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Arthur Forey

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Université Pierre et Marie Curie

École doctorale de sciences mathématiques de Paris centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Arthur Forey

Invariants motiviques dans les corps valués

---

dirigée par François Loeser

Soutenue le 7 décembre 2017 devant le jury composé de :

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Parce que les mathématiques, ça sert à faire de la physique. La physique, ça sert à faire des frigidaires. Les frigidaires, ça sert à y mettre des langoustes, et les langoustes, ça sert aux mathématiciens, qui les mangent et sont alors dans de bonnes dispositions pour faire des mathématiques, qui servent à la physique, qui sert à faire des frigidaires, qui...

Laurent Schwartz

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Résumé

Cette thèse est consacrée à définir et étudier des invariants motiviques associés aux ensembles semi-algébriques dans les corps valués. Ceux-ci sont les combinaisons booléennes d'ensembles définis par des inégalités valuatives. L'outil principal que nous utilisons est l'intégration motivique, une forme de théorie de la mesure à valeurs dans le groupe de Grothendieck des variétés définies sur le corps résiduel.

Dans une première partie, on définit la notion de densité locale motivique. C'est un analogue valuatif du nombre de Lelong complexe, de la densité réelle de Kurdyka-Raby et de la densité p-adique de Cluckers-Comte-Loeser. C'est un invariant métrique à valeurs dans un localisé du groupe de Grothendieck des variétés. Notre résultat principal est que cet invariant se calcule sur le cône tangent muni de multiplicités motiviques. On établit aussi un analogue de la formule de Cauchy-Crofton locale. On montre enfin que dans le cas d'un germe de courbe plane définie sur le corps résiduel, la densité locale motivique est égale à la somme des inverses des multiplicités des branches formelles.

L'objet de la seconde partie est de définir un morphisme d'anneau du groupe de Grothendieck des ensembles semi-algébriques sur un corps valué K vers le groupe de Grothendieck de la catégorie d'Ayoub des motifs rigides analytiques sur K. On montre qu'il étend le morphisme qui envoie la classe d'une variété algébrique sur la classe de son motif cohomologique à support compact. Cela fournit donc une notion virtuelle de motif cohomologique à support compact pour les variétés rigides analytiques. On montre également un théorème de dualité permettant de comparer le motif cohomologique de la fibre de Milnor analytique avec la fibre de Milnor motivique.

Mots-clés

Intégration motivique, corps valués, densité locale, motifs rigides analytiques, fibre de Milnor motivique, cycles proches.

Motivic invariants in valued fields

Abstract

This thesis is devoted to define and study some motivic invariants associated to semi-algebraic sets in valued fields. They are boolean combinations of sets defined by valuative inequalities. Our main tool is the theory of motivic integration, which is a kind of measure theory with values in the Grothendieck group of varieties defined over the residue field.

In the first part, we define the notion of motivic local density. It is a valuative analog of complex Lelong number, Kurdyka-Raby real density and p-adic density of Cluckers-Comte-Loeser. It is a metric invariant with values in a localization of the Grothendieck group of varieties. Our main result is that it can be computed on the tangent cone with motivic multiplicities. We also establish an analog of the local Cauchy-Crofton formula. We finally show that the density of a germ of plane curve defined over the residue field is equal to the sum of the inverses of the multiplicities of the formal branches of the curve.

The goal of the second part is to define a ring morphism from the Grothendieck group of semi-algebraic sets defined over a valued field K to the Grothendieck group of Ayoub's category of rigid analytic motives over K. We show that it extends the morphism sending the class of an algebraic variety to the class of its cohomological motive with compact support. This gives a notion of virtual cohomological motive with compact support for rigid analytic varieties. We also show a duality theorem allowing us to compare the cohomological motive of the analytic Milnor fiber with the motivic Milnor fiber.

Keywords

Motivic integration, valued fields, local density, rigid analytic motives, motivic Milnor fiber, nearby cycles.

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Introduction

L'objet de cette thèse est de définir et d'étudier des invariants motiviques métriques et additifs dans les corps valués. On s'intéresse donc en particulier aux ensembles semi-algébriques dans un corps K muni d'une valuation v, qui sont des combinaisons booléennes d'ensembles de la forme

$$\{x \in K^m \mid \mathrm{v}(f(x)) \geq \mathrm{v}(g(x))\},$$

avec f et g des polynômes à coefficients dans K. Nos invariants sont motiviques dans le sens où ils sont à valeurs dans des groupes de Grothendieck mais également parce qu'il sont construits en utilisant l'intégration motivique. C'est une forme de théorie de la mesure pour les corps valués, à valeurs dans le groupe de Grothendieck des variétés définies sur le corps résiduel de K.

Groupe de Grothendieck

Le groupe de Grothendieck des k-variétés $\mathbf{K}(\mathrm{Var}_k)$ est le groupe abélien libre engendré par les classes d'isomorphismes $[X]$ de k-variétés X soumis aux relations $[X] = [Y] + [U]$ pour Y une sous variété fermée de X et U l'ouvert complémentaire. Le produit cartésien induit une structure d'anneau commutatif sur $\mathbf{K}(\mathrm{Var}_k)$. On notera $\mathbb{L} = [\mathbb{A}^1_k]$ la classe de la droite affine. On a par exemple $[\mathbb{P}^1_k] = \mathbb{L} + 1$, où $\mathbb{P}^1_k$ est la droite projective. Cet anneau est intéressant grâce à ses réalisations, c'est-à-dire les morphismes d'anneaux de source $\mathbf{K}(\mathrm{Var}_k)$. Citons par exemple si k est fini la mesure de comptage induite par le comptage de points

$$X \in \mathrm{Var}_k \mapsto \sharp X(k) \in \mathbb{Z}.$$

Pour un k général on a la caractéristique d'Euler induite par

$$X \in \mathrm{Var}_k \mapsto \sum_{i \geq 0} (-1)^i \dim(H^i_{\text{ét},c}(X \times \overline{k}, \mathbb{Q}_l)) \in \mathbb{Z},$$

ou encore le polynôme de Poincaré et le polynôme de Hodge-Deligne HD dans le cas où k est de caractéristique nulle. On pourra consulter l'article de Nicaise et Sebag [82] pour plus de détails.

Dans sa lettre à Serre du 16 août 1964 [26], Grothendieck postule l'existence d'un mor-

phisme d'anneaux de $\mathbf{K}(\mathrm{Var}_k)$ vers $\mathbf{K}(\mathrm{Mot}_{k,\mathbb{Q}})$ qui envoie la classe d'une variété projective lisse sur la classe de son motif associé. La catégorie $\mathrm{Mot}_{k,\mathbb{Q}}$ est la catégorie des motifs de Chow sur k à coefficients rationnels. Si k est de caractéristique nulle, Gillet et Soulé [48], Guillen et Navarro Aznar [49] puis Bitter [8] montrent son existence. L'approche de cette dernière fournit une présentation de $\mathbf{K}(\mathrm{Var}_k)$ dont les relations proviennent d'éclatements. Le formalisme des six opérations de Grothendieck développé par Ayoub [1] dans les catégories stables homotopiques $\mathrm{SH}(k)$ de Morel et Voevodsky [78] permet à Ivorra et Sebag [59] de construire un morphisme similaire vers $\mathbf{K}(\mathrm{SH}(k))$.

L'un des objectifs de cette thèse est de construire un morphisme similaire pour les ensembles semi-algébriques de K, en utilisant du côté motifs la catégorie $\mathrm{RigSH}(K)$ des motifs des variétés rigides analytiques d'Ayoub [3]. On utilisera pour cela l'intégration motivique.

Intégration motivique

Différentes théories d'intégration motivique ont été développées durant les vingt dernières années. Essayons de donner un aperçu de ce qu'on entend par intégration motivique. Une mise en garde : bien qu'on utilise le terme *intégration*, aucune de ces théories n'est une théorie de la mesure au sens classique de Lebesgue. Les mesures motiviques jouissent néanmoins de propriétés similaires à une mesure classique, comme un théorème de changement de variable. Comme déjà évoqué, le terme *motivique* vient du fait que ces mesures sont à valeurs dans un localisé du groupe de Grothendieck des variétés.

Soit k un corps de caractéristique nulle. Notons $K = k((t))$ et $\mathcal{O}_K = k[[t]]$ son anneau d'entiers. Les ensembles que l'on peut mesurer comprennent les ensembles semi-algébriques dans des corps valués. Ceux-ci sont vus soit comme des sous-ensembles de l'espace des arcs de variétés définies sur le corps résiduel, c'est le point de vue de Kontsevich [61] et Denef-Loeser [36], soit comme des ensembles définisables dans une théorie du premier ordre (voir Hrushovski-Kazdhan [54] et Cluckers-Loeser [23]), soit comme des sous-ensembles de schémas de Greenberg par Loeser-Sebag dans [73].

Notons μ_{mot} la mesure motivique. Par exemple, si X est une k-variété lisse, alors $\mu_{\mathrm{mot}}(X(\mathcal{O}_K)) = [X]$, où $[X]$ est la classe de X dans le groupe de Grothendieck $\mathbf{K}(\mathrm{Var}_k)$.

Afin d'établir une formule de changement de variables, on voudrait également pouvoir intégrer des fonctions. Guidé par l'analogie avec la mesure de Haar sur $\mathbb{Q}_p$, on veut définir l'intégrale de fonctions de la forme $x \in Y \mapsto \mathbb{L}^{-\alpha(x)}$, où $\alpha(x) = \mathrm{v}(P(x))$, avec P un polynôme à coefficients dans K. On pose donc

$$\int_Y^{\mathrm{mot}} \mathbb{L}^{-\alpha} = \sum_{n\in\mathbb{N}} \mu_{\mathrm{mot}}(\{x \in Y \mid \alpha x = n\})\mathbb{L}^{-n}.$$

Il faut tout de même s'assurer que le terme de droite à un sens. Une possibilité est de le

considérer dans la complétion $\mathcal{M}_k$ de $\mathbf{K}(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ par rapport à la filtration dimensionnelle. En analysant plus finement la sommation ci-dessus, Cluckers et Loeser montrent dans [23] qu'il suffit de se placer dans $\mathbf{K}(\mathrm{Var}_k)_{\mathrm{loc}} = \mathbf{K}(\mathrm{Var}_k)\left[\mathbb{L}^{-1}, \left(\frac{1}{1-\mathbb{L}^{-k}}\right)_{k\in\mathbb{N}^*}\right]$.

Cette définition posée, on peut maintenant énoncer le théorème de changement de variables. Soit $h : Y \to X$ une bijection entre deux ensembles mesurables. Alors on a l'égalité suivante :

$$\int_X^{\mathrm{mot}} \mathbb{L}^{-\alpha} = \int_Y^{\mathrm{mot}} \mathbb{L}^{-\alpha\circ h - \mathrm{v}(\mathrm{Jac}(h))}.$$

La fonction $\mathrm{v}(\mathrm{Jac}(h))$, pour ordre du jacobien, est la valuation du déterminant jacobien de h. Cette formule est l'ingrédient clef du résultat suivant.

Théorème 0.0.1 (Batyrev-Kontsevich)**.** *Soient X et X' deux variétés de Calabi-Yau propres et lisses birationellement équivalentes. Alors X et X' ont mêmes classes dans la complétion $\mathcal{M}_{\mathbb{C}}$ du groupe de Grothendieck. En particulier, X et X' ont mêmes nombres de Betti et de Hodge.*

Batyrev a montré l'égalité des nombres de Betti en utilisant des intégrales p-adiques et les conjectures de Weil pour les fonctions zeta. Konstevich a construit l'intégrale motivique afin de généraliser ce résultat, voir [61].

Donnons l'idée de sa preuve. Par la résolution des singularités d'Hironaka, il existe une variété lisse Y et deux morphismes propres birationels $h : Y \to X$ et $h' : Y \to X'$. Notons K_Y un diviseur canonique de Y, $K_{Y/X}$ et $K_{Y/X'}$ les diviseurs canoniques relatifs à h et h'. Si D est un diviseur de Y, l'ordre d'annulation selon D induit une fonction $\mathrm{ord}_D : Y(\mathcal{O}_K) \to \mathbb{Z}$. Comme X est Calabi-Yau, $K_{Y/X}$ et $K_{Y/X'}$ sont linéairement équivalents à K_Y. Comme $\dim(H^0(Y, K_Y)) = 1$, $\mathrm{ord}_{K_{Y/X}} = \mathrm{ord}_{K_{Y/X'}}$. Comme h est propre birationel, h induit une bijection $\widetilde{h} : Y(\mathcal{O}_K) \to X(\mathcal{O}_K)$. Par la formule de changement de variable, on a

$$[X] = \int_{X(\mathcal{O}_K)}^{\mathrm{mot}} \mathbb{1}_X = \int_{Y(\mathcal{O}_K)}^{\mathrm{mot}} \mathbb{L}^{-\mathrm{ord}_{K_{Y/X}}}$$

Par symétrie, on a aussi $[X'] = \int_{Y(\mathcal{O}_K)}^{\mathrm{mot}} \mathbb{L}^{-\mathrm{ord}_{K_{Y/X'}}} = \int_{Y(\mathcal{O}_K)}^{\mathrm{mot}} \mathbb{L}^{-\mathrm{ord}_{K_{Y/X}}}$.

Cette construction est publiée par Denef et Loeser dans [36] et [38] qui la généralisent au cas des variétés singulières. Ils appliquent également le théorème de changement de variable afin de montrer la rationalité de la fonction zeta motivique et de définir la fibre de Milnor motivique.

Fibre de Milnor

Avant de définir les fonctions zeta p-adiques et motiviques, rappelons la construction de la fibre de Milnor. Milnor introduit dans [77] un outil fondamental dans l'étude des singularités d'une variété analytique, qu'on appellera par la suite la fibre de Milnor. Soit g

une application analytique de $\mathbb{C}^n \to \mathbb{C}$ non constante, et $x \in \mathbb{C}^n$ vérifiant $g(x) = 0$. Soit $B = B(x, \varepsilon)$ la boule ouverte dans $\mathbb{C}^n$ de centre x et de rayon ε, $D = D(0, \eta)$ le disque ouvert dans $\mathbb{C}$ de rayon η; on note $D^* := D \setminus \{0\}$. Le théorème de Milnor affirme alors que pour $0 < \eta \ll \varepsilon \ll 1$, l'application

$$g_x : g^{-1}(D^*) \cap B \to D^*$$

est une fibration localement triviale. On considère le revêtement universel de D^*

$$\widetilde{D^*} = \{z \in \mathbb{C} | \Im(z) > -\log \eta\} \to D^* : z \mapsto \exp(iz),$$

et on pose

$$F_x := g^{-1}(D^*) \cap B \times_{D^*} \widetilde{D^*}.$$

C'est la fibre universelle de la fibration g_x, appelée la fibre de Milnor de g en x. Comme g_x est une fibration localement triviale, et que $\widetilde{D^*}$ est contractile, il y a une équivalence d'homotopie entre F_x et la fibre de g_x au-dessus d'un point quelconque de D^*.

La translation $t \mapsto t + 2\pi$ sur $\widetilde{D^*}$ induit un automorphisme de F_x, la monodromie. Il induit sur les espaces de cohomologie singulière $H^i_{sing}(F_x, \mathbb{Z})$ un automorphisme également appelé monodromie et noté M_x ; ses valeurs propres sont des racines de l'unité.

Cette construction se traduit dans le monde non-archimédien. Rappelons que $K = k((t))$. On note $\overline{K}$ sa clôture algébrique et v la valuation sur $\overline{K}$.

On peut définir

$$\mathcal{F}_{f,x} = \left\{ y \in \overline{K}^n \mid \mathrm{v}(y) \geq 0, f(y) = t, \mathrm{res}(y) = x \right\},$$

qu'on appelle fibre de Milnor analytique. On peut aussi la voir comme une variété rigide analytique sur K.

La condition $\mathrm{res}(y) = x$ signifie que l'on regarde les points dans la boule ouverte centrée en x de rayon valuatif 0. L'élément t s'interprète dans K comme un infiniment petit. On a donc bien l'analogie avec la construction classique de la fibre de Milnor.

Notons μ_n le groupe des racines n-ièmes de l'unité, $\hat{\mu} = \lim_{\leftarrow n} \mu_n$ et φ un générateur topologique de $\hat{\mu}$. L'action de φ sur $\mathcal{F}_{f,x} \times \overline{K}$ correspond à l'action de la monodromie M sur F_x.

On s'attend donc à retrouver à partir de $\mathcal{F}_{f,x}$ les invariants associés à la fibre de Milnor. En utilisant une théorème de comparaison de Berkovich [6], Nicaise et Sebag montrent dans [81] qu'il existe un isomorphisme canonique

$$H^i_{\mathrm{sing}}(F_x, \mathbb{Q}_l) \simeq H^i_{\mathrm{Ber}}(\mathcal{F}_{f,x} \times_K \overline{K}, \mathbb{Q}_l)$$

compatible avec les action de M et φ. Ci-dessus, H^i_{Ber} désigne le i-ème groupe de coho-

mologie étale de Berkovich.

Fonction zêta d'Igusa

Soit F un corps de nombres, $f \in F[x_1, ..., x_d]$ et L une complétion p-adique de F. Soit $\mathcal{O}_L$ son anneau d'entiers et $\mathcal{M}_L$ l'idéal maximal de $\mathcal{O}_L$. Notons μ_{Haar} la mesure de Haar sur L, normalisée par $\mu_{\text{Haar}}(\mathcal{O}_L) = 1$. Soit q le cardinal du corps résiduel de L et $|x|_L = q^{-v_L(x)}$ la valeur absolue L-adique. La fonction zêta locale d'Igusa est alors définie comme suit :

$$Z_{f,L}(s) = \int_{\mathcal{O}_L^d} |f|_L^s \, \mathrm{d}\mu_{\text{Haar}}.$$

Supposons que f est à coefficients dans $\mathcal{O}_L$, ce qui est le cas pour toutes les complétions de F sauf un nombre fini. On pose $T = q^{-s}$ et

$$N_n(f) = \sharp \left\{ x \in (\mathcal{O}_L/\mathcal{M}_L^n)^d \mid f(x) = 0 \mod \mathcal{O}_L/\mathcal{M}_L^n \right\}.$$

La série de Poincaré associée à f est

$$P_{f,L} = \sum_{n \in \mathbb{N}} q^{-dn} N_n(f) T^n.$$

On a alors $P_{f,L}(s) = \frac{1 - T Z_{f,L}(s)}{1-T}$.

Igusa étudie extensivement ces fonctions et en particulier montre que $Z_{f,L}$ est une fonction rationnelle en T. On pourra consulter son livre [58] ainsi que l'exposé de Denef [34]. Ses calculs le conduisent à formuler la conjecture de monodromie.

Conjecture 0.0.2. *Pour presque toutes les complétions p-adiques L de F, si s est un pôle de $Z_{f,L}$, alors $\exp(2\pi i \Re(s))$ est une valeur propre de la monodromie M_x de la fibre de Milnor F_x en un point complexe $x \in f^{-1}(0) \subset \mathbb{C}^n$.*

Une version plus forte de la conjecture affirme que sous les conditions ci-dessus, $\Re(s)$ est une racine du polynôme de Bernstein de f.

Les cas connus de cette conjecture sont notamment $d = 2$ par Loeser [70], $d = 3$ et f homogène par Rodrigues et Veys [88], f non-dégénérée par rapport à son polygone de Newton et non-résonnante par Loeser [71], $d = 3$ et f non-dégénérée par Lemahieu et Van Proeyen [66] pour la variante de la conjecture pour la fonction zêta topologique, puis Bories et Veys [10] pour la fonction zêta p-adique.

Fonction zêta motivique

Soit X un schéma de type fini sur k, un corps de caractéristique nulle. Le foncteur défini sur la catégorie des k-algèbres

$$R \mapsto \hom(\mathrm{Spec}(R[t]/(t^{n+1})), X)$$

est représentable par un schéma $\mathcal{L}_n(X)$ appelé l'espace des n-jets de X. En particulier, $\mathcal{L}_0(X) \simeq X$, $\mathcal{L}_1(X) \simeq TX$, l'espace tangent de X.

Soit $f \in k[x_1, ..., x_d]$. On note $X_\sigma = f^{-1}(0)$. On considère les ensembles

$$\mathcal{X}_n(f) = \left\{ \varphi(t) \in \mathcal{L}_n(\mathbb{A}_k^d)(k) \mid f \circ \varphi(t) = t^n + o(t^n) \right\}.$$

Ce sont des sous-ensembles constructibles du k-schéma $\mathcal{L}_n(\mathbb{A}_k^d)$. On peut donc en particulier considérer leur classe dans $\mathbf{K}(\mathrm{Var}_k)$ ou $\mathbf{K}(\mathrm{Var}_{X_\sigma})$.

La fonction zêta motivique de Denef et Loeser est définie comme suit :

$$Z_f^{\mathrm{mot}}(T) = \sum_{n \in \mathbb{N}^*} [\mathcal{X}_n(f)] \mathbb{L}^{-dn} T^n \in \mathbf{K}(\mathrm{Var}_{X_\sigma})[\mathbb{L}^{-1}][[T]].$$

En utilisant la formule de changement de variable et une résolution plongée des singularités de X_σ, Denef et Loeser montrent que Z_f^{mot} est une fonction rationnelle en T et formulent un analogue de la conjecture de monodromie. Cette nouvelle conjecture, dite de monodromie motivique, implique en particulier celle d'Igusa.

En utilisant la formule explicite pour Z_f^{mot}, ils constatent que cette fonction rationnelle est de degré négatif ou nul et donc donnent un sens à la limite

$$\psi_f = -\lim_{T \to +\infty} Z_f^{\mathrm{mot}}(T) \in \mathbf{K}(\mathrm{Var}_{X_\sigma})[\mathbb{L}^{-1}].$$

L'élément ψ_f est appelé le cycle proche motivique, pour $x \in X_\sigma$, on appelle $\psi_{f,x} = x^* \psi_f \in \mathcal{M}_k$ la fibre de Milnor motivique.

Cette définition se justifie notamment par le fait que si k est un sous-corps de $\mathbb{C}$, alors $\chi(\psi_{f,x}) = \chi(F_x)$ et $\mathrm{HD}(\psi_{f,x})$ est égal au polynome de Hodge-Deligne de la structure de Hodge mixte construite par Steenbrink sur les groupes de cohomologie de F_x, voir [90]. Cette égalité s'étend au groupe de Grothendieck des structures de Hodge avec action d'un endomorphisme quasi-unipotent, voir [35].

Le lien entre fibre de Milnor analytique et fonction zêta motivique est fait par Nicaise et Sebag dans [81] en utilisant l'intégration sur les schémas formels. Hrushovski et Loeser [55] obtiennent également des résultats en ce sens, en se passant de la résolution des singularités.

Théorie des modèles des corps valués

Les théories d'intégration motivique dont nous allons nous servir dans cette thèse, celle de Cluckers et Loeser [23] et celle de Hrushovski et Kazdhan [54], se placent toutes deux dans le contexte de la théorie des modèles des corps valués.

Ce domaine s'est développé dans les cinquante dernières années. Le premier résultat est le principe d'Ax-Kochen-Eršov. Il stipule que deux corps valués henséliens de caractéristique nulle sont élémentairement équivalents si et seulement si leurs corps résiduels et groupes de valeurs le sont. Il implique en particulier un principe de transfert assurant qu'une formule du langage des anneaux est vraie sur $\mathbb{F}_p((t))$ pour tout p assez grand si et seulement si elle est vraie sur $\mathbb{Q}_p$ pour tout p assez grand. Cela permet à Ax de répondre à la conjecture d'Artin.

Le résultat suivant est l'élimination des quantificateurs dans la théorie des corps valués algébriquement clos ACVF, par Robinson [87]. Macintyre prouve dans [75] un résultat d'élimination des quantificateurs pour les corps p-adiques.

Ensuite van den Dries et Scowcroft [89] puis Denef et van den Dries [40] étudient plus en détail la structure des ensembles semi-algébriques et sous-analytiques p-adiques et montrent en particulier un analogue du lemme de sélection des courbes réelles.

Pas montre dans [84] et [85] des théorèmes de décomposition cellulaire pour les définissables avec composante angulaire dans les corps valués Henséliens de caractéristique nulle, de ramification finie dans le cas de caractéristique résiduelle positive. Ces théorèmes sont l'outil de base de la théorie de Cluckers et Loeser.

Des structures enrichies avec des fonctions analytiques sont également étudiées. Nous avons déjà cité [40] pour la structure p-adique analytique. Lipshitz prouve un résultat d'élimination des quantificateurs pour les corps valués avec structure analytique dans [68]. Suivrons des travaux de Lipshitz et Robinson pour des versions uniformes d'élimination, citons notamment [69], puis de Cluckers et Lipshitz [20] pour un résultat d'élimination et de décomposition cellulaire.

Haskell, Hrushovski et Macpherson étudient dans [51] plus en détail la théorie ACVF et montrent en particulier qu'elle élimine les imaginaires dans le langage géométrique. Cela signifie que les quotients d'ensembles définissables par une relation d'équivalence définissable sont encore définissables. Leur notion de type stablement dominé permet à Hrushovski et Loeser dans [56] de donner une construction modèle-théorique de l'espace topologique sous-jacent à un espace de Berkovich. Ils prouvent en particulier que tout espace de Berkovich quasi-projectif admet un rétracte fort sur un polytope rationnel.

D'autres résultats d'élimination des imaginaires dans des corps valués vont suivre, citons notamment Hrushovski, Martin et Rideau [57] dans les corps p-adiques.

L'intégrale de Cluckers et Loeser

Après cette aparté modèle-théorique, revenons à l'intégration motivique. L'intégration de Cluckers et Loeser [23], [24] , [25] dont il sera question dans la première partie de cette thèse mesure les ensembles définissables dans la théorie des corps valués Henséliens de caractéristique nulle, dans le langage à trois sortes de Denef et Pas. Ce langage comporte une sorte pour le corps valué, une pour le corps résiduel et une pour le groupe de valeur. Outre la valuation v, on a une fonction composante angulaire $\overline{\mathrm{ac}}$ du corps valué vers le corps résiduel définie comme suit si $K = k((t))$:

$$\overline{\mathrm{ac}}\left(\sum_{i=i_0}^{+\infty} a_i t^i\right) = a_{i_0} \quad \text{si} \quad a_{i_0} \neq 0.$$

Le théorème de décomposition cellulaire de Pas [84] présente les ensembles définissables sous une forme adéquate pour définir leur mesure.

Cluckers et Loeser associent à tout ensemble définissable X l'anneau $\mathcal{C}(X)$ des fonctions constructibles motiviques sur X, puis définissent leurs intégrales. Outre le théorème de changement de variable, cela permet d'obtenir un théorème de Fubini et de traiter d'intégrales à paramètres : le résultat sera encore une fonction constructible motivique.

Cette théorie a des applications arithmétiques. Tout d'abord à travers le principe de spécialisation elle montre le caractère uniforme en p des intégrales p-adiques. Elle fournit aussi un principe de transfert entre intégrales sur $\mathbb{Q}_p$ et sur $\mathbb{F}_p((t))$, généralisant le principe d'Ax-Kochen-Eršov : on a égalité entre des intégrales sur $\mathbb{Q}_p$ pour tout p assez grand si et seulement si on a égalité entre les intégrales correspondantes sur $\mathbb{F}_p((t))$ pour tout p assez grand. D'après Cluckers, Hales et Loeser [18], cela s'applique notamment aux égalités d'intégrales orbitales impliquant le lemme fondamental, permettant de retrouver un résultat montré indépendamment par Waldspurger [97]. Cluckers, Gordon et Halupczok [17] prouvent également que l'intégrabilité peut se transférer et montrent l'intégrabilité locale des caractères d'Harish-Chandra en caractéristique positive.

L'intégrale de Hrushovski et Kazhdan

Hrushovski et Kazhdan construisent dans [54] une théorie d'intégration pour les définissables de théories de corps valués algébriquement clos, principalement ACVF et $\mathrm{ACVF}^{\mathrm{an}}$ où l'on rajoute des fonctions analytiques au langage. Ils se placent dans un langage à deux sortes. L'une pour le corps valué, nommée VF. L'autre, appelée RV, est définie comme suit. Soit L un corps valué, d'anneau de valuation $\mathcal{O}_L$ dont l'idéal maximal est $\mathcal{M}_L$, de corps résiduel k_L, de groupe de valeur Γ_L, on pose $\mathrm{RV}(L) = L^\times/(1+\mathcal{M}_L)$. On a donc une suite exacte de groupes abéliens

$$1 \to k_L^\times \to \mathrm{RV}_L \to \Gamma_L \to 0.$$

La sorte RV est donc un moyen de mêler corps résiduel et groupe de valeur. Cette suite est scindée si la valuation $v : L^\times \to \Gamma_L$ a une section, mais on ne le suppose pas en général.

Soit $\mathcal{C}$ une catégorie dont les objets sont des ensembles définissables et les flèches les bijections définissables. Le groupe de Grothendieck $\mathbf{K}(\mathcal{C})$ de $\mathcal{C}$ est le groupe abélien libre engendré par les classes d'isomorphismes $[X]$ d'objets de $\mathcal{C}$ soumis aux relations $[X] = [U]+[V]$ si X est l'union disjointe des ensembles définissables U et V. Le produit cartésien munit $\mathbf{K}(\mathcal{C})$ d'une structure d'anneau.

On a donc $\mathbf{K}(\mathrm{VF}_K)$ et $\mathbf{K}(\mathrm{RV}_K)$, l'indice K indiquant que l'on considère les ensembles définissables à paramètres dans K. En pratique, c'est plutôt une version graduée $\mathbf{K}(\mathrm{RV}_K[*])$ qu'on va utiliser. Grâce à l'élimination des quantificateurs, les ensembles définissables de VF_K sont précisément les ensembles semi-algébriques à paramètres dans K, et donc $\mathbf{K}(\mathrm{VF}_K)$ est le groupe de Grothendieck des ensembles semi-algébriques définis sur le corps valué K.

Hrushovski et Kazhdan construisent donc un isomorphisme

$$\oint : \mathbf{K}(\mathrm{VF}_K) \simeq \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}},$$

où I_{sp} est un idéal de $\mathbf{K}(\mathrm{RV}_K[*])$ explicitement décrit. Cet isomorphisme permet de construire des invariants additifs des ensembles semi-algébriques valués. L'isomorphisme d'intégration motivique

$$\mathbf{K}(\mu\mathrm{VF}_K) \simeq \mathbf{K}(\mu\mathrm{RV}_K[*])/\mathrm{I}'_{\mathrm{sp}}$$

s'en déduit. Le μ signifie que l'on enrichit les catégories, les objets sont maintenant des ensembles définissables avec forme volume et les morphismes doivent respecter ces formes. On peut maintenant trouver des morphismes de $\mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}}$ (ou $\mathbf{K}(\mu\mathrm{RV}_K[*])/\mathrm{I}'_{\mathrm{sp}}$) vers $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$, le $\hat{\mu}$ signifiant qu'on considère le groupe de Grothendieck des variétés avec $\hat{\mu}$-action. Cela a été en particulier utilisé par Hrushovski et Loeser [55] pour remontrer que les nombres de Lefschetz de la monodromie, définis par

$$\Lambda(M_x^n) = \mathrm{tr}(M_x^n \mid H^\bullet(F_x, \mathbb{Q}))$$

sont égaux à $\chi_c(x^*\mathcal{X}_n(f))$. Cela avait déjà été montré par Denef et Loeser dans [39], mais cette nouvelle preuve n'utilise plus la résolution des singularités.

Dans une autre direction, Nicaise, Payne et Schroeter utilisent cette intégrale dans [80] et [79] dans le cadre de problèmes de comptage en géométrie tropicale.

Contenu détaillé de la thèse

Comme on l'a vu ci-dessus, l'intégration motivique permet d'associer des invariants tant métriques qu'additifs aux ensembles semi-algébriques et de comparer ceux-ci. On se pro-

pose dans cette thèse de fournir deux exemples de ce principe, l'un métrique, l'autre additif. Les deux parties sont indépendantes.

Densité locale motivique

L'objet du premier chapitre de cette thèse est de définir une notion de densité locale dans les corps valués de caractéristique nulle. Rappelons les contextes où une telle notion a été définie.

Soit K un corps muni d'une distance et d'une mesure μ. La densité locale d'un ensemble $X \subseteq K^m$ de dimension d, en un point $x \in K^m$ est la limite si elle existe

$$\Theta_d(X, x) = \lim_{r \to 0} \frac{\mu_d(X \cap B(x, r))}{\mu_d(B(x, r))},$$

où $B(x, r)$ est la boule de centre x et rayon r. Si X est une variété analytique complexe ou réelle, la densité existe, par les résultats respectifs de Lelong [65] puis Kurdyka et Raby [63]. Dans le cas complexe, c'est un entier égal à la multiplicité de X en x, d'après Thie [93] et Draper [41]. Dans le cas réel, ce n'est plus un entier, mais Kurdyka et Raby montrent que la densité s'exprime comme somme des densités du cône tangent avec multiplicités.

Cluckers, Comte et Loeser étudient ensuite le cas p-adique dans [15]. Le corps K est un extension finie de $\mathbb{Q}_p$, la mesure μ la mesure de Haar et X un ensemble semi-algébrique ou sous-analytique. La limite définie ci-dessus n'existe plus, mais si l'on pose $\theta_n = \frac{\mu_d(X \cap B(x,n))}{\mu_d(B(x,n))}$, où $B(x, n)$ est la boule de centre x et rayon valuatif n, il existe un entier $e > 0$ tel que les suites extraites $(\theta_{ne+i})_{n \geq 0}$ convergent toutes, vers des $d_i \in \mathbb{Q}$. La densité locale de X en x est alors définie comme la moyenne arithmétique $\frac{1}{e} \sum_{0 \leq i \leq e-1} d_i$.

On va généraliser cette notion pour K un corps discrètement valué Hensélien de caractéristique nulle, en utilisant la mesure motivique de Cluckers et Loeser. On se limite dans cette introduction au cas des ensembles semi-algébriques de K, mais dans le corps du texte on travaillera plus généralement avec des ensembles définissables dans une théorie modérée de corps valués, au sens de Cluckers et Loeser [25]. Fixons donc X un sous-ensemble semi-algébrique de K^n, de dimension d. On définit pour $n \in \mathbb{Z}$ comme précédemment $\theta(n) = \frac{\mu_{\mathrm{mot}}(X \cap B(x,n))}{\mathbb{L}^{-nd}}$. La fonction θ appartient à $\mathcal{C}(\mathbb{Z})$, l'anneau des fonctions constructibles motiviques sur $\mathbb{Z}$. En utilisant l'orthogonalité entre corps résiduel et groupe de valeur, ainsi que l'élimination des quantificateurs pour l'arithmétique de Presburger, on montre qu'il existe un entier $e > 0$ tel que les suites extraites $(\theta(ne + i))_{n \geq 0}$ convergent toutes, vers des $d_i \in \mathbf{K}(\mathrm{Var}_k)_{\mathrm{loc}}$. La topologie sur $\mathbf{K}(\mathrm{Var}_k)_{\mathrm{loc}}$ est induite par le $\mathbb{L}$-degré. La densité locale motivique de X en x est alors définie par

$$\Theta_d(X, x) := \frac{1}{e} \sum_{0 \leq i \leq e-1} d_i \in \mathbb{Q} \otimes \mathbf{K}(\mathrm{Var}_k)_{\mathrm{loc}}.$$

On peut généraliser cette définition pour toute fonction $\varphi \in \mathcal{C}(K^n)$ supportée sur un

ensemble de dimension d et bornée au voisinage de x. Remarquons aussi que ce phénomène de périodicité apparait pour des ensembles algébriques, comme par exemple le cusp défini par l'équation $x^2 = y^3$.

On s'emploie ensuite à établir un analogue des théorèmes de Thie, Kurdyka et Raby puis de Cluckers, Comte et Loeser qui montrent que la densité locale se calcule sur le cône tangent avec multiplicités.

Soit Λ un sous-groupe multiplicatif définissable de $K^\times$. Le Λ-cône tangent de X en x est défini comme

$$C_x^\Lambda(X) = \{u \in K^m \mid \forall n \in \mathbb{N}, \exists y \in X, \exists \lambda \in \Lambda, \mathrm{v}(y-x) \geq n, \mathrm{v}(\lambda(y-x)-u) \geq n\}.$$

On notera l'analogie avec le cône tangent positif, avec Λ dans le rôle de $\mathbb{R}_+^*$. En pratique on n'aura à considérer uniquement des Λ de la forme

$$\Lambda_n = \left\{\lambda \in K^\times \mid \mathrm{v}(\lambda) = 0 \mod n, \overline{\mathrm{ac}}(\lambda) = 1\right\}$$

si la caractéristique résiduelle est nulle. Si la caractéristique résiduelle est positive, on considère uniquement des Λ de la forme

$$\Lambda_{n,r} = \left\{\lambda \in K^\times \mid \mathrm{v}(\lambda) = 0 \mod n, \overline{\mathrm{ac}}_r(\lambda) = 1\right\},$$

où $\overline{\mathrm{ac}}_r$ est la r-ème composante angulaire.

Afin d'attribuer des multiplicités au cône tangent, on utilise le résultat suivant. Il existe un ensemble définissable $Y \subset K^m$ de dimension strictement inférieure à d et une fonction définissable $g : X \backslash Y \to k^s$ telle que pour tout $\xi \in k^s$, $g^{-1}(\xi)$ est le graphe d'une application localement 1-Lipschitzienne $\varphi_\xi : U_\xi \subset K^d \to K^{m-d}$, quitte à permuter les coordonnées.

Le Λ-cône tangent avec multiplicités est alors défini par

$$CM_x^\Lambda(X) = [\{(y,\xi) \in K^m \times k^s \mid y \in C_x^\Lambda(g^{-1}(\xi))\}] \in \mathrm{C}^d(K^m).$$

Ici, $\mathrm{C}^d(K^m)$ est le quotient de l'anneau des fonctions constructibles motiviques de support de dimension au plus d par l'idéal des fonctions de support de dimension strictement plus petite que d. Négliger ces fonctions implique que $CM_x^\Lambda(X)$ ne dépend pas du choix de la fonction g.

Le résultat principal du premier chapitre est le théorème suivant, qui montre la stabilisation des cône tangents et que le cône tangent avec multiplicité permet de calculer la densité locale.

Théorème A. *Soit $X \subseteq K^m$ un ensemble définissable de dimension d. Il existe $\Lambda = \Lambda_n$*

tel que pour tout $\Lambda' \subset \Lambda$ *et tout* $x \in K^m$,

$$C_x^{\Lambda}(X) = C_x^{\Lambda'}(X)$$

et

$$\Theta_d(X, x) = \Theta_d(CM_x^{\Lambda}(X), 0).$$

Afin de montrer ce théorème, on développe une notion de stratification régulière satisfaisant un analogue de la condition w_f de Verdier. On utilise également (et étend pour la caractéristique résiduelle positive) des méthodes de découpage qui permettent de passer de fonctions localement 1-Lipschitzienne à globalement 1-Lipschitzienne, voir [19] et [16].

Grâce au principe de spécialisation, ce théorème nous permet d'obtenir une version uniforme du théorème de Cluckers, Comte et Loeser sur la densité locale p-adique.

Le court chapitre deux est constitué de compléments sur la densité locale. On établit tout d'abord une version motivique de la formule de Cauchy-Crofton locale :

Théorème B. *Soit* $X \subseteq K^n$ *un ensemble définissable de dimension* d *et* $x \in K^n$. *On a l'égalité suivante :*

$$\Theta_d(X, x) = \int_{V \in \Omega \subseteq G(n,n-d)} \Theta_d(p_{V!,x}(\mathbb{1}_X), 0)\omega_{n,n-d}(V).$$

Ci-dessus, $G(n, n-d)$ est la grassmannienne des sous-espaces vectoriels de K^n de dimension $n-d$ avec pour origine x, $\omega_{n,n-d}$ la fonction provenant de la mesure uniforme sur $G(n, n-d)$, p_V la projection linéaire $K^n \to K^n/V$ et la fonction $p_{V!,x}$ prend en compte les multiplicités motiviques des fibres.

On présente ensuite deux manières de calculer la densité locale de courbes algébriques. On fixe k un corps de caractéristique nulle algébriquement clos et $f \in k[x, y]$ un polynôme sans terme constant et sans facteurs carrés. Soit $f = f_1 \cdot ... \cdot f_r$ la décomposition en facteurs irréductibles de f dans $k[[x, y]]$, notons N_i la multiplicité de f_i. Soit $C = \{(x, y) \in K^2 \mid f(x, y) = 0\}$ où $K = k((t))$.

On montre alors la proposition suivante.

Proposition C. *La densité locale de* C *en 0 est égale à*

$$\Theta_1(C, 0) = \sum_{i=1}^{r} \frac{1}{N_i}.$$

En particulier, si $k = \mathbb{C}$, cela montre que la densité locale est un invariant topologique des singularités de courbes planes. On montre cette proposition d'abord dans le cas particulier où f est non-dégénéré par rapport à son polygone de Newton. Dans ce cas, on peut généraliser le calcul à des dimensions supérieures. On traite le cas général en utilisant des paramétrisations de Newton-Puiseux et la formule de changement de variable.

Motifs rigides virtuels

Le chapitre trois est consacré à définir et étudier un morphisme d'anneaux noté $\chi_{\mathrm{Rig}} : \mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RigSH}(K))$, où $\mathrm{RigSH}(K)$ est la catégorie des motifs des K-variétés rigides analytiques.

On fixe un corps k de caractéristique nulle contenant les racines de l'unité et on note $K = k((t))$. Ayoub construit dans [3] la catégorie $\mathrm{RigSH}(K)$ des motifs des K-variétés rigides analytiques au sens de Tate [91]. Elle est construite de manière similaire à la catégorie $\mathrm{SH}(K)$, mais en partant des préfaiscaux simpliciaux sur la catégorie des variétés rigides analytiques lisses. À toute K-variété rigide lisse X, on peut associer son motif rigide homologique $\mathrm{M}_{\mathrm{Rig}}(X)$ et cohomologique $\mathrm{M}^{\vee}_{\mathrm{Rig}}(X)$.

Ayoub prouve qu'il y a une équivalence de catégories entre celle-ci et la catégorie des motifs quasi-unipotents sur k notée $\mathrm{QUSH}(k)$. C'est une sous-catégorie triangulée de la catégorie $\mathrm{SH}(\mathbb{G}_{mk})$ engendrée par les motifs des variétés de la forme $X[T, T^{-1}, V]/(V^r - Tf)$ où X est une k-variété lisse et f une fonction régulière inversible.

On a déjà évoqué le morphisme $\chi_k : \mathbf{K}(\mathrm{Var}_k) \to \mathbf{K}(\mathrm{SH}(k))$ envoyant la classe d'une variété lisse sur la classe de son motif cohomologique à support compact. On peut le généraliser en un morphisme d'anneaux $\chi_{\hat{\mu}} : \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{QUSH}(k))$.

Le résultat principal du chapitre trois est le suivant.

Théorème D. *Il existe un unique morphisme d'anneaux*

$$\chi_{\mathrm{Rig}} : \mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RigSH}(K))$$

tel que pour toute K-variété rigide quasi-compacte lisse et connexe de dimension d, on a $\chi_{\mathrm{Rig}}([X]) = [\mathrm{M}_{\mathrm{Rig}}(X)(-d)]$.

De plus, le diagramme suivant commute :

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_K) \arrow[r] \arrow[d, "\chi_K"'] & \mathbf{K}(\mathrm{VF}_K) \arrow[r, "\oint", "\simeq"'] \arrow[d, "\chi_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}} \arrow[r, "\Theta\circ\mathcal{E}_c"] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[r] \arrow[d, "\chi_{\hat{\mu}}"] & \mathbf{K}(\mathrm{Var}_k) \arrow[d, "\chi_k"] \\
\mathbf{K}(\mathrm{SH}(K)) \arrow[r, "\mathrm{Rig}^*"'] & \mathbf{K}(\mathrm{RigSH}(K)) \arrow[rrr, "\simeq", "\mathfrak{R}"'] & & \mathbf{K}(\mathrm{QUSH}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}(k)).
\end{tikzcd}$$

En particulier le morphisme χ_{Rig} étend celui qui envoie la classe d'une K-variété algébrique sur la classe de son motif cohomologique à support compact. On peut donc qualifier $\chi_{\mathrm{Rig}}(X)$ de motif virtuel cohomologique à support compact. On prouve ce théorème en utilisant un calcul explicite dans le cas où X est la fibre générique d'un $k[[t]]$-schéma formel semi-stable.

On prouve également un théorème de dualité.

Théorème E. *Soit X une variété rigide lisse quasi-compacte, $\mathcal{X}$ un modèle formel de X, D un sous-ensemble localement fermé et propre de sa fibre spéciale $\mathcal{X}_\sigma$. Soit $]D[$ le tube de*

D dans X. C'est une sous-variété rigide (possiblement non quasi-compacte) de X. On a

$$\chi_{\mathrm{Rig}}(]D[) = [\mathrm{M}^{\vee}_{\mathrm{Rig}}(]D[)].$$

En particulier, si X est une K-variété rigide propre et lisse,

$$\chi_{\mathrm{Rig}}(X) = [\mathrm{M}^{\vee}_{\mathrm{Rig}}(X)].$$

Ce théorème montre en particulier que pour les variétés propres et lisses, le motif virtuel cohomologique à support compact coïncide avec le motif virtuel cohomologique, comme attendu.

On peut également l'appliquer pour un résultat de comparaison de la fibre de Milnor. On note toujours $\mathcal{F}_{f,x}$ la fibre de Milnor analytique, qu'on considère soit comme une K-variété rigide analytique, soit comme un ensemble semi-algébrique. On note également $x^*\psi_f$ la fibre de Milnor motivique de Denef-Loeser, qu'on considère dans $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$ ou $\mathbf{K}(\mathrm{Var}_k)$. On a que $\Theta \circ \mathcal{E}_c \circ \oint(\mathcal{F}_{f,x}) = x^*\psi_f$. En combinant les deux théorèmes ci-dessus, on a le corollaire suivant.

Corollaire F. *Pour tout polynôme $f \in k[X]$ et $x \in f^{-1}(0)$ un point fermé, on a*

$$[\mathfrak{R}(M^{\vee}(\mathcal{F}_{f,x})] = \chi_{\hat{\mu}}(x^*\psi_f) \in \mathbf{K}(\mathrm{QUSH}(k)).$$

C'est une généralisation équivariante du résultat d'Ayoub, Ivorra et Sebag [4] qui prouvent que

$$[1^* \circ \mathfrak{R}(M^{\vee}(\mathcal{F}_{f,x})] = \chi_k(x^*\psi_f) \in \mathbf{K}(\mathrm{SH}(k)).$$

Chapter 1

Motivic local density

This chapter appears in [44].

1.1 Introduction

Notions of local densities have been developed in various contexts since their introduction by Lelong in [65]. We recall what is known in the complex and real cases. Denote by λ_d the Lebesgue measure on $\mathbb{R}^n$ of dimension d and $B^n(a,r)$ the ball around $a \in \mathbb{R}^n$ of radius r. If $X \subseteq \mathbb{R}^n$ of dimension d and $a \in X$, the local density of X at a is the limit, if it makes sense,

$$\Theta_d(X,a) = \lim_{r \to 0} \frac{\lambda_d(X \cap B^n(a,r))}{\lambda_d(B^d(0,r))}.$$

In [65], Lelong first shows the existence of the limit if X is a complex analytic manifold of real dimension d. In this complex case, Thie in [93] expresses the local density as a sum of the local densities of components of the tangent cone of X at a with multiplicities, which implies that $\Theta_d(X,a)$ is a positive integer. Then in [41], Draper interprets the local density as the algebraic multiplicity of the local ring of X at a.

In [63], Kurdyka and Raby study the real subanalytic case and show the existence of the local density if X is real subanalytic. They prove an analogous of Thie's result by expressing it as a finite sum of densities of components of the positive tangent cone, with multiplicities, but the local density is no longer an integer. Lion gives in [67] a geometric interpretation of the local density using the Cauchy-Crofton formula and proves existence of the density of semi-pfaffian sets. Comte, Lion and Rolin show in [30] that the local density viewed as a function of the base point is a log-analytic function. In fact Kurdyka and Raby approach works in any real o-minimal setting, see for example [31] for a proof of the existence of local density in this setting.

Cluckers, Comte and Loeser study in [15] the local density in a p-adic context. They show the existence of a notion of local density for a semi-algebraic or subanalytic subset $X \subseteq K^n$, where K is a finite extension of the field of p-adic numbers $\mathbb{Q}_p$. The definition

is more subtle than just the analogous of the real one, with Lebesgue measure replaced by Haar measure, because the limit does not exist in general. However, one can consider the following sequence of normalized volumes

$$\theta_n = \frac{\mu_{d,K}(X \cap B(a,n))}{\mu_{d,K}(B(0,n))},$$

where $\mu_{d,K}$ is the Haar measure on K^d, extended to K^n and

$$B(a,n) = \{x \in K^n \mid \mathrm{ord}(x-a) \geq n\}$$

is the ball around a of valuative radius n. Then they show that there exists an integer $e > 0$ such that for any $i = 0, ..., e-1$, the sequence $(\theta_{i+ke})_{k\geq 0}$ converges to some d_i. Then they define the local density of X at a as the mean value

$$\Theta_d^K(X,a) = \frac{1}{e}\sum_{i=0}^{e-1} d_i.$$

In this p-adic setting, there is no natural notion of tangent cone, therefore they are led to study for any Λ, a multiplicative subgroup of $K^\times$ of finite index, a Λ-tangent cone. They show that for a given X, for Λ small enough, all the Λ-tangent cones coincide, hence there exists a distinguished tangent cone. Then they assign multiplicities to the tangent cone and show as in the real case that the local density can be computed on this tangent cone with multiplicities. One should point out that in [15], two different notions of multiplicities are introduced and claimed to coincide but it is not the case, see Section 1.6.1 ; only the most naive one using cardinality of fibers over the tangent cone leads to the correct notion of multiplicities.

In this paper, we generalize this notion. Fix $L = \mathbb{C}((t))$ the field of formal Laurent series over the complex numbers. In this introduction, we limit ourself to this field for simplicity, but in the next sections we work in greater generality, considering some Henselian fields of characteristic zero and all residue characteristic. We also do not work with a single fixed field but with a tame or mixed tame theory of fields, see Section 1.2.1. Consider the three sorted language of Denef-Pas, with one sort for the valued field L, with the language $\{0,1,+,-,\cdot,t\}$, one sort for the residue field $k_L = \mathbb{C}$ with the language $\{0,1,+,-,\cdot\}$ and one sort for the value group $\mathbb{Z}$ with the Presburger language $\left\{0,+,\leq,\{\equiv_n\}_{n\geq 1}\right\}$, where $\equiv_n$ is a binary relation for congruence modulo n. We also add a function $\mathrm{ord} : L^\times \to \mathbb{Z}$ which is interpreted as the valuation on $K^\times$ and a function $\overline{\mathrm{ac}} : L^\times \to k_L$ which we interpret by the angular component

$$\overline{\mathrm{ac}}\left(\sum_{i\geq i_0} a_i t^i\right) = a_{i_0} \text{ if } a_{i_0} \neq 0.$$

In this introduction, by definable we mean definable in this first order language. Consider $K_0(\mathrm{Var}_{\mathbb{C}})$, the Grothendieck group of varieties over $\mathbb{C}$. By variety, we mean a separated scheme of finite type over $\mathbb{C}$ but a more naive definition would lead to the same group. Recall that $K_0(\mathrm{Var}_{\mathbb{C}})$ is the abelian group generated by symbols $[X]$ for X a variety with relations

$$[X] = [Y] + [X \backslash Y],$$

where Y is a closed subvariety of X. The cartesian product induces a product on $K_0(\mathrm{Var}_{\mathbb{C}})$, making it a commutative ring (with unity the class of a point). Denote $\mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}]$ the class of the affine line and consider the localization

$$\mathcal{M}_{\mathbb{C}} = K_0(\mathrm{Var}_{\mathbb{C}}) \left[\mathbb{L}^{-1}, \left\{ \frac{1}{1 - \mathbb{L}^{-i}} \right\}_{i>0} \right].$$

In [23], Cluckers and Loeser assign to some definable subsets X of L^n an element of $\mathcal{M}_{\mathbb{C}}$, called the motivic volume of X. They define more generally a whole class of so-called motivic constructible functions that are stable under integration and with good properties like Fubini's theorem, change of variable formula and integration with respect to parameters. Denote μ_d the d-dimensional measure defined in [23], and let $B^n(a, m) = \{x \in L^n \mid \mathrm{ord}(x - a) \geq m\}$, the n-dimensional ball around $a \in L^n$ of valuative radius $m \in \mathbb{Z}$. Fix $X \subseteq L^n$ definable of dimension d, $a \in L^n$ and define the sequence of normalized volumes by

$$\theta_m = \frac{\mu_d(X \cap B^n(a, m))}{\mu_d(B^d(0, m))} \in \mathcal{M}_{\mathbb{C}}.$$

We prove that there exists an integer $e > 0$ such that for any $i = 0, 1, ..., e - 1$, the subsequence $(\theta_{i+ke})_{k \geq 0}$ converges to some $d_i \in \mathcal{M}_{\mathbb{C}}$ for the dimension topology. We then define the motivic local density of X at a by

$$\Theta_d(X, a) = \frac{1}{e} \sum_{i=0}^{e-1} d_i \in \mathbb{Q} \otimes \mathcal{M}_{\mathbb{C}}.$$

This periodic convergence can occur even if X is an algebraic set, for example the set $X = \left\{x^2 = y^3\right\}$ has a density of $1/2$.

In order compute the local density of the tangent cone, we have to consider Λ-tangent cones $C_a^{\Lambda}(X)$ for all Λ for the form

$$\Lambda_n = \left\{\lambda \in L^{\times} \mid \overline{\mathrm{ac}}(\lambda) = 1, \mathrm{ord}(\lambda) \equiv 0 \mod n\right\}.$$

We are able to show that there is a distinguished tangent cone, that is, there is a Λ such that for any smaller Λ', $C_a^{\Lambda'}(X) = C_a^{\Lambda}(X)$. This distinguished tangent cone captures the local properties of X at a, in the sense that one can compute the local density of X at a on this cone. To do so, one needs to assign multiplicities to this cone. Whereas in the

p-adic case the multiplicities are integers, here the multiplicities need to be some motives. In order to define the multiplicities, we partition X into graphs of 1-Lipschitz functions parametrized by (a definable of) the residue field. This allows us to define the tangent cone with multiplicities which is a motivic constructible function of support $C_a^\Lambda(X)$. Our main Theorem 1.3.25 is an analogue of respective theorems of Thie, Kurdyka-Raby and Cluckers-Comte-Loeser, stating that the local density of X is the local density of its tangent cone with multiplicities. By the specialization principle of Cluckers-Loeser, this implies a uniformity result for p-adic local density.

In Section 1.2, we define the framework of (mixed-)tame theories, prove some results about Lipschitz continuous function, a weak curve selection lemma and recall the setting of Cluckers-Loeser theory of motivic integration. In Section 1.3, we define the motivic local density and the tangent cone with multiplicities in order to state our main theorem. Section 1.4 is devoted to establishing the existence of regular stratification satisfying analogies of Verdier condition (w_f). Section 1.5 is devoted to the proof of the main theorem. In Section 1.6, we first show how our results specialize to the p-adic case and then prove a uniformity theorem for p-adic local density.

