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Vertex partition of sparse graphs

Présentée par François DROSS

Le 27 Juin 2018

Sous la direction de Mickael Montassier et Alexandre Pinlou

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Résumé

Dans cette thèse, nous traiterons de partitions des sommets de graphes peu denses. Pour tout k, une coloration propre de k couleurs d'un graphe est une partition des sommets du graphe en k ensembles indépendants. Le premier résultat de ce style fut le théorème des quatre couleurs. Il indique que tout graphe planaire admet une coloration propre d'au plus 4 couleurs.

Un *sous-graphe* d'un graphe G est un graphe qui contient uniquement des sommets et des arêtes de G. À noter que comme un sous-graphe de G est un graphe, il ne peut contenir une arête de G que s'il contient ses deux extrémités. Un *sous-graphe induit* d'un graphe G est un sous-graphe H de G qui contient toutes les arêtes de G dont les deux extrémités sont des sommets de H. Pour G un graphe et S un ensemble de sommets de G, on note $G[S]$ le sous-graphe induit de G dont l'ensemble de sommets est S. On appelle le graphe $G[S]$ le sous-graphe de G induit par S. Une *forêt* est un graphe sans cycles, et une *forêt induite* d'un graphe G est un sous-graphe induit de G sans cycles.

Albertson et Bermann [3] ont fait en 1979 la conjecture suivante :

Conjecture 1 (Albertson et Bermann [3])**.** *Tout graphe planaire admet une forêt induite contenant au moins la moitié de ses sommets.*

Cette conjecture impliquerait que tout graphe planaire admet un ensemble indépendant contenant au moins le quart de ses sommets, résultat dont la seule preuve connue actuellement repose sur le théorème des quatre couleurs.

Une notion peut-être plus connue est celle de coupe-cycle de sommets (feedback vertex set). Dans un graphe G, un coupe-cycle de sommets est un ensemble de sommets qui intersectent tous les cycles de G. Notons qu'un ensemble $S \subseteq V(G)$ est un coupe-cycle de sommets si et seulement si $G[V(G) \setminus S]$ est une forêt. Par conséquent, une formulation équivalente de la conjecture 1 est que tout graphe planaire d'ordre n admet un coupe-cycle de sommets contenant au plus $\frac{n}{2}$ sommets.

Si la conjecture 1 est vraie, elle ne peut être renforcée, comme on peut le voir grâce à une union disjointe de copies de K_4, le graphe complet à quatre sommets. Le meilleur résultat connu est que tout graphe planaire a une forêt induite sur au moins deux cinquièmes de ses sommets. C'est une conséquence directe du théorème de Borodin [9] comme quoi tout graphe planaire admet une partition de ses sommets en cinq ensembles indépendants tels que toute union de deux de ces ensembles induit une forêt.

La conjecture 1 a été prouvée, et même renforcée dans des sous-classes des graphes planaires. Par exemple, Hosono [37] a montré que tout graphe planaire extérieur admet une forêt induite contenant au moins deux tiers des sommets. Il a également donné des exemples montrant que ce résultat ne peut pas être amélioré. D'autres résultats ont été dérivés de résultats sur les colorations acycliques. Fertin, Godard et Raspaud [33] ont donné de tels résultats. Ceux qui concernent les graphes peu denses sont résumés dans la table 1.

Akiyama et Watanabe [1], ainsi qu'Albertson et Haas [2] ont indépendamment proposé la conjecture suivante :

Conjecture 2 (Akiyama et Watanabe [1], Albertson et Haas [2])**.** *Tout graphe planaire biparti admet une forêt induite contenant au moins $\frac{5}{8}$ de ses sommets.*

Dans l'optique de prouver cette conjecture, Alon, Mubayi et Thomas [4] ont prouvé les résultats suivants :

Theorème 3 (Alon et al. [4])**.** *Tout graphe sans triangles avec n sommets et m arêtes admet une forêt induite contenant au moins $n - \frac{m}{4}$ sommets.*

Famille $\mathcal{F}$	ordre d'une plus grande forêt induite: minorant	majorant
Planaire	$\frac{2n}{5}$	$\lceil \frac{n}{2} \rceil$
Planaire de maille 5 ou 6	$\frac{n}{2}$	$\frac{7n}{10} + 2$
Planaire de maille au moins 7	$\frac{2n}{3}$	$\frac{5n}{6} + 1$

Table 1: Minorants et majorants sur l'ordre d'une plus grande forêt induite pour certaines familles $\mathcal{F}$ de graphes planaires [33].

