crude Monte Carlo unbiased estimator \bar{X}_n .

Note that the optimal λ^* involves moments of X and Y that are generally unknown. Hence λ is often chosen to be 1. This choice makes sense if the control variate Y is very similar to X , and thus if ρ is close to 1 and $\text{Var}[X] \approx \text{Var}[Y]$. The optimal λ^* could also be estimated from the simulated data, but one should take into account that this introduces bias of order $O(1/n)$ to the estimator $\bar{X}_n(\lambda)$. A straightforward way to overcome this problem is to use different samples for the estimation of λ and $E[X]$.

2.3 Comonotonic Control Variate

The concept of comonotonicity has received a lot of interest in the recent actuarial and financial literature, mainly due to its interesting properties that can be used

to facilitate various complicated problems, see Dhaene et al. (2014); Deelstra et al. (2011); Liu et al. (2013); Tsuzuki (2013). In the following sections we describe the properties of comonotonicity that can be used to construct a comonotonic control variate for a multivariate Monte Carlo simulation.

2.3.1 Comonotonic Upper Bound

Consider a random vector $\underline{X} = (X_1, \dots, X_n)$ for which the marginal distributions of X_i 's are known. In order to determine the distribution function of the sum of random variables, $\mathbb{S} = \sum_{i=1}^n X_i$, it is often assumed that the individual random variables X_i 's are mutually independent. However, the assumption of mutual independence might be violated and may result in underestimating the sum \mathbb{S} . To avoid this underestimating, we need to consider the dependence structure of the random vector \underline{X} . If the joint distribution of \underline{X} is unspecified or less tractable, we can derive an upper bound for the sum \mathbb{S} in convex order².

Dhaene et al. (2002) proved that the convex-largest sum of the components of a random vector \underline{X} with given marginal distributions will be obtained in the case that the random vector \underline{X} has a comonotonic distribution, which means that each two possible outcomes (x_1, \dots, x_n) and (y_1, \dots, y_n) of (X_1, \dots, X_n) are ordered component-wise.

Definition 2.3.1. *A random vector $\underline{X} = (X_1, \dots, X_n)$ is comonotonic if and only if it has a comonotonic copula i.e. for all $\underline{x} = (x_1, \dots, x_n)$, we have*

$$F_{\underline{X}}(\underline{x}) = \min \{F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_n}(x_n)\}. \quad (2.1)$$

Proposition 2.3.2. *If \underline{X} has a comonotonic copula then for $U \sim \text{Uniform}(0, 1)$, we have*

$$\underline{X} \stackrel{d}{=} (F_{X_1}^{-1}(U), (F_{X_2}^{-1}(U), \dots, (F_{X_n}^{-1}(U)). \quad (2.2)$$

Proof: See Dhaene et al. (2002). □

²A random variable X is said to precede a random variable Y in the convex order sense, written $X \leq_{cx} Y$, if and only if

$$\begin{cases} E(X) = E(Y) \\ E(X - d)_+ \leq E(Y - d)_+ \quad \text{for all real } d \end{cases}$$

According to Proposition 2.3.2, for any random vector $\underline{X} = (X_1, \dots, X_n)$, not necessarily comonotonic, we can construct its comonotonic counterpart which will be denoted by $\underline{X}^c = (X_1^c, \dots, X_n^c)$ as follows

$$\underline{X}^c := (F_{X_1}^{-1}(U), (F_{X_2}^{-1}(U), \dots, (F_{X_n}^{-1}(U)).$$

Clearly \underline{X}^c and \underline{X} have the same marginal distributions F_{X_i} , but they have a different copula. Also the sum of its components, $\mathbb{S}^c = \sum_{i=1}^n X_i^c$, gives an upper bound for the sum \mathbb{S} . In fact, replacing the copula by a comonotonic copula yields the largest sum in the convex order, see Dhaene et al. (2002).

2.3.2 Additivity property

Here we discuss the additivity property of the quantile function and any distortion risk measure for a sum of comonotonic random variables. The additivity property will be used to compute the comonotonic upper bound.

Proposition 2.3.3. *The quantile function $F_{\mathbb{S}^c}^{-1}$ of a sum \mathbb{S}^c of comonotonic random variables with distribution functions F_{X_1}, \dots, F_{X_n} is additive*

$$F_{\mathbb{S}^c}^{-1}(p) = \sum_{i=1}^n F_{X_i}^{-1}(p), \quad 0 < p < 1. \quad (2.3)$$

Proof: See Dhaene et al. (2002). □

By the additivity property exhibited in Proposition 2.3.3, calculating the distribution function of \mathbb{S}^c is straightforward. The distribution of \mathbb{S}^c simply follows from inverting its quantile function. This makes the comonotonic upper bound \mathbb{S}^c a natural control variate, namely comonotonic control variate, in a Monte Carlo simulation.

In the following propositions it will be shown that any distortion risk measure has the additivity property for comonotonic variables. This property facilitates deriving the comonotonic control variate for estimating the Tail Value-at-Risk (TVaR) and option pricing in a so-called comonotonic Monte Carlo (CoMC) framework.

Definition 2.3.4. *The distorted expectation of a random variable X is defined by*

$$\rho_g[X] = \int_{-\infty}^0 (g(\bar{F}_X(x)) - 1) dx + \int_0^{\infty} g(\bar{F}_X(x)) dx, \quad (2.4)$$

where $\bar{F}_X(x) = 1 - F_X(x)$ denotes the tail function of $F_X(x)$ and the function $g(\cdot)$ is

a so-called *distortion function*, i.e. a non-decreasing function $g : [0, 1] \rightarrow [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$.

According to Wang (1996), ρ_g is known as the distortion risk measure associated with distortion function g . Note that the distortion function g is assumed to be independent of the distribution function of X .

Proposition 2.3.5. *The distortion risk measure for a sum of comonotonic variables is additive i.e. for any distortion function g and all random variables X_i we have*

$$\rho_g[\mathbb{S}^c] = \sum_{i=1}^n \rho_g[X_i]. \quad (2.5)$$

Proof: This result is proved in Wang (1996) for non-negative random variables, but it can be easily extended to all real-valued variables. Substituting $g(\bar{F}_X(x))$ by $\int_0^{\bar{F}_X(x)} dg(q)$ in (2.4) and changing the order of the integrations, we find that

$$\rho_g[X] = \int_0^1 \bar{F}_X^{-1}(q) dg(q) = \int_0^1 F_X^{-1}(1-q) dg(q) \quad (2.6)$$

for any distortion function g and any random variable X . Combining equations (2.3) and (2.6) yields

$$\rho_g[\mathbb{S}^c] = \int_0^1 F_{\mathbb{S}^c}^{-1}(1-q) dg(q) = \int_0^1 \sum_{i=1}^n F_{X_i}^{-1}(q) dg(q) = \sum_{i=1}^n \rho_g[X_i],$$

which completes the proof. □

Corollary 2.3.6. *The Tail Value-at-Risk, $TVaR_X(p)$, at level $p \in (0, 1)$ given by*

$$TVaR_X(p) = \frac{1}{1-p} \int_p^1 F_X^{-1}(q) dq \quad (2.7)$$

is a distortion risk measure with distortion function

$$g(x) = \min\left(\frac{x}{1-p}, 1\right), 0 \leq x \leq 1,$$

hence it is additive for comonotonic random variables.

We remark that risk measures which can be written as a linear combination of distortion risk measures satisfy the additivity property as well. For instance the Expected

Shortfall (ESF) defined as

$$ESF_X(p) = E[(X - F_X^{-1}(p))_+]$$

is not a distortion risk measure, but it is also additive for comonotonic random variables.

Corollary 2.3.7. *The ESF can be written as a linear combination of distortion risk measures given by*

$$TVaR_X(p) = F_X^{-1}(p) + \frac{1}{1-p}ESF_X(p),$$

see Dhaene et al. (2006), thus it follows

$$\begin{aligned} ESF_{\mathbb{S}^c}(p) &= (1-p)(TVaR_{\mathbb{S}^c}(p) - F_{\mathbb{S}^c}^{-1}(p)) \\ &= (1-p) \left(\sum_{i=1}^n TVaR_{X_i}(p) - \sum_{i=1}^n F_{X_i}^{-1}(p) \right) \\ &= \sum_{i=1}^n ESF_{X_i}(p), \quad 0 < p < 1. \end{aligned}$$

It is worth noting that the Expected Shortfall basically consists of a stop-loss premium, so it is very closely related to the pricing of options. More generally, for the stop-loss transform of a sum of comonotonic variables we have the following result.

Corollary 2.3.8. *By choosing $p = F_{\mathbb{S}^c}(K)$ in Corollary 2.3.7, it follows that the stop-loss premium $E[(\mathbb{S}^c - K)_+]$ of a sum \mathbb{S}^c of comonotonic random variables with strictly increasing distribution functions F_{X_1}, \dots, F_{X_n} can be written as*

$$E[(\mathbb{S}^c - K)_+] = \sum_{i=1}^n [(X_i - F_{X_i}^{-1}(F_{\mathbb{S}^c}(K)))_+], \quad \forall K \in \mathbb{R}. \quad (2.8)$$

The additivity property of distortion risk measures for comonotonic variables reduces the multivariate problem to univariate ones.

