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Raphaël Belliard

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Université Pierre et Marie Curie
ED564 - Physique en Ile-de-France

GEOMETRY OF INTEGRABLE SYSTEMS

FROM TOPOLOGICAL LAX SYSTEMS TO CONFORMAL
FIELD THEORIES

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PhD thesis under the supervision of Bertrand Eynard^{1,2}

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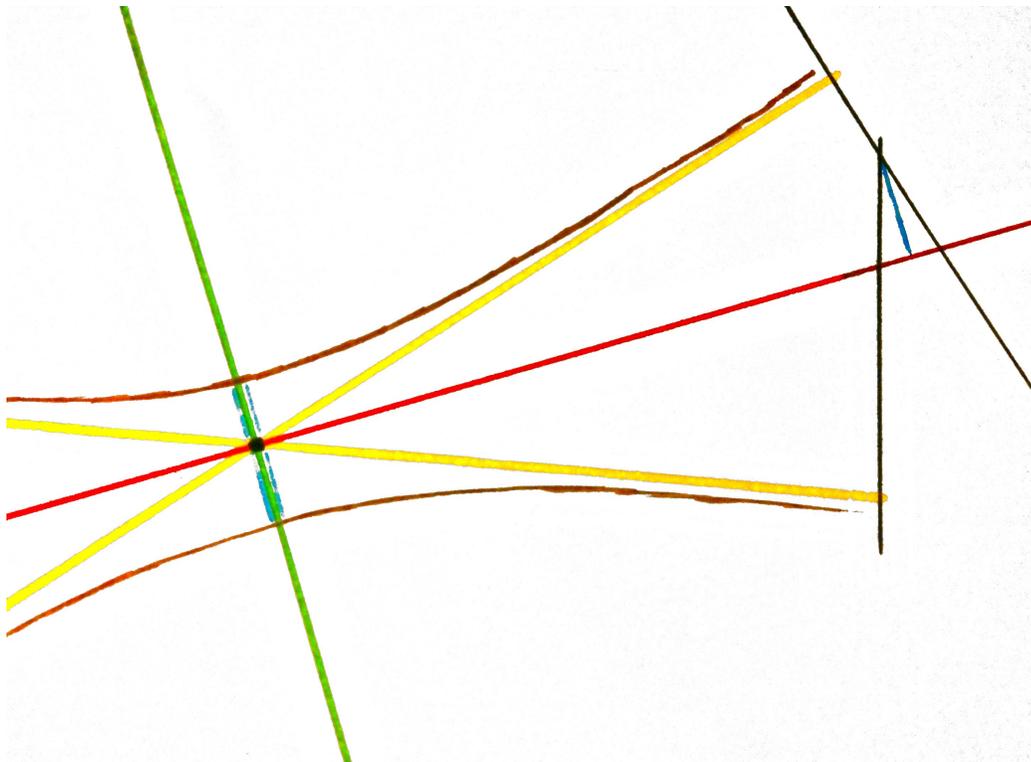


Figure 1: Real section of the complex surface of integrable quantizations, dispersionless limits and perturbative reconstruction.

To my father.

“If a man does not keep pace with his companions, perhaps it is because he hears a different drummer. Let him step to the music which he hears however distant or far away.”

Henry David Thoreau

Abstract

This PhD thesis is about a framework in complex geometry and methods thereof for solving sets of compatible differential equations arising from integrable systems, classical or quantum, in the context of the geometry of moduli spaces of connections over complex curves, or Riemann surfaces.

It is based on the idea in mathematical Physics that integrable systems possess symmetries that impose algebro-differential constraints, so-called loop equations, on the objects of interest (e.g. partition or correlation functions). In turn, we intend to solve these constraints recursively in certain topological regimes using a particular procedure called the topological recursion. Their solutions are in general generating functions of enumerative-geometric quantities. Since they are for the most part determined by the initial data of the recursive process, it realizes in the making an algebro-geometric classification of the family of integrable models under consideration.

Résumé

Cette thèse de doctorat traite d'un cadre en géométrie complexe et de méthodes pouvant y être développées pour résoudre des ensembles d'équations différentielles compatibles venant de systèmes intégrables, classiques ou quantiques, dans le contexte de la géométrie d'espaces de modules de connexions au-dessus de courbes complexes, ou surfaces de Riemann. Elle vient de l'idée en physique mathématique que les symétries des systèmes intégrables imposent aux objets d'intérêt (fonctions de partitions ou de corrélations) des contraintes algebro-différentielles nommées équations de boucles. Le but est par la suite de résoudre ces contraintes par récurrence dans des régimes dits topologiques en utilisant une procédure nommée récurrence topologique. Leurs solutions sont en général les fonctions génératrices de quantités issues de problèmes de géométrie énumérative. Étant principalement déterminées par les conditions initiales de la récurrence, on produit au passage une classification algebro-géométrique de la famille de systèmes intégrables considérée.

Merci à vous !

Je remercie chaleureusement toutes les personnes qui m'ont accompagné, soutenu et aidé de quelque manière que ce soit lors du déroulement de cette thèse.

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Introduction

1 Integrability : a marriage

Theoretical Physics tries to give objective mathematical descriptions of not necessarily visible physical phenomena. To do so, it uses mathematical modeling to logically derive equations that measurable quantities of interest should satisfy. True objectivity not being possible to obtain, the theoretical approach would not have any meaning without its experimental counterpart. Statements or claims will then be called objectively true if they can be checked up to satisfying precisions in experimental devices. Its range of application is limitless and as such, it caused a lot of people to think that a *theory of everything* could be constructed. Unfortunately, simple reasoning gives a heuristic to the contrary. Indeed, a choice of theoretical framework carries its own limits although whichever phenomenon we are trying to study will admit a description in terms of a mathematical model. Knowing what a model under consideration does not describe is of the utmost importance. A counterpart to this remark from logic theory can be found in the mathematician and logician Gödel's completeness and incompleteness theorems that we will loosely state as

- *Any theory admits a model.*
- *Any model admits an assertion that can neither be proved nor disproved*

That being written, we can at most hope for a theory describing objectively every phenomenon we know exists *right now*.

The equations arising from this method appear to be of various really different natures. In particular, they can be exactly solvable, or integrable, namely one can give (an algorithm to compute) an expression of their

solutions, or not. If not, we have other ways to extract information from the model, from approximations and numerics to chaos theory. In this thesis we will study exclusively those problems that are integrable in this given sense.

One of the most important facts about exactly solvable equations arising from theoretical Physics is that they are not at all the generic situation. Indeed as we shall see, they are those whose solutions satisfy many relations. Intuitively, these relations enlarge the vocabulary at our disposition and this is what allows to formulate how to compute the corresponding solutions.

In the early XIXth century, Liouville defined *classical integrable systems* as a subclass of models whose equations of motions are Hamiltonian flows on finite dimensional symplectic manifolds. They are those that possess *enough* independent quantities that are conserved by the dynamics, enough being half the dimension of the underlying space of states accessible to the system, namely the phase space. The reader can find all the needed definitions to understand these statements in the next chapter. This defining theorem then shows that their equations of motions can be solved by means of algebraic operations and integrals (periods). Moreover, by use of so-called generating functions, these equations become those of uniform translations on invariant tori. We will generalize this notion of integrability throughout what follows. Moreover, in 1918, Nøether related smooth symmetries of the corresponding action to conserved quantities. These smooth symmetries are typically Lie group actions leaving the dynamics unchanged.

At this point one should note that in the past, **all major advances in theoretical Physics were always achieved by using and developing contemporary Mathematics.** The study of integrable systems explores this tight relationship in all its glory. Developing such theories requires some of the latest advances in virtually all fields of the mathematical universe and gives rise to beautiful new structures among them. The reason is simple, the quest for objectivity that turned out to be so successful in Physics and the relative truth Mathematics provide are always compatible as being part of the same temporality and thus encompassing similar paradigm. An immediate consequence of this remark is that such paral-

lels can be made outside of those fields, with other sciences and arts. As mentioned, the core aim of the theoretical approach to Physics is to relate our reasoning and logical understanding to our perceptions, experiences and observations. The pre-Socratic Greek philosopher Protagoras defined humanity as “*the measure of all things*”.

Topology, geometry and algebra are those parts of Mathematics that deal respectively with the notions of shapes, distances and angles, and mathematical symbols and the rules to manipulate them. In our way to use mathematical modeling, the objects that are considered in any one of these broad topics are interpreted in relation with the underlying Physics. In particular, all the quantities one wishes to consider in a theory, fixed constants of nature or not, can be treated as variables parameterizing deformations of the studied systems.

This brings us to the notion of deformation of theories, through the variations of their so-called *moduli*, namely their parameters. A natural question is then whether two given theories can be deformed one into the other in this sense. The answer is of course no in general. We know already however from elementary Physics textbooks some instances where this is the case. Let us mention two of those.

- the series RLC circuit is equivalent to the dissipative dynamics of a one dimensional spring, indeed they are both second order linear filters,
- incompressible vorticity in the hydrodynamics of smoke rings is equivalent to the magnetism of toroidal solenoids, indeed they both describe rotational propagation of waves.

In addition, these are examples of integrable models, and as we shall see, the ability to switch from different viewpoints using deformations of geometrical structures is a key feature of integrability.

In 1931, Bethe proposed an ansatz he used to solve exactly for accessible states and their levels of energy in the so-called *one-dimensional anti-ferromagnetic Heisenberg model* [24]. This is a problem treated quantum mechanically to study critical points and phase transitions of certain magnetic systems of spins. Thus was born the notion of quantum integrability.

Since then there has been a tremendous amount of work done to understand the geometry behind the algebraic structures whose presence allowed for exact solvability in quantum problems.

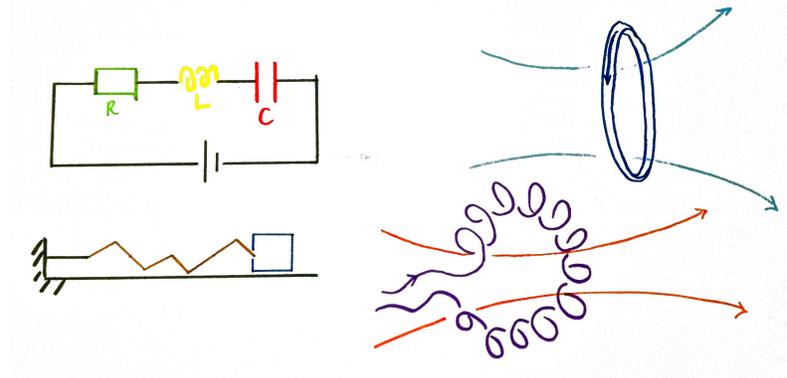


Figure 2: Elementary Physics correspondences.

In the last chapter of this thesis we will be interested in a particular type of quantum integrable system, namely conformal field theory, possessing huge symmetry algebras. Let us therefore review some fundamentals of quantum Physics from an oriented geometric point of view, starting by a bit of history.

2 Quantum Physics and theory

At the dawn of the XXth century, long after the industrial revolution, a short list of problems stood apart from the seemingly global understanding of the world that had been offered by thermodynamics and electromagnetism. These unsolved problems had all been formulated during the previous century

- 1838 : Faraday discovers cathode rays
- 1860 : Kirchhoff states the black-body radiation problem
- 1877 : Boltzmann suggests that the accessible energy levels of a system might be discrete
- 1887 : Hertz discovers the photo-electric effect

In 1900, Planck made the quantum hypothesis, stipulating that energy-radiating systems can be decomposed as fundamental *energy blocks* with energies proportional to the frequencies to which they respectively radiate. If we denote the energy of a building block by ϵ and its frequency by ν , this relation is written

$$\epsilon = h\nu, \quad (2-1)$$

thus defining the universal constant $h \simeq 6.626 \cdot 10^{-34} m^2 kg/s$, called Planck's constant and measured since with great accuracy.

In 1905, as part of what has been called later the *Annus Mirabilis* papers of Einstein, he extended Planck's hypothesis to light by introducing the photon, quantum of light, and explained in this way the photo-electric effect.

- 1913 : Bohr quantizes the angular momentum of electrons in atoms
- 1923 : de Broglie introduces matter-waves
- 1925 : Heisenberg, Born and Jordan develop matrix mechanics, Schrödinger defines wave-mechanics and his famous equation as an approximation to de Broglie's theory, Dirac writes his equation for the dynamics of electrons
- 1932 : von Neumann lays the rigorous mathematical basis for quantum mechanics as the theory of linear operators on Hilbert spaces

The quantum paradigm that came with these discoveries is a change of logic. Indeed, experiments such as *Young's experiment* in 1801, that was generalized as the *double slit experiment* of Davisson and Germer in 1927, show that light and matter (with matter particles as big as the fullerene C_{60} whose molecule contains 60 carbon atoms) behave sometimes as waves and sometimes as particles, according to the observer. The quantum theory therefore redefined the notions of observation and measure in the sense that it identified them as fundamental interactions. On one hand subparts of the system under consideration observe one another while on the second hand, the experimentalists are themselves an interacting part of the system.

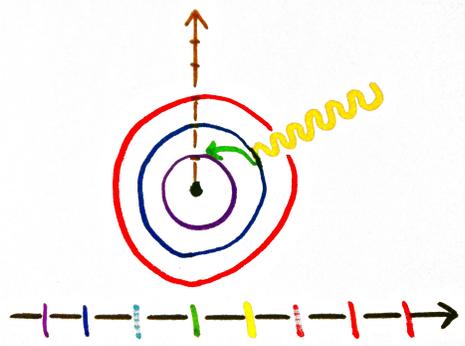


Figure 3: Both energy and angular momentum levels are quantized in an atom.

By a quantum system we will from now on mean any collection of objects, visible or not, that we may want to study from a quantum perspective.

Rather than the usual cat of Schrödinger's thought experiment, let us consider human beings, part of a society where they can sometimes express their opinion through vote. At all time they may have several possibly contradictory opinions but the closer we get to the measure of that opinion the less multiple we appear, until the moment when a choice has to be made, choice from which we will have to evolve once again.

Quantum systems behave as voters : at all time in a superposition of states until one of these states is singled out by an observation or measurement.

As my dear friend Remi Jaoui once told me, we knew human beings were as such, we just had not realized matter as well. At least we had not taken it into account in the way we had tried to describe the world through Physics. Mystics from all around the globe and from as far back in time as one can imagine had already expressed this idea in their own words. The reason is probably that the European middle-age caused for a strong materialistic back-reaction that lasted around three hundred years in science and is still the predominant doctrine *e.g.* in economy or agriculture.

In 1941, Feynman introduced the notion of *path integral*, a way of relating phenomena in quantum systems to their classical counterparts that tells us

that the outcome of a quantum event is an average over all the possible classical outcomes with respect to a (complexified) probability measure.

Let us briefly and schematically describe this procedure. Let us denote by \mathfrak{X} the (possibly infinite dimensional symplectic) phase space, or space of states accessible to a classical system. According to the *principle of least action*, there exists a function

$$\begin{aligned} S_{cl}: \pi'_1(\mathfrak{X}) &\longrightarrow \mathbb{R} \\ \Gamma &\longmapsto S_{cl}[\Gamma] \end{aligned} \quad (2-2)$$

where $\pi'_1(\mathfrak{X})$ is the set of all possible trajectories in \mathfrak{X} , called the action functional and such that the classical trajectories followed by the system are those minimizing this action. They are in particular critical trajectories satisfying

$$\frac{\delta S_{cl}}{\delta \Gamma} = 0 \quad (2-3)$$

Feynman's method of the path integral is then to introduce the wavefunction $\psi: \mathfrak{L} \longrightarrow \mathbb{C}$ defined on a Lagrangian subspace $\mathfrak{L} \subset \mathfrak{X}$ such that for any initial and final states $\varphi_i, \varphi_f \in \mathfrak{L}$, ψ is given in terms of an integral over all possible trajectories $\gamma \subset \mathfrak{L}$, that is over one dimensional paths drawn on the space \mathfrak{L} , starting at φ_i and ending at φ_f , as

$$\psi(\varphi_i, \varphi_f) \stackrel{def}{=} \int_{\substack{\Gamma \in \pi'_1(\mathfrak{L}) \\ \partial \Gamma = \varphi_f - \varphi_i}} [\mathcal{D}\Gamma] e^{i \frac{S_{cl}[\Gamma]}{\hbar}} \quad (2-4)$$

where we introduced Planck's reduced constant $\hbar = \frac{h}{2\pi}$ and the Boltzmann-type weight $e^{i \frac{S_{cl}[\Gamma]}{\hbar}}$ on the space of trajectories. The wavefunction written as such is not, in general, a well-defined mathematical object but we will very soon see an alternative way to construct the physical quantities of interest. For the time being, let us consider these equalities as notations, a formal way to encode the algebraic properties the wavefunction is assumed to have by definition.

We may not be willing to chose an initial and a final state for the system in which case we would define the so-called *partition function* by summing over all possible coinciding initial and final conditions.

$$\mathcal{Z} \stackrel{def}{=} \int_{\mathcal{L}} [\mathcal{D}\varphi] \int_{\substack{\Gamma \in \pi_1(\mathcal{L}) \\ \varphi_f = \varphi_i = \varphi}} [\mathcal{D}\Gamma] e^{i \frac{S_{cl}[\Gamma]}{\hbar}} \quad (2-5)$$

$$= \int_{\Gamma \in \pi_1(\mathcal{L})} [\mathcal{D}\Gamma] e^{i \frac{S_{cl}[\Gamma]}{\hbar}} \quad (2-6)$$

Classically, physical quantities of interest, or observables, are functions $\mathcal{O} : \mathcal{X} \rightarrow \mathbb{R}$. In the quantum theory, their measure becomes uncertain and the previous construction allows to define their vacuum expected value at a state $\varphi \in \mathcal{L}$ as

$$\langle \mathcal{O}(\varphi) \rangle \stackrel{def}{=} \frac{1}{\mathcal{Z}} \int_{\substack{\Gamma \in \pi_1(\mathcal{L}) \\ \varphi \in \Gamma}} [\mathcal{D}\Gamma] e^{i \frac{S_{cl}[\Gamma]}{\hbar}} \mathcal{O}(\varphi) \quad (2-7)$$

and similarly, for any integer $n \in \mathbb{N}^*$, define the correlation of n observables $\mathcal{O}_1, \dots, \mathcal{O}_n : \mathcal{X} \rightarrow \mathbb{R}$ in the classical states $\varphi_1, \dots, \varphi_n \in \mathcal{L}$ by the formula

$$\langle \mathcal{O}_1(\varphi_1) \cdots \mathcal{O}_n(\varphi_n) \rangle \stackrel{def}{=} \frac{1}{\mathcal{Z}} \int_{\substack{\Gamma \in \pi_1(\mathcal{L}) \\ \varphi_1, \dots, \varphi_n \in \Gamma}} [\mathcal{D}\Gamma] e^{i \frac{S_{cl}[\Gamma]}{\hbar}} \mathcal{O}_1(\varphi_1) \cdots \mathcal{O}_n(\varphi_n) \quad (2-8)$$

Solving the theory then amounts to computing all such correlation functions between observables of interest. Let us now investigate a possible way for one to do so.

Assuming the Lagrangian $\mathcal{L} \subset \mathcal{X}$ to be such that

$$|S_{cl}[\Gamma]| \xrightarrow{\Gamma \rightarrow \partial\pi_1(\mathcal{L})} \infty \quad (2-9)$$

when the trajectory Γ goes to the boundary $\partial\pi_1(\mathcal{L})$, yields

$$\int_{\substack{\Gamma \in \pi_1(\mathcal{L}) \\ \varphi_1, \dots, \varphi_n \in \Gamma}} [\mathcal{D}\Gamma] \frac{\delta}{\delta\Gamma} \left(e^{i \frac{S_{cl}[\Gamma]}{\hbar}} \mathcal{O}_1(\varphi_1) \cdots \mathcal{O}_n(\varphi_n) \right) = 0 \quad (2-10)$$

for generic values of all the other arguments and using Leibniz rule, this can be rewritten as the so-called Schwinger-Dyson equations

$$\left\langle \frac{\delta S_{cl}}{\delta \Gamma} \mathcal{O}_1(\varphi_1) \cdots \mathcal{O}_n(\varphi_n) \right\rangle = i\hbar \sum_{j=1}^n \left\langle \mathcal{O}_1(\varphi_1) \cdots \left(\frac{\delta \mathcal{O}_j(\varphi_j)}{\delta \Gamma} \right) \cdots \mathcal{O}_n(\varphi_n) \right\rangle \quad (2-11)$$

The hope is then to be able to find a set of observables $\{\mathcal{O}_j\}_{j \in J}$ such that all possible Schwinger-Dyson equations one can write in this way form a complete set of compatible equations, by which we exactly means that they can be solved together.

Notice that in the limit where $\hbar \rightarrow 0$, one can rewrite the leading order of the Schwinger-Dyson equations as

$$\left\langle \frac{\delta S_{cl}}{\delta \Gamma} \mathcal{O}_1(\varphi_1) \cdots \mathcal{O}_n(\varphi_n) \right\rangle^{(0)} = 0 \quad (2-12)$$

for any number n of observables $\mathcal{O}_1, \dots, \mathcal{O}_n$ in any insertion states $\varphi_1, \dots, \varphi_n \in \mathcal{L}$. This implies that in this limit any physical measure would yield the classical equations of motion $\frac{\delta S_{cl}}{\delta \Gamma} = 0$.

We verify the fundamental fact that Planck's reduced constant \hbar measures quantization and that one recovers the classical Physics in the limit $\hbar \rightarrow 0$. We call it taking the classical limit of a quantum model.

The path integral formulation of a theory is not well-defined in general and a way to circumvent this problem is to define directly a theory by the Schwinger-Dyson equations the correlation functions of interest should satisfy. In this thesis, we will study a certain type of Schwinger-Dyson equations arising from some (classical and quantum) two-dimensional field theories, namely Fuchsian differential systems and Casimir conformal field theories at the classical level and abelian W-symmetric conformal field theories at the quantum level.

3 Categorification and topological string theory

In the process of constructing a theory describing a physical phenomenon, the first two basic questions to be answered are who are the actors at

play and what information exchange defines their interaction. The answers to these questions given in mathematical terms define the nature and the scope of a theory. Let us give a few examples :

- Newtonian Physics : leaving electromagnetism and gravity aside since they cannot be properly described in this framework, it deals with the mechanical contact interactions of a finite given number of massive objects. The information is exchanged in the form of mechanical energy (kinetic or potential).
- Thermodynamics : when the given number of objects considered in newtonian physics becomes large, their individual dynamics becomes irrelevant and even though the objects are of the same nature than before, interacting by colliding one with another, statistical quantities such as pressure, temperature, entropy and free energy emerge as the right variables to consider.

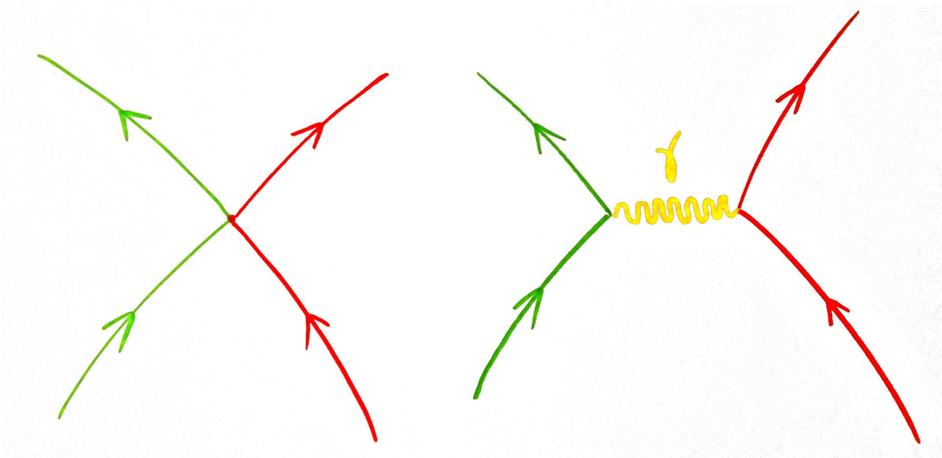


Figure 4: Classical contact interaction vs. quantum long-range correlation.

- Quantum electrodynamics : following Feynman, this theory can be interpreted as describing charged electrons interacting by the exchange of virtual photons. The information exchanged is in this case encoded in the quantized electromagnetic field.

The right mathematical notion to describe this idea for systems at equilibrium is that of cobordism. It is an equivalence relation on the class of compact manifolds with a given dimension, say $d \in \mathbb{N}$. Two such manifolds

are said to be cobordant if and only if their disjoint union is the boundary of a $d + 1$ -dimensional manifold. It is a fundamental equivalence relation. Indeed, the word problem of the fundamental group of topological manifolds of dimensions higher than 4 cannot be solved, hence such manifolds cannot be classified up to homeomorphism. They can however be classified up to cobordism. This is at the root of the functorial definition of topological quantum field theory.

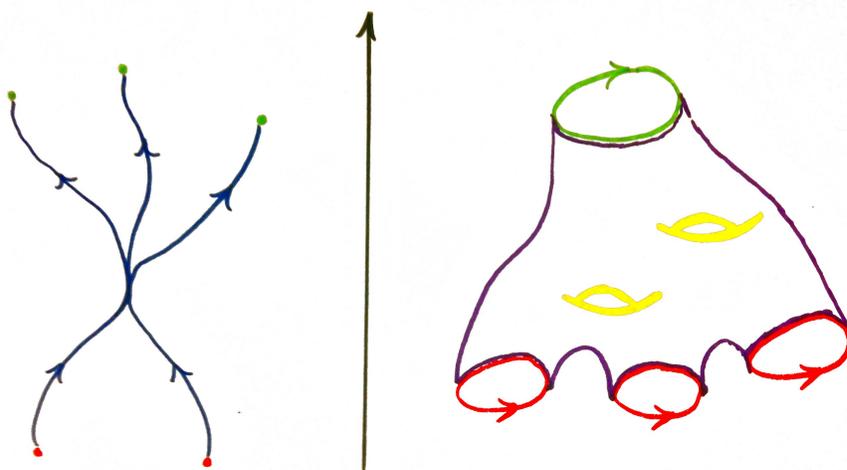


Figure 5: Cobordism in $0 + 1d$ and $1 + 1d$. Time flows upward from incoming to outgoing.

Topological quantum field theories were first defined as those quantum field theories whose partition functions do not depend on the choice of metric on the physical space on which the fields are defined [73]. With an action that typically takes the form

$$S_{cl}[g, \varphi] = \int_M d^{d+1}x \sqrt{g} \mathcal{L}_\varphi(x) \quad (3-1)$$

where φ is now some section of a bundle over a Riemannian $d + 1$ -dimensional manifold (M, g) , \sqrt{g} denotes the square root of the determinant of the metric on M and \mathcal{L}_φ , called the Lagrangian density, is typically a scalar-valued (differential) polynomial expression of φ .

The metric-independence hypothesis means that one can scale the metric $g \mapsto t \cdot g$ for some non-zero complex number $t \in \mathbb{C}^*$ without changing the value of the partition function. Then taking the particular limit $t \rightarrow \infty$ would localize the path integral on its saddle points.

They were later on redefined as functors from cobordism categories to the category of vector spaces satisfying a certain set of axioms we will not be describing here but we refer the reader to [2] for more details.

Let us consider the particular example of 1+1d. In this case, the only one-dimensional closed, compact, connected manifold is the circle \mathbb{S}^1 to which a topological quantum field theory functor associates a vector space that we denote by V . The non-cited axioms imply that V has the structure of a Frobenius algebra (see the next chapter's definitions).

Let us upgrade the definition of a 1+1d topological quantum field theory to that of a so-called *topological conformal field theory* where the topological surfaces used as cobordism are now endowed with conformal structures and are thus Riemann surfaces.

In the spirit of how we have defined physical theories so far, following the mathematical treatment of [37], let us define topological string theory on a compact almost Kähler manifold X , called the target space, as a theory of branes interacting by cobordism. More precisely, the branes are represented by cohomology classes in $H^*(X, \mathbb{Q})$, that we suppose such that

$$H^{\text{odd}}(X, \mathbb{C}) = 0 \quad (3-2)$$

for simplicity. The correlation functions of the theory are called Gromov-Witten invariants and can be defined as intersection numbers of certain cycles on the sequence of *moduli stacks* $\overline{\mathcal{M}}_{g,M}(X, \beta)$ of stable maps of a given degree $\beta \in H_2(X, \mathbb{Z})/T$ (T is the torsion), defined for $g, M \in \mathbb{N}$ such that $2g - 2 + M > 0$ by

$$\overline{\mathcal{M}}_{g,M}(X, \beta) \stackrel{\text{def}}{=} \{f : (\Sigma; z_1, \dots, z_M) \longrightarrow X \mid f_*[\Sigma] = \beta\} / \text{equivalence} \quad (3-3)$$

Σ is an algebraic complex curve of genus g with M pairwise distinct marked points z_1, \dots, z_M . Two such maps are equivalent when they have identical images and are identical on the marked points.

The M -point correlation functions are then defined for any choice of cohomology classes $\varphi_1, \dots, \varphi_M \in H^*(X, \mathbb{Q})$ and integers p_1, \dots, p_M as

$$\langle \tau_{p_1}(\varphi_1) \cdots \tau_{p_M}(\varphi_M) \rangle_{g,\beta} \stackrel{def}{=} \int_{[\overline{\mathcal{M}}_{g,M}(X,\beta)]^{virt}} \text{ev}_1^*(\varphi_1) \psi_1^{p_1} \cdots \text{ev}_M^*(\varphi_M) \psi_M^{p_M} \quad (3-4)$$

where $\text{ev}_i : \overline{\mathcal{M}}_{g,M}(X,\beta) \rightarrow X$ is the evaluation map at the i^{th} marked point, $\psi_i \stackrel{def}{=} c_1(\mathcal{L}_i)$ is the first (and only) Chern class of the tautological line bundle \mathcal{L}_i over $\overline{\mathcal{M}}_{g,M}(X,\beta)$ whose fiber over $f : (\Sigma; z_1, \dots, z_M) \rightarrow X$ is the cotangent plane $T_{z_i}^* \Sigma$. Finally, since the compactification $\overline{\mathcal{M}}_{g,M}(X,\beta)$ is made of strata of various dimensions, one cannot define integration properly in the usual way and we need to introduce the *virtual fundamental class*, an element of the Chow ring

$$[\overline{\mathcal{M}}_{g,M}(X,\beta)]^{virt} \in A_* \left(\overline{\mathcal{M}}_{g,M}(X,\beta) \right) \quad (3-5)$$

$$\dim [\overline{\mathcal{M}}_{g,M}(X,\beta)]^{virt} = (1-g)(\dim X - 3) + M + \langle \beta, c_1(TX) \rangle \quad (3-6)$$

that has the right expected dimension. The correlation functions are defined to vanish whenever the integrated cohomology class do not have matching degree.

If now $(\gamma_1 \stackrel{def}{=} 1, \gamma_2, \dots, \gamma_n)$ is a basis of $H^*(X, \mathbb{C})$, with $\gamma_i \in H^{2q_i}(X, \mathbb{C})$ (in particular $q_1 = 0$ and $q_n = d$), let us define the generating function of correlation functions, or free energy, in genus g to be the series

$$\mathcal{F}_g(\mathbf{t}, \mathbf{q}) \stackrel{def}{=} \sum_{M=0}^{\infty} \sum_{(i_1, p_1), \dots, (i_M, p_M)} \frac{t_{p_1}^{i_1} \cdots t_{p_M}^{i_M}}{M!} \sum_{\beta \in H_2(X, \mathbb{Z})} \langle \tau_{p_1}(\gamma_{i_1}) \cdots \tau_{p_M}(\gamma_{i_M}) \rangle_{g,\beta} q^\beta \quad (3-7)$$

where the sums over i_j 's run from 1 to n , the sums over p_j 's run from 0 to ∞ , $\mathbf{t} = \{t_p^i\}_{i,p}$ are indeterminates and

$$q^\beta \stackrel{def}{=} q_1^{m_1} \cdots q_l^{m_l} \quad (3-8)$$

is an element of the Novikov ring for $\beta = m_1 \beta_1 + \cdots + m_l \beta_l$ in a basis β_1, \dots, β_l of $H_2(X, \mathbb{Z})/T$.

The total Gromov-Witten potential is defined by summing over all genera as

$$\mathcal{F}(\mathbf{t}, q, \varepsilon) \stackrel{\text{def}}{=} \sum_{g=0}^{\infty} \varepsilon^{2g-2} \mathcal{F}_g(\mathbf{t}, q) \quad (3-9)$$

$$\stackrel{\text{def}}{=} \ln \mathfrak{T}(\mathbf{t}, q, \varepsilon) \quad (3-10)$$

where we defined the τ -function \mathfrak{T} by the last equality.

This theory is identified with the topological conformal field theory obtained by choosing the vector space associated to the boundary circles to be the Frobenius algebra $H^*(X, \mathbb{Q})$. Recall that we assumed the odd part of this cohomology ring to vanish otherwise we would have had to take into account its Frobenius super-algebra structure.

In [37] Dubrovin associates to this data a dispersive integrable hierarchy by first using the genus 0 free energy $\mathcal{F}_0(\mathbf{t}, q)$, satisfying the WDVV relations, as prepotential to define a Frobenius manifold. Then, as we shall review in the next chapter, to any Frobenius manifold can be associated a principal hierarchy of compatible hamiltonian equations to which a quasi-triviality transformation can be applied to obtain the corresponding dispersive integrable hierarchy [38].

Its original definition from the A-twisted nonlinear sigma model is far more involved but a localization phenomenon similar to the one described before shows that its partition function can be reduced drastically and eventually coincides with that of the topological conformal field theory just defined.

4 Integrable dispersive field theories

In [64], Lax defined those integrable systems in which the information needed to describe the dynamics with respect to a time evolution parameter $t \in \mathcal{T}$ can be encoded in an object called the Lax pair and denoted $(\mathcal{L}, \mathcal{R})$. The Lax operators $\mathcal{L}(t)$ and $\mathcal{R}(t)$ depend locally holomorphically in the time variable t and they are typically elements of the space $\mathbb{C}[\mathcal{X}] \otimes \mathcal{A}$, where $\mathbb{C}[\mathcal{X}]$ is the coordinate ring of (or ring of functions on) the phase

space \mathfrak{X} and \mathcal{A} is a (non-necessarily finite dimensional) Lie algebra or an associative algebra endowed with the corresponding Lie algebra structure.

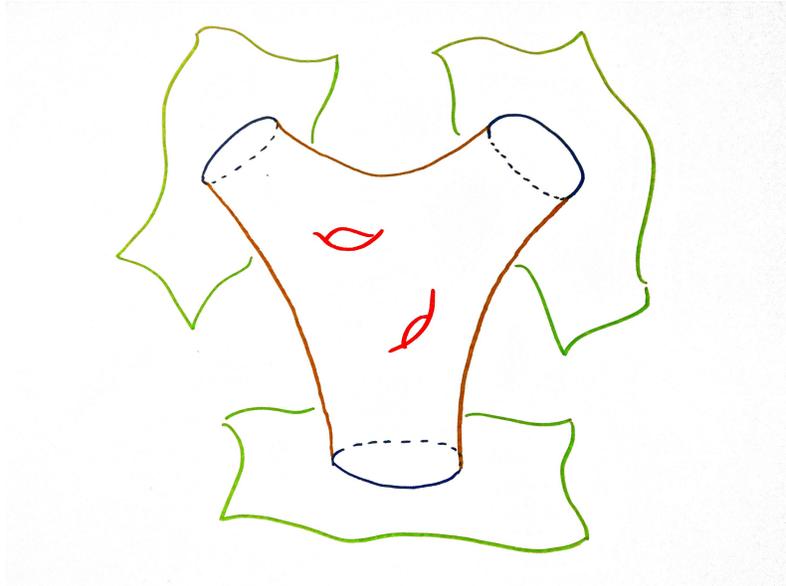


Figure 6: Branes interacting by quantum intersection, or cobordism.

They define a Lax pair whenever the corresponding equations of motions take the form

$$\frac{d}{dt} \mathcal{L}(t) = [\mathcal{R}(t), \mathcal{L}(t)] \quad (4-1)$$

Notice that in an infinite dimensional situation, \mathfrak{X} could be a space of functions in which case $\mathbb{C}[\mathfrak{X}]$ would typically be the corresponding ring of differential polynomials and similarly \mathcal{A} could be a ring of pseudo-differential operators endowed with the Lie algebra structure coming from the fact that it is an associative algebra although in this case \mathcal{L} is assumed to be differential (no negative powers of the formal derivation).

From our perspective, the procedure of the inverse-scattering method then goes as follows : for a given complex number $x \in \text{Spec } \mathcal{L} \subset \mathbb{C}$, consider the eigenvalue equation

$$\mathcal{L} \cdot \psi = x \psi \quad (4-2)$$

The Lax form ensures that the eigenvalues $x \in \text{Spec } \mathcal{L}$ of the Lax operator do not depend on the parameter $t \in \mathcal{T}$. Such evolutions are thus called *isospectral*.

If \mathcal{A} is finite dimensional, ψ will typically take its values in a chosen vector representation of the associative algebra. If it is infinite dimensional [13] ψ will typically be a function acted upon by the pseudo-differential operators.

In any case, one can define a *fundamental matrix solution* Ψ therefore depending on both $t \in \mathcal{T}$ and $x \in \text{Spec } \mathcal{L}$. It is a function on $\mathfrak{X} \times \mathcal{T} \times \mathcal{L}$ valued in G , a compact connected reductive complex Lie group. Notice that if \mathcal{A} is a ring of pseudo-differential operators, then Ψ will also depend on the formal complex parameter on which they act.

We will for simplicity and from now on forget about the \mathfrak{X} part of the objects that were used to define the setup, just remember that the objects at stake may have this additional dependence. It brings no loss of generality as the whole construction is done at fixed value of the $\mathbb{C}[\mathfrak{X}]$ part and as such it can be thought of as evaluated at fixed given points in \mathfrak{X} .

The topology of $\text{Spec } \mathcal{L}$ depends on the choice of Lax operator \mathcal{L} and it carries a complex structure with respect to which $\Psi(\cdot, t)$ is holomorphic at fixed $t \in \mathcal{T}$. $\text{Spec } \mathcal{L}$ together with this complex structure defines a Riemann surface that we will denote $\overset{o}{\Sigma}$. This will be our notation throughout the text for the so-called *base curve*.

In turn, one can choose a t -dependent family of connections $\{\nabla(t)\}_{t \in \mathcal{T}}$ acting only in the $x \in \overset{o}{\Sigma}$ direction in a principal G -bundle over $\overset{o}{\Sigma}$ and such that $\nabla(t)\Psi(x, t) = 0$, where we now included the x dependence in the notation for Ψ . Locally these connections take the usual form

$$\nabla(t) \sim d_x - \phi(x, t) \tag{4-3}$$

By studying the connection ∇ with respect to a (once and for all) fixed reference connection ∇_0 , one can associate [9] a sequence $\{W_n\}_{n \in \mathbb{N}^*}$ of so-called correlators that were shown to satisfy constraints called *loop equations*, generalizing the constructions of [15], [16]. As we will recall, they take the form of symmetrizations of the W_n 's, with respect to Casimir elements of the Lie algebra \mathfrak{g} , that end up having nice analytic properties in

the variable $x \in \overset{o}{\Sigma}$. It was conjectured that when a set of hypothesis called the *Topological Type* are satisfied (it includes the fact that the correlators admit topological expansions $W_n = \sum_{g \geq 0} \varepsilon^{2g-2+n} \omega_{g,n}$ in terms of a formal *small* parameter $\varepsilon \neq 0$), then the topological recursion procedure [43], [15] allows to reconstruct perturbatively these expansions. It is a recursive algorithm to compute the $\omega_{g,n}$'s from complex geometry of a covering space of $\overset{o}{\Sigma}$ called the *spectral curve* associated to the $\varepsilon \rightarrow 0$ limit of the setup. It was put in practice and proved in [57], [58], [10] for special cases (in particular $\overset{o}{\Sigma}$ had genus 0 and \mathfrak{g} was a matrix Lie algebra).

5 On the use of conformal field theory

A phase transition in a physical system is a transformation of its macroscopic properties due the variation of one of its parameters across a threshold. Phase transitions can arise from classical statistical systems in the thermodynamic limit as well as from quantum systems at zero temperature. Different phases of a system do not always have different symmetries but the converse is somehow true, a change of symmetry properties is a phase transition.

For the classical case of statistical systems admitting thermal fluctuations, phase transitions can be obtained by crossing a certain critical value with the temperature. They occur in the thermodynamic limit (collective effects) when the free energy (first order) or its derivative (second order) has a singularity. If a symmetry breaking occurs, the most symmetric phase is often the stable one at high temperature.

At the singularity, the renormalization procedure averages over the small scales to yield *universal exponents*. They are universal in the sense that they do not depend on the microscopic structure over which the averaging was done. Finite size effects can however change the macroscopic behavior even at critical points. Let us assume their absence.

Second order phase transitions exhibit scale invariance at the critical points where the transition occurs. Which can often be lifted to a full conformal invariance. Let us mention the example of critical opalescence of water at its boiling point where one can easily see that there would be

air bubbles of all sizes if the water recipient did not introduce an upper length scale.

Quantum phases are quantum states of matter at zero temperature. They might be described by some microscopic (short-range) Hamiltonian and two such Hamiltonians will belong to the same class if they yield the same macroscopic properties for the system, assumed to be described by some effective quantum field theory.

The quantum fluctuations then might depend on some extra parameters of the problem and therefore phase transitions can occur.

Take for example the quantum Hall effect. One can show that its large scale properties are well described by Chern-Simons theory in three dimensions which is known to be equivalent to a conformal field theory located at the boundary of its physical space, namely a WZW model.

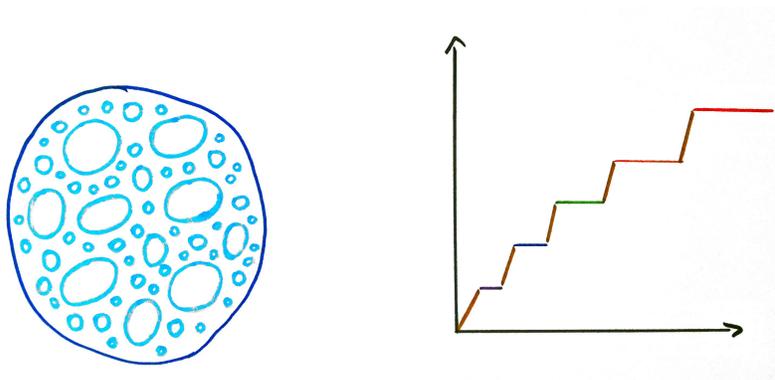


Figure 7: Classical opalescence of water and quantum Hall plateaux.

In this thesis we will not be dealing with what would happen away from equilibrium. Indeed, one might expect conformal symmetry to be broken in that situation, in the presence for example of disorder introducing its own length scale.

Last but not least, let us mention that conformal field theory appears naturally in two ways in string theory, the first one being through the conformal symmetry of the world-sheet of the string and the second one being through holography and the fact that, similarly as was mentioned in the case of Chern-Simons theory and a WZW model on its boundary, quantum gravity theories are expected to be dual to (super-) conformal field theories.

6 Plan of the thesis

- In chapter 1, we will recall some mathematical notions that will be need either for technical support or for inspiration in order to sometimes achieve their generalization. We will describe some features of the complex geometry first of Riemann surfaces and then of Kähler manifolds, recall some definitions and results of the theory of (Kac-Moody) Lie algebras, introduce the notion of Frobenius manifolds and their relation with integrable hierarchies [38], define the Hitchin integrable system and its realization of the non-abelian Hodge correspondence and we will then end the chapter by some notions of symplectic geometry and the corresponding methods of quantization.
- In chapter 2, we will study the geometry of the Fuchsian (integrable) system, defined as a fibration of the moduli space of Fuchsian differential systems over Riemann surfaces $(P \rightarrow \overset{o}{\Sigma}, \nabla) \in \mathcal{M}_{Fuchs}$. To do so, we will first construct the non-perturbative spectral curve $\widehat{\Sigma}$ associated to a Fuchsian differential system, the correlators W_n 's satisfying the loop equations and the corresponding homology of cycles $\widehat{\mathcal{H}}_1$. It will be related to deformations $\delta \in T^* \mathcal{M}_{Fuchs}$ and the underlying special geometry gives a natural conjectural definition of a non-perturbative τ -function of the theory and we will mention how it should relate to enumerative geometry. What is understood however is the perturbative reconstruction of the correlators by the topological recursion in a topological regime. This regime consists in promoting the considered connections to ε -connections ∇_ε for some *small* complex parameter $\varepsilon \in \mathbb{C}^*$ and study the WKB asymptotics of the construction when $\varepsilon \rightarrow 0$. There, a cameral curve $\Sigma_{\mathcal{H}(\Phi^{(0)})}$ emerges, associated to the Higgs field obtained by the limit $\Phi^{(0)} = \lim_{\varepsilon \rightarrow 0} (\varepsilon \nabla_0 - \nabla_\varepsilon)$. The cameral geometry allows to define the *cameral curve topological recursion* reconstruction procedure. This gives a natural scheme to compute perturbatively the generating functions for derivatives of the τ -function of any integrable hierarchy given in Lax form [8]. We apply this scheme to propose a spectral curve for the KdV hierarchy. We end the chapter by studying the Topological Type property ensuring that one can

reconstruct from the usual topological recursion [43] and apply it to the six Painlevé equations and the (p, q) minimal models of Liouville gravity [10]

- In chapter 3, we turn towards defining the quantum geometry of conformal field theories with extended algebras of symmetry given by W-algebras. We start by recalling basic definitions and facts about conformal field theories with W-algebra symmetry defined from a chiral spin-one current \mathbf{J} valued in a dual \mathfrak{g}^* of a Lie algebra \mathfrak{g} of ADE type, including their operator product expansions and Ward identities. We then define the corresponding quantum spectral curve $\widehat{\mathcal{E}}$ and show that the *usual* topological recursion procedure allows to reconstruct the correlation functions *with* a non-zero number of currents inserted [5]. We then see that what changes is actually that the initial data of the recursion has to be defined from the quantum spectral curve. We then introduce a special geometry ansatz (Seiberg-Witten relations) [35] to propose a scheme to reconstruct perturbatively chiral correlation functions of the theory *without* insertions of currents, that is M -point correlation functions of Toda lattice quantum field theory.
- We will then conclude and give a list of possible projects for the future.

To conclude this introduction, note that this text is based on the published paper [10], the pre-prints [9] and [11] and the articles still in preparation [12], [8], [13], [5] and [6] that will hopefully be available soon. [9], [10] and [11] are collected at the end of the text after the bibliography.

Chapter 1

Mathematical preliminaries

In this section we will present in an informal way some needed mathematical material. It is not needed in the sense that we will use all of the following definitions and results in the developments of this work but rather because it allows one to get much more context around the objects at stake. Almost no proof will be given since everything can be found quite easily in textbooks or in the references that we give here. A few of them will be given nevertheless when they contain useful insights for what will follow.

1 Complex geometry

1.1 Complex curves

Definition 1.1 *Riemann surfaces*

A Riemann surface Σ is a complex algebraic curve. It is defined as the zero locus of a polynomial expression of two variables $P \in \mathbb{C}[T_1, T_2]$, that is

$$\Sigma = \{(x, y) \in \mathbb{C}^2 \mid P(x, y) \stackrel{\text{def}}{=} \sum_{i,j} P_{i,j} x^i y^j = 0\} \quad (1-1)$$

immersed in \mathbb{C}^2 .

Proposition 1.2 *Space of holomorphic functions*

The only holomorphic functions on a Riemann surface are the constants.



Figure 1.1: A compact Riemann surface.

As a real two-dimensional surface, the topology of the curve is classified by its genus, an integer $g \in \mathbb{N}$ such that its first homology group is given by

$$H_1(\Sigma, \mathbb{Z}) = \mathbb{Z}^{2g} \quad (1-2)$$

Fundamental theorems

The Riemann-Hurwitz formula helps computing genera of covers.

Theorem 1.3 *Riemann-Hurwitz*

Let $\Sigma \rightarrow \overset{\circ}{\Sigma}$ be a holomorphic map. It realizes a finite branched cover. Let $d \in \mathbb{N}^*$ be the number of sheets of the cover and $\overset{\circ}{g}, g \in \mathbb{N}$ be the genera of $\overset{\circ}{\Sigma}$ and Σ respectively. They are related by

$$2g - 2 = d(2\overset{\circ}{g} - 2) + B \quad (1-3)$$

where B is the total index of the branch points, that is the sum of the number of sheets coalescing at each branch point minus the number of branch points itself.

Definition 1.4 *Divisors on curves*

A divisor on the curve Σ is a formal sum of points with multiplicities,

$$D = n_1 P_1 + \cdots + n_r P_r \quad (1-4)$$

$n_1, \dots, n_r \in \mathbb{Z}$, $P_1, \dots, P_r \in \Sigma$. Its degree is the number $\deg(D) = \sum_i n_i$.

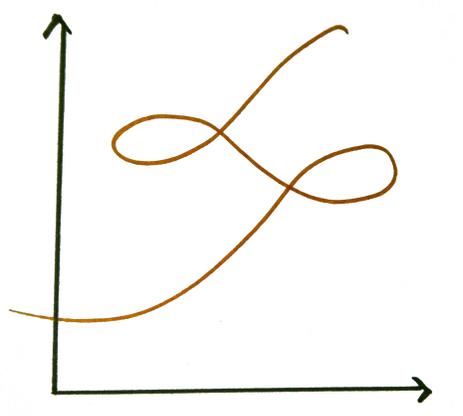


Figure 1.2: A branched covering.

For any function f on Σ , denote by (f) the divisor of its poles and zeroes counted with multiplicities (positive for zeroes and negative for poles). For any divisor D on Σ , let $\mathcal{M}^0(D)$ (resp. $\mathcal{M}^1(D)$) denote the space of meromorphic functions f (resp. forms ω) whose poles are at most the ones specified by D and whose zeroes are at least the ones specified by D (denoted $D \leq (f)$ resp. $D \leq (\omega)$).

The Riemann-Roch formula helps computing dimensions of spaces of meromorphic functions and forms with given zeroes and poles on a Riemann surface in terms of its genus.

Theorem 1.5 *Riemann-Roch*

Let D be a divisor on a Riemann surface of genus $g \in \mathbb{N}^$.*

$$\dim \mathcal{M}^0(-D) = \dim \mathcal{M}^1(D) + \deg(D) - g + 1 \quad (1-5)$$

Example 1.6 *Space of holomorphic differentials*

When $D = 0$, since the only holomorphic functions on Σ are constants, we get that the space of holomorphic differentials on Σ has dimension equal to g .

Theorem 1.7 *Riemann's bilinear identity*

Let $\{\mathcal{A}_i\}_{i=1,\dots,2g}$ be an integer basis of $H_1(\Sigma, \mathbb{Z})$ with intersection product defined as

$$\mathcal{A}_i \cap \mathcal{A}_j \stackrel{\text{def}}{=} I_{i,j} \quad (1-6)$$

for any $i, j \in \{1, \dots, 2g\}$. Then for any holomorphic one-forms $\omega, \omega' \in H^1(\Sigma, \mathcal{O}_\Sigma)$, and any generic point $z_0 \in \Sigma$,

$$\begin{aligned} \sum_{x \in p(\omega)} \operatorname{Res}_{z=x} \left(\omega(z) \int_{z_0}^z \omega' \right) - \sum_{x \in p(\omega')} \operatorname{Res}_{z=x} \left(\omega'(z) \int_{z_0}^z \omega \right) \\ = \frac{1}{2\pi i} \sum_{i,j=1}^{2g} \left(\oint_{\mathcal{A}_i} \omega \right) (I^{-1})_{i,j} \left(\oint_{\mathcal{A}_j} \omega' \right) \end{aligned} \quad (1-7)$$

where we introduced the notation $p(\omega)$ (respectively $p(\omega')$) for the set of poles of ω (respectively ω').

Newton's polygon**Definition 1.8** *Newton's polygon*

Newton's polygon $N(P)$ associated to the Riemann surface defined by a polynomial $P \in \mathbb{C}[T_1, T_2]$ is the convex hull of the set $\{(i, j) \in \mathbb{Z}^2 \mid P_{i,j} \neq 0\}$.

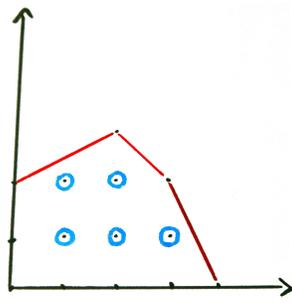


Figure 1.3: The Newton polygon of $Y^3 X^2 + (1 + Y + Y^2)(1 + X + X^2 + X^3) + X^4$.

Proposition 1.9 *Critical exponents*

In a local coordinate z , generic asymptotic directions of the curve immersed in \mathbb{C}^2 are of the form $(x \sim_\infty z^p, y \sim_\infty z^q)$ with $p, q \in \mathbb{N}$ such that $-\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ is the slope of one of the edges of $N(P)$.

proof:

In a local coordinate in an asymptotic direction of the curve,

$$\begin{cases} x \sim_{\infty} z^p \\ y \sim_{\infty} z^q \end{cases} \quad (1-8)$$

for some integers $p, q \in \mathbb{N}$. In this vicinity, introducing the number

$$m_{p,q} = \max_{(i,j)} \{ip + jq\}, \quad (1-9)$$

the leading term of the equation defining the curve reads

$$\sum_{\{(i,j) \in N(P) \mid ip + jq = m_{p,q}\}} P_{i,j} = 0 \quad (1-10)$$

The sum can't reduce to one single term otherwise it would contradict the fact that the corresponding (i, j) pair belongs to $N(P)$. Therefore, the sum generically contains two terms thus yielding that the straight line $\{(i, j) \in \mathbb{Z}^2 \mid m_{p,q} = ip + jq\}$ is tangent to $N(P)$, assertion that contains the wanted result. ■

The genus of a Riemann surface can in most case be computed from its Newton's polygon but in general, the number of interior points of Newton's polygon only gives an upper bound to the genus through the following

Proposition 1.10 $g \leq \# \overset{o}{N}(P)$

with equality if and only if the singularities of the projectivization of the curve in \mathbb{P}^2 are non-degenerate and located at (001) , (010) , (100) .

proof:

To prove this inequality we will exhibit a generating family of $H^1(\Sigma, \mathcal{O}_{\Sigma})$ consisting of $\# \overset{o}{N}(P)$ holomorphic one-forms.

By differentiating the defining equation along the curve we get the equality

$$P'_x dx + P'_y dy = 0 \quad (1-11)$$

but generically we can't have P'_x and P'_y vanishing at the same point (which would correspond to a cusp singularity) so that $\frac{dx}{P'_y} = -\frac{dy}{P'_x}$ cannot have a pole at any zero of P'_x (or P'_y either by symmetry) and therefore is a holomorphic form on Σ . This is the starting point and now, for any $(i, j) \in \mathbb{Z}^2$, define the one-form $\omega_{i,j}$ on Σ by $\omega_{i,j}(x, y) = x^i y^j \frac{dx}{P'_y(x,y)}$. By the previous argument, this differential form is regular everywhere except maybe at infinity in an asymptotic direction. In such a direction ($x \sim_\infty z^p, y \sim_\infty z^q$), in a local coordinate $z \in \mathbb{C}$ and with the notations of the previous proposition,

$$\omega_{i,j}(x, y) \underset{z \rightarrow \infty}{\sim} z^{i p + j q} \frac{z^{p-1} dz}{\sum_{k,l} l P_{k,l} z^{k p + (l-1) q}} \quad (1-12)$$

$$\underset{z \rightarrow \infty}{\sim} \frac{z^{(i+1)p + j q - 1} dz}{z^{m_{p,q} - q} \sum_{k p + l q = m_{p,q}} l P_{k,l}} \quad (1-13)$$

As seen before, there are generically two distinct terms (k, l) and (k', l') in the sum of the denominator for which we know that $P_{k,l} + P_{k',l'} = 0$. The condition for $l P_{k,l} + l' P_{k',l'}$ to vanish as well is therefore $l = l'$ which is generically equivalent to $k = k'$ and is thus contradicting the fact that there are two distinct terms. Therefore $l P_{k,l} + l' P_{k',l'} \neq 0$ and

$$\omega_{i,j}(x, y) \underset{z \rightarrow \infty}{\sim} z^{(i+1)p + (j+1)q - m_{p,q} - 1} dz \quad (1-14)$$

$$\underset{z' \rightarrow 0}{\sim} (z')^{m_{p,q} - (i+1)p - (j+1)q - 1} dz' \quad (1-15)$$

where we have changed the local coordinate to $z' = \frac{1}{z}$. From this last expression we can conclude that $\omega_{i,j}$ is holomorphic on Σ if and only if $(i+1)p + (j+1)q < m_{p,q}$, that is $(i+1, j+1) \in \overset{o}{N}(P)$ which concludes the proof. ■

Jacobi variety and theta-functions

Definition 1.11 *Symplectic basis of cycles*

A basis $\{\mathcal{A}_i, \mathcal{B}_i\}_{i=1, \dots, g}$ of $H_1(\Sigma, \mathbb{Z})$ such that the intersection product is given for any pair of indices $i, j \in \{1, \dots, g\}$ by

$$\mathcal{A}_i \cap \mathcal{A}_j = 0, \quad \mathcal{B}_i \cap \mathcal{B}_j = 0, \quad \text{and} \quad \mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}, \quad (1-16)$$

is called a symplectic basis or a Torelli marking of Σ . It admits a dual basis of $H^1(\Sigma, \mathcal{O}_\Sigma)$, $\{\omega_i\}_{i=1,\dots,g}$ normalized on \mathcal{A} -cycles, that is for any $i, j \in \{1, \dots, g\}$,

$$\oint_{\mathcal{A}_i} \omega_j = \delta_{i,j}, \quad \oint_{\mathcal{B}_i} \omega_j \stackrel{\text{def}}{=} \tau_{i,j} \quad (1-17)$$

where we introduced the period matrix τ .

