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Mathematical and numerical study of the inverse problem of electro-seismicity in porous media

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dans l'**École Doctorale Mathématiques, Sciences et**
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**Etude mathématique et numérique du
problème inverse de l'électro-sismique en
milieu poreux**

**Mathematical and numerical study of the
inverse problem of electro-seismicity in
porous media**

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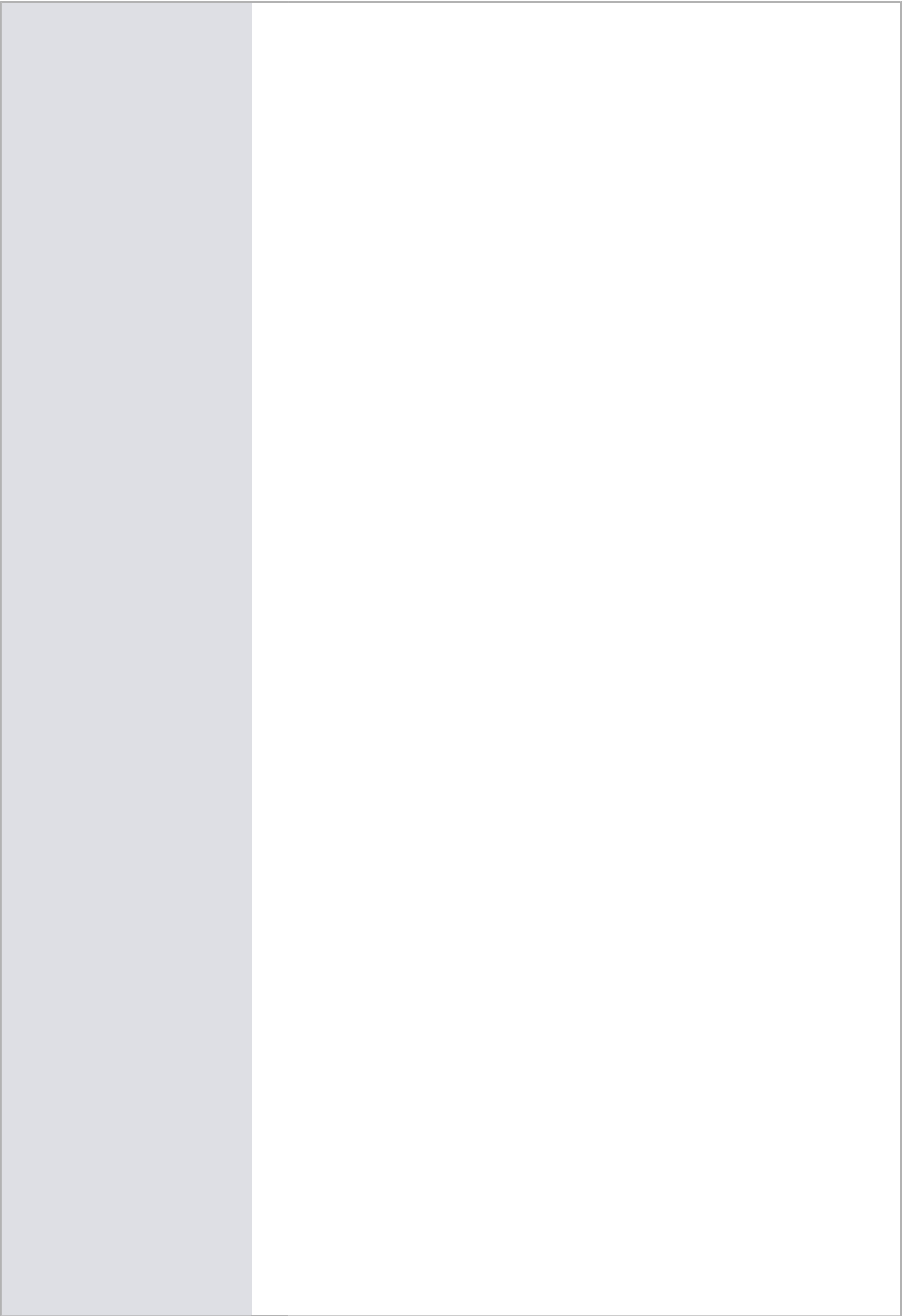
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Mathematical and numerical study of the inverse problem of electro-seismicity in porous media



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Abstract

In this thesis, we study the inverse problem of the coupling phenomenon of electromagnetic (EM) and seismic waves. Partial differential equations governing the coupling phenomenon are composed of Maxwell and Biot equations. Since the coupling phenomenon is rather weak, in low frequency we only consider the transformation from EM waves to seismic waves. We use electroseismic model to refer to this transformation. In the model, the electric field becomes the source of Biot equations. A coupling coefficient is used to denote the efficiency of the transformation.

Chapter 2, we consider the existence and uniqueness of the forward problem in both frequency domain and time domain. In the frequency domain, we propose the suitable Sobolev space to consider the electrokinetic problem. We prove that the weak formula satisfies a Garding's inequality using Helmholtz decomposition. The Fredholm alternative can be applied, which shows that the existence is equivalent to the uniqueness. In the time domain, the weak solution is defined and the existence and uniqueness of the weak solution is proved.

The stability of the inverse problem is considered in Chapter 3. We first prove Carleman estimates for both Biot equations and electroseismic equations. Based on the Carleman estimates for electroseismic equations, we prove a Holder stability to inverse all the parameters in Maxwell equation and the coupling coefficient. To simply the problem, we use electrostatic equations to replace Maxwell equations. The inverse problem is decomposed into two steps: the inverse source problem for Biot equations and the inverse parameter problem for the electrostatic equation. We can prove the stability of the inverse source problem for Biot equations based on the Carleman estimate for Biot equations. Then the conductivity and the coupling coefficient can be reconstructed with the information from the first step.

In Chapter 4, we solve the electroseismic equations numerically. The electrostatic equation is solved by the Matlab PDE toolbox. Biot equations are

solved with a staggered finite difference method. To decrease the computation consumption, we only deal with the two dimensional problem. To simulate waves propagating in unbounded domain, we use PML to absorb waves reaching the cut-off boundary.

Chapter 5 deals with the numerical inverse source problem for Biot equations. The method we are going to use is a variant of the time reversal method. The first step of the method is to transform the source problem into an initial value problem without any source. Then the application of the time reversal method recovers the initial value. Numerical examples demonstrate that this method works well even for Biot equations with a small damping term. But if the damping term is too large, the inverse process is not symmetric with the forward process and the reconstruction results degenerate.

Abstract

Dans cette thèse, nous étudions le problème inverse du phénomène de couplage des ondes électromagnétiques (EM) et sismiques. Maxwell et Biot. Comme le phénomène de couplage est plutôt faible, nous considérons la basse fréquence comme la transformation des ondes électromagnétiques en ondes sismiques. Nous utilisons le modèle électrosismique pour se référer à cette transformation. Dans le modèle, le champ électrique devient la source des équations de Biot. Un coefficient de couplage est utilisé pour désigner l'efficacité de la transformation.

Chapitre 2, nous considérons l'existence et l'unicité du problème directe dans le domaine fréquentiel et dans le domaine temporel. Dans le domaine fréquentiel, nous proposons l'espace de Sobolev approprié pour considérer le problème électrocinétique. Nous prouvons que la formule faible satisfait l'inégalité de Garding en utilisant la décomposition de Helmholtz. L'alternative de Fredholm peut être appliquée, ce qui montre que l'existence est équivalente à l'unicité. Dans le domaine temporel, la solution faible est définie et l'existence et l'unicité de la solution faible est démontrée.

La stabilité du problème inverse est examinée au chapitre 3. Nous prouvons d'abord les estimations de Carleman pour les deux équations et les équations électrosismiques. Sur les estimations de Carleman pour les équations électrosismiques, nous prouvons une stabilité de Holder pour inverser tous les paramètres de l'équation de Maxwell et le coefficient de couplage. Pour simplifier le problème, nous utilisons des équations électrostatiques pour remplacer les équations de Maxwell. Le problème inverse est décomposé en deux étapes : le problème de source inverse pour les équations de Biot et le paramètre inverse pour l'équation électrostatique. Nous pouvons prouver la stabilité du problème de source inverse pour les équations de Biot sur la base de l'estimation de Carleman pour les équations de Biot. Ensuite, la conductivité et le coefficient de couplage peuvent être reconstitués avec les informations de la première étape.

Dans le chapitre 4, nous résolvons les équations électrosismiques numériquement. L'équation électrostatique est résolue par la boîte à outils Matlab PDE. Les équations de Biot sont résolues avec un schéma de différences finies quinconce. Pour réduire la consommation de calcul, nous ne traitons que du problème bidimensionnel. Pour simuler des ondes se propageant dans un domaine non borné, nous utilisons le PML pour absorber les ondes atteignant la limite de modèle.

Le chapitre 5 traite du problème de source inverse numérique pour les équations de Biot. La méthode est une variante de la méthode d'inversion temporelle. La première étape de la méthode consiste à transformer le problème source en une valeur initiale sans aucune source. Ensuite, l'application de la méthode d'inversion de temps récupère la valeur initiale. Des exemples numériques montrent que cette méthode fonctionne bien même pour des équations de Biot avec un petit terme d'amortissement. Si le processus inverse n'est pas symétrique avec le processus d'anticipation et que les résultats de la reconstruction dégénèrent.

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Chapter 1

Background

In this thesis we are interested in the coupling phenomenon of electromagnetic (EM) and seismic waves in fluid-filled media. In general we speak of the *electrokinetic* phenomenon. The *electroseismic* effect is used to refer to the transformation of EM into seismic waves while the transformation from seismic to EM waves is called the *seismo-electric* effect.

The electro-kinetic phenomenon can be explained as follows. The sediment layers of the earth are in fact porous media filled with fluid electrolytes. The solid grains of the porous medium have extra charges (usually negative) on their surfaces due to chemical reactions between the ions in the fluid and the crystals that compose the solid. These charges are balanced by ions of opposite sign in the fluid, forming an electrical double layer (Debye layer). See Figure 1.1¹. When seismic waves propagate through such porous medium, the relative solid-fluid motion induces an electrical current which is the source of EM waves. Vice versa, when EM waves pass through the porous medium, ions in the fluid move. The fluid moves as well due to viscous traction.

The electrokinetic phenomenon has been studied since 1944 by Frenkel [46] and has been observed during earthquakes [23, 44, 32]. Field and laboratory experiments [13, 43, 57, 25, 28] have further demonstrated the coupling of EM and seismic waves. This effect rose interest in the geophysics community, as the coupling of EM and seismic waves may provide an efficient tool for imaging the subsoil in view of oil prospection: In electroseismic exploration, one probes the ground with EM waves and measures the resulting seismic waves on the surface. In seismoelectric exploration, the ground is shaken and one takes surface measurements of the resulting EM fields. In both cases, the information at disposal for reconstructing the medium constitutive parameters results from the interaction of the EM and seismic waves.

1. [https://en.wikipedia.org/wiki/Double_layer_\(surface_science\)](https://en.wikipedia.org/wiki/Double_layer_(surface_science))

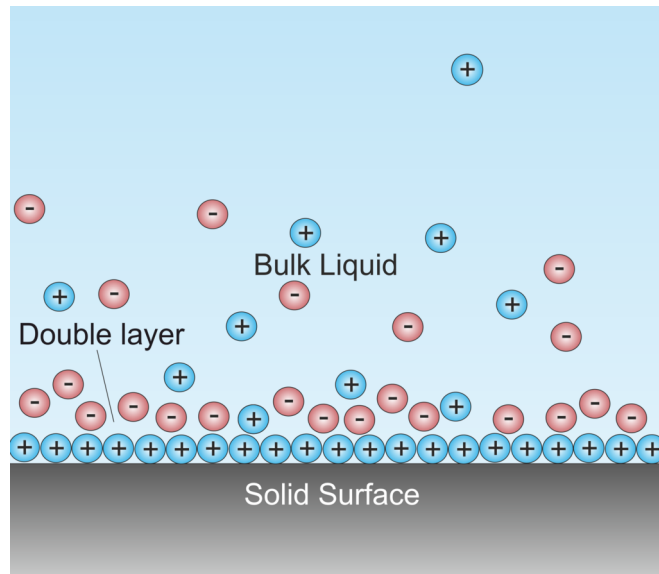


Figure 1.1 – Debye layer.

Such an imaging technique, and the associated inverse problem of reconstructing the constitutive parameters of the subsoil, fall into the category of multi-physics inverse problems, where a medium is probed using two types of waves. One type of waves is very sensitive to media parameters (the fluid dielectric constant in our case) however, these waves are usually very diffusive and therefore very weak at the receivers. The other type, on the contrary, is not very sensitive to medium properties, but is able to carry information through the medium with high resolution and little distortion.

Multiphysics (or hybrid) inverse problems have gained popularity about 10 years ago, when the first experimental results on the photo-acoustic coupling produced spectacularly accurate images [10]. Since then, several modalities have been studied : Photo-acoustic tomography, thermo-acoustic tomography, electrical impedance tomography by perturbation. In most cases, the reconstruction of the medium is a two-step procedure. One solves a first inverse problem to recover the initial pressure of a wave equation for the field that is not sensitive to fluid properties (in our case, the seismic waves). The outcome of this step produces internal data for a second inverse problem where one determines some of the constitutive coefficients of the medium (in our case, electric permittivity, conductivity and magnetic permeability).

1.1 Biot equations

The equations that govern the propagation of seismic waves in fluid-filled porous medium are given by Frenkel [24] and Biot [8, 9, 7]. Because of the outstanding work of Biot, the equations are called the Biot equations.

We are interested in porous media, where at the microscopic (pore) level, both a liquid and a solid phase co-exist. Such media are described by the displacement field \mathbf{u} (the solid displacement) and by the relative displacement \mathbf{w} between the solid and the fluid.

Throughout the thesis, we denote by ∂_t and ∂_j the partial derivatives of a function with respect to t and x_j respectively, and by ∇ (resp. $\nabla_{\mathbf{x},t}$) the gradients with respect to the variable \mathbf{x} (resp. \mathbf{x} and t). By gradient of a vector-valued function, we mean the transpose of the Jacobian matrix. The notations div or $\nabla \cdot$ (resp. curl or $\nabla \times$) stands for the divergence (resp. curl) of a vector-valued function.

The stress-strain relation

The body force fields are modeled in the solid by a stress tensor and a pressure term. In isotropic media, these fields are related to the kinematic fields via the stress-strain relations. We consider the following model constitutive equations:

$$\begin{cases} \boldsymbol{\tau} &= (\lambda \nabla \cdot \mathbf{u} + C \nabla \cdot \mathbf{w}) \mathbf{I} + G(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \\ -p &= C \nabla \cdot \mathbf{u} + M \nabla \cdot \mathbf{w}, \end{cases} \quad (1.1)$$

where $\boldsymbol{\tau}$ denotes the stress tensor of the fluid-filled media, p the pressure, and λ, G, C, M the elastic moduli.

Momentum equations

The momentum equations in the frequency domain read

$$\begin{cases} -\omega^2(\rho \mathbf{u} + \rho_f \mathbf{w}) &= \nabla \cdot \boldsymbol{\tau}, \\ -\omega^2(\rho_f \mathbf{u} + \tilde{\rho} \mathbf{w}) &= -\nabla p. \end{cases} \quad (1.2)$$

Here ρ_f is the fluid density, ρ_s the solid grain density, and $\rho = \phi \rho_f + (1 - \phi) \rho_s$ the drained bulk density where ϕ is the porosity. The parameter $\tilde{\rho}$ is frequency dependent

$$\tilde{\rho} = \frac{i\eta}{\omega \kappa(\omega)},$$

where η is the fluid viscosity and $\kappa(\omega)$ is the fluid flow permeability.

Low frequency time domain equations

The Biot equations are of interest to us in both the time domain and the frequency domain. When we go from frequency domain to time domain, the frequency dependence of $\kappa(\omega)$ results in a convolution term in the PDE. At low frequency, we can use a Taylor expansion of $\kappa(\omega)$ to approximate this convolution by a polynomial.

Johnson et al. [31] proposed a formula for $\kappa(\omega)$

$$\frac{\kappa(\omega)}{\kappa_0} = \left[\left(1 - i \frac{4}{m} \frac{\omega}{\omega_t} \right)^{1/2} - i \frac{\omega}{\omega_t} \right]^{-1}.$$

Here κ_0 is the frequency independent static flow permeability and ω_t is a frequency threshold (we refer to [31] for the meaning of the other parameters). In practice, $\omega \ll \omega_t$ and therefore $\kappa(\omega)$ has the Taylor expansion

$$\begin{aligned} \frac{\eta}{\kappa(\omega)} &= \frac{\eta}{\kappa_0} \left(1 - i(1 + \Phi) \frac{\omega}{\omega_t} + O\left(\frac{\omega}{\omega_t}\right)^2 \right) \\ &\simeq \frac{\eta}{\kappa_0} - i\omega\rho_e \end{aligned}$$

where $\Phi = 2/m$ and $\rho_e = (1 + \Phi) \frac{\eta}{\kappa_0 \omega_t}$. The second equation of (1.2) becomes

$$-\omega^2(\rho_f \mathbf{u} + \rho_e \mathbf{w}) - i\omega \frac{\eta}{\kappa_0} \mathbf{w} = -\nabla p.$$

Now all the parameters in the Biot equations are frequency independent. Applying the Fourier transform yields the time domain Biot equations at low frequency

$$\begin{cases} \rho \partial_t^2 \mathbf{u} + \rho_f \partial_t^2 \mathbf{w} &= \nabla \cdot \boldsymbol{\tau}, \\ \rho_f \partial_t^2 \mathbf{u} + \rho_e \partial_t^2 \mathbf{w} &= -\frac{\eta}{\kappa_0} \partial_t \mathbf{w} - \nabla p, \\ \boldsymbol{\tau} &= (\lambda \nabla \cdot \mathbf{u} + C \nabla \cdot \mathbf{w}) \mathbf{I} + G(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \\ -p &= C \nabla \cdot \mathbf{u} + M \nabla \cdot \mathbf{w}. \end{cases} \quad (1.3)$$

1.2 Pride equations

Pride [45] has derived macroscopic equations that control the coupling of EM and seismic waves in fluid-filled porous medium by averaging microscopic properties. See also [47, 46]. In isotropic media, Pride equations in the frequency domain read

$$\begin{cases} \nabla \times \mathbf{E} &= i\omega\mu\mathbf{H}, \\ \nabla \times \mathbf{H} &= (\sigma(\omega) - i\omega\varepsilon)\mathbf{E} + L(\omega)(-\nabla p + \omega^2 \rho_f \mathbf{u} + \mathbf{f}) + \mathbf{J}, \\ -\omega^2(\rho \mathbf{u} + \rho_f \mathbf{w}) &= \nabla \cdot \boldsymbol{\tau} + \mathbf{F}, \\ -i\omega \mathbf{w} &= L(\omega)\mathbf{E} + (\kappa(\omega)/\eta)(-\nabla p + \omega^2 \rho_f \mathbf{u} + \mathbf{f}), \\ \boldsymbol{\tau} &= (\lambda \nabla \cdot \mathbf{u} + C \nabla \cdot \mathbf{w}) \mathbf{I} + G(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \\ -p &= C \nabla \cdot \mathbf{u} + M \nabla \cdot \mathbf{w}. \end{cases} \quad (1.4)$$

The first two equations of this system are the Maxwell equations and the remaining ones are the Biot equations with the electric field as the source. Here $\mathbf{F}, \mathbf{f}, \mathbf{J}$ are external sources, and \mathbf{E}, \mathbf{H} denote the electric and magnetic fields respectively. The physical meaning of all the parameters in the above equations is given in Table 1.1.

σ	electric conductivity
ε	electric permittivity
μ	magnetic permeability
ω	frequency
ρ	bulk density
L	electrokinetic parameter
ρ_f	fluid density
κ	fluid flow permeability
η	fluid viscosity

Table 1.1 – Physical meanings of parameters in Pride equations.

Low frequency time domain equations

Pride [46] has given analytical frequency dependent expressions of the parameters, namely, the fluid permeability $\kappa(\omega)$, the electro-kinetic coupling coefficient $L(\omega)$, and the electric conductivity $\sigma(\omega)$ (for more accurate expressions see [45])

$$\begin{aligned}\frac{\kappa(\omega)}{\kappa_0} &= \left(\left(1 - i \frac{\Lambda^2 \omega}{m \omega_t} \right)^{1/2} - i \frac{\omega}{\omega_t} \right)^{-1}, \\ \frac{L(\omega)}{L_0} &= \left(1 - i \frac{\omega}{\omega_t} \right)^{-1/2}, \\ \sigma(\omega) &= \sigma_0,\end{aligned}$$

where

$$\begin{aligned}\omega_t &= \frac{\eta}{F \kappa_0 \rho_f}, \quad m = \frac{\Lambda^2}{F \kappa_0}, \\ L_0 &= -\frac{\varepsilon_f \zeta}{\eta F}, \quad \sigma_0 = \frac{\sigma_f}{F}.\end{aligned}$$

Here Λ is the characteristic pore-throat radius,

$$\zeta = 0.01 + 0.025 \log C$$

the electric potential in the double layer, F the electric formation factor, κ_0 the steady-state flow permeability, ε_f the fluid electric permittivity, and σ_f the fluid conductivity.

In practice, the low frequency hypothesis $\omega \ll \omega_t$ is relevant so that. We may then expand all the frequency dependent parameters

$$\begin{aligned}
\frac{\eta}{\kappa(\omega)} &= \frac{\eta}{\kappa_0} \left(\left(1 - i \frac{4}{m} \frac{\omega}{\omega_t}\right)^{1/2} - i \frac{\omega}{\omega_t} \right) \\
&= \frac{\eta}{\kappa_0} \left(1 - i(1 + 2/m) \frac{\omega}{\omega_t} + O\left(\frac{\omega}{\omega_t}\right)^2 \right) \\
&\approx \frac{\eta}{\kappa_0} - i\omega\rho_e, \\
L(\omega) &= -\frac{\varepsilon_f \zeta}{\eta F} \left(1 - i \frac{\omega}{\omega_t}\right)^{-1/2} \\
&= -\frac{\varepsilon_f \zeta}{\eta F} \left(1 - O\left(\frac{\omega}{\omega_t}\right)\right) \\
&\approx L_0, \\
\sigma(\omega) &= \sigma_0,
\end{aligned}$$

where

$$\rho_e = \rho_f(1 + 2/m)F.$$

Note that here ρ_e is exactly the same as in (1.3).

Assume that only the EM source is active, i.e., that

$$\mathbf{F} = \mathbf{f} = \mathbf{0}.$$

Since the coupling coefficient L_0 is rather small (10^{-13}), we neglect the transformation from seismic to EM waves. In other words,

$$\mathbf{J} \gg L(\omega)(-\nabla p + \omega^2 \rho_f \mathbf{u})$$

and the term $L(\omega)(-\nabla p + \omega^2 \rho_f \mathbf{u})$ in the second equation of (1.4) is neglected: The Maxwell equations are decoupled from the Biot equations.

Multiplying both sides of the fourth equation of (1.4) by $\frac{\eta}{\kappa(\omega)}$ results in

$$-i\omega \frac{\eta}{\kappa(\omega)} \mathbf{w} = \frac{L_0 \eta}{\kappa(\omega)} \mathbf{E} - \nabla p + \omega^2 \rho_f \mathbf{u},$$

i.e.,

$$-\omega^2(\rho_f \mathbf{u} + \tilde{\rho} \mathbf{w}) = \frac{L_0 \eta}{\kappa(\omega)} \mathbf{E} - \nabla p.$$

In other words, the electric field becomes a source term in the Biot equations.

Since σ_0 is frequency independent, the frequency domain Maxwell equations in (1.4) can easily be transformed into the time domain. Combining with the time domain Biot

equations (1.3), one obtains the full electroseismic model, in the form

$$\left\{ \begin{array}{l} \varepsilon \partial_t \mathbf{E} = \nabla \times \mathbf{H} - \sigma_0 \mathbf{E} - \mathbf{J}, \\ \mu \partial_t \mathbf{H} = -\nabla \times \mathbf{E}, \\ \rho \partial_t^2 \mathbf{u} + \rho_f \partial_t^2 \mathbf{w} = \nabla \cdot \boldsymbol{\tau}, \\ \rho_f \partial_t^2 \mathbf{u} + \rho_e \partial_t^2 \mathbf{w} = -\frac{\eta}{\kappa_0} \partial_t \mathbf{w} - \nabla p + \frac{L_0 \eta}{\kappa_0} \mathbf{E}, \\ \boldsymbol{\tau} = (\lambda \nabla \cdot \mathbf{u} + C \nabla \cdot \mathbf{w}) \mathbf{I} + G(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \\ -p = C \nabla \cdot \mathbf{u} + M \nabla \cdot \mathbf{w}. \end{array} \right. \quad (1.5)$$

1.3 The forward problem

To the author's best knowledge, the existence and uniqueness of the frequency domain electro-kinetic problem has not been studied so far. In the first part of Chapter 2, we introduce the weak formulation of the problem in a suitable functional setting. We show that the weak formula satisfies a Garding inequality. The Fredholm alternative tells us that the existence of a weak solution is then equivalent to its uniqueness. If the frequency ω is not an eigenvalue of the electrokinetic equations, we show that the solution is unique and therefore that the solution also exists.

In the time domain, the electro-seismic equations are decoupled into the Maxwell equations and the Biot equations. The electric field is a source term for the latter. The existence and uniqueness of the solutions to the Maxwell equations relies on classical results [1] since in our approximate model, they are decoupled from the Biot equations. Concerning the time domain Biot equations, Santos [49] proves the existence and uniqueness of solutions in 2 dimensions (2D). The 3D case is addressed in [5] with different boundary conditions than ours. We follow the same technique and adapt the proof of [5] to our boundary conditions in 3D. To this end, we introduce the weak formulation of the Biot system. Since Sobolev spaces are separable and thus have a countable basis, we approximate the solution by projecting it on an increasing sequence of finite dimensional subspaces spanned by the basis. We show that the approximate solutions are bounded and therefore have a weak* convergent subsequence, which proves to converge to the solution of the weak formulation.

In chapter 4 of this thesis, we present a numerical method to solve the electro-seismic equation. We restrict our computational domain to a region whose side lengths are of the order of several hundred meters. The frequencies of the electric source range from several Hertz to hundreds of Hertz. Due to the CFL condition, solving the Maxwell equations in the time domain would thus require extremely small time steps and would prove

extremely time consuming. Since our computational domain is much smaller than the wavelength, we solve the following electrostatic equation instead of the Maxwell equations

$$-\operatorname{div}(\sigma \nabla \Phi) = f(\mathbf{x}) \quad \text{in } \mathbb{R}^3.$$

Note that we use Φ to denote the electric potential from now on. The electric field is approximated by

$$\mathbf{E} = -\nabla \Phi.$$

In the time domain, we assume that the source is given by $f(\mathbf{x})g(t)$. From the linearity of the problem, the electric field is given by

$$\mathbf{E} = -\nabla \Phi g(t).$$

A similar approximation of the solution to the Maxwell equations by that of the electrostatic equation for the numerical solution of the seismoelectric equations can be found in [27].

The numerical resolution of the Biot equations, also called computational poroelasticity, has been well studied, see the nice review in [14]. We follow the staggered-grid Finite Difference (FD) method from [41]. The staggered-grid method is proposed by Yee [54] to solve Maxwell equations. The difference between staggered-grid FD and the normal FD method is that the fields are recorded at different points. Staggered-grid FD method in seismology were pioneered by [51, 52, 38].

One of the difficulties we face, is that we are interested in the propagation of waves in an unbounded domain. Careless truncation of the computational domain may result in artificial reflection on the boundary. The method of Perfectly Matched Layers (PML) [6] has been introduced to avoid these spurious reflections. The idea of PMLs is to add a thin layer outside the computational domain in such a way that no wave is reflected from the interface, and such that waves decrease rapidly in the PML layer. PMLs have been implemented for the Maxwell equations [19, 17, 37, 48]. PML for elastic and Biot equations can be found in [18, 39, 20, 35] and [56, 55, 40].

1.4 The inverse problem

Multiwave imaging methods, also called hybrid methods, have attracted a lot of interest, as they try to combine the high resolution of one type of waves with the high sensitivity to material parameter contrast of another. In the electroseismic model, the

elastic waves provide high resolution, since the wave length is small, while EM waves are more sensitive to fluid properties. In the case of oil exploration, where the subsoil is described as a porous medium, one tracks areas of porous rock filled with oil. As porous rock filled with water and petroleum have similar elastic parameters, the Biot waves are not sensitive enough to discriminate between regions filled with water or oil. On the other hand, the parameter that couples the Biot waves to EM waves varies dramatically, and could be a good indicator of the presence of petroleum.

Generally speaking, the solution of the inverse problem in hybrid methods requires two steps. The first step consists in the inversion of the first wave to obtain internal information about the second wave. Combining boundary measurements and internal information allows then the reconstruction of parameters sensed by the second wave. A lot of work has been devoted to photoacoustic tomography and thermoacoustic tomography [3, 36, 50].

The electroseismic model falls into the general framework of hybrid methods. To our best knowledge, only two papers have been considering the inverse electroseismic problem so far [16, 15]. These two references only consider the second step of the inversion, i.e., they assume that LE is known everywhere inside the domain and they want to recover σ and L at the same time. Carleman estimates [34, 29, 53] are a useful tool to prove the stability of inverse parameter or inverse source problems. Based on a Carleman estimate of the scalar wave equation [30, 29], we derive Carleman estimates for the Biot equations and the electroseismic equations in Chapter 3.

Assuming that we know all the parameters in the Biot equations except the coupling coefficient. In the first part of Chapter 3, we consider the inverse electroseismic problem as a whole. Based on the Carleman estimate of electroseismic equations, we prove that it is possible to recover all the parameters corresponding to the Maxwell equations and the coupling coefficient at the same time. In the second part of Chapter 3, we consider the electrostatic equation instead of Maxwell's. We prove the stability of the inverse source problem for the Biot equations and the stability of the inverse conductivity problem for the electrostatic equation with internal data.

In chapter 5 we deal with the first step of the inverse electroseismic problem numerically. We assume that there is a small region in which the coupling coefficient L is much larger than in the rest of the domain, and we seek to locate this region. This in fact corresponds to finding the source term in the Biot equations. To this end, we use the *source time reversal method* given by [12], and to which we now give a short introduction :

The source time reversal method

The source time reversal method is proposed to reconstruct the source term $f(\mathbf{x})$ of the scalar wave equation

$$\begin{cases} \partial_t^2 u(\mathbf{x}, t) - c^2(\mathbf{x})\Delta u(\mathbf{x}, t) = f(\mathbf{x})g(t) & \text{in } \mathbb{R}^n \times (0, T), \\ u(\mathbf{x}, 0) = 0 & \text{in } \mathbb{R}^n, \\ \partial_t u(\mathbf{x}, 0) = 0 & \text{in } \mathbb{R}^n, \end{cases} \quad (1.6)$$

from measurements of $u(\mathbf{x}, t)$ on the boundary of a bounded domain Ω . The term $g(t)$ is assumed to be known.

Let $v(\mathbf{x}, t)$ solve

$$\begin{cases} \partial_t^2 v(\mathbf{x}, t) - c^2(\mathbf{x})\Delta v(\mathbf{x}, t) = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ v(\mathbf{x}, 0) = 0 & \text{in } \mathbb{R}^n, \\ \partial_t v(\mathbf{x}, 0) = f(\mathbf{x}) & \text{in } \mathbb{R}^n. \end{cases} \quad (1.7)$$

It is easy to show that u given by

$$u(\mathbf{x}, t) = (v * g)(t) = \int_0^t v(\mathbf{x}, t - \tau)g(\tau) d\tau \quad (1.8)$$

is a solution to (1.6). The homogeneous initial value problem with nonzero source is transformed into a non-homogeneous initial value problem without source. If we have measurements of u on $\partial\Omega$, we can compute v from the integral equation (1.8). Let us choose T large enough such that waves exit the region of interest, i.e.,

$$v(\mathbf{x}, T) = \partial_t v(\mathbf{x}, T) = 0.$$

From the symmetry of the wave, the following problem

$$\begin{cases} \partial_t^2 v(\mathbf{x}, t) - c^2(\mathbf{x})\Delta v(\mathbf{x}, t) = 0 & \text{in } \Omega \times (T, 2T), \\ v(\mathbf{x}, T) = 0 & \text{in } \Omega, \\ \partial_t v(\mathbf{x}, T) = 0 & \text{in } \Omega, \\ w(\mathbf{x}, t) = v(\mathbf{x}, 2T - t) & \text{on } \partial\Omega \times (T, 2T), \end{cases}$$

has the final values

$$v(\mathbf{x}, 2T) = 0, \quad \partial_t v(\mathbf{x}, 2T) = f(\mathbf{x}).$$

For Biot equations, we show in Chapter 5 how to transform the source problem into an initial value problem without source. In the case that Biot equations have no damping term, i.e., $\frac{\eta}{\kappa} = 0$, the symmetry of waves is valid for Biot equations. We can reverse the propagation of waves with measurements on the boundary. If $\frac{\eta}{\kappa}$ is far from zero, we don't know what kind of equations should be used to have the symmetry of waves. If we use the same Biot equations with damping for the inversion, the source time reverse

method is not very successful, although one may still distinguish the area, where L is large, from the background.

At last, let us give a short summary of the content of each chapter. In Chapter 2, the existence and uniqueness of the solution to the Pride equations is proved in both frequency and time domain. Chapter 3 deals with the stability of the inverse problem using Carleman estimates. The electrostatic equation and the Biot equations are solved numerically in Chapter 4. We use PML to decrease reflections from the boundary. Chapter 5 shows numerical experiments of the inverse source problem for Biot equations.

Chapter 2

Existence and uniqueness of the forward problem

In this chapter, we will consider the existence and uniqueness of the forward problem in both frequency and time domains. As far as we know, no one has considered the existence and uniqueness of the electrokinetic problem in the frequency domain. In the first part of this chapter we propose the proper Sobolev space to consider this problem. Then we define the variational formula of Pride equations in the frequency domain. To study the compactness of the weak formula, a Helmholtz decomposition is introduced to split the field. We show that the Fredholm alternative is applicable to the weak formula. That's to say the existence is equivalent to the uniqueness.

In the time domain, the electroseismic model is in fact separated into two parts: Maxwell equations and Biot equations. The existence and uniqueness of Maxwell equations have been well understood long time ago. There are few papers considering Biot equations. To the author's best knowledge, the existence and uniqueness for Biot equations in 2D was first proved in [49]. In [5], the 3D case is studied, but with different boundary conditions to those considered in the thesis. Although the general arguments are similar, we prove the existence and uniqueness of solutions to our version of Biot equations in 3D for the sake of completeness.