Furthermore, replacing the copula by a comonotonic copula leaves the marginal distributions intact. Therefore the simulated samples in the univariate cases are readily available from the main simulation routine. Considering the mentioned properties, the comonotonic upper bound is an obvious control variate choice. In the next section we apply the CoMC method to Asian and Basket option pricing and to estimating

the TVaR of a portfolio.

2.4 Comonotonic Control Variate for Asian Options, Basket Options and Tail Value-at-Risk

2.4.1 Asian Option

An Asian option is a path dependent option, for which the payoff depends on the average price of the underlying risky asset in the considered time interval. We consider a discrete set of n time points along the time interval $[0, T]$ such that the asset price, S_t , is observed at time points $0 = t_0 < t_1 < \dots < t_n = T$.

In a complete market, the no-arbitrage price of the Asian option at time 0 is its expected discounted pay-off under a martingale measure Q given by

$$AC(n, K, T) = e^{-rT} E^Q \left[\left(\frac{1}{n} \sum_{i=1}^n S_{t_i} - K \right)_+ \right],$$

where r is the risk-free rate.

Since in general the distribution of the average $\frac{1}{n} \sum_{i=1}^n S_{t_i}$ of dependent random variables is not available, it is not possible to derive a closed-form expression for the Asian option price. Therefore the comonotonic Monte Carlo simulation is a useful method for estimating the price of Asian option. In the following, we derive the comonotonic control variate for this estimation in the CoMC framework.

The comonotonic upper bound of $AC(n, K, T)$, which is obtained by replacing the price vector $(S_{t_1}, \dots, S_{t_n})$ by its comonotonic counterpart $(S_{t_1}^c, \dots, S_{t_n}^c)$, reads

$$\begin{aligned} AC_{com}(n, K, T) &= e^{-rT} E^Q \left[\left(\frac{1}{n} \sum_{i=1}^n S_{t_i}^c - K \right)_+ \right] \\ &= \frac{e^{-rT}}{n} E^Q \left[(\mathbb{S}^c - nK)_+ \right], \end{aligned}$$

where $\mathbb{S}^c = \sum_{i=1}^n S_{t_i}^c$.

Note that from Proposition 2.3.2 we have $\sum_{i=1}^n S_{t_i}^c = \sum_{i=1}^n F_{S_{t_i}}^{-1}(U)$. Using Corollary 2.3.8, we have

$$AC_{com}(n, K, T) = \frac{e^{-rT}}{n} \sum_{i=1}^n E^Q \left[\left(S_{t_i} - F_{S_{t_i}}^{-1}(F_{\mathbb{S}^c}(nK)) \right)_+ \right].$$

Hence, $AC_{com}(n, K, T)$ can be rewritten in terms of prices of European call options $EC(k_i, t_i)$ at time 0 with exercising times t_i and strike prices k_i

$$AC_{com}(n, K, T) = \frac{1}{n} \sum_{i=1}^n e^{-r(T-t_i)} EC(k_i, t_i), \quad (2.9)$$

where $k_i = F_{S_{t_i}}^{-1}(F_{\mathbb{S}^c}(nK))$, see also Simon et al. (2000).

For the practical determination of the strike prices k_i , the distribution function of the comonotonic sum $F_{\mathbb{S}^c}$ has to be calculated and evaluated at nK by Proposition 2.3.3. Under the risk-neutral probability, this can be done numerically in a straightforward way. The k_i 's are then obtained by evaluating the inverse distribution function of the marginals at $F_{\mathbb{S}^c}(nK)$.

Considering the Lévy market model for asset prices we derive the comonotonic upper bound (2.9). We assume that the price S_t of the risky asset follows a variance gamma process $\{X_t^{(VG)}, t \geq 0\}$, which is a popular class of Lévy process. The risk-neutral model for the asset price is then given by

$$S_t = S_0 \frac{\exp((r-q)t)}{E[\exp(X_t)]} \exp(X_t).$$

The factor $\exp((r-q)t)/E[\exp(X_t)]$ guarantees that the risk-neutral setting holds by considering a mean correcting argument, see Albrecher et al. (2005).

The price $EC(K, T)$ of a European call option with strike price K and maturity date T under the variance gamma model can be calculated by the Carr and Madan formula, see Madan et al. (1998); Albrecher et al. (2005), which formulates the price of European call option in terms of the characteristic function of the underlying Lévy process.

Let α be a positive constant such that the α th moment of the stock price exists and let ϕ be the characteristic function of the variance gamma process. Then we have

$$EC(K, T) = \frac{\exp(-\alpha \log(K))}{\pi} \int_0^{+\infty} \exp(-iv \log(K)) \varrho(v) dv, \quad (2.10)$$

where

$$\begin{aligned} \varrho(v) &= \frac{\exp(-rT) E[\exp(i(v - (\alpha + 1)i) \log(S_T))]}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} \\ &= \frac{\exp(-rT) \phi(v - (\alpha + 1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}. \end{aligned}$$

Hence the comonotonic upper bound can be obtained using the European option pricing formula (2.10) of Carr and Madan and (2.9). We consider this comonotonic upper bound as a control variate in the CoMC method for estimating the price of Asian options in a variance gamma model.

Numerical example

We illustrate the performance of the CoMC method to estimate the price of an Asian option when the underlying asset follows a variance gamma process. We consider an arithmetic Asian option with maturity of 1 year and averaging every month (i.e. 12 averaging dates). The initial value of the stock price is normalized to be 100 and the yearly risk free interest rate is $r = 0.02$. The parameters of the variance gamma process that were used to generate the price paths are those from Albrecher et al. (2005). Five values (80, 90, 100, 110 and 120) are assumed for the strike price K .

In Table 2.1 we compare the performance of the crude Monte Carlo (MC) method and the CoMC method based on 10,000,000 simulated paths. The estimated price based on MC and CoMC is represented by AC_{MC} and AC_{CoMC} respectively. The performance of CoMC method is examined by comparing its computation time and obtained variance with the crude Monte Carlo method.

The ratio of computation times (T_{MC}/T_{CoMC}) and Variances (V_{MC}/V_{CoMC}) are depicted for each of strike prices in Table 2.1.

K	AC_{MC}	AC_{CoMC}	V_{MC}/V_{CoMC}	T_{MC}/T_{CoMC}
80	20.7295	20.7441	161.7179	0.5175
90	11.8211	11.7605	94.3999	0.5176
100	4.5661	4.5684	54.5796	0.5164
110	0.9405	0.9295	21.3384	0.5143
120	0.2039	0.2006	11.3167	0.5157

Table 2.1: Performance of the CoMC method in Asian option pricing

We observe that by increasing the strike price, K , the ratio of variance reduction V_{MC}/V_{CoMC} decreases while the ratio of computation time T_{MC}/T_{CoMC} is almost constant. In other words, the CoMC method performs well when the option is in the money.

Since \mathbb{S}^c is larger than \mathbb{S} in convex order, they have the same expectation value, $E(\mathbb{S}^c) = E(\mathbb{S})$, but \mathbb{S}^c has heavier tails than \mathbb{S} , see Vyncke et al. (2001). Therefore, the difference of $E[(\mathbb{S} - K)_+]$ and $E[(\mathbb{S}^c - K)_+]$ is smaller for the in the money cases compared to the other cases where the strike price is comparatively larger. This results in a higher correlation between $(\mathbb{S} - K)_+$ and $(\mathbb{S}^c - K)_+$ when K is small. Consequently the comonotonic control variate method performs better for the in the money cases.

The efficiency of the method can be quantified by comparing the number of samples required for the crude Monte Carlo method to achieve the same degree of accuracy. For the different strike prices $K = 80, 90, 100, 110, 120$ the number of samples required for the crude Monte Carlo to reach the same level of precision as the CoMC varies between 11 to 160 times the original number of samples.

Considering that the required computation time for the comonotonic control variate Monte Carlo method is only twice the computation time of crude Monte Carlo method for the same number of samples, it can be concluded that employing the CoMC method significantly increases the computation performance and efficiency.

2.4.2 Basket Option

A Basket option is an option on a portfolio (or basket) of several underlying assets whose payoff is dependent on the value of a weighted sum of the underlying assets. Consider a portfolio of n risky assets with price process $\{S_i(t), t \geq 0\}$, $i = 1, \dots, n$ and positive weights a_i , $\sum_{i=1}^n a_i = 1$.

In a complete market, the no-arbitrage price of a Basket call option with maturity date T and strike price K at time 0 is given by

$$BC(n, K, T) = e^{-rT} E^Q \left[\left(\sum_{i=1}^n a_i S_i(T) - K \right)_+ \right], \quad (2.11)$$

which is the expected payoff of the call option under a martingale measure Q , discounted at the risk-free rate r .

In the classical Black-Scholes model, the price process of assets are assumed to follow the risk-neutral stochastic differential equations

$$dS_i(t) = (r - q_i)S_i(t)dt + \sigma_i S_i(t)dB_i(t), \quad (2.12)$$

where the $B_i(t)$ are Brownian motions, q_i and σ_i denote the dividend rate and the volatility of the underlying asset i respectively. Given the above dynamics, the price of the i th asset at time T equals

$$S_i(T) = S_i(0)e^{(r-q_i-\sigma_i^2/2)T+\sigma_i B_i(T)}.$$

Thus, the random variable $S_i(T)/S_i(0)$ is lognormally distributed with parameters $(r - q_i - \sigma_i^2/2)T$ and $\sigma_i^2 T$. We assume that the Brownian motions B_i and B_j are correlated with a constant correlation ρ_{ij} .