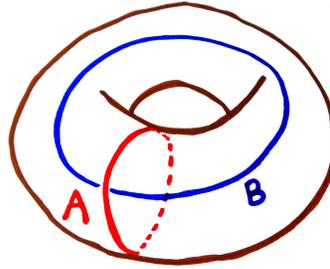


Figure 1.4: A symplectic basis of cycles on a torus.

Definition 1.12 *Jacobian of curves*

The Jacobian variety of Σ is defined as the g -dimensional torus defined by the quotient

$$Jac(\Sigma) \stackrel{\text{def}}{=} \mathbb{C}^g / (\mathbb{Z}^g + \tau \cdot \mathbb{Z}^g) \quad (1-18)$$

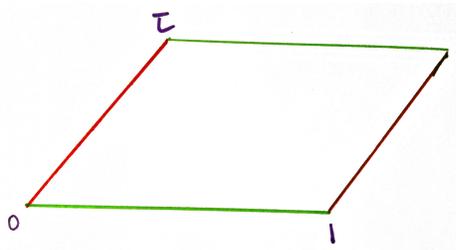


Figure 1.5: The Jacobian of a genus 1 Riemann surface.

We can embed the Riemann surface Σ into its Jacobian $Jac(\Sigma)$ by

Definition 1.13 *The Abel map*

The Abel map at a point $z_0 \in \Sigma$ is the embedding defined by

$$\begin{aligned} \mathcal{A}_{z_0}: \Sigma &\longrightarrow Jac(\Sigma) \\ z &\longmapsto \mathcal{A}_{z_0}(z) = \left(\int_{z_0}^z \omega_1, \dots, \int_{z_0}^z \omega_g \right) \end{aligned} \quad (1-19)$$

Changing the base point z_0 amounts to a translation in the Jacobian.

Proposition 1.14 *Matrix of periods*

τ is symmetric and the real matrix $\mathcal{I}m(\tau)$ is positive definite.

This allows for the definition of the following fundamental objects

Definition 1.15 *Riemann theta-functions*

It is the locally analytic function Θ defined for any $u \in \mathbb{C}^g$ by

$$\Theta(u; \tau) \stackrel{\text{def}}{=} \sum_{m \in \mathbb{Z}^g} e^{2\pi i \langle m, u \rangle + \pi i \langle \tau m, m \rangle} \quad (1-20)$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical scalar product on \mathbb{C}^g .

It has simple automorphic properties with respect to the period lattice of the Riemann surface.

Property 1.16 *Automorphic of theta-functions*

For any $u \in \mathbb{C}^g$ and $l \in \mathbb{Z}^g$,

$$\Theta(u + l; \tau) = \Theta(u; \tau) \quad (1-21)$$

$$\Theta(u + \tau \cdot l; \tau) = e^{-\pi i \langle \tau l, l \rangle - 2\pi i \langle l, u \rangle} \Theta(u; \tau) \quad (1-22)$$

Definition 1.17 *Theta divisors*

The divisor of the theta-function is the zero locus of the theta-function in the Jacobian of the Riemann surface Σ . It is a well defined $g-1$ dimensional subvariety in $Jac(\Sigma)$ thanks to the automorphic properties.

Definition 1.18 *Theta characteristics*

It is a point $\chi \in \mathbb{C}^g$ such that $2\chi = a + \tau \cdot b \in \mathbb{Z}^g + \tau \cdot \mathbb{Z}^g$. It is said to be odd if and only if $\langle a, b \rangle$ is odd. It is otherwise said to be even.

Remark 1.19 *A theta characteristic endows a spin structure on Σ , namely a line bundle whose tensor square is the canonical bundle of the curve. Indeed, χ has vanishing class in $\text{Jac}(\Sigma)$ which is an abelian group isomorphic to the Picard group of line bundles. This correspondence and the integrality of the characteristic yield the spin structure.*

Definition 1.20 *Siegel theta-functions*

Let $\chi = \frac{a}{2} + \tau \cdot \frac{b}{2}$ be a odd theta characteristic on Σ . The corresponding Siegel theta-function is the analytic function Θ_χ defined for any $u \in \mathbb{C}^g$ by

$$\Theta_\chi(u; \tau) \stackrel{\text{def}}{=} \sum_{m \in \mathbb{Z}^g} e^{2\pi i \langle m + \frac{a}{2}, u + \frac{b}{2} \rangle + \pi i \langle m + \frac{a}{2}, \tau \cdot (m + \frac{b}{2}) \rangle} \quad (1-23)$$

Remark 1.21 *Regularity of odd theta-characteristics*

The theta-characteristics such that Θ_χ does not vanish identically are called regular theta-characteristics. We will assume from now on χ to be regular.

Prime forms and twists

Let $z_0 \in \Sigma$ be a point on the Riemann surface and let ϕ_{z_0} be the map that sends any point $z \in \Sigma$ to the complex number $\phi_{z_0}(z) = \Theta_\chi(\mathcal{A}_{z_0}(z); \tau)$. ϕ_{z_0} has exactly g zeroes (including q). Then the holomorphic one-form defined by

$$h \stackrel{\text{def}}{=} \sum_{i=1, \dots, g} \left(\frac{\partial \Theta_\chi}{\partial u_i} \right)_{|u=0} \omega_i \quad (1-24)$$

has exactly $g - 1$ double zeroes such that \sqrt{h} is a well defined holomorphic spinor and has exactly $g - 1$ simple zeroes. Moreover, let $\chi \stackrel{\text{def}}{=} \frac{1}{2}(n + \tau \cdot m) \in \mathbb{C}^g$ be an odd theta-characteristic. Following [45],

Definition 1.22 *Fay's prime form*

The prime form \mathcal{E}_χ is the skew-symmetric $(-\frac{1}{2}, -\frac{1}{2})$ spinor defined on the squared universal cover $\tilde{\Sigma} \times \tilde{\Sigma}$ by

$$\mathcal{E}_\chi(\tilde{p}, \tilde{q}) \stackrel{\text{def}}{=} \frac{\phi_q(p)}{\sqrt{h(p)h(q)}} \quad (1-25)$$

It vanishes linearly on the diagonal and nowhere else. For any choice of coordinates,

$$\mathcal{E}_\chi(\tilde{p}, \tilde{q}) = \frac{\Theta_\chi(\mathcal{A}_q(p); \tau)}{\sqrt{h(p)h(q)}} \stackrel{p \sim q}{=} \frac{p - q}{\sqrt{d p d q}} (1 + \mathcal{O}(p - q)^2) \quad (1-26)$$

It can be shown not to depend on the choice of odd spin characteristics χ and we will therefore drop the subscript χ from now on.

Remark 1.23 *it is not clear from this local definition that the prime form is indeed defined on the universal cover but one can easily check that although \mathcal{E} has no monodromy around \mathcal{A} -cycles, it satisfies*

$$\mathcal{E}(\tilde{p} + \mathcal{B}_i, \tilde{q}) = \mathcal{E}(\tilde{p}, \tilde{q}) e^{-2i\pi(\mathcal{A}_q(p)_i + \chi_i)} e^{-i\pi\tau_{i,i}} \quad (1-27)$$

for any choice of index $i \in \{1, \dots, g\}$.

In our following constructions we will need a prime form defined globally on our base curve and not its universal cover. This is the reason to introduce a twist. It consists in compensating the \mathcal{B} -cycle monodromies of \mathcal{E} by use of a meromorphic one-form.

Definition 1.24 *Twisted prime forms*

Let f be a meromorphic one-form on Σ with vanishing \mathcal{A} -cycle integrals

$$\oint_{\mathcal{A}_i} f = 0, \quad \text{and} \quad \frac{1}{2\pi i} \oint_{\mathcal{B}_i} f \stackrel{\text{def}}{=} \zeta_i \quad (1-28)$$

such that $\zeta \stackrel{\text{def}}{=} (\zeta_1, \dots, \zeta_g) \in \mathbb{C}^g - (\mathbb{Z}^g + \tau \cdot \mathbb{Z}^g)$. ζ is called a polarization and its required property implies that f has singularities. Define the f -twisted prime form by

$$\mathcal{E}_f(p, q) \stackrel{\text{def}}{=} \mathcal{E}(\tilde{p}, \tilde{q}) \frac{\Theta_\chi(\zeta; \tau)}{\Theta_\chi(\mathcal{A}_q(p); \tau)} e^{-\int_q^p f} \quad (1-29)$$

with $\int_q^p f$ defined as integrating along the unique homology chain not intersecting \mathcal{A} nor \mathcal{B} -cycles with boundary $p - q$.

Remark 1.25 The dependence in the odd theta-characteristic χ was reintroduced in the twisted prime form but in all uses that we will make of prime forms, that is cyclic products of the form $\mathcal{E}_f(p_1, p_2) \cdots \mathcal{E}_f(p_n, p_1)$ for a given number $n \in \mathbb{N}^*$ of generic points $p_1, \dots, p_n \in \Sigma$, the dependence in χ will disappear and the dependence in f will only be through the polarization ζ . Moreover, the skew-symmetry is broken by the twist.

Proposition 1.26 *Globality of the twisted prime form*

The twisted prime form \mathcal{E}_f is well-defined $(-\frac{1}{2}, -\frac{1}{2})$ spinor on Σ with essential singularities at the poles of the meromorphic differential one-form f of order ≥ 2 .

Theorem 1.27 *Fay's identities*

Let $n \in \mathbb{N}^*$ be a positive integer and consider $2n$ generic points $p_1, q_1, \dots, p_n, q_n \in \Sigma$. We have the equality

$$\frac{\Theta_\chi(\mathcal{A}_{q_i}(p_i) + \zeta)}{\Theta_\chi(\zeta)} \text{Det}_{1 \leq i, j \leq n} \left(\frac{1}{\mathcal{E}_f(p_i, q_j)} \right) = \frac{\prod_{i < j} \mathcal{E}_f(p_i, p_j) \mathcal{E}_f(q_i, q_j)}{\prod_{i, j} \mathcal{E}_f(p_i, q_j)} \quad (1-30)$$

Remark 1.28 In the genus $g = 0$ situation, Fay's identities reduce to Cauchy's identity

$$\text{Det}_{1 \leq i, j \leq n} \left(\frac{1}{p_i - q_j} \right) = \frac{\prod_{i < j} (p_i - p_j)(q_i - q_j)}{\prod_{i, j} (p_i - q_j)} \quad (1-31)$$

We also define the evaluation of this twisted prime form on a degree 0 divisor on the Riemann surface $D \stackrel{\text{def}}{=} \sum_i \alpha_i [p_i]$ to be

$$\mathcal{E}_f(D) \stackrel{\text{def}}{=} \prod_{i \neq j} \mathcal{E}_f(p_i, p_j)^{-\alpha_i \alpha_j} \quad (1-32)$$

such that when the divisor is taken to be $D = p - q$, $\mathcal{E}_f(p - q)$ coincides with $\mathcal{E}_f(p, q)$ for any pair of distinct points (p, q) on the surface.

Klein form, Bergman kernel and third kind differentials

Definition 1.29 Klein form

The Klein form is defined from a twisted prime form \mathcal{E}_f by the formula

$$B_\zeta(p, q) \stackrel{\text{def}}{=} -\frac{1}{\mathcal{E}_f(p, q)\mathcal{E}_f(q, p)} \quad (1-33)$$

Remark 1.30 The Klein form can be shown as mentioned before to depend only on the polarization ζ hence the subscript.

Definition 1.31 Bergman kernel

The fundamental second kind differential, or Bergman kernel B_m associated to the symplectic basis of cycles $m = \{\mathcal{A}_i, \mathcal{B}_i\}_{1 \leq i \leq g}$, is the unique symmetric bi-differential such that

$$\oint_{\mathcal{A}_i} B(z, \cdot) = 0, \quad \oint_{\mathcal{B}_i} B(z, \cdot) = \omega_i, \quad B(p, q) \underset{p \sim q}{\sim} \frac{dp dq}{(p - q)^2} \quad (1-34)$$

for any index $i \in \{1, \dots, g\}$ and generic point $z \in \Sigma$.

It is related to the Klein form by the identity

$$B(p, q) = B_\zeta(p, q) - 2\pi i \sum_{i,j=1}^g \left(\partial_{i,j}^2 \ln \Theta_\chi(\zeta; \tau) \right) \omega_i(p) \omega_j(q) \quad (1-35)$$

and as a consequence of this identity, it's right hand side does not depend on the choice of polarization ζ .

Definition 1.32 *Third-kind differentials*

Given two generic points $p, q \in \Sigma$, define accordingly two corresponding third-kind differentials to be

- *Klein* :

$$\omega_{\zeta}^{p-q}(x) \stackrel{\text{def}}{=} \int_q^p B_{\zeta}(x, \cdot) \quad (1-36)$$

- *Bergman* :

$$\omega^{p-q}(x) \stackrel{\text{def}}{=} \int_q^p B(x, \cdot) \quad (1-37)$$

where the integral is once again computed along the only homology chain with vanishing intersection with the symplectic basis \mathfrak{m} and boundary equal to $p - q$. They are both meromorphic one-forms with a simple pole at p (resp. q) with residue $+1$ (resp. -1).

1.2 Kähler geometry

Definition 1.33 *Hermitian manifolds*

A hermitian manifold is a pair (M, g) where M is a complex manifold and g is a hermitian metric, that is a riemannian metric on M as a differentiable manifold which is invariant under the action of the complex structure I . It has a canonical 2-form called the Kähler form of (M, g) and defined by

$$\Omega(\xi, \zeta) = g(I \cdot \xi, \zeta) \quad (1-38)$$

It is also invariant under the action of I .

We will write M instead of (M, g) for a hermitian manifold, without specifying the metric g , whenever no confusion is possible.

Property 1.34 *Let Ω be the Kähler form of a hermitian manifold M with a given dimension $\dim_{\mathbb{C}} M = d \in \mathbb{N}^*$, then $\Omega^{\wedge d}$ is a nowhere vanishing $2d$ -form.*

Definition 1.35 *Kähler manifolds*

A Kähler manifold is a hermitian manifold whose Kähler form is closed.

Every Kähler manifold is also a symplectic manifold but the converse is not true and a general symplectic manifold does not have an integrable complex structure compatible with the symplectic form.

Definition 1.36 *Hyperkähler metrics*

A hyperkähler metric on a 4d-dimensional manifold M is a riemannian metric g which is kählerian with respect to three complex structures I, J and K that satisfy the algebraic identities of the quaternions,

$$I^2 = J^2 = K^2 = -1, \quad (1-39)$$

$$IJ = -JI = K, \quad (1-40)$$

$$JK = -KJ = I, \quad (1-41)$$

$$KI = -IK = J. \quad (1-42)$$

Corresponding to each complex structure is a Kähler form,

$$\omega_1(\xi, \zeta) = g(I \cdot \xi, \zeta), \quad \omega_2(\xi, \zeta) = g(J \cdot \xi, \zeta), \quad \omega_3(\xi, \zeta) = g(K \cdot \xi, \zeta), \quad (1-43)$$

and furthermore this set of symplectic (Kähler) forms determines the metric uniquely.

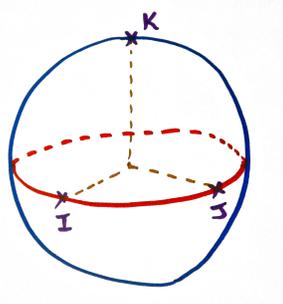


Figure 1.6: A hyperkähler manifold has a $\mathbb{C}\mathbb{P}^1$ of complex structures.

2 Algebras of symmetries

2.1 Definitions and examples

Definition 2.1 *Lie algebras*

A Lie algebra $(\mathfrak{g}, [.,.])$ is a vector space \mathfrak{g} together with a skew-symmetric bilinear operation $[.,.]: \mathfrak{g}^2 \rightarrow \mathfrak{g}$ called the Lie bracket and satisfying the Jacobi identity

$$\forall A, B, C \in \mathfrak{g}, [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \quad (2-1)$$

we will most of the time omit the bracket and simply write \mathfrak{g} when no confusion is possible.

Example 2.2 • Let \mathcal{A} be an associative algebra, then $[a, b] = ab - ba$ is a Lie bracket in \mathcal{A} and $(\mathcal{A}, [., .])$ is a Lie algebra.

- Let X be a differential manifold and $\Gamma(TX)$ be the set of its vector fields. For any two vector fields $\chi, \eta \in \Gamma(TX)$, define $[\chi, \eta]$ to be the vector field such that for any function $f \in C^\infty(X)$, $[\chi, \eta](f) = \chi[\eta(f)] - \eta[\chi(f)]$. Then $(\Gamma(TX), [., .])$ is a Lie algebra.
- Let V be a vector space and let $\phi: V \rightarrow \mathfrak{g}$ be an injective linear map into a Lie algebra \mathfrak{g} such that $[\phi(V), \phi(V)] \subset \phi(V)$. Then $(V, \phi^*[., .])$ is a Lie algebra. In particular, every subspace of \mathfrak{g} stable by the bracket with itself has a Lie algebra structure.

Definition 2.3 *Ideals of Lie algebras*

An ideal \mathcal{I} of a Lie algebra \mathfrak{g} is a Lie sub-algebra such that $[\mathcal{I}, \mathfrak{g}] \subset \mathcal{I}$.

Definition 2.4 *Simple, semi-simple and reductive Lie algebras*

- A simple Lie algebra is a non-abelian Lie algebra whose only ideals are 0 and itself.
- A Lie algebra is said to be semi-simple if it has no non-zero abelian ideal.
- A Lie algebra is said to be reductive if it is a direct sum of semi-simple Lie algebras.

Proposition 2.5 *A Lie algebra is reductive if and only if it admits a faithful, completely reducible, finite-dimensional representation.*

2.2 Lie algebras and Lie groups

Definition 2.6 Lie groups

A Lie group is a differential manifold G with a compatible group structure, that is a product $G^2 \rightarrow G$ and an inverse map $G \rightarrow G$ that are both smooth with respect to the differential structure.

Definition 2.7 Interior automorphisms, Adjoint and adjoint actions

- G acts on itself by interior automorphism, that is for any $g \in G$, define $i_g \in \text{Aut}(G)$ such that for any $h \in G$ by $i_g(h) = ghg^{-1}$.
- G acts on T_eG by the Adjoint action, $\text{Ad} : G \rightarrow \text{GL}(T_eG)$ defined for any $g \in G$ by $\text{Ad}_g = T_e i_g \in \text{GL}(T_eG)$.
- T_eG acts on itself by the adjoint action, $\text{ad} : T_eG \rightarrow \text{End}(T_eG)$ defined by $\text{ad} = T_e \text{Ad}$.

Theorem 2.8 Cartan

A Lie algebra is semi-simple if and only if its Killing form defined by

$$K(E, F) \stackrel{\text{def}}{=} \text{Tr}(\text{ad}_E \circ \text{ad}_F) \quad (2-2)$$

is non-degenerate.

Proposition 2.9 Left-invariant vector fields

Let G be a Lie group, e be its identity element and $L_G \subset \Gamma(TG)$ be the set of its left-invariant vector fields. Then $L_G \simeq T_eG$ and $(T_eG, [\cdot, \cdot])$, where

$$[A, B] \stackrel{\text{def}}{=} \text{ad}_A(B) \quad (2-3)$$

is a Lie algebra. We will denote it $\mathfrak{g} = \text{Lie}(G)$.

To each Lie group G we can therefore canonically associate a Lie algebra \mathfrak{g} but the converse is only partially true. Indeed, there exists an application, called the exponential map and denoted by \exp allowing us to associate to any Lie algebra \mathfrak{g} a connected Lie group $\exp \mathfrak{g}$ isomorphic to the connected part of G containing the identity element. It is defined as follows

Definition 2.10 *Exponential map*

Let G be a Lie group and let $\mathfrak{g} = \text{Lie}(G)$ be its corresponding Lie algebra. For any element $A \in \mathfrak{g} \simeq T_e G$. Let us denote by $\gamma_A : \mathbb{R} \rightarrow G$ the unique one-parameter subgroup of G such that $\gamma'_A(0) = A$. Then define the exponential map $\exp : \mathfrak{g} \rightarrow G$ by $\exp(A) = \gamma_A(1)$ for any Lie algebra element $A \in \mathfrak{g}$. Its image lie by definition in the connected component of the identity in G and it actually surjects onto it.

2.3 Root systems and semi-simple Lie algebras**Definition 2.11** *Cartan subalgebras*

A Cartan subalgebra \mathfrak{h} of a given reductive Lie algebra \mathfrak{g} is a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ that is equal to its normalizer

$$\mathfrak{h} = N(\mathfrak{h}) \stackrel{\text{def}}{=} \{E \in \mathfrak{g} \mid \text{ad}_E(\mathfrak{h}) \subset \mathfrak{h}\} \quad (2-4)$$

All Cartan subalgebras of \mathfrak{g} have the same dimension denoted $r \stackrel{\text{def}}{=} \text{rk } \mathfrak{g}$ and called the rank of the Lie algebra. Since we assumed the Lie algebra to be reductive, they are maximal abelian Lie subalgebras.

Remark 2.12 *Cartan subalgebras $\mathfrak{h} \subset \mathfrak{g}$ generically correspond to centralizers, or commutants, of elements of the Lie algebra. Any non-trivial element whose centralizer is maximal in this sense is called regular.*

Let us now assume the Lie algebra to be semi-simple. \mathfrak{h} acts on \mathfrak{g} by restriction of the adjoint representation ad and for any linear form $\lambda \in \mathfrak{h}^*$, define

$$\mathfrak{g}_\lambda \stackrel{\text{def}}{=} \{E \in \mathfrak{g} \mid \forall H \in \mathfrak{h}, \text{ad}_H(E) = \lambda(H)E\} \quad (2-5)$$

Lemma 2.13 $\mathfrak{g}_0 = \mathfrak{h}$ and for any $\lambda, \mu \in \mathfrak{h}^*$, $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$.

Definition 2.14 *Roots and Weyl group*

A root of \mathfrak{h} in \mathfrak{g} is a linear form $\lambda \in \mathfrak{h}^*$ such that $\mathfrak{g}_\lambda \neq 0$. Let us denote by \mathfrak{R} the set of all roots. It is independent of the choice of Cartan subalgebra.

Let $\mathfrak{w} \stackrel{\text{def}}{=} \langle s_\lambda \rangle_{\lambda \in \mathfrak{R}} \subset \text{Aut}(\mathfrak{h}^*)$ the so-called Weyl group generated by the maps s_λ defined for any root $\lambda \in \mathfrak{R}$ by

$$\begin{aligned} s_\lambda: \mathfrak{h}^* &\longrightarrow \mathfrak{h}^* \\ \mu &\longmapsto \mu - 2 \frac{\langle \lambda, \mu \rangle}{\langle \lambda, \lambda \rangle} \lambda \end{aligned} \quad (2-6)$$

where $\langle \cdot, \cdot \rangle$ is the so-called minimal bilinear form on \mathfrak{h}^* . Choose a set of simple roots $\mathfrak{R}_0 \stackrel{\text{def}}{=} \{\tau_1, \dots, \tau_r\}$ (an integer basis of the root lattice \mathfrak{R}). It is then defined as the unique bilinear form such that $\langle \tau_i, \tau_j \rangle = 2\delta_{i,j}$.

s_λ is precisely the orthogonal reflexion with respect to the hyperplane of \mathfrak{h}^* defined by λ .

Lemma 2.15 *The root system \mathfrak{R} spans \mathfrak{h}^* and it is invariant under the action of the Weyl group. Namely, $\mathfrak{w} \cdot \mathfrak{R} = \mathfrak{R}$.*

Proposition 2.16 *Dimension of root spaces*

For any root $\lambda \in \mathfrak{R}$, $\dim \mathfrak{g}_\lambda = 1$ and $\{n \in \mathbb{Z} \mid n\lambda \in \mathfrak{R}\} = \{\pm 1\}$.

Corollary 2.17 *For any $\lambda \in \mathfrak{R}$, $\mathfrak{g}_\lambda \oplus [\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}] \oplus \mathfrak{g}_{-\lambda} \simeq \mathfrak{sl}_2(\mathbb{C})$*

Theorem 2.18 *Construction of semi-simple Lie algebras*

i) Let \mathfrak{g} be a semi-simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and \mathfrak{R} the corresponding root system. Let $\mathfrak{R}_0 = \{\tau_1, \dots, \tau_r\} \subset \mathfrak{R}$ be a basis of simple roots and define the dual basis $\{H_1, \dots, H_r\}$ of \mathfrak{h} by the requirement $\tau_i(H) = 2 \frac{K(H_i, H)}{K(H_i, H_i)}$ for any $H \in \mathfrak{h}$.

Then \mathfrak{g} is generated by $\{E_i, F_i, H_i\}_{1 \leq i \leq r}$ required for any $i, j \in \{1, \dots, r\}$ to satisfy

- $[H_i, H_j] = 0$
- $[E_i, F_j] = \delta_{i,j} H_i$
- $[H_i, E_j] = \tau_j(H_i) E_j$ and $[H_i, F_j] = -\tau_j(H_i) F_j$
- $(\text{ad}_{E_i})^{-\tau_j(H_i)+1}(\mathfrak{g}_{\tau_j}) = 0$ and $(\text{ad}_{F_i})^{-\tau_j(H_i)+1}(\mathfrak{g}_{-\tau_j}) = 0$

ii) The previous presentation defines a semi-simple Lie algebra.

iii) This correspondence between semi-simple Lie algebras and reduced root systems is bijective up to isomorphism.

Definition 2.19 *Cartan matrix*

The Cartan matrix associated to a set of simple roots $\mathfrak{R}_0 = \{\tau_1, \dots, \tau_r\} \subset \mathfrak{R}$ is defined as

$$\kappa \stackrel{\text{def}}{=} \left(\kappa_{i,j} \stackrel{\text{def}}{=} \tau_j(H_i) \right)_{1 \leq i,j \leq r} \tag{2-7}$$

Lemma 2.20 \mathfrak{R} is determined by κ up to isomorphism and if \mathfrak{R}'_0 is another basis of simple roots of \mathfrak{g} , then its associated Cartan matrix κ' is conjugated to κ .

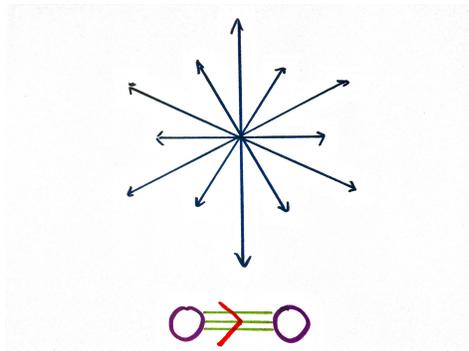


Figure 1.7: Root system and Dynkin diagram of the exceptional Lie algebra of type G_2 .

Definition 2.21 *Dynkin diagram*

The Dynkin diagram associated to a Cartan matrix κ of a basis of simple roots \mathfrak{R}_0 of the Lie algebra \mathfrak{g} is the graph satisfying the following :

- its set of vertices is \mathfrak{R}_0
- the number of edges relating τ_i to τ_j is $n_{i,j} \stackrel{\text{def}}{=} \kappa_{i,j} \kappa_{j,i}$
- if $n_{i,j} > 1$ and $|\kappa_{i,j}| = 1$, then the edges are oriented from τ_i to τ_j .

Proposition 2.22 *This construction of the Dynkin diagram is a bijective process.*

Definition 2.23 *Simply laced Lie algebra*

The Lie algebra is said to be simply laced if its Dynkin diagram only contains simple links.

Remark 2.24 *This implies that all non-zero roots have same length.*

Proposition 2.25 *Lie algebras of A, D and E type are simply laced while those of type B, C, F and G are not.*

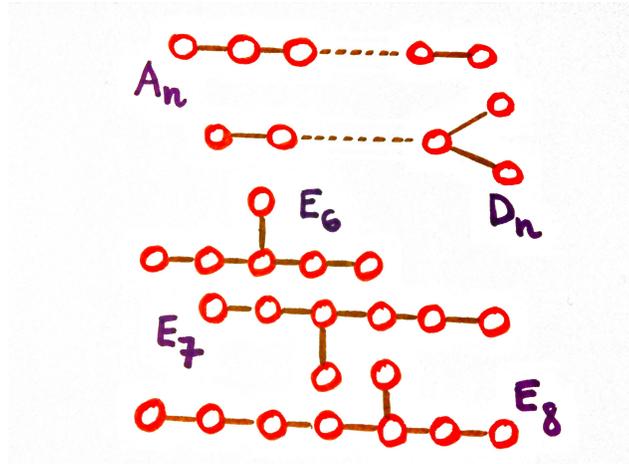


Figure 1.8: Dynkin diagrams corresponding to simply laced Lie algebras (ADE).

2.4 Representation theory

Definition 2.26 *Representations and group actions*

Let G be a Lie group and let V be a vector space (not necessarily of finite dimension). A representation of G into V is equivalently defined as a map $\rho : G \rightarrow \text{Aut}(V)$, where $\text{Aut}(V)$ is the set of automorphisms of the vector space V , or as the corresponding action $G \times V \rightarrow V$ mapping a pair (g, v) to the vector $\rho(g)v$. We call (V, ρ) a G -module.

Moreover ρ pushes forward to a Lie algebra homomorphism $\rho_ : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ and we call (V, ρ_*) a representation of the Lie algebra \mathfrak{g} or equivalently a \mathfrak{g} -module.*

Theorem 2.27 *Weyl*

A Lie algebra \mathfrak{g} is semi-simple if and only if all its finite dimensional representations are entirely reducible.

Definition 2.28 *Weights of \mathfrak{g} in a representation*

Similarly as we did in the case of the adjoint action of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ on the Lie algebra, consider for any $\lambda \in \mathfrak{g}^$ the eigenspace*

$$V_\lambda \stackrel{\text{def}}{=} \{v \in V \mid \forall E \in \mathfrak{g}, \rho(E)v = \lambda(E)v\} \quad (2-8)$$

If $V_\lambda \neq 0$ we say that λ is a weight of \mathfrak{g} in the representation (V, ρ) . Let us denote by $\mathfrak{X}(\rho)$ the set of weights of \mathfrak{g} in representation ρ .

Example 2.29 *Roots are weights of \mathfrak{h} in the \mathfrak{h} -module $(\mathfrak{g}, \text{ad})$.*

2.5 Universal enveloping algebras, casimirs and central extensions

The universal enveloping algebra of a given Lie algebra is in essence an algebra containing it and such that every true relation will hold in any representation. It is defined as follows:

Definition 2.30 *Universal enveloping algebra*

Let \mathfrak{g} be a Lie algebra. Its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is the smallest unital associative algebra containing \mathfrak{g} and such that for any two elements $A, B \in \mathfrak{g}$,

$$[A, B] = AB - BA \quad (2-9)$$

for the product in $\mathcal{U}(\mathfrak{g})$.

Proposition 2.31 *Explicit realization*

For any Lie algebra \mathfrak{g} ,

$$\mathcal{U}(\mathfrak{g}) \simeq \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k} / \{A \otimes B - B \otimes A = [A, B]\} \quad (2-10)$$

Example 2.32 *If \mathfrak{g} is an abelian Lie algebra, then its universal enveloping algebra is identified with its symmetric algebra $\mathcal{S}(\mathfrak{g})$.*

Remark 2.33 *Let us notice that the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ possesses more structure than just that of an associative algebra. Indeed the map*

$$\begin{aligned} \varphi : \mathfrak{g} &\longrightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \\ E &\longmapsto E \otimes 1 + 1 \otimes E \end{aligned} \tag{2-11}$$

satisfies the co-associativity axiom $(\varphi \otimes \text{Id}) \circ \varphi = (\text{Id} \otimes \varphi) \circ \varphi$ showing that $\mathcal{U}(\mathfrak{g})$ is a bi-algebra. Without going into any details, it is in fact a Hopf algebra with a unique one-dimensional deformation quantization family $(\mathcal{U}_q(\mathfrak{g}))_q$ (with formal complex parameter q) conserving this property. $\mathcal{U}_q(\mathfrak{g})$ is called the quantum group associated to the Lie algebra \mathfrak{g} . Although nothing related is described in this work, the q -deformation of the presented constructions will be investigated in the near future.

Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{M}_d(\mathbb{C})$ be a matrix realization of a Lie algebra \mathfrak{g} . The trace Tr_ρ and the determinant Det_ρ in this representation canonically define two applications $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathbb{C}$.

Definition 2.34 *Casimirs of a representation*

Let \mathfrak{g} be a reductive Lie algebra, $d \in \mathbb{N}^$ be a non-negative integer and let $\rho : \mathfrak{g} \longrightarrow \mathfrak{M}_d(\mathbb{C})$ be a faithful matrix realization of \mathfrak{g} . The casimirs of ρ are defined as the symmetric maps $C_k^{(\rho)} \in \text{Hom}(\mathfrak{g}^{\otimes k}, \mathbb{C})$, for $k \in \{1, \dots, d\}$, such that for any $A \in \mathfrak{g}$ and formal variable $y \in \mathbb{C}$,*

$$\text{Det}_\rho(y - A) = \sum_{k=0}^d (-1)^k y^{d-k} C_k^{(\rho)}(A^{\otimes k}) \tag{2-12}$$

They depend on a choice of representation but we will drop the upper-script ρ whenever no confusion is possible.

Proposition 2.35 *Casimirs and center of universal enveloping algebras*

The casimirs of any faithful matrix realization of a Lie algebra generate the center of its universal enveloping algebra.

Remark 2.36 *There is an alternative definition of the casimirs $\{C_k\}$ by use of differential equations. Indeed, identifying $\mathfrak{g} \simeq \mathfrak{g}^*$ with its dual a Casimir can be defined as a function $C \in C^\infty(\mathfrak{g}^*)$ that Poisson commutes (for the Kirillov-Kostant bracket associated to \mathfrak{g}^* , see the later paragraph on symplectic geometry) with all the others, namely $\{C, F\} = 0$ for any $F \in C^\infty(\mathfrak{g}^*)$. It takes the form of a differential operator in terms of coordinate variables in \mathfrak{g}^* .*

2.6 Affine Kac-Moody algebras

Let \mathfrak{g} be a reductive complex Lie algebras and $\langle \cdot, \cdot \rangle$ a non-degenerate symmetric bilinear bracket.

Definition 2.37 Affine Lie algebras

For any complex number $c \in \mathbb{C}$, The corresponding affine Lie algebra with central charge c is defined as a central extension of the infinite dimensional Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ with one dimensional center $\mathbb{C}c$

$$\mathfrak{g}_c = \mathbb{C}c \oplus \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \quad (2-13)$$

where $\mathbb{C}[t, t^{-1}]$ is the space of Laurent polynomials in the variable t and the Lie bracket is defined for any elements $E_1, E_2 \in \mathfrak{g}$ and $f_1, f_2 \in \mathbb{C}[t, t^{-1}]$ by

$$[E_1 \otimes f_1, E_2 \otimes f_2] = [E_1, E_2] \otimes f_1 f_2 + \langle E_1, E_2 \rangle \operatorname{Res}_{t=0} (f_1 \, d f_2) \quad (2-14)$$

In such a central extension \mathfrak{g}_c there exists a distinguished vector field (or derivation) $\delta = t \frac{d}{dt} \in \operatorname{End}(\mathfrak{g}_c)$ which in turn allows to define the associated

Definition 2.38 Affine Kac-Moody algebras

The affine Kac-Moody algebra associated to the affine Lie algebra \mathfrak{g}_c is defined as the semi-direct product $\widehat{\mathfrak{g}}_c = \mathfrak{g}_c \times \mathbb{C}d$ where the extra generator d is defined by the differential action $[d, M] = \delta(M)$ for any $M \in \mathfrak{g}_c$.

We will come back to Kac-Moody algebras as they will be the starting point to define the generators of the W-algebras of symmetry of the conformal field theories we will study.

2.7 Examples of infinite dimensional Lie algebras

Canonical quantization

Let us consider the canonical quantization of a classical particle of mass m moving in a one dimensional space in some potential $V(x) \in \mathbb{R}[|x|]$.

Classically, the phase space corresponding to this system is the symplectic space $T^*\mathbb{R} = \mathbb{R}^2$ with canonical symplectic structure and symplectic form $\omega = dp \wedge dx$. The energy of a given state (p, x) is $E = \frac{p^2}{2m} + V(x)$.

Canonical quantization in the Heisenberg picture consists in studying one-parameter families of non trivial hermitian representations $(\rho_t)_{t \in \mathbb{R}}$ of a Lie algebra \mathfrak{g} containing a non-trivial central element \hbar , and two elements X and P satisfying $[P, X] = i\hbar$ (quantization of the Poisson structure of Hamiltonian dynamics). It is called the Heisenberg algebra.

Introducing the notation $A_t = \rho_t(A)$ for any $A \in \mathfrak{g}$ and $t \in \mathbb{R}$, we ask that it has the Ehrenfest time dependence

$$i\hbar_t \frac{dA_t}{dt} = [H_t, A_t] \quad (2-15)$$

where we introduced the Hamiltonian operator $H = \frac{p^2}{2m} + V(X)$ (notice that in this formulation there can be no explicit time dependence in the elements of \mathfrak{g}). In particular this immediately implies the conservations $\frac{d\hbar_t}{dt} = \frac{dH_t}{dt} = 0$ such that we will denote (abusively) these elements as \hbar and H respectively. The Ehrenfest equation then rewrites

$$i\hbar \frac{dA_t}{dt} = [H, A_t] \quad (2-16)$$

Property 2.39 *Isospectrality*

For any $A \in \mathfrak{g}$, $t \in \mathbb{R}$ and $k \in \mathbb{N}^*$,

$$\frac{d}{dt} \text{Tr } A_t^k = 0 \quad (2-17)$$

Corollary 2.40 For any $A \in \mathfrak{g}$, $\text{Det}(A_t)$ is a constant.

We will from now on assume that $\text{Det}(\hbar) \neq 0$. Furthermore, Ehrenfest equation applied to X and P gives

$$\frac{dX_t}{dt} = \frac{1}{i\hbar} \left[\frac{P_t^2}{2m}, X_t \right] = \frac{P_t}{m} \quad \text{and} \quad \frac{dP_t}{dt} = \frac{1}{i\hbar} [V(X_t), P_t] = -V'(X_t) \quad (2-18)$$

where we can formally recognize the equations we would have expected from Hamiltonian mechanics at the classical level by making the replacement $\frac{1}{i\hbar}[\cdot, \cdot] \longrightarrow \{\cdot, \cdot\}$ to extract the Hamiltonian dynamics we shall describe in the next section.

Proposition 2.41 *Infinite dimensionality of ρ*

For any $t \in \mathbb{R}$, ρ_t is infinite dimensional and \hbar acts as a constant scalar.

proof:

The Heisenberg algebra admits a unique (up to isomorphism) faithful 3 dimensional minimally non-trivial nilpotent representation in the space of upper triangular matrices given by

$$(\alpha P + \beta X + i\gamma \hbar) \longmapsto \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix} \quad (2-19)$$

This shows that any non-trivial hermitian representation of the Heisenberg algebra has to be infinite dimensional. The fact that \hbar acts as a scalar is then a consequence of the Stone-von Neumann theorem. ■

The Virasoro algebra

The Virasoro algebra is the Lie algebra of the group of conformal (or holomorphic) transformations of the plane. It is the fundamental algebra of symmetries of conformal field theories. We will not elaborate too much on these for now but simply state that they are quantum field theories in which the symmetry algebra contains the following

Definition 2.42 *Virasoro algebra*

For any complex number $c \in \mathbb{C}$, the Virasoro algebra with central charge c is

the infinite dimensional Lie algebra generated by the family $(L_n)_{n \in \mathbb{Z}}$ of elements satisfying the following commutation relations

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0} \quad (2-20)$$

Let us notice at this point that the Virasoro algebra contains an $\mathfrak{sl}_2(\mathbb{C})$ subalgebra generated by (L_{-1}, L_0, L_1) .

As is usual in physics, the information of the possible field content of a quantum field theory endowed with certain symmetries is contained in the representation theory of the corresponding algebra. We will come back to conformal field theories in 2D later on when exploring an associated quantum geometry.

3 Symplectic geometry

3.1 Symplectic structures

Definition 3.1 *Symplectic manifolds*

A symplectic manifold (X, ω) is a holomorphic complex (resp. smooth real) manifold X endowed with an exact skew-symmetric non-degenerate holomorphic (resp. smooth) two-form ω .

Being non-degenerate, a symplectic form induces a smooth bundle automorphism of TX . Allowing to introduce the following

Definition 3.2 *Symplectic orthogonal*

Let (X, ω) be a symplectic manifold, then the symplectic orthogonal of a distribution $\mathcal{D} \subset TX$, that is a holomorphic (resp. smooth) subbundle of TX with constant dimension k over X , is defined as $\mathcal{D}^\perp = \text{Ker } \omega(\mathcal{D}, \cdot)$.

Theorem 3.3 *Darboux*

Let (X, ω) be a symplectic manifold of dimension $2d \in \mathbb{N}^*$. There exists an atlas of X (Darboux atlas) in which ω is the pullback of the standard symplectic form on \mathbb{C}^{2d} (resp. \mathbb{R}^{2d}). We write it in local coordinates $(q_i, p_i) \in X$ as

$$\omega = \sum_{i=1}^d dp_i \wedge dq_i = d\alpha \quad (3-1)$$

where we introduced the Liouville 1-form $\alpha = \sum_{i=1}^d p_i dq_i$. Such coordinates are called Darboux coordinates.

For any vector fields χ, ψ on X , denote by i_χ the interior product by (or insertion of) χ , L_χ the Lie derivative in the χ direction and $[\chi, \psi]$ the Lie bracket of χ and ψ such that

$$L_\chi = i_\chi d + d i_\chi, \quad (3-2)$$

$$L_{[\chi, \psi]} = L_\chi L_\psi - L_\psi L_\chi, \quad (3-3)$$

$$i_{[\chi, \psi]} = L_\chi i_\psi - i_\psi L_\chi. \quad (3-4)$$

Definition 3.4 *Canonical transformations*

Changes of Darboux coordinates are called canonical transformations.

Assuming that the variety X is compact and connected implies that

Property 3.5 *de Rham cohomology classes*

For any $k \in \{1, \dots, d\}$, $[\omega^k] \in H_{dR}^{2k}(X)$ is nonzero and the de Rham cohomology class $[\omega^d]$ generates the one-dimensional vector space $H_{dR}^{2d}(X)$.

Definition 3.6 *Hamiltonian vector field*

Given a symplectic manifold (X, ω) and for any function $H \in H^0(X, \mathbb{C})$ (resp. $C^\infty(X)$), there exists a unique vector field χ_H on X such that for any other vector field Y on X ,

$$dH(Y) \stackrel{def}{=} \omega(\chi_H, Y) \quad (3-5)$$

It is called the Hamiltonian vector field associated to the function H . The converse remains only locally true, for any vector field χ on X , there locally exists around each point a function H such that for any other local vector field ψ , $dH(\psi) = \omega(\chi, \psi)$. H is called a Hamiltonian of χ . Moreover, (X, ω, H) for any choice of smooth function H will be called a Hamiltonian manifold.

Definition 3.7 *Poisson bracket*

Let (X, ω) be a symplectic manifold, then the Poisson bracket is the bilinear antisymmetric operation on functions on X defined for any $f, g \in H^0(X, \mathbb{C})$ (resp. $C^\infty(X)$) by $\{f, g\} = \omega(\chi_f, \chi_g) \in H^0(X, \mathbb{C})$ (resp. $C^\infty(X)$).

3.2 Symplectic actions of compact Lie groups

We will from now on only consider complex symplectic manifolds.

Definition 3.8 *Moment map*

Let \mathfrak{g} be a finite dimensional Lie algebra. Suppose that it is the Lie algebra of a connected compact Lie group $G = \exp \mathfrak{g}$ acting on a symplectic manifold (X, ω) by isometries that preserve ω . If all the fundamental vector fields corresponding to this action are Hamiltonian vector fields, then define a moment map to be an equivariant application $\mu : X \rightarrow \mathfrak{g}^$ defined for any $x \in X$ and $E \in \mathfrak{g}$ by $\mu(x)(E) = f(x)$, where f is a Hamiltonian of the fundamental vector field of E .*

Notice that without the equivariance requirement, a moment map would be far from unique. Indeed, there would be as many possibilities as there are choices for the Hamiltonian representations of the fundamental vector fields of the Lie algebra. In this situation, since two Hamiltonians of the same vector field differ by a constant, this amounts to an affine line of Hamiltonians for each independent direction in \mathfrak{g} , in turn the space of moment maps is of dimension $\dim \mathfrak{g}$.

Given a moment map μ , the group G acts on both the definition space X and the target space \mathfrak{g}^* via the coadjoint representation. The equivariance of μ is the fact that this map intertwines between the two actions. It reads

$$\mu(g \cdot x) = \text{Ad}_g^* \mu(x) \tag{3-6}$$

for any elements $x \in X$ and $g \in G$ and requiring this property fixes the integration constant.

Moreover, it yields that the level set $\mu^{-1}(\{0\})$ is preserved by the action of G and if this action is free and proper, the quotient $\mu^{-1}(\{0\})/G$ will be a smooth manifold. In general, it may only be an orbifold (the generalization of a manifold where the data of local finite automorphism groups is added). In turn we have the following

Theorem 3.9 *Marsden-Weinstein's symplectic reduction*

If $\mu^{-1}(\{0\})/G$ is a manifold, then it is symplectic.

The usual notation for the symplectic manifold thus obtained is $\mu^{-1}(\{0\})//G$.

Example 3.10 *When $X = T^*\mathbb{R}^n$ and $G = \mathbb{R}^n$ acts by affine translations, the moment map is the linear momentum and the symplectic reduction is the space of constant momenta \mathbb{R}^n .*

CoAdjoint orbits

Definition 3.11 CoAdjoint action and orbits

G acts on the dual of its Lie algebra \mathfrak{g}^* by the coAdjoint action Ad^* defined for any $g \in G$, $\Xi \in \mathfrak{g}^*$ and $E \in \mathfrak{g}$ by $[\text{Ad}_g^* \Xi](E) = \Xi(\text{Ad}_{g^{-1}} E)$. We will denote by $\text{Ad}_G^* \Xi$ the corresponding orbit of an element $\Xi \in \mathfrak{g}^*$.

By differentiation, it descends to the coadjoint action of \mathfrak{g} on \mathfrak{g}^* defined for any $E, E' \in \mathfrak{g}$ and $\Xi \in \mathfrak{g}^*$ by $[\text{ad}_E^* \Xi](F) = -\Xi([E, E'])$.

Denote by $\mathcal{F}(\mathfrak{g}^*)$ the set of smooth functions (not necessarily linear) on \mathfrak{g}^* . For any $F \in \mathcal{F}(\mathfrak{g}^*)$, its differential $dF \in \text{Hom}(\mathfrak{g}^*, \mathbb{C})$ can be identified with an element of \mathfrak{g} , thus leading to this next

Definition 3.12 Kirilov-Kostant bracket

For any $F, G \in \mathcal{F}(\mathfrak{g}^*)$ define their Poisson bracket to be the function denoted $\{F, G\}_{KK} \in \mathcal{F}(\mathfrak{g}^*)$ and defined for any $\Xi \in \mathfrak{g}^*$ by

$$\{F, G\}_{KK}(\Xi) = \Xi([dF, dG]) \quad (3-7)$$

Property 3.13 *The Kirilov-Kostant bracket is in general degenerate.*

Example 3.14 *If the structure constants of the Lie algebra \mathfrak{g} are totally anti-symmetric, then the function $\Xi \mapsto \Xi^2 = \sum_a \Xi_a^2$ Poisson commutes with every element of $\mathcal{F}(\mathfrak{g}^*)$.*

Proposition 3.15 Center of the Kirilov-Kostant bracket

The center of $\{.,.\}_{KK}$ consists in all the Ad^ -invariant functions.*

Corollary 3.16 Coadjoint orbits are symplectic

The Kirilov-Kostant bracket is non-degenerate on any coadjoint orbit.

3.3 Hamiltonian actions of loop groups

Definition 3.17 Loop groups and loop algebras

Let \mathfrak{g} be a reductive complex Lie algebra, G its associated connected Lie group and let Σ be a Riemann surface. The corresponding loop group at a point $z_0 \in \Sigma$ is the group of invertible G valued power series expansions around $z = z_0$. Let us denote it by $\mathcal{L}_{z_0}G$. Its elements are regular series $g(\zeta) = \sum_{k=0}^{\infty} g_k \zeta^k$ in a local coordinate ζ around $z = z_0$. The product law is given by the pointwise product $(gh)(\zeta) = g(\zeta)h(\zeta)$ for any $g, h \in \mathcal{L}_{z_0}G$. Accordingly, its Lie algebra $\mathcal{L}_{z_0}\mathfrak{g}$ consists of Lie algebra valued power series of the form $M(\zeta) = \sum_{k=0}^{\infty} M_k \zeta^k$. The corresponding Lie bracket is given by the pointwise Lie bracket.

Property 3.18 Linear dual of the loop algebra

Given a faithful representation ρ of \mathfrak{g} , the linear dual $\mathcal{L}_{z_0}^*\mathfrak{g} = (\mathcal{L}_{z_0}\mathfrak{g})^*$ of the loop algebra at the point $z_0 \in \Sigma$ can be identified with the set of polar Lie algebra valued elements $\Xi(\zeta) = \sum_{1 \leq k} \Xi_k \zeta^{-k}$, where the sum contains a finite but arbitrary large number of terms, by the pairing

$$\langle \Xi, M \rangle = \text{Tr}_{\rho} \left(\text{Res}_{\zeta=0} \Xi(\zeta) M(\zeta) \right) = \sum_{k \in \mathbb{Z}} \text{Tr}_{\rho} (\Xi_{k+1} X_k) \quad (3-8)$$

for any two elements $\Xi \in \mathcal{L}_{z_0}^*\mathfrak{g}$, $M \in \mathcal{L}_{z_0}\mathfrak{g}$, where the residue is defined as taking the coefficient of ζ^{-1} and the last sum is actually finite.

The coAdjoint action of $g \in \mathcal{L}_{z_0}G$ on $\Xi \in \mathcal{L}_{z_0}^*\mathfrak{g}$ is defined as usual for any $M \in \mathcal{L}_{z_0}\mathfrak{g}$ by $[\text{Ad}_g^* \Xi](M) = \Xi(g^{-1}Mg)$ which by invariance of the pairing $\langle \cdot, \cdot \rangle$ (following from the cyclicity of the trace) yields

$$\text{Ad}_g^* \Xi = (g \Xi g^{-1})_- \quad (3-9)$$

where $(\cdot)_-$ indicates that we keep only the polar part of the expression.

3.4 Liouville integrability

Definition 3.19 Liouville's integrability

Let (X, ω, H) be a Hamiltonian manifold of dimension $2d \in \mathbb{N}^*$ and consider

the one-parameter dynamical system $\gamma' = \gamma^* \chi_H$ defined by the Hamiltonian flow of H with unknown curve $\gamma: I \subset \mathbb{R} \rightarrow X$. We will say that it is Liouville integrable if and only if there exists a set $\{F_1, \dots, F_d\}$ of d independent smooth functions on X conserved by the dynamics and in involution. That is for any pair of indices $i, j \in \{1, \dots, d\}$,

$$\{H, F_i\} = 0 \quad \text{and} \quad \{F_i, F_j\} = 0 \quad (3-10)$$

where $\{.,.\}$ is the Poisson bracket associated to the symplectic form ω .

Proposition 3.20 *Hamilton's equations*

In any set of Darboux coordinates (p, q) , denoting $\gamma(t) = (P(t), Q(t))$ for any $t \in I$, the differential equation $\gamma' = \gamma^* \chi_H$ is equivalent to Hamilton's equations

$$\frac{dP_j}{dt} = -\frac{\partial H}{\partial q_j}(P, Q) \quad (3-11)$$

$$\frac{dQ_j}{dt} = \frac{\partial H}{\partial p_j}(P, Q) \quad (3-12)$$

for any $j \in \{1, \dots, d\}$.

Theorem 3.21 *Geometric interpretation*

Such a Liouville integrable system on (X, ω) defines a Lagrangian foliation $X \rightarrow \mathbb{C}^d$ (resp. \mathbb{R}^d) whose fiber over any vector $f = (f_1, \dots, f_d) \in \mathbb{C}^d$ (resp. \mathbb{R}^d) is the level manifold defined by

$$X_f \stackrel{\text{def}}{=} \{x \in X \mid \forall i \in \{1, \dots, d\}, F_i(x) = f_i\} \quad (3-13)$$

proof:

The only thing to prove is that X_f is indeed a Lagrangian submanifold of (X, ω) for any choice of f . Let χ_i be the Hamiltonian vector field associated to F_i for any index $i \in \{1, \dots, d\}$. From involutiveness of the conserved quantities we get that these vector fields are tangent to the manifold. Indeed, this is equivalent to $\chi_i(F_j) = \{F_i, F_j\} = 0$. Therefore, since the F_j 's are assumed to be independent functions, the tangent space to X_f is generated

by $\{\chi_i, i \in \{1, \dots, d\}\}$. In turn, $\omega(F_i, F_j) = 0$ implies $\omega|_{X_f} = 0$ which is the wanted result. ■

In particular, denoting by α a Liouville 1-form such that $d\alpha = \omega$, its restriction to X_f is closed and for any $x_1, x_2 \in X_f$, $\int_{x_1}^{x_2} \alpha|_{X_f}$ is a well defined number.

Definition 3.22 *Generating function of canonical transformations*

For any $f \in \mathbb{C}^d$ (resp. \mathbb{R}^d) and points $x_1, x_2 \in X_f$, define the corresponding generating function to be the number

$$S(f, x_2; x_1) \stackrel{\text{def}}{=} \int_{x_1}^{x_2} \alpha|_{X_f} = \int_{q_1}^{q_2} \sum_{i=1}^d p_i(f, q) dq_i = S(f, q; q_0) \quad (3-14)$$

where the integration path lies in X_f and the last two equalities are loosely written in Darboux coordinates such that $q(x_1) = q_1$ and $q(x_2) = q_2$, assuming that we can invert the relations $F_i(p, q) = f_i$ into functions $p_i(f, q)$. Moreover, define $\psi_j = \frac{\partial}{\partial f_j} S(f, x_2; x_1)$ for any $j \in \{1, \dots, d\}$.

Theorem 3.23 *Equations of motion*

Introducing the classical action

$$S_{cl}[\gamma] \stackrel{\text{def}}{=} \int_{\gamma} \alpha - \int_{t_1}^{t_2} \gamma^* H(t) dt \quad (3-15)$$

for any continuous path $\gamma : I = [t_1, t_2] \longrightarrow X$ starting from $\gamma(t_1) = x_1 \in X$ and ending at $\gamma(t_2) = x_2 \in X$, it is extremal for paths satisfying Hamilton's equations.

We do not prove it here but just say that it follows from the Euler-Lagrange equations associated to an integral functional. It sits at the foundation of classical mechanics. We now get to the main result of this paragraph

Theorem 3.24 *Liouville*

Any Liouville integrable system is solvable by quadrature.

proof:

Starting from a set of Darboux coordinates (p, q) , the change of coordinates $(p, q) \mapsto (f, \psi)$ is a canonical transformation in which the dynamical equations of the position variables will be particularly simple. Indeed, by definition of the generating function S we have $dS = \sum_{j=1}^d \psi_j df_j + p_j dq_j$ and exactness of this 1-form implies

$$d^2 S = 0 = - \sum_{j=1}^d d\psi_j \wedge df_j + \sum_{j=1}^d dp_j \wedge dq_j \quad (3-16)$$

In turn, with loose notations, Hamilton's equations now read for any $j \in \{1, \dots, d\}$,

$$\frac{df_j}{dt} = \{H, F_j\} = \{H, F_j\} = 0 \quad (3-17)$$

$$\frac{d\psi_j}{dt} = \{H, \psi_j\} = \frac{\partial H}{\partial f_j} = \Omega_j(F) \quad (3-18)$$

where $\Omega_j(F)$ is a constant for all j . ■

3.5 Noether's theorem

Liouville's theorem ensures that when studying Hamiltonian dynamical systems, finding conserved quantities can be decisive in the process of finding the solution to the problem. The next natural question is then whether one can find enough of these conserved quantities when given such equations of motion. Unfortunately the result is no in general. Noether however brought some positive answer.