2.1 The frequency domain

Let us consider the frequency domain Pride equations in a bounded Lipschitz domain Ω . We can rewrite the fourth equation of (1.4) into

$$-\nabla p + \omega^2 \rho_f \mathbf{u} + \mathbf{f} = \frac{\eta}{\kappa} (-i\omega \mathbf{w} - L(\omega) \mathbf{E}). \quad (2.1)$$

From the first equation of (1.4), we see that

$$\mathbf{H} = \frac{1}{i\omega\mu} \operatorname{curl} \mathbf{E}. \quad (2.2)$$

Substitution (2.1) and (2.2) into the second equation of (1.4) results

$$\operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} \mathbf{E} \right) - \omega^2 (-i\omega\tilde{\rho}L\mathbf{w} + \tilde{\varepsilon}\mathbf{E}) = \frac{i}{\omega} \mathbf{J}$$

where

$$\tilde{\rho} = \frac{i\eta}{\omega\kappa}, \quad \tilde{\varepsilon} = \varepsilon + \frac{i\sigma}{\omega} - \tilde{\rho}L^2.$$

For simplicity we will still use \mathbf{J} instead of $\frac{i}{\omega}\mathbf{J}$ to represent the source. Substitution the fifth equation of (1.4) into the third one, we can cancel out $\boldsymbol{\tau}$ and obtain

$$-\nabla(\lambda \operatorname{div} \mathbf{u} + C \operatorname{div} \mathbf{w}) - \operatorname{div}(G(\nabla \mathbf{u} + \nabla \mathbf{u}^T)) - \omega^2(\rho \mathbf{u} + \rho_f \mathbf{w}) = \mathbf{F}.$$

We use the same process to cancel out p in the fourth equation of (1.4) to obtain

$$-\nabla(C \operatorname{div} \mathbf{u} + M \operatorname{div} \mathbf{w}) - \omega^2(\rho_f \mathbf{u} + \tilde{\rho} \mathbf{w} - \frac{i\tilde{\rho}L}{\omega} \mathbf{E}) = \mathbf{f}.$$

We conclude that

$$\begin{cases} -\nabla(\lambda \operatorname{div} \mathbf{u} + C \operatorname{div} \mathbf{w}) - \operatorname{div}(G(\nabla \mathbf{u} + \nabla \mathbf{u}^T)) - \omega^2(\rho \mathbf{u} + \rho_f \mathbf{w}) & = \mathbf{F}, \\ -\nabla(C \operatorname{div} \mathbf{u} + M \operatorname{div} \mathbf{w}) - \omega^2(\rho_f \mathbf{u} + \tilde{\rho} \mathbf{w} - \frac{i\tilde{\rho}L}{\omega} \mathbf{E}) & = \mathbf{f}, \\ \operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} \mathbf{E} \right) - \omega^2(-i\omega\tilde{\rho}L\mathbf{w} + \tilde{\varepsilon}\mathbf{E}) & = \mathbf{J}. \end{cases} \quad (2.3)$$

From now on we use z_r to represent the real part and z_i the imaginary part of a complex variable z , e.g.,

$$z_r = \operatorname{Re} z, \quad z_i = \operatorname{Im} z.$$

The notation \bar{z} means the complex conjugate of z . For a typical fluid-filled porous rock, we have

$$\tilde{\rho}_i \gg \tilde{\rho}_r > 0, \quad \tilde{\varepsilon}_i \gg \tilde{\varepsilon}_r > 0, \quad L_r \gg L_i > 0 \text{ and } \lambda M > C^2.$$

Taking $\omega = 10$, the parameter $\tilde{\rho}$ is usually at the scale of $10^6 - 10^9$, $\tilde{\varepsilon}$ at the scale of $10^{-8} - 10^{-5}$ and L at the scale of $10^{-13} - 10^{-11}$. All the other coefficients are assumed to be real, piecewise smooth and positive. Assume the source terms $\mathbf{F}, \mathbf{f}, \mathbf{J} \in [L^2(\Omega)]^3$. From time to time, we use c_0 and C_0 to denote positive constants which may vary in different places.

Boundary conditions

First we introduce some notations. For a given complex Hilbert space H , $(u, v)_H$ denotes the inner product for $u, v \in H$ and $\|u\|_H$ the corresponding norm. The convention is that the inner product is linear with respect to the first variable and conjugate linear with respect to the second one. We use $[H]^m$ to denote the vector-valued Hilbert space

$$\mathbf{u} = (u_1, \dots, u_m) \quad \text{such that} \quad u_j \in H, \quad 1 \leq j \leq m.$$

The inner product on this space is defined by

$$(\mathbf{u}, \mathbf{v})_H = \sum_i (u_i, v_i)_H.$$

Here the inner product notation $(\cdot, \cdot)_H$ is ambiguous used for the scalar case and the vector case. The dual space of H is denoted as H^* and $\langle u, f \rangle$ the duality pair for $u \in H, f \in H^*$. We use similar definitions for spaces of matrix-valued functions.

When we consider the space $L^2(\Omega)$ or $[L^2(\Omega)]^m$, we usually omit all subscripts, e.g., for $\mathbf{u}, \mathbf{v} \in [L^2(\Omega)]^3$,

$$(\mathbf{u}, \mathbf{v}) := (\mathbf{u}, \mathbf{v})_{L^2} = \int_{\Omega} \sum_{i=1}^3 u_i \bar{v}_i \, d\mathbf{x}.$$

Let $H^1(\Omega)$ be the usual Sobolev space with the norm

$$\|u\|_{H^1} = (\|u\|^2 + \|\nabla u\|^2)^{1/2}.$$

The Sobolev space $H(\text{div}, \Omega)$ is given by

$$H(\text{div}, \Omega) = \{\mathbf{u} \in [L^2(\Omega)]^3 : \text{div} \mathbf{u} \in L^2(\Omega)\}$$

with the norm

$$\|\mathbf{u}\|_{H(\text{div})} = (\|\mathbf{u}\|^2 + \|\text{div} \mathbf{u}\|^2)^{1/2}.$$

The Sobolev space $H(\text{curl}, \Omega)$ is given by

$$H(\text{curl}, \Omega) = \{\mathbf{u} \in [L^2(\Omega)]^3 : \text{curl} \mathbf{u} \in [L^2(\Omega)]^3\}$$

with the norm

$$\|\mathbf{u}\|_{H(\text{curl})} = (\|\mathbf{u}\|^2 + \|\text{curl} \mathbf{u}\|^2)^{1/2}.$$

The Sobolev space $H^{1/2}(\Omega)$ is defined to be the image of the trace operator γ_0 (Lemma 2.1.1) and $H^{-1/2}(\Omega)$ the dual space. Similar definitions apply for the vector case.

Let $\mathcal{D}(\Omega)$ be the set of infinitely differentiable functions with support inside Ω , and $\mathcal{D}(\bar{\Omega})$ the set of infinitely differentiable functions up to the boundary. Let us recall the trace theorem [26, 11].

Lemma 2.1.1. *Trace operators*

$$\gamma_0 : [\mathcal{D}(\bar{\Omega})]^3 \rightarrow [\mathcal{D}(\partial\Omega)]^3,$$

$$\gamma_n : [\mathcal{D}(\bar{\Omega})]^3 \rightarrow \mathcal{D}(\partial\Omega),$$

$$\gamma_\tau : [\mathcal{D}(\bar{\Omega})]^3 \rightarrow [\mathcal{D}(\partial\Omega)]^3$$

given by

$$\gamma_0 \mathbf{u} = \mathbf{u}|_{\partial\Omega},$$

$$\gamma_n \mathbf{w} = \mathbf{w} \cdot \mathbf{n}|_{\partial\Omega},$$

$$\gamma_\tau \mathbf{E} = \mathbf{E} \times \mathbf{n}|_{\partial\Omega}$$

have unique extensions to bounded linear operators

$$\gamma_0 : [H^1(\Omega)]^3 \rightarrow [H^{1/2}(\partial\Omega)]^3,$$

$$\gamma_n : H(\operatorname{div}, \Omega) \rightarrow H^{-1/2}(\partial\Omega),$$

$$\gamma_\tau : H(\operatorname{curl}, \Omega) \rightarrow [H^{-1/2}(\partial\Omega)]^3,$$

respectively and the operators γ_0 and γ_n are surjective. Here \mathbf{n} is the unit outward vector normal to the boundary.

The kernel spaces of $\gamma_0, \gamma_n, \gamma_\tau$ are denoted $[H_0^1(\Omega)]^3, H_0(\operatorname{div}, \Omega)$ and $H_0(\operatorname{curl}, \Omega)$ respectively. $\mathcal{D}(\Omega)^3$ is dense in each of the three kernel spaces.

The boundary conditions for the equation system (2.3) can be given by

$$\gamma_0 \mathbf{u} \in [H^{1/2}(\partial\Omega)]^3,$$

$$\gamma_n \mathbf{w} \in H^{-1/2}(\partial\Omega),$$

$$\gamma_\tau \mathbf{E} \in \operatorname{Ran}(\gamma_\tau).$$

Here $\operatorname{Ran}(\gamma_\tau)$ means the image space of the operator γ_τ . Then there exist

$$\mathbf{u}_0 \in [H^1(\Omega)]^3, \quad \mathbf{w}_0 \in H(\operatorname{div}, \Omega) \quad \text{and} \quad \mathbf{E}_0 \in H(\operatorname{curl}, \Omega)$$

such that

$$\gamma_0 \mathbf{u} = \gamma_0 \mathbf{u}_0, \quad \gamma_n \mathbf{w} = \gamma_n \mathbf{w}_0, \quad \gamma_\tau \mathbf{E} = \gamma_\tau \mathbf{E}_0.$$

Define

$$\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_0, \quad \tilde{\mathbf{w}} = \mathbf{w} - \mathbf{w}_0 \quad \text{and} \quad \tilde{\mathbf{E}} = \mathbf{E} - \mathbf{E}_0.$$

From the properties of the trace operators, we have

$$\tilde{\mathbf{u}} \in [H_0^1(\Omega)]^3, \quad \tilde{\mathbf{w}} \in H_0(\operatorname{div}, \Omega), \quad \tilde{\mathbf{E}} \in H_0(\operatorname{curl}, \Omega).$$

After substituting $\tilde{\mathbf{u}}, \tilde{\mathbf{w}}$ and $\tilde{\mathbf{E}}$ into (2.3), we have the following equation system with homogeneous boundary conditions

$$\begin{cases} -\nabla(\lambda \operatorname{div} \tilde{\mathbf{u}} + C \operatorname{div} \tilde{\mathbf{w}}) - \operatorname{div}(G(\nabla \tilde{\mathbf{u}} + \nabla \tilde{\mathbf{u}}^T)) - \omega^2(\rho \tilde{\mathbf{u}} + \rho_f \tilde{\mathbf{w}}) & = \tilde{\mathbf{F}}, \\ -\nabla(C \operatorname{div} \tilde{\mathbf{u}} + M \operatorname{div} \tilde{\mathbf{w}}) - \omega^2(\rho_f \tilde{\mathbf{u}} + \tilde{\rho} \tilde{\mathbf{w}} - \frac{i\rho L}{\omega} \tilde{\mathbf{E}}) & = \tilde{\mathbf{f}}, \\ \operatorname{curl}(\frac{1}{\mu} \operatorname{curl} \tilde{\mathbf{E}}) - \omega^2(-i\omega \tilde{\rho} L \tilde{\mathbf{w}} + \tilde{\varepsilon} \tilde{\mathbf{E}}) & = \tilde{\mathbf{J}}, \end{cases}$$

where

$$\begin{aligned} \tilde{\mathbf{F}} &= \mathbf{F} + \nabla(\lambda \operatorname{div} \mathbf{u}_0 + C \operatorname{div} \mathbf{w}_0) + \operatorname{div}(G(\nabla \mathbf{u}_0 + \nabla \mathbf{u}_0^T)) + \omega^2(\rho \mathbf{u}_0 + \rho_f \mathbf{w}_0), \\ \tilde{\mathbf{f}} &= \mathbf{f} + \nabla(C \operatorname{div} \mathbf{u}_0 + M \operatorname{div} \mathbf{w}_0) + \omega^2(\rho_f \mathbf{u}_0 + \rho \mathbf{w}_0 - \frac{i\rho L}{\omega} \mathbf{E}_0), \\ \tilde{\mathbf{J}} &= \mathbf{J} - \operatorname{curl}(\frac{1}{\mu} \operatorname{curl} \mathbf{E}_0) + \omega^2(-i\omega \rho L \mathbf{w}_0 + \varepsilon \mathbf{E}_0). \end{aligned}$$

For simplicity, we drop all the tilde notations on the variables $(\mathbf{u}, \mathbf{w}, \mathbf{E})$, i.e., we consider

$$(\mathbf{u}, \mathbf{w}, \mathbf{E}) \in [H_0^1(\Omega)]^3 \times H_0(\operatorname{div}, \Omega) \times H_0(\operatorname{curl}, \Omega)$$

in the equation system (2.3). Now we have given proper spaces to consider the electrokinetic problem. Before defining a weak formula of the original problem and studying the property of the weak formula, we first introduce some decomposition and embedding lemmas of the Sobolev spaces.

The Helmholtz decomposition and compact embedding

To deal with Maxwell equations, it is necessary to introduce the Helmholtz decomposition [33].

Lemma 2.1.2. *Let $\Omega \subset \mathbb{R}^3$ be open and bounded and $\mathbf{A} \in L^\infty(\Omega, \mathbb{C}^{3 \times 3})$ such that $\mathbf{A}(\mathbf{x})$ is symmetric for almost all \mathbf{x} . Furthermore, assume that there exists a constant c_0 such that*

$$\operatorname{Re}(\bar{\mathbf{z}} \cdot \mathbf{A}(\mathbf{x}) \cdot \mathbf{z}) \geq c_0 \|\mathbf{z}\|^2$$

for all $\mathbf{z} \in \mathbb{C}^3$ and almost all $\mathbf{x} \in \Omega$. Then the spaces $H_0(\operatorname{curl}, \Omega)$ and $[L^2(\Omega)]^3$ have the following decompositions respectively:

$$H_0(\operatorname{curl}, \Omega) = H_0(\operatorname{curl}_0, \Omega) \oplus V,$$

$$[L^2(\Omega)]^3 = H_0(\operatorname{curl}_0, \Omega) \oplus \tilde{V},$$

where

$$\begin{aligned} H_0(\operatorname{curl}_0, \Omega) &= \{\mathbf{u} \in H_0(\operatorname{curl}, \Omega) : \operatorname{curl} \mathbf{u} = 0\}, \\ V &= \{\mathbf{u} \in H_0(\operatorname{curl}, \Omega) : (\mathbf{v}, \mathbf{A}\mathbf{u}) = 0 \text{ for } \forall \mathbf{v} \in H_0(\operatorname{curl}_0, \Omega)\}, \\ \tilde{V} &= \{\mathbf{u} \in [L^2(\Omega)]^3 : (\mathbf{v}, \mathbf{A}\mathbf{u}) = 0 \text{ for } \forall \mathbf{v} \in H_0(\operatorname{curl}_0, \Omega)\}. \end{aligned}$$

Here \oplus means the direct sum. Furthermore, all the projection operators are bounded and all the subspaces are closed.

Proof. The closeness of $H_0(\operatorname{curl}_0, \Omega)$ in $H_0(\operatorname{curl}, \Omega)$ is obvious. The closeness of $H_0(\operatorname{curl}_0, \Omega)$ in $[L^2(\Omega)]^3$ is shown below. Let $\mathbf{u}_n \in H_0(\operatorname{curl}_0, \Omega)$ be a Cauchy sequence in the $[L^2(\Omega)]^3$ norm and assume \mathbf{u}_n converges to $\mathbf{u} \in [L^2(\Omega)]^3$. We only need to prove $\mathbf{u} \in H_0(\operatorname{curl}_0, \Omega)$. Since $\operatorname{curl} \mathbf{u}_n = 0$, the sequence $\{\mathbf{u}_n\}$ is also a Cauchy sequence in $H(\operatorname{curl})$ norm. This implies that $\mathbf{u} \in H_0(\operatorname{curl}_0, \Omega)$, which completes the proof. The closeness of V and \tilde{V} is easy to prove from the continuity of L^2 inner product.

It holds that

$$\tilde{V} \cap H_0(\operatorname{curl}_0, \Omega) = \{0\}$$

because

$$u \in \tilde{V} \cap H_0(\operatorname{curl}_0, \Omega)$$

implies $(u, \mathbf{A}u) = 0$, that is,

$$c_0 \|u\|^2 \leq \operatorname{Re}(u, \mathbf{A}u) = 0.$$

Similarly we have

$$V \cap H_0(\operatorname{curl}_0, \Omega) = \{0\}.$$

Next we prove that $[L^2(\Omega)]^3 = H_0(\operatorname{curl}_0, \Omega) \oplus \tilde{V}$. We define the sesquilinear form

$$a : H_0(\operatorname{curl}_0, \Omega) \times H_0(\operatorname{curl}_0, \Omega) \rightarrow \mathbb{C}$$

by $a(\psi, v) = (\psi, \mathbf{A}v)$. Then a is coercive since

$$\operatorname{Re} a(v, v) = \operatorname{Re}(v, \mathbf{A}v) \geq c_0 \|v\|_{L^2}^2 = c_0 \|v\|_{H(\operatorname{curl})}^2.$$

For fixed $u \in [L^2(\Omega)]^3$, let us define linear form

$$l(\psi) = (\psi, \mathbf{A}u) \text{ for } \psi \in H_0(\operatorname{curl}_0, \Omega).$$

It's obvious that the linear operator l is bounded. From the Lax-Milgram theorem, there exists a unique $u_0 \in H_0(\operatorname{curl}_0, \Omega)$ such that

$$a(\psi, u_0) = l(\psi) \text{ for all } \psi \in H_0(\operatorname{curl}_0, \Omega),$$

that is $(\psi, \mathbf{A}u_0) = (\psi, \mathbf{A}u)$. Therefore $u - u_0 \in \tilde{V}$. Similarly we have the decomposition

$$H_0(\text{curl}, \Omega) = H_0(\text{curl}_0, \Omega) \oplus V.$$

The boundedness of all the projections are from general properties of the direct sum. ■

Let us recall two embedding lemmas [33].

Lemma 2.1.3. *V is compactly embedded into $[L^2(\Omega)]^3$.*

Lemma 2.1.4. *$[H_0^1(\Omega)]^3$ is compactly embedded into $[L^2(\Omega)]^3$.*

The weak formula

Set

$$\tilde{\mathbf{w}} = -\frac{i\tilde{\rho}L}{\tilde{\varepsilon}}\mathbf{w}, \quad \tilde{\mathbf{J}} = \frac{1}{\tilde{\varepsilon}}\mathbf{J}.$$

With these new notations, the third equation of (2.3) is written as

$$\text{curl}\left(\frac{1}{\mu}\text{curl}\mathbf{E}\right) - \omega^2\tilde{\varepsilon}(\omega\tilde{\mathbf{w}} + \mathbf{E}) = \tilde{\varepsilon}\tilde{\mathbf{J}} \quad (2.4)$$

Choosing $\mathbf{A} = \tilde{\varepsilon}$ in Lemma 2.1.2, we have

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2, \quad \tilde{\mathbf{w}} = \mathbf{w}_1 + \mathbf{w}_2, \quad \tilde{\mathbf{J}} = \mathbf{J}_1 + \mathbf{J}_2$$

where

$$\mathbf{E}_1, \mathbf{w}_1, \mathbf{J}_1 \in H_0(\text{curl}_0, \Omega) \text{ and } \mathbf{E}_2 \in V, \mathbf{w}_2, \mathbf{J}_2 \in \tilde{V}.$$

The weak formula for the equation (2.4) is

$$(\text{curl}\boldsymbol{\theta}, \frac{1}{\mu}\text{curl}\mathbf{E}) - \omega^2(\boldsymbol{\theta}, \tilde{\varepsilon}(\omega\tilde{\mathbf{w}} + \mathbf{E})) = (\boldsymbol{\theta}, \tilde{\varepsilon}\tilde{\mathbf{J}}), \quad (2.5)$$

for $\forall \boldsymbol{\theta} \in H_0(\text{curl}, \Omega)$. First choosing $\boldsymbol{\theta} \in H_0(\text{curl}_0, \Omega)$, we obtain

$$(\boldsymbol{\theta}, \omega^2\tilde{\varepsilon}(\omega\tilde{\mathbf{w}} + \mathbf{E}) + \tilde{\varepsilon}\tilde{\mathbf{J}}) = 0.$$

The orthogonality property of the decomposition results

$$(\boldsymbol{\theta}, \omega^2\tilde{\varepsilon}(\omega\mathbf{w}_1 + \mathbf{E}_1) + \tilde{\varepsilon}\mathbf{J}_1) = 0,$$

that's to say,

$$\omega^2(\omega\mathbf{w}_1 + \mathbf{E}_1) + \mathbf{J}_1 = 0. \quad (2.6)$$

After substituting (2.6) into the weak formula (2.5), we have the reduced weak formula

$$\left(\operatorname{curl} \boldsymbol{\theta}, \frac{1}{\mu} \operatorname{curl} \mathbf{E}_2 \right) - \omega^2 \left(\boldsymbol{\theta}, \tilde{\varepsilon}(\omega \mathbf{w}_2 + \mathbf{E}_2) \right) = (\boldsymbol{\theta}, \tilde{\varepsilon} \mathbf{J}_2), \quad (2.7)$$

for $\forall \boldsymbol{\theta} \in V$.

The second equation of (2.3) has the following weak formula

$$\left(\operatorname{div} \boldsymbol{\psi}, C \operatorname{div} \mathbf{u} + M \operatorname{div} \mathbf{w} \right) - \omega^2 \left(\boldsymbol{\psi}, \rho_f \mathbf{u} + \tilde{\rho} \mathbf{w} - \frac{i \tilde{\rho} L}{\omega} \mathbf{E} \right) = (\boldsymbol{\psi}, \mathbf{f}), \quad (2.8)$$

for $\forall \boldsymbol{\psi} \in H_0(\operatorname{div}, \Omega)$. Similar to the weak formula (2.7), we could eliminate \mathbf{E}_1 from the weak formula (2.8) by substituting (2.6). Therefore we have

$$\left(\operatorname{div} \boldsymbol{\psi}, C \operatorname{div} \mathbf{u} + M \operatorname{div} \mathbf{w} \right) - \omega^2 \left(\boldsymbol{\psi}, \rho_f \mathbf{u} + \tilde{\rho} \mathbf{w} - \frac{i \tilde{\rho} L}{\omega} (\mathbf{E}_2 - \omega \mathbf{w}_1) \right) = (\boldsymbol{\psi}, \mathbf{f} + \frac{i \tilde{\rho} L}{\omega} \mathbf{J}_1) \quad (2.9)$$

for $\forall \boldsymbol{\psi} \in H_0(\operatorname{div}, \Omega)$.

The weak formula for the first equation of (2.3) is written as

$$\left(\operatorname{div} \boldsymbol{\varphi}, \lambda \operatorname{div} \mathbf{u} + C \operatorname{div} \mathbf{w} \right) + \left(e(\boldsymbol{\varphi}), 2Ge(\mathbf{u}) \right) - \omega^2 \left(\boldsymbol{\varphi}, \rho \mathbf{u} + \rho_f \mathbf{w} \right) = (\boldsymbol{\varphi}, \mathbf{F}), \quad (2.10)$$

for $\forall \boldsymbol{\varphi} \in H_0^1(\Omega)$. Here

$$e(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T).$$

Summing up the weak formulas (2.7), (2.9) and (2.10), the weak formula for the equation system (2.3) reads

$$A(\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\theta}; \mathbf{u}, \mathbf{w}, \mathbf{E}_2) = b(\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\theta}) \quad (2.11)$$

for $\forall (\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\theta}) \in [H_0^1(\Omega)]^3 \times H_0(\operatorname{div}, \Omega) \times V$, where

$$\begin{aligned} A(\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\theta}; \mathbf{u}, \mathbf{w}, \mathbf{E}_2) &= \left(\operatorname{div} \boldsymbol{\varphi}, \lambda \operatorname{div} \mathbf{u} + C \operatorname{div} \mathbf{w} \right) + \left(\operatorname{div} \boldsymbol{\psi}, C \operatorname{div} \mathbf{u} + M \operatorname{div} \mathbf{w} \right) \\ &\quad + \left(\operatorname{curl} \boldsymbol{\theta}, \frac{1}{\mu} \operatorname{curl} \mathbf{E}_2 \right) + \left(e(\boldsymbol{\varphi}), 2Ge(\mathbf{u}) \right) \\ &\quad - \omega^2 \left(\boldsymbol{\varphi}, \rho \mathbf{u} + \rho_f \mathbf{w} \right) - \omega^2 \left(\boldsymbol{\psi}, \rho_f \mathbf{u} + \tilde{\rho} \mathbf{w} - \frac{i \tilde{\rho} L}{\omega} (\mathbf{E}_2 - \omega \mathbf{w}_1) \right) \\ &\quad - \omega^2 \left(\boldsymbol{\theta}, \tilde{\varepsilon}(\omega \mathbf{w}_2 + \mathbf{E}_2) \right), \end{aligned} \quad (2.12)$$

$$b(\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\theta}) = (\boldsymbol{\varphi}, \mathbf{F}) + \left(\boldsymbol{\psi}, \mathbf{f} + \frac{i \tilde{\rho} L}{\omega} \mathbf{J}_1 \right) + (\boldsymbol{\theta}, \tilde{\varepsilon} \mathbf{J}_2). \quad (2.13)$$

The Fredholm alternative

We study the properties of the operators $A(\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\theta}; \mathbf{u}, \mathbf{w}, \mathbf{E}_2)$ and $b(\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\theta})$. Obviously $A(\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\theta}; \mathbf{u}, \mathbf{w}, \mathbf{E}_2)$ is a bounded sesquilinear form, linear with respect to $(\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\theta})$ and conjugate linear with respect to $(\mathbf{u}, \mathbf{w}, \mathbf{E}_2)$, and $b(\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\theta})$ is a bounded linear functional. The following theorem shows that A satisfies the Garding's inequality.

Theorem 2.1.1. *If parameters in the electrokinetic system (2.3) satisfy*

$$\frac{1}{\sqrt{2}}(\min \rho_i - \max \rho_r) > s \max |\tilde{\rho}L|,$$

where

$$s = \frac{\max |\tilde{\varepsilon}| \max \left| \frac{\tilde{\rho}L}{\tilde{\varepsilon}} \right|}{\min |\tilde{\varepsilon}|},$$

then there exist constants α and β such that

$$|A(\mathbf{u}, \mathbf{w}, \mathbf{E}_2; \mathbf{u}, \mathbf{w}, \mathbf{E}_2)| \geq \beta (\|\mathbf{u}\|_{H^1}^2 + \|\mathbf{w}\|_{H(\text{div})}^2 + \|\mathbf{E}_2\|_{H(\text{curl})}^2) - \alpha (\|\mathbf{u}\|^2 + \|\mathbf{E}_2\|^2).$$

Proof. We estimate $A(\mathbf{u}, \mathbf{w}, \mathbf{E}_2; \mathbf{u}, \mathbf{w}, \mathbf{E}_2)$ term by term. Since

$$\lambda(\mathbf{x})M(\mathbf{x}) > C^2(\mathbf{x}),$$

the matrix

$$\begin{pmatrix} \lambda(\mathbf{x}) & C(\mathbf{x}) \\ C(\mathbf{x}) & M(\mathbf{x}) \end{pmatrix} \quad (2.14)$$

is symmetric positive definite. Therefore there exists a positive constant β which is smaller than the smallest eigenvalue of the matrix (2.14) for all $\mathbf{x} \in \Omega$. Thus

$$\begin{aligned} & (\text{div } \mathbf{u}, \lambda \text{ div } \mathbf{u} + C \text{ div } \mathbf{w}) + (\text{div } \mathbf{w}, C \text{ div } \mathbf{u} + M \text{ div } \mathbf{w}) \\ &= \int_{\Omega} (\text{div } \mathbf{u} \quad \text{div } \mathbf{w}) \begin{pmatrix} \lambda(\mathbf{x}) & C(\mathbf{x}) \\ C(\mathbf{x}) & M(\mathbf{x}) \end{pmatrix} \begin{pmatrix} \text{div } \bar{\mathbf{u}} \\ \text{div } \bar{\mathbf{w}} \end{pmatrix} d\mathbf{x} \\ &\geq \beta (\|\text{div } \mathbf{u}\|^2 + \|\text{div } \mathbf{w}\|^2). \end{aligned}$$

Let us choose β small enough such that

$$\min_{\mathbf{x}} \frac{1}{\mu(\mathbf{x})} \geq \beta \text{ and } \min_{\mathbf{x}} G(\mathbf{x}) \geq \beta.$$

Therefore

$$(\text{curl } \mathbf{E}_2, \frac{1}{\mu} \text{curl } \mathbf{E}_2) \geq \beta \|\text{curl } \mathbf{E}_2\|^2, \quad (e(\mathbf{u}), 2Ge(\mathbf{u})) \geq \beta \|e(\mathbf{u})\|^2.$$

There exists a positive constant α such that

$$\max_{\mathbf{x}} \rho(\mathbf{x}) \leq \alpha \text{ and } \max_{\mathbf{x}} |\tilde{\varepsilon}(\mathbf{x})| \leq \alpha.$$

Therefore

$$(\mathbf{u}, \rho \mathbf{u}) \leq \alpha \|\mathbf{u}\|^2, \quad |(\mathbf{E}_2, \tilde{\varepsilon} \mathbf{E}_2)| \leq \alpha \|\mathbf{E}_2\|^2.$$

We can choose α large enough such that the cross product terms have the following estimations

$$\begin{aligned} |(\mathbf{u}, \rho_f \mathbf{w})| &\leq \alpha(\delta \|\mathbf{w}\|^2 + \frac{1}{\delta} \|\mathbf{u}\|^2), \\ |(\mathbf{w}, \rho_f \mathbf{u})| &\leq \alpha(\delta \|\mathbf{w}\|^2 + \frac{1}{\delta} \|\mathbf{u}\|^2), \\ |(\mathbf{w}, -\frac{i\tilde{\rho}L}{\omega} \mathbf{E}_2)| &\leq \alpha(\delta \|\mathbf{w}\|^2 + \frac{1}{\delta} \|\mathbf{E}_2\|^2), \\ |(\mathbf{E}_2, \omega \tilde{\mathbf{w}})| &\leq \alpha(\delta \|\mathbf{w}_2\|^2 + \frac{1}{\delta} \|\mathbf{E}_2\|^2), \end{aligned}$$

where δ is a positive constant which could be as small as possible. Let us note that

$$\begin{aligned} \|\tilde{\mathbf{w}}\| &\leq \max \left| \frac{\tilde{\rho}L}{\tilde{\varepsilon}} \right| \|\mathbf{w}\|, \\ \|\mathbf{w}_1\|^2 &\leq \frac{1}{\min |\tilde{\varepsilon}|} |(\mathbf{w}_1, \tilde{\varepsilon} \mathbf{w}_1)|, \\ |(\mathbf{w}_1, \tilde{\varepsilon} \tilde{\mathbf{w}})| &\leq \max |\tilde{\varepsilon}| \|\mathbf{w}_1\| \|\tilde{\mathbf{w}}\|. \end{aligned}$$

From Lemma 2.1.2, we know that $(\mathbf{w}_1, \tilde{\varepsilon} \mathbf{w}_1) = (\mathbf{w}_1, \tilde{\varepsilon} \mathbf{w})$. Therefore

$$\begin{aligned} \|\mathbf{w}_1\|^2 &\leq \frac{1}{\min |\tilde{\varepsilon}|} |(\mathbf{w}_1, \tilde{\varepsilon} \mathbf{w}_1)| \\ &= \frac{1}{\min |\tilde{\varepsilon}|} |(\mathbf{w}_1, \tilde{\varepsilon} \tilde{\mathbf{w}})| \\ &\leq s \|\mathbf{w}\| \|\mathbf{w}_1\| \end{aligned}$$

where

$$s = \frac{\max |\tilde{\varepsilon}| \max \left| \frac{\tilde{\rho}L}{\tilde{\varepsilon}} \right|}{\min |\tilde{\varepsilon}|}.$$

That's to say, $\|\mathbf{w}_1\| \leq s \|\mathbf{w}\|$. Since

$$\begin{aligned} \|\mathbf{w}_2\|^2 &\leq \frac{1}{\min |\tilde{\varepsilon}|} |(\mathbf{w}_2, \tilde{\varepsilon} \mathbf{w}_2)|, \\ |(\mathbf{w}_2, \tilde{\varepsilon} \tilde{\mathbf{w}})| &\leq \max |\tilde{\varepsilon}| \|\mathbf{w}_2\| \|\tilde{\mathbf{w}}\|, \end{aligned}$$

we obtain

$$\begin{aligned} \|\mathbf{w}_2\|^2 &\leq \frac{1}{\min |\tilde{\varepsilon}|} |(\mathbf{w}_2, \tilde{\varepsilon} \mathbf{w}_2)| \\ &= \frac{1}{\min |\tilde{\varepsilon}|} |(\mathbf{w}_2, \tilde{\varepsilon} \tilde{\mathbf{w}})| \\ &\leq s \|\mathbf{w}\| \|\mathbf{w}_2\|, \end{aligned}$$

i.e., $\|\mathbf{w}_2\| \leq s \|\mathbf{w}\|$. The Schwartz inequality tells us

$$\begin{aligned} |(\mathbf{w}, i\tilde{\rho}L\mathbf{w}_1)| &\leq \max |\tilde{\rho}L| \|\mathbf{w}\| \|\mathbf{w}_1\| \\ &\leq s \max |\tilde{\rho}L| \|\mathbf{w}\|^2. \end{aligned}$$

Set

$$\begin{aligned}
A_1(\mathbf{u}, \mathbf{w}, \mathbf{E}_2) &= (\operatorname{div} \mathbf{u}, \lambda \operatorname{div} \mathbf{u} + C \operatorname{div} \mathbf{w}) + (\operatorname{div} \mathbf{w}, C \operatorname{div} \mathbf{u} + M \operatorname{div} \mathbf{w}) \\
&\quad + (\operatorname{curl} \mathbf{E}_2, \frac{1}{\mu} \operatorname{curl} \mathbf{E}_2) + (e(\mathbf{u}), 2Ge(\mathbf{u})) - \omega^2(\mathbf{w}, \tilde{\rho} \mathbf{w}), \\
A_2(\mathbf{u}, \mathbf{w}, \mathbf{E}_2) &= \omega^2(\mathbf{u}, \rho \mathbf{u} + \rho_f \mathbf{w}) + \omega^2(\mathbf{w}, \rho_f \mathbf{u} - \frac{i\tilde{\rho}L}{\omega}(\mathbf{E}_2 - \omega \mathbf{w}_1)) + \omega^2(\mathbf{E}_2, \tilde{\epsilon}(\omega \mathbf{w}_2 + \mathbf{E}_2)).
\end{aligned}$$

The real part and the imaginary part of A_1 are given by

$$\begin{aligned}
\operatorname{Re} A_1 &= (\operatorname{div} \mathbf{u}, \lambda \operatorname{div} \mathbf{u} + C \operatorname{div} \mathbf{w}) + (\operatorname{div} \mathbf{w}, C \operatorname{div} \mathbf{u} + M \operatorname{div} \mathbf{w}) \\
&\quad + (\operatorname{curl} \mathbf{E}_2, \frac{1}{\mu} \operatorname{curl} \mathbf{E}_2) + (e(\mathbf{u}), 2Ge(\mathbf{u})) - \omega^2(\mathbf{w}, \tilde{\rho}_r \mathbf{w}), \\
\operatorname{Im} A_1 &= -\omega^2(\mathbf{w}, \tilde{\rho}_i \mathbf{w}).
\end{aligned}$$

The term A_1 can be bounded from below as follows

$$\begin{aligned}
|A_1| &\geq \frac{1}{\sqrt{2}} (|\operatorname{Re} A_1| + |\operatorname{Im} A_1|) \\
&\geq \frac{\beta}{\sqrt{2}} (\|\operatorname{div} \mathbf{u}\|^2 + \|\operatorname{div} \mathbf{w}\|^2 + \|\operatorname{curl} \mathbf{E}_2\|^2 + \|e(\mathbf{u})\|^2) \\
&\quad + \frac{\omega^2}{\sqrt{2}} (\min \rho_i - \max \rho_r) \|\mathbf{w}\|^2.
\end{aligned}$$

The term A_2 can be bounded from above as follows

$$\begin{aligned}
|A_2| &\leq \omega^2 \left(\|(\mathbf{u}, \rho \mathbf{u} + \rho_f \mathbf{w})\| + \|(\mathbf{w}, \rho_f \mathbf{u} - \frac{i\tilde{\rho}L}{\omega}(\mathbf{E}_2 - \omega \mathbf{w}_1))\| + \|(\mathbf{E}_2, \tilde{\epsilon}(\omega \mathbf{w}_2 + \mathbf{E}_2))\| \right) \\
&\leq \omega^2 \alpha \left(1 + \frac{2}{\delta}\right) \|\mathbf{u}\|^2 + \omega^2 \alpha \left(1 + \frac{2}{\delta}\right) \|\mathbf{E}_2\|^2 \\
&\quad + 4\omega^2 \alpha \delta \|\mathbf{w}\|^2 + \omega^2 s \max |\tilde{\rho}L| \|\mathbf{w}\|^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
|A| &\geq |A_1| - |A_2| \\
&\geq \frac{1}{\sqrt{2}} \beta \|e(\mathbf{u})\|^2 - \omega^2 \alpha \left(1 + \frac{2}{\delta}\right) \|\mathbf{u}\|^2 \\
&\quad + \frac{1}{\sqrt{2}} \beta \|\operatorname{curl} \mathbf{E}_2\|^2 - \omega^2 \alpha \left(1 + \frac{2}{\delta}\right) \|\mathbf{E}_2\|^2 \\
&\quad + \frac{1}{\sqrt{2}} \beta \|\operatorname{div} \mathbf{w}\|^2 + \zeta \|\mathbf{w}\|^2
\end{aligned}$$

where

$$\zeta = \frac{\omega^2}{\sqrt{2}} (\min \rho_i - \max \rho_r) - 4\omega^2 \alpha \delta - \omega^2 s \max |\tilde{\rho}L|.$$

From the assumption we know that

$$\frac{1}{\sqrt{2}} (\min \rho_i - \max \rho_r) > s \max |\tilde{\rho}L|. \tag{2.15}$$

Let us choose δ small enough such that $\zeta > 0$. With the help of Korn's inequality,

$$\|e(\mathbf{u})\|^2 \geq \beta \|\mathbf{u}\|_{H^1}^2 - \alpha \|\mathbf{u}\|^2,$$

we can find constants α and β such that

$$|A(\mathbf{u}, \mathbf{w}, \mathbf{E}_2; \mathbf{u}, \mathbf{w}, \mathbf{E}_2)| \geq \beta (\|\mathbf{u}\|_{H^1}^2 + \|\mathbf{w}\|_{H(\text{div})}^2 + \|\mathbf{E}_2\|_{H(\text{curl})}^2) - \alpha (\|\mathbf{u}\|^2 + \|\mathbf{E}_2\|^2).$$

■

Set

$$\begin{aligned} C(\boldsymbol{\varphi}, \boldsymbol{\theta}; \mathbf{u}, \mathbf{E}_2) &= \zeta(\boldsymbol{\varphi}, \mathbf{u}) + \zeta(\boldsymbol{\theta}, \mathbf{E}_2), \\ B(\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\theta}; \mathbf{u}, \mathbf{w}, \mathbf{E}_2) &= A(\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\theta}; \mathbf{u}, \mathbf{w}, \mathbf{E}_2) + C(\boldsymbol{\varphi}, \boldsymbol{\theta}; \mathbf{u}, \mathbf{E}_2). \end{aligned}$$

Here ζ is a positive constant which is big enough such that B is coercive, i.e.,

$$|B(\mathbf{u}, \mathbf{w}, \mathbf{E}_2; \mathbf{u}, \mathbf{w}, \mathbf{E}_2)| \geq \beta (\|\mathbf{u}\|_{H^1}^2 + \|\mathbf{w}\|_{H(\text{div})}^2 + \|\mathbf{E}_2\|_{H(\text{curl})}^2).$$

Such a constant exists because of Theorem 2.1.1. The next theorem shows that the operator $C(\boldsymbol{\varphi}, \boldsymbol{\theta}; \mathbf{u}, \mathbf{E}_2)$ is compact.