Since the distribution of a sum of lognormally distributed random variables is not log-normal, the distribution of the weighted sum $\sum_{i=1}^n a_i S_i(T)$ is not known analytically and hence determining the price of the Basket option is not straightforward.

In order to estimate the price of a Basket option in the comonotonic Monte Carlo framework, the corresponding comonotonic control variate can be constructed as follows.

By replacing the weighted average $\sum_{i=1}^n a_i S_i(T)$ with the comonotonic weighted average $\sum_{i=1}^n a_i S_i^c(T)$ in (2.11), the comonotonic upper bound of $BC(n, K, T)$ is then given by

$$BC_{com}(n, K, T) = e^{-rT} E^Q \left[(\mathbb{S}^c - K)_+ \right], \quad (2.13)$$

where $\mathbb{S}^c = \sum_{i=1}^n a_i S_i^c(T) = \sum_{i=1}^n a_i F_{S_i(T)}^{-1}(U)$, see Proposition 2.3.2.

Note that by using Corollary 2.3.8, the comonotonic upper bound (2.13) can be written in terms of a weighted sum of European call options,

$$\begin{aligned} BC_{com}(n, K, T) &= e^{-rT} \sum_{i=1}^n a_i E^Q \left[\left(S_i(T) - F_{S_i(T)}^{-1}(F_{\mathbb{S}^c}(K)) \right)_+ \right] \\ &= \sum_{i=1}^n a_i EC_i(k_i, T), \end{aligned} \quad (2.14)$$

where $k_i = F_{S_i(T)}^{-1}(F_{\mathbb{S}^c}(K))$.

We know from Proposition 2.3.3 that the quantile function of a sum of comonotonic random variables is simply the sum of the quantile functions of the marginal distributions. Moreover, in case of strictly increasing and continuous marginals, the

cumulative distribution function $F_{\mathbb{S}^c}(x)$ is uniquely determined by

$$F_{\mathbb{S}^c}^{-1}(F_{\mathbb{S}^c}(x)) = \sum_{i=1}^n a_i F_{S_i(T)}^{-1}(F_{\mathbb{S}^c}(x)) = x \quad F_{\mathbb{S}^c}^{-1}(0) < x < F_{\mathbb{S}^c}^{-1}(1), \quad (2.15)$$

see Kaas et al. (2000). Hence using the inverse distribution function of $S_i(T)$ given by

$$F_{S_i(T)}^{-1}(p) = S_i(0)e^{(r-q_i-\sigma_i^2/2)T+\sigma_i\sqrt{T}\Phi^{-1}(p)}, \quad \forall p \in (0, 1), \quad (2.16)$$

where Φ is the cdf of the standard normal distribution, (2.15) results in

$$\sum_{i=1}^n a_i S_i(0)e^{(r-q_i-\sigma_i^2/2)T+\sigma_i\sqrt{T}\Phi^{-1}(F_{\mathbb{S}^c}(K))} = K, \quad (2.17)$$

from which $F_{\mathbb{S}^c}(K)$ can be obtained numerically. Therefore the strike prices k_i for asset i can be determined by evaluating (2.16) at $F_{\mathbb{S}^c}(K)$.

Having obtained the k_i 's, the price of a European call option with strike price k_i and maturity date T at time 0 reads

$$EC_i(k_i, T) = S_i(0)\Phi(d_{i,1}) - k_i e^{-rT}\Phi(d_{i,2}),$$

where

$$d_{i,1} = \frac{\ln(S_i(0)/k_i) + (r_i + \sigma_i^2/2)T}{\sigma_i\sqrt{T}}, \quad d_{i,2} = d_{i,1} - \sigma_i\sqrt{T}.$$

Thus, the comonotonic control variate for a Basket option pricing in Black-Scholes setting can be determined by the weighted summation of $EC_i(k_i, T)$ in (2.14).

Note that an alternative control variate can be obtained by replacing the weighted arithmetic average with the geometric average. Since the geometric average of the lognormally distributed variables is also lognormally distributed, obtaining the closed-form formulation for the geometric control variate is trivial, see Kemna and Vorst (1990).

Numerical example

In this section, the performance of the CoMC method is evaluated for pricing basket options. We consider a Basket option consisting of seven assets. The data used for this purpose is based on the basket of seven stock indices underlying the G-7 index-linked guaranteed investment certificates offered by Canada Trust Co, see Milevsky

and Posner (1998a,b).

The risk free interest rate is $r = 0.063$ and the maturity date is set to 1 year. The initial value of each asset in the basket is normalized to be 100. The other considered parameters are given in Table 2.2 and 2.3.

country	index	weight (in%)	volatility (in%)	dividend yield (in%)
Canada	TSE 100	10	11.55	1.69
Germany	DAX	15	14.53	1.36
France	CAC 40	15	10.68	2.39
U.K.	FSTE 100	10	14.62	3.62
Italy	MIB 30	5	17.99	1.92
Japan	Nikkei 225	20	15.59	0.81
U.S.	S&P 500	25	15.68	1.66

Table 2.2: G-7 index linked guaranteed investment certificate weightings

	Canada	Germany	France	U.K.	Italy	Japan	U.S.
Canada	1.00	0.35	0.10	0.27	0.04	0.17	0.71
Germany	0.35	1.00	0.39	0.27	0.50	-0.08	0.15
France	0.10	0.39	1.00	0.53	0.70	-0.23	0.09
U.K.	0.27	0.27	0.53	1.00	0.45	-0.22	0.32
Italy	0.04	0.50	0.70	0.45	1.00	-0.29	0.13
Japan	0.17	-0.08	-0.23	-0.22	-0.29	1.00	-0.03
U.S.	0.71	0.15	0.09	0.32	0.13	-0.03	1.00

Table 2.3: Correlation structure of the G-7 index

The performance of the CoMC method is examined by comparing its computation time and obtained variance with the crude Monte Carlo method. The ratio of variances V_{MC}/V_{CoMC} is depicted for different strike prices in Table 2.4. The estimated

prices for the considered Basket option based on MC and CoMC methods are represented by BC_{MC} and BC_{CoMC} respectively. The obtained results for both methods are based on 10,000,000 simulated paths.

K	BC_{MC}	BC_{CoMC}	V_{MC}/V_{CoMC}	T_{MC}/T_{CoMC}
80	23.1366	23.1387	273.6565	0.5191
90	13.8112	13.8166	35.2039	0.5778
100	5.6312	5.6440	10.7839	0.5577
110	1.2320	1.2387	3.2520	0.5260
120	0.1334	0.1336	1.4792	0.5339

Table 2.4: Performance of the CoMC method in Basket option pricing

According to Table 2.4, the variance reduction capability of CoMC decreases by increasing the strike price, while the required computation resources for the CoMC method is only twice the crude Monte Carlo method. Therefore the method is best suited for the in the money cases with the same reasoning given in numerical example for Asian option.

In this example it is observed that, based on the estimation error, the number of samples required for the crude MC to reach the same level of accuracy as the precision of the CoMC, varies between 3 to 273 times the original number of samples.

Considering that the comonotonicity assumption induces the strongest positive dependency, it is expected that the correlation structure has a strong influence on the performance of the CoMC method. Therefore it is worth to examine these effects quantitatively in Basket option pricing. To this end, we consider a Basket option consisting of the first two assets of Table 2.2 with equal weights. The performance of the CoMC method is evaluated for different strike prices and correlations ρ .

ρ	K	BC_{MC}	BC_{CoMC}	V_{MC}/V_{CoMC}
0.75	80	23.4258	23.4196	2910.4
	90	14.5317	14.5374	534.91
	100	7.3315	7.3320	147.17
0.25	80	23.3801	23.3843	706.49
	90	14.2747	14.2701	87.210
	100	6.6607	6.6828	19.961
-0.25	80	23.3826	23.3734	313.20
	90	14.0618	14.0633	30.654
	100	5.9161	5.9256	6.6869
-0.75	80	23.3657	23.3714	100.95
	90	13.9747	13.9855	9.4026
	100	4.9893	4.9947	2.7801

Table 2.5: Influence of the correlation on the efficiency of CoMC

For the constant strike price K , the variance reduction ratio V_{MC}/V_{CoMC} increases for higher level of positive assets correlation ρ , see Table 2.5. On the other hand, in case of a negative correlation, which is highly contrasting with the comonotonicity assumption, the variance reduction ratio V_{MC}/V_{CoMC} is considerably decreased.