When our problem possesses some continuous symmetries, namely when the symplectic action of a Lie group generated by independent vector fields commutes with the Hamiltonian flow, then we have a conserved quantity for each of these independent vector fields.

Theorem 3.25 *Nøther*

Consider the symplectic action of a Lie group G on a symplectic manifold (X, ω) with Liouville one-form $\alpha = p \, dq$ (in a set of Darboux coordinates). For any smooth path $\gamma : I \rightarrow X$, let us introduce the associated Lagrangian function

$$\mathcal{L}_\gamma \stackrel{\text{def}}{=} \gamma^* p \frac{d}{dt} \gamma^* q - \gamma^* H \quad (3-19)$$

defined on $I \stackrel{\text{def}}{=} [t_1, t_2]$. Suppose there exists a function K such that for any $g \in G$, and any smooth path $\gamma : I \rightarrow X$,

$$g \cdot \mathcal{L}_\gamma(t) = \mathcal{L}_\gamma(t) + \frac{dK}{dt} \quad (3-20)$$

then any choice F_1, \dots, F_r of functions such that the corresponding Hamiltonian vector fields χ_1, \dots, χ_r coincide with the fundamental vector fields associated to a basis E_1, \dots, E_r of $\mathfrak{g} = \text{Lie}(G)$ yields $r = \text{rk } \mathfrak{g}$ conserved quantities in involution.

Corollary 3.26 *Enough symmetries imply integrability*

With the hypotheses of last theorem, if $r = \text{rk } \mathfrak{g} \geq d$ then the system is Liouville integrable.

Liouville's and Nøther's theorem sit together at the foundation of modern theoretical Physics.

4 Frobenius manifolds and integrable hierarchies

A Frobenius manifold as introduced by Dubrovin and Zhang [38] axiomatizes the structure that physical parameters' spaces should bear. An underlying notion is the one of a Frobenius algebra which is the mathematical counterpart of the algebras of scalar fields as they appear in field theories.

4.1 Physical Motivation

Here we show formally that whenever correlators are defined in a physical theory, some particular structure emerges on the corresponding space of parameters.

Consider a physical system in which observable $\{\phi_k\}_{k \in I}$ fluctuate. Solving the theory consists in determining all correlation functions of the type $\langle \phi_{k_1} \dots \phi_{k_n} \rangle$, $k_1, \dots, k_n \in I$, that is computing the amplitude of probability for the fields to simultaneously have given values at given points of the manifold. We arrange them into a formal generating function of an infinite set of formal variables $\{t_k\}_{k \in I}$ called times (or coupling constants).

Definition 4.1 *Partition function*

Denoting $\mathbf{t} = (t_k)_{k \in I}$, let $Z \in \mathbb{C}[[\mathbf{t}]]$ be the generating function, called the *partition function of the theory*, of all such correlation functions

$$Z(\mathbf{t}) \stackrel{\text{def}}{=} \left\langle \exp \left(\sum_{k \in I} t_k \phi_k \right) \right\rangle \quad (4-1)$$

$$= \int \exp \left(\sum_{k \in I} t_k \phi_k \right) \quad (4-2)$$

It is a generating function in the sense that

$$\langle \phi_{k_1} \dots \phi_{k_n} \rangle = \frac{\partial^n Z}{\partial t_{k_1} \dots \partial t_{k_n}}(\mathbf{t} = 0) \quad (4-3)$$

Now consider the parameter space $M = \{\mathbf{t} = (t_k)_{k \in I}\}$. The data of a partition function Z implies the following

- A flat metric is defined on M by the formula

$$\eta(\partial_{t_k}, \partial_{t_l}) = \langle \phi_k \phi_l \rangle, \quad (4-4)$$

- An operator algebra structure is defined on any of the tangent spaces to M by the totally symmetric tensor

$$c(\partial_{t_k}, \partial_{t_l}, \partial_{t_m}) = \langle \phi_k \phi_l \phi_m \rangle \quad (4-5)$$

4.2 Frobenius structures

The first structure to introduce here is the one of a Frobenius algebra, characterizing tangent spaces to Frobenius manifolds. Following [38]

Definition 4.2 Frobenius algebras

A Frobenius algebra A is a commutative, associative unital algebra (over \mathbb{C}) of finite dimension $\dim_{\mathbb{C}} A < \infty$ together with a non-degenerate symmetric bilinear form denoted $\eta : A \times A \rightarrow \mathbb{C}$ satisfying $\eta(uv, w) = \eta(u, vw)$ for any three elements $u, v, w \in A$.

Property 4.3 A Frobenius algebras admits the following

- a projection onto \mathbb{C} defined by $A \rightarrow \mathbb{C} / u \mapsto \eta(u, 1)$
- a dualilty isomorphism $\lambda : A \xrightarrow{\sim} A^* / u \mapsto \eta(u, \cdot)$
- a comultiplication μ defined as the only map making the following diagram commutative

$$\begin{array}{ccc} A & \xrightarrow{\mu} & A \otimes A \\ \lambda \downarrow & & \downarrow \lambda \otimes \lambda \\ A^* & \xrightarrow{m^*} & A^* \otimes A^* \end{array} \quad (4-6)$$

where m^* is the dual map to the usual multiplication. In any basis $\{e_1, \dots, e_{\dim_{\mathbb{C}} A}\}$ of A , and for any $\alpha \in A$

$$\mu(\alpha) = \sum_{a,b,i,j=1}^{\dim_{\mathbb{C}} A} \eta(\alpha, e_a e_b) \eta^{ai} \eta^{bj} e_i \otimes e_j \quad (4-7)$$

where we introduced the inverse matrix notation $(\eta^{ab})_{a,b} = (\eta(e_a, e_b))_{a,b}^{-1}$.

Definition 4.4 Frobenius manifolds

A Frobenius manifold is a complex manifold whose tangent space at each point is endowed with the structure of a Frobenius algebra such that

- η defines a flat metric on M ,

- the unity vector field e is covariantly constant w.r.t. the Levi-Civita connection ∇ of η ,
- if $c(u, v, w) = \eta(u \cdot v, w)$, the 4-tensor $(\nabla_z c)(u, v, w)$ is symmetric in the four vector fields u, v, w, z ,
- there exists a vector field E such that $\nabla(\nabla E) = 0$ and the corresponding one-parameter group of diffeomorphisms acts by conformal transformations of η and rescalings on tangent spaces $T_t M$ for $t \in M$.

The associated deformed (or Dubrovin) connection is then defined to be the symmetric connection

$$\tilde{\nabla}_u(z)v \stackrel{def}{=} \nabla_u v + z u \times v \quad (4-8)$$

depending on the complex parameter z .

Theorem 4.5 *Prepotential [Dubrovin-Zhang]*

Identical flatness in $z \in \mathbb{C}$ is then equivalent to the existence of a function $F_0 : M \rightarrow \mathbb{C}$, the prepotential, such that

$$\eta(\partial_\alpha \times \partial_\beta, \partial_\gamma) = \partial_\alpha \partial_\beta \partial_\gamma F_0 \quad (4-9)$$

here $\partial_\alpha = \frac{\partial}{\partial \theta^\alpha}$ with $(\theta^\alpha)_{1 \leq \alpha \leq d}$ a basis of flat coordinates for ∇ (i.e. such that $d\theta^\alpha$'s are nowhere vanishing flat sections).

Proposition 4.6 *WDVV equations*

The associativity of the Frobenius structure is equivalent to the set of equations

$$\frac{\partial^3 F_0}{\partial v^\alpha \partial v^\beta \partial v^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F_0}{\partial v^\mu \partial v^\gamma \partial v^\delta} = \frac{\partial^3 F_0}{\partial v^\delta \partial v^\beta \partial v^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F_0}{\partial v^\mu \partial v^\gamma \partial v^\alpha} \quad (4-10)$$

for the prepotential. They are called the WDVV equations on the Riemannian manifold (M, η) .

Corollary 4.7 *Alternative definition*

If F_0 is a solution of the WDVV equations on (M, η) , then it endows M with a Frobenius manifold structure.

4.3 Integrable hierarchies

The principal hierarchy associated to the Frobenius manifold M is a hierarchy of partial differential systems linear in derivatives (so-called hydrodynamic type) for an M -valued function $\mathbf{v}(\mathbf{t}) = (v^\alpha(\mathbf{t}))_{1 \leq \alpha \leq d}$ of an infinite set of times $\mathbf{t} = (t^{\alpha,p})_{\substack{1 \leq \alpha \leq d \\ p \geq 0}}$,

$$\partial_{t^{\alpha,p}} v^\beta = (c_{\alpha,p})_\gamma^\beta \partial_x v^\gamma, \quad (4-11)$$

where $(c_{\alpha,p})_\gamma^\beta$ are some matrix coefficients and $x = t^{1,0}$, called the cosmological constant, plays a special role as it will be considered as the variable on which the unknown function depends and the other times are viewed as parameters. As a consequence, introducing the loop space $\mathcal{L}(M)$ of M -valued functions $\mathbf{v}(X)$ and the Poisson bracket

$$\{v^\alpha(X), v^\beta(Y)\} = \eta^{\alpha,\beta} \delta'(X - Y), \quad (4-12)$$

this system has Hamiltonian form

$$\partial_{t^{\alpha,p}} v^\beta(X) = \{v^\beta(X), H_{\alpha,p}\}, \quad (4-13)$$

where the Hamiltonians are of the form $H_{\alpha,p} = \int h_{\alpha,p+1}(\mathbf{v}(X)) dX$ with Hamiltonian densities such that $h_\alpha(\mathbf{v}, z) = \sum_{p=0}^{\infty} h_{\alpha,p}(\mathbf{v}) z^p$ coincide with flat coordinates $\{\tilde{\theta}(z)_\alpha\}_{1 \leq \alpha \leq d}$ of the deformed connection $\tilde{\nabla}(z)$. These Hamiltonians commute pairwise. Then defining the density of flux of the Hamiltonian $H_{\alpha,p-1}$ along the flow $t^{\beta,q}$, denoted $f_{\alpha,p;\beta,q}$, by the equation $\partial_{t^{\beta,q}} h_{\alpha,p} = \partial_x f_{\alpha,p;\beta,q}$ allows to define the

Definition 4.8 τ -function of the principal hierarchy

$$\mathcal{F}(\mathbf{t}) = \log \tau(\mathbf{t}) = \frac{1}{2} \sum_{\substack{\alpha,p \\ \beta,q}} f(\mathbf{t})_{\alpha,p;\beta,q} t^{\alpha,p} t^{\beta,q}. \quad (4-14)$$

\mathcal{F} is the corresponding tree-level free energy.

All known integrable hierarchies admit such a principal hierarchy as *semi-classical* (or dispersionless) limit and conversely, so-called *quasi-triviality* transformations introduce a dispersion parameter back into a principal hierarchy [38]. They are Miura-type deformations of the corresponding Poisson pencils.

We will argue in the following that some of the quasi-triviality transformations turning dispersionless into dispersive integrable systems can be realized by the topological recursion of [43]. They are the ones given in a Lax form corresponding to very special solutions of the associated loop equations, in the spirit of [16] and [39]. We will accordingly give a simple scheme of study using WKB techniques expected to work away from semi-simple points.

5 Hitchin system and moduli spaces of flat connections

5.1 Geometric setting

In 1986, N. Hitchin was studying the condition under which a connection in a principal differential G -bundle over \mathbb{R}^4 , where G is a compact connected Lie group, written as $A = \sum_{i=1}^4 A_i dx_i$, has a self dual curvature tensor $F = dA + \frac{1}{2}[A, A] = \sum_{i<j} F_{ij} dx_i \wedge dx_j$, that is satisfying the Yang-Mills self-duality equations $*F = F|_{\mathbb{R}^4}$ where $*$ is the Hodge star operator [55]. He assumed furthermore that the connection potential A did not depend on the coordinates (x_3, x_4) . This translation invariance in half the directions has no clear physical interpretation a priori (though intuition for two dimensional systems lie in condensed matter). Indeed, imposing translation invariance in one direction yields instanton solutions related to magnetic monopoles. Going one step further and considering solutions with translation invariance along two independent directions will in term lead to a conformally invariant theory on a Riemann surface.

Introducing the complex parameter $z = x_1 + ix_2$ over the complex plane and changing variables to

$$D \stackrel{\text{def}}{=} A_1 dx_1 + A_2 dx_2 = \frac{A_1 - iA_2}{2} d(x_1 + ix_2) + \frac{A_1 + iA_2}{2} d(x_1 - ix_2) \quad (5-1)$$

$$= D^{1,0} + D^{0,1} \quad (5-2)$$

$$\phi \stackrel{\text{def}}{=} \frac{A_3 - iA_4}{\sqrt{-2i}} d(x_1 + ix_2) \quad (5-3)$$

he obtained the following

Theorem 5.1 *Hitchin's equation*

The Yang-Mills self-duality equations are in this case equivalent to the following system:

$$F_D + [\phi, \phi^\dagger] = 0 \quad (5-4)$$

$$D^{0,1}\phi = 0 \quad (5-5)$$

where F_D is the curvature associated to the 2-dimensional connection D , \dagger is complex conjugation and $D^{0,1}$ acts on ϕ by partial covariant derivation.

Proposition 5.2 *Let $\overset{\circ}{\Sigma}$ be a Riemann surface and denote by $\mathcal{M}_H \stackrel{\text{def}}{=} \{(D, \phi, \mu)\}$*

the moduli space of solutions to the Hitchin equations on $\overset{\circ}{\Sigma}$ taken together with a hermitian metric on the surface, it is diffeomorphic to the Dolbeau moduli space $\mathcal{M}_{Dol} \stackrel{\text{def}}{=} \{(P, \phi)\}$ of Higgs pairs, that is stable holomorphic principal G -bundle over Σ together with a Higgs field $\phi \in H^0(\Sigma, \text{Ad}(P) \otimes K_\Sigma)$, where K is the canonical bundle functor. \mathcal{M}_{Dol} then comes equipped with a complex structure that we will denote I and coming from that of $\overset{\circ}{\Sigma}$.

5.2 The Hitchin system

When $\mathfrak{g} = \mathfrak{gl}_r(\mathbb{C})$, it is usual to associate to a Higgs field its spectral cover, the family of characteristic polynomials of the values it takes over the base curve. For general finite dimensional reductive Lie algebras, taking the determinant requires to choose a representation and the whole construction therefore depends upon this choice. In [36] was understood that a

generalization of the spectral cover, namely the cameral cover, was more fundamental and independent of a choice of representation.

Here we introduce the Hitchin fibration [55], it is the mathematical framework that will describe the moduli space of possible classical (or $\varepsilon \rightarrow 0$) limit. To emphasize this fact we will from now on denote Higgs fields by $\Phi^{(0)}$ instead of ϕ . It will indeed appear as the limit of the difference of two ε -connections when $\varepsilon \rightarrow 0$.

Let \mathfrak{g} be the finite-dimensional reductive complex Lie algebra of a connected Lie group G , and choose a complex curve $\overset{o}{\Sigma}$, with genus $\overset{o}{g}$ and $M \in \mathbb{N}$ punctures such that the stability condition $2\overset{o}{g} - 2 + M \geq 0$ is satisfied.

Definition 5.3 *Hitchin's fibration*

The Hitchin system [55] is the algebraically completely integrable system defined by the so-called Hitchin fibration

$$\mathcal{H}: \mathcal{M}_{Dol} \longrightarrow \mathcal{B} \stackrel{def}{=} \bigoplus_{i=1}^r H^0(\overset{o}{\Sigma}, K_{\overset{o}{\Sigma}}^{\otimes d_i}) \quad (5-6)$$

$$(P, \Phi^{(0)}) \longmapsto (\sigma_1(\Phi^{(0)}), \dots, \sigma_r(\Phi^{(0)})) \quad (5-7)$$

where d_i is the degree of σ_i in a basis $\sigma_1, \dots, \sigma_r \in \mathbb{C}[\mathfrak{g}]^G$ of Ad-invariant polynomials. This map, sometimes also called the abelianization of the Hitchin moduli space, has fibers that are Lagrangian polarized abelian varieties (so-called distinguished Pryms), thus defining an algebraically completely integrable system.

5.3 Cameral covers

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of \mathfrak{g} and denote the associated Weyl group by the symbol $\mathfrak{w} \subset \text{Aut}(\mathfrak{h}^*)$. By Chevalley's theorem, the restriction of polynomials yields an isomorphism $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^{\mathfrak{w}}$, thus identifying G -invariant polynomial functions on \mathfrak{g} with \mathfrak{w} -invariant polynomial functions on the subalgebra \mathfrak{h} .

In turn, twisting the quotient map $\mathfrak{h}^* \rightarrow \mathfrak{h}^*/\mathfrak{w}$ by the canonical bundle yields a \mathfrak{w} -Galois cover $\pi_{\mathfrak{w}}: \mathfrak{h}^* \otimes K_{\overset{o}{\Sigma}} \rightarrow [\mathfrak{h}^* \otimes K_{\overset{o}{\Sigma}}]/\mathfrak{w}$, the image of which is isomorphic to the Hitchin base $[\mathfrak{h}^* \otimes K_{\overset{o}{\Sigma}}]/\mathfrak{w} \simeq \mathcal{B}$.

Definition 5.4 *Cameral cover*

The cameral cover associated to any regular point $b \in \mathcal{B}_{reg}$ of the Hitchin map is then defined as the fiber $\Sigma_b = \pi_w^{-1}(b) \subset \mathfrak{h}^* \otimes K_\Sigma^o$ and projecting onto the base induces a w -Galois cover $\Sigma_b \rightarrow \overset{o}{\Sigma}$, where the Weyl group acts naturally on the first factor by what we choose to be a right action.

If $\mathfrak{R}_o \subset \mathfrak{h}^*$ is a set of simple roots, then for any $\alpha \in \mathfrak{R}_o$ and any point $b \in \mathcal{B}$ in the base, evaluation by the Chevalley basis element E_α dual to the simple root α defines a projection $p_\alpha : \Sigma_b \rightarrow K_\Sigma^o$ whose image is realized as a Lagrangian immersion that we will denote $p_\alpha(\Sigma_b) = \mathcal{S}_{b,\alpha} \subset T^*\overset{o}{\Sigma}$.

$$\begin{array}{ccccc} \Sigma_b & \xrightarrow{p_\alpha} & \mathcal{S}_{b,\alpha} & \xrightarrow{i_{b,\alpha}} & T^*\overset{o}{\Sigma} \\ & \searrow & \downarrow & \swarrow & \\ & & \overset{o}{\Sigma} & & \end{array}$$

Definition 5.5 *Spectral curve*

For a generic cameral cover and a generic pair of simple roots $\alpha, \beta \in \mathfrak{R}_o$, the curves $\mathcal{S}_{b,\alpha}$ and $\mathcal{S}_{b,\beta}$ have the same desingularization that we denote \mathcal{S}_b and that we shall call the underlying spectral curve. The various projection maps introduced here are equivariant in the sense that for any $y \in \Sigma_b$ and $\alpha, \beta \in \mathfrak{R}_o$, $p_\alpha(y) = p_{\sigma_{\alpha\beta}\alpha}(\sigma_{\alpha\beta} \cdot y)$, where we denoted the element of the Weyl group acting as the transposition exchanging the simple roots α and β by $\sigma_{\alpha\beta} \in w \subset \text{Aut}(\mathfrak{h}^*)$.

Definition 5.6 *Branch points*

The set of branch points $\text{br}(\Sigma_b)$ of this covering map is the set of points $a \in \Sigma_b$ that are fixed by the action of a Weyl group element, that is

$$\text{br}(\Sigma_b) \stackrel{\text{def}}{=} \{a \in \Sigma_b \mid \exists \sigma \in w, \sigma \cdot a = a\} \quad (5-8)$$

Remark 5.7 The fiberwise group action of a Galois cover being transitive, all the preimages of a given basepoint are alike in the sense that either all the points of a fiber are ramification points or it contains no branch point at all.

At one of these branch points $a \in \text{br}(\Sigma_b)$, generically simple, one has a unique order 2 Weyl group element $\sigma_a \in \mathfrak{w}$ that fixes it.

In the case $\mathfrak{g} = \mathfrak{gl}_r(\mathbb{C})$ in its fundamental representation, these evaluation correspond to the coefficients of the characteristic polynomial, valued in the ring of differential forms over the curve, seen as sections of the tensor products of the canonical bundle over Σ with itself, they can be collected in $\text{Det}(y - \Phi^{(0)}) \in \mathcal{B}$ where y is a formal variable.

5.4 Deformation of Higgs bundles

Let us denote by $\mathcal{U}_{G, \overset{\circ}{g}, M}$ the moduli space of stable holomorphic principal G -bundles P over stable curves $\overset{\circ}{\Sigma}$ of genus $\overset{\circ}{g} \in \mathbb{N}$ with $M \in \mathbb{N}$ punctures, that is such that $2 - 2\overset{\circ}{g} - M < 0$, up to isomorphism. It is a bundle over the corresponding moduli space of stable curves $\mathcal{M}_{\overset{\circ}{g}, M}$. We then have the following

Proposition 5.8 *The Hitchin moduli space via cohomology*

The fiber of the deformation space, the total space of the cotangent bundle, $T^\mathcal{U}_{G, \overset{\circ}{g}, M}$ over a point $P \longrightarrow \overset{\circ}{\Sigma} \in \mathcal{U}_{G, \overset{\circ}{g}, M}$ is isomorphic to $H^0(\overset{\circ}{\Sigma}, \text{Ad}(P) \otimes K_{\overset{\circ}{\Sigma}})$.*

proof:

Isomorphism classes of holomorphic principal G -bundles are uniquely determined by isomorphism classes of transition functions on charts and that are valued in G . These transition functions should satisfy inverse, cocycle and restriction relations such that their isomorphism classes define unique elements of $H^1(\overset{\circ}{\Sigma}, G)$. This correspondence is a biholomorphism and by deformation (in this case going to the tangent bundle) yields that, for any choice of principal bundle $P \longrightarrow \overset{\circ}{\Sigma} \in \mathcal{U}_{G, \overset{\circ}{g}, M}$, $T_P\mathcal{U}_{G, \overset{\circ}{g}, M} = H^1(\overset{\circ}{\Sigma}, \text{Ad}(P))$ which allows to conclude by Serre duality and identifying $\text{Ad}(P) \simeq \text{Ad}(P)^*$.

■

In this study, we will thoroughly rely on the fact that the Hitchin system is very closely related to the moduli space of meromorphic connections in principal G -bundles over curves. When describing the geometry of Fuchsian differential systems we will count the following dimension.

Proposition 5.9 *Generic dimension of a fiber of the Hitchin map*
Considering Higgs fields with at most simple poles at the punctures (so-called Fuchsian case), we have

$$\dim H^0(\overset{o}{\Sigma}, \text{Ad}(P) \otimes K_{\overset{o}{\Sigma}}) = \frac{1}{2} [(2\overset{o}{g} - 2 + M) \dim \mathfrak{g} - M \text{rk} \mathfrak{g}] \quad (5-9)$$

where $\text{rk} \mathfrak{g}$ is the dimension of any Cartan subalgebra of \mathfrak{g} .

In the special cases where the Lie algebra of the structural group is $\mathfrak{g} = \mathfrak{sl}_r(\mathbb{C})$ with no punctures (in particular with $\overset{o}{g} \geq 2$ and globally holomorphic Higgs fields),

$$\dim \mathcal{M}_{Dol} = 2(r^2 - 1)(\overset{o}{g} - 1) \quad (5-10)$$

$$\dim H^0(\overset{o}{\Sigma}, \text{Ad}(P) \otimes K_{\overset{o}{\Sigma}}) = (r^2 - 1)(\overset{o}{g} - 1) \quad (5-11)$$

$$\dim \mathcal{B} = (r^2 - 1)(\overset{o}{g} - 1) \quad (5-12)$$

5.5 Non-abelian Hodge correspondence

$\mathcal{M}_{Dol} \simeq T^* \mathcal{U}_{G, \overset{o}{g}, M}$ is the total space of a cotangent bundle and therefore comes equipped with a natural symplectic structure such that in particular it is a Poisson manifold. Moreover,

Property 5.10 \mathcal{M}_{Dol} has a natural hermitian metric for which it is a Kähler manifold.

Property 5.11 The natural metric on \mathcal{M}_{Dol} is hyperkähler.

We now relate the moduli space \mathcal{M}_{Dol} to \mathcal{M}_{dR} , the moduli space of holomorphic connections in G -bundles over stable curves with genus g and number of punctures M satisfying $2\overset{o}{g} - 2 + M > 0$.

Proposition 5.12 *Connection counterparts to Hitchin pairs*
 \mathcal{M}_{Dol} is diffeomorphic to \mathcal{M}_{dR} .

5.6 Relation to integrability

Recall that the Hitchin map $\mathcal{M}_{\text{Dol}} \longrightarrow \mathcal{B}$ sends a Higgs field to its spectral invariants.

Definition 5.13 *Vertical deformations*

Families of element in \mathcal{M}_{Dol} that have the same image by the Hitchin map will be called vertical, or isospectral, deformation families of Hitchin pairs.

Theorem 5.14 *Integrability of Hitchin systems*

The Hitchin map induces an abelian Lagrangian fibration of \mathcal{M}_{Dol} in which the Hamiltonian flows with respect to the (Poisson commuting) spectral invariants are integrable isospectral deformations of Hitchin pairs.

Remark 5.15 *Although we are not proving the theorem here, we just note that in this formalism, the Poisson commutation of the Hamiltonians is a rather easy result to get. Indeed they are the spectral invariant functions of the Higgs field and therefore do not depend on a (polarized) half of a Darboux set of coordinates on \mathcal{M}_{Dol} , namely the dependence on the moduli of the principal bundle. In turn, their Poisson brackets vanish.*

Chapter 2

Geometry of the Fuchsian system

We have argued in the introduction that in order to study integrable systems given in convenient Lax pairs, we would consider a corresponding G -equivariant local system on a base Riemann surface $\overset{o}{\Sigma}$. These are in one to one correspondence with connections ∇ in principal G -bundles P over the same curve, where G is a connected complex reductive Lie group. This space of connections is an affine space and we will pick a reference holomorphic connection ∇_0 .

For the time being, we abstract ourselves from the underlying Lax formulation and consider the triplet (P, ∇_0, ∇) allowing to associate to any choice of faithful representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}_d(\mathbb{C})$ [9] a sequence $\{W_n\}_{n \in \mathbb{N}^*}$ of so-called correlators defined on a bundle $\widehat{\Sigma} \rightarrow \overset{o}{\Sigma}$. They will be shown to satisfy constraints called *loop equations* [16], [30], [15], [9]. These take the form of symmetrizations of the W_n 's, with respect to Casimir elements of the Lie algebra \mathfrak{g} in some representation and they end up having nice analytic properties in the variable $x \in \overset{o}{\Sigma}$.

From these correlators we will associate a notion of cycles to $\widehat{\Sigma}$ that will allow to describe deformations of $\Phi \stackrel{def}{=} \nabla_0 - \nabla$ and derive special geometry relations of Seiberg-Witten type [11], [12]. This construction is to be understood as introducing dispersion into the Riemannian geometry of complex curves, generalizing it non-perturbatively. The space of cycles carries a symplectic structure such that to any Lagrangian subspace of cycles one can associate a τ -function. We conjecture that when the triplet (P, ∇_0, ∇) comes from an integrable hierarchy, then it coincides with

its non-perturbative τ -function as defined in [38]. To motivate solving the loop equations let us mention that it was shown in many cases [21], [23] that generating functions for derivatives of the τ -function with respect to the various times of the hierarchy take the form of correlators $\{W_n\}_{n \in \mathbb{N}^*}$.

When a set of hypothesis called the *Topological Type* are satisfied (including in particular the fact that the correlators have to admit a topological expansions of the form $W_n = \sum_{g \geq 0} \varepsilon^{2g-2+n} \omega_{g,n}$ in terms of a formal *small* parameter $\varepsilon \neq 0$), then a topological recursion procedure [43], [15], [8] allows to reconstruct perturbatively the expansions. It is a recursive algorithm to compute the $\omega_{g,n}$'s from complex geometry of a cover of $\overset{o}{\Sigma}$. Several situations can then occur.

First, independently of the choice of representation of \mathfrak{g} , one can introduce [8] a new version of the topological recursion that we call the *cameral curve topological recursion*. To do so we will have to extract from the starting data an object called a Higgs bundle [55] to which is canonically associated an algebraic covering called the *cameral cover*. We will then associate a so-called *cameral geometry* that will be of use in the developments, recall the usual formulation of the topological recursion and then its cameral version. After this is done we will reconstruct solutions to the loop equations in the same way as in [9] to show in turn that the cameral curve topological recursion allows to solve them perturbatively when the topological type property is satisfied. We will then end investigating a possible application to integrable hierarchies, where the parameter space \mathcal{T} is infinite dimensional.

Second, we will use the choice of representation ρ to associate a projection of the cameral cover called the *spectral curve* associated to the $\varepsilon \rightarrow 0$ limit of the setup. We will then give necessary conditions for the Topological Type property to be satisfied [10] for a special case (where in particular $\overset{o}{\Sigma}$ has genus 0 and \mathfrak{g} is a matrix Lie algebra).

1 Non-perturbative aspects

1.1 Correlation functions and loop equations

Let $P \longrightarrow \overset{o}{\Sigma}$ be a holomorphic principal G -bundle with a holomorphic reference connection ∇_0 and a meromorphic connection ∇ , and let $\underline{\pi}_1 \longrightarrow \overset{o}{\Sigma}$ be the family of fundamental groups, it has fiber $\pi_1(\overset{o}{\Sigma}, x)$ over a generic point $x \in \overset{o}{\Sigma}$.

Define the quotient space

Definition 1.1 *Non-perturbative spectral curve as a $\underline{\pi}_1$ -invariant bundle*

$$\widehat{\Sigma} \stackrel{def}{=} \left[\underline{\pi}_1 \times_{\overset{o}{\Sigma}} \text{Ad}(P) \right] / \underline{\pi}_1 \quad (1-1)$$

It is an orbifold of dimension $\dim \mathfrak{g}$ and is naturally immersed in the total space of $\text{Ad}(P)$ by the map

$$\begin{aligned} M: \quad \widehat{\Sigma} &\longrightarrow \text{Ad}(P) \\ X = [\gamma \cdot E] &\longmapsto \text{Hol}_{\gamma}^{\nabla} E \end{aligned} \quad (1-2)$$

where $X \in \widehat{\Sigma}$ over a point $x \in \overset{o}{\Sigma}$ is represented by choosing a homology class of closed paths $\gamma \in \pi_1(\overset{o}{\Sigma}, x)$ and an element $E \in \text{Ad}(P)_x$. This definition is easily seen to be independent of the choice of representative and fits into the short exact sequence of bundles

$$1 \longrightarrow \underline{\pi}_1 \longrightarrow \underline{\pi}_1 \times_{\overset{o}{\Sigma}} \text{Ad}(P) \xrightarrow{M} \widehat{\Sigma}_{\Psi} \longrightarrow 0 \quad (1-3)$$

Let ρ be a faithful matrix realization of \mathfrak{g} in $\mathfrak{M}_d(\mathbb{C})$ for a certain integer $d \in \mathbb{N}^*$. The trace in this representation yields a bracket on the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$

$$\langle . \rangle = \text{Tr}_{\rho}: \mathcal{U}(\mathfrak{g}) \longrightarrow \mathbb{C} \quad (1-4)$$

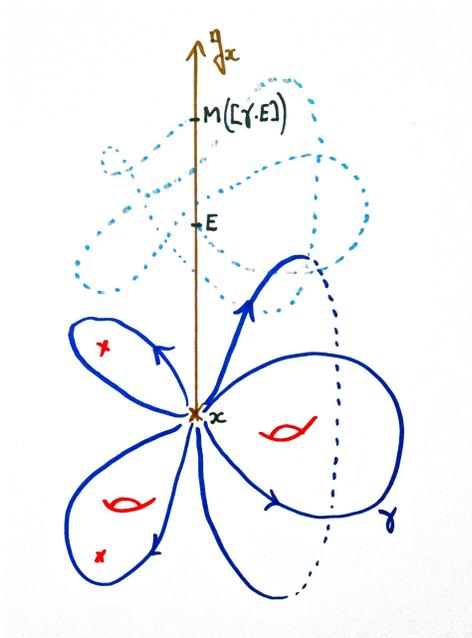


Figure 2.1: The map M realizes the holonomy pairing.

The difference of the two connections is an $\text{Ad}(P)$ -valued meromorphic branch point on $\overset{o}{\Sigma}$. We will denote it by

$$\Phi_{def} = \nabla_0 - \nabla \in H^0(\overset{o}{\Sigma}, \text{Ad}(P) \otimes K_{\overset{o}{\Sigma}}). \quad (1-5)$$

We restrict our study to the Fuchsian case, that is when Φ has at most regular singularities at the punctures ($M \in \mathbb{N}$ non-compact directions on $\overset{o}{\Sigma}$) and denote the moduli space of all such equivariant connections as \mathcal{M}_{Fuchs} (it has many strata such as the ones consisting of the cases when the base curve $\overset{o}{\Sigma}$ is assumed to be any Riemann surface with fixed genus and number of punctures). This hypothesis is less restrictive than it seems. Indeed, connections with higher order poles, so-called *irregular singularities* [25], [26], can be realized as Fuchsian connections with collapsing tuples of singularities which correspond to non-compact directions of \mathcal{M}_{Fuchs} .

The differential system associated to the principal G -bundle P by the connection $\nabla \in \mathcal{M}_{Fuchs}$ in this context takes the form of the linear differential equation for flat local sections $\nabla_0 \Psi = \Phi \cdot \Psi$ over $\overset{o}{\Sigma}$ (here \cdot denotes

Lie derivation from the left). If $o \in \overset{o}{\Sigma}$ denotes a reference point and \mathfrak{g}_o its adjoint fiber, then the local solutions to the connection problem can be analytically continued starting at o to global sections of the principal G -bundle P pulled back to the universal cover $\overset{o}{\Sigma}$ of the curve. We will denote the set of these ∇ -flat local sections by $H_{loc}^0(\overset{o}{\Sigma}, P)_0$ and choosing such a solution yields the isomorphism

Lemma 1.2 *Coordinate representation*

$$\widehat{\Sigma} \simeq [\overset{o}{\Sigma} \times \mathfrak{g}_o] / \pi_1(\overset{o}{\Sigma}, o) \stackrel{def}{=} \widehat{\Sigma}_\Psi \quad (1-6)$$

where the quotient is relative to the monodromy action of ∇ at the reference point $o \in \overset{o}{\Sigma}$. It is the natural action given for any $\gamma \in \pi_1(\overset{o}{\Sigma}, o)$ and $(\tilde{x}, F) \in \overset{o}{\Sigma} \times \mathfrak{g}_o$ by

$$\gamma \cdot (\tilde{x}, F) = (\tilde{x} + \gamma, \text{Hol}_{\gamma^{-1}}^\nabla F) \quad (1-7)$$

where $\tilde{x} + \gamma$ means concatenation of homology classes in reading order. Identifying $\widehat{\Sigma}_\Psi$ with the image of the map M_Ψ , it fits into the short exact sequence

$$1 \longrightarrow \pi_1(\overset{o}{\Sigma}, o) \longrightarrow \overset{o}{\Sigma} \times \mathfrak{g}_o \xrightarrow{M_\Psi} \widehat{\Sigma}_\Psi \longrightarrow 0 \quad (1-8)$$

Corollary 1.3 *The choice of Ψ yields the isomorphism*

$$\begin{aligned} M_\Psi : \quad \widehat{\Sigma}_\Psi &\xrightarrow{\sim} \text{Ad}(P) \\ [\tilde{x} \cdot F] &\longmapsto \text{Ad}_{\Psi(\tilde{x})} F \end{aligned} \quad (1-9)$$

allowing the identification of $[\gamma \cdot E] \in \widehat{\Sigma}$ and $[\tilde{x} \cdot F] \in \widehat{\Sigma}_\Psi$ whenever the equalities $M(X) = \text{Hol}_\gamma^\nabla E = \text{Ad}_{\Psi(\tilde{x})} F = M_\Psi([\tilde{x} \cdot F])$ hold.

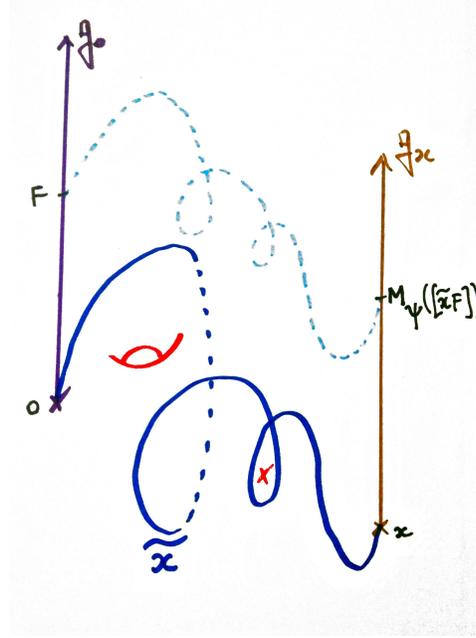


Figure 2.2: The map M_Ψ realizes parallel transport along continuous paths starting at o .

Remark 1.4 *This map yields a nice geometrical interpretation of $\widehat{\Sigma}_\Psi$. Indeed, M_Ψ satisfies almost by definition the differential system*

$$\nabla_0 M_\Psi([\tilde{x} \cdot F]) = [\Phi(x), M_\Psi([\tilde{x} \cdot F])] \quad (1-10)$$

at any generic $\tilde{x} \in \overset{o}{\widehat{\Sigma}}$ and for any fixed $F \in \mathfrak{g}_o$. Conversely, any ∇ -flat local section of the adjoint bundle $\mathfrak{s} \in H_{loc}^0(\overset{o}{\widehat{\Sigma}}, \text{Ad}P)_0$ is easily shown to satisfy

$$\mathfrak{s}(\tilde{x}) = M_\Psi([\tilde{x} \cdot \text{Ad}_{\Psi(\{o\})^{-1}} \mathfrak{s}(\{o\})]) \quad (1-11)$$

such that $\widehat{\Sigma}_\Psi$ gives a ∇ -flat parametrization of $\text{Ad}P$ in terms of values of the corresponding flat local sections.

Definition 1.5 *Monodromy matrices*

For any choice of ∇ -flat local section Ψ , the corresponding monodromy matrices are defined for any loop $\gamma \in \pi(\overset{o}{\widehat{\Sigma}}, o)$ by

$$S_\gamma \stackrel{\text{def}}{=} \Psi(\tilde{x})^{-1} \Psi(\tilde{x} + \gamma) \quad (1-12)$$

Remark 1.6 *This allows to rewrite the definition of the $\pi(\overset{o}{\Sigma}, o)$ -action as*

$$\gamma \cdot (\tilde{x}, E) = (\tilde{x} + \gamma, \text{Ad}_{S_\gamma^{-1}} E) \quad (1-13)$$

To this data is now associated a sequence of so-called connected correlators, or n -points function $\{\widehat{W}_n\}_{n \in \mathbb{N}^*}$ by

Definition 1.7 *Connected correlators*

$$\begin{aligned} \widehat{W}_1 : \widehat{\Sigma}_\Psi &\longrightarrow \mathbb{K}_\Sigma^o \\ X &\longmapsto \langle M_\Psi(X) \Phi(x) \rangle \\ \forall n \geq 2, \widehat{W}_n : \widehat{\Sigma}_\Psi^{\boxtimes n} &\longrightarrow S^{\boxtimes n}(\mathbb{K}_\Sigma^o) \\ X_1, \dots, X_n &\longmapsto (-1)^{n-1} \sum_{\tau \in \mathfrak{S}_n^c} \frac{\langle M_\Psi(X_1) \cdots M_\Psi(X_{\tau^{n-1}(1)}) \rangle_o}{\mathcal{E}_f(\tilde{x}_1, \tilde{x}_{\tau(1)}) \cdots \mathcal{E}_f(\tilde{x}_{\tau^{n-1}(1)}, \tilde{x}_1)} \end{aligned} \quad (1-14)$$

$$(1-15)$$

where the sum is over so-called connected permutations \mathfrak{S}_n^c (containing a single element in their cycle decomposition) and this formula for $\widehat{W}_n(X_1, \dots, X_n)$ is extended linearly to

$$\widehat{\Sigma}_\Psi^{\boxtimes n} = \text{Span}_{\mathbb{C}} \{X_1 \otimes \cdots \otimes X_n \mid \forall i, j \in \{1, \dots, n\}, \tilde{x}_i \neq \tilde{x}_j\} \quad (1-16)$$

the n^{th} tensor power of $\widehat{\Sigma}_\Psi$ at different base points (remove a codimension 1 subspace, namely the diagonal divisor), S^{\boxtimes} is the symmetric tensor product over distinct base points, $\langle \cdot \rangle_o$ means that all objects are parallel transported one to the base point of the next by the reference connection ∇_0 through the reference point o , and \mathcal{E}_f is a twisted prime-form over $\overset{o}{\Sigma}$. Recall that the twist consists in the introduction of essential singularities in Fay's (untwisted) prime-form to annihilate its monodromies around non-contractible cycles of the curve [45]. These singularities appear as $e^{\int_x^y f}$ for a certain choice of a meromorphic 1-form f . In turn they vanish in the expressions of the W_n 's yielding that the correlators are meromorphic (globally defined) objects on $\overset{o}{\Sigma}$.

These correlators naturally appear in various contexts such as random matrix models, conformal field theories [35], Painlevé equations [17], [57], [58] or even cohomological field theories [22].

Remark 1.8 *The \widehat{W}_n 's seem to depend heavily on a choice of Ψ however, switching to another solution of the flat section equation $\nabla \cdot \Psi = 0$ amounts to multiplying Ψ from the right by a given element $C \in G_o$, $\Psi' \stackrel{\text{def}}{=} \Psi C$, in the fiber G_o of the principal bundle over the reference point $o \in \overset{\circ}{\Sigma}$. Such different solutions have conjugate monodromy data in the sense that*

$$(\Psi(\tilde{x})C)^{-1}(\Psi(\tilde{x} + \gamma)C) = \text{Ad}_{C^{-1}}(\Psi(\tilde{x})^{-1}\Psi(\tilde{x} + \gamma)) \quad (1-17)$$

$$= \text{Ad}_{C^{-1}}S_\gamma \quad (1-18)$$

which immediately implies

$$M_{\Psi C}([\tilde{x} + \gamma \cdot \text{Ad}_{C^{-1}S_\gamma^{-1}} E]) = M_\Psi([\tilde{x} \cdot E]) \quad (1-19)$$

for any $C \in G_o$ and $\gamma \in \pi_1(\overset{\circ}{\Sigma}, o)$. As a consequence, the whole construction is $G \times \pi_1(\overset{\circ}{\Sigma}, o)$ equivariant on the trivial G -bundle of flat sections Ψ over $\mathcal{M}_{\text{Fuchs}}$ (actually over the pullback of $\mathcal{M}_{\text{Fuchs}}$ to the universal curve where the reference point o lives).

One of the fundamental and most important properties of the correlators, apart from being globally defined meromorphic on the base curve $\overset{\circ}{\Sigma}$, is their symmetry.

Proposition 1.9 *Symmetry of the correlators*

For any $n \in \mathbb{N}^$, W_n is a symmetric function of its arguments.*

proof:

The cyclic symmetry of the bracket $\langle \cdot \rangle_o$ implies that W_n can be rewritten as a sum over the whole set of permutations \mathfrak{S}_n as

$$W_n(X_1, \dots, X_n) = \frac{(-1)^{n-1}}{n} \sum_{\tau \in \mathfrak{S}_n} \frac{\langle M_\Psi(X_{\tau(1)}) \cdots M_\Psi(X_{\tau(n)}) \rangle_o}{\mathcal{E}_f(\tilde{x}_{\tau(1)}, \tilde{x}_{\tau(2)}) \cdots \mathcal{E}_f(\tilde{x}_{\tau(n)}, \tilde{x}_{\tau(1)})}, \quad (1-20)$$

expression through which the symmetry is now explicit. ■

Definition 1.10 *Non-connected correlators*

$$W_n(X_1, \dots, X_n) = \sum_{\mu \vdash \{X_1, \dots, X_n\}} \prod_{i=1}^{\ell(\mu)} \widehat{W}_{|\mu_i|}(\mu_i) \quad (1-21)$$

for any integer $n \in \mathbb{N}^*$.

For example

$$W_1(X_1) = \widehat{W}_1(X_1), \quad (1-22)$$

$$W_2(X_1, X_2) = \widehat{W}_1(X_1)\widehat{W}_1(X_2) + \widehat{W}_2(X_1, X_2), \quad (1-23)$$

$$\begin{aligned} W_3(X_1, X_2, X_3) &= \widehat{W}_1(X_1)\widehat{W}_1(X_2)\widehat{W}_1(X_3) + \widehat{W}_1(X_1)\widehat{W}_2(X_2, X_3) \\ &\quad + \widehat{W}_1(X_2)\widehat{W}_2(X_1, X_3) + \widehat{W}_1(X_3)\widehat{W}_2(X_1, X_2) \\ &\quad + \widehat{W}_3(X_1, X_2, X_3) \end{aligned} \quad (1-24)$$

and so on ...

An important remark at this point is that these definitions are singular whenever two of the X_i 's have coinciding base points. Their behaviour near the diagonal divisor is described by the following lemma that will be useful in particular when comparing the correlators to conformal field theory amplitudes in the last chapter.

Lemma 1.11 *Coinciding base-points singularities*

Let $n \in \mathbb{N}^*$ and consider $n+1$ points $(X, X_1, \dots, X_n) \in \widehat{\Sigma}^{\boxtimes n+1}$ of the non-perturbative spectral curve with distinct base points. Then $\widehat{W}_{n+1}(X, X_1, \dots, X_n)$ is a meromorphic 1-form of the variable defined by $x \stackrel{\text{def}}{=} \pi(X) \in \widehat{\Sigma}$ and for $n > 1$, it has a simple pole at each point $x_i \stackrel{\text{def}}{=} \pi(X_i)$, for $i \in \{1, \dots, n\}$, with residue

$$\text{Res}_{x=x_i} \widehat{W}_{n+1}(X, X_1, \dots, X_n) = \widehat{W}_n(\dots X_{i-1}, [X, X_i], X_{i+1}, \dots) \quad (1-25)$$

where $[X, X_i]$ in \widehat{W}_n is defined by replacing $M(X_i)$ by $[\lim_{x \rightarrow x_i} M(X), M(X_i)]$ in the expression of $\widehat{W}_n(X_1, \dots, X_n)$. For $n = 1$, $W_2(X, X_1)$ is a meromorphic one-form of x with a double pole at x_1 with bi-residue 1 and residue given by the same formula.

proof:

For $n = 1$ it can be read from the definition. For $n > 1$, it is a straightforward computation in local coordinates around $x \sim x_i$, denoting $x = x_0$ and considering \mathfrak{S}_{n+1}^c as 1-cycle permutations of $\{0, 1, \dots, n\}$,

$$\begin{aligned} \widehat{W}_{n+1}(X, X_1, \dots, X_n) &\underset{x \rightarrow x_i}{\sim} \sum_{\substack{\tau \in \mathfrak{S}_{n+1}^c \\ \tau(0)=i}} (-1)^\tau \frac{\langle \dots M(X)M(X_i) \dots \rangle_o}{\dots (x-x_i) \dots} \sqrt{dx dx_i} \\ &\quad + \sum_{\substack{\tau \in \mathfrak{S}_{n+1}^c \\ \tau^{-1}(0)=i}} (-1)^\tau \frac{\langle \dots M(X_i)M(X) \dots \rangle_o}{\dots (x_i-x) \dots} \sqrt{dx dx_i} \end{aligned} \quad (1-26)$$

$$= \sum_{\substack{\tau \in \mathfrak{S}_{n+1}^c \\ \tau(0)=i}} (-1)^\tau \frac{\langle \dots M(X_{\tau^{-1}(0)}) [M(X)M(X_i)] M(X_{\tau(i)}) \dots \rangle_o}{\dots \mathcal{E}(x_{\tau^{-1}(0)}, x)(x-x_i) \mathcal{E}(x_i, x_{\tau(i)}) \dots} \sqrt{dx dx_i} \quad (1-27)$$

where the last equality is obtained by associating term by term the dummy permutation τ to $(0i)\tau^{-1}(0i)$, $(0i)$ being the transposition exchanging 0 and i . Notice that the last sum is realized as a sum over $\sigma' \in \mathfrak{S}_n^c$ where σ' is such that $\sigma'(\tau^{-1}(0)) = i$ and for any $k \in \{1, \dots, n\}$, $k \neq \tau^{-1}(0)$, $\sigma'(k) = \tau(k)$. Reading the residue yields the wanted result. ■

In what follows, and in particular to write the loop equations, we will need a *prescription* to evaluate the correlators at these singular points. The prescription goes as follows :

Definition 1.12 *Normal ordering /coinciding base-points prescription*

For any given $n \in \mathbb{N}^*$, starting from the explicit definition of the connected correlator $\widehat{W}_n(X_1, \dots, X_n)$ for a generic n -tuple $(X_1, \dots, X_n) \in \widehat{\Sigma}_\Psi$, make the replacement

$$\frac{1}{\mathcal{E}_f(\tilde{x}_i, \tilde{x}_j)} \xrightarrow{x_i=x_j} \Phi(x) \quad (1-28)$$

explicitly in the definition when the base points x_i and x_j , of X_i and X_j respectively, are assumed identical. It practically amounts to removing the singular part and then taking the wanted limit, using the differential equation satisfied by M_Ψ . It extends the definition of the correlators (connected as well as non-connected) to the diagonal divisor and is closely related to the normal ordering of operators in conformal field theories.

Recall that the center $\mathcal{Z}(\mathcal{U}(\mathfrak{g})) \subset \mathcal{U}(\mathfrak{g})$ of the universal enveloping algebra of \mathfrak{g} is generated by $d + 1$ elements $\{\mathcal{C}_k\}_{0 \leq k \leq d}$ called the Casimirs of \mathfrak{g} in representation ρ and defined by

$$\sum_{k=0}^d (-1)^k y^{d-k} \mathcal{C}_k(\nu) \stackrel{\text{def}}{=} \underset{\rho}{\text{Det}}(y - \nu) \quad (1-29)$$

where $y \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ is a formal variable and \mathcal{C}_0 is the identity element of $\mathcal{U}(\mathfrak{g})$. As such, $\mathcal{C}_k \in (\mathfrak{g}^*)^{\otimes k} \simeq \mathfrak{g}^{\otimes k}$ has a class in $\mathcal{U}(\mathfrak{g})$ that we still denote \mathcal{C}_k (some of them may be trivial).

Definition 1.13 *Insertion of Casimirs*

This allows to define the insertion of Casimirs in the correlation functions for any $n, k \in \mathbb{N}$, generic $(X_1, \dots, X_n) \in \widehat{\Sigma}_{\Psi}^{\boxtimes n}$ and $\tilde{x} \in \widehat{\Sigma}$ such that $x \neq x_i$ for any given $i \in \{1, \dots, n\}$, by

$$W_{k;n}(\tilde{x}; X_1, \dots, X_n) \stackrel{\text{def}}{=} W_{k+n}([\tilde{x} \cdot \mathcal{C}_k], X_1, \dots, X_n) \quad (1-30)$$

with $[\tilde{x} \cdot \mathcal{C}_k]$ is defined by expanding \mathcal{C}_k in any tensor basis of $\mathfrak{g}^{\otimes k}$ and where we use thoroughly the prescription for evaluating correlators at coinciding base points. We can sometime denote formally the object thus obtained by

$$W_{k;n}(\tilde{x}; X_1, \dots, X_n) = \mathcal{C}_k(\tilde{x}) \cdot W_n(X_1, \dots, X_n) \quad (1-31)$$

to put emphasis on an action by the Casimir element.

The insertion of the k^{th} Casimir element \mathcal{C}_k into a correlator W_n can be viewed as a symmetrization of the k variables involved. Note eventually that the resulting object $\mathcal{C}_k \cdot W_n$ is still symmetric in the n unaffected variables.

These *insertions* of Casimirs together with the behaviour of the correlators near the diagonal divisor in $\widehat{\Sigma}$ will be the starting point to investigate the parallels between Fuchsian differential systems and conformal field theories with Casimir algebra symmetry [11], [12] where they will correspond to insertions of W-algebra generators. Their main property is that they satisfy an infinite set of relations called *loop equations* [16], [30], [15], [9].

Theorem 1.14 *Loop equations [B.-Eynard-Marchal]*

$$\sum_{k=0}^d (-1)^k y^{d-k} W_{k;n}(\tilde{x}; X_1, \dots, X_n) \stackrel{def}{=} P_n(x, y; X_1, \dots, X_n) \quad (1-32)$$

$$= [\epsilon_1 \dots \epsilon_n] \underset{\rho}{\text{Det}_o} (y - \Phi(x) - \mathcal{M}_\epsilon(x; X_1, \dots, X_n)) \quad (1-33)$$

$$\stackrel{def}{=} \sum_{k=0}^d (-1)^k y^{d-k} P_{k;n}(x; X_1, \dots, X_n) \quad (1-34)$$

where $y \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})) \otimes \mathbb{K}_{\Sigma}^o$ is a formal variable and for $\epsilon = (\epsilon_1, \dots, \epsilon_n)$,

$$\mathcal{M}_\epsilon(x; X_1, \dots, X_n) \stackrel{def}{=} \sum_{\substack{k=1 \\ 1 \leq i_1 < \dots < i_k \leq n}}^d \epsilon_{i_1} \dots \epsilon_{i_k} \frac{M_\Psi(X_{i_1}) \dots M_\Psi(X_{i_k})}{\mathcal{E}_f(\tilde{x}, \tilde{x}_{i_1}) \dots \mathcal{E}_f(\tilde{x}_{i_k}, \tilde{x})} \quad (1-35)$$

Moreover one has to be careful here and the subscript o in Det_o means that all the factors appearing in the determinant in the right hand side have to be parallel transported through the reference point $o \in \Sigma$ by connection ∇_0 before evaluation. Lastly, the symbol $[\epsilon_{i_1} \dots \epsilon_{i_k}]$ for any k -tuple of indices (i_1, \dots, i_k) is defined for any polynomial expression $p(\epsilon)$ of $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ by

$$p(\epsilon) \stackrel{def}{=} \sum_{k=0}^{\infty} \sum_{1 \leq i_1 < \dots < i_k \leq n} \epsilon_{i_1} \dots \epsilon_{i_k} [\epsilon_{i_1} \dots \epsilon_{i_k}] p(\epsilon) \quad (1-36)$$

where the sum has by definition only finitely many non-zero terms.

To prove this theorem we first need the following lemma which is obtained by straightforward computation using the definition of the correlators.

Lemma 1.15 *Determinantal formulae*

For any $n \in \mathbb{N}^*$ and generic $X_1, \dots, X_n \in \widehat{\Sigma}$,

$$W_n(X_1, \dots, X_n) = \left\langle \underset{1 \leq i, j \leq n}{\text{Det}} (E_i K_\Psi(\tilde{x}_i, \tilde{x}_j)) \right\rangle_o \quad (1-37)$$

with determinant defined as a sum over permutations, $X_i = [\tilde{x}_i \cdot E_i]$ is any choice of representative for $i \in \{1, \dots, n\}$ and where we introduced the formal kernel

$$K_\Psi(\tilde{x}_i, \tilde{x}_j) \stackrel{\text{def}}{=} \frac{\Psi(\tilde{x}_j)^{-1} \Psi(\tilde{x}_i)}{\mathcal{E}_f(\tilde{x}_i, \tilde{x}_j)} \quad (1-38)$$

to be inserted into brackets $\langle \cdot \rangle_o$. In particular, the factors in the numerator being evaluated over different basepoints, one has to use the normal ordering prescription thoroughly.