Theorem 2.1.2. *The operator*

$$C : [H_0^1(\Omega)]^3 \times V \rightarrow [H_0^{-1}(\Omega)]^3 \times V^*$$

which maps $(\mathbf{u}, \mathbf{E}_2)$ into $C(\cdot, \cdot; \mathbf{u}, \mathbf{E}_2)$ is compact.

Proof. From Lemma 2.1.3 and Lemma 2.1.4, the embedding from $[H_0^1(\Omega)]^3 \times V$ into $[L^2(\Omega)]^6$ is compact and therefore the embedding from $[L^2(\Omega)]^6$ into $[H_0^{-1}(\Omega)]^3 \times V^*$ is compact. Here we identify $[L^2(\Omega)]^6$ with its dual but not $[H_0^1(\Omega)]^3 \times V$. Therefore $C : [H_0^1(\Omega)]^3 \times V \rightarrow [H_0^{-1}(\Omega)]^3 \times V^*$ defined by

$$C(\boldsymbol{\varphi}, \boldsymbol{\theta}; \mathbf{u}, \mathbf{E}_2) = \zeta(\boldsymbol{\varphi}, \mathbf{u}) + \zeta(\boldsymbol{\theta}, \mathbf{E}_2), \quad \text{for all } \boldsymbol{\varphi} \in [H_0^1(\Omega)]^3, \boldsymbol{\theta} \in V,$$

is compact. ■

As a conclusion, the operator $A = B - C$ is Fredholm of index 0 and the Fredholm theorem applies [42], i.e., the weak formula (2.11) has a unique solution if and only if the homogeneous weak formula

$$A(\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\theta}; \mathbf{u}, \mathbf{w}, \mathbf{E}_2) = 0$$

has only the trivial solution $(\mathbf{u}, \mathbf{w}, \mathbf{E}_2) = 0$.

2.2 The time domain

In this section, we deal with the existence and uniqueness of the time domain electroseismic problem. As stated before, in the electroseismic system the term coupling the electromagnetic fields and the solid and fluid displacements are neglected in Maxwell equations, so that they are totally independent of the Biot equations. Therefore, the question of the existence and uniqueness of solutions to the electroseismic system reduces to showing the existence and uniqueness of solutions to Biot equations. As we have stated before, this has been done in [49] and [5]. Since our Biot equations and boundary conditions are a little bit different from those in the references, we will prove the existence and uniqueness of the time domain Biot equations in this section with the same technique from the references.

For two matrices $\mathbf{E} = (E_{ij})$, $\mathbf{F} = (F_{ij})$ of the same size, we define

$$\mathbf{E} : \mathbf{F} = \sum_{i,j} E_{ij} F_{ij}.$$

We use $L^p(0, T; H)$ to denote the space of functions $f : (0, T) \rightarrow H$ satisfying

$$\|f\|_{L^p(0, T; H)} = \left(\int_0^T \|f\|_H^p dt \right)^{1/p} < \infty$$

for $1 \leq p < \infty$ and

$$\|f\|_{L^\infty(0, T; H)} = \operatorname{ess\,sup}_t \|f\|_H < \infty.$$

The weak formula

Denoting $V = [H^1(\Omega)]^3 \times H(\operatorname{div}, \Omega)$, $\mathbf{v}_1 = \mathbf{u}$, $\mathbf{v}_2 = \mathbf{w}$,

$$\mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \mathbf{0} \\ \xi \mathbf{D} \end{pmatrix},$$

$$\mathbf{A} = \begin{pmatrix} \rho \mathbf{I}_3 & \rho_f \mathbf{I}_3 \\ \rho_f \mathbf{I}_3 & \rho_e \mathbf{I}_3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\eta}{\kappa} \mathbf{I}_3 \end{pmatrix}, \quad \mathcal{L}\mathbf{v} = \begin{pmatrix} -\operatorname{div} \boldsymbol{\tau} \\ \nabla p \end{pmatrix},$$

Biot equations in the electroseismic model can be compactly written in the form

$$\begin{cases} \mathbf{A} \partial_t^2 \mathbf{v} + \mathbf{B} \partial_t \mathbf{v} + \mathcal{L}\mathbf{v} = \mathbf{F}, & \text{in } \Omega \times (0, T), \\ \mathbf{v}(\mathbf{x}, 0) = \mathbf{0}, & \text{in } \Omega, \\ \partial_t \mathbf{v}(\mathbf{x}, 0) = \mathbf{0}, & \text{in } \Omega, \\ \mathbf{n} \cdot \boldsymbol{\tau} = \mathbf{0}, \quad p = 0, & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (2.16)$$

Integration by parts and using the boundary conditions, we have

$$\begin{aligned}
(\mathcal{L}\mathbf{v}, \mathbf{v}') &= \int_{\Omega} (-\operatorname{div} \boldsymbol{\tau} \cdot \mathbf{v}'_1 + \nabla p \cdot \mathbf{v}'_2) \\
&= \int_{\Omega} (\boldsymbol{\tau} : \nabla \mathbf{v}'_1 - p \operatorname{div} \mathbf{v}'_2) \\
&= (\operatorname{div} \mathbf{v}_1, \lambda \operatorname{div} \mathbf{v}'_1 + C \operatorname{div} \mathbf{v}'_2) + (\operatorname{div} \mathbf{v}_2, C \operatorname{div} \mathbf{v}'_1 + M \operatorname{div} \mathbf{v}'_2) + (2Ge(\mathbf{v}_1), e(\mathbf{v}'_1)).
\end{aligned}$$

Define

$$\mathcal{B}(\mathbf{v}, \mathbf{v}') = (\operatorname{div} \mathbf{v}_1, \lambda \operatorname{div} \mathbf{v}'_1 + C \operatorname{div} \mathbf{v}'_2) + (\operatorname{div} \mathbf{v}_2, C \operatorname{div} \mathbf{v}'_1 + M \operatorname{div} \mathbf{v}'_2) + (2Ge(\mathbf{v}_1), e(\mathbf{v}'_1))$$

where $e(\mathbf{v}_1) = \frac{1}{2}(\nabla \mathbf{v}_1 + \nabla \mathbf{v}_1^T)$. It's obvious that \mathcal{B} is a symmetric bounded bilinear form. We recall the Korn inequality

$$(e(\mathbf{v}_1), e(\mathbf{v}_1)) \geq C_0 \|\mathbf{v}_1\|_{H^1}^2 - \|\mathbf{v}_1\|^2,$$

where C_0 is a positive constant. From the Korn inequality, we obtain

$$\begin{aligned}
\mathcal{B}(\mathbf{v}, \mathbf{v}) &\geq \int_{\Omega} (\operatorname{div} \mathbf{v}_1 \quad \operatorname{div} \mathbf{v}_2) \begin{pmatrix} \lambda & C \\ C & M \end{pmatrix} \begin{pmatrix} \operatorname{div} \mathbf{v}_1 \\ \operatorname{div} \mathbf{v}_2 \end{pmatrix} d\mathbf{x} + 2 \min\{G\} (e(\mathbf{v}_1), e(\mathbf{v}_1)) \\
&\geq \lambda_* \|\operatorname{div} \mathbf{v}_1\|^2 + \lambda_* \|\operatorname{div} \mathbf{v}_2\|^2 + 2C_0 \min\{G\} \|\mathbf{v}_1\|_{H^1}^2 - 2 \min\{G\} \|\mathbf{v}_1\|^2 \\
&\geq C_0 \|\mathbf{v}\|_V^2 - \theta \|\mathbf{v}\|^2,
\end{aligned}$$

where θ is a positive constant independent of \mathbf{v} and λ_* is the smallest eigenvalue of the matrix

$$\begin{pmatrix} \lambda & C \\ C & M \end{pmatrix}.$$

We define $\mathcal{B}_\theta(\mathbf{v}, \mathbf{v}') = \mathcal{B}(\mathbf{v}, \mathbf{v}') + \theta(\mathbf{v}, \mathbf{v}')$. The bilinear form \mathcal{B}_θ is symmetric, bounded, and it satisfies the following elliptic condition

$$\mathcal{B}_\theta(\mathbf{v}, \mathbf{v}) \geq C_0 \|\mathbf{v}\|_V^2.$$

Definition 2.2.1. We call $\mathbf{r} \in L^\infty(0, T; V)$ a generalized solution to (2.16) if it satisfies

$$(\mathbf{A}\partial_t^2 \mathbf{r}(t), \mathbf{v}) + (\mathbf{B}\partial_t \mathbf{r}(t), \mathbf{v}) + \mathcal{B}(\mathbf{r}(t), \mathbf{v}) = (\mathbf{F}(t), \mathbf{v}) \quad \text{a.e. } t \in (0, T) \quad (2.17)$$

for any $\mathbf{v} \in V$.

The existence and uniqueness in the time domain

In this section, we prove the existence and uniqueness of Biot equations in the time domain.

Theorem 2.2.1. *Let $\mathbf{F} \in H^1(0, T; [L^2(\Omega)]^6)$. Then the system (2.16) has a unique weak solution $\mathbf{r}(\mathbf{x}, t)$ such that*

$$\mathbf{r}, \partial_t \mathbf{r} \in L^\infty(0, T; V(\Omega)), \text{ and } \partial_t^2 \mathbf{r} \in L^\infty(0, T; [L^2(\Omega)]^6).$$

Proof. Since V is separable, there exist a sequence of linearly independent functions $\{\mathbf{v}^{(n)}\}_{n \geq 1}$ which form a basis of V . Let us define

$$S_m = \text{span}\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \dots, \mathbf{v}^{(m)}\},$$

and choose

$$\mathbf{r}^{(m)}(t) = \sum_{j=1}^m g_{jm}(t) \mathbf{v}^{(j)}$$

satisfying

$$\mathbf{r}^{(m)}(0) \rightarrow \mathbf{0}, \quad \partial_t \mathbf{r}^{(m)}(0) \rightarrow \mathbf{0}.$$

The functions $g_{jm}(t)$ are determined by the system of ordinary differential equations

$$(\mathbf{A} \partial_t^2 \mathbf{r}^{(m)}, \mathbf{v}) + (\mathbf{B} \partial_t \mathbf{r}^{(m)}, \mathbf{v}) + \mathcal{B}(\mathbf{r}^{(m)}, \mathbf{v}) = (\mathbf{F}, \mathbf{v}), \quad \mathbf{v} \in S_m. \quad (2.18)$$

In fact we construct an approximate solution $\mathbf{r}^{(m)}(t)$ to (2.17) in S_m .

Next we prove two a priori estimates of $\mathbf{r}^{(m)}(t)$. By Choosing $\mathbf{v} = \partial_t \mathbf{r}^{(m)}$ in (2.18), we obtain

$$(\mathbf{A} \partial_t^2 \mathbf{r}^{(m)}, \partial_t \mathbf{r}^{(m)}) + (\mathbf{B} \partial_t \mathbf{r}^{(m)}, \partial_t \mathbf{r}^{(m)}) + \mathcal{B}(\mathbf{r}^{(m)}, \partial_t \mathbf{r}^{(m)}) = (\mathbf{F}, \partial_t \mathbf{r}^{(m)}). \quad (2.19)$$

Let

$$\Lambda(t) = \|\mathbf{A}^{1/2} \partial_t \mathbf{r}^{(m)}(t)\|^2 + \mathcal{B}_\theta(\mathbf{r}^{(m)}(t), \mathbf{r}^{(m)}(t)).$$

Since \mathcal{B}_θ is elliptic, $\Lambda(t)$ can be lower bounded as follows

$$\Lambda(t) \geq C_0 (\|\mathbf{r}^{(m)}(t)\|_V^2 + \|\partial_t \mathbf{r}^{(m)}(t)\|^2),$$

Taking the time derivative of $\Lambda(t)$ shows

$$\frac{d}{dt} \Lambda(t) = 2(\mathbf{A} \partial_t^2 \mathbf{r}^{(m)}, \partial_t \mathbf{r}^{(m)}) + 2\mathcal{B}_\theta(\mathbf{r}^{(m)}, \partial_t \mathbf{r}^{(m)}).$$

From (2.19), we know that

$$\begin{aligned}\frac{d}{dt}\Lambda(t) &= 2\theta(\mathbf{r}^{(m)}, \partial_t \mathbf{r}^{(m)}) + 2(\mathbf{F}, \partial_t \mathbf{r}^{(m)}) - 2(\mathbf{B}\partial_t \mathbf{r}^{(m)}, \partial_t \mathbf{r}^{(m)}) \\ &\leq C_0(\|\mathbf{F}(t)\|^2 + \|\mathbf{r}^{(m)}(t)\|^2 + \|\partial_t \mathbf{r}^{(m)}(t)\|^2) \\ &\leq C_0(\|\mathbf{F}(t)\|^2 + \|\mathbf{r}^{(m)}(t)\|_V^2 + \|\partial_t \mathbf{r}^{(m)}(t)\|^2).\end{aligned}$$

Integrating from 0 to t , we have

$$\Lambda(t) \leq C_0 \int_0^t \|\mathbf{F}(\tau)\|^2 + \Lambda(0) + C_0 \int_0^t (\|\mathbf{r}^{(m)}(\tau)\|_V^2 + \|\partial_\tau \mathbf{r}^{(m)}(\tau)\|^2).$$

Since

$$\Lambda(0) = \|\mathbf{A}^{1/2} \partial_t \mathbf{r}^{(m)}(0)\|^2 + \mathcal{B}_\theta(\mathbf{r}^{(m)}(0), \mathbf{r}^{(m)}(0)) \quad (2.20)$$

and

$$\mathbf{r}^{(m)}(0), \partial_t \mathbf{r}^{(m)}(0) \rightarrow 0,$$

$\Lambda(0)$ is bounded by a constant C_0 independent of m . We conclude that

$$\begin{aligned}\|\mathbf{r}^{(m)}(t)\|_V^2 + \|\partial_t \mathbf{r}^{(m)}(t)\|^2 &\leq C_0 \Lambda(t) \\ &\leq C_0(\|\mathbf{F}\|_{L^2(0,t;V)}^2 + 1) \\ &\quad + C_0 \int_0^t (\|\mathbf{r}^{(m)}(\tau)\|_V^2 + \|\partial_\tau \mathbf{r}^{(m)}(\tau)\|^2)\end{aligned} \quad (2.21)$$

and by the Gronwall inequality

$$\|\mathbf{r}^{(m)}(t)\|_V^2 + \|\partial_t \mathbf{r}^{(m)}(t)\|^2 \leq C_0 \quad (2.22)$$

where C_0 is independent of t and m . Let us choose $\mathbf{v} = \partial_t^2 \mathbf{r}^{(m)}$ in (2.18) and let $t = 0$. The term $\partial_t^2 \mathbf{r}^{(m)}(0)$ can be bounded by

$$\begin{aligned}\|\partial_t^2 \mathbf{r}^{(m)}(0)\|^2 &\leq C_0(\mathbf{A}\partial_t^2 \mathbf{r}^{(m)}(0), \partial_t^2 \mathbf{r}^{(m)}(0)) \\ &\leq C_0\|\partial_t^2 \mathbf{r}^{(m)}(0)\|(\|\partial_t \mathbf{r}^{(m)}(0)\|^2 + \|\mathbf{r}^{(m)}(0)\|^2 + \|\mathbf{F}(0)\|^2).\end{aligned}$$

and therefore

$$\|\partial_t^2 \mathbf{r}^{(m)}(0)\| \leq C_0,$$

where C_0 is a constant independent of m . By take the time derivative of (2.18), we get

$$(\mathbf{A}\partial_t^3 \mathbf{r}^{(m)}, \mathbf{v}) + (\mathbf{B}\partial_t^2 \mathbf{r}^{(m)}, \mathbf{v}) + \mathcal{B}(\partial_t \mathbf{r}^{(m)}, \mathbf{v}) = (\partial_t \mathbf{F}, \mathbf{v}), \quad \mathbf{v} \in S_m. \quad (2.23)$$

Choosing $\mathbf{v} = \partial_t^2 \mathbf{r}^{(m)}$ in (2.23), we obtain

$$(\mathbf{A}\partial_t^3 \mathbf{r}^{(m)}, \partial_t^2 \mathbf{r}^{(m)}) + (\mathbf{B}\partial_t^2 \mathbf{r}^{(m)}, \partial_t^2 \mathbf{r}^{(m)}) + \mathcal{B}(\partial_t \mathbf{r}^{(m)}, \partial_t^2 \mathbf{r}^{(m)}) = (\partial_t \mathbf{F}, \partial_t^2 \mathbf{r}^{(m)}). \quad (2.24)$$

Let

$$\Lambda(t) = \|\mathbf{A}^{1/2} \partial_t^2 \mathbf{r}^{(m)}(t)\|^2 + \mathcal{B}_\theta(\partial_t \mathbf{r}^{(m)}(t), \partial_t \mathbf{r}^{(m)}(t)).$$

Since \mathcal{B}_θ is elliptic, $\Lambda(t)$ can be lower bounded as follows

$$\Lambda(t) \geq C_0 (\|\partial_t \mathbf{r}^{(m)}(t)\|_V^2 + \|\partial_t^2 \mathbf{r}^{(m)}(t)\|^2),$$

and from (2.24)

$$\begin{aligned} \frac{d}{dt} \Lambda(t) &= 2(\mathbf{A} \partial_t^3 \mathbf{r}^{(m)}, \partial_t^2 \mathbf{r}^{(m)}) + 2\mathcal{B}_\theta(\partial_t \mathbf{r}^{(m)}, \partial_t^2 \mathbf{r}^{(m)}) \\ &\leq C_0 (\|\partial_t \mathbf{F}(t)\|^2 + \|\partial_t \mathbf{r}^{(m)}(t)\|^2 + \|\partial_t^2 \mathbf{r}^{(m)}(t)\|^2) \\ &\leq C_0 (\|\partial_t \mathbf{F}(t)\|^2 + \|\partial_t \mathbf{r}^{(m)}(t)\|_V^2 + \|\partial_t^2 \mathbf{r}^{(m)}(t)\|^2). \end{aligned}$$

Integrating from 0 to t , we have

$$\Lambda(t) \leq C_0 \int_0^t \|\partial_\tau \mathbf{F}(\tau)\|^2 + \Lambda(0) + C_0 \int_0^t (\|\partial_\tau \mathbf{r}^{(m)}(\tau)\|_V^2 + \|\partial_\tau^2 \mathbf{r}^{(m)}(\tau)\|^2).$$

We conclude that

$$\begin{aligned} \|\partial_t \mathbf{r}^{(m)}(t)\|_V^2 + \|\partial_t^2 \mathbf{r}^{(m)}(t)\|^2 &\leq C_0 \Lambda(t) \\ &\leq C_0 (\|\partial_t \mathbf{F}\|_{L^2(0,T;V)}^2 + 1) \\ &\quad + C_0 \int_0^t (\|\partial_\tau \mathbf{r}^{(m)}(\tau)\|_V^2 + \|\partial_\tau^2 \mathbf{r}^{(m)}(\tau)\|^2) \end{aligned} \quad (2.25)$$

and by the Gronwall inequality

$$\|\partial_t \mathbf{r}^{(m)}(t)\|_V^2 + \|\partial_t^2 \mathbf{r}^{(m)}(t)\|^2 \leq C_0 \quad (2.26)$$

where C_0 is independent of t and m . Consequently

$$\mathbf{r}^{(m)}, \partial_t \mathbf{r}^{(m)} \in L^\infty(0, T; V), \quad \partial_t^2 \mathbf{r}^{(m)} \in L^\infty(0, T; [L^2(\Omega)]^6).$$

Note the following duality relationships

$$L^\infty(0, T; V) = L^1(0, T; V)^*, \quad L^\infty(0, T; [L^2(\Omega)]^6) = L^1(0, T; [L^2(\Omega)]^6)^*.$$

The sequences $\{\mathbf{r}^{(m)}(t)\}$, $\{\partial_t \mathbf{r}^{(m)}(t)\}$ and $\{\partial_t^2 \mathbf{r}^{(m)}(t)\}$ are uniformly bounded. It is possible to extract a subsequence from $\{\mathbf{r}^{(m)}\}$, still denoted by $\{\mathbf{r}^{(m)}\}$, such that

$$\begin{aligned} \mathbf{r}^{(m)} &\rightarrow \mathbf{r} && \text{in the weak* topology in } L^\infty(0, T; V), \\ \partial_t \mathbf{r}^{(m)} &\rightarrow \partial_t \mathbf{r} && \text{in the weak* topology in } L^\infty(0, T; V), \\ \partial_t^2 \mathbf{r}^{(m)} &\rightarrow \partial_t^2 \mathbf{r} && \text{in the weak* topology in } L^\infty(0, T; [L^2(\Omega)]^6). \end{aligned}$$

It's obvious that, for any $\mathbf{v} \in V$,

$$\begin{aligned} (\mathbf{A}\partial_t^2 \mathbf{r}^{(m)}, \mathbf{v}) &\rightarrow (\mathbf{A}\partial_t^2 \mathbf{r}, \mathbf{v}) && \text{in the weak* topology in } L^\infty(0, T), \\ (\mathbf{B}\partial_t \mathbf{r}^{(m)}, \mathbf{v}) &\rightarrow (\mathbf{B}\partial_t \mathbf{r}, \mathbf{v}) && \text{in the weak* topology in } L^\infty(0, T), \\ \mathcal{B}(\mathbf{r}^{(m)}, \mathbf{v}) &\rightarrow \mathcal{B}(\mathbf{r}, \mathbf{v}) && \text{in the weak* topology in } L^\infty(0, T). \end{aligned}$$

Let $s(t) \in L^1(0, T)$. Multiplying (2.18) by $s(t)$ and integrating from 0 to T result

$$\int_0^T \left((\mathbf{A}\partial_t^2 \mathbf{r}^{(m)}, \mathbf{v}) + (\mathbf{B}\partial_t \mathbf{r}^{(m)}, \mathbf{v}) + \mathcal{B}(\mathbf{r}^{(m)}, \mathbf{v}) \right) s(t) = \int_0^T (\mathbf{F}, \mathbf{v}) s(t),$$

for all $\mathbf{v} \in S_m$. By letting m go to infinity, we obtain

$$\int_0^T \left((\mathbf{A}\partial_t^2 \mathbf{r}, \mathbf{v}) + (\mathbf{B}\partial_t \mathbf{r}, \mathbf{v}) + \mathcal{B}(\mathbf{r}, \mathbf{v}) \right) s(t) = \int_0^T (\mathbf{F}, \mathbf{v}) s(t). \quad (2.27)$$

for all $\mathbf{v} \in S_m$. Since $\cup S_m$ is dense in V , (2.27) is also true for all $\mathbf{v} \in V$, that's to say

$$(\mathbf{A}\partial_t^2 \mathbf{r}(t), \mathbf{v}) + (\mathbf{B}\partial_t \mathbf{r}(t), \mathbf{v}) + \mathcal{B}(\mathbf{r}(t), \mathbf{v}) = (\mathbf{F}(t), \mathbf{v}) \quad \text{a.e. } t \in (0, T),$$

for all $\mathbf{v} \in V$.

To prove the uniqueness, we only need to show that when $\mathbf{F} = \mathbf{0}$, the Biot system (2.16) has only the trivial weak solution. Let us choose $\mathbf{v} = \partial_t \mathbf{r}(t)$ in the weak formula (2.17). As a result, we get

$$(\mathbf{A}\partial_t^2 \mathbf{r}, \partial_t \mathbf{r}) + (\mathbf{B}\partial_t \mathbf{r}, \partial_t \mathbf{r}) + \mathcal{B}(\mathbf{r}, \partial_t \mathbf{r}) = 0. \quad (2.28)$$

Let us set

$$\Lambda(t) = \|\mathbf{A}^{1/2} \partial_t \mathbf{r}(t)\|^2 + \mathcal{B}_\theta(\mathbf{r}(t), \mathbf{r}(t)),$$

and we have the lower bound for $\Lambda(t)$

$$\Lambda(t) \geq C_0 (\|\mathbf{r}(t)\|_V^2 + \|\partial_t \mathbf{r}(t)\|^2). \quad (2.29)$$

Since the initial values are zero, $\Lambda(0)$ is also zero. Differentiating $\Lambda(t)$ results

$$\begin{aligned} \frac{d}{dt} \Lambda(t) &= 2(\mathbf{A}\partial_t^2 \mathbf{r}, \partial_t \mathbf{r}) + 2\mathcal{B}_\theta(\mathbf{r}, \partial_t \mathbf{r}) \\ &\leq C_0 (\|\mathbf{r}(t)\|^2 + \|\partial_t \mathbf{r}(t)\|^2) \\ &\leq C_0 (\|\mathbf{r}(t)\|_V^2 + \|\partial_t \mathbf{r}(t)\|^2). \end{aligned}$$

Integrating from 0 to T , we get an upper bound for $\Lambda(t)$

$$\Lambda(t) \leq C_0 \int_0^t (\|\mathbf{r}(\tau)\|_V^2 + \|\partial_\tau \mathbf{r}(\tau)\|^2).$$

Combining with (2.29), we obtain

$$\|\mathbf{r}(t)\|_V^2 + \|\partial_t \mathbf{r}(t)\|^2 \leq C_0 \int_0^t (\|\mathbf{r}(\tau)\|_V^2 + \|\partial_\tau \mathbf{r}(\tau)\|^2).$$

The Gronwall inequality tells us that

$$\|\mathbf{r}(t)\|_V^2 + \|\partial_t \mathbf{r}(t)\|^2 \leq 0,$$

i.e., $\mathbf{r}(t) = 0$. ■

Chapter 3

Stability of the Inverse Electrostatic Problem

Let us recall the electrostatic model

$$\partial_t \mathbf{D} - \text{curl}(\alpha \mathbf{B}) + \gamma \mathbf{D} = \mathbf{0}, \quad (3.1)$$

$$\partial_t \mathbf{B} + \text{curl}(\beta \mathbf{D}) = \mathbf{0}, \quad (3.2)$$

$$\rho \partial_t^2 \mathbf{u} + \rho_f \partial_t^2 \mathbf{w} - \text{div} \boldsymbol{\tau} = \mathbf{0}, \quad (3.3)$$

$$\rho_f \partial_t^2 \mathbf{u} + \rho_e \partial_t^2 \mathbf{w} + \nabla p + \frac{\eta}{\kappa} \partial_t \mathbf{w} - \xi \mathbf{D} = \mathbf{0}, \quad (3.4)$$

$$(\lambda \text{div} \mathbf{u} + C \text{div} \mathbf{w}) \mathbf{I} + G(\nabla \mathbf{u} + \nabla \mathbf{u}^T) = \boldsymbol{\tau}, \quad (3.5)$$

$$C \text{div} \mathbf{u} + M \text{div} \mathbf{w} = -p. \quad (3.6)$$

Here we introduce some new notations

$$\alpha = \frac{1}{\mu}, \quad \beta = \frac{1}{\varepsilon}, \quad \gamma = \frac{\sigma}{\varepsilon}, \quad \xi = \frac{L\eta}{\kappa\varepsilon}.$$

Note that all the parameters are real and positive. To close the Maxwell system (3.1)-(3.2), we assume that the media do not contain any free charges, i.e.,

$$\text{div} \mathbf{D} = \text{div} \mathbf{B} = 0. \quad (3.7)$$

We consider the system of equations (3.1)-(3.7) in $Q = \Omega \times (-T, T)$ where $\Omega \in \mathbb{R}^3$ is a bounded domain with C^∞ boundary $\partial\Omega$. For simplicity we choose homogeneous boundary conditions

$$\mathbf{n} \times \mathbf{D} = \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{B} = 0, \quad \mathbf{n} \cdot \boldsymbol{\tau} = \mathbf{0}, \quad p = 0, \quad \text{on } \partial\Omega, \quad (3.8)$$

where \mathbf{n} is the outer normal vector on $\partial\Omega$. As we are interested in the electrostatic model, we consider zero initial values for the solid displacement \mathbf{u} and for the relative fluid displacement \mathbf{w}

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{0}, \quad \mathbf{w}(\mathbf{x}, 0) = \mathbf{0}, \quad \partial_t \mathbf{u}(\mathbf{x}, 0) = \mathbf{0}, \quad \partial_t \mathbf{w}(\mathbf{x}, 0) = \mathbf{0}, \quad (3.9)$$

while we impose nonzero electric and magnetic fluxes in Ω

$$\mathbf{D}(\mathbf{x}, 0) = \mathbf{D}_0(\mathbf{x}), \quad \mathbf{B}(\mathbf{x}, 0) = \mathbf{B}_0(\mathbf{x}). \quad (3.10)$$

In accordance with the physical properties of underground immersed porous rocks, we assume that the matrices

$$\begin{pmatrix} \rho & \rho_f \\ \rho_f & \rho_e \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & C \\ C & M \end{pmatrix} \quad (3.11)$$

are symmetric positive definite and that $\rho_e > \rho_f, \rho > \rho_f$.

We also assume that all the parameters in Biot equations (3.3)-(3.6) are known, except the coupling coefficient ξ . The main object of this chapter is to deal with the inverse problem of determining the parameters $(\alpha, \beta, \gamma, \xi)$ from measurements of $(\mathbf{D}, \mathbf{B}, \mathbf{u}, \mathbf{w})$ in Q_ω , where $Q_\omega = \omega \times (-T, T)$ and $\omega \subset \Omega$ is a fixed neighborhood of the boundary. To the authors' best knowledge, [16] and its following work [15] are the only papers considering the inverse electroseismic problem. In those papers, the authors assume that LE is known everywhere and focus on the inverse of (L, σ) . Their method is based on the CGO solution [21] of frequency domain Maxwell equations. Different from their work, our method is based on a Carleman estimate to the electroseismic model and we try to inverse the problem with only near boundary measurements.

In Section 3.1 we prove a Carleman estimate for the electroseismic system, from which we prove, in section 3.2, the Hölder stability of the inverse problem with measurements of $(\mathbf{D}, \mathbf{B}, \mathbf{u}, \mathbf{w})$ near the boundary.

3.1 A Carleman estimate for the electroseismic model

In this section we first show how to derive Carleman estimates for Maxwell equations and Biot equations, respectively. Then the combination of the two estimates gives the Carleman estimate for the electroseismic equations. To derive a Carleman estimate for a system of equations, the usual process consists of diagonalizing the system and then of applying a Carleman estimate for each scalar equation that composes the diagonalized system [29]. We first recall a Carleman estimate for the scalar wave equation [30, 29].

Lemma 3.1.1. *Let us consider a point $\mathbf{x}_* \in \mathbb{R}^3 \setminus \bar{\Omega}$ and a strictly positive function $c(\mathbf{x}) \in C^1(\bar{\Omega})$. Assume that they satisfy*

$$\frac{\nabla c(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_*)}{2c(\mathbf{x})} < 1 - c_0, \quad \text{for all } \mathbf{x} \in \bar{\Omega}, \quad (3.12)$$

where $c_0 \in (0, 1)$ is a constant. Then there exist constants $\zeta, \theta, C_0 > 0$, such that the function $\varphi = e^{\theta\psi}$, where

$$\psi = |\mathbf{x} - \mathbf{x}_*|^2 - \zeta|t|^2,$$

satisfies

$$\varphi(\mathbf{x}, T) = \varphi(\mathbf{x}, -T) < 1, \quad \varphi(\mathbf{x}, 0) \geq 1,$$

and

$$\int_Q e^{2\tau\varphi} (\tau^3 |u|^2 + \tau |\nabla_{\mathbf{x},t} u|^2) \leq C_0 \int_Q e^{2\tau\varphi} |f|^2,$$

for all τ larger than a positive constant τ_0 and for any $u \in C_0^2(Q)$ that solves

$$\partial_t^2 u - c(\mathbf{x}) \Delta u = f,$$

The notation $|\cdot|^2$ means the sum of the squares of all the components of vectors or matrices.

Remark 1. For any $\epsilon > 0$ sufficiently small, there exists a constant δ such that

$$\varphi(\mathbf{x}, t) > 1 - \epsilon, \text{ for } |t| < \delta$$

and

$$\varphi(\mathbf{x}, t) < 1 - 2\epsilon, \text{ for } t > T - \delta \text{ or } t < -T + \delta.$$

We denote

$$\varphi_0(\mathbf{x}) = \varphi(\mathbf{x}, 0), \quad \Phi = \max_{(\mathbf{x}, t) \in Q} \varphi.$$

For Maxwell equations with $\sigma = 0$, Carleman estimates can be found, for example, in [29, 4]. The arguments in these references can be easily generalized to the case when $\sigma \neq 0$.