Corollaire 4 (Alon et al. [4])**.** *Tout graphe cubique (i.e. dont tous les sommets sont de degré* 3*), sans triangles et avec n sommets admet une forêt induite contenant au moins $\frac{5n}{8}$ sommets.*

Corollaire 5 (Alon et al. [4])**.** *Tout graphe planaire sans triangles et avec n sommets admet une forêt induite contenant au moins $\frac{n}{2}$ sommets.*

Cette dernière minoration a été améliorée pour $n \geq 1$ par Salavatipour [49] :

Theorème 6 (Salavatipour [49])**.** *Tout graphe planaire sans triangles, avec n sommets et m arêtes admet une forêt induite contenant au moins $\frac{29n-6m}{32}$ sommets et donc au moins $\frac{17n+24}{32} \approx 0.531n$ sommets.*

En nous basant sur les méthodes développées par Kowalik, Lužar et Škrekovski [43], nous améliorons le résultat précédent :

Theorème 7. *Tout graphe planaire sans triangles, avec n sommets et avec m arêtes admet une forêt induite contenant au moins* $\max\{\frac{38n-7m}{44}, n - \frac{m}{4}\}$ *sommets et donc au moins $\frac{6n+7}{11} \approx 0.545n$ sommets.*

La preuve s'appuie sur l'existence d'une série de configurations interdites dans un contre-exemple minimum, donnant des contraintes locales sur la structure d'un tel contre-exemple. Ces contraintes locales nous permettent d'aboutir à une contradiction, grâce à un argument de double comptage des sommets apparaissant dans les frontières de certaines faces.

Kowalik, Lužar et Škrekovski [43] ont fait la conjecture suivante et ont donné des exemples pour prouver qu'elle ne peut pas être améliorée.

Conjecture 8 (Kowalik et al. [43])**.** *Tout graphe planaire de maille au moins* 5 *et avec n sommets admet une forêt induite contenant au moins $\frac{7n}{10}$ sommets.*

Utilisant une méthode similaire à celle utilisée pour la preuve du Théorème 7, nous avons prouvé que tout graphe planaire de maille au moins 5, avec n sommets et m arêtes admet une forêt induite contenant au moins $n - \frac{5m}{23}$ sommets et donc au moins $\frac{44n+50}{69} \approx 0.638n$ sommets. Ce résultat a par la suite été amélioré indépendamment, d'une part par Shi et Xu [50] et d'autre part par Kelly et Liu [41] :

Theorème 9 (Shi et Xu [50], Kelly et Liu [41])**.** *Tout graphe planaire connexe de maille au moins* 5*, avec n sommets et m arêtes admet une forêt induite contenant au moins $\frac{8n-2m-2}{7}$ sommets et donc au moins $\frac{2n-2}{3} \approx 0.667n$ sommets.*

Pour les grandes mailles, nous faisons la conjecture suivante :

Conjecture 10. *Pour tout entier $g \geq 3$, tout graphe planaire de maille au moins g, avec n sommets et m arêtes admet une forêt induite contenant au moins $n - \frac{m}{g}$ sommets et donc au moins $n - \frac{n-2}{g-2}$ sommets.*

Nous donnons un exemple montrant que cette conjecture ne peut pas être renforcée. Avec cette conjecture pour objectif, nous prouvons le théorème suivant.