As mentioned in the theoretical background, the geometric control variate is an alternative control variate in Basket option pricing. The second numerical example is aimed at comparing the performance of the comonotonic control variate with its competent alternative, the geometric control variate. For this purpose, similar to the previous example, we consider a two asset basket and compute the efficiency of the methods for different weights a_i and initial prices S_0 while the strike price is the initial value of the portfolio and correlation coefficient is considered to be constant, $\rho = 0.35$.

a_1/a_2	(S_{01}, S_{02})	V_{MC}/V_{CoMC}	V_{MC}/V_G
1	(100, 100)	29.7870	539.7020
	(100, 50)	34.1525	7.2490
	(50, 100)	50.0775	8.1827
0.25	(100, 100)	105.2138	2251.6
	(100, 50)	48.7746	17.9618
	(50, 100)	377.3903	41.6317

Table 2.6: Comparison of CoMC and geometric control variate

Table 2.6 compares the variance reduction V_{MC}/V_G obtained by the geometric control variate to the variance reduction V_{MC}/V_{CoMC} of the CoMC method. The results show that for the cases where the initial prices are equal the geometric control variate performs much better than the CoMC method. In the other cases, the performance of the comonotonic control variate method surpasses the variance reduction obtained by the geometric control variate method. We conclude that for non-equal initial prices, the CoMC method has real added value.

2.4.3 Tail Value-at-Risk

The Tail Value-at-Risk (TVaR) of a portfolio at a given level of probability $p \in (0, 1)$, is defined as the arithmetic average of its quantiles from the threshold p up to 1, see Corollary 2.3.6.

Consider a portfolio consisting of n risky assets where each asset price $S_i(t)$ follows the risk-neutral stochastic differential equation in (2.12). The value of the portfolio at time T equals $\mathbb{S} = \sum_{i=1}^n a_i S_i(T)$. Since the distribution function of \mathbb{S} is unknown, determining the Tail Value-at-Risk of the loss of the portfolio, $TVaR_{-\mathbb{S}}(p)$, is not straightforward. Therefore, the comonotonic Monte Carlo method can be employed for estimating TVaR.

As already mentioned in Corollary 2.3.6, the Tail Value-at-Risk is additive for a sum of comonotonic random variables. Hence, the comonotonic control variate for estimating the TVaR for the loss of portfolio in the CoMC framework is given by

$$TVaR_{com} = TVaR_{-\mathbb{S}^c}(p) = \frac{1}{1-p} \int_p^1 F_{-\mathbb{S}^c}^{-1}(q) dq$$

$$= \sum_{i=1}^n a_i \left(-\frac{1}{1-p} \int_p^1 F_{S_i(T)}^{-1}(1-q) dq \right), \quad (2.18)$$

where $\mathbb{S}^c = \sum_{i=1}^n a_i S_i^c(T) = \sum_{i=1}^n a_i F_{S_i(T)}^{-1}(U)$, see Proposition 2.3.2.

Considering that the price $S_i(T)$ of each asset at time T is lognormally distributed, we have for (2.18)

$$-\frac{1}{1-p} \int_p^1 F_{S_i(T)}^{-1}(1-q) dq = -\frac{E(S_i(T))}{1-p} \left(\Phi(\Phi^{-1}(1-p) - \sigma_i \sqrt{T}) \right), \quad (2.19)$$

where Φ denotes the standard normal distribution function and σ_i is the volatility of asset i , see Sandström (2010).

Numerical example

The performance of the CoMC method is evaluated here for the calculation of the TVaR risk measure. We consider a portfolio consisting of the first two assets, Canada and Germany, of Table 2.2. We generate the price paths in a Black-Scholes setting using the parameters given in Tables 2.2 and 2.3.

risk measure	$TVaR_{MC}$	$TVaR_{CoMC}$	V_{MC}/V_{CoMC}	T_{MC}/T_{CoMC}
TVaR(0.90)	86.4584	86.4631	3.5193	0.5321
TVaR(0.95)	83.6405	83.6446	2.6940	0.5227
TVaR(0.99)	78.4445	78.4150	1.8013	0.5600

Table 2.7: The performance of CoMC method for TVaR

The results of the CoMC method are compared with the ones obtained from the crude Monte Carlo method for the different levels of probability p , see Table 2.7. For this specific correlation structure, the variance reduction ratio, V_{MC}/V_{CoMC} , obtained by the CoMC method is rather limited.

2.5 Conclusion

In this paper, we presented a novel control variate Monte Carlo method based on the concept of comonotonicity. The CoMC method is explained for basket options, Asian options and TVaR.

We evaluated the performance of the method in realistic cases by illustrative numerical examples. The realistic benchmark examples show that the precision of estimating the price of Asian and Basket options is drastically increased by employing the CoMC method while the computation time is not increased considerably compared to the crude Monte Carlo method. The parametric study revealed the strong dependence of the method performance on the correlation between assets for Basket option pricing. Moreover, we showed that decreasing the strike price increases the efficiency of the method in Asian option and Basket option pricing. Thus the CoMC method is best suited for in the money options.

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Chapter 3

Optimality of deductible for Yaari's model: a reappraisal.

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Abstract

The main purpose of this paper is to show that left monotone risk aversion, a meaningful refinement of strong risk aversion, characterizes Yaari's decision makers for whom deductible insurance is optimal. A second goal is to offer a detailed proof of the deductible's computation, which proves the tractability of Yaari's model under left-monotone risk aversion.

Keywords : Yaari's model, Jewitt's left-monotone risk aversion, optimality of deductible.

JEL classification: D80, D81.

3.1 Introduction

In the framework of EU model, Arrow (1965) proved that for a given premium, the optimal insurance contract for a EU risk averse decision maker is a contract with deductible. Gollier and Schlesinger (1996) obtained a nice generalization of this result by proving that this result holds also under strong aversion, whatever be the decision maker's decision model under risk.

Vergnaud (1997) refined this result by proving that for any left monotone risk averse decision maker (not necessarily strongly risk averse), whatever be the decision model under risk, the optimal contract for a given premium is a deductible policy.

This last result is important since strong risk aversion is disputable in some situations, while Jewitt (1989)'s refinement i.e. left monotone risk aversion appears to be better adapted to insurance. This adds further justification to RDEU (rank-dependent expected utility) models and in particular to Yaari (1987)'s model that allow the decision maker to be left monotone risk averse without being strongly risk averse, which is impossible in the EU model, see Chateauneuf et al. (2004).

The goal of the present paper is to revisit the optimality of deductible in the framework of Yaari's model. Actually we show in section 3.4 that while left-monotone risk aversion is sufficient for a Yaari's decision maker to prefer deductible (a known result since Vergnaud (1997), for which we propose what we hope to be a more direct proof, actually as *Vergnaud* whatever be the decision maker's decision model), it turns out that for Yaari's decision maker left-monotone risk aversion is also a necessary condition for optimality of deductible.

In actual fact a main new result of this paper is to prove that optimality of deductible characterizes left-monotone risk averse Yaari's decision makers.

Moreover it is clear that once the optimality of a deductible policy for the insurer has been established, the question of computing her optimal level of deductible has to be settled.

As pointed out in Chateauneuf et al. (1997), Doherty and Eeckhoudt (1995) have shown that this question is very tractable in Yaari's model, when dealing with strongly risk averse insurers. It turns out that Chateauneuf et al. (1997) stated a theorem proving that the computation of the deductible remains very tractable for Yaari's decision maker merely assumed to be left monotone risk averse, but in fact the proof of this theorem has never been published, so a second goal of this paper is to fill this gap, see Section 3.5.

The paper is organized as follows: section 3.2 describes the framework and gives the main definitions, section 3.3 introduces left monotone increase in risk, sections 3.4 and 3.5 are devoted to our main results respectively characterization of left monotone risk averse Yaari's decision maker through optimality of deductible and computation of the optimal level of deductible. Finally, section 3.6 concludes the paper.

3.2 Framework and Definitions

In this section, we describe the decision maker's preference and the structure of insurance contract. We assume the decision maker chooses his preference through Yaari's

model, a particular case of rank-dependent expected utility.

3.2.1 Yaari's Model

Under the rank-dependent expected utility (RDEU) model a decision maker is characterized by a utility function u and an increasing probability-transformation function $f : [0, 1] \rightarrow [0, 1]$ that satisfies $f(0) = 0$, $f(1) = 1$. Such a decision maker prefers a random variable X to a random variable Y if and only if $V(X) > V(Y)$ where the functional V is given by

$$\begin{aligned} V(Z) = V_{u,f}(Z) &= - \int_{-\infty}^{\infty} u(x) df(P(Z > x)) = - \int_{-\infty}^{\infty} u(x) df(1 - F(x)) \\ &= \int_{-\infty}^0 [f(P(u(Z) > t)) - 1] + \int_0^{\infty} f(P(u(Z) > t)) dt, \end{aligned}$$

see Quiggin (1982); Chateauneuf et al. (1997). The Yaari functional is the special case of $V(Z)$ where $V(Z) = V_{I,f}(Z)$. In fact the utility under certainty is the identity function $u(x) \equiv x$, see Yaari (1987).

In the context of insurance, we address prospects of the form $W - D$, such that W is a wealth endowment and D is a risky but insurable damage (defined on the support $[0, W]$). Therefore, the Yaari functional V is defined in terms of the damage distribution $F(D)$ as follows:

$$V = \int_{-\infty}^{+\infty} (W - D) df(F(D)). \quad (3.1)$$

3.2.2 Insurance contracts with Deductible structure

Consider a decision maker with initial deterministic wealth $W > 0$ and possible random damage $D \geq 0$ with distribution function F buying an insurance i.e. an indemnity $I(D)$ such that:

1. $0 \leq I(D) \leq D$, $\forall D \in \mathbb{R}^+$
2. It has a price π given by $\pi = (1 + m)E(I(D))$ where $m \geq 0$ is the loading factor.