Lemma 3.1.2. *Assume that*

$$\alpha \in C^2(\bar{\Omega}), \quad \beta \in C^2(\bar{\Omega}), \quad \gamma \in C^1(\bar{\Omega}),$$

satisfy

$$\alpha > c_0 > 0, \quad \beta > c_0 > 0, \quad \gamma \geq 0,$$

where c_0 is a positive constant. Assume additionally that the wave speed c defined by $c = \alpha\beta$ satisfies (3.12). Then there exists a constant C_0 such that

$$\begin{aligned} & \int_Q e^{2\tau\varphi} \left(\tau^3 (|\mathbf{D}|^2 + |\mathbf{B}|^2) + \tau (|\nabla_{\mathbf{x},t} \mathbf{D}|^2 + |\nabla_{\mathbf{x},t} \mathbf{B}|^2) \right) \\ & \leq C_0 \int_Q e^{2\tau\varphi} (|\mathbf{J}_1|^2 + |\mathbf{J}_2|^2 + |\nabla_{\mathbf{x},t} \mathbf{J}_1|^2 + |\nabla_{\mathbf{x},t} \mathbf{J}_2|^2), \end{aligned}$$

for all τ larger than a positive constant τ_0 and for any $\mathbf{D}, \mathbf{B} \in C_0^2(Q)$ that solve

$$\begin{cases} \partial_t \mathbf{D} - \operatorname{curl}(\alpha \mathbf{B}) + \gamma \mathbf{D} &= \mathbf{J}_1, \\ \partial_t \mathbf{B} + \operatorname{curl}(\beta \mathbf{D}) &= \mathbf{J}_2, \\ \operatorname{div} \mathbf{D} = \operatorname{div} \mathbf{B} &= 0. \end{cases} \quad (3.13)$$

Proof. From the second equation of (3.13), we know that

$$\partial_t \mathbf{B} = -\operatorname{curl}(\beta \mathbf{D}) + \mathbf{J}_2.$$

Taking the time derivative of the first equation of (3.13) and substituting $\partial_t \mathbf{B}$ result

$$\partial_t^2 \mathbf{D} - \operatorname{curl}(\alpha(-\operatorname{curl}(\beta \mathbf{D}) + \mathbf{J}_2)) + \gamma \partial_t \mathbf{D} = \partial_t \mathbf{J}_1,$$

that is

$$\partial_t^2 \mathbf{D} - \alpha \beta \Delta \mathbf{D} = \partial_t \mathbf{J}_1 + \operatorname{curl}(\alpha \mathbf{J}_2) - \mathcal{R}_1. \quad (3.14)$$

where

$$\mathcal{R}_1 = \nabla(\alpha \beta) \times \operatorname{curl} \mathbf{D} + \operatorname{curl}(\alpha \nabla \beta \times \mathbf{D}) + \gamma \partial_t \mathbf{D}.$$

Similarly we can cancel out \mathbf{D} from (3.13) to get

$$\partial_t^2 \mathbf{B} - \alpha \beta \Delta \mathbf{B} = \partial_t \mathbf{J}_2 - \operatorname{curl}(\beta \mathbf{J}_1) - \mathcal{R}_2, \quad (3.15)$$

where

$$\mathcal{R}_2 = \nabla(\alpha \beta) \times \operatorname{curl} \mathbf{B} + \operatorname{curl}(\beta \nabla \alpha \times \mathbf{B}) - \operatorname{curl}(\beta \gamma \mathbf{D}).$$

Applying Lemma 3.1.1 to each component of the equation (3.14) and (3.15), we have

$$\begin{aligned} \int_Q e^{2\tau\varphi} (\tau^3 |\mathbf{D}|^2 + \tau |\nabla_{\mathbf{x},t} \mathbf{D}|^2) &\leq C_0 \int_Q e^{2\tau\varphi} (F + |\mathbf{D}|^2 + |\nabla_{\mathbf{x},t} \mathbf{D}|^2), \\ \int_Q e^{2\tau\varphi} (\tau^3 |\mathbf{B}|^2 + \tau |\nabla_{\mathbf{x},t} \mathbf{B}|^2) &\leq C_0 \int_Q e^{2\tau\varphi} (F + |\mathbf{D}|^2 + |\nabla_{\mathbf{x},t} \mathbf{D}|^2 + |\mathbf{B}|^2 + |\nabla_{\mathbf{x},t} \mathbf{B}|^2), \end{aligned}$$

where

$$F = |\mathbf{J}_1|^2 + |\mathbf{J}_2|^2 + |\nabla_{\mathbf{x},t} \mathbf{J}_1|^2 + |\nabla_{\mathbf{x},t} \mathbf{J}_2|^2.$$

Adding these two inequalities, we obtain

$$\begin{aligned} &\int_Q e^{2\tau\varphi} (\tau^3 (|\mathbf{D}|^2 + |\mathbf{B}|^2) + \tau (|\nabla_{\mathbf{x},t} \mathbf{D}|^2 + |\nabla_{\mathbf{x},t} \mathbf{B}|^2)) \\ &\leq C_0 \int_Q e^{2\tau\varphi} (F + |\mathbf{D}|^2 + |\nabla_{\mathbf{x},t} \mathbf{D}|^2 + |\mathbf{B}|^2 + |\nabla_{\mathbf{x},t} \mathbf{B}|^2). \end{aligned}$$

Taking τ large enough, the right hand side term

$$|\mathbf{D}|^2 + |\nabla_{\mathbf{x},t} \mathbf{D}|^2 + |\mathbf{B}|^2 + |\nabla_{\mathbf{x},t} \mathbf{B}|^2$$

can be absorbed by the left side, which completes the proof. \blacksquare

Before deriving a Carleman estimate for Biot equations, we study the property of the matrices associated with the parameters. Define

$$\begin{aligned}
\rho_0 &= \rho\rho_e - \rho_f^2, \\
c &= \frac{\rho_e}{\rho_0}G, \\
\mathbf{a} &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\rho_0}{\rho_e} & \rho_f \\ 0 & \rho_e \end{pmatrix}^{-1} \begin{pmatrix} \lambda + G - \frac{\rho_f}{\rho_e}C & C \\ C - \frac{\rho_f}{\rho_e}M & M \end{pmatrix} \\
&= \begin{pmatrix} \frac{\rho_e}{\rho_0}(\lambda + G) - 2\frac{\rho_f}{\rho_0}C + \frac{\rho_f^2}{\rho_0\rho_e}M & \frac{\rho_e}{\rho_0}C - \frac{\rho_f}{\rho_0}M \\ \frac{1}{\rho_e}C - \frac{\rho_f}{\rho_e^2}M & \frac{1}{\rho_e}M \end{pmatrix}.
\end{aligned} \tag{3.16}$$

From the positive definite of the matrices (3.11), we know that $\rho_0 > 0$. Let us denote

$$\tilde{\mathbf{a}} = \begin{pmatrix} c + a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{3.17}$$

The two eigenvalues of $\tilde{\mathbf{a}}$ are

$$\frac{(c + a_{11} + a_{22}) \pm \sqrt{(c + a_{11} - a_{22})^2 + 4a_{12}a_{21}}}{2}.$$

Since

$$a_{12}a_{21} = \frac{1}{\rho_0}(C - \frac{\rho_f}{\rho_e}M)^2 \geq 0,$$

$\tilde{\mathbf{a}}$ has two real eigenvalues. Since

$$\tilde{\mathbf{a}} = \begin{pmatrix} \frac{\rho_e}{\rho_0}(\lambda + 2G) - 2\frac{\rho_f}{\rho_0}C + \frac{\rho_f^2}{\rho_0\rho_e}M & \frac{\rho_e}{\rho_0}C - \frac{\rho_f}{\rho_0}M \\ \frac{1}{\rho_e}C - \frac{\rho_f}{\rho_e^2}M & \frac{1}{\rho_e}M \end{pmatrix},$$

the determinant of $\tilde{\mathbf{a}}$ is

$$\begin{aligned}
\det \tilde{\mathbf{a}} &= \frac{1}{\rho_0}((\lambda + 2G) + \frac{\rho_f^2}{\rho_e^2}M - 2\frac{\rho_f}{\rho_e}C)M - \frac{1}{\rho_0}(C - \frac{\rho_f}{\rho_e}M)^2 \\
&= \frac{1}{\rho_0}(\lambda M - C^2 + 2GM) > 0,
\end{aligned}$$

which is larger than zero and the trace of $\tilde{\mathbf{a}}$ is

$$\begin{aligned}
\text{tr} \tilde{\mathbf{a}} &= \frac{1}{\rho_0}(\rho_e(\lambda + 2G) + \rho M - 2\rho_f C) \\
&\geq \frac{\rho_f}{\rho_0}(\lambda + 2G + M - 2C).
\end{aligned}$$

From the positive definite of the matrices (3.11), we have $(\lambda + M)^2 \geq 4\lambda M > 4C^2$ and hence $\text{tr} \tilde{\mathbf{a}} > 0$. Therefore $\tilde{\mathbf{a}}$ is similar to a diagonal matrix and it has two positive eigenvalues.

In the following, we will derive a Carleman estimate for Biot equations (3.3)-(3.6). The idea is similar to Maxwell equations. We emphasize that the Carleman estimate from [5] does not apply directly to our Biot system, since our Biot equations are different from those in [5] due to the presence of the term $\partial_t \mathbf{w}$. We will explain in detail the difference in the proof of the following lemma.

Lemma 3.1.3. *Assume that all the parameters in Biot equations are in $C^3(\bar{\Omega})$. Assume that $c = \frac{\rho_e}{\rho_0} G$ and two eigenvalues of the matrix $\tilde{\mathbf{a}}$ given by (3.17) satisfy the condition (3.12). Then there exists a constant C_0 such that*

$$\begin{aligned} & \int_Q e^{2\tau\varphi} \left(\tau^3 (|\mathbf{u}|^2 + |\operatorname{div} \mathbf{u}|^2 + |\operatorname{div} \mathbf{w}|^2 + |\operatorname{curl} \mathbf{u}|^2) \right. \\ & \quad \left. + \tau (|\nabla_{\mathbf{x},t} \mathbf{u}|^2 + |\nabla_{\mathbf{x},t} (\operatorname{div} \mathbf{u})|^2 + |\nabla_{\mathbf{x},t} (\operatorname{div} \mathbf{w})|^2 + |\nabla_{\mathbf{x},t} (\operatorname{curl} \mathbf{u})|^2) \right) \\ & \leq C_0 \int_Q e^{2\tau\varphi} (|\mathbf{F}_1|^2 + |\mathbf{F}_2|^2 + |\mathbf{D}|^2 + |\nabla \mathbf{F}_1|^2 + |\nabla \mathbf{F}_2|^2 + |\nabla \mathbf{D}|^2), \end{aligned}$$

for all τ larger than a positive constant τ_0 and for any $\mathbf{u}, \mathbf{w} \in C_0^3(Q)$ that are symmetric with respect to t and solve

$$\begin{cases} \rho \partial_t^2 \mathbf{u} + \rho_f \partial_t^2 \mathbf{w} - \operatorname{div} \boldsymbol{\tau} & = \mathbf{F}_1, \\ \rho_f \partial_t^2 \mathbf{u} + \rho_e \partial_t^2 \mathbf{w} + \nabla p + \frac{\eta}{\kappa} \partial_t \mathbf{w} - \xi \mathbf{D} & = \mathbf{F}_2, \\ (\lambda \operatorname{div} \mathbf{u} + C \operatorname{div} \mathbf{w}) \mathbf{I} + G(\nabla \mathbf{u} + \nabla \mathbf{u}^T) & = \boldsymbol{\tau}, \\ C \operatorname{div} \mathbf{u} + M \operatorname{div} \mathbf{w} & = -p. \end{cases} \quad (3.18)$$

Proof. Let $\mathbf{v} = \mathbf{w} + \frac{\rho_f}{\rho_e} \mathbf{u}$ and replace \mathbf{w} by \mathbf{u}, \mathbf{v} in the above system to obtain an equation system of \mathbf{u} and \mathbf{v}

$$\begin{aligned} & \frac{\rho_0}{\rho_e} \partial_t^2 \mathbf{u} + \rho_f \partial_t^2 \mathbf{v} - \operatorname{div} \boldsymbol{\tau} = \mathbf{F}_1, \\ & \rho_e \partial_t^2 \mathbf{v} + \nabla p + \frac{\eta}{\kappa} \left(\partial_t \mathbf{v} - \frac{\rho_f}{\rho_e} \partial_t \mathbf{u} \right) - \xi \mathbf{D} = \mathbf{F}_2, \\ & \left((\lambda - C \frac{\rho_f}{\rho_e}) \operatorname{div} \mathbf{u} + C \operatorname{div} \mathbf{v} - C \mathbf{u} \cdot \nabla \frac{\rho_f}{\rho_e} \right) \mathbf{I} + G(\nabla \mathbf{u} + \nabla \mathbf{u}^T) = \boldsymbol{\tau}, \\ & (C - M \frac{\rho_f}{\rho_e}) \operatorname{div} \mathbf{u} + M \operatorname{div} \mathbf{v} - M \mathbf{u} \cdot \nabla \frac{\rho_f}{\rho_e} = -p, \end{aligned}$$

where \mathbf{I} is the identity matrix of order 3. After the substitution of $\boldsymbol{\tau}$ and p , we have

$$\frac{\rho_0}{\rho_e} \partial_t^2 \mathbf{u} + \rho_f \partial_t^2 \mathbf{v} - G \Delta \mathbf{u} - \left(\lambda + G - \frac{\rho_f}{\rho_e} C \right) \nabla \operatorname{div} \mathbf{u} - C \nabla \operatorname{div} \mathbf{v} = \mathbf{F}_1 + \mathcal{P}_1, \quad (3.19)$$

$$\rho_e \partial_t^2 \mathbf{v} + \frac{\eta}{\kappa} \partial_t \mathbf{v} - \left(C - \frac{\rho_f}{\rho_e} M \right) \nabla \operatorname{div} \mathbf{u} - M \nabla \operatorname{div} \mathbf{v} = \mathbf{F}_2 + \mathcal{P}_2, \quad (3.20)$$

where

$$\begin{aligned} \mathcal{P}_1 &= (\operatorname{div} \mathbf{u}) \nabla \left(\lambda - \frac{\rho_f}{\rho_e} M \right) + (\operatorname{div} \mathbf{v}) \nabla C + (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \cdot \nabla G - \nabla \left(C \mathbf{u} \cdot \nabla \frac{\rho_f}{\rho_e} \right), \\ \mathcal{P}_2 &= (\operatorname{div} \mathbf{u}) \nabla \left(C - \frac{\rho_f}{\rho_e} M \right) + (\operatorname{div} \mathbf{v}) \nabla M - \nabla \left(M \mathbf{u} \cdot \nabla \frac{\rho_f}{\rho_e} \right) + \frac{\rho_f \eta}{\rho_e \kappa} \partial_t \mathbf{u} + \xi \mathbf{D}. \end{aligned}$$

Set

$$r = \operatorname{div} \mathbf{u}, \quad s = \operatorname{div} \mathbf{v}, \quad \mathbf{m} = \operatorname{curl} \mathbf{u}, \quad \mathbf{n} = \operatorname{curl} \mathbf{v}$$

and

$$\mathbf{K} = \begin{pmatrix} \frac{\rho_0}{\rho_e} \mathbf{I} & \rho_f \mathbf{I} \\ \mathbf{0} & \rho_e \mathbf{I} \end{pmatrix}^{-1}.$$

We multiply the equation system (3.19)-(3.20) by \mathbf{K} to obtain

$$\partial_t^2 \mathbf{u} - c \Delta \mathbf{u} - \frac{\rho_f \eta}{\rho_e \kappa} \partial_t \mathbf{v} - a_{11} \nabla \operatorname{div} \mathbf{u} - a_{12} \nabla \operatorname{div} \mathbf{v} = \mathbf{G}_1 + \mathcal{P}_3, \quad (3.21)$$

$$\partial_t^2 \mathbf{v} + \frac{\eta}{\rho_e \kappa} \partial_t \mathbf{v} - a_{21} \nabla \operatorname{div} \mathbf{u} - a_{22} \nabla \operatorname{div} \mathbf{v} = \mathbf{G}_2 + \mathcal{P}_4, \quad (3.22)$$

where

$$c = \frac{\rho_e}{\rho_0} G, \quad \begin{pmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{pmatrix} = \mathbf{K} \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathcal{P}_3 \\ \mathcal{P}_4 \end{pmatrix} = \mathbf{K} \begin{pmatrix} \mathcal{P}_1 \\ \mathcal{P}_2 \end{pmatrix},$$

and

$$\mathbf{a} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is given by (3.16). Note that \mathcal{P}_3 and \mathcal{P}_4 are composed of

$$r, \quad s, \quad \mathbf{u}, \quad \nabla_{\mathbf{x}, t} \mathbf{u}, \quad \mathbf{D}.$$

The equations (3.21) and (3.22) can be rewritten as

$$\partial_t^2 \mathbf{u} - c \Delta \mathbf{u} = \mathbf{G}_1 + \mathcal{Q}_1, \quad (3.23)$$

$$\partial_t^2 \mathbf{v} + \frac{\eta}{\rho_e \kappa} \partial_t \mathbf{v} = \mathbf{G}_2 + \mathcal{Q}_2, \quad (3.24)$$

where \mathcal{Q}_1 and \mathcal{Q}_2 are first order differential operators of $r, s, \mathbf{u}, \mathbf{D}$. The operator \mathcal{Q}_1 also contains $\partial_t \mathbf{v}$. We can see that the equation (3.24) is not a wave equation anymore. Taking the divergence on both sides of the equation (3.21) and (3.22) and with the help of the equality

$$\Delta \mathbf{u} = \nabla r - \operatorname{curl} \mathbf{m},$$

we obtain

$$\partial_t^2 r - (c + a_{11}) \Delta r - a_{12} \Delta s = \operatorname{div} \mathbf{G}_1 + \mathcal{S}_1, \quad (3.25)$$

$$\partial_t^2 s - a_{21} \Delta r - a_{22} \Delta s = \operatorname{div} \mathbf{G}_2 + \mathcal{S}_2, \quad (3.26)$$

where \mathcal{S}_1 and \mathcal{S}_2 are first order differential operators of $r, s, \mathbf{D}, \mathbf{u}, \mathbf{m}$. Besides, they also contain $\partial_t \mathbf{v}$. Taking the curl on both sides of the equations (3.21) and (3.22) gives

$$\partial_t^2 \mathbf{m} - c \Delta \mathbf{m} = \operatorname{curl} \mathbf{G}_1 + \mathcal{T}_1, \quad (3.27)$$

$$\partial_t^2 \mathbf{n} + \frac{\eta}{\rho_e \kappa} \partial_t \mathbf{n} = \operatorname{curl} \mathbf{G}_2 + \mathcal{T}_2, \quad (3.28)$$

where \mathcal{T}_1 and \mathcal{T}_2 are first order differential operators of $r, s, \mathbf{D}, \mathbf{u}, \mathbf{m}$. The expression of \mathcal{T}_1 also involves the terms $\partial_t \mathbf{v}, \partial_t \mathbf{n}$ and \mathcal{T}_2 also contains $\partial_t \mathbf{v}$. We can see that equation (3.28) is not a wave equation anymore.

We emphasize that the presence of $\partial_t \mathbf{v}$ and $\partial_t \mathbf{n}$ on the right-hand side of \mathcal{Q}_1 and \mathcal{T}_1 prevents us from using the Carleman estimate in [5]. From (3.24) and (3.28), we see that \mathbf{v} and \mathbf{n} are wave equations. We cannot apply the Carleman estimate on these two equations. We need to design a new bound for them. In the following we are going to show how to bound $\mathbf{u}, \mathbf{m}, r, s, \mathbf{v}, \mathbf{n}$ one by one. Applying Lemma 3.1.1 to (3.23) and (3.27), we obtain the following Carleman estimates for \mathbf{u} and \mathbf{m}

$$\begin{aligned}
& \int_Q e^{2\tau\varphi} \left(\tau^3 |\mathbf{u}|^2 + \tau |\nabla_{\mathbf{x},t} \mathbf{u}|^2 \right) \\
& \leq C_0 \int_Q e^{2\tau\varphi} \left(|\mathbf{G}_1|^2 + |\mathbf{D}|^2 + |r|^2 + |\nabla_{\mathbf{x},t} r|^2 + |s|^2 + |\nabla_{\mathbf{x},t} s|^2 + |\partial_t \mathbf{v}|^2 \right), \quad (3.29) \\
& \int_Q e^{2\tau\varphi} \left(\tau^3 |\mathbf{m}|^2 + \tau |\nabla_{\mathbf{x},t} \mathbf{m}|^2 \right) \\
& \leq C_0 \int_Q e^{2\tau\varphi} \left(|\operatorname{curl} \mathbf{G}_1|^2 + |\mathbf{D}|^2 + |\nabla \mathbf{D}|^2 + |\mathbf{u}|^2 + |\nabla_{\mathbf{x},t} \mathbf{u}|^2 \right. \\
& \quad \left. + |r|^2 + |\nabla_{\mathbf{x},t} r|^2 + |s|^2 + |\nabla_{\mathbf{x},t} s|^2 + |\partial_t \mathbf{v}|^2 + |\partial_t \mathbf{n}|^2 \right). \quad (3.30)
\end{aligned}$$

The Carleman estimate for r and s is given by Lemma 2.2 from [5].

Lemma 3.1.4. *There exists a constant C_0 such that*

$$\begin{aligned}
& \int_Q e^{2\tau\varphi} \left(\tau^3 (|r|^2 + |s|^2) + \tau (|\nabla_{\mathbf{x},t} r|^2 + |\nabla_{\mathbf{x},t} s|^2) \right) \\
& \leq C_0 \int_Q e^{2\tau\varphi} \left(|\operatorname{div} \mathbf{G}_1|^2 + |\operatorname{div} \mathbf{G}_2|^2 + |\mathbf{D}|^2 + |\nabla \mathbf{D}|^2 \right. \\
& \quad \left. + |\mathbf{u}|^2 + |\nabla_{\mathbf{x},t} \mathbf{u}|^2 + |\mathbf{m}|^2 + |\nabla_{\mathbf{x},t} \mathbf{m}|^2 + |\partial_t \mathbf{v}|^2 \right), \quad (3.31)
\end{aligned}$$

for all τ large than a constant τ_0 and for any $r, s \in C_0^2(Q)$ that solve (3.25) and (3.26).

We will prove this lemma at the end of this section. Combining (3.29)-(3.31) shows that

$$\begin{aligned}
& \int_Q e^{2\tau\varphi} \left(\tau^3 (|r|^2 + |s|^2 + |\mathbf{u}|^2 + |\mathbf{m}|^2) \right. \\
& \quad \left. + \tau (|\nabla_{\mathbf{x},t} r|^2 + |\nabla_{\mathbf{x},t} s|^2 + |\nabla_{\mathbf{x},t} \mathbf{u}|^2 + |\nabla_{\mathbf{x},t} \mathbf{m}|^2) \right) \\
& \leq C_0 \int_Q e^{2\tau\varphi} \left(|\mathbf{G}_1|^2 + |\nabla \mathbf{G}_1|^2 + |\mathbf{G}_2|^2 + |\nabla \mathbf{G}_2|^2 \right. \\
& \quad \left. + |\mathbf{D}|^2 + |\nabla \mathbf{D}|^2 + |\partial_t \mathbf{v}|^2 + |\partial_t \mathbf{n}|^2 \right). \quad (3.32)
\end{aligned}$$

Next, we bound $\partial_t \mathbf{v}$ and $\partial_t \mathbf{n}$. Let us multiply (3.24) by $e^{2\tau\varphi} \partial_t \mathbf{v}$ and integrate on $(-T, T)$, to get

$$\int_{-T}^T e^{2\tau\varphi} \partial_t^2 \mathbf{v} \cdot \partial_t \mathbf{v} + \frac{\eta}{\rho e^{\mathcal{K}}} \int_{-T}^T e^{2\tau\varphi} |\partial_t \mathbf{v}|^2 = \int_{-T}^T e^{2\tau\varphi} (\mathbf{G}_2 + \mathcal{Q}_2) \cdot \partial_t \mathbf{v}$$

for $\mathbf{v} \in C_0^3(Q)$. Since $\partial_t \varphi = -2\theta\zeta t\varphi$ is odd and \mathbf{v} is even with respect to t ,

$$\begin{aligned} \int_{-T}^T e^{2\tau\varphi} \partial_t^2 \mathbf{v} \cdot \partial_t \mathbf{v} &= - \int_{-T}^T \tau \partial_t \varphi e^{2\tau\varphi} |\partial_t \mathbf{v}|^2 + \left(\frac{1}{2} e^{2\tau\varphi} |\partial_t \mathbf{v}|^2 \right) \Big|_{-T}^T \\ &= 0. \end{aligned}$$

Therefore

$$\frac{\eta}{\rho e^{\mathcal{K}}} \int_{-T}^T e^{2\tau\varphi} |\partial_t \mathbf{v}|^2 = \int_{-T}^T e^{2\tau\varphi} (\mathbf{G}_2 + \mathcal{Q}_2) \cdot \partial_t \mathbf{v}.$$

Integrating on Ω and applying the Hölder inequality yield

$$\int_Q e^{2\tau\varphi} |\partial_t \mathbf{v}|^2 \leq C_0 \int_Q e^{2\tau\varphi} (|\mathbf{G}_2|^2 + |\mathcal{Q}_2|^2).$$

Let us multiply (3.28) by $\partial_t \mathbf{n}$ and integrate from $-T$ to T , to get

$$\int_{-T}^T e^{2\tau\varphi} \partial_t^2 \mathbf{n} \cdot \partial_t \mathbf{n} + \frac{\eta}{\rho e^{\mathcal{K}}} \int_{-T}^T e^{2\tau\varphi} |\partial_t \mathbf{n}|^2 = \int_{-T}^T e^{2\tau\varphi} (\text{curl } \mathbf{G}_2 + \mathcal{F}_2) \cdot \partial_t \mathbf{n}.$$

Similar to $\partial_t \mathbf{v}$, we have

$$\frac{\eta}{\rho e^{\mathcal{K}}} \int_{-T}^T e^{2\tau\varphi} |\partial_t \mathbf{n}|^2 = \int_{-T}^T e^{2\tau\varphi} (\text{curl } \mathbf{G}_2 + \mathcal{F}_2) \cdot \partial_t \mathbf{n}.$$

Integrating on Ω and applying the Hölder inequality yield

$$\int_Q e^{2\tau\varphi} |\partial_t \mathbf{n}|^2 \leq C_0 \int_Q e^{2\tau\varphi} (|\text{curl } \mathbf{G}_2|^2 + |\mathcal{F}_2|^2).$$

Let us recall that \mathcal{Q}_2 is a first order differential operator of $r, s, \mathbf{u}, \mathbf{D}$. That's to say $\partial_t \mathbf{v}$ can be bounded by $r, s, \mathbf{D}, \mathbf{u}$ and their first order derivatives. Note that \mathcal{F}_2 is composed of $r, s, \mathbf{D}, \mathbf{u}, \mathbf{m}$, their first order derivatives and $\partial_t \mathbf{v}$. Therefore $\partial_t \mathbf{n}$ can be bounded by $r, s, \mathbf{D}, \mathbf{u}, \mathbf{m}$ and their first order derivatives. The estimate (3.32) becomes

$$\begin{aligned} & \int_Q e^{2\tau\varphi} \left(\tau^3 (|r|^2 + |s|^2 + |\mathbf{u}|^2 + |\mathbf{m}|^2) \right. \\ & \quad \left. + \tau (|\nabla_{\mathbf{x},t} r|^2 + |\nabla_{\mathbf{x},t} s|^2 + |\nabla_{\mathbf{x},t} \mathbf{u}|^2 + |\nabla_{\mathbf{x},t} \mathbf{m}|^2) \right) \\ & \leq C_0 \int_Q e^{2\tau\varphi} (|\mathbf{G}_1|^2 + |\nabla \mathbf{G}_1|^2 + |\mathbf{G}_2|^2 + |\nabla \mathbf{G}_2|^2 + |\mathbf{D}|^2 + |\nabla \mathbf{D}|^2 \\ & \quad + |r|^2 + |s|^2 + |\mathbf{u}|^2 + |\mathbf{m}|^2 + |\nabla_{\mathbf{x},t} r|^2 + |\nabla_{\mathbf{x},t} s|^2 + |\nabla_{\mathbf{x},t} \mathbf{u}|^2 + |\nabla_{\mathbf{x},t} \mathbf{m}|^2). \end{aligned} \quad (3.33)$$

Taking τ large enough to absorb the right-hand side terms related to $r, s, \mathbf{u}, \mathbf{m}$, we get

$$\begin{aligned} & \int_Q e^{2\tau\varphi} \left(\tau^3 (|r|^2 + |s|^2 + |\mathbf{u}|^2 + |\mathbf{m}|^2) + \tau (|\nabla_{\mathbf{x},t} r|^2 + |\nabla_{\mathbf{x},t} s|^2 + |\nabla_{\mathbf{x},t} \mathbf{u}|^2 + |\nabla_{\mathbf{x},t} \mathbf{m}|^2) \right) \\ & \leq C_0 \int_Q e^{2\tau\varphi} (|\mathbf{G}_1|^2 + |\nabla \mathbf{G}_1|^2 + |\mathbf{G}_2|^2 + |\nabla \mathbf{G}_2|^2 + |\mathbf{D}|^2 + |\nabla \mathbf{D}|^2). \end{aligned}$$

From the relationship between \mathbf{F} and \mathbf{G} , we see that

$$\begin{aligned} |\mathbf{G}_1|^2 + |\mathbf{G}_2|^2 & \leq C_0 (|\mathbf{F}_1|^2 + |\mathbf{F}_2|^2), \\ |\nabla \mathbf{G}_1|^2 + |\nabla \mathbf{G}_2|^2 & \leq C_0 (|\mathbf{F}_1|^2 + |\mathbf{F}_2|^2 + |\nabla \mathbf{F}_1|^2 + |\nabla \mathbf{F}_2|^2). \end{aligned}$$

Since $\mathbf{v} = \mathbf{w} + \frac{\rho_f}{\rho_e} \mathbf{u}$, we have

$$\begin{aligned} \operatorname{div} \mathbf{w} & = \operatorname{div} \mathbf{v} - \nabla \left(\frac{\rho_f}{\rho_e} \right) \cdot \mathbf{u} + \frac{\rho_f}{\rho_e} \operatorname{div} \mathbf{u} \\ & = s + \frac{\rho_f}{\rho_e} r - \nabla \left(\frac{\rho_f}{\rho_e} \right) \cdot \mathbf{u}, \end{aligned}$$

and hence

$$\begin{aligned} |\operatorname{div} \mathbf{w}|^2 & \leq C_0 (|r|^2 + |s|^2 + |\mathbf{u}|^2), \\ |\nabla_{\mathbf{x},t} (\operatorname{div} \mathbf{w})|^2 & \leq C_0 (|r|^2 + |\nabla_{\mathbf{x},t} r|^2 + |\nabla_{\mathbf{x},t} s|^2 + |\mathbf{u}|^2 + |\nabla_{\mathbf{x},t} \mathbf{u}|^2). \end{aligned}$$

This completes the proof. ■

Combining Lemma 3.1.2 and 3.1.3 yields a Carleman estimate for the electroseismic equations.

Theorem 3.1.1. *Assume that all the parameters in the electroseismic system satisfy the hypotheses of Lemma 3.1.2 and Lemma 3.1.3. Then, there exists a constant C_0 such that*

$$\begin{aligned} & \int_Q e^{2\tau\varphi} \left(\tau^3 (|\mathbf{D}|^2 + |\mathbf{B}|^2 + |\mathbf{u}|^2 + |\operatorname{div} \mathbf{u}|^2 + |\operatorname{div} \mathbf{w}|^2 + |\operatorname{curl} \mathbf{u}|^2) \right. \\ & \quad \left. + \tau (|\nabla_{\mathbf{x},t} \mathbf{D}|^2 + |\nabla_{\mathbf{x},t} \mathbf{B}|^2 + |\nabla_{\mathbf{x},t} \mathbf{u}|^2 + |\nabla_{\mathbf{x},t} (\operatorname{div} \mathbf{u})|^2 + |\nabla_{\mathbf{x},t} (\operatorname{div} \mathbf{w})|^2 + |\nabla_{\mathbf{x},t} (\operatorname{curl} \mathbf{u})|^2) \right) \\ & \leq C_0 \int_Q e^{2\tau\varphi} (|\mathbf{F}_1|^2 + |\mathbf{F}_2|^2 + |\mathbf{J}_1|^2 + |\mathbf{J}_2|^2 + |\nabla \mathbf{F}_1|^2 + |\nabla \mathbf{F}_2|^2 + |\nabla_{\mathbf{x},t} \mathbf{J}_1|^2 + |\nabla_{\mathbf{x},t} \mathbf{J}_2|^2), \end{aligned}$$

for all τ larger than a positive constant τ_0 and for any $\mathbf{D}, \mathbf{B} \in C_0^2(Q)$, $\mathbf{u}, \mathbf{w} \in C_0^3(Q)$ that solve (3.13) and (3.18).

At the end of this section, we prove Lemma 3.1.4 for completeness.

Proof of Lemma 3.1.4. From the discussion of the property of $\tilde{\mathbf{a}}$, we know that $\tilde{\mathbf{a}}$ has two real eigenvalues, denoted as $\lambda_1(\mathbf{x})$ and $\lambda_2(\mathbf{x})$. There exists an invertible matrix $\mathbf{P}(\mathbf{x})$ such that

$$\mathbf{P}^{-1}(\mathbf{x})\tilde{\mathbf{a}}\mathbf{P}(\mathbf{x}) = \begin{pmatrix} \lambda_1(\mathbf{x}) & 0 \\ 0 & \lambda_2(\mathbf{x}) \end{pmatrix}.$$

Let us define

$$\begin{pmatrix} \tilde{r} \\ \tilde{s} \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} r \\ s \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

$$\begin{pmatrix} G_3 \\ G_4 \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} \operatorname{div} \mathbf{G}_1 \\ \operatorname{div} \mathbf{G}_2 \end{pmatrix}, \quad \begin{pmatrix} \mathcal{S}_3 \\ \mathcal{S}_4 \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \end{pmatrix}.$$

The equation system (3.25)-(3.26) is equivalent to

$$\begin{pmatrix} \partial_t^2 \tilde{r} \\ \partial_t^2 \tilde{s} \end{pmatrix} - \begin{pmatrix} \lambda_1 \Delta \tilde{r} \\ \lambda_2 \Delta \tilde{s} \end{pmatrix} = \begin{pmatrix} G_3 \\ G_4 \end{pmatrix} + \begin{pmatrix} \mathcal{S}_3 \\ \mathcal{S}_4 \end{pmatrix} + \mathcal{B}(r, s), \quad (3.34)$$

where $\mathcal{B}(r, s)$ is a first order differential operator of r and s . Since we assume that λ_1 and λ_2 satisfy (3.12), we can apply Lemma 3.1.1 to the equation system (3.34) to obtain

$$\begin{aligned} & \int_Q e^{2\tau\varphi} \left(\tau^3 (|\tilde{r}|^2 + |\tilde{s}|^2) + \tau (|\nabla_{\mathbf{x},t} \tilde{r}|^2 + |\nabla_{\mathbf{x},t} \tilde{s}|^2) \right) \\ & \leq C_0 \int_Q e^{2\tau\varphi} (|G_3|^2 + |G_4|^2 + |\mathcal{S}_3|^2 + |\mathcal{S}_4|^2 + |r|^2 + |s|^2 + |\nabla r|^2 + |\nabla s|^2). \end{aligned}$$

From the relationship between r, s and \tilde{r}, \tilde{s} , we know that

$$\begin{aligned} |r|^2 + |s|^2 & \leq C_0 (|\tilde{r}|^2 + |\tilde{s}|^2), \\ |\nabla_{\mathbf{x},t} r|^2 + |\nabla_{\mathbf{x},t} s|^2 & \leq C_0 (|\tilde{r}|^2 + |\tilde{s}|^2 + |\nabla_{\mathbf{x},t} \tilde{r}|^2 + |\nabla_{\mathbf{x},t} \tilde{s}|^2). \end{aligned}$$

This completes the proof. ■

3.2 The inverse electroseismic problem

Assume that we know all the parameters in Biot equations (3.3)-(3.6) except the coupling coefficient ξ . In this section we prove the stability of the inverse parameter problem in electroseismic equations using the Carleman estimate given by Theorem 3.1.1. We now state our main result: a stability theorem for the inverse problem

Theorem 3.2.1. *Let $(\alpha_1, \beta_1, \gamma_1, \xi_1)$ and $(\alpha_2, \beta_2, \gamma_2, \xi_2)$ denote two sets of parameters, which satisfy the hypotheses of Theorem 3.1.1. Assume that the values of these parameters coincide in a set $\bar{\omega}$ where $\omega \subset \Omega$ is a neighborhood of $\partial\Omega$. Let $(\mathbf{D}_0^{(1)}, \mathbf{B}_0^{(1)})$ and $(\mathbf{D}_0^{(2)}, \mathbf{B}_0^{(2)})$ denote*

two sets of initial values, such that the matrix $\mathbf{M}(\mathbf{x})$ defined by

$$\mathbf{M}(\mathbf{x}) = \begin{pmatrix} \mathbf{e}_1 \times \mathbf{B}_0^{(1)} & \mathbf{e}_2 \times \mathbf{B}_0^{(1)} & \mathbf{e}_3 \times \mathbf{B}_0^{(1)} & -\mathbf{D}_0^{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{e}_1 \times \mathbf{D}_0^{(1)} & -\mathbf{e}_2 \times \mathbf{D}_0^{(1)} & -\mathbf{e}_3 \times \mathbf{D}_0^{(1)} \\ \mathbf{e}_1 \times \mathbf{B}_0^{(2)} & \mathbf{e}_2 \times \mathbf{B}_0^{(2)} & \mathbf{e}_3 \times \mathbf{B}_0^{(2)} & -\mathbf{D}_0^{(2)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{e}_1 \times \mathbf{D}_0^{(2)} & -\mathbf{e}_2 \times \mathbf{D}_0^{(2)} & -\mathbf{e}_3 \times \mathbf{D}_0^{(2)} \end{pmatrix}$$

has a nonzero 7×7 minor on Ω . Here

$$\mathbf{e}_1 = (1 \ 0 \ 0), \quad \mathbf{e}_2 = (0 \ 1 \ 0), \quad \mathbf{e}_3 = (0 \ 0 \ 1).$$

Denote by

$$\mathbf{D}_k^{(j)}, \mathbf{B}_k^{(j)} \in C^5(\bar{Q}), \quad \mathbf{u}_k^{(j)}, \mathbf{w}_k^{(j)} \in C^6(\bar{Q}) \quad j = 1, 2,$$

the solutions to the system (3.1)-(3.7), where $\mathbf{v}_k^{(j)}$ represents the field \mathbf{v} corresponding to the parameters $(\alpha_k, \beta_k, \gamma_k, \xi_k)$ and the j -th initial values. Then there exist constants C_0 and $c_0 \in (0, 1)$ such that

$$\int_{\Omega} \Lambda \leq C_0 (\mathfrak{D}^{(1)} + \mathfrak{D}^{(2)})^{c_0}$$

where

$$\begin{aligned} \Lambda &= \tilde{\Lambda} + |\xi|^2 + |\nabla \xi|^2, \\ \tilde{\Lambda} &= |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\nabla \alpha|^2 + |\nabla \beta|^2 + |\nabla \gamma|^2 + |\nabla \nabla \alpha|^2 + |\nabla \nabla \beta|^2, \\ \mathfrak{D}^{(j)} &= \|\mathbf{D}^{(j)}\|_{H^4(Q_\omega)}^2 + \|\mathbf{B}^{(j)}\|_{H^4(Q_\omega)}^2 + \|\mathbf{u}^{(j)}\|_{H^5(Q_\omega)}^2 + \|\mathbf{w}^{(j)}\|_{H^5(Q_\omega)}^2, \end{aligned}$$

and

$$\alpha = \alpha_2 - \alpha_1, \quad \beta = \beta_2 - \beta_1, \quad \gamma = \gamma_2 - \gamma_1, \quad \xi = \xi_2 - \xi_1,$$

$$\mathbf{D} = \mathbf{D}_2 - \mathbf{D}_1, \quad \mathbf{B} = \mathbf{B}_2 - \mathbf{B}_1, \quad \mathbf{u} = \mathbf{u}_2 - \mathbf{u}_1, \quad \mathbf{w} = \mathbf{w}_2 - \mathbf{w}_1.$$

Remark 2. If we choose $\mathbf{B}_0^{(1)} = \mathbf{e}_1$, $\mathbf{D}_0^{(1)} = \mathbf{e}_2$, $\mathbf{B}_0^{(2)} = \mathbf{D}_0^{(2)} = \mathbf{e}_3$, the matrix $\mathbf{M}(\mathbf{x})$ formed by rows (2,3,4,5,8,9,10) and by all the columns of $\mathbf{M}(\mathbf{x})$ is nonsingular. That's to say, it's possible to choose two sets of initial values such that the matrix $\mathbf{M}(\mathbf{x})$ has a nonzero 7×7 minor.