Theorème 11. *Pour tout entier $g \geq 3$, tout graphe planaire de maille au moins g, avec n sommets et m arêtes admet une forêt induite contenant au moins $n - \frac{4m}{3g}$ sommets et donc an moins $n - \frac{4n}{3g-6}$ sommets.*

La preuve du Théorème 11 est basée sur une réduction vers les graphes 2-connexes de degré maximum au plus 3. Elle repose sur le résultat suivant :

Theorème 12. *Tout graphe 2-connexe de degré maximum au plus 3 avec au plus n sommets admet une forêt induite contenant au moins $\frac{2n-2}{3}$ sommets.*

La conjecture 1 a aussi mené à des recherches similaires sur les forêts linéaires induites. Une forêt linéaire (induite) est une forêt (induite) dont le degré maximum est au plus 2. Chappell a émis la conjecture suivante (voir Pelsmajer [45]) :

Conjecture 13 (Chappell)**.** *Tout graphe planaire d'ordre n admet une forêt linéaire induite contenant au moins $\frac{4n}{9}$ sommets.*

Cette conjecture ne peut pas être renforcée. La meilleure approche à ce jour est une conséquence du théorème suivant de Poh [47].

Theorème 14 (Poh [47])**.** *Tout graphe planaire admet une partition en trois forêts linéaires.*

Corollaire 15. *Tout graphe planaire à n sommets admet une forêt linéaire induite contenant au moins $\frac{n}{3}$ sommets.*

Ce problème a été résolu sur les graphes planaires extérieurs.

Theorème 16 (Pelsmajer [45])**.** *Tout graphe planaire extérieur avec n sommets admet une forêt linéaire induite contenant au moins $\frac{4n+2}{7}$ sommets.*

Le théorème précédent ne peut être amélioré, comme le montrent les exemples de Chappell (voir Pelsmajer [45]).

Nous nous concentrons à présent sur les graphes planaires sans triangles, et faisons la conjecture suivante :

Conjecture 17. *Tout graphe planaire sans triangles avec n sommets admet une forêt linéaire induite contenant au moins $\frac{n}{2}$ sommets.*

Nous donnons des exemples prouvant que cette conjecture ne peut pas être renforcée. Nous faisons un premier pas vers cette conjecture :

Theorème 18. *Tout graphe planaire sans triangles avec n sommets et m arêtes admet une forêt linéaire induite contenant au moins $\frac{9n-2m}{11}$ sommets et donc au moins $\frac{5n+8}{11} \approx 0.455n$ sommets.*

Ce théorème a une preuve du même type que celle du théorème 7. Le problème de trouver une plus grande forêt induite de degré maximum au plus d a été résolu pour tout $d \geq 2$ dans les graphes de largeur arborescente (treewidth) au plus k pour tout k par Chappell et Pelsmajer [16]. À noter que le cas $d = 2$ correspond à la plus grande forêt linéaire induite. Ils ont également fait la conjecture suivante :

Conjecture 19 (Chappell et Pelsmajer [16])**.** *Soit $d \geq 2$ un entier. Tout graphe planaire à n sommets admet une forêt induite de degré maximum au plus d avec au moins $\frac{2dn}{4d+1}$ sommets.*

Pour $d = 2$, cela donne la conjecture 13, et pour $d \to \infty$, cela donne la conjecture 1. Chappell et Pelsmajer ont donné des exemples montrant que cette conjecture ne peut être renforcée. À la connaissance de l'auteur, aucun progrès n'a été fait vers la conjecture 19 mis à part les cas $d = 2$ et $d \to \infty$. Notons tout

de même que la minoration pour $d = 2$ donne indirectement des minorations pour les autres valeurs de d, donc la meilleure approche vers la conjecture 19 est le corollaire 15.