1.2 Preliminaries

1.2.1 Tame theories of valued fields

A discretely valued field K is a field with a surjective group homomorphism

$$\mathrm{ord} : K^\times \to \mathbb{Z},$$

extended by $\mathrm{ord}(0) = \infty$ and satisfying for all $x, y \in K^\times$,

$$\mathrm{ord}(x + y) \geq \min \{\mathrm{ord}(x), \mathrm{ord}(y)\}.$$

A ball in K is a set of the form $\{x \in K \mid \mathrm{ord}(x - a) \geq \alpha\}$ for some $a \in K$ and $\alpha \in \mathbb{Z}$. The collection of all balls of K forms a basis of the so-called valuation topology on K. The field K is called Henselian if the valuation ring $\mathcal{O}_K$ is Henselian. Write $\mathfrak{M}_K$ for the maximal ideal of $\mathcal{O}_K$.

Definition 1.2.1. Fix p to be either 0 or a prime number, and an integer e. A $(0, p, e)$-field is a Henselian discretely valued field K of characteristic 0, of residue characteristic p, together with a chosen uniformizer ϖ_K of the valuation ring $\mathcal{O}_K$ of K, such that either $0 = p = e$ or $p > 0$ and the ramification degree is e, that is, $\mathrm{ord}(\varpi_K^e) = \mathrm{ord}(p) = e$.

We will always identify the value group of a $(0, p, e)$-field with $\mathbb{Z}$. For example, $\mathbb{Q}_p$ with p as uniformizer is a $(0, p, 1)$-field. A $(0, p, e)$-field comes with higher order angular

components maps, defined for $n \geq 1$,

$$\overline{\mathrm{ac}}_n : K^\times \to R_{n,K} := \mathcal{O}_K/\mathfrak{M}_K^n,\ x \mapsto \varpi_K^{-\mathrm{ord}(x)} x \mod \mathfrak{M}_K^n,$$

extended by $\overline{\mathrm{ac}}_n(0) = 0$. Write $\overline{\mathrm{ac}} = \overline{\mathrm{ac}}_1$, note that the maps $\overline{\mathrm{ac}}_n$ are multiplicative on K and coincide on $\mathcal{O}_K$ with the natural projection $\mathcal{O}_K \to R_{n,K}$. Note also that $R_{1,K} = k_K$ is the residue field of K. We call the $R_{n,K}$ the residue rings of K.

Definition 1.2.2 (First order language)**.** We define now the semi-algebraic language that we will use to describe subsets of $(0,p,e)$-fields. Let $\mathcal{L}_{\mathrm{high}}$ the first order language consisting of one sort for the valued field, one sort for the value group, and one sort for each residue ring. On these sorts, $\mathcal{L}_{\mathrm{high}}$ consists of the language of rings $(0,1,+,-,\cdot)$ and a symbol π for the uniformizer on the valued field, the language of Presburger arithmetics $(0,1,+,-,\leq, \{\cdot \equiv \cdot \mod n\}_{n>1})$ on the value group, the language of rings on each residue ring, a symbol ord for the valuation map, symbols $\overline{\mathrm{ac}}_n$ for the higher order angular components, and for $n \geq m$, the symbol $p_{n,m}$ for natural projection between residue rings R_n and R_m. We will also use the Denef-Pas language $\mathcal{L}_{\mathrm{DP}}$ which is the restriction of $\mathcal{L}_{\mathrm{high}}$ with only three sorts : one for the valued field, one for the value group and one for the residue field and only one angular component $\overline{\mathrm{ac}}$.

Any $(0,p,e)$-field has a natural interpretation as $\mathcal{L}_{\mathrm{high}}$-structure. Let $\mathcal{T}_{(0,p,e)}$ the $\mathcal{L}_{\mathrm{high}}$-theory of sentences true in every $(0,p,e)$-field. As concrete list of axioms that implies this theory can be found in the work of Pas [85], in particular, $\mathcal{T}_{(0,p,e)}$ eliminates field quantifiers.

Let $\mathcal{L}$ a language with the same sorts as $\mathcal{L}_{\mathrm{high}}$, containing $\mathcal{L}_{\mathrm{high}}$, and $\mathcal{T}$ be an $\mathcal{L}$-theory containing $\mathcal{T}_{(0,p,e)}$.

The following axioms will be about properties of every model $\mathcal{K}$ of $\mathcal{T}$, even those which are not $(0,p,e)$-fields (this happens if the value group is not isomorphic to $\mathbb{Z}$). By abuse of notation, we will often identify $\mathcal{K}$, a model of $\mathcal{T}$, with the associated valued field K.

We call the sort for the valued field the main sort, and the other ones the auxiliary sorts. We call the collection of all the sorts for the residue rings the residue sorts.

We will need the following notion of **k**-partition, in order to work piecewise over the residue rings.

Definition 1.2.3 (Definable **k**-partition)**.** If X is a definable set of a model K of $\mathcal{T}$, a definable **k**-partition of X is an injective definable function

$$g : X \to \prod_{i=1}^{s} R_{n_i,K},$$

for some integers $n_i \geq 1$. A fiber of g is called a **k**-part of X.

Given two definable $\mathbf{k}$-partitions g and g' of X, g' is said to be a refinement of g if every $\mathbf{k}$-part of X for g' is included into a $\mathbf{k}$-part of X for g. It is clear that given two definable $\mathbf{k}$-partitions, we can find a third one which is a refinement of the two others.

If $X \subseteq A \times B$, and $b \in B$, we denote the fiber of X over b by

$$X_b := \{a \in A \mid (a, b) \in X\}.$$

Definition 1.2.4 (Jacobian property for functions)**.** Let K be a model of $\mathcal{T}_{(0,p,e)}$. Let $F : X \subseteq K \to K$ be a function. We say that F has the Jacobian property on X if for every ball $B \subseteq X$, either F is constant on B, or the following properties hold :

1. F is injective on B and $F(B)$ is a ball,
2. F is C^1 on B with derivative F',
3. F' is nonzero on B and $\mathrm{ord}(F')$ is constant on B,
4. for all $x, y \in B$ with $x \neq y$, we have

$$\mathrm{ord}(F') + \mathrm{ord}(x - y) = \mathrm{ord}(F(x) - F(y)).$$

Moreover, if $n > 0$ is an integer, we say that F has the n-Jacobian property if the following additional conditions hold :

6. $\overline{\mathrm{ac}}_n(F')$ is constant on B,
7. for all $x, y \in B$ with $x \neq y$, we have

$$\overline{\mathrm{ac}}_n(F') \cdot \overline{\mathrm{ac}}_n(x - y) = \overline{\mathrm{ac}}_n(F(x) - F(y)).$$

Definition 1.2.5 (Jacobian property for $\mathcal{T}$)**.** Say that the Jacobian property holds for the $\mathcal{L}$-theory $\mathcal{T}$ if any model K satisfies the following. For any integer $n \geq 1$, any definable function $F : X \subset K \times Y \to K$, there is a definable $\mathbf{k}$-partition of X such that for any $y \in Y$ and any $\mathbf{k}$-part of X_y, the restriction of F to this set has the Jacobian property. In more explicit terms, if g is the function witnessing the $\mathbf{k}$-partition, then for every ξ and every $y \in Y$, then

$$x \in g^{-1}(\xi)_y \mapsto F(x, y)$$

has the Jacobian property.

The following definition translates the stable embedding of the value group and orthogonality between value group and residue rings.

Definition 1.2.6 (Split)**.** Say that the theory $\mathcal{T}$ splits if the following holds. For any model $\mathcal{K}$, with valued field K, value group Γ, and residue rings $R_{n,K}$, we have :

1. any $\mathcal{K}$-definable subset of Γ^r is Γ-definable in the language $(+, -, 0, <)$,
2. any definable subset $X \subseteq \Gamma^r \times \prod_{i=1}^{s} R_{n_i,K}$ is equal to a finite disjoint union of $Y_i \times Z_i$ where the Y_i are definable subsets of Γ^r and the Z_i are definable subsets of $\prod_{i=1}^{s} R_{n_i,K}$.

The notion of b-minimality has been introduced in [22], it is closely related to the concept of cell decomposition. It is shown in [25], Corollary 3.8 that the following notion of finite b-minimality implies b-minimality.

Definition 1.2.7 (Finite b-minimality)**.** Call $\mathcal{T}$ finitely b-minimal if the following holds for any model K. Each locally constant definable function $g : K \to K$ has finite image, and for any definable set $X \subseteq K \times Y \to K$, where $Y \subseteq K^n$ is definable, there exists an integer $\ell \geq 1$, a **k**-partition of X

$$g : X \to A,$$

a definable function

$$c : Y \times A \to K,$$

and a definable $B \subseteq \Gamma \times Y$ such that for every $\xi \in A$ and $y \in Y$ such that if $g^{-1}(\xi)_y$ is nonempty, $g^{-1}(\xi)_y$ is either

1. equal to the singleton $\{c(y, \xi)\}$,
2. equal to the ball $\{x \in K \mid \overline{\mathrm{ac}}_\ell(x - c(y, \xi)) = \zeta, \mathrm{ord}(x - c(y, \xi)) \in B_y\}$ where ζ is one of the component of ξ.

In the first case, we say X is a 0-cell, in the second we say it is a 1-cell. We call c the center of the cell, although it is not necessarily unique and does not need to lie in the cell. We say that the set $g^{-1}(\xi)_y$ is in $c(y, \xi)$-config.

The abstract notion of tame theories is introduced by Cluckers and Loeser in [25], but examples of such theories were already previously considered, see Example 1.2.9 below.

Definition 1.2.8 (Tame theory)**.** Let $\mathcal{T}$ a theory containing $\mathcal{T}_{(0,p,e)}$ in a language $\mathcal{L}$ with the same sorts as $\mathcal{L}_{\text{high}}$, which splits, is finitely b-minimal, has the Jacobian property and has at least one $(0, p, e)$-field as a model. Then we call $\mathcal{T}$ a tame theory of fields if $e = 0$ and a mixed tame theory of fields if $e > 0$.

If $\mathcal{T}$ is an $\mathcal{L}$-theory with he same sorts as $\mathcal{L}_{\text{DP}}$, contains $\mathcal{T}_{(0,0,0)}$, splits, is finitely b-minimal and has the Jacobian property, we also call $\mathcal{T}$ a tame theory of fields. A $\mathcal{T}$-field is a model of $\mathcal{T}$ that is a $(0, p, e)$-field.

Example 1.2.9. 1. The first examples of (mixed) tame theory are given by Pas in [84] and [85] : the theory $\mathcal{T}_{(0,p,e)}$ is a (mixed) tame theory of fields. This is the semi-algebraic example.

2. We now define an analytic language. Let K a $(0,p,e)$-field. Let $K\{x_1,...,x_n\}$ the ring of formal power series $\sum_{i\in\mathbb{N}} a_i x^i$ over K such that $\mathrm{ord}(a_i)$ goes to $+\infty$ when the i goes to $+\infty$. If K is complete, this coincide with power series converging on $\mathcal{O}_K^n$. Then let $\mathcal{L}_{K,\mathrm{an}}$ the language $\mathcal{L}_{\mathrm{high}}$ with a function symbol for each element of $K\{x_1,...,x_n\}$ (for all $n>0$). Then each $(0,p,e)$-complete extension L of K has a natural $\mathcal{L}_{K,\mathrm{an}}$-structure, a function $f\in K\{x_1,...,x_n\}$ is interpreted as the corresponding function $\mathcal{O}_L^n\to L$ and extended by 0 outside $\mathcal{O}_L^n$. Let $\mathcal{T}_{K,\mathrm{an}}$ the theory of all such fields L. Then $\mathcal{T}_{K,\mathrm{an}}$ is a (mixed) tame theory by [21].

3. More generally, for every analytic structure defined in [20], with associated language $\mathcal{L}$ and theory $\mathcal{T}$, if $\mathcal{T}$ has a $(0,p,e)$ model, then $\mathcal{T}$ is a (mixed) tame theory.

4. Let $\mathcal{T}$ a (mixed) tame theory in a language $\mathcal{L}$, and R be a subring of a $\mathcal{T}$-field. Let $\mathcal{L}(R)$ the language $\mathcal{L}$ with additional constant symbols for elements of R. Any $\mathcal{T}$-field that is an algebra over R and with ord and $\overline{\mathrm{ac}}_n$ extending the one on R, can be viewed as an $\mathcal{L}(R)$-structure. Let $\mathcal{T}(R)$ the $\mathcal{L}(R)$-theory of such structures. Then $\mathcal{T}(R)$ is a (mixed) tame theory, by [25], Proposition 3.11.

1.2.2 Definable subassignments

Fix a (mixed-)tame theory of fields $\mathcal{T}$. Fix n,ℓ,s some nonnegative integers, and $r=r_1,\dots,r_\ell$ a tuple of ℓ nonnegative integers. Define $h[n,r,s]$ to be the functor form the category of $\mathcal{T}$-fields to the category of sets sending any $\mathcal{T}$-field K to

$$h[n,r,s](K)=K^n\times R_{r_1,K}\times\cdots\times R_{r_\ell,K}\times\mathbb{Z}^s.$$

A definable subassignment of $h[n,r,s]$ is the data for every $\mathcal{T}$-field K of a subset X_K of $h[n,r,s](K)$ such that for some $\mathcal{L}$-formula φ (with free variables of corresponding lengths), $X_K=\varphi(K)$. Most often, we will just say that X is definable and call X_K a definable subset. A definable function $f:X\to Y$ is a definable subassignment G such that $G(K)$ is the graph of a function $f_K:X_K\to Y_K$.

As usual in model theory, we fix a $\mathcal{T}$-field K with residue rings sufficiently saturated, such that the study of definable subassignments X in $\mathcal{T}$ is reduced to the study of definable subsets X_K

1.2.3 Dimension theory

A (mixed-)tame theory is b-minimal in the sense of [22], see [25, Corollary 3.8]. In particular, one can use dimension theory for definable in b-minimal theories. Fix a (mixed-)tame theory $\mathcal{T}$ and a $\mathcal{T}$-field K. The dimension of a definable set $X \subseteq K^n$ is the biggest d such that for some coordinate projection $p : K^n \to K^d$, $p(X)$ contains an open ball of K^d (with convention $\dim(\emptyset) = -\infty$).

If $X \subseteq K^n \times S$, where S is a product of residue rings and $\mathbb{Z}^s$, then the dimension of X is the dimension of $p(X)$ with p the projection to K^n.

If $f : X \to Y$ is definable, then the dimension of X over Y (also called dimension of X relatively to Y) is the maximum of the dimensions of the fibers $f^{-1}(y)$ over $y \in Y$. Denote $\dim(X)$ the dimension of X and $\dim(X/Y)$ the dimension of X over Y (which depends implicitly on f).

If X is a definable subassigment, it may happen that $\dim(X_K) \neq \dim(X_L)$ for K and L two $\mathcal{T}$-fields. For example, for $X = \{x \mid x^2 + 1 = 0\}$, $K = \mathbb{R}((t))$, $L = \mathbb{C}((t))$, then $\dim(X_K) = -\infty$ and $\dim(X_L) = 0$.

Hence we define $\dim(X)$ to be the maximum of the $\dim(X_K)$ for K a $\mathcal{T}$-field. However, if K has its residue rings sufficiently saturated, then $\dim(X) = \dim(X_K)$, hence we will ignore this subtlety in the rest of the paper.

Here are some properties of the dimension that we will use thorough the paper.

Proposition 1.2.10 ([22, Propositions 4.2 and 4.3])**.** *Let W, Y, Z be definable, X, X', C be definable over Y, with $X, X' \subseteq C$, and $f : X \to X'$ definable over Y (that is, commuting to projections to Y). Then*

- $\dim(A \cup B/Y) = \max\{\dim(A/Y), \dim(B/Y)\}$,
- $\dim(W \times Z) = \dim(W) + \dim(Z)$,
- $\dim(X) \geq \dim(f(X)$,
- *For each integer $d \geq 0$, the set $S_f(d) = \{x' \in X' \mid \dim(f^{-1}(x')/Y) = d)\}$ is definable and*
$$\dim(f^{-1}(S_f(d))/Y) = \dim(S_f(d)/Y) + d,$$
with convention $-\infty + d = -\infty$,
- *if Y is contained in some auxiliary sorts, then* $\dim(X/Y) = \dim(X)$.

1.2.4 Graph partitions

We fix a (mixed-)tame theory of fields $\mathcal{T}$ and K a $\mathcal{T}$-field.

We will use either additive or multiplicative notation for the value group, denoting respectively $\mathrm{ord}(x)$ and $|x| = e^{-\mathrm{ord}(x)}$ the valuation and the absolute value of an element

$x \in K$. It is a purely conventional matter; depending on the context one might be more readable than the other. If $x = (x_i)_i \in K^n$, we set $\mathrm{ord}(x) = \min_i \mathrm{ord}(x_i)$, this defines a topology on K^n. If $X \subseteq K^n$, $\overline{X}$ the closure of X, $\mathring{X}$ the interior of X and ∂X the boundary of X are all definable. Define $B(a, m) = \{x \in K^n \mid \mathrm{ord}(x - a) \geq m\}$ to be the ball around a of valuative radius m.

Definition 1.2.11. Consider a function $f : A \to B$. Then f is said globally C-Lipschitz if for all $x, y \in A$,

$$|f(x) - f(y)| \leq C\,|x - y|\,.$$

The function f is locally C-Lipschitz at $u \in A$ if there is an $n \in \mathbb{Z}$ such that f is globally C-Lipschitz on $U \cap B(u, n)$.

The function f is locally C-Lipschitz if it is locally C-Lipschitz at all $u \in A$.

In particular, if f is differentiable, then f is locally 1-Lipschitz if $\mathrm{ord}(\partial f/\partial x) \geq 0$.

Proposition 1.2.12 (Injectivity versus constancy [16, 2.2.5])**.** *Let $F : X \subseteq K \to K$ be definable. Then there exists a* $\mathbf{k}$*-partition of X such that the restriction of F on each* $\mathbf{k}$*-part is either injective or constant.*

The following property is used in [25] to define motivic integral. In positive residue characteristic, one can already see the need of using higher angular components.

Lemma 1.2.13. *Let $X \subseteq K^n$ be finite and definable. Then X is in definable bijection with a subset of some residue rings.*

Proof. We work by induction on the number n of points. If $X = \{x_1, ..., x_n\}$, set $y = \frac{1}{n}\sum_{i=1}^n x_i$, then there are $e > 0$ and $i \neq j$ such that $\overline{\mathrm{ac}}_e(x_i - y) \neq \overline{\mathrm{ac}}_e(x_j - y)$, then this reduces to the case of less than n points. □

The following proposition will be useful in the study of the tangent cone, allowing us consider only tangent cones of graphs of 1-Lipschitz functions, which are easier to study.

Proposition 1.2.14. *Let $X \subseteq K^n \times S$ be definable of dimension d relatively to S. Then there exists a definable $Y \subseteq X$ of dimension $< d$ relatively to S, and a definable* $\mathbf{k}$*-partition of $X \setminus Y$ such that for any $s \in S$, on each* $\mathbf{k}$*-part $X_{s,\xi}$, there is an (s, ξ)-definable open $U_{s,\xi} \subseteq K^d$ and a (s, ξ)-definable function $\phi_{s,\xi} : U_{s,\xi} \to K^{n-d}$ of graph $\Gamma_{s,\xi}$ that is differentiable and locally 1-Lipschitz and such that for a permutation of coordinates $\gamma_{s,\xi}$, $X_{s,\xi} = \gamma_{s,\xi}(\Gamma_{s,\xi})$.*

Proof. Working piecewise in the fibers of the projection of $X \to S$ and using logical compactness, one can suppose $X \subseteq K^n$. One proceeds by induction on n. If $n = d$ one proceeds $Y = X \setminus \mathring{X}$, and we are done. If $n > d$, for $i = 1, ..., n$, one considers the projection $\pi_i : K^n \to K^{n-1}$ that omits the i-th coordinate. Then X is partitioned into two definable

parts X_0^i and X_1^i, such that $\pi_{i|X}$ has 0-dimensional fibers on X_0^i and 1-dimensional fibers on X_1^i. The set $\pi_i(X_1^i) \subseteq K^{n-1}$ is then of dimension $d-1$, so one is done by the induction hypothesis. On X_0^i, the fibers are 0-dimensional, so they are in definable bijection with a definable in the auxiliary sorts. Up to adding some **k**-parameters, one can assume the fibers are one point, *i.e.* $\pi_i : X_0^i \to \pi_i(X_0^i)$ is a **k**-definable bijection. One is then reduced to considering the case $X = \cap_i X_0^i$, *i.e.* $\pi_i : X \to \pi_i(X)$ is a **k**-definable bijection for all $i = 1, ..., n$. Denote h_i the i-th coordinate function of π_i^{-1}. By the Jacobian property, up to adding some additional **k**-parameters and neglecting a set of dimension $< d$, one can assume all h_i have partial derivatives on the whole of $\pi_i(X)$. Up to cutting X into finitely many pieces, one can also assume that the order of the partial derivatives of the functions h_i is either nonnegative or negative on the whole $\pi_i(X)$. It suffices to find an i_0 such that all the partial derivatives of h_{i_0} are nonnegative and then we are done by induction. But using the chain rule for derivation, this is clear that such an i_0 exists. □

1.2.5 Some results about Lipschitz continuity

We recall now some results of [16] about reaching global Lipschitz continuity from local Lipschitz continuity.

Theorem 1.2.15 ([16], Theorem 2.2.3)**.** *Let $f : X \to K$ a definable function which is locally 1-Lipschitz. Then there is a* **k***-definable partition of X and a real $C \geq 1$ such that on each part, f is globally C-Lipschitz.*

Theorem 1.2.16. *Let $f : A \subseteq K^n \to K$ a definable function which is locally 1-Lipschitz. Then there is a finite definable partition of A such that for each part X, the following holds. There is a coordinate projection $p : K^n \to K^{n-1}$ and some definable functions $g : X \to k^r$, $c : k^r \times K^{n-1} \to K$ and $d : k^r \times K^{n-1} \to K$ such that for each $\eta \in k^r$, the restrictions of $c(\eta, \cdot)$ and $d(\eta, \cdot)$ to $p(g^{-1}(\eta))$ are locally 1-Lipschitz, and for each $w \in p(K^n)$, the set $g^{-1}(\eta)_w$ is in $c(\eta, w)$-config and the image of $g^{-1}(\eta)_w$ under f_w is in $d(\eta, w)$-config.*

This theorem is not explicitly stated in [16], but it follows from the proof of Theorem 2.1.8 of [16]. The idea is the same as Proposition 1.2.14 : use the Jacobian property on the centers of a cell decomposition to show that they are differentiable, then switch the order of the coordinates to bound the derivatives by 1, in the new coordinates, the centers are now locally 1-Lipschitz.

In the residue characteristic zero case, Theorem 2.1.8 of [16] shows that one can take $C = 1$ in Theorem 1.2.15. In [19], Cluckers and Halupczok show that the same is true if K is a finite extension of $\mathbb{Q}_p$. We now generalize this result but only in the one dimensional case.

Theorem 1.2.17. *Let $f : X \subseteq K^n \times Y \to K$ a definable function such that X_y is one dimensional for every $y \in Y$ and the restriction of f to X_y is locally 1-Lipschitz. Then*

there is a $\mathbf{k}$*-definable partition of* X *such that for any* $y \in Y$*, on each* $\mathbf{k}$*-part* $X_{y,\xi}$*,* f *is globally* 1*-Lipschitz on* $X_{y,\xi}$*.*

To prove the theorem, we need the following lemma.

Lemma 1.2.18. *Let* $f : X \subseteq K \to K$ *a definable differentiable function with* $f' = 0$*. Then the image of* f *is finite, there is a finite definable partition of* X *in parts* X_i *such that* $f_{|X_i}$ *is constant.*

Proof. By the Jacobian property, f is locally constant hence it follows from the definition of finite b-minimality. □

If will also be convenient to use the following notion of thin cell.

Definition 1.2.19. Let Y be definable, and $A \subseteq K \times Y$ be a 1-cell over Y. Then, for each $(t, y) \in A$, there is a unique maximal (for the inclusion) ball $B_{t,y}$ containing t and satisfying $B_{t,y} \times \{y\} \subseteq A$. For $y_0 \in Y$, we call the collection of balls $\{B_{t,y_0}\}_{(t,y_0)\in A}$ the balls of the cell A above y_0.

We call a 1-cell A over Y thin if the collection of balls of A above any $y \in Y$ consists of at most one ball.

Proof of Theorem 1.2.17. Suppose first that $n = 1$. We have a definable function $f : X \subseteq K \times Y \to K$ such that $f(\cdot, y)$ is locally 1-Lipschitz on X_y for every $y \in Y$. Apply Theorem 1.2.16 to f, working on a part, call c and d the functions that parametrize the centers of the domain and range. After translating, assume $c = d = 0$. Working on a $\mathbf{k}$-part $X_{y,\eta}$, for some $\eta \in \mathbf{k}$, assume $X_{y,\eta}$ is a 1-cell (otherwise the result is trivial) with center 0, f is injective on $X_{y,\eta}$, f has the Jacobian property on $X_{y,\eta}$, and $f(X_{y,\eta})$ is a 1-cell with center 0. Fix $x_1, x_2 \in X_{y,\eta}$. Note that if $|x_1| = |x_2|$, *i.e.* they both lie in the same ball above (y, η), then by the Jacobian property

$$\left|\frac{\partial f}{\partial x}(x_1, y) \cdot (x_1 - x_2)\right| = |f(x_1, y) - f(x_2, y)|,$$

hence we are done because $|\partial f(x_1, y)/\partial x| \leq 1$.

Assume now $|x_1| \neq |x_2|$. Then by our assumption, there exist $a_1, a_2, b_1, b_2 \in \mathbb{Z}$, $a_1 \neq a_2$, $b_1 \neq b_2$, $\ell, \ell' \in \mathbb{N}^*$, $\xi, \xi' \in \mathbf{k}$ such that for $i = 1, 2$ the balls

$$B_{x_1,y} = \{x \in K \mid \operatorname{ord}(x) = a_i, \overline{\operatorname{ac}}_\ell(x) = \xi\}$$

are included in $X_{y,\eta}$, $x_i \in B_{x_i,y}$, and

$$f(B_{x_i,y}) = \left\{x \in K \mid \operatorname{ord}(x) = b_i, \overline{\operatorname{ac}}_{\ell'}(x) = \xi'\right\}.$$

We have the equalities

$$\operatorname{ord}(f(x_1, y) - f(x_2, y)) = \min_{i=1,2} b_i,$$

$$\mathrm{ord}(x_1 - x_2) = \min_{i=1,2} a_i,$$

and by the Jacobian property, comparing the sizes of the balls $B_{x_i,y}$ and $f(B_{x_i,y})$,

$$\ell + \mathrm{ord}(\partial f(x_i, y)/\partial x) + a_i = \ell' + b_i.$$

Hence we are done if $\ell' \leq \ell$, or if $\mathrm{ord}(\partial f(x_i, y)/\partial x) \geq \ell' - \ell$. Therefore by taking a finite partition of X, we can assume $\mathrm{ord}(\partial f(x, y)/\partial x)$ is constant on $X_{y,\eta}$. If $X_{y,\eta}$ is a thin cell, then once again we are done by the Jacobian property. By injectivity versus constancy (Proposition 1.2.12), up to taking another $\mathbf{k}$-partition, we can assume $\partial f/\partial x$ is either constant or injective on $X_{y,\eta}$.

Assume first that $\partial f/\partial x$ is constant on $X_{y,\eta}$. Then the function

$$(x, y) \mapsto f(x, y) - \frac{\partial f(x, y)}{\partial x} \cdot x$$

has derivative 0 on $X_{y,\eta}$, hence it has a finite image by Lemma 1.2.18, so by refining the partition, we can assume it is constant on $X_{y,\eta}$. Hence on $X_{y,\eta}$,

$$f(x, y) = a(y)x + b(y),$$

for a, b some definable functions. As $|a(y)| \leq 1$, f is then 1-Lipschitz on $X_{y,\eta}$.

Consider now the last case, where $\partial f/\partial x$ is injective on $X_{y,\eta}$. By Theorem 1.2.16 applied to $\partial f/\partial x$ (for $n = 1$ the Lipschitz assumption is superfluous), we get a refinement of the $\mathbf{k}$-partition, and some definable functions d, e such that each $\mathbf{k}$-part $X_{y,\eta}$ is a 1-cell in $d(y)$-config, its image by $\partial f/\partial x$ is in $e(y)$-config, and $\partial f/\partial x$ has the Jacobian property on $X_{y,\eta}$. Once again, assume $X_{y,\eta}$ is not a thin cell otherwise we are done. By injectivity of $\partial f/\partial x$, and the fact that $\mathrm{ord}(\partial f/\partial x) \geq 0$, up to removing finitely many thin cells that we can treat separately, we can assume $\mathrm{ord}(\partial f(x, y)/\partial x - e(y)) \geq \alpha$, where α is an integer chosen as follows. Consider the function

$$g(x, y) = f(x, y) - e(y) \cdot x.$$

Apply the beginning of the proof to the function g. We get a refinement of the decomposition, and call $\tilde{\ell}, \tilde{\ell}'$ the depth of embedding of the decomposition adapted to g. It follows that g is globally 1-Lipschitz on the $\mathbf{k}$-parts if $\mathrm{ord}(\partial g/\partial x) \geq \tilde{\ell}' - \tilde{\ell}$. Now set $\alpha = \tilde{\ell}' - \tilde{\ell}$, as

$$\frac{\partial g}{\partial x}(x, y) = \frac{\partial f}{\partial x}(x, y) - e(y),$$

$g(\cdot, y)$ is globally 1-Lipschitz on $X_{y,\eta}$. The function f is then a sum of two globally 1-Lipschitz functions of $X_{y,\eta}$ (because $\mathrm{ord}(e(y)) \geq 0$), hence by ultrametric inequality, it is itself globally 1-Lipschitz.

The general case reduces to the previous one, using Proposition 1.2.14 on X and the chain rule for derivation. $\square$

1.2.6 Curve selection

We prove a weak form of the curve selection lemma. A full curve selection lemma is proved in the p-adic semi-algebraic case in [89] and sub-analytic case in [40]. In our context, because the residue field can be infinite, there are no Skolem functions, hence definable choice is not available. Hence we cannot find a truly definable arc, however, we can find some uniformly definable family of arcs. Our result is weak in the sense that the arcs we get are Lipschitz continuous, but not analytic, this is the drawback of working with an abstract theory. In the the semi-algebraic language for equicharacteristic zero fields, Nowak shows in [83] existence of definable analytic arcs, using resolution of singularities.

Before stating the result, we prove some lemmas about limit values of Lipschitz definable functions needed for the proof.

Lemma 1.2.20. *Let $f : U \to K$ a bounded function, $x \in \overline{U}$ and Γ the graph of f. Then $\overline{\Gamma} \cap (\{x_0\} \times K^{n-d})$ is non-empty.*

Proof. One can assume $x = 0$, and let β a bound from below for $\mathrm{ord}(f)$. Let

$$X = \{(z, \gamma) \in K \times \mathbb{Z} \mid \exists u \in U, \mathrm{ord}(u) \geq \gamma, \mathrm{ord}(f(u) - z) \geq \gamma\}.$$

Then X is non-empty and projects surjectively on its second coordinate. By tameness, X can be assumed of the form

$$\{(z, \gamma) \in K \times \mathbb{Z} \mid \overline{\mathrm{ac}}(z - c) \in D_1, \mathrm{ord}(z - c) \in D_2(\gamma)\},$$

where D_1 is a definable of k, and $D_2(\gamma)$ is a γ-definable of $\mathbb{Z}$. By definable choice for Presburger sets, there is a definable Presburger function $\alpha : \mathbb{Z} \to \mathbb{Z}$ with $\alpha(\gamma) \in D_2(\gamma)$ for every $\gamma \in \mathbb{Z}$.

Let show by contradiction that $\alpha(\gamma)$ is bounded from below when γ goes to $+\infty$. If not, we can find a γ with $\alpha(\gamma') < d < \gamma$, for $d = \min\{\mathrm{ord}(c), \beta\}$. By the description of X above, there is a $z \in X$ with $\mathrm{ord}(z - c) = \alpha(\gamma)$. Then by definition of X, there is a $u \in U$ such that $\mathrm{ord}(u) \geq \gamma$, $\mathrm{ord}(f(u) - z) \geq \gamma$. Using the ultrametric inequality, we get

$$\begin{aligned} \alpha(\gamma) = \mathrm{ord}(z - c) &= \mathrm{ord}(z - f(u) + f(u) - c) \\ &\geq \min\{\mathrm{ord}(f(u) - z), \mathrm{ord}(f(u) - c)\} \geq \min\{\gamma, \beta\} \geq d. \end{aligned}$$

Contradiction, hence $\alpha(\gamma)$ is bounded from below when γ goes to $+\infty$. Then one can extract a sequence of γ such that either $\alpha(\gamma) = \alpha_0$ is constant, either $\alpha(\gamma)$ goes to $+\infty$. In the first case, any z with $\mathrm{ord}(z - c) = \alpha_0$ satisfy $(0, z) \in \overline{\Gamma}$, in the second one, $(0, c) \in \overline{\Gamma}$. $\square$

Lemma 1.2.21. *Let $f : U \subseteq K^d \to K^{n-d}$ a globally C-Lipschitz function, of graph Γ. Then for any $x_0 \in \overline{U}$, there are only finitely many points of $\overline{\Gamma}$ which projects to x_0 under the coordinate projection $K^d \times K^{n-d} \to K^d$, i.e. $X := \overline{\Gamma} \cap \{x_0\} \times K^{n-d}$ is finite.*

For the proof we will use the tangent cone that will be studied in more details in section 1.3.3. The $K^\times$-tangent cone of a definable subset Y of K^n is the set $C_x^{K^\times}(X)$ defined by

$$\left\{u \in K^n \mid \forall i \in \mathbb{Z}, \exists y \in X, \exists \lambda \in K^\times, \mathrm{ord}(y-x) \geq i, \mathrm{ord}(\lambda(y-x)-u) \geq i\right\}.$$

Proof. Suppose by contradiction that $\dim(X) \geq 1$. Then X contains a one dimensional ball. For a z in such a ball, $\dim(C_z^{K^\times}(X)) \geq 1$.

As f is globally C-Lipschitz, if $z = (z_0, z_1)$, then for any $x \in U$, $|f(x) - z_1| \leq C\,|x - z_0|$, therefore

$$C_z^{K^\times}(\Gamma) \subseteq \left\{(x,y) \in K^d \times K^{n-d} \mid |y| \leq C\,|x|\right\}.$$

As $C_z^{K^\times}(X) \subseteq C_z^{K^\times}(\overline{\Gamma}) \cap C_z^{K^\times}(\{x_0\} \times K^{n-d})$, we then have

$$C_z^{K^\times}(X) \subseteq \left\{(x,y) \in K^d \times K^{n-d} \mid |y| \leq C\,|x|\right\} \cap (\{0\} \times K^{n-d}) = \{0\},$$

contradiction. □

Corollary 1.2.22. *Let $f : U \subseteq K^n \to K$ a locally 1-Lipschitz function. Then f has a nonzero finite number of limit values at any $z \in \overline{U}$.*

Proof. By Theorem 1.2.15, there is a $C \geq 1$ and $\mathbf{k}$-partition of U such that f is globally C-Lipschitz on each of the $\mathbf{k}$-parts of U. For any $z \in \overline{U}$, by the following Lemma 1.2.23 there is a $\mathbf{k}$-part such that z is in the closure of this $\mathbf{k}$-part. Moreover, as any definable bijection between the residues sorts and the valued field sort is finite, it suffices to show that the result holds on each of the $\mathbf{k}$-parts. But then f is globally C-Lipschitz, hence bounded on some neighborhood of z, then by Lemma 1.2.20 f has a limit value at z. By Lemma 1.2.21, f has only a finite number of limit values. □

Lemma 1.2.23. *Let $X \subseteq K^n$ be definable and $g : X \to A$ a $\mathbf{k}$-partition of X. Then the closure of X is equal to the union of the closures of its $\mathbf{k}$-parts.*

Proof. Fix $u \in \overline{X}$. Consider the u-definable set

$$D = \{(n,\xi) \in \mathbb{Z} \times A \mid \exists z \in X, \mathrm{ord}(z-u) \geq n, g(z) = \xi\}.$$

By definition of u, D projects surjectively on $\mathbb{Z}$. Because the theory splits, D is a finite union of cartesian products of definable sets of $\mathbb{Z}$ and A, hence there is a $\xi \in A$ such that for all $n \in \mathbb{Z}, (n,\xi) \in D$, which means $u \in \overline{g^{-1}(\xi)}$. □

Proposition 1.2.24 (Weak curve selection)**.** *Let $X \subseteq K^n$ be some definable, and $a \in \overline{X} \setminus X$. Then for some integer $e \geq 1$, there exists a definable function $\sigma : B_e \times D \to X \cup \{a\}$ for some nonempty definable D, where $B_e = \{x \in K \mid |x| \leq 1, \overline{\mathrm{ac}}_e(x) = 1\}$ and $B_e^* = B_e \setminus \{0\}$, and such that if we denote for any $d \in D$*

$$\sigma_d : t \in B_e \mapsto \sigma(t, d),$$

the function σ_d is 1-Lipschitz, $\sigma_d(0) = a$ and $\sigma_d(B_e^*) \subseteq X$. In the tame case, that is, when the residue characteristic is zero, one can take $e = 1$.

To simplify the notations we will write down the proof in the equicharacteristic zero case, with $B = B_1$. For the mixed case, replace $k = k_K$ by some residue rings, and at the very end, e is the greatest s that appears in the $R^s_{s,K}$.

Proof. Observe first that we only need to show the existence of a locally 1-Lipschitz function σ satisfying the other requirements of the proposition, then use Theorem 1.2.17 to get a globally 1-Lipschitz function.

We proceed by induction over n. Up to translating, assume $a = 0$. By Theorem 1.2.16, there is a finite partition of X into definable parts, such that on each part A, the following holds. There exists an $s \geq 0$, a coordinate projection $p : K^n \to K^{n-1}$, and some $\mathcal{L}$-definable functions

$$g : A \to k^s \text{ and } c : k^s \times K^{n-1} \to K$$

such that for all $\eta \in k^s$, the restriction of $c(\eta, \cdot)$ to $(g^{-1}(\eta))$ is locally 1-Lipschitz, and for each w in $p(K^n)$, the set $g^{-1}(\eta)_w$ is in $c(\eta, w)$-config. There is a part A such that $a \in \overline{A}$, and one can assume $X = A$. If $p(a) \in p(X)$ and $p(a)$ is isolated in $p(X)$, one is done by using the induction hypothesis with $n = 1$ for $p^{-1}(a) \cap X$. Otherwise up to reducing X, one can assume that $p(a) \in \overline{p(X)} \setminus p(X)$.

Let's fix an η such that $g^{-1}(\eta)$ is nonempty, $a \in \overline{g^{-1}(\eta)}$ and $p(a) \in \overline{p(g^{-1}(\eta))} \setminus p(g^{-1}(\eta))$. Such an η exists because the theory splits. Up to reducing X again, using Corollary 1.2.22 one can assume $\lim_{w \to 0} c(\eta, w) = c_0$ exists. One can assume $X = g^{-1}(\eta)$. The definable X is now η-definable, and η will be one of the parameters of the function we are defining.

By the induction hypothesis with $n-1$, there is a definable with parameters 1-Lipschitz function $\sigma : B \to p(X) \cup \{a\}$ such that $\sigma(0) = p(a)$ and $\sigma(B \setminus \{0\}) \subseteq p(X)$.

If $c_0 \neq 0$, as $0 \in \overline{X}$, for all $w \in p(X)$ $0 \in X_w$, hence the function $\sigma' = (\sigma, 0)$ answer the problem.

Let's consider the case $c_0 = 0$. We know that X is in $c(\eta, w)$-config. If it is in graph-config, then the function $\sigma' = (\sigma, c \circ \sigma)$ works. Otherwise

$$X_w = \{x \in K \mid \overline{\mathrm{ac}}(x - c(\eta, w)) = \eta_{i_0}, \mathrm{ord}(x - c(\eta, w)) \in D_2\},$$

with D_2 an (η, w)-definable of Γ and η_{i_0} is one of the coordinates of η. As the theory splits,

D_2 is in fact w-definable, denote it by $D_2(w)$. By Presburger quantifier elimination and compactness, up to reducing X again, one can assume $D_2(w)$ is a finite union of cells of the form $\{\alpha \in \Gamma \mid \alpha_1(w)\square_1\alpha\square_2\alpha_2(w), \alpha \equiv_e i\}$, for some definable functions α_1, α_2, integers e, k and $\square_j$ either $\leq$ or and empty condition, and $\alpha_j(w) \in D_2(w)$. As $0 \in \overline{g^{-1}(\eta)}$, there is a cell with α_2 unbounded when w goes to 0 or with the condition $\square_2$ empty.

Suppose we are in the first case. Let Δ the projection on the last two variables of

$$\left\{(w, \alpha, \beta) \in p(X) \times \mathbb{Z}^2 \mid |w| = \alpha, \alpha_2(w) = \beta\right\}.$$

Because $\alpha_2(w)$ is unbounded when w goes to 0, the point (∞, ∞) is in the closure of Δ for the order topology. By Presburger cell decomposition and existence of Skolem functions in Presburger arithmetic, there exists integers $\alpha_0, a_0, b_0, e_0, i_0$ with $a_0 \geq 1$ and $e_0 \geq 1$ such that Δ contains the definable

$$\left\{(\alpha, \beta) \in \mathbb{Z}^2 \mid \alpha_0 \leq \alpha, \alpha \equiv_{e_0} i_0, \beta = a_0\frac{\alpha - i_0}{e_0} + b_0\right\}.$$

By the induction hypothesis there is a definable with parameters 1-Lipschitz function $\sigma : B \to p(X) \cup \{p(a)\}$ such that $\sigma(0) = p(a)$ and $\sigma(B \setminus \{0\}) \subseteq p(X)$. By tameness, the function $t \mapsto \mathrm{ord}(\sigma(t))$ is of the form $f(\mathrm{ord}(t))$, with f a Presburger function, hence there are some $\alpha_1, a_1, b_1, e_1, i_1$ with $a_1 \geq 1$ and $e_1 \geq 1$ such that for $\alpha_1 \leq \mathrm{ord}(t)$ and $\mathrm{ord}(t) \equiv_{e_1} i_1$, $\mathrm{ord}(\sigma(t)) = a_1\frac{\mathrm{ord}(t)-i_1}{e_1} + b_1$, and by composition, there are some integers i_2, e_2, a_2, b_2 with $a_2 \geq 1$, $e_2 \geq 1$ such that for all $t \in B$ and $\sigma'(t) := \sigma(\varpi_K^{i_2}t^{e_2})$, $\alpha_2(\sigma'(t)) = a_2\mathrm{ord}(t) + b_2$. Choose a $d \in X_{\sigma'(1)}$. Then the function $t \mapsto (\sigma'(t), c(\eta, \sigma'(t)) + dt^{a_2})$ maps 0 to 0 and $B \setminus \{0\}$ to X and is locally 1-Lipschitz. The latter case, where the cell of $D_2(w)$ is of the form $\alpha_1(w) \leq \alpha$ is similar. $\square$

We get the following parametrized curve selection lemma by working over parameters and using logical compactness.

Proposition 1.2.25 (Parametrized curve selection)**.** *Let $X \subseteq K^n \times Y$ be definable and for every $y \in Y$, a y-definable point $a_y \in \overline{X_y} \setminus X_y$. Then for some integer $e \geq 1$, there exists a definable function $\sigma : B_e \times D \times Y \to K^n$ for some nonempty definable D, where $B_e = \{x \in K \mid |x| \leq 1, \overline{\mathrm{ac}}_e(x) = 1\}$ and $B_e^* = B_e \setminus \{0\}$, and such that if we denote for any $(d, y) \in D \times Y$*

$$\sigma_{d,y} : t \in B_e \mapsto \sigma(t, d, y),$$

the function σ_d is 1-Lipschitz, $\sigma_d(0) = a_y$ and $\sigma_d(B_e^*) \subseteq X_y$. In the tame case, that is, when the residue characteristic is zero, one can take $e = 1$.

The following immediate corollary will be useful in the study of regular stratifications. In addition to the use of the curve selection, one has to notice that any definable function

is differentiable almost everywhere, that is, everywhere but on a definable set of lower dimension, which is guaranteed by the Jacobian property.

Corollary 1.2.26. *Let $U \subseteq K^m$ be a nonempty open definable and $S \subseteq K^n$, such that for all $u \in U$, $(u,0) \in \overline{S_u} \setminus S_u$. Then there is a nonempty open $V \subseteq U$ $\mathbf{k}$-definable and a definable with parameters*

$$\begin{array}{rcl} \overline{\sigma} : V \times B_e & \to & Y \cup (V \times \{0\}^{n-m}) \\ (u,s) & \mapsto & (u, \sigma(u,s)) \end{array}$$

such that $\overline{\sigma}(u,0) = 0$, $\overline{\sigma}(V \times B_e^*) \subseteq V$, for any fixed $u \in V$, $\sigma(u,\cdot)$ is 1-Lipschitz, and for any $s \in B_e$, $\sigma(\cdot,s)$ is differentiable.

The following proposition will be of crucial use in the study of tangent cones. In the p-adic case, the analogous result is proven using analytic curve selection and Whitney's lemma. As we do not have analytic curve selection lemma, we cannot use Whitney's lemma but we will instead use our Theorem 1.2.17 to get a globally Lipschitz function.

Proposition 1.2.27. *Let $f : X \subseteq K^d \to K^{n-d}$ a definable function, Γ be its graph and suppose f is locally C-Lipschitz. Fix $\Lambda \in \mathcal{D}$ and $z = (z_1, z_2) \in \overline{\Gamma}$. Then there is an open neighborhood U of z_1 in K^d such that the following inclusions hold :*

$$\{(x - z_1, f(x) - z_2) \mid x \in U\} \subseteq \left\{(x,y) \in K^d \times K^{n-d} \mid |y| \leq C\,|x|\right\}$$

and

$$C_z^\Lambda(\Gamma) \subseteq \left\{(x,y) \in K^d \times K^{n-d} \mid |y| \leq C\,|x|\right\}.$$

For the definition of the tangent cone $C_z^\Lambda(\Gamma)$, see Definition 1.3.15.

Proof. After translating and rescaling, one can assume $z = 0$ and $C = 1$ and it suffices to show the existence of a neighborhood U of z_1 such that the first inclusion holds. Assume by contradiction that there is a $\delta > 0$ such that the set

$$\Gamma' := \{(x, f(x)) \mid x \in X, |f(x)| \geq (1+\delta)\,|x|\}$$

is nonempty and satisfies $0 \in \overline{\Gamma'} \setminus \Gamma'$. By the curve selection lemma 1.2.24, there is a definable function $\sigma : B_e \times D \to K^n$ with σ_d 1-Lipschitz, $\sigma_d(0) = 0$ and $\sigma_d(B_e^*) \subseteq \Gamma'$. Denote $\tilde{\sigma} : B_e \times D \to K^d$ the first d components of σ and set

$$X' := \left\{(x,d) \in K^d \times D \mid \exists t \in B_e^*, x = \tilde{\sigma}(t,d)\right\}.$$

For any $d \in D$, the set X'_d is one dimensional and $(0,0)$ lies in the closure of the graph of f on X'_d. By Proposition 1.2.17, up to taking a $\mathbf{k}$-partition, one can assume f is globally

1-Lipschitz on X'_d. Fix a $x \in X'_d$. There is a $y \in X'_d$ with $|y| < |x|$ and $|f(y)| \leq |f(x)|$. Then by 1-Lipschitz continuity of f,

$$|f(x) - f(y)| \leq |x - y|,$$

but $|x| = |x - y|$ and $|f(x)| = |f(x) - f(y)|$, hence $|f(x)| \leq |x|$. Contradiction with $x \in \Gamma'$. □

The following proposition is a partial converse to Proposition 1.2.27.

Proposition 1.2.28. *Let $f : U \to K^{n-d}$ a differentiable definable function on a nonempty open $U \subseteq K^d$, and let Γ be its graph. Assume*

$$C^{\Lambda}_{(x_0, f(x_0))}(\Gamma) \subseteq \left\{(x, y) \in K^d \times K^{n-d} \mid |y| \leq C\,|x|\right\}.$$

Then for all $i = 1, ..., d$, $|\partial f/\partial x_i(x_0)| \leq C$, and in particular, f is locally C-Lipschitz.

Proof. For $i = 1, ..., d$, the vector $(0, ..., 0, 1, 0, ..., \partial f/\partial x_i(x_0))$ (with a 1 in i-th position) is in $C^{\Lambda}_{(x_0, f(x_0))}(\Gamma)$. □

1.2.7 Recap on motivic integration

We use Cluckers-Loeser theory of motivic integration defined in [25], see also [23] for the residue characteristic zero case. We review here the main steps of the construction in order to fix the notation.

Fix either a tame theory or a mixed tame theory $\mathcal{T}$ in a language $\mathcal{L}$ and X a definable subassignment.

Value group Let $\mathbb{A}$ be the ring

$$\mathbb{A} := \mathbb{Z}\left[\mathbb{L}, \mathbb{L}^{-1}, \left(\frac{1}{1 - \mathbb{L}^{-n}}\right)_{n \geq 1}\right].$$

For every real $q > 1$, define $\theta_q : \mathbb{A} \to \mathbb{R}$ to be the ring morphism generated by $\mathbb{L} \mapsto q$, and $\mathbb{A}$ the sub-semiring $\mathbb{A}_+ := \{a \in \mathbb{A} \mid \forall q > 1, \theta_q(a) \geq 0\}$. The ring $\mathbb{A}$ is equipped with a partial order defined by $a \leq b$ if and only if $a - b \in \mathbb{A}_+$.

The application degree $\mathbb{Z}[\mathbb{L}] \to \mathbb{N}$ extends uniquely as a map $\deg_{\mathbb{L}} : \mathbb{A} \to \mathbb{Z}$ satisfying $\deg_{\mathbb{L}}(ab) = \deg_{\mathbb{L}}(a) + \deg_{\mathbb{L}}(b)$. A sequence $(a_n)_n \geq 0$ of elements of $\mathbb{A}$ is said to converge to 0 if $\deg_{\mathbb{L}}(a_n)$ goes to $-\infty$. This defines a topology on $\mathbb{A}$.

Let $\mathcal{P}(X)$ the ring generated by functions

- $\alpha : X \to \mathbb{Z}$ definable,
- $\mathbb{L}^{\alpha}$, with $\alpha : X \to \mathbb{Z}$ definable,

- constants of $\mathbb{A}$.

The sub-semiring $\mathcal{P}_+(X)$ of $\mathcal{P}(X)$ is constituted of functions with value in $\mathbb{A}_+$.

In the definition above, we allow a slight abuse of notation by identifying $\mathbb{Z}$ with the value group sort, but it is harmless since we only consider models of $\mathcal{T}$ that are $\mathcal{T}$-fields, hence with value group $\mathbb{Z}$.

The partial order on $\mathbb{A}$ induces a partial order on $\mathcal{P}_+(X)$ and $\mathcal{P}(X)$: for $\varphi, \psi \in \mathcal{P}(X)$, define $\varphi \geq \psi$ if $\varphi - \psi \in \mathcal{P}_+(X)$.

For any $\mathcal{T}$-field K, $\varphi \in \mathcal{P}(X)$, define $\theta_{q,K}(\varphi)$ to be the function

$$\theta_{q,K}(\varphi) : x \in X(K) \mapsto \theta_q(\varphi(x)).$$

A function $\varphi \in \mathcal{P}(X[0,0,s])$ is said to be X-integrable if for any $\mathcal{T}$-field K and any $x \in X(K)$ the family $(\theta_{q,K}(\varphi)(x,i))_{i\in\mathbb{Z}^s}$ is absolutely summable (as a family of real numbers).

Denote by $I_X\mathcal{P}(X[0,0,s])$ (resp. by $I_X\mathcal{P}_+(X[0,0,s])$) the $\mathcal{P}(X)$-module (resp. the $\mathcal{P}_+(X)$-semimodule) of X-integrable $\mathcal{P}(X)$ functions (resp. $\mathcal{P}_+(X)$ functions).

Proposition 1.2.29 ([25], Theorem 5.1)**.** *Let $\varphi \in \mathcal{P}(X[0,0,s])$ an X-integrable function, then there is a unique $\mu_{/X}(\varphi) \in \mathcal{P}(X)$ such that for all $q > 1$, K a $\mathcal{T}$-field and $x \in X(K)$,*

$$\theta_{q,K}(\mu_{/X}(\varphi))(x) = \sum_{i\in\mathbb{Z}^s} \theta_{q,K}(\varphi)(x,i).$$

Moreover, the map $\varphi \mapsto \mu_{/X}(\varphi)$ is a morphism of $\mathcal{P}(X)$-modules

$$\mu_{/X} : I_X\mathcal{P}(X[0,0,s]) \to \mathcal{P}(X)$$

and send $I_X\mathcal{P}_+(X[0,0,s])$ to $\mathcal{P}_+(X)$.

Define also $\mathcal{P}^0(X)$ (resp. $\mathcal{P}^0_+(X)$) as the sub-ring of $\mathcal{P}(X)$ (resp. as the sub-semiring of $\mathcal{P}_+(X)$) generated by characteristic functions $\mathbf{1}_Y$ of every definable $Y \subseteq X$ and the constant function $\mathbb{L} - 1$.

For any $f : X \to Y$ definable, define the pullback by

$$\begin{array}{rcl} f^* : \mathcal{P}_+(Y) & \to & \mathcal{P}_+(X) \\ \varphi & \mapsto & \varphi \circ f. \end{array}$$

Residue ring Let $\mathcal{Q}_+(X)$ the quotient of the free abelian semigroup generated by symbols $[Y_{/X}]$, where $Y \subseteq X[0,r,0]$ is definable for some $r = (r_1, ..., r_s) \in \mathbb{N}^s$, by the relations :

- $[Y_{/X}] = [Y'_{/X}]$ if Y and Y' are definably isomorphic over the projection to X,
- $[Y_{/X}] + [Y'_{/X}] = [(Y \cup Y')_{/X}] + [(Y \cap Y')_{/X}]$,

- $[\emptyset_{/X}] = 0$
- $[Y_{/X}] = [Y'_{/X}]$ if for some definable Z of $X[0,r,0]$, one has

$$Y' = p^{-1}(Z) \text{ and } Y = Z[0,1,0],$$

  where $p : X[0,(r_1+1,r_2,...,r_s),0] \to X[0,r,0]$ is the natural projection between residue rings.

In particular, the classes of $X[0,r,0]$ and $X[0,r',0]$ in $\mathcal{Q}_+(X)$ are identified for any tuples $(r_i)_i$ and $(r'_j)_j$ such that $\sum_i r_i = \sum_j r'_j$. Note also that the last relation is vacuous if the residue field is perfect.

The semigroup $\mathcal{Q}_+(X)$ carries a semiring structure with multiplication for subsets $Y \subseteq X[0,m,0]$ and $Y' \subseteq X[0,m',0]$ given by

$$[Y_{/X}] \cdot [Y'_{/X}] := [Y \times_X Y'_{/X}].$$

We denote $\mathcal{Q}(X)$ the groupification of $\mathcal{Q}_+(X)$. Then the canonical morphism $\mathcal{Q}_+(X) \to \mathcal{Q}(X)$ has no reason in general to be injective. If the theory of the residue field is the theory of algebraically closed fields of characteristic zero, Karzhemanov shows in [60] that it is not injective.