Remark 3. From the structure of $\mathbf{M}(\mathbf{x})$, the existence of a nonzero 7×7 minor indicates that there exists a positive constant c_0 such that

$$|\mathbf{B}_0^{(1)}|^2 + |\mathbf{B}_0^{(2)}|^2 > c_0, \quad |\mathbf{D}_0^{(1)}|^2 + |\mathbf{D}_0^{(2)}|^2 > c_0.$$

We prove Theorem 3.2.1 in 3 steps in the following subsections.

A modified Carleman estimate

Since our Carleman estimate is applicable for functions compactly supported in Q , in the first step we cut off the functions. The near boundary part corresponds to the measurements and the inner part can be bounded by the Carleman estimate.

It's easy to show that the fields $(\mathbf{D}, \mathbf{B}, \mathbf{u}, \mathbf{w})$ satisfy the following system of equations in Q

$$\partial_t \mathbf{D} - \operatorname{curl}(\alpha_2 \mathbf{B}) + \gamma_2 \mathbf{D} = \operatorname{curl}(\alpha \mathbf{B}_1) - \gamma \mathbf{D}_1, \quad (3.35)$$

$$\partial_t \mathbf{B} + \operatorname{curl}(\beta_2 \mathbf{D}) = -\operatorname{curl}(\beta \mathbf{D}_1), \quad (3.36)$$

$$\rho \partial_t^2 \mathbf{u} + \rho_f \partial_t^2 \mathbf{w} - \operatorname{div} \boldsymbol{\tau} = \mathbf{0}, \quad (3.37)$$

$$\rho_f \partial_t^2 \mathbf{u} + \rho_e \partial_t^2 \mathbf{w} + \nabla p + \frac{\eta}{\kappa} \partial_t \mathbf{w} - \xi_2 \mathbf{D} = \xi \mathbf{D}_1, \quad (3.38)$$

with zero initial conditions. Define $\chi(\mathbf{x}, t) = \chi_1(\mathbf{x})\chi_2(t)$ with

$$\chi_1 \in C_0^\infty(\Omega), \quad \chi_2 \in C_0^\infty(-T, T), \quad 0 \leq \chi_1, \chi_2 \leq 1$$

such that

$$\chi_1 = 1 \text{ in } \overline{\Omega}_0, \quad \chi_2 = 1 \text{ in } [-T + \delta, T - \delta],$$

where δ is chosen as in Remark 1 and $\Omega_0 = \Omega \setminus \overline{\omega}$. Let us denote

$$\tilde{\mathbf{D}} = \chi \mathbf{D}, \quad \tilde{\mathbf{B}} = \chi \mathbf{B}, \quad \tilde{\mathbf{u}} = \chi \mathbf{u}, \quad \tilde{\mathbf{w}} = \chi \mathbf{w}.$$

Then it's easy to show that $(\tilde{\mathbf{D}}, \tilde{\mathbf{B}}, \tilde{\mathbf{u}}, \tilde{\mathbf{w}})$ satisfy the following equations

$$\partial_t \tilde{\mathbf{D}} - \operatorname{curl}(\alpha_2 \tilde{\mathbf{B}}) + \gamma_2 \tilde{\mathbf{D}} = \chi (\operatorname{curl}(\alpha \mathbf{B}_1) - \gamma \mathbf{D}_1) + \mathcal{P}_1, \quad (3.39)$$

$$\partial_t \tilde{\mathbf{B}} + \operatorname{curl}(\beta_2 \tilde{\mathbf{D}}) = -\chi \operatorname{curl}(\beta \mathbf{D}_1) + \mathcal{P}_2, \quad (3.40)$$

$$\rho \partial_t^2 \tilde{\mathbf{u}} + \rho_f \partial_t^2 \tilde{\mathbf{w}} - \operatorname{div} \tilde{\boldsymbol{\tau}} = \mathcal{P}_3, \quad (3.41)$$

$$\rho_f \partial_t^2 \tilde{\mathbf{u}} + \rho_e \partial_t^2 \tilde{\mathbf{w}} + \nabla \tilde{p} + \frac{\eta}{\kappa} \partial_t \tilde{\mathbf{w}} - \xi_2 \tilde{\mathbf{D}} = \chi \xi \mathbf{D}_1 + \mathcal{P}_4, \quad (3.42)$$

where

$$\mathcal{P}_1 = (\partial_t \chi) \mathbf{D} - \nabla \chi \times (\alpha_2 \mathbf{B}),$$

$$\mathcal{P}_2 = (\partial_t \chi) \mathbf{B} + \nabla \chi \times (\beta_2 \mathbf{D}),$$

$\mathcal{P}_3, \mathcal{P}_4$ are first order differential operators of \mathbf{u}, \mathbf{w} . Let us note that $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ vanish in $Q_0(\delta) = \Omega_0 \times (-T + \delta, T - \delta)$, because $\chi \equiv 0$ in $Q_0(\delta)$. The right hand side of (3.39)-(3.42)

can be divided into two groups: one group corresponding to parameters $(\alpha, \beta, \gamma, \xi)$, another group \mathcal{P}_1 - \mathcal{P}_4 corresponding to fields $(\mathbf{D}, \mathbf{B}, \mathbf{u}, \mathbf{w})$. Applying Theorem 3.1.1 to (3.39)-(3.42), we can bound the fields $(\tilde{\mathbf{D}}, \tilde{\mathbf{B}}, \tilde{\mathbf{u}})$ by near boundary measurements and the parameters as follows

$$\begin{aligned} & \int_Q e^{2\tau\varphi} \left(\tau^3 (|\tilde{\mathbf{D}}|^2 + |\tilde{\mathbf{B}}|^2 + |\tilde{\mathbf{u}}|^2) + \tau (|\nabla_{\mathbf{x},t} \tilde{\mathbf{D}}|^2 + |\nabla_{\mathbf{x},t} \tilde{\mathbf{B}}|^2 + |\nabla_{\mathbf{x},t} \tilde{\mathbf{u}}|^2) \right) \\ \leq & C_0 \int_Q e^{2\tau\varphi} \Lambda \\ & + C_0 \int_{Q_\omega} e^{2\tau\varphi} \Pi + C_0 \int_{\Omega \times (-T, -T+\delta)} e^{2\tau\varphi} \Pi + C_0 \int_{\Omega \times (T-\delta, T)} e^{2\tau\varphi} \Pi, \end{aligned} \quad (3.43)$$

where

$$\begin{aligned} \Pi = & |\mathbf{D}|^2 + |\mathbf{B}|^2 + |\mathbf{u}|^2 + |\mathbf{w}|^2 + |\nabla_{\mathbf{x},t} \mathbf{D}|^2 + |\nabla_{\mathbf{x},t} \mathbf{B}|^2 \\ & + |\nabla_{\mathbf{x},t} \mathbf{u}|^2 + |\nabla_{\mathbf{x},t} \mathbf{w}|^2 + |\nabla \nabla_{\mathbf{x},t} \mathbf{u}|^2 + |\nabla \nabla_{\mathbf{x},t} \mathbf{w}|^2. \end{aligned}$$

We can see that the bound is divided into two parts: the parameter part Λ and the field part Π . Since $\chi \equiv 1$ in $Q_0(\delta)$, we observe that $(\tilde{\mathbf{D}}, \tilde{\mathbf{B}}, \tilde{\mathbf{u}})$ and $(\mathbf{D}, \mathbf{B}, \mathbf{u})$ are exactly the same in $Q_0(\delta)$. From Remark 1, we know that $\varphi < 1 - 2\epsilon$ near T and $-T$ and therefore

$$\int_{\Omega \times (-T, -T+\delta)} e^{2\tau\varphi} \Pi \quad \text{and} \quad \int_{\Omega \times (T-\delta, T)} e^{2\tau\varphi} \Pi$$

can be bounded by $e^{2\tau(1-2\epsilon)}$. Let us denote $\Phi = \max_{(\mathbf{x},t) \in Q} \varphi$. As a conclusion, we have

$$\begin{aligned} & \int_{Q_0(\delta)} e^{2\tau\varphi} \left(\tau^3 (|\mathbf{D}|^2 + |\mathbf{B}|^2 + |\mathbf{u}|^2) + \tau (|\nabla_{\mathbf{x},t} \mathbf{D}|^2 + |\nabla_{\mathbf{x},t} \mathbf{B}|^2 + |\nabla_{\mathbf{x},t} \mathbf{u}|^2) \right) \\ \leq & C_0 \int_Q e^{2\tau\varphi} \Lambda + C_0 e^{2\tau\Phi} \mathfrak{D} + C_0 e^{2\tau(1-2\epsilon)}. \end{aligned} \quad (3.44)$$

We want to have similar estimations for high order time derivatives of $(\mathbf{D}, \mathbf{B}, \mathbf{u})$. Let us denote $(\dot{\mathbf{D}}, \dot{\mathbf{B}}, \dot{\mathbf{u}}, \dot{\mathbf{w}})$ the first order time derivatives of $(\mathbf{D}, \mathbf{B}, \mathbf{u}, \mathbf{w})$. Taking the derivative with respect to t on both sides of (3.35)-(3.38) yields a equation system of $(\dot{\mathbf{D}}, \dot{\mathbf{B}}, \dot{\mathbf{u}}, \dot{\mathbf{w}})$

$$\begin{aligned} \partial_t \dot{\mathbf{D}} - \text{curl}(\alpha_2 \dot{\mathbf{B}}) + \gamma_2 \dot{\mathbf{D}} &= \text{curl}(\alpha \partial_t \dot{\mathbf{B}}_1) - \gamma \partial_t \dot{\mathbf{D}}_1, \\ \partial_t \dot{\mathbf{B}} + \text{curl}(\beta_2 \dot{\mathbf{D}}) &= -\text{curl}(\beta \partial_t \dot{\mathbf{D}}_1), \\ \rho \partial_t^2 \dot{\mathbf{u}} + \rho_f \partial_t^2 \dot{\mathbf{w}} - \text{div} \dot{\boldsymbol{\tau}} &= \mathbf{0}, \\ \rho_f \partial_t^2 \dot{\mathbf{u}} + \rho_e \partial_t^2 \dot{\mathbf{w}} + \nabla \dot{p} + \frac{\eta}{\kappa} \partial_t \dot{\mathbf{w}} - \xi_2 \dot{\mathbf{D}} &= \xi \partial_t \dot{\mathbf{D}}_1, \end{aligned}$$

We can do the same cut-off process for $(\dot{\mathbf{D}}, \dot{\mathbf{B}}, \dot{\mathbf{u}}, \dot{\mathbf{w}})$ to have compactly supported functions. The application of the Carleman estimate to the cut-off functions results the fol-

lowing inequalities

$$\begin{aligned}
& \int_{Q_0(\delta)} e^{2\tau\varphi} \left(\tau^3 (|\partial_t^j \mathbf{D}|^2 + |\partial_t^j \mathbf{B}|^2 + |\partial_t^j \mathbf{u}|^2) \right. \\
& \quad \left. + \tau (|\nabla_{\mathbf{x},t} \partial_t^j \mathbf{D}|^2 + |\nabla_{\mathbf{x},t} \partial_t^j \mathbf{B}|^2 + |\nabla_{\mathbf{x},t} \partial_t^j \mathbf{u}|^2) \right) \\
& \leq C_0 \int_Q e^{2\tau\varphi} \Lambda + C_0 e^{2\tau\Phi} \mathfrak{D} + C_0 e^{2\tau(1-2\epsilon)}, \tag{3.45}
\end{aligned}$$

where $j = 1$. Similarly we can show that this is also true for $j = 2, 3$.

Bounding parameters by initial values

Letting t goes to 0 in (3.35)-(3.38) shows that

$$\partial_t \mathbf{D}(\mathbf{x}, 0) = \text{curl}(\alpha \mathbf{B}_0) - \gamma \mathbf{D}_0, \tag{3.46}$$

$$\partial_t \mathbf{B}(\mathbf{x}, 0) = -\text{curl}(\beta \mathbf{D}_0), \tag{3.47}$$

$$\rho \partial_t^2 \mathbf{u}(\mathbf{x}, 0) + \rho_f \partial_t^2 \mathbf{w}(\mathbf{x}, 0) = \mathbf{0}, \tag{3.48}$$

$$\rho_f \partial_t^2 \mathbf{u}(\mathbf{x}, 0) + \rho_e \partial_t^2 \mathbf{w}(\mathbf{x}, 0) = \xi \mathbf{D}_0. \tag{3.49}$$

Expanding the curl in (3.46) and (3.47) yields

$$\nabla \alpha \times \mathbf{B}_0 + \alpha \text{curl} \mathbf{B}_0 - \gamma \mathbf{D}_0 = \partial_t \mathbf{D}(\mathbf{x}, 0),$$

$$-\nabla \beta \times \mathbf{D}_0 - \beta \text{curl} \mathbf{D}_0 = \partial_t \mathbf{B}(\mathbf{x}, 0).$$

Substituting (3.48) into (3.49) to eliminate \mathbf{w} gives

$$\mathbf{D}_0 \xi = -\rho_1 \partial_t^2 \mathbf{u}(\mathbf{x}, 0),$$

where $\rho_1 = \frac{\rho_0}{\rho_f}$. Considering the two sets of initial values, we write the above equations together into the matrix form

$$\mathbf{M}(\mathbf{x}) \begin{pmatrix} \nabla \alpha \\ \gamma \\ \nabla \beta \end{pmatrix} = \mathbf{N}(\mathbf{x}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \mathbf{b}(\mathbf{x}), \tag{3.50}$$

$$\mathbf{D}_0^{(j)} \xi = -\rho_1 \partial_t^2 \mathbf{u}^{(j)}(\mathbf{x}, 0), \tag{3.51}$$

where

$$\mathbf{N}(\mathbf{x}) = \begin{pmatrix} -\text{curl} \mathbf{B}_0^{(1)} & \mathbf{0} \\ \mathbf{0} & \text{curl} \mathbf{D}_0^{(1)} \\ -\text{curl} \mathbf{B}_0^{(2)} & \mathbf{0} \\ \mathbf{0} & \text{curl} \mathbf{D}_0^{(2)} \end{pmatrix}, \quad \mathbf{b}(\mathbf{x}) = \begin{pmatrix} \partial_t \mathbf{D}^{(1)}(\mathbf{x}, 0) \\ \partial_t \mathbf{B}^{(1)}(\mathbf{x}, 0) \\ \partial_t \mathbf{D}^{(2)}(\mathbf{x}, 0) \\ \partial_t \mathbf{B}^{(2)}(\mathbf{x}, 0) \end{pmatrix}.$$

Since $\mathbf{M}(\mathbf{x})$ has a 7×7 nonzero minor, we have

$$|\nabla\alpha|^2 + |\nabla\beta|^2 + |\gamma|^2 \leq C_0(|\alpha|^2 + |\beta|^2 + |\mathbf{b}|^2), \quad (3.52)$$

$$|\xi|^2 \leq C_0(|\partial_t^2 \mathbf{u}^{(1)}(\mathbf{x}, 0)|^2 + |\partial_t^2 \mathbf{u}^{(2)}(\mathbf{x}, 0)|^2). \quad (3.53)$$

Taking the derivative with respect to the variable x_k on both sides of (3.51), shows that

$$\mathbf{D}_0^{(j)} \partial_k \xi = -(\partial_k \rho_1) \partial_t^2 \mathbf{u}^{(j)}(\mathbf{x}, 0) - \rho_1 \partial_k \partial_t^2 \mathbf{u}^{(j)}(\mathbf{x}, 0) - (\partial_k \mathbf{D}_0^{(j)}) \xi,$$

and hence

$$|\nabla \xi|^2 \leq C_0 \sum_{j=1}^2 (|\partial_t^2 \mathbf{u}^{(j)}(\mathbf{x}, 0)|^2 + |\nabla \partial_t^2 \mathbf{u}^{(j)}(\mathbf{x}, 0)|^2). \quad (3.54)$$

Therefore

$$\int_{\Omega} e^{2\tau\varphi_0} (|\xi|^2 + |\nabla \xi|^2) \leq C_0 \int_{\Omega_0} e^{2\tau\varphi_0} \left(\sum_{j=1}^2 (|\partial_t^2 \mathbf{u}^{(j)}(\mathbf{x}, 0)|^2 + |\nabla \partial_t^2 \mathbf{u}^{(j)}(\mathbf{x}, 0)|^2) \right). \quad (3.55)$$

In addition, taking the derivative with respect to the variable x_k on both sides of (3.50), we obtain

$$\mathbf{M}(\mathbf{x}) \begin{pmatrix} \nabla \partial_k \alpha \\ \partial_k \gamma \\ \nabla \partial_k \beta \end{pmatrix} = \partial_k \mathbf{N}(\mathbf{x}) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \mathbf{N}(\mathbf{x}) \begin{pmatrix} \partial_k \alpha \\ \partial_k \beta \end{pmatrix} + \partial_k \mathbf{b}(\mathbf{x}) - \partial_k \mathbf{M}(\mathbf{x}) \begin{pmatrix} \nabla \alpha \\ \gamma \\ \nabla \beta \end{pmatrix},$$

and hence

$$|\nabla \nabla \alpha|^2 + |\nabla \nabla \beta|^2 + |\nabla \gamma|^2 \leq C_0(|\alpha|^2 + |\beta|^2 + |\mathbf{b}|^2 + |\nabla \mathbf{b}|^2). \quad (3.56)$$

Since α, β are supported in Ω_0 , the Poincaré inequality tells us

$$\int_{\Omega_0} (|\alpha|^2 + |\beta|^2) \leq C_0 \int_{\Omega_0} (|\nabla \alpha|^2 + |\nabla \beta|^2).$$

Therefore

$$\begin{aligned} & \int_{\Omega} e^{2\tau\varphi_0} (|\nabla \nabla \alpha|^2 + |\nabla \nabla \beta|^2 + |\nabla \gamma|^2) \\ & \leq C_0 \int_{\Omega} e^{2\tau\varphi_0} (|\alpha|^2 + |\beta|^2) + C_0 \int_{\Omega_0} e^{2\tau\varphi_0} (|\mathbf{b}|^2 + |\nabla \mathbf{b}|^2), \\ & \leq C_0 \int_{\Omega} e^{2\tau\varphi_0} (|\nabla \alpha|^2 + |\nabla \beta|^2) + C_0 \int_{\Omega_0} e^{2\tau\varphi_0} (|\mathbf{b}|^2 + |\nabla \mathbf{b}|^2), \end{aligned} \quad (3.57)$$

To cancel out the right had side term

$$\int_{\Omega} e^{2\tau\varphi_0} (|\nabla \alpha|^2 + |\nabla \beta|^2),$$

we recall one lemma from [22].

Lemma 3.2.1. *There exist constants $\tau_0 > 0$ and $C_0 > 0$ such that, for all $\tau > \tau_0$ and $\mathbf{v} \in H_0^1(\Omega)$,*

$$\tau \int_{\Omega} e^{2\tau\varphi_0} |\mathbf{v}|^2 \leq C_0 \int_{\Omega} e^{2\tau\varphi_0} (|\operatorname{curl} \mathbf{v}|^2 + |\operatorname{div} \mathbf{v}|^2).$$

Applying Lemma 3.2.1 with $\mathbf{v} = \nabla \alpha$, we have

$$\tau \int_{\Omega} e^{2\tau\varphi_0} |\nabla \alpha|^2 \leq C_0 \int_{\Omega} e^{2\tau\varphi_0} |\Delta \alpha|^2 \leq C_0 \int_{\Omega} e^{2\tau\varphi_0} |\nabla \nabla \alpha|^2,$$

and hence

$$\tau \int_{\Omega} e^{2\tau\varphi_0} (|\nabla \alpha|^2 + |\nabla \beta|^2) \leq C_0 \int_{\Omega} e^{2\tau\varphi_0} (|\nabla \nabla \alpha|^2 + |\nabla \nabla \beta|^2).$$

Combining with (3.52) and (3.57), we finally obtain the bound

$$\int_{\Omega} e^{2\tau\varphi_0} \tilde{\Lambda} \leq C_0 \int_{\Omega_0} e^{2\tau\varphi_0} (|\mathbf{b}|^2 + |\nabla \mathbf{b}|^2) \quad (3.58)$$

for τ large enough.

End of the proof of Theorem 3.2.1

We have shown how to bound the fields by measurements and parameters, and how to bound the parameters by initial values. We need to connect the fields with initial values. One lemma from [5] is helpful.

Lemma 3.2.2. *There exist constants $\tau_0 > 0$ and $C_0 > 0$ such that, for all $\tau > \tau_0$ and $\mathbf{v} \in C^1(Q_0(\delta))$,*

$$\int_{\Omega_0} |\mathbf{v}(\mathbf{x}, 0)|^2 \leq C_0 \tau \int_{Q_0(\delta)} |\mathbf{v}(\mathbf{x}, t)|^2 + C_0 \tau^{-1} \int_{Q_0(\delta)} |\partial_t \mathbf{v}(\mathbf{x}, t)|^2.$$

We recall that $\Omega_0 = \Omega \setminus \bar{\omega}$ and $Q_0(\delta) = \Omega_0 \times (-T + \delta, T - \delta)$.

We will prove this lemma at the end of the section. By taking $\mathbf{v} = e^{\tau\varphi_0} \partial_t \mathbf{D}^{(j)}(\mathbf{x}, 0)$ in the above estimate and invoking (3.45) to control the derivatives of $\mathbf{D}^{(j)}(\mathbf{x}, 0)$, we see that

$$\begin{aligned} \int_{\Omega_0} e^{2\tau\varphi_0} |\partial_t \mathbf{D}^{(j)}(\mathbf{x}, 0)|^2 &\leq C_0 \tau \int_{Q_0(\delta)} e^{2\tau\varphi} |\partial_t \mathbf{D}^{(j)}|^2 + C_0 \tau^{-1} \int_{Q_0(\delta)} e^{2\tau\varphi} |\partial_t^2 \mathbf{D}^{(j)}|^2 \\ &\leq C_0 \tau^{-2} \mathfrak{E}^{(j)}, \end{aligned}$$

where

$$\mathfrak{E}^{(j)} = \int_Q e^{2\tau\varphi} \Lambda + e^{2\tau\Phi} \mathfrak{D}^{(j)} + e^{2\tau(1-2\epsilon)}.$$

We proceed similarly for the higher-order derivatives of $\mathbf{D}^{(j)}(\mathbf{x}, 0)$ and for the other fields, to obtain

$$\begin{aligned}
\int_{\Omega_0} e^{2\tau\varphi_0} |\partial_k \partial_t \mathbf{D}^{(j)}(\mathbf{x}, 0)|^2 &\leq C_0 \tau \int_{Q_0(\delta)} e^{2\tau\varphi} |\partial_k \partial_t \mathbf{D}^{(j)}|^2 + C_0 \tau^{-1} \int_{Q_0(\delta)} e^{2\tau\varphi} |\partial_k \partial_t^2 \mathbf{D}^{(j)}|^2 \\
&\leq C_0 \mathfrak{E}^{(j)}, \\
\int_{\Omega_0} e^{2\tau\varphi_0} |\partial_t \mathbf{B}^{(j)}(\mathbf{x}, 0)|^2 &\leq C_0 \tau \int_{Q_0(\delta)} e^{2\tau\varphi} |\partial_t \mathbf{B}^{(j)}|^2 + C_0 \tau^{-1} \int_{Q_0(\delta)} e^{2\tau\varphi} |\partial_t^2 \mathbf{B}^{(j)}|^2 \\
&\leq C_0 \tau^{-2} \mathfrak{E}^{(j)}, \\
\int_{\Omega_0} e^{2\tau\varphi_0} |\partial_k \partial_t \mathbf{B}^{(j)}(\mathbf{x}, 0)|^2 &\leq C_0 \tau \int_{Q_0(\delta)} e^{2\tau\varphi} |\partial_k \partial_t \mathbf{B}^{(j)}|^2 + C_0 \tau^{-1} \int_{Q_0(\delta)} e^{2\tau\varphi} |\partial_k \partial_t^2 \mathbf{B}^{(j)}|^2 \\
&\leq C_0 \mathfrak{E}^{(j)}, \\
\int_{\Omega_0} e^{2\tau\varphi_0} |\partial_t^2 \mathbf{u}^{(j)}(\mathbf{x}, 0)|^2 &\leq C_0 \tau \int_{Q_0(\delta)} e^{2\tau\varphi} |\partial_t^2 \mathbf{u}^{(j)}|^2 + C_0 \tau^{-1} \int_{Q_0(\delta)} e^{2\tau\varphi} |\partial_t^3 \mathbf{u}^{(j)}|^2 \\
&\leq C_0 \tau^{-2} \mathfrak{E}^{(j)}, \\
\int_{\Omega_0} e^{2\tau\varphi_0} |\partial_k \partial_t^2 \mathbf{u}^{(j)}(\mathbf{x}, 0)|^2 &\leq C_0 \tau \int_{Q_0(\delta)} e^{2\tau\varphi} |\partial_k \partial_t^2 \mathbf{u}^{(j)}|^2 + C_0 \tau^{-1} \int_{Q_0(\delta)} e^{2\tau\varphi} |\partial_k \partial_t^3 \mathbf{u}^{(j)}|^2 \\
&\leq C_0 \mathfrak{E}^{(j)}.
\end{aligned}$$

It follows that

$$\int_{\Omega_0} e^{2\tau\varphi_0} (|\mathbf{b}|^2 + |\nabla \mathbf{b}|^2) \leq C_0 (\mathfrak{E}^{(1)} + \mathfrak{E}^{(2)}), \quad (3.59)$$

$$\int_{\Omega_0} e^{2\tau\varphi_0} \left(\sum_{j=1}^2 (|\partial_t^2 \mathbf{u}^{(j)}(\mathbf{x}, 0)|^2 + |\nabla \partial_t^2 \mathbf{u}^{(j)}(\mathbf{x}, 0)|^2) \right) \leq C_0 (\mathfrak{E}^{(1)} + \mathfrak{E}^{(2)}). \quad (3.60)$$

From (3.55) and (3.58), we infer that

$$\int_{\Omega} e^{2\tau\varphi_0} \Lambda - C_0 \int_Q e^{2\tau\varphi} \Lambda \leq C_0 e^{2\tau\Phi} (\mathfrak{D}^{(1)} + \mathfrak{D}^{(2)}) + C_0 e^{2\tau(1-2\epsilon)}. \quad (3.61)$$

Since $\varphi - \varphi_0 < 0$ for $|t| > 0$, by choosing τ_0 large enough we can make $\int_{-T}^T e^{2\tau(\varphi - \varphi_0)}$ so small that for all $\tau > \tau_0$,

$$\int_Q e^{2\tau\varphi} \Lambda = \int_{\Omega} e^{2\tau\varphi_0} \Lambda \int_{-T}^T e^{2\tau(\varphi - \varphi_0)} \ll \int_{\Omega} e^{2\tau\varphi_0} \Lambda.$$

Combining this estimate with (3.61) and using the fact that $\varphi_0 \geq 1 - \epsilon$, it follows that

$$\int_{\Omega} \Lambda \leq e^{-2\tau(1-\epsilon)} \int_{\Omega} e^{2\tau\varphi_0} \Lambda \leq C_0 e^{2\tau\Phi} (\mathfrak{D}^{(1)} + \mathfrak{D}^{(2)}) + C_0 e^{-2\tau\epsilon}$$

for all $\tau > \tau_0$. Taking

$$\tau - \tau_0 = \frac{-\ln(\mathfrak{D}^{(1)} + \mathfrak{D}^{(2)})}{2(\Phi + \epsilon)}.$$

we finally obtain

$$\begin{aligned}
& C_0 e^{2\tau\Phi} (\mathfrak{D}^{(1)} + \mathfrak{D}^{(2)}) + C_0 e^{-2\tau\epsilon} \\
&= C_0 e^{2\tau_0\Phi} e^{2(\tau-\tau_0)\Phi} (\mathfrak{D}^{(1)} + \mathfrak{D}^{(2)}) + C_0 e^{-2\tau_0\epsilon} e^{-2(\tau-\tau_0)\epsilon} \\
&\leq C_0 (\mathfrak{D}^{(1)} + \mathfrak{D}^{(2)})^{\frac{\epsilon}{\epsilon+\Phi}},
\end{aligned}$$

which completes the proof.

Proof of Lemma 3.2.2. Let us choose a cut-off function $\chi \in C_0^\infty(-T+\delta, T-\delta)$ that satisfies

$$0 \leq \chi \leq 1, \quad \chi = 1 \text{ in } [-T+2\delta, T-2\delta].$$

Direct computation shows

$$\begin{aligned}
\int_{\Omega_0} |\mathbf{v}(\mathbf{x}, 0)|^2 &= \int_{\Omega_0} \chi^2(0) |\mathbf{v}(\mathbf{x}, 0)|^2 \\
&= \int_{-T+\delta}^0 \frac{d}{dt} \left(\int_{\Omega_0} \chi^2(t) |\mathbf{v}(\mathbf{x}, t)|^2 \right) \\
&= 2 \int_{-T+\delta}^0 \int_{\Omega_0} (\chi^2(t) \mathbf{v}(\mathbf{x}, t) \cdot \partial_t \mathbf{v}(\mathbf{x}, t) + \chi'(t) \chi(t) |\mathbf{v}(\mathbf{x}, t)|^2).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\int_{\Omega_0} |\mathbf{v}(\mathbf{x}, 0)|^2 &\leq C_0 \int_{Q_0(\delta)} (2|\mathbf{v}(\mathbf{x}, t) \cdot \partial_t \mathbf{v}(\mathbf{x}, t)| + |\mathbf{v}(\mathbf{x}, t)|^2) \\
&\leq C_0 \int_{Q_0(\delta)} (2|\mathbf{v}(\mathbf{x}, t)| |\partial_t \mathbf{v}(\mathbf{x}, t)| + |\mathbf{v}(\mathbf{x}, t)|^2) \\
&\leq C_0 \int_{Q_0(\delta)} (\tau |\mathbf{v}(\mathbf{x}, t)|^2 + \tau^{-1} |\partial_t \mathbf{v}(\mathbf{x}, t)|^2),
\end{aligned}$$

for τ large enough and for all $\mathbf{v} \in C^1(Q_0(\delta))$. ■

3.3 The inversion of the simplified electroseismic problem

As we have shown before, Maxwell equations can be approximated by the electrostatic equation

$$-\operatorname{div}(\sigma \nabla \Phi) = f(\mathbf{x}).$$

In the time domain, assume that the source is given by $f(\mathbf{x})g(t)$. From the linearity of the problem, the electric field is given by

$$\mathbf{E}(\mathbf{x}, t) = -\nabla \Phi(\mathbf{x})g(t).$$

The simplified electroseismic problem reads

$$-\operatorname{div}(\sigma \nabla \Phi) = f(\mathbf{x}), \quad (3.62)$$

$$\rho \partial_t^2 \mathbf{u} + \rho_f \partial_t^2 \mathbf{w} - \operatorname{div} \boldsymbol{\tau} = \mathbf{0}, \quad (3.63)$$

$$\rho_f \partial_t^2 \mathbf{u} + \rho_e \partial_t^2 \mathbf{w} + \nabla p + \frac{\eta}{\kappa} \partial_t \mathbf{w} + \xi \nabla \Phi g(t) = \mathbf{0}, \quad (3.64)$$

$$(\lambda \operatorname{div} \mathbf{u} + C \operatorname{div} \mathbf{w}) \mathbf{I} + G(\nabla \mathbf{u} + \nabla \mathbf{u}^T) = \boldsymbol{\tau}, \quad (3.65)$$

$$C \operatorname{div} \mathbf{u} + M \operatorname{div} \mathbf{w} = -p. \quad (3.66)$$

Here $\xi = \frac{L\eta}{\kappa}$. We use homogeneous boundary condition for Φ

$$\Phi|_{\partial\Omega} = 0.$$

Then the inverse problem becomes: is that possible to obtain σ and ξ from measurements of \mathbf{u} , \mathbf{w} and Φ near the boundary?

The inverse problem can be decomposed into two steps. The first step is the inversion of the source term $\mathbf{F}(\mathbf{x}) = -\xi \nabla \Phi$ of Biot equations from the boundary measurements of \mathbf{u} and \mathbf{w} , and the second step is the inverse medium of the electrostatic equation with the knowledge of $\mathbf{F}(\mathbf{x})$.