On appelle $\mathcal{I}$ la classe des graphes vides (i.e. sans arêtes) et $\mathcal{F}$ la classe des forêts. Pour tout $d \in \mathbb{N}$, on appelle Δ_d la classe des graphes de degré maximum au plus d, $\mathcal{F}_d$ la classe des forêts de degré maximum au plus d, et $\mathcal{D}_d$ la classe des graphes d-dégénérés, c'est-à-dire des graphes dont tous les sous-graphes ont un degré minimum d'au plus d.

Une façon de voir une forêt induite d'un graphe G est de la voir comme une partition des sommets en un ensemble induisant une forêt et un autre ensemble. Dans ce qui précède, nous avons tenté d'assurer que ce deuxième ensemble de sommets n'était pas trop grand. Une autre approche est d'imposer d'autres propriétés pour cet ensemble de sommets, par exemple qu'il induise un graphe particulier. Dans la suite, nous allons nous intéresser à cette approche.

Soient $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_k$ k classes de graphes. Une $(\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_k)$-partition d'un graphe G, notée $(H_1, H_2, ..., H_k)$, est une partition des sommets du graphe G en k ensembles H_1, H_2, ..., H_k tels que pour tout $i \in \{1, ..., k\}$, $G[H_i] \in \mathcal{C}_i$.

L'étude des partitions des sommets des graphes planaires a commencé avec le théorème des quatre couleurs, qui peut être reformulé de la manière suivante :

Theorème 20. *Tout graphe planaire admet une $(\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I})$-partition.*

De nombreux théorèmes peuvent également être reformulés en terme de partitions en forêts. On rappelle le théorème de Poh, qui peut être reformulé comme suit :

Theorème 14 (Poh [47], reformulé)**.** *Tout graphe planaire admet une $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_2)$-partition.*

De plus, une conséquence du théorème de Borodin [9] (tout graphe planaire admet une partition de ses sommets en cinq ensembles indépendants tels que toute union de deux de ces ensembles induit une forêt) est que tout graphe planaire admet une $(\mathcal{F}, \mathcal{F}, \mathcal{I})$-partition.

Thomassen a étudié les $(\mathcal{D}_i, \mathcal{D}_j)$-partitions des graphes planaires. Il a prouvé que tout graphe planaire admet une $(\mathcal{F}, \mathcal{D}_2)$-partition [51] et une $(\mathcal{I}, \mathcal{D}_3)$-partition [52] (notez que $\mathcal{F} = \mathcal{D}_1$ et $\mathcal{I} = \mathcal{D}_0$).

Toutefois, il y a des graphes planaires qui n'ont pas de $(\mathcal{F}, \mathcal{F})$-partition [17], et même des graphes planaires qui n'ont pas de $(\mathcal{F}, \mathcal{I}, \mathcal{I})$-partition (Wegner [53]). Borodin et Glebov [11] ont montré le théorème suivant :

Theorème 21 (Borodin et Glebov [11])**.** *Tout graphe planaire de maille au moins 5 admet une $(\mathcal{I}, \mathcal{F})$-partition.*

Raspaud et Wang [48] ont prouvé que tout graphe planaire où deux triangles sont à une distance d'au moins 2 (et donc tout graphe planaire sans triangles) admet une $(\mathcal{F}, \mathcal{F})$-partition.

Le résultat comme quoi tout graphe planaire sans triangle admet une $(\mathcal{F}, \mathcal{F})$-partition est du folklore, et peut être prouvé aisément. Cependant, l'existence d'un graphe planaire sans triangles n'admettant pas de $(\mathcal{I}, \mathcal{F})$-partition est inconnue. Nous posons les questions suivantes :

Question 22. *Tout graphe planaire sans triangle admet-il une $(\mathcal{I}, \mathcal{F})$-partition ?*

Question 23. *Quel est le plus petit entier d tel que tout graphe planaire sans triangles admet une $(\mathcal{F}, \mathcal{F}_d)$-partition ?*

Notons que prouver que $d = 0$ à la question 23 reviendrait à répondre à la question 22 par l'affirmative. On prouve le théorème suivant :

Theorème 24. *Tout graphe planaire sans triangles admet une $(\mathcal{F}, \mathcal{F}_5)$-partition.*

Ce théorème implique que $d \leq 5$ dans la question 23. La preuve du théorème précédent utilise la méthode du déchargement, avec en particulier trois configurations réductibles assez complexes. La preuve est constructive et il en découle immédiatement un algorithme en temps polynomial pour obtenir une $(\mathcal{F}, \mathcal{F}_5)$-partition d'un graphe planaire sans triangles.