The decision maker will be said to express preference for deductible if for a given premium π among all the possible indemnities I satisfying (1) and (2), he prefers the indemnity $I_d(D)$,

$$I_d(D) = (D - d)_+$$

where d is the level of the deductible.

In such a contract, the future wealth W_d of the decision maker is $W_d = W - \pi - D + (D - d)_+$. Therefore, the utility $u(d)$ of this contract is given under Yaari's model by:

$$u(d) = W - (1 + m) \int_d^{+\infty} (1 - F(t))dt - d + \int_0^d f(F(t))dt. \quad (3.2)$$

3.3 Left monotone increase in risk

In the framework of expected utility (EU), a random variable Y is a mean preserving spread (MPS) of a random variable X if and only if all risk averse expected utility maximizers prefer X to Y . It should be noted that even in EU, there are some counter-intuitive examples. For instance, if a risk averse expected utility maximizer D_1 is ready to pay c to exchange Y for a less risky X (i.e. Y MPS X), and if an expected utility maximizer D_2 is more risk averse than D_1 , it can happen that D_2 is ready to pay only $c' < c$ for the same exchange. This single example proved that this notion of increasing risk, MPS, is not universal.

The left monotone order have been constructed initially by Jewitt, see Jewitt (1989), to solve the problem arised in insurance by MPS. This notion of increasing risk seems to be linked with the EU model but in fact there is a model-free equivalent definition of left monotone increase in risk.

Definition 3.3.1. *For random variables X and Y with the same mean, Y is a left monotone increase in risk of X if $\int_{-\infty}^{F_Y^{-1}(p)} F_Y(p) \geq \int_{-\infty}^{F_X^{-1}(p)} F_X(p)$, $\forall p \in [0, 1]$. Let us recall that for any distribution F i.e. any mapping $F : \mathbb{R} \rightarrow \mathbb{R}$ non-decreasing, right-continuous such that $\lim_{t \rightarrow -\infty} F(t) = 0$, $\lim_{t \rightarrow +\infty} F(t) = 1$, $F^{-1} : [0, 1] \rightarrow \mathbb{R}$ is defined $\forall p \in [0, 1]$ by $F^{-1}(p) = \inf \{t \in \bar{\mathbb{R}}, F(t) \geq p\}$. Note that $F^{-1}(0) = -\infty$.*

In this definition, the upper limits of integration are arbitrary quantiles corresponding to equal probability level p . In fact, Y is a left monotone increase in risk of X if Y has more weight in the lower tail than X .

Chateauneuf et al. (2004) have been proved that when X and Y are discrete, with the same mean, left monotone increase in risk can be obtained by a finite sequence of following corresponding Pigou-Dalton transfers. To elaborate such a transformation, we consider the following generating process.

Let X and Y be two discrete random variables with distributions $\mathcal{L}(X) = (x_1, p_1; x_2, p_2; x_3, p_3; x_4, p_4)$ where $x_1 < x_2 < x_3 < x_4$ and

$$\mathcal{L}(Y) = (x_1 - \epsilon p_3, p_1; x_2, p_2; x_3 + \epsilon p_1, p_3; x_4, p_4), \quad (3.3)$$

where the outcomes are again in non-decreasing order.

One can prove that $E(X) = E(Y)$ and Y is a left monotone increase in risk of X . In fact for any X and Y such that $E(X) = E(Y)$, Y is a left monotone increase in risk of X if and only if Y can be obtained from X to a finite sequence of Pigou-Dalton transfers as above. In this spread, the minimal outcome is always spread out, but not necessarily the maximal outcome.

Lemma 3.3.2. *For every pair (X, Y) of discrete random variables with $E(X) = E(Y)$ such that Y is a left monotone increase in risk of X , Y can be reached from X by a finite sequence of transfers as in (3.3).*

Proof: See Chateauneuf et al. (2004). □

The following Definition 3.3.3 and Lemma 3.3.4 taken from Landsberger and Meilijson (1994) will be of great help for some proofs.

Definition 3.3.3. *Distribution G is a left-monotone simple spread of F if*

1. $E(G) = E(F)$

2. $\exists p_0 \in (0, 1)$ such that:

$$p \leq p_0 \implies (2.1) \quad G^{-1}(p) \leq F^{-1}(p)$$

$$(2.2) \quad d(p) = F^{-1}(p) - G^{-1}(p) \text{ is non-increasing on } (0, p_0]$$

$$p > p_0 \implies (2.3) \quad G^{-1}(p) \geq F^{-1}(p).$$

Lemma 3.3.4. *If G is a left-monotone simple spread of F then F is left-monotone less risky than G .*

3.3.1 Left monotone risk aversion

A decision maker is left monotone risk averse if and only if for every X and Y such that Y is a left monotone increase in risk of X then he prefers X to Y . The left monotone risk aversion is a weaker notion of risk aversion compare to the mean preserving spreads (strong risk aversion) but stronger than the one based on the preference for

the expected value of a random variable to the random variable itself. Next we discuss the left monotone increasing in risk in the Yaari's framework.

Lemma 3.3.5. *Any Yaari decision maker is a left monotone increase in risk if and only if the probability transformation function is star shaped¹ at 1 i.e. $\frac{1-f(p)}{1-p}$ is an increasing function of p on $[0, 1)$.*

Proof: See Chateauneuf et al. (1997). □

3.4 Optimality of deductible characterizes left monotone risk averse Yaari's decision maker

We first show that preference for deductible implies left-monotone risk aversion.

Theorem 3.4.1. *Any Yaari's decision maker who has preference for deductibles with any given premium is a left-monotone risk averse.*

Proof: As we have already mentioned, Chateauneuf et al. (2004) have been shown that a decision maker is left-monotone risk averse if and only if the decision maker prefers any discrete random variable X such that $\mathcal{L}(X) = (x_1, p_1; x_2, p_2; x_3, p_3; x_4, p_4)$ to any random variable Y such that $\mathcal{L}(Y) = (x_1 - \epsilon p_3, p_1; x_2, p_2; x_3 + \epsilon p_1, p_3; x_4, p_4)$ where $\epsilon \geq 0$. Therefore in order to prove that a decision maker who has preference for deductibles turns out to be a left-monotone decision maker, it is sufficient to prove the following Lemma.

Lemma 3.4.2. *Any decision maker who exhibits preference for deductible will prefer $\mathcal{L}(X) = (x_1, p_1; x_2, p_2; x_3, p_3; x_4, p_4)$ to $\mathcal{L}(Y) = (x_1 - \epsilon p_3, p_1; x_2, p_2; x_3 + \epsilon p_1, p_3; x_4, p_4)$, [Recall that through the definitions of the " \mathcal{L} ", one has $p_i \geq 0$, $\sum_{i=1}^4 p_i = 1$ and $x_1 < x_2 < x_3 < x_4$ and $x_1 - \epsilon p_3 < x_2 < x_3 + \epsilon p_1 < x_4$].*

Proof: Indeed it is enough to show that $\exists w > 0, D \geq 0, m \geq 0$ and $d \geq 0$ such that:

$$X = W - D - \pi + I_d(D)$$

$$Y = W - D - \pi + I(D)$$

Let $\mathcal{L}(D) = (0, p_4; x_4 - x_3, p_3; x_4 - x_2, p_2; d_1, p_1)$, $d = x_4 - x_1$ and d_1 be such that $d_1 - d - \epsilon p_3 > 0$.

¹A function $f \in F$ is star-shaped at m , if:
 $\frac{f(m)-f(p)}{m-p}$
 is an increasing function of p on $[0, m) \cup (m, 1]$.

Note that d_1 is consistent since $d_1 > x_4 - x_2$, actually $d_1 > x_4 - x_1 + \epsilon p_3$. Note also that $d_2 = x_4 - x_2 < d = x_4 - x_1 < d_1$. Therefore $E((D - d)_+) = p_1(d_1 - d)$ and π is necessarily of the type $\pi = (1 + m)p_1(d_1 - d)$ for some $m > 0$.

We need now to see if there exists some W such that our "new" $X = W - D - \pi + I_d(D)$ actually satisfies:

$$\mathcal{L}(X) = (x_1, p_1; x_2, p_2; x_3, p_3; x_4, p_4).$$

Considering the four states s_i related to p_i we must have:

- (i) $x_1 = W - d_1 - \pi + d_1 - d$
- (ii) $x_2 = W - (x_4 - x_2) - \pi + 0$
- (iii) $x_3 = W - (x_4 - x_3) - \pi + 0$
- (iv) $x_4 = W - 0 - \pi + 0$

It is immediate that $W = \pi + x_4$ is convenient. So for such a choice of W, D, π, d and m we actually get that our initial X satisfies $X = W - D - \pi + I_d(D)$.

It remains to check if one has actually that our initial Y is equal to $W - D - \pi + I(D)$ where $I(D)$ is a convenient indemnity idem est satisfying (i) and (ii).

Since (iii): $I(D) = Y + D + \pi - W$, one gets $E(I(D)) = E(Y) + E(D) + \pi - W$ but $E(Y) = p_1x_1 + p_2x_2 + p_3y_3 + p_4x_4 = E(X)$ so $E(I(D)) = E(X) + E(D) + \pi - W = E(I_d(D))$, therefore (ii) is satisfied.