Inverse source problem for Biot equations

In this section, we prove the stability of the inverse source problem for Biot equations. In fact, a similar inverse source problem for Biot equations has been done in [5]. We have emphasized that our Biot equations are different from those considered in [5] because of the existence of the term $\partial_t \mathbf{w}$. We will prove the stability of the inverse source problem for Biot equations in this section.

Theorem 3.3.1. *Assume that all the parameters in Biot equations*

$$\rho \partial_t^2 \mathbf{u} + \rho_f \partial_t^2 \mathbf{w} - \operatorname{div} \boldsymbol{\tau} = \mathbf{0}, \quad (3.67)$$

$$\rho_f \partial_t^2 \mathbf{u} + \rho_e \partial_t^2 \mathbf{w} + \nabla p + \frac{\eta}{\kappa} \partial_t \mathbf{w} = \mathbf{F}(\mathbf{x}) g(t), \quad (3.68)$$

$$(\lambda \operatorname{div} \mathbf{u} + C \operatorname{div} \mathbf{w}) \mathbf{I} + G(\nabla \mathbf{u} + \nabla \mathbf{u}^T) = \boldsymbol{\tau}, \quad (3.69)$$

$$C \operatorname{div} \mathbf{u} + M \operatorname{div} \mathbf{w} = -p. \quad (3.70)$$

satisfy the hypothesis of Lemma 3.1.3. Assume that $g(t) \in C^3[-T, T]$ with $g(0) \neq 0$ and $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \bar{\omega}$ where $\omega \subset \Omega$ is a neighborhood of $\partial\Omega$, $Q_\omega = \omega \times (-T, T)$. Denote by $\mathbf{u}, \mathbf{w} \in C^6(\bar{Q})$ the solutions to the system (3.67)-(3.70). Then there exist constants C_0 and $c_0 \in (0, 1)$ such that

$$\int_{\Omega} \Lambda \leq C_0 \mathfrak{D}^{c_0}$$

where

$$\begin{aligned}\Lambda &= |\mathbf{F}(\mathbf{x})|^2 + |\nabla \mathbf{F}(\mathbf{x})|^2, \\ \mathfrak{D} &= \|\mathbf{u}\|_{H^5(Q_\omega)}^2 + \|\mathbf{w}\|_{H^5(Q_\omega)}^2.\end{aligned}$$

Proof. The first step of the proof is to cut off the fields and apply the Carleman estimate. Define $\chi(\mathbf{x}, t) = \chi_1(\mathbf{x})\chi_2(t)$ with

$$\chi_1 \in C_0^\infty(\Omega), \quad \chi_2 \in C_0^\infty(-T, T), \quad 0 \leq \chi_1, \chi_2 \leq 1$$

such that

$$\chi_1 = 1 \text{ in } \overline{\Omega_0}, \quad \chi_2 = 1 \text{ in } [-T + \delta, T - \delta],$$

where δ is chosen as in Remark 1 and $\Omega_0 = \Omega \setminus \bar{\omega}$. Let us denote

$$\tilde{\mathbf{u}} = \chi \mathbf{u}, \quad \tilde{\mathbf{w}} = \chi \mathbf{w}.$$

Then it's easy to show that $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}})$ satisfy

$$\begin{aligned}\rho \partial_t^2 \tilde{\mathbf{u}} + \rho_f \partial_t^2 \tilde{\mathbf{w}} - \operatorname{div} \tilde{\boldsymbol{\tau}} &= \mathcal{P}_1, \\ \rho_f \partial_t^2 \tilde{\mathbf{u}} + \rho_e \partial_t^2 \tilde{\mathbf{w}} + \nabla \tilde{p} + \frac{\eta}{\kappa} \partial_t \tilde{\mathbf{w}} &= \chi \mathbf{F}(\mathbf{x}) g(t) + \mathcal{P}_2.\end{aligned}$$

where $\mathcal{P}_1, \mathcal{P}_2$ are first order differential operators of \mathbf{u}, \mathbf{w} which vanish inside $Q_0(\delta) = \Omega_0 \times (-T + \delta, T - \delta)$. Applying Lemma 3.1.3 yields

$$\begin{aligned}\int_Q e^{2\tau\varphi} \left(\tau^3 |\tilde{\mathbf{u}}|^2 + \tau |\nabla_{\mathbf{x},t} \tilde{\mathbf{u}}|^2 \right) &\leq C_0 \int_Q e^{2\tau\varphi} \Lambda + C_0 \int_{Q_\omega} e^{2\tau\varphi} \Pi \\ &\quad + C_0 \int_{\Omega \times (-T, -T+\delta)} e^{2\tau\varphi} \Pi + C_0 \int_{\Omega \times (T-\delta, T)} e^{2\tau\varphi} \Pi,\end{aligned}$$

where

$$\Pi = |\mathbf{u}|^2 + |\mathbf{w}|^2 + |\nabla_{\mathbf{x},t} \mathbf{u}|^2 + |\nabla_{\mathbf{x},t} \mathbf{w}|^2 + |\nabla \nabla_{\mathbf{x},t} \mathbf{u}|^2 + |\nabla \nabla_{\mathbf{x},t} \mathbf{w}|^2.$$

Since $\chi \equiv 1$ in $Q_0(\delta)$, we have $\tilde{\mathbf{u}} \equiv \mathbf{u}$ inside $Q_0(\delta)$, and hence

$$\int_{Q_0(\delta)} e^{2\tau\varphi} \left(\tau^3 |\mathbf{u}|^2 + \tau |\nabla_{\mathbf{x},t} \mathbf{u}|^2 \right) = \int_Q e^{2\tau\varphi} \left(\tau^3 |\tilde{\mathbf{u}}|^2 + \tau |\nabla_{\mathbf{x},t} \tilde{\mathbf{u}}|^2 \right)$$

From Remark 1, we know that

$$\varphi(\mathbf{x}, t) < 1 - 2\epsilon, \text{ for } t > T - \delta \text{ or } t < -T + \delta,$$

and therefore

$$\int_{\Omega \times (-T, -T+\delta)} e^{2\tau\varphi} \Pi \leq C_0 e^{2\tau(1-2\epsilon)}, \quad \int_{\Omega \times (T-\delta, T)} e^{2\tau\varphi} \Pi \leq C_0 e^{2\tau(1-2\epsilon)}.$$

We conclude that

$$\int_{Q_0(\delta)} e^{2\tau\varphi} \left(\tau^3 |\mathbf{u}|^2 + \tau |\nabla_{\mathbf{x},t} \mathbf{u}|^2 \right) \leq C_0 \int_Q e^{2\tau\varphi} \Lambda + C_0 e^{2\tau\Phi} \mathfrak{D} + C_0 e^{2\tau(1-2\epsilon)}.$$

Similarly we have

$$\int_{Q_0(\delta)} e^{2\tau\varphi} \left(\tau^3 |\partial_t^j \mathbf{u}|^2 + \tau |\nabla_{\mathbf{x},t} \partial_t^j \mathbf{u}|^2 \right) \leq C_0 \int_Q e^{2\tau\varphi} \Lambda + C_0 e^{2\tau\Phi} \mathfrak{D} + C_0 e^{2\tau(1-2\epsilon)} \quad (3.71)$$

for $j = 1, 2, 3$.

The second step is to bound the source by the initial values. By letting t goes to 0 in (3.67)-(3.68), we obtain

$$\begin{aligned} \rho \partial_t^2 \mathbf{u}(\mathbf{x}, 0) + \rho_f \partial_t^2 \mathbf{w}(\mathbf{x}, 0) &= \mathbf{0}, \\ \rho_f \partial_t^2 \mathbf{u}(\mathbf{x}, 0) + \rho_e \partial_t^2 \mathbf{w}(\mathbf{x}, 0) &= \mathbf{F}(\mathbf{x}) g(0). \end{aligned}$$

We cancel out $\partial_t^2 \mathbf{w}(\mathbf{x}, 0)$ to obtain

$$\mathbf{F}(\mathbf{x}) = -\rho_1 \partial_t^2 \mathbf{u}(\mathbf{x}, 0),$$

where $\rho_1 = \frac{\rho_0}{\rho_f g(0)}$. Differentiating with respect to x_k yields

$$\partial_k \mathbf{F}(\mathbf{x}) = -(\partial_k \rho_1) \partial_t^2 \mathbf{u}(\mathbf{x}, 0) - \rho_1 \partial_k \partial_t^2 \mathbf{u}(\mathbf{x}, 0).$$

As a conclusion, we have

$$\begin{aligned} |\mathbf{F}(\mathbf{x})|^2 + |\nabla \mathbf{F}(\mathbf{x})|^2 &\leq C_0 (|\partial_t^2 \mathbf{u}(\mathbf{x}, 0)|^2 + |\partial_k \partial_t^2 \mathbf{u}(\mathbf{x}, 0)|^2), \\ \int_{\Omega_0} e^{2\tau\varphi_0} (|\mathbf{F}(\mathbf{x})|^2 + |\nabla \mathbf{F}(\mathbf{x})|^2) &\leq C_0 \int_{\Omega} e^{2\tau\varphi_0} (|\partial_t^2 \mathbf{u}(\mathbf{x}, 0)|^2 + |\nabla \partial_t^2 \mathbf{u}(\mathbf{x}, 0)|^2). \end{aligned} \quad (3.72)$$

At last we complete the proof with the help of Lemma 3.2.2. By taking $\mathbf{v} = \partial_t^2 \mathbf{u}$ and $\mathbf{v} = \partial_k \partial_t^2 \mathbf{u}$, we get

$$\begin{aligned} \int_{\Omega} e^{2\tau\varphi_0} (|\partial_t^2 \mathbf{u}(\mathbf{x}, 0)|^2 + |\nabla \partial_t^2 \mathbf{u}(\mathbf{x}, 0)|^2) &\leq C_0 \tau \int_{Q_0(\delta)} (|\partial_t^2 \mathbf{u}(\mathbf{x}, 0)|^2 + |\nabla \partial_t^2 \mathbf{u}(\mathbf{x}, 0)|^2) \\ &\quad + C_0 \tau^{-1} \int_{Q_0(\delta)} (|\partial_t^3 \mathbf{u}(\mathbf{x}, 0)|^2 + |\nabla \partial_t^3 \mathbf{u}(\mathbf{x}, 0)|^2). \end{aligned}$$

Combining with (3.71), we obtain

$$\int_{\Omega} e^{2\tau\varphi_0} (|\partial_t^2 \mathbf{u}(\mathbf{x}, 0)|^2 + |\nabla \partial_t^2 \mathbf{u}(\mathbf{x}, 0)|^2) \leq C_0 \int_Q e^{2\tau\varphi} \Lambda + C_0 e^{2\tau\Phi} \mathfrak{D} + C_0 e^{2\tau(1-2\epsilon)}.$$

From (3.72), we know that

$$\int_{\Omega} e^{2\tau\varphi_0} \Lambda \leq C_0 \int_Q e^{2\tau\varphi} \Lambda + C_0 e^{2\tau\Phi} \mathfrak{D} + C_0 e^{2\tau(1-2\epsilon)}.$$

Since $\varphi - \varphi_0 < 0$ for $|t| > 0$,

$$\int_Q e^{2\tau\varphi} \Lambda = \int_\Omega e^{2\tau\varphi_0} \Lambda \int_{-T}^T e^{2\tau(\varphi-\varphi_0)} \ll \int_\Omega e^{2\tau\varphi_0} \Lambda,$$

for τ large enough. Since $\varphi_0 \geq 1 - \varepsilon$, it follows that

$$\int_\Omega \Lambda \leq e^{-2\tau(1-\varepsilon)} \int_\Omega e^{2\tau\varphi_0} \Lambda \leq C_0 e^{2\tau\Phi} \mathfrak{D} + C_0 e^{-2\tau\varepsilon}$$

By taking

$$\tau - \tau_0 = \frac{-\ln \mathfrak{D}}{2(\Phi + \varepsilon)}.$$

we finally obtain

$$\begin{aligned} & C_0 e^{2\tau\Phi} \mathfrak{D} + C_0 e^{-2\tau\varepsilon} \\ \leq & C_0 e^{2\tau_0\Phi} e^{2(\tau-\tau_0)\Phi} \mathfrak{D} + C_0 e^{-2\tau_0\varepsilon} e^{-2(\tau-\tau_0)\varepsilon} \\ = & C_0 \mathfrak{D}^{\frac{\varepsilon}{\varepsilon+\Phi}}, \end{aligned}$$

which completes the proof. ■

Inverse σ and ξ from internal data

From now on we only consider the problem in 2D. After the reconstruction of $\mathbf{F}(\mathbf{x})$, we want to recover σ and ξ . Since $\mathbf{F}(\mathbf{x}) = \xi \nabla \Phi$, the area where $\xi \equiv 0$ is known from $\mathbf{F}(\mathbf{x})$. It's not possible to recover σ in such kind of area. From now on, let us consider a sub-domain Ω_0 of Ω in which $\xi \neq 0$ and $f(\mathbf{x}) = 0$. For simplicity, we assume that Ω_0 is open connected. We measure Φ on $\partial\Omega_0$. We cite a result from [2].

Lemma 3.3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected open set, whose boundary $\partial\Omega$ is a simple closed curve. Let $\boldsymbol{\phi} = (\phi_1, \phi_2)$, $\boldsymbol{\phi} : \partial\Omega \rightarrow \mathbb{R}^2$ be a homeomorphism of $\partial\Omega$ onto a convex closed curve Γ and let D be the bounded convex domain bounded by Γ . Let $\boldsymbol{\Phi} = (\Phi_1, \Phi_2)$ be the σ -harmonic mapping whose components are the solutions of the Dirichlet problems*

$$\begin{cases} -\operatorname{div}(\sigma \nabla \Phi) = 0 & \text{in } \Omega_0, \\ \Phi_i = \phi_i & \text{on } \partial\Omega_0. \end{cases}$$

Then $\boldsymbol{\Phi}$ is a homeomorphism of $\overline{\Omega}$ onto \overline{D} and $\det(\nabla \Phi_1, \nabla \Phi_2) \neq 0$.

Now we present the result of the inversion of σ and ξ .

Lemma 3.3.2. *Assume that the two boundary conditions (ϕ_1, ϕ_2) satisfy the conditions in Lemma 3.3.1 and we know the value of $\frac{\sigma}{\xi}$ on a point $\mathbf{x}_0 \in \partial\Omega_0$. Then we can recover $\frac{\sigma}{\xi}$ from $\mathbf{F}_i = -\xi \nabla \Phi_i$ for $i = 1, 2$.*

Proof. It's obvious that

$$\operatorname{div}\left(\frac{\sigma}{\xi}\mathbf{F}\right)=0.$$

We expand the equation to obtain

$$\nabla\frac{\sigma}{\xi}\cdot\mathbf{F}+\frac{\sigma}{\xi}\operatorname{div}\mathbf{F}=0,$$

and therefore

$$(\mathbf{F}_1, \mathbf{F}_2) \cdot \nabla \log \frac{\sigma}{\xi} = (-\operatorname{div}\mathbf{F}_1, -\operatorname{div}\mathbf{F}_2)^T.$$

We deduce from Lemma 3.3.1 that the matrix $(\mathbf{F}_1, \mathbf{F}_2)$ is invertible. We complete the proof by inverting the matrix $(\mathbf{F}_1, \mathbf{F}_2)$ and integrating. ■

We can recover $\sigma\nabla\Phi$ using the following formula

$$\sigma\nabla\Phi=-\frac{\sigma}{\xi}\mathbf{F}.$$

Since $\sigma\nabla\Phi$ is divergence free, the Helmholtz decomposition of $\sigma\nabla\Phi$ in two dimension reads

$$\sigma\nabla\Phi=\mathbf{J}\nabla\Psi,$$

where $\mathbf{J}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a rotation matrix. Note that for any function α , $\mathbf{J}\nabla\alpha$ is divergence free. Multiplying both sides by \mathbf{J} yields

$$\frac{1}{\sigma}\nabla\Psi=-\mathbf{J}\nabla\Phi.$$

Taking the divergence on both sides, we obtain

$$\operatorname{div}\left(\frac{1}{\sigma}\nabla\Psi\right)=0.$$

Now the problem becomes: is that possible to reconstruct σ knowing $\nabla\Psi$ everywhere in Ω_0 ? Assume that we can obtain two sets of boundary conditions (ψ_1, ψ_2) of Ψ satisfying the conditions in Lemma 3.3.2. Then we can use Lemma 3.3.2 again to recover σ . We summarize the discussion above in the following theorem.

Theorem 3.3.2. *Assume that we have two sets of positive upper and lower bounded parameters (σ, ξ) and $(\tilde{\sigma}, \tilde{\xi})$ in Ω_0 , which are equal on the boundary. Corresponding to each set of parameters, we have two sets of boundary conditions satisfying the conditions in Lemma 3.3.2. Then*

$$\|\sigma-\tilde{\sigma}\|^2\leq C_0\sum_{i=1}^2\|\mathbf{F}_i-\tilde{\mathbf{F}}_i\|_{H^1}^2, \quad \|\xi-\tilde{\xi}\|^2\leq C_0\sum_{i=1}^2\|\mathbf{F}_i-\tilde{\mathbf{F}}_i\|_{H^1}^2.$$

Proof. From Lemma 3.3.2, we know that

$$\begin{aligned} \nabla \left(\log \frac{\sigma}{\xi} - \log \frac{\tilde{\sigma}}{\tilde{\xi}} \right) &= (\mathbf{F}_1, \mathbf{F}_2)^{-1} \cdot \begin{pmatrix} -\operatorname{div} \mathbf{F}_1 \\ -\operatorname{div} \mathbf{F}_2 \end{pmatrix} - (\tilde{\mathbf{F}}_1, \tilde{\mathbf{F}}_2)^{-1} \cdot \begin{pmatrix} -\operatorname{div} \tilde{\mathbf{F}}_1 \\ -\operatorname{div} \tilde{\mathbf{F}}_2 \end{pmatrix} \\ &= (\mathbf{F}_1, \mathbf{F}_2)^{-1} \cdot \begin{pmatrix} -\operatorname{div}(\mathbf{F}_1 - \tilde{\mathbf{F}}_1) \\ -\operatorname{div}(\mathbf{F}_2 - \tilde{\mathbf{F}}_2) \end{pmatrix} \\ &\quad - (\mathbf{F}_1, \mathbf{F}_2)^{-1} (\tilde{\mathbf{F}}_1 - \mathbf{F}_1, \tilde{\mathbf{F}}_2 - \mathbf{F}_2) (\tilde{\mathbf{F}}_1, \tilde{\mathbf{F}}_2)^{-1} \cdot \begin{pmatrix} -\operatorname{div} \tilde{\mathbf{F}}_1 \\ -\operatorname{div} \tilde{\mathbf{F}}_2 \end{pmatrix}. \end{aligned}$$

Therefore we have

$$\left\| \nabla \left(\log \frac{\sigma}{\xi} - \log \frac{\tilde{\sigma}}{\tilde{\xi}} \right) \right\|^2 \leq C_0 \sum_{i=1}^2 \|\mathbf{F}_i - \tilde{\mathbf{F}}_i\|_{H^1}^2.$$

Since (σ, ξ) and $(\tilde{\sigma}, \tilde{\xi})$ are positive upper and lower bounded in Ω_0 , we have

$$\left\| \frac{\sigma}{\xi} - \frac{\tilde{\sigma}}{\tilde{\xi}} \right\|^2 \leq C_0 \left\| \log \frac{\sigma}{\xi} - \log \frac{\tilde{\sigma}}{\tilde{\xi}} \right\|^2.$$

Recall that (σ, ξ) and $(\tilde{\sigma}, \tilde{\xi})$ are equal on the boundary. Poincaré lemma shows that

$$\begin{aligned} \left\| \frac{\sigma}{\xi} - \frac{\tilde{\sigma}}{\tilde{\xi}} \right\|^2 &\leq \left\| \nabla \left(\log \frac{\sigma}{\xi} - \log \frac{\tilde{\sigma}}{\tilde{\xi}} \right) \right\|^2 \\ &\leq C_0 \sum_{i=1}^2 \|\mathbf{F}_i - \tilde{\mathbf{F}}_i\|_{H^1}^2. \end{aligned} \tag{3.73}$$

We can also show that

$$\begin{aligned} \left\| \nabla \left(\frac{\sigma}{\xi} - \frac{\tilde{\sigma}}{\tilde{\xi}} \right) \right\|^2 &= \left\| \frac{\sigma}{\xi} \nabla \log \frac{\sigma}{\xi} - \frac{\tilde{\sigma}}{\tilde{\xi}} \nabla \log \frac{\tilde{\sigma}}{\tilde{\xi}} \right\|^2 \\ &\leq C_0 \sum_{i=1}^2 \|\mathbf{F}_i - \tilde{\mathbf{F}}_i\|_{H^1}^2. \end{aligned}$$

Similar to (3.73) we can prove

$$\|\sigma - \tilde{\sigma}\|^2 \leq C_0 \sum_{i=1}^2 \|\nabla \Psi_i - \nabla \tilde{\Psi}_i\|_{H^1}^2.$$

Since $\nabla \Psi = \mathbf{J} \cdot \left(\frac{\sigma}{\xi} \mathbf{F} \right)$, we obtain

$$\begin{aligned} \|\sigma - \tilde{\sigma}\|^2 &\leq C_0 \left\| \frac{\sigma}{\xi} \mathbf{F}_i - \frac{\tilde{\sigma}}{\tilde{\xi}} \tilde{\mathbf{F}}_i \right\|_{H^1}^2 \\ &\leq C_0 \sum_{i=1}^2 \|\mathbf{F}_i - \tilde{\mathbf{F}}_i\|_{H^1}^2. \end{aligned}$$

Finally we can bound $\xi - \tilde{\xi}$ by

$$\begin{aligned} \|\xi - \tilde{\xi}\|^2 &= \left\| \frac{\xi}{\sigma} \sigma - \frac{\tilde{\xi}}{\tilde{\sigma}} \tilde{\sigma} \right\|^2 \\ &\leq C_0 \left(\left\| \frac{\xi}{\sigma} - \frac{\tilde{\xi}}{\tilde{\sigma}} \right\|^2 + \|\sigma - \tilde{\sigma}\|^2 \right) \\ &\leq C_0 \sum_{i=1}^2 \|\mathbf{F}_i - \tilde{\mathbf{F}}_i\|_{H^1}^2. \end{aligned}$$

■

Chapter 4

Numerical Solution of the Forward Problem

For shallow electroseismic surveys, the model area is several hundred meters and the frequency ranges from 10 Hz to 1kHz. In this case, $\omega\epsilon/\sigma \ll 10^{-4}$. Therefore, the term $\epsilon\partial_t\mathbf{E}$ can be neglected relative to the term $\sigma\mathbf{E}$. The wave length of the EM wave is much larger than the area of interest. This means $\nabla \times \mathbf{E}$ is almost 0 throughout the model region, and thus $\mathbf{E} = -\nabla\Phi$, where Φ is the electric potential. Now the full Maxwell equations can be simplified to the electrostatic equation, i.e.,

$$-\operatorname{div}(\sigma\nabla\Phi) = f(x)g(t) \quad \text{in } \Omega.$$

From CFL condition, we know that the time step is inversely proportional to wave speed. The EM speed is so large that solving the full Maxwell equations numerically for several seconds becomes much too time consuming. This is the reason why we solve the electrostatic equation instead of the full Maxwell equations. The approximation of the full Maxwell equations by the electrostatic equation for seismo-electric modeling is applied and justified in [27]. The time domain solution of Maxwell equations is approximated by

$$\mathbf{E}(\mathbf{x}, t) = -\nabla\Phi(\mathbf{x})g(t).$$

The rest of the chapter focuses on the solution of Biot equations in time domain, including the staggered-grid Finite Difference (FD) method and absorbing boundary conditions. The staggered-grid method was first proposed by Yee [54] to solve Maxwell equations. Virieux [52] and Levander [38] extended this method to the elastic equation. A similar technique was applied to Biot equations by Masson and Pride [41].

Solving Biot equations in infinite domain is not possible numerically. The computational domain must be truncated. Special boundary conditions must be constructed

to decrease artificial reflections from the boundary. In this thesis we consider a method, Perfectly Matched Layer (PML), proposed by Berenger [6]. This method is first used to simulate Maxwell equations in infinite domain and then extended to the elastic equation [18] and Biot equations [56, 40].

The idea of PML is to add a special thin layer outside the computational domain. The parameters in the PML layer is specially chosen such that no reflection happens on the interface and waves attenuate rapidly in the PML layer. Numerical experiments find out that PML works bad for evanescent waves and waves almost parallel to the interface. Kuzuogh and Mittra [37] proposed the Complex Frequency Shifted PML (CFS-PML) to improve the performance of PML. They introduce frequency dependent parameters in the PML layer. We are especially interested in Biot equations in time domain. When we go from the frequency domain to the time domain, the frequency dependent parameters in the PML layer result in convolution between parameters and wave fields. Roden and Gedney [48] found out the convolution can be implemented locally by adding another field. The new PML is call C-PML and is the one that we will use in the numerical simulation.

4.1 Numerical solution of the electrostatic equation

In this section we discuss how to solve the electrostatic equation and demonstrate some results. Let us recall the electrostatic equation

$$-\operatorname{div}(\sigma \nabla \Phi) = f(x) \quad \text{in } \Omega.$$

For simplicity we choose the vanishing Dirichlet boundary condition, i.e.,

$$\Phi|_{\partial\Omega} = 0.$$

The proper Sobolev space to consider this problem is $H_0^1(\Omega)$. The corresponding weak formula is

$$(\nabla \varphi, \sigma \nabla \Phi) = (\varphi, f),$$

for all $\varphi \in H_0^1(\Omega)$. We use Matlab toolbox to solve this equation.

Set up the problem

To decrease the computational consumption, let us consider the 2D problem. Let us take Ω a square of 250 meters times 250 meters. The source $f(\mathbf{x})$ is a Gaussian function with center $(120, -100)$ and standard deviation $5\sqrt{2}$

$$f(\mathbf{x}) = e^{\frac{-(x_1-120)^2-(x_2+100)}{100}}.$$

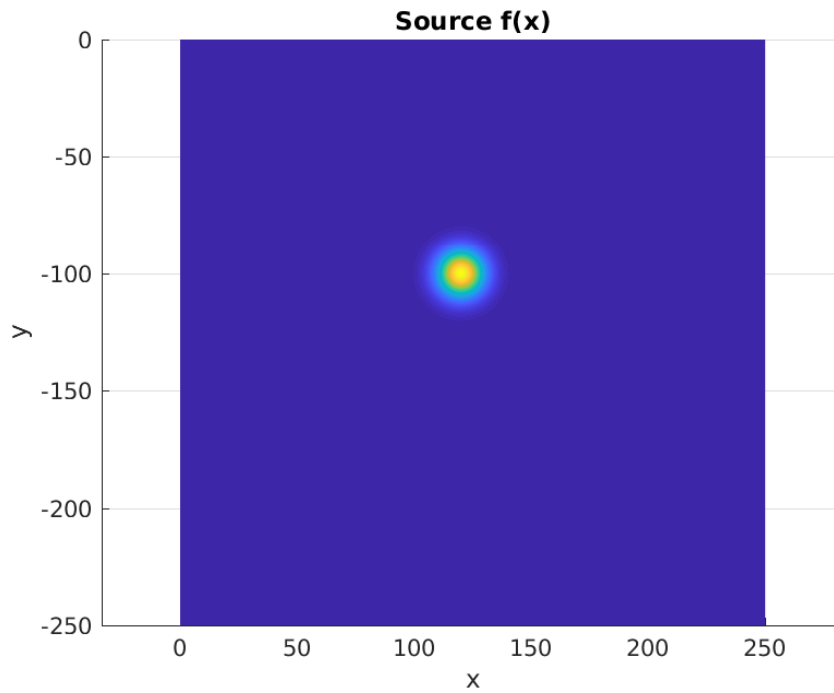


Figure 4.1 – The source $f(\mathbf{x})$ of the electrostatic equation. It is a Gaussian function with center $(120, -100)$ and stand deviation $5\sqrt{2}$.

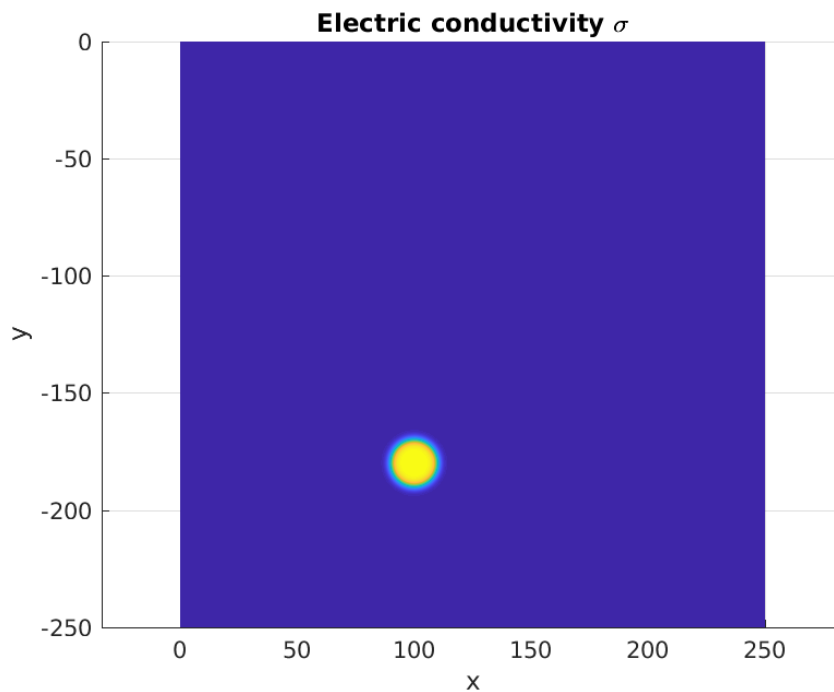


Figure 4.2 – The conductivity $\sigma(\mathbf{x})$ of the electrostatic equation. The conductivity has a homogeneous background with value 7.6×10^{-3} and a circle inclusion with value 2.1×10^{-2} . The edge of the inclusion is smoothed using the function \tanh .

For the conductivity, we choose a uniform background with a circle inclusion of radius 10 meters with the center located at $(100, -180)$. The background has a value of 7.6×10^{-3} and the inclusion has a value of 2.1×10^{-2} . To avoid sharp edge between the background and the inclusion, we use the function \tanh to obtain a smooth area between these two values. The smoothness of the conductivity will stabilize the numerical solution of the electrostatic equation. Figure 4.2 shows the image of the conductivity.

Numerical solution

We demonstrate the numerical solution of the electrostatic equation in this subsection. Figure 4.3 shows the image of the electric potential. The peak of the potential corresponds to the source area. From the potential image, we could hardly distinguish the inclusion. Figure 4.4 demonstrates the contour of the potential. We can easily observe that on the left bottom side of the image there is an area with sparse contour. That's the inclusion area. Based on the contour, we can compute the electric field through $\mathbf{E} = \nabla\Phi$. Figure 4.5 is the image of the field E_x and E_y . Figure 4.6 is the corresponding potential image.

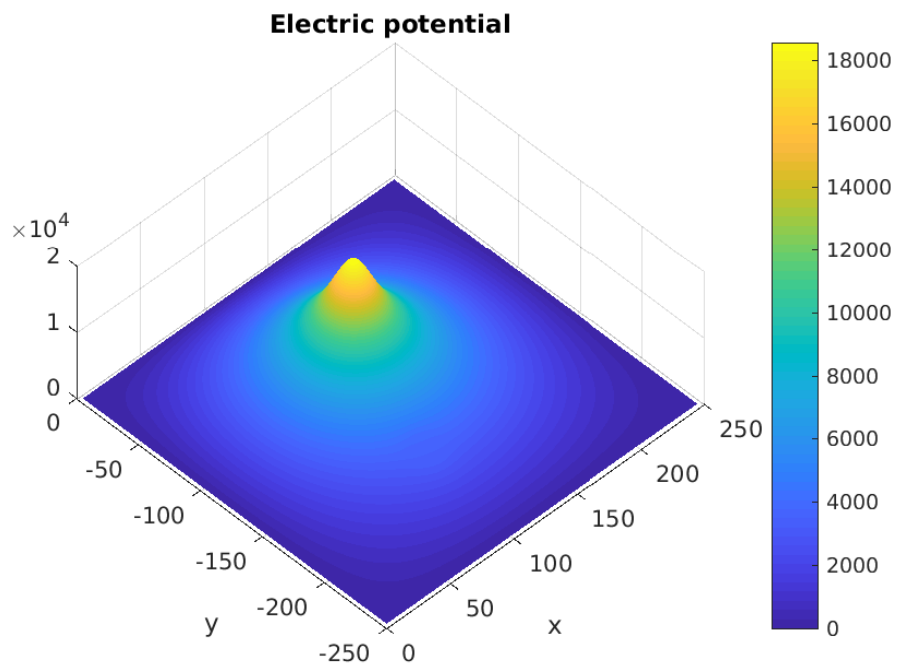


Figure 4.3 – The electric potential $\Phi(\mathbf{x})$. The peak of the potential corresponds to the source.

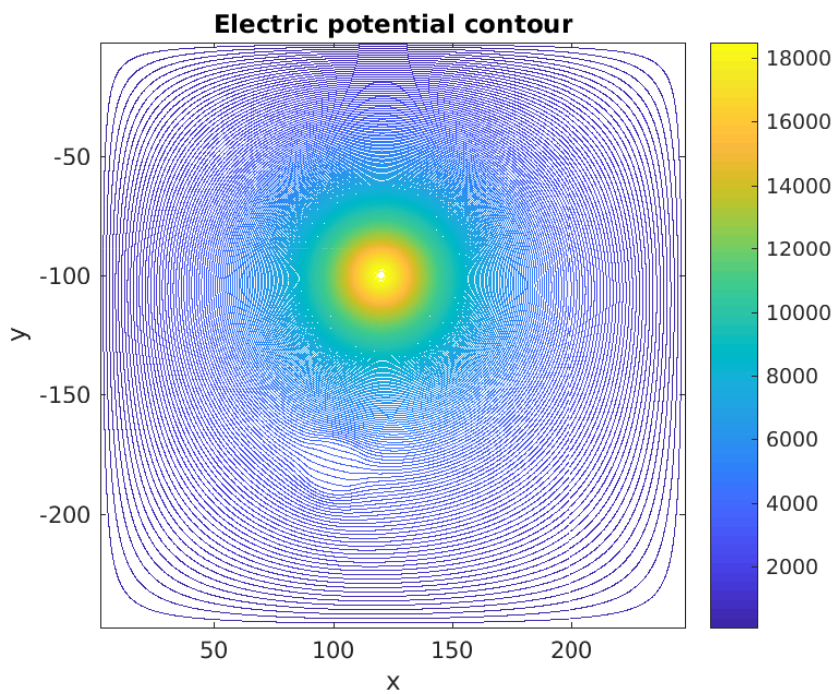


Figure 4.4 – The contour of the electric potential $\Phi(\mathbf{x})$. On the left bottom side there is a small area with sparse contour. That area is the inclusion.