Montassier et Ochem [44] donnent, pour tout d, un graphe planaire sans triangles ne pouvant pas être partitionné en deux sous-graphes induits de degré maximum au plus d, ce qui montre que la forêt dans le théorème 24 ne peut pas être remplacée par une forêt de degré borné (ou même par un sous-graphe de degré borné).

Nous montrons également que pour tout d, s'il existe un graphe planaire sans triangles qui n'admet pas de $(\mathcal{F}, \mathcal{F}_d)$-partition, alors le problème consistant à décider si un graphe planaire sans triangles donné admet une $(\mathcal{F}, \mathcal{F}_d)$-partition est NP-complet. La preuve est une réduction vers Planar 3-Sat.

Le degré moyen maximum d'un graphe G, noté $\mathrm{mad}(G)$, est le maximum, sur tous les sous-graphes H de G, de deux fois le nombre d'arêtes de H sur le nombre de sommets de H. Il s'agit d'une mesure locale de la densité d'un graphe, à comprendre que si un graphe a un degré moyen maximum borné, il n'est en quelques sortes dense nulle part. La formule d'Euler nous indique que les graphes planaires ont un degré moyen maximum strictement inférieur à 6, et plus généralement, que pour tout entier g, les graphes planaires de maille au moins g ont un degré moyen maximum strictement inférieur à $\frac{2g}{g-2}$.

Borodin et Kostochka [15] ont montré le théorème suivant :

Theorème 25 (Borodin et Kostochka [15])**.** *Pour tout $j \geq 0$ et tout $k \geq 2j+2$, tout graphe G avec $\mathrm{mad}(G) < 2\left(2 - \frac{k+2}{(j+2)(k+1)}\right)$ admet une (Δ_j, Δ_k)-partition.*

En particulier, le théorème précédent implique que tout graphe G avec $\mathrm{mad}(G) < \frac{8}{3}$ admet une $(\mathcal{I}, \Delta_2)$-partition, et que tout graphe G avec $\mathrm{mad}(G) < \frac{14}{5}$ admet une $(\mathcal{I}, \Delta_4)$-partition. Avec la formule d'Euler, on obtient le résultat suivant :

Corollaire 26. *Tout graphe planaire de maille au moins 7 admet une $(\mathcal{I}, \Delta_4)$-partition, et tout graphe planaire de maille au moins 8 admet une $(\mathcal{I}, \Delta_2)$-partition.*

Borodin et Kostochka [14] ont montré le théorème suivant :

Theorème 27 (Borodin et Kostochka [14])**.** *Tout graphe G avec $\mathrm{mad}(G) < \frac{12}{5}$ admet une $(\mathcal{I}, \Delta_1)$-partition.*

Ce dernier théorème implique que tout graphe planaire de maille au moins 12 admet une $(\mathcal{I}, \Delta_1)$-partition, ce qui a été amélioré par Kim, Kostochka et Zhu [42] :

Theorème 28 (Kim, Kostochka et Zhu [42])**.** *Tout graphe G sans triangles avec $\mathrm{mad}(G) < \frac{11}{9}$ admet une $(\mathcal{I}, \Delta_1)$-partition.*

Corollaire 29. *Tout graphe planaire de maille au moins 11 admet une $(\mathcal{I}, \Delta_1)$-partition.*

À l'inverse, Borodin, Ivanova, Montassier, Ochem et Raspaud [13] ont exhibé, pour tout entier d, un graphe planaire de maille au moins 6 qui n'admet pas de $(\mathcal{