Let us come to (i), from (iii) we get denoting d_i the damage if state s_i (related to probability p_i) occurs: $I(d_1) = x_1 - \epsilon p_3 + d_1 + \pi - x_4 - \pi = d_1 - (x_4 - x_1 + \epsilon p_3)$.

Note that since d_1 has been chosen such that $d_1 > (x_4 - x_1 + \epsilon p_3)$ we actually have $0 \leq I(d_1) \leq d_1$, $I(d_2) = x_2 + x_4 - x_2 - x_4$ so $0 \leq I(d_2) \leq d_2$ and $I(d_3) = x_3 + \epsilon p_1 + x_4 - x_3 - x_4$ i.e. $I(d_3) = \epsilon p_1$ but $x_3 + \epsilon p_1 < x_4$ hence $0 \leq I(d_3) \leq d_3$, also $I(d_4) = x_4 + 0 - x_4$ hence $0 \leq I(d_4) \leq d_4$ which complete the proof of Lemma 3.4.2. \square

Remark 3.4.3. *Note that if we had required that indemnities should satisfy the Moral Hazard requirement i.e. that what remains to be paid by the decision maker namely $D - I(D)$ should increase with the amount of the damage our Lemma 4 would remain valid. Actually: $d_4 - I(d_4) = 0 < d_3 - I(d_3) = x_4 - x_3 - \epsilon p_1 < d_2 - I(d_2) = x_4 - x_2 < d_1 - I(d_1) = x_4 - x_1 + \epsilon p_3$.*

Remark 3.4.4. *The proof of Theorem 3.4.1 shows that it is enough that a Yaari's decision maker has preference for deductible only in case of finite discrete random losses, in order to be a left-monotone risk averter.*

Theorem 3.4.5. (*Vergnaud (1997)*) *Any left-monotone risk-averse decision maker has preference for deductible.*

Proof: Consider a left-monotone decision maker with initial deterministic wealth W and possible stochastic loss $D \geq 0$, buying an insurance with indemnity $I(\cdot)$ where $0 \leq I(t) \leq t \forall t \in \mathbb{R}$ at price $\pi = (1 + m)E(I(D))$.

We intend to show that this decision maker will actually buy the insurance with deductible d where indeed $E((D - d)_+) = \frac{\pi}{1 + m}$.

From Lemma 3.3.4 it is enough to prove that Z is a left-monotone spread of Z_d where $Z = W - \pi - D + I(D)$, $Z_d = W - \pi - D + I_d(D)$ and $I(D) = (D - d)_+$.

Since for any random variable T and any $a \in \mathbb{R}$ one has $F_{T+a}^{-1} = F_T^{-1} + a$, it is enough to prove that $-Y = -D + I(D)$ is a left monotone spread of $-Y_d = -D + (D - d)_+$. Let $p_0 = F_{-Y}(-d)$ the proof will be completed if we show that $p_0 \in (0, 1)$ and that:

$$\begin{aligned} \forall p \leq p_0 \quad & F_{-Y}^{-1}(p) \leq F_{-Y_d}^{-1}(p) \\ & \text{and } F_{-Y_d}^{-1}(p) - F_{-Y}^{-1}(p) \text{ is non-increasing on } (0, p_0] \\ \forall p > p_0 \quad & F_{-Y}^{-1}(p) \geq F_{-Y_d}^{-1}(p). \end{aligned}$$

Note that $-Y_d = \text{Max}(-D, -d)$, and that $-Y = -D + I(D) \geq -D$ therefore $F_{-Y}(t) \leq F_{-D}(t) \forall t \in \mathbb{R}$. Hence

$$\begin{aligned} t < -d \quad & F_{-Y_d}(t) = 0 \leq F_{-Y}(t) \\ t \geq -d \quad & F_{-Y}(t) \leq F_{-D}(t) = F_{-Y_d}(t) \end{aligned} \tag{3.4}$$

It turns out that $p_0 > 0$, otherwise $p_0 = 0$ implies $F_{-Y_d}(t) \geq F_{-Y}(t) \forall t$ and $E(-Y_d) = E(-Y)$ entails $-Y_d = -Y$ a contradiction.

Similarly $p_0 = 1$ is impossible otherwise one would have $F_{-Y_d}(t) \leq F_{-Y}(t) \forall t$, hence $-Y_d = -Y$ a contradiction. So from the above single-crossing of F_{-Y_d} and F_{-Y} we obtain that $\exists p_0 \in (0, 1)$ namely $p_0 = F_{-Y}(-d)$ such that

$$\begin{aligned} \forall p \leq p_0 \quad & F_{-Y}^{-1}(p) \leq F_{-Y_d}^{-1}(p) \\ \forall p > p_0 \quad & F_{-Y}^{-1}(p) \geq F_{-Y_d}^{-1}(p). \end{aligned}$$

It remains to prove that $F_{-Y_d}^{-1}(p) - F_{-Y}^{-1}(p)$ is non-increasing on $(0, p_0]$. Since F_{-Y}^{-1} is

non-decreasing it is enough to see that $F_{-Y_d}^{-1}(p) = -d \quad \forall p \in (0, p_0]$.

From (3.4) we have $F_{-Y_d}(-d) \geq p_0$ but $-Y_d = \text{Max}(-D, -d)$ implies $F_{-Y_d}(t) = 0 \quad \forall t < -d$ hence $F_{-Y_d}^{-1}(p_0) = -d$. Furthermore, if $0 < p < p_0$ indeed $F_{-Y_d}^{-1}(p) \leq -d$, but since $F_{-Y_d}(t) = 0 \quad \forall t < -d$, this implies finally that $F_{-Y_d}^{-1}(p) = -d \quad \forall 0 < p \leq p_0$ which completes the proof. \square

3.5 Computing the optimal level of deductible for a left monotone Yaari decision maker

Once the optimality of a deductible contract for the decision maker has been established, the question of computing his optimal level of deductible has to be settled.

Consider a left monotone Yaari decision maker with an initial wealth $w \in \mathbb{R}_{++}$, facing an insurable risky loss L with distribution function $F : F(L) = P(L \leq \ell), \forall \ell \in \mathbb{R}$. Assume that $[0, \bar{\ell}]$ is the support of the random loss L and that the decision maker buys an indemnity I_d at price π given by $\pi(I_d) = (1 + m)E(I_d)$.

Theorem 3.5.1. *A strict left monotone risk averse Yaari decision maker will purchase full insurance if*

$$(1 + m)(1 - F(0)) - (1 - f(F(0))) < 0. \quad (3.5)$$

Otherwise, \bar{d} is an optimal level of deductible if and only if it satisfies

$$(1 + m)(1 - F(\bar{d}_-)) - (1 - f(F(\bar{d}_-))) \geq 0 \geq (1 + m)(1 - F(\bar{d})) - (1 - f(F(\bar{d}))). \quad (3.6)$$

Remark 3.5.2. *If F is continuous, indeed the inequality (3.6) in theorem 3.5.1 reduces to the following simple equation:*

$$(1 + m)(1 - F(\bar{d})) - (1 - f(F(\bar{d}))) = 0.$$

Proof: Let us consider the simplest case which is named case 1.

Case 1: The distribution function F of the loss L is assumed to be strictly increasing on $[0, \bar{\ell}]$ and continuous on \mathbb{R} .

We know that the decision maker aims at maximizing over $[0, \bar{\ell}]$, the function:

$$u(d) = w - d - (1 + m) \int_d^\infty (1 - F(t))dt + \int_0^d f(F(t))dt$$

where $u(\cdot)$ is well defined and continuous on $[0, \bar{\ell}]$, so $\text{Max } u(d)$ over $[0, \bar{\ell}]$ exists. Furthermore from continuity of f and F , we have $u'(d)$ exists on $[0, \bar{\ell}]$ and $u'(d) = (1 + m)(1 - F(d)) - (1 - f(F(d)))$, $\forall d \in [0, \bar{\ell}]$. Since F is strictly increasing and $F(\bar{\ell}) = 1$ one has $F(d) < 1 \forall d \in [0, \bar{\ell})$, hence $u'(d) = (1 - F(d))((1 + m) - \frac{1-f(F(d))}{1-F(d)})$, $\forall d \in [0, \bar{\ell})$.

Therefore on $[0, \bar{\ell})$ the sign of the derivative is equal to the sign of $g(d) = (1 + m) - \frac{1-f(F(d))}{1-F(d)}$ i.e. $\text{sign } u'(d) = \text{sign } g(d)$.

Let us assume that the decision maker is a strict left-monotone risk averter so $h(d) = \frac{1-f(F(d))}{1-F(d)}$ is strictly increasing on $[0, \bar{\ell})$ and consequently $g(d)$ is strictly decreasing on $[0, \bar{\ell})$, note that here $F(0) = 0$ so $g(0) = m$. Therefore if $m = 0$, $u'(0) = 0$ and $u'(d) < 0 \forall d \in [0, \bar{\ell})$.

Since u is continuous on $[0, \bar{\ell}]$ this implies that $\text{Max } u(d)$ over $[0, \bar{\ell}]$ is obtained at the unique point $d = 0$. That is if $m = 0$, the decision maker buys full insurance.