To keep information at the level of residue rings, we define formally integration of residue variables. Let r be a tuple of integers, the integral of $[Y_{/X}] \in \mathcal{Q}_+(X[0,r,0])$ with respect to X is

$$\mu_{/X} : [Y_{X[0,r,0]} \in \mathcal{Q}_+(X[0,r,0]) \mapsto [Y_{/X}] \in \mathcal{Q}_+(X).$$

Motivic constructible functions We identify $[X \times \mathbb{G}_m(k)_{/X}]$ with $\mathbb{L}-1$, and for $Y \subseteq X$ definable, $[Y_{/X}]$ with $\mathbf{1}_Y$, hence $\mathbb{L}$ is identified with $[X \times \mathbb{A}^1_k] = [X[0,(1),0]]$. Therefore there are inclusions

$$\mathcal{P}^0_+(X) \hookrightarrow \mathcal{Q}_+(X) \text{ and } \mathcal{P}^0(X) \hookrightarrow \mathcal{Q}(X).$$

Define the semiring of positive motivic constructible functions as the tensor product

$$\mathcal{C}_+(X) := \mathcal{P}_+(X) \otimes_{\mathcal{P}^0_+(X)} \mathcal{Q}_+(X)$$

and the ring of motivic constructible functions

$$\mathcal{C}(X) := \mathcal{P}_+(X) \otimes_{\mathcal{P}^0_+(X)} \mathcal{Q}_+(X).$$

For any $f : X \to Y$ definable, the pullback morphisms on $\mathcal{P}_+(Y)$ and $\mathcal{Q}_+(Y)$ extend canonically to a pullback morphism $f^* : \mathcal{C}_+(Y) \to \mathcal{C}_+(X)$.

There is also a canonical morphism $\iota : \mathcal{C}_+(X) \to \mathcal{C}(X)$, but it has no reason to be injective.

Using the fact that the theory splits, we get canonical isomorphisms

$$\mathcal{C}_+(X[0,r,s]) \simeq \mathcal{P}_+(X[0,0,s]) \otimes_{\mathcal{P}^0_+(X)} \mathcal{Q}_+(X[0,r,0])$$

and

$$\mathcal{C}(X[0,r,s]) \simeq \mathcal{P}(X[0,0,s]) \otimes_{\mathcal{P}^0(X)} \mathcal{Q}(X[0,r,0]).$$

These isomorphisms allow one to define canonically integrals over auxiliary variables. Define

$$I_X\mathcal{C}_+(X[0,r,s]) := I_X\mathcal{P}_+(X[0,0,s]) \otimes_{\mathcal{P}^0_+(X)} \mathcal{Q}_+(X[0,r,0])$$

and

$$\begin{aligned} \mu_{/X} : I_X\mathcal{C}_+(X[0,r,s]) &\to \mathcal{C}_+(X) \\ a \otimes b &\mapsto \mu_{/X}(a) \otimes \mu_{/X}(b). \end{aligned}$$

To integrate over valued field variables, one needs to work component by component, using cell decomposition. Fix K a $\mathcal{T}$-field, for any ball $B = a + b\mathcal{O}_K \subseteq K$ and real $q > 1$, define $\theta_{q,K}(B)$ to be the real $q^{-\mathrm{ord}(b)}$. A countable disjoint union of balls of K, all of different radii, is called a step domain. A function $\varphi : K \to \mathbb{R}_{\geq 0}$ is called a step function if there is a unique step domain S such that φ is constant and non-zero on each ball of S and zero outside $S \cup \{a\}$ for some $a \in K$.

Let $q > 1$ be a real. A step-function $\varphi : K \to \mathbb{R}_{\geq 0}$ on step-domain S is said to be q-integrable if

$$\sum_{B \subseteq S} \theta_{q,K}\varphi(B) \leq +\infty,$$

the summation being on the balls B of S. This value is the q-integral of φ.

Let $\varphi \in \mathcal{P}_+(X[1,0,0])$, we say φ is an X-integrable step function if for any $\mathcal{T}$-field K, any $x \in X(K)$ and $q > 1$, the function

$$\theta_{q,K}(\varphi)(x,\cdot) : K \to \mathbb{R}_{\geq 0} t \mapsto \theta_{q,K}(\varphi)(x,t)$$

is a q-integrable step-function. For such a function, there is a unique function $\psi \in \mathcal{P}_+(X)$ such that for any $x \in X(K)$, $\theta_{q,K}(\psi)(x)$ equals the q-integral of $\theta_{q,K}(\varphi)(x,\cdot)$. We say that φ is X-integrable and set

$$\mu_{/X}(\varphi) := \psi,$$

the integral of φ over X.

We can now define the integral of a general motivic constructible function in relative dimension 1.

Lemma 1.2.30 ([25], Lemma-Definition 8.2)**.** *Let $\varphi \in \mathcal{P}_+(X[1,0,0])$. Say that φ is X-integrable if there is a $\psi \in \mathcal{P}_+(X[1,r,0])$ with $\mu_{/X[1,0,0]}(\psi) = \varphi$ such that ψ is $X[0,r,0]$-integrable in the previous sense and set*

$$\mu_{/X}(\varphi) = \mu_{/X}(\mu_{/X[0,r,0]}(\psi)) \in \mathcal{C}_+(X)$$

This definition is independent of the choices, it is the integral of φ over X.

Using cell decomposition, for any $\varphi \in \mathcal{C}_+(X[1,0,0])$ one can find a step function $\psi \in \mathcal{P}_+(X[1,r,0])$ for some tuple r such that $\mu_{/X[1,0,0]}(\psi) = \varphi$, but one needs to show that this definition is independent of the choice of ψ.

One can now define general integration by induction on the relative dimension.

Lemma 1.2.31 ([25], Lemma-Definition 9.1)**.** *Let $\varphi \in \mathcal{C}_+(Z)$, for some $Z = X[n,r,s]$ and X definable. Say φ is X-integrable if there is a definable $Z' \subseteq Z$ of complement in Z of relative dimension $< n$ over X and a choice of the order of coordinates on Z such that $\varphi' := \mathbf{1}_{Z'} \cdot \varphi$ is $X[n-1,r,s]$-integrable and $\mu_{/X[n-1,r,s]}(\varphi')$ is X-integrable. In this case, set*

$$\mu_{/X}(\varphi) := \mu_{/X}(\mu_{/X[n-1,r,s]}(\varphi')) \in \mathcal{C}_+(X).$$

This definition is independent of the choice of the order of coordinates. When $X = h[0,0,0]$ is the final object, one denote $\mu := \mu_{/X}$.

We can now integrate functions $\varphi \in \mathcal{C}_+(h[n,r,s])$, but the integral $\mu(\varphi)$ is zero if (the support) of φ is of dimension $< n$. By analogy with classical measure theory, one now define the integral of dimension n of functions embedded in higher dimensional spaces, using a change of variable formula. Fix K a $\mathcal{T}$-field. For any $f : A \subset K^n \to K^n$, let $\mathrm{Jac} f : A \to K$ the determinant of the Jacobian matrix when it is defined, 0 otherwise. In the relative case, for a function $f : A \subset C \times K^n \to C \times K^n$ such that the diagram of projections to C commutes, one define $\mathrm{Jac}_{/C} : A \to K$ to be the function satisfying for every $c \in C$,

$$(\mathrm{Jac}_{/C} f)(c,z) = (\mathrm{Jac} f_c)(z),$$

with f_c the function sending z to t such that $f(c,z) = (c,t)$.

If f is definable, the (relative) Jacobian is definable, by definability of partial derivatives, and one sets $\mathrm{Jac}_{/C} : A \to h[1,0,0]$ the unique definable function compatible for any $\mathcal{T}$-field K with the definition above.

One can now state the change of variable formula. The proof relies on the Jacobian property in dimension 1.

Proposition 1.2.32 ([25], Theorem 10.3)**.** *Let $f : Z \subseteq X[n,0,0] \to Z' \subseteq X[n,0,0]$ a definable isomorphism over X, and $\varphi \in \mathcal{C}_+(Z')$. Then there is a definable $Y \subseteq Z$ with complement in Z of dimension $< n$ relatively to X and such that the relative Jacobian*

$\mathrm{Jac}_{/X} f$ *is nonzero on* Y. *Then* $\mathbb{L}^{-\mathrm{ord}(\mathrm{Jac}_{/X} f)} f^*(\varphi)$ *is* X*-integrable if and only if* φ *is* X'*-integrable, and then*

$$\mu_{/X}(\mathbb{L}^{-\mathrm{ord}(\mathrm{Jac}_{/X} f)} f^*(\varphi)) = \mu_{/X}(\varphi) \in \mathcal{C}_+(X),$$

with the convention $\mathbb{L}^{-\mathrm{ord}(0)} = 0$.

We define now the graded ring of positive motivic constructible $\mathcal{F}$unctions. For any integer $d \geq 0$, define $\mathcal{C}_+^{\leq d}(X)$ to be the ideal of $\mathcal{C}_+(X)$ generated by characteristic functions $\mathbf{1}_Y$ for $Y \subseteq X$ of dimension $\leq d$. One also sets $\mathcal{C}_+^{\leq 1}(X) = \{0\}$. For any $d \geq 0$, set

$$\mathrm{C}_+^d(X) := \mathcal{C}_+^{\leq d}(X)/\mathcal{C}_+^{\leq d-1}(X),$$

$$\mathrm{C}_+(X) := \bigoplus_{d \geq 0} \mathrm{C}_+^d(X).$$

The nonzero elements of $\mathrm{C}_+^d(X)$ are functions of support of dimension d, defined up to sets of dimension $< d$.

One can define integrals of $\mathcal{F}$unctions in $\mathrm{C}_+(X)$ that take into account dimension. Rather than a formal definition, we explain how to compute them. Suppose $X = h[n, 0, 0]$. If $[\varphi] \in \mathrm{C}_+^d(X)$. Using functoriality properties and working with parameters, one can suppose using Proposition 1.2.14 that X is a graph of a 1-Lipschitz function $f : U \subseteq h[d, 0, 0] \to h[n-d, 0, 0]$, where U is open in K^d. By abuse of notation, write $f : u \in U \mapsto (u, f(u)) \in X$ and define

$$\mu_d(\varphi) := \mu(f^*(\varphi)).$$

This does not depend on the chosen representative of $[\varphi]$, because if $\psi \in \mathcal{C}_+(U)$ has support of dimension $< d$, then $\mu(\psi) = 0$.

1.3 Local density

By analogy with complex and real cases, it would be tempting to define the local density of a definable $X \subseteq K^m$ at $x \in K^m$ by

$$\lim_{n \to +\infty} \frac{\mu_d(X \cap S(x, n))}{\mu_d(S_d(0, n))},$$

where $S(x, n) = \{y \in K^m \mid \min_i \{\mathrm{ord}(y_i - x_i)\} = n\}$ is the (non-archimedean) sphere of valuative radius n. However this limit does not exist in general, for example, if we take

$$X = \{x \in K \mid \mathrm{ord}(x) = 0 \mod 2\},$$

then $X \cap S(0, 2k+1) = \emptyset$ and $X \cap S(0, 2k) = S(0, 2k)$, hence the sequence oscillate between 0 and 1. In this case, we will define the local density of X to be $1/2$. More generally, we will show that for every sequence of the form

$$\theta_n = \frac{\mu_d(X \cap S(x, n))}{\mu_d(S_d(0, n))},$$

there is an integer e such that the subsequences $(\theta_{ke+i})_{k \geq 0}$ converge to some d_i and one can define the density to be the arithmetic mean $\frac{1}{e}\sum_{i=1}^{e} d_i$. Note that this phenomenon is not anecdotic, if we compute the density of the cusp $x^2 = y^3$ at 0, the same oscillation appears, and we get a density of $1/2$.

1.3.1 Mean value at infinity

Let X, Y be definable. If $x \in X$, we write i_x the inclusion $\{x\} \times Y \to X \times Y$. Recall that there is a canonical pullback morphism $i_x^* : \mathcal{P}_+(X \times Y) \to \mathcal{P}(Y)$, we will use sometimes the convenient notation $\varphi(x) = i_x^*(\varphi)$.

We define also $\widetilde{\mathcal{P}_+}(X) := \mathcal{P}_+(X) \otimes \mathbb{Q}$ and $\widetilde{\mathcal{C}_+}(X) := \mathcal{C}_+(X) \otimes \mathbb{Q}$.

Definition 1.3.1. We say that a function $\varphi \in \mathcal{P}_+(\mathbb{Z})$ admits a mean value at infinity, denoted by $\mathrm{MV}_\infty(\varphi)$ if there is a positive integer e such that for all $i = 0, ..., e-1$, the subsequence $\varphi(i + ne)_{n \in \mathbb{N}}$ converges to a $d_i \in \mathbb{A}_+$. One sets $\mathrm{MV}_\infty(\varphi) = \frac{1}{e}\sum_{i=0}^{e-1} d_i \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{A}_+$.

More generally, we say that a function $\varphi \in \mathcal{P}_+(X \times \mathbb{Z})$ has a mean value at infinity $\mathrm{MV}_\infty(\varphi) \in \widetilde{\mathcal{P}_+}(X)$ if and only if for every $x \in X$, $i_x^*(\varphi)$ has a mean value at infinity and one set $\mathrm{MV}_\infty(\varphi) \in \mathcal{P}_+(X)$ such that for every $x \in X$, $i_x^*(\mathrm{MV}_\infty(\varphi)) = \mathrm{MV}_\infty(i_x^*(\varphi))$.

Definition 1.3.2. Let X be definable. A function $\varphi \in \mathcal{P}_+(\mathbb{Z})$ is said to be bounded if there is an $M \in \mathbb{A}_+$ such that $\varphi < M$. A function $\varphi \in \mathcal{P}_+(X \times \mathbb{Z})$ is said to be X-bounded if for all $x \in X$, $i_x^*(\varphi)$ is bounded.

Lemma 1.3.3. *Let $\varphi \in \mathcal{P}(X \times \mathbb{Z})$ an X-bounded function. Then φ admits a mean value at infinity.*

Proof. By Presburger cell decomposition, one can assume there are some $e, k \in \mathbb{N}^*$, a definable function $b : X \to \mathbb{N}$, $h_{i,j} \in \mathcal{P}_+(X)$, $a_i \in \mathbb{N}, b_i \in \mathbb{Z}$ for $i = 1, ..., k$ and $j = 0, ..., e-1$ such that for all $x \in X$, $j \in \{0, ..., e-1\}$, $n \equiv j \mod e$ and $n \geq b(x)$, one has

$$i_{x,n}^*(\varphi) = \sum_{i=1}^{k} h_{i,j}(x) \left(\frac{n-j}{e}\right)^{a_i} \mathbb{L}^{b_i(\frac{n-j}{e})}.$$

One can moreover assume that the pairs (a_i, b_i) are pairwise distinct, and that the functions $h_{i,j}$ are not identically zero (unless perhaps the one corresponding to (0,0)). As $i_x^*(\varphi)$ is

bounded, for all i, $b_i \leq 0$, and if $b_i = 0$, then $a_i = 0$. Let i_0 such that $(a_{i_0}, b_{i_0}) = (0,0)$. One then has $\lim_{n\to+\infty} i^*_{x,j+ne}(\varphi) = h_{i_0,j}(x)$. This implies that φ has a mean value at infinity which is

$$\mathrm{MV}_\infty(\varphi) = \frac{1}{e}\sum_{j=0}^{e-1} h_{i_0,j} \in \widetilde{\mathcal{P}_+}(X).$$

□

Definition 1.3.4. Let $\varphi \in \mathcal{C}_+(X \times \mathbb{Z})$. One says that φ has a mean value at infinity if it can be written $\varphi = \sum_i a_i \otimes b_i$ with $a_i \in \mathcal{Q}_+(X)$, $b_i \in \mathcal{P}_+(X \times \mathbb{Z})$, such that the b_i have mean values at infinity. One sets

$$\mathrm{MV}_\infty(\varphi) = \sum_i a_i \otimes \mathrm{MV}_\infty(b_i) \in \widetilde{\mathcal{C}_+}(X).$$

By using some refined partition of X, one can check that this definition does not depend on the chosen representation of φ.

One says that $\varphi \in \mathcal{C}_+(X \times \mathbb{Z})$ is X-bounded if it can be written $\varphi = \sum_i a_i \otimes b_i$ with $a_i \in \mathcal{Q}_+(X)$, $b_i \in \mathcal{P}_+(X \times \mathbb{Z})$, such that the b_i are X-bounded.

Lemma 1.3.5. *Let $\varphi \in \mathcal{C}_+(X \times \mathbb{Z})$ an X-bounded function. Then φ admits a mean value at infinity.*

Proof. It follows from Lemma 1.3.3 and the definitions, using the canonical isomorphism

$$\mathcal{C}_+(X \times \mathbb{Z}) \simeq \mathcal{Q}_+(X) \otimes_{\mathcal{P}^0_+(X)} \mathcal{P}_+(X \times \mathbb{Z}).$$

□

Let A be definable, and $\varphi \in \mathcal{C}_+(A[m,0,0])$, locally bounded, of support of dimension $\leq d$ relatively to A. Let $p : A[2m,0,1] \to A[m,0,0]$ the projection, and

$$\mathcal{S} = A \times \left\{ (x,y,z) \in h[m+m,0,1] \mid \min_i \{\mathrm{ord}(y_i - x_i)\} = n \right\}.$$

Then $\mathbf{1}_{\mathcal{S}} \cdot p^*(\varphi) \in \mathcal{C}_+(A[2m,0,1])$, and define

$$\gamma_d(\varphi) := \mu_{d,A[m,0,1]}(\mathbf{1}_{\mathcal{S}} \cdot p^*(\varphi)) \in \mathcal{C}_+(A[m,0,1]).$$

Then one normalizes by the volume $(1-\mathbb{L}^{-d})\mathbb{L}^{-nd}$ of the sphere $S^d(0,n)$ of dimension d and radius n. One defines $\theta_d(\varphi) \in \mathcal{C}_+(A[m,0,1])$ such that

$$\theta_d(\varphi)(n) = \frac{\gamma_d(\varphi)(n)}{(1-\mathbb{L}^{-d})\mathbb{L}^{-nd}}.$$

Proposition-Definition 1.3.6. If $\varphi \in \mathcal{C}_+(A[m,0,0])$ is locally bounded, of support of dimension $\leq d$ relatively to A, then $\theta_d(\varphi)$ is well defined and admits a mean value at infinity. One sets

$$\Theta_d(\varphi) := \mathrm{MV}_\infty(\theta_d(\varphi)) \in \widetilde{\mathcal{C}_+}(A[m,0,0]),$$

the motivic local density of φ.

If $\varphi = \mathbf{1}_X$, and $x \in K^m$, one denote $\Theta_d(X,x) := i_x^*(\Theta_d(\mathbf{1}_X)) \in \widetilde{\mathcal{C}_+}(\{x\})$.

Proof. As φ is locally bounded, at least for n big enough, $\mathbf{1}_{\mathcal{S}} p^*(\varphi)$ is $A[m,0,1]$-integrable, and $\theta_d(\varphi)$ is $A[m,0,0]$ bounded, hence by Lemma 1.3.5 it has a mean value at infinity. □

In more explicit terms, one has simply

$$\Theta_d(X,x) = \mathrm{MV}_\infty\left(n \mapsto \frac{\mu_d(X \cap S^m(x,n))}{\mu_d(S^d(0,n))}\right).$$

The following lemma shows that in the definition of local density we can replace the spheres by balls.

Lemma 1.3.7. *We keep the notations of 1.3.6. Let*

$$\mathcal{B} := A \times \left\{(x,y,z) \in h[m+m,0,1] \mid \min_i \{\mathrm{ord}(y_i - x_i)\} \geq n\right\},$$

$\gamma'_d(\varphi) := \mu_{d,A[m,0,1]}(\mathbf{1}_{\mathcal{B}} \cdot p^*(\varphi))$, *and* $\theta'_d(\varphi)(n) := \frac{\gamma'_d(\varphi)(n)}{\mathbb{L}^{-nd}}$. *Then* $\theta'_d(\varphi)$ *has a mean value at infinity and*

$$\mathrm{MV}_\infty(\theta'_d(\varphi)) = \Theta_d(\varphi).$$

Proof. The limit $\mathrm{MV}_\infty(\theta'_d(\varphi))$ exists for the very same reasons that $\mathrm{MV}_\infty(\theta_d(\varphi))$. One has for all $n \in \mathbb{Z}$

$$\gamma_d(\varphi)(n) = \gamma'_d(\varphi)(n) - \gamma'_d(\varphi)(n+1),$$

and

$$\theta_d(\varphi)(n) = \frac{1}{1-\mathbb{L}^{-d}}(\theta'_d(\varphi)(n) - \mathbb{L}^{-d}\theta'_d(\varphi)(n+1)).$$

By linearity of MV_∞, one has the result. □

Here are some basic properties of the local density.

Proposition 1.3.8. *Let* $\varphi, \psi \in \mathcal{C}_+(A[m,0,0])$ *be locally bounded, of supports of dimension* $\leq d$ *relatively to* A. *Let* $X, Y \subseteq K^m$ *definable of dimension* d, *with* $X \cap Y$ *of dimension* $< d$.

- $\Theta_d(\varphi + \psi) = \Theta_d(\varphi) + \Theta_d(\psi)$,
- $\Theta_d(X \cup Y) = \Theta_d(X) + \Theta_d(Y)$,

- $\Theta_d(X) = \Theta_d(\overline{X})$.

The following lemma explains why we can use definable **k**-partitions to compute the local density. It follows essentially from orthogonality between the value group and the residue rings.

Lemma 1.3.9. *Let S and A be definable, with $A \subseteq S[0, r, 0]$. Then the following diagram commutes :*

$$\begin{tikzcd}
\mathcal{C}_+(A[m,0,1]) \arrow[d, "\mu_{d,/A[0,0,1]}"] \arrow[dd, bend right=60, "\mu_{d,/S[0,0,1]}"'] & \\
\mathcal{C}_+(A[0,0,1]) \arrow[r, "\mathrm{MV}_\infty"] \arrow[d, "\mu_{/S[0,0,1]}"] & \widetilde{\mathcal{C}_+}(A) \arrow[d, "\mu_{/S}"] \\
\mathcal{C}_+(S[0,0,1]) \arrow[r, "\mathrm{MV}_\infty"'] & \widetilde{\mathcal{C}_+}(S)
\end{tikzcd}$$

Proof. The left hand side commutes by definition of motivic integration. For the right hand side, one use the canonical isomorphism

$$\mathcal{C}_+(A[0,0,1]) \simeq \mathcal{Q}_+(A) \otimes_{\mathcal{P}^0_+(S)} \mathcal{P}_+(S[0,0,1])$$

and the definition of MV_∞. □

1.3.2 Cones

Kurdyka and Raby show in [63] that the real local density of real subanalytic sets can be computed on the positive tangent cone. In our case, there is no canonical multiplicative subgroup of $K^\times$ anymore. One needs to consider a whole class of subgroups of $K^\times$, similarly to the p-adic case studied in [15].

Definition 1.3.10. Let Λ a definable multiplicative subgroup of $K^\times$. A Λ-cone with origin x is a definable $C \subseteq K^m$, with $x \in K^m$, such that for all $\lambda \in \Lambda$, and $y \in C$,

$$\lambda(y - x) \in C.$$

We say that $X \subseteq K^m$ is a local Λ-cone with origin x if for some $\alpha \in \mathbb{Z}$, $X = C \cap B(x, \alpha)$, with C a Λ-cone with origin x.

If $X \subseteq K^n$ and $x \in K^n$, we call the definable set

$$\{\lambda(y - x) \mid y \in X, \lambda \in \Lambda\}$$

the Λ-cone with origin x generated by X. It is the smallest Λ-cone with origin x that contains X.

It will be sufficient to consider Λ-cones for $\Lambda \in \mathcal{D}$, where

$$\mathcal{D} = \{\Lambda_{n,m} \mid n, m \in \mathbb{N}^*\},$$

$$\Lambda_{n,m} = \left\{\lambda \in K^\times \mid \overline{\mathrm{ac}}_m(\lambda) = 1, \mathrm{ord}(\lambda) = 0 \mod n\right\}.$$

Remark 1.3.11. 1. If X and X' are respectively a $\Lambda_{n,m}$-cone and a $\Lambda_{n',m'}$-cone with origin x, then $X \cup Y$ is a $\Lambda_{\mathrm{lcm}(n,n'),\max(m,m')}$-cone with origin x.

2. If C is a $\Lambda_{n,m}$-cone with origin x, then

$$\Theta_d(C, x) = \frac{1}{n} \sum_{i=0}^{n-1} \frac{\mu_d(C \cap S(x, i))}{\mathbb{L}^{-di}(1 - \mathbb{L}^{-d})}.$$

Definition 1.3.12. Let $X \subseteq K^n$ be definable and $x \in K^n$. Let

$$\pi_x : K^n \setminus \{x\} \to \mathbb{P}^{n-1}(K)$$

denote the function that send a point $z \neq x$ to the line between z and x, and π_x^X its restriction to $X \setminus \{0\}$.

We now show the local conic property of definable in ambient dimension 1. This is false in higher dimension, as is shown by the parabola example.

Lemma 1.3.13. *For every definable $X \subseteq K$, there is a $\Lambda \in \mathcal{D}$ and a definable function $\gamma : X \to \mathbb{Z}$ such that $X \cap B(y, \gamma(y))$ is a local Λ-cone with origin y for all $y \in Y$.*

Proof. By Remark 1.3.11, one can partition X into finitely many definable parts. By cell decomposition, one can then assume that

$$X = \{x \in K \mid \mathrm{ord}(x - c) \in A, \overline{\mathrm{ac}}_m(x - c) \in B\},$$

for a $c \in K$ and $A \subseteq \mathbb{Z}$, $B \subseteq k_K$ definable. For $y \neq c$, we set $\gamma(y) := \mathrm{ord}(y - c)$, and $X \cap B(y, \gamma(y))$ is a local $\Lambda_{1,1}$-cone. For $y = c$, by Presburger quantifier elimination, up to cutting A into a finite number of parts, there is a $b \in \mathbb{Z}$ definable, $e \in \mathbb{N}^*$, $i \in \{0, ..., e - 1\}$ such that either

$$A \cap \{\alpha \mid \alpha \geq b\} = \{\alpha \mid \alpha \geq b, \alpha = i \mod e\},$$

either this set is empty. In both cases, we set $\gamma(y) = b$, and $X \cap B(y, \gamma(y))$ is a local $\Lambda_{e,m}$-cone. □

By working with parameters and using logical compactness, we get the following corollary.

Corollary 1.3.14. *Let X be a definable of K^n and $x \in K^n$. Then there is a $\Lambda \in \mathcal{D}$ and a definable function $\alpha_x : \mathbb{P}^{n-1}(K) \to \mathbb{Z}$ (*i.e. *definable in affine charts of $\mathbb{P}^{n-1}(K)$) such that*

$$(\pi_x^X)^{-1}(\ell) \cap B(x, \alpha_x(\ell))$$

is a local Λ-cone with origin x for all $\ell \in \mathcal{P}^{n-1}(K)$. Moreover, one can choose Λ independent from x, such that $(x, \ell) \mapsto \alpha_x(\ell)$ is definable.

We call a $\Lambda \in \mathcal{D}$ satisfying the first condition adapted to (X, x). If Λ is adapted to (X, x) for all x, then we say that Λ is adapted to X.

1.3.3 Tangent cone

Definition 1.3.15. Let $\Lambda \in \mathcal{D}$, $X \subseteq K^n$ be definable and $x \in K^n$. The tangent Λ-cone of X at x is

$$C_x^\Lambda(X) = \{u \in K^n \mid \forall i \in \mathbb{Z}, \exists y \in X, \exists \lambda \in \Lambda, \mathrm{ord}(y - x) \geq i, \mathrm{ord}(\lambda(y - x) - u) \geq i\}.$$

By definition, $C_x^\Lambda(X)$ is definable.

For X,Y definable in K^n, and $\Lambda' \subseteq \Lambda \in \mathcal{D}$, we have the following properties.

$$C_x^\Lambda(X \cup Y) = C_x^\Lambda(X) \cup C_x^\Lambda(Y),$$

$$C_x^\Lambda(\overline{X}) = C_x^\Lambda(X),$$

$$C_x^{\Lambda'}(X) \subseteq C_x^\Lambda(X).$$

Remark 1.3.16. The inclusion above can be strict, but we have the following result. If X is a local Λ-cone with origin x, then for all $\Lambda' \subseteq \Lambda$, $C_x^{\Lambda'}(X) = C_x^\Lambda(X)$. Indeed, as X is a local Λ-cone, there is a Λ-cone C with origin x and an $n \in \mathbb{Z}$ such that $X = C \cap B(x, n)$. We have $C_x^\Lambda(X) = C_x^\Lambda(C)$ and $C_x^{\Lambda'}(X) = C_x^{\Lambda'}(C)$. Let show the inclusion $C_x^\Lambda(C) \subseteq C_x^{\Lambda'}(C)$. Let $u \in C_x^\Lambda(C)$, there are sequences (y_i) and (λ_i) of elements of respectively C and Λ such that $y_i \to x$ and $\lambda_i(y_i - x) \to u$. As $\Lambda', \Lambda \in \mathcal{D}$, for all i there are $\lambda_i' \in \Lambda'$, $\mu_i \in \Lambda$ such that $|\mu_i| = 1$ and $\lambda_i = \lambda_i' \cdot \mu_i$. Set $y_i' = \mu_i(y_i - x) + x$. Because C is a Λ-cone $y_i' \in C$, and we have $y_i' \to x$, $\lambda_i'(y_i - x) \to u$ hence $u \in C_x^{\Lambda'}(C)$.

The following lemma justifies the use of **k**-partitions to compute the tangent cone, it follows from orthogonality between value group and residue rings.

Lemma 1.3.17. *Let X be a definable of K^m, and $g : X \to A$ a* **k***-partition of X. Then for any $x \in K^m$ and $\Lambda \in \mathcal{D}$,*

$$C_x^\Lambda(X) = \bigcup_{\xi \in A} C_x^\Lambda(X_\xi),$$

where $X_\xi = g^{-1}(\xi)$.

Proof. Fix $u \in C_x^\Lambda(X)$. Let D be the u-definable

$$\{(n,\xi) \in \mathbb{Z} \times A \mid \exists y \in X, \exists \lambda \in \Lambda, \mathrm{ord}(y-x) \geq n, \mathrm{ord}(\lambda(y-x)-u) \geq n, g(y) = \xi\}.$$

By definition of $C_x^\Lambda(X)$, D projects surjectively on its first coordinate. Because the theory splits, D is a union of products of definable sets of $\mathbb{Z}$ and A, hence there is a $\xi \in A$ such that for all $n \in \mathbb{Z}$, $(n,\xi) \in D$. This means that $u \in C_x^\Lambda(X_\xi)$. □

Proposition 1.3.18. *Let $X \subseteq K^m$ be definable, $x \in K^m$, and $\Lambda' \subseteq \Lambda \in \mathcal{D}$. Then $C_x^\Lambda(X)$ is the Λ-cone with origin 0 generated by $C_x^{\Lambda'}(X)$.*

Proof. Suppose $\Lambda' = \Lambda_{n,e}$. Consider the **k**-partition of X defined by

$$g : X \to R_{e,K}^m, y \mapsto \overline{\mathrm{ac}}_e(y-x).$$

Let $u \in C_x^\Lambda(X)$ be nonzero. By Lemma 1.3.17 above, we can assume $u \in C_x^\Lambda(X_\xi)$, for some **k**-part X_ξ. Then there are some sequences $(y_k)_{k\geq 0} \in X_\xi$ and $(\lambda_k)_{k\geq 0} \in \Lambda$ such that (y_k) converges to x and $(\lambda_k(y_k - x))$ converges to u. By definition of the **k**-partition, $\overline{\mathrm{ac}}_e(y_k - x)$ is constant. Because $(\lambda_k(y_k - x))$ converges to u and $u \neq 0$, $\overline{\mathrm{ac}}_e(\lambda_k(y_k - x))$ is also constant for k big enough. Hence $\overline{\mathrm{ac}}_e(\lambda_k)$ is constant for k big enough, so we can write (for k big enough) $\lambda_k = \mu\lambda'_k$ where $\mu \in \Lambda$ does not depend on k and $\overline{\mathrm{ac}}_e(\lambda'_k) = 1$. Up to extracting a subsequence, we can also assume that $\mathrm{ord}(\lambda'_k) \mod n$ is constant, hence up to multiplying μ and λ'_k by some constant, we have $\lambda_k = \mu\lambda'_k$ where $\mu \in \Lambda$ and $\lambda'_k \in \Lambda'$. Then $\lambda'_k(y_k - x)$ converges to some $u' \in C_x^{\Lambda'}(X)$, and $u = \mu u'$. □

Corollary 1.3.19. *The dimension of $C_x^\Lambda(X)$ is independent of $\Lambda \in \mathcal{D}$.*

Proof. As every $\Lambda \in \mathcal{D}$ is included in $\Lambda_{1,1}$, we can assume we are in the situation of Proposition 1.3.18, with $\Lambda' \subseteq \Lambda \in \mathcal{D}$, and we have to prove that

$$\dim(C_x^\Lambda(X)) = \dim(C_x^{\Lambda'}(X)).$$

By Proposition 1.3.18, $C_x^\Lambda(X)$ is the Λ-cone generated by $C_x^{\Lambda'}(X)$. Suppose $\Lambda = \Lambda_{n,e}$ and $\Lambda' = \Lambda_{n',e'}$. Then n divides n' and let

$$A = \left\{\xi \in R_{e',K} \mid p_{e',e}(\xi) = 1\right\} \times \left\{0, n, 2n, ..., n'-n\right\} \subseteq R_{e',K} \times \mathbb{Z}$$

and

$$\begin{aligned} B \;=\; & \{(\xi,\alpha,z) \in A \times K^m \mid \exists y \in C_x^{\Lambda'}(X), \exists \mu \in \Lambda, \\ & \overline{\mathrm{ac}}_e(\mu) = \xi, \mathrm{ord}(\mu) = \alpha, z = \mu y\}. \end{aligned}$$

Because $C_x^{\Lambda'}(X)$ is a Λ'-cone, the fiber $B_{(\xi,\alpha)}$ of B above $(\xi,\alpha) \in A$ is in bijection with

$C_x^{\Lambda'}(X)$. It follows that D is of dimension $\dim(C_x^{\Lambda'}(X))$, because A lives in the auxiliary sorts, hence is of dimension 0. Moreover, as $C_x^{\Lambda}(X)$ is the Λ-cone generated by $C_x^{\Lambda'}(X)$, the projection of D on K^m has image $C_x^{\Lambda}(X)$, hence

$$\dim(C_x^{\Lambda'}(X)) \geq \dim(C_x^{\Lambda}(X)).$$

The other inequality being clear by the inclusion $C_x^{\Lambda'}(X) \subseteq C_x^{\Lambda}(X)$, the corollary is proven. □

Definition 1.3.20. Let X a definable of K^n, $x \in K^n$ and $\Lambda \in \mathcal{D}$. The deformation of X is

$$\mathcal{D}(X, x, \Lambda) := \{(y, \lambda) \in K^n \times \Lambda \mid x + \lambda y \in X\}$$

Let $\overline{\mathcal{D}(X, x, \Lambda)}$ be its closure in K^{n+1}, we have $\overline{\mathcal{D}(X, x, \Lambda)} \cap K^n \times \{0\} = C_x^{\Lambda}(X) \times \{0\}$. We identify the latter with $C_x^{\Lambda}(X)$.

Proposition 1.3.21. *For every definable set X of K^n, $x \in K^n$ and $\Lambda \in \mathcal{D}$, we have*

$$\dim(\mathcal{D}(X, x, \Lambda)) = \dim(X) + 1 \quad \textit{and} \quad \dim(C_x^{\Lambda}(X)) \leq \dim(X).$$

Proof. The projection $(y, \lambda) \in \mathcal{D}(X, x, \Lambda) \mapsto x + \lambda y \in X$ has its fibers isomorphic to Λ, hence of dimension 1. By dimension theory, $\dim(\mathcal{D}(X, x, \Lambda)) = \dim(X) + 1$. As $C_x^{\Lambda}(X) \subseteq \overline{\mathcal{D}(X, x, \Lambda)} \setminus \mathcal{D}(X, x, \Lambda)$, it is of dimension less than $\dim(\mathcal{D}(X, x, \Lambda))$. □

We need to define a notion of multiplicity on the tangent cone. As shown by the following example, the multiplicity cannot be an integer.

Example 1.3.22. Consider $K = \mathbb{C}((t))$, and the cusp $X = \{y^2 = x^3\}$. The density of X at 0 is $1/2$. The tangent Λ_2-cone is

$$\begin{aligned} C_0^{\Lambda_2}(X) &= \{(0, y) \in K^2 \mid \operatorname{ord}(y) = 0 \mod 2, \exists \eta \in \mathbb{C}^\times, \overline{\mathrm{ac}}(y)^3 = \eta^2\} \\ &= \{(0, y) \in K^2 \mid \operatorname{ord}(y) = 0 \mod 2\}, \end{aligned}$$

because $\mathbb{C}$ is algebraically closed. It seems that there are two fibers over each point $(0, y)$ of the tangent cone, defined by the two roots of $\overline{\mathrm{ac}}(y)^3 = \eta^2$, hence it seems we should attribute a multiplicity of 2 at every point of the tangent cone. But this is too naive, as it leads to a multiplicity of 1. A way to proceed is to view the multiplicity as a motive, the density m_y of the tangent cone at $(0, y)$ is

$$[\{\eta \in \mathbb{C} \mid \eta^2 = \overline{\mathrm{ac}}(y)^3\}] \in \mathcal{C}_+(\{y\})$$

and the tangent cone with multiplicities is

$$CM_0^{\Lambda_2}(X) = [\{(0, y, \eta) \in K^2 \times \mathbb{C} \mid \mathrm{ord}(y) = 0 \mod 2, \eta^2 = \overline{\mathrm{ac}}(y)^3\}] \in \mathcal{C}_+(K^2),$$

this leads to the correct value

$$\Theta_1(CM_0^{\Lambda_2}(X))(0) = \frac{1}{2}\frac{[\{(\xi, \eta) \in k^2 \mid \xi^2 = \eta^3\}] - 1}{\mathbb{L} - 1} = \frac{1}{2}.$$

This motivates the following definition.

Definition 1.3.23. Let $X \subseteq K^n$ be definable of dimension d. Let $g : X \to A$ a definable $\mathbf{k}$-partition of X as in Proposition 1.2.14. The definable X is then partitioned into graphs X_ξ of 1-Lipschitz functions defined on open sets of K^d, up to a set of dimension $< d$. For any $\Lambda \in \mathcal{D}$, we define the tangent Λ-cone with multiplicities as

$$CM_x^\Lambda(X) := [\{(y, \xi) \in K^n \times A \mid y \in C_x^\Lambda(g^{-1}(\xi))\}]_d \in \mathrm{C}_+^d(K^n).$$

Remark 1.3.24. This depends a priori of the chosen partition. However, if we have two such $\mathbf{k}$-partitions, by taking a refinement, we see that the class of

$$m_y = \{\xi \mid y \in C_x^\Lambda(g^{-1}(\xi))\}$$

does not depend on the chosen partition, up to a set of dimension $< d$. Hence $CM_x^\Lambda(X)$ is well defined (recall that $\mathcal{F}$unctions in $\mathrm{C}_+^d(K^n)$ are defined up to a set of dimension $< d$).

We can now state our main theorem, that will be proved in Section 1.5.

Theorem 1.3.25. *Let $X \subseteq h[n, 0, 0]$ a definable subassignment of dimension d. Then there is a $\Lambda \in \mathcal{D}$ such that for all $x \in K^d$*

$$\iota(\Theta_d(X, x)) = \iota(\Theta_d(CM_x^\Lambda(X), 0)) \in \widetilde{\mathcal{C}}(\{x\}).$$

Because the map from the Grothendieck semi-group to the group is not expected to be injective, the canonical induced map

$$\iota : \mathcal{C}_+(X) \to \mathcal{C}(X)$$

is not expected to be injective either. As we will prove Theorem 1.3.25 by a double inequality, this gives the equality only at the level of the Grothendieck group. This should not be seen as a problem, because every known realization of the Grothendieck semi-group factorizes through the Grothendieck group.

1.4 Regular stratifications

Regular stratifications satisfying the Whitney condition (a_f) have been introduced by Thom [94] and studied by Hironaka [53] in the complex case, Kurdyka-Raby [63] in the real case. Afterward, the stronger Verdier condition (w_f) has been introduced by Henry-Merle-Sabbah [52], and studied by Bekka [5] and Kurdyka-Parusinski [62] in the real case, Loi [74] in the o-minimal case and Cluckers-Comte-Loeser [15] in the p-adic setting. This section is devoted to defining regular stratification satisfying analogous of conditions (a_f) and (w_f) in our context of tame Henselian fields.

Note that Halupczok [50] has already defined a notion of regular stratification in Henselian fields of equicharacteristic zero, the so-called t-stratifications. Instead of starting from the classical definition of (a_f)-stratification as we do, he starts by the property of local trivialization. This leads to two different notions of regular stratification in this context, and it is unknown whether a common generalization can be found, see [50], open question 9.1.

Because we work in an abstract (mixed) tame theory, we do not have a notion of analytic manifold, hence we only require in the following definition the strata to be differentiable. However, if one works in a concrete tame theory where a notion of analytic manifold is defined and an analytic cell decomposition theorem is available, then one can require the strata to be also analytic.

Definition 1.4.1. A definable stratification of $Y \subseteq K^n$ is a definable $\mathbf{k}$-partition of Y such that each $\mathbf{k}$-part is in definably diffeomorphic to some open subset of K^d for some d (depending on the chosen $\mathbf{k}$-part). The $\mathbf{k}$-parts are called strata and required to satisfy the so-called frontier condition :

$$\text{for each stratum } X,\ \overline{X} \setminus X \text{ is a union of strata.}$$

If $f : Y \to K$ is a definable function, an f-stratification of Y is a stratification of Y such that for every stratum X, $f_{|X}$ is differentiable of constant rank.

Denote $T_{x,f}(X) = T_x(f_{|X}^{-1}(f(x)))$ the tangent space in x of the fiber of $f_{|X}$ above $f(x)$.

If V, V' are two K-vector spaces of finite dimension with $\dim(V) \leq \dim(V')$, set

$$\delta(V, V') = \sup_{v \in V, |v|=1} \{ \inf_{v' \in V', |v'|=1} \left| v - v' \right| \}.$$

It is the Hausdorff distance between the units balls of V and V'.

It satisfies $\delta(V, V') = 0$ if and only if $V \subseteq V'$, and for $V'' \subseteq V'$ with $\dim(V'') \geq \dim(V)$, $\delta(V, V'') \geq \delta(V, V')$.

A couple of strata (X, X') is said to satisfy the condition (a_f) at $x_0 \in X \subseteq \overline{X'}$ if for every sequence (x_n) of points of X' converging to x_0,

$$\delta(T_{x_0,f}X, T_{x_n,f}X') \to 0. \tag{a_f}$$

A couple of strata (X,X') is said to satisfy the condition (w_f) at $x_0 \in X \subseteq \overline{X'}$ if there is a constant C such that for all neighborhoods W of x_0, all $x \in W \cap X$ and $x' \in W \cap X'$, one has

$$\delta(T_{x,f}X, T_{x',f}X') \leq C\,|x-y|\,. \tag{w_f}$$

A stratification Y is said to be (w_f)-regular if for all strata (X, X') and $x_0 \in X \subseteq \overline{X'}$, the condition (w_f) is satisfied at x_0.

If V and V' are definable, then the condition $\delta(V, V') \leq \varepsilon$ is definable by the formula

$$\forall v \in V, |v| = 1, \exists v' \in V', \left|v'\right| = 1, \left|v - v'\right| \leq \varepsilon.$$

It follows that if X, X' are definably diffeomorphic to some open subset of K^d (for different d), then

$$W_f(X, X') = \left\{x \in X \mid (X, X') \text{ satisfies } (w_f) \text{ at } x\right\}$$

is definable.

Theorem 1.4.2. *Let $Y \subseteq K^n$ a definable, and $f : Y \to K$ a continuous definable function. Then there exists a (w_f)-regular definable stratification of Y.*

The proof of this theorem will follow closely the approach of Loi [74] and uses as a key ingredient the curve selection lemma. We start by some lemmas.

Lemma 1.4.3. *Let $M : \Omega \to K$ be a differentiable and definable function on $\Omega \subseteq K^m \times K$. Suppose $\overline{\Omega} \cap K^m \times \{0\}$ has non-empty interior $U \subseteq K^m$, and M is bounded on Ω. Then there exist a non-empty open $V \subseteq U$, $\varepsilon > 0$, and $d \in K^\times$ such that for all $(x, t) \in \Omega$, $x \in V$, and $|t| \leq \varepsilon$, one has*

$$|D_x M(x, t)| \leq |d|\,.$$

Here and later, we use the notation $D_x M(x, t) = (\partial M(x, t)/\partial x_1, ..., \partial M(x, t)/\partial x_m)$.

Proof. By cell decomposition, we can partition Ω into $\mathbf{k}$-definable parts such that on each cell A such that $\overline{A} \cap K^m \times \{0\} \neq \emptyset$, one has

$$|D_{x_i} M(x, t)| = |D_x M(x, t)| = |h(x)| \cdot |t|^a\,,$$

for some $a \in \mathbb{Q}$, h a definable function, and $i \in \{1, ..., m\}$. If all the a are nonnegative, then we are done because the $|h(x)|$ and the function bounding $|t|$ from below are constant on small enough open subsets $V \subseteq U$.

Assume that one a is negative, say on a cell where $|D_x M(x, t)| = |\partial M(x, t)/\partial x_1|$. By the Jacobian property applied to $x_1 \mapsto M(x_1, y)$, there is a $(\mathbf{k}, y)$-definable of K such that the Jacobian property holds on each ball in it for the function $x_1 \mapsto M(x_1, y)$. Up to another cell decomposition, one can assume that there are some $\mathbf{k}$-definable balls $B_1 \subseteq K$, $B \subseteq K^{m-1}$ and $C \subseteq K$ a $\mathbf{k}$-definable with $0 \in \overline{C}$ such that for all $y \in B \times C$, the Jacobian

property holds on B_1 for $x_1 \mapsto M(x_1, y)$. We then have for all $y = (x_2, ..., x_n, t) \in B \times C$ and $x_1, x_1' \in B_1$,

$$\left|M(x_1', y) - M(x_1, y)\right| = \left|x_1' - x_1\right| \cdot |\partial M(x,t)/\partial x_1| = \left|x_1' - x_1\right| \cdot |h(x)| \cdot |t|^a$$

As M is bounded, the left term is bounded but the right one goes to $+\infty$ when t goes to 0, this is a contradiction. □

Lemma 1.4.4. *Let $V \subseteq K^m$, $S \subseteq K^n$ be definable with V nonempty open in K^m, and $V \times \{0\}^{m-n} \subseteq \overline{S} \setminus S$. Let $\pi : K^n \to K^m$ be the natural projection, and for every $u \in V$, $S_u = \{x \in S \mid \pi(x) = u\}$. Then there is a nonempty open $U \subseteq V$ such that for all $u \in U$, $(u, 0) \in \overline{S_u} \setminus S_u$.*

Proof. Let

$$A = \{(x, y, \delta) \in S \times V \times \Gamma \mid \pi(x) = y, |x - y| \leq \delta\}.$$

Let π_1 the projection defined by $\pi_1(x, y, \delta) = y$. The definable $\pi_1(A)$ is dense in V. Indeed, let $y_0 \in V$, and $\varepsilon > 0$. By openness of V, there is an ε' with $0 < \varepsilon' < \varepsilon$ and $B(y_0, \varepsilon') \cap K^m \times \{0\}^{n-m} \subseteq V \times \{0\}^{n-m}$. Since $y_0 \in \overline{S} \setminus S$, there is a $x \in B(y_0, \varepsilon') \cap S$. Then $y := \pi(x) \in \pi_1(A)$ and $|y - y_0| \leq \varepsilon$.

Let π_2 be the projection defined by $\pi_2(x, y, \delta) = (y, \delta)$ and set

$$\pi_2(A)_y := \{\delta \mid (y, \delta) \in \pi_2(A)\}, \ \varepsilon(y) = \inf \pi_2(A)_y, \ y \in \pi_1(A)$$

Then ε is definable, and it suffices to show that $\dim(\{y \in \pi_1(A) \mid \varepsilon(y) > 0\}) < m$. Suppose the contrary. Then by cellular decomposition there is a $W \subseteq \pi_1(A)$ open on which ε is greater than a $c > 0$. But then for all $y_0 \in W$, and δ with $0 < \delta < c$ and $B(y_0, \delta) \cap K^m \times \{0\}^{n-m} \subseteq W \times \{0\}^{n-m}$. Then $|x - y| > c$ for all $y \in B(y_0, \delta) \cap K^m$, $x \in S$ with $\pi(x) = y$. Contradiction with the argument above. □

Lemma 1.4.5. *Let $f : Y \subseteq K^n \to K$ a definable continuous function. Assume that for all couples (X, X') of definable differentiable sets, with $X \subseteq \overline{X'} \setminus X'$, we have $\dim(X \setminus W_f(X, X')) \leq \dim(X)$. Then Y admits a (w_f)-regular f-stratification.*

Proof. There exists S an f-stratification of Y by Proposition 1.2.14 applied inductively. By decreasing induction on $d \in \{0, ..., n\}$, one builds some successive refinements S_d of S satisfying :

$$\text{for all } X, X' \in S_d, X \subset \overline{X'}, \dim(X) \geq d, \text{ one has } W_f(X, X') = X,$$

using the lemma hypothesis. Then S_0 is (w_f)-regular. □

Proof of Theorem 1.4.2. By Lemma 1.4.5, it suffices to show that for a pair (X, X') of definable manifolds of Y, with $X \subseteq \overline{X'} \setminus X'$, one has $\dim(X \setminus W_f(X, X')) < \dim(X)$. Suppose

by contradiction that for such a pair, $\dim(X \setminus W_f(X, X')) = \dim(X)$. Up to reducing X and X', assume $W_f(X, X')$ is empty, that f is of constant rank on respectively X and X'. We can also replace X by $f_{|X}^{-1}(a)$, for a $a \in K$ such that this set is nonempty, therefore assume f is constant on X. By invariance of the condition (w_f) under diffeomorphisms, one can assume $X = V \times \{0\}^{n-m}$, for $V \subseteq K^m$ open definable, therefore for $x \in X$, $T_x X = K^m \times \{0\}^{n-m}$.

There are two cases to consider.

• **Case 1 :** f is of rank 0 on X'. In this case, one can assume f constant on X'. As $W_f(X, X') = \emptyset$, the definable

$$D = \left\{(y, t) \in X' \times K^\times \mid \delta(K^m \times \{0\}^{n-m}, T_y X') \geq |y_1| \, |t|^{-1}\right\}$$

satisfies $V \times \{0\}^{n-m+1} \subseteq \overline{D} \setminus D$ (where $y = (y_0, y_1)$, with y_1 of length $n - m$). By Lemma 1.4.4, there is a nonempty open $U \subseteq V$ such that for all $u \in U$, $(u, 0) \in \overline{D_u} \setminus D_u$. By the curve selection lemma 1.2.24, there is a definable (with parameters) function $\overline{\rho} : U \times B_e \to \overline{D}$ with $\overline{\rho}(u, s) \in \overline{D}_u$, $\overline{\rho}(u, 0) = (u, 0)$, $\overline{\rho}(u, s) \in D$ if $s \neq 0$, $\overline{\rho}(u, \cdot)$ C-Lipschitz. By the Jacobian property, up to restricting ourself to an open subset of U one can moreover assume that $\overline{\rho}(\cdot, s)$ is differentiable. Also denote $\overline{\rho}(u, s) = (u, \rho(u, s), r(u, s))$. We have for all $u \in U$ and $s \in B_e^*$,

$$\delta(K^m \times \{0\}^{n-m}, T_{u,\rho(u,s)} X') \geq |\rho(u, s)| \, |r(u, s)|^{-1} .$$

By definition of Hausdorff distance,

$$\delta(K^m \times \{0\}^{n-m}, T_{u,\rho(u,s)} X') \leq |D_u \rho(u, s)| .$$

By cell decomposition, up to replacing $(u, s) \mapsto \overline{\rho}(u, s)$ by $(u, s) \mapsto \overline{\rho}(u, s^k)$, and U by a nonempty open subset, there are an $a \in K^\times$ and an integer $\ell > 0$ such that for all $(u, s) \in U \times B$,

$$|\rho(u, s)| = |a| \, |s|^\ell .$$

Since the function $(u, s) \in U \times B^* \to \rho(u, s)/s^\ell$ is bounded, by Lemma 1.4.3, there is a $d \in K^\times$ such that up to replacing U by a nonempty open subset, for all $(u, s) \in U \times B^*$,

$$|D_u \rho(u, s)| \leq |d| , \text{ hence } \frac{|D_u \rho(u, s)|}{|\rho(u, s)|} \leq |ad| .$$

On the other hand,

$$\frac{|D_u \rho(u, s)|}{|\rho(u, s)|} \geq |r(u, s)|^{-1} ,$$

and $r(u, s)$ goes to 0 when s goes to 0, contradiction.