Remark 1.16 Since the determinant is that of a matrix with coefficients in a non-commutative space, an ordering prescription has to be chosen. We choose

$$\text{Det}_{n \times n}(A) \stackrel{\text{def}}{=} \sum_{\tau \in \mathfrak{S}_n} (-1)^\tau A_{1, \tau(1)} \cdots A_{n, \tau(n)} \quad (1-39)$$

We can now prove the theorem.

proof:

Let $(e_1, \dots, e_{\dim \mathfrak{g}})$ be a basis of \mathfrak{g}_o and consider its dual basis $(e^1, \dots, e^{\dim \mathfrak{g}})$ defined by requiring $\langle e_i, e^j \rangle = \delta_i^j$ and let $\Psi \in H_{loc}^0(\overset{o}{\Sigma}, \text{Ad}(P))_0$. On one hand,

$$\sum_{k=0}^d (-1)^k y^{d-k} W_{k;n}(\tilde{x}; X_1, \dots, X_n) = \sum_{k=0}^d (-1)^k y^{d-k} W_{k+n}([\tilde{x} \cdot \mathcal{C}_k], X_1, \dots, X_n) \quad (1-40)$$

$$= \sum_{k=0}^d (-1)^k y^{d-k} \sum_{1 \leq m_1, \dots, m_k \leq \dim \mathfrak{g}} C_{k; m_1, \dots, m_k} \left\langle \text{Det}_{1 \leq i, j \leq n+k} (E_i K_\Psi(\tilde{x}_i, \tilde{x}_j)) \right\rangle_o \quad (1-41)$$

where we used the determinantal formulae and for any $l \in \{1, \dots, n\}$, $X_l = [\tilde{x}_l \cdot E_l]$ while we decomposed

$$[\gamma_x \cdot \mathcal{C}_k] = \sum_{1 \leq m_1, \dots, m_k \leq \dim \mathfrak{g}} C_{k; m_1, \dots, m_k} [\tilde{x} \cdot e^{m_1}] \otimes \cdots \otimes [\tilde{x} \cdot e^{m_k}] \quad (1-42)$$

in $\widehat{\Sigma}^{\otimes k}$ and denoted for any $l \in \{1, \dots, k\}$, $\tilde{x}_{n+l} = \tilde{x}$ and $E_{n+l} = e^{m_l}$ in each determinant term of the second sum.

On the other hand, in representation ρ and with the same notations,

$$\begin{aligned} \text{Ad}_{\Psi(\tilde{x})^{-1}} [\Phi(x) + \mathcal{M}_\epsilon(x; X_1, \dots, X_n)] &= K_\Psi(\tilde{x}, \tilde{x}) + \sum_{i=1}^n \epsilon_i K_\Psi(\tilde{x}, \tilde{x}_i) E_i K_\Psi(\tilde{x}_i, \tilde{x}) \\ &+ \sum_{k=2}^n \sum_{1 \leq i_1 \neq \dots \neq i_k \leq n} \epsilon_{i_1} \cdots \epsilon_{i_k} K_\Psi(\tilde{x}, \tilde{x}_{i_1}) E_{i_1} K_\Psi(\tilde{x}_{i_1}, \tilde{x}_{i_2}) \cdots E_{i_k} K_\Psi(\tilde{x}_{i_k}, \tilde{x}) \end{aligned} \quad (1-43)$$

such that the coefficient of $\epsilon_1 \dots \epsilon_n$ in the right hand side of the loop equations gives exactly the sum over permutations of $k+n$ elements

$$\begin{aligned} [\epsilon_1 \cdots \epsilon_n] \text{Det}_\rho \left(y - (\Phi(x) + \mathcal{M}_\epsilon(x; X_1, \dots, X_n)) \right) \\ = [\epsilon_1 \cdots \epsilon_n] \text{Det}_\rho \left(y - \text{Ad}_{\Psi(\tilde{x})^{-1}} [\Phi(x) + \mathcal{M}_\epsilon(x; X_1, \dots, X_n)] \right) \end{aligned} \quad (1-44)$$

$$= \sum_{k=0}^d (-1)^k y^{d-k} W_{k;n}(\tilde{x}; X_1, \dots, X_n) \quad (1-45)$$

which is the wanted result. ■

The set of loop equations satisfied by the correlation functions can be interpreted as meromorphic conditions on the objects obtained by inserting the Casimirs of the Lie algebra \mathfrak{g} in the representation ρ . Indeed, the right hand side depends polynomially in the formal variable $y \in \mathcal{Z}(\mathcal{U}(\mathfrak{g})) \otimes K_\Sigma^o$ and depends meromorphically on $x \in \overset{o}{\Sigma}$. In particular, $W_{k;n}(\tilde{x}; X_1, \dots, X_n)$ depends in fact in $\tilde{x} \in \overset{o}{\Sigma}$ only through its base projection $x \in \overset{o}{\Sigma}$.

With previous notations we can rewrite them formally as

$$\mathcal{C}(x, y) \cdot W_n \stackrel{\text{def}}{=} \left(\sum_{k=0}^d (-1)^k y^{d-k} \mathcal{C}_k(x) \right) \cdot W_n = P_n(x, y) \quad (1-46)$$

where we have introduced the notation $\mathcal{C}(x, y) = \sum_{k=0}^d (-1)^k y^{d-k} \mathcal{C}_k(x)$ and not written explicitly the dependence in the X_i 's. We immediately get

Corollary 1.17 *Multi-insertions of Casimirs*

For any $n, m \in \mathbb{N}$, $m \neq 0$, and choice of indices $k_1, \dots, k_m \in \{1, \dots, d\}$, the symbol

$$\mathcal{C}_{k_1}(x_1) \cdots \mathcal{C}_{k_m}(x_m) \cdot W_n \quad (1-47)$$

defines a meromorphic differential form of the variables $x_1, \dots, x_m \in \overset{o}{\Sigma}$ with respective corresponding degrees k_1, \dots, k_m .

These equations are in general hard to solve but they happen to be affine relations in terms of the non-connected correlators. Such affine relations admit an underlying linear structure generated by some very special solutions to the loop equations. Dimension counting shows that these are the solutions that can be computed perturbatively – under a precise set of assumptions called the *Topological Type* property – by the spectral curve topological recursion [43], [15]. We will recall what this property amounts to when studying from a WKB perspective what this setup becomes in a topological asymptotic regime.

1.2 Forms, cycles, intersections and integration

The correlators that we associated to the triplet (P, ∇_0, ∇) are defined as maps from tensor products of copies of $\widehat{\Sigma}_\Psi$ to symmetrized spaces of differentials over the base Riemann surface $\overset{o}{\Sigma}$. We will now define a notion of differential forms on $\widehat{\Sigma}_\Psi$ to interpret the correlators in this way.

Definition 1.18 *Non-perturbative differentials*

Let $n \in \mathbb{N}$ be an integer. A holomorphic n -differential on $\widehat{\Sigma}_\Psi$ is a holomorphic map

$$\omega : \widehat{\Sigma}_\Psi^{\boxtimes n} \longrightarrow \mathbb{K}_{\overset{o}{\Sigma}}^{\boxtimes n} \quad (1-48)$$

such that the for any $E, E' \in \mathfrak{g}_o$ and generic $\tilde{x} \in \overset{o}{\Sigma}$,

$$\omega([\tilde{x} \cdot (E + E')], \dots) = \omega([\tilde{x} \cdot E], \dots) + \omega([\tilde{x} \cdot E'], \dots) \quad (1-49)$$

In other words, we require differential forms on $\widehat{\Sigma}_\Psi$ to depend linearly on the Lie algebra component of $X = [\tilde{x} \cdot E]$. We denote the space of all holomorphic n -differentials on $\widehat{\Sigma}_\Psi$ by $\widehat{\Omega}_\Psi^n$. In particular, each n -differential on $\widehat{\Sigma}_\Psi$ is a 1-differential in each of its variables.

This definition immediately yields

Proposition 1.19 *Correlators as non-perturbative differentials*

For any $n \in \mathbb{N}^*$, W_n and \widehat{W}_n are meromorphic n -differentials on $\widehat{\Sigma}_\Psi$. That is

$$W_n, \widehat{W}_n \in \widehat{\Omega}_\Psi^n \quad (1-50)$$

We know from the study of dispersionless completely algebraically integrable systems [48] that they exhibit so-called *special geometry*, or equivalently the data of Seiberg-Witten type relations. We will now show that taking into account the dispersion does not alter this feature. Indeed, there exists a notion of homology associated to $\widehat{\Sigma}_\Psi$, pulled-back from that of $\overset{o}{\Sigma}$, such that the corresponding period integrals of the 1-point function W_1 (of the first, second and third kind) overdetermine the base of an integrable system [11], [12]. As stated, let us therefore define the relevant notion of cycles, boundaries and pairing with differentials on $\widehat{\Sigma}_\Psi$.

Definition 1.20 *Arcs and their boundaries*

Let an arc in $\widehat{\Sigma}$ be an equivalence class of the form $\Gamma = [\gamma' \cdot E]$ under the action of π_1 , where γ' is an element of the Poincaré groupoid of the universal cover $\overset{o}{\Sigma}$. Recall that the Poincaré groupoid $\pi_1'(Y)$ of a topological manifold Y can be defined as the total space of the bundle over Y whose fiber over a given point $y \in Y$ is the space of all oriented Jordan arcs drawn on the manifold starting from y and considered up to homotopy.

In the case where $Y = \overset{o}{\Sigma}$ is the universal cover, it carries a natural free π_1 -action that we twist by the monodromy representation of the connection ∇ to define the equivalence classes denoted $\Gamma = [\gamma \cdot E]$. Accordingly, define the boundary operator by the formula

$$\partial_\Psi[(\tilde{x}, \tilde{y}) \cdot E] \stackrel{def}{=} y M_\Psi([\tilde{y} \cdot E]) - x M_\Psi([\tilde{x} \cdot E]) \quad (1-51)$$

Remark 1.21 *We could have adopted the equivalent convention that consists in defining the boundary operator without evaluating by the bundle map M_Ψ , it would then have taken the form $\partial_\Psi[(\tilde{x}, \tilde{y}) \cdot E] = [\tilde{y} \cdot E] - [\tilde{x} \cdot E]$ of a linear combination of points of $\widehat{\Sigma}_\Psi$. Instead we chose to represent the boundaries of the arcs just defined as \mathfrak{g} -valued divisors on $\widehat{\Sigma}$.*

Definition 1.22 *First kind cycles*

Define a first kind cycle on $\widehat{\Sigma}$ to be a formal linear combination of homotopy classes of arcs with vanishing boundary. Namely, a cycle of the first kind is of the form

$$\Gamma = \sum_i c_i [(\tilde{x}_i, \tilde{y}_i) \cdot E_i] \quad (1-52)$$

$$\partial_\Psi \Gamma = \sum_i c_i (y_i M_\Psi([\tilde{y}_i \cdot E_i]) - x_i M_\Psi([\tilde{x}_i \cdot E_i])) = 0 \quad (1-53)$$

where the vanishing of the boundary has to be implemented at each of the points x_i or y_i involved. We denote this homology space of 1-cycles over $\widehat{\Sigma}$ by $\widehat{H}'_{1, def} = \text{Ker}(\partial_\Psi)$ and call it the space of cycles of the first kind.

We now wish to compute the dimension of \widehat{H}'_1 and extend it to a larger space (understand higher dimensional) of *similar* cycles satisfying a *form-cycle duality*. To do so let us first denote by $\mathcal{P}_{def} = \{z_1, \dots, z_M\}$ the set of (simple) poles of Φ and define, for any point $p \in \widehat{\Sigma}$ and Lie algebra element $E \in \mathfrak{g}_o$, the class

$$\mathcal{A}(p \cdot E) \stackrel{def}{=} [\mathcal{C}_p \cdot E] \quad (1-54)$$

where \mathcal{C}_p is equivalent to a small circle surrounding p . Notice then that \widehat{H}'_1 contains an M -dimensional subspace consisting of elements of the form $\mathcal{A}(z_j \cdot E)$ for some puncture index $j \in \{1, \dots, M\}$. For Γ to be a cycle, the Lie algebra element $E \in \mathfrak{g}_o$ has to satisfy

$$\partial_\Psi[\mathcal{C}_{z_j} \cdot E] = oM_\Psi \left([\{o\} \cdot (E - \text{Hol}_{\mathcal{C}_{z_j}}^\nabla E)] \right) = 0 \quad (1-55)$$

where we used the $\pi_1(\hat{\Sigma}, o)$ -equivariance of M_Ψ and therefore E has to belong to the linear subspace

$$\mathfrak{h}_j \stackrel{\text{def}}{=} \{E \in \mathfrak{g}_o \mid \text{Hol}_{\mathcal{C}_{z_j}}^\nabla E = E\} \quad (1-56)$$

which is generically a Cartan subalgebra $\mathfrak{h}_{z_j} \subset \mathfrak{g}_o$ and thus has dimension $\text{rk } \mathfrak{g}$.

Definition 1.23 *First homology*

Define the first homology \hat{H}_1 of the bundle $\hat{\Sigma}_\Psi$ as the coset

$$\hat{H}_1 \stackrel{\text{def}}{=} \hat{H}'_1 / \bigoplus_{j=1}^M \mathcal{A}(z_j \cdot \mathfrak{h}_j) \quad (1-57)$$

or equivalently by the short exact sequence of vector spaces

$$0 \longrightarrow \bigoplus_{j=1}^M \mathcal{A}(z_j \cdot \mathfrak{h}_j) \longrightarrow \hat{H}'_1 \longrightarrow \hat{H}_1 \longrightarrow 0 \quad (1-58)$$

In the spirit of a form-cycle duality, a cycle is viewed as a linear form on the vector space of $\pi_1(\hat{\Sigma}, o)$ -equivariant differential forms acting by an integration procedure to be explicated.

Definition 1.24 *Second kind cycles*

For any puncture index $j \in \{1, \dots, M\}$, define the associated second kind cycle by

$$\mathcal{B}_j'' \stackrel{\text{def}}{=} ev_{[\tilde{z}_j \cdot \text{Ad}_{C_j^{-1}} \Phi_j]} \quad (1-59)$$

where for any $X \in \hat{\Sigma}_\Psi$ of the form $X = [z_j \cdot E]$, ev_X is defined as the evaluation linear form at $[z_j \cdot E_0]$, E_0 being the orthogonal projection of E on \mathfrak{h}_j

with respect to a root decomposition $\mathfrak{g}_o = \mathfrak{h}_j \oplus \bigoplus_{\tau \in \mathfrak{R}_j} \mathfrak{g}_\tau$ corresponding to a root system $\mathfrak{R}_j \in \text{Aut } \mathfrak{h}_j^*$. We define $\Phi_j \stackrel{\text{def}}{=} \text{Res}_{z_j} \Phi$ and lastly the constant element C_j is defined uniquely by the asymptotics

$$\Psi(\tilde{x}) \underset{x \sim z_j}{=} (1 + \mathcal{O}(\xi)) \xi^{\Phi_j} C_j \quad (1-60)$$

in a local coordinate $\xi \stackrel{\text{def}}{=} x - z_j$ near the puncture z_j . We denote the vector space generated by these M independent elements as \widehat{H}_1'' and call it the space of second kind cycles.

Remark 1.25 *Second kind cycles are defined with the help of an orthogonal projection. This ensures that correlators, whose dependence in $\widehat{\Sigma}_\Psi$ is through the bundle map M_Ψ will have convergent integrals on second kind cycles as we shall see.*

Moreover, in the spirit of generalizing the geometry of Riemann surfaces recalled in the introduction, we want to define the third kind cycles as arcs on $\widehat{\Sigma}_\Psi$ whose boundary might be located at the punctures.

Definition 1.26 *Third kind cycles*

A cycle of the third kind is an arc \mathcal{B}''' whose boundary $\partial_\Psi \mathcal{B}'''$ satisfies

$$\partial_\Psi \mathcal{B}''' \in \bigoplus_{j=1}^M z_j \mathfrak{h}_j \quad (1-61)$$

which is the condition for $M_\Psi(\partial_\Psi \mathcal{B}''')$ to be well defined even though it involves terms located at the punctures. We denote the space of third kind cycles by \widehat{H}_1''' . In particular, for any $j \in \{1, \dots, M\}$ define cycles of the form

$$\mathcal{B}_{j,E}''' \stackrel{\text{def}}{=} [(\tilde{p}, \tilde{z}_j) \cdot E] + [(\tilde{p}, \tilde{p}') \cdot E'] + [(\tilde{p}', \tilde{p}'') \cdot E''] \quad (1-62)$$

where $p = p' = p''$, $E \in \text{Ad}_{C_j}^{-1} \mathfrak{h}_j$ and $E', E'' \in \mathfrak{g}_o$ are Lie algebra elements chosen such that $\delta_\Psi \mathcal{B}_{j,E}''' = z_j \text{Ad}_{C_j} E$ (this is not unique). Moreover we have

$$\widehat{H}_1' \subset \widehat{H}_1''' \quad (1-63)$$

Definition 1.27 *Generalized cycles*

Define the finite dimensional vector space

$$\widehat{\mathcal{H}}_1 \stackrel{\text{def}}{=} \widehat{H}_1'' \oplus \widehat{H}_1''', \quad (1-64)$$

that we call the space of generalized 1-cycles on $\widehat{\Sigma}_\Psi$.

Theorem 1.28 *Dimensions of the spaces of cycles [B.-Eynard-Ribault]*

Let us define the number

$$\widehat{g} \stackrel{\text{def}}{=} (2\overset{\circ}{g} - 2 + M) \frac{\dim \mathfrak{g} - 3\text{rk } \mathfrak{g}}{2} + (3\overset{\circ}{g} - 3 + M)\text{rk } \mathfrak{h} \quad (1-65)$$

Then

$$\dim \widehat{H}_1 = 2\widehat{g}, \quad \dim \widehat{H}_1' = 2\widehat{g} + M\text{rk } \mathfrak{g}, \quad (1-66)$$

$$\dim \widehat{H}_1'' = M, \quad \dim \widehat{H}_1''' = 2\widehat{g} + 2M\text{rk } \mathfrak{g} \quad (1-67)$$

and therefore

$$\dim \widehat{\mathcal{H}}_1 = 2\widehat{g} + 2M\text{rk } \mathfrak{h} + M \quad (1-68)$$

The only non-trivial case is the first one, that is computing $\dim \widehat{H}_1$. Once this dimension is computed, all the others will follow directly from their definitions. We will exhibit a linear basis of this space and to do so we will need the following definition and property.

Definition 1.29 *Channels on Riemann surfaces*

A channel on the Riemann surface $\overset{\circ}{\Sigma}$ is a trivalent unicellular graph $\mathcal{G} \subset \overset{\circ}{\Sigma}$ considered up to homotopy, with boundary equal to the pole divisor of Φ

$$\partial^c \mathcal{G} = \sum_{j=1}^M z_j \quad (1-69)$$

and such that its complement on the complex curve $\overset{o}{\Sigma} - \mathcal{G}$ is simply connected

$$\pi_1(\overset{o}{\Sigma} - \mathcal{G}, o) = 0 \quad (1-70)$$

where we supposed that generically the reference point does not belong to the channel.

Proposition 1.30 *Combinatorics of channels*

Any channel \mathcal{G} on $\overset{o}{\Sigma}$ has

$$\underset{def}{\mathfrak{v}} = 4\overset{o}{g} - 2 + M \quad \text{vertices,} \quad (1-71)$$

$$\underset{def}{\mathfrak{e}} = 6\overset{o}{g} - 3 + M \quad \text{internal edges} \quad (1-72)$$

$$\text{and } M \text{ external edges ending at the } z_j' \text{'s} \quad (1-73)$$

proof:

This is a standard counting procedure from graph theory. The graph \mathcal{G} has $2\mathfrak{e} = 3\mathfrak{v} + M$ half-edges and its Euler characteristic is the topological invariant $2 - 2\overset{o}{g} = M + \mathfrak{v} - \mathfrak{e} + 1$. A 2×2 linear system that easily yields the wanted result. ■

A choice of channel $\mathcal{G} \subset \overset{o}{\Sigma}$ defines a fundamental domain on the base Riemann surface, allowing for example for any pair of points $p, q \notin \mathcal{G}$ to define a unique homology chain $\gamma_{q,p}$ with boundary $\partial\gamma_{q,p} = p - q$ and such that $\gamma_{q,p} \cap \mathcal{G} = \emptyset$.

Remark 1.31 *There are numerous possible choices of channels on a given Riemann surface. We didn't count how many but one easily sees that if we choose a channel $\mathcal{G} \subset \overset{o}{\Sigma}$, then they can all be obtained from \mathcal{G} by so-called π -moves (related to the crossing symmetry of conformal field theories). Moreover, they are dual to pants decompositions of the base curve (identifying the punctures with length zero boundaries).*

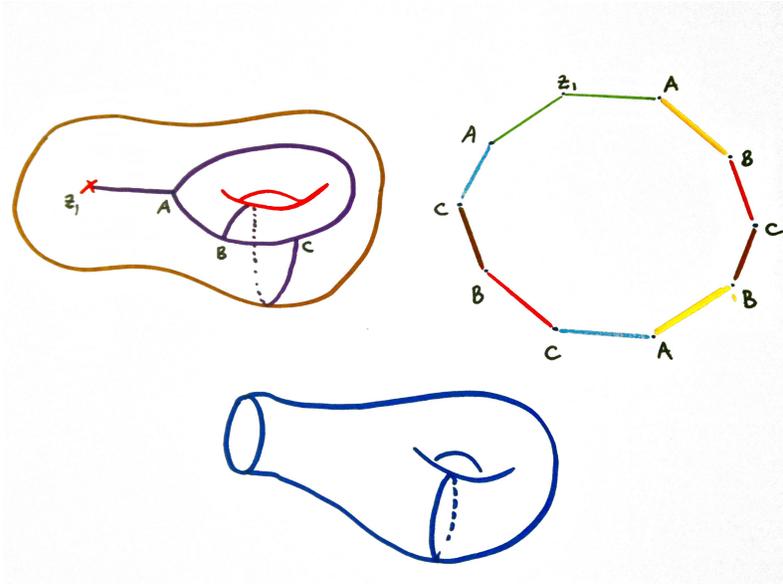


Figure 2.3: A channel on a punctured torus, the corresponding fundamental domain and the associated pant decomposition.

We can now use this notion to prove the theorem.

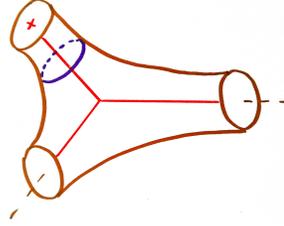
proof:

Let $\mathcal{G} \subset \overset{o}{\Sigma}$ be a channel and consider the dual pant decomposition. We will now associate to this choice a basis of \widehat{H}_1 . Recall that this vector space was defined as a coset of the space of arcs with vanishing boundaries $\widehat{H}_1 = \text{Ker}(\partial_\Psi) / \bigoplus_{j=1}^M \mathcal{A}_j(\mathfrak{h}_j)$. Let us draw and describe the different generic possibilities.

- Each boundary of a pair of pants that is not a boundary of the surface crosses an internal edge e . First we associate to each of those $\text{rk}g$ independent cycles of the form

$$\mathcal{A}_{e,E} \stackrel{\text{def}}{=} [\gamma_e \cdot E] \in \widehat{H}_1 \quad (1-74)$$

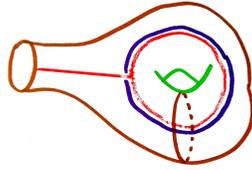
with $\text{Ad}_{S_{\gamma_e}} E = E$ (the space of such elements E is generically $\text{rk } \mathfrak{g}$ -dimensional) and where γ_e denotes the unique homology class on $\overset{o}{\Sigma}$ intersecting \mathcal{G} exactly once through the edge e . In total we therefore get $(3\overset{o}{g} - 3 + M)\text{rk } \mathfrak{g}$ elements. Indeed there are 3 edges not crossing any boundary of a pair of pants for each hole of the surface.



Second, to each of the edges e that can be extended to loops $\tilde{\gamma}_e$ going once around the holes of the surface we associate cycles of the form

$$\mathcal{B}_{e,E} \stackrel{def}{=} [\tilde{\gamma}_e \cdot E] \tag{1-75}$$

with Lie algebra element $E \in \mathfrak{g}_o$ again satisfying $\text{Ad}_{S_{\tilde{\gamma}}} E = E$ thus providing with $\overset{o}{g} \times \text{rk } \mathfrak{g}$ new independent elements to the basis.

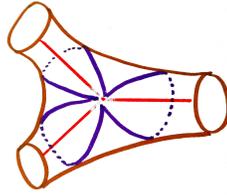


- Similarly to each pair of pants of the decomposition corresponds a unique vertex ν , denoting the corresponding adjacent edges by e_1, e_2 and e_3 , such that we can associate a trefoil cycle (not crossing the edges of channel) of the type

$$\Gamma_\nu = [\gamma_{e_1} \cdot E_1] + [\gamma_{e_2} \cdot E_2] + [\gamma_{e_3} \cdot E_3] \tag{1-76}$$

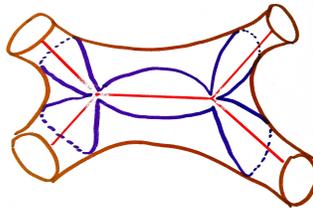
To count the dimension of such cycles with vanishing boundaries, notice that we have three Lie algebra elements E_1, E_2 and E_3 constrained

by the two relations in \mathfrak{g} , namely, $\partial_\Psi \Gamma_\nu = 0$ and $\gamma_{e_1} + \gamma_{e_2} + \gamma_{e_3} = 0$ such that only $\dim \mathfrak{g}$ independent directions remain. Removing the $3rk \mathfrak{g}$ dimensions already counted, we have $(2\overset{o}{g} - 2 + M)(\dim \mathfrak{g} - 3rk \mathfrak{g})$ additional elements. Indeed, there are 2 vertices for each hole of the surface from which such a cycle cannot be drawn without intersecting \mathcal{G} .



- Finally, to each internal edge crossing a boundary where two distinct pairs of pants meet, we associate a cycle of the form

$$\Gamma_e^\perp = \sum_{i=1}^6 [\gamma_i \cdot E_i] \quad (1-77)$$



It involves six Lie algebra elements. However, the vanishing boundary condition, taking into account that only four of the six arcs $\gamma_1, \dots, \gamma_6$ are independent and avoiding redundancy of the previously introduced cycles yields

$$6 \dim \mathfrak{g} - 2 \dim \mathfrak{g} - 2 \dim \mathfrak{g} - 2(\dim \mathfrak{g} - 3rk \mathfrak{g}) - 5rk \mathfrak{g} = rk \mathfrak{g} \quad (1-78)$$

remaining degrees of freedom.

We will denote these cycles as $\mathcal{B}_{e,E} = [\widehat{\gamma}_e \cdot E] + \dots$, where $\widehat{\gamma}_e$ surrounds e and the dots are contributions from the other arcs involved in the cycle (such as the light blue cycle in Fig. 2.4).

The number of such internal edges is $2\overset{\circ}{g} - 3 + M$, it is equal to the dimension $3\overset{\circ}{g} - 3 + M$ of the Teichmuller space of the topological surface underlying $\overset{\circ}{\Sigma}$ minus 1 for each pant surrounding a hole (since they are glued to themselves).

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Summing all these contributions gives the wanted dimension. Moreover, one can check that no independent cycle can be added and that these cycles therefore form a basis of \widehat{H}_1 . ■

We will from now on assume that we have made a choice of channel $\mathfrak{G} \subset \overset{\circ}{\Sigma}$. As we have seen it allows to construct a basis of $\widehat{\mathcal{H}}_1$. As in dispersionless limit, there exists a notion of intersection of cycles

Definition 1.32 *Intersection form*

The space of arcs on $\widehat{\Sigma}_\Psi$ carries a natural (skew-symmetric) intersection form given for any generic arcs, denoted $\Gamma = [\gamma \cdot E]$ and $\Gamma' = [\gamma' \cdot E']$, by

$$\Gamma \cap \Gamma' \stackrel{def}{=} \sum_{x \in \gamma \cap \gamma'} (\gamma, \gamma')_x \langle E, E' \rangle \quad (1-79)$$

where $(\gamma, \gamma') \stackrel{def}{=} +1$ (resp. -1) if the tangents of the arcs γ and γ' intersecting at $x \in \overset{\circ}{\Sigma}$ define a positively (resp. negatively) oriented basis of the corresponding tangent plane.

This intersection form is skew-symmetric, stable under homotopic deformations and extends bilinearly to whole space of cycles on $\widehat{\Sigma}_\Psi$.

Theorem 1.33 *Symplectic structure [B.-Eynard-Ribault]*

The intersection form vanishes on $\bigoplus_{j=1}^M \mathcal{A}(z_j \cdot \mathfrak{h}_j)$ and is non-degenerate on \widehat{H}_1 , therefore endowing it with a symplectic structure.

proof:

The intersection form vanishes by definition on $\bigoplus_{j=1}^M \mathcal{A}(z_j \cdot \mathfrak{h}_j)$ and writing its matrix in the basis exhibited for computing the dimension of \widehat{H}_1 shows that it is non-degenerate. ■

Corollary 1.34 *Symplectic basis*

There exists a symplectic basis of cycles $\{\mathcal{A}_h, \mathcal{B}_h\}_{1 \leq h \leq \widehat{g}}$ in \widehat{H}_1 ,

$$\mathcal{A}_h \cap \mathcal{A}_{h'} = 0, \quad \mathcal{B}_h \cap \mathcal{B}_{h'} = 0 \quad \text{and} \quad \mathcal{A}_h \cap \mathcal{B}_{h'} = \delta_{h,h'} \quad (1-80)$$

These notions of cycles and differentials are dual in the sense that they can be paired in a non-degenerate way by integration.

Definition 1.35 *First and third kind integration pairing*

For any generic cycle $\Gamma = [\sum_i \gamma_i \cdot E_i] \in \widehat{H}_1' \oplus \widehat{H}_1'''$ and any differential $\omega \in \widehat{\Omega}^1$ having no singularity along Γ ,

$$\int_{\Gamma} \omega \stackrel{\text{def}}{=} \sum_i \int_{x \in \pi(\gamma_i)} \omega([\tilde{x} \cdot E_i]) \quad (1-81)$$

which does not depend on a choice of representative of Γ and is bilinear in (Γ, ω) .

proof:

We have to check that $\int_{\Gamma} \omega$ does not depend on a choice of representative for Γ . Indeed, if p is an end point of some homology cycle appearing in $\Gamma = [\sum_i \gamma_i \cdot E_i]$ but is not a puncture,

$$d_p \int_{\Gamma} \omega = \sum_{\substack{i \\ \pi(p_i) = \pi(p)}} \omega([\tilde{p}_i \cdot E_i]) \quad (1-82)$$

which vanishes since the boundary $\partial_{\Psi} \Gamma$ of the cycle Γ is located at the z_j 's. ■

Remark 1.36 *If we wish to integrate W_1 on a third kind cycle we need a regularization procedure since this differential has the same singularities than Φ at the punctures. We use the prescription*

$$\int_{[(\tilde{x}_0, \tilde{z}_j) \cdot E]} W_1 \stackrel{def}{=} \int_{x_0}^{z_j} \left(W_1([\tilde{x} \cdot E]) - \langle \text{Ad}_{C_j} E, \Phi_j \rangle \omega^{z_j - o}(x) \right) \quad (1-83)$$

for any point in the universal cover $\tilde{x}_0 \in \overset{o}{\Sigma}$ over a point $x_0 \neq o$, any puncture index $j \in \{1, \dots, M\}$ and any Lie algebra element $E \in \text{Ad}_{C_j^{-1}} \mathfrak{h}_j$.

Definition 1.37 *Second kind integration pairing*

Similarly, let us extend the definition to the second kind cycles by

$$\int_{\mathcal{B}_j''} \omega \stackrel{def}{=} \omega([\tilde{z}_i \cdot \text{Ad}_{C_j^{-1}} \Phi_j]) \quad (1-84)$$

where for any puncture index $j \in \{1, \dots, M\}$, \tilde{z}_j is the unique homology chain with boundary $z_j - o$ that does not intersect the channel \mathcal{G} .

1.3 Non-perturbative special geometry

Let us end this section on the non-perturbative aspects of the Fuchsian system by introducing the form-cycle duality and by stating some (almost proved) conjectures about the underlying geometry. Recall that we defined \mathcal{M}_{Fuchs} as the moduli space of equivariant connections (P, ∇) . Our point-wise study involves the choice of a reference connection ∇_0 and this can be done holomorphically only locally in the moduli space. A consequence of this is that deformations of such equivariant connections in the ∇ direction are isomorphic to deformations of the moduli of $\Phi = \nabla_0 - \nabla$, keeping ∇_0 constant, namely $[\delta, \nabla_0] = 0$. We also allow for deformations of the flat local section Ψ .

In local coordinates, one can easily study how the various elements of the construction change under deformations.

For any deformation $\delta \in T^* \mathcal{M}_{Fuchs}$, one can define

$$F_{\delta} \stackrel{def}{=} \delta \Psi \cdot \Psi^{-1} \quad (1-85)$$

such that $(\delta - F_{\delta}) \cdot \Psi = 0$. This implies in particular the zero curvature equation

$$[\delta - F_{\delta}, \nabla_0 - \Phi] = 0 \quad (1-86)$$

which translates the fact that the two differential systems defined by these operators admit a common solution. On the punctured Riemann sphere $\overset{o}{\Sigma} = \mathbb{C}\mathbb{P}^1 - \{z_1, \dots, z_M\}$, choosing only isomonodromic deformations, namely a deformation such that $\delta S_{\gamma_e} = 0$ for any edge e of the channel, would yield the Schlesinger system of differential equations [68].

Recall that we have the asymptotic behaviour $\Psi(\tilde{x}) \underset{x \sim z_j}{=} (1 + \mathcal{O}(\xi)) \xi^{\Phi_j} C_j$ in local coordinates $\xi = x - z_j$ around the j^{th} puncture and this implies immediately

Lemma 1.38 *Singularities of F_{δ}*

For any puncture index $j \in \{1, \dots, M\}$,

$$F_{\delta}(\tilde{x}) \underset{x \sim z_j}{=} -\frac{\delta z_j}{x - z_j} \Phi_j + \ln(\xi) \delta \Phi_j + M_{\Psi}([\tilde{x} \cdot C_j^{-1} \delta C_j]) + \mathcal{O}(1) \quad (1-87)$$

where $\mathcal{O}(1)$ means terms that are analytic at $x = z_j$.

Remark 1.39 *We see from the last asymptotics that F_{δ} is automatically singular at z_j if the deformation affects the position of this puncture. Nevertheless, although the term $M_{\Psi}([\tilde{x} \cdot C_j^{-1} \delta C_j])$ is generically divergent at a given puncture, the condition*

$$\forall j \in \{1, \dots, M\}, \forall \mathfrak{r} \in \mathfrak{R}, \Re(\mathfrak{r}(\Phi_j)) > -1 \quad (1-88)$$

ensures that $\xi F_{\delta}(z_j + \xi)$ is bounded near z_j ($\xi = 0$).

Lemma 1.40 *Monodromies of F_δ*

For any deformation $\delta \in T^* \mathcal{M}_{\text{Fuchs}}$, F_δ is analytic in the fundamental domain $\overset{o}{\Sigma} - \mathcal{G}$ defined by the channel \mathcal{G} with possible singularities at the punctures and its discontinuity across an edge e at a point $x \in e$, denoted $\Delta_{x \in e} F_\delta$, is

$$\Delta_{x \in e} F_\delta = M_\Psi \left([\tilde{x} \cdot \delta S_{\gamma_e} S_{\gamma_e}^{-1}] \right) \quad (1-89)$$

proof:

One can easily compute the monodromy of F_δ around a loop $\gamma \in \pi_1(\overset{o}{\Sigma}, x)$

$$F_\delta(\tilde{x} + \gamma) = \delta \Psi(\tilde{x} + \gamma) \cdot \Psi(\tilde{x} + \gamma)^{-1} \quad (1-90)$$

$$= (\delta \Psi(\tilde{x}) S_\gamma + \Psi(\tilde{x}) \delta S_\gamma) \cdot S_\gamma^{-1} \Psi(\tilde{x})^{-1} \quad (1-91)$$

such that

$$F_\delta(\tilde{x} + \gamma) - F_\delta(\tilde{x}) = M_\Psi \left([\tilde{x} \cdot \delta S_\gamma S_\gamma^{-1}] \right) \quad (1-92)$$

and this last equality implies the wanted result. ■

Lemma 1.41 *For any deformation $\delta \in T^* \mathcal{M}_{\text{Fuchs}}$ and any meromorphic one-form ω on $\overset{o}{\Sigma}$ such that $\omega''' - \omega^{x-o}$ is holomorphic,*

$$F_\delta(x) - F_\delta(o) = \int_{\delta^*} M_\Psi(X') \omega'''(x') \quad (1-93)$$

where

$$\delta^* = - \sum_{j=1}^M \delta z_j \mathcal{B}_j'' + \sum_{j=1}^M \mathcal{B}_{j, (\delta S_{\gamma z_j} S_{\gamma z_j}^{-1})}''' \pmod{\delta \Psi} \quad (1-94)$$

meaning equality up to boundary terms, or equality of the corresponding classes in $\widehat{\mathcal{H}}_1 / \widehat{\mathcal{H}}_1'$.

proof:

Applying Cauchy's formula around a point $x \in \overset{o}{\Sigma} - \mathcal{G}$ in the fundamental domain to the analytic function

$$\tilde{F}(x) \stackrel{def}{=} F_\delta(x) - F_\delta(o) + \sum_{j=1}^M \delta z_j \int_{\mathcal{B}_j''} M_\Psi(X') \omega'''(x') \quad (1-95)$$

and then moving the small contour around that point to the boundary we get

$$F_\delta(x) - F_\delta(o) = - \sum_{j=1}^M \delta z_j \int_{\mathcal{B}_j''} M_\Psi(X') \omega(x') - \frac{1}{2\pi i} \oint_{x' \in \partial(\overset{o}{\Sigma} - \mathcal{G})} \tilde{F}(x') \omega'''(x') \quad (1-96)$$

where since each edge appears twice at the boundary of the fundamental domain, the last integral is equal to the integral of the discontinuity of \tilde{F} along the channel. The second kind integrals are meromorphic in the variable x (it appears as the location of the first order pole of ω) and therefore we are left with the integral of the discontinuity of F_δ alone, that is $M_\Psi([\tilde{x} \cdot \delta S_{\gamma_e} S_{\gamma_e}^{-1}])$ for each edge e .

It reads

$$\begin{aligned} F_\delta(x) - F_\delta(o) &= - \sum_{j=1}^M \delta z_j \int_{X' \in \mathcal{B}_j''} M_\Psi(X') \omega(x') \\ &\quad - \frac{1}{2\pi i} \sum_e \int_{x' \in e} M_\Psi([\tilde{x}' \cdot (\delta S_{\gamma_e} \cdot S_{\gamma_e}^{-1})]) \omega'''(x') \end{aligned} \quad (1-97)$$

$$= \int_{X' \in \delta^*} M_\Psi(X') \omega'''(x') \quad (1-98)$$

where $\delta^* \stackrel{def}{=} - \sum_{j=1}^M \delta z_j \mathcal{B}_j'' + \Gamma$, with $\Gamma \stackrel{def}{=} \frac{1}{2\pi i} \sum_e e (\delta S_{\gamma_e} \cdot S_{\gamma_e}^{-1})$. Since the interior vertices of the channel are not singularities of Φ , for any interior vertex v we have the identity $S_{\gamma_{e_1}} S_{\gamma_{e_2}} S_{\gamma_{e_3}} = \text{Id}$, where we labelled e_1, e_2 and e_3 the adjacent edges in direct order (up to cyclic permutation), we get that the only contributions to the boundary of Γ come from the punctures

$$\delta_{\Psi} \left(\Gamma - \sum_{j=1}^M \mathcal{B}_{j,(\delta S_{\gamma z_j} \cdot S_{\gamma z_j})}''' \right) = 0 \quad (1-99)$$

which concludes the proof. ■

Corollary 1.42 *For any holomorphic one-form $\omega \in H^1(\overset{\circ}{\Sigma}, \mathbb{C})$,*

$$\int_{X \in \delta^*} M_{\Psi}(X) \omega(x) = 0 \quad (1-100)$$

proof:

The previous lemma being true for any third-kind differential that has the wanted simple poles with residues ± 1 , the space of differences of two such one-forms is the space of holomorphic one-forms and this implies the result. ■

Similarly to the cycles introduced by Bertola in [19], [20], we therefore introduce

Proposition 1.43 *Deformation dual cycle*

To each deformation can be associated a dual cycle by the injective mapping

$$\begin{aligned} * : T^* \mathcal{M}_{Fuchs} &\longrightarrow \widehat{\mathcal{H}}_1 \\ \delta &\longmapsto \delta_{def}^* = - \sum_{j=1}^M \delta z_j \mathcal{B}_j'' + \sum_e [e \cdot \delta S_{\gamma_e} S_{\gamma_e}^{-1}] \end{aligned} \quad (1-101)$$

Corollary 1.44 *Symplectic structure*

The intersection product on $\widehat{H}_1 \oplus \widehat{H}'_1$ restricts to a symplectic structure on $T^ \mathcal{M}_{Fuchs}$ defined for any pair of deformations $\delta_1, \delta_2 \in T^* \mathcal{M}_{Fuchs}$ by*

$$\Omega_{Fuchs}(\delta_1, \delta_2) =_{def} \delta_1^* \cap \delta_2^* \quad (1-102)$$

Theorem 1.45 *Seiberg-Witten relations [B.-Eynard-Ribault]*

The correlators satisfy Seiberg-Witten relations. That is for any positive integer $n \in \mathbb{N}^$ and any deformation $\delta \in T^* \mathcal{M}_{Fuchs}$,*

$$\delta W_n = \int_{\delta^*} W_{n+1} \quad (1-103)$$

proof:

Let us start from $K_\Psi(\tilde{x}, \tilde{x}') = \frac{\Psi(\tilde{x}')^{-1} \Psi(\tilde{x})}{\mathcal{E}_f(\tilde{x}, \tilde{x}')}$ (remember that this expression only makes sense inside brackets $\langle \cdot \rangle_o$).

$$\delta K_\Psi(\tilde{x}, \tilde{x}') = \Psi(\tilde{x}')^{-1} \frac{F_\delta(\tilde{x}) - F_\delta(\tilde{x}')}{\mathcal{E}_f(\tilde{x}, \tilde{x}')} \Psi(\tilde{x}) \quad (1-104)$$

$$= \int_{X'' \in \delta^*} \Psi(\tilde{x}')^{-1} \frac{M(X'')}{\mathcal{E}_f(\tilde{x}, \tilde{x}') \mathcal{E}_f(\tilde{x}, \tilde{x}')} \Psi(\tilde{x}) \quad (1-105)$$

$$= - \int_{[\tilde{x}'' \cdot E''] \in \delta^*} K_\Psi(\tilde{x}, \tilde{x}'') E'' K_\Psi(\tilde{x}'', \tilde{x}') \quad (1-106)$$

Now recall the determinantal formula

$$W_n(X_1, \dots, X_n) = \left\langle \text{Det}_{1 \leq i, j \leq n} (E_i K_\Psi(\tilde{x}_i, \tilde{x}_j)) \right\rangle_o \quad (1-107)$$

and the Leibniz rule together with the integral formula for δK_Ψ inserts an additional factor of K_Ψ in all possible ways

$$\delta \left(\sum_{\tau \in \mathfrak{S}_n} (-1)^\tau \left\langle \prod_{i=1}^{\rightarrow n} K_\Psi(\tilde{x}_i, \tilde{x}_{\tau(i)}) \right\rangle_o \right) = \int_{[\tilde{x}_{n+1} \cdot E''] \in \delta^*} \sum_{\tau \in \mathfrak{S}_{n+1}} (-1)^\tau \left\langle \prod_{i=1}^{\rightarrow n} K_\Psi(\tilde{x}_i, \tilde{x}_{\tau(i)}) \right\rangle_o \quad (1-108)$$

which concludes the proof. ■

1.4 τ -functions and enumerative geometry

Theorem 1.46 τ -function [B.-Eynard-Ribault]

For any Lagrangian submanifold $\mathcal{L} \subset T^* \mathcal{M}_{\text{Fuchs}}$, there exists a function $\mathfrak{T}_{\mathcal{L}}$ such that for any $\delta \in \mathcal{L}$,

$$\delta \ln \mathfrak{T}_{\mathcal{L}} = \int_{\delta^*} W_1 \quad (1-109)$$

It is a generalization of the isomonodromic τ -function of [59].

proof:

Using the Seiberg-Witten relation for deformations of W_1 ,

$$[\delta_1, \delta_2] \ln \mathfrak{T}_{\mathcal{L}} = \delta_1 \int_{\delta_2^*} W_1 - \delta_2 \int_{\delta_1^*} W_1 \quad (1-110)$$

$$= \left[\int_{\delta_2^*}, \int_{\delta_1^*} \right] W_2 + \int_{\delta_1(\delta_2) - \delta_2(\delta_1)} W_1 \quad (1-111)$$

$$= 2\pi i \delta_1^* \cap \delta_2^* \quad (1-112)$$

$$= 0 \quad (1-113)$$

where the intersection of cycles appearing in the line before the last comes from shrinking the contours to the diagonal where W_2 has its singularities and vanishes by definition of the Lagrangian submanifold. ■

Remark 1.47 *Note that in the heuristics of the introduction and in the dispersionless classical case of Liouville's theorem, a choice of Lagrangian was needed in order to define the action, in both case to ensure single-valuedness of an integral. We recover this feature for the non-perturbative τ -function.*

Conjecture 1.48 *Flat period coordinates [B.-Eynard-Ribault]*

There exists a symplectic basis of cycles $\{\mathcal{A}_h, \mathcal{B}_h\}_{1 \leq h \leq \widehat{g}}$ in \widehat{H}_1 such that for any index $h \in \{1, \dots, \widehat{g}\}$ and positive integer $n \in \mathbb{N}^$,*

$$\oint_{\mathcal{A}_h} W_n = \varepsilon_h \delta_{n,1} \quad (1-114)$$

The ε_h 's are called the flat period coordinates of the Fuchsian system and denoting by δ_{ε_h} the corresponding deformation, it satisfies

$$(\delta_{\varepsilon_h})^* = \mathcal{B}_h \quad (1-115)$$

Their dual coordinates are defined similarly for any $h \in \{1, \dots, \widehat{g}\}$ by

$$\varepsilon_h^D \stackrel{def}{=} \oint_{\mathcal{B}_h} W_1 \quad (1-116)$$

$$= \delta_{\varepsilon_h} \ln \mathfrak{T}_{\mathcal{L}} \quad (1-117)$$

Remark 1.49 *The last relation with the dual period coordinates shows that $\mathcal{T}_{\mathcal{L}}$ would play the role of an effective superpotential similarly to that of [66]. There, starting from a six dimensional $(0,2)$ super conformal field theory on the product $\overset{o}{\Sigma} \times \mathbb{R}_{\epsilon_1}^2 \times \mathbb{R}_{\epsilon_2}^2$ in the so-called $\Omega_{\epsilon_1, \epsilon_2}$ -background [65], they define an effective superpotential by*

$$\mathcal{W}^{eff}(\epsilon_1) = \lim_{def \ \epsilon_2 \rightarrow 0} (\epsilon_2 \mathcal{F}(\epsilon_1, \epsilon_2)) \quad (1-118)$$

However, the Non-perturbative Fuchsian geometry we are describing here corresponds to $\epsilon_1 = -\epsilon_2 = 1$. We will now deform this to get $\epsilon_1 = -\epsilon_2 \stackrel{def}{=} \epsilon$ to study the $\epsilon \rightarrow 0$ limit using WKB techniques. In general, and it will be the case when we investigate the geometry of the Ward identities of W -symmetric conformal field theories, there are two deformation parameters, the background charge $Q \stackrel{def}{=} \epsilon_1 + \epsilon_2 \neq 0$ and the WKB expansion parameter ϵ satisfying the relation $\epsilon^2 = -\epsilon_1 \epsilon_2$. We keep $Q = 0$ for now.

2 Perturbative aspects

2.1 Topological expansions and cameral geometry

Let us now introduce the "small" deformation parameter ϵ . The previous construction can be carried without any additional trouble when the connection ∇ is replaced by an ϵ -connection ∇_ϵ . Redefining

$$\nabla_0 - \frac{1}{\epsilon} \nabla_\epsilon \stackrel{def}{=} \frac{1}{\epsilon} \Phi_\epsilon, \quad (2-1)$$

$$\Phi_\epsilon \stackrel{def}{=} \sum_{k \geq 0} \epsilon^k \Phi^{(k)} \quad (2-2)$$

is now a *formal* power series in ϵ valued in the vector space $H^0(\overset{o}{\Sigma}, \text{Ad}(P) \otimes K_{\overset{o}{\Sigma}})$.

As was noted before, the principal G -bundle $P \rightarrow \overset{o}{\Sigma}$ together with the leading order term of the ϵ -expansion of Φ_ϵ , $\Phi_\epsilon|_{\mathcal{O}(\epsilon)^0} = \Phi^{(0)}$ define the structure of a Higgs bundle $(P, \Phi^{(0)}) \in \mathcal{M}$.

Theorem 2.1 *WKB asymptotics*

Denoting by $\mathfrak{h}_o \subset \mathfrak{g}_o$ a Cartan subalgebra of the adjoint fiber at the reference point o and diagonalizing the Higgs field potential as

$$\Phi^{(0)} = \text{Ad}_V Y, \quad (2-3)$$

Y is a local \mathfrak{h}_o -valued meromorphic 1-form (its Weyl group valued monodromies define a cameral curve as we shall later see). WKB analysis considers Ψ_ε , defined as a formal fundamental solution of the connection problem $\nabla_\varepsilon \Psi = 0$ admitting for any $\tilde{x} \in \overset{o}{\Sigma}$ an asymptotic expansion of the form

$$\Psi_\varepsilon(\tilde{x}) \underset{\varepsilon \rightarrow 0}{=} V(x) \widehat{\Psi}_\varepsilon(\tilde{x}) e^{\frac{1}{\varepsilon} \int_{\tilde{x}} Y} C_{\tilde{x}} \quad (2-4)$$

where $\widehat{\Psi}_\varepsilon$ is a solution of the linear equation

$$(\text{Ad}_{V^{-1}} \nabla_0) \widehat{\Psi} = [Y, \widehat{\Psi}] = Y \cdot \widehat{\Psi} - \widehat{\Psi} \cdot Y, \quad (2-5)$$

denoting respectively Lie derivation from the left and from the right, and such that it satisfies $\widehat{\Psi}_\varepsilon(\tilde{x}) = 1 + \mathcal{O}(\varepsilon)$. $C_{\tilde{x}} \in P_o \simeq G$ is an integration constant that does not depend on ε and depends on the point $\tilde{x} \in \overset{o}{\Sigma}$ through Stokes' phenomenon only [69].

Indeed, if we introduce the so-called Stokes and anti-Stokes lines on $\overset{o}{\Sigma}$ respectively defined for any root $\alpha \in \mathfrak{R}$ by

$$\Gamma_\alpha \underset{def}{=} \left\{ \tilde{x} \in \overset{o}{\Sigma} \mid \Re e \left(\int_{\tilde{x}} \alpha(Y) \right) = 0 \right\} \quad (2-6)$$

$$\bar{\Gamma}_\alpha \underset{def}{=} \left\{ \tilde{x} \in \overset{o}{\Sigma} \mid \Im m \left(\int_{\tilde{x}} \alpha(Y) \right) = 0 \right\} \quad (2-7)$$

then from Stokes's theorem, $C_{\tilde{x}}$ is piecewise constant in any connected component of $\overset{o}{\Sigma} - \bigcup_{\alpha \in \mathfrak{R}} \Gamma_\alpha \cup \bar{\Gamma}_\alpha$, where we removed the Stokes' network

$\bigcup_{\alpha \in \mathfrak{R}} \Gamma_\alpha \cup \bar{\Gamma}_\alpha$ from the universal cover. By definition, the reference point o is a vertex of the network and we will furthermore consider paths not ending on the network, that is such that $\tilde{x} \notin \bigcup_{\alpha \in \mathfrak{R}} \Gamma_\alpha \cup \bar{\Gamma}_\alpha$.

Theorem 2.2 *Emergent cameral geometry [B.-Eynard-Hurtubise]*
The cameral cover $\Sigma_{\mathcal{H}(\Phi)}$ is the semi-classical limit $\varepsilon \rightarrow 0$ of $\widehat{\Sigma}_{\Psi_\varepsilon}$.

proof:

Let $[\tilde{x} \cdot E] \in \widehat{\Sigma}_{\Psi_\varepsilon}$ and let us write $\text{Ad}_{C_{\tilde{x}}} E = E_o + \sum_{\alpha \in \mathfrak{R}} E_\alpha$ introducing to the direct sum decomposition $\mathfrak{g}_o = \mathfrak{h}_o \oplus \bigoplus_{\alpha \in \mathfrak{R}} \mathfrak{g}_\alpha$, where \mathfrak{R} denotes the root system. We then get the asymptotic expression

$$M_{\Psi_\varepsilon}([\tilde{x} \cdot E]) \underset{\varepsilon \rightarrow 0}{=} \text{Ad}_{V(\tilde{x})[1+\mathcal{O}(\varepsilon)]} \text{Ad}_{e^{\frac{1}{\varepsilon} \int_{\tilde{x}} Y}} \text{Ad}_{C_{\tilde{x}}} E \quad (2-8)$$

$$\underset{\varepsilon \rightarrow 0}{=} \text{Ad}_{V(\tilde{x})[1+\mathcal{O}(\varepsilon)]} e^{\frac{1}{\varepsilon} \int_{\tilde{x}} \text{Ad}_Y} \left(E_o + \sum_{\alpha \in \mathfrak{R}} E_\alpha \right) \quad (2-9)$$

$$\underset{\varepsilon \rightarrow 0}{=} \text{Ad}_{V(\tilde{x})[1+\mathcal{O}(\varepsilon)]} \left(E_o + \sum_{\alpha \in \mathfrak{R}} e^{\frac{1}{\varepsilon} \int_{\tilde{x}} \alpha(Y)} E_\alpha \right) \quad (2-10)$$

from which we conclude that $M_{\Psi_\varepsilon}([\tilde{x} \cdot E])$ will admit a $\varepsilon \rightarrow 0$ limit if and only if for any $\alpha \in \mathfrak{R}$ such that $\Re(\int_{\tilde{x}} \alpha(Y)) > 0$ we have $E_\alpha = 0$ and for any $\alpha \in \mathfrak{R}$ such that $\Re(\int_{\tilde{x}} \alpha(Y)) = 0$ we also have $\Im(\int_{\tilde{x}} \alpha(Y)) = 0$. $\text{Ad}_{C_{\tilde{x}}} E$ thus has to belong to the Borel subalgebra $\mathfrak{h}_o \oplus \bigoplus_{\alpha \in \mathfrak{R}_+(\tilde{x})} \mathfrak{g}_\alpha$, where

$$\mathfrak{R}_+(\tilde{x}) \underset{def}{=} \left\{ \alpha \in \mathfrak{R} \mid \Re \left(\int_{\tilde{x}} \alpha(Y) \right) \leq 0 \right\} \quad (2-11)$$

or equivalently that E has to belong to $\mathfrak{B}_{\tilde{x}} \underset{def}{=} \text{Ad}_{C_{\tilde{x}}}^{-1} [\mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{R}_+} \mathfrak{g}_\alpha]$. In turn, introducing

$$\overset{o}{\widetilde{\Sigma}}' \underset{def}{=} \left[\overset{o}{\widetilde{\Sigma}} - \bigcup_{\alpha \in \mathfrak{R}} \Gamma_\alpha \cup \bar{\Gamma}_\alpha \right] \quad (2-12)$$

and the Borel bundle

$$\mathfrak{B}' \stackrel{def}{=} \bigsqcup_{\tilde{x} \in \overset{o}{\tilde{\Sigma}'}} \mathfrak{B}_{\tilde{x}} \subset \overset{o}{\tilde{\Sigma}'} \times \mathfrak{g} \quad (2-13)$$

we get that for any $\tilde{x} \in \overset{o}{\tilde{\Sigma}'}$ and $E \in \mathfrak{B}_{\tilde{x}}$, $M_{\Psi_\varepsilon}([\tilde{x} \cdot E])$ has a well defined $\varepsilon \rightarrow 0$ limit given by

$$\widetilde{M}_{cl}([\tilde{x} \cdot E]) \stackrel{def}{=} \lim_{\varepsilon \rightarrow 0} M_{\Psi_\varepsilon}([\tilde{x} \cdot E]) = \text{Ad}_{V(\tilde{x})} E_o \quad (2-14)$$

where the *cl* subscript stands for *classical* and as previously $E_o \in \mathfrak{h}_o$ denotes the Cartan part of $\text{Ad}_{C_{\tilde{x}}} E$. Note that there could a priori be a non-Cartan part that would converge to

$$\sum_{\substack{\alpha \in \mathfrak{R} \\ \alpha(Y(\tilde{x}))=0}} E_\alpha$$

but since $\tilde{x} \in \overset{o}{\tilde{\Sigma}'}$, it is not on the Stokes network and this sum is therefore over the empty set. In turn, outside of Stokes' network, only the Cartan part of $\text{Ad}_{C_{\tilde{x}}} E$ contributes to the limit

$$M_{\Psi_\varepsilon}(\mathfrak{B}_{\tilde{x}}) \xrightarrow{\varepsilon \rightarrow 0} \text{Ad}_{V(\tilde{x})C_{\tilde{x}}} \mathfrak{h}_o \quad (2-15)$$

thus allowing to define the quotient space

$$\widehat{\mathfrak{B}}_o \stackrel{def}{=} \mathfrak{B}' / \widetilde{M}_{cl} \quad (2-16)$$

by identifying the elements of the Borel subbundle whose images by M_{Ψ_ε} admit the same $\varepsilon \rightarrow 0$ limit. Similarly, the map

$$\begin{aligned} M_{cl} : \bigsqcup_{\tilde{x} \in \overset{o}{\tilde{\Sigma}'}} \text{Ad}_{C_{\tilde{x}}^{-1}} \mathfrak{h}_o &\longrightarrow \text{Ad}(P) \\ [\tilde{x} \cdot E_o] &\longmapsto \text{Ad}_{\Psi_{cl}(\tilde{x})} E_o \end{aligned} \quad (2-17)$$

with $\Psi_{cl}(\tilde{x}) = V(\tilde{x})C_{\tilde{x}}$ allows, by identifying preimages of a same element, to take the corresponding quotient

$$\widehat{\mathcal{F}}_{cl} \stackrel{def}{=} \left(\bigsqcup_{\tilde{x} \in \overset{o}{\Sigma}'} \text{Ad}_{C_{\tilde{x}}^{-1}} \mathfrak{h}_o \right) / M_{cl} \quad (2-18)$$

It is such that for any $E \in \mathfrak{B}_{\tilde{x}}$, $M_{\Psi_\varepsilon}([\tilde{x} \cdot E]) \longrightarrow M_{cl}([\tilde{x} \cdot \Pi_{\text{Ad}_{C_{\tilde{x}}^{-1}} \mathfrak{h}_o} E])$, where we introduced the projector $\Pi_{\text{Ad}_{C_{\tilde{x}}^{-1}} \mathfrak{h}_o}$ on $\text{Ad}_{C_{\tilde{x}}^{-1}} \mathfrak{h}_o$ with respect to a root decomposition corresponding to this Cartan subalgebra.