Assume now that $m > 0$ therefore $u'(0) > 0$, either $g(d) > 0 \forall d \in [0, \bar{\ell})$ i.e. $\lim_{p \rightarrow 1} \frac{1-f(p)}{1-p} \leq 1 + m$ and $\text{Max } u(d)$ over $[0, \bar{\ell}]$ will be obtained for $d = \bar{\ell}$, so the decision maker will not buy insurance, or $\lim_{p \rightarrow 1} \frac{1-f(p)}{1-p} > 1 + m$, and therefore u' will be first positive and then negative and by continuity of u' on $[0, \bar{\ell})$, there exist a unique $\bar{d} \in (0, \bar{\ell})$ where $u'(\bar{d}) = 0$ and at that point one gets the optimal deductible of the decision maker.

Summary of the results in case 1:

If $m = 0$ i.e. fair insurance, the decision maker will buy full insurance.

If $m > 0$ denoting $h(1_-) = \lim_{p \rightarrow 1} \frac{1-f(p)}{1-p}$ (note that this limit exists and is finite since $\frac{1-f(p)}{1-p}$ is increasing on $[0, 1)$). Either $h(1_-) \leq 1 + m$ and so the decision maker will not buy insurance or $h(1_-) > 1 + m$ and the decision maker will choose the level of deductible \bar{d} which is the unique solution of $u'(\bar{d}) = 0$ i.e. \bar{d} such that:

$$(1 + m)(1 - F(\bar{d})) = 1 - f(F(\bar{d})). \quad \square$$

Case 2: We now switch to the general case which needs the preliminary lemmas 3.5.3 and 3.5.4.

Lemma 3.5.3. *Let $u : [a, b] \rightarrow \mathbb{R}$ be continuous and such that $u'_+(\cdot)$ exists on (a, b) with $u'_+(x) \leq 0 \forall x \in (a, b)$ then u is non-increasing on $[a, b]$.*

Proof: Let us first prove that $\forall x \in (a, b)$ and $y \in (x, b]$ one has $u(y) \leq u(x)$. So take

$x \in (a, b)$. By hypothesis $u'_+(x) \leq 0$.

• Take $\epsilon > 0$ arbitrary, from $u'_+(x) \leq 0$, it comes that there exists $y_0 \in (x, b]$ such that $\forall y \in (x, y_0]$ one has:

$$u(y) - u(x) \leq \epsilon(y - x)$$

or else $f(y) = u(y) - \epsilon y \leq u(x) - \epsilon x = f(x)$.

• Let us prove that in fact $f(y) \leq f(x) \forall y \in (x, b]$. Let us define E :

$E = \{z \in (x, b] \text{ s.t. } y \in (x, z] \Rightarrow f(y) \leq f(x)\}$, $E \neq \emptyset$ since $y_0 \in E$. E is bounded above by b , so $\text{Sup}E$ exists. Denote $M =: \text{Sup}E$.

Let us prove that $M \in E$. Actually by definition of M , for any $z \in (x, M)$ one has $f(z) \leq f(x)$, take $z_n \in (x, M)$ with $z_n \uparrow M$ one has $f(z_n) \leq f(x) \forall n$, since f is continuous one gets $f(M) = \lim f(z_n) \leq f(x)$, so $M \in E$. The proof will be completed if we show that $M = b$.

• Assume $M < b$ and let us show a contradiction. Since $u'_+(M) \leq 0$, $\exists y_1(\epsilon) > M$ where $y_1(\epsilon) \in (M, b]$ such that $u(y) - u(M) \leq \epsilon(y - M) \forall y \in (M, y_1]$ hence $f(y) \leq f(M)$, and therefore since $f(M) \leq f(x)$ one gets $f(y) \leq f(x) \forall y \in (x, y_1]$ a contradiction since $y_1 > \text{Sup}E$.

Therefore $\forall \epsilon > 0$ one has $u(y) - \epsilon y \leq u(x) - \epsilon x \forall y \in (x, b]$ so $u(y) \leq u(x) \forall y \in (x, b]$, i.e. $\forall x \in (a, b)$ and $y \in (x, b]$ one has $u(y) \leq u(x)$. Remains to show $u(x) \leq u(a) \forall x \in [a, b]$. But let $x \in (a, b]$ and take $a < x_n < x$ one has $u(x) \leq u(x_n)$ let $x_n \downarrow a$, but u continuous implies $u(a) = \lim u(x_n) \geq u(x)$ which completes the proof. \square

Lemma 3.5.4. *Let $u : [a, b] \rightarrow \mathbb{R}$ be continuous and such that $u'_+(\cdot)$ exists and strictly negative on $J = (a, b)$ where $J \neq \emptyset$ then u is strictly decreasing on $[a, b]$.*

Proof: Let us first prove that for any given $x \in (a, b)$ one has $y \in (a, b]$ $y > x$ implies $u(y) < u(x)$. From $u'_+(x) < 0$ i.e. $\lim_{h \rightarrow 0^+} \frac{u(x+h) - u(x)}{h} < 0$ where $x + h \in (x, b]$, it comes that there exists $y_0 \in (x, b]$ such that $u(y) < u(x) \forall y \in (x, y_0]$. If $y_0 = b$ the proof is completed. If $y_0 < b$ it is enough to show that u is non-increasing on $(x, b]$ since $y \in (x, b]$ $y > y_0$ will imply $u(y) \leq u(y_0) < u(x)$ which implies $u(y) < u(x)$.

Let $E = \{z \in (x, b] \text{ s.t. } u(t) \leq u(x) \forall t \in (x, z]\}$. One has $E \neq \emptyset$ since $y_0 \in E$. E is bounded above by b , so $\text{Sup}E$ exists. Denote $M =: \text{Sup}E$.

Let us prove that $M \in E$. Actually by definition of M , for any $z \in (x, M)$ one has $u(z) \leq u(x)$, take $z_n \in (x, M)$ with $z_n \uparrow M$ one has $u(z_n) \leq u(x) \forall n$, since u is continuous one gets $u(M) = \lim u(z_n) \leq u(x)$, so $M \in E$.

Suppose $M < b$, then from $u'_+(M) < 0$ there exists $z_1 \in (M, b]$ such that $u(t) < u(M) \forall t \in (M, z_1]$, so since $u(M) \leq u(x)$, one gets $u(t) \leq u(x) \forall t \in (x, z_1]$ and $z_1 > M$ contradicts the definition of M , so $M = b$.

It remains to prove that $u(a) > u(x) \forall x \in (a, b]$. Let $a < x_n < x$, $x_n \downarrow a$, one has $u(x) < u(x_n) \forall n$, so $u(x) < u(x_{n_0}) \leq u(x_n) n \geq n_0$, $u(x_n) \uparrow u(a)$ since u is continuous so $u(x) < u(a)$ which completes the proof. \square

Theorem 3.5.1 is proved based on the results of lemma 3.5.3 and lemma 3.5.4 as follows.

Proof of Theorem 3.5.1: Let $[0, \bar{\ell}]$ be the support of the random variable of losses L where the c.d.f is F so $\bar{\ell} = \inf \{\ell \geq 0, F(\ell) = 1\}$. One has:

$$u'_+(d) = (1+m)(1-F(d)) - (1-f(F(d))) \text{ on } [0, \bar{\ell}),$$

$$u'_-(d) = (1+m)(1-F(d_-)) - (1-f(F(d_-))) \text{ on } (0, \bar{\ell}].$$

By hypothesis $F(\ell) < 1 \forall \ell \in [0, \bar{\ell})$, f is strictly increasing and continuous on $[0, 1]$ with $f(0) = 0$, $f(1) = 1$ and $\frac{1-f(p)}{1-p}$ in non-decreasing on $[0, 1]$.

Therefore:

$$u'_+(d) = g(d)(1-F(d)) \quad \forall d \in (0, \bar{\ell})$$

$$u'_-(d) = h(d)(1-F(d_-)) \quad \forall d \in (0, \bar{\ell})$$

$$\text{where } g(d) = 1+m - \frac{1-f(F(d))}{1-F(d)}, \quad h(d) = g(d_-).$$

Note that $h(d) \geq g(d) \forall d \in (0, \bar{\ell})$, hence $h(d) \leq 0$ implies $g(d) \leq 0$ and $g(d) \geq 0$ implies $h(d) \geq 0$.

So on $(0, \bar{\ell})$, $u'_+(d) \leq 0 \Leftrightarrow g(d) \leq 0$ since $1-F(d) \geq 0$ and $u'_-(d) \geq 0 \Leftrightarrow h(d) \geq 0$ since $1-F(d_-) \geq 0$ as well.

Note that $u'_+(0)$ exist and $u'_+(0) = (1-F(0))g(0)$. Note that g is defined on $[0, \bar{\ell})$ and is non-increasing.

$$\underline{\text{Case 1}} : 1+m < \frac{1-f(F(0))}{1-F(0)} \tag{3.7}$$

Let us show that in such a case $\text{Max } u(d)$ over $[0, \bar{\ell}]$ (which exists since u is continuous) is obtained for $\bar{d} = 0$. So in case 1 $\bar{d} = 0$ i.e. there exists a unique optimal deductible which proves to be full insurance. Note that this is (3.5) of theorem 3.5.1.