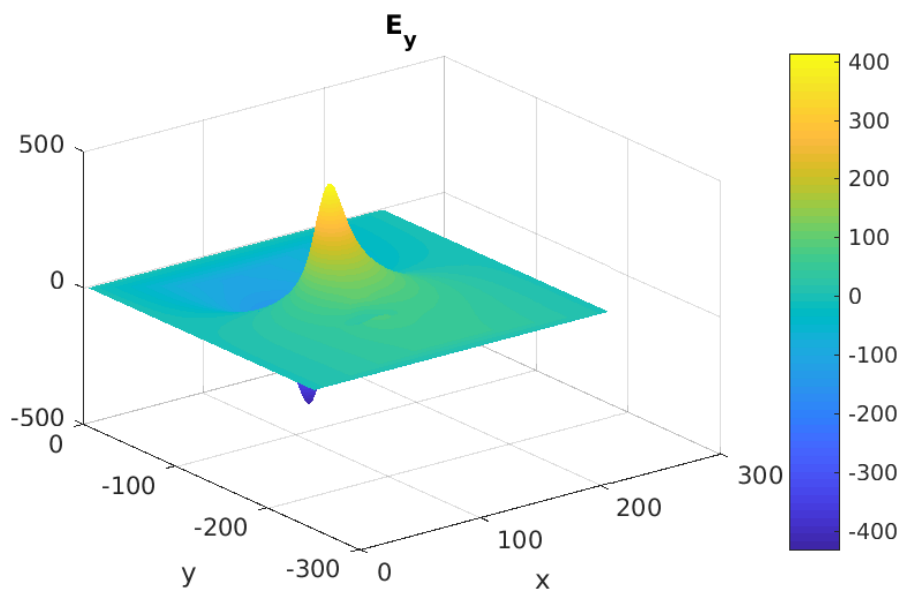
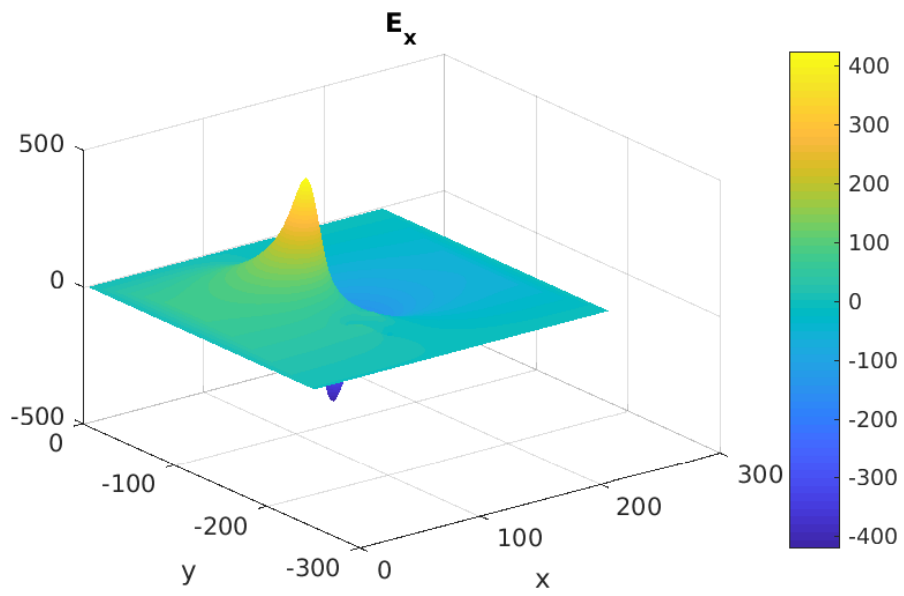


Figure 4.5 – The electric field E_x and E_y .

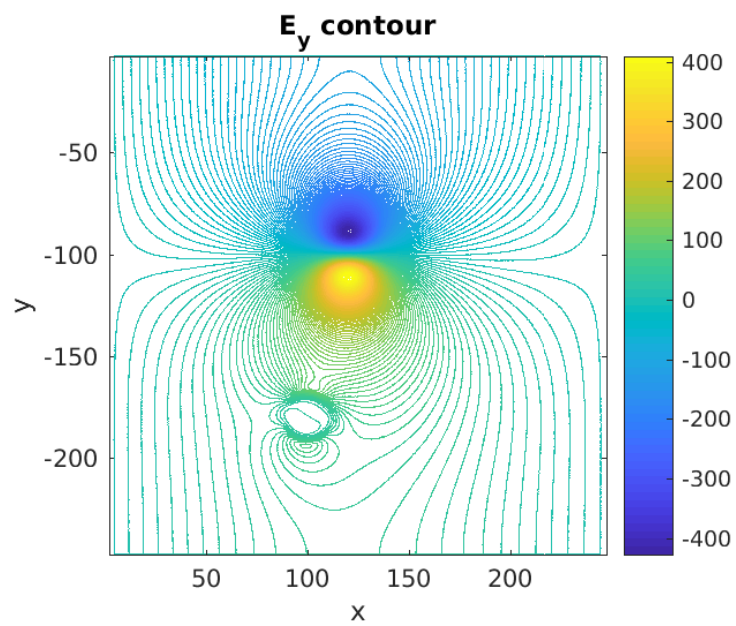
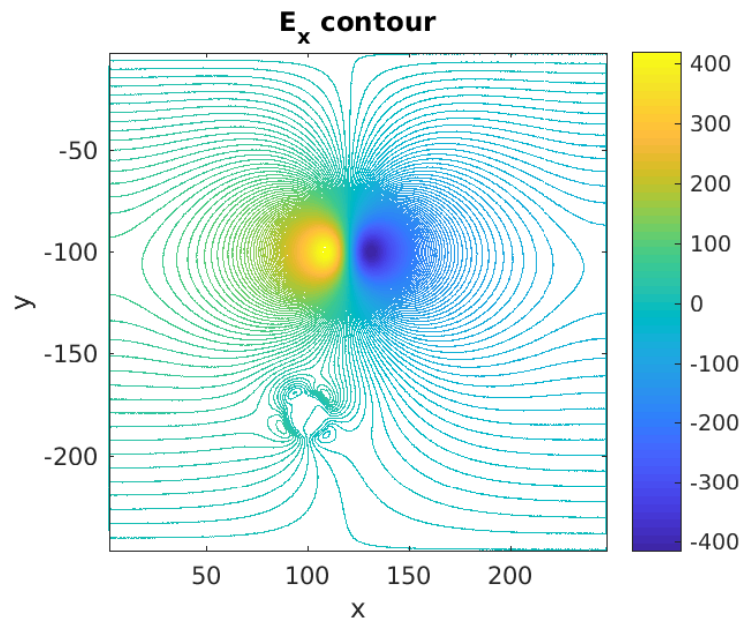


Figure 4.6 – The contour of the electric field E_x and E_y .

4.2 Staggered-grid finite difference method for Biot equations

The staggered-grid finite difference method is based on the **velocity-pressure** form of the partial differential equations. The velocity-pressure form use the velocity instead of the displacement as the variable. The second order partial differential equations decreases into a set of first order ones. Let us recall the time domain Biot equations

$$\begin{cases} \rho \partial_t^2 \mathbf{u} + \rho_f \partial_t^2 \mathbf{w} = \nabla \cdot \boldsymbol{\tau} + \mathbf{F}, \\ \rho_f \partial_t^2 \mathbf{u} + \rho_e \partial_t^2 \mathbf{w} = -\frac{\eta}{\kappa} \partial_t \mathbf{w} - \nabla p + \mathbf{f}, \\ \boldsymbol{\tau} = (\lambda \nabla \cdot \mathbf{u} + C \nabla \cdot \mathbf{w}) \mathbf{I} + G(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \\ -p = C \nabla \cdot \mathbf{u} + M \nabla \cdot \mathbf{w}. \end{cases} \quad (4.1)$$

Let us denote $\mathbf{v} = \partial_t \mathbf{u}$, $\mathbf{q} = \partial_t \mathbf{w}$. Then (4.1) can be rewritten in \mathbf{v} , \mathbf{q} as follows

$$\begin{cases} \rho \partial_t \mathbf{v} + \rho_f \partial_t \mathbf{q} = \nabla \cdot \boldsymbol{\tau} + \mathbf{F}, \\ \rho_f \partial_t \mathbf{v} + \rho_e \partial_t \mathbf{q} = -\frac{\eta}{\kappa} \mathbf{q} - \nabla p + \mathbf{f}, \\ \partial_t \boldsymbol{\tau} = (\lambda \nabla \cdot \mathbf{v} + C \nabla \cdot \mathbf{q}) \mathbf{I} + G(\nabla \mathbf{v} + \nabla \mathbf{v}^T), \\ -\partial_t p = C \nabla \cdot \mathbf{v} + M \nabla \cdot \mathbf{q}. \end{cases} \quad (4.2)$$

Instead of considering the 3D problem, let us consider the 2D one. Assume no variance along z axis, namely, all partial derivatives with respect to z are 0. The whole system is decomposed into two parts: SH (shear waves orthogonal to the $x - y$ plane) and P-SV (pressure and shear waves parallel to the $x - y$ plane)

— **SH**

$$\begin{cases} \rho \partial_t v_z + \rho_f \partial_t q_z = \partial_x \tau_{xz} + \partial_y \tau_{yz} + F_z, \\ \rho_f \partial_t v_z + \rho_e \partial_t q_z = -\frac{\eta}{\kappa} q_z + f_z, \\ \partial_t \tau_{xz} = G \partial_x v_z, \\ \partial_t \tau_{yz} = G \partial_y v_z. \end{cases} \quad (4.3)$$

— **P-SV**

$$\begin{cases} \rho \partial_t v_x + \rho_f \partial_t q_x = \partial_x \tau_{xx} + \partial_y \tau_{xy} + F_x, \\ \rho_f \partial_t v_x + \rho_e \partial_t q_x = -\partial_x p - \frac{\eta}{\kappa} q_x + f_x, \\ \rho \partial_t v_y + \rho_f \partial_t q_y = \partial_x \tau_{xy} + \partial_y \tau_{yy} + F_y, \\ \rho_f \partial_t v_y + \rho_e \partial_t q_y = -\partial_y p - \frac{\eta}{\kappa} q_y + f_y, \\ \partial_t \tau_{xx} = (\lambda + 2G) \partial_x v_x + \lambda \partial_y v_y + C(\partial_x q_x + \partial_y q_y), \\ \partial_t \tau_{xy} = G(\partial_x v_y + \partial_y v_x), \\ \partial_t \tau_{yy} = \lambda \partial_x v_x + (\lambda + 2G) \partial_y v_y + C(\partial_x q_x + \partial_y q_y), \\ -\partial_t p = C(\partial_x v_x + \partial_y v_y) + M(\partial_x q_x + \partial_y q_y). \end{cases} \quad (4.4)$$

From now on we only pay attention to the more complex P-SV equation system.

The difference between the staggered-grid finite difference method and the normal one is that the fields are not recorded at the same position. This is illustrated in Fig 4.7.

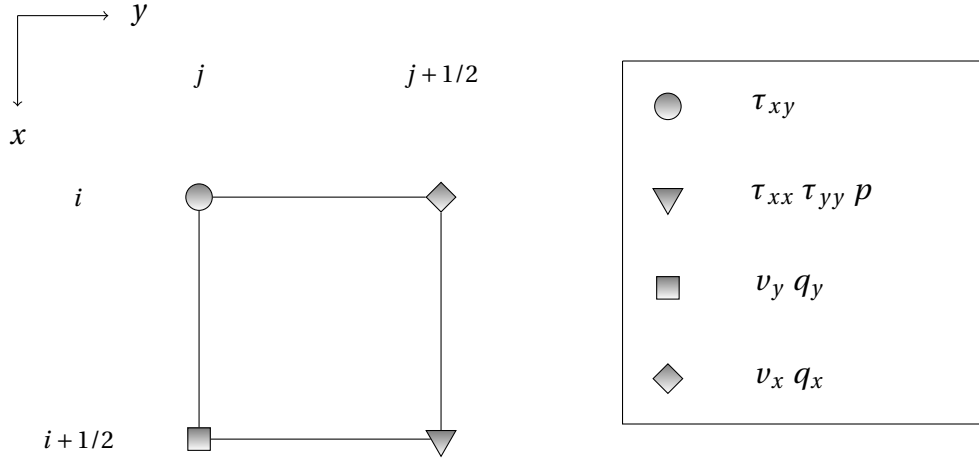


Figure 4.7 – Two dimensional space discrete strategy for the P-SV wave. τ_{xy} is posed at (i, j) , τ_{xx}, τ_{yy}, p at $(i + \frac{1}{2}, j + \frac{1}{2})$, v_y, q_y at $(i + \frac{1}{2}, j)$ and v_x, q_x at $(i, j + \frac{1}{2})$.

From the figure we can see that τ_{xy} is posed at (i, j) , τ_{xx}, τ_{yy}, p at $(i + \frac{1}{2}, j + \frac{1}{2})$, v_y, q_y at $(i + \frac{1}{2}, j)$ and v_x, q_x at $(i, j + \frac{1}{2})$. The advantage of the staggered-grid method is shown below. To update v_x at the point $(i, j + \frac{1}{2})$, we need $\partial_x \tau_{xx}$ which can be approximated by τ_{xx} at $(i - \frac{1}{2}, j + \frac{1}{2})$ and $(i + \frac{1}{2}, j + \frac{1}{2})$. Since we just record τ_{xx} at such points, we have exactly these data. We don't need to approximate these values from nearby points.

In practice we use the fourth order finite difference method instead of the second order one to have higher precision. Let us denote $u(i, j)$ a field at the space position (x_i, y_j) . The fourth order approximation of the spatial derivative $\partial_y u$ at (i, j) is

$$\partial_y u(i, j) \simeq -\frac{1}{24\Delta y} \left(u(i, j + \frac{3}{2}) - u(i, j - \frac{3}{2}) \right) + \frac{9}{8\Delta y} \left(u(i, j + \frac{1}{2}) - u(i, j - \frac{1}{2}) \right).$$

With the help of the Taylor expansion, it's easy to show that the error is fourth order of the space step.

The time discretization strategy is that we update \mathbf{q}, \mathbf{v} at time t_k and $p, \boldsymbol{\tau}$ at time $t_{k+1/2}$. We use second order formula, i.e., to approximate $\partial_t u$ at time $t_{k+1/2}$ we use

$$\partial_t u^{k+\frac{1}{2}} \simeq \frac{1}{\Delta t} \left(u^{k+1} - u^k \right).$$

The benefit of this scheme is shown below. For example, to update τ_{xy} at $t_{k+1/2}$, we need τ_{xy} at $t_{k-1/2}$ and $\partial_x v_y, \partial_y v_x$ at t_k . Since \mathbf{v}, \mathbf{q} are recorded at t_k , we don't need to do approximation for these values. We summarize the updating strategy in the following

— At time step $k + \frac{1}{2}$. Update $p^{k+\frac{1}{2}}, \tau_{xx}^{k+\frac{1}{2}}, \tau_{yy}^{k+\frac{1}{2}}, \tau_{xy}^{k+\frac{1}{2}}$ by

$$\begin{aligned} p^{k+\frac{1}{2}} &= p^{k-\frac{1}{2}} - \Delta t \left(C(\partial_x v_x^k + \partial_y v_y^k) + M(\partial_x q_x^k + \partial_y q_y^k) \right), \\ \tau_{xx}^{k+\frac{1}{2}} &= \tau_{xx}^{k-\frac{1}{2}} + \Delta t \left((\lambda + 2G)\partial_x v_x^k + \lambda\partial_y v_y^k + C(\partial_x q_x^k + \partial_y q_y^k) \right), \\ \tau_{yy}^{k+\frac{1}{2}} &= \tau_{yy}^{k-\frac{1}{2}} + \Delta t \left(\lambda\partial_x v_x^k + (\lambda + 2G)\partial_y v_y^k + C(\partial_x q_x^k + \partial_y q_y^k) \right), \\ \tau_{xy}^{k+\frac{1}{2}} &= \tau_{xy}^{k-\frac{1}{2}} + \Delta t \left(G(\partial_x v_y^k + \partial_y v_x^k) \right). \end{aligned}$$

— At time step $k + 1$. Update $q_x^{k+1}, q_y^{k+1}, v_x^{k+1}, v_y^{k+1}$ by

$$\begin{aligned} q_x^{k+1} &= \frac{1}{1 + \frac{\rho_0 \eta}{2\rho\kappa} \Delta t} \left(\left(1 - \frac{\rho_0 \eta}{2\rho\kappa} \Delta t\right) q_x^k - \frac{\rho_f}{\rho_0} \Delta t (\partial_x \tau_{xx}^{k+\frac{1}{2}} + \partial_y \tau_{xy}^{k+\frac{1}{2}} + F_x) \right. \\ &\quad \left. + \frac{\rho_0}{\rho} \Delta t (-\partial_x p^{k+\frac{1}{2}} + f_x) \right), \\ v_x^{k+1} &= v_x^k + \frac{\rho_e}{\rho_0} \Delta t (\partial_x \tau_{xx}^{k+\frac{1}{2}} + \partial_y \tau_{xy}^{k+\frac{1}{2}} + F_x) \\ &\quad - \frac{\rho_f}{\rho_0} \Delta t \left(-\partial_x p^{k+\frac{1}{2}} - \frac{\eta}{\kappa} q_x^{k+\frac{1}{2}} + f_x \right), \\ q_y^{k+1} &= \frac{1}{1 + \frac{\rho_0 \eta}{2\rho\kappa} \Delta t} \left(\left(1 - \frac{\rho_0 \eta}{2\rho\kappa} \Delta t\right) q_y^k - \frac{\rho_f}{\rho_0} \Delta t (\partial_x \tau_{xy}^{k+\frac{1}{2}} + \partial_y \tau_{yy}^{k+\frac{1}{2}} + F_y) \right. \\ &\quad \left. + \frac{\rho_0}{\rho} \Delta t (-\partial_y p^{k+\frac{1}{2}} + f_y) \right), \\ v_y^{k+1} &= v_y^k + \frac{\rho_e}{\rho_0} \Delta t (\partial_x \tau_{xy}^{k+\frac{1}{2}} + \partial_y \tau_{yy}^{k+\frac{1}{2}} + F_y) \\ &\quad - \frac{\rho_f}{\rho_0} \Delta t \left(-\partial_y p^{k+\frac{1}{2}} - \frac{\eta}{\kappa} q_y^{k+\frac{1}{2}} + f_y \right). \end{aligned}$$

4.3 PML

The idea of PML is to put a special layer outside the computational domain such that no reflection is generated across the interface and waves attenuate rapidly inside the PML layer. It is physically impossible to have such kind of special media. In [6], the author computed the plane wave reflection on the interface. By splitting the wave in the PML layer into two, the author found out that it's possible to have zero reflections for any frequency and any incident angle. In fact, in the PML layer wave equations are changed to have the non-reflection property. Later in [19] authors showed that the split-PML could be written into more elegant stretched-coordinate formula. Taking the Maxwell

equations for example, equations in the PML layer are

$$\begin{aligned} i\omega\epsilon\mathbf{E} &= \tilde{\nabla} \times \mathbf{H} \\ i\omega\mu\mathbf{H} &= \tilde{\nabla} \times \mathbf{E} \end{aligned}$$

where

$$\tilde{\nabla} = \left(\frac{1}{s_x} \partial_x, \frac{1}{s_y} \partial_y, \frac{1}{s_z} \partial_z \right).$$

The choice of s totally depends on the direction of the PML layer. If the PML layer is normal to the x direction, then s_y and s_z are set to be 1 to avoid reflections. The term s_x is set as follows

$$s_x = 1 + \frac{\sigma}{i\omega\epsilon}$$

so that waves are damped in the PML layer. To obtain better numerical result, σ increases gradually from the interface to the outside boundary.

Researchers find out that PML doesn't work well for evanescent waves and waves almost parallel to the interface. Kuzuoglu and Mittra [37] provide complex frequency shifted PML (CFS-PML) to deal with this problem. The idea is to choose s as follows

$$\begin{aligned} s &= 1 + \frac{\sigma}{\alpha + i\omega\epsilon} \\ \sigma(x) &= \left(\frac{x}{d} \right)^m \sigma_{\max}, \quad \sigma_{\max} = -\frac{(m+1)\ln(R(0))}{2\eta d} \\ \alpha(x) &= \left(1.1 - \left(\frac{x}{d} \right)^m \right) \alpha_{\max}, \quad \alpha_{\max} = \pi f_0 \end{aligned}$$

where m is chosen to be between 3 and 4, d the length of PML, f_0 the peak frequency. To balance the theoretical reflection error and the numerical discretization error, an optimal choice for a 10-cell-thick PML is $R(0) = e^{-16}$ and $R(0) = e^{-8}$ for the 5-cell-thick case.

Time domain implementation

We want to know how to obtain PML for time domain equations. The most obvious way is to use Fourier transform. The problem is that the dependence of s on frequency results in convolution in the equations.

In the case of stretched-coordinate PML with $s = 1 + \frac{\sigma}{i\omega\epsilon}$, the split-PML can be applied to avoid the convolution. Taking the equation of H_z in 2D for example, recall that

$$i\omega\mu H_z = -\frac{1}{s_x} \partial_x E_y + \frac{1}{s_y} \partial_y E_x.$$

Let us split H_z into two parts, $H_z = H_{zx} + H_{zy}$ such that

$$\begin{aligned} i\omega\mu H_{zx} &= -\frac{1}{s_x}\partial_x E_y, \\ i\omega\mu H_{zy} &= \frac{1}{s_y}\partial_y E_x. \end{aligned}$$

Let us consider the case that the PML layer is normal to the x direction. In this case, $s_y = 1$. Therefore we have

$$\mu\partial_t H_{zy} = \partial_y E_x.$$

Substituting $s_x = 1 + \frac{\sigma}{i\omega\varepsilon}$ shows

$$i\omega\mu H_{zx} + \frac{\sigma\mu}{\varepsilon} H_{zx} = -\partial_x E_y.$$

The time domain equation for H_{zx} follows

$$\mu\partial_t H_{zx} + \frac{\sigma\mu}{\varepsilon} H_{zx} = -\partial_x E_y.$$

For CFS-PML, the split field technique doesn't work any more, because of the existence of α in the choice of s . The time domain PML will involve a convolution term. It was found out in [48] that the convolution can be implemented recursively by adding a few auxiliary functions. The new implementation of PML is called C-PML. We will demonstrate the implementation below.

Let us recall the equation for E_x in 2D

$$i\omega\varepsilon E_x = \frac{1}{s_y}\partial_y H_z - \frac{1}{s_z}\partial_z H_y.$$

Using Fourier transform, we obtain the time domain equation

$$\varepsilon\partial_t E_x = \mathcal{F}^{-1}\left(\frac{1}{s_y}\right) * \partial_y H_z - \mathcal{F}^{-1}\left(\frac{1}{s_z}\right) * \partial_z H_y.$$

Here $f * g$ means the convolution of the function f and g . Let us note that we use the same notations for fields in the frequency domain and fields in the time domain. Let us recall the Laplace transform

$$F(l) := \mathcal{L}(f(t)) = \int_0^{\infty} e^{-lt} f(t) dt$$

where t is real and l is complex. The inverse Laplace transform is

$$f(t) = \mathcal{L}^{-1}(F(l)) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\gamma-iT}^{\gamma+iT} e^{lt} F(l) ds.$$

The integration is along the line $\text{Re } l = \gamma$ such that γ is larger than any singularity of $F(s)$. Let us recall Laplace transforms for two basic functions

$$\mathcal{L}(\delta(t)) = 1, \quad \mathcal{L}(e^{at}) = \frac{1}{l-a}.$$

Since

$$\frac{1}{s} = 1 - \frac{\sigma}{\varepsilon} \frac{1}{i\omega + \left(\frac{\sigma}{\varepsilon} + \frac{\alpha}{\varepsilon}\right)},$$

the inverse Fourier transform of $\frac{1}{s}$ is

$$\begin{aligned} \mathcal{F}^{-1}\left(\frac{1}{s}\right) &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{s} e^{i\omega t} d\omega \\ &= \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{iT}^{iT} \left(1 - \frac{\sigma}{\varepsilon} \frac{1}{l + \left(\frac{\sigma}{\varepsilon} + \frac{\alpha}{\varepsilon}\right)}\right) e^{lt} dl \\ &= \delta(t) + \zeta(t), \end{aligned}$$

where

$$\zeta(t) = -\frac{\sigma}{\varepsilon} e^{-\left(\frac{\sigma}{\varepsilon} + \frac{\alpha}{\varepsilon}\right)t}.$$

Therefore the time domain equation for E_x reads

$$\varepsilon \partial_t E_x = \partial_y H_z - \partial_z H_y + \zeta_y(t) * \partial_y H_z - \zeta_z(t) * \partial_z H_y. \quad (4.5)$$

The convolution of ζ with another function can be implemented recursively. Let

$$g(t) = \int_0^t f(t') e^{-a(t-t')} dt'.$$

Then

$$\begin{aligned} g(t + \Delta t) &= \int_0^{t+\Delta t} f(t') e^{-a(t+\Delta t-t')} dt' \\ &\approx e^{-a\Delta t} g(t) + f\left(t + \frac{\Delta t}{2}\right) \frac{1 - e^{-a\Delta t}}{a}. \end{aligned}$$

We can discretize (4.5) to obtain the update formula for E_x in the PML layer

$$\begin{aligned} E_x^{n+1} &= E_x^n + \frac{\Delta t}{\varepsilon} \left(\partial_y H_z^{n+1/2} + \psi_{Hz}^{n+1/2} - \partial_z H_y^{n+1/2} - \psi_{Hy}^{n+1/2} \right), \\ \psi_{Hz}^{n+1/2} &= p_y \psi_{Hz}^{n-1/2} + q_y \partial_y H_z^{n+1/2}, \\ \psi_{Hy}^{n+1/2} &= p_z \psi_{Hy}^{n-1/2} + q_z \partial_z H_y^{n+1/2}, \end{aligned}$$

where

$$\begin{aligned} p_y &= e^{-\left(\frac{\sigma_y}{\varepsilon} + \frac{\alpha_y}{\varepsilon}\right)\Delta t}, \quad q_y = \frac{\sigma_y}{\sigma_y + \alpha_y} (p_y - 1), \\ p_z &= e^{-\left(\frac{\sigma_z}{\varepsilon} + \frac{\alpha_z}{\varepsilon}\right)\Delta t}, \quad q_z = \frac{\sigma_z}{\sigma_z + \alpha_z} (p_z - 1). \end{aligned}$$

C-PML for Biot equations

We can apply the C-PML strategy on Biot equations (4.4). Let's take the following equation for example

$$\partial_t \tau_{yy} = \lambda \partial_x v_x + (\lambda + 2G) \partial_y v_y + C(\partial_x q_x + \partial_y q_y).$$

In the PML layer, the equation is

$$\begin{aligned} \partial_t \tau_{yy} = & \lambda \mathcal{F}^{-1}\left(\frac{1}{s_x}\right) * \partial_x v_x + (\lambda + 2G) \mathcal{F}^{-1}\left(\frac{1}{s_y}\right) * \partial_y v_y \\ & + C\left(\mathcal{F}^{-1}\left(\frac{1}{s_x}\right) * \partial_x q_x + \mathcal{F}^{-1}\left(\frac{1}{s_x}\right) * \partial_y q_y\right). \end{aligned}$$

The updating formula is

$$\begin{aligned} \tau_{yy}^{k+\frac{1}{2}} &= \tau_{yy}^{k-\frac{1}{2}} + \Delta t \left(\lambda (\partial_x v_x^k + \psi_{vx}^k) + (\lambda + 2G) (\partial_y v_y^k + \psi_{vy}^k) \right. \\ &\quad \left. + C(\partial_x q_x^k + \psi_{qx}^k + \partial_y q_y^k + \psi_{qy}^k) \right), \\ \psi_{vx}^k &= m_x \psi_{vx}^{k-1} + n_x \partial_x v_x^k, \\ \psi_{vy}^k &= m_y \psi_{vy}^{k-1} + n_y \partial_y v_y^k, \\ \psi_{qx}^k &= m_x \psi_{qx}^{k-1} + n_x \partial_x q_x^k, \\ \psi_{qy}^k &= m_y \psi_{qy}^{k-1} + n_y \partial_y q_y^k, \end{aligned}$$

where

$$\begin{aligned} m_x &= e^{-(\sigma_x + \alpha_x) \Delta t}, & n_x &= \frac{\sigma_x}{\sigma_x + \alpha_x} (m_x - 1), \\ m_y &= e^{-(\sigma_y + \alpha_y) \Delta t}, & n_y &= \frac{\sigma_y}{\sigma_y + \alpha_y} (m_y - 1). \end{aligned}$$

Numerical solution of Biot equations

In the following we will show some images of the solution to Biot equations. Let us recall that the source of Biot equations (4.4) in the electroseismic model are

$$\mathbf{F}(\mathbf{x}, t) = \mathbf{0}, \quad \mathbf{f}(\mathbf{x}, t) = -\frac{L\eta}{\kappa} \nabla \Phi(\mathbf{x}) g(t).$$

We choose that L equals to 0 except a small ball centered at $(100, -180)$ with radius 10 in which L takes the value of 2.1×10^{-12} . For simplicity we take constants for all the other parameters in Biot equations. Table 4.1 shows our choice of values.

ρ	2.74×10^3
ρ_f	1.10×10^3
ρ_e	6.11×10^4
λ	3.30×10^{10}
C	3.98×10^9
M	2.48×10^{10}
G	2.20×10^{10}

Table 4.1 – Values of parameters in Biot equations

We show in Figure 4.8 the space term $-\frac{L\eta}{\kappa}\nabla\Phi(\mathbf{x})$ of the source of Biot equations. For the time part of the source, we choose

$$g(t) = -\frac{1}{400}e^{-a(t-t_0)^2}, \quad (4.6)$$

where $a = 150^2\pi^2$ and $t_0 = -\sqrt{\frac{1}{2a}}$.

Let us recall one equation of Biot equations

$$\rho_f\partial_t\mathbf{v} + \rho_e\partial_t\mathbf{q} = -\frac{\eta}{\kappa}\mathbf{q} - \nabla p + \mathbf{f}.$$

The parameter $\frac{\eta}{\kappa}$ before $\partial_t\mathbf{q}$ controls the damping amount. Biot equations with different damping quantity behave quite differently. In Figure 4.9 we show the propagation of v_x with low damping coefficient $\frac{\eta}{\kappa} = 1.5 \times 10^5$. We can observe three wave fronts with different speed. Figure 4.10 demonstrates the propagation of v_x with high damping $\frac{\eta}{\kappa} = 1.5 \times 10^9$. Only two wave fronts are observed. The third one is absorbed by the medium because of the high damping term. The C-PML works well for both cases. There is little reflection from the boundary for any angle of incident wave.

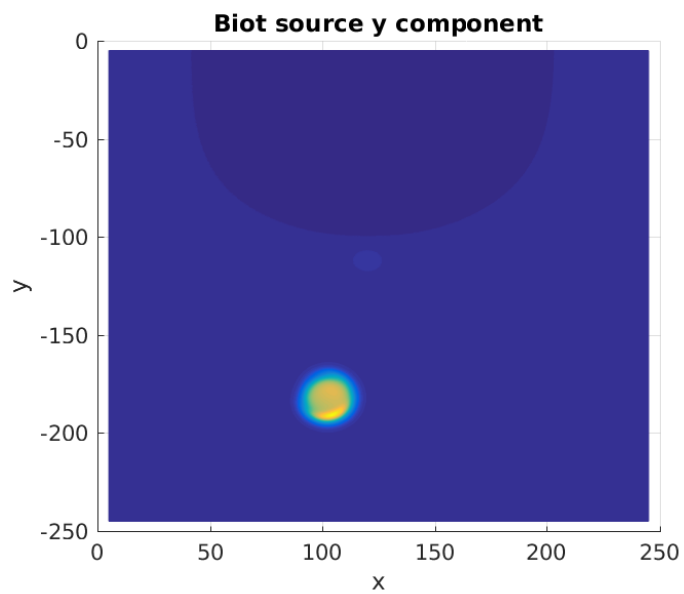
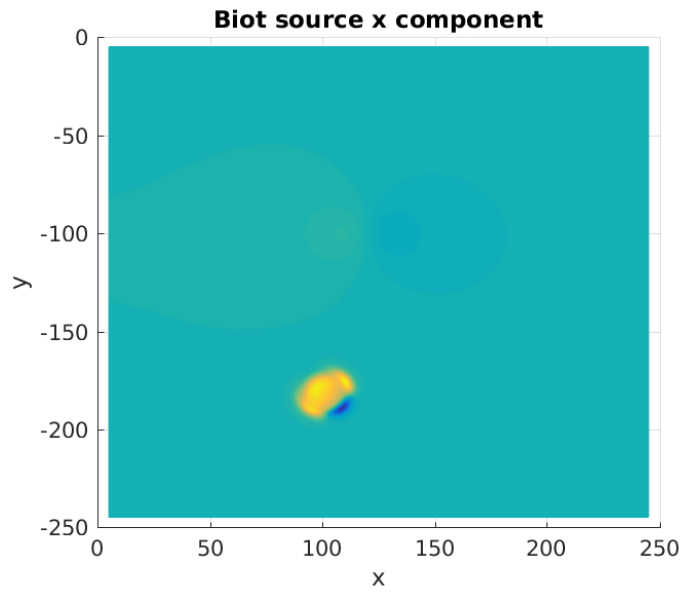


Figure 4.8 – Biot source $-\frac{L\eta}{\kappa}\nabla\Phi(\mathbf{x})$.

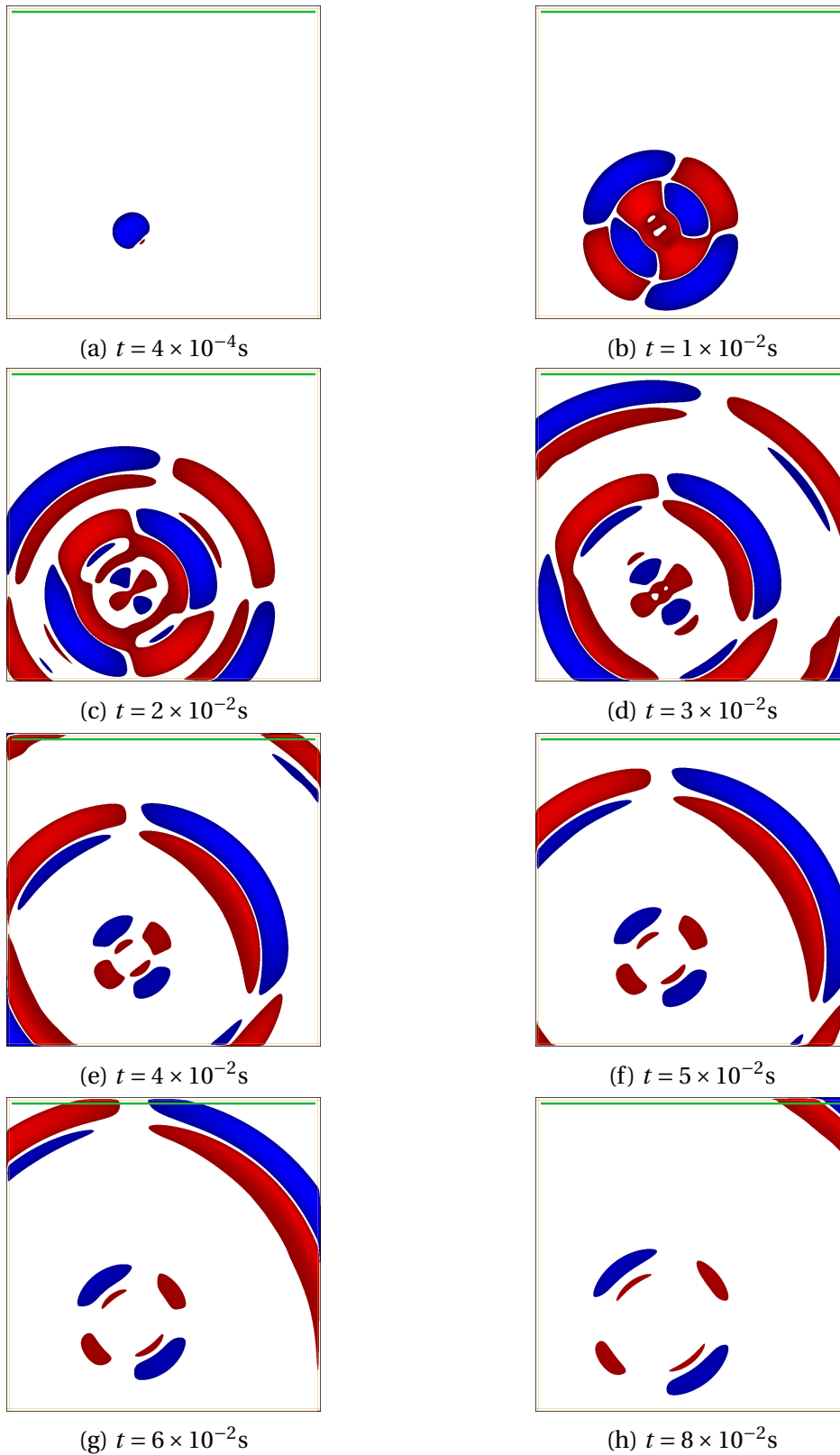


Figure 4.9 – Propagation of v_x for Biot equations with low damping term. We could observe three wave fronts with different speed.