Define $\Phi_{cl} \in H_{loc}^0(\overset{o}{\Sigma}, K_\Sigma \otimes \mathfrak{g})$ by

$$\Phi_{cl}(\tilde{x}) = \nabla_0 \Psi_{cl}(\tilde{x}) \cdot \Psi_{cl}(\tilde{x})^{-1} = \nabla_0 V(\tilde{x}) \cdot V(\tilde{x})^{-1}, \quad (2-19)$$

a multivalued expression on the base curve $\overset{o}{\Sigma}$ but with meromorphic singularities, the last equality coming from the fact that the symbol $C_{\tilde{x}}$ is constant in every connected component of $\overset{o}{\Sigma}'$.

Let us now consider a point $\tilde{x} \in \overset{o}{\Sigma}'$ and a closed loop $\gamma_x \in \pi_1(\overset{o}{\Sigma}, x)$. We define the following monodromy matrices,

$$\Psi_\varepsilon(\tilde{x} + \gamma_x) \stackrel{def}{=} \Psi_\varepsilon(\tilde{x}) S_{\gamma_x}^\varepsilon \quad (2-20)$$

$$\Psi_{cl}(\tilde{x} + \gamma_x) \stackrel{def}{=} \Psi_{cl}(\tilde{x}) S_{\gamma_x}^{cl} \quad (2-21)$$

The previous arguments show that in the $\varepsilon \longrightarrow 0$ limit, the relevant monodromy matrices are in fact $S_{\gamma_x}^{cl}$, that is the monodromy matrices of the connection defined by $\nabla^{cl} = \nabla_0 - \Phi_{cl}$. In turn, representing the monodromy of V , $S_{\gamma_x}^{cl}$ is an element of the Weyl group \mathfrak{w} and as such allows to define a representation

$$\begin{aligned} S^{cl} : \pi_1(\overset{o}{\Sigma}, o) &\longrightarrow \mathfrak{w} \\ \gamma_o &\longmapsto S_{\gamma_o}^{cl} \end{aligned} \quad (2-22)$$

We thus obtained that in the appropriate $\varepsilon \rightarrow 0$ limit, that consists in removing the singular part of the connection, the monodromy representation of the connection ∇^ε reduces to a Weyl action on the Cartan subalgebra \mathfrak{h}_o of \mathfrak{g}_o . Such a representation contains the information of an algebraic cover of $\overset{o}{\Sigma}$ as the monodromy representation of a \mathfrak{w} -cover of this curve which is isomorphic to the cameral cover denoted $x : \Sigma_{\mathcal{H}(\Phi^{(0)})} \rightarrow \overset{o}{\Sigma}$. Indeed, the Weyl action arising as a gauge freedom when diagonalizing the Higgs field $\Phi^{(0)}$ is precisely the right action of which the map $\pi_{\mathfrak{w}} : \mathfrak{h}^* \otimes K_\Sigma \rightarrow [\mathfrak{h}^* \otimes K_\Sigma] / \mathfrak{w}$ is the quotient projection. Which concludes the proof. ■

The cameral curve $\Sigma_{\mathcal{H}(\Phi^{(0)})}$ canonically associated to the Higgs pair $(P_0, \Phi^{(0)} = \text{Ad}_V Y)$ has a local parametrization given for any $\tilde{x} \in \overset{o}{\Sigma}$ and $\sigma \in \mathfrak{w}$ by

$$\overset{\sigma}{\tilde{x}} = R_\sigma(Y(\tilde{x})) \quad (2-23)$$

where we introduced the dual representative $R_\sigma \in \text{Aut}(\mathfrak{h}_o)$ of the Weyl group element $\sigma \in \mathfrak{w} \subset \text{Aut}(\mathfrak{h}_o^*)$.

We will now define non-perturbative correlators defined on copies the cameral curve. First, let us define new 1-point and 2-points classical correlators by

$$\begin{aligned} W_1^{cl} : \widehat{\mathcal{F}}_{cl} &\longrightarrow K_\Sigma \\ [\tilde{x} \cdot E] &\longmapsto \langle \text{Ad}_{C_{\tilde{x}}} E \cdot Y(x) \rangle \end{aligned} \quad (2-24)$$

$$\begin{aligned} \widehat{W}_2^{cl} : \widehat{\mathcal{F}}_{cl}^{\boxtimes 2} &\longrightarrow S^{\boxtimes 2} K_\Sigma \\ [\tilde{x} \cdot E] \otimes [\tilde{x}' \cdot E'] &\longmapsto - \frac{\langle M_{cl}([\tilde{x} \cdot E]) M_{cl}([\tilde{x}' \cdot E']) \rangle_o}{\mathcal{E}_f(\tilde{x}, \tilde{x}') \mathcal{E}_f(\tilde{x}', \tilde{x})} \\ &= - \frac{\langle \text{Ad}_{C_{\tilde{x}}} E \cdot \text{Ad}_{V(x)^{-1} V(x')} (\text{Ad}_{C_{\tilde{x}'}} E') \rangle_o}{\mathcal{E}_f(\tilde{x}, \tilde{x}') \mathcal{E}_f(\tilde{x}', \tilde{x})} \end{aligned} \quad (2-25)$$

$$(2-26)$$

from which the input of the cameral curve topological recursion, to be

defined in the next section, can be defined for any generic $\tilde{x}, \tilde{x}' \in \overset{o}{\Sigma}'$ and Weyl group elements $\sigma, \sigma' \in \mathfrak{w}$ as

$$\tilde{\Omega}^{\sigma}(\tilde{x}) = \sum_{\alpha \in \mathfrak{R}_o} W_1^{cl}([\tilde{x} \cdot R_{\sigma}(H_{\alpha})]) \alpha \quad (2-27)$$

$$\tilde{B}^{\sigma, \sigma'}(\tilde{x}, \tilde{x}') = \sum_{\alpha, \beta \in \mathfrak{R}_o} \widehat{W}_2^{cl}([\tilde{x} \cdot R_{\sigma}(H_{\alpha})], [\tilde{x}' \cdot R_{\sigma'}(H_{\beta})]) \alpha \otimes \beta \quad (2-28)$$

where we denoted by $\{H_{\alpha}\}_{\alpha \in \mathfrak{R}_o}$ the standard Chevalley basis associated to the set of simple roots \mathfrak{R}_o . These definitions are consistent with the input data described previously thanks to the identities

$$S_{\gamma}^{cl} \cdot \tilde{\Omega}^{\sigma}(\tilde{x} + \gamma) = \sum_{\alpha \in \mathfrak{R}_o} W_1^{cl}([\tilde{x} + \gamma \cdot R_{\sigma}(H_{\alpha})]) S_{\gamma}^{cl}(\alpha) \quad (2-29)$$

$$= \sum_{\alpha \in \mathfrak{R}_o} W_1^{cl}([\tilde{x} \cdot (R_{S_{\gamma}^{cl}} R_{\sigma}) H_{\alpha}]) \alpha \quad (2-30)$$

$$= \tilde{\Omega}^{\sigma S_{\gamma}^{cl}}(\tilde{x}) \quad (2-31)$$

which is the definition of an \mathfrak{h}_o^* -valued \mathfrak{w} -equivariant differential form on the cameral curve $\Sigma_{\mathcal{H}(\Phi^{(o)})}$. Similarly, \tilde{B} is symmetric and satisfies

$$(S_{\gamma}^{cl} \otimes 1) \cdot \tilde{B}^{\sigma, \sigma'}(\tilde{x} + \gamma, \tilde{x}') = \tilde{B}^{\sigma S_{\gamma}^{cl}, \sigma'}(\tilde{x}, \tilde{x}') \quad (2-32)$$

together with the singular behavior

$$\tilde{B}^{\sigma, \sigma'}(\tilde{x}, \tilde{x}') \underset{x \sim x'}{\sim} -\frac{1}{2} \sum_{\alpha, \beta \in \mathfrak{R}_o} \frac{\kappa_{\sigma \cdot \alpha, \sigma' \cdot \beta}}{\mathcal{E}_f(\tilde{x}, \tilde{x}') \mathcal{E}_f(\tilde{x}', \tilde{x})} \alpha \otimes \beta \quad (2-33)$$

where κ denotes the Cartan matrix. $\tilde{\Omega}$ and \tilde{B} are then both well-defined \mathfrak{h}_o^* -valued (resp. $(\mathfrak{h}_o^*)^{\otimes 2}$ -valued) \mathfrak{w} -equivariant meromorphic forms on the cameral cover $\Sigma_{\mathcal{H}(\Phi^{(o)})} \longrightarrow \overset{o}{\Sigma}$ (by additionally identifying the tangent spaces at a base point and one of its pre-images by this covering map).

2.2 Cameral curve topological recursion and reconstruction

Topological recursion for spectral curves

We start by reminding the usual formulation of the spectral curve topological recursion [43], [30].

Definition 2.3 *Spectral curves*

Define a spectral curve to be the data (\mathcal{S}, Ω, B) consisting of a Lagrangian immersion

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{i} & T^*\overset{o}{\Sigma} \\ \downarrow x & \swarrow & \\ \overset{o}{\Sigma} & & \end{array}$$

together with a meromorphic one-form $\Omega = i^*\eta$ on Σ with Liouville one-form η on $T^*\overset{o}{\Sigma}$ and a symmetric bi-differential B on $\Sigma \boxtimes \Sigma$ having a double pole on the diagonal divisor

$$B(z, z') \underset{z \sim z'}{\sim} \frac{dz dz'}{(z - z')^2} \quad (2-34)$$

with no residue and no other singularity. When a choice of Torelli marking, a symplectic basis of cycles in $H_1(\overset{o}{\Sigma}, \mathbb{Z})$, is made, the associated Bergman kernel is a natural candidate for B . Similarly, when a choice of polarization is made, the Klein form is an other candidate.

The Eynard-Orantin topological recursion procedure was introduced to solve perturbatively the Schwinger-Dyson, or loop equations, arising in random matrix models. It associates to the data of a spectral curve (\mathcal{S}, Ω, B) a sequence of so-called invariants denoted $\{\omega_{g,n}\}_{g,n \in \mathbb{N}}$ with numerous algebraic and modular properties. In particular, $\omega_{g,n}$ is a symmetric n -differential on $\overset{o}{\Sigma}$ called the n -points correlator in genus g .

Denoting by $\text{br}(\mathcal{S})$ the set of critical points of the projection x and generically by σ_a , for any $a \in \text{br}(\mathcal{S})$, its corresponding local Galois involution, the procedure goes as follows :

Definition 2.4 *Global spectral curve topological recursion*

First define $\omega_{0,1} \stackrel{\text{def}}{=} \Omega$ and $\omega_{0,2} \stackrel{\text{def}}{=} B$, then for any stable $(g, n) \in \mathbb{N}$, that is such that $2 - 2g - n < 0$, and any generic n -tuple of points $(z_1, \dots, z_n) \in \mathcal{S}^n$,

$$\begin{aligned} \omega_{g,n}(z_1, \dots, z_n) = & \sum_{a \in \text{br}(\mathcal{S})} \text{Res}_{z=a} K_a(z_1, z) [\omega_{g-1, n+1}(z, \sigma_a(z), z_2, \dots, z_n) \\ & + \sum'_{\substack{h+h'=g \\ I \sqcup J = \{2, \dots, n\}}} \omega_{h, 1+|I|}(z, z_I) \omega_{h', 1+|J|}(\sigma_a(z), z_J)] \end{aligned} \quad (2-35)$$

where $K_a(z_1, z) \stackrel{\text{def}}{=} \frac{\int_{\sigma_a(z)}^z \omega_{0,2}(z_1, \cdot)}{\omega_{0,1}(z) - \omega_{0,1}(\sigma_a(z))}$ is called the kernel at branch point a and \sum' means that in the sum over (h, I) and (h', J) we omit the terms corresponding to $(h, I) = (0, \emptyset)$ or $(h, I) = (g, \{2, \dots, n\})$.

Definition 2.5 *Free energies*

The free energies associated to this topological recursion procedure are defined in the following way :

- for any $g \geq 2$,

$$\omega_{g,0} \stackrel{\text{def}}{=} F_g \stackrel{\text{def}}{=} \frac{1}{2-2g} \sum_{a \in \text{br}(\mathcal{S})} \text{Res}_{z=a} \phi(z) \omega_{g,1}(z) \quad (2-36)$$

where $\phi \stackrel{\text{def}}{=} \int \omega_{0,1}$ is a choice of primitive of $\omega_{0,1} = \Omega$ from which this definition can be shown to be independent.

- for $g = 1$,

$$F_1 \stackrel{\text{def}}{=} -\frac{1}{2} \ln \tau_B - \frac{1}{24} \ln \left(\prod_{a \in \text{br}(\mathcal{S})} y'(a) \right) \quad (2-37)$$

where the symbol τ_B is the so-called Bergmann τ -function defined such that for any branch point $a \in \text{br}(\mathcal{S})$, $\frac{\partial \ln \tau_B}{\partial x(a)} \stackrel{\text{def}}{=} \text{Res}_{z \rightarrow a} \frac{B(z, \bar{z})}{dx(z)}$. It is well defined

up to a non-relevant multiplicative constant since the Rauch variational formula implies that the right hand side of the last equality is a closed form. Moreover, defining a good local coordinate $\zeta_a(z) \stackrel{\text{def}}{=} \sqrt{x(z) - x(a)}$ near the simple branch point a , such that $y'(a) = \frac{d\Omega(\cdot, \frac{\partial}{\partial x})}{d\zeta_a} \Big|_{z=a}$.

- for $g = 0$, introducing a Torelli marking $\{A_i, B_i\}_{1 \leq i \leq g} \subset H_1(\Sigma, \mathbb{Z})$,

$$F_0 \stackrel{\text{def}}{=} \frac{1}{2} \left(\sum_{i=1}^g \varepsilon_i \oint_{B_i} \omega_{0,1} + \sum_{s \in p(\omega_{0,1})} t_s \int_{B_{s,o}'''} \omega_{0,1} + \sum_{\substack{k \in \mathbb{N}^* \\ s \in p(\omega_{0,1})}} t_{s,k} \oint_{B_{s,k}''} \omega_{0,1} \right) \quad (2-38)$$

where we introduced the periods $\varepsilon_i \stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint_{B_i} \omega_{0,1}$, the set $p(\omega_{0,1})$ of poles of the one-form $\omega_{0,1}$, $t_s \stackrel{\text{def}}{=} \text{Res}_s \omega_{0,1}$, the homology chain $B_{s,o}'''$ uniquely defined by requiring that its boundary is $\partial B_{s,o}''' = s - o$ and that its intersection with the marking vanishes, with $o \in \Sigma$ a reference point from which this definition can be shown to be independent, similarly define the times $t_{s,k} \stackrel{\text{def}}{=} \text{Res}_{z=s} [(x(z) - x(s))^k \omega_{0,1}(z)]$ and their duals $\oint_{B_{s,k}''} \omega_{0,1} \stackrel{\text{def}}{=} \frac{1}{k} \text{Res}_{z=s} [(x(z) - x(s))^{-k} \omega_{0,1}(z)]$. It is a quadratic expression in the input data $\omega_{0,1} = \Omega$.

The general solution of the topological recursion procedure involves intersection numbers on certain moduli spaces of coloured, stable, nodal, Riemann surfaces [41]. Without going into too much detail, it takes the form

Theorem 2.6 *General solution [Eynard]*

For any $g, n \in \mathbb{N}$ such that $2g - 2 + n > 0$, the general solution of the spectral curve topological recursion has the form

$$\omega_{g,n}(z_1, \dots, z_n) = 2^{3g-3+n} \sum_{\substack{d_1, \dots, d_n \\ a_1, \dots, a_n}} \left(\prod_{i=1}^n d\xi_{a_i, d_i}(z_i) \right) \int_{\overline{\mathcal{M}}_{g,n}^{\{a_1, \dots, a_n\}}} \psi_1^{d_1} \dots \psi_n^{d_n} \Lambda(\mathcal{S})^{\{a_1, \dots, a_n\}} \quad (2-39)$$

where the sum is over branch point indices $a_1, \dots, a_n \in \text{br}(\mathcal{S})$ and non-negative integers $d_1, \dots, d_n \in \mathbb{N}$. $\overline{\mathcal{M}}_{g,n}^{\{a_1, \dots, a_n\}}$ denotes the compactified moduli space of stable Riemann surfaces each stable component of which is labelled by a branch point index in $\{a_1, \dots, a_n\}$. It is realized as

$$\begin{aligned} \overline{\mathcal{M}}_{g,n}^{\{a_1, \dots, a_n\}} &\stackrel{\text{def}}{=} \{(C; z_1, \dots, z_n; \varphi) \mid (C; z_1, \dots, z_n) \in \overline{\mathcal{M}}_{g,n}, \\ &\quad \varphi : C - \{\text{nodes}\} \longrightarrow \{a_1, \dots, a_n\} \text{ continuous}\} \end{aligned} \quad (2-40)$$

where $\overline{\mathcal{M}}_{g,n}$ is the moduli space of nodal Riemann surfaces of genus g with n marked points. Note that the function φ being continuous from a discrete set means that it is constant on each stable component of the nodal curve. This is what we call a colouring of the Riemann surface.

For any $p \in \{1, \dots, n\}$, ψ_p is the p^{th} tautological class, namely the Chern class $\psi_p \stackrel{\text{def}}{=} c_1(\mathcal{L}_p)$, where $\mathcal{L}_p \longrightarrow \overline{\mathcal{M}}_{g,n}^{\{a_1, \dots, a_n\}}$ is the tautological line bundle whose fiber over a coloured Riemann surface with n marked points $(C; z_1, \dots, z_n) \in \overline{\mathcal{M}}_{g,n}^{\{a_1, \dots, a_n\}}$ is the cotangent plane $\mathcal{L}_p|_{(C; z_1, \dots, z_n)} \stackrel{\text{def}}{=} T_{z_p}^* C$. The symbol $\Lambda(\mathcal{S})^{\{a_1, \dots, a_n\}}$ is the Eynard class (related to the Chiodo class), it is a generating function of cohomology classes on $\overline{\mathcal{M}}_{g,n}^{\{a_1, \dots, a_n\}}$ defined in [41] for any spectral curve \mathcal{S} and last but not least,

$$d\xi_{a,d} \stackrel{\text{def}}{=} -(2d-1)!! 2^{-d} \text{Res}_{z' \rightarrow a} (B(z, z') \zeta_a(z')^{-2d-1}) \quad (2-41)$$

Cameral curves and the recursion

In the spirit of the previous framework, we now define a cameral curve to be the data $(\Sigma_b, \tilde{\Omega}, \tilde{B})$ consisting of a cameral cover $x: \Sigma_b \rightarrow \overset{o}{\Sigma}$ over a point $b \in \mathcal{B}_{reg}$ in the regular locus of the Hitchin fibration, an \mathfrak{h}^* -valued \mathfrak{w} -equivariant one-form $\tilde{\Omega}$ on Σ_b of the form

$$\tilde{\Omega} \stackrel{def}{=} \sum_{\alpha \in \mathfrak{R}_0} \tilde{\Omega}_\alpha \alpha \quad (2-42)$$

for a given set of simple roots \mathfrak{R}_0 . Given the cameral cover Σ_b , a natural candidate could be

$$\tilde{\Omega} = \sum_{\alpha \in \mathfrak{R}_0} [(p_\alpha)^* (i_{b,\alpha})^* \eta] \alpha \quad (2-43)$$

where we used the notations of the introduction

$$\begin{array}{ccccc} \Sigma_b & \xrightarrow{p_\alpha} & \mathcal{S}_{b,\alpha} & \xrightarrow{i_{b,\alpha}} & T^* \overset{o}{\Sigma} \\ & \searrow & \downarrow & \swarrow & \\ & & \overset{o}{\Sigma} & & \end{array}$$

and where η is still the Liouville one-form on $T^* \overset{o}{\Sigma}$, and a choice of symmetric $(\mathfrak{h}^*)^{\otimes 2}$ -valued \mathfrak{w} -equivariant bi-differential \tilde{B} on Σ_b required to have a double pole on the pullback of the diagonal in the squared base curve $(\overset{o}{\Sigma})^2$ with no residue and no other singularity, namely we choose it such that

$$\tilde{B}(z, z') \underset{z \sim \sigma \cdot z'}{\sim} \frac{1}{2} \kappa_{\alpha\beta} \overset{o}{B}(x(z), x(z')) \alpha \otimes \beta \quad (2-44)$$

where $\sigma \in \mathfrak{w}$ is a Weyl group element sending the simple root β , labelling the sheet to which z' belongs, to $\alpha = \sigma(\beta)$ where z is. Also introducing the Cartan matrix $\kappa = (\kappa_{\alpha\beta})_{\alpha, \beta \in \mathfrak{R}_0}$ of the Lie algebra \mathfrak{g} viewed as an element of $(\mathfrak{h}^*)^{\otimes 2}$ and the Bergman kernel $\overset{o}{B}$ associated to a given choice of Torelli

marking on the base curve that we denote by $\overset{o}{\mathfrak{m}} = \{A_i, B_i\}_{1 \leq i \leq g} \subset H_1(\overset{o}{\Sigma}, \mathbb{Z})$ on the base curve.

This choice is the one of bi-residues $\frac{1}{2}\kappa_{\alpha\beta}$ when the base points z and z' are, in the limit $x(z) \sim x(z')$, in the sheets labelled by some simple roots α and β respectively. In particular, since the Cartan matrix has entry 2 at any diagonal position, this bi-residue is 1 when $z' \sim z$ (or $x(z') \sim x(z)$ with z and z' located in the same sheet of the cameral cover).

We will now associate to any cameral curve $(\Sigma_b, \tilde{\Omega}, \tilde{B})$ a sequence of invariants $\{\tilde{\omega}_{g,n}\}_{g,n \in \mathbb{N}}$, $\tilde{\omega}_{g,n}$ being a $(\mathfrak{h}^*)^{\otimes n}$ -valued \mathfrak{w} -equivariant n -differential on Σ_b . For any integer $n > 0$ and any generic points $z_1, \dots, z_n \in \Sigma_b$ we will use the notation

$$\tilde{\omega}_{g,n}(z_1, \dots, z_n) \stackrel{def}{=} \sum_{\alpha_1, \dots, \alpha_n \in \mathfrak{R}_o} \tilde{\omega}_{g,n}^{\alpha_1, \dots, \alpha_n}(z_1, \dots, z_n) \alpha_1 \otimes \dots \otimes \alpha_n \quad (2-45)$$

To do so, the remaining ingredient to introduce is the recursion kernel. For a given generic branch point $a \in \text{br}(\Sigma_b)$ fixed by the order two element $\sigma_a \in \mathfrak{w}$, it is defined in a similar fashion than in the case of spectral curves as

$$\tilde{K}_a(z_1, z) \stackrel{def}{=} - \frac{\int_{H_\alpha \cdot (z, \sigma_a(z))} \tilde{\omega}_{0,2}(z_1, \cdot)}{\tilde{\omega}_{0,1}(z)(H_\alpha) - \tilde{\omega}_{0,1}(\sigma_a(z))(H_\alpha)} \cdot \frac{H_\alpha \otimes H_\alpha}{4} \quad (2-46)$$

where we used the pairing between equivariant differential forms and generalized cycles and $H_\alpha \in \mathfrak{h}$ is dual to the simple root $\alpha \in \mathfrak{R}_o$ corresponding to branch point a (that is such that σ_a is the orthogonal reflexion with respect to the hyperplane orthogonal to root α).

Definition 2.7 *Cameral curve topological recursion [B.-Eynard-Hurtubise]*
 Define the invariants $\tilde{\omega}_{g,n}$ for any stable $g \in \mathbb{N}$ and $n \in \mathbb{N}^*$, $2 - 2g - n < 0$, by the following recursion with respect to $-\chi_{g,n}$ and which is the same as the one for spectral curves.

$$\begin{aligned}
\tilde{\omega}_{g,n+1}(z_1, \dots, z_{n+1}) &= \frac{r}{|\mathfrak{w}|} \sum_{a \in \text{br}(\Sigma_b)} \text{Res}_{z \rightarrow a} \tilde{K}_a(z_1, z) [\tilde{\omega}_{g-1, n+2}(z, \sigma_a(z), z_2, \dots, z_n) \\
&\quad + \sum'_{\substack{I \sqcup I' = \{2, \dots, n+1\} \\ h+h'=g}} \tilde{\omega}_{h, |I|+1}(z, z_I) \otimes \tilde{\omega}_{h', |I'|+1}(\sigma_a(z), z_{I'})]
\end{aligned} \tag{2-47}$$

where \tilde{K}_a is contracted with the two first tensors of all the terms between brackets $[\cdot]$, $z_{\{i_1, \dots, i_k\}} = \{z_{i_1}, \dots, z_{i_k}\}$ for any choice of pairwise distinct indices that we denote $\{i_1, \dots, i_k\} \subset \{2, \dots, n+1\}$ and Σ' means that we omit in the sum the terms with $(h, I) = \emptyset$ or $(h, I) = (g, \{2, \dots, n+1\})$.

Notice at this point that the sum over the set of branch points has many more terms in the present case of cameral covers compared to that of spectral curves, hence the symmetry factor.

Reconstruction

The cameral curve topological recursion associated to the triplet $(\Sigma_{\mathcal{H}(\Phi^{(0)})}, \tilde{\Omega}, \tilde{B})$ yields a sequence $\{\tilde{\omega}_{g,n}\}_{2g-2+n>0}$, where $\tilde{\omega}_{g,n}$ is an $(\mathfrak{h}_o^*)^{\otimes n}$ -valued \mathfrak{w} -equivariant meromorphic n -differential on $\Sigma_{\mathcal{H}(\Phi^{(0)})}$. Defining

$$\begin{aligned}
\widehat{W}_n(\tilde{x}_1, \dots, \tilde{x}_n) &\stackrel{\text{def}}{=} \\
&\sum_{\alpha_1, \dots, \alpha_n \in \mathfrak{X}_o} \widehat{W}_n([\tilde{x}_1 \cdot R_{\sigma_1}(H_{\alpha_1})], \dots, [\tilde{x}_n \cdot R_{\sigma_n}(H_{\alpha_n})]) \alpha_1 \otimes \dots \otimes \alpha_n
\end{aligned} \tag{2-48}$$

The proof has not been completed yet but we wish to show the following equality of formal power series in ε

Conjecture 2.8

$$\widehat{W}_n = \sum_{g=0}^{\infty} \varepsilon^{2g-2+n} \tilde{\omega}_{g,n} \tag{2-49}$$

Notice that for these definitions we used the connected correlators \widehat{W}_n and not the non-connected ones W_n . Similarly, we can define the non-connected cameral n -points function by

$$\widetilde{W}_n^{\sigma_1, \dots, \sigma_n}(\tilde{x}_1, \dots, \tilde{x}_n) = \sum_{\mu \vdash \{\tilde{x}_1, \dots, \tilde{x}_n\}} \bigotimes_{i=1}^{\ell(\mu)} \widehat{W}_{|\mu_i|}(\mu_i) \quad (2-50)$$

and show that they also satisfy a set of loop equations. Indeed, if we choose a Weyl group element $\tau \in \mathfrak{w}$, we can write a twisted k^{th} Casimir element \mathcal{C}_k^τ , for each $k \in \{1, \dots, d\}$ (restricted to the Cartan subalgebra \mathfrak{h}_o)

$$\mathcal{C}_k^\tau \stackrel{\text{def}}{=} \sum_{\alpha_1, \dots, \alpha_k \in \mathfrak{R}_o} \mathcal{C}_k^{\alpha_1, \dots, \alpha_k} R_\tau^{-1}(H_{\alpha_1}) \otimes \dots \otimes R_\tau^{-1}(H_{\alpha_k}) \quad (2-51)$$

$$\stackrel{\text{def}}{=} \sum_{\alpha_1, \dots, \alpha_k \in \mathfrak{R}_o} \mathcal{C}_k^{\tau; \alpha_1, \dots, \alpha_k} H_{\alpha_1} \otimes \dots \otimes H_{\alpha_k} \quad (2-52)$$

and the set of twisted Casimirs $\{\mathcal{C}_k^\tau\}_{1 \leq k \leq d}$ still generate the center $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$. We then write the loop equations with this set of twisted Casimirs

$$\sum_{k=0}^d (-1)^k y^{d-k} \sum_{\alpha_1, \dots, \alpha_k \in \mathfrak{R}_o} \mathcal{C}_k^{\tau; \alpha_1, \dots, \alpha_k} W_{k+n}([\tilde{x} \cdot H_{\alpha_1}], \dots, [\tilde{x} \cdot H_{\alpha_k}], X_1, \dots, X_n) = P_n(x, y; X_1, \dots, X_n) \quad (2-53)$$

in which we can make the replacement $X_i = [\tilde{x}_i \cdot R_{\sigma_i}(H_{\beta_i})]$, $i \in \{1, \dots, n\}$, for any generic $(\tilde{x}_1, \dots, \tilde{x}_n) \in \widetilde{\Sigma}^n$ and any n -tuples of Weyl group elements $\sigma_1, \dots, \sigma_n \in \mathfrak{w}$ and simple roots $\beta_1, \dots, \beta_n \in \mathfrak{R}_o^n$ to obtain

$$\begin{aligned} & \sum_{k=0}^d (-1)^k y^{d-k} \sum_{\alpha_1, \dots, \alpha_k \in \mathfrak{R}_o} \mathcal{C}_k^{\tau; \alpha_1, \dots, \alpha_k} W_{k+n}([\tilde{x} \cdot H_{\alpha_1}], \dots, [\tilde{x} \cdot H_{\alpha_k}], \\ & \quad [\tilde{x}_1 \cdot R_{\sigma_1}(H_{\beta_1})], \dots, [\tilde{x}_n \cdot R_{\sigma_n}(H_{\beta_n})]) \\ & = P_n(x, y; [\tilde{x}_1 \cdot R_{\sigma_1}(H_{\beta_1})], \dots, [\tilde{x}_n \cdot R_{\sigma_n}(H_{\beta_n})]) \end{aligned}$$

$$= \sum_{k=0}^d (-1)^k y^{d-k} \sum_{\alpha_1, \dots, \alpha_k \in \mathfrak{R}_o} \mathcal{C}_k^{\tau\sigma^{-1}; \alpha_1, \dots, \alpha_k} W_{k+n}([\tilde{x} \cdot R_\sigma(H_{\alpha_1})], \dots, [\tilde{x} \cdot R_\sigma(H_{\alpha_k})], \\ [\tilde{x}_1 \cdot R_{\sigma_1}(H_{\beta_1})], \dots, [\tilde{x}_n \cdot R_{\sigma_n}(H_{\beta_n})])$$

which we can now multiply by $\beta_1 \otimes \dots \otimes \beta_n$. Summing over $\beta_1, \dots, \beta_n \in \mathfrak{R}_o^n$ and relabeling $\tau\sigma^{-1} \rightarrow \tau$ then yields the set of cameral loop equations

$$\sum_{k=0}^d (-1)^k y^{d-k} \mathcal{C}_k^\tau(\tilde{x}) \cdot \widetilde{W}_n^{\sigma_1, \dots, \sigma_n}(\tilde{x}_1, \dots, \tilde{x}_n) \\ = \sum_{\beta_1, \dots, \beta_n \in \mathfrak{R}_o} P_n(x, y; [\tilde{x}_1 \cdot R_{\sigma_1}(H_{\beta_1})], \dots, [\tilde{x}_n \cdot R_{\sigma_n}(H_{\beta_n})]) \beta_1 \otimes \dots \otimes \beta_n \quad (2-58)$$

$$\stackrel{def}{=} \widetilde{P}_n(x, y; \tilde{x}_1^{\sigma_1}, \dots, \tilde{x}_n^{\sigma_n}) \stackrel{def}{=} \sum_{k=0}^d (-1)^k y^{d-k} \widetilde{P}_{k;n}(x; \tilde{x}_1^{\sigma_1}, \dots, \tilde{x}_n^{\sigma_n}) \quad (2-59)$$

for any Weyl group elements $\tau, \sigma \in \mathfrak{w}$, where $\mathcal{C}_k^\tau \cdot \widetilde{W}_n$ means contraction of \mathcal{C}_k^τ with the k first tensor elements of each term in the definition of \widetilde{W}_{k+n} . Note that the right hand side of the last equalities/definitions are independent of the choice of $\tau, \sigma \in \mathfrak{w}$. Symbolically we may write

$$\mathcal{C}_k^\tau(\tilde{x}) \cdot \widetilde{W}_n(J) \stackrel{def}{=} \langle \mathcal{C}_k^\tau | \widetilde{W}_{k+n}(\tilde{x}, \dots, \tilde{x}, J) \rangle \quad (2-60)$$

where the bracket in the right hand side is the contraction of tensors over \mathfrak{h}_o .

The next steps are then to decompose the non-connected correlators into connected ones and use their ε expansions to derive the perturbative loop equations. Notice at this point that we generically only need the *linear and quadratic loop equation* to show that the correlators are indeed reconstructed from topological recursion. Let us therefore consider the coefficient of y^{r-2} in the cameral loop equations. It states that

$$\mathcal{C}_2^\tau(\tilde{x}) \cdot \widetilde{W}_n^{\sigma_1, \dots, \sigma_n}(\tilde{x}_1, \dots, \tilde{x}_n) = \widetilde{P}_{2;n}(x; \tilde{x}_1^{\sigma_1}, \dots, \tilde{x}_n^{\sigma_n}) \quad (2-61)$$

is a meromorphic quadratic differential in the variable x defined on the base curve $\overset{o}{\Sigma}$ with possible singularities at the x_i 's and z_j 's but nowhere else.

Let us now assume the existence of topological expansions for the connected correlators given in the form

$$\widehat{W}_n = \sum_{g=0}^{\infty} \varepsilon^{2g-2+n} \omega_{g,n} \quad (2-62)$$

defined on products of the cameral cover. The quadratic loop equations can then be expanded in powers of ε and extracting the coefficient of ε^{2g-2+n} yields

$$\begin{aligned} \widetilde{P}_{2;n}^{(g)}(x; J)^{\beta_1, \dots, \beta_n} &\stackrel{def}{=} \langle H_{\beta_1} \otimes \dots \otimes H_{\beta_n} | \widetilde{P}_{2;n}^{(g)}(x; J) \rangle \\ &\stackrel{def}{=} \left\langle \mathcal{C}_2^\tau \otimes H_{\beta_1} \otimes \dots \otimes H_{\beta_n} \mid \omega_{g-1, n+2}(\overset{\sigma}{\tilde{x}}, \overset{\sigma}{\tilde{x}}, J) \right. \\ &\quad \left. + \sum_{\substack{I \sqcup I' = J \\ h+h'=g}} \omega_{h, 1+\#I}(\overset{\sigma}{\tilde{x}}, I) \otimes \omega_{h', 1+\#I'}(\overset{\sigma}{\tilde{x}}, I') \right\rangle \\ &\quad + \dots \end{aligned} \quad (2-63)$$

for any generic $\tilde{x} \in \overset{o}{\Sigma}$, $\sigma \in \mathfrak{w}$ and $\beta_1, \dots, \beta_n \in \mathfrak{X}_o$, where $J \stackrel{def}{=} \{\overset{\sigma_1}{\tilde{x}}_1, \dots, \overset{\sigma_n}{\tilde{x}}_n\}$ and the dots ... in the right hand side hide terms that already have the wanted regularity (implied from quadratic loop equations at lower values of $2g-2+n$). Accordingly to previous notations, we also introduced

$$\begin{aligned} \widetilde{P}_{2;n}^{(g)}(x; J) &\stackrel{def}{=} \sum_{\alpha_1, \dots, \alpha_n \in \mathfrak{X}_o} \widetilde{P}_{2;n}^{(g)}(x; J)^{\alpha_1, \dots, \alpha_n} \alpha_1 \otimes \dots \otimes \alpha_n \quad (2-64) \\ &\stackrel{def}{=} \left\langle \mathcal{C}_2^\tau \mid \omega_{g-1, n+2}(\overset{\sigma}{\tilde{x}}, \overset{\sigma}{\tilde{x}}, J) \right. \\ &\quad \left. + \sum_{\substack{I \sqcup I' = J \\ h+h'=g}} \omega_{h, 1+\#I}(\overset{\sigma}{\tilde{x}}, I) \otimes \omega_{h', 1+\#I'}(\overset{\sigma}{\tilde{x}}, I') \right\rangle \\ &\quad + \dots \end{aligned} \quad (2-65)$$

For any given sequence of simple roots $(\beta_n)_{n \in \mathbb{N}} \in \mathfrak{R}_0^{\mathbb{N}}$, we obtained a set of expressions $\left(\langle H_{\beta_1} \otimes \cdots \otimes H_{\beta_n} | \tilde{P}_{2;n}^{(g)} \rangle \right)_{\substack{g, n \in \mathbb{N} \\ 2g-2+n > 0}}$ identical to those appearing in the perturbative expansion of quadratic loop equations of [43], [30].

2.3 A scheme for solving integrable hierarchies given in Lax form

From Lax presentations to cameral curves

Keeping the notations of the previous section, let us consider a family of Fuchsian differential systems $(P_\varepsilon, d_x - \frac{1}{\varepsilon} \Phi_\varepsilon) \in \mathcal{M}_{Fuchs}$ with "small" deformation parameter ε and let us assume that this family also depends on a set of (complex) parameters $\{t_k\}_{k \in \mathcal{I}}$ called times for some (possibly infinite) set \mathcal{I} . We will furthermore assume that they are part of an integrable hierarchy

$$[\varepsilon d_x - \Phi_\varepsilon, \varepsilon d_{t_k} - A_\varepsilon^{(k)}] = [\varepsilon d_{t_k} - A_\varepsilon^{(k)}, \varepsilon d_{t_l} - A_\varepsilon^{(l)}] = 0 \quad (2-66)$$

for any indices $k, l \in I$. In this section, all the considered objects have time-dependences but we will never write these explicitly.

For generic values of ε , the pair $(P_\varepsilon, d_x - \frac{1}{\varepsilon} \Phi_\varepsilon)$ defines a Fuchsian differential system. Following the previous constructions, we can associate to any choice of Lagrangian submanifold $\mathcal{L} \subset \widehat{\mathcal{H}}_1$ a non-perturbative τ -function $\mathfrak{T}_\mathcal{L}$ such that for any $k \in I$,

$$d_{t_k} \ln \mathfrak{T}_\mathcal{L} = \int_{\Gamma_k} W_1 \quad (2-67)$$

where the cycle $\Gamma_k \stackrel{def}{=} (d_{t_k})^* \in \widehat{\mathcal{H}}_1$ is dual to the deformation d_{t_k} and is given by the formula

$$\Gamma_k = - \sum_{j=1}^M (d_{t_k} z_j) \mathcal{B}_j'' + \frac{1}{2\pi i} \sum_e e (d_{t_k} S_{\gamma_e} \cdot S_{\gamma_e}) \quad (2-68)$$

In principle, assuming all the correlators to admit topological expansions that can be computed from topological recursion would reconstruct perturbatively the τ -function $\mathfrak{T}_\mathcal{L}$.

Assuming they all admit a well-defined $\varepsilon \rightarrow 0$ limit, we then get in particular that Φ_ε and the $A_\varepsilon^{(k)}$ are codiagonalizable at leading order in the expansion parameter ε . That is there exists a decomposition

$$\Phi_\varepsilon|_{\mathcal{O}(\varepsilon)^0} = \text{Ad}_V Y, \quad A_\varepsilon^{(k)}|_{\mathcal{O}(\varepsilon)^0} = \text{Ad}_V \alpha_k \quad (2-69)$$

where Y and the α_k 's are \mathfrak{h}_o -valued meromorphic 1-forms.

These commutation relations also imply that there exists a common globally flat local solution $\Psi_\varepsilon \in H_{loc}^0(\overset{o}{\Sigma}, P)_0$ to the integrable hierarchy of equations

$$\varepsilon d_x \Psi_\varepsilon = \Phi_\varepsilon \Psi_\varepsilon \quad (2-70)$$

$$\forall k \in \mathcal{J}, \quad \varepsilon d_{t_k} \Psi_\varepsilon = A_\varepsilon^{(k)} \Psi_\varepsilon \quad (2-71)$$

In this case and for any $\tilde{x} \in \overset{o}{\Sigma}$, Ψ_ε admits an asymptotic $\varepsilon \rightarrow 0$ description of the form

$$\Psi_\varepsilon(\tilde{x}) \underset{\varepsilon \rightarrow 0}{=} V(x) \widehat{\Psi}_\varepsilon(\tilde{x}) e^{\frac{1}{\varepsilon} S(\tilde{x})} C_{\tilde{x}} \quad (2-72)$$

where $S: \overset{o}{\Sigma} \rightarrow \mathfrak{h}_o \otimes \mathbb{K}_\Sigma^o$ is an \mathfrak{h} -valued function on the universal cover and is set to satisfy

$$\partial_x S = Y, \quad \partial_{t_k} S = \alpha_k \quad (2-73)$$

$\widehat{\Psi}_\varepsilon$ is a solution of the linear equation

$$(\text{Ad}_{V^{-1}} d_x) \widehat{\Psi} = [Y, \widehat{\Psi}] = Y \cdot \widehat{\Psi} - \widehat{\Psi} \cdot Y, \quad (2-74)$$

\cdot denoting respectively Lie derivation from the left and from the right, and such that it satisfies $\widehat{\Psi}_\varepsilon(\tilde{x}) = 1 + \mathcal{O}(\varepsilon)$. Once again, $C_{\tilde{x}} = C \in P_o$ is sectorwise constant where now Stokes sectors are defined as being the

connected components of $\overset{o}{\tilde{\Sigma}'} = [\overset{o}{\tilde{\Sigma}} - \bigcup_{\alpha \in \mathfrak{R}} \Gamma_\alpha \cup \bar{\Gamma}_\alpha]$, where the Stokes and anti-Stokes lines are now defined as

$$\Gamma_\alpha \stackrel{def}{=} \{\tilde{x} \in \overset{o}{\tilde{\Sigma}} \mid \Re e S(\tilde{x}) = 0\} \quad (2-75)$$

$$\bar{\Gamma}_\alpha \stackrel{def}{=} \{\tilde{x} \in \overset{o}{\tilde{\Sigma}} \mid \Im m S(\tilde{x}) = 0\} \quad (2-76)$$

This Stoke's network can be used once again to show the emergence of cameral geometry in the WKB limit. The corresponding cameral cover is still that associated to the Higgs pair $(P_0, \Phi^{(0)})$ but now also depends on the times $\{t_k\}_{k \in \mathcal{T}}$ of the hierarchy.

In the last chapter, we will investigate how quantum W-algebras can be comprised in this setup. They are indeed the natural quantum analog of the Fuchsian system since their classical counterparts, namely classical W-algebras, are the Poisson structures relevant for the study of Gelfand-Dickey hierarchies [3] or more generally Drinfeld-Sokolov hierarchy [51]. The background charge Q will play the role of the quantization parameter and the $Q = 0$ case will be identified with a Fuchsian system.

3 A spectral curve for the KdV hierarchy

3.1 The Korteweg-de Vries equation and its applications

The Korteweg-de Vries equation, or KdV, originated in the study of shallow dispersive water waves. It allows for solitary wave solutions, or *solitons*, that are localized but propagate through space and time. It is a non-linear equation that possess two contributions that have clear physical interpretations. In its dimensionless form, it can be written as

$$\partial_t u = u \partial_x u + \frac{\varepsilon^2}{12} \partial_x^3 u \quad (3-1)$$

The first term of the right hand side is the non-linear term and accounts for the *finite size* effects (depth of water) while its second term is directly related to *dispersion*. The typical hydrodynamic situation in which these waves can be observed is the propagation of waves on the surface of a

canal (*cf.* the canals of Amsterdam for which it was written down). In this situation, the finite depth of the water slows the front of the wave down and accelerates its tail while dispersion acts the other way around. This heuristics gives a hint to understand how such solitary waves that keep the same size over time can emerge.

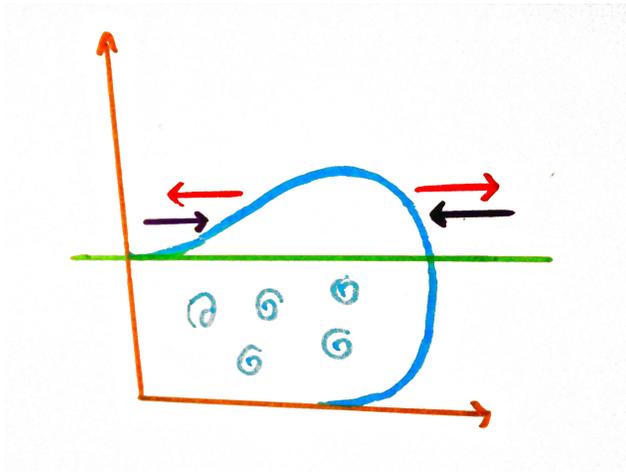


Figure 2.4: Finite size effects (black) vs. dispersion (red).

Hydrodynamics being scale invariant (what allows for example for the use of wind tunnels in engineering), the KdV equation can be applied to

- tidal waves caused by the competition between the gravitational pulls of celestial bodies (*e.g.* earth vs. moon)
- tsunamis triggered by large (often underwater) earthquakes, volcanic eruptions, (possibly submarine) landslides, etc.
- both of the above but in the sky !

as long as the *shallow* hypothesis can be applied, namely that the height of the wave is negligible compared to its width. Furthermore, there are other fields of Physics in which such competing features as non-linear finite size effects and dispersion appear and where one can therefore hope

to use this equation as a descriptive tool. Let us mention that it goes from acoustics (*e.g.* acoustic waves in crystal lattices or ionic plasmas) to electromagnetism, where it was hypothesized that the KdV equation could describe ball lightning although there exists no consensus whatsoever on the subject. In any case, one would still have to compare numerically the predictions from the KdV equation to the particular phenomena under study and sometimes this can rule it out from bearing responsibility (*e.g.* in [70] the author shows the *innocence* of the KdV dynamics regarding the Chilean tsunami of 1960 by studying its magnitude).

Now this equation is also interesting mathematically speaking as it possesses an infinite dimensional space of solutions. In other words, complexifying the problem, the generic solution of this equation depends holomorphically on an infinite number of additional parameters and this infinite set of flows commute in the space of solutions. This gives rise to the KdV hierarchy. It is a special case, namely the 2-reduced, of a more general construction of integrable hierarchies called the Kadomtsev-Petiashvili hierarchy or KP. Moreover, it fits in the framework of [38] as the most simple non-trivial integrable hierarchy one can think of.

In what follows we will start by introducing the notion of τ -function of the KdV hierarchy. It is known that τ -functions of the KdV hierarchy is a generating function for intersection numbers of ψ and κ -classes on the Deligne-Mumford moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable algebraic curves. After following their introduction to the subject, we state results by Bertola, Dubrovin and Yang [21] expressing in a closed way the generating functions of such intersection numbers.

After noticing that they are correlators associated with an $\mathfrak{sl}_2(\mathbb{C})$ Fuchsian system of topological type, we will determine the spectral curve underlying the topological recursion satisfied by their WKB genus expansions (notice that $\mathrm{rk} \mathfrak{sl}_2(\mathbb{C}) = 1$ and therefore the cameral geometry coincides with the usual geometry of spectral curves [42]).

3.2 KdV hierarchy and higher Weil-Petersson volumes

Hierarchy and τ -function

As we have seen, the KdV hierarchy is an infinite set of compatible integrable Lax equations, integrable in the sense that they possess an infinite number of Poisson commuting conserved quantities. In the pseudo-differential operators formalism it was constructed as

$$\partial_{t_k} L = [((L^{k/2})_+, L)] \quad (3-2)$$

where $L \stackrel{\text{def}}{=} \partial^2 - u$ and $L^{1/2} = \partial - \frac{1}{2}u\partial^{-1} - \frac{3}{4}(\partial u)\partial^{-2} + \dots$ is determined recursively by requiring that no negative powers of ∂ occur in $(L^{1/2})^2$. With these notations, the general solution u of the hierarchy depends on the odd times $(t_{2k+1})_{k \in \mathbb{N}}$ and of course on a family of parameters for the initial conditions $(s_k)_{k \in \mathbb{N}^*}$ with respect to which we will specify the dependence soon enough (they are the coupling constants generating the κ -classes). Here we will change our conventions to a normalization better fit to study the relationship with intersection theory on $\overline{\mathcal{M}}_{g,n}$. That is we consider the Lax operator $L = \partial^2 + 2u$ satisfying the hierarchy

$$\partial_{t_k} L = [A_k, L] \quad (3-3)$$

$$A_k \stackrel{\text{def}}{=} \frac{1}{(2k+1)!!} (L^{\frac{1+2k}{2}})_+ \quad (3-4)$$

With these conventions, the KdV equation is $\partial_t u = u\partial_x u + \frac{1}{12}\partial_x^3 u$ where we identified $t_0 = x$ as the space variable and $t_1 = t$ as the first time. We will denote $\mathbf{t} = (t_k)_{k \in \mathbb{N}}$ and $\mathbf{s} = (s_k)_{k \in \mathbb{N}^*}$

Definition 3.1 τ -function

A Sato τ -function of the KdV hierarchy is a formal power series expansion $\tau \in \mathbb{C}[[\mathbf{t}, \mathbf{s}]]$ satisfying the Hirota bilinear identities

$$\text{Res}_{z=\infty} (\psi(z; \mathbf{t}; \mathbf{s}) \psi^*(z; \tilde{\mathbf{t}}; \mathbf{s}) z^{2p} dz) = 0 \quad (3-5)$$

for any \mathbf{t} and $\tilde{\mathbf{t}}$, where we respectively defined the wave and dual wave-functions by

$$\psi(z; \mathbf{t}; \mathbf{s}) \stackrel{\text{def}}{=} \frac{\tau(t_k - \frac{(2k-1)!!}{z^{2k+1}}; \mathbf{s})}{\tau(t_k; \mathbf{s})} e^{\vartheta(z, \mathbf{t})} \quad (3-6)$$

$$\text{and} \quad \psi^*(z; \mathbf{t}; \mathbf{s}) = \psi(-z; \mathbf{t}; \mathbf{s}) \stackrel{\text{def}}{=} \frac{\tau(t_k + \frac{(2k-1)!!}{z^{2k+1}}; \mathbf{s})}{\tau(t_k; \mathbf{s})} e^{-\vartheta(z, \mathbf{t})} \quad (3-7)$$

with a phase ϑ given by $\vartheta(z, \mathbf{t}) \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} t_j \frac{z^{2j+1}}{(2j+1)!!}$.