• **Case 2 :** f is of rank 1 on X'. We can assume $f(X) = 0$ and $f(X') \neq 0$. As

$W_f(X, X') = \emptyset$, the definable

$$D = \left\{(y,t) \in X' \times K^\times \mid \delta(K^m \times \{0\}^{n-m}, T_{y,f}X') \geq |y_1|\,|t|^{-1}\right\}$$

satisfies $V \times \{0\}^{n-m+1} \subseteq \overline{D_u} \backslash D_u$ (where $y = (y_0, y_1)$, with y_1 of length $n-m$). By Lemma 1.4.4, there is a nonempty open $U \subseteq V$ such that for all $u \in U$, $(u,0) \in \overline{D_u} \backslash D_u$. By Corollary 1.2.26, up to restricting U to a $\mathbf{k}$-definable subpart, there is a definable with parameters function $\overline{\rho} : U \times B_e \to \overline{D}$ with $\overline{\rho}(u,s) \in \overline{D_u}$, $\overline{\rho}(u,0) = (u,0)$, $\overline{\rho}(u,s) \in D$ if $s \neq 0$ and $\overline{\rho}(u,\cdot)$ C-Lipschitz for all $u \in U$. Denote $\overline{\rho}(u,s) = (u, \rho(u,s), r(u,s))$. By Proposition 1.2.12, we can moreover assume (up to restricting again U and taking e bigger) that $f(u, \rho(u,\cdot))$ is injective (it cannot be constant because f is of rank one on X'). Consider $f_0(u,s) = f(u, \rho(u,s))$. By cell decomposition, there is an integer e' and a definable with parameters continuous function $h : B_{e'} \to f_0(U \times B_e)$ with $h(0) = 0$ and $h(B^*_{e'}) \subseteq K^\times$. For $(u,s') \in U \times B_{e'}$, there is an unique $s \in B_e$ such that $f_0(u,s) = h(s')$, denote $h_0(u,s') = s$. The function h_0 is definable (with parameters) and $h_0(u,s')$ goes to 0 when s' goes to 0 (for fixed u). Replace $\overline{\rho}$ by

$$\overline{\rho'} : \begin{cases} U \times B_{e'} & \to & \overline{D} \\ (u,s) & \mapsto & \overline{\rho}(u, h_0(u,s)) \end{cases}$$

The function $\overline{\rho'}$ satisfies $\overline{\rho'}(u,s) \in \overline{D_u}$, $\overline{\rho'}(u,0) = (u,0)$, $\overline{\rho'}(u,s) \in D$ if $s \neq 0$ and $\lim_{s\to 0} \overline{\rho'}(u,s) = 0$ for all $u \in U$. Up to reducing U, assume $\overline{\rho'}(\cdot, s)$ is differentiable with respect to u at all $u \in U$. Denote $\overline{\rho'}(u,s) = (u, \rho'(u,s), r'(u,s))$. For all $(u,s) \in U \times B_{e'}$,

$$\delta(K^m \times \{0\}^{n-m}, T_{(u,\rho(u,s)),f}X') \geq \left|\rho'(u,s)\right| \left|r'(u,s)\right|^{-1}$$

and

$$f(u, \rho'(u,s)) = f_0(u,s) = h_0(s).$$

The vector space spanned by vectors $(0, ..., 0, 1, 0, ..., 0, \partial\rho'(u,s)/\partial u_i)$ (with 1 in i-th position) is a subspace of $T_{(u,\rho'(u,s)),f}X'$, hence

$$\left|D_u\rho'(u,s)\right| \geq \delta(K^m \times \{0\}^{n-m}, W) \geq \delta(K^m \times \{0\}^{n-m}, T_{(u,\rho(u,s)),f}X').$$

We are now in a situation similar to case 1, up to replacing $\overline{\rho'}(u,s)$ by $\overline{\rho'}(u,s^k)$ and reducing U, we can assume

$$\left|\rho'(u,s)\right| = |a|\,|s|^\ell,$$

and by Lemma 1.4.3,

$$|D_u\rho((u,s))| \leq |d|.$$

Hence

$$\left|r'(u,s)\right|^{-1} \leq \frac{|D_u\rho(u,s)|}{|\rho(u,s)|} \leq |ad|\,,$$

contradiction. □

1.5 Proof of Theorem 1.3.25

1.5.1 Codimension 0 case

The proof of Theorem 1.3.25 goes along the same lines as the proof of the main theorem of [15]. We start by proving it in the case $d = n$, using a monotone convergence lemma.

Lemma 1.5.1 (Monotone convergence)**.** *Let $\varphi \in \mathcal{C}_+(A \times \mathbb{N})$ integrable. Then*

$$\lim_{n\to+\infty} \sum_{k=0}^{n} \mu(i_n^*(\varphi)) = \mu(\varphi).$$

In particular, if $(X_n)_{n\geq 0}$ is a bounded uniformly definable sequence of sets of K^m, increasing for the inclusion (resp. decreasing), then

$$\lim_{n\to\infty} \mu(\bigcup_{0\leq k\leq n} X_k) = \mu(\bigcup_{k=0}^{+\infty} X_k),$$

$$\text{(resp.)} \quad \lim_{n\to\infty} \mu(\bigcap_{0\leq k\leq n} X_k) = \mu(\bigcap_{k=0}^{+\infty} X_k).$$

Proof. The sequence $\sum_{k=0}^{n} \mu(i_n^*(\varphi))$ is bounded, definable, and increasing. By Presburger cell decomposition, as in Lemma 1.3.3, we have the existence of the limit. By definition of the motivic integral with respect to a value group variable, we have the result.

We consider now the definable $X = \bigcup_{0\leq k} X_k$, where $(X_k)_k$ is an increasing sequence. Let $f : X \to \mathbb{N}$ the definable function that maps x to the least n such that $x \in X_n$. Let $\widetilde{X}_n = f^{-1}(n) = X_n \setminus X_{n-1}$. We see X as a definable of $X \times \mathbb{N}$, and by the first point,

$$\lim_{n\to+\infty} \sum_{k=0}^{n} \mu(\widetilde{X}_k) = \mu(X) = \mu(\bigcup_{k=0}^{+\infty} X_k).$$

By linearity of the integral,

$$\sum_{k=0}^{n} \mu(\widetilde{X}_k) = \mu(\bigcup_{0\leq k\leq n} \widetilde{X}_k) = \mu(\bigcup_{0\leq k\leq n} X_k),$$

hence we have the required equality ; the other case is similar. □

Proposition 1.5.2. *Let X be definable of K^d. Then for all $\Lambda \in \mathcal{D}$ small enough and $x \in K^d$,*

$$\iota(\Theta_d(X,x)) = \iota(\Theta_d(C_x^\Lambda(X),0)).$$

Proof. We have to show the two inequalities

$$\Theta_d(C_x^\Lambda(X),0) \geq \Theta_d(X,x)$$

and

$$\Theta_d(C_x^\Lambda(X),0) \leq \Theta_d(X,x).$$

By Corollary 1.3.14, there is a $\Lambda = \Lambda_{n,m} \in \mathcal{D}$, a definable function $\alpha : \mathbb{P}^{d-1}(K) \to \mathbb{Z}$ such that for all $\ell \in \mathbb{P}^{d-1}(K), (\pi_x^X)^{-1}(\ell) \cap B(x,\alpha(\ell))$ is a local Λ-cone with origin x. This remains true for any $\Lambda' \in \mathcal{D}$, $\Lambda' \subseteq \Lambda$, hence the following reasoning holds also for any such Λ'.

For all $n \geq 0$, let C_n the Λ-cone with origin x generated by $X \cap B(x,n)$ and $W = \bigcap_{n\geq 0} C_n$, W is also a Λ-cone. By monotone convergence lemma 1.5.1, for any $i = 0, ..., n-1$

$$\mu_d(W \cap S(x,i)) = \lim_{n\to+\infty} \mu_d(C_n \cap S(x,i)),$$

hence by Remark 1.3.11 and linearity of the integral,

$$\Theta_d(W,x) = \lim_{n\to+\infty} \Theta_d(C_n,x).$$

Moreover, the tangent cone satisfies by definition $x + C_x^\Lambda(X) = \cap_{n\geq 0}\overline{C_n}$. The $\overline{C_n}$ being also Λ-cones, for the same reason we have

$$\Theta_d(C_x^\Lambda(X),0) = \lim_{n\to+\infty} \Theta_d(\overline{C_n},x).$$

For all n we have $\Theta_d(\overline{C_n},x) = \Theta_d(C_n,x)$, so

$$\Theta_d(W,x) = \Theta_d(C_x^\Lambda(X),0).$$

Because Θ_d is increasing, for all $n \geq 0$, we have

$$\Theta_d(X,x) = \Theta_d(X \cap B(x,n)) \leq \Theta_d(C_n,x),$$

hence $\Theta_d(X,x) \leq \Theta_d(C_x^\Lambda(X),0)$.

For the other inequality, for all $n \geq 0$, set W_n defined as the $w \in W$ such that $\alpha(\pi_x^X(w)) \leq n$. By definition of α, $\bigcup_{n\geq 0} W_n = W \setminus \{0\}$, and $W_n \cap B(x,n)$ is a local

Λ-cone. As before, by the monotone convergence lemma 1.5.1

$$\Theta_d(W,x) = \lim_{n\to+\infty} \Theta_d(W_n,x).$$

Moreover, for all $w \in W_n \cap B(x,n)$, there is a $y \in X \cap B(x,n)$ and a $\lambda \in \Lambda$ such that $w = \lambda(y-x)+x$ and $\alpha(\pi_x^X(w)) \leq n$, so $w \in X$. We showed that $W_n \cap B(x,n) \subseteq X \cap B(x,n)$, hence

$$\Theta_d(W_n,x) \leq \Theta_d(X,x),$$

and by taking the limit,

$$\Theta_d(C_x^\Lambda(X),0) = \Theta_d(W,x) \leq \Theta_d(X,x).$$

□

1.5.2 Deformation to the tangent cone.

The above proof does not extend to the case $X \subseteq K^n$ of dimension d, for $d < n$. The reason is that the Λ-cone generated by a definable of dimension d is in general of dimension n. However, using Proposition 1.2.14, we will restrict ourself to the case of a graph of a 1-Lipschitz function of domain U included in K^d. By the case we just studied, it suffices to show that for some Λ,

$$\Theta_d(C_u^\Lambda(U),0) = \Theta_d(C_x^\Lambda(X),0).$$

To do so, using deformation to the tangent cone we will build a 1-Lipschitz function ψ of domain (included in) $C_u^\Lambda(U)$ such that the graph of ψ is a dense subset of $C_x^\Lambda(X)$. The displayed equality follows from the existence of such a ψ. We need first some preparatory lemmas.

Remark 1.5.3. In the following lemmas, at several occasions we will have to use a **k**-partition of the studied definable. By applying Corollary 1.3.14 with parameters, we can assume that the same Λ is adapted to all the **k**-parts.

Let U an open of K^d. By Corollary 1.3.14, there is a definable function

$$\alpha : \mathbb{P}^{d-1}(K) \to \mathbb{Z} \cup \{\infty\}$$

such that for all $\ell \in \mathbb{P}^{d-1}(K)$, $(\pi_u^U)^{-1}(\ell) \cap B(u,\alpha(\ell))$ is a local Λ-cone with origin u, with the convention that $\alpha(\ell) = \infty$ if and only if $(\pi_u^U)^{-1}(\ell) = \emptyset$. Because α is definable, it is continuous on a definable dense open subset Ω_0 of $\mathbb{P}^{d-1}(K)$. By dense, we mean that the dimension of its complement into $\mathbb{P}^{d-1}(K)$ is strictly less than $d-1$. Let fix an $u \in K^d$ and let Ω_1 the definable constituted by the $\ell \in \Omega_0$ such that for all neighborhood V of u

in K^d, $(\pi_u^U)^{-1}(\ell) \cap V$ is nonempty.

Lemma 1.5.4. *Suppose $C_u^\Lambda(U)$ is of maximal dimension,* i.e. *d. Then Ω_1 contains a nonempty open definable of $\mathbb{P}^{d-1}(K)$.*

Proof. Let Ω_1^c the complement of Ω_1 in $\mathbb{P}^{d-1}(K)$. We need to show that Ω_1^c is not dense. Assume it is, for the sake of contradiction. By definability and density, $\Omega_1^c \cap \Omega_0$ is also dense in $\mathbb{P}^{d-1}(K)$. For every $\ell \in \Omega_1^c \cap \Omega_0$, by definition of the tangent cone, $(\pi_u^U)^{-1}(\ell) \cap (u + C_u^\Lambda(U)) = \emptyset$. As $(\pi_u^U)^{-1}(\Omega_1^c \cap \Omega_0)$ is dense and definable in K^d, this implies that $C_u^\Lambda(U)$ is contained into a definable of dimension less than d, contradiction. □

For all definable O of $\mathbb{P}^{d-1}(K)$, we consider the definable

$$C_u^{\Lambda,O}(U) := (\pi_u)^{-1}(O) \cap C_u^\Lambda(U).$$

Lemma 1.5.5. *We still suppose* $\dim(C_u^\Lambda(U)) = d$. *Let O a dense open definable of Ω_1. Then $C_u^{\Lambda,O}(U)$ is dense open in $C_u^\Lambda(U)$, hence of dimension d.*

Proof. As π_u is continuous, $(\pi_u)^{-1}(O)$ is open, so $C_u^{\Lambda,O}(U)$ is open in $C_u^\Lambda(U)$. Let show that it is dense. Its complement is the disjoint union

$$(C_u^\Lambda(U) \cap (\pi_u)^{-1}(\Omega_1 \setminus O)) \cup (C_u^\Lambda(U) \cap (\pi_u)^{-1}(\Omega_1^c)).$$

By density of O in Ω_1, $C_u^\Lambda(U) \cap (\pi_u)^{-1}(\Omega_1 \setminus O)$ is of dimension less than d. For all $z \in C_u^\Lambda(U)$, $\pi_u(z) \in \overline{\Omega_1}$, hence

$$C_u^\Lambda(U) \cap (\pi_u)^{-1}(\Omega_1^c) \subseteq (\pi_u)^{-1}(\overline{\Omega_1} \setminus \Omega_1),$$

this shows that it is of dimension less that d. □

Lemma 1.5.6. *For almost all $\ell \in \Omega_1$ (*i.e. *in a dense subset of Ω_1),*

$$C_u^\Lambda(U) \cap \ell \cap B(u, \alpha(\ell)) = \overline{(\pi_u^U)^{-1}(\ell)} \cap B(u, \alpha(\ell)).$$

Proof. Suppose $u = 0$ to simplify the notations. For any $\ell \in \Omega_1$, we have

$$\begin{aligned} \overline{(\pi_u^U)^{-1}(\ell)} \cap B(u, \alpha(\ell)) &= C_u^\Lambda((\pi_u^U)^{-1}(\ell)) \cap B(0, \alpha(\ell)) \\ &\subseteq C_u^\Lambda(U) \cap \ell \cap B(u, \alpha(\ell)), \end{aligned}$$

hence one inclusion is proven. For the other one, let X be the definable set

$$X := \left\{ (x,t) \in K^d \setminus \{0\} \times K \mid u + tx \in U \right\},$$

parametrizing lines $\ell \cap U$. Let us apply a cell decomposition of X relatively to the t variable. We recall the notation X_x for the fiber of X above $x \in K^d \setminus \{0\}$. There are three cases. Either 0 is in the interior of X_x, or is in its (definable) boundary ∂X_x, or it is outside $\overline{X_x}$. We do not need to consider the last case, because $\ell \in \Omega_1$. For the first one, $(\pi_u^U)^{-1}(\ell) \cap B(u, \alpha(\ell)) = B(u, \alpha(\ell))$, hence the inclusion is satisfied. For the case $0 \in \partial X_x$, up to a set of ℓ of dimension less than $d-1$, we also have the inclusion, because the function (of the variable x) describing the centers of the cell is almost everywhere continuous. □

Lemma 1.5.7. *We suppose* $\dim(C_u^\Lambda(U)) = d$. *Then there is a dense open definable subset of* Ω_1 *such that for all* $z \in C_u^{\Lambda,\Omega}(U)$, *with* $\ell := \pi_u(z)$, *and all* $\lambda \in \Lambda$ *small enough (depending on* z*),*

$$u + \lambda z \in (\pi_u^U)^{-1}(\ell).$$

Proof. For all $\ell \in \Omega_1$, $z \in \ell \cap C_u^\Lambda(U)$, and $\lambda \in \Lambda$, we have $u + \lambda z \in C_u^\Lambda(U)$. From the inclusion $\subseteq$ of Lemma 1.5.6, we have that for all $\lambda \in \Lambda$ with $\mathrm{ord}(\lambda z) \geq \alpha(\ell)$,

$$u + \lambda z \in \overline{(\pi_u^U)^{-1}(\ell)}.$$

As $\overline{(\pi_u^U)^{-1}(\ell)} \setminus (\pi_u^U)^{-1}(\ell)$ is finite, we have for every $\lambda \in \Lambda$ small enough $u + \lambda z \in (\pi_u^U)^{-1}(\ell)$. □

Corollary 1.5.8. *Let* $d > 0$, *and* U *a definable nonempty open of* K^d, *and* $\Lambda \in \mathcal{D}$ *as in Corollary 1.3.14. Then for all* $u \in \overline{U}$ *such that* $C_u^\Lambda(U)$ *is of maximal dimension* d, *for every* $\Lambda' \in \mathcal{D}$ *with* $\Lambda' \subseteq \Lambda$, *we have*

$$C_u^{\Lambda'}(U) = C_u^\Lambda(U).$$

Proof. Let us show the inclusion $C_u^{\Lambda,\Omega}(U) \subseteq C_u^{\Lambda',\Omega}(U)$, with Ω given by Lemma 1.5.7. Let $z \in C_u^{\Lambda,\Omega}(U)$, and $\ell = \pi_u(z)$. We have $z \in C_u^\Lambda(\overline{U \cap \ell}) = C_u^\Lambda(U \cap \ell)$, hence by Lemma 1.5.7, $U \cap \ell \cap B(u, \alpha(\ell))$ is a local Λ-cone with origin u, hence by Remark 1.3.16,

$$z \in C_u^\Lambda(U \cap \ell) = C_u^{\Lambda'}(U \cap \ell) \subseteq C_u^{\Lambda'}(U).$$

The other inclusion being clear, we showed that $C_u^{\Lambda,\Omega}(U) = C_u^{\Lambda',\Omega}(U)$, hence that $\overline{C_u^{\Lambda,\Omega}(U)} = \overline{C_u^{\Lambda',\Omega}(U)}$. By Lemma 1.5.5, $\overline{C_u^{\Lambda,\Omega}(U)} = C_u^\Lambda(U)$. But Lemmas 1.5.4 and 1.5.5 are also true for Λ', provided that $\dim(C_u^{\Lambda'}(U)) = \dim(C_u^\Lambda(U)) = d$ by Corollary 1.3.19. Hence

$$C_u^{\Lambda'}(U) = \overline{C_u^{\Lambda',\Omega}(U)} = \overline{C_u^{\Lambda,\Omega}(U)} = C_u^\Lambda(U).$$

□

Proposition 1.5.9. *Let* U *an open definable of* K^d, *and* $\varphi : U \to K^{m-d}$ *definable, differentiable and locally 1-Lipschitz. Let* $u \in \overline{U}$ *be fixed,* $\Lambda \in \mathcal{D}$ *adapted to* (U, u). *We suppose*

$\lim_{x\to u}\varphi(x) = v$, *and set* $w = (u, v)$. *We suppose* $\dim(C_u^{\Lambda}(U)) = d$, *and consider* Ω *and* $C_u^{\Lambda,\Omega}(U)$ *as given by Lemma 1.5.7.*

Then there exists a $\mathbf{k}$*-partition of* U *and* $\Lambda' \in \mathcal{D}$, *such that for each* $\mathbf{k}$*-part* U_ξ *with* $\dim(U_\xi) = d$, φ *is globally* C*-Lipschitz on* U_ξ, Λ' *is adapted to* (U_ξ, u), *there is a* Ω_ξ *given by Lemma 1.5.7. For all* $z \in C_u^{\Lambda',\Omega_\xi}(U_\xi)$, *the limit*

$$\lim_{\substack{\lambda\to 0\\ \lambda\in\Lambda'}} \lambda^{-1}(\varphi(u+\lambda z) - v)$$

exists, yielding a definable function $\psi : C_u^{\Lambda',\Omega_\xi}(U_\xi) \to K^{m-d}$.

Proof. By Lemma 1.5.7, for all $z \in C_u^{\Lambda,\Omega}(U)$, for all $\lambda \in \Lambda$ small enough, $u + \lambda z \in U$. Define $U' \subseteq U$ as the set of all such $u + \lambda z$. By definition, $C_u^{\Lambda,\Omega}(U) \subseteq C_u^{\Lambda}(U')$, hence $\dim(C_u^{\Lambda}(U \setminus U')) < d$, we can neglect it and suppose $U' = U$. By Theorem 1.2.15, assume that φ is globally C-lispchitz on $\mathbf{k}$-parts of a $\mathbf{k}$-partition of U. The function

$$f_z : \lambda \mapsto \frac{\varphi(u+\lambda z) - v}{\lambda}$$

is bounded, indeed otherwise there would exist a $y \in K^{m-d}\setminus\{0\}$ such that $(0, y) \in C_u^{\Lambda}(\Gamma(\varphi))$, in contradiction with Proposition 1.2.27 and the fact that φ is C-Lipschitz. By Lemma 1.2.20, f_z has at least one limit value when $\lambda \to 0$. As $\dim(\overline{\Gamma(f_z)} \setminus \Gamma(f_z)) = 0$, this set is finite, so the z-definable $A_z \subseteq K^{m-d}$ of adherence values at 0 of f_z is nonempty and finite. By Lemma 1.2.13, there is a z-definable function $g_z : K^{m-d} \to R_{s,K}^s$ sending bijectively A_z to some B_z and we can assume g_z is constant on a neighborhood of each point of A_z. We can moreover assume that if $z' = \lambda' z$ for some $\lambda' \in \Lambda$, then $g_{z'}(a) = g_z(a/\lambda')$. The definable function

$$u + \lambda z \in U \mapsto g_z\left(\frac{\varphi(u+\lambda z) - v}{\lambda}\right)$$

is then well defined, and yields a $\mathbf{k}$-partition of U. We can then apply Corollary 1.3.14, Remark 1.5.3, and Lemmas 1.5.4 to 1.5.7 to get $\Lambda' \subseteq \Lambda$ and Ω_ξ as required. For any $z \in C_u^{\Lambda',\Omega_\xi}(U_\xi)$ and $\lambda \in \Lambda'$ small enough such that $u + \lambda z \in U_\xi$, we have $g_z(\frac{\varphi(u+\lambda z)-v}{\lambda})$ constant, so $\frac{\varphi(u+\lambda z)-v}{\lambda}$ has a limit when $\lambda \in \Lambda'$ goes to 0. □

Proposition 1.5.10. *Let* U *be an open definable of* K^d, *and* $\varphi : U \to K^{m-d}$ *definable, differentiable and locally 1-Lipschitz and globally C-Lipschitz. Let* $u \in \overline{U}$ *be fixed,* $\Lambda \in \mathcal{D}$ *adapted to* (U, u). *We suppose* $\lim_{x\to u}\varphi(x) = v$, *and note* $w := (u, v)$. *We suppose* $\dim(C_u^{\Lambda}(U)) = d$, *and consider* Ω *and* $C_u^{\Lambda,\Omega}(U)$ *as given by Lemma 1.5.7. We suppose that the conclusion of Proposition 1.5.9 holds,* i.e.

$$\lim_{\substack{\lambda\to 0\\ \lambda\in\Lambda}} \lambda^{-1}(\varphi(u+\lambda z) - v)$$

is a well defined definable function from $C_u^{\Lambda,\Omega}(U)$ *to* K^{m-d}.

Then ψ *is a locally* 1*-Lipschitz function on a dense subset of* $C_u^{\Lambda,\Omega}(U)$*, and* $\Gamma(\psi)$ *the graph of* ψ *is dense in* $C_w^{\Lambda}(\Gamma(\varphi))$*, where* $\Gamma(\varphi)$ *is the graph of* φ.

Proof. By the Jacobian property, ψ is differentiable on a dense subset of $C_u^{\Lambda,\Omega}(U)$. To show that ψ is locally 1-Lipschitz, by Proposition 1.2.28, it suffices to show that

$$T_{(x_0,\psi(x_0))}(\Gamma(\psi)) \subseteq \left\{(x,y) \in K^d \times K^{n-d} \mid |y| \leq |x|\right\} =: A$$

for x_0 in a dense subset of $C_u^{\Lambda,\Omega}(U)$. By definition of ψ, $\Gamma(\psi) \subseteq C_w^{\Lambda}(\Gamma(\varphi))$. As $\dim(\Gamma(\psi)) = \dim(C_w^{\Lambda}(\Gamma(\varphi)))$, it suffices to show that $T_y C_w^{\Lambda}(\Gamma(\varphi)) \subseteq A$ for every y in a dense subset of $C_w^{\Lambda}(\Gamma(\varphi))$. To do so, consider the deformation $h : \overline{\mathcal{D}(\Gamma(\varphi), w, \Lambda)} \to K$ introduced in definition 1.3.20. We identify the fiber $h^{-1}(0)$ with $C_w^{\Lambda}(\Gamma(\varphi))$ and for $\lambda \in \Lambda$,

$$h^{-1}(\lambda) = \{(z,\lambda) \mid w + \lambda z \in \Gamma(\varphi)\}$$

with the set

$$\{z \in K^m \mid w + \lambda z \in \Gamma(\varphi)\}.$$

As φ is locally 1-Lipschitz, for all $\lambda \in \Lambda$ and $y \in h^{-1}(\lambda)$, $T_y h^{-1}(\lambda) \subseteq A$ by Proposition 1.2.27. By Theorem 1.4.2, there is a (w_h)-stratification, hence for all y in a dense subset of $C_w^{\Lambda}(\Gamma(\varphi))$, there is a sequence $(y_n, \lambda_n) \to (y, 0)$ such that $T_y h^{-1}(0)$ is the limit of the $T_{y_n} h^{-1}(\lambda)$, hence

$$T_y C_w^{\Lambda}(\Gamma(\varphi)) = T_y h^{-1}(0) \subseteq A.$$

We prove now the second point. Let $z \in C_w^{\Lambda}(\Gamma(\varphi))$, and $(w_n)_n \in \Gamma(\varphi)$, $(\lambda_n)_n \in \Lambda$ be two sequences such that $w_n \to w$ and $\lambda_n(w_n - w) \to z$. Denote by π the projection of $\Gamma(\varphi)$ to U, and $u_n = \pi(w_n)$. Then (u_n) converges to u and $(\lambda_n(u_n - u))$ to $a = \pi(z) \in C_u^{\Lambda}(U)$. We fix an $\varepsilon > 0$. By Lemma 1.5.7, there is an $a' \in C_u^{\Lambda,\Omega}(U)$ with $|a' - a| \leq \varepsilon$ and such that $u + \lambda a' \in U$ for every $\lambda \in \Lambda$ small enough. Because φ is globally C-Lipschitz, we have

$$\left|\lambda_n[(\varphi(\lambda_n^{-1}a' + u) - v) - (\varphi(u_n) - v)]\right| \leq C\left|a' - \lambda_n(u_n - u)\right|.$$

By taking the limit, we have

$$\lim_{n\to+\infty} \left|\lambda_n(\lambda_n^{-1}a', \varphi(\lambda_n^{-1}a' + u) - v) - \lambda_n(w_n - w)\right| \leq C\left|a' - a\right|,$$

$$\left|(a', \psi(a')) - z\right| \leq C\varepsilon,$$

which shows that $\Gamma(\psi)$ is dense in $C_w^{\Lambda}(\Gamma(\varphi))$. □

We can now finish the proof of Theorem 1.3.25.

Proof of Theorem 1.3.25. We have $X \subseteq K^m$, definable of dimension less or equal to d.

By Proposition 1.2.14, there is a definable Y of dimension less than d and a $\mathbf{k}$-definable partition of $X \setminus Y$ such that for each part X_ξ, there is a ξ-definable open $U_\xi \subseteq K^d$ and a ξ-definable differentiable locally 1-Lipschitz function $\phi_\xi : U_\xi \to K^{m-d}$ of graph Γ_ξ such that for some permutation of coordinates γ_ξ, $X_\xi = \gamma_\xi(\Gamma_\xi)$. As the γ_ξ are isometries, we have the equalities

$$\Theta_d(\gamma_\xi(\Gamma_\xi), x) = \Theta_d(\Gamma_\xi, \gamma_x i^{-1}(x)) \text{ and } \Theta_d(C_x^\Lambda(\gamma_\xi(\Gamma_\xi)), 0) = \Theta_d(C^\Lambda_{\gamma_\xi^{-1}(x)}(\Gamma_\xi), 0)$$

By Lemma 1.3.9, it suffices to find a $\Lambda \in \mathcal{D}$ such that for all ξ,

$$\iota(\Theta_d(\Gamma_\xi, \gamma_\xi^{-1}(x))) = \iota(\Theta_d(C^\Lambda_{\gamma_\xi^{-1}(x)}(\Gamma_\xi), 0)).$$

Suppose $w := \gamma_\xi^{-1}(x) = (u, v)$. By Corollary 1.2.22, up to refining the partition we can assume $\lim_{y\to u} \phi_\xi(y)$ exists. If this limit is not v, both sides of the equality we need to show are 0, so we can assume the limit is v. Using Remark 1.5.3, Lemmas 1.5.4 to 1.5.7, and Proposition 1.5.9, we refine the partition and assume the hypotheses of Proposition 1.5.10 hold. To simplify, we keep the notations of Proposition 1.5.10, we have a $\Lambda \in \mathcal{D}$, U an open definable of K^d, $\varphi : U \to K^{m-d}$, $\lim_{y\to u} \varphi(y) = v$, $w = (u, v)$, $\dim(C_u^\Lambda(U)) = d$, Ω such that $C_u^{\Lambda,\Omega}(U)$ is dense in $C_u^\Lambda(U)$, and $\psi : C_u^{\Lambda,\Omega}(U) \to K^{m-d}$ the definable function

$$\psi(z) = \lim_{\substack{\lambda\to 0\\ \lambda\in\Lambda}} \lambda^{-1}(\varphi(u + \lambda z) - v).$$

As φ is locally 1-Lipschitz, $z \in U \mapsto (z, \varphi(z))$ preserves the motivic measure. By Proposition 1.2.27, for z small enough, $|\varphi(z)| \leq |z|$, hence

$$\Theta_d(U, u) = \Theta_d(\Gamma(\varphi), w).$$

By Proposition 1.5.10, $\psi : C_u^{\Lambda,\Omega}(U) \to K^{m-d}$ a definable locally 1-Lipschitz function on a dense subset of $C_u^{\Lambda,\Omega}$, and $\Gamma(\psi)$ is dense in $C_w^\Lambda(\Gamma(\varphi))$. By the Jacobian property, $z \mapsto (z, \psi(z))$ preserve the motivic d-dimensional measure and as before by Proposition 1.2.27, for z small enough $|\psi(z)| \leq |z|$, hence

$$\Theta_d(C_u^\Lambda(U), 0) = \Theta_d(C_u^{\Lambda,\Omega}(U), 0) = \Theta_d(\Gamma(\psi), 0) = \Theta_d(C_w^\Lambda(\Gamma(\varphi), 0)),$$

the first and last equality begin obtained by density. Finally, by Proposition 1.5.2, we have

$$\iota(\Theta_d(U, u)) = \iota(\Theta_d(C_u^\Lambda(U), 0)).$$

Putting the three equalities together, we get

$$\iota(\Theta_d(\Gamma(\varphi)), w) = \iota(\Theta_d(C_w^\Lambda(\Gamma(\varphi)), 0)).$$

□

Remark 1.5.11. In the above proof, we implicitly used that if $\dim(C_u^\Lambda(U)) < d$, then $\dim(C_w^\Lambda(\Gamma(\varphi)) < d$. If not, then there would be some coordinate projection $\pi : K^m \to K^d$ such that for $U' = \pi(\Gamma(\varphi))$, $\dim(C_{\pi(w)}^\Lambda(U')) = d$, but this would imply that $\Theta_d(\Gamma(\varphi), w) \neq 0$.

1.5.3 Stabilisation of the tangent cone

We prove in this section that the tangent cone stabilizes, that is, the existence of a distinguished tangent cone.

Theorem 1.5.12. *Let X a definable of K^m of dimension d. Then there is a $\Lambda \in \mathcal{D}$ such that for all $\Lambda' \in \mathcal{D}$, $\Lambda' \subseteq \Lambda$ and $x \in K^n$, we have*

$$C_x^\Lambda(X) = C_x^{\Lambda'}(X).$$

Proof. The proof goes by induction on the dimension d of X. If $d = 0$, then $C_x^\Lambda(X) = \{0\}$ for any Λ. We suppose then that $\dim(X) = d \geq 1$ and that the proposition is true for lower dimensions.

By Proposition 1.2.14, up to taking a **k**-partition and working on one of the **k**-parts, we can suppose X is a graph of a locally 1-Lipschitz function φ defined on an open $U \subseteq K^d$. Note also that only finitely many $\Lambda \in \mathcal{D}$ will occur, hence at the end we can pick a $\Lambda \in \mathcal{D}$ adapted to all the parts. In what follows, we fix an $x \in K^n$ to simplify the notations, but we are in fact working uniformly in x. Let u be the projection of x on K^d. Let $\Lambda \in \mathcal{D}$ be as given by Corollary 1.3.14. Suppose first that $C_u^\Lambda(U)$ is of dimension d and fix a $\Lambda' \in \mathcal{D}$, $\Lambda' \subseteq \Lambda$. Then by Corollary 1.5.8,

$$C_u^{\Lambda'}(U) = C_u^\Lambda(U).$$

By Propositions 1.5.9 and 1.5.10 applied to φ and Λ, there is an Ω and a function

$$\psi : C_u^{\Lambda,\Omega}(U) \to K^{m-d}$$

with graph dense in $C_x^\Lambda(X)$. These propositions also apply for Λ', yielding an Ω' and a function

$$\psi' : C_u^{\Lambda',\Omega'}(U) \to K^{m-d}$$

with graph dense in $C_x^{\Lambda'}(X)$. By taking their intersections, we can suppose $\Omega = \Omega'$. Then $C_u^{\Lambda,\Omega}(U) = C_u^{\Lambda',\Omega'}(U)$ and ψ and ψ' agree. Hence

$$C_x^\Lambda(X) = \overline{\Gamma(\psi)} = \overline{\Gamma(\psi')} = C_x^{\Lambda'}(X).$$

Suppose now that $C_x^\Lambda(X)$ is of dimension $< d$ (or equivalently, $C_u^\Lambda(U)$ of dimension $< d$ by Remark 1.5.11). We will show the existence of a definable (with additional parameters) $Y \subseteq X$ such that $\dim(Y) = \dim(C_x^\Lambda(X))$ and $C_x^\Lambda(X) = C_x^\Lambda(Y)$. If we do so, we are done by induction.

Apply the parametrized curve selection 1.2.25 to $\mathcal{D}(X, x, \Lambda) \times C_x^\Lambda(X)$, with for every $y \in C_x^\Lambda(X)$,

$$a_y = y \in \overline{\mathcal{D}(X, x, \Lambda)} \backslash \mathcal{D}(X, x, \Lambda).$$

Hence there is a nonempty definable D, and a definable function

$$\sigma : B_e \times D \times C_x^\Lambda(X)$$

such that for any $(d_0, y) \in D \times C_x^\Lambda(X)$, $\sigma_{d_0,y}(B_e^*) \subseteq \mathcal{D}(X, x, \Lambda)$ and $\lim_{t\to 0} \sigma_{d_0,y}(t) = \sigma_{d_0,y}(0) = y$.

Then if we fix a $d \in D$, the d-definable set $\sigma(B_e^*, d, C_x^\Lambda(X)) \subseteq \mathcal{D}(X, x, \Lambda)$ is of dimension $d + 1$, and we can map it to X using

$$(z, \lambda) \in \mathcal{D}(X, x, \Lambda) \mapsto x + \lambda \cdot z \in X.$$

The image $Y(d_0)$ is then a d_0-definable of dimension d, such that $C_x^\Lambda(Y(d_0)) = C_x^\Lambda(X)$. $\square$

1.6 Application to the p-adic case

1.6.1 p-adic density

In this section we show how our results extend the work of Cluckers, Comte and Loeser [15] on p-adic local density.

For the remainder of the section, we fix K a finite extension of $\mathbb{Q}_p$. Let $\mathcal{L}'_{\text{Mac}} = \{0, 1, +, -, \cdot, (P_n)_n > 0\}$ the language of Macintyre, where P_n is a predicate for n-th powers in $K^\times$. Let $\mathcal{L}'_{\text{an}} = \mathcal{L}'_{\text{Mac}} \cup \left\{^{-1}, (K\{x_1, ..., x_n\})n \geq 1\right\}$ the analytic language, where $^{-1}$ is the field inverse and $K\{x_1, ..., x_n\}$ the ring of restricted power series (*i.e.*power series converging on $\mathcal{O}_K^n$). The definable of K in the language $\mathcal{L}'_{\text{Mac}}$ (resp. $\mathcal{L}'_{\text{an}}$) are exactly the definable of K in the three sorted language $\mathcal{L}_{\text{high}}$ (resp. in the language $\mathcal{L}_{K,\text{an}}$ of Example 1.2.9.2). For the remainder of this section, let $\mathcal{L}'$ be either $\mathcal{L}'_{\text{Mac}}$ or $\mathcal{L}'_{\text{an}}$ and (accordingly) $\mathcal{L}$ be either $\mathcal{L}_{\text{high}}$ or $\mathcal{L}_{K,\text{an}}$. We work in the mixed tame theory $\mathcal{T} = \text{Th}(K)$ in the language $\mathcal{L}$.

Note that the residue rings $R_n = \mathcal{O}_K/\mathcal{M}_K^n$ are finite, because the residue field k_K of K is finite, hence a $\mathbf{k}$-partition of some definable of $X \subseteq K^n$ is in fact a finite partition of X. Counting points induces an isomorphism $\mathcal{Q}(h[0, 0, 0]) \simeq \mathbb{Z}$. More generally, let $\mu_{n,K}$ be the Haar measure on K^n, normalized by $\mu_{n,K}(\mathcal{O}_K^n) = 1$. This measure extends to a measure on definable sets of K^m of dimension at most n, still denoted $\mu_{n,K}$. By [25, Proposition

9.2], if $\varphi \in \mathcal{C}_+(h[m,0,0])$ is of dimension at most n and integrable, then

$$\mu_n(\varphi) = \mu_{n,K}(\varphi_K).$$

In particular, the definition of motivic local density specializes to the p-adic case. That is, if $X \subseteq K^n$ of dimension d and $x \in K^n$, the motivic local density $\Theta_d(X,x) \in \widetilde{\mathcal{C}_+}(\{x\}) \simeq \mathbb{Q}_+$ equals the local density of X at x of [15, Definition 2.3.2].

Our definition of tangent cone also coincides with the definitions of [15, Section 3.3]. Note also that the definition of cone with multiplicities $CM_x^\Lambda(X)$ given in [15, Section 5.4] coincides with our Definiton 1.3.23, because a **k**-partition in this case is a finite partition and of the isomorphism $\mathcal{Q}(h[0,0,0]) \simeq \mathbb{Z}$.

However, unlike what is claimed in [15], this does not coincide with the other definition of multiplicities on the tangent cone, given by [15, Section 3.6] and denoted $SC_x^\Lambda(X)$, as shown by the following example.

Example 1.6.1. Fix $K = \mathbb{Q}_p$ for $p \neq 2$, consider the cusp $X = \left\{x^2 = y^3\right\}$ and $\Lambda = \{\lambda \in K^\times \mid P_2(\lambda)\}$ the subgroup of squares in $K^\times$. The group Λ is of finite index in $K^\times$ and $[K^\times/\Lambda] = 4$.

One has $C_0^\Lambda(X) = \{(0,y) \mid P_2(y)\}$. Fix $(0,y_0) \in C_0^\Lambda(X)$, $y_0 \neq 0$. We need to compute $\Theta_2(\mathcal{D}(X,0,\Lambda),(0,y_0,0))$. For $n \geq 3\mathrm{ord}(y_0)$, we have

$$\begin{aligned}\mathcal{D}(X,0,\Lambda) \cap B((0,y_0,0),n) \quad &= \\ &\left\{(x,y,\lambda) \in K^2 \times \Lambda \mid \min\{\mathrm{ord}(x), \mathrm{ord}(y-y_0)\} \geq n, x^2 = \lambda y^3\right\}.\end{aligned}$$

We have $\mathrm{ord}(\lambda) = \mathrm{ord}(x^2/y^3) = 2\mathrm{ord}(x) - 3\mathrm{ord}(y_0)$ and $\overline{\mathrm{ac}}(\lambda) = \overline{\mathrm{ac}}(x^2)/\overline{\mathrm{ac}}(y_0)^3$. As y_0 is a square, $\overline{\mathrm{ac}}(y_0)$ is a square and $\mathrm{ord}(y_0)$ is even, hence for any $(x,y) \in (K^\times)2$ with $\mathrm{ord}(x) \geq n$, $\mathrm{ord}(y-y_0) \geq n$, x^2/y^3 is a square, *i.e.*in Λ. Hence $\mathcal{D}(X,0,\Lambda) \cap B((0,y_0,0),n)$ is a graph of a 1-Lipschitz function. Hence

$$\begin{aligned}\mu_{2,K}(\mathcal{D}(X,0,\Lambda) \cap B((0,y_0,0),n)) \quad &= \\ &\mu_{2,K}(\left\{(x,y) \in (K^\times)^2 \mid \mathrm{ord}(x) \geq n, \mathrm{ord}(y-y_0) \geq n\right\}) = p^{-2n},\end{aligned}$$

then $\Theta_2(\mathcal{D}(X,0,\Lambda),(0,y_0,0)) = 1$ when y_0 is a square and

$$SC_0^\Lambda(X)(0,y_0) = \begin{cases} 4 & \text{if } P_2(y_0) \\ 0 & \text{otherwise} \end{cases}.$$

This differs from the correct value of the multiplicity

$$CM_0^\Lambda(X)(0, y_0) = \begin{cases} 2 & \text{if } P_2(y_0) \\ 0 & \text{otherwise} \end{cases}$$

that one computes by partitioning X into graphs of 1-Lipschitz functions using

$$g : (x, y) \in X \mapsto (\overline{\mathrm{ac}}(x), \overline{\mathrm{ac}}(y)).$$

Hence in the paper [15], each occurrence of $SC_x^\Lambda(X)$ must be replaced by $CM_x^\Lambda(X)$. With this modification, the first part of Theorem 3.6.1 of [15] remains true, as it is exactly our Theorem 1.3.25 applied to the theory of K.

Note that it is unclear how to attach multiplicities in the case of non locally constant constructible functions. We do not know how to suitably modify the definition of ν_x^Λ in [15, Section 3.6] in order to make true the second part of Theorem 3.6.1 of [15].

Note also that one can easily correct the proof of this theorem. Instead of [15, Corollary 5.3.8] and [15, Section 5.6], one can copy our proof of Theorem 1.3.25, but quoting [15, Proposition 1.5.3] instead of Proposition 1.2.14, [15, Lemma 5.3.3] instead of Lemma 1.5.7, [15, Proposition 5.3.7] instead of 1.5.10 and [15, Proposition 3.4.1] instead of Proposition 1.2.27.

We mention also that Proposition 2.4.2 of [15] is false in general. For example, take $\varphi_n = \mathbf{1}_{A_n}$, where $A_n = \{x \in K \mid 0 \leq \mathrm{ord}(x) \leq n\}$. We have $\Theta_1(\varphi_n)(0) = 0$ but $\sup(\varphi_n) = \mathbf{1}_{B^*}$, with B^* the unit ball without 0 and $\Theta_1(\mathbf{1}_{B^*})(0) = 1$.

It is used in [15, Section 5.1] to reduce the second part of [15, Theorem 3.6.1] to the first part. As we have seen that the second part is meaningless without a way to attach multiplicities to constructible functions, this is not a problem. It is also used in the proof of [15, Section 5.2]. Instead of this Proposition 2.4.2, one can use in this proof the monotone convergence theorem from classical measure theory, similarly to what we have done in the proof of Proposition 1.5.2.

1.6.2 Uniformity in p.

Since the pioneering work of Denef on the degree of Igusa's local zeta functions [33], many results on model theory of the p-adics fields and p-adic integration have been improved into some versions uniform in p. This started with the uniform p-adic cell decompositions theorems of (Denef-)Pas [84], [85] and was one of the motivation for the development of motivic integration. Denef and Loeser [35] show existence of motivic zeta functions specializing for almost every p to Igusa's zeta functions and later Cluckers and Loeser [24] prove a general specialization theorem for integrals of motivic constructible functions with parameters.

In this line of ideas, we prove in this section a uniform version of Theorem 3.6.1 of [15]. Let us recall the setting of [24, Section 9.1], in order to use the specialization principle. Fix a number field k with ring of integers $\mathcal{O}$. Let $\mathcal{A}_{\mathcal{O}}$ be the collection of all p-adic completions of finite field extensions of k and for $N > 0$, let $\mathcal{C}_{\mathcal{O},N}$ be the collection of all K in $\mathcal{A}_{\mathcal{O}}$ such that the caracteristic of the residue field of K is greater than N and set $\mathcal{C}_{\mathcal{O}} = \mathcal{C}_{\mathcal{O},1}$. For K in $\mathcal{C}_{\mathcal{O}}$, we set $\mathcal{O}_K$ its valuation ring, $\mathcal{M}_K$ its maximal ideal, k_K its residue field and $q_K = \operatorname{card}(k_K)$. If we fix a uniformizing parameter ϖ_K of $\mathcal{O}_K$, there is a unique multiplicative map $\overline{\mathrm{ac}} : K^\times \to k_K^\times$ extending the projection $\mathcal{O}_K^\times \to k_K^\times$ and sending φ_K to 1. We use the three sorted language $\mathcal{L}_{\mathcal{O}} := \mathcal{L}_{DP}(R)$, where R is any subring of $\mathcal{O}[[t]]$ containing $\mathcal{O}[t]$. Any $K \in \mathcal{C}_{\mathcal{O}}$ can be seen as a $\mathcal{L}_{\mathcal{O}}$-structure by interpreting t by ϖ_K. We work in the tame theory $\mathcal{T}$ of all $(0,0,0)$-fields in the language $\mathcal{L}_{\mathcal{O}}$ and definable subassignments are considered up to equivalence in this theory. If $X \subseteq h[n,r,s]$ is a definable subassignment, there is a $N > 0$ such that for any $K \in \mathcal{C}_{\mathcal{O},N}$, any $\mathcal{L}_{\mathcal{O}}$-formula defining X defines the same set $X_K \subseteq K^n \times {k_K}^r \times \mathbb{Z}^s$ which does not depends on the choice of φ_K. This follows from logical compatness and Ax-Kochen-Eršov principle, see for example [37].

A motivic constructible function $\varphi \in \mathcal{C}(Z)$ gives rise for some $N > 0$ and any $K \in \mathcal{C}_{\mathcal{O},N}$ to some constructible function $\varphi_K : Z_K \to \mathbb{Q}$ defined in the following way. Suppose first that $\varphi = [Y_{/Z}] \in \mathcal{Q}(Z)$. Then Y_K is a subset of $Z_k \times k_K^\ell$ for a $\ell \geq 0$ and one set

$$\varphi_K : \begin{cases} Z_K \to \mathbb{Q} \\ x \mapsto \operatorname{card}\left(\pi_K^{-1}(x)\right), \end{cases}$$

where $\pi : Y \to Z$ is the canonical projection. This extends by linearity to general functions in $\mathcal{Q}(Z)$.

If $\varphi \in \mathcal{P}(Z)$, then φ can be written in terms of $\mathbb{L}$ and definable morphisms $\alpha : Z \to \mathbb{Z}$. We replace formally each occurrence of $\mathbb{L}$ by q_K and each function α by $\alpha_K : Z_K \to \mathbb{Z}$. This yields a function $\varphi_K : Z_K \to \mathbb{Z}$.

By tensor product, this defines a function $\varphi_K : Z_K \to \mathbb{Q}$ for any $\varphi \in \mathcal{C}(Z)$, well defined for $K \in \mathcal{C}_{\mathcal{O},N}$ for some $N > 0$ (depending on φ).

With this definitions, we can now state the specialization principle [24, Theorem 9.1.4]. Fix a definable morphism $f : Z \to S$. Recall that we have a relative version of the measures, denoted μ_n^S and $\mu_{n,K}^{S_K}$, that assign measure to constructible function of dimension n relatively to S, in order to deal with integrals with parameters. For any motivic constructible function $\varphi \in \mathcal{C}(Z)$, of dimension n relatively to S and integrable relatively to S, there is a $N > 0$ such that for any $K \in \mathcal{C}_{\mathcal{O},N}$,

$$\mu_n^S(\varphi)_K = \mu_{n,K}^{S_K}(\varphi_K).$$

Using this result, we can prove the following uniformity theorem. For all $K \in \mathcal{C}_{\mathbb{Z}}$, we write

MV_∞^K and Θ_d^K for the mean value at infinity and the local density on K defined in [15].

Theorem 1.6.2. *Let $X \subseteq h[n,0,0]$ a definable subassignment of dimension d. Then there is a $\Lambda \in \mathcal{D}$ and a integer $N > 0$ such that for all $K \in \mathcal{C}_{\mathbb{Z},N}$,*

$$\Theta_d(X)_K = \Theta_d^K(X_K),$$

$$\Theta_d(CM^\Lambda(X))_K = \Theta_d^K(CM^{\Lambda_K}(X_K)),$$

and for every $x \in K^n$,

$$\Theta_d^K(X_K, x) = \Theta_d^K(CM_x^{\Lambda_K}(X_K), 0).$$

Proof. Given the definition of Θ_d, the first two equalities follow from the specialization principle and the fact that $\mathrm{MV}_\infty(\varphi)_K = \mathrm{MV}_\infty^K(\varphi_K)$ which follow from the definition. The last equality follows from Theorem 1.3.25 and the first two equalities. □

Chapter 2

Cauchy-Crofton formula and density of curves

2.1 A motivic local Cauchy-Crofton formula

The results of this section appear in [43].

The aim of this section is to establish a motivic analogue of the local Cauchy-Crofton formula. The classical Cauchy-Crofton formula is a geometric measure theory result stating that the volume of a set X of dimension d can be recovered by integrating over the Grassmannian the number of points of intersection of X with affine spaces of codimension d, see for example [42]. It has been used by Lion [67] to show the existence of the local density of semi-Pfaffian sets. Comte [27], [28] has established a local version of the formula for sets $X \subseteq \mathbb{R}^n$ definable in an o-minimal structure. The formula states that the local density of such a set X can be recovered by integrating over a Grassmannian the density of the projection of X on subspaces. This allows him to show the continuity of the real local density along Verdier's strata in [28].

The local Cauchy-Crofton formula appears as a first step toward comparing the local Lipschitz-Killing curvature invariants and the polar invariants of a germ of a definable set $X \subseteq \mathbb{R}^n$. It is shown by Comte and Merle in [31] that one can recover one set of invariants by linear combination of the other, see also [29].

A p-adic analogue has been developed by Cluckers, Comte and Loeser in [15, Section 6]. We will follow closely their approach. Our precise result appears as Theorem 2.1.1 at the end of Section 2.1.1.

Consider a definable function $\pi : X \to Y$ between definable sets X and Y of dimension n. Recall the notation $\pi_!(\varphi) \in \mathcal{C}(Y)$ for any constructible function $\varphi \in \mathcal{C}(X)$.

If X is a definable subset of K^n and $x \in K^n$, define the ring of germs of constructible motivic functions at x by $\mathcal{C}(X)/\sim$, where $\varphi \sim \psi$ if there is an $r \in \mathbb{N}$ such that $\mathbb{1}_{B(x,r)}\varphi = \mathbb{1}_{B(x,r)}\psi$. in particular, if $\varphi \sim \psi$ are locally bounded, then $\Theta_d(\varphi, x) = \Theta_d(\psi, x)$ hence the

local motivic density is defined on $\mathcal{C}(K^n)/\sim$.

Consider now a linear projection $\pi : K^n \to K^d$ and let $X \subseteq K^n$ a definable set of dimension d. Say that condition $(*)$ is satisfied if $\pi_{|X\cap B(x,r)}$ is finite-to-one for some $r \geq 0$.

Then define $\pi_{!,x}(\varphi) = \pi_!(\mathbb{1}_{B(x,r)}\varphi) \in \mathcal{C}(Y)_{\pi(x)}$ for $\varphi \in \mathcal{C}(X)_x$. It does not depend on the large enough r chosen. Indeed, as $\pi : X \cap B(x,r) \to Y$ is finite-to-one, then there is $W_r \subseteq R^s_s$ finite with $X \cap B(x,r) \simeq W_r \times Y$ and the composite

$$W \times Y \xrightarrow{\pi} X \cap B(x,r) \to Y$$

being the projection on Y. Hence (up to taking a finite definable partition of X), for some r_0, for any $r \geq r_0$, $W_r = W_{r_0}$ and $\pi(X \cap B(x,r) = \pi(Y) \cap B(\pi(x), \alpha r)$, which means that $\pi_!(\mathbb{1}_{B(x,r)}\varphi) \in \mathcal{C}(Y)/\sim$ is independent of r.

2.1.1 Grassmannians

Fix a point $x \in K^n$ and view K^n as a K-vector space with origin 0. Then denote by $G(n,d)$ the Grassmannian of dimension d subvector spaces of K^n. The canonical volume form on $G(n,d)$ invariant under $GL_n(\mathcal{O}_K)$-transformations induce a constuctible function $\omega_{n,d}$ on $G(n,d)$ invariant under $GL_n(\mathcal{O}_K)$ transformations, see [23, Section 15] for details. As $G(n,d)$ is proper, the (motivic) volume of $G(n,d)$ is finite, hence up to multiplying $\omega_{n,d}$ by a constant, we can suppose it is normalized to 1.

For $V \in G(n, n-d)$, define $p_V : K^n \to K^n/V$ the canonical projection. We identify K^n/V to K^d as follows. There is some $g \in GL_n(\mathcal{O}_K)$ such that $g(K^{n-d} \times \{0\}^d) = V$ We identify K^n/V to $g(\{0\}^{n-d} \times K^{n-d})$ The particular choice of g does not matter thanks to the change of variable formula. If X is a dimension d definable subset of K^n then there is an dense definable subset $\Omega = \Omega(X,x)$ of $G(n, n-d)$ such that for every $V \in \Omega$, π_V satisfies condition $(*)$. Indeed the tangent $K^\times$-cone of X is of dimension at most d and it suffices to set

$$\Omega = \left\{ V \in G(n, n-d) \mid V \cap C_x^{K^\times}(X) = \{0\} \right\},$$

which is indeed dense in $G(n, n-d)$ In particular, for any $V \in \Omega$, $p_{!,x}(\varphi)$ is well defined for any $\varphi \in \mathcal{C}(X)/\sim$

With these notations, we can now state our motivic local Cauchy-Crofton formula.

Theorem 2.1.1 (local Cauchy-Crofton)**.** *Let $X \subseteq K^n$ a definable set of dimension d and $x \in K^n$. Then*

$$\Theta_d(X,x) = \int_{V \in \Omega \subseteq G(n,n-d)} \Theta_d(p_{V!,x}(\mathbb{1}_X), 0)\omega_{n,n-d}(V).$$

By Proposition 1.3.8, we may assume $X = \overline{X}$. We can also assume $x = 0$ and $0 \in X$. Indeed, if $0 \notin X$, then both sides of the formula are 0.

2.1.2 Tangential Crofton formula

We start by proving the theorem in the particular case where X is a Λ-cone.

Lemma 2.1.2. *Let X be a definable Λ-cone with origin 0 contained in some $\Pi \in G(n,d)$. Then*

$$\Theta_d(X,0) = \int_{V\in\Omega\subseteq G(n,n-d)} \Theta_d(p_V(\mathbb{1}_X),0)\omega_{n,n-d}(V).$$

Proof. Assume $\Lambda = \Lambda_{e,r}$. Fix some $V \in G(n,n-d)$ such that $\pi_V : \Pi \to K^d$ is bijective. As X is a Λ-cone and π_V is linear, $\pi_V(X)$ is also a Λ-cone. From the definition of local density, see also Remark 1.3.11, we have

$$\Theta_d(X,0) = \frac{1}{e(1-\mathbb{L}^{-d})}\sum_{i=0}^{e-1}\mathbb{L}^{di}\mu_d(X\cap S(0,i)) \tag{2.1.1}$$

and

$$\Theta_d(\pi_V(X),0) = \frac{1}{e(1-\mathbb{L}^{-d})}\sum_{j=0}^{e-1}\mathbb{L}^{dj}\mu_d(X\cap S(0,j)). \tag{2.1.2}$$

Let

$$A_i = \{y\in\pi_V(X) \mid \exists x\in X\cap S(0,i), \exists\lambda\in\Lambda, y=\lambda\pi_V(x)\}$$

Then since X is a Λ-cone and π_V is bijective, we have a disjoint union

$$\pi_V(X)\setminus\{0\} = \dot{\bigcup}_{i=0}^{e-1} A_i.$$

Now set $B_i^j = A_i\cap S(0,j)$ and $D_i^V = \dot{\bigcup}_{j=0}^{e-1} t^{i-j}B_i^j$. The set C_j is indeed a disjoint union since it is the image of $X\cap S(0,i)$ by the application

$$\varphi_V : x\in X \mapsto t^{\mathrm{ord}(x)-\mathrm{ord}(\pi_V(x)}\pi_V(x).$$

Indeed the function φ_V restricted to $X\cap S(0,i)$ is a definable bijection of image D_i^V since π_V is linear and bijective on Π.