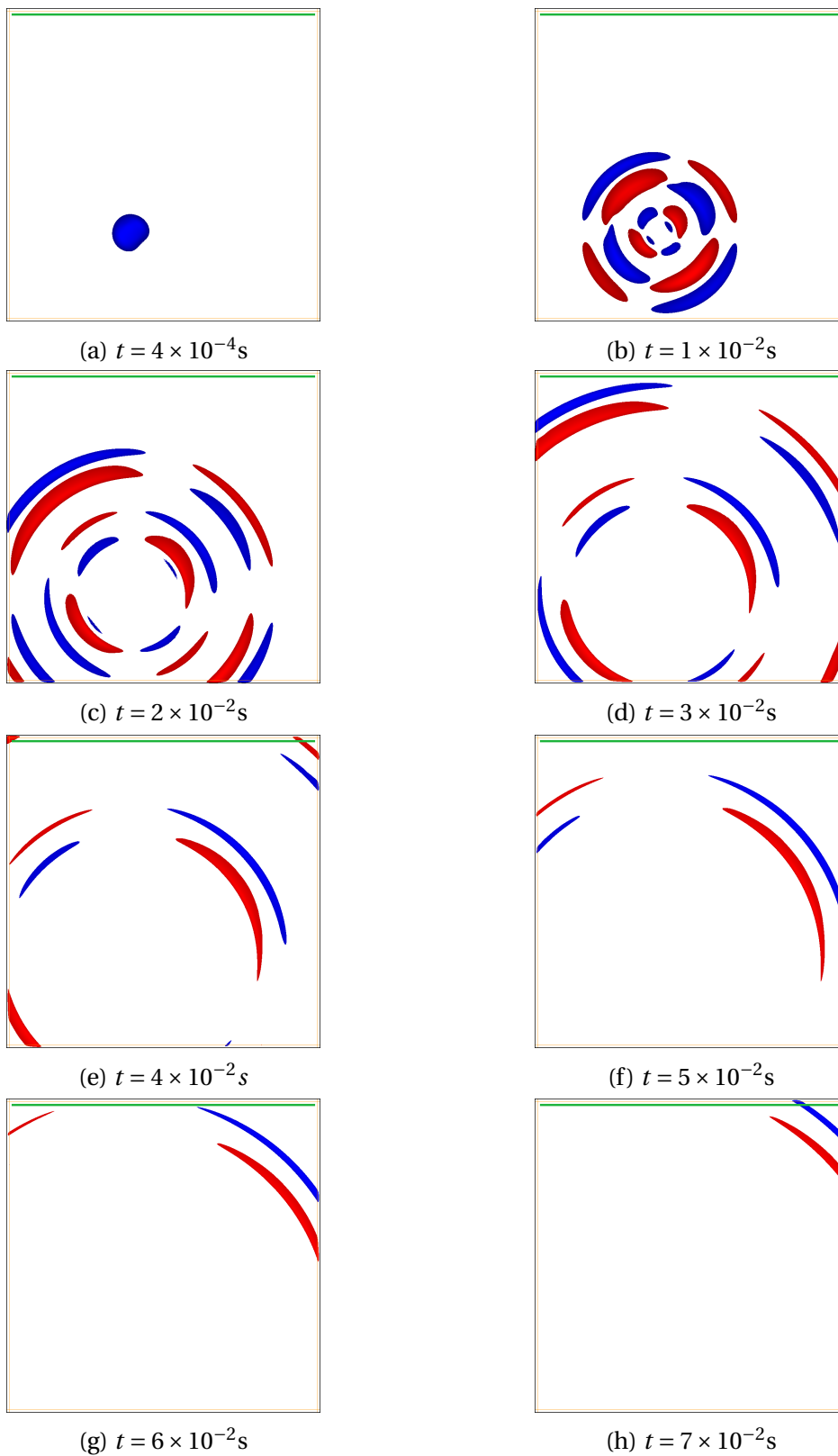


Figure 4.10 – Propagation of v_x for Biot equations with high damping term. Two wave fronts are observed.

Chapter 5

Numerical Inverse Source Problem for Biot Equations

In this chapter, we will show some numerical result of the inverse source problem for Biot equations. This corresponds to the first step of the hybrid inverse electroseismic problem. The method we are going to use is the time source reverse method. In Chapter 1, We have had a short introduction of the time source reverse method to reconstruct the space source of the acoustic equation. During the process of the transformation from source problem into initial value problem, we need to solve an integral equation (1.8). The choice of $g(t)$ plays an important role in the numerical solution of the integral equation. We find out that if we choose $g(t)$ like the one in the reference [12], we meet severe stability problem. We will choose a different $g(t)$ to solve this problem.

5.1 Source time reversal method for the acoustic equation

Let us consider the acoustic equation in \mathbb{R}^n

$$\begin{cases} \partial_t^2 u(\mathbf{x}, t) - c^2(\mathbf{x})\Delta u(\mathbf{x}, t) = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) & \text{in } \mathbb{R}^n, \\ \partial_t u(\mathbf{x}, 0) = u_1(\mathbf{x}) & \text{in } \mathbb{R}^n, \end{cases} \quad (5.1)$$

with initial conditions supported in the ball $B_{\rho_1} = \{\mathbf{x} : |\mathbf{x}| < \rho_1\}$ of radius $0 < \rho_1 < \infty$. Assume that the observation region Ω is bounded and contains the ball B_{ρ_1} . Assume also that Ω is contained in a bigger ball B_{ρ_2} . We measure the acoustic equation (5.1) on $\partial\Omega \times (0, T)$. We introduce the time reverse method

$$\begin{cases} \partial_t^2 u_r(\mathbf{x}, t) - c^2(\mathbf{x})\Delta u_r(\mathbf{x}, t) = 0 & \text{in } \Omega \times (T, 2T), \\ u_r(\mathbf{x}, T) = u(\mathbf{x}, T) & \text{in } \Omega, \\ \partial_t u_r(\mathbf{x}, T) = -\partial_t u(\mathbf{x}, T) & \text{in } \Omega, \\ u_r(\mathbf{x}, t) = u(\mathbf{x}, 2T - t) & \text{on } \partial\Omega \times (T, 2T). \end{cases} \quad (5.2)$$

From the symmetry of the wave, we know that

$$u_r(\mathbf{x}, 2T) = u_0(\mathbf{x}), \text{ and } \partial_t u_r(\mathbf{x}, 2T) = -u_1(\mathbf{x}).$$

In practice it's usually impossible to obtain $u(\mathbf{x}, T)$ and $\partial_t u(\mathbf{x}, T)$ inside Ω . We need to choose T large enough such that the wave propagate outside Ω , i.e.,

$$u(\mathbf{x}, T) = 0, \quad \partial_t u(\mathbf{x}, T) = 0.$$

Next we will show how to transform the source problem into the initial value problem. Let us consider two problems:

$$\begin{cases} \partial_t^2 u(\mathbf{x}, t) - c^2(\mathbf{x})\Delta u(\mathbf{x}, t) = f(\mathbf{x})g(t) & \text{in } \mathbb{R}^n \times (0, T), \\ u(\mathbf{x}, 0) = 0 & \text{in } \mathbb{R}^n, \\ \partial_t u(\mathbf{x}, 0) = 0 & \text{in } \mathbb{R}^n, \end{cases} \quad (5.3)$$

and

$$\begin{cases} \partial_t^2 v(\mathbf{x}, t) - c^2(\mathbf{x})\Delta v(\mathbf{x}, t) = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ v(\mathbf{x}, 0) = 0 & \text{in } \mathbb{R}^n, \\ \partial_t v(\mathbf{x}, 0) = f(\mathbf{x}) & \text{in } \mathbb{R}^n. \end{cases} \quad (5.4)$$

Assume that $v(\mathbf{x}, t)$ solves (5.4). Denote

$$u(\mathbf{x}, t) = (v * g)(t) = \int_0^t v(\mathbf{x}, t - \tau)g(\tau) d\tau. \quad (5.5)$$

Then

$$\begin{aligned} \partial_t u(\mathbf{x}, t) &= v(\mathbf{x}, 0)g(t) + \int_0^t \partial_t v(\mathbf{x}, t - \tau)g(\tau) d\tau \\ &= \int_0^t \partial_t v(\mathbf{x}, t - \tau)g(\tau) d\tau, \\ \partial_t^2 u(\mathbf{x}, t) &= \partial_t v(\mathbf{x}, 0)g(t) + \int_0^t \partial_t^2 v(\mathbf{x}, t - \tau)g(\tau) d\tau \\ &= f(\mathbf{x})g(t) + \int_0^t \partial_t^2 v(\mathbf{x}, t - \tau)g(\tau) d\tau. \end{aligned}$$

Therefore $u(\mathbf{x}, 0) = 0$, $\partial_t u(\mathbf{x}, 0) = 0$ and

$$\begin{aligned} \partial_t^2 u(\mathbf{x}, t) - c^2(\mathbf{x})\Delta u(\mathbf{x}, t) &= f(\mathbf{x})g(t) + \int_0^t (\partial_t^2 v(\mathbf{x}, t - \tau) - c^2\Delta v(\mathbf{x}, t - \tau))g(\tau) d\tau \\ &= f(\mathbf{x})g(t), \end{aligned}$$

that's to say $u(\mathbf{x}, t)$ solves (5.3). We conclude with the following proposition.

Proposition 5.1.1. *If $v(\mathbf{x}, t)$ solves (5.4), then $u(\mathbf{x}, t)$ defined by (5.5) solves (5.3).*

Let $\dot{u} = \partial_t u$ and $\dot{v} = \partial_t v$. We have shown that \dot{u} and \dot{v} have the following relation

$$\dot{u}(\mathbf{x}, t) = (\dot{v} * g)(t) = \int_0^t \dot{v}(\mathbf{x}, t - \tau) g(\tau) d\tau. \quad (5.6)$$

We differentiate (5.4) by t to obtain

$$\begin{cases} \partial_t^2 \dot{v}(\mathbf{x}, t) - c^2(\mathbf{x}) \Delta \dot{v}(\mathbf{x}, t) = 0 & \text{in } \mathbb{R}^n \times (0, T), \\ \dot{v}(\mathbf{x}, 0) = f(\mathbf{x}) & \text{in } \mathbb{R}^n. \end{cases} \quad (5.7)$$

By applying the time reverse method (5.2), we can reconstruct $f(x)$. During the time reverse method, we need $\dot{v}(\mathbf{x}, t)$ on the boundary. Assume we have measurement of $\dot{u}(\mathbf{x}, t)$ on the boundary. We need to solve the integral equation (5.6). Let us divide the interval $[0, T]$ into n equal segments with length Δt . The rectangle numerical integration rule can be used to solve the integral equation

$$\begin{aligned} \dot{u}(\mathbf{x}, k\Delta t) &= \int_0^{k\Delta t} \dot{v}(\mathbf{x}, n\Delta t - \tau) g(\tau) d\tau \\ &\approx \Delta t \sum_{i=0}^{k-1} \dot{v}(\mathbf{x}, (k-i)\Delta t) g(i\Delta t) \end{aligned}$$

For $k = 1, \dots, n$, we can assemble all these approximations into a linear system

$$\begin{pmatrix} g(0) & & & & \\ g(\Delta t) & g(0) & & & \\ \vdots & \vdots & \ddots & & \\ g((n-1)\Delta t) & g((n-2)\Delta t) & \dots & g(0) & \end{pmatrix} \begin{pmatrix} \dot{v}(\mathbf{x}, \Delta t) \\ \dot{v}(\mathbf{x}, 2\Delta t) \\ \vdots \\ \dot{v}(\mathbf{x}, n\Delta t) \end{pmatrix} = \begin{pmatrix} \dot{u}(\mathbf{x}, \Delta t) \\ \dot{u}(\mathbf{x}, 2\Delta t) \\ \vdots \\ \dot{u}(\mathbf{x}, n\Delta t) \end{pmatrix} \quad (5.8)$$

We see that the matrix is lower triangular with diagonal element $g(0)$. In the numerical experiment, we will choose $g(t)$ with a nonzero initial value to stabilize the calculation of \dot{v} from \dot{u} .

Numerical examples

Let us choose

$$\begin{aligned} f(\mathbf{x}) &= e^{-\frac{(x-200)^2 + (y+100)^2}{128\pi}} \\ g(t) &= -\frac{1}{400} e^{-a(t-t_0)^2}, \end{aligned}$$

where $a = 150^2 \pi^2$ and $t_0 = -\sqrt{\frac{1}{2a}}$. The forward problem 5.3 is solved by finite difference method. To decrease reflections from the boundary, C-PML is applied. Figure 5.1 shows the propagation of u . We measure \dot{u} on the boundary $\partial\Omega$ and compute \dot{v} from (5.8). To construct $f(\mathbf{x})$, we solve (5.7) in the bounded domain Ω with knowledge of \dot{v} on $\partial\Omega$. Figure 5.2 demonstrates the inverse process.

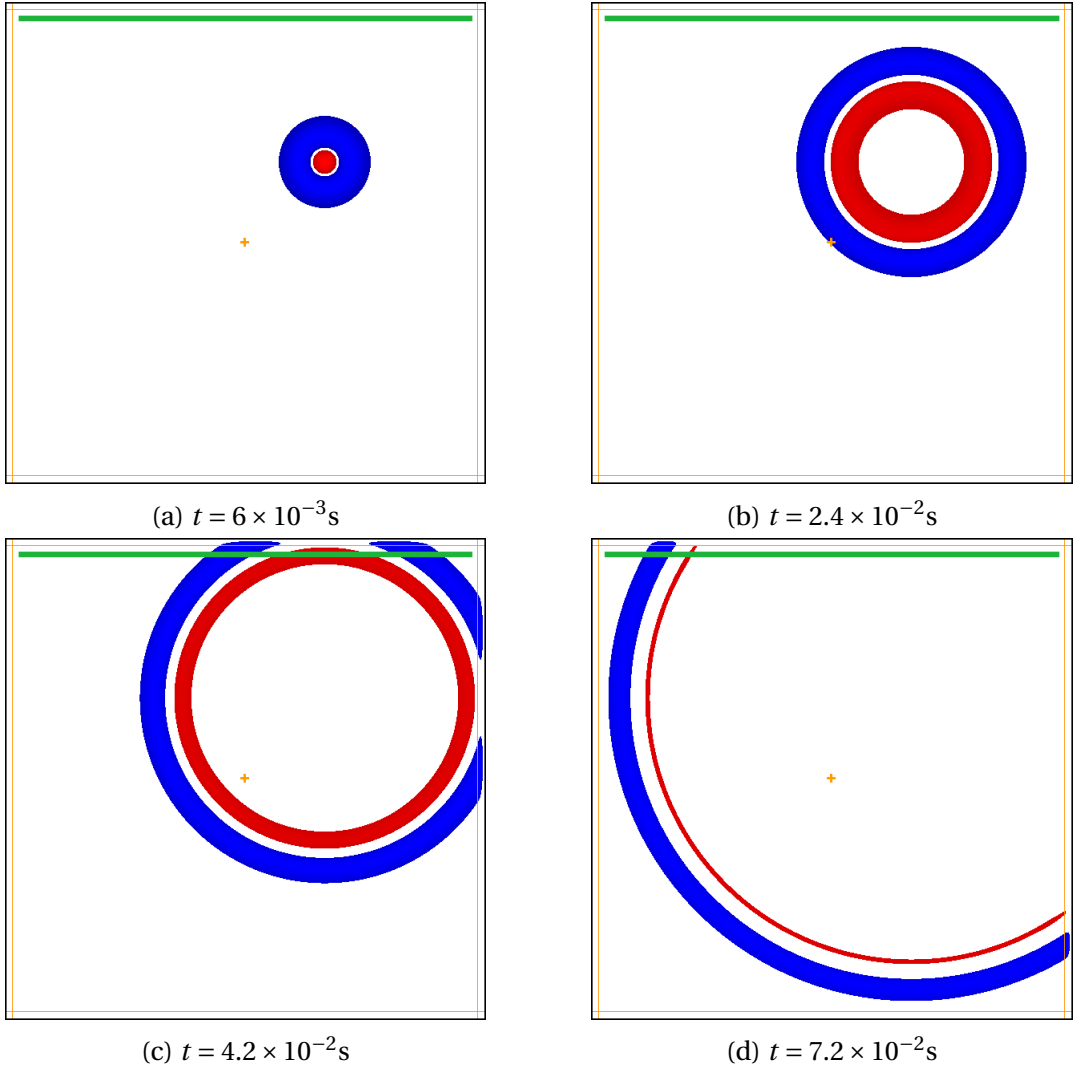


Figure 5.1 – Propagation of solution u to the acoustic equation.

5.2 Source time reversal method for Biot equations

Biot equations with homogeneous initial conditions and a nonzero source in $\mathbb{R}^3 \times (0, T)$ read

$$\begin{cases}
 \rho \partial_t^2 \mathbf{u} + \rho_f \partial_t^2 \mathbf{w} = \operatorname{div} \boldsymbol{\tau}(\mathbf{u}, \mathbf{w}), \\
 \rho_f \partial_t^2 \mathbf{u} + \rho_e \partial_t^2 \mathbf{w} = -\frac{\eta}{\kappa} \partial_t \mathbf{w} - \nabla p(\mathbf{u}, \mathbf{w}) + \mathbf{f}(\mathbf{x}) g(t), \\
 \boldsymbol{\tau} = (\lambda \operatorname{div} \mathbf{u} + C \operatorname{div} \mathbf{w}) \mathbf{I} + G(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \\
 -p = C \operatorname{div} \mathbf{u} + M \operatorname{div} \mathbf{w}, \\
 \mathbf{u}(\mathbf{x}, 0) = \partial_t \mathbf{u}(\mathbf{x}, 0) = 0, \\
 \mathbf{w}(\mathbf{x}, 0) = \partial_t \mathbf{w}(\mathbf{x}, 0) = 0.
 \end{cases} \quad (5.9)$$

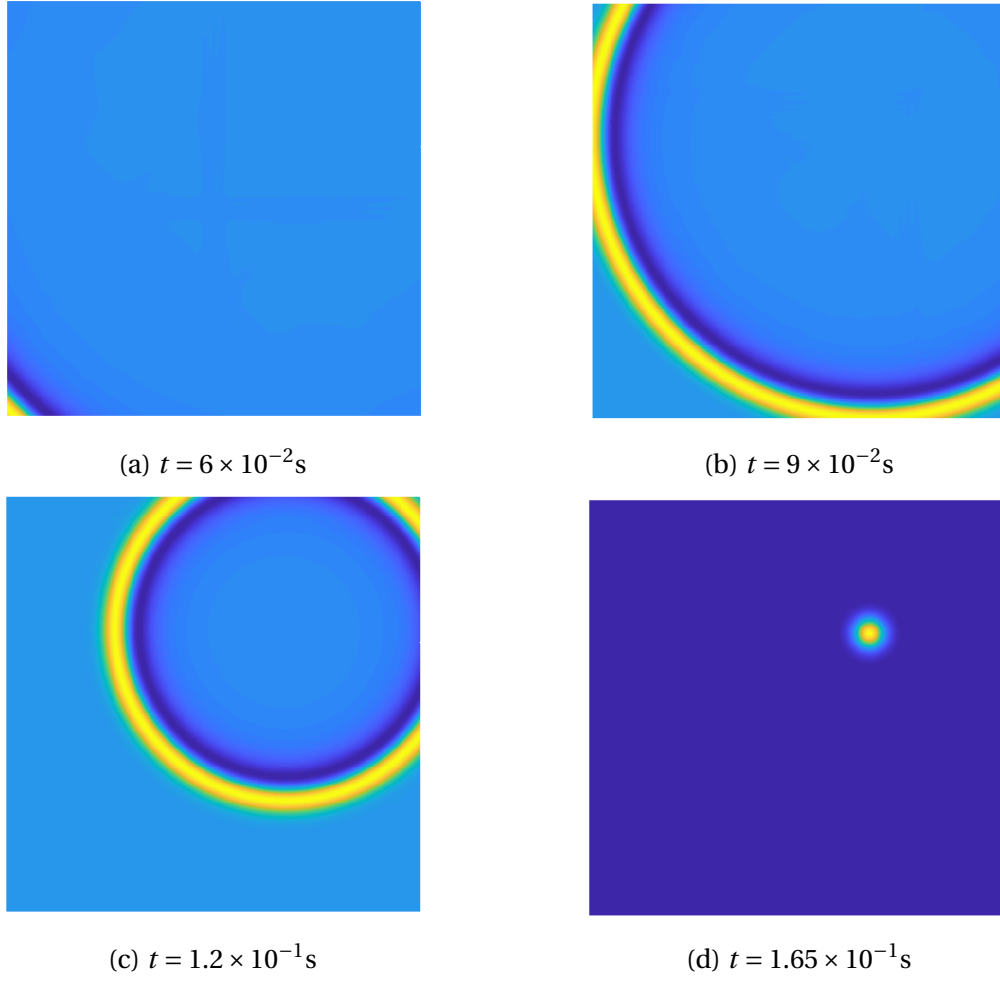


Figure 5.2 – Time reversal process of the acoustic equation.

Similar to the acoustic equation, we will transform this problem into a non-homogeneous initial value problem without a source. Let us consider

$$\begin{cases} \rho \partial_t^2 \mathbf{v} + \rho_f \partial_t^2 \mathbf{q} = \text{div } \boldsymbol{\tau}(\mathbf{v}, \mathbf{q}), \\ \rho_f \partial_t^2 \mathbf{v} + \rho_e \partial_t^2 \mathbf{q} = -\frac{\eta}{\kappa} \partial_t \mathbf{q} - \nabla p(\mathbf{v}, \mathbf{q}), \\ \boldsymbol{\tau}(\mathbf{v}, \mathbf{q}) = (\lambda \text{ div } \mathbf{v} + C \text{ div } \mathbf{q}) \mathbf{I} + G(\nabla \mathbf{v} + \nabla \mathbf{v}^T), \\ -p(\mathbf{v}, \mathbf{q}) = C \text{ div } \mathbf{v} + M \text{ div } \mathbf{q}, \\ \mathbf{v}(\mathbf{x}, 0) = 0, \quad \partial_t \mathbf{v}(\mathbf{x}, 0) = \alpha(\mathbf{x}) \mathbf{f}(\mathbf{x}), \\ \mathbf{q}(\mathbf{x}, 0) = 0, \quad \partial_t \mathbf{q}(\mathbf{x}, 0) = \beta(\mathbf{x}) \mathbf{f}(\mathbf{x}). \end{cases} \quad (5.10)$$

Define

$$\mathbf{u} = (\mathbf{v} * g)(t) = \int_0^t \mathbf{v}(\mathbf{x}, t - \tau) g(\tau) d\tau,$$

$$\mathbf{w} = (\mathbf{q} * g)(t) = \int_0^t \mathbf{q}(\mathbf{x}, t - \tau) g(\tau) d\tau.$$

Then

$$\begin{aligned}
\partial_t \mathbf{u} &= \int_0^t \partial_t \mathbf{v}(\mathbf{x}, t - \tau) g(\tau) d\tau, \\
\partial_t^2 \mathbf{u} &= \alpha(\mathbf{x}) \mathbf{f}(\mathbf{x}) g(t) + \int_0^t \partial_t^2 \mathbf{v}(\mathbf{x}, t - \tau) g(\tau) d\tau, \\
\partial_t \mathbf{w} &= \int_0^t \partial_t \mathbf{q}(\mathbf{x}, t - \tau) g(\tau) d\tau, \\
\partial_t^2 \mathbf{w} &= \beta(\mathbf{x}) \mathbf{f}(\mathbf{x}) g(t) + \int_0^t \partial_t^2 \mathbf{q}(\mathbf{x}, t - \tau) g(\tau) d\tau,
\end{aligned}$$

and therefore

$$\begin{aligned}
\rho \partial_t^2 \mathbf{u} + \rho_f \partial_t^2 \mathbf{w} &= (\rho \alpha + \rho_f \beta) \mathbf{f}(\mathbf{x}) g(t) + \int_0^t (\rho \partial_t^2 \mathbf{v}(\mathbf{x}, t - \tau) + \rho_f \partial_t^2 \mathbf{q}(\mathbf{x}, t - \tau)) g(\tau) d\tau \\
&= (\rho \alpha + \rho_f \beta) \mathbf{f}(\mathbf{x}) g(t) + \int_0^t \operatorname{div} \boldsymbol{\tau}(\mathbf{v}(\mathbf{x}, t - \tau), \mathbf{q}(\mathbf{x}, t - \tau)) g(\tau) d\tau \\
&= (\rho \alpha + \rho_f \beta) \mathbf{f}(\mathbf{x}) g(t) + \operatorname{div} \boldsymbol{\tau}(\mathbf{u}, \mathbf{w}).
\end{aligned}$$

Similarly, we obtain

$$\rho_f \partial_t^2 \mathbf{u} + \rho_e \partial_t^2 \mathbf{w} = -\frac{\eta}{\kappa} \partial_t \mathbf{w} - \nabla p + (\rho_f \alpha + \rho_e \beta) \mathbf{f}(\mathbf{x}) g(t).$$

If we choose $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ satisfying

$$\begin{cases} \rho(\mathbf{x}) \alpha(\mathbf{x}) + \rho_f(\mathbf{x}) \beta(\mathbf{x}) = 0, \\ \rho_f(\mathbf{x}) \alpha(\mathbf{x}) + \rho_e(\mathbf{x}) \beta(\mathbf{x}) = 1, \end{cases}$$

i.e.,

$$\alpha = -\frac{\rho_f}{\rho \rho_e - \rho_f^2} \text{ and } \beta = \frac{\rho}{\rho \rho_e - \rho_f^2},$$

then \mathbf{u}, \mathbf{w} solve (5.9).

Let $\mathbf{m} = \partial_t \mathbf{v}$ and $\mathbf{n} = \partial_t \mathbf{q}$. Then \mathbf{m} and \mathbf{n} satisfy

$$\begin{cases} \rho \partial_t^2 \mathbf{m} + \rho_f \partial_t^2 \mathbf{n} = \operatorname{div} \boldsymbol{\tau}(\mathbf{m}, \mathbf{n}), \\ \rho_f \partial_t^2 \mathbf{m} + \rho_e \partial_t^2 \mathbf{n} = -\frac{\eta}{\kappa} \partial_t \mathbf{m} - \nabla p(\mathbf{m}, \mathbf{n}), \\ \boldsymbol{\tau}(\mathbf{m}, \mathbf{n}) = (\lambda \operatorname{div} \mathbf{m} + C \operatorname{div} \mathbf{n}) \mathbf{I} + G(\nabla \mathbf{m} + \nabla \mathbf{m}^T), \\ -p(\mathbf{m}, \mathbf{n}) = C \operatorname{div} \mathbf{m} + M \operatorname{div} \mathbf{n}, \\ \mathbf{m}(\mathbf{x}, 0) = \alpha(\mathbf{x}) \mathbf{f}(\mathbf{x}), \\ \mathbf{n}(\mathbf{x}, 0) = \beta(\mathbf{x}) \mathbf{f}(\mathbf{x}), \end{cases} \quad (5.11)$$

in $\mathbb{R}^3 \times (0, T)$. Assume that we have measurements of \mathbf{m} and \mathbf{n} on $\partial\Omega \times (0, T)$. Let us choose T large enough such that the wave exits the region Ω . In $\Omega \times (T, 2T)$, we consider

the inverse problem

$$\left\{ \begin{array}{ll} \rho \partial_t^2 \mathbf{m} + \rho_f \partial_t^2 \mathbf{n} = \operatorname{div} \boldsymbol{\tau}(\mathbf{m}, \mathbf{n}) & \text{in } \Omega \times (T, 2T), \\ \rho_f \partial_t^2 \mathbf{m} + \rho_e \partial_t^2 \mathbf{n} = -\frac{\eta}{\kappa} \partial_t \mathbf{m} - \nabla p(\mathbf{m}, \mathbf{n}) & \text{in } \Omega \times (T, 2T), \\ \boldsymbol{\tau}(\mathbf{m}, \mathbf{n}) = (\lambda \operatorname{div} \mathbf{m} + C \operatorname{div} \mathbf{n}) \mathbf{I} + G(\nabla \mathbf{m} + \nabla \mathbf{m}^T) & \text{in } \Omega \times (T, 2T), \\ -p(\mathbf{m}, \mathbf{n}) = C \operatorname{div} \mathbf{m} + M \operatorname{div} \mathbf{n} & \text{in } \Omega \times (T, 2T), \\ \mathbf{m}(\mathbf{x}, T) = \partial_t \mathbf{m}(\mathbf{x}, T) = 0 & \text{in } \Omega, \\ \mathbf{n}(\mathbf{x}, T) = \partial_t \mathbf{n}(\mathbf{x}, T) = 0 & \text{in } \Omega, \\ \mathbf{m}(\mathbf{x}, t) = \mathbf{m}(\mathbf{x}, 2T - t) & \text{on } \partial\Omega \times (T, 2T), \\ \mathbf{n}(\mathbf{x}, t) = \mathbf{n}(\mathbf{x}, 2T - t) & \text{on } \partial\Omega \times (T, 2T). \end{array} \right. \quad (5.12)$$

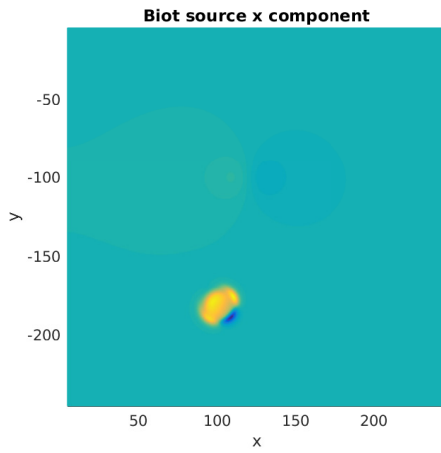
When $\frac{\eta}{\kappa} = 0$, the equation system (5.12) is just the change of variable of the equation system (5.11), and therefore we have

$$\mathbf{m}(\mathbf{x}, 2T) = \alpha(\mathbf{x})\mathbf{f}(\mathbf{x}) \text{ and } \mathbf{n}(\mathbf{x}, 2T) = \beta(\mathbf{x})\mathbf{f}(\mathbf{x}).$$

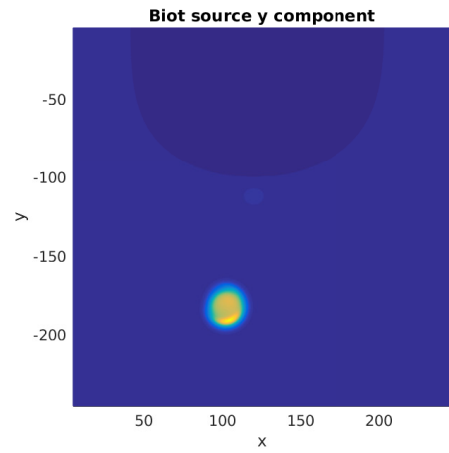
In the case of a nonzero damping, this is not true anymore. In the numerical experiment, we show that the result degenerates with the increasing of the damping term.

Numerical example

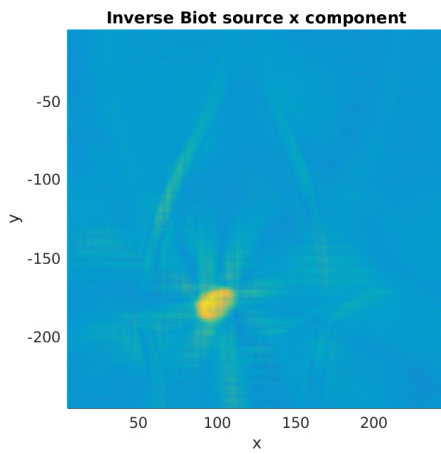
We show two numerical examples of the inverse source for Biot equations. In the first example, Fig 5.3c-5.3d, the damping coefficient is set to be small, with the value $\frac{\eta}{\kappa} = 1.5 \times 10^5$. In the second example, Fig 5.3e-5.3f, the damping coefficient is much larger, with the value 1.5×10^9 . We see that with the increasing of the damping coefficient the inverse result degenerates. This is because the equation system (5.11) and (5.12) are only symmetric if there is no damping term. With large damping term, the time reversal method is not correct.



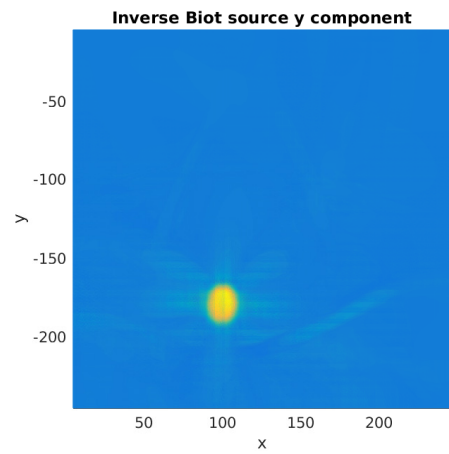
(a) The x component of the source.



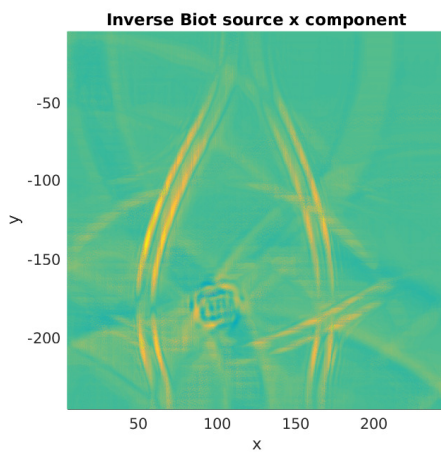
(b) The y component of the source.



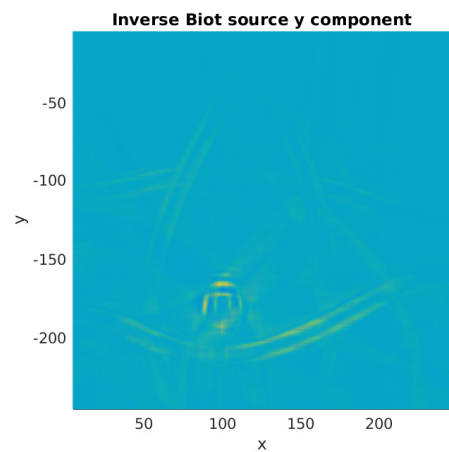
(c) The inversion of the x component of the source with low damping coefficient.



(d) The inversion of the y component of the source with low damping coefficient.



(e) The inversion of the x component of the source with high damping coefficient.



(f) The inversion of the y component of the source with high damping coefficient.

Figure 5.3 – Time reversal process of Biot equations.

Chapter 6

Conclusion

We have a summary of what we have done in this thesis. We proved some existence and uniqueness results for the forward problem. We proved a Fredholm alternative for the frequency domain electrokinetic equations. The existence is equivalent to the uniqueness. It's difficult to study the uniqueness because the parameters are frequency dependent. We don't address this problem in this thesis. In the time domain, we solved the problem in a series of finite dimensional subspaces. Then we show that these approximated solutions converge to the solution of the original equations.

The whole electroseismic equations has so many parameters. It's impossible reconstruct all of them. Here we concentrated on the parameters related to the fluid properties and we assumed that we know all the parameters in Biot equations. We want to reconstruct electric parameters and the coupling coefficient. We derive Carleman estimates for Biot equations and electroseismic equations. Based on the Carleman estimate, we prove the stability of the inverse parameter problem. To simplify the problem, we use the electrostatic equation instead of the full Maxwell equations. Then the inverse problem is decomposed into two steps. The first step is to inverse the source of Biot equations which can be done using the Carleman estimate. The second step is to inverse the parameter of the electrostatic equation. We prove that we can recover the conductivity and coupling coefficient in two dimensions with two sets of measurements.

The staggered finite difference method is used to solve Biot equations. We want to simulate the propagation of seismic waves in unbounded domain. We discuss the application of PML to decrease reflections from the boundary. We only consider the two dimensional problem. The three dimensional problem is much too time consuming and remains a problem.

At last we have shown how to do the inverse source problem for Biot equations numerically. The method we use is the time reversal method. Time reversal method is only valid for waves without damping. If a damping term exists, the inverse process fails.

The inverse source problem for Biot equations with a high damping term is still under research.

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