Proposition 3.2 τ -function of KdV $\tau \in \mathbb{C}[[\mathbf{t}, \mathbf{s}]]$ is a τ -function of the KdV hierarchy if and only if $u = \partial_x^2 \log \tau$ is a solution of the KdV hierarchy and

$$\tau(\mathbf{t}; \mathbf{s}) = \exp \left(\sum_{g, n, l \geq 0} \frac{1}{n!} \sum_{\substack{k_1, \dots, k_n \geq 0 \\ d_1, \dots, d_l \geq 0}} \langle \kappa_1^{d_1} \cdots \kappa_l^{d_l} \tau_{k_1} \cdots \tau_{k_n} \rangle_{g, n} t_{k_1} \cdots t_{k_n} \frac{s_1^{d_1} \cdots s_l^{d_l}}{d_1! \cdots d_l!} \right) \quad (3-8)$$

where

$$\langle \kappa_1^{d_1} \cdots \kappa_l^{d_l} \tau_{k_1} \cdots \tau_{k_n} \rangle_{g, n} = \int_{\mathcal{M}_{g, n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \kappa_1^{d_1} \cdots \kappa_l^{d_l} \quad (3-9)$$

Definition 3.3 n -point correlation functions and generating functions
For any τ -function of the KdV hierarchy and any integer $n \in \mathbb{N}^*$, define the n -point correlation functions as

$$\langle \langle \tau_{k_1} \cdots \tau_{k_n} \rangle \rangle (\mathbf{t}; \mathbf{s}) = \frac{\partial^n \log \tau}{\partial t_{k_1} \cdots \partial t_{k_n}} (\mathbf{t}; \mathbf{s}) \quad (3-10)$$

and their generating function to be

$$F_n(z_1, \dots, z_n; \mathbf{t}; \mathbf{s}) = \sum_{k_1, \dots, k_n \in \mathbb{N}} \langle \langle \tau_{k_1} \cdots \tau_{k_n} \rangle \rangle (\mathbf{t}; \mathbf{s}) \frac{(2k_1 + 1)!!}{z_1^{2k_1+2}} \cdots \frac{(2k_n + 1)!!}{z_n^{2k_n+2}} \quad (3-11)$$

Lax presentation of the problem and expression of the generating functions

The wave functions are formal eigenfunctions of the Lax operator $L\psi = z^2\psi$ and $L\psi^* = z^2\psi^*$ and their dependence in the times is specified by the compatible system $\partial_{t_k}\psi = A_k\psi$, $\partial_{t_k}\psi^* = -A_k\psi^*$ for any integer $k \geq 0$. They fit into the fundamental matrix solution

$$\Psi(z; \mathbf{t}; \mathbf{s}) = \begin{pmatrix} \psi(z; \mathbf{t}; \mathbf{s}) & \psi^*(z; \mathbf{t}; \mathbf{s}) \\ -\partial_x \psi(z; \mathbf{t}; \mathbf{s}) & -\partial_x \psi^*(z; \mathbf{t}; \mathbf{s}) \end{pmatrix} \quad (3-12)$$

Theorem 3.4 *Closed form for the generating functions [Bertola, Dubrovin, Yang]*

Defining the matrix Θ by $\Theta(z; \mathbf{t}; \mathbf{s}) = z\Psi(z; \mathbf{t}; \mathbf{s})\sigma_3\Psi(z; \mathbf{t}; \mathbf{s})^{-1}$, where we used the usual definition $\sigma_3 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a Pauli matrix, then for any integer $n \geq 2$, the generating functions for multi-point correlation functions take the form

$$F_1(z; \mathbf{t}; \mathbf{s}) = \frac{1}{2} \text{Tr}(\Psi(z; \mathbf{t}; \mathbf{s})^{-1} \partial_z \Psi(z; \mathbf{t}; \mathbf{s}) \sigma_3) - \partial_z \vartheta(z, \mathbf{t}) \quad (3-13)$$

$$F_n(z_1, \dots, z_n; \mathbf{t}; \mathbf{s}) = -\frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n} \frac{\text{Tr}(\Theta(z_{\sigma(1)}; \mathbf{t}; \mathbf{s}) \cdots \Theta(z_{\sigma(n)}; \mathbf{t}; \mathbf{s}))}{\prod_{j=1}^n (z_{\sigma(j)}^2 - z_{\sigma(j+1)}^2)} - \delta_{n,2} \frac{z_1^2 + z_2^2}{(z_1^2 - z_2^2)^2} \quad (3-14)$$

3.3 WKB analysis

Introducing the "small" parameter ε , we apply our scheme using standard WKB analysis (up to first order only !) applied to the formal eigenfunction relations (they are second order ordinary differential equations of the variable x). This yields the asymptotics

$$\psi(\pm z; \mathbf{t}; \mathbf{s}) \underset{\hbar \rightarrow 0}{\sim} \left(1 - \frac{2u_0(\mathbf{t}; \mathbf{s})}{z^2}\right)^{-1/4} e^{\pm \frac{1}{\varepsilon} \int_0^x d\chi \sqrt{2u_0(t_0=\chi, t_1, \dots; \mathbf{s}) - z^2}} \quad (3-15)$$

Straightforward computations of limits from the definitions of the generating functions of the 1-point and 2-points generating functions and the topological expansions

$$F_1 = \sum_{g=0}^{\infty} \varepsilon^{2g-1} \omega_{g,1}, \text{ and } F_2 = \sum_{g=0}^{\infty} \varepsilon^{2g} \omega_{g,2}, \quad (3-16)$$

then allow to find the expressions

$$\begin{aligned} \omega_{0,1}(z) = & \, dz[(1-t_1)z^2 - \sum_{j \geq 2} (t_j - h_{j-1}(-\mathbf{s})) \frac{z^{2j}}{(2j-1)!!} \\ & - \int_0^x \frac{d\chi}{\sqrt{1 - \frac{2u_0(t_0=\chi, t_1, \dots; \mathbf{s})}{z^2}}}] \end{aligned} \quad (3-17)$$

with the coefficients $(h_j)_{j \in \mathbb{N}^*}$ being obtained from the parameters $(s_j)_{j \in \mathbb{N}^*}$ by solving the formal equation

$$1 + \sum_{j \geq 1} h_j(\mathbf{s}) z^j = \exp\left(\sum_{j \geq 1} s_j z^j\right) \quad (3-18)$$

$$\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2 [z_1^2 + z_2^2 - 4u_0(\mathbf{t}; \mathbf{s})]}{(z_1^2 - z_2^2)^2 \sqrt{1 - \frac{2u_0(\mathbf{t}; \mathbf{s})}{z_1^2}} \sqrt{1 - \frac{2u_0(\mathbf{t}; \mathbf{s})}{z_2^2}}} \quad (3-19)$$

$$\omega_{1,1}(z_1) = \left(\frac{\partial_x u_0(\mathbf{t}; \mathbf{s})}{8z^4 \left(1 - \frac{2u_0(\mathbf{t}; \mathbf{s})}{z_1^2}\right)^{5/2}} + \frac{\partial_x^2 u_0(\mathbf{t}; \mathbf{s})}{24z^2 \partial_x u_0(\mathbf{t}; \mathbf{s}) \left(1 - \frac{2u_0(\mathbf{t}; \mathbf{s})}{z_1^2}\right)^{3/2}} \right) dz_1 \quad (3-20)$$

KdV spectral curve

Definition 3.5 *KdV spectral curve [B.-Yang]*

Let $\mathcal{S} = ((\lambda, \eta), \Omega, B)$ be the genus 0 spectral curve given by

$$\lambda(z) \stackrel{\text{def}}{=} \frac{1}{2}z^2 \quad (3-21)$$

$$\eta(z) \stackrel{\text{def}}{=} (1-t_1)z - \sum_{j \geq 2} (t_j - h_{j-1}(-\mathbf{s})) \frac{z^{2j-1}}{(2j-1)!!} - \int_0^x \frac{d\chi}{\sqrt{z^2 - 2u_0(t_0 = \chi, t_1, \dots; \mathbf{s})}} \quad (3-22)$$

$$\Omega(z) \stackrel{\text{def}}{=} \eta(z) d\lambda(z) \quad (3-23)$$

$$B(z_1, z_2) \stackrel{\text{def}}{=} \frac{dz_1 dz_2 [z_1^2 + z_2^2 - 4u_0(\mathbf{t}; \mathbf{s})]}{(z_1^2 - z_2^2)^2 \sqrt{1 - \frac{2u_0(\mathbf{t}; \mathbf{s})}{z_1^2}} \sqrt{1 - \frac{2u_0(\mathbf{t}; \mathbf{s})}{z_2^2}}} \quad (3-24)$$

where z is a local parameter around $0 \in \mathbb{C}$, $t_0 = x$ and $u_0(\mathbf{t}; \mathbf{s})$ is the genus 0 part of the general solution $u(\mathbf{t}; \mathbf{s}; \hbar) = \sum_{g \geq 0} \varepsilon^{2g} u_g(\mathbf{t}, \mathbf{s})$ to the KdV hierarchy, it is the unique solution to the dispersionless KdV hierarchy

$$\partial_{t_k} v = \frac{v^k}{k!} \partial_{t_0} v, \quad k \in \mathbb{N} \quad (3-25)$$

such that the initial data $u_0(t_0 = x, 0, 0, \dots; \mathbf{s}) = f(x; \mathbf{s})$ satisfies

$$x = f(x; \mathbf{s}) + \sum_{k \geq 2} h_{k-1}(-\mathbf{s}) \frac{f(x; \mathbf{s})^k}{k!}, \quad (3-26)$$

WKB analysis gives the only natural candidate for a spectral curve to run the topological recursion to compute the correlators and therefore yields the following reconstruction conjecture

Conjecture 3.6 *Higher W-P volumes by topological recursion [B.-Yang]*
Define the symplectic invariants $(\omega_{g,n})_{2g-2+n \geq 0}$ of \mathcal{S} by the topological recursion

$$\begin{aligned}
\omega_{g,n}(z_1, \dots, z_n; \mathbf{t}; \mathbf{s}) &\stackrel{\text{def}}{=} \sum_{a \in \{\pm\sqrt{2v(\mathbf{t}; \mathbf{s})}\}} \operatorname{Res}_{z=a} \frac{\int_a^z \omega_{0,2}(z_1, \cdot; \mathbf{t}; \mathbf{s})}{2\omega_{0,1}(z; \mathbf{t}; \mathbf{s})} \\
&\times \left(\omega_{g-1, n+1}(z, -z, z_2, \dots, z_n; \mathbf{t}; \mathbf{s}) \right. \\
&\quad \left. + \sum'_{\substack{g_1+g_2=g \\ I \sqcup J = \{2, \dots, n\}}} \omega_{g_1, 1+|I|}(z, z_I; \mathbf{t}; \mathbf{s}) \omega_{g_2, 1+|J|}(-z, z_J; \mathbf{t}; \mathbf{s}) \right),
\end{aligned} \tag{3-27}$$

where i_a is the local involution around $z = a$ sending $a + \xi$ to $a - \xi$ and Σ' means omitting the terms with $(g_1, I) = (0, \emptyset)$ and $(g_1, I) = (g, \{2, \dots, n\})$.

They are the generating functions of the higher Weil-Petersson volumes. Namely, for any $g, n \in \mathbb{N}$ such that $2g - 2 + n > 0$,

$$\begin{aligned}
\omega_{g,n}(z_1, \dots, z_n; \mathbf{t}; \mathbf{s}) &= \sum_{k_1, \dots, k_n \geq 0} (2k_1 + 1)!! \frac{dz_1}{z_1^{2k_1+2}} \cdots (2k_n + 1)!! \frac{dz_n}{z_n^{2k_n+2}} \\
&\times \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{j_1, \dots, j_m \geq 0} \int_{\mathcal{M}_{g, n+m}} \psi_1^{k_1} \cdots \psi_n^{k_n} \psi_{n+1}^{j_1} \cdots \psi_{n+m}^{j_m} t_{j_1} \cdots t_{j_m} \exp\left(\sum_{i \geq 1} s_i \kappa_i\right)
\end{aligned} \tag{3-28}$$

3.4 A non semi-simple cohomological field theory

In the case where $s_1 = s_2 = \cdots = 0$, the spectral curve becomes

$$\lambda = \frac{1}{2}z^2, \tag{3-29}$$

$$\eta = z - \int_0^{t_0} \frac{dx}{(z^2 - 2v(x, t_1, t_2, \dots))^{\frac{1}{2}}} - t_1 z - \sum_{k \geq 2} t_k \frac{z^{2k-1}}{(2k-1)!!}. \tag{3-30}$$

The input data for the topological recursions become

$$\omega_{0,1}(z; \mathbf{t}; 0) = \left(z^2 - \int_0^{t_0} \frac{dx}{\left(1 - \frac{2v(x, t_1, t_2, \dots)}{z^2}\right)^{\frac{1}{2}}} - t_1 z^2 - \sum_{k \geq 2} t_k \frac{z^{2k}}{(2k-1)!!} \right) dz, \quad (3-31)$$

$$\omega_{0,2}(z_1, z_2; \mathbf{t}; 0) = \frac{z_1^2 + z_2^2 - 4v(\mathbf{t})}{(z_1^2 - z_2^2)^2 \left(1 - \frac{2v(\mathbf{t})}{z_1^2}\right)^{\frac{1}{2}} \left(1 - \frac{2v(\mathbf{t})}{z_2^2}\right)^{\frac{1}{2}}} dz_1 dz_2, \quad (3-32)$$

$$\omega_{1,1}(z; \mathbf{t}; 0) = \frac{v_{t_0 t_0}(\mathbf{t}) dz}{24z^2 \left(1 - \frac{2v(\mathbf{t})}{z^2}\right)^{3/2} v_{t_0}(\mathbf{t})} + \frac{v_{t_0}(\mathbf{t}) dz}{8z^4 \left(1 - \frac{2v(\mathbf{t})}{z^2}\right)^{5/2}}. \quad (3-33)$$

The conjecture takes the following simpler form

Conjecture 3.7 *Vanishing initial conditions for the KdV hierarchy [B.-Yang]*
For any integers $g, n \in \mathbb{N}$ such that $2g - 2 + n > 0$, the following formula holds true

$$\omega_{g,n}(z_1, \dots, z_n; \mathbf{t}; 0) = \sum_{k_1, \dots, k_n \geq 0} \prod_{\ell=1}^n (2k_\ell + 1)!! \frac{dz_\ell}{z_\ell^{2k_\ell+2}} \sum_{k_1, \dots, k_n \geq 0} \int_{\mathcal{M}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \Omega_{g,n}(\mathbf{t}) \quad (3-34)$$

where $\Omega_{g,n}(\mathbf{t})$ is an analogue of a CohFT given by

$$\Omega_{g,n}(\mathbf{t}) \stackrel{def}{=} \sum_{m \geq 0} \frac{1}{m!} \sum_{j_1, \dots, j_m \geq 0} \sum_{\tau \in S_n} \kappa_{\tau, (j_1, \dots, j_m)} t_{j_1} \dots t_{j_m}. \quad (3-35)$$

and $\kappa_{\tau, (j_1, \dots, j_m)}$ is a product of Mumford classes associated to certain cycle decompositions of the permutation τ .

Example 3.8 *If furthermore $t_0 = t_1 = t_2 = \dots = 0$ the KdV curve becomes*

$$\lambda = \frac{1}{2} z^2, \quad (3-36)$$

$$\eta = z + \sum_{k \geq 2} h_{k-1}(-\mathbf{s}) \frac{z^{2k-1}}{(2k-1)!!}. \quad (3-37)$$

This curve coincides with Eynard–Orantin–Zhou’s curve [43], [75]. We have in this case

$$\omega_{0,1}(z;0;\mathbf{s}) = \left(z^2 + \sum_{k \geq 2} h_{k-1}(-\mathbf{s}) \frac{z^{2k}}{(2k-1)!!} \right) dz, \quad (3-38)$$

$$\omega_{0,2}(z_1, z_2;0;\mathbf{s}) = \frac{z_1^2 + z_2^2}{(z_1^2 - z_2^2)^2} dz_1 dz_2, \quad (3-39)$$

$$\omega_{1,1}(z;0;\mathbf{s}) = \frac{dz}{8z^4} + \frac{s_1 dz}{24z^2} \quad (3-40)$$

It has been proved by Eynard–Orantin and by Zhou from different approaches that

$$\begin{aligned} \omega_{g,n}(z_1, \dots, z_n;0;\mathbf{s}) &= \sum_{k_1, \dots, k_n \geq 0} \prod_{\ell=1}^n (2k_\ell + 1)!! \frac{dz_\ell}{z_\ell^{2k_\ell+2}} \\ &\quad \times \sum_{k_1, \dots, k_n \geq 0} \int_{\mathcal{M}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} e^{\sum_{k \geq 1} s_k \kappa_k} \end{aligned} \quad (3-41)$$

Back to the conjecture, one can easily check that the sequence of cohomology classes $(\Omega_{g,n}(\mathbf{t}))_{2g-2+n>0}$ do not satisfy the most elementary axioms of cohomological field theories [61] and this relates to works that have studied cases corresponding to semi-simple Frobenius algebras [39] or semi-simple cohomological field theories [1] and remarks thereof that these were not the general situation where the topological recursion could appear.

4 The Topological Type property

Following the work of [16] and [15] we define the following topological type property:

The ε -connection $\nabla_\varepsilon = \varepsilon \nabla_0 - \Phi_\varepsilon$ is said to be of “*topological type*” if and only if all the following conditions are met:

1. Asymptotic expansion: There exists some simply connected open domain of $\overset{o}{\Sigma}$ and an Cartan subalgebra \mathfrak{h} of \mathfrak{g} (this allows for the local identifications $\overset{o}{\Sigma} \simeq \overset{o}{\Sigma}$ and $\widehat{\Sigma} \simeq \overset{o}{\Sigma} \times \mathfrak{g}$), in which the connected correlators $\widehat{W}_n(X_1, \dots, X_n)$'s with each $X_i \in \overset{o}{\Sigma} \times \mathfrak{h}$, have a Poincaré asymptotic ε expansion

$$\widehat{W}_n(X_1, \dots, X_n) = \frac{\delta_{n,1}}{\varepsilon} \widehat{W}_1^{(0)}(X_1) + \sum_{k=0}^{\infty} \varepsilon^k \widehat{W}_n^{(k)}(X_1, \dots, X_n), \quad (4-1)$$

such that each $\widehat{W}_n^{(k)}([x_1.E_1], \dots, [x_n.E_n])$ is, at fixed $E_i \in \mathfrak{h}$, an algebraic symmetric n -form of the variables x_1, \dots, x_n . In other words, there must exist a (possibly nodal) Riemann surface \mathcal{S} independent of k and n , which is a ramified cover of $\overset{o}{\Sigma}$, such that the pullbacks, at fixed $E_i \in \mathfrak{h}$, of $\widehat{W}_n^{(k)}([x_1.E_1], \dots, [x_n.E_n])$ to \mathcal{S}^n are meromorphic symmetric n -forms.

2. Pole only at branch points: For any pair of integers $(k, n) \notin \{(0, 1), (0, 2)\}$ and any n -tuple $(E_1, \dots, E_n) \in \mathfrak{h}^n$, the k^{th} order connected correlation functions $\widehat{W}_n^{(k)}([x_1.E_1], \dots, [x_n.E_n])$ pulled back to \mathcal{S} , may only have poles at the ramification points of $\mathcal{S} \rightarrow \overset{o}{\Sigma}$. In particular they cannot have singularities at nodal points of \mathcal{S} , or at the punctures, i.e. the pullbacks of singularities of Φ . Moreover $\widehat{W}_2^{(0)}([x_1.E_1], [x_2.E_2])$ may only have a double pole along the pullback of the diagonal in $\overset{o}{\Sigma} \times \overset{o}{\Sigma}$ to $\mathcal{S} \times \mathcal{S}$ of the form $\frac{dx_1 dx_2 \langle E_1, E_2 \rangle}{(x_1 - x_2)^2}$ but no other singularities.

3. Parity: Under the involution $\varepsilon \rightarrow -\varepsilon$:

$$\forall n \geq 1 : \widehat{W}_n|_{-\varepsilon} = (-1)^n \widehat{W}_n|_{\varepsilon} \quad (4-2)$$

for $n \geq 1$. This is equivalent to say that for any generic $E_1, \dots, E_n \in \mathfrak{h}_o$, the series expansion of $W_n([x_1.E_1], \dots, [x_n.E_n])$ is even (resp. odd) when n is even (resp. odd).

4. Leading order: For all $n \geq 1$, the leading order term of the series expansion in ε of the correlation function \widehat{W}_n is at least of order ε^{n-2} . In other words:

$$\forall n \geq 1, \forall 0 \leq k \leq n-3 : \widehat{W}_n^{(k)} = 0 \quad (4-3)$$

If the system satisfies the topological type property, we denote

$$\widehat{\omega}_{g,n} = \widehat{W}_n^{(2g-2+n)} \quad (4-4)$$

and we have

$$\widehat{W}_n = \sum_{g=0}^{\infty} \varepsilon^{2g-2+n} \widehat{\omega}_{g,n} \quad (4-5)$$

All those conditions are non-trivial, and there exist plenty of examples of Φ_ε for which they are not met. Fortunately, there are also plenty of very interesting examples for which they are. We now recall a set of simplifying assumptions that is shown in [10] in Appendix to imply the Topological Type property.

4.1 Sufficient assumptions

Here we shall restrict ourselves to the case where the Lie algebra is the set of $d \times d$ matrices $\mathfrak{g} \stackrel{def}{=} \mathfrak{M}_d(\mathbb{C})$ and the base curve is the punctured Riemann sphere

$$\overset{o}{\Sigma} \stackrel{def}{=} \mathbb{CP}^1 - \{z_1, \dots, z_M\} \quad (4-6)$$

In particular we are allowed to choose the usual de Rham differential as reference connection $\nabla_0 \stackrel{def}{=} dx \partial_x$

Setting : compatible linear differential systems

- In the spirit of the WKB study of Lax systems we shall consider a compatible system of linear equations of the form :

$$\varepsilon \partial_x \Psi_\varepsilon(x, t) = \Phi_\varepsilon(x, t) \Psi_\varepsilon(x, t) \quad (4-7)$$

$$\varepsilon \partial_t \Psi_\varepsilon(x, t) = R_\varepsilon(x, t) \Psi_\varepsilon(x, t) \quad (4-8)$$

where the $d \times d$ matrix therefore belongs for any value of its arguments to the group of invertible matrices $\Psi_\varepsilon(x, t) \in G \stackrel{def}{=} GL_d(\mathbb{C})$. The $d \times d$ matrices L_ε and R_ε are assumed to be rational functions of x for any values of t and ε . Even though they appear to play symmetric roles, x is usually called the *spectral parameter* and t the *time parameter*. To shorten notations, we will often not write explicitly the time dependence when no confusion is possible.

We shall give a set of sufficient conditions on Φ_ε and R_ε for the system to be of *Topological Type*. We will then apply this to two examples, namely the (p, q) minimal models et the six Painlevé equations.

- The compatibility relation of the two equations is called the Lax equation :

$$\varepsilon \partial_t \Phi_\varepsilon(x, t) - \varepsilon \partial_x R_\varepsilon(x, t) = [R_\varepsilon(x, t), \Phi_\varepsilon(x, t)]. \quad (4-9)$$

- Recall that the construction that was presented associated to a given solution Ψ_ε of the differential system is associated a Lie algebra-valued solution M_ε of the adjoint system :

$$\varepsilon \partial_x M_\varepsilon(x, t) = [\Phi_\varepsilon(x, t), M_\varepsilon(x, t)] \quad (4-10)$$

$$\varepsilon \partial_t M_\varepsilon(x, t) = [R_\varepsilon(x, t), M_\varepsilon(x, t)] \quad (4-11)$$

whose solutions are of the form

$$M_\varepsilon(x, t) = \Psi_\varepsilon(x, t) E \Psi_\varepsilon(x, t)^{-1} \stackrel{def}{=} M_{\Psi_\varepsilon}([x \cdot E]) \quad (4-12)$$

where E is a fixed Lie algebra element (it is constant in the sense $\partial_x E = 0$).

Recall that any another solution of the Lax compatible system of differential equations is obtained from Ψ through the right multiplication $\Psi(x) \rightarrow \Psi(x)C$ by a constant matrix $C \in G$, $\partial_x C = 0$, which can be re-absorbed in a global conjugation of the Lie algebra \mathfrak{g} . Moreover, Ψ_ε and M_ε are multivalued on $\overset{o}{\Sigma}$ and globally defined only on its universal cover. The construction of the non-perturbative bundle $\widehat{\Sigma}_{\Psi_\varepsilon}$ allowed in the general framework to have a uniquely valued map M_{Ψ_ε} taking into account different choices of Lie algebra element E .

We shall now describe our assumptions on the Lax pair $(\Phi_\varepsilon, R_\varepsilon)$. These assumptions are described in terms of algebraic geometry and the notion of spectral curve. There are in total 6 assumptions that are presented in the following paragraphs. Each assumption allows new definitions and/or implies new properties that are presented in several lemmas and propositions. Although they may appear technical, these assumptions have been proved to hold in many cases like Painlevé Lax pairs [17], [57], [58]. We prove them for all (p, q) minimal models in the next section.

Spectral curve(s)

Assumption 4.1 (ε expansion) *We make the assumption that Φ_ε and R_ε admit a well defined limit when $\varepsilon \rightarrow 0$:*

$$\lim_{\varepsilon \rightarrow 0} \Phi_\varepsilon(x, t) = \Phi^{(0)}(x, t) \quad , \quad \lim_{\varepsilon \rightarrow 0} R_\varepsilon(x, t) = R^{(0)}(x, t), \quad (4-13)$$

and that both limits are rational functions of x . Furthermore, we assume that they admit a ε expansion (formal or asymptotic) of the form :

$$\Phi_\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k \Phi^{(k)} \quad , \quad R_\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k R^{(k)} \quad (4-14)$$

where for any $k \geq 0$, $\Phi^{(k)}$ and $R^{(k)}$ are rational functions of x .

The spectral curve is defined as the zero locus of the characteristic polynomial of the matrix $\Phi^{(0)}$, i.e. the eigenvalue locus, whence the name

“spectral” curve. In the general Lie group context, this corresponds to Hitchin’s map.

Definition 4.2 *Spectral curve*

The (family of) spectral curve of the differential system is the zero locus of the characteristic polynomial in $\mathbb{C} \times \mathbb{C}$:

$$\mathcal{S}_t \stackrel{\text{def}}{=} \{(x, y) \in \mathbb{C}^2 \mid E_t(x, y) \stackrel{\text{def}}{=} \text{Det}(y - \Phi^{(0)}(x, t)) = 0\} \quad (4-15)$$

This defines an (a family of) algebraic plane curve immersed into $\mathbb{C} \times \mathbb{C}$. We define the two meromorphic functions corresponding to the x and y projection in $\mathbb{C} \times \mathbb{C}$:

$$\begin{aligned} x_t: \mathcal{S}_t &\longrightarrow \mathbb{C} \\ (x, y) &\longmapsto x \end{aligned} \quad (4-16)$$

$$\begin{aligned} y_t: \mathcal{S}_t &\longrightarrow \mathbb{C} \\ (x, y) &\longmapsto y \end{aligned} \quad (4-17)$$

The plane curve can be desingularized. Its desingularization is a smooth compact Riemann surface noted Σ_t , and the functions x_t and y_t can be identified with meromorphic functions $\Sigma_t \longrightarrow \mathbb{C}$. This allows to redefine the spectral curve as the triplet :

$$\mathcal{S}_t = (\Sigma_t, x_t, y_t), \quad (4-18)$$

given by a (family of) compact Riemann surface $\Sigma \equiv \Sigma_t$, equipped with two meromorphic functions $x_t: \Sigma_t \longrightarrow \mathbb{C}$ and $y_t: \Sigma_t \longrightarrow \mathbb{C}$. On a compact curve, any two meromorphic functions are related by an algebraic equation :

$$\forall z \in \Sigma_t \quad , \quad E_t(x_t(z), y_t(z)) = 0 \quad \text{where } E_t \text{ is a polynomial} \quad (4-19)$$

thus giving an alternative definition of the spectral curve directly from (4-18). All these objects come in time dependent families

$$x \stackrel{\text{def}}{=} (x_t)_t, \quad y \stackrel{\text{def}}{=} (y_t)_t, \quad E \stackrel{\text{def}}{=} (E_t)_t \quad \text{and} \quad \mathcal{S} \stackrel{\text{def}}{=} (\mathcal{S}_t)_t \quad (4-20)$$

We shall also be interested in the time dependent family of one-forms $\omega_1^{(0)}$, defined at any time t on the corresponding Riemann surface Σ_t defined by $\omega_1^{(0)}(t) = y_t dx_t$, sometimes referred to as the Liouville form. It is indeed the pullback of the tautological form of $\mathbb{C} \times \mathbb{C}$ to the spectral curves.

The characteristic polynomial has degree in the variable y equal to the size of the matrices (the dimension of the fundamental representation of $\mathfrak{gl}_d(\mathbb{C})$):

$$\deg_y E_t(x, y) = d \quad (4-21)$$

and thus for a given generic $x \in \mathbb{C}$, the equation $E_t(x, y) = 0$ has d solutions, namely the d eigenvalues $Y_1(x, t), \dots, Y_d(x, t)$ of $\Phi^{(0)}(x, t)$. They are the images by the function y_t , of the d pre-images of x by x_t :

$$x_t^{-1}(x) = \{z \in \Sigma_t \mid x_t(z) = x\} \stackrel{\text{def}}{=} \{z^1(x, t), \dots, z^d(x, t)\} \quad (4-22)$$

gives

$$Y_i(x, t) = y_t(z^i(x, t)). \quad (4-23)$$

Here we took an arbitrary ordering of the eigenvalues and this ordering can always be chosen locally to be analytic within some open simply connected domain of the punctured sphere $\mathbb{C} - \{\text{Singularities of } x_t \text{ and } x_t^{-1}\}$. This will turn out to be irrelevant for our purpose.

Definition 4.3 Auxiliary spectral curve

Similarly we define the family of auxiliary spectral curves $\widetilde{\mathcal{F}} \stackrel{\text{def}}{=} (\widehat{\mathcal{F}}_t)_t$ by the zero locus of the characteristic polynomial of $R^{(0)}$. That is at any time t by

$$\widetilde{\mathcal{F}}_t \stackrel{\text{def}}{=} \{(x, s) \in \mathbb{C}^2 \mid \widetilde{E}_t(x, s) \stackrel{\text{def}}{=} \text{Det}(s - R^{(0)}(x, t)) = 0\} \quad (4-24)$$

that we shall encode as the triple

$$\widetilde{\mathcal{F}}_t = (\widetilde{\Sigma}_t, \widetilde{x}_t, s_t), \quad (4-25)$$

given by a Riemann surface $\widetilde{\Sigma} \stackrel{\text{def}}{=} (\widetilde{\Sigma}_t)_t$, equipped at each time t with two meromorphic functions $\widetilde{x}_t : \widetilde{\Sigma}_t \rightarrow \mathbb{C}$ and $s_t : \widetilde{\Sigma}_t \rightarrow \mathbb{C}$, related by the algebraic equation

$$\forall z \in \tilde{\Sigma}_t \quad , \quad \tilde{E}_t(\tilde{x}_t(z), s_t(z)) = 0. \quad (4-26)$$

Similarly, for a given x there exists a set of d solutions denoted $(S_1(x, t), \dots, S_d(x, t))$ of the equation of the auxiliary curve $\tilde{E}_t(x, s) = 0$. They are the d eigenvalues of $R^{(0)}(x, t)$, and they satisfy $S_i(x, t) = s_t(\tilde{z}_t^i(x))$ with $\tilde{x}_t(\tilde{z}_t^i(x)) = x$.

Lemma 4.4 *The matrices $\Phi^{(0)}(x, t), R^{(0)}(x, t)$ commute thus they generically have a common basis of eigenvectors and their eigenvalues are not algebraically independent. In particular the spectral curves $\mathcal{S}_t, \tilde{\mathcal{S}}_t$ have the same desingularization : $\Sigma_t = \tilde{\Sigma}_t$ and the same x -projection to $\bar{\mathbb{C}} : \tilde{x}_t = x_t$.*

proof:

At order ε^0 , the Lax compatibility condition (4-7) reads :

$$[\Phi^{(0)}(x, t), R^{(0)}(x, t)] = 0. \quad (4-27)$$

For generic x , all the eigenvalues of $R^{(0)}(x, t)$ are distinct. It implies that the set of matrices commuting with $R^{(0)}(x, t)$ is the algebra of polynomials of $R^{(0)}(x, t)$. Consequently there exists a polynomial $Q(x, s)$ (the interpolating Lagrange polynomial) such that $\Phi^{(0)}(x, t) = Q(x, R^{(0)}(x, t))$, and $Y_i(x) = Q(x, S_i(x))$, i.e. $y(z) = Q(\tilde{x}(z), s(z))$ for all $z \in \tilde{\Sigma}_t$. This implies that y_t is a meromorphic function on $\tilde{\Sigma}_t$. Exchanging the roles of $R^{(0)}$ and $\Phi^{(0)}$ also shows that s_t is a meromorphic function on Σ_t . Therefore $\Sigma_t = \tilde{\Sigma}_t$, and $x_t = \tilde{x}_t$. ■

Geometry of the spectral curve

Genus 0 assumption From now on, we shall assume that our system is such that :

Assumption 4.5 (Genus zero Spectral Curve) *The compact Riemann surface Σ_t has genus equal to 0. This implies that it is isomorphic to the Riemann sphere*

$$\Sigma_t \stackrel{\text{def}}{=} \mathbb{C}P^1 \quad (4-28)$$

and that, for any given t in an open domain, the functions x_t, y_t, s_t , are rational functions of a variable $z \in \overline{\mathbb{C}}$:

$$x_t(z), y_t(z), s_t(z) \in \mathbb{C}(z) = \{\text{rational functions of } z\} \quad (4-29)$$

Remark 4.6 *The issue of determining if this genus zero hypothesis can be lifted is mostly open. In fact in the example of matrix models, it is known that the TT property is generically not satisfied when the genus is strictly positive. But a generalization of the TT property can be found by allowing the coefficients in the ε expansion, to be “oscillatory”, i.e. bounded quasi-periodic functions of $\frac{1}{\varepsilon}$. In that case, the oscillatory terms are themselves found by the topological recursion. See [40].*

Besides, in knot theory, the TT property happens to hold with spectral curves (A-polynomial) of strictly positive genus. This is due to a miracle that the $\frac{1}{\varepsilon}$ term is exactly a period of the oscillatory term, and thus can be treated as a constant coefficient, see [40], and then the TT property holds. The general situation is still unclear.

Remark 4.7 *The choice of the parameterizing variable z is arbitrary up to Möbius transformations (automorphisms of the Riemann sphere) :*

$$z \longmapsto \frac{az + b}{cz + d}. \quad (4-30)$$

In particular, we may chose the coefficients of the Möbius transformation a, b, c, d to be time dependent : $a(t), b(t), c(t), d(t)$.

The functions x, y and s depend on both z and t . They are moreover rational in z . Let us mention that their dependence on t is not required to be rational. On the contrary, there are case for which they are transcendental functions of t , like for example solutions of Painlevé equations.

We shall denote for any function $f = f(z, t)$,

$$f'(z, t) \stackrel{\text{def}}{=} \frac{\partial f}{\partial z}, \quad \dot{f}(z, t) \stackrel{\text{def}}{=} \frac{\partial f}{\partial t}. \quad (4-31)$$

Note that taking a time derivative at a fixed value $x(z, t)$, the chain rule yields a Poisson bracket $\{f, x\} = \dot{f} x' - \dot{x} f'$:

$$\left. \frac{df(z, t)}{dt} \right|_{x(z, t)} = \dot{f} - \dot{x} \frac{f'}{x'} = \frac{\dot{f} x' - \dot{x} f'}{x'} = \frac{1}{x'} \{f, x\}, \quad (4-32)$$

thus reflecting the symplectic structure of $\mathbb{C} \times \mathbb{C}$ of which the family $\Sigma = (\Sigma_t)_t$ defines a Lagrangian foliation.

Behavior at poles

Lemma 4.8 *Bound on the orders of the singularities of the eigenvalue functions*
The poles of the eigenvalue functions y_t (resp. s_t) are poles of $\Phi^{(0)}(x(z), t)$ (resp. $R^{(0)}$) of at least the same order.

proof:

Let α be a pole of $y_t(z)$ of order $d_\alpha > 0$ so that $y_t(z) = O((z - \alpha)^{-d_\alpha})$. Let us assume that $(z - \alpha)^{d_\alpha} \Phi^{(0)}(x(z), t) = o(1)$. This would imply that

$$0 = \text{Det}(y_t(z)\text{Id} - \Phi^{(0)}(x(z), t)) = y_t(z)^d (1 + o(1)) \quad (4-33)$$

which is a contradiction. This implies that $\Phi^{(0)}(x(z), t)$ has a pole of order at least d_α . Obviously, the same holds for $R^{(0)}(x, t)$. ■

Lemma 4.9 Poisson structure

The eigenvalues $(Y_i(x, t))_{1 \leq i \leq d}$ of $\Phi^{(0)}(x, t)$ and $(S_i(x, t))_{1 \leq i \leq d}$ of $R^{(0)}(x, t)$ are related by the following Schwartz relation :

$$\frac{\partial Y_i(x, t)}{\partial t} = \frac{\partial S_i(x, t)}{\partial x}. \quad (4-34)$$

Equivalently, the functions $x_t(z), y_t(z), s_t(z)$ satisfy :

$$\frac{\partial y_t(z)}{\partial t} \frac{\partial x_t(z)}{\partial z} - \frac{\partial x_t(z)}{\partial t} \frac{\partial y_t(z)}{\partial z} = \frac{\partial s_t(z)}{\partial z},$$

or written in the notations of (4-31) :

$$\{y_t, x_t\} = \dot{y}_t x'_t - \dot{x}_t y'_t = s'_t.$$

proof:

Since $\Phi^{(0)}(x, t)$ and $R^{(0)}(x, t)$ commute, they generically have a common basis of eigenvectors, let us denote $V(x, t)$ the matrix whose i^{th} column is the eigenvector of $\Phi^{(0)}(x, t)$ with eigenvalue $Y_i(x, t)$ and of $R^{(0)}(x, t)$ with eigenvalue $S_i(x, t)$. Denoting $Y(x, t) = \text{diag}(Y_1(x, t), \dots, Y_d(x, t))$ and $S(x, t) = \text{diag}(S_1(x, t), \dots, S_d(x, t))$, we have

$$\Phi^{(0)}(x, t) = V(x, t)Y(x, t)V(x, t)^{-1} \quad , \quad R^{(0)}(x, t) = V(x, t)S(x, t)V(x, t)^{-1}. \quad (4-35)$$

Now write the Lax equation to order ε^1 and conjugate by $V(x, t)$:

$$[S(x, t), V(x, t)^{-1}\Phi^{(1)}(x, t)V(x, t)] + [V(x, t)^{-1}R^{(1)}(x, t)V(x, t), Y(x, t)] \\ = \partial_t Y(x, t) - \partial_x S(x, t) \quad (4-36)$$

The left hand side is a sum of commutators with diagonal matrices, hence has vanishing entries on the diagonal. On the contrary, the right hand side is a diagonal matrix and evaluating its diagonal entries gives the wanted result. ■

As an immediate corollary we get :

Corollary 4.10 *Auxiliary singularities and their orders*

Finite (i.e at $x \neq \infty$) singularities of S are also singularities of Y , of at least the same degree. And if S has a singularity of order d_∞ at $x = \infty$, then Y has a singularity at $x = \infty$ of order at least $d_\infty + 1$.

Note that the converse is not true : some singularities of Y may be time independent and may not be singularities of S . In some sense, we can say that $R^{(0)}$ is less singular than $\Phi^{(0)}$. We here see how having a Lax pair simplifies the discussion. Indeed, when having a set of compatible differential systems, consider the less singular.

branch points and double points

Definition 4.11 *Branch points*

We define the branch points $(a_i)_{1 \leq i \leq r}$ as the points of Σ around which the

projection map $z \mapsto x(z)$ is no longer locally invertible. There may be two kinds of branch points :

- *Finite branch points, at which $x(a_i) \neq \infty$. They are zeros of the differential dx :*

$$dx(a_i) = 0.$$

Moreover, they are among the simultaneous solutions of $E(x, y) = 0$ and $E_y(x, y) \equiv \partial_y E(x, y) = 0$.

- *branch points at poles of x of order ≥ 2 .*

A branch point a_i of the spectral curve \mathcal{S} (resp. $\widetilde{\mathcal{S}}$) is called regular if it is not a branch point of y (resp. s). Generic finite branch points of x have order 2, i.e. are simple zeros of dx , and regularity means that they are not zeros of dy (resp. ds).

Note that the branch points may depend on time t . However, the number of branch points $r \geq 1$ does not locally depend on t . We will also need the following definition:

Definition 4.12 *Self-intersections*

We define the double points $((b_i, \bar{b}_i))_{1 \leq i \leq r''}$ (resp. $((\tilde{b}_i, \tilde{\bar{b}}_i))_{1 \leq i \leq \tilde{r}''}$) of the spectral curve $\mathcal{S}_t = (\Sigma_t, x_t, y_t)$ (resp. of $\widetilde{\mathcal{S}}_t = (\widetilde{\Sigma}_t, \tilde{x}_t, s_t)$), as the pairs $(b_i, \bar{b}_i) = (z, z')$ (resp. $(\tilde{b}_i, \tilde{\bar{b}}_i) = (z, z')$) solutions of

$$\left\{ \begin{array}{l} x(z) = x(z') \\ y(z) = y(z') \\ z \neq z' \end{array} \right. , \quad \left(\text{resp.} \left\{ \begin{array}{l} x(z) = x(z') \\ s(z) = s(z') \\ z \neq z' \end{array} \right. \right) \quad (4-37)$$

These double points $(x, y) = (x(b_i), y(b_i)) = (x(\bar{b}_i), y(\bar{b}_i)) \in \mathbb{C} \times \mathbb{C}$ of the spectral curve (resp. $(x, s) = (x(b_i), s(b_i)) = (x(\bar{b}_i), s(\bar{b}_i)) \in \mathbb{C} \times \mathbb{C}$), are then solutions of the system

$$\left\{ \begin{array}{l} E(x, y) = 0 \\ E_y(x, y) = 0 \\ E_x(x, y) = 0 \end{array} \right. , \quad \left(\text{resp.} \left\{ \begin{array}{l} \tilde{E}(x, s) = 0 \\ \tilde{E}_s(x, s) = 0 \\ \tilde{E}_x(x, s) = 0 \end{array} \right. \right) \quad (4-38)$$

We shall make the following assumption regarding the double points of the auxiliary spectral curve :

Assumption 4.13 (Regularity of \mathcal{S}_t and no double points for $\widetilde{\mathcal{F}}_t$) *Let us make the assumption that for any value of the time parameter t , the auxiliary spectral curve $\widetilde{\mathcal{F}}_t$ is regular and has no double points. In other words, $\widetilde{\mathcal{F}}_t$ is a smooth embedding into $\mathbb{C} \times \mathbb{C}$ (rather than an immersion) with no self-intersection. Moreover we assume that \mathcal{S}_t is regular.*

Note that the last assumption does not exclude the possibility that the spectral curve \mathcal{S}_t admits double points. Moreover, the auxiliary spectral curves $\widetilde{\mathcal{F}}_t = (\widetilde{\Sigma}_t, \widetilde{x}_t, s_t)$ satisfying assumptions 4.5 and 4.13 are the same as the ones described in [27]. We will sometimes lighten notations by dropping the subscript t when no confusion is possible.

We have the following lemma :

Lemma 4.14 *The meromorphic one-form*

$$\frac{dx(z)}{E_y(x(z), y(z))} \quad (4-39)$$

is holomorphic at all branch points (finite or infinite). It has poles only at double points (generically simple poles at b_i and \bar{b}_i with opposite residues) and/or at simple poles of x .

proof:

This is a classical algebro-geometric result, we refer to [45]. Let us sketch the proof as the method is quite instructive. Near a finite branch point a of given order $k \geq 2$, $z \stackrel{\text{def}}{=} (x - x(a))^{1/k}$ can be used as a local coordinate. Consider the case $y(a) \neq \infty$. Since the branch point is regular, dy does not vanish at that point, i.e.

$$y(z) = y_0 + y_1 z + O(z^2), \quad y_1 \neq 0. \quad (4-40)$$

This gives :

$$E(x, y) = ((y - y_0)^k - y_1^k (x - x(a))) \times (1 + o(1)), \quad (4-41)$$

and

$$\frac{dx}{E_y(x, y)} = \frac{kz^{k-1} dz}{k(y - y_0)^{k-1}} \times (1 + o(1)) \quad (4-42)$$

and thus $\frac{dx}{E_y(x, y)}$ is analytic at $z = 0$, i.e. at $x = a$. The other cases where $x(a) = \infty$ or $y(a) = \infty$ can be treated similarly in a local variable. In other words, at a finite regular branch point, both $dx(z)$ and $E_y(x(z), y(z))$ vanish at the same order such that the ratio remains finite.

For double points, $E_y(x(z), y(z))$ vanishes but not $dx(z)$, so the ratio has a pole. Factor the polynomial E as

$$E(x, y) = \prod_{k=1}^d (y - y(z^k(x))) \quad (4-43)$$

We then have that when $z \rightarrow b_i$ and $z' \rightarrow \bar{b}_i$ simultaneously, and

$$E_y(x(z), y(z)) \sim (y(z) - y(z')) E_{y,y}(x(b_i), y(b_i)) \quad (4-44)$$

$$\sim (z - z') \frac{dy(z)}{dz} E_{y,y}(x(b_i), y(b_i)) \quad (4-45)$$

Assuming that the double point is generic, i.e. $E_{y,y} dy \neq 0$, we get :

$$\operatorname{Res}_{z=b_i} \frac{dx(z)}{E_y(x(z), y(z))} = -\operatorname{Res}_{z=\bar{b}_i} \frac{dx(z)}{E_y(x(z), y(z))} \quad (4-46)$$

$$= \frac{dx(b_i)}{dy(b_i) E_{y,y}(x(b_i), y(b_i))} \quad (4-47)$$

■

Eigenvectors

Let $\operatorname{div}_\infty x = \sum_{k=1}^p d_k \alpha_k$ be the divisor of poles of the rational map $z \mapsto x(z)$, d_k being the degree of α_k (α_k may depend on t). The total degree is then defined as the size of the matrix

$$\sum_{k=1}^p d_k \stackrel{\text{def}}{=} d. \quad (4-48)$$

Up to a Möbius change of variable on z , we may assume that none of the α_k 's is located at ∞ . We can thus write the rational function $x(z)$ uniquely as :

$$x(z) = X_{\infty,0} + \sum_{k=1}^p \sum_{l=1}^{d_k} \frac{X_{k,l}}{(z - \alpha_k)^l}. \quad (4-49)$$

where $d_k \geq 1$. Moreover, if $d_k \geq 2$ then α_k is a (non-finite) branch point. Note that if one of the α_k is at $\alpha_\infty = \infty$ we would rather write :

$$x(z) = \sum_{l=0}^{d_\infty} X_{\infty,l} z^l + \sum_{k=1}^p \sum_{l=1}^{d_k} \frac{X_{k,l}}{(z - \alpha_k)^l}, \quad (4-50)$$

But to avoid useless notation complications, upon changing z by a Möbius transformation, we shall assume that all poles of $x(z)$ are finite.

The generalized Vandermonde matrix $\mathcal{V}(x)$

Definition 4.15 *Generalized Vandermonde matrix*

For generic points $z \in \Sigma$, in particular away from the branch points, let us define the d -dimensional vector $\vec{\mathcal{V}}(z)$ with entries labeled by all possible pairs (k, l) with $1 \leq k \leq p$ and $1 \leq l \leq d_k$:

$$\vec{\mathcal{V}}(z) = (\mathcal{V}_{k,l}(z))_{k,l} \quad , \quad \text{where } \mathcal{V}_{k,l}(z) \stackrel{\text{def}}{=} \frac{1}{(z - \alpha_k)^l \sqrt{x'(z)}}. \quad (4-51)$$

In addition we define these matrix entries to be ordered as follows

$$\vec{\mathcal{V}}(z) \stackrel{\text{def}}{=} (\mathcal{V}_{1,1}(z), \dots, \mathcal{V}_{1,d_1}(z), \dots, \mathcal{V}_{p,1}(z), \dots, \mathcal{V}_{p,d_p}(z)). \quad (4-52)$$

Let $\mathcal{V}(x)$ be the $d \times d$ square matrix whose columns are the vectors $\vec{\mathcal{V}}(z^j(x))$:

$$\forall 1 \leq k \leq p, 1 \leq l \leq d_k, 1 \leq j \leq d : (\mathcal{V}(x))_{k,l;j} \stackrel{\text{def}}{=} \mathcal{V}_{k,l}(z^j(x)) \quad (4-53)$$

It is analytic locally in some open simply connected domain, in which the z^i and the square root are defined.

Remark 4.16 *The sign of the square root, chosen arbitrarily, is well defined and analytic only locally in some open simply connected domain – the same domain in which we defined the ordering of $z^i(x)$'s. This will eventually be irrelevant as the square root will always appear to even powers.*

Remark 4.17 *Note that if x has only one pole ($p = 1$ and $d_1 = d$) then the previous matrix is a Vandermonde matrix multiplied by $\frac{1}{(z-\alpha_1)\sqrt{x'(z)}}$, hence the denomination “generalized Vandermonde matrix”.*

The matrix $\mathcal{V}(x)$ satisfies remarkable properties.

Lemma 4.18 *Charge conjugation*

There exists an invertible $d \times d$ matrix $C \equiv C(t)$ (independent of x), such that

$$\mathcal{V}(x)^T C \mathcal{V}(x) = \text{Id}, \quad (4-54)$$

where \mathcal{V}^T denotes the transpose of the matrix \mathcal{V} . Its coefficients are given by

$$C_{k,l;k',l'} = -\delta_{k,k'} X_{k,l+l'-1} \quad (4-55)$$

proof:

From (4-49) we have

$$\frac{x(z) - x(z')}{z - z'} = \sum_{k,l;k',l'} \frac{C_{k,l;k',l'}}{(z - \alpha_k)^l (z' - \alpha_{k'})^{l'}} \quad , \quad C_{k,l;k',l'} = -\delta_{k,k'} X_{k,l+l'-1} \quad (4-56)$$

The matrix C is made of triangular blocks because $X_{k,l+l'-1} = 0$ if $l+l' > d_k + 1$. C is invertible because the antidiagonals of each triangular block is $-C_{k,d_k} \neq 0$ by definition of d_k . We have :

$$\forall 1 \leq i, j \leq d : (\mathcal{V}(x)^T C \mathcal{V}(x'))_{i,j} = \frac{x - x'}{z^i(x) - z^j(x')} \frac{1}{\sqrt{x'(z^i(x))x'(z^j(x'))}} \quad (4-57)$$

Taking the x -differential of (4-60) and using the fact that C is independent of x directly shows that $\mathcal{V}(x)^{-1} d\mathcal{V}(x)$ is antisymmetric. Starting from (4-57) and differentiating with respect to x' we get :

$$\begin{aligned} (\mathcal{V}(x)^{-1} d\mathcal{V}(x'))_{i,j} &= \frac{(x-x') dz^j(x')}{(z^i(x)-z^j(x'))^2} \frac{1}{\sqrt{x'(z^i(x))x'(z^j(x'))}} \\ &\quad - \frac{x'(z^j(x')) dz^j(x')}{z^i(x)-z^j(x')} \frac{1}{\sqrt{x'(z^i(x))x'(z^j(x'))}} \\ &\quad - \frac{1}{2} \frac{x''(z^j(x')) dz^j(x')}{x'(z^j(x'))} \frac{x-x'}{z^i(x)-z^j(x')} \frac{1}{\sqrt{x'(z^i(x))x'(z^j(x'))}} \end{aligned} \quad (4-62)$$

We now take the limit $x \rightarrow x'$. Since $x = x(z^i(x)) = x(z^j(x))$ we get the equalities $dx = x'(z^i(x)) dz^i(x) = x'(z^j(x)) dz^j(x)$. When $i \neq j$, the denominator does not vanish and only the terms without $x-x'$ in the numerator survive thus giving the claimed result. When $i = j$, the first two terms are computed by Taylor expansion up to the second order, i.e. involve the second derivative of x , which is exactly canceled by the last term. ■

Corollary 4.20 *With $G = GL_d(\mathbb{C})$ and the Cartan subalgebra \mathfrak{h} given by the set of diagonal matrices, defining the canonical basis of \mathfrak{h} by the usual expression $e_i = \text{diag}(0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$ implies the following identity :*

$$\frac{\text{Tr} \mathcal{V}(x) e_i \mathcal{V}(x)^{-1} \mathcal{V}(x') e_j \mathcal{V}(x')^{-1}}{(x-x')^2} dx dx' = \frac{dz^i(x) dz^j(x')}{(z^i(x)-z^j(x'))^2} = B(z^i(x), z^j(x')), \quad (4-63)$$

where $B(z, z') = \frac{dz dz'}{(z-z')^2}$ is the fundamental second-kind bi-differential of the Riemann sphere.

Moreover, we get the following property :

Proposition 4.21 *The matrix $x \mapsto \mathcal{V}(x) e_i \mathcal{V}(x)^{-1}$ is a rational function of $z^i(x)$. It is only singular when $z^i(x)$ is at the branch points (i.e. finite branch points where $x'(z) = 0$ and poles of $x(z)$ of degree at least 2).*

proof:

Use $\mathcal{V}(x)^{-1} = \mathcal{V}(x)^T C$ and the definition of $\mathcal{V}(x)$:

$$\left(\mathcal{V}(x)e_i\mathcal{V}(x)^T\right)_{(k,l),(k',l')} = \frac{1}{(z^i(x) - \alpha_k)^l (z^i(x) - \alpha_{k'})^{l'} x'(z^i(x))}$$

This function has poles when $x'(z^i(x))$ vanishes, i.e. at branch points, and also possibly at the punctures $z^i(x) = \alpha_k$.

If α_j is a puncture (i.e. a pole of $x(z)$) but not a branch point we must have $d_j = 1$ and thus $l = 1$. We get that :

$$\left(\mathcal{V}(x)e_i\mathcal{V}(x)^T\right)_{(k,l),(k',l')} = O\left((z^i(x) - \alpha_j)^{d_j+1-l\delta_{k,j}-l'\delta_{k',j}}\right). \quad (4-64)$$

In the worst case, $k = k' = j$ implying $l = l' = 1$, and the exponent then vanishes. This shows that $\mathcal{V}(x)e_i\mathcal{V}(x)^T$ is regular. ■

We will now use the matrix $\mathcal{V}(x)$ and its properties to formulate our next assumption.

Decomposition of the matrix of eigenvectors at order ε^0

Assumption 4.22 (Eigenvector decomposition) *We assume that there exists an invertible $d \times d$ matrix $v(t)$, independent of x , such that*

$$V(x, t) = v(t)\mathcal{V}(x) \quad (4-65)$$

is an invertible matrix whose columns are the eigenvectors of $\Phi^{(0)}$ (and thus of $R^{(0)}$). Consequently we have (not writing the t dependence to lighten notations) :

$$\Phi^{(0)}(x) = v\mathcal{V}(x)Y(x)\mathcal{V}(x)^T C v^{-1}, \quad (4-66)$$

$$R^{(0)}(x) = v\mathcal{V}(x)S(x)\mathcal{V}(x)^T C v^{-1}. \quad (4-67)$$

In coordinates it is equivalent to :

$$(\Phi^{(0)}(x))_{i,j} = \sum_{k,l,k',l',m} \frac{-y(z^m(x)) v_{i;k,l} X_{k',l'+l''-1}(v^{-1})_{k',l'';j}}{(z^m(x) - \alpha_k)^l (z^m(x) - \alpha_{k'})^{l'} x'(z^m(x))} \quad (4-68)$$

$$(R^{(0)}(x))_{i,j} = \sum_{k,l,k',l',m} \frac{-s(z^m(x)) v_{i;k,l} X_{k',l'+l''-1}(v^{-1})_{k',l'';j}}{(z^m(x) - \alpha_k)^l (z^m(x) - \alpha_{k'})^{l'} x'(z^m(x))} \quad (4-69)$$

Notice that the last assumption implies that :

$$v(t)^{-1} \Phi^{(0)}(x, t) v(t) C(t)^{-1} \quad \text{and} \quad v(t)^{-1} R^{(0)}(x, t) v(t) C(t)^{-1} \quad (4-70)$$

are symmetric matrices at all times.

Remark 4.23 *This is a very strong assumption on $\Phi^{(0)}(x, t)$ and $R^{(0)}(x, t)$. It implies that the x -dependent part of $\Phi^{(0)}(x, t)$ (resp. $R^{(0)}(x, t)$) has in fact only $\frac{d(d+1)}{2}$ degrees of freedom, rather than d^2 . In other words it imposes $\frac{d(d-1)}{2}$ constraints on $\Phi^{(0)}(x, t)$ (resp. $R^{(0)}(x, t)$). However, most (if not all) well-known integrable systems satisfy it and examples of Painlevé systems and (p, q) minimal models are given at the end of the chapter.*

Remark 4.24 *The purpose of assumption 4.22 is to match the (defined below) correlator $W_2^{(0)}$ with the fundamental 2nd kind bi-differential $B(z_1, z_2)$, defined in Corollary 4.20, as it is necessary for the system to satisfy the topological type property.*

Classification of admissible systems

From (4-68) we must have :

$$(v^{-1} \Phi^{(0)}(x) v C^{-1})_{k,l;k',l'} = \sum_{j=1}^d \mathcal{V}_{k,l}(z^j(x)) \mathcal{V}_{k',l'}(z^j(x)) y(z^j(x)) \quad (4-71)$$

$$= \sum_{j=1}^d \frac{1}{(z^j(x) - \alpha_k)^l} \frac{1}{(z^j(x) - \alpha_{k'})^{l'}} \frac{y(z^j(x))}{x'(z^j(x))} \quad (4-72)$$

$$= \sum_{j=1}^d \operatorname{Res}_{z \rightarrow z^j(x)} \frac{1}{(z - \alpha_k)^l} \frac{1}{(z - \alpha_{k'})^{l'}} \frac{y(z)}{x(z) - x} \quad (4-73)$$

$$= - \sum_{p \in \{\text{poles of } x \text{ and } y\}} \operatorname{Res}_{z \rightarrow p} \frac{1}{(z - \alpha_k)^l} \frac{1}{(z - \alpha_{k'})^{l'}} \frac{y(z)}{x(z) - x} \quad (4-74)$$

The pole at $z = \alpha_i$ gives a polynomial of x of degree lower or equal to $\frac{l\delta_{k,i} + l'\delta_{k',i} - 2d_i + \deg_{\alpha_i} y}{d_i}$. Thus if y has no pole at α_i , this gives at most an x independent term, and only for $k = k' = i$, $l = l' = d_i$.

If p is a pole of y which is not a pole of x , we get a pole $(x(p) - x)^m$ with $m \leq \frac{\deg_p y}{1 + \text{ord}_p x'}$.

Decomposition on z^r Any rational function $y(z)$ can be uniquely written as

$$y(z) = \sum_{r=0}^{d-1} z^r f_r(x(z)). \quad (4-75)$$

where $f_r(x)$ is a rational function of x . Since functions of x go through (4-71), it is sufficient to study the cases $y(z) = z^r$.

So let us substitute $y(z) \rightarrow z^r$ in (4-71), with $0 \leq r \leq d-1$, and we assume (up to a Möbius transformation of z) that x is regular at $z = \infty$ (i.e. none of the α_i 's are located at ∞). The contribution to (4-71) of poles at α_i 's is a constant matrix $\widehat{A}_{i,r}$, which is a triangular block of size d_i , which we denote :

$$A_{r,0} = \sum_i \widehat{A}_{i,r}, \quad (\widehat{A}_{i,r})_{k,l;k',l'} = \delta_{k,i} \delta_{k',i} \widehat{A}_{i,r,l+l'} \quad (4-76)$$

that is non vanishing only if $l + l' \geq d_i + 1$.

On the anti-diagonal we get :

$$\widehat{A}_{i,r,d_i+1} = \frac{-\alpha_i^r}{X_{i,d_i}}. \quad (4-77)$$

Example :

$$r = 2 : A_{2,2} = \begin{array}{|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot & * & * & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & * & * & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & * & * & * & * & \cdot & \cdot & \cdot \\ \hline * & * & * & * & * & * & * & * & * \\ \hline * & * & * & * & * & * & * & * & * \\ \hline \cdot & \cdot & \cdot & \cdot & * & * & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & * & * & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & * & * & \cdot & \cdot & \cdot \\ \hline \end{array}, \quad A'_{2,2} = \begin{array}{|c|c|c|} \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \cdot & * & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & * & * & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \cdot & \cdot \\ \hline \end{array}$$

and so on. For larger value of r , the non-vanishing off-diagonal blocks have size $r \times d_i$, and the non-vanishing entries are some *universal* functions of the $X_{i,k}$'s, in the sense that these functions themselves do not depend on the choice of differential system as long as it satisfies the assumptions we made so far. Eventually we have

$$\nu^{-1} \Phi^{(0)}(x) \nu C^{-1} = \sum_{i,r} \mathcal{Y}_{i,r}(x) (x A'_{i,r} + A_{i,r}) \quad (4-87)$$

Again we obtain a very restrictive class of matrices $\Phi^{(0)}(x)$.