By the change of variable formula, we have

$$\mu_d(D_i^V) = \int_{X\cap S(0,i)} \mathbb{L}^{-\mathrm{ord}(\mathrm{Jac}(\varphi_V(x)))}. \tag{2.1.3}$$

By Fubini theorem we get

$$\int_{V\in G(n,n-d)} \mu_d(D_i^V)\omega_{n,n-d}(V) = \int_{x\in X\cap S(0,i)}\int_{V\in G(n,n-d)} \mathbb{L}^{-\mathrm{ord}(\mathrm{Jac}(\varphi_V(x)))}\omega_{n,n-d}(V).$$

Set $C_i(x) = \int_{V\in G(n,n-d)} \mathbb{L}^{-\mathrm{ord}(\mathrm{Jac}(\varphi_V(x)))}\omega_{n,n-d}(V)$. We claim that $C_i(x)$ is independent of $x\in S(0,i)$. Indeed, if $x,x'\in S(0,i)$, we can find some $g\in GL_n(\mathcal{O}_K)$ such that $x'=gx$.

Since $\omega_{n,n-d}$ is invariant under $GL_n(\mathcal{O}_K)$-transformations and $\varphi_V(x) = \varphi_{gV}(x')$, by the change of variable formula we get

$$C_i(x) = \int_{V\in G(n,n-d)} \mathbb{L}^{-\mathrm{ord}(\mathrm{Jac}(\varphi_V(x)))}\omega_{n,n-d}(V) = \int_{V'\in G(n,n-d)} \mathbb{L}^{-\mathrm{ord}(\mathrm{Jac}(\varphi_{V'}(gx)))}\omega_{n,n-d}(V').$$

Moreover, it is independent of i by linearity of π_V, hence we denote it by C and we have

$$\int_{V\in G(n,n-d)} \mu_d(D_i^V)\omega_{n,n-d}(V) = C\mu_d(X\cap S(0,i). \tag{2.1.4}$$

We also have

$$\mathbb{L}^{dj}\mu_d(B_i^j) = \mathbb{L}^{di}\mu_d(t^{i-j}B_i^j), \tag{2.1.5}$$

hence

$$\begin{aligned}\sum_{j=0}^{e-1}\mathbb{L}^{dj}\mu_d(\pi_V(X)\cap S(0,j)) &= \sum_{j=0}^{e-1}\sum_{i=0}^{e-1}\mathbb{L}^{dj}\mu_d(B_i^j)\\ &= \sum_{i=0}^{e-1}\sum_{j=0}^{e-1}\mathbb{L}^{di}\mu_d(t^{i-j}B_i^j)\\ &= \sum_{i=0}^{e-1}\mathbb{L}^{di}\mu_d(D_i^V). \end{aligned}\tag{2.1.6}$$

Combining Equations 2.1.2, 2.1.6 and 2.1.4, we get

$$\int_{V\in G(n,n-d)} \Theta_d(\pi_V(X),0)\omega_{n,n-d}(V) = C\frac{1}{e(1-\mathbb{L}^{-d})}\sum_{i=0}^{e-1}\mathbb{L}^{di}\mu_d(X\cap S(0,i)) = C\Theta_d(X,0).$$

We find $C = 1$ by computing both sides of the previous equality with $X = \Pi$. □

This following lemma is the motivic analogous of the classical spherical Crofton formula, see for example [42, Theorem 3.2.48]. See also [15, Remark 6.2.4] for a reformulation in the p-adic case.

Lemma 2.1.3. *Let X be a definable Λ-cone with origin 0. Then*

$$\Theta_d(X,0) = \int_{V\in G(n,n-d)} \Theta_d(p_{V!,0}(\mathbb{1}_X),0)\omega(V).$$

Proof. We only have to modify slightly the proof of Lemma 2.1.2. Indeed, we use now the function

$$\varphi_V : x\in X \mapsto t^{\mathrm{ord}(x)-\mathrm{ord}(\pi_V(x)}\pi_V(x).$$

restricted to the smooth part of X. It is now longer injective on $X\cap S(0,i)$, however the

motivic volume of the fibers is taken into account in $p_{V!,0}(X)$. Hence we get similarly

$$\int_{V\in G(n,n-d)} \Theta_d(\pi_{V!,0}(X),0)\omega_{n,n-d}(V) = C\frac{1}{e(1-\mathbb{L}^{-d})}\sum_{i=0}^{e-1}\mathbb{L}^{di}\mu_d(X\cap S(0,i)) = C\Theta_d(X,0)$$

and then $C=1$ by computing both sides for X a vector space of dimension d. □

2.1.3 General case

Before proving Theorem 2.1.1, we need a technical lemma.

Lemma 2.1.4. *Let $X \subseteq K^n$ be a definable set of dimension d and $V \in G(n,n-d)$ such that the projection $p_V : C_0^\Lambda(X) \to K^d$ is finite-to-one. Then there is a definable* **k***-partition of X such that for each* **k***-part X_ξ, there is a ξ-definable set C_ξ of dimension less than d such that p_V is injective on $C_0^\Lambda(X_\xi)\backslash C_\xi$.*

Proof. We can assume $\Lambda = K^\times$. As the projection $p_V : C_0^\Lambda(X) \to K^d$ is finite-to-one, by finite b-minimality one can find a **k**-partition of $C_0^\Lambda(X)$ such that p_V is injective on each **k**-part of $C_0^\Lambda(X)$. For a **k**-part $C_0^\Lambda(X)_\xi$, define B_ξ to be the ξ-definable subset of K^n defined as the union of lines ℓ passing through 0 such that the distance between $\ell\cap S(0,0)$ and $C_0^\Lambda(X)_\xi \cap S(0,0)$ is strictly smaller than the distance between $\ell\cap S(0,0)$ and $C_0^\Lambda(X)_{\xi'}\cap S(0,0)$ for every $\xi'\neq\xi$. Set $X_\xi = X\cap B_\xi$. Then setting $Y = X\backslash \cup_\xi X_\xi$, we have $C_0^\Lambda(Y)$ empty and $C_0^\Lambda(X_\xi) \subseteq \overline{C_0^\Lambda(X)_\xi}$. Hence we set $C_\xi = \overline{C_0^\Lambda(X)_\xi}\backslash C_0^\Lambda(X)_\xi$ and we have X_ξ and C_ξ as required. □

Proof of Theorem 2.1.1. From Theorem 1.3.25, there is a $\Lambda\in\mathcal{D}$ such that $\Theta_d(X,0) = \Theta_d(CM_0^\Lambda(X),0)$. As in the proof of Theorem 1.3.25, by Lemma 1.3.9 we can assume X is the graph of a 1-Lipschitz function defined on some definable set $U\subset K^d$. In this case, $CM_0^\Lambda(X) = \mathbb{1}_{C_0^\Lambda(X)}$. From Lemma 2.1.3, we have

$$\Theta_d(C_0^\Lambda(X),0) = \int_{V\in G(n,n-d)} \Theta_d(p_{V!,0}(C_0^\Lambda(X)),0)\omega_{n,n-d}(V).$$

Hence we need to show that for every V in a dense subset of $G(n,n-d)$,

$$\Theta_d(p_{V!,0}(C_0^\Lambda(X)),0) = \Theta_d(p_{V!,0}(X),0).$$

We can find a **k**-partition of X such that p_V is injective on the **k**-parts. Replace X by one of the **k**-parts and suppose then that p_V is injective on X.

Fix a $V\in G(n,n-d)$ such that p_V is injective on $C_0^\Lambda(X)$. By Lemma 2.1.4, there is a **k**-partition of X (depending on V) such that for each **k**-part X_ξ there is a ξ-definable set C_ξ of dimension less that d such that p_V is injective on $C_0^\Lambda(X_\xi)\backslash C_\xi$. By a new use of

Lemma 1.3.9, it suffices to show that

$$\Theta_d(p_{V!,0}(C_0^\Lambda(X_\xi)),0) = \Theta_d(p_{V!,0}(X_\xi),0).$$

As p_V is injective on X and $C_0^\Lambda(X_\xi)\backslash C_\xi$, we have $p_{V!,0}(X) = \mathbb{1}_{p_V(X)}$ and

$$p_{V!,0}(C_0^\Lambda(X_\xi)) = \mathbb{1}_{p_V(C_0^\Lambda(X_\xi)\backslash C_\xi)} + p_{V!,0}(C_\xi).$$

We have $\Theta_d(p_{V!,0}(C_\xi),0) = 0$ for dimensional reasons. As $C_0^\Lambda(p_V(X)) = p_V(C_0^\Lambda(X))$ the result follow from Proposition 1.5.2, the particular case of Theorem 1.3.25 for sets of codimension zero. □

2.2 Local density of curves

In this section, we compute the motivic local density of a plane curve singularity using two diffrent methods.

2.2.1 Non-degenerated with respect to Newton polygon

Let $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha x^I \in k[x_1,...,x_n]$ be a polynomial. Define the support of f as $\mathrm{Supp}(f) = \{\alpha \in \mathbb{N}^n \mid a_\alpha \neq 0\}$ and the Newton diagram of f as the convex hull $\mathcal{N}(f) = \mathrm{Conv}(\mathrm{Supp}(f)+\mathbb{N}^n)$. The Newton boundary $\mathcal{N}_0(f)$ of f is the set of compact faces of $\mathcal{N}(f)$. The polynomial f is said to be convenient if $\mathcal{N}_0(f)$ meets each of the coordinate axes.

Definition 2.2.1 (Non-degenerate)**.** For $S \in \mathcal{N}_0(f)$, let

$$f_l S = \sum_{\alpha\in S\cap\mathrm{Supp}(f)} a_\alpha x^\alpha.$$

We say that f is non-degenerate on $\mathcal{S}$ if the polynomials

$$\partial f_S/\partial x_1, ..., \partial f_S/\partial x_n$$

have no common roots on $(k^*)^n$

We concentrate now on the plane case. Let $f \in k[X,Y]$. A face $S \in \mathcal{N}_0(f)$ is the edge joining two points of coordinates (x_1,x_2) and (y_1,y_2), that we order choosing $x_1 < y_1$ hence $y_2 < x_2$. Set $a_S = x_1$, $b_S = y_2$, $|S|_1 = y_1 - x_1$ and $|S|_2 = x_2 - y_2$. Denote also by r_S the greatest common divisor of $|S|_1$ and $|S|_2$ and set $m_S = |S|_1/r_S$ and $n_S = |S|_2/r_S$.

The polynomial f_S can be written as $f_S = X^{a_S}Y^{b_S}g$ with g a homogeneous polynomial, which can itself be factored, hence there are for $i = 1,...,l \in \mathbb{N}$, $a_i \in k$ two by two distinct,

$d_i \in \mathbb{N}^*$ and $c \in k$ such that

$$f_S = cX^{a_S}Y^{b_S} \prod_{i=1,\dots,l_S} (Y^{n_S} - a_i X^{m_S})^{d_i}.$$

Then f is non-degenerate on S if and only if $d_1 = ... = d_l = 1$.

The polynomial f can be factored as $f = f_1 ... f_r$ with $f_i \in k\{X, Y\}$, we call r the number of branches of f. Denote by N_i the multiplicity of the branch defined by $f_i = 0$.

We work now in the theory $\mathrm{Th}(k((t)))$ in the Denef-Pas language $\mathcal{L}_{\mathrm{DP}}(k((t)))$ and use Cluckers and Loeser's theory of motific integration. The target ring of motivic integration in this case is $\mathcal{C}(\mathrm{Spec}(k)) = \mathbf{K}(\mathrm{Var}_k)_{\mathrm{loc}}$, with $\mathbf{K}(\mathrm{Var}_k)$ the Grothendieck group of k-varieties and

$$\mathbf{K}(\mathrm{Var}_k)_{\mathrm{loc}} = \mathbf{K}(\mathrm{Var}_k)\left[\mathbb{L}^{-1}, \left(\frac{1}{1-\mathbb{L}^{-\alpha}}\right)_{\alpha \in \mathbb{N}^*}\right].$$

Hence the motivic local density has values in $\widetilde{\mathbf{K}(\mathrm{Var}_k)_{\mathrm{loc}}} = \mathbf{K}(\mathrm{Var}_k)_{\mathrm{loc}} \otimes \mathbb{Q}$.

Let $f \in k[X, Y]$ be a polynomical without constant term and non-degenerate. Let $C = \left\{(x, y) \in K^2 \mid f(x, y) = 0\right\}$. We want to compute the motivic local density of C at 0.

We will prove the following proposition.

Proposition 2.2.2. *Let $f \in k[X, Y]$ be a polynomial without constant term and non-degenerate. Let $f = f_1 \cdots f_r$ with $f_j \in k[[X, Y]]$ be the decomposition of f into irreducible factors. Denote by N_j the multiplicity of the branch defined by $f_j = 0$. Then the motivic local density at 0 of the curve C defined by $f = 0$ is*

$$\Theta(C, 0) = \sum_{j=1}^{r} \frac{1}{N_j}.$$

Proof. As the motivic density is additive, we can suppose f is convenient.

For $S \in \mathcal{N}_0(f)$, set

$$C_S = \left\{(x, y) \in K^2 \mid m_s \mathrm{ord}(x) = n_S \mathrm{ord}(y)\right\}.$$

Fix some $n \in \mathbb{N}^*$. Then $C \cap S(0, n)$ is a disjoint union

$$C \cap S(0, n) = \dot{\cup}_{S \in \mathcal{N}_0(f)} C_S \cap S(0, n).$$

By additivity of the local density, it suffices to compute the density of C_S.

As f is non-degenerate on S, we can partition $(k^*)^2$ as $(k^*)^2 = U_1 \dot{\cup} U_2$, where for $(x, y) \in U_1$, $\partial f_S / \partial x(x, y) \neq 0$ and for $(x, y) \in U_2$, $\partial f_S / \partial y(x, y) \neq 0$. Partition $C_S = C_{S,1} \dot{\cup} C_{S,2}$ with

$$C_{S,i} = \{(x, y) \in C_S \mid (\overline{\mathrm{ac}}(x), \overline{\mathrm{ac}}(y)) \in U_i\}.$$

Up to exchanging x and y, assume $m_S \geq n_S$.

For any $(x, y) \in C_{S,1} \cap S(0, n)$. We have $m_S \mathrm{ord}(x) = n_S \mathrm{ord}(y)$, hence $\mathrm{ord}(x) = n$. In particular $C_{S,1} \cap S(0, n)$ is non-empty if and only if n_S divides n (since n_S and m_S are coprime). In what follows, we suppose it is the case.

Denote by γ the common valuation of the monomials in f_S. Set $y' = t^{-n}y$ and $x' = t^{-nn_S/m_S}x$ and

$$C_1' = \left\{(x', y') \mid (x, y) \in C_{S,1} \cap S(0, n)\right\}.$$

The order of jacobian of the bijection from C_1' to $C_{S,1} \cap S(0, n)$ is n, since $m_S \geq n_S$.

As f_S is quasi-homogeneous, we have

$$f(x, y) = t^{\gamma} f_S(x', y') + t^{\gamma+1} R_S(x', y'),$$

with $R_S \in k[[t]][X, Y]$.

Set

$$W_1 = \{(\nu, y) \in k^* \times K \mid \mathrm{ord}(y) = m_S n/n_S, (\nu, \overline{\mathrm{ac}}(y)) \in U_1, f_S(\nu, \overline{\mathrm{ac}}(y)) = 0\}$$

and

$$W_1' = \left\{(\nu, y') \in k^* \times K \mid \mathrm{ord}(y') = 0, (\nu, \overline{\mathrm{ac}}(y')) \in U_1, f_S(\nu, \overline{\mathrm{ac}}(y')) = 0\right\}.$$

The order of jacobian of the bijection from W_1 to W_1' is $-m_S n/n_S$.

Since $(\overline{\mathrm{ac}}(x), \overline{\mathrm{ac}}(y)) \in U_1$, by Hensel's lemma we have a bijection

$$\varphi : (\nu, y') \in W_1' \mapsto (x', y') \in C'.$$

By chain rule for derivation, $\mathrm{ord}_{\mathrm{Jac}}(\varphi)$ has constant value 0.

By composing these three bijections, we get a bijection

$$W_1 \to C_{S,1} \cap S(0, n)$$

with order of jacobian of constant value $-m_S n/n_S + n$.

By the change of variable formula, we have

$$\mu(C_{S,1} \cap S(0, n)) = \mathbb{L}^{m_S n/n_S - n} \mu(W) = \mathbb{L}^{-n-1}[\overline{W}],$$

with $\overline{W_1} = \{(\nu, \xi) \in U_1 \mid f_S(\nu, \xi) = 0\}$.

Proceed similarly for $C_{S,2}$ to show that

$$\mu(C_{S,2} \cap S(0, n)) = \mu(W_2) = \mathbb{L}^{-n-1}[\overline{W_2}],$$

with

$$W_2 = \{(x, \xi) \in K \times k^* \mid \mathrm{ord}(x) = n, (\overline{\mathrm{ac}}(x), \xi) \in U_2, f_S(\overline{\mathrm{ac}}(x), \xi) = 0\}$$

and $\overline{W_2} = \{(\nu, \xi) \in U_2 \mid f_S(\nu, \xi) = 0\}$.

Since U_1 and U_2 partition $(k^*)^2$,

$$[\overline{W_1}] + [\overline{W_2}] = [\{(\nu, \xi) \in (k^*)^2 \mid f_S(\nu, \xi) = 0\}].$$

As $f_S = cX^{a_S}Y^{b_S}\prod_{i=1,\dots,l_S}(Y^{n_S} - a_iX^{m_S})$ with the a_i two by two disjoint,

$$[\{(\nu, \xi) \in (k^*)^2 \mid f_S(\nu, \xi) = 0\}] = l_S(\mathbb{L} - 1).$$

We have then

$$\Theta(C_S, 0) = \frac{\mathbb{L}^{-n-1}[\{(\nu, \xi) \in (k^*)^2 \mid f_S(\nu, \xi) = 0\}]}{n_S\mathbb{L}^{-n}(1 - \mathbb{L}^{-1})} = \frac{l_S}{n_S}.$$

and

$$\Theta(C, 0) = \sum_{S \in \mathcal{N}_0(f)} \frac{l_S}{n_S} = \sum_{j=1}^{r} \frac{1}{N_j}.$$

□

2.2.2 Analytic branches

We now follow a diffrent approch and we will compute the local density of curves using Newton-Puiseux parametrizations.

Here is the parametrization theorem that we will use.

Theorem 2.2.3 ([92, Theorem 3.2]). *Let $f \in k[[x, y]]$ be a power series without constant term such that $f(0, y) \neq 0$. Let $f = uf_1^{a_1} \cdots f_r^{a_r}$ be the decomposition of f into Weierstrass polynomials f_i with u a unit in $k[[x, y]]$. Let m_i be the degree of the polynomial f_i. For each index i, there is a uniquely determined power series $h_i \in k[[s]]$ such that $f_i(s^{m_i}, h_i(s)) = 0$. Moreover, if we write $h_i = cs^l + \dots$ with $c \in k^\times$, then the Newton diagram of f is the sum*

$$\mathcal{N}(f) = \sum_{i=1}^{r} \left\{ \frac{a_i m_i}{a_i l_i} \right\}.$$

In the above statement, we use the notation $\left\{\frac{a}{b}\right\}$ for the Newton diagram with vertices $(0, b)$ and $(a, 0)$. The sum of two subsets A and B of $\mathbb{R}^2$ is the Minkowski sum $A + B = \{x + y \mid x \in A, y \in B\}$.

Observe that greatest common divisor of all the indexes of the non-zero coefficients of h_i and m_i is 1. Indeed, otherwise one can find another parametrization $(s^{m'_i}, h'_i(s))$ with $m'_i < m_i$. This parametrization satisfy an equation of the form $g'_i(x, y) = 0$, with $g'_i \in k[[x]][y]$ a polynomial in the variable y of degree m'_i. But then g'_i would divide g_i, contradicting the irreducibility of g_i.

In particular, we can apply the theorem to f_i to see that the multiplicity N_i of the branch associated to f_i is equal to $\min(m_i, l_i)$.

In this section we compute the motivic local density of a plane curve using Newton-Puiseux parametrizations.

We keep a fixed algebraically closed field k of characteristic zero and $K = k((t))$. We work here in the theory $\mathcal{T}_{\mathrm{an}}(K)$ in the analytic language $\mathcal{L}_{\mathrm{an}}(K)$. Recall the ring of converging power series $K\{x_1, ..., x_n\}$. We have $f = \sum_{I \in \mathbb{N}^n} a_I x^I \in K\{x_1, ..., x_n\}$ if $\mathrm{ord}(a_I)$ converges to $+\infty$ when $|I|$ goes to $+\infty$. The language $\mathcal{L}_{\mathrm{an}}(K)$ is the extension of the Denef-Pas language with a function symbol for each element of $K\{x_1, ..., x_n\}$ (for each $n \in \mathbb{N}^*$). We interpret such functions on complete valued field extensions of K as the corresponding function on $B(0,0)$ extended by zero outside $B(0,0)$.

Embed $k[[x,y]]$ in $K\{x,y\}$ by sending x to tx and y to ty. Denote by $\tilde{f} \in K\{x,y\}$ the image of $f \in k[[x,y]]$. For any $f \in k[[x,y]]$, the function $(x,y) \in B(0,1) \mapsto f(x,y)$ is well defined and definable since $\tilde{f} \in \mathcal{L}_{\mathrm{an}}(K)$ and $f(x,y) = \tilde{f}(t^{-1}x, t^{-1}y)$. In particular, the functions f_i, u, h_i are well defined of domain the maximal ideal (or its square) and are all definable.

Proposition 2.2.4. *Let $f \in k[x,y]$ be a power series without square factor. Let $f = f_1 \cdots f_r$ the decomposition in $k[[x,y]]$ of f into irreducible factors. Denote by N_i the multiplicity of f_i. Then the motivic density of the curve defined by $C = \{(x,y) \in K^2 \mid f(x,y) = 0\}$ is*

$$\Theta(C, 0) = \sum_{i=1}^{r} \frac{1}{N_i}.$$

In the case $k = \mathbb{C}$, we can compare with the complex local density at the origin of the complex curve defined by $f = 0$, also known as Lelong's number. It is equal to the multiplicity of f, $\sum_{j=1}^{r} N_j$ by Drapper's result [41].

Corollary 2.2.5. *The motivic density is an invariant of plane curves singularities.*

Proof. Let $B_i = \{(x,y) \in B(0,1) \mid f_i(x,y) = 0\}$ Recall that for all $(x,y) \in B(0,1)$, $f(x,y) = x^{\varepsilon} u(x,y) \prod_i f_i(x,y)$. Since u is a unit, $u(x,y) \neq 0$. Hence by additivity of the motivic density,

$$\Theta(C, 0) = \varepsilon + \sum_i \Theta(B_i, 0).$$

We then need to show that $\Theta(B_i, 0) = 1/N_i$.

If $f_i = y$, we are done since $B_i = \{(x,0) \in B(0,1)\}$. Assume then that $g_i \neq y$. As noted above, the multiplicity of the branch associated to g_i is $N_i = \min(m_i, l_i)$, where l_i is the order of h_i. Let

$$\varphi : s \in B(0,1) \mapsto (s^{m_i}, h_i(s)) \in B_i.$$

Then φ is a definable bijection. Indeed, it is surjective by definition of h_i. It is injective

since the greatest common divisor of all the indexes of the non-zero coefficients of h_i and m_i is 1.

Observe that $B_i \cap S(0,n)$ is non-empty if and only if n is divisible by N_i.

The order of jacobian of φ is equal to $(N_i - 1)\mathrm{ord}(s)$. Fix some $k \in \mathbb{N}^*$. By the change of variable formula along φ, the motivic volume of $C_i \cap S(0, N_i k)$ is

$$\mu(B_i \cap S(0, N_i k)) = \mathbb{L}^{-(N_i-1)k}\mu(S(0,k)) = \mathbb{L}^{-N_i k}(1 - \mathbb{L}^{-1}).$$

Hence the motivic density is

$$\Theta(B_i, 0) = 1/N_i.$$

□

Chapter 3

Virtual rigid motives

GASQUET : *Vous êtes content ce matin ?*
CÉZANNE : *Je tiens mon motif...* (Il joint les mains)
Un motif, voyez-vous, c'est ça...
GASQUET : *Comment ?*
CÉZANNE : *Eh ! Oui...* (Il refait son geste, écarte les mains, les dix doigts ouverts, les rapproche lentement, lentement, puis les joints, les serre, les crispe, les fait pénétrer l'une dans l'autre.) *Voilà ce qu'il faut atteindre.*

Joaquim Gasquet, *Conversations avec Cézanne*

3.1 Introduction

The results of this chapter appear in [45].

Let k be a field of characteristic zero containing all roots of unity and $K = k((t))$ the field of Laurent series. Morel and Voevodsky build in [78] the category $\mathrm{SH}(k)$ of stable $\mathbb{A}^1$-invariant motivic sheaves without transfers over k. More generally for S a k-scheme they build the category of S-motives $\mathrm{SH}(S)$. Following insight by Voevodsky, see Deligne's notes [32], Ayoub developed in [1] a six functors formalism for the categories $\mathrm{SH}(-)$, mimicking Grothendieck's six functors formalism for étale cohomology. See also in [14] an alternative construction by Cisinski and Déglise. For $f : X \to Y$ a morphism of schemes, in addition to the direct image $f_* : \mathrm{SH}(X) \to \mathrm{SH}(Y)$ and pull-back $f^* : \mathrm{SH}(Y) \to \mathrm{SH}(X)$, one has the extraordinary direct image $f_! : \mathrm{SH}(X) \to \mathrm{SH}(Y)$ and extraordinary pull-back $f^! : \mathrm{SH}(Y) \to \mathrm{SH}(X)$. It allows in particular to define for any S-scheme $f : X \to S$ an object $\mathrm{M}^\vee_{S,c}(X) = f_! f^* \mathbb{1}_k \in \mathrm{SH}(S)$, the so-called cohomological motive with compact support of X.

Denote by $\mathbf{K}(\mathrm{Var}_k)$ the Grothendieck group of k-varieties. It is the abelian group generated by isomorphism classes of k-varieties, with the scissors relations

$$[X] = [Y] + [X \backslash Y]$$

for Y a closed subvariety of X. Cartesian product induces a ring structure on $\mathbf{K}(\mathrm{Var}_k)$.

As $\mathrm{SH}(k)$ is a triangulated category, we can consider its Grothendieck group $\mathbf{K}(\mathrm{SH}(k))$, which is the abelian group generated by isomorphism classes of its compact (also called constuctible) objects, with relations $[B] = [A] + [C]$ whenever there is a distinguished triangle

$$A \to B \to C \overset{+1}{\to} .$$

Elements of $\mathbf{K}(\mathrm{SH}(k))$ are called virtual motives and tensor product on $\mathrm{SH}(k)$ induces a ring structure on $\mathbf{K}(\mathrm{SH}(k))$. The locality principle implies that the assignment $X \in \mathrm{Var}_k \mapsto [\mathrm{M}^{\vee}_{k,c}(X)] \in \mathbf{K}(\mathrm{SH}(k))$ satisfies the scissors relations, hence induces a morphism

$$\chi_k : \mathbf{K}(\mathrm{Var}_k) \to \mathbf{K}(\mathrm{SH}(k))$$

which is a ring morphism. Such a morphism was first considered by Ivorra and Sebag in [59].

Ayoub builds in [3] the category $\mathrm{RigSH}(K)$ of rigid analytic motives over K, in a similar fashion of $\mathrm{SH}(K)$ but instead of K-schemes, he starts with rigid analytic K-varieties in the sense of Tate. The analytification functor from algebraic K-varieties to rigid K-varieties induce a functor

$$\mathrm{Rig}^* : \mathrm{SH}(K) \to \mathrm{RigSH}(K).$$

For any rigid K-variety X, Ayoub defines $\mathrm{M}_{\mathrm{Rig}}(X)$ and $\mathrm{M}^{\vee}_{\mathrm{Rig}}(X)$ respectively the homological and cohomological rigid motives of X. However to our knowledge there is no general notion of cohomological rigid motive with compact support.

One can also consider $\mathbf{K}(\mathrm{VF}_K)$ the Grothendieck ring of semi-algebraic sets over K. If $X = \mathrm{Spec}(A)$ is an affine variety over K, a semi-algebraic subset of X^{an} is a boolean combination of subsets of the form $\{x \in X^{\mathrm{an}} \mid \mathrm{v}(f(x)) \leq \mathrm{v}(g(x))\}$, for $f, g \in A$ (where v is the valuation on K). The group $\mathbf{K}(\mathrm{VF}_K)$ is then the abelian group of isomorphism classes of semi-algebraic sets (for semi-algebraic bijections) with relations $[X] = [U] + [V]$ if X is the disjoint union of U and V. Locally closed semi-algebraic sets inherits a structure of rigid K-varieties and $\mathbf{K}(\mathrm{VF}_K)$ is generated as a group by classes of such sets that are smooth rigid K-varieties. We could also consider $\mathbf{K}(\mathrm{VF}^{\mathrm{an}}_K)$, the Grothendieck group of subanalytic sets over K but they turn out to be isomorphic as a byproduct of Hrushovski and Kazhdan's theory of motivic integration [54].

In this situation it is rather natural to ask about the existence of a ring morphism

$$\chi_{\mathrm{Rig}} : \mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RigSH}(K))$$

extending the morphism $\chi_K : \mathbf{K}(\mathrm{Var}_K) \to \mathbf{K}(\mathrm{SH}(K))$. If X is an algebraic K-variety smooth and connected of dimension d, then $[\mathrm{M}^{\vee}_{K,c}(X)] = [\mathrm{M}_K(X)(-d)]$, where $(-d)$ is the Tate twist (iterated d times). We would like to define for X a quasi-compact rigid K-variety smooth and connected of dimension d, $\chi_{\mathrm{Rig}}([X]) = [\mathrm{M}_{\mathrm{Rig}}(X)(-d)]$. Such classes generate $\mathbf{K}(\mathrm{VF}_K)$. If χ_{Rig} is well-defined, it will be the unique morphism satisfying such conditions. The main objective of this paper is to show the existence of such a morphism.

The strategy of proof is to use alternative descriptions of $\mathbf{K}(\mathrm{VF}_K)$ and $\mathrm{RigSH}(K)$, the former being established by Hrushovski and Kazhdan, the latter by Ayoub. Let us describe them briefly.

From a model-theoretic point of view, semi-algebraic sets over K are definable sets in the (first order) theory of algebraically closed valued fields over K. If L is a valued field, with ring of integers $\mathcal{O}_L$ of maximal ideal $\mathcal{M}_L$, we set $\mathrm{RV}(L) = L^{\times}/(1+\mathcal{M}_L)$. Observe that RV fits in the following exact sequence, where $\mathbf{k}$ is the residue field and Γ the value group :

$$1 \to \mathbf{k}^{\times} \to \mathrm{RV} \to \Gamma \to 0.$$

Working in a two sorted language, with one sort VF for the valued field and one sort RV, Hrushovski and Kazhdan establish in [54] the following isomorphism of rings :

$$\oint : \mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}},$$

where $\mathbf{K}(\mathrm{RV}_K[*])$ is the Grothendieck ring of definable sets of RV, the $[*]$ meaning that some grading is taken into account and I_{sp} is an ideal generated by a single explicit relation, see Section 3.2.1. Set $\hat{\mu} = \lim_{\leftarrow n} \mu_n$, with μ_n the group of n-th roots of unity in k and $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$ the Grothendieck of varieties equipped with a good $\hat{\mu}$-action, good meaning that the action factors through a good μ_n-action for some n.

The ring $\mathbf{K}(\mathrm{RV}_K[*])$ can be further decomposed into a part generated by $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$ and a part generated by definable subsets of the value group. The latter being polytopes, one can apply Euler characteristic with compact supports to get a ring morphism

$$\Theta \circ \mathcal{E}_c : \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}} \to \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}).$$

Ayoub on his side defines the category of quasi-unipotent motives $\mathrm{QUSH}(k)$ as the triangulated subcategory of $\mathrm{SH}(\mathbb{G}_{mk})$ with infinite sums generated by homological motives (and their twists) of $\mathbb{G}_{mk}$-varieties of the form

$$X[T,T^{-1},V]/(V^r - Tf) \to \mathrm{Spec}(k[T,T^{-1}]) = \mathbb{G}_{mk}$$

where X is a smooth k-variety, $r \in \mathbb{N}^*$, and $f \in \Gamma(X, \mathcal{O}_X^\times)$. Let $q : \mathrm{Spec}(K) \to \mathbb{G}_{mk}$ be the morphism defined by $T \in k[T, T^{-1}] \mapsto t \in K = k((t))$. Ayoub shows in [3] that the functor

$$\mathfrak{F} : \mathrm{QUSH}(k) \xrightarrow{q^*} \mathrm{SH}(K) \xrightarrow{\mathrm{Rig}^*} \mathrm{RigSH}(K)$$

is an equivalence of categories, denote by $\mathfrak{R}$ a quasi-inverse.

We will define a morphism

$$\chi_{\hat{\mu}} : \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{QUSH}(k))$$

compatible with χ_k in the sense that it commutes with the morphism $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{Var}_k)$ induced by the forgetful functor and $1^* : \mathbf{K}(\mathrm{SH}(\mathbb{G}_{mk})) \to \mathbf{K}(\mathrm{SH}(k))$, where $1 : \mathrm{Spec}(k) \to \mathbb{G}_{mk}$ is the unit section, see section 3.3.4.

Here is our main theorem.

Theorem 3.1.1. *Let k be a field of characteristic zero containing all roots of unity and let $K = k((t))$. Then there exists a unique ring morphism*

$$\chi_{\mathrm{Rig}} : \mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RigSH}(K))$$

such that for any quasi-compact rigid K-variety X, smooth and connected of dimension d, $\chi_{\mathrm{Rig}}(X) = [\mathrm{M}_{\mathrm{Rig}}(X)(-d)]$.

Moreover, all the squares in the following diagram commute :

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_K) \arrow[r] \arrow[d, "\chi_K"'] & \mathbf{K}(\mathrm{VF}_K) \arrow[r, "\oint", "\simeq"'] \arrow[d, "\chi_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}} \arrow[r, "\Theta\circ\mathcal{E}_c"] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[r] \arrow[d, "\chi_{\hat{\mu}}"] & \mathbf{K}(\mathrm{Var}_k) \arrow[d, "\chi_k"] \\
\mathbf{K}(\mathrm{SH}(K)) \arrow[r, "\mathrm{Rig}^*"'] & \mathbf{K}(\mathrm{RigSH}(K)) \arrow[rr, "\simeq", "\mathfrak{R}"'] & & \mathbf{K}(\mathrm{QUSH}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}(k)).
\end{tikzcd}$$

Observe that with this diagram in mind, defining χ_{Rig} is easy since $\mathfrak{R}$ is an isomorphism, it is showing that it satisfies $\chi_{\mathrm{Rig}}(X) = [\mathrm{M}_{\mathrm{Rig}}(X)(-d)]$ that we will have to prove. We will rely for this on an explicit computation of $\oint[X]$ when a semi-stable formal $R = k[[t]]$-model of X is chosen.

Two choices are made in this construction. The first is when applying compactly supported Euler characteristic $\mathcal{E}_c$, where we also could have used Euler characteristic $\mathcal{E}$, the second is when we apply the morphism $\chi_{\hat{\mu}}$, where we can also consider the mophism sending the class of a variety to its homological motive with compact support. Varying these choices leads to define three other ring morphisms

$$\chi'_{\mathrm{Rig}}, \widetilde{\chi_{\mathrm{Rig}}}, \widetilde{\chi'_{\mathrm{Rig}}} : \mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

satisfying properties analog to Theorem 3.1.1. In particular, we will show that $\widetilde{\chi'_{\mathrm{Rig}}}$ also

extends the morphism χ_K.

We claim that $\chi_{\mathrm{Rig}}(X)$ is the virtual incarnation of an hypothetical cohomological rigid motive with compact support of X. Hence we expect some duality to appear. Here is what we prove in this direction.

Theorem 3.1.2. *Let X be a quasi-compact smooth rigid variety, $\mathcal{X}$ an formal R-model of X, D a locally closed and proper subset of its special fiber $\mathcal{X}_\sigma$. Consider the tube $]D[$ of D in $\mathcal{X}$, it is a (possibly non quasi-compact) rigid subvariety of X. Then*

$$\chi_{\mathrm{Rig}}(]D[) = [\mathrm{M}^\vee_{\mathrm{Rig}}(]D[)].$$

In particular, if X is a smooth and proper rigid variety,

$$\chi_{\mathrm{Rig}}(X) = [\mathrm{M}^\vee_{\mathrm{Rig}}(X)].$$

To prove this theorem, we will once again rely on a choice of a semi-stable formal R-model of X and compute explicitly $[\mathrm{M}^\vee_{\mathrm{Rig}}(]D[)]$ in terms of homological motives of $]D[$ and some subsets of $]D[$. Our approach is inspired by parts of Bittner's works [8] and [9] where she defines duality involutions in $\mathbf{K}(\mathrm{Var}_k)[\mathbb{L}^{-1}]$ and shows that a toric variety associated to a simplicial fan satisfies an instance of Poincare's duality.

Theorem 3.1.2 allows us to answer partially a question asked by Ayoub, Ivorra and Sebag in [4, Remark 8.13] in relation to the motivic Milnor fiber. Fix X a smooth connected k-variety and let $f : X \to \mathbb{A}^1_k$ be a non constant morphism. Set X_σ to be the closed subvariety of X defined by the vanishing of f. Jan Denef and François Loeser define in [35, 36, 38], see also [72], the motivic nearby cycle of f as an element $\psi_f \in \mathbf{K}(\mathrm{Var}^{\hat{\mu}}_{X_\sigma})$. If $x : \mathrm{Spec}(k) \to X_\sigma$ is a closed point of X_σ, fiber product induces a morphism $x^* : \mathbf{K}(\mathrm{Var}^{\hat{\mu}}_{X_\sigma}) \to \mathbf{K}(\mathrm{Var}^{\hat{\mu}}_k)$, and $\psi_{f,x} = x^*\psi_f \in \mathbf{K}(\mathrm{Var}^{\hat{\mu}}_k)$ is the motivic Milnor fiber of f at x.

Denef and Loeser justify their definition by showing that known additive invariants associated to the classical nearby cycle functor can be recovered from ψ_f and $\psi_{f,x}$, the Euler characteristic for example.

Ivorra and Sebag study a new instance of such a principle in [59] where they show (with our notations) that $\chi_{X_\sigma}(\psi_f) = [\Psi_f \mathbb{1}] \in \mathbf{K}(\mathrm{SH}(X_\sigma))$, where Ψ_f is the motivic nearby cycle functor constructed by Ayoub in [2, Chapitre 3]. Literally speaking they only prove it in $\mathbf{K}(\mathrm{DA}^{\text{ét}}(X_\sigma, \mathbb{Q}))$, but it is observed in [4, Section 8.2] that their result generalize to $\mathbf{K}(\mathrm{SH}(X_\sigma))$.

It was first observed by Nicaise and Sebag in [81] that one can relate the motivic Milnor fiber to a rigid analytic variety. Consider the morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(\mathbb{A}^1_k)$ induced by $T \in k[T] \mapsto t \in k[[t]]$. Still denote $X \to \mathrm{Spec}(R)$ the base change of f along this morphism, and let $\mathcal{X}$ be the formal t-adic completion of X. For $x \in X_\sigma$ a closed point, set $\mathcal{F}^{\mathrm{an}}_{f,x}$ the tube of $\{x\}$ in $\mathcal{X}$. It is the analytic Milnor fiber.

Ayoub, Ivorra and Sebag show in [4] that

$$[1^* \circ \mathfrak{R}\mathrm{M}^{\vee}_{\mathrm{Rig}}(\mathcal{F}^{\mathrm{an}}_{f,x})] = \chi_k(\psi_{f,x}) \in \mathbf{K}(\mathrm{SH}(k)).$$

In our context, we have $\Theta \circ \mathcal{E}_c \circ \oint \mathcal{F}^{\mathrm{an}}_{f,x} = \psi_{f,x} \in \mathbf{K}(\mathrm{Var}^{\hat{\mu}}_k)$, we can see it either by a direct computation using resolution of singularities as in [80] and [79] or by adapting results by Hrushovski and Loeser [55]. Now Theorem 3.1.2 shows that $\chi_{\mathrm{Rig}}(\mathcal{F}^{\mathrm{an}}_{f,x}) = [\mathrm{M}^{\vee}_{\mathrm{Rig}}(\mathcal{F}^{\mathrm{an}}_{f,x})]$ hence by Theorem 3.1.1,

$$[\mathfrak{R}\mathrm{M}^{\vee}_{\mathrm{Rig}}(\mathcal{F}^{\mathrm{an}}_{f,x})] = \chi_{\hat{\mu}}(\psi_{f,x}) \in \mathbf{K}(\mathrm{QUSH}(k)).$$

We then have refined the result of Ayoub, Ivorra and Sebag to an equivariant setting.

The paper is organized as follows. See the beginning of each section for the precise content. Section 3.2 is devoted to what we need from Hruskovski and Kazhdan's motivic integration. In Section 3.3, we settle what we will use on motives, rigid analytic geometry and rigid motives. In Section 3.4 we build the realization map χ_{Rig} and prove Theorem 3.1.1. The last Section 3.5 is devoted to duality and the proof of Theorem 3.1.2.

3.2 Premilinaries on motivic integration

In this section we will introduce Hrushovski and Kazhdan's theory of motivic integration in Section 3.2.1 and use it to define two maps from the Grothendieck group of semi-algebraic sets over K to the equivariant Grothendieck group of varieties over k in Section 3.2.2.

3.2.1 Recap on Hrushovski and Kazhdan's integration in valued fields

We outline here the construction of Hrushovski and Kazhdan's motivic integration [54], focusing on the universal additive invariant since this is the only part that we will use. See also the papers [99] and [100] by Yin who gives an account of the theory in ACVF.

We will work in the first order theory ACVF of algebraically closed valued fields of equicharacteristic zero in the two-sorted language $\mathcal{L}$. The two sorts are VF and RV. We put the ring language on VF, with symbols $(0, 1, +, -, \cdot)$, on RV we put the group language $(\cdot, ()^{-1})$, a unary predicate $\mathbf{k}^\times$ for a subgroup, and operations $+ : \mathbf{k}^2 \to \mathbf{k}$ where $\mathbf{k}$ is the union of $\mathbf{k}^\times$ and a symbol 0. We add also a unary function $\mathrm{rv} : \mathrm{VF}^\times = \mathrm{VF} \setminus \{0\} \to \mathrm{RV}$.

We will also consider the imaginary sort Γ defined by the exact sequence

$$1 \to \mathbf{k}^\times \to \mathrm{RV} \to \Gamma \to 0,$$

together with maps $\mathrm{v}_{\mathrm{rv}} : \mathrm{RV} \to \Gamma$ and $\mathrm{v} : \mathrm{VF}^\times \to \Gamma$.

If L is a valued field, with valuation ring $\mathcal{O}_L$ and maximal ideal $\mathcal{M}_L$, define an $\mathcal{L}$-structure by $\mathrm{VF}(L) = L$, $\mathrm{RV}(L) = L^\times/(1 + \mathcal{M}_L)$, $\mathbf{k}(L) = \mathcal{O}_L/\mathcal{M}_L$, $\Gamma(L) = L^\times/\mathcal{O}_L^\times$. Note

that the valuation ring is definable in this language because $\mathcal{O}_L^\times = \mathrm{rv}^{-1}(\mathbf{k}^\times(L))$.

Fix a field k of characteristic zero containing all roots of unity and set $K = k((t))$. View K as a fixed base structure, for the rest of the paper, we will only consider $\mathcal{L}(K)$-structures, where $\mathcal{L}(K)$ is the language obtained by adjoining to $\mathcal{L}$ constants symbols for elements of K. Any valued field extending K can be interpreted as a $\mathcal{L}(K)$-structure. Denote ACVF_K the $\mathcal{L}(K)$-theory of such algebraically closed valued fields. The theory ACVF_K admits quantifier elimination in the language $\mathcal{L}(K)$. Quantifier elimination was first proven by Robinson using a two sorted language, with one sort VF and one sort Γ for the value group, see for example [98].

We will use the notation $\mathrm{VF}^\bullet$ for VF^n for some n. The $\mathcal{L}(K)$-definable subsets of $\mathrm{VF}^\bullet$ are semi-algebraic sets, that is boolean combinations of sets of the form

$$\{x \in \mathrm{VF}^n \mid \mathrm{v}(f(x)) \geq \mathrm{v}(g(x))\},$$

where f and g are polynomials with coefficients in K. Observe that constructible sets are semi-algebraic, since one can take $g = 0$ in the definition.

Denote by $\mathbf{K}(\mathrm{VF}_K)$ the free group of $\mathcal{L}(K)$-definable subsets of $\mathrm{VF}^\bullet$, with the following relations :

- $[X] = [Y]$ if there is a semi-algebraic bijection $X \to Y$
- $[X] = [U] + [V]$ if X is a the disjoint union $X = U \dot\cup V$.

Cartesian product endows $\mathbf{K}(\mathrm{VF}_K)$ with a ring structure.

Remark 3.2.1. Note that this framework allows us to consider general semi-algebraic subsets of K-varieties as studied for example by Florent Martin in [76]. We say that S is a semi-algebraic subset X^{an}, for X a k-scheme, if S is a finite union $S = \cup S_i$ such that for every i, there is an open affine subset $U_i = \mathrm{Spec}(A_i)$ of U such that $S_i \subseteq U_i^{\mathrm{an}}$ is defined in U_i^{an} by boolean combination of subsets of the form $\{y \in U_i^{\mathrm{an}} \mid \mathrm{v}(f(y)) \leq r \cdot \mathrm{v}(g(y))\}$, with $f, g \in A_i$, $r \in \mathbb{Q}$. Hence we can consider its class $[S] \in \mathbf{K}(\mathrm{VF}_K)$.

Remark 3.2.2. Hrushovski and Kazdhan use a slightly different definition for $\mathbf{K}(\mathrm{VF}_K)$. They define it as the group generated by isomorphism classes of definable sets $X \subseteq \mathrm{VF}^\bullet \times \mathrm{RV}^\bullet$, such that for some $n \in \mathbb{N}$, there is some definable function $f : X \to \mathrm{VF}^n$ with finite fibers, with cut-and-paste relations (the function f is not part of the data). We can show that for such an X, there is some definable $X' \subseteq \mathrm{VF}^\bullet$, with a definable bijection $X \simeq X'$, see [54, Lemma 8.1]. Hence both definitions leads to the same group. Nevertheless, this alternative presentation is very usefull for defining motivic integration, since it amounts to relate $\mathbf{K}(\mathrm{VF}_K)$ and some group related to definable sets in the sort RV, that we will now define.

Define RV_K to be the category of objects $Y \subseteq \mathrm{RV}$, with definable functions as morphisms. The category $\mathrm{RV}_K[n]$ is the category of pairs (Y, f), with $Y \subseteq \mathrm{RV}^\bullet$ definable

and $f : Y \to \mathrm{RV}^n$ a definable finite-to-one function. An morphism between (Y, f) and (Y', f') is a definable function $g : Y \to Y'$. The category RES_K is the full subcategory of RV_K whose objects Y satisfy $\mathrm{v}_{\mathrm{rv}}(Y)$ finite. One defines similarly $\mathrm{RES}_K[n]$ to be the full subcategory of $\mathrm{RV}[n]$ whose objects (Y, f) satisfy $\mathrm{v}_{\mathrm{rv}}(Y)$ finite.

Note that the definition of morphisms in $\mathrm{RV}_K[n]$ implies that $\mathrm{RV}_K[n]$ is equivalent to the category of definable sets $X \subseteq \mathrm{RV}^\bullet$ such that there exist a definable function $f : X \to \mathrm{RV}^n$ with finite fibers.

From those categories one form the graded categories

$$\mathrm{RV}_K[\leq n] := \coprod_{0 \leq i \leq n} \mathrm{RV}_K[i],$$

$$\mathrm{RV}_K[*] := \coprod_{i \in \mathbb{N}} \mathrm{RV}_K[i],$$

$$\mathrm{RES}_K[\leq n] := \coprod_{0 \leq i \leq n} \mathrm{RES}_K[i],$$

$$\mathrm{RES}_K[*] := \coprod_{i \in \mathbb{N}} \mathrm{RES}_K[i].$$

For later purpose, we will also need a category related to the value group. One defines $\Gamma[n]$ to be the category with objects subsets of Γ^n defined by piecewise linear equations and inequations with $\mathbb{Z}$-coefficients and parameters in $\mathbb{Q}$. A morphism between Y and Y' is a bijection defined piecewise by composite of $\mathbb{Q}$-translations and $\mathrm{GL}_n(\mathbb{Z})$ morphisms. From this one forms

$$\Gamma[*] := \coprod_{n \in \mathbb{N}} \Gamma[n].$$

One defines also $\Gamma^{\mathrm{fin}}[n]$ and $\Gamma^{\mathrm{fin}}[*]$ to be the full subcategories of $\Gamma[n]$ and $\Gamma[*]$ whose objects are finite.

Each of these categories $\mathcal{C}$ has disjoint union, induced by disjoint union of definable sets. We can form the associated Grothendieck group $\mathbf{K}(\mathcal{C})$. It is the abelian group generated by isomorphism classes of objects of $\mathcal{C}$, with relations induced by disjoint union.

Let us remark that for a fixed definable set $X \subseteq \mathrm{RV}^m$, we can view X as an object in $\mathrm{RV}_K[n]$, for any $n \geq m$. Hence for each $n \geq m$, X induce a class denoted $[X]_n \in \mathbf{K}(\mathrm{RV}_K[n])$. We have

$$[X]_n \in \mathbf{K}(\mathrm{RV}_K[n]) \subset \mathbf{K}(\mathrm{RV}_K[*]) = \bigoplus_{k \geq 0} \mathbf{K}(\mathrm{RV}_K[k]).$$

If X is non-empty, we then have $[X]_n \neq [X]_{n'}$ for $n \neq n'$.

The Cartesian product induces ring structure on $\mathbf{K}(\mathrm{RV}_K)$ and $\mathbf{K}(\mathrm{RES}_K)$. If $(X, f) \in \mathrm{RV}[n]$ and $(X', f') \in \mathrm{RV}[n']$, then $(X \times X', f \times f') \in \mathrm{RV}[n + n']$, inducing a structure of

graded ring on $\mathbf{K}(\mathrm{RV}_K[*])$ and $\mathbf{K}(\mathrm{RES}_K[*])$.

Let $(X, f) \in \mathrm{RV}_K[n]$. Define $\mathfrak{L}(X, f)$ as a fiber product

$$\mathfrak{L}(X, f) = \{(x, y) \in \mathrm{VF}^n \times X \mid \mathrm{rv}(x) = f(y)\}.$$

As f is finite-to-one, the projection of $\mathfrak{L}(X, f)$ to VF^n is finite-to-one, hence we can view it as an object in VF_K by Remark 3.2.2.

It turns out that if $(X, f), (X', f') \in \mathrm{RV}_K[n]$, with a definable bijection $X \simeq X'$, then there is a definable bijection $\mathfrak{L}(X, f) \simeq \mathfrak{L}(X', f')$ by [54, Proposition 6.1], hence we have a ring morphism

$$\mathbf{K}(\mathrm{RV}_K[*]) \to \mathbf{K}(\mathrm{VF}_K).$$

For example, $\mathfrak{L}[1]_n$ is the class of an open polydisc of dimension n.

Hrushovski and Kazhdan show that $\mathfrak{L}$ is surjective and study the kernel. Let $\mathrm{RV}^{>0} = \{x \in \mathrm{RV} \mid v_{\mathrm{rv}}(x) > 0\}$. Then $\mathfrak{L}[\mathrm{RV}^{>0}]_1 = [\{x \in \mathrm{VF}^\times \mid \mathrm{v}(x) > 0\}]$, the open unit ball without zero. Hence in $\mathbf{K}(\mathrm{VF}_K)$, we have

$$\mathfrak{L}[\mathrm{RV}^{>0}]_1 + \mathfrak{L}[1]_0 = \mathfrak{L}[1]_1.$$

This relation generate the whole kernel of $\mathfrak{L}$. Denote by I_{sp} the ideal of $\mathbf{K}(\mathrm{RV}_K[*])$ generated by $[\mathrm{RV}^{>0}] + [1]_0 - [1]_1$.

The main theorem of [54] is the following.

Theorem 3.2.3. *The morphism $\mathfrak{L}$ is surjective and its kernel is I_{sp}.*

Denote by $\oint$ its inverse :

$$\oint : \mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}}.$$

This morphism is what we call (additive) motivic integration. Hrushovski and Kazhdan prove this isomorphism at the level of semi-rings, but we will not need it here. They also show that we can add volume forms to the various categories and still get an isomorphism.

Remark 3.2.4. We will also consider the theory $\mathrm{ACVF}_K^{\mathrm{an}}$ in the language $\mathcal{L}_{\mathrm{an}}(K)$. This language is an enrichment of $\mathcal{L}$ where we add symbols for restricted analytic functions with coefficients in K, see [69] and [20] for details. A maximally complete algebraically closed valued field containing K can be enriched as an $\mathcal{L}_{\mathrm{an}}(K)$-structure. Denote $\mathrm{ACVF}_K^{\mathrm{an}}$ their $\mathcal{L}_{\mathrm{an}}(K)$-theory.

We shall refer to $\mathcal{L}_{\mathrm{an}}(K)$-definable subsets of $\mathrm{VF}^\bullet$ as subanalytic sets. We can form similarly the Grothendieck ring of sub-analytic sets $\mathbf{K}(\mathrm{VF}_K^{\mathrm{an}})$.

As $\mathrm{ACVF}_K^{\mathrm{an}}$ is an enrichment of ACVF_K, we have a canonical map

$$\mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{VF}_K^{\mathrm{an}}),$$

which turns out to be an isomorphism.

Hrushovski and Kazhdan establish the isomorphism for any first order theory T which is V-minimal. The theory $\mathrm{ACVF}^{\mathrm{an}}$ being an example of such a theory, we get also an isomorphism

$$\oint^{\mathrm{an}} : \mathbf{K}(\mathrm{VF}_K^{\mathrm{an}}) \to \mathbf{K}(\mathrm{RV}_K^{\mathrm{an}}[*])/\mathrm{I}_{\mathrm{sp}}.$$

Quantifier elimination shows that $\mathbf{K}(\mathrm{RV}_K^{\mathrm{an}}[*]) \simeq \mathbf{K}(\mathrm{RV}_K[*])$, hence in particular $\mathbf{K}(\mathrm{VF}_K) \simeq \mathbf{K}(\mathrm{VF}_K^{\mathrm{an}})$.

This isomorphism is also valid at the level of semi-rings, showing in particular that for any subanalytic bijection between two semi-algebraic sets, we can find a semi-algebraic bijection between them.

The above isomorphism allows to consider the class of any subanalytic set in $\mathbf{K}(\mathrm{VF}_K)$. From now on, we will implicitly use it when referring to classes of subanalytic sets. This convention is for example used in the statement of Theorem 3.1.1.