Proof: By hypothesis $g(0) < 0$ but g is non-increasing on $[0, \bar{\ell})$, hence $g(d) < 0 \forall d \in (0, \bar{\ell})$ so $u'_+(d) < 0 \forall d \in (0, \bar{\ell})$. Since u is continuous on $[0, \bar{\ell}]$, from Lemma 3.5.4 u is strictly decreasing on $[0, \bar{\ell}]$ hence the maximum of u on $[0, \bar{\ell}]$ is uniquely obtained

for $d = 0$, so $\bar{d} = 0$ is the optimal deductible. \square

$$\text{Case 2 : } \lim_{d \uparrow \bar{\ell}} \frac{1 - f(F(d))}{1 - F(d)} < 1 + m^1 \quad (3.8)$$

In such a case there exists a unique optimal deductible which is $\bar{d} = \bar{\ell}$, so the decision maker will ask for no insurance, and it is straightforward that (3.6) is satisfied with $\bar{d} = \bar{\ell}$.

Proof: From case 2, $h(\bar{\ell})$ exists finite strictly positive, and since h is non-increasing on $(0, \bar{\ell}]$, one gets $h(d) > 0$ on $(0, \bar{\ell})$, hence $u'_-(d) > 0$ on $(0, \bar{\ell})$. Since u is continuous on $[0, \bar{\ell}]$, it turns out from Lemma 3.5.4' (the version of Lemma 3.5.4 where $u'_-(\cdot)$ exists on (a, b) and $u'_-(x) > 0 \forall x \in (a, b)$) that u is strictly increasing on $[0, \bar{\ell}]$ and therefore that $\text{Max } u(d)$ over $[0, \bar{\ell}]$ is uniquely obtained at $\bar{d} = \bar{\ell}$.

Case 3 : Note that $r(d) = \frac{1-f(F(d))}{1-F(d)}$ is well-defined $\forall d \in [0, \bar{\ell})$ (indeed we eliminate the case when $F(0) = 1$, since in this case the decision maker would suffer for no loss, hence would not like to insure) and positive, hence since $r(d)$ is non-decreasing with d , we obtain that $\lim_{d \uparrow \bar{\ell}} r(d)$ exists eventually equal to $+\infty$.

This last case 3 occurs when case 1 and case 2 are falsified, i.e. when :

$$\frac{1 - f(F(0))}{1 - F(0)} \leq 1 + m \leq \lim_{d \uparrow \bar{\ell}} \frac{1 - f(F(d))}{1 - F(d)} \quad (3.9)$$

Let us now consider $E = \left\{ d \in [0, \bar{\ell}), \frac{1-f(F(d))}{1-F(d)} > 1 + m \right\}$.

• Either $E \neq \emptyset$ Since E is bounded from below, $\text{Inf}(E)$ exists, let $d_0 := \text{Inf}(E)$. From the definition of d_0 , one has $\frac{1-f(F(d))}{1-F(d)} > 1 + m \forall d \in [0, \bar{\ell})$ such that $d > d_0$, since $r(\cdot)$ is right-continuous on $[0, \bar{\ell})$ one gets $\frac{1-f(F(d_0))}{1-F(d_0)} \geq 1 + m$. If $d_0 = 0$ one has $\frac{1-f(F(0))}{1-F(0)} = 1 + m$ and $\frac{1-f(F(d))}{1-F(d)} > 1 + m \forall d \in (0, \bar{\ell})$, hence $g(d) < 0$ on $(0, \bar{\ell})$, so u is strictly decreasing on $[0, \bar{\ell}]$, and therefore there exists a unique optimal deductible for the decision maker i.e. $\bar{d} = 0$. To summarize the first sub-case is:

$$\text{Case 3.1 : } \frac{1 - f(F(0))}{1 - F(0)} = 1 + m < \frac{1 - f(F(d))}{1 - F(d)} \quad \forall d \in (0, \bar{\ell}) \quad (3.10)$$

and then the unique optimal deductible for the decision maker is $\bar{d} = 0$ i.e. the

¹Note that $\forall p \in [0, 1) \frac{1-f(p)}{1-p} \geq \frac{1-0}{1-0} = 1$, that $\frac{1-f(p)}{1-p}$ is non-decreasing on $[0, 1)$ so this means that here $\lim_{p \uparrow 1} \frac{1-f(p)}{1-p}$ exists and is finite.

decision maker choose full insurance. Again (3.6) is satisfied with $\bar{d} = 0$.

Let us switch to the second case, where $d_0 > 0$. In such a case since $d_0 = \text{Inf}(E)$, one has $\forall d \in [0, \bar{\ell})$, $d < d_0$, $\frac{1-f(F(d))}{1-F(d)} \leq 1+m$, hence $\frac{1-f(F(d_0-))}{1-F(d_0-)} \leq 1+m$. Therefore the second sub-case is:

Case 3.2: $\exists d_0 \in (0, \bar{\ell})$ such that :

$$\frac{1-f(F(d_0-))}{1-F(d_0-)} \leq 1+m \leq \frac{1-f(F(d_0))}{1-F(d_0)} \quad (3.11)$$

In such a case $\bar{d} = d_0$ is an optimal deductible. This optimal deductible is not unique if and only if there exists $d'_0 > 0$, $d'_0 < d_0$ such that:

$$\frac{1-f(F(d_-))}{1-F(d_-)} = \frac{1-f(F(d_0-))}{1-F(d_0-)} = 1+m \quad \forall d \in [d'_0, d_0]$$

in which case any $\bar{d} \in [d'_0, d_0]$ is an optimal deductible. Note that (3.6) is satisfied for any $\bar{d} \in [d'_0, d_0]$

Proof: For $d \in (0, d_0)$, we have $h(d) \geq 0$, therefore $u'_-(d) \geq 0$, hence Lemma 3.5.3' (the dual Lemma of Lemma 3.5.3) implies u is non-decreasing over $[0, d_0]$. Similarly, for $d \in [d_0, \bar{\ell})$, we have $g(d) \leq 0$, therefore, $u'_+(d) \leq 0$, hence Lemma 3.5.3 implies u is non-increasing over $[d_0, \bar{\ell}]$. Therefore d_0 is an optimal deductible.

Since $u(d)$ is non-decreasing over $[0, d_0]$, if the optimal deductible d_0 is not unique, there exists $d'_0 < d_0$ such that $u(d) = u(d_0)$ for $d \in [d'_0, d_0]$, the interval is closed due to the continuity of $u(\cdot)$. Hence $u'_-(d) = 0 \forall d \in (d'_0, d_0)$ and we have $h(d) = 0$ for $d \in (d'_0, d_0]$.

On the other hand if there exists $d'_0 < d_0$ such that $h(d) = 0$ for $d \in (d'_0, d_0]$ then $u'_-(d) \geq 0$ and $u'_+(d) \leq 0$ for all $d \in (d'_0, d_0]$. Therefore, u is both non-decreasing and non-increasing over this interval. Hence $u(d) = u(d_0)$ for $d \in [d'_0, d_0]$, in this case all the points $d \in [d'_0, d_0]$ are optimal. \square

We now need to consider the case:

• Or $E = \emptyset$ In such a case we have:

$$\frac{1-f(F(d))}{1-F(d)} \leq 1+m \quad \forall d \in [0, \bar{\ell}) \quad (3.12)$$

hence $\lim_{d \uparrow \bar{\ell}} \frac{1-f(F(d))}{1-F(d)} \leq 1+m$, therefore from (3.9):

$$\text{Case 3.3 : } \lim_{d \uparrow \bar{\ell}} \frac{1 - f(F(d))}{1 - F(d)} = 1 + m \quad (3.13)$$

Proof: In this case, by the assumption, we have $h(\bar{\ell}) = 0$. Since h is non-increasing on $(0, \bar{\ell}]$, we can conclude that $h(d) \geq 0$ on $(0, \bar{\ell}]$. Lemma 3.5.3' (the version of Lemma 3.5.3 where $u'_-(\cdot)$ exists on (a, b) and $u'_-(x) \geq 0 \forall x \in (a, b)$) implies that u is non-decreasing on $[0, \bar{\ell}]$. Therefore, $\bar{\ell}$ is an optimal deductible in this case.

Similar to the case 3.2, given that $u(d)$ is non-decreasing and continuous, $\bar{\ell}$ is not the unique optimal, if and only if there exists d' such that $0 \leq d' < \bar{\ell}$ and $\frac{1-f(F(d))}{1-F(d)} = 1 + m \forall d \in [d', \bar{\ell}]$ in which case any $d \in [d', \bar{\ell}]$ is an optimal deductible. Again it is straightforward that (3.6) is satisfied $\forall d \in [d', \bar{\ell}]$. This completes the proof of theorem 3.5.1.

3.6 Conclusion

While it is known since Vergnaud (1997) that whatever be the decision maker's model under risk, left-monotone risk aversion, a meaningful refinement of strong risk aversion introduced by Jewitt (1989), implies optimality of deductible, the main purpose of this paper is to show that in fact within the Yaari's model left-monotone risk aversion does characterize the optimality of deductible insurance.

A second main goal of this paper is to show that for such left-monotone Yaari's risk averters, the computation of the deductible is very tractable. Chateauneuf et al. (1997) stated a theorem related to this point, but in fact the proof of this theorem has never been published, this paper aims to fill this gap.

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