Classification of $R^{(0)}(x)$ The previous results also hold for $R^{(0)}$ with y replaced by s . However, the requirement for the auxiliary curve not to have any double points makes the combine to make the assumptions even more restrictive.

We may uniquely write

$$s(z) = \sum_{j=0}^m f_j(x(z)) z^j, \quad m \leq d-1. \quad (4-88)$$

If $m = 1$, then it is obvious that there can be no double points, in that case

$$s(z) = f_0(x(z)) + f_1(x(z))z. \quad (4-89)$$

In other words, $R^{(0)}(x, t)$

$$R^{(0)}(x, t) = f_0(x, t) \nu(t) \widehat{A}_{0,0}(t) C(t) \nu(t)^{-1}$$

$$+ f_1(x, t) \nu(t) \left(\widehat{A}_{1,0}(t)C(t) + \frac{\widehat{A}_{1,1}(t)C(t)}{x - x(\infty, t)} \right) \nu(t)^{-1}. \quad (4-90)$$

Up to a Möbius transformation on z we could have chosen $z = \infty$ to be a pole of x , and then we would have obtained

$$R^{(0)}(x, t) = f_\infty(x, t) \nu(t) \left(A_{\infty,1}(t)C(t) + x A'_{\infty,1}(t)C(t) \right) \nu(t)^{-1}. \quad (4-91)$$

Notice that if $d_\infty > 1$, then $A'_{\infty,1}(t)C(t)$ is a nilpotent matrix :

$$\left(A'_{\infty,1}(t)C(t) \right)_{k,l;k',l'} = \frac{1}{X_{\infty,d_\infty}} \delta_{k,\infty} \delta_{k',\infty} \delta_{l,d_\infty} \delta_{l',1}. \quad (4-92)$$

Assumptions regarding the higher orders in the ε expansions

In order to prove the topological type property and in addition to assumptions 4.5 and 4.22, we make the following sufficient assumptions regarding the spectral curve and the possible singularities of the system. We shall need the notion that $L^{(k \geq 1)}$ has to be “less singular” than $\Phi^{(0)}$ – symbolically denoted $\Phi^{(k)} < \Phi^{(0)}$ –. Our precise statement is the following :

Assumption 4.25 (Analytic behavior $\Phi^{(k)} < \Phi^{(0)}$) *We assume that :*

- for every $k \geq 1$, all poles of $\Phi^{(k)}(x, t)$ are among the poles of $\Phi^{(0)}(x, t)$.
- for any matrix \tilde{C} , and any generic distinct x_0, x_1 , the following ε -formal series whose coefficients are bi-rational functions of x and y :

$$\frac{\text{Det} \left(y - \Phi_\varepsilon(x) - \frac{\tilde{C}}{(x-x_0)(x-x_1)} \right) - \text{Det} \left(y - \Phi^{(0)}(x) \right)}{E_y(x, y)} dx \quad (4-93)$$

is, when restricted to the spectral curve, a one-form $\Omega(z)$ that is analytic (at each order in ε) at all singularities of $\Phi^{(0)}(x)$.

Equivalently, its only singularities can either be poles over $x = x_0$ and $x = x_1$, due to the $\frac{\tilde{C}}{(x-x_0)(x-x_1)}$ term, or at double points of $\mathcal{S} : (b_i, \tilde{b}_i)$.

$$\Omega(z) = \text{Det} \left(y(z) - \Phi_\varepsilon(x(z)) - \frac{\tilde{C}}{(x(z) - x_0)(x(z) - x_1)} \right) \frac{dx(z)}{E_y(x(z), y(z))}$$

$$= \sum_i \beta_i \left(\frac{dz}{z-b_i} - \frac{dz}{z-\tilde{b}_i} \right) + \sum_{i \in \{0,1\}} \sum_{j=1}^d \sum_{k=1}^d \frac{c_{i,j,k} dz}{(z-z^j(x_i))^k} \quad (4-94)$$

where the coefficients β_i , $c_{i,j,k}$ are formal power series of ε , starting at $O(\varepsilon)$.

In other words, the ε corrections do not change the Newton's polygon of $E(x, y)$. They may only change the interior coefficients, as well as possibly adding poles over $x = x_0$ or $x = x_1$.

Evaluating this one-form at $z^i(x)$, inserting and subtracting the diagonal term $Y(x) = V(x)^{-1}\Phi^{(0)}(x)V(x)$ and then expanding the determinant, we get after simplification that it is equal to

$$\Omega(z^i(x)) = dx \sum_{I \subset \{1, \dots, d\}, i \in I} \frac{\text{Det}_{I \times I} \left(V(x)^{-1} \left(\Phi_\varepsilon(x) - \Phi^{(0)}(x) + \frac{\tilde{C}}{(x-x_0)(x-x_1)} \right) V(x) \right)}{\prod_{j \in I, j \neq i} (y(z^i(x)) - y(z^j(x)))}. \quad (4-95)$$

In particular, to order ε we must have $\forall i$:

$$dx \left(V(x)^{-1} \Phi^{(1)}(x) V(x) \right)_{i,i} = 0 \quad (4-96)$$

(which implies $W_1^{(0)}(x.e_i) = 0$, as we will see below). This is equivalent to say that $\Phi^{(1)}(x)$ must be derived from $\Phi^{(0)}(x)$, i.e. $\exists \tilde{\Phi}^{(1)}(x)$ such that

$$\Phi^{(1)}(x) = [\tilde{\Phi}^{(1)}(x), \Phi^{(0)}(x)]. \quad (4-97)$$

At order ε^2 we get that

$$dx \left(V(x)^{-1} \Phi^{(2)}(x) V(x) \right)_{i,i} - \sum_{j \neq i} dx \frac{\left(V(x)^{-1} \Phi^{(1)}(x) V(x) \right)_{i,j} \left(V(x)^{-1} \Phi^{(1)}(x) V(x) \right)_{j,i}}{(y(z^i(x)) - y(z^j(x)))}$$

is analytic at all poles of y .

Remark 4.26 *Assumption 4.25 may appear technical but it can be proved easily in many cases. For example :*

1. *Assumption 4.25 is trivially verified if for all $k \geq 1$, $\Phi^{(k)}$ is independent of x . (This happens for the Airy Lax pair for example).*

2. Assumption 4.25 is verified if Φ_ε defines a Fuchsian system, i.e. has only simple poles $c_i(t)$ independent of ε , and residues $C_i(t, \varepsilon)$ whose eigenvalues are independent of ε

$$\Phi_\varepsilon = \sum_{i=1}^p \frac{C_i(t, \varepsilon)}{x - c_i(t)} \quad (4-98)$$

Indeed, in that case the poles of $L^{(k)}$ are the same as those of $\Phi^{(0)}$. The eigenvalues of Φ_ε have only simple poles above $x = c_i(t)$, with residues the eigenvalues of $C_i(t, \varepsilon)$, and thus all the singular behavior of the eigenvalues of Φ_ε , is independent of ε , showing that the characteristic polynomials of Φ_ε and $\Phi^{(0)}(x, t)$ can differ only by the interior of their Newton's polygon.

Parity Assumption

In order to prove sufficient conditions for the topological type property, we need (as proposed in [15]) another assumption :

Assumption 4.27 (Parity) *We assume that there exists a matrix*

$$\Gamma(t, \varepsilon) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \varepsilon^k \Gamma^{(k)}(t), \quad (4-99)$$

independent of x , such that

$$\Phi_{-\varepsilon}(x, t) = \Gamma(t, \varepsilon)^{-1} \Phi_\varepsilon^T \Gamma(t, \varepsilon). \quad (4-100)$$

with

$$\Gamma^{(0)} = (\nu^T(t))^{-1} C \nu(t)^{-1} = \Gamma^{(0)T}. \quad (4-101)$$

This assumption can be shown to hold for many well-known integrable systems. Moreover, its leading order in ε is a consequence of assumption 4.22. It was made in [15] and yields one of the Topological Type requirements.

Notice that we have

$$\Gamma(t, -\varepsilon) = \Gamma(t, \varepsilon)^T, \quad (4-102)$$

i.e. for all $k \geq 0$:

$$\Gamma^{(k)}(t) = (-1)^k \Gamma^{(k)}(t)^T. \quad (4-103)$$

In other words, the coefficients of the matrices appearing in the series expansion of $\Gamma(t, \varepsilon)$ are either symmetric or antisymmetric matrices depending on the parity of their index.

As is proved in Appendix [10], these assumptions imply that the Topological Type property and therefore that one can reconstruct the correlators from topological recursion. We will now end this chapter by presenting as examples various cases in which the assumptions are met and the method applies. The first one deals with (p, q) minimal models that were studied in [15]. The second one deals with the Painlevé Lax pairs and was developed in [57]. For clarity we will only focus on the Painlevé VI case though all other Painlevé systems can be treated similarly (details can be found in [58]).

4.2 (p, q) minimal models

These were studied with the topological recursion in [15]. However the proof presented in this article was incomplete (the proof of the leading order property used an insertion operator. A part of the definition of this operator was missing. The gap was completed for $q = 2$ in [40] but the general case remained incomplete). This new proof doesn't use insertion operators, it uses our general loop equations method. We will here follow the standard notations of [15] taking in particular $q = d$.

In (p, q) minimal models (p and q are coprime strictly positive integers, see [15] for details), $R_\varepsilon(x, t)$ is a $q \times q$ companion matrix :

$$R_\varepsilon(x, t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ & & 0 & 1 & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \dots & 0 & & & 1 \\ u_{d-1}(t, \varepsilon) & \dots & u_1(t, \varepsilon) & u_0(t, \varepsilon) - x & & \end{pmatrix} \quad (4-104)$$

The matrix $\Psi(x, t)$, described in [?], is given by :

$$\Psi_\varepsilon(x, t) = \begin{pmatrix} \psi_1(x, t) & \dots & \psi_q(x, t) \\ (\varepsilon\partial_t)\psi_1(x, t) & \dots & (\varepsilon\partial_t)\psi_q(x, t) \\ \vdots & & \vdots \\ (\varepsilon\partial_t)^{q-1}\psi_1(x, t) & \dots & (\varepsilon\partial_t)^{q-1}\psi_q(x, t) \end{pmatrix} \quad (4-105)$$

where $(\psi_i)_{1 \leq i \leq q}$ are linearly independent solutions of the system :

$$x\psi(x, t) = Q\psi(x, t), \quad \varepsilon\partial_t\psi(x, t) = -P\psi(x, t) \quad \text{and} \quad [P, Q] = \varepsilon \quad (4-106)$$

where the operator (P, Q) are of the form :

$$P = \sum_{k=0}^p v_k(t)(\varepsilon\partial_t)^k, \quad v_p = 1, v_{p-1} = 0, v_{p-2} = -pu(t) \quad (4-107)$$

$$Q = \sum_{l=0}^q u_l(t)(\varepsilon\partial_t)^l, \quad u_q = 1, u_{q-1} = 0, u_{q-2} = -qu(t) \quad (4-108)$$

In particular, the condition $[P, Q] = \varepsilon$ determines all functions $(v_i)_{1 \leq i \leq p}$ and $(u_i)_{1 \leq i \leq q}$ in terms of $u(t)$ and its derivatives. The matrix denoted $\Phi(x, t) = (\Phi_{k,j}(x, t))_{1 \leq k, j \leq q}$ is determined by decomposing the operators $(\Phi_k)_{k \geq 0}$ on the basis $((\varepsilon\partial_t)^i)_{i \geq 0}$:

$$\Phi_k(x, t) = \sum_{j=0}^q \Phi_{k,j}(x, t)(\varepsilon\partial_t)^j \quad (4-109)$$

where the operators $(\Phi_k)_{k \geq 0}$ are defined recursively as :

$$\Phi_0(x, t) = -\sum_{l=0}^p v_l(t)F_l(x, t), \quad \Phi_{k+1}(x, t) = (\varepsilon\partial_t)\Phi_k(x, t) + \Phi_{k,q-1}(x, t)(x - Q) \quad (4-110)$$

with $F_l(x, t) = \sum_{j \geq 0} F_{l,j}(x, t)(\varepsilon\partial_t)^j$ defined recursively by :

$$F_0(x, t) = 1, \quad F_{l+1}(x, t) = (\varepsilon\partial_t)F_l(x, t) + F_{l,q-1}(x, t)(x - Q) \quad (4-111)$$

In particular, it is obvious from the definitions that $L(x, t)$ is a polynomial in x .

In the context of (p, q) minimal models, one is interested in formal expansion in ε . Since the functions $(u_i(t, \varepsilon))_{i \geq 0}$ and $(v_i(t, \varepsilon))_{i \geq 0}$ admit a formal expansion in ε , we get that assumption 4.1 is verified. Moreover, the spectral curve is of genus 0, so assumption 4.5 is verified. It is given by (see Proposition 5.2 of [15]) :

$$x_t(z) = \sum_{k=0}^q u_k^{(0)}(t) z^k \quad (4-112)$$

$$y_t(z) = \sum_{l=0}^p v_l^{(0)}(t) z^l \quad (4-113)$$

The auxiliary spectral curve is given by the characteristic polynomial of the companion matrix $R_\varepsilon(x, t)$:

$$\tilde{E}_{\varepsilon=0}(x, s, t) = \text{Det}(s - R^{(0)}(x, t)) = x_t(s) - x \quad (4-114)$$

The set of solutions of $\tilde{E}_t(x, s; t, 0) = 0$ is thus the set of all $(x_t(z), z)$ for points $z \in \overset{o}{\Sigma} = \mathbb{C}\mathbb{P}^1 - \{\text{singularities}\}$. Therefore the auxiliary spectral curve is equivalent to the triplet :

$$\tilde{\mathcal{F}}_t = (\overset{o}{\Sigma}, \tilde{x}_t, s_t) \quad (4-115)$$

with the function s_t is the identity map $s_t : z \mapsto z$. The auxiliary spectral curve obviously does not admit any double points and the spectral curve (4-112) is regular so assumption 4.13 is verified. Note that in our setting, the poles of the x_t function correspond to $k = 1$, $d_1 = q$ and $\alpha_1 = \infty$. In other words, $z \mapsto x_t(z)$ has only one pole at infinity of order q (in the general theory developed above, the point $z = \infty$ was assumed not to be a pole of x . This means that some of the above formulas require some basic adaptations to accommodate this particular case). Since the $R(x, t)$ matrix is a companion matrix, its eigenvectors are given by a Vandermonde-like matrix and we obtain :

$$V(x, t) = \mathcal{V}(x) \Rightarrow v(t) = I_q \quad (4-116)$$

In particular, assumption 4.22 is trivially satisfied.

Notice that by definition, $\Phi_\varepsilon(x, t)$ depends polynomially in the variable x whose coefficients admit an ε -expansion. Thus, assumption 4.25 is satisfied.

Assumption 4.27 was partly proved in [15]. Indeed, the authors proved that the matrix $\Gamma(t)$ given by (note that there is a change of convention from the cited article where the Γ matrix is defined as the inverse of our present matrix and with a global $(-1)^{q-1}$ constant) :

$$\Gamma(t) = \gamma(t)^{-1} \text{ with } \gamma(t) = (-1)^{q-1} \Phi(x, t) \Psi(x, t)^T \quad (4-117)$$

satisfy (4-100). A precise definition of $\Phi(x, t)$ is given in [15] and in particular, Theorem 5.2 in this paper shows that the matrix $\gamma(t)$ does not depend on x . Therefore the only remaining issue to prove assumption 4.27 is to match $\Gamma^{(0)}$ with $(\nu(t)^T)^{-1} C \nu(t)^{-1}$ to satisfy (4-101). We observe that by definition, the generalized Vandermonde matrix $\mathcal{V}(x)$ leads to :

$$C = \begin{pmatrix} u_1^{(0)}(t) & u_2^{(0)}(t) & \dots & u_{q-2}^{(0)}(t) & 0 & 1 \\ u_2^{(0)}(t) & \dots & \dots & \dots & 1 & 0 \\ \vdots & \dots & \dots & \dots & \dots & 0 \\ u_{q-2}^{(0)}(t) & \dots & \dots & \dots & \dots & \vdots \\ 0 & 1 & \dots & \dots & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad (4-118)$$

In other words : we have both $C_{i,j} = 0$ if $i+j > q+1$ and $C_{i,j} = u^{(0)}(t)_{i+j-1}$ if $i+j \leq q+1$. Its inverse is given by :

$$C^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & \dots & \dots & 1 & a_2 \\ \vdots & \dots & \dots & \dots & \dots & a_3 \\ 0 & \dots & \dots & \dots & \dots & \vdots \\ 0 & 1 & \dots & \dots & \dots & a_{q-1} \\ 1 & a_2 & a_3 & \dots & a_{q-1} & a_q \end{pmatrix} \quad (4-119)$$

In other words, $(C^{-1})_{i,j} = 0$ if $i+j < q+1$ and $(C^{-1})_{i,j} = a_{i+j-q}$ if $i+j \geq q+1$. The coefficients $(a_i)_{1 \leq i \leq q}$ are determined by the following recursion

(obtained by looking at the term $(C^{-1}C)_{i,1} = \delta_{i,1}$ with $1 \leq i \leq q$) :

$$a_1 = 1, a_2 = 0 \text{ and } a_{i+1} = -\sum_{j=1}^{i-1} a_j u_{j+q-i+1}^{(0)}(t) \text{ for } 2 \leq i \leq q-1 \quad (4-120)$$

Since $v(t) = I_q$, condition (4-101) is equivalent to prove that $C^{-1} = \gamma^{(0)}(t)$. The matrix $\gamma(t)$ (unfortunately denoted \mathbf{C} with entries labeled from 0 to $q-1$ in [?]) is described in equations 5.77, 5.78 and 5.79 of [?]. It satisfies $\gamma_{i,j} = 0$ if $i+j < q+1$ and

$$\begin{aligned} \gamma_{1,j} &= \delta_{j,q} \text{ for } 1 \leq j \leq q \\ \varepsilon \partial_t \gamma_{i,j} &= \gamma_{i,j+1} - \gamma_{i+1,j} \text{ for } 1 \leq i, j \leq q-1 \\ \varepsilon \partial_t \gamma_{i,q-1} &= -\gamma_{i,q} - \sum_{l=0}^{q-2} u_l(t) \gamma_{i,l+1} \text{ for } 1 \leq i \leq q-1 \end{aligned} \quad (4-121)$$

Let us denote for clarity $B = \gamma^{(0)}$. Projecting the last set of equations at order ε^0 gives $B_{i,j} = 0$ if $i+j < q+1$ and :

$$\begin{aligned} B_{1,j} &= \delta_{j,q} \text{ for } 1 \leq j \leq q \\ B_{i+1,j} &= B_{i,j+1} \text{ for } 1 \leq i, j \leq q-1 \\ B_{i,q} &= -\sum_{l=0}^{q-2} u_l^{(0)}(t) B_{i,l+1} \text{ for } 1 \leq i \leq q-1 \end{aligned} \quad (4-122)$$

The second equation is equivalent to say that B is a Hankel matrix of the same form as C^{-1} . In other words, $B_{i,j} = 0$ if $i+j < q+1$ and $B_{i,j} = b_{i+j-q}$ if $i+j \geq q+1$. The coefficients $(b_i)_{1 \leq i \leq q}$ are determined by the first and last equations of (4-122). We get :

$$b_1 = 1, b_2 = 0 \text{ and } b_{i+1} = -\sum_{l=1}^{q-2} b_l u_l^{(0)}(t) \text{ for } 2 \leq i \leq q-1 \quad (4-123)$$

Hence we recover the same recursion as (4-120). This finally proves that $C^{-1} = \gamma^{(0)}$ so that assumption 4.27 is verified.

In conclusion, we have proved all required assumptions for the (p, q) minimal models that therefore satisfy the Topological Type property.

4.3 Painlevé VI case

Painlevé equations were studied with the topological recursion in [57] (Painlevé II) and [58] (all six Painlevé equations). A simpler method (only valid in the case $d = 2$) was used to prove that the Painlevé Lax pairs satisfy the topological type property. We propose here to show that our generalization also applies directly to these cases. We will only carry out the Painlevé VI case (which is the most difficult) in details but all results presented here can be easily adapted to the other Painlevé cases using computations presented in the cited articles.

In the Painlevé 6 system we have :

$$\Phi_\varepsilon^{\text{VI}}(x, t) = \frac{A_0(t, \varepsilon)}{x} + \frac{A_1(t, \varepsilon)}{x-1} + \frac{A_t(t, \varepsilon)}{x-t} \quad (4-124)$$

$$R_\varepsilon^{\text{VI}}(x, t) = -\frac{A_t(t, \varepsilon)}{x-t} - \frac{(q-t)(\theta_\infty - \varepsilon)}{2t(t-1)}\sigma_3 \quad (4-125)$$

$$A_0 = \begin{pmatrix} z_0 + \frac{\theta_0}{2} & -\frac{q}{t} \\ \frac{tz_0(z_0 + \theta_0)}{q} & -\left(z_0 + \frac{\theta_0}{2}\right) \end{pmatrix}, \quad A_1 = \begin{pmatrix} z_1 + \frac{\theta_1}{2} & \frac{q-1}{t-1} \\ -\frac{(t-1)z_1(z_1 + \theta_1)}{q-1} & -\left(z_1 + \frac{\theta_1}{2}\right) \end{pmatrix} \quad (4-126)$$

$$A_t = \begin{pmatrix} z_t + \frac{\theta_t}{2} & -\frac{q-t}{t(t-1)} \\ \frac{t(t-1)z_t(z_t + \theta_t)}{q-t} & -\left(z_t + \frac{\theta_t}{2}\right) \end{pmatrix}, \quad A_\infty = \begin{pmatrix} \frac{\theta_\infty}{2} & 0 \\ 0 & -\frac{\theta_\infty}{2} \end{pmatrix} = -(A_0 + A_1 + A_t) \quad (4-127)$$

Here, $z_0(t), z_1(t)$ and $z_t(t)$ are auxiliary functions of t that can be expressed in terms $q(t)$ and a function $p(t)$ defined by :

$$p = \frac{z_0 + \theta_0}{q} + \frac{z_1 + \theta_1}{q-1} + \frac{z_t + \theta_t}{q-t} \quad (4-128)$$

The explicit expression for z_0, z_1 and z_t in terms of q can be found in [58] where $q(t)$ is shown to satisfy a ε -deformed version of the Painlevé 6 equation. Note that the matrix form $\Phi_\varepsilon^{\text{VI}}(x, t) dx$ has simple poles located at $x \in \{0, 1, \infty, t\}$ while $R_\varepsilon^{\text{VI}}(x, t) dx$ only has simple poles at $x \in \{\infty, t\}$. Existence of an ε -expansion is discussed in [58] where assumption 4.1 is proved. At first order in ε it is shown in [58] that the spectral and auxiliary curves are of genus 0 :

$$\begin{aligned} y^2 &= \frac{\theta_\infty^2 (x - q_0)^2 P_2(x)}{4x^2 (x - 1)^2 (x - t)^2} \\ s^2 &= \frac{(q_0 - t)^2 \theta_\infty^2 P_2(x)}{4t^2 (t - 1)^2 (x - t)^2} \end{aligned} \quad (4-129)$$

where $P_2(x) = x^2 + \left(-1 - \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2} + \frac{\theta_1^2 (t-1)^2}{\theta_\infty^2 (q_0-1)^2}\right) x + \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2} = (x - a)(x - b)$ that can be written equivalently $P_2(x) = x^2 + \left(-\frac{\theta_0^2 t(t+1)}{\theta_\infty^2 q_0^2} + \frac{\theta_1^2 t(t-1)}{\theta_\infty^2 (q_0-1)^2} - \frac{\theta_t^2 t(t-1)}{\theta_\infty^2 (q_0-t)^2}\right) x + \frac{\theta_0^2 t^2}{\theta_\infty^2 q_0^2}$. Here q_0 stands for $q^{(0)}(t)$ the leading order in ε of $q(t)$. It satisfies an algebraic equation of degree 6 that can be found explicitly in [58]. Inserting this result in the definition of $R^{(0)}(x, t)$, we get an expression of $z_t^{(0)}$ and q_0 in terms of a, b and t (and the monodromy parameters):

$$\begin{aligned} z_t^{(0)} &= -\frac{\theta_t}{2} + \frac{1}{4} + \frac{\theta_\infty (q_0 - t) \left(t - \frac{a+b}{2}\right)}{2t(t-1)} = -\frac{\theta_t}{2} \pm \frac{t - \frac{a+b}{2}}{2\sqrt{(t-a)(t-b)}} \\ (q_0 - t) &= \pm \frac{t(t-1)\theta_t}{\theta_\infty \sqrt{(t-a)(t-b)}} \end{aligned} \quad (4-130)$$

so that we get :

$$\begin{aligned} R^{(0)}(x, t) &= \pm \begin{pmatrix} -\frac{\theta_t \left(x - \frac{a+b}{2}\right)}{2(x-t)\sqrt{(t-a)(t-b)}} & \frac{\theta_t}{\theta_\infty (x-t)\sqrt{(t-a)(t-b)}} \\ -\frac{(b-a)^2 \theta_t \theta_\infty}{16(x-t)\sqrt{(t-a)(t-b)}} & \frac{\theta_t \left(x - \frac{a+b}{2}\right)}{2(x-t)\sqrt{(t-a)(t-b)}} \end{pmatrix} \\ \Phi^{(0)}(x, t) &= \frac{(x - q_0) t (t - 1)}{x(x - 1)(q_0 - t)} R^{(0)}(x, t) \end{aligned} \quad (4-131)$$

The spectral curve (4-129) is of genus 0 with two finite branch points located at the two simple zeros of the polynomial P_2 denoted a and b . Thus assumption 4.5 is satisfied. Note that there is also a double point at $x = q_0$ for the spectral curve but it is absent in the auxiliary curve. Since the spectral curve is of genus 0, it can be parametrized globally on $\overline{\mathbb{C}}$ and we choose a parametrization suitable with the convention that $z = \infty$ is not a pole of $x(z)$ (so that it slightly differs from the usual Zhukovski parametrization of [58]). We take:

$$\begin{aligned}
x(z) &= \frac{a+b}{2} + \frac{b-a}{2} \left(1 + \frac{1}{z-1} - \frac{1}{z+1} \right) = b + \frac{b-a}{(z+1)(z-1)} \\
y(z) &= \frac{\theta_\infty(x(z) - q_0)(b-a)z}{2(z-1)(z+1)x(z)(x(z)-1)(x(z)-t)} \\
s(z) &= \frac{(q_0 - t)\theta_\infty(b-a)z}{2(z-1)(z+1)t(t-1)(x(z)-t)} \\
&= \pm \frac{(b-a)z\theta_t}{2(z-1)(z+1)(x(z)-t)\sqrt{(t-a)(t-b)}}
\end{aligned} \tag{4-132}$$

Note that $x'(z) = -\frac{2z(b-a)}{(z+1)^2(z-1)^2}$. In the z variable, the two branch points are located at $z=0$ and $z=\infty$ while the poles are located at $z=\pm 1$. The involution (corresponding to $x(z) = x(\bar{z})$) is given by $\bar{z} = -z$. Inverting the relation between x and z leads to :

$$z^1(x) = \sqrt{\frac{x-a}{x-b}} \text{ and } z^2(x) = -\sqrt{\frac{x-a}{x-b}} \tag{4-133}$$

so that

$$\begin{aligned}
S_1(x) &= \frac{\theta_\infty(q_0 - t)\sqrt{(x-a)(x-b)}}{t(t-1)(x-t)} = \pm \frac{\theta_t}{(x-t)} \sqrt{\frac{(x-a)(x-b)}{(t-a)(t-b)}} \\
S_2(x) &= -\frac{\theta_\infty(q_0 - t)\sqrt{(x-a)(x-b)}}{t(t-1)(x-t)} = \mp \frac{\theta_t}{(x-t)} \sqrt{\frac{(x-a)(x-b)}{(t-a)(t-b)}} \\
Y_1(x) &= \frac{\theta_\infty(x - q_0)\sqrt{(x-a)(x-b)}}{x(x-1)(x-t)} \\
Y_2(x) &= -\frac{\theta_\infty(x - q_0)\sqrt{(x-a)(x-b)}}{x(x-1)(x-t)}
\end{aligned} \tag{4-134}$$

In particular, from the last identities it is straightforward to verify that the auxiliary curve has no double points, i.e. that assumption 4.13 is satisfied. Moreover, application of the previous formulas leads to :

$$\vec{\gamma}(z) = \left(-\frac{i(z+1)}{\sqrt{2(b-a)}\sqrt{z}}, -\frac{i(z-1)}{\sqrt{2(b-a)}\sqrt{z}} \right) \tag{4-135}$$

and thus :

$$\begin{aligned}
\mathcal{V}(x) &= \begin{pmatrix} -\frac{i(z^1(x)+1)}{\sqrt{2(b-a)}} \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} & -\frac{(z^2(x)+1)}{\sqrt{2(b-a)}} \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} \\ -\frac{i(z^1(x)-1)}{\sqrt{2(b-a)}} \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} & -\frac{(z^2(x)-1)}{\sqrt{2(b-a)}} \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{i}{\sqrt{2(b-a)}} \frac{\sqrt{x-a}+\sqrt{x-b}}{((x-a)(x-b))^{\frac{1}{4}}} & -\frac{1}{\sqrt{2(b-a)}} \frac{\sqrt{x-b}-\sqrt{x-a}}{((x-a)(x-b))^{\frac{1}{4}}} \\ -\frac{i}{\sqrt{2(b-a)}} \frac{\sqrt{x-a}-\sqrt{x-b}}{((x-a)(x-b))^{\frac{1}{4}}} & \frac{1}{\sqrt{2(b-a)}} \frac{\sqrt{x-b}+\sqrt{x-a}}{((x-a)(x-b))^{\frac{1}{4}}} \end{pmatrix} \\
&= \frac{1}{\sqrt{2(b-a)}} \begin{pmatrix} -i \left(\left(\frac{x-a}{x-b}\right)^{\frac{1}{4}} + \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} \right) & \left(\frac{x-a}{x-b}\right)^{\frac{1}{4}} - \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} \\ i \left(\left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} - \left(\frac{x-a}{x-b}\right)^{\frac{1}{4}} \right) & \left(\frac{x-a}{x-b}\right)^{\frac{1}{4}} + \left(\frac{x-b}{x-a}\right)^{\frac{1}{4}} \end{pmatrix} \quad (4-136)
\end{aligned}$$

It is then straightforward to verify that :

$$\begin{aligned}
\mathcal{V}(x)\mathcal{V}(x)^T &= \begin{pmatrix} \frac{(z^2(x)-z^1(x))}{(b-a)} \left(\frac{x-b}{x-a}\right)^{\frac{1}{2}} & \frac{((z^2(x))^2-(z^1(x))^2)}{(b-a)} \left(\frac{x-b}{x-a}\right)^{\frac{1}{2}} \\ \frac{((z^2(x))^2-(z^1(x))^2)}{(b-a)} \left(\frac{x-b}{x-a}\right)^{\frac{1}{2}} & -\frac{(z^2(x)-z^1(x))}{(b-a)} \left(\frac{x-b}{x-a}\right)^{\frac{1}{2}} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{2}{b-a} & 0 \\ 0 & \frac{2}{b-a} \end{pmatrix} \quad (4-137)
\end{aligned}$$

Hence we get $C = \frac{b-a}{2} \text{diag}(-1, 1)$ as claimed from (4-56). Note that we also get :

$$\begin{aligned}
\mathcal{V}(x, t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathcal{V}(x, t)^T &= \begin{pmatrix} -\frac{\sqrt{\frac{x-a}{x-b}} + \sqrt{\frac{x-b}{x-a}} + 2}{2(b-a)} & \frac{\sqrt{\frac{x-b}{x-a}} - \sqrt{\frac{x-a}{x-b}}}{2(b-a)} \\ \frac{\sqrt{\frac{x-b}{x-a}} - \sqrt{\frac{x-a}{x-b}}}{2(b-a)} & -\frac{\sqrt{\frac{x-a}{x-b}} + \sqrt{\frac{x-b}{x-a}} - 2}{2(b-a)} \end{pmatrix} \\
\mathcal{V}(x, t) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathcal{V}(x, t)^T &= \begin{pmatrix} \frac{\sqrt{\frac{x-a}{x-b}} + \sqrt{\frac{x-b}{x-a}} - 2}{2(b-a)} & -\frac{\sqrt{\frac{x-b}{x-a}} - \sqrt{\frac{x-a}{x-b}}}{2(b-a)} \\ -\frac{\sqrt{\frac{x-b}{x-a}} - \sqrt{\frac{x-a}{x-b}}}{2(b-a)} & \frac{\sqrt{\frac{x-a}{x-b}} + \sqrt{\frac{x-b}{x-a}} + 2}{2(b-a)} \end{pmatrix} \quad (4-138)
\end{aligned}$$

Computing $\mathcal{V}(x)S(x)\mathcal{V}(x)^T C$ leads to :

$$\mathcal{V}(x)S(x)\mathcal{V}(x)^T C = \begin{pmatrix} \frac{\theta_\infty(q_0-t)}{2t(t-1)} + \frac{\theta_\infty(q_0-t)\left(t-\frac{a+b}{2}\right)}{2(x-t)} & -\frac{\theta_\infty(q_0-t)(b-a)}{4t(t-1)(x-t)} \\ \frac{\theta_\infty(q_0-t)(b-a)}{4t(t-1)(x-t)} & -\frac{\theta_\infty(q_0-t)}{2t(t-1)} - \frac{\theta_\infty(q_0-t)\left(t-\frac{a+b}{2}\right)}{2(x-t)} \end{pmatrix}$$

$$= \pm \begin{pmatrix} \frac{\theta_t}{\sqrt{(t-a)(t-b)}} + \frac{\left(t - \frac{a+b}{2}\right)t(t-1)\theta_t}{2\sqrt{(t-a)(t-b)(x-t)}} & -\frac{\theta_t(b-a)}{4\sqrt{(t-a)(t-b)(x-t)}} \\ \frac{\theta_t(b-a)}{4\sqrt{(t-a)(t-b)(x-t)}} & -\frac{\theta_t}{\sqrt{(t-a)(t-b)}} - \frac{\left(t - \frac{a+b}{2}\right)t(t-1)\theta_t}{2\sqrt{(t-a)(t-b)(x-t)}} \end{pmatrix} \quad (4-139)$$

where we used (4-130) to replace q_0 . Eventually a direct computation from (4-136) and (4-131) shows that :

$$V(x, t) = v(t)\mathcal{V}(x) \text{ with } v(t) = \begin{pmatrix} 0 & \frac{4}{\theta_\infty(b-a)} \\ 1 & 0 \end{pmatrix} \quad (4-140)$$

$$\Phi^{(0)}(x, t) = v(t)\mathcal{V}(x)Y(x)\mathcal{V}(x)^T C v(t)^T \quad (4-141)$$

$$R^{(0)}(x, t) = v(t)\mathcal{V}(x)S(x)\mathcal{V}(x)^T C v(t)^T \quad (4-142)$$

so that assumption 4.22 is verified.

Eventually since

$$\Phi_\varepsilon^{\text{VI}}(x, t) = \frac{A_0(t, \varepsilon)}{x} + \frac{A_1(t, \varepsilon)}{x-1} + \frac{A_t(t, \varepsilon)}{x-t} \quad (4-143)$$

$$R_\varepsilon^{\text{VI}}(x, t) = -\frac{A_t(t, \varepsilon)}{x-t} - \frac{(q-t)(\theta_\infty - \varepsilon)}{2t(t-1)}\sigma_3 \quad (4-144)$$

we see that there is no mixing between the x -dependence and the ε -expansion. In particular, $\Phi^{(k)}$ has poles only at $x \in \{0, 1, t\}$ and assumption 4.25 is trivially verified. Finally, the symmetry condition is answered in [58] where it is proved that

$$\Gamma_{\text{VI}}(t) = \begin{pmatrix} -\frac{t^2 z_0(z_0 + \theta_0)}{q} + \frac{(t-1)^2 z_1(z_1 + \theta_1)}{q-1} & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies assumption 4.27. Note that at order ε^0 computations from [58] gives :

$$\Gamma_{\text{VI}}^{(0)}(t) = \begin{pmatrix} -\frac{\theta_\infty^2(b-a)^2}{16} & 0 \\ 0 & 1 \end{pmatrix} \quad (4-145)$$

Since Γ is only determined up to a global multiplication by a constant, we can easily match it with the direct computation of :

$$(\nu^T)^{-1}C\nu^{-1} = \frac{b-a}{2} \begin{pmatrix} -\frac{\theta_\infty^2(b-a)^2}{16} & 0 \\ 0 & 1 \end{pmatrix} \quad (4-146)$$

and thus assumption [4.27](#) is satisfied.

Chapter 3

W-symmetric conformal field theories

1 Lightning review of conformal field theory

Conformal field theories in two dimensions have appeared in the physics literature as powerful tools to study numerous systems, from critical (possibly quantum) statistical models in two dimensions (in some thermodynamic limit) to the worldsheet conformal symmetry of string theories [7]. They have the particularity to exhibit infinite dimensional conformal algebras of symmetries, namely extensions of the Virasoro algebra (that are not necessarily Lie algebras as we shall see). It was in turn argued in some cases [4] that there exists an underlying structure of quantum integrable system with commuting transfer matrices and such.

Mathematically speaking, all these constructions assume the existence of a certain set of functions called M -point correlation functions, for some integer $M \in \mathbb{N}^*$, defined on M copies of a given connected Riemann surface Σ , and denoted formally as

$$\left\langle \prod_{i=1}^M \Phi_{\alpha_i, \bar{\alpha}_i}(z_i, \bar{z}_i) \right\rangle \quad (1-1)$$

for distinct generic points $z_1, \dots, z_M \in \Sigma$, called the punctures, to which are associated labels $\alpha_1, \bar{\alpha}_1, \dots, \alpha_M, \bar{\alpha}_M$, called the charges, in the linear dual \mathfrak{h}^* of a Cartan subalgebra \mathfrak{h} of the considered considered reductive complex Lie algebra \mathfrak{g} . They are moreover assumed to be smooth on the generic locus of Σ^M and to satisfy a set of axioms, written here for our purpose.

- *Axiom 1 : Holomorphic factorization.* For any $M \in \mathbb{N}^*$, much like a Hodge decomposition, or a separation of variables, there exists a sequence of objects called conformal blocks $\{\mathcal{F}_\gamma\}_{\gamma \in B_{\mathcal{G}}}$ such that

$$\left\langle \prod_{i=1}^M \Phi_{\alpha_i, \bar{\alpha}_i}(z_i, \bar{z}_i) \right\rangle = \sum_{\gamma, \gamma' \in B_{\mathcal{G}}} C^{\gamma, \gamma'} \mathcal{F}_\gamma(\mathbf{z}) \mathcal{F}_{\gamma'}(\bar{\mathbf{z}}) \quad (1-2)$$

where we introduced the notation $\mathbf{z} = (z_1, \dots, z_M)$. $B_{\mathcal{G}}$ is the set of labels parametrizing this basis of conformal blocks and it contains the data of \mathcal{G} , a channel, namely a certain choice of unicellular trivalent graph on the considered Riemann surface satisfying $\partial\mathcal{G} = \{z_1, \dots, z_M\}$ and $\pi_1(\Sigma - \mathcal{G}, o) = 0$ with respect to a chosen reference point $o \in \Sigma$. In particular, the labelling γ (resp. γ') of Γ contains the data of $(\alpha_1, \dots, \alpha_M)$ (resp. $(\bar{\alpha}_1, \dots, \bar{\alpha}_M)$). This allows to reduce the problem to its holomorphic (often called chiral) and anti-holomorphic (anti-chiral) parts.

In Physics, one wishes the correlation functions that are reconstructed in this way to be modular invariant and this constrains admissible root geometries for the Lie algebras. It is known that in the case of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ this requires the corresponding Dynkin diagram to be simply laced (ADE) but such a statement does not exist for higher rank Lie algebras.

In this chapter we will be interested solely in studying chiral, or holomorphic, conformal blocks denoted $\mathcal{F}_\gamma(\mathbf{z}) = \left\langle \prod_{i=1}^M V_{\alpha_i}(z_i) \right\rangle$, where we introduced the vertex operators V_{α_i} merely as a notation (although the vertex operator formalism [28] is the right way to give a precise meaning to the bracket $\langle \cdot \rangle$ and operator product expansions to come). To do so (and as is customary in quantum mechanics, to mimic the interaction of an observer with the system) we introduce a probe, a so-called chiral spin-one current $\mathbf{J}(\tilde{x})$ valued in the dual \mathfrak{g}^* of the Lie algebra and defined for points $\tilde{x} \in \tilde{\Sigma}$ in the universal cover $\tilde{\Sigma} \rightarrow \Sigma$. It can be seen as multivalued on Σ and generically having essential singularities at the z_j 's.

In this quantum theory, the vertex operators are interpreted as the matter content with which the current interacts. This interaction is such that to

configurations of points on the Riemann surface, where the operators and currents are inserted, are associated correlations, describing the entanglement of the particles.

The following axioms are analytic and algebraic requirements these correlations should satisfy as functions of these configurations of points.

Let us choose once and for all a set of simple roots $\mathfrak{R}_0 \stackrel{\text{def}}{=} \{\mathfrak{r}_1, \dots, \mathfrak{r}_r\}$ in \mathfrak{h}^* , where $r = \text{rk } \mathfrak{g}$ is the rank of \mathfrak{g} . Then, introducing the *minimal* invariant bilinear form (\cdot, \cdot) on $\mathfrak{g}^* \times \mathfrak{g}^*$ (giving length 2 to maximal roots), let us consider the algebra generated by a central element K together with the harmonics $(\mathbf{J}^{(n)})_{n \in \mathbb{Z}}$, or modes, obtained by decomposing the chiral current around any generic point $\tilde{x}_0 \in \Sigma$ (with local coordinate $t = x - x_0$) as

$$\mathbf{J}(\tilde{x}) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}} \frac{\mathbf{J}^{(n)}(\tilde{x}_0)}{(x - x_0)^{n+1}} \quad (1-3)$$

Remark 1.1 *We will drop the explicit writing of the dependence of the modes in the generic point x_0 when no confusion is possible. Modes can only be compared when taken at the same point.*

These generators satisfy the commutation relations

$$[\mathbf{J}^{(n)}, \mathbf{J}^{(m)}] = [\mathbf{J}, \mathbf{J}]^{(n+m)} + (\mathbf{J}^{(n)}, \mathbf{J}^{(m)}) \delta_{n+m,0} K \quad (1-4)$$

where the \mathbf{J} symbols in $[\mathbf{J}, \mathbf{J}]$ are generically evaluated at different Lie algebra elements and therefore have non-trivial Lie bracket. The so-called affine Kac-Moody algebra at level $\kappa \in \mathbb{C}$ denoted $\widehat{\mathfrak{g}}_\kappa$ is then defined as the Lie algebra $\widehat{\mathfrak{g}}_\kappa \stackrel{\text{def}}{=} \widehat{\mathfrak{g}} / (\kappa - K)$, where $\widehat{\mathfrak{g}}$, called the generic affine Kac-Moody algebra associated to \mathfrak{g} , is defined as the central extension of vector spaces

$$0 \longrightarrow \mathbb{C}K \longrightarrow \widehat{\mathfrak{g}}_\kappa \longrightarrow \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}\partial \longrightarrow 0 \quad (1-5)$$

where $\mathcal{L}(\mathfrak{g})$ is the *loop algebra* of \mathfrak{g} denoted $\mathcal{L}(\mathfrak{g}) \stackrel{\text{def}}{=} \mathfrak{g}((t))$ (endowed with the natural Lie algebra structure coming from \mathfrak{g}) and the extra generator ∂ is defined to satisfy

$$[\partial, M] = \frac{d}{dt}M, \quad (\text{and thus } [\partial, K] = 0) \quad (1-6)$$

for any $M \in \mathcal{L}(\mathfrak{g}) = \mathfrak{g}((t))$. K is a central element assumed to act trivially as multiplication by κ (using the fact that the short exact sequence splits, this is equivalent to focusing on diagonalizable $\widehat{\mathfrak{g}}$ -module with finite weight spaces [54] and restricting ourselves to the highest weight representations they define). The levels κ are in general not constrained but they are for example in the case where the conformal field theory can be extended to a certain three dimension topological field theory named Cern-Simons theory on a given 3-manifold M whose boundary $\partial M = \Sigma$ is the Riemann surface. Then the levels are often required to make two copies of Cern-Simons on M equivalent if they yield the same conformal field theory on ∂M . They are then parametrized by maps $M_{\Sigma}M \rightarrow G$ considered up to homotopy, where $M_{\Sigma}M$ denotes gluing of the copies of M along their identical boundary Σ with matching of orientations. In the case where $G = SU(2)$, Σ is the Riemann sphere and $M \subset \mathbb{R}^3$ is the unit ball, since gluing in this case yields a 3-sphere, the levels of the Kac-Moody algebras of interest are then parametrized by the third homotopy group given by $\pi_3(SU(2)) \simeq \mathbb{Z}$. Constraints can also arise from the representation theory of the Kac-Moody algebra, indeed, when the levels under consideration are positive integers, $\widehat{\mathfrak{g}}$ admits unitary highest weight representations whose highest weights are dominant integral.

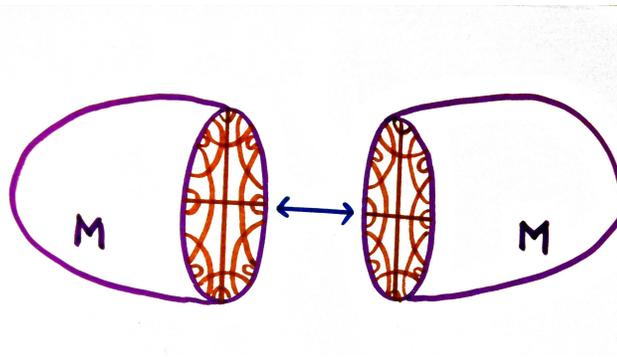


Figure 3.1: Gluing two copies of Chern-Simons theory on M along their identical boundary Σ is parametrized by maps $\Psi : M_{\Sigma}M \rightarrow G$.

Definition 1.2 *Insertions of currents*

By insertions of currents into chiral correlation functions we mean that we

consider an infinite countable set of additional $(\mathfrak{g}^*)^{\otimes n}$ -valued functions of interest denoted

$$\langle\langle \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \rangle\rangle \stackrel{\text{def}}{=} \left\langle \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \prod_{i=j}^M V_{\alpha_j}(z_j) \right\rangle. \quad (1-7)$$

This will allow us to define our algebra of symmetries $\mathcal{W}(\mathfrak{g})$ by requiring some closure condition under Lie bracket. It takes the form of some prescriptions for the singular behaviour the obtained functions are assumed to exhibit near the divisor where two of the points to which currents are inserted have base point projections that come together, or when one of them goes to one of the punctures z_1, \dots, z_M (where the *vertex operators* are located).

- *Axiom 2 : Operator product expansion.*

Keeping the notation (\cdot, \cdot) for the form on $\mathfrak{g} \times \mathfrak{g}$ dual to minimal invariant bilinear form on $\mathfrak{g}^* \times \mathfrak{g}^*$,

$$\begin{aligned} \mathbf{J}(\tilde{x} \cdot E) \mathbf{J}(\tilde{y} \cdot F) &\stackrel{x \sim y}{=} -\kappa \frac{(E, F)}{(x-y)^2} + \frac{\mathbf{J}(\tilde{y} \cdot [E, F])}{x-y} \\ &+ : \mathbf{J}(\tilde{x} \cdot E) \mathbf{J}(\tilde{y} \cdot F) :_{x=y} + \mathcal{O}(x-y) \end{aligned} \quad (1-8)$$

$$\mathbf{J}(\tilde{x} \cdot E) V_{\alpha_j}(z_j) \stackrel{x \sim z_j}{=} b \frac{\alpha_j(E)}{x-z_j} V_{\alpha_j}(z_j) + : \mathbf{J}(\tilde{x} \cdot E) V_{\alpha_j}(z_j) :_{x=z_j} + \mathcal{O}(x-z_j) \quad (1-9)$$

for $\tilde{x}, \tilde{y} \in \tilde{\Sigma}$, with some Lie algebra elements $E, F \in \mathfrak{g}$ and some puncture index $j \in \{1, \dots, M\}$. We also introduced the parameter $b \in \mathbb{C}^*$ and the notation $\mathbf{J}(\tilde{x} \cdot E) \stackrel{\text{def}}{=} \mathbf{J}(\tilde{x})(E)$ to relate with our notations in the study of Fuchsian differential systems for the evaluation. We denote the normal ordering operation as $: A(\tilde{x})B(\tilde{y}) :_{x=y}$ defined as the next to singular term when the basepoints of $\tilde{x}, \tilde{y} \in \tilde{\Sigma}$ come together (but

are not necessarily such that $\tilde{x} = \tilde{y}$). The last asymptotic equality defines V_{α_j} as a primary field, that is an eigenvector of the zero mode $\mathbf{J}^{(0)}$ that is annihilated by all positive modes $\mathbf{J}^{(n)}$, $n > 0$, and moreover, the presence of a simple pole means that we consider only regular singularities. Irregular singularities in the $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ case of Liouville theory were studied in [52].

Definition 1.3 *Background charge*

Define the background charge as the number $Q \stackrel{def}{=} b + b^{-1}$

These asymptotic relations are to be understood as holding when inserted into correlation functions, that is to say that they are meromorphic conditions on the functions we denoted $\langle\langle \mathbf{J} \cdots \mathbf{J} \rangle\rangle$. They are strong requirements as \mathbf{J} contains for example both data of the Lie bracket and the minimal invariant bilinear form of \mathfrak{g} .

Recall that the Virasoro algebra Vir is the infinite dimensional Lie algebra that generates the conformal transformations of the complex plane. It is defined as the central extension of vector spaces

$$0 \longrightarrow \mathbb{C}c \longrightarrow Vir \longrightarrow \text{Der}_{\mathbb{C}} \longrightarrow 0 \quad (1-10)$$

where we introduced the Lie algebra $\text{Der}_{\mathbb{C}}$ of holomorphic derivations of the field of Laurent series on the complex plane as well as the central element c (c stands for Casimir) called the central charge. Vir is generated by c together with the elements $(L_n)_{n \in \mathbb{Z}}$ satisfying the famous commutation relations

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}, \quad (n, m) \in \mathbb{Z}^2 \quad (1-11)$$

where for any $n \in \mathbb{Z}$, L_n generates the one-parameter family of local conformal transformations $(z \mapsto tz^{n+1})_{t \in \mathbb{C}}$ e.g. L_0 is the dilation operator. It goes to the Witt algebra in the zero central charge limit $c \rightarrow 0$. The generators of the Virasoro algebra can be gathered into a meromorphic

stress-energy tensor (anticipating on the next axiom by denoting the variable by x and not one of its preimages \tilde{x} by the universal covering map) around a base point $x_0 \in \Sigma$ by

$$T(x) \underset{x \sim x_0}{=} \sum_{n \in \mathbb{Z}} \frac{L_n(x_0)}{(x - x_0)^{n+2}}, \quad (1-12)$$

such that the Virasoro canonical commutation relations are translated in the following *operator product expansion*

$$T(x)T(x') \underset{x \sim x'}{=} \frac{c/2}{(x - x')^4} + \frac{2T(x)}{(x - x')^2} + \frac{\partial T(x)}{x - x'} + \mathcal{O}(1) \quad (1-13)$$

and similarly

$$T(x)\mathbf{J}(\tilde{y}) \underset{x \sim y}{=} \frac{\mathbf{J}(\tilde{y})}{(x - y)^2} + \frac{\partial \mathbf{J}(\tilde{y})}{x - y} + \mathcal{O}(1) \quad (1-14)$$

to be again understood as identities holding when inserted into correlation functions. The coefficient 1 in front of the second order pole in the last expression tells us that the current \mathbf{J} has spin 1.

Remark 1.4 *One might be afraid that such a decomposition for the stress-energy tensor would create singularities of infinite order in some operator product expansion appearing in the theory but a requirement of the vertex operator algebra formalism is that any admissible field V_α should be annihilated by all high enough modes of T , see [28] for details. In particular, define an admissible ground state as a vector $|0\rangle \in \mathcal{A}$ in the considered representation satisfying the so-called Virasoro constraints*

$$\forall n \geq -1, \quad L_n|0\rangle = 0 \quad (1-15)$$

In particular, if we were to assume that $L_n^\dagger = L_{-n}$, then the Virasoro constraints would yield that the expected value of the stress-energy tensor vanishes

$$\langle 0|T(x)|0\rangle = 0 \quad (1-16)$$

namely that we have conformal symmetry in this ground state at the quantum level.

The next axiom is at the heart of the method we adopt to study conformal field theories. As was mentioned in the introduction, a path integral formulation of the problem with a Lagrangian allows for the derivation of Schwinger-Dyson equations. Their counterparts in this non-perturbative definition of conformal field theories are the following conformal Ward identities.

- *Axiom 3 : Conformal Ward identities.*

For any generic $\tilde{x}_1, \dots, \tilde{x}_n \in \tilde{\Sigma}$, any holomorphic one-form η on Σ and small circle \mathcal{C}_t surrounding a point $t \in \Sigma - \{z_1, \dots, z_M\}$,

$$\oint_{\mathcal{C}_t} \eta(x) \langle\langle T(x) \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \rangle\rangle = \oint_{\mathcal{C}_t} \eta(x) \left\langle T(x) \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \prod_{j=1}^M V_{\alpha_j}(z_j) \right\rangle \quad (1-17)$$

$$= 0 \quad (1-18)$$

This is (again !) an axiom prescribing some analytic conditions for the functions of interest. It can be rewritten by decomposing the integration homology class on a basis of cycles in $\Sigma - \{z_1, \dots, z_M\}$. We will be applying similar ideas for the generating series of generators of $\mathcal{W}(\mathfrak{g})$ for which the conformal Ward identities together with the operator products expansions yield the so-called loop equations.

The two last axioms deal with how one can reconstruct the full theory from its chiral and anti-chiral parts. We will not be needing them in the context of this work but we still state them for completeness.

- *Axiom 4 : Single-valuedness.*

The M -point correlation functions have no monodromy around cycles in the moduli space of configurations of M distinct points on the Riemann surface Σ .

- *Axiom 5 : Fusion and crossing symmetries.*

The decomposition of the real correlation functions in terms of the conformal blocks requires in particular a choice of channel, a unicellular trivalent graph, on the base Riemann surface Σ and different choices of such channels should lead to the same correlation function after reconstruction. This is often referred to as the *associativity* of the operator product expansions.

There is no general proof that all these axioms are actually compatible. We will therefore work at the formal level, assuming these axioms to be compatible and satisfied, to define the algebra $\mathcal{W}(\mathfrak{g})$ and the so-called insertions of W -generators in these chiral correlation functions with currents. In turn, this will yield $\mathcal{W}(\mathfrak{g})$ -symmetric conformal Ward identities. In the second part we will define the associated quantum geometry through the quantum spectral curve. This will turn out to be the initial data needed to run the topological recursion of [43] in this context and we will show that it solves the $\mathcal{W}(\mathfrak{g})$ -symmetric conformal Ward identities perturbatively in the topological regime.

2 **W-algebras and associated conformal field theories**

2.1 **From Virasoro to W-algebras**

A conformal field theory is a quantum field theory defined on a Riemann surface Σ and endowed with an action of the product $\mathcal{A} \times \mathcal{A}'$ of two extensions $Vir \subset \mathcal{A}$, $Vir \subset \mathcal{A}'$ of the Virasoro algebra (they need not be the same).

Let us stress at this point that these two extensions \mathcal{A} and \mathcal{A}' act respectively upon the holomorphic and the anti-holomorphic dependence of the observables defined on the Riemann surface. We will here only be interested in the chiral theory, that is in the action of \mathcal{A} and in the meromorphic properties of the soon to be defined chiral correlation functions.

We will be interested particularly in the extension $Vir \subset \mathcal{A} \stackrel{def}{=} \mathcal{W}(\mathfrak{g})$ defined from a finite dimensional Lie algebra \mathfrak{g} , using a higher rank generalization of the Sugawara construction [71], namely the quantum Miura trans-

form, and defining generating functions whose modes generate $\mathcal{W}(\mathfrak{g})$. Let us just mention that another way to define the algebra $\mathcal{W}_c(\mathfrak{g})$ is by canonical quantization of the Poisson algebra underlying the Drinfeld-Sokolov hierarchy associated to \mathfrak{g} [46].