3.2.2 Landing in $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$

Our goal here is to relate the target ring of motivic integration $\mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}}$ to something more algebraic, namely the Grothendieck group of k-varieties equipped with a $\hat{\mu}$-action, where $\hat{\mu} = \lim_{\underset{n}{\leftarrow}} \mu_n$ and μ_n the group of n-th roots of unity.

Let first show that $\mathbf{K}(\mathrm{RV}_K[*])$ splits into a RES part and a Γ part.

The map $Y \in \Gamma[n] \mapsto \mathrm{v}_{\mathrm{rv}}^{-1}(Y) \in \mathrm{RV}_K[n]$ induces a functor $\Gamma[n] \to \mathrm{RV}_K[n]$, because the morphisms in $\Gamma[n]$ are piecewise $\mathrm{GL}_n(\mathbb{Z})$ transformations, hence lift to $\mathrm{RV}_K[n]$ morphisms. One also gets a functor $\Gamma^{\mathrm{fin}}[n] \to \mathrm{RV}_K[n]$, whose image lies in $\mathrm{RES}_K[n]$. Hence one gets also functor $\Gamma^{\mathrm{fin}}[*] \to \mathrm{RES}_K[*]$. At the level of Grothendieck groups, one then can see $\mathbf{K}(\Gamma^{\mathrm{fin}}[n])$ as a subgroup of $\mathbf{K}(\Gamma[n])$ and $\mathbf{K}(\mathrm{RES}_K[n])$. Hence the map $\mathbf{K}(\mathrm{RES}_K[*]) \otimes \mathbf{K}(\Gamma[*]) \to \mathbf{K}(\mathrm{RV}_K[*])$ defined by $[X] \otimes [Y] \mapsto [X \times \mathrm{v}_{\mathrm{rv}}^{-1}(Y)]$ induces a map

$$\mathbf{K}(\mathrm{RES}_K[*]) \otimes_{\mathbf{K}(\Gamma^{\mathrm{fin}}[*])} \mathbf{K}(\Gamma[*]) \to \mathbf{K}(\mathrm{RV}_K[*]).$$

Proposition 3.2.5 ([54, Corollary 10.3])**.** *The map* $\mathbf{K}(\mathrm{RES}_K[*]) \otimes_{\mathbf{K}(\Gamma^{\mathrm{fin}}[*])} \mathbf{K}(\Gamma[*]) \to \mathbf{K}(\mathrm{RV}_K[*])$ *is an isomorphism of rings.*

As the theory of Γ is o-minimal, one can use o-minimal Euler characteristic to get an additive map $\mathrm{eu} : \mathbf{K}(\Gamma[n]) \to \mathbb{Z}$. Any $X \subseteq \Gamma^n$ can be finitely partitioned into pieces definably isomorphic to open cubes $\prod_{i=1,\dots,k}(\alpha_i, \beta_i)$, with $\alpha_i, \beta_i \in \Gamma \cup \{-\infty, +\infty\}$. One sets $\mathrm{eu}((\alpha, \beta)^k) = (-1)^k$ and then defines $\mathrm{eu}(X)$ by additivity. One can show that this does not depends on the chosen partition of X, see [95, Chapter 4]. One can also show that when $M \to +\infty$, $\mathrm{eu}(X \cap [-M, M]^n)$ stabilizes and one defines the bounded Euler characteristic to be

$$\mathrm{eu}_c(X) := \lim_{M \to +\infty} \mathrm{eu}(X \cap [-M, M]^n).$$

The Euler characteristics eu and eu_c do coincide on bounded sets, but not in general. For example, $eu((0,+\infty)) = -1$ but $\mathrm{eu}_c((0,+\infty)) = 0$.

Using those Euler characteristics, one can now get rid of the Γ-part in $\mathbf{K}(\mathrm{RV}_K[*])$.

Recall that I_{sp} is the ideal of $\mathbf{K}(\mathrm{RV}_K[*])$ spanned by the element $[1]_1 - [1]_0 - [\mathrm{RV}^{>0}]_1$. For $a \in \mathbb{Q}$, denote also $e_a = [\mathrm{v}_{\mathrm{rv}}^{-1}(a)]_1 \in \mathbf{K}(\mathrm{RES}_K[1])$. Let $!\mathrm{I}$ the ideal of $\mathbf{K}(\mathrm{RES}_K)$ spanned by all differences $e_a - e_0$ and denote $!\mathbf{K}(\mathrm{RES}_K) := \mathbf{K}(\mathrm{RES}_K)/!\mathrm{I}$. Define also $\mathbb{L} = [\mathbb{A}^1_k]$.

Proposition 3.2.6 ([54, Theorem 10.5 (2)]). *There is a ring morphism*

$$\mathcal{E} : \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}} \to !\mathbf{K}(\mathrm{RES}_K)[\mathbb{L}^{-1}],$$

with $\mathcal{E}([X]_k) = [X]/\mathbb{L}^k$ *for* $X \in \mathrm{RES}_K[n]$, $\mathcal{E}([\mathrm{RV}^{>0}]_1) = [\mathbb{G}_m]/\mathbb{L}$.

Proof. From the ring isomorphism $\mathbf{K}(\mathrm{RES}_K[*]) \otimes_{\mathbf{K}(\Gamma^{\mathrm{fin}}[*])} \mathbf{K}(\Gamma[*]) \to \mathbf{K}(\mathrm{RV}_K[*])$, it suffices to define the image of $a \otimes b$, for $a \in \mathbf{K}(\mathrm{RES}_K[r])$ and $b \in \mathbf{K}(\Gamma[s])$. We set $\mathcal{E}(a \otimes b) = \mathrm{eu}(b)a \cdot [\mathbb{G}_m]^s/\mathbb{L}^{r+s} \in !\mathbf{K}(\mathrm{RES}_K)[\mathbb{L}^{-1}]$. As the product is defined by Cartesian product, we check that this indeed defines a ring morphism. If $a \in \mathbf{K}(\Gamma^{\mathrm{fin}}[*])$, the relations involved in $!I$ show that $\mathcal{E}(a \otimes 1) = \mathcal{E}(1 \otimes a)$, hence this defines a map $\mathbf{K}(\mathrm{RES}_K[*]) \otimes_{\mathbf{K}(\Gamma^{\mathrm{fin}}[*])} \mathbf{K}(\Gamma[*]) \to !\mathbf{K}(\mathrm{RES}_K)[\mathbb{L}^{-1}]$. Moreover, as

$$\mathcal{E}([\mathrm{RV}^{>0}]_1) + \mathcal{E}([1]_0) = -\frac{[\mathbb{G}_m]}{\mathbb{L}} + 1 = \mathbb{L}^{-1} = \mathcal{E}([1]_1),$$

the generator of I_{sp} is sent to 0 hence it induces a map

$$\mathcal{E} : \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}} \to !\mathbf{K}(\mathrm{RES}_K)[\mathbb{L}^{-1}].$$

□

Proposition 3.2.7 ([54, Theorem 10.5 (4)]). *There is a ring morphism*

$$\mathcal{E}_c : \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}} \to !\mathbf{K}(\mathrm{RES}_K),$$

with $\mathcal{E}_c([X]_k) = [X]$ *for* $X \in \mathrm{RES}_K[n]$ *and* $\mathcal{E}_c(\mathrm{v}^{-1}(\Delta)) = \mathrm{eu}_c(\Delta)$.

Proof. From the ring isomorphism $\mathbf{K}(\mathrm{RES}_K[*]) \otimes_{\mathbf{K}(\Gamma^{\mathrm{fin}}[*])} \mathbf{K}(\Gamma[*]) \to \mathbf{K}(\mathrm{RV}_K[*])$, it suffices to define the image of $a \otimes b$, for $a \in \mathbf{K}(\mathrm{RES}_K[r])$ and $b \in \mathbf{K}(\Gamma[s])$. We set $\mathcal{E}_c(a \otimes b) = \mathrm{eu}(b)a \cdot [\mathbb{G}_s]^s \in !\mathbf{K}(\mathrm{RES}_K)$. As the product is defined by Cartesian product, we check that this indeed defines a ring morphism. If $a \in \mathbf{K}(\Gamma^{\mathrm{fin}}[*])$, the relations involved in $!I$ show that $\mathcal{E}_c(a \otimes 1) = \mathcal{E}_c(1 \otimes a)$, hence this defines a map $\mathbf{K}(\mathrm{RES}_K[*]) \otimes_{\mathbf{K}(\Gamma^{\mathrm{fin}}[*])} \mathbf{K}(\Gamma[*]) \to !\mathbf{K}(\mathrm{RES}_K)$. Moreover, as $\mathcal{E}_c([\mathrm{RV}^{>0}]_1) = 0$,

$$\mathcal{E}_c([\mathrm{RV}^{>0}]_1 + [1]_0) = 1 = \mathcal{E}_c([1]_1),$$

the generator of I_{sp} is sent to 0 hence it induces a map

$$\mathcal{E}_c : \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}} \to !\mathbf{K}(\mathrm{RES}_K).$$

□

Remark 3.2.8. If one mods out by $[\mathbb{A}^1] - 1$, then the two morphisms $\mathcal{E}$ and $\mathcal{E}_c$ coincide. However, in our situation, one cannot change one for the other.

Let μ_n the group of n-th roots of unity and $\hat{\mu} = \varprojlim_{n} \mu_n$. Define $\mathrm{Var}_k^{\hat{\mu}}$ to be the category of quasi-projective k-varieties equipped with a good $\hat{\mu}$-action, that is, a $\hat{\mu}$-action that factors through some μ_n-action. Since the varieties are assumed to be quasi-projective, such an action is automatically good in the usal sense, *i.e.* the orbit of every point is contained in an affine open subset stable by the action. Let $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat})$ the abelian group generated by isomorphism classes of quasi-projective k-varieties X equipped with good $\hat{\mu}$-action, with the scissors relations. Let $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$ the quotient of $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat})$ by additional relations $[(V,\rho)] = [(V,\rho')]$ if V is a finite dimensional k-vector space and ρ, ρ' two good linear $\hat{\mu}$-actions on V. Note that Cartesian product induces ring structures on $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat})$ and $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$.

We want to define a map $\mathbf{K}(\mathrm{RES}_K) \to \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat})$. Fix a set of parameters $t_a \in K((t))^{\mathrm{alg}}$ for $a \in \mathbb{Q}$ such that $t_1 = t$ and $t_{ab} = t_b^a$ for $a \in \mathbb{N}^*$ and denote $\mathbf{t}_a := \mathrm{rv}(t_a)$. Set $V_\gamma^* = \mathrm{v}^{-1}(\gamma)$ and $V_\gamma = V_\gamma^* \cup \{0\}$. If $X \in \mathrm{RES}$, then $X \subseteq \mathrm{RV}^n$ and the image of $\mathrm{v}_{\mathrm{rv}} : X \to \Gamma^n$ is finite. Working piecewize we can suppose this image is a singleton. In this case, there are $m, k_1, ..., k_n \in \mathbb{N}^*$ such that $X \subseteq V_{k_1/m} \times ... \times V_{k_n/m}$. The function $g : (x_1, ..., x_n) \in X \mapsto (x_1/\mathbf{t}_{k_1/m}, ..., x_n/\mathbf{t}_{k_n/m}) \in k^n$ is $K((t^{1/m}))$-definable and its image $g(X)$ inherits a μ_n-action from the one on X. Moreover $g(X)$ is a definable subset of k^n, hence constructible by quantifier elimination. So we get a map $\Theta : \mathbf{K}(\mathrm{RES}_K) \to \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat})$, and it induces also a map $!\mathbf{K}(\mathrm{RES}_K) \to \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$. Hrushovski and Loeser prove in [55] the following proposition.

Proposition 3.2.9. *The ring morphisms*

$$\Theta : \mathbf{K}(\mathrm{RES}_K) \to \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat})$$

and

$$\Theta : !\mathbf{K}(\mathrm{RES}_K) \to \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$$

are isomorphisms.

Proof. As a linear μ_m-action on k^n is diagonalisable, the relations added when dropping the $\flat$ corresponds to $!I$, hence it suffices to show the first isomorphism. Let first prove that it is injective. By scissors relations, it suffices to show that if $\Theta(X) = \Theta(Y)$ for some $X \subseteq \mathrm{RV}^n$ and $Y \subseteq \mathrm{RV}^{n'}$ such that $\mathrm{v}_{\mathrm{rv}}(X)$ and $\mathrm{v}_{\mathrm{rv}}(Y)$ are singletons, then $[X] = [Y]$.

Consider g, g' as in the definition of Θ, suppose $g(X)$ and $g(Y)$ are equipped with μ_m actions and pick f a μ_m-invariant definable bijection between $g(X)$ and $g'(Y)$. Then $g'^{-1} \circ f \circ g$ is a $k((t^{1/m}))$-definable bijection between X and Y that is invariant under $\mu_n = \mathrm{Gal}(k((t^{1/m}))/k((t)))$-action, hence it is $k((t))$-definable, that is $[X] = [Y]$.

Now we prove surjectivity. Fix X a quasi-projective k-variety with a μ_n-action. By induction on the dimension of X, it suffices to find $W \in \mathrm{RES}_K$ with $\Theta(W)$ dense in X. Up to partitioning, one can moreover assume the kernel of the action is trivial, that is the μ_n action is faithful (for possibly a smaller m). Consider the quotient $U = X/\mu_m$. Then the Galois group of $k(X)$ over $k(U)$ is μ_m and by Kummer theory there is an $f \in k(U)$ such that $k(X) = k(U)(f^{1/m})$. Up to neglecting a part of smaller dimension, one can moreover assume that f is regular and non-vanishing, that is $f \in \Gamma(U, \mathcal{O}_U^\times)$. If we set $W = \left\{(u, v) \in U \times V_{1/m} \mid v^m = \mathbf{t} f(u)\right\}$, then we have $\Theta(W) = X$. □

Set $\mathbf{t} = \mathrm{rv}(t)$. If $U \subseteq \mathbb{A}_k^n$ is a smooth subvariety of $\mathbb{A}_k^n$, $f \in \Gamma(U, \mathcal{O}_U^\times)$ an invertible regular function on U and $r \in \mathbb{N} \setminus \{0\}$, denote

$$Q_r^{\mathrm{RV}}(U, f) = \left\{(u, v) \in V_0^n \times V_{1/r} \mid u \in U, v^r = \mathbf{t} f(u)\right\}.$$

Corollary 3.2.10. *The ring $\mathbf{K}(\mathrm{RES}_K)$ is generated by classes of $Q_r^{\mathrm{RV}}(U, f)$. The ring $\mathbf{K}(\mathrm{RES}_K[*])$ is generated by classes of $[Q_r^{\mathrm{RV}}(U, f)]_n \in \mathbf{K}(\mathrm{RES}_K[n])$.*

Corollary 3.2.11. *There is an map*

$$\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{Var}_{\mathbb{G}_{mk}}).$$

Proof. From the proof of Proposition 3.2.9, $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat})$ is generated by classes of the form

$$Y = X[V]/(V^m - f),$$

for X a k-variety, $f \in \Gamma(X, \mathcal{O}_X^\times)$ with the μ_m-action induced by multiplication of V by $\zeta \in \mu_m$. To such an class, one associate the class of

$$Z = X[V, V^{-1}, T, T^{-1}]/(V^m - Tf) \to \mathbb{G}_{mk} = \mathrm{Spec}(k[T, T^{-1}]).$$

in $\mathbf{K}(\mathrm{Var}_{\mathbb{G}_{mk}})$. By induction on the dimension as in the proof of proposition, this leads to a well-defined map

$$\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}^\flat}) \to \mathbf{K}(\mathrm{Var}_{\mathbb{G}_{mk}}).$$

Indeed, given Y as above, X and m are uniquely determined. The function f is only determined up to a factor in ${\mathcal{O}_X^\times}^n$, but all different choices of representatives will lead to isomorphic Z.

Finally, note that the relations added when dropping the flat are in the kernel of the

above map. Indeed, once again, because a linear action of μ_n on k^r is diagonalisable, it suffices to show that the image of $[(k, \mu_n)]$, where μ_n acts on k by multiplication by n-th roots of unity, is independent of n. As 0 is a fixed point, we can restrict the action on $k^\times$. The image in $\mathbf{K}(\mathrm{Var}_{\mathbb{G}_{mk}})$ is then $[\mathrm{Spec}(k[U, U^{-1}, T, T^{-1}, V]/(V^n - TU))]$. But this variety is isomorphic to $\mathrm{Spec}(k[V, V^{-1}, T, T^{-1}])$ over $\mathbb{G}_{mk}$, by the map defined by $U \mapsto V^n T^{-1}$, $V \mapsto V$. □

3.3 Preliminaries on Motives

This section is devoted to recall what we need on motives. After a brief recap on triangulated categories in Section 3.3.1, we will outline the construction of the Λ-linear category of motives in Section 3.3.2, then explain what we need about the six functors formalism in Section 3.3.3. We then built a map from the equivariant Grothendieck group of varieties to the Grothendieck group of quasi-unipotent motives in Section 3.3.4. Finally, we give some background on rigid analytic geometry and formal schemes in Sections 3.3.5 and 3.3.6, before giving the definitions of rigid motives in Section 3.3.7.

3.3.1 Triangulated categories

A triangulated category, as introduced by Verdier in his thesis [96], is an additive category endowed with an autoequivalence, denoted $-[1]$ and called the suspension, and a class of distinguished triangles, of the form

$$A \to B \to C \overset{+1}{\to},$$

satisfying some axioms.

To every arrow $f : A \to B$, we can find a $g : B \to C$ such that

$$A \to B \to C \overset{+1}{\to}$$

is a distinguished triangle. Such a C is unique up to non-cannonical isomorphism. We call C the (mapping) cone of f, written $\mathrm{Cone}(f)$. We have that f is an isomorphism if and only $\mathrm{Cone}(f) = 0$.

If $\mathcal{T}$ is a triangulated category, and $\mathcal{S}$ a full subcategory closed under suspension, desuspension and stable by cones, Verdier shows that one can define a quotient triangulated category $\mathcal{T}/\mathcal{S}$ which is universal with respect to the following properties :

- There is a canonical triangulated functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ which is the identity on objects.
- For every $A \in \mathcal{S}$, $A \simeq 0$ in $\mathcal{T}/\mathcal{S}$.

The idea is to consider the class of arrows $\mathrm{Ar}(\mathcal{S}) = \{\alpha : A \to B \mid \mathrm{Cone}(\alpha) \in \mathcal{S}\}$ and formally invert them inside $\mathcal{T}$. We call $\mathcal{T}/\mathcal{S}$ the Verdier localization of $\mathcal{T}$ with respect to $\mathcal{S}$.

Assume $\mathcal{T}$ is a triangulated category admitting direct sums indexed by arbitrary sets. An object A of $\mathcal{T}$ is said to be compact if the canonical morphism

$$\hom_{\mathcal{T}}(A, \bigoplus_{s\in S} B_s) \to \bigoplus_{s\in S} \hom_{\mathcal{T}}(A, B_s)$$

is an isomorphism for any set $\{B_s\}_{s\in S}$ of objects of $\mathcal{T}$. Let $\mathcal{T}_{\mathrm{cp}}$ the full subcategory of compact objects of $\mathcal{T}$. It is a triangulated subcategory of $\mathcal{T}$.

If $\mathcal{T}$ is a triangulated category with infinite sums, we define its Grothendieck group $\mathbf{K}(\mathcal{T})$ as the free abelian group generated by isomorphism classes of objects of $\mathcal{T}_{\mathrm{cp}}$ with relations $[B] = [A] + [C]$ for every distinguished triangle

$$A \to B \to C \overset{+1}{\to}.$$

Note that the class $[\mathrm{Cone}(\alpha)] \in \mathbf{K}(\mathcal{T})$ of the cone of an arrow $\alpha : A \to B$ is now canonically defined.

Remark 3.3.1. We must restrict to compact objects when we define $\mathbf{K}(\mathcal{T})$ to prevent the group to be trivial. Indeed, otherwise if A is an object of $\mathcal{T}$, we would have $[\bigoplus_{n\in\mathbb{N}} A] = [A] + [\bigoplus_{n\in\mathbb{N}} A]$ hence $[A] = 0$.

As for every compact object A, the triangle

$$A \to 0 \to A[1] \overset{+1}{\to}$$

is distinguished, $[A[1]] = -[A] \in \mathbf{K}(\mathcal{T})$, hence the suspension is indempotent in $\mathbf{K}(\mathcal{T})$.

Moreover, as we have an distinguished triangle for every $A, B \in \mathcal{T}_{\mathrm{cp}}$

$$A \to A \oplus B \to B \overset{+1}{\to},$$

we have $[A \oplus B] = [A] + [B]$.

If $\mathcal{T}$ is moreover a monoidal triangulated category, then $\mathbf{K}(\mathcal{T})$ inherits of a ring structure induced by tensor product.

3.3.2 Constuction of the stable category of motives

In this section, we decribe briefely the construction of Morel-Voevodsky motives without transfers, together with Grothendieck's six functors formalism as developped by Ayoub [1].

All schemes will be separated and of finite type. Fix a base scheme S.

Denote $\mathrm{SH}_{\mathfrak{M}}(S)$ the category of motivic sheaves over S for the Nisnevich topology and

coefficients $\mathfrak{M}$, as studied by Ayoub in [2, Définition 4.5.21].

The two main examples are if $\mathfrak{M}$ is the category of simplicial spectra, in which case $\mathrm{SH}_{\mathfrak{M}}(S)$ is the stable homotopy category (without transfers) of Morel-Voevodsky introduced in [78]. The other one is if $\mathfrak{M}$ is the category of complexes of Λ modules, for some ring Λ. In this case we denote $\mathrm{SH}_{\mathfrak{M}}(S) = \mathrm{DA}(S, \Lambda)$. We will only recall the construction of the latter.

Fix also some ring Λ, the ring of coefficients. For any set E, define $\Lambda \otimes E = \bigoplus_{e \in E} \Lambda$.

Definition 3.3.2. A family $\mathcal{R} = (u_i : U_i \to X)_{i \in I}$ is a Nisnevich cover of X if it is an étale cover and for every point $x \in X$, there is some $i \in I$ such that x lift to $x_u \in U_i$ and the induced map of residue fields $k(x) \to k(x_u)$ is an isomorphism.

Nisnevich covers form a Grothendieck topology.

Let Sm_S be the category of smooth S-schemes. Endow Sm_S with the Nisnevich topology, and denote $\mathrm{Shv}_{\mathrm{Nis}}(S, \Lambda)$ the category of Nisnevich sheaves on Sm_S.

For any $X \in \mathrm{Sm}_S$, define $\Lambda(X)$ as the the (Nisnevich) sheaf associated to the presheaf

$$U \in \mathrm{Sm}_S \to \Lambda \otimes \mathrm{Hom}(U, X).$$

In particular, $\Lambda(S)$ is the constant sheaf, we will sometimes denote it $\mathbb{1}_S$.

Let $T = \mathbb{A}^1_S / \mathbb{G}_{mS}$ be the cokernel of the inclusion $\mathbb{G}_{mS} \to \mathbb{A}^1_S$. One should think of T as a pointed sphere, and could use instead $L = \mathbb{P}^1_S / \infty_S$.

Consider the category of T-spectra $\mathrm{Spt}_T(\mathrm{Shv}_{\mathrm{Nis}}(S, \Lambda))$. It is the category whose objects are $\mathcal{E} = (\mathcal{E}_n, \gamma_n)_{n \in \mathbb{N}}$, with $\mathcal{E}_n \in \mathrm{Shv}_{\mathrm{Nis}}(S)$ and $\gamma_n : \mathcal{E}_n \otimes T \to \mathcal{E}_{n+1}$. An arrow between $\mathcal{E} = (\mathcal{E}_n, \gamma_n)_{n \in \mathbb{N}}$ and $\mathcal{E}' = (\mathcal{E}'_n, \gamma'_n)_{n \in \mathbb{N}}$ is a collection of maps $f_n : \mathcal{E}_n \to \mathcal{E}'_n$ such that $\gamma'_n \circ f_n = f_{n+1} \circ \gamma_n$. For $r \in \mathbb{N}$, define the suspension functor

$$\mathrm{Sus}^r_T : \mathrm{Shv}_{\mathrm{Nis}}(S, \Lambda) \to \mathrm{Spt}_T(\mathrm{Shv}_{\mathrm{Nis}}(S, \Lambda))$$

sending a sheaf $\mathcal{F}$ to $\mathrm{Sus}^r_T(\mathcal{F})_n = 0$ if $n < r$ and $\mathrm{Sus}^r_T(\mathcal{F})_n = \mathcal{F} \otimes T^{\otimes n-r}$ otherwise.

The category $\mathrm{Spt}_T(\mathrm{Shv}_{\mathrm{Nis}}(S, \Lambda))$ is abelian, hence we can consider its derived category $\mathbf{D}(\mathrm{Spt}_T(\mathrm{Shv}_{\mathrm{Nis}}(S, \Lambda))$ which is triangulated.

Consider $\mathcal{S}$, the smallest triangulated full subcategory of $\mathbf{D}(\mathrm{Spt}_T(\mathrm{Shv}_{\mathrm{Nis}}(S, \Lambda))$ closed under direct sums and containing for any smooth S-scheme U and $r \in \mathbb{N}$ the complexes of spectra

$$[\cdots \to 0 \to \mathrm{Sus}^r_T(\mathbb{A}^1 \times U) \to \mathrm{Sus}^r_T(U) \to 0 \to \dots]$$

and

$$[\cdots \to 0 \to \mathrm{Sus}^r_T(U \otimes T) \to \mathrm{Sus}^{r+1}_T(U) \to 0 \to \dots].$$

The first condition is refereed as $\mathbb{A}^1$-invariance, the second one as T-stabilization.

The category $\mathrm{DA}(S,\Lambda)$ is the Verdier localization of $\mathbf{D}(\mathrm{Spt}_T(\mathrm{Shv}_{\mathrm{Nis}}(S,\Lambda)))$ with respect to $\mathcal{S}$. It is the category of motives with Λ-coefficients.

3.3.3 Six functors formalism

We go back now to the general case and expose some of the properties of $\mathrm{SH}_{\mathfrak{M}}(S)$.

If X is a smooth S-scheme, the homological motive of X is $\mathrm{M}_S(X) = \mathrm{Sus}^0_T(X \otimes \mathbb{1}) \in \mathrm{SH}_{\mathfrak{M}}(S)$.

For any $A \in \mathrm{SH}_{\mathfrak{M}}(S)$, the tensor product $- \otimes A$ admits a right adjoint $\underline{\mathrm{Hom}}(A,-)$, the internal Hom. Denote by $\mathrm{M}^\vee_S(X) = \underline{\mathrm{Hom}}(\mathrm{M}_S(X), \mathbb{1}_S)$. It is the cohomological motive of X.

For $r \in \mathbb{N}$, denote by $\mathbb{1}_S(r) := \mathrm{Sus}_T(T^{\otimes r})[-2r]$ and $-(-r)$ the functor $- \otimes \mathbb{1}_S(r)$. It is the Tate twist. As there is a canonnical isomorphism $\mathrm{Sus}^r_T(-) \otimes \mathrm{Sus}^s_T(-) \simeq \mathrm{Sus}^{r+s}_T(- \otimes -)$ in the category $\mathrm{SH}_{\mathfrak{M}}(S)$, we see that $(-r) \circ (-s) = (-r-s)$. Set also $-(r) = - \otimes \mathrm{Sus}^r(\mathbb{1}_S)[2r]$. Thanks to the stabilization, (r) is the inverse of $(-r)$, because $\mathrm{Sus}^0_T(T^{\otimes r}) \otimes \mathrm{Sus}^r_T(\mathbb{1}_S) = \mathrm{Sus}^r_T(T^{\otimes r}) = \mathrm{Sus}^0_T(\mathbb{1}_S)$.

The categories $\mathrm{SH}_{\mathfrak{M}}(-)$ possess various fonctorialities. If $f : X \to Y$ is a morphism of schemes, then the pull-back f^* and the push-forward f_* defined at the level of sheaves induce functors $f^* : \mathrm{SH}_{\mathfrak{M}}(Y) \to \mathrm{SH}_{\mathfrak{M}}(X)$ and $f_* : \mathrm{SH}_{\mathfrak{M}}(X) \to \mathrm{SH}_{\mathfrak{M}}(Y)$, f_* is a right adjoint to f^*. If f is smooth, then f^* also possesses a left adjoint denoted $f_\sharp$, moreover, for any commutative diagram

$$\begin{tikzcd} \bullet \arrow[r, "g'"] \arrow[d, "f'"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "g"'] & \bullet, \end{tikzcd}$$

there is a 2-isomorphism (*i.e.* an invertible natural transformation) between $f'_\sharp g'^*$ and $g^* f_\sharp$.

Locality. Let X some S-scheme, $i : Z \to X$ a closed immersion and $j : U \to X$ the complementary open immersion. Then the pair (i^*, j^*) is conservative, that is for any $A \in \mathrm{SH}_{\mathfrak{M}}(X)$, $A = 0$ if and only if $i^*A = 0$ and $j^*A = 0$. Moreover, the counit of the adjuction $i^*i_* \to \mathrm{id}$ is an isomorphism.

From this property one can deduce that for any $A \in \mathrm{SH}_{\mathfrak{M}}(X)$, there is a distinguished triangle

$$j_\sharp j^* A \to A \to i_* i^* A \overset{+1}{\to} . \tag{3.3.1}$$

In this situation, we set $i_! = i_*$ and $j_! = j_\sharp$, such that the triangle 3.3.1 becomes

$$j_! j^* A \to A \to i_! i^* A \overset{+1}{\to} . \tag{3.3.2}$$

We also set $j^! = j^*$ and $i^!(A) = i^* \mathrm{Cone}(A \to j_* j^* A)[-1]$ (one needs to check that this construction is functorial), such that we have the dual triangle

$$i_* i^! A \to A \to j_* j^! A \overset{+1}{\to} . \tag{3.3.3}$$

Thom equivalences. Let $\mathcal{E}$ be a locally free sheaf over X. Identifying $\mathcal{E}$ with the total space of the vector bundle associated to it, set $p : \mathcal{E} \to X$ the projection and $s : X \to \mathcal{E}$ the zero section. Define $\mathrm{Th}(\mathcal{E}) = p_\sharp \circ s_*$ the Thom equivalence associated to $\mathcal{E}$. Thanks to the stabilization, it is an autoequivalence of $\mathrm{SH}_{\mathfrak{M}}(X)$, hence we can consider an quasi-inverse $\mathrm{Th}^{-1}(\mathcal{E})$. The exact triangle given by locality also show that $\mathrm{Th}(\mathcal{E})(\mathbb{1}_X) = \mathcal{E}/(\mathcal{E}\backslash s(X))$, the maybe more familiar definition of Thom space.

If $\mathcal{E} \simeq \mathcal{O}_S^n$ is free, then $\mathrm{Th}(\mathcal{E})[-2n]$ is the Tate twist (n).

For f smooth, denote Ω_f the sheaf of relative differentials and set $f_! = f_\sharp \circ \mathrm{Th}^{-1}(\Omega_f)$ and $f^! = \mathrm{Th}(\Omega_f) \circ f^*$.

For f quasi-projective, we can factor (non-uniquely) f as $f = g \circ i$ where g is smooth and i is a closed immersion. In this situation, we set $f_! = g_! i_*$ and $f^! = i^! g^!$. One of the main results of [1] is that such a construction is independent of the choice of the factorization $f = g \circ i$.

For any f, $f_!$ is a left adjoint to $f^!$. For f projective, we have $f_! = f_*$.

One can drop the assuption of quasi-projectivity. Using Nagata compactification theorem, if f is a morphism separated of finite type, we can factor f as $f = p \circ j$ with j an open immersion and p proper and define $f_! = p_* \circ j_\sharp$. Once again, one can show that this does not depend on the choice of the compactification, see [14] for details.

Definition 3.3.3. We can now define for any $f : X \to S$ the homological motive of X as

$$\mathrm{M}_S(X) = f_! f^!(\mathbb{1}_S),$$

the cohomological motive of X as

$$\mathrm{M}_S^\vee(X) = f_* f^*(\mathbb{1}_S),$$

the cohomological motive with compact support of X as

$$\mathrm{M}_{S,c}^\vee(X) = f_! f^*(\mathbb{1}_S)$$

and the homological motive with compact support of X as

$$\mathrm{M}_{S,c}(X) = f_* f^!(\mathbb{1}_S).$$

Observe that such a definition is compatible with the one given above for f smooth. Indeed, in that case we have $f_! = f_\sharp \mathrm{Th}^{-1}(\Omega_f)$ and $f^! = \mathrm{Th}(\Omega_f) f^*$ hence $f_! f^! = f_\sharp f^*$.

If f is proper, observe $\mathrm{M}_{S,c}^\vee(X) = \mathrm{M}_S^\vee(X)$ because $f_! = f_*$.

Proposition 3.3.4 ([1, Scholie 1.4.2])**.** *Suppose there is a Cartesian square*

$$\begin{tikzcd} \bullet \arrow[r, "g'"] \arrow[d, "f'"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$$

Then there is a 2-isomorphism between $f'_! g'^*$ and $g^* f_!$.

The following proposition is the projection formula.

Proposition 3.3.5 ([1, Proposition 2.3.40])**.** *Let $f : Y \to X, A \in \mathrm{SH}_{\mathfrak{M}}(X), B \in \mathrm{SH}_{\mathfrak{M}}(Y)$. Then*

$$f_!(f^* A \otimes_Y B) = A \otimes_X f_! B.$$

The subcategory $\mathrm{SH}_{\mathfrak{M}}(S)_{\mathrm{cp}}$ of compact objects of $\mathrm{SH}_{\mathfrak{M}}(S)$ is the smallest triangulated subcategory of $\mathrm{SH}_{\mathfrak{M}}(S)$ stable by direct factors and generated by homological motives of smooth quasi-projective varieties and their twists.

The following motivic realisation has already been considered by Ivorra and Sebag in [59, Lemma 2.1].

Proposition 3.3.6. *Let S be a k-scheme. There is a unique ring morphism*

$$\chi_S : \mathbf{K}(\mathrm{Var}_S) \to \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(S))$$

such that $\chi_S([X]) = \mathrm{M}^\vee_{S,c}(X)$ for any S-scheme $f : X \to S$.

Proof. To show that χ_S is a group morphism well defined on $\mathbf{K}(\mathrm{Var}_S)$, we need to check that for any S-scheme X, if $i : Y \hookrightarrow X$ is a closed immersion, with $j : U \hookrightarrow X$ the open complement immersion, then $[\mathrm{M}_{S,c}(X)] = [\mathrm{M}_{S,c}(Y)] + [\mathrm{M}_{S,c}(U)]$ in $\mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(S))$. For any object $A \in \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(X))$ we have the distinguished triangle

$$j_! j^* A \to A \to i_! i^* A \overset{+1}{\to}$$

By applying $f_!$ to this triangle and substituting A by $f^* \mathbb{1}_S$, we get the required distinguished triangle

$$\mathrm{M}^\vee_{S,c}(U) \to \mathrm{M}^\vee_{S,c}(X) \to \mathrm{M}^\vee_{S,c}(Y) \overset{+1}{\to}.$$

It remains to show that it is compatible with ring structure. Pick $f : X \to S$ and $g : Y \to S$ two S-schemes. Form the fiber product $r : X \times_S Y \to S$. From the exchange property applied to the Cartesian square

$$\begin{tikzcd} X \times_S Y \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, "r"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & S \end{tikzcd}$$

and the projection formula, we have

$$\begin{aligned}\chi_S([X]\cdot[Y]) &= \chi_S([X\times_S Y]) = [\mathrm{M}_{S,c}(X\times_S Y)] = [r_!\mathbb{1}_{X\times_S Y}] = [g_!f'_!g'^*f^*\mathbb{1}_S]\\ &= [g_!g^*f_!f^*\mathbb{1}_S] = [g_!g^*\mathrm{M}_{S,c}(X)] = [g_!(g^*\mathrm{M}_{S,c}(X)\otimes_Y \mathbb{1}_Y)]\\ &= [\mathrm{M}^\vee_{S,c}(X)\otimes_S g_!\mathbb{1}_Y)] = [\mathrm{M}^\vee_{S,c}(X)\otimes_S \mathrm{M}_{S,c}(Y)]\\ &= \chi_S([X])\cdot\chi_S([Y]).\end{aligned}$$

□

Proposition 3.3.7. *Let $f : X \to S$ a smooth morphism of pure relative dimension d. Then*

$$[\mathrm{M}^\vee_{S,c}(X)] = [\mathrm{M}_S(X)(-d)] \in \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(S)).$$

Proof. By definition, $\mathrm{M}^\vee_{S,c}(X) = f_!f^*(\mathbb{1}_S)$ and $f_! = f_*\mathrm{Th}^{-1}(\Omega_f)$. As $\mathrm{M}_{S,c}(-)$ is additive and Ω_f is locally free, we can assume Ω_f is free (of rank d). In that case, $\mathrm{Th}^{-1}(\Omega_f) = (-d)[-2d]$. The result now follows because suspension function is idempotent in the Grothendieck group. □

3.3.4 From $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$ to $\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k))$

Let $X = \mathrm{Spec}(A)$ a k-scheme of finite type, $r \in \mathbb{N}^*$ and $f \in A^\times$. We denote $Q_r^{gm}(X,f)$ the $\mathbb{G}_{mk}$-scheme

$$\mathrm{Spec}(A[T,T^{-1},V]/(V^r - fT)) \to \mathbb{G}_{mk} = \mathrm{Spec}(k[T,T^{-1}]).$$

More generally, we define by gluing for $X = \mathrm{Spec}(A)$ k-scheme of finite type, $r \in \mathbb{N}\setminus\{0\}$, $f \in \Gamma(X,\mathcal{O}_X^\times)$.

$$Q_r^{gm}(X,f) \to \mathbb{G}_{mk}.$$

Let $\mathrm{QUSH}_{\mathfrak{M}}(k)$ the triangulated subcategory with infinite sums of $\mathrm{SH}_{\mathfrak{M}}(\mathbb{G}_{mk})$ spanned by objects $\mathrm{Sus}_T^p(Q_r^{gm}(X,f)\otimes A_{\mathrm{cst}})$ for X smooth k-scheme and $A \in \mathcal{E}$. Here, $\mathcal{E}$ is a set of homotopically compacts objects of $\mathfrak{M}$ generating the homotopy category of $\mathfrak{M}$, see [3, Définition 1.2.31] for details.

Let $q : \mathbb{G}_{mk} \to \mathrm{Spec}(k)$ the structural projection and $1 : \mathrm{Spec}(k) \to \mathbb{G}_{mk}$ its unit section.

Proposition 3.3.8. *There is a unique ring morphism*

$$\chi_{\hat{\mu}} : \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k))$$

such that the class of $X[V]/(V^r - f)$, for X a smooth k-scheme, $f \in \Gamma(X,\mathcal{O}_X^\times)$, $r \in \mathbb{N}\setminus\{0\}$ (with the μ_n action on V), is send to $[\mathrm{M}^\vee_{\mathbb{G}_{mk},c}(Q_r^{gm}(X,f))]$.

Proof. The ring morphism

$$\chi_{\hat{\mu}} : \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{Var}_{\mathbb{G}_{mk}}) \to \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(\mathbb{G}_{mk})).$$

is defined by composition of maps from Corollary 3.2.11 and Proposition 3.3.6.

Hence it suffices to show that the image of this morphism lies in $\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k))$. From the proof of 3.2.9, $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})$ is generated by classes of $X[V, V^{-1}]/V^r = f$ as in the statement of the proposition. Hence it suffices to show that

$$[\mathrm{M}_{\mathbb{G}_{mk},c}(Q_r^{gm}(X, f))] \in \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)).$$

But $\mathrm{QUSH}_{\mathfrak{M}}(k)$ is the triangulated subcategory with infinite sums generated by the set of objects $\mathrm{Sus}_T^p(Q_r^{gm}(X, f) \otimes A_{\mathrm{cst}})$, which is stable by Tate twist, hence by Proposition 3.3.7, $[\mathrm{M}_{\mathbb{G}_{mk},c}(Q_r^{gm}(X, f))] \in \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k))$. □

Lemma 3.3.9. *We have a commutative diagram :*

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[r] \arrow[d, "\chi_{\hat{\mu}}"'] & \mathbf{K}(\mathrm{Var}_k) \arrow[d, "\chi_k"] \\
\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(k)),
\end{tikzcd}$$

where $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{Var}_k)$ *is induced by the forgetful functor and*

$$1^* : \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)) \to \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(k))$$

is the composite

$$\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)) \longrightarrow \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(\mathbb{G}_{mk})) \xrightarrow{1^*} \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(k)).$$

Proof. Recall that the composition of

$$\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{Var}_{\mathbb{G}_{mk}}) \to \xrightarrow{1^*} \mathbf{K}(Vark)$$

is the forgetful map. Hence it suffices to show that the following diagram is commutative :

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_{\mathbb{G}_{mk}}) \arrow[r] \arrow[d, "\chi_{\mathbb{G}_{mk}}"'] & \mathbf{K}(\mathrm{Var}_k) \arrow[d, "\chi_k"] \\
\mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(\mathbb{G}_{mk})) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(k)),
\end{tikzcd}$$

where the upper map is induced by picking the fiber above 1 of a $\mathbb{G}_{mk}$-variety. For $X \in$

$\mathrm{Var}_{\mathbb{G}_{mk}}$, we consider the following Cartesian square :

$$\begin{tikzcd} X' \arrow[r, "1'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ k \arrow[r, "1"'] & \mathbb{G}_{mk}. \end{tikzcd}$$

One needs to show that $1^*\mathrm{M}_{\mathbb{G}_{mk},c}(X) \simeq \mathrm{M}_{k,c}(X')$. By [1, Scholie 1.4.3], there is a 2-isomorphism $f'_!1'^* \cong 1^*f_!$. Hence

$$1^*\mathrm{M}_{\mathbb{G}_{mk},c}(X) = 1^*f_!f^*\mathbb{1}_{\mathbb{G}_{mk}} \simeq f'_!1'^*f^*\mathbb{1}_{\mathbb{G}_{mk}} \simeq f'_!f'^*1^*\mathbb{1}_{\mathbb{G}_{mk}} = \mathrm{M}_{k,c}(X').$$

□

3.3.5 Rigid analytic geometry

We will use the formalism of Tate's rigid geometry. For details and proofs, see [12] and [46].

An ultrametric multiplicative semi-norm (resp. norm) on some commutative ring A is an application

$$|\cdot| : A \to \mathbb{R}_{\geq 0}$$

such that

- $|1| \leq 1$, $|0| = 0$ (resp. $|x| = 0 \iff x = 0$),
- $|x + y| \leq \max\{|x|, |y|\}$,
- $|xy| \leq |x|\,|y|$.

As we will only consider utrametric mutliplicative semi-norms, we will juste call them by semi-norms. Note that to every norm on A we can associate a rank one valuation and conversely.

Denote by $A^\circ = \{a \in A \mid |x| \leq 1\}$, which is a subring of A.

For $(A, |\cdot|)$ a ring equipped with a norm, define the ring of converging power series by

$$A\{T_1, ..., T_n\} := \left\{ \sum_{i=(i_1,...,i_n)\in\mathbb{N}^n} a_i T_1^{i_1} ... T_n^{i_n} \mid \underset{i\to+\infty}{|a_i|} \to 0 \right\}.$$

Because the norm is ultrametric, for $f \in A\{T_1, ..., T_n\}$ and $x \in (A^\circ)^n$ the power series $f(x)$ converges and we have $f(x) \in A$.

Endow $A\{T_1, ..., T_n\}$ with the norm

$$|f| = \left|\sum a_i T^i\right| := \max |a_i|.$$

We say that $(A, |\cdot|)$ is complete if A is complete for the distance induced by $|\cdot|$.

In particular, if A is complete, then $A\{T\}$ is also complete.

For any $(A, |\cdot|)$, we have a canonical map $(A, |\cdot|) \to (\hat{A}, |\cdot|)$ with $\hat{A}$ complete.

If B, B' are (normed) A-algebras, denote by $B\hat{\otimes}_A B'$ the completion of $B \otimes_A B'$.

For T and T' some tuples of variables and A a complete ring, observe that $A\{T, T'\} = A\{T\}\hat{\otimes}_A A\{T'\}$.

Recall that $K = k((t))$ for some field k of characteristic zero. In particular, it is complete for the norm associated to the valuation on K.

A Banach K-algebra is a K-algebra complete for some norm extending the one on K.

An affinoid K-algebra is a K-algebra A such that there is a surjective morphism $\pi : K\{T_1, ..., T_n\} \to A$. The norm $|\cdot|$ on $K\{T_1, ..., T_n\}$ induce a norm $|\cdot|_\pi$ on A, the residue norm, defined by

$$|a|_\pi = \inf \{|x| \mid \pi(x) = a\}.$$

Such a semi-norm induces on A a structure of Banach K-algebra and one can show that the topology defined by $|\cdot|_\pi$ does not depend on the choice of the semi-norm.

Let A be an affinoid K-algebra. We define $\mathrm{Spm}(A)$ as the set of maximal ideals of A. For any $x \in \mathrm{Spm}(A)$, $k(x) := A/x$ is a finite extension of K hence the norm $|\cdot|$ extends uniquely to $k(x)$. For any $f \in A$, denote by $|f(x)|$ the norm of the image of f in $k(x)$.

Let $f_1, ..., f_n \in A$ such that the ideal generated by $f_1, ..., f_n$ is A. Then consider

$$\mathrm{D}(f_1 \mid f_2, ..., f_n) = \{x \in \mathrm{Spm}(A) \mid |f_2(x)| \leq |f_1(x)|, ..., |f_2(x)| \leq |f_1(x)|\}$$

Such a set is called a rational domain of $\mathrm{Spm}(A)$ associated of $f_1, ...f_n$.

Observe that

$$\mathrm{D}(f_1 \mid f_2, ..., f_n) = \mathrm{Spm}(A < f_1 \mid f_2, ..., f_n >).$$

using the notation

$$A < f_1 \mid f_2, ..., f_n >= A\{T_2, ..., T_n\}/(f_i T_i = f_1)_{i=2,...,n}.$$

A subset $D \subset \mathrm{Spm}(A)$ is a domain if it is a finite union of rational domains of $\mathrm{Spm}(A)$. We say that $U \subset \mathrm{Spm}(A)$ is open if it is an arbitrary union of rational domains of $\mathrm{Spm}(A)$.

Definition 3.3.10.

- For $f_1, ..., f_n \in A$ generating the ideal A, we say that the covering of $\mathrm{Spm}(A)$

$$(\mathrm{D}(f_i \mid f_1, ..., \hat{f}_i, ..., f_n))_{i=1,...,n}$$

  is the standard rational covering associated to $f_1, ..., f_n$.

- Fix some open $U \subset \mathrm{Spm}(A)$ and $\mathcal{U} = (U_i)_{i \in I}$ some cover of U by open subsets. We say that $\mathcal{U}$ is an admissible cover of U if for any rational domain D of U, the cover $(U_i \cap D)_{i \in I}$ of D can be refined by some standard rational covering of D.

- An open $U \subset \mathrm{Spm}(A)$ is said to be admissible open if for any morphism of affinoid K-algebra $f : A \to B$ such that $\mathrm{Spm}(f)(\mathrm{Spm}(B)) \subset U$, there exists some domain D of $\mathrm{Spm}(A)$ such that $\mathrm{Spm}(f)(\mathrm{Spm}(B)) \subset D \subset U$.

Example 3.3.11. Denote $\mathbb{B}(o,1) = \mathrm{Spm}(K\{T\})$ the Tate ball, the thin annuli $\partial\mathbb{B}(o,1) = \mathrm{Spm}(K\{T,T^{-1}\})$ and $U = \{x \in \mathbb{B}(o,1) \mid |x| < 1\}$. Then both $\partial\mathbb{B}(o,1)$ and U are admissible open subsets of $\mathbb{B}(o,1)$, they form an set-theoretical cover of $\mathbb{B}(o,1)$ but not an admissible cover.

Let $\mathrm{Op}(\mathrm{Spm}(A))$ the set of admissible open subsets of $\mathrm{Spm}(A)$. Observe that if $(U_i)_{i\in I}$ is an admissible cover of an admissible open U, the U_i are admissible open. Hence the admissible covers define a Grothendieck topology on $\mathrm{Op}(\mathrm{Spm}(A)$. Such a topology satisfies axioms of G-topology, that we recall now.

Definition 3.3.12. A G-topology on a set X is the data of a subset $\mathrm{Op}(X)$ of subsets of X, stable by finite union and intersection, and a Grothendieck topology τ_X on $\mathrm{Op}(X)$ satisfying the following condition. For any $V \in \mathrm{Op}(X)$ and $P \subset V$, then $P \in \mathrm{Op}(X)$ if for some covering $(U_i)_{i\in I} \in \tau_X$ of U we have $P \cap U_i \in \mathrm{Op}(X)$ for any $i \in I$.

The G-topology on $\mathrm{Op}(\mathrm{Spm}(A))$ induces a G-topology on $\mathrm{Rat}(\mathrm{Spm}(A))$, the set of rational domains of $\mathrm{Spm}(A)$. The presheaf

$$D = \mathrm{D}(f_i \mid f_1, ..., \hat{f}_i, ..., f_n) \in \mathrm{Rat}(\mathrm{Spm}(A)) \mapsto A\{T_2, ..., T_n\}/(f_iT_i = f_1)_{i=2,...,n}$$

is a sheaf for the G-topology and can be canonically extended as a sheaf on $\mathrm{Op}(\mathrm{Spm}(A))$, it is the structural sheaf $\mathcal{O}_{\mathrm{Spm}(A)}$ on $\mathrm{Spm}(A)$.

We call such a pair $(\mathrm{Spm}(A), \mathcal{O}_{\mathrm{Spm}(A)})$ an K-affinoid space.

A rigid K-variety is a locally ringed space with a G-topology that is locally for the G-topology a K-affinoid space. A morphism of rigid spaces is a morphism that is locally of the form $\mathrm{Spm}(f)$ for some morphism of K-affinoid algebra f.

A rigid K-variety is said to be quasi-compact if it can be covered by finitely many affinoid spaces.

In analogy with algebraic geometry, we can define notions of smoothness, properness.

If $X = \mathrm{Spm}(A)$ is a K-affinoid space, $f, g \in A$ and, $p, q \in \mathbb{N}^*$ such that $|f|^{1/p} \leq |g|^{1/q}$, let $\mathbb{B}_X(o, \left|f^{1/p}\right|) = \mathrm{Spm}(A\{f^{-1/p}T\})$ where

$$A\{f^{-1/p}T\} = \left\{ h = \sum_i a_i t^i \in A[[t]] \mid \lim_i \left|a_i^p f^i\right| = 0 \right\}.$$

We define similarly annuli $\mathbb{C}r_X(o, |f|^{1/p}, |g|^{1/q}) = \mathrm{Spm}(A\{g^{-1/q}T\} < T^p \mid f >)$ and thin annuli $\partial\mathbb{B}_X(o, |f|^{1/p}) = \mathbb{C}r_X(o, |f|^{1/p}, |f|^{1/p})$.

The variety $\mathbb{B}_X(o, |f|^{1/p})$ represents a variety of closed balls parametrized by X and of radii $|f|^{1/p}$. Similarly, $\mathbb{C}r_X(o, |f|^{1/p}, |g|^{1/q})$ is a family of closed annuli of radii between $|f|^{1/p}$ and $|g|^{1/q}$.

Such constructions can be glued in order to define similar notions for any rigid variety X.

If $X = \mathrm{Spm}(K)$, we will drop the X to ease the notation. If $p/q \in \mathbb{Q}$ and $r = |t^p|^{1/q}$ we will denote $\mathbb{B}(o, r) = \mathbb{B}(o, |t^p|^{1/q})$ the closed ball of radius r.

In particular, $\cup_{r<1}\mathbb{B}(o, r)$ is the unit open ball. Denote $\mathbb{A}^{1,\mathrm{an}}_K = \cup_{r>0}\mathbb{B}(o, r)$ the rigid affine line.

Observe that neither the unit open ball or $\mathbb{A}^{1,\mathrm{an}}_K$ are quasi-compact.

Let K be an affinoid K-algebra. There is an analytfication functor

$$(-)^{\mathrm{an}} : \mathrm{Sch}_A \to \mathrm{VarRig}_A,$$

such that

$$\mathrm{Hom}_{\mathrm{Spec}(A)}(\mathrm{Spec}(B), X) \simeq \mathrm{Hom}_{\mathrm{Spm}(A)}(\mathrm{Spm}(A), X^{\mathrm{an}})$$

for X a separated scheme. It preserves smoothness, properness.

3.3.6 Formal schemes

We will use Raynaud's viewpoint [86] on formal schemes in order to make a bridge between algebraic varieties over k and rigid varieties over $K = k((t))$. See [7] or [64] for details on tubes. Denote by $R = k[[t]]$ the valuation ring of K.

An R-algebra A is topologically of finite type if it is complete for the t-adic topology and admits a presentation $\pi : R\{T_1, ..., T_n\} \twoheadrightarrow A$. It is topologically of finite presentation if moreover π can be chosen such that its kernel is of finite type. Here since $R\{T_1, ..., T_n\}$ is Noetherian topologically of finite type implies topologically of finite presentation. We say that A is separated if it is separated for the t-adic topology. Denote by $\mathrm{Spf}(A)$ the formal spectrum of A, it is the set of open prime ideals $\mathfrak{p} \subset A$, *i.e.* such that $(t^n) \subset \mathfrak{p}$ for some $n \in \mathbb{N}$. Observe that it is in bijection with $\mathrm{Spec}(A/(t))$.

For $f \in A$, consider $D(f)$ the non-vanishing locus of f in $\mathrm{Spf}(A)$. Then the presheaf

$$D(f) \mapsto A < f^{-1} >= \lim_{\substack{\leftarrow \\ n}} A/(t^n)[f^{-1}]$$

is a sheaf that extends to a sheaf on Zariski open subsets of $\mathrm{Spf}(A)$, the structural sheaf on $\mathrm{Spf}(A)$. A formal R-scheme of finite type is a quasi-compact locally ringed space that is locally the formal spectrum of a separated topologically of finite type R-algebra equipped with its structural sheaf. As we will only consider formal R-scheme of finite type, we will abusively call them formal R-scheme.

A formal R-scheme that is flat over R will be said to be admissible. At the level of R-algebra, it means that there is no t-torsion.

Given an affine formal R-scheme $\mathcal{X} = \mathrm{Spf}(A)$, the K-algebra $K \otimes_R A$ is affinoid. Denote by $\mathcal{X}_\eta$ its maximal spectrum. It is the generic fiber of $\mathcal{X}$. By gluing affine pieces, such a construction gives rise to the generic fiber functor

$$(-)_\eta : \mathrm{FSch}_R \to \mathrm{VarRig}_K.$$

If X is a rigid K-variety, a formal model of X is a formal R-scheme $\mathcal{X}$ together with an isomorphism $X \simeq \mathcal{X}_\eta$.

Proposition 3.3.13. *[13, Theorem 4.1] Any separated quasi-compact rigid K-variety admits an admissible formal R-model.*

We also have a functor

$$(-)_\sigma : \mathrm{FSch}_R \to \mathrm{Var}_k$$

induced by $\mathcal{X} \mapsto \mathcal{X} \times_R k$, it is the special fiber of $\mathcal{X}$ and can be though as the reduction modulo t of $\mathcal{X}$. At the level of topological spaces, $\mathcal{X}$ and $\mathcal{X}_\sigma$ are in bijection, because $\mathrm{Spf}(A) \simeq \mathrm{Spec}(A/(t))$ for any R-algebra A topologically of finite type.

Proposition 3.3.14. *Given an admissible formal R-scheme $\mathcal{X}$, there is a canonical map, called the specialization map (or the reduction map),*

$$\mathrm{sp} : \mathcal{X}_\eta \to \mathcal{X}_\sigma.$$

It is defined at the level of topological spaces and is surjective on the closed points of $\mathcal{X}_\sigma$.

The construction of sp is local. Assume $\mathcal{X} = \mathrm{Spf}(A)$. For $\mathfrak{m} \in \mathrm{Spm}(A \times_R K)$, set $\mathfrak{p} = \mathfrak{m} \cap A$. It induces a closed imersion of formal schemes $\mathrm{Spf}(A/\mathfrak{p}) \to \mathrm{Spf}(A)$ hence a closed immersion of k-schemes $\mathrm{Spec}(A/\mathfrak{p} \otimes_R k) \to \mathrm{Spf}(A \otimes_R k)$. One shows that $\mathrm{Spec}(A/\mathfrak{p} \otimes_R k)$ is in this case the spectrum of a field hence its image is a closed point $\mathrm{sp}(\mathfrak{m}) \in \mathcal{X}_\sigma$.

Proposition 3.3.15. *Given an R-scheme X, we can consider its formal t-adic completion $\mathcal{X}$. We can also base change to K and take the analytification. Then there is an open immersion of rigid varieties*

$$\mathcal{X}_\eta \to (X \times_R K)^{\mathrm{an}}.$$

Assuming X is flat, this immersion is an isomorphism if and only if X is proper on R.

Definition 3.3.16. Let $\mathcal{X}$ a formal R-scheme. If D is a locally closed subset of the special fiber, the tube of D in $\mathcal{X}$ is the inverse image $]D[_{\mathcal{X}} := \mathrm{sp}^{-1}(D)$, with its reduced rigid variety structure. It is an open rigid analytic subvariety of $\mathcal{X}_\eta$. When there is no possible confusion, we will denote $]D[=]D[_{\mathcal{X}}$.

If $\mathcal{U}$ is an open formal subscheme of $\mathcal{X}$ such that $D \subset \mathcal{U}_\sigma$, then $]D[_{\mathcal{X}}=]D[_{\mathcal{U}}$. In particular, $]\mathcal{U}_\sigma[_{\mathcal{X}} = \mathcal{U}_\eta$.

Proposition 3.3.17 ([7, Proposition 1.1.14])**.** *Let $\mathcal{X}$ a formal R-scheme, locally of finite type, and $D \subset \mathcal{X}_\sigma$.*

1. *For any open cover $(V_i)_i$ of V, the tubes $]V_i[$ form an admissible cover of $]D[$.*
2. *If $(D_i)_i$ is a finite closed cover of D, then the tubes $]D_i[$ form an admissible cover of $]D[$.*

Proposition 3.3.18 ([7, Proposition 1.3.1])**.** *Let D a k-scheme of finite type, $\mathcal{X}$, $\mathcal{X}'$ two formal R-schemes of finite type, two immersions $i : D \hookrightarrow \mathcal{X}_\sigma$, $i' : D \hookrightarrow \mathcal{X}'_\sigma$, $u : \mathcal{X}' \to \mathcal{X}$ an étale morphism such that $i = i' \circ u$. Then*

$$u_K :]D[_{\mathcal{X}'} \to]D[_{\mathcal{X}}$$

is an isomorphism.

Definition 3.3.19. Let $\mathcal{X}$ a formal R-scheme of finite type. Say that $\mathcal{X}$ is semi-stable if for every $x \in \mathcal{X}_\sigma$, there is a regular open formal subscheme $\mathcal{U} \subset \mathcal{X}$ containing x and elements $u, t_1, ..., t_r \in \mathcal{O}(\mathcal{U})$ such that the following properties hold:

1. u is invertible and there are positive integers $N_1, ..., N_r$ such that $t = ut_1^{N_1} \cdots t_r^{N_r}$,
2. for every non empty $I \subset \{1, ..., r\}$, the subscheme $D_I \subseteq \mathcal{U}_\sigma$ defined by equations $t_i = 0$ for $i \in I$ is smooth over k, has codimension $|I| - 1$ in $\mathcal{U}_\sigma$ and contains x.

If $\mathcal{X}$ is a formal R-scheme, $f \in \Gamma(\mathcal{X}, \mathcal{O}_X)$, $\underline{N} \in \mathbb{N}^k$, we define

$$\mathrm{St}^f_{\mathcal{X},\underline{N}} = \mathcal{X}\{T_1, ..., T_k\}/(T_1^{N_1} ... T_k^{N_k} - f).$$

Proposition 3.3.20 ([3, Proposition 1.1.62])**.** *Let X a semi-stable R-scheme of type $\underline{N}$ around some $x \in X$. Then for some Zariski neighborhood U of $x[V, V^{-1}]$ in $X[V, V^{-1}]$, there is an étale R-morphism*

$$U \to \mathrm{St}^{Ut}_{\mathrm{Spf}(R[U,U^{-1},S_1,...,S_r)],\underline{N}}.$$

3.3.7 Rigid motives

Ayoub builds in [3] a category $\mathrm{RigSH}_{\mathfrak{M}}(K)$ of rigid motives over K, in an analogous manner of $\mathrm{SH}_{\mathfrak{M}}(K)$, but with starting point rigid varieties instead of schemes.

As in the algebraic case, one needs to choose a category of coefficients $\mathfrak{M}$, the main examples being $\mathrm{RigSH}(K)$ and $\mathrm{RigDA}(K, \Lambda)$.

We need to use a topology on rigid varieties, however for technical reasons, it is not the straightforward translation of Nisnevich topology that will be used.

Definition 3.3.21.

- First we define the notion of Nisnevich cover of formal schemes. A family $\mathcal{R} = (u_i : \mathcal{U}_i \to \mathcal{X})_{i\in I}$ is a Nisnevich cover of a formal scheme $\mathcal{X}$ if the u_i are étale morphisms and the reduction $\mathcal{R}_\sigma = (u_{i\sigma} : \mathcal{U}_{i\sigma} \to \mathcal{X}_\sigma)_{i\in I}$ is a Nisnevich cover of $\mathcal{X}_\sigma$ (in the algebraic sense).
- Let X a quasi-compact smooth rigid K-variety. Then a family $(U_i \to X)_{i\in I}$ is a Nisnevich cover of X if there is a formal R-model $\mathcal{X}$ of X and a Nisnevich cover $\mathcal{R}$ of $\mathcal{X}$ such that $\mathcal{R}_\eta$ is a refinement of $(U_i \to X)_{i\in I}$.
- A Nisnevich cover of a (non-quasi-compact) rigid variety X is a family of étale morphisms $(U_i \to X)_{i\in I}$ such that for any rigid quasi-compact smooth rigid X-variety $V \to X$, the family $(U_i \times_X V \to V)_{i\in I}$ is a Nisnevich cover of V.

An admissible cover of some rigid variety X is in particular a Nisnevich cover.

The other ingredient we need to change in the construction of $\mathrm{RigDA}(K, \Lambda)$ is what we want to localize. We replace $\mathbb{A}^1$-invariance by $\mathbb{B}^1$-invariance, and T-spectra by T^{an}-spectra, where T^{an} is the cokernel of the inclusion (of sheaves represented by) $\partial\mathbb{B}(o, 1) \to \mathbb{B}(o, 1)$. The category $\mathrm{RigDA}(K, \Lambda)$ is now defined as the Verdier localization of the derived category of T^{an}-spectra of Λ-sheaves of smooth rigid varieties for the Nisnevich topology, with respect to the subcategory generated by complexes

$$[\cdots \to 0 \to \mathrm{Sus}^r_{T^{\mathrm{an}}}(\mathbb{B}_U(o, 1)) \to \mathrm{Sus}^r_{T^{\mathrm{an}}}(U) \to 0 \to \dots]$$

and

$$[\cdots \to 0 \to \mathrm{Sus}^r_{T^{\mathrm{an}}}(U \otimes T^{\mathrm{an}}) \to \mathrm{Sus}^{r+1}_{T^{\mathrm{an}}}(U) \to 0 \to \dots].$$

As in the algebraic case, we have suspension functor $\mathrm{Sus}^r_{T^{\mathrm{an}}}(-)$, the tensor $- \otimes A$ has a right adjoint $\underline{\mathrm{Hom}}(A, -)$, the Tate twist $(-)(n)$ is also defined as in the algebraic case.

Definition 3.3.22. We define for X a smooth rigid K-variety its homological motive by $\mathrm{M}_{\mathrm{Rig}}(X) = \mathrm{Sus}^0_{T^{\mathrm{an}}}(X \otimes \mathbb{1}_K)$ and dually, its cohomological motive by $\mathrm{M}^\vee_{\mathrm{Rig}}(X) = \underline{\mathrm{Hom}}(\mathrm{M}_{\mathrm{Rig}}(X), \mathbb{1}_K)$.

A relative version $\mathrm{RigSH}_{\mathfrak{M}}(S)$ of motives of rigid S-varieties exists and satisfies various fonctorialities, like the existence of a pair of adjoint functors (f^*, f_*) for any $f : S' \to S$. To our knowledge a full six functor formalism is not available in this context, the missing ingredients being $f_!$ and $f^!$. Hence there is no already defined notion of compactly supported rigid motive.

The analytification functor induces a (monoidal triangulated) functor

$$\mathrm{Rig}^* : \mathrm{SH}_{\mathfrak{M}}(K) \to \mathrm{RigSH}_{\mathfrak{M}}(K).$$

Such a functor is compatible in a strong sense with the six operations defined on $\mathrm{SH}_{\mathfrak{M}}(-)$, see [3, Théorème 1.4.40], but we do not need it.

The first point of the following proposition is direct consequence of $\mathbb{B}^1$-invariance imposed in the construction of $\mathrm{RigSH}_{\mathfrak{M}}(K)$, the others follows after some reductions.

Proposition 3.3.23 ([3, Proposition 1.3.4])**.** *Let X a rigid K-variety, $f, g, h \in \Gamma(X, \mathcal{O}_X^\times)$ and $p, q, r \in \mathbb{N}\setminus\{0\}$ such that $|f(x)|^{1/p} \leq |g(x)|^{1/q} \leq |h(x)|^{1/r}$. Then we have the following isomorphisms in $\mathrm{RigSH}_{\mathfrak{M}}(K)$:*

- $\mathrm{M}_{\mathrm{Rig}}(\mathbb{B}_X(o, f^{1/p})) \simeq \mathrm{M}_{\mathrm{Rig}}(X)$,
- $\mathrm{M}_{\mathrm{Rig}}((\mathbb{A}^1_K)^{\mathrm{an}} \times_K X) \simeq \mathrm{M}_{\mathrm{Rig}}(X)$,
- $\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}_X(o, g^{1/q})) \simeq \mathrm{M}_{\mathrm{Rig}}(\mathbb{C}r_X(o, f^{1/p}, h^{1/r}))$,
- $\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}_X(o, g^{1/q})) \simeq \mathrm{M}_{\mathrm{Rig}}(\mathbb{G}_{m}{}_K^{\mathrm{an}} \times_K X)$,
- $\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}_X(o, g^{1/q})) \simeq \mathrm{M}_{\mathrm{Rig}}(\mathbb{B}_X(o, g^{1/q})\setminus\{o_X\})$.

Let X a smooth k-scheme, $f \in \Gamma(X, \mathcal{O}_X^\times)$ and $p \in \mathbb{N}^*$. Then define $Q_p^{\mathrm{for}}(X, f)$ as the t-adic completion of the R-scheme $X \times_k R[V]/(V^p - tf)$, and $Q_p^{\mathrm{Rig}}(X, f)$ the generic fiber of $Q_p^{\mathrm{for}}(X, f)$.

Define also $Q_p^{\mathrm{an}}(X, f)$ as the analytification of $X \times_k K[V]/(V^p - tf)$. By Proposition 3.3.15, there is an open immersion of rigid K-varieties

$$Q_p^{\mathrm{Rig}}(X, f) \to Q_p^{\mathrm{an}}(X, f).$$

Theorem 3.3.24 ([3, Théorème 1.3.11])**.** *Let X a smooth k-scheme, $f \in \Gamma(X, \mathcal{O}_X^\times)$, and p a positive integer. Then the inclusion*

$$Q_p^{\mathrm{Rig}}(X, f) \to Q_p^{\mathrm{an}}(X, f)$$

induces an isomorphism

$$\mathrm{M}_{\mathrm{Rig}}(Q_p^{\mathrm{Rig}}(X, f)) \simeq \mathrm{M}_{\mathrm{Rig}}(Q_p^{\mathrm{an}}(X, f)).$$

Define a functor $\mathfrak{F} : \mathrm{QUSH}_{\mathfrak{M}}(k) \to \mathrm{RigSH}_{\mathfrak{M}}(K)$ as the composite

$$\mathfrak{F} : \mathrm{QUSH}_{\mathfrak{M}}(k) \to \mathrm{SH}_{\mathfrak{M}}(\mathbb{G}_{mk}) \underset{\pi^*}{\to} \mathrm{SH}_{\mathfrak{M}}(K) \underset{\mathrm{Rig}^*}{\to} \mathrm{RigSH}_{\mathfrak{M}}(K),$$

where $\pi : \mathrm{Spec}(K) \to \mathbb{G}_{mk}$ corresponds to the ring morphism $k[T, T^{-1}] \to K = k((t))$ sending T to t. Observe that $\mathfrak{F}$ sends the generators $\mathrm{M}_{\mathbb{G}_{mk}}(Q_p^{\mathrm{gm}}(X, f)) \in \mathrm{QUSH}_{\mathfrak{M}}(k)$ to $\mathrm{M}_{\mathrm{Rig}}(Q_p^{\mathrm{an}}(X, f))$.

One of the main results of Ayoub in [3] is the following theorem.

Theorem 3.3.25 ([3, Scholie 1.3.26])**.** *The functor*

$$\mathfrak{F} : \mathrm{QUSH}_{\mathfrak{M}}(k) \to \mathrm{RigSH}_{\mathfrak{M}}(K)$$

is an equivalence of categories, denote by $\mathfrak{R}$ *a quasi-inverse.*

3.4 Realization map for definable sets

This aim of this section is to define a morphism $\chi_{\mathrm{Rig}} : \mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$. We will first define it on $\mathbf{K}(\Gamma[*])$ and $\mathbf{K}(\mathrm{RES}_K[*])$ in Sections 3.4.1 and 3.4.2, before using Hrushovski and Kazhdan isomorphism of define it on $\mathbf{K}(\mathrm{VF}_K)$ in Section 3.4.3. We will in fact define two morphisms χ_{Rig} and χ'_{Rig}. Section 3.4.4 is devoted to the proof of Theorem 3.1.1 via the study of motives of tubes in a semi-stable situation, the main results are grouped in Section 3.4.5. The last Section 3.4.6 is devoted to the definition of two other realization maps $\mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$ and the statement of analogous of Theorem 3.1.1 for them.

3.4.1 The Γ part

Recall that $\Gamma[n]$ is the category with objects subsets of Γ^n defined by piecewise linear equations and inequations with $\mathbb{Z}$-coefficients. A morphism between Y and Y' is a bijection piecewise induced by functions of the form $x \mapsto Ax + B$ with $A \in GL_n(\mathbb{Z})$ and $B \in \mathbb{Z}^n$. From this one forms the graded

$$\Gamma[*] := \coprod_{n\in\mathbb{N}} \Gamma[n].$$

One defines also $\Gamma^{\mathrm{fin}}[n]$ and $\Gamma^{\mathrm{fin}}[*]$ to be the full subcategories of $\Gamma[n]$ and $\Gamma[*]$ whose objects are finite.

Note that morphisms in $\Gamma[*]$ must respect the graduation, in particular, the class $[\{0\}]_1 \in \Gamma[1]$ is not equal to the class $[\{0\}]_n \in \Gamma[n]$ for $n \neq 1$. Recall the o-minimal Euler characteristics eu and eu_c defined above Proposition 3.2.6.

Definition 3.4.1. If $[X] \in \mathbf{K}(\Gamma[*])$, with $X \in \Gamma[d]$, define

$$\chi^{\Gamma}_{\mathrm{Rig}}(X) = \mathrm{eu}_c(X)[\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o,1)^d)(-d)] \in \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

and

$$\chi'^{\Gamma}_{\mathrm{Rig}}([X]) = \mathrm{eu}(X)[\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o,1)^d)] \in \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)).$$

Hence we get two ring morphisms

$$\chi^{\Gamma}_{\mathrm{Rig}}, \chi'^{\Gamma}_{\mathrm{Rig}} : \mathbf{K}(\Gamma[*]) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)).$$

It is well defined because $\mathbf{K}(\Gamma[*])$ is naturally graded and by additivity of Euler characteristic.

Proposition 3.4.2. *Let $X \subseteq \Gamma^n$ be a convex bounded polytope. If X is closed, then*

$$\chi^{\Gamma}_{\mathrm{Rig}}([X]) = [\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})(-n)] \text{ and } \chi'^{\Gamma}_{\mathrm{Rig}}([X]) = [\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})].$$

If X is open, then

$$\chi^{\Gamma}_{\mathrm{Rig}}([X]) = (-1)^n[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})(-n)] \text{ and } \chi^{\Gamma}_{\mathrm{Rig}}([X]) = (-1)^n[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})].$$

Proof. If X is empty, $\mathrm{eu}(X) = \mathrm{eu}_c(X) = 0$ hence the proposition is verified. Hence we can suppose X in nonempty. We have $\mathrm{eu}(X) = \mathrm{eu}_c(X) = 1$ if X is closed, and $\mathrm{eu}(X) = \mathrm{eu}_c(X) = (-1)^n$ if X is open. Hence the result follows from the following Lemma 3.4.3. □

Lemma 3.4.3. *Let $X \subseteq \Gamma^n$ be a non-empty convex polytope, either closed or open. Then*

$$\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}}) \simeq \mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o,1)^n).$$

Proof. We first assume that X is closed. We work by induction on n. If $n = 1$, then

$$X = \{x \mid \alpha \leq px \leq \beta\}$$

for $\alpha, \beta \in \mathbb{Z}, p \in \mathbb{N}$. Hence

$$\mathrm{v}^{-1}(X)^{\mathrm{Rig}} = \mathbb{C}r(o, \left|\pi^{\beta}\right|^{1/p}, |\pi^{\alpha}|^{1/p}).$$

By Proposition 3.3.23, $\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}}) = \mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o,1))$.

Suppose now the result is known for $n-1$. There are finitely many affine functions h_i, $i \in I_0$ with $\mathbb{Z}$-coefficients such that

$$X = \{x \in \Gamma^n \mid h_i(x) \geq 0\}.$$

We can rewrite X as

$$X = \left\{(x,y) \in \Gamma \times \Gamma^{n-1} \mid p_i x \leq f_i(y), q_j x \geq g_j(y), i \in I, j \in J\right\}$$

for some (possibly empty) finite sets I, J integers $p_i, q_j \in \mathbb{N}$, affine functions $f_i, g_j : \Gamma^{n-1} \to \Gamma$ with $\mathbb{Z}$ coefficients. Now observe that the projection of X on the last $n-1$-th coordinates is

$$Y = \left\{y \in \Gamma^{n-1} \mid \forall (i,j) \in I \times J, p_i g_j(y) \leq q_j f_i(y)\right\}.$$

It satisfies the hypotheses of the proposition hence we get that $[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(Y)^{\mathrm{Rig}})] = [\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o,1)^{n-1})]$ by induction.

We set $I' = \{i \in I \mid p_i > 0\}$ and $J' = \{j \in J \mid q_j > 0\}$ and observe that

$$X = \left\{(x,y) \in \Gamma \times Y \mid p_i x \leq f_i(y), q_j x \geq g_j(y), i \in I', j \in J'\right\}.$$

Set $X_{i,j} = \mathbb{C}r_{Y^{\mathrm{Rig}}}(o, \left|\tilde{f}_i\right|^{1/p_i}, |\tilde{g_j}|^{1/q_j})$, where we used the notation that if

$$f(y_1, ..., y_{n-1}) = b + a_1 y_1 + ... + a_{n-1} y_{n-1},$$

then

$$\tilde{f}(x_1, ..., x_{n-1}) = t^b \cdot x_1^{a_1} \cdot ... \cdot x_{n-1}^{a_{n-1}}.$$

We have now

$$X^{\mathrm{Rig}} = \bigcap_{(i,j) \in I' \times J'} X_{i,j}.$$

Set

$$Y_{i,j} = \left\{y \in Y^{\mathrm{Rig}} \mid \forall (i',j') \in I \times J, \left|\tilde{f}_i\right|^{1/p_i} \geq \left|\tilde{f}_{i'}\right|^{1/p_{i'}}, |\tilde{g_j}|^{1/q_j} \leq \left|\tilde{g_{j'}}\right|^{1/q_{j'}}\right\}.$$

The $(Y_{i,j})_{(i,j) \in I \times J}$ form an admissible cover of Y^{Rig}, indeed, $Y_{i,j}$ is defined in Y^{Rig} by some non-strict valuatives inequalities, if D is a rational domain, the standard cover of D induced by functions used to defined D and the functions used to defines all $Y_{i,j}$ gives the required refinement of $(D \cap X_{i,j})_{(i,j) \in I \times J}$.

Then it suffices to show the result for $X^{\mathrm{Rig}} \cap (Y_{i,j} \times_K \mathbb{A}_K^{1,\mathrm{an}})$. But we have then

$$X^{\mathrm{Rig}} \cap (Y_{i,j} \times_K \mathbb{A}_K^{1,\mathrm{an}}) = \mathbb{C}r_{Y_{i,j}}(o, \left|\tilde{f}_i\right|^{1/p_i}, |\tilde{g_j}|^{1/q_j})$$

hence the result follows from Proposition 3.3.23, which gives

$$\mathrm{M}_{\mathrm{Rig}}(\mathbb{C}r_{Y_{i,j}}(o, \left|\tilde{f}_i\right|^{1/p_i}, |\tilde{g_j}|^{1/q_j})) \simeq \mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}_{Y_{i,j}}(o,1)).$$

Suppose now that X is an open polyhedron. We work similarly by induction on n. If $n = 1$,

$$X = \{x \mid \alpha < px < \beta\}$$

for $\alpha, \beta \in \mathbb{Z}, p \in \mathbb{N}$. Hence

$$\mathrm{v}^{-1}(X)^{\mathrm{Rig}} = \bigcup_{r_0 \leq r < 1} \mathbb{C}r(o, r^{-1}\left|\pi^{\beta}\right|^{1/p} r, r\left|\pi^{\alpha}\right|^{1/p}),$$

for some $r_0 < 1$ close enough to 1. It is thus enough to show that the inclusion

$$\partial\mathbb{B}(o, r_1) \hookrightarrow \mathbb{C}r(o, r^{-1}\left|\pi^\beta\right|^{1/p}, r\,|\pi^\alpha|^{1/p})$$

induces an isomorphism in $\mathrm{RigSH}_{\mathfrak{M}}(K)$, for $r_0 \leq r < 1$ and $r_1 \in \mathbb{R}_+^*$ such that $r_0^{-1}\left|\pi^\beta\right|^{1/p} \leq r_1 \leq r_0|\pi^\alpha|^{1/p}$. This is Proposition 3.3.23.

Suppose now the result is known for $n-1$. There are finitely many affine functions h_i, $i \in I_0$ with $\mathbb{Z}$-coefficients such that

$$X = \{x \in \Gamma^n \mid h_i(x) > 0\}.$$

Proceed now as in the closed case, denote by Y the projection of X on the last $n-1$-th coordinates. By induction, it suffices to show that $\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}}) \simeq \mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}_Y(o,1))$. Define $I, J, Y_{i,j}, X_{i,j}$ as above, replacing large inequalities by strict ones where needed. We will show that $\mathrm{M}_{\mathrm{Rig}}(X_{i,j}) \simeq \mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}_{Y_{i,j}}(o,1))$, with these isomorphisms compatible on $X_{i,j} \cap X_{i',j'}$. We can find $r_0 \in \mathbb{Q}$ with $0 < r_0 < 1$ such that for each $(i,j) \in I \times J$, and $y \in Y_{i,j}$, ${r_0}^{-1}\left|\tilde{f}_i(y)\right|^{1/p_i} \leq r_0\,|\tilde{g_j}(y)|^{1/q_j}$. We can moreover choose for each $(i,j) \in I \times J$ a monomial function $\tilde{h_{i,j}}$ on $Y_{i,j}$ such that for all $y \in Y_{i,j}$,

$${r_0}^{-1}\left|\tilde{f}_i(y)\right|^{1/p_i} \leq \left|\tilde{h_{i,j}}(y)\right| \leq r_0\,|\tilde{g_j}(y)|^{1/q_j}$$

and assume that these functions coincide on $Y_{i,j} \cap Yi', j'$.

We now have

$$X_{i,j} \cap (Y_{i,j} \times_K \mathbb{A}_K^{1,\mathrm{an}} = \cup_{r_0<r<1}\mathbb{C}r_{Y_{i,j}}(o, r^{-1}\left|\tilde{f}_i(y)\right|^{1/p_i}, r\,|\tilde{g_j}(y)|^{1/q_j}),$$

hence it suffices to show that the immersion

$$\partial\mathbb{B}_{Y_{i,j}}(o, \tilde{h_{i,j}}) \hookrightarrow \mathbb{C}r_{Y_{i,j}}(o, r^{-1}\left|\tilde{f}_i(y)\right|^{1/p_i}, r\,|\tilde{g_j}(y)|^{1/q_j})$$

induces an isomorphism in in $\mathrm{RigSH}_{\mathfrak{M}}(K)$, which follows once again from Proposition 3.3.23. □

We will not use it, but Proposition 3.4.2 can be extended to all closed polyhedral complexes.

Proposition 3.4.4. *Let $X \subseteq \Gamma^n$ be (the realization of) a bounded closed polyhedral complex. Then*

$$\chi_{\mathrm{Rig}}^{\Gamma}([X]) = [\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})(-d)].$$

Proof. In view of the definition of $\chi^{\Gamma}_{\mathrm{Rig}}$, we need to show that

$$[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})] = \mathrm{eu}(X)[\partial\mathbb{B}(o,1)^n].$$

We work by double induction on the maximal dimension of simplex in X and the number of simplex of maximal dimension. Let $\Delta \subset X$ a simplex of maximal dimension. Let $Y = X\backslash\Delta^\circ$, with Δ° the interior of Δ, $\partial\Delta = \Delta\backslash\Delta^\circ$.

Then $(\mathrm{v}^{-1}(\Delta)^{\mathrm{Rig}}, \mathrm{v}^{-1}(Y)^{\mathrm{Rig}})$ is an admissible cover of $\mathrm{v}^{-1}(X)^{\mathrm{Rig}}$, with intersection $\mathrm{v}^{-1}(\partial\Delta)^{\mathrm{Rig}}$ hence

$$[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(X)^{\mathrm{Rig}})] = [\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)^{\mathrm{Rig}})] + [\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(Y)^{\mathrm{Rig}})] - [\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(\partial\Delta)^{\mathrm{Rig}})].$$

By Lemma 3.4.3, $[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)^{\mathrm{Rig}})] = [\partial\mathbb{B}(o,1)^n]$. Apply the induction hypothesis to get $[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(\partial\Delta)^{\mathrm{Rig}})] = (1-(-1)^d)([\partial\mathbb{B}(o,1)^n]$ and $[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(Y)^{\mathrm{Rig}})] = \mathrm{eu}(Y)[\partial\mathbb{B}(o,1)^n]$. We have the result, since $\mathrm{eu}(X) = \mathrm{eu}(Y) + (-1)^d$. □

3.4.2 The RES part

Recall that the category $\mathrm{RES}_K[*]$ is defined as the full subcategory of $\mathrm{RV}_K[*]$ such that for $X \in \mathrm{RV}_K[*]$, $X \in \mathrm{RES}_K[*]$ if and only if $\mathrm{v}(X)$ is finite. Alternatively, one can define RES_K as the category of definable sets consisting of V_γ for $\gamma \in \mathbb{Q}$, the residue field $\mathbf{k}$ together with the functions $H : V_{\overline{\gamma}} \to \mathbf{k}$ associated to every $\overline{\gamma}$-polynomial H.

Fix $\overline{\gamma} = (\gamma_1, ..., \gamma_n) \in \mathbb{Z}^n$. A $\overline{\gamma}$-weighted monomial is a term $a_\alpha X^\alpha = a_\alpha \Pi_i X_i^{\alpha_i}$ with $a_\alpha \in \mathrm{RV}(K)$, $\alpha_i \in \mathbb{N}$ such that $\mathrm{v}(a_\alpha) + \sum_i \alpha_i\gamma_i = 0$. A $\overline{\gamma}$-polynomial H is a finite sum of $\overline{\gamma}$-monomials. It leads to a well defined function $V_{\overline{\gamma}} \to \mathbf{k}$.

Definition 3.4.5. Define ring morphisms

$$\chi^{\mathrm{RES}}_{\mathrm{Rig}} : \mathbf{K}(\mathrm{RES}_K)[*] \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

and

$$\chi'^{\mathrm{RES}}_{\mathrm{Rig}} : \mathbf{K}(\mathrm{RES}_K)[*] \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

by the formulas, for X a smooth k-variety of pure dimension r, $f \in \Gamma(X, \mathcal{O}_X^\times)$, $m \in \mathbb{N}^*$,

$$\chi^{\mathrm{RES}}_{\mathrm{Rig}}([Q^{\mathrm{RV}}_m(X,f)]_n) = [Q^{\mathrm{Rig}}_m(X,f)(-r)]$$

and

$$\chi'^{\mathrm{RES}}_{\mathrm{Rig}}([Q^{\mathrm{RV}}_m(X,f)]_n) = [Q^{\mathrm{Rig}}_m(X,f)(n-r)].$$

As the $[Q^{\mathrm{RV}}_m(X,f)]_n$ generate $\mathbf{K}(\mathrm{RES}_K[*])$ by Corollary 3.2.10, one only need to show that the maps are well defined. But we can check that they coincide with the composite

$$\mathbf{K}(\mathrm{RES}_K[*]) \to !\mathbf{K}(\mathrm{RES}_K)[\mathbb{L}^{-1}] \xrightarrow{\Theta} \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})[\mathbb{L}^{-1}] \xrightarrow{\chi_{\hat{\mu}}} \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)) \xrightarrow{\mathfrak{F}} \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)).$$

where the map $\mathbf{K}(\mathrm{RES}_K[*]) \to !\mathbf{K}(\mathrm{RES}_K)[\mathbb{A}_k^1]^{-1}$ is $[X]_n \mapsto [X]$ for $\chi_{\mathrm{Rig}}^{\mathrm{RES}}$ and $[X]_n \mapsto [X]\mathbb{L}^{-n}$ for $\chi_{\mathrm{Rig}}'^{\mathrm{RES}}$. The maps Θ, $\chi_{\hat{\mu}}$, $\mathfrak{F}$ are respectively defined in Propositions 3.2.9, 3.3.8 and Theorem 3.3.25. Note that this also implies that it is a morphism of rings.

3.4.3 Definition of $\chi_{\mathrm{Rig}}^{\mathrm{RV}}$

Recall the isomorphism $\mathbf{K}(\mathrm{RES}_K[*]) \otimes_{\mathbf{K}(\Gamma^{\mathrm{fin}}[*])} \mathbf{K}(\Gamma[*]) \to \mathbf{K}(\mathrm{RV}_K[*])$ from Proposition 3.2.5. Using this isomorphism, to define a ring morphism $\mathbf{K}(\mathrm{RV}_K[*]) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$, it suffices to specify two ring morphisms $\mathbf{K}(\mathrm{RES}_K[*]) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$ and $\mathbf{K}(\Gamma[*]) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$ that coincide on $\mathbf{K}(\Gamma^{\mathrm{fin}}[*])$.

Definition 3.4.6. Define

$$\chi_{\mathrm{Rig}}^{\mathrm{RV}} : \mathbf{K}(\mathrm{RV}_K[*]) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

using the morphisms $\chi_{\mathrm{Rig}}^{\mathrm{RES}}$ and $\chi_{\mathrm{Rig}}^{\Gamma}$ and

$$\chi_{\mathrm{Rig}}'^{\mathrm{RV}} : \mathbf{K}(\mathrm{RV}_K[*]) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

using the morphisms $\chi_{\mathrm{Rig}}'^{\mathrm{RES}}$ and $\chi_{\mathrm{Rig}}'^{\Gamma}$.

To show that is is well defined, one needs to check that if $A \subseteq \Gamma^n$ is definable and finite, then $\chi_{\mathrm{Rig}}^{\Gamma}([A]) = \chi_{\mathrm{Rig}}^{\mathrm{RES}}([\mathrm{v}^{-1}(A)]_n)$ and $\chi_{\mathrm{Rig}}'^{\Gamma}([A]) = \chi_{\mathrm{Rig}}'^{\mathrm{RES}}([\mathrm{v}^{-1}(A)]_n)$. By additivity, one can assume $A = \{\alpha\}$. Hence it follows from the following lemma.

Lemma 3.4.7. *Let $\alpha = (\alpha_1, ..., \alpha_n) \in \Gamma^n$ be definable. Then*

$$\chi_{\mathrm{Rig}}^{\mathrm{RES}}([\mathrm{v}^{-1}(\{\alpha\})] = [\partial\mathbb{B}(o,1)^n(-n)]$$

and

$$\chi_{\mathrm{Rig}}'^{\mathrm{RES}}([\mathrm{v}^{-1}(\{\alpha\})] = [\partial\mathbb{B}(o,1)^n]$$

Proof. We have $\mathrm{v}^{-1}(\{\alpha\}) = V_{\alpha_1}^* \times ... \times V_{\alpha_n}^*$. Because $\chi_{\mathrm{Rig}}^{\mathrm{RES}}$ and $\chi_{\mathrm{Rig}}'^{\mathrm{RES}}$ are ring morphisms and $[\partial\mathbb{B}(o,1)^n(-n)] = [\partial\mathbb{B}(o,1)(-1)]^n$, we can assume $n = 1$. Suppose $\alpha = k/m$, with $k \in \mathbb{Z}$ and $m \in \mathbb{N}^*$ relatively prime. Let $a, b \in \mathbb{Z}$ such that $am + bk = 1$. In this case, we have

$$V_{k/m}^* \simeq \left\{(z,u) \in V_{1/m}^* \times V_0^* \mid z^m = \mathbf{t}u^b\right\} = Q_m^{\mathrm{RV}}(\mathbb{G}_{mk}, u^b)$$

via the isomorphism $w \in V^*_{k/m} \mapsto (\mathbf{t}^a w^b, \mathbf{t}^{-k} w^m)$. But now, $Q^{\mathrm{Rig}}_m(\mathbb{G}_{mk}, u^b) \simeq \partial\mathbb{B}(o, k/m)$ via the isomorphism $(z, u) \mapsto z^k u^a$ and $\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o, k/m)) \simeq \mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o, 1))$ by Proposition 3.3.23. □

Remark 3.4.8. If $X \subseteq \mathrm{RV}^n$ is definable, then $\chi_{\mathrm{Rig}}([X]_m) = \chi_{\mathrm{Rig}}([X]_n)$ for any $m \geq n$, hence χ_{Rig} does not depends on the grading in $\mathbf{K}(\mathrm{RV}[*])$, it is in fact defined on $\mathbf{K}(\mathrm{RV})$

Proposition 3.4.9. *The ring morphisms $\chi^{\mathrm{RV}}_{\mathrm{Rig}}$ and $\chi'^{\mathrm{RV}}_{\mathrm{Rig}}$ of Definition 3.4.6 induce ring morphisms*

$$\chi^{\mathrm{RV}}_{\mathrm{Rig}}, \chi'^{\mathrm{RV}}_{\mathrm{Rig}} : \mathbf{K}(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}} \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)).$$

Proof. We need to check that the generator of I_{sp} vanish under $\chi^{\mathrm{RV}}_{\mathrm{Rig}}$ and $\chi'^{\mathrm{RV}}_{\mathrm{Rig}}$.

We have $\mathrm{M}_{\mathrm{Rig}}(\{1\}) = \mathrm{M}_{\mathrm{Rig}}(\mathrm{Spm}(K)) = \mathbb{1}_K$, hence $\chi^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_1) = \chi^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_0) = \chi'^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_0) = [\mathbb{1}_K]$ and $\chi'^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_1) = [\mathbb{1}_K(1)]$.

Moreover, $\mathrm{RV}^{>0} = \mathrm{v}^{-1}((0, +\infty))$ and $\mathrm{eu}_c((0, +\infty)) = 0$, $\mathrm{eu}((0, +\infty)) = -1$. Hence $\chi^{\mathrm{RV}}_{\mathrm{Rig}}([\mathrm{RV}^{>0}]_1) = 0$ and $\chi^{\mathrm{RV}}_{\mathrm{Rig}}([\mathrm{RV}^{>0}]_1) = -[\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o, 1))]$.

We already see that

$$\chi^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_1) = \chi^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_0 + [\mathrm{RV}^{>0}]_1).$$

For $\chi'^{\mathrm{RV}}_{\mathrm{Rig}}$, we have by construction

$$\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o, 1)) = \mathrm{M}_{\mathrm{Rig}}(\mathrm{Spm}(K)) \oplus \mathrm{M}_{\mathrm{Rig}}(\mathrm{Spm}(()K))(1)[1].$$

Hence we get in $\mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$ the equality

$$[\mathrm{M}_{\mathrm{Rig}}(\mathrm{Spm}(K))] = [\mathrm{M}_{\mathrm{Rig}}(\mathrm{Spm}(K))(1)] + [\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o, 1))],$$

which is

$$\chi'^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_0] = \chi'^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_1] - \chi'^{\mathrm{RV}}_{\mathrm{Rig}}([\mathrm{RV}^{>0}]_1),$$

hence

$$\chi'^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_1) = \chi'^{\mathrm{RV}}_{\mathrm{Rig}}([\{1\}]_0 + [\mathrm{RV}^{>0}]_1).$$

□

Recall the isomorphism $\oint : K(\mathrm{VF}_K) \to K(\mathrm{RV}_K[*])/\mathrm{I}_{\mathrm{sp}}$.

Definition 3.4.10. Define χ_{Rig} and χ'_{Rig} :

$$\mathbf{K}(\mathrm{VF}) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

by $\chi_{\mathrm{Rig}} = \chi^{\mathrm{RV}}_{\mathrm{Rig}} \circ \oint$ and $\chi'_{\mathrm{Rig}} = \chi'^{\mathrm{RV}}_{\mathrm{Rig}} \circ \oint$.

For any irreducible smooth k-variety X of dimension d, $f \in \mathcal{O}_X^\times(X)$ and $r \in \mathbb{N}^*$, denote

$$Q_r^{\mathrm{VF}}(X,f) = \{(x,y) \in X(\mathrm{VF}) \times \mathrm{VF} \mid y^r = tf(x)\}.$$

Proposition 3.4.11. *For any irreducible smooth k-variety X of dimension d, $f \in \mathcal{O}_X^\times(X)$ and $r \in \mathbb{N}^*$,*

$$\chi_{\mathrm{Rig}}(Q_r^{\mathrm{VF}}(X,f)) = [\mathrm{M}_{\mathrm{Rig}}(Q_r^{\mathrm{Rig}}(X,f))(-d)].$$

Proof. Since $\oint[Q_r^{\mathrm{VF}}(X,f)] = [Q_r^{\mathrm{RV}}(X,f)]_r$, it simply follows from the definition of $\chi_{\mathrm{Rig}}^{\mathrm{RES}}$. □

Proposition 3.4.12. *Let $\Delta \subset \Gamma^n$ defined by linearly independent affine equations $l_i > 0$ for $i = 0, ..., n$. Then*

$$\chi_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)) = (-1)^n[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)^{\mathrm{Rig}})(-n)].$$

Proof. It follows from Proposition 3.4.2. □

Theorem 3.4.13. *There are commutative squares*

$$\begin{tikzcd}
\mathbf{K}(\mathrm{VF}_K) \arrow[r, "\Theta\circ\mathcal{E}_c\circ\oint"] \arrow[d, "\chi_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[d, "\chi_{\hat{\mu}}"] \\
\mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)) \arrow[r, "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k))
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{K}(\mathrm{VF}_K) \arrow[r, "\Theta\circ\mathcal{E}\circ\oint"] \arrow[d, "\chi'_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})[\mathbb{A}_k^1]^{-1} \arrow[d, "\chi_{\hat{\mu}}"] \\
\mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)) \arrow[r, "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)).
\end{tikzcd}$$

Proof. We will only show the commutativity of the first diagram, the second being similar.

We need to show that the following diagram is commutative :

$$\begin{tikzcd}
\mathbf{K}(\mathrm{RV}[*])/\mathrm{I}_{\mathrm{sp}} \arrow[r, "\Theta\circ\mathcal{E}_c"] \arrow[d, "\chi_{\mathrm{Rig}}^{\mathrm{RV}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[d, "\chi_{\hat{\mu}}"] \\
\mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)) \arrow[r, "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)).
\end{tikzcd}$$

It suffices to show that the following diagrams are commutative :

$$\begin{tikzcd}
\mathbf{K}(\mathrm{RES}_K[*]) \arrow[r, "\Theta\circ\mathcal{E}_c"] \arrow[d, "\chi^{\mathrm{RES}}_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[d, "\chi_{\hat{\mu}}"] \\
\mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)) \arrow[r, "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)).
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{K}(\Gamma) \arrow[r, "\Theta\circ\mathcal{E}_c"] \arrow[d, "\chi^{\Gamma}_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[d, "\chi_{\hat{\mu}}"] \\
\mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)) \arrow[r, "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)).
\end{tikzcd}$$

For the first one, we already observed that $\chi^{\mathrm{RES}}_{\mathrm{Rig}}$ does not depend on the grading, hence we need to show that $\mathfrak{R}\circ\chi^{\mathrm{RES}}_{\mathrm{Rig}} = \chi_{\hat{\mu}}\circ\Theta$, as morphisms from $!\mathbf{K}(\mathrm{RES}_K)$ to $\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k))$. By Corollary 3.2.10, $!\mathbf{K}(\mathrm{RES}_K)$ is generated by classes of $Q^{\mathrm{RV}}_r(X,f)$, for X a k-variety smooth of pure dimension d, $r\in\mathbb{N}\setminus\{0\}$ and $f\in\Gamma(X,\mathcal{O}_X^\times)$. The definition of $\chi_{\hat{\mu}}\circ\Theta$ show that $\chi_{\hat{\mu}}\circ\Theta(Q^{\mathrm{RV}}_r(X,f)) = [\mathrm{M}_{\mathbb{G}_{mk}}(Q^{\mathrm{gm}}_r(X,f))(-\dim(X))]$. From the definition of $\chi^{\mathrm{RES}}_{\mathrm{Rig}}$, $\chi^{\mathrm{RES}}_{\mathrm{Rig}}(Q^{\mathrm{RV}}_r(X,f)) = [\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_r(X,f)(-d))]$; from Theorem 3.3.24, $\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_r(X,f)\simeq \mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{an}}_r(X,f)$, and from the definition of $\mathfrak{R}$, $\mathfrak{R}(\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{an}}_r(X,f)) = \mathrm{M}_{\mathbb{G}_{mk}}(Q^{\mathrm{gm}}_r(X,f))$. For the second square, for any $X\subset\Gamma^n$, $\chi^{\Gamma}_{\mathrm{Rig}}(X) = \mathrm{eu}_c(X)[\partial\mathbb{B}(o,1)^n(-n)]$ and $\mathcal{E}_c(X) = \mathrm{eu}_c(X)[\mathbb{G}_{mk}{}^n]$, so it follows from the fact that $\mathfrak{R}[\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o,1)] = [\mathrm{M}_{\mathbb{G}_{mk}}(\mathbb{G}_{mk}\times_k\mathbb{G}_{mk})]$. $\square$

3.4.4 Motives of tubes

The aim of this section is to compute χ_{Rig} for a quasi-compact smooth rigid K-variety. We will use semi-stable formal models, and in particular tubes of their branches.

Let $\mathcal{X}$ a semi-stable formal R-scheme and $(D_i)_{i\in J}$ be the branches of its special fiber $\mathcal{X}_\sigma$. For any non empty $I\subset J$, let $D_I = \cap_{i\in I}D_i$ and $D(I) = \cup_{i\in I}D_i$. Denote also for $I'\subset J\backslash I$, $D_I^{\circ I'} = D_I\backslash D(I')$ and if $I' = J\backslash I$, simply $D_I^\circ = D_I\backslash D(J\backslash I)$.

Ayoub, Ivorra and Sebag prove the following proposition.

Proposition 3.4.14 ([4, Theorem 5.1])**.** *For any non empty $I\subset J$ and $I'\subset I''\subset J\backslash I$, the inclusion $]D_I^{\circ I''}[\hookrightarrow]D_I^{\circ I'}[$ induces an isomorphism*

$$\mathrm{M}_{\mathrm{Rig}}(]D_I^{\circ I''}[) \simeq \mathrm{M}_{\mathrm{Rig}}(]D_I^{\circ I'}[).$$

We will mostly use this proposition in the following particular case.

Corollary 3.4.15. *For any non empty $I\subseteq J$, there is an isomorphism*

$$\mathrm{M}_{\mathrm{Rig}}(]D_I^\circ[) \simeq \mathrm{M}_{\mathrm{Rig}}(]D_I[).$$

Proposition 3.4.16. *Let $\mathcal{X}$ a semi-stable formal R-scheme and $D = \cup_{i\in J'} D_i$ a union of branches. Then the following equalities holds in* $\mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$

$$[\mathrm{M}_{\mathrm{Rig}}(]D[)] = \sum_{I\subset J'} (-1)^{|I|-1}[\mathrm{M}_{\mathrm{Rig}}(]D_I[)]$$

and

$$[\mathrm{M}^{\vee}_{\mathrm{Rig}}(]D[)] = \sum_{I\subset J'} (-1)^{|I|-1}[\mathrm{M}^{\vee}_{\mathrm{Rig}}(]D_I[)].$$

Proof. The collection $(D_i)_{i\in J'}$ is a closed cover of D, hence by Proposition 3.3.17, the $(]D_i[)_{i\in J'}$ is an admissible cover of $]D[$. Hence by Mayer-Vietoris distinguished triangle and induction on the cardinal of I, we have the result. □

Using Corollary 3.4.15, we deduce the following formula.

Corollary 3.4.17. *Under the hypotheses of Proposition 3.4.16, we have*

$$[\mathrm{M}_{\mathrm{Rig}}(]D[)] = \sum_{I\subseteq J'} (-1)^{|I|-1}[\mathrm{M}_{\mathrm{Rig}}(]D^{\circ}_I[)]$$

and

$$[\mathrm{M}^{\vee}_{\mathrm{Rig}}(]D[)] = \sum_{I\subseteq J'} (-1)^{|I|-1}[\mathrm{M}^{\vee}_{\mathrm{Rig}}(]D^{\circ}_I[)].$$

Theorem 3.4.18. *Let $\mathcal{X}$ a semi-stable formal R-scheme of dimension d. Then*

$$\chi_{\mathrm{Rig}}(\mathcal{X}^{\mathrm{VF}}_{\eta}) = [\mathrm{M}_{\mathrm{Rig}}(\mathcal{X}_{\eta}(-d))].$$

Still denoting $\mathcal{X}_{\sigma} = \cup_{i\in J} D_i$ the irreducible components of $\mathcal{X}_{\sigma}$, the special fiber of X, we can write $\mathcal{X}^{\mathrm{VF}}_{\eta}$ as a disjoint union $\mathcal{X}^{\mathrm{VF}}_{\eta} = \dot{\bigcup}_{I\subset J}]D^{\circ}_I[$, hence

$$\chi_{\mathrm{Rig}}(]D[^{\mathrm{VF}}_{\mathcal{X}}) = \sum_{I\subset J} \chi_{\mathrm{Rig}}(]D^{\circ}_I[).$$

In view of the formula of Corollary 3.4.17, to prove Theorem 3.4.18, it suffices to prove the following proposition.

Proposition 3.4.19. *Let $\mathcal{X}$ be a semi-stable R-scheme. Then*

$$\chi_{\mathrm{Rig}}(]D^{\circ}_I[^{\mathrm{VF}}) = (-1)^{|I|-1}[\mathrm{M}_{\mathrm{Rig}}(]D^{\circ}_I[)(-d)],$$

where $d = \dim(\mathcal{X}_{\sigma})$.

Before proving the proposition, we need a reduction.

Lemma 3.4.20. *To prove Proposition 3.4.19, we can assume $\mathcal{X} = \mathrm{St}^{u^{-1}t}_{D^{\circ}_I\times_k R,\underline{N}}$, where $\underline{N} = (N_1, ..., N_r) \in (\mathbb{N}^{\times})^r$ (where $r = |I|$), $u_I \in \mathcal{O}^{\times}(D^{\circ}_I \times_k R)$.*

Proof. Using Mayer-Vietoris distinguished triangles, we can also work Zariski locally, hence suppose by Proposition 3.3.20 that there is an étale R-morphism

$$e : \mathcal{X}\{V, V^{-1}\} \to \mathcal{S} = \mathrm{St}^{Ut}_{\mathrm{Spec}(R[U,U^{-1}]),\underline{N}}[S_1, ..., S_r],$$

where $\underline{N}$ is the type of $\mathcal{X}$ at $x \in]D_I^\circ[$. The irreducible components of $\mathcal{S}_\sigma$ are defined by equations $T_i = 0$, denote by C their intersection. We have $C = \mathrm{Spec}(k[U, U^{-1}, S_1, ..., S_r])$. Up to permuting the D_i, we can assume D_i is defined in $\mathcal{X}_\sigma$ by $T_i \circ e = 0$, inducing an étale morphism $e_\sigma : D_I[V, V^{-1}] \to C$ and a Cartesian square of R-schemes

$$\begin{tikzcd} D_I[V, V^{-1}] \arrow[r] \arrow[d, "e_\sigma"'] & X \arrow[d, "e"] \\ C \arrow[r] & S. \end{tikzcd}$$

The morphism e_σ induces an étale morphism of R-schemes

$$D_I[V, V^{-1}] \times_k R \to C \times_k R,$$

which itself induces an étale R-morphism

$$e' : \mathcal{X}' = \mathrm{St}^{u_I^{-1}t}_{D_I[V,V^{-1}]\times_k R,\underline{N}} \to \mathcal{S},$$

together with a Cartesian square

$$\begin{tikzcd} D_I[V, V^{-1}] \arrow[r] \arrow[d, "e_\sigma"'] & X' \arrow[d, "e'"] \\ C \arrow[r] & S. \end{tikzcd}$$

The fiber product $\mathcal{X}\{V, V^{-1}\} \times_{\mathcal{S}} \mathcal{X}'$ hence satisfies $\mathcal{X}\{V, V^{-1}\} \times_{\mathcal{S}} \mathcal{X}' \times_{\mathcal{S}} C \simeq D_I[V, V^{-1}] \times_C D_I[V, V^{-1}]$. Because $e_\sigma : D_I[V, V^{-1}] \to C$ is étale, the diagonal embedding $D_I[V, V^{-1}] \to D_I[V, V^{-1}] \times_C D_I[V, V^{-1}]$ is an open and closed immersion, hence induces a decomposition $D_I[V, V^{-1}] \times_C D_I[V, V^{-1}] \simeq D_I[V, V^{-1}] \cup F$. Set $\mathcal{X}'' = \mathcal{X}\{V, V^{-1}\} \times_{\mathcal{S}} \mathcal{X}' \backslash F$. We have two étale morphisms $f : \mathcal{X}'' \to \mathcal{X}\{V, V^{-1}\}$ and $f' : \mathcal{X}'' \to \mathcal{X}'$ such that $f^{-1}(D_I[V, V^{-1}]) \simeq D_I[V, V^{-1}]$ and $f'^{-1}(D_I[V, V^{-1}]) \simeq D_I[V, V^{-1}]$. We can apply twice Proposition 3.3.18 to get that

$$]D_I^\circ[V, V^{-1}][_{\mathcal{X}\{V,V^{-1}\}} \simeq]D_I^\circ[V, V^{-1}][_{\mathcal{X}''}$$

and

$$]D_I^\circ[V, V^{-1}][_{\mathcal{X}'} \simeq]D_I^\circ[V, V^{-1}][_{\mathcal{X}''}.$$

By the choice of F, at the ring level both f and f' send V to the same element. Hence the

isomorphism $]D_I^\circ[V,V^{-1}][_{\mathcal{X}\{V,V^{-1}\}} \simeq]D_I^\circ[V,V^{-1}][_{\mathcal{X}'}$ induces an isomorphism

$$]D_I^\circ[_{\mathcal{X}} \simeq]D_I^\circ[_{\mathrm{St}^{u_I^{-1}t}_{D_I\times_k R,\underline{N}}},$$

where the above map is the composition

$$]D_I^\circ[_{\mathcal{X}} \hookrightarrow]D_I^\circ[V,V^{-1}][_{\mathcal{X}\{V,V^{-1}\}} \simeq]D_I^\circ[V,V^{-1}][_{\mathcal{X}'} \twoheadrightarrow]D_I^\circ[_{\mathrm{St}^{u_I^{-1}t}_{D_I\times_k R,\underline{N}}},$$

with the first map the inclusion of the unit section and the last one the projection forgetting the V variable. □

Remark 3.4.21. The proof of Lemma 3.4.20 also gives a definable bijection $]D_I^\circ[^{\mathrm{VF}}_{\mathcal{X}} \simeq]D_I^\circ[^{\mathrm{VF}}_{\mathcal{X}'}$, see also [79, Theorem 2.6.1] for an alternative approach.

Proof of Proposition 3.4.19. We can suppose that we are in the situation of Lemma 3.4.20, with $\mathcal{X} = \mathrm{St}^{u^{-1}t}_{D_I^\circ\times_k R,\underline{N}}$. Let N_I be the greatest common divisor of the N_i for $i \in I$. Let $N_i' = N_i/N_I$. As the N_i' are coprime, we can form an $r\times r$ matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ which first row is constituted by the N_i'. The matrices A and A^{-1} define automorphisms of $\mathbb{G}_{m_K}^{r,\mathrm{an}}$, hence of $G = D_I^\circ(R)\times \mathbb{G}_{m_K}^{r,\mathrm{an}}$. As $]D_I^\circ[_{\mathcal{X}}$ is a rigid subvariety of G, we can consider W, its image by A. Then W is the locally closed semi-algebraic subset of G defined by

$$\left\{(x,w)\in D_I(R)\times (K^\times)^r \mid w_1^{N_I}u_I(x) = t, l_1(\mathrm{v}(w)) > 0, ..., l_{r-1}(\mathrm{v}(w)) > 0\right\},$$

where the $l_i : \Gamma^r \to \Gamma$ are linearly independent affine functions of linear parts with integer coefficients. Hence $W = Q^{\mathrm{VF}}_{N_I}(D_I^\circ, u_I)\times \mathrm{v}^{-1}(\Delta)$, where $\Delta \subset \Gamma^{r-1}$ is defined by equations $l_i > 0$ for $i = 1, ..., r$.