The idea behind W-algebras is that they allow for a better encoding of some representations of Vir . Indeed, there are spaces representing both the W-algebra and the Virasoro algebra that decompose as an infinite direct sums of irreducible representations of Vir but as a finite direct sums of irreducible representations of $\mathcal{W}(\mathfrak{g})$. In particular, the operator product expansions they satisfy should be expressible in terms of these generators only. We will see two different situations in which this is possible but we will not get any further in studying the representation theory of W-algebras and refer the reader the [31], [32].

We will start by defining the Casimir algebra $\mathcal{W}_0(\mathfrak{g})$ with integer central charge $c = rk\mathfrak{g}$ and consider $\mathcal{W}_0(\mathfrak{g})$ -symmetric conformal field theory in the presence of currents in the affine Lie algebra at level 1 denoted $\widehat{\mathfrak{g}}_1$. We will show how to compute the correlation functions with insertions of such currents in terms of an associated Fuchsian differential system will then argue that it can be reconstructed by the topological recursion in some situations. This corresponds to the non-generic situation of a more general construction of the algebra $\mathcal{W}(\mathfrak{g})$, where a parameter called the background charge and denoted Q , is sent to 0.

We will then consider the generic situation $Q \neq 0$. In this case, the definition of the W-algebra generators involves non-commutative geometry and the algebra does not close if we do not restrict the current \mathbf{J} to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. The choice of level then becomes irrelevant, as it can be reabsorbed in the definition of \mathbf{J} , and $\mathcal{W}(\mathfrak{g})$ will then appear as a subalgebra $\mathcal{W}(\mathfrak{g}) \subset \overline{\mathcal{U}(\widehat{\mathfrak{h}})}$ of a completion of the universal enveloping algebra of the affine commutative Kac-Moody Lie algebra $\widehat{\mathfrak{h}}$ which is defined from with the same root system as \mathfrak{g} .

The background charge plays the role of a quantization parameter and noticing that the W-algebra for generic values of Q reduces to the Casimir algebra in the limit $Q \rightarrow 0$ will allow for the interpretation of $\mathcal{W}(\mathfrak{g})$ -symmetric conformal field theory as the quantization of the Fuchsian

system.

Before defining these algebras, let us review a few generalities on W-algebras.

2.2 Operator product expansions

Let us now make, once and for all, the choice of a faithful d -dimensional representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}_d(\mathbb{C})$ of the considered Lie algebra. Similarly to the case of the Virasoro algebra, introducing the rank $r = \text{rk } \mathfrak{g}$, the soon to be defined generators of $\mathcal{W}(\mathfrak{g})$, denoted $\{\mathbf{W}_n^{(d_p)}\}_{\substack{n \in \mathbb{Z} \\ 1 \leq p \leq r}}$, fit for a given $p \in \{1, \dots, r\}$, into a generating function defined around a base point $x_0 \in \Sigma$ by

$$\mathbf{W}^{(d_p)}(\tilde{x}) = \sum_{n \in \mathbb{Z}} \frac{\mathbf{W}_n^{(d_p)}}{(x - x_0)^{n+d_p}} \quad \text{for } p \in \{1, \dots, r\} \quad (2-1)$$

where the d_p 's are integer indices defined as follows : since \mathfrak{h} is a commutative Lie algebra, we have an isomorphism $\mathcal{U}(\mathfrak{h}^*) \simeq \mathbb{C}[\mathfrak{h}]$ and moreover, by a theorem of Chevalley, the subspace of this last ring invariant under the action of the Weyl group is actually a polynomial ring $\mathbb{C}[\mathfrak{h}]^{\text{inv}} \simeq \mathbb{C}[\sigma_1, \dots, \sigma_r]$ where for any $p \in \{1, \dots, r\}$, $d_p \in \mathbb{N}^*$ is then defined as the degree of the invariant polynomial σ_p .

We will assume the algebra $\mathcal{W}(\mathfrak{g})$ to be an extension of the Virasoro algebra and will define its generators, denoted $\mathbf{W}^{(k)}$, for any $k \in \{1, \dots, d\}$ although some of them might actually vanish. Let us mention the classical rank $r = d - 1$ example of $\mathfrak{g} = \mathfrak{sl}_d(\mathbb{C})$ in its fundamental representation for which $\mathbf{W}^{(1)} = 0$. For $\mathfrak{sl}_2(\mathbb{C})$ we then have $r = 1$ and $d_1 = 2$.

Following the introduction of [32], the corresponding operator product expansions can be presented schematically as

$$\begin{aligned}
\mathbf{W}^{(d_p)}(\tilde{x})\mathbf{W}^{(d_q)}(\tilde{y}) &\stackrel{x\sim y}{=} \frac{g^{p,q}}{(x-y)^{d_p+d_q}} \\
&+ \sum_{s=1}^r f_{(1),s}^{p,q} \frac{\mathbf{W}^{(d_s)}(\tilde{y}) + g_s^{p,q} \partial \mathbf{W}^{(d_s)}(\tilde{y}) + \dots}{(x_y)^{d_p+d_q-d_s}} \\
&+ \sum_{s,t=1}^r f_{(2),s,t}^{p,q} \frac{:\mathbf{W}^{(d_s)}(\tilde{y})\mathbf{W}^{(d_t)}(\tilde{y}): + \dots}{(x-y)^{d_p+d_q-d_s-d_t}} + \dots
\end{aligned} \tag{2-2}$$

Moreover we assume these generating functions to be primary fields with respect to the Virasoro algebra, in turn they are required to satisfy the operator product expansions

$$T(x)\mathbf{W}^{(d_p)}(\tilde{y}) \stackrel{x\sim y}{=} d_p \frac{\mathbf{W}^{(d_p)}(\tilde{y})}{(x-y)^2} + \frac{\partial \mathbf{W}^{(d_p)}(\tilde{y})}{x-y} + \mathcal{O}(1) \tag{2-3}$$

In particular this implies the commutation relations

$$[L_n, \mathbf{W}_m^{(d_p)}] = [(d_p - 1)n - m] \mathbf{W}_{n+m}^{(d_p)} \tag{2-4}$$

Let us mention that the $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ case was investigated in [74] and the algebra implied by the corresponding operator product expansions is

$$[L_n, \mathbf{W}_m^{(3)}] = (2n - m) \mathbf{W}_{n+m}^{(3)} \tag{2-5}$$

where we identified the modes of $\mathbf{W}^{(2)}$ with some Virasoro generators and

$$\begin{aligned}
[\mathbf{W}_m^{(3)}, \mathbf{W}_m^{(3)}] &= (n-m) \left[\frac{1}{15} (n+m+2)(n+m+3) - \frac{1}{6} (n+2)(m+2) \right] L_{n+m} \\
&+ \frac{c}{3 \cdot 5!} n(n^2-1)(n^2-4) \delta_{n+m,0} + \frac{16}{22+c} (n-m) \Lambda_{n+m}
\end{aligned} \tag{2-6}$$

where we introduced the symbols

$$\Lambda_n \stackrel{def}{=} \sum_{k \in \mathbb{Z}} :L_k L_{n-k}: + \frac{1}{5} \nu_n L_n \quad (2-7)$$

$$\text{with } \nu_{2l} \stackrel{def}{=} (1+l)(1-l) \quad (2-8)$$

$$\text{and } \nu_{2l+1} \stackrel{def}{=} (2+l)(1-l) \quad (2-9)$$

2.3 Casimir algebras

The Casimir algebra we shall define in this paragraph could in principle be defined as the $Q \rightarrow 0$ limit of the general construction we will present afterwards but it has another definition which is purely algebraic and we shall present it in this way first. Assume $b = i$ such that $Q = 0$.

Definition 2.1 Casimir algebra

The Casimir algebra, denoted $\mathcal{W}_0(\mathfrak{g})$, is the extension of the Virasoro algebra generated by the modes of fields $\{\mathbf{W}_0^{(d_p)}\}_{1 \leq p \leq r}$ defined through the formula

$$\mathbf{W}_0^{(d_p)} \stackrel{def}{=} : \sigma_p(\mathbf{J}) : \quad \text{for any } p \in \{1, \dots, r\}, \quad (2-10)$$

with the algebraic basis $(\sigma_1, \dots, \sigma_r)$ of the Ad-invariant polynomials $\mathbb{C}[\mathfrak{g}]^{\text{Ad}} \simeq \mathbb{C}[\mathfrak{h}]^{\text{inv}}$ with respective degrees $\deg \sigma_p = d_p$ that was introduced before, and again the normal ordering prescription has been thoroughly used to evaluate products of components of the chiral spin-one field at coinciding points. Notice that this definition is purely algebraic.

The W-algebra $\mathcal{W}_0(\mathfrak{g})$ embeds in a completion of $\mathcal{U}(\widehat{\mathfrak{g}}_\kappa)$ and the Virasoro algebra in particular since after a rescaling, $T^S \stackrel{def}{=} \frac{1}{2(\kappa + h^\vee)} \mathbf{W}^{(2)}$, called the Segal-Sugawara vector, satisfies the same operator product expansions as the ones of the stress-energy tensor of the Virasoro algebra-symmetric conformal field theory but with the central charge

$$c(\mathfrak{g}, \kappa) = \frac{\kappa \dim \mathfrak{g}}{\kappa + h^\vee} \quad (2-11)$$

where recall that the level $\kappa \in \mathbb{C}$ is the complex number by which the central extension of the Lie algebra \mathfrak{g} is chosen to act trivially and h^\vee is the dual coxeter number. $\kappa = -h^\vee$ is called the critical level [14]. In the language of vertex operator algebras, $\mathcal{W}(\mathfrak{g})$ is said to be endowed with the conformal structure coming from the conformal vector $\mathbf{W}^{(2)}$.

We are now ready to state a theorem that follows straight from the axiomatics of the Casimir algebra-symmetric conformal field theory.

Theorem 2.2 *Casimir CFT from Fuchsian systems [B.-Eynard-Ribault]*

Consider a Fuchsian differential system $(P \rightarrow \Sigma, \nabla)$ where ∇ is a meromorphic connection in a principal G -bundle P over the base curve Σ . Choose a solution Ψ to the flat section equation $\nabla \cdot \Psi = 0$. Let ∇_0 be a reference holomorphic connection and assume that $\Phi \stackrel{\text{def}}{=} \nabla_0 - \nabla$ has a simple pole at each z_j , $j \in \{1, \dots, M\}$, with corresponding residue $\Phi_j \stackrel{\text{def}}{=} \text{Res}_{z_j} \Phi$ such that

$$i\alpha_j(E) = \left\langle \Phi_j, \pi_j \left(\text{Ad}_{C_j} E \right) \right\rangle \quad (2-12)$$

where we introduced the projector $\pi_j : \mathfrak{g} \rightarrow \mathfrak{h}_j$ on the comutant of Φ_j . Then the sequence of correlators $\{\widehat{W}_n\}_{n \in \mathbb{N}^}$ defines a chiral Casimir algebra-symmetric conformal field theory at level $\kappa = 1$ in the sense that we can make the identification*

$$\widehat{W}_n(X_1, \dots, X_n) = \frac{\left\langle \mathbf{J}(X_1) \cdots \mathbf{J}(X_n) \prod_{j=1}^M V_{\alpha_j}(z_j) \right\rangle}{\left\langle \prod_{j=1}^M V_{\alpha_j}(z_j) \right\rangle} \quad (2-13)$$

for all $n \in \mathbb{N}^$ and generic arguments $X_1, \dots, X_n \in \widehat{\Sigma}_\Psi$.*

proof:

The proof consist in checking that the correlators W_n satisfy all the desired axioms from conformal field theory and indeed this is true, the operator product expansions can be checked from the definitions of the correlators and the Ward identities are identical to the loop equations. In particular, the singularity profile of W_2 imposes that the level should be $\kappa = 1$ ■

As a consequence, both the non-perturbative and perturbative studies of Fuchsian differential systems of the previous chapter can be applied here and to Casimir algebra-symmetric conformal field theories. In particular,

Conjecture 2.3 *τ -function and conformal blocks*

For any choice of Lagrangian submanifold $\mathcal{L} \subset \widehat{\mathcal{H}}_1$, $\mathfrak{T}_{\mathcal{L}}$ is a linear combination of conformal blocks of $\mathcal{W}_0(\mathfrak{g})$ conformal field theory.

The Casimir algebra is the special case $\mathcal{W}(\mathfrak{g})$ with $\mathfrak{c} = \mathfrak{rkg}$ of the construction we shall now describe.

2.4 Generic background charge Q

Let us now return to the generic situation where $Q = b + b^{-1} \neq 0$. As mentioned earlier, for generic values of Q , the W-algebra with non-abelian currents does not close. It does however when we restrict ourselves to abelian currents defined by the following.

Definition 2.4 *W-algebra generators*

Let $H_1, \dots, H_d \in \mathfrak{h}$ be a set of Cartan elements dual to a set of highest weights of the representation ρ . The generating functions of generators of the algebra $\mathcal{W}(\mathfrak{g})$ are expressed through the quantum Miura transform

$$\widehat{\mathcal{E}} = \sum_{k=0}^d (-1)^k \mathbf{W}^{(k)} \widehat{y}^{d-k} \underset{def}{=} : (\widehat{y} - \mathbf{J}_1) \cdots (\widehat{y} - \mathbf{J}_d) : \quad (2-14)$$

where $\widehat{y} \underset{def}{=} Q\partial$ and the subscripts $1, \dots, d$ of $\mathbf{J}_1, \dots, \mathbf{J}_d$ denote the evaluation of highest weights by the current as $\mathbf{J}_i \underset{def}{=} \mathbf{J}(H_i)$. The non-commutative prescription for evaluating these products at coinciding points has been used. These generating functions could in principle be multivalued on Σ .

This definition is to be understood as the identification of the coefficients of the polynomial expression in \widehat{y} obtained by commuting all the derivative symbols to the right.

Example 2.5

$$\mathbf{W}^{(1)}(\tilde{x}) = \sum_{i=1}^d \mathbf{J}(\tilde{x} \cdot H_i) \quad (2-15)$$

$$\mathbf{W}^{(2)}(\tilde{x}) = \sum_{1 \leq i < j \leq d} : \mathbf{J}(\tilde{x} \cdot H_i) \mathbf{J}(\tilde{x} \cdot H_j) : - Q \sum_{i=2}^d (i-1) \partial \mathbf{J}(\tilde{x} \cdot H_i) \quad (2-16)$$

Lemma 2.6 For any $k \in \{1, \dots, d\}$, the k^{th} generator $\mathbf{W}^{(k)}$ of $\mathcal{W}(\mathfrak{g})$ is equal to

$$\sum_{p=1}^k (-1)^{k-p} Q^{k-p} \sum_{\substack{1 \leq i_1 < \dots < i_p \leq d \\ k \leq i_p}} \sum_{\substack{\forall l \in \llbracket 1, p \rrbracket \\ 0 \leq q_l \leq i_l - i_{l-1} - 1 \\ p + \sum_{l=1}^p q_l = k}} \prod_{l=1}^p \binom{i_l - i_{l-1} - 1}{q_l} : \partial^{q_1} (\mathbf{J}_{i_1} \dots \partial^{q_p} \mathbf{J}_{i_p}) : \quad (2-17)$$

proof:

The proof consists in using non-commutative algebra in the ring of differential operators \mathcal{D}_Σ over the base curve to commute all $Q\partial$ symbols to the right before identifying the coefficients of the differential operators. To do so we first identify holomorphic functions $f \in \mathcal{O}_\Sigma$ with the degree 0 differential operators $f \cdot \in \mathcal{D}_\Sigma$ of multiplication by f on the left. For any function $f \in \mathcal{O}_\Sigma$ we then have the commutation relation $[Q\partial, f] = Q(\partial f)$ and it recursively yields the non-commutative version of Leibniz formula

$$(Q\partial)^p f = Q^p \sum_{q=0}^p \binom{p}{q} (\partial^q f) \partial^{p-q} \quad (2-18)$$

where the equality takes place in \mathcal{D}_Σ . It is then a straightforward computation to derive the wanted result. ■

Example 2.7 Straightforward computation for example yields, for $d = 2, 3$,

$$\bullet \quad \widehat{\mathcal{E}}_{d=2} = (Q\partial)^2 - [\mathbf{J}_1 + \mathbf{J}_2](Q\partial) + : \mathbf{J}_1 \mathbf{J}_2 : - Q(\partial \mathbf{J}_2) \quad (2-19)$$

$$\begin{aligned}
\bullet \quad \widehat{\mathcal{E}}_{d=3} &= (Q\partial)^3 - [\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3](Q\partial)^2 \\
&+ [:\mathbf{J}_1\mathbf{J}_2 + \mathbf{J}_2\mathbf{J}_3 + \mathbf{J}_1\mathbf{J}_3: - Q\partial\mathbf{J}_2 - 2Q\partial\mathbf{J}_3](Q\partial) \\
&+ Q^2(\partial^2\mathbf{J}_3) + Q : [\mathbf{J}_1 + \mathbf{J}_2] (\partial\mathbf{J}_3) : + Q : (\partial\mathbf{J}_2)\mathbf{J}_3 : \quad (2-20)
\end{aligned}$$

and for any $d \in \mathbb{N}^*$, for $k = 1, 2, 3$,

$$\bullet \quad \mathbf{W}^{(1)} = \sum_{i=1}^d \mathbf{J}_i \quad (2-21)$$

$$\bullet \quad \mathbf{W}^{(2)} = \sum_{1 \leq i < j \leq d} : \mathbf{J}_i \mathbf{J}_j : - Q \sum_{i=2}^d (i-1) \partial \mathbf{J}_i \quad (2-22)$$

$$\begin{aligned}
\bullet \quad \mathbf{W}^{(3)} &= \sum_{1 \leq i < j < k \leq d} : \mathbf{J}_i \mathbf{J}_j \mathbf{J}_k : \\
&- Q \sum_{\substack{1 \leq i < j \leq d \\ 3 \leq j}} [(j-i-1) : \mathbf{J}_i \partial \mathbf{J}_j : + (i-1) \partial (: \mathbf{J}_i \mathbf{J}_j :)] \\
&+ Q^2 \sum_{i=3}^d \binom{i-1}{2} \partial^2 \mathbf{J}_i \quad (2-23)
\end{aligned}$$

For $l \in \{1, \dots, r\}$, $\mathbf{W}^{(d_l)}$ therefore involves at most terms of degree d_l as differential polynomials in d copies of a chosen so-called ‘‘chiral \mathfrak{h}^* -valued spin-one field’’ $\mathbf{J}(\tilde{x})$ as described before. We require as stated in *Axiom 2* that it satisfies

$$\mathbf{J}(\tilde{x} \cdot E) \mathbf{J}(\tilde{y} \cdot F) \underset{x \sim y}{=} - \frac{(E, F)}{(x-y)^2} + : \mathbf{J}(\tilde{x} \cdot E) \mathbf{J}(\tilde{y} \cdot F) :_{x=y} + \mathcal{O}(x-y) \quad (2-24)$$

for Cartan elements $E, F \in \mathfrak{h}$, where (\cdot, \cdot) still denotes the corresponding minimal invariant bilinear form and we redefined the current by a factor of $\kappa^{-1/2}$.

2.5 Ward identities

We now generalize *Axiom 3* to the algebra $\mathcal{W}(\mathfrak{g})$ (not necessarily a Lie algebra) defined by the operator coefficients of the expansions of the generators $\mathbf{W}^{(k)}$, $k \in \{1, \dots, d\}$, around a base point $x_0 \in \Sigma$. We then get that the chiral spin-one current \mathbf{J} should be chosen such that it satisfies

Definition 2.8 *The Ward identities of this $\mathcal{W}(\mathfrak{g})$ -symmetric conformal field theory is the set of relations defined for any $k \in \{1, \dots, d\}$, any generic $\tilde{x}_1, \dots, \tilde{x}_n \in \tilde{\Sigma}$, any holomorphic one-form η on Σ and small circle \mathcal{C}_t surrounding a point $t \in \Sigma - \{z_1, \dots, z_M\}$ by*

$$\oint_{\mathcal{C}_t} \eta(x) \langle\langle \mathbf{W}^{(k)}(\tilde{x}) \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \rangle\rangle = \oint_{\mathcal{C}_t} \eta(x) \left\langle \mathbf{W}^{(k)}(\tilde{x}) \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \prod_{j=1}^M V_{\alpha_j}(z_j) \right\rangle \quad (2-25)$$

$$= 0 \quad (2-26)$$

This definition yields that for an admissible chiral current \mathbf{J} , the insertion $\langle\langle \mathbf{W}^{(k)}(\tilde{x}) \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \rangle\rangle$ of any of the fields $\mathbf{W}^{(k)}$, $k \in \{1, d\}$, should be uniquely valued as a holomorphic function of $x \in \Sigma - \{z_1, \dots, z_M\}$. We can therefore drop the upperscript in \tilde{x} and simply write $\mathbf{W}^{(k)}(x)$ when evaluating the insertion of such a generator. Notice that they are trivially satisfied when $k \neq d_l$ for all $l \in \{1, \dots, r\}$ since we then have $\mathbf{W}^{(k)} = 0$.

Replacing the previously computed expression for $\mathbf{W}^{(k)}$ in terms of the current \mathbf{J} in $\langle\langle \mathbf{W}^{(k)}(x) \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \rangle\rangle$ yields that

Proposition 2.9 *Ward identities as loop equations*

$$\begin{aligned} & \langle\langle \mathbf{W}^{(k)}(x) \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \rangle\rangle \\ &= \sum_{p=1}^k (-1)^{k-p} Q^{k-p} \sum_{\substack{1 \leq i_1 < \dots < i_p \leq d \\ k \leq i_p}} \sum_{\substack{\forall l \in [1, p] \\ 0 \leq q_l \leq i_l - i_{l-1} - 1 \\ p + \sum_{l=1}^p q_l = k}} \left(\prod_{l=1}^p \binom{i_l - i_{l-1} - 1}{q_l} \right) \\ & \quad \times \langle\langle : \partial^{q_1} (\mathbf{J}_{i_1} \dots \partial^{q_p} \mathbf{J}_{i_p}) (\tilde{x}) : \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \rangle\rangle \end{aligned} \quad (2-27)$$

is a holomorphic function of $x \in \Sigma - \{z_1, \dots, z_M\}$.

For a generic value of Q and specializing to the cases $k = 1, 2$, the expressions for $\mathbf{W}^{(1)}$ and $\mathbf{W}^{(2)}$ yield that

Corollary 2.10

$$\sum_{i=1}^d \langle \langle \mathbf{J}_i(\tilde{x}) \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \rangle \rangle \quad \text{and} \quad (2-28)$$

$$\sum_{1 \leq i < j \leq d} \langle \langle \mathbf{J}_i \mathbf{J}_j(\tilde{x}) : \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \rangle \rangle - Q \sum_{i=2}^d (i-1) \partial_x \langle \langle \mathbf{J}_i(\tilde{x}) \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \rangle \rangle \quad (2-29)$$

are holomorphic functions of $x \in \Sigma - \{z_1, \dots, z_M\}$.

2.6 Classical limit and quantization

The definition of the generators of $\mathcal{W}(\mathfrak{g})$ involved identifying the coefficients of two differential operators. We get the following classical limit

Theorem 2.11 *Quantization of Fuchsian differential systems [B.-Eynard]*

$$\sum_{k=0}^d (-1)^k \mathbf{W}_0^{(k)} y^{r-k} = \text{Symb} \left(\sum_{k=0}^d (-1)^k \mathbf{W}^{(k)} \hat{y}^{r-k} \right) \quad (2-30)$$

where the symbol of a differential operator $P(x, \hat{y}) \in \mathbb{C}(x)[\hat{y}]$ is defined as

$$\text{Symb}(P(x, \hat{y})) = \lim_{\text{def } Q \rightarrow 0} (e^{-xy/Q} P(x, \hat{y}) \cdot e^{xy/Q}) \quad (2-31)$$

and therefore the $\mathcal{W}(\mathfrak{g})$ -symmetric conformal field theory quantizes the Fuchsian differential system corresponding to this classical limit $\mathcal{W}_0(\mathfrak{g})$.

More explicitly, putting $Q = 0$ in the Ward identities, the only remaining term of the expression of last proposition is for $p = k$ and is equal to

$$\langle \langle \mathbf{W}_0^{(k)}(x) \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \rangle \rangle = \sum_{1 \leq i_1 < \cdots < i_k \leq r} \langle \langle (\mathbf{J}_{i_1} \cdots \mathbf{J}_{i_k})(\tilde{x}) : \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \rangle \rangle \quad (2-32)$$

that is the sum over all possible ways to insert k of the d copies of the chiral current \mathbf{J} . This is exactly what one would obtain by writing the Ward identities for a Casimir algebra-symmetric conformal field theory and we can read the loop equations on the right hand side.

3 Quantum geometry

We shall now define the quantum geometry associated to the $\mathcal{W}(\mathfrak{g})$ -symmetric conformal field theory we are considering [5]. It first consists in a definition of the quantum spectral curve and quantum complex structure, encoded in the 2-points function, when we assume the existence of a topological regime. We will then introduce the relevant topological recursion and show one of the main results of the chapter, namely that it solves the Ward identities.

We will for simplicity restrict ourselves to the case where the Riemann surface is the Riemann sphere $\Sigma = \mathbb{CP}^1$, although most of the reasoning is local and could be generalized to an arbitrary Riemann surface.

3.1 Topological regime and quantum spectral curve

This is the main assumption of this study. Let us suppose that all the functions appearing in our construction are now formal series in an expansion parameter $\varepsilon \rightarrow 0$.

This could for instance be the limit $b \rightarrow i0$ (0 along the imaginary axis) keeping $Q = b + b^{-1}$ fixed to a generic value and taking b and the expansion parameter ε to be proportionnal one to the other $\varepsilon \sim b$.

Let us now assume that the chiral correlation functions with current insertions admit $\varepsilon \rightarrow 0$ asymptotic expansions of the form

$$\langle\langle \mathbf{J}(\tilde{x}_1 \cdot H_{i_1}) \cdots \mathbf{J}(\tilde{x}_n \cdot H_{i_n}) \rangle\rangle \stackrel{\text{def}}{=} \sum_{g=0}^{\infty} \varepsilon^{2g-2+n} \left(W_{g,n}(\tilde{x}_1^{i_1}, \dots, \tilde{x}_n^{i_n}) - \delta_{n,2} \delta_{g,0} \frac{(H_{i_1}, H_{i_2})}{(x_1 - x_2)^2} \right) \quad (3-1)$$

for all $n \in \mathbb{N}^*$ for which $W_{g,n}$, with $g \in \mathbb{N}$, is a multi-valued meromorphic function on n copies of the Riemann sphere.

As a consequence, the differential operators obtained by inserting $\widehat{\mathcal{E}}(x)$ into a chiral correlation function with current insertions $\langle\langle \mathbf{J}(\tilde{x}_1) \cdots \mathbf{J}(\tilde{x}_n) \rangle\rangle$ also admit asymptotic expansions of a similar form

$$\langle\langle \widehat{\mathcal{E}}(x) \mathbf{J}(\tilde{x}_1 \cdot H_{i_1}) \cdots \mathbf{J}(\tilde{x}_n \cdot H_{i_n}) \rangle\rangle \stackrel{def}{=} \sum_{g=0}^{\infty} \varepsilon^{2g-1+n} \mathcal{E}_n^{(g)}(x; \tilde{x}_1^{i_1}, \dots, \tilde{x}_n^{i_n}) \quad (3-2)$$

$$\begin{aligned} &\stackrel{def}{=} \sum_{k=0}^d \sum_{g=0}^{\infty} (-1)^{d-k} \varepsilon^{2g-1+n} \\ &\quad \times P_{n,d-k}^{(g)}(x; \tilde{x}_1^{i_1}, \dots, \tilde{x}_n^{i_n}) \widehat{y}^k \end{aligned} \quad (3-3)$$

Definition 3.1 *The differential operator $\mathcal{E} \stackrel{def}{=} \mathcal{E}_0^{(0)}$ is called the quantum spectral curve.*

The Ward identities extended to these formal ε -expansions imply that the operator $\mathcal{E}_n^{(g)}(x; \tilde{x}_1^{i_1}, \dots, \tilde{x}_n^{i_n})$ is, for all $g, n \in \mathbb{N}$, $n \neq 0$, a meromorphic function of the variable $x \in \mathbb{C}$ with possible singularities at $x = \infty$, $x = x_i$ for some $i \in \{1, \dots, d\}$ or $x = z_j$ for some $j \in \{1, \dots, M\}$ and nowhere else.

This definition can be interpreted as exhibiting the quantization of a classical spectral curve, perturbatively this time. Indeed, define a function E of the variables $x, y \in \mathbb{C}$ by the generic assignment

$$E(x, y) \stackrel{def}{=} \text{Symb}(\mathcal{E}_0^{(0)}(x)) \quad (3-4)$$

The Riemann surface defined by the equation $E(x, y) = 0$, the character variety of the quantum spectral curve, embeds in \mathbb{C}^2 and defines a $d : 1$ cover of the complex plane by a meromorphic projection $x : \mathcal{S} \rightarrow \mathbb{C}$ called the classical spectral curve of the $\mathcal{W}(\mathfrak{g})$ -symmetric conformal field theory.

Recall that to any classical integrable systems presented in Lax form can be associated its corresponding spectral curve, a meromorphic covering of complex curves, and that this is the starting point to run the topological recursion of [43], [30] in order for example to compute recursively the expansion coefficients of generating functions of derivatives of the τ -function. We wish to upgrade these techniques to the non-commutative, or quantum, case using this operator formalism arising from conformal field theory.

3.2 Fermionic description and notion of sheets

A dual point of view to describe the quantum spectral curve is through the solutions of the linear differential equation it defines. Let us therefore consider a set of d independent functions ψ_j , for $j \in \{1, \dots, d\}$, satisfying

$$\psi_j \cdot \mathcal{E} = 0 \quad (3-5)$$

where the differential operator acts from the right.

Theorem 3.2 *The quantum spectral curve decomposes as*

$$\mathcal{E} = (\hat{y} - Y_1(x)) \cdots (\hat{y} - Y_d(x)) \quad (3-6)$$

where for any $k \in \{1, \dots, d\}$,

$$Y_k \stackrel{\text{def}}{=} Q\partial \left(\ln \frac{D_{k-1}}{D_k} \right) = \frac{Q\partial D_{k-1}}{D_{k-1}} - \frac{Q\partial D_k}{D_k} \quad (3-7)$$

with

$$D_k \stackrel{\text{def}}{=} \text{Det}_{0 \leq i, j \leq k-1} ((-Q\partial)^i \psi_{j+1}) \quad (3-8)$$

and the convention that $D_0 \stackrel{\text{def}}{=} 1$.

proof:

This is a straightforward corollary of the following

Lemma 3.3 *For any $k \in \{1, \dots, d\}$,*

$$(\hat{y} - Y_1(x)) \cdots (\hat{y} - Y_k(x)) = \text{Det}_{k+1} \begin{pmatrix} 1 & \psi_1(x) & \cdots & \psi_k(x) \\ \hat{y} & (-Q\partial)\psi_1(x) & \cdots & (-Q\partial)\psi_k(x) \\ \hat{y}^2 & (-Q\partial)^2\psi_1(x) & \cdots & (-Q\partial)^2\psi_k(x) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{y}^k & (-Q\partial)^k\psi_1(x) & \cdots & (-Q\partial)^k\psi_k(x) \end{pmatrix} \frac{1}{D_k(x)} \quad (3-9)$$

and in particular for any $j \in \{1, \dots, d\}$,

$$\psi_j \cdot (\hat{y} - Y_0(x)) \cdots (\hat{y} - Y_j(x)) = 0 \quad (3-10)$$

Indeed, since then the differential operator of the right hand side and the quantum spectral curve have same degree and space of solutions they are thus equal. ■

Definition 3.4 *Master loop equation*

Define the auxiliary operator

$$\mathcal{U} \stackrel{\text{def}}{=} (\hat{y} - Y_2(x)) \cdots (\hat{y} - Y_d(x)) \quad (3-11)$$

such that the following identity holds

$$(\hat{y} - Y_1(x)) \mathcal{U} = \mathcal{E} \quad (3-12)$$

and is called the master loop equation.

Remark 3.5 *This terminology comes from the similarities existing between the quantum geometry of the $\mathcal{W}(\mathfrak{g})$ -symmetric conformal field theory we are studying and that of the $\beta(\mathfrak{g})$ -deformed two-matrix model, a generalization of that of [34] and [18]. These similarities hide a full correspondence between the theories that is being described and should appear soon [6].*

Remark 3.6 *The geometrical interpretation of this factorization is that there exists a function Y that is actually multivalued on the punctured sphere in such a way that $Y(\overset{i}{x}) = Y_i(x)$ is its value at a generic point $x \in \Sigma$ taken in sheet $i \in \{1, \dots, d\}$. The sheet labelled by $i = 1$ is often called the physical sheet. The quantum sheets therefore label solutions of the quantum spectral curve and in this sense, the function Y is defined on the quantum spectral curve.*

This allows to define the meromorphic multivalued potential V of the theory defined by the deformed multi-Penner type formula

$$V(x) \stackrel{\text{def}}{=} \sum_{k=0}^{d+1} t_k \frac{x^{k+1}}{k+1} + \sum_{j=1}^M (\alpha_j, H_i) \ln(x - z_j) \quad (3-13)$$

for generic values of the arguments. The parameters t_k , $k \in \{0, \dots, d+1\}$, are chosen such that the following identity holds

$$W_{0,1}(x) + Y(x) = \partial V(x) = \sum_{k=0}^{d+1} t_k x^k + \sum_{j=1}^M \frac{(\alpha_j, H_i)}{x - z_j} \quad (3-14)$$

From a quantum field theory perspective, this potential is interpreted as the one to which are submitted Toda fields φ_α related locally to our currents by $Q\partial\varphi_\alpha \sim \mathbf{J}_\alpha$, that is the sum of a polynomial interaction with the vacuum and a Coulomb type potential with the fields inserted at the z_j 's. Note that the currents cannot globally be such derivatives since they have non-trivial monodromy in general.

Although the multi-penner potential is what we expect to be needed to reconstruct perturbatively correlation functions of Toda quantum field theory with the right deformation properties, we do not need this precise form for the present construction.

3.3 2-points function

Recall that the 2-points function has the asymptotic expansion

$$\langle\langle \mathbf{J}(\tilde{x}_1 \cdot H_{i_1}) \mathbf{J}(\tilde{x}_2 \cdot H_{i_2}) \rangle\rangle = \sum_{g=0}^{\infty} \varepsilon^{2g-2+n} \left(W_{g,2}(x_1^{i_1}, x_2^{i_2}) - \delta_{n,2} \delta_{g,0} \frac{(H_{i_1}, H_{i_2})}{(x_1 - x_2)^2} \right) \quad (3-15)$$

In the classical formalism of [43], [30], the initial data needed to run the topological recursion included a symmetric bi-differential $\omega_{0,2}$ on two copies of this curve having a double pole with no residue and bi-residue 1 on the diagonal divisor and no other singularities. In the special instance where the spectral curve is a complex curve embedded in \mathbb{C}^2 , a natural candidate then was the Bergman kernel, or second-kind fundamental form,

associated to a choice of Torelli marking, that is to a symplectic basis of real codimension 1 homology cycles. The wanted singularities plus the requirement of vanishing periods on a given half of the symplectic basis fixes the Bergman kernel uniquely.

Our quantum situation is a direct generalization of that of [33] where the construction can be interpreted as solving the Ward identities of a $\mathcal{W}(\mathfrak{sl}_2(\mathbb{C}))$ -symmetric conformal field theory. The existence of an hyperelliptic involution then simplified the discussion and a structure of quantum Riemann surface with cuts, cycles, holomorphic differential forms and their mutual pairing was defined. In particular, the periods of the quantum Bergman kernel were vanishing on a corresponding half of a symplectic basis of quantum cycles.

This geometric construction is expected to extend to our setup and will be further investigated in subsequent work. By anticipation

Definition 3.7 *Quantum Bergman kernel*

We interpret the leading order of the 2-points function of the theory as the quantum Bergman kernel, or second-kind fundamental form, and denote it by

$$B \stackrel{\text{def}}{=} W_{0,2} \tag{3-16}$$

Accordingly, we define the third-kind differential form G (up to a constant additive factor) by the formula

$$\partial_z G(\overset{i}{x}, \overset{j}{z}) = 2B(\overset{i}{x}, \overset{j}{z}) \tag{3-17}$$

proof:

The only thing to check is that since $W_{0,2}$ is symmetric, any function of the variable x that one can add to the last equality without changing the property is actually constant. ■

4 Topological recursion

4.1 Ward identities in the $\varepsilon \rightarrow 0$ expansion

To solve the Ward identities recursively, we must rewrite them order by order in the topological regime. An exceptional feature of the structure of these equations is that only the two lowest order Ward identities, that we will call *linear and quadratic loop equation*, are needed to reconstruct the chiral correlation functions with current insertions perturbatively. This illustrates the *over-determination* of integrable systems.

Recall that using the multi-sheet notation, we denoted the insertion of the k^{th} $\mathcal{W}(\mathfrak{g})$ -algebra generator $\mathbf{W}^{(k)}$ at a generic point $x \in \Sigma$ into a chiral correlation function with $n \in \mathbb{N}^*$ current insertions at generic points $x_1, \dots, x_n \in \Sigma$ by

$$\langle\langle \mathbf{W}^{(k)}(x) \mathbf{J}(\tilde{x}_1 \cdot H_{i_1}), \dots, \mathbf{J}(\tilde{x}_n \cdot H_{i_n}) \rangle\rangle \stackrel{\text{def}}{=} \sum_{g=0}^{\infty} (-1)^k \varepsilon^{2g-1+n} P_{n,k}^{(g)}(x; \tilde{x}_1^{i_1}, \dots, \tilde{x}_n^{i_n}) \quad (4-1)$$

Replacing this expression in the two first conformal Ward identities yields

Theorem 4.1 *Linear and quadratic loop equations*

The axioms of the $\mathcal{W}(\mathfrak{g})$ -symmetric conformal field theory require the linear and quadratic loop equations in the topological limit. Namely for any choice of integers $n, g \in \mathbb{N}$, $n \neq 0$, and any generic choice of points and sheet indices $J = \{\tilde{x}_1^{i_1}, \dots, \tilde{x}_n^{i_n}\}$,

$$P_{n;0}^{(g)}(x, J) = \sum_{i=1}^d W_{n+1}^{(g)}(x, J) \quad (4-2)$$

and

$$P_{n;1}^{(g)}(x, J) = \sum_{1 \leq i < j \leq d} \left(W_{n+2}^{(g-1)}(x, x, J) + \sum_{\substack{I \sqcup I' = J \\ h+h'=g}} W_{1+\#I}^{(h)}(x, I) W_{1+\#I'}^{(h')}(x, I') \right) - Q \sum_{i=2}^d (i-1) \partial_x W_{n+1}^{(g)}(x, J) \quad (4-3)$$

are holomorphic functions of $x \in \Sigma - \{z_1, \dots, z_M\}$.

proof:

The proof is done by induction on $2g-2+n$ and it is a straightforward computation. ■

4.2 Bethe roots and kernel

Generically, a zero of D_i for a given $i \in \{1, \dots, d\}$ is both a pole of Y_i and Y_{i+1} , with residue ± 1 and is not a zero of any other D_j and therefore not a pole of any other Y_j . This statement is the quantum analog to that saying that generically, there are only two sheets meeting at a branch point of an algebraic curve.

Definition 4.2 Bethe roots

Let us denote by $S \stackrel{\text{def}}{=} \{s \in \mathbb{C} \mid \exists j, D_j(s) = 0\}$ the set of all roots of the D_j 's and call them the Bethe roots. We will moreover generically denote by $s_\alpha \in S$ the root such that there exists exactly two functions Y_{α_+} and Y_{α_-} , of which s_α is a pole with $|\alpha_+ - \alpha_-| = 1$.

Definition 4.3 Recursion kernel

Let us define the "recursion kernel" $K_\alpha(x_0, x)$ as the solution of the following differential equation which is analytic at $x = s_\alpha$:

$$(Y_{\alpha_+}(x) - Y_{\alpha_-}(x) + (\alpha_+ - \alpha_-) Q \partial_x) K_\alpha(x_0, x) = G(x_0, x)^{\alpha_+} - G(x_0, x)^{\alpha_-} \quad (4-4)$$

Notice that K_α is symmetric under the exchange $\alpha_+ \leftrightarrow \alpha_-$.

If $\alpha_+ - \alpha_- = 1$, we get

$$\left(-2 \frac{QD'_{\alpha_-}}{D_{\alpha_-}} + \frac{QD'_{\alpha_{-+1}}}{D_{\alpha_{-+1}}} + \frac{QD'_{\alpha_{--1}}}{D_{\alpha_{--1}}} + Q\partial_x \right) K_\alpha(x_0^i, x) = G(x_0^i, \alpha_+) - G(x_0^i, \alpha_-) \quad (4-5)$$

4.3 Recursion

Theorem 4.4 *Reconstruction by topological recursion [B.-Eynard]*

We have the topological recursion

$$W_{n+1}^{(g)}(x, J) = \sum_{\alpha} \operatorname{Res}_{x \rightarrow s_\alpha} K_\alpha(x_0^i, x) \left(W_{n+2}^{(g-1)}(\alpha_+, \alpha_-, J) + \sum_{\substack{I \sqcup I' = J \\ h+h'=g}} W_{1+\#I}^{(h)}(\alpha_+, I) W_{1+\#I'}^{(h')}(\alpha_-, I') \right) \quad (4-6)$$

where the recursion kernel is the solution of

$$\left(Y_{\alpha_+}(x) - Y_{\alpha_-}(x) + (\alpha_+ - \alpha_-) Q\partial_x \right) K_\alpha(x_0^i, x) = G(x_0^i, \alpha_+) - G(x_0^i, \alpha_-) \quad (4-7)$$

which is analytic at the roots $S = \{s \in \mathbb{C} \mid \exists j, D_j(s) = 0\}$ with the third-kind differential G defined by

$$\partial_x G(x_0^i, \alpha) = 2B(x_0^i, \alpha) \quad (4-8)$$

proof:

Let us compute the expression

$$\sum_{\alpha} \operatorname{Res}_{x \rightarrow s_\alpha} K_\alpha(x_0^i, x) \left(W_{n+2}^{(g-1)}(\alpha_+, \alpha_-, J) + \sum_{\substack{I \sqcup I' = J \\ h+h'=g}} W_{1+\#I}^{(h)}(\alpha_+, I) W_{1+\#I'}^{(h')}(\alpha_-, I') \right) \quad (4-9)$$

where $\sum'_{h, h', I, I'}$ means that we exclude both the cases $(h = 0, I = \emptyset)$ and $(h' = 0, I' = \emptyset)$ from the sum. Let us define the same quantity as the one between parentheses but without the prime symbol ' :

$$\mathcal{Q}_{i,j} \stackrel{def}{=} W_{n+2}^{(g-1)}(\overset{i}{x}, \overset{j}{x}, J) + \sum_{\substack{I \sqcup I' = J \\ h+h'=g}} W_{1+\#I}^{(h)}(\overset{i}{x}, I) W_{1+\#I'}^{(h')}(\overset{j}{x}, I') \quad (4-10)$$

for root indices $i, j \in S$. We thus have to compute:

$$\sum_{\alpha} \operatorname{Res}_{x \rightarrow s_{\alpha}} K_{\alpha}(\overset{i_0}{x_0}, x) \left(\mathcal{Q}_{\alpha, \alpha+1} - W_1^{(0)}(\overset{\alpha+1}{x}) W_{n+1}^{(g)}(\overset{\alpha}{x}, J) - W_1^{(0)}(\overset{\alpha}{x}) W_{n+1}^{(g)}(\overset{\alpha+1}{x}, J) \right) \quad (4-11)$$

Let us rewrite

$$\begin{aligned} 2\mathcal{Q}_{\alpha, \alpha+1} &= \mathcal{Q}_{\alpha+1, \alpha} + \mathcal{Q}_{\alpha, \alpha+1} \\ &= \sum_{i \neq j} \mathcal{Q}_{i,j} - \sum_{j \neq \alpha, \alpha+1} (\mathcal{Q}_{\alpha, j} + \mathcal{Q}_{\alpha+1, j}) - \sum_{i \neq \alpha, \alpha+1} (\mathcal{Q}_{i, \alpha} + \mathcal{Q}_{i, \alpha+1}) \\ &\quad - \sum_{\substack{i \neq j, i \neq \alpha, \alpha+1, j \neq \alpha, \alpha+1}} \mathcal{Q}_{i,j} \\ &= 2 \sum_{i < j} \mathcal{Q}_{i,j} - \sum_{\substack{i \neq j, (i,j) \neq (\alpha, \alpha+1), (i,j) \neq (\alpha+1, \alpha)}} \mathcal{Q}_{i,j} \\ &= 2P_{n,1}^{(g)}(x, J) + 2Q \sum_j j \partial_x W_{n+1}^{(g)}(\overset{j}{x}, J) - \sum_{\substack{i \neq j, (i,j) \neq (\alpha, \alpha+1), (i,j) \neq (\alpha+1, \alpha)}} \mathcal{Q}_{i,j} \\ &= +2Q\alpha \partial_x W_{n+1}^{(g)}(\overset{\alpha}{x}, J) + 2Q(\alpha+1) \partial_x W_{n+1}^{(g)}(\overset{\alpha+1}{x}, J) + \text{analytic at } s_{\alpha} \end{aligned} \quad (4-12)$$

where we used the $i \longleftrightarrow j$ symmetry of the symbol $\mathcal{Q}_{i,j}$ and the linear loop equation after we have introduced $P_{n,1}^{(g)}$. Multiplying by the recursion kernel and summing over residues at the Bethe roots imply that

$$\begin{aligned} &\sum_{\alpha} \operatorname{Res}_{x \rightarrow s_{\alpha}} K_{\alpha}(\overset{i_0}{x_0}, x) \left(\mathcal{Q}_{\alpha, \alpha+1} - W_1^{(0)}(\overset{\alpha+1}{x}) W_{n+1}^{(g)}(\overset{\alpha}{x}, J) - W_1^{(0)}(\overset{\alpha}{x}) W_{n+1}^{(g)}(\overset{\alpha+1}{x}, J) \right) \\ &= \sum_{\alpha} \operatorname{Res}_{x \rightarrow s_{\alpha}} K_{\alpha}(\overset{i_0}{x_0}, x) \left(-W_1^{(0)}(\overset{\alpha+1}{x}) W_{n+1}^{(g)}(\overset{\alpha}{x}, J) - W_1^{(0)}(\overset{\alpha}{x}) W_{n+1}^{(g)}(\overset{\alpha+1}{x}, J) \right. \\ &\quad \left. + Q\alpha \partial_x W_{n+1}^{(g)}(\overset{\alpha}{x}, J) + Q(\alpha+1) \partial_x W_{n+1}^{(g)}(\overset{\alpha+1}{x}, J) \right) \end{aligned} \quad (4-13)$$

Since $W_{n+1}^{(g)}(\overset{\alpha}{x}, J) + W_{n+1}^{(g)}(\overset{\alpha+1}{x}, J)$ is analytic at s_{α} , we may rewrite:

$$\begin{aligned}
& \sum_{\alpha} \operatorname{Res}_{x \rightarrow s_{\alpha}} K_{\alpha}(\dot{x}_0, x) \left(\mathcal{Q}_{\alpha, \alpha+1} - W_1^{(0)}(\dot{x}^{\alpha+1}) W_{n+1}^{(g)}(\dot{x}^{\alpha}, J) - W_1^{(0)}(\dot{x}^{\alpha}) W_{n+1}^{(g)}(\dot{x}^{\alpha+1}, J) \right) \\
&= \sum_{\alpha} \operatorname{Res}_{x \rightarrow s_{\alpha}} K_{\alpha}(\dot{x}_0, x) \left(W_1^{(0)}(\dot{x}^{\alpha}) - W_1^{(0)}(\dot{x}^{\alpha+1}) + Q \partial_x \right) W_{n+1}^{(g)}(\dot{x}^{\alpha}, J) \quad (4-14)
\end{aligned}$$

Integrating by parts then yields

$$\begin{aligned}
& \sum_{\alpha} \operatorname{Res}_{x \rightarrow s_{\alpha}} K_{\alpha}(\dot{x}_0, x) \left(\mathcal{Q}_{\alpha, \alpha+1} - W_1^{(0)}(\dot{x}^{\alpha+1}) W_{n+1}^{(g)}(\dot{x}^{\alpha}, J) - W_1^{(0)}(\dot{x}^{\alpha}) W_{n+1}^{(g)}(\dot{x}^{\alpha+1}, J) \right) \\
&= - \sum_{\alpha} \operatorname{Res}_{x \rightarrow s_{\alpha}} W_{n+1}^{(g)}(\dot{x}^{\alpha}, J) \left(W_1^{(0)}(\dot{x}^{\alpha+1}) - W_1^{(0)}(\dot{x}^{\alpha}) + Q \partial_x \right) K_{\alpha}(\dot{x}_0, x) \quad (4-15)
\end{aligned}$$

and using the defining differential equations of each K_{α} we get

$$\begin{aligned}
& \sum_{\alpha} \operatorname{Res}_{x \rightarrow s_{\alpha}} K_{\alpha}(\dot{x}_0, x) \left(\mathcal{Q}_{\alpha, \alpha+1} - W_1^{(0)}(\dot{x}^{\alpha+1}) W_{n+1}^{(g)}(\dot{x}^{\alpha}, J) - W_1^{(0)}(\dot{x}^{\alpha}) W_{n+1}^{(g)}(\dot{x}^{\alpha+1}, J) \right) \\
&= - \sum_{\alpha} \operatorname{Res}_{x \rightarrow s_{\alpha}} W_{n+1}^{(g)}(\dot{x}^{\alpha}, J) \left(G(\dot{x}_0, \dot{x}^{\alpha+1}) - G(\dot{x}_0, \dot{x}^{\alpha}) \right) \\
&= - \frac{1}{2} \sum_{\alpha} \operatorname{Res}_{x \rightarrow s_{\alpha}} \left(W_{n+1}^{(g)}(\dot{x}^{\alpha}, J) - W_{n+1}^{(g)}(\dot{x}^{\alpha+1}, J) \right) \left(G(\dot{x}_0, \dot{x}^{\alpha+1}) - G(\dot{x}_0, \dot{x}^{\alpha}) \right) \quad (4-16)
\end{aligned}$$

Using again the linear loop equation we have

$$\begin{aligned}
& \sum_{\alpha} \operatorname{Res}_{x \rightarrow s_{\alpha}} K_{\alpha}(\dot{x}_0, x) \left(\mathcal{Q}_{\alpha, \alpha+1} - W_1^{(0)}(\dot{x}^{\alpha+1}) W_{n+1}^{(g)}(\dot{x}^{\alpha}, J) - W_1^{(0)}(\dot{x}^{\alpha}) W_{n+1}^{(g)}(\dot{x}^{\alpha+1}, J) \right) \\
&= \frac{1}{2} \sum_{\alpha} \operatorname{Res}_{x \rightarrow s_{\alpha}} \left(W_{n+1}^{(g)}(\dot{x}^{\alpha}, J) G(\dot{x}_0, \dot{x}^{\alpha}) + W_{n+1}^{(g)}(\dot{x}^{\alpha+1}, J) G(\dot{x}_0, \dot{x}^{\alpha+1}) \right) \\
&= \frac{1}{2} \sum_{\alpha} \left(\delta_{\alpha, i_0} W_{n+1}^{(g)}(\dot{x}_0, J) + \delta_{\alpha+1, i_0} W_{n+1}^{(g)}(\dot{x}_0, J) \right) \\
&= W_{n+1}^{(g)}(\dot{x}_0, J) \quad (4-17)
\end{aligned}$$

where we used the fact that G has simple poles along the diagonal (third kind differential). ■

Remark 4.5 Notice that the proof of the topological recursion in this quantum setup is almost the same as the proof that was done in the previous chapter in the context of cameral curves.

4.4 Special geometry and free energies

This topological recursion procedure applied to a quantum spectral curve is expected to allow the perturbative reconstruction of the correlation function

$$\left\langle \prod_{j=1}^M V_{\alpha_j}(z_j) \right\rangle \quad (4-18)$$

viewed as a function on the moduli space of all such W -symmetric conformal field theories on the Riemann sphere

$$\mathcal{M}_{W(\mathfrak{g})} \stackrel{def}{=} \{ \{(z_j, \alpha_j) \in \mathbb{C}\mathbb{P}^1 \times \mathfrak{h}^*\}_{1 \leq j \leq M}, \mathbf{t} \} \quad (4-19)$$

where the potential V , with $\partial V = W_{0,1} + Y$, is defined as a function of the times \mathbf{t} through the multi-Penner formula of section 3.2.

We would then like to define the τ -function $\mathfrak{T}_{\mathbf{z}, \alpha, \mathbf{t}}$ associated to the W -symmetric conformal field theory as the function on the moduli space $\mathcal{M}_{W(\mathfrak{g})}$ by

$$\ln \mathfrak{T}_{\mathbf{z}, \alpha, \mathbf{t}} \stackrel{def}{=} \sum_{g=0}^{\infty} \varepsilon^{2g-2} \mathcal{F}_g \quad (4-20)$$

where the genus g free energy \mathcal{F}_g is such that for any deformation $\delta \in T^* \mathcal{M}_{W(\mathfrak{g})}$, we have the special geometry relations

$$\delta \mathcal{F}_g = \int_{\delta^*} W_{g,1} \quad (4-21)$$

with a cycle δ^* dual to the deformation and defined such that

$$\delta W_{g,n} = \int_{\delta^*} W_{g,n+1} \quad (4-22)$$

for any $g, n \in \mathbb{N}$, $n \neq 0$. Such a definition needs a systematic definition and study of deformations and associated cycles that should generalize the

one introduced in [33] and [35]. In particular, the additional choices that have to be made to define form-cycle type dualities should correspond to parameters of bases of conformal blocks and the natural conjecture

Conjecture 4.6 *τ -function as conformal block*

$$\ln \mathfrak{T}_{\mathbf{z}, \alpha, \mathbf{t}} = \left\langle \prod_{j=1}^M V_{\alpha_j}(z_j) \right\rangle \quad (4-23)$$

would then go one step further in performing the conformal bootstrap of *W*-symmetric conformal field theories. Indeed, the remaining problem would then be to recollect these conformal blocks into single-valued smooth correlation functions.

Similarly, the wave function reconstructed from topological recursion applied to the quantum spectral curve of the $\mathcal{W}(\mathfrak{g})$ -symmetric conformal field theory is defined as

$$\Psi(D) \stackrel{def}{=} \exp \left(\sum_{n=0}^{\infty} \sum_{g=0}^{\infty} \frac{\varepsilon^{2g-2+n}}{n!} \int_D \cdots \int_D W_{g,n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \right) \quad (4-24)$$

and is expected to be related to correlation functions of the theory with insertions of degenerate fields, in the sense that it should satisfy some BPZ equation.

Conclusions

In this thesis we have worked out a unifying geometric framework for some classical as well as some quantum integrable systems. In this setting, integrability is encoded in the form of symmetry and analytic constraints called *loop equations*.

The classical context we started from was the one of Fuchsian differential systems. They are classes of local systems on Riemann surfaces up to gauge transformations. From there were defined correlators satisfying some loop equations that one can hope to solve by topological recursion when in presence of nice topological expansions. Nevertheless we showed that the special geometry they exhibit allows for the definition of a non-perturbative τ -function. Investigation of the KdV hierarchy suggests once more [40] that the topological recursion procedure should indeed reproduce the τ -functions of integrable hierarchies with initial data given by the WKB asymptotics of the differential system. The next steps in this direction are to finish the proofs of the conjectures regarding the cameral curve topological recursion and the KdV hierarchy and then apply the method in the enhanced setup of the Drinfeld-Sokolov hierarchy as presented in [23]. Unfortunately for the dream of solving Gromov-Witten theory by this method, a Lax formulation of the corresponding integrable hierarchy [37] is still missing. Finding such a Lax formulation would therefore mean tremendous progress towards reaching the dream.

The corresponding quantum context is that of W -symmetric conformal field theories, where the W -algebra is defined from a reductive Lie algebra \mathfrak{g} . It realizes a quantization of the Poisson structure [31] [32] [72] associated to equivariant integrable hierarchies of Drinfeld-Sokolov type and in our setup this quantization is identified with the presence of a deformation parameter Q called the *background charge*. From this point of view,

one recovers Casimir algebras by appropriately extracting the $Q \rightarrow 0$ limit from the non-commutative algebra. We then showed that the quantum spectral curve allows for a perturbative reconstruction of the correlation functions with current insertions in the topological regime. An important feature of this part was that the recursion commutes with the variation of the background charge in the sense that the initial data is quantized while the recursion stays the same as the one used in the classical theory. The next steps in this direction would be to give explicit computations of the corresponding free energies in higher ranks examples. Moreover, even though it was virtually absent of this work, the q -deformation, or second quantization of these constructions seems to be a promising direction toward using topological recursion *e.g.* for quantum integrable systems in the context of quantum groups.

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Papers

Here are the three already available papers that were used to write this thesis [9], [10] and [11].