By Propositions 3.4.11 and 3.4.12, we know that

$$\chi_{\mathrm{Rig}}(Q^{\mathrm{VF}}_{N_I}(D_I^\circ, u_I)) = [\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_I}(D_I^\circ, u_I))(-d+r-1)]$$

and

$$\chi_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)) = (-1)^{r-1}[\mathrm{M}_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)^{\mathrm{Rig}})(-r+1)].$$

Hence as χ_{Rig} is multiplicative,

$$\chi_{\mathrm{Rig}}(W) = (-1)^{r-1}[\mathrm{M}_{\mathrm{Rig}}(W^{\mathrm{Rig}})(-d)]$$

and applying the isomorphism A^{-1}, we get as required

$$\chi_{\mathrm{Rig}}(]D_I^\circ[^{\mathrm{VF}}_X) = (-1)^{|I|-1}[\mathrm{M}_{\mathrm{Rig}}(]D_I^\circ[)(-d)].$$

□

The proof of Theorem 3.4.18 is now complete.

For later use, note that the proofs of Proposition 3.4.19 and Lemma 3.4.20 gives the following equality.

Corollary 3.4.22. *With the notation of Proposition 3.4.19, we have*

$$\mathrm{M}_{\mathrm{Rig}}(]D_I^\circ[) \simeq \mathrm{M}_{\mathrm{Rig}}(Q_{N_I}^{\mathrm{Rig}}(D_I^\circ, u_I) \times \partial\mathbb{B}(o,1)^{|I|-1}).$$

3.4.5 Compatibilities of χ_{Rig}

We will now derive consequences of Theorem 3.4.18.

Theorem 3.4.23. *The morphism χ_{Rig} is the unique ring morphism*

$$\mathbf{K}(\mathrm{VF}_K) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K))$$

such that for any quasi-compact smooth rigid K-variety of pure dimension d,

$$\chi_{\mathrm{Rig}}(X^{\mathrm{VF}}) = [\mathrm{M}_{\mathrm{Rig}}(X)(-d)].$$

Proof. By quantifier elimination in the theory ACVF_K, $\mathbf{K}(\mathrm{VF}_K)$ is generated by classes of smooth affinoid rigid K-varieties, which shows uniqueness. For the existence, fix X a quasi-compact smooth rigid K-variety of pure dimension d. We can find $\mathcal{X}$, a formal R-model of X and by Hironaka's resolution of singularities, we can assume $\mathcal{X}$ is semi-stable. We can now apply Theorem 3.4.18. □

Theorem 3.4.24. *There is a commutative diagram*

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_K) \arrow[r] \arrow[d, "\chi_K"'] & \mathbf{K}(\mathrm{VF}_K) \arrow[d, "\chi_{\mathrm{Rig}}"] \\
\mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(K)) \arrow[r, "\mathrm{Rig}"'] & \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)).
\end{tikzcd}$$

Proof. By Hironaka's resolution of singularities, the ring $\mathbf{K}(\mathrm{Var}_K)$ is generated by classes of smooth projective varieties, hence it suffices to check the compatibility for such a K-variety X. Denoting $f : X \to K$ the structural morphism, one has by definition

$$\chi_K([X]) = [f_! f^* \mathbb{1}_K] = [f_\sharp f^* \mathbb{1}_K(-d)] = [\mathrm{M}_K(X)(-d)],$$

where $d = \dim(X)$ and we used $f_! = f_\sharp \circ \mathrm{Th}^{-1}(\Omega_f)$ for f smooth. Applying the functor Rig, one needs to show that $\chi_{\mathrm{Rig}}([X]^{\mathrm{VF}}) = [\mathrm{M}_{\mathrm{Rig}}(X^{\mathrm{an}})(-d)]$. As X is projective, X^{an} is a quasi-compact rigid K-variety hence one can find a semi-stable formal model of X^{an}

over R, denote it $\widetilde{X}$. Hence $X \simeq \widetilde{X}_K \simeq]\widetilde{X}_\sigma[_{\widetilde{X}}$, so by Theorem 3.4.18, $\chi_{\mathrm{Rig}}([X]^{\mathrm{VF}}) = [\mathrm{M}_{\mathrm{Rig}}(X^{\mathrm{Rig}})(-d)]$. $\square$

Note that combining Theorems 3.4.23 and 3.4.24 gives Theorem 3.1.1.

3.4.6 A few more realization maps

In this section we construct in addition to χ_{Rig} and χ'_{Rig} two more realization maps $\widetilde{\chi_{\mathrm{Rig}}}$ and $\widetilde{\chi'_{\mathrm{Rig}}}$ obtained by considering homological motives with compact support instead of cohomological motives with compact support.

Recall the ring morphism of Proposition 3.3.6 $\chi_S : \mathbf{K}(\mathrm{Var}_S) \to \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(S))$ sending $[f : X \to S]$ to $[\mathrm{M}^\vee_{S,c}(X)] = [f_! f^* \mathbb{1}_S]$.

Working dually, we can define also a morphism $\widetilde{\chi_S} : \mathbf{K}(\mathrm{Var}_S) \to \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(S))$ sending $[f : X \to S]$ to $[\mathrm{M}_{S,c}(X)] = [f_* f^! \mathbb{1}_S]$. The proof that it respect the scissors relations is similar, using exact triangle 3.3.3 instead of 3.3.2. We can also use duality involutions of the following Section 3.5.

Composing with the morphism $\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{Var}_{\mathbb{G}_{mk}})$, we get a morphism $\widetilde{\chi_{\hat{\mu}}} : \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \to \mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(\mathbb{G}_{mk}))$, fitting in the following commutative square :

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[r] \arrow[d, "\widetilde{\chi_{\hat{\mu}}}"'] & \mathbf{K}(\mathrm{Var}_k) \arrow[d, "\widetilde{\chi_k}"] \\
\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(k)).
\end{tikzcd}$$

Recall that

$$\chi_{\mathrm{Rig}} = \mathfrak{F} \circ \chi_{\hat{\mu}} \circ \Theta \circ \mathcal{E}_c \circ \oint \text{ and } \chi'_{\mathrm{Rig}} = \mathfrak{F} \circ \chi_{\hat{\mu}} \circ \Theta \circ \mathcal{E} \circ \oint .$$

We can now define

$$\widetilde{\chi_{\mathrm{Rig}}} = \mathfrak{F} \circ \widetilde{\chi_{\hat{\mu}}} \circ \Theta \circ \mathcal{E}_c \circ \oint \text{ and } \widetilde{\chi'_{\mathrm{Rig}}} = \mathfrak{F} \circ \widetilde{\chi_{\hat{\mu}}} \circ \Theta \circ \mathcal{E} \circ \oint .$$

Unraveling the definitions, we see that if X is a smooth connected k-variety of dimension d, $f \in \Gamma(X, \mathcal{O}_X^\times)$, $r \in \mathbb{N}^*$ and $\Delta \subset \Gamma^n$ an open simplex of dimension n,

$$\widetilde{\chi_{\mathrm{Rig}}}(Q_r^{\mathrm{VF}}(X, f)) = [\mathrm{M}^\vee_{\mathrm{Rig}}(Q_r^{\mathrm{Rig}}(X, f))(d)],$$

$$\widetilde{\chi'_{\mathrm{Rig}}}(Q_r^{\mathrm{VF}}(X, f)) = [\mathrm{M}^\vee_{\mathrm{Rig}}(Q_r^{\mathrm{Rig}}(X, f))],$$

$$\widetilde{\chi_{\mathrm{Rig}}}(\mathrm{v}^{-1}(\Delta)) = (-1)^d [\mathrm{M}^\vee_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)^{\mathrm{Rig}})(d)],$$

and

$$\widetilde{\chi'_{\mathrm{Rig}}}(Q_r^{\mathrm{VF}}(X, f)) = (-1)^d [\mathrm{M}^\vee_{\mathrm{Rig}}(\mathrm{v}^{-1}(\Delta)^{\mathrm{Rig}})].$$

See the proofs of Propositions 3.4.11 and 3.4.12 for details.

Hence the proof of Theorem 3.4.18 can be adapted to χ'_{Rig}, $\widetilde{\chi_{\mathrm{Rig}}}$ and $\widetilde{\chi'_{\mathrm{Rig}}}$, showing in particular that if X is a quasi-compact smooth connected rigid K-variety of dimension d,

$$\chi_{\mathrm{Rig}}(X^{\mathrm{VF}}) = [\mathrm{M}_{\mathrm{Rig}}(X)(-d)], \quad \chi'_{\mathrm{Rig}}(X^{\mathrm{VF}}) = [\mathrm{M}_{\mathrm{Rig}}(X)],$$

$$\widetilde{\chi_{\mathrm{Rig}}}(X^{\mathrm{VF}}) = [\mathrm{M}^{\vee}_{\mathrm{Rig}}(X)(d)], \quad \widetilde{\chi'_{\mathrm{Rig}}}(X^{\mathrm{VF}}) = [\mathrm{M}^{\vee}_{\mathrm{Rig}}(X)].$$

If X is a proper algebraic K-variety of structural morphism f, since $f_* = f_!$, we have $\mathrm{M}^{\vee}_{K,c}(X) = \mathrm{M}^{\vee}_K(X)$ and $\mathrm{M}^{\vee}_{K,c}(X) = \mathrm{M}_K(X)$, hence we can adapt the proof of Theorem 3.4.24 to get commutative diagrams similar of Theorem 3.1.1, the first one being the statement of Theorem 3.1.1.

Proposition 3.4.25. *The squares in the following diagrams commutes :*

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_K) \arrow[r] \arrow[d, "\chi_K"'] & \mathbf{K}(\mathrm{VF}_K) \arrow[r, "\Theta\circ\mathcal{E}_c\circ\oint"] \arrow[d, "\chi_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[r] \arrow[d, "\chi_{\hat{\mu}}"] & \mathbf{K}(\mathrm{Var}_k) \arrow[d, "\chi_k"] \\
\mathbf{K}(\mathrm{SH}(K)) \arrow[r, "\mathrm{Rig}^*"'] & \mathbf{K}(\mathrm{RigSH}(K)) \arrow[r, "\simeq", "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}(k)),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_K) \arrow[r] \arrow[d, "\widetilde{\chi_K}"'] & \mathbf{K}(\mathrm{VF}_K) \arrow[r, "\Theta\circ\mathcal{E}\circ\oint"] \arrow[d, "\chi'_{\mathrm{Rig}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})[\mathbb{L}^{-1}] \arrow[r] \arrow[d, "\chi_{\hat{\mu}}"] & \mathbf{K}(\mathrm{Var}_k)[\mathbb{L}^{-1}] \arrow[d, "\chi_k"] \\
\mathbf{K}(\mathrm{SH}(K)) \arrow[r, "\mathrm{Rig}^*"'] & \mathbf{K}(\mathrm{RigSH}(K)) \arrow[r, "\simeq", "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}(k)),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_K) \arrow[r] \arrow[d, "\widetilde{\chi_K}"'] & \mathbf{K}(\mathrm{VF}_K) \arrow[r, "\Theta\circ\mathcal{E}_c\circ\oint"] \arrow[d, "\widetilde{\chi_{\mathrm{Rig}}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}) \arrow[r] \arrow[d, "\widetilde{\chi_{\hat{\mu}}}"] & \mathbf{K}(\mathrm{Var}_k) \arrow[d, "\widetilde{\chi_k}"] \\
\mathbf{K}(\mathrm{SH}(K)) \arrow[r, "\mathrm{Rig}^*"'] & \mathbf{K}(\mathrm{RigSH}(K)) \arrow[r, "\simeq", "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}(k)),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{K}(\mathrm{Var}_K) \arrow[r] \arrow[d, "\chi_K"'] & \mathbf{K}(\mathrm{VF}_K) \arrow[r, "\Theta\circ\mathcal{E}\circ\oint"] \arrow[d, "\widetilde{\chi'_{\mathrm{Rig}}}"'] & \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})[\mathbb{L}^{-1}] \arrow[r] \arrow[d, "\widetilde{\chi_{\hat{\mu}}}"] & \mathbf{K}(\mathrm{Var}_k)[\mathbb{L}^{-1}] \arrow[d, "\widetilde{\chi_k}"] \\
\mathbf{K}(\mathrm{SH}(K)) \arrow[r, "\mathrm{Rig}^*"'] & \mathbf{K}(\mathrm{RigSH}(K)) \arrow[r, "\simeq", "\mathfrak{R}"'] & \mathbf{K}(\mathrm{QUSH}(k)) \arrow[r, "1^*"'] & \mathbf{K}(\mathrm{SH}(k)).
\end{tikzcd}$$

In particular, we see that χ_{Rig} and $\widetilde{\chi'_{\mathrm{Rig}}}$ agree on the image of $\mathbf{K}(\mathrm{Var}_K)$ and similarly χ'_{Rig} and $\widetilde{\chi_{\mathrm{Rig}}}$ agree on the image of $\mathbf{K}(\mathrm{Var}_K)$.

Remark 3.4.26 (Volume forms). In addition of the additive morphism $\oint$, Hrushovski and Kazhdan also study the Grothendieck ring of definable sets with volume forms $\mathbf{K}(\mu_\Gamma \mathrm{VF}_K)$. Objects in $\mu_\Gamma \mathrm{VF}_K$ are pairs (X, ω) with $X \subseteq \mathrm{VF}^\bullet$ a definable set and $\omega : X \to \Gamma$ a

definable function. Morphisms are measure preserving definable bijections (up to a set of lower dimension). In this context, they build an isomorphism

$$\oint^{\mu} : \mathbf{K}(\mu_\Gamma \mathrm{VF}_K) \to \mathbf{K}(\mu_\Gamma \mathrm{RV}_K)/\mu \mathrm{I}_{\mathrm{sp}},$$

see [54, Theorem 8.26].

One can further decompose $\mathbf{K}(\mu_\Gamma \mathrm{RV}_K)$ similarly to Proposition 3.2.5. Using this Hruskovski and Loeser define in [55] for $m \in \mathbb{N}$ morphisms

$$h_m : \mathbf{K}(\mu_\Gamma \mathrm{RV}_K^{\mathrm{bdd}})/\mu \mathrm{I}_{\mathrm{sp}} \to \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})_{\mathrm{loc}}$$

with the m related to considering rational points in $k((t^{1/m}))$. Here, bdd means we consider only bounded sets. Note that there is an inacurracy in the definition of h_m in [55] since they use [54, Proposition 10.10 (2)] which happens to be incorrect. Using the category of bounded sets with volume forms deals with the issue.

We can further compose with the morphism $\mathfrak{F} \circ \chi_{\hat{\mu}}$ in order to get for each $m \in \mathbb{N}^*$ a morphism

$$\mathbf{K}(\mu_\Gamma \mathrm{VF}_K^{\mathrm{bdd}}) \to \mathbf{K}(\mathrm{RigSH}_{\mathfrak{M}}(K)).$$

Such morphisms do not seem to satisfy properties similar to those of χ_{Rig}.

3.5 Duality

The goal of this section is to prove Theorem 3.1.2. We will adapt Bittner's results on duality in the Grothendieck group of varieties in Section 3.5.1 in order to be able to compute in Section 3.5.2 explicitly the cohomological motive of some tubes in terms of homological motives. The last Section 3.5.3 is devoted to an application to the motivic Milnor fiber and analytic Milnor fiber.

3.5.1 Duality involutions

Bittner developed in [8] an abstract theory of duality in the equivariant Grothendieck group of varieties. We recall here some of her results and show that they imply similar results for $\mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(K))$.

Using the weak factorization theorem, Bittner prove the following alternative description of $\mathbf{K}(\mathrm{Var}_X)$. We state if for a variety S above K, but it holds for varieties above any field of characteristic zero.

Proposition 3.5.1 ([8, Theorem 5.1])**.** *Fix S, a K-variety. The group $\mathbf{K}(\mathrm{Var}_S)$ is isomorphic to the abelian group generated by classes of S-varieties which are smooth over K, proper over S, subject to the relations $[\emptyset]_S = 0$ and $[\mathrm{Bl_Y(X)}]_S - [E]_S = [X]_S - [Y]_S$, where*

X is smooth over K, proper over S, $Y \subset X$ a closed smooth subvariety, $\mathrm{Bl_Y(X)}$ *is the blow-up of X along Y and E is the exceptional divisor of this blow-up.*

For $f : X \to Y$ a morphism of S-varieties, composition with f induce a (group) morphism $f_! : \mathbf{K}(\mathrm{Var}_X) \to \mathbf{K}(\mathrm{Var}_Y)$ and pull-back along f induces a (group) morphism $f^* : \mathbf{K}(\mathrm{Var}_Y) \to \mathbf{K}(\mathrm{Var}_X)$. Both induces $\mathcal{M}_S$-linear morphisms $f_! : \mathcal{M}_X \to \mathcal{M}_Y$ and $f^* : \mathcal{M}_Y \to \mathcal{M}_X$, where $\mathcal{M}_X = \mathbf{K}(\mathrm{Var}_X)[\mathbb{L}^{-1}]$.

Definition 3.5.2. We define now a duality operator $\mathcal{D}_X : \mathbf{K}(\mathrm{Var}_X) \to \mathcal{M}_X$ for any K-variety X. Define $\mathcal{D}_X([Y]) = [Y]\mathbb{L}^{-\dim(Y)}$ if Y is an X-variety proper over X, connected and smooth over K. In view of Proposition 3.5.1, to show that it induces an unique (group) morphism $\mathbf{K}(\mathrm{Var}_X) \to \mathcal{M}_X$, it suffices to show that if $Y \subset Z$ is a closed immersion of X-varieties, proper over X, smooth and connected over K,

$$[\mathrm{Bl}_Y(Z)]\mathbb{L}^{-\dim(Z)} - [E]\mathbb{L}^{-\dim(Z)+1} = [Z]\mathbb{L}^{-\dim(Z)} - [Y]\mathbb{L}^{-\dim(Y)},$$

it holds since $(\mathbb{L} - 1)[E] = (\mathbb{L}^{\dim(Z)-\dim(Y)} - 1)[Y]$.

See [8, Definition 6.3] for details.

Observe that $\mathcal{D}_X(\mathbb{L}) = \mathbb{L}^{-1}$ and that $\mathcal{D}_X$ is $\mathcal{D}_S$-linear, hence $\mathcal{D}_X$ can be extended as a $\mathcal{D}_K$-linear morphism $\mathcal{D}_X : \mathcal{M}_X \to \mathcal{M}_X$, which is an involution.

Although $\mathcal{D}_X$ is not in general a ring morphism, $\mathcal{D}_K$ is a ring morphism.

For $f : X \to Y$, set $f^! = \mathcal{D}_X f^* \mathcal{D}_Y$ and $f_* = \mathcal{D}_Y f_! \mathcal{D}_X$. Observe that if f is proper, $f_! = f_*$ and if f is smooth of relative dimension d over S, $f^! = \mathbb{L}^{-d} f^*$.

Such a duality operator can also be defined in the Grothendieck ring of varieties equipped with a good action of some finite group G, see [8, Sections 7,8] for details.

In $\mathrm{SH}_{\mathfrak{M}}(K)$, the internal hom gives also notion of duality, define the duality functor $\mathbb{D}_K(A) = \underline{\mathrm{Hom}}_K(A, \mathbb{1}_K)$. As $\mathbb{D}_K$ is triangulated, it induces a morphism on $\mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(S))$, still denoted $\mathbb{D}_K$. By [1, Théorème 2.3.75], $\mathbb{D}_K$ is an autoequivalence on constructible objects and its own quasi-inverse. In particular, (compactly supported) (co)homological motives of S-varieties are constructible, hence $\mathbb{D}_K$ is an involution. One can also define more generally for $a : X \to \mathrm{Spec}(K)$, $\mathbb{D}_X(A) = \underline{\mathrm{Hom}}_X(A, a^!\mathbb{1}_K)$, but we will not use those. By [1, Théorème 2.3.75], $\mathbb{D}_K(\mathrm{M}_K(X)) = \mathrm{M}_K^\vee(X)$ for any K-variety X.

The following proposition shows the compatibility between those two duality operators.

Proposition 3.5.3. *There is a commutative diagram*

$$\begin{tikzcd}
\mathcal{M}_K \arrow[r, "\mathcal{D}_K"] \arrow[d, "\chi_K"'] & \mathcal{M}_K \arrow[d, "\chi_K"] \\
\mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(K)) \arrow[r, "\mathbb{D}_K"'] & \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(K)).
\end{tikzcd}$$

Proof. It suffices to show that $\chi_K\mathcal{D}_K([X]) = \mathbb{D}_K\chi_K([X])$ for X a connected, smooth and proper K-variety, of dimension d. Denote $f : X \to \mathrm{Spec}(K)$. As f is smooth and proper, $[\mathrm{M}^\vee_{K,c}(X)] = [\mathrm{M}^\vee_K(X)] = [\mathrm{M}_K(X)(-d)]$. We then have

$$\chi_K\mathcal{D}_K([X]) = \chi_K([X]\mathbb{L}^{-d}) = [\mathrm{M}^\vee_{K,c}(X)(d)] = [\mathrm{M}_K(X)]$$

and

$$\mathbb{D}_K\chi_K([X]) = \mathbb{D}_K([\mathrm{M}^\vee_{K,c}(X)]) = \mathbb{D}_K([\mathrm{M}^\vee_K(X)]) = [\mathrm{M}_K(X)].$$

□

All our duality results will ultimately boils down to the following lemma, which states that normal toric varieties satisfy Poincare duality. It is due to Bittner, see [9, Lemma 4.1]. The proof rely on toric resolution of singularities and the Dehn-Sommerville equations, see for example [47].

Lemma 3.5.4. *Let X an affine toric K-variety associated to a simplicial cone, $X \to Y$ be a proper morphism, G a finite group acting on X via the torus with trivial action on Y. Then*

$$\mathcal{D}_Y([X]) = [X]\mathbb{L}^{-\dim(X)} \in \mathcal{M}^G_Y.$$

For the rest of the section, we fix a semi-stable formal R-scheme $\mathcal{X}$ and let $(D_i)_{i\in J}$ be the branches of its special fiber $\mathcal{X}_\sigma$. Fix $I \subset J$, up to reordering the coordinates, suppose $I = \{1, ..., k\}$. Recall that around every closed point $x \in D^\circ_I$, there is a Zariski open neighborhood $\mathcal{U}$ and regular functions $u_I, x_1, ..., x_k$ such that $u_I \in \mathcal{O}^\times(\mathcal{U})$ and $t = u_I x_1^{N_1} ... x_k^{N_k}$, with the branch D_i defined by $x_i = 0$. Still denote $u_I, x_1, ..., x_k$ their reductions to $U = \mathcal{U}_\sigma$. The various u_I glue to define a section $u_I \in \Gamma(D^\circ_I, \mathcal{O}^\times_{D^\circ_I}/(\mathcal{O}^\times_{D^\circ_I})^{N_I})$. Recall that N_I is the greatest common divisor of the N_i, $i \in I$.

We already considered (the analytification of) the K-variety

$$Q^{\mathrm{geo}}_{N_I}(D^\circ_I, u_I) = D^\circ_I \times_k K[V]/(V^{N_I} - tu_I).$$

In this section, we will denote $\widetilde{D^\circ_I} = Q^{\mathrm{geo}}_{N_I}(D^\circ_I, u_I)$ to simplify the notations. We will also abuse the notations and still denote D°_I, D_I, U the base change to K of those varieties.

Let $\widetilde{D_I}$ be the normalization of D_I in $\widetilde{D^\circ_I}$.

We also set for $K \subset I$, $\widetilde{D_I}_{|D_K} = \widetilde{D_I} \times_{D_I} D_K$

Proposition 3.5.5. *For every $I \subset K \subset J$, we have $\widetilde{D_I}_{|D_K} \simeq \widetilde{D_K}$ and $\mathcal{D}_{D_I}[\widetilde{D_I}] = \mathbb{L}^{|I|-d+1}[\widetilde{D_I}]$.*

Observe that Bittner's Lemma 5.2 in [9] is analogous, but holds in $\mathbf{K}(\mathrm{Var}^{\hat{\mu}}_k)$. Since it is not a priori clear that dualities on $\mathbf{K}(\mathrm{Var}^{\hat{\mu}}_k)$ and $\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k))$ are compatible, we cannot apply directly her Lemma. We will nevertheless follow closely her proof.

Combination of Propositions 3.5.5 and 3.5.3 yields the following corollary.

Corollary 3.5.6. *For any $I \subset J$ such that D_I is proper, we have the equality in $\mathbf{K}(\mathrm{Var}_K)$*

$$[\widetilde{D_I}] = \sum_{I \subset K \subset J} [\widetilde{D_K^\circ}],$$

and $[\mathbb{D}_K(\mathrm{M}^\vee_{K,c}(\widetilde{D_I}))] = [\mathrm{M}^\vee_{K,c}(\widetilde{D_I})(d - |I| + 1)] \in \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(K))$.

Proof. The first equality is the first point of Proposition 3.5.5. For the second equality, since D_I is proper, denoting $f : D_I \times_k \mathbb{G}_{mk} \to \mathbb{G}_m$, we have $f_! \mathcal{D}_{D_I} = \mathcal{D}_K f_!$ hence by the second point of Proposition 3.5.5,

$$\mathcal{D}_K([\widetilde{D_I}]) = [\widetilde{D_I}]\mathbb{L}^{-d+|I|-1}.$$

Since $\widetilde{D_I}$ is proper over K, we have $\mathrm{M}^\vee_K(\widetilde{D_I}) = \mathrm{M}^\vee_{K,c}(\widetilde{D_I})$, hence the result follows from Proposition 3.5.3 after applying a Tate twist. □

Proof of Proposition 3.5.5. Working inductively on the codimension of D_K in D_I and up to reordering the branches, we can suppose $I = \{1, ..., k-1\} \subset K = \{1, ..., k\}$. As the statement is local we can work on an open neighborhood U of some closed point of D_K and choose a system of local coordinates $x_1, ..., x_d$ on U such that $u_I x_1^{N_1} ... x_{k-1}^{N_{k-1}} = u_K x_1^{N_1} ... x_k^{N_k} = u x_1^{N_1} ... x_d^{N_d}$, where $u \in \mathcal{O}^\times(U)$ and for $i = 1, ..., k$, D_i is defined by equation $x_i = 0$. Define a μ_{N_1}-étale cover of U by

$$V = U[Z]/(Z^{N_1} - tu).$$

We consider V as a variety with a μ_{N_1}-action induced by multiplication of z by $\zeta \in \mu_{N_1}$.

Then $y_1 = sx_1, y_2 = x_2, ..., x_d = y_{d+1}$ is a system of local coordinates on V. Shrinking U, we can assume the morphism $V \to \mathbb{A}_k^{d+1}$ induced by $y_1, ..., y_{d+1}$ is étale. Denote by F_I, F_I°, F_K, F_K° the pull-backs of D_I, D_I°, D_K, D_K°.

Denote by $\widetilde{F_I^\circ}$ the following étale cover of F_I° :

$$\widetilde{F_I^\circ} = F_I^\circ[W]/(W^{N_I} - y_{k+1}^{N_{k+1}} ... y_d^{N_d}).$$

Observe that $\widetilde{F_I^\circ}$ is isomorphic to the fiber product $(F_I^\circ) \times_{D_I^\circ} \widetilde{D_I^\circ}$. The variety $\widetilde{F_I^\circ}$ is equipped with a μ_{N_1}-action, with $\zeta \in \mu_{N_1}$ acting on w by multiplication by ζ^{N_1/N_I}.

Denote by $\widetilde{F_I}$ the normalization of F_I in $\widetilde{F_I^\circ}$ and consider the following diagram:

$$\begin{tikzcd} p^*\widetilde{D_I} \arrow[r] \arrow[d] & \widetilde{D_I} \arrow[d] \\ F_I \arrow[r, "p"'] & D_I. \end{tikzcd}$$

As $p : F_I \to D_I$ is smooth and $\widetilde{D_I}$ is normal, $p^*\widetilde{D_I}$ is normal. As $p^*\widetilde{D_I} \to F_I$ is finite

and surjective, $p^*\widetilde{D_I}$ is isomorphic to $\widetilde{F_I}$. Denoting $\widetilde{F_I}/\mu_{N_1}$ the quotient of $\widetilde{F_I}$ by the μ_{N_1}-action, we then have $\widetilde{D_I} \simeq \widetilde{F_I}/\mu_{N_1}$ and similarly $\widetilde{D_K} \simeq \widetilde{F_K}/\mu_{N_1}$. Hence it suffices to show that $\widetilde{F_I}_{|F_K} \simeq \widetilde{F_K}$ and $\mathcal{D}_{F_I}[\widetilde{F_I}] = \widetilde{F_I}\mathbb{L}^{-n+k-1}$, both equalities being compatible with the μ_{N_1}-actions.

Now consider the étale morphism $\pi : V \to \mathbb{A}^{d+1}_K$. Denoting $z_1, ..., z_{d+1}$ the coordinates of $\mathbb{A}^{d+1}_K$, define $C_I \subseteq \mathbb{A}^{d+1}_{\mathbb{G}_{mk}}$ by the equations $z_1 = ... = z_{k-1} = 0$ and $C_I^\circ = C_I \setminus \cup_{j=k,...d+1} \{z_j = 0\}$, define similarly C_K and C_K°.

Define an étale cover of C_I° by

$$\widetilde{C_I^\circ} = C_I^\circ[S]/(S^{N_I} - z_k^{N_k}...z_{d+1}^{N_{d+1}}),$$

define similarly $\widetilde{C_K^\circ}$ and let $\widetilde{C_I}$ and $\widetilde{C_K}$ be the normalizations of C_I and C_K in $\widetilde{C_I^\circ}$ and $\widetilde{C_K^\circ}$.

We then have a Cartesian diagram

$$\begin{tikzcd} \widetilde{F_I} \arrow[r] \arrow[d] & \widetilde{C_I} \arrow[d] \\ F_I \arrow[r, "\pi"'] & C_I \end{tikzcd}$$

with π étale, hence it suffices to show that $\widetilde{C_I}_{|C_K} \simeq \widetilde{C_K}$ and $\mathcal{D}_{C_I}[\widetilde{C_I}] = \widetilde{C_I}\mathbb{L}^{-n+k-1}$, both equalities being compatible with the μ_{N_1}-actions ($\zeta \in \mu_{N_1}$ acts on $\widetilde{C_I^\circ}$ by multiplication of s by ζ^{N_1/N_I}). Indeed, since π is étale, we have $\pi * \mathcal{D}_{C_I} = \mathcal{D}_{F_I \times_k \mathbb{G}_{mk}} \pi^*$.

Since the projection $\mathbb{A}^{d+1} \to \mathbb{A}^{d+1-k+1}$ is smooth, the result now follows from the following Lemmas 3.5.7 and 3.5.8 which correspond to Lemmas 5.3 and 5.4 in [9]. □

Lemma 3.5.7. *The restriction of the normalization $\widetilde{S}$ of $S = \left\{s^N = x_1^{a_1}...x_d^{a_d}\right\} \subset \mathbb{A}^1_k \times_k \mathbb{A}^d_{\mathbb{G}_{mk}}$ to $\{x_1 = 0\} \subset \mathbb{A}^d_{\mathbb{G}_{mk}}$ is isomorphic to the normalization $\widetilde{S'}$ of $S' = \left\{s^{N'} = x_2^{a_2}...x_d^{N_d}\right\}$, where $N' = \gcd(N, a_1)$. If N divides some $q \in \mathbb{N}^*$, then the isomorphism is compatible with the μ_q-actions on $\widetilde{S}$ and $\widetilde{S'}$ where $\zeta \in \mu_q$ acts on S by multiplication of s by $\zeta^{q/N}$ and on S' by multiplication of s by $\zeta^{q/N'}$.*

Proof. Assume first that $N, a_1, ..., a_d$ are coprime. Then S is irreducible. Let M be the lattice of $\mathbb{R}^d$ spanned by $\mathbb{Z}^d$ and $v = (a_1/N, ..., a_d/N)$. Set $M^+ = M \cap \mathbb{R}^d_+$ Then $\widetilde{S} \simeq \mathrm{Spec}(k[M^+])$. If $M_1 := \{u \in M \mid u_1 = 0\}$ and $M_1^+ = M_1 \cap \mathbb{R}^d_+$, then $\mathrm{Spec}(k[M_1^+])$ is the restriction of $\widetilde{S}$ to $\{x_1 = 0\}$.

Now consider the lattice M' generated by $v' = (0, a_2/N', ..., a_d/N')$ and $\{0\} \times \mathbb{Z}^{d-1}$, and set $M'^+ = M' \cap \mathbb{R}^d_+$. We have $\widetilde{S'} \simeq \mathrm{Spec}(k[M'^+])$, hence it suffices to show that $M' \simeq M_1$.

Denote $e_1, ..., e_d$ the canonical basis of $\mathbb{Z}^d$. Set $k = a_1/N'$ and $l = N/N'$. Observe that $v' = N/N'v - a_1/N'e_1 = lv - ke_1$, hence $M' \subseteq M_1$. Reciprocally, if $u = \sum_{i=1}^d \lambda_i e_i + \mu v \in M_1$, then $\lambda_1 + \mu k/l \in \mathbb{Z}$, hence $\mu' = \mu/l \in \mathbb{Z}$ (since k and ℓ are coprime), hence

$u = \sum_{i=2}^{d} \lambda_i e_i + \mu' v' \in M'$.

The μ_q-actions are compatible, since $s'^{N'} = s^N x_1^{-a_1}$.

Back to the general case, let c be the greatest common divisor of $N, a_1, ..., a_d$. Define $e = N/c$, $a_i' = a_i/c$, $e' = N'/c$. Let $\widetilde{T}$ be the normalization of $T = \left\{ s^e = x_1^{a_1'} ... x_d^{a_d'} \right\}$ and $\widetilde{T'}$ be the normalization of $T' = \left\{ s^{e'} = x_2^{a_2'} ... x_d^{a_d'} \right\}$. Both $\widetilde{T}$ and $\widetilde{T'}$ are equipped with a μ_e action as in the statement of the lemma.

The mapping

$$(\zeta, s, x) \in \mu_N \times T \to (\zeta s, x) \in S$$

induce an isomorphism

$$(\mu_N \times \widetilde{T})/\mu_e \simeq \widetilde{S},$$

where the μ_N action on $\widetilde{S}$ correspond to the action on $(\mu_N \times \widetilde{T})/\mu_e$ given by multiplication on μ_N. Similarly, the mapping

$$(\zeta, s, x) \in \mu_N \times T' \to (\zeta^{N/N'} s, x) \in S'$$

induce an isomorphism

$$(\mu_N \times \widetilde{T'})/\mu_e \simeq \widetilde{S'}.$$

Hence we can apply the first case to $\widetilde{T}$ and $\widetilde{T'}$ to get $\widetilde{S}_{|x_1=0} \simeq \widetilde{S'}$ and check that the actions correspond. □

Lemma 3.5.8. *Denote again by $\widetilde{S}$ the normalization of $S = \left\{ s^N = x_1^{a_1} ... x_d^{a_d} \right\}$. Then $\mathcal{D}_{\mathbb{A}_K^d}[\widetilde{S}] = [\widetilde{S}]\mathbb{L}^{-d} \in \mathcal{M}_{\mathbb{A}_K^d}^{\mu_q}$, $\zeta \in \mu_q$ acting again by multiplication of s by $\zeta^{q/N}$.*

Proof. It is a particular case of Lemma 3.5.4 when $N, a_1, ..., a_d$ are coprime, and the general case follows as in the proof of Lemma 3.5.7. □

3.5.2 Computation of cohomological motives

Using Corollary 3.5.6, we can now compute the cohomological motives of $\widetilde{D_I^\circ}$ in terms of their homological motives.

Proposition 3.5.9. *For any $I \subset J$ such that D_i is proper for $i \in I$, we have*

$$[\mathrm{M}_K^\vee(\widetilde{D_I^\circ})] = \sum_{I \subset L \subset J} (-1)^{|L|-|I|} [\mathrm{M}_K(\widetilde{D_L^\circ} \times_K \mathbb{G}_m{}_K^{|L|-|I|})(-d + |I| - 1)] \in \mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(K)).$$

We first prove an auxiliary formula.

Lemma 3.5.10. *For any $I \subset J$ such that D_i is proper for $i \in I$, we have $[\mathrm{M}_K^\vee(\widetilde{D_I^\circ})] =$*

$$[\mathrm{M}_K(\widetilde{D_I^\circ})(-d + |I| - 1)] + \sum_{I \subsetneq L \subset J} ([\mathrm{M}_K(\widetilde{D_L^\circ})(-d + |L| - 1)] - [\mathrm{M}_K^\vee(\widetilde{D_L^\circ})(|I| - |L|)]).$$

Proof. By the first point of Corollary 3.5.6 and additivity of $\mathrm{M}^\vee_{K,c}(-)$, we have

$$[\mathrm{M}^\vee_{K,c}(\widetilde{D^\circ_I})] = [\mathrm{M}^\vee_{K,c}(\widetilde{D_I})] - \sum_{I \subsetneq L \subset J} [\mathrm{M}^\vee_{K,c}(\widetilde{D^\circ_L})]. \tag{3.5.1}$$

As each of the $\widetilde{D^\circ_L}$ is smooth of pure dimension $d-|L|+1$, by Proposition 3.3.7, $[\mathrm{M}^\vee_{K,c}(\widetilde{D^\circ_L})] = [\mathrm{M}_K(\widetilde{D^\circ_L})(-d+|L|-1)]$. We apply $\mathbb{D}_K$ to equation 1. By linearity of $\mathbb{D}_K$, the fact that $\mathbb{D}_K\mathrm{M}_K(\widetilde{D^\circ_L})(-d+|L|-1) = \mathrm{M}^\vee_K(\widetilde{D^\circ_L})(+d-|L|+1)$ and second point of Corollary 3.5.6, we get

$$[\mathrm{M}^\vee_K(\widetilde{D^\circ_I})(d-|I|+1)] = [\mathrm{M}^\vee_{K,c}(\widetilde{D_I})(d-|I|+1)] - \sum_{I \subsetneq L \subset J} [\mathrm{M}^\vee_K(\widetilde{D^\circ_L})(d-|L|+1)]. \tag{3.5.2}$$

Twisting this equation $d-|I|+1$ times and applying again Corollary 3.5.6 gives the desired equation. □

Proof of Proposition 3.5.9. We work by induction on $d-|I|+1$. If $d-|I|+1<0$, then $\widetilde{D^\circ_I}$ is empty and there is nothing to show. If $d-|I|+1=0$, the the formula boils down to

$$[\mathrm{M}^\vee_K(\widetilde{D^\circ_I})] = [\mathrm{M}_K(\widetilde{D^\circ_I})],$$

which holds since $\widetilde{D^\circ_I}$ is of dimension 0.

Suppose now the proposition holds for any L with $|L|>|I|$. Let $r=d-|I|+1$.

By Lemma 3.5.10,

$$[\mathrm{M}^\vee_K(\widetilde{D^\circ_I})] = [\mathrm{M}_K(\widetilde{D^\circ_I})(-r)] + \sum_{I \subsetneq L \subset J} ([\mathrm{M}_K(\widetilde{D^\circ_L})(-d+|L|-1)] - [\mathrm{M}^\vee_K(\widetilde{D^\circ_L})(|I|-|L|)]).$$

Applying the induction hypothesis to the $[\mathrm{M}^\vee_K(\widetilde{D^\circ_L})]$, we get

$$\begin{aligned}[\mathrm{M}^\vee_K(\widetilde{D^\circ_I})] = [\mathrm{M}_K(\widetilde{D^\circ_I})(-r)] + \sum_{I \subsetneq L \subset J} \Bigg(& [\mathrm{M}_K(\widetilde{D^\circ_L})(-d+|L|-1)] \\ & - \sum_{L \subset L' \subset J} (-1)^{|L'|-|L|}[\mathrm{M}_K(\widetilde{D^\circ_{L'}} \times_K \mathbb{G}_m{}_K^{|L'|-|L|})(-d+|I|-1)]\Bigg).\end{aligned}$$

Interverting the sums, we get

$$\begin{aligned}[\mathrm{M}^\vee_K(\widetilde{D^\circ_I})] = [\mathrm{M}_K(\widetilde{D^\circ_I})(-r)] &+ \sum_{I \subsetneq L \subset J} [\mathrm{M}_K(\widetilde{D^\circ_L})(-d+|L|-1)] \\ &- \sum_{I \subsetneq L' \subset J} \sum_{i=0}^{|L'|-|I|-1} \binom{|L'|-|I|}{|L'|-|I|-i} (-1)^i [\mathrm{M}_K(\widetilde{D^\circ_{L'}} \times_K \mathbb{G}_m{}_K^i)(-r)].\end{aligned}$$

Regrouping the terms, we get

$$[\mathrm{M}_K^\vee(\widetilde{D_I^\circ})] = [\mathrm{M}_K(\widetilde{D_I^\circ})(-r)] + \sum_{I\subsetneq L'\subset J} [\mathrm{M}_K(\widetilde{D_{L'}^\circ})(-r)]$$
$$\cdot \left([\mathrm{M}_K(\mathbb{1})(|L'|-|I|)] - \sum_{i=0}^{|L'|-|I|-1} \binom{|L'|-|I|}{i}(-1)^i[\mathrm{M}_K(\mathbb{G}_{m\,K}^{\,i})]\right). \quad (3.5.3)$$

We need to compute the expression inside the big brackets. We have

$$[\mathrm{M}_K(\mathbb{1})(|L'|-|I|)] - \sum_{i=0}^{|L'|-|I|-1} \binom{|L'|-|I|}{i}(-1)^i[\mathrm{M}_K(\mathbb{G}_{m\,K}^{\,i})]$$
$$= [\mathrm{M}_K(\mathbb{1})(|L'|-|I|)] + (-1)^{|L'|-|I|}[\mathrm{M}_K(\mathbb{G}_{m\,K}^{\,|L'|-|I|})] - \big([\mathrm{M}_K(\mathbb{1})] - [\mathrm{M}_K(\mathbb{G}_{m\,K})]\big)^{|L'|-|I|}$$
$$= (-1)^{|L'|-|I|}[\mathrm{M}_K(\mathbb{G}_{m\,K}^{\,|L'|-|I|})]$$

because $[\mathrm{M}_K(\mathbb{G}_{m\,K})] = [\mathrm{M}_K(\mathbb{1})] - [\mathrm{M}_K(\mathbb{1})(1)]$. Injecting in 3.5.3 gives the required equation for $[\mathrm{M}_K^\vee(\widetilde{D_I^\circ})]$. □

Proposition 3.5.11. *Let $\mathcal{X}$ be a semi-stable formal R-scheme and $(D_i)_{i\in I}$ the reduced irreducible components of its special fiber. Let $J' \subset J$ such that for every $i \in J'$, D_i is proper. Then setting $D = \bigcup_{i\in J'} D_i$, we have*

$$\chi_{\mathrm{Rig}}(]D[_{\mathcal{X}}^{\mathrm{VF}}) = [\mathrm{M}_{\mathrm{Rig}}^\vee(]D[_{\mathcal{X}})].$$

Proof. By additivity of χ_{Rig}, Proposition 3.4.19 and Corollary 3.4.22, we have

$$\begin{aligned}\chi_{\mathrm{Rig}}(]D[_{\mathcal{X}}^{\mathrm{VF}}) &= \sum_{\substack{I\subset J\\ I\cap J'\neq\emptyset}} \chi_{\mathrm{Rig}}(]D_I^\circ[^{\mathrm{VF}})\\ &= \sum_{\substack{I\subset J\\ I\cap J'\neq\emptyset}} (-1)^{|I|-1}[\mathrm{M}_{\mathrm{Rig}}(]D_I^\circ[)(-d)]\\ &= \sum_{\substack{I\subset J\\ I\cap J'\neq\emptyset}} (-1)^{|I|-1}[\mathrm{M}_{\mathrm{Rig}}(Q_{N_I}^{\mathrm{Rig}}(D_I^\circ, u_I)\times \partial\mathbb{B}(o,1)^{|I|-1})(-d)]. \quad (3.5.4)\end{aligned}$$

We will relate the cohomological motive of the tube to this formula using the duality relations proven above. By Corollary 3.4.17, we have

$$[\mathrm{M}_{\mathrm{Rig}}^\vee(]D[)] = \sum_{I\subseteq J'} (-1)^{|I|-1}[\mathrm{M}_{\mathrm{Rig}}^\vee(]D_I^\circ[)]. \quad (3.5.5)$$

By Corollary 3.4.22,

$$\begin{aligned}[\mathrm{M}^\vee_{\mathrm{Rig}}(]D_I^\circ[)] &= [\mathrm{M}^\vee_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_I}(D_I^\circ, u_I) \times \partial\mathbb{B}(o,1)^{|I|-1})] \\ &= [\mathrm{M}^\vee_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_I}(D_I^\circ, u_I))] \cdot [\mathrm{M}^\vee_{\mathrm{Rig}}(\partial\mathbb{B}(o,1)^{|I|-1})]. \end{aligned} \tag{3.5.6}$$

Combining Equations 3.5.5 and 3.5.6, we get

$$[\mathrm{M}^\vee_{\mathrm{Rig}}(]D[)] = \sum_{I\subseteq J'} (-1)^{|I|-1}[\mathrm{M}^\vee_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_I}(D_I^\circ, u_I))][\mathrm{M}^\vee_{\mathrm{Rig}}(\partial\mathbb{B}(o,1^{|I|-1})]. \tag{3.5.7}$$

The analytifcation of $\widetilde{D_I^\circ}$ is $Q^{\mathrm{an}}_{N_I}(D_I^\circ, u_I)$, hence by Theorem 3.3.24, $\mathrm{Rig}^*\mathrm{M}_K(\widetilde{D_I^\circ}) = \mathrm{M}^\vee_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_I}(D_I^\circ, u_I))$ and similarly for cohomological motives.

As D_I is proper, we can apply Proposition 3.5.9 which gives after applying Rig^*

$$\begin{aligned}&[\mathrm{M}^\vee_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_I}(D_I^\circ, u_I))] = \\ &\qquad \sum_{I\subset L\subset J} (-1)^{|L|-|I|}[\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_L}(D_L^\circ, u_L) \times_K \partial\mathbb{B}(o,1)^{|L|-|I|})(-d+|I|-1)]. \end{aligned} \tag{3.5.8}$$

We also know that $[\mathrm{M}^\vee_{\mathrm{Rig}}(\partial\mathbb{B}(o,1))] = -[\mathrm{M}_{\mathrm{Rig}}(\partial\mathbb{B}(o,1))(-1)]$. With these two remarks, Equation 3.5.7 yields

$$\begin{aligned}[\mathrm{M}^\vee_{\mathrm{Rig}}(]D[)] &= \sum_{\emptyset\neq I\subset J'} \sum_{I\subset L\subset J} (-1)^{|L|-|I|}[\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_L}(D_L^\circ, u_L) \times_K \partial\mathbb{B}(o,1)^{|L|-1})(-d)] \\ &= \sum_{\substack{L\subset J\\ L\cap J'\neq\emptyset}} (-1)^{|L|-1}[\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_L}(D_L^\circ, u_L) \times_K \partial\mathbb{B}(o,1)^{|L|-1})(-d)] \\ &\qquad \cdot \sum_{i=1}^{|L\cap J'|} \binom{|L\cap J'|}{i}(-1)^{i-1} \\ &= \sum_{\substack{L\subset J\\ L\cap J'\neq\emptyset}} (-1)^{|L|-1}[\mathrm{M}_{\mathrm{Rig}}(Q^{\mathrm{Rig}}_{N_L}(D_L^\circ, u_L) \times_K \partial\mathbb{B}(o,1)^{|L|-1})(-d)]. \end{aligned} \tag{3.5.9}$$

Comparing to 3.5.4 gives the desired

$$\chi_{\mathrm{Rig}}(]D[^{\mathrm{VF}}) = [\mathrm{M}^\vee_{\mathrm{Rig}}(]D[)].$$

□

Proposition 3.5.11 imply the following theorem, which is Theorem 3.1.2 of the introduction. All we need to do is choosing an semi-stable formal R-scheme $\mathcal{Y}$ over $\mathcal{X}$ such that $\mathcal{Y} \to \mathcal{X}$ is a composition of admissible blow-ups. Hence the induced morphism at the level of special fibers is proper and we can apply Proposition 3.5.11.

Theorem 3.5.12. *Let X be a quasi-compact smooth rigid variety, $\mathcal{X}$ an formal R-model of X, D a locally closed and proper subset of its special fiber $\mathcal{X}_\sigma$. Then*

$$\chi_{\mathrm{Rig}}(]D[^{\mathrm{VF}}) = [\mathrm{M}^{\vee}_{\mathrm{Rig}}(]D[)].$$

In particular, if X is a smooth and proper rigid variety,

$$\chi_{\mathrm{Rig}}([X^{\mathrm{VF}}]) = [\mathrm{M}^{\vee}_{\mathrm{Rig}}(X)].$$

3.5.3 Analytic Milnor fiber

Let X a smooth k-variety and $f : X \to \mathbb{A}^1_k$ a non constant regular function. Base change to R makes of X an R-scheme. Denote $\mathcal{X}_f$ the formal completion of X with respect to (t). Its special fiber $\mathcal{X}_{f,\sigma}$is the zero locus of f in X. For any closed point $x \in \mathcal{X}_{f,\sigma}$, denote by $\mathcal{F}_{f,x}$ the tube of $\{x\}$ in $\mathcal{X}_f$. It is the analytic Milnor fiber. It is a rigid subvariety of $\mathcal{X}_{f,\eta}$, the analytic nearby cycles.

Consider an embedded resolution of singularities of $\mathcal{X}_{f,\sigma}$ in X. It is a proper birationnal morphism $h : Y \to X$ such that $h^{-1}(\mathcal{X}_{f,\sigma})$ is a smooth normal crossing divisor. Denote by $(E_i)_{i\in J}$ the reductions of its (smooth) irreducible components and $N_i \in \mathbb{N}^*$ the multiplicity of E_i in $h^{-1}(\mathcal{X}_{f,\sigma})$.

For any non empty $I \subset J$, denote by $E_I = \cap_{i\in I} E_i$ and $E_I^\circ = E_I \setminus \cup_{j\in J\setminus I} E_j$.

Define as follows the étale cover $\widetilde{E_I^\circ}$ of E_I°. Let N_I be the greatest common divisor of the N_i, for $i \in I$. Working locally on some open neighborhood U of E_I° in Y, we can assume that E_i is defined by equation $t_i = 0$, for some $t_i \in \mathcal{O}(U)$ and that on U, $f = u_I t_{i_1}^{N_{i_1}} ... t_{i_r}^{N_{i_r}}$, with $u_I \in \mathcal{O}(U)^\times$. Then set $\widetilde{E_I^\circ \cap U} = \big\{(v, x) \mid x \in U, v^{N_I} = u\big\}$.

Recall the motivic Milnor fiber, defined by Denef and Loeser, see for example [39]. In an equivariant setting, it is defined for any closed point $x \in X_0$ by the formula

$$\psi_{f,x} = \sum_{\emptyset\neq I\subseteq J} (-1)^{|I|-1}[\mathbb{G}_{m_k}^{|I|-1}][\widetilde{E_I^\circ} \cap h^{-1}(x)] \in \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}}).$$

In particular, they show that this formula is independent of the chosen resolution h.

Proposition 3.5.13. *For any closed point $x \in X_0$,*

$$\Theta \circ \mathcal{E}_c \circ \oint (\mathcal{F}^{\mathrm{VF}}_{f,x}) = \psi_{f,x} \in \mathbf{K}(\mathrm{Var}_k^{\hat{\mu}})[(\mathbb{L}-1)^{-1}].$$

Proof. The embedded resolution of X_0 induces an admissible morphism $h : Y \to X$, hence $]h^{-1}(x)[_Y \simeq]\{x\}[_X$. Up to changing h, we can suppose $h^{-1}(x)$ is a divisor $E = \cup_{i\in J'} E_i$

in $\mathcal{Y}_\sigma = \cup_{i\in J} D_i$, with $I' \subset I$. Then we have

$$\mathcal{F}^{\mathrm{VF}}_{f,x} = \dot{\bigcup_{\substack{I\subset J\\ I\cap J'\neq\emptyset}}}]E^\circ_I[^{\mathrm{VF}}.$$

We want to show that

$$\Theta\circ\mathcal{E}_c\circ\oint(]E^\circ_I[_X) = (-1)^{|I|-1}[\mathbb{G}_{m_k}{}^{|I|-1}\times_k \widetilde{E^\circ_I}].$$

By Remark 3.4.21 following Lemma 3.4.20, we can suppose $X = \mathrm{St}^{u_I^{-1}t}_{E^\circ_I\times_k R,\underline{N}}$, where $\underline{N} = (N_1,...,N_r)\in(\mathbb{N}^\times)^r$ (where $r=|I|$), $u_I\in\mathcal{O}(E^\circ_I\times_k R)^\times$. But now, as in the proof of Proposition 3.4.19, $]E^\circ_I[_X$ is definably isomorphic to $Q^{\mathrm{VF}}_{N_I}(E^\circ_I,u)\times \mathrm{v}^{-1}(\Delta)$. Hence $\oint]E^\circ_I[_X = [Q^{\mathrm{RV}}_{N_I}(E^\circ_I,u_I)\times \mathrm{v}^{-1}_{\mathrm{rv}}(\Delta)]_d$. Now $\Theta Q^{\mathrm{RV}}_{N_I}(E^\circ_I,u_I) = [\widetilde{E^\circ_I}]$ and

$$\Theta\circ\mathcal{E}_c\mathrm{v}^{-1}_{\mathrm{rv}}(\Delta) = \mathrm{eu}_c(\Delta)[\mathbb{G}_{mk}]^{|I|-1} = (-1)^{|I|-1}[\mathbb{G}_{mk}]^{|I|-1},$$

so putting pieces together by linearity of $\Theta\circ\oint$,

$$\cdot\Theta\circ\mathcal{E}_c\circ\oint\mathcal{F}^{\mathrm{VF}}_{f,x} = \sum_{\substack{I\subset J\\ I\cap J'\neq\emptyset}}(-1)^{|I|-1}[\mathbb{G}_{m_k}{}^{|I|-1}\times_k\widetilde{E^\circ_I}] = \psi_{f,x}\in\mathbf{K}(\mathrm{Var}^{\hat\mu}_k).$$

□

Remark 3.5.14. In the literature, the motivic Milnor is defined in the localization of $\mathbf{K}(\mathrm{Var}^{\hat\mu}_k)$ by $\mathbb{L} = [\mathbb{A}^1_k]$. Such a localization in non injective if $k=\mathbb{C}$, see Borisov [11]. However, Proposition 3.5.13 show that it is well defined in $\mathbf{K}(\mathrm{Var}^{\hat\mu}_k)$. The same fact is proven in [79, Corollary 2.6.2] using a computation similar to ours.

From Theorem 3.5.12, we deduce the following corollary.

Corollary 3.5.15. *For any closed point $x\in X_\sigma$,*

$$\chi_{\mathrm{Rig}}(\mathcal{F}^{\mathrm{VF}}_{f,x}) = [\mathrm{M}^\vee_{\mathrm{Rig}}(\mathcal{F}_{f,x})].$$

Combining Corollary 3.5.15 and Proposition 3.5.13 with Theorem 3.4.13 gives the following result.

Corollary 3.5.16. *For any closed point $x\in X_\sigma$, we have*

$$[\mathfrak{R}\mathrm{M}^\vee_{\mathrm{Rig}}(\mathcal{F}_{f,x})] = \chi_{\hat\mu}(\psi_{f,x})\in\mathbf{K}(\mathrm{QUSH}_{\mathfrak{M}}(k)).$$

Recall that $\psi_{f,x}$ is the Denef-Loeser motivic Milnor fiber. It is a generalization of Corollary 8.8 of Ayoub, Ivorra and Sebag [4] at an equivariant level. They show the same

equality, but in $\mathbf{K}(\mathrm{SH}_{\mathfrak{M}}(k))$, hence one deduce their result from Corollary 3.5.15 using Lemma 3.3.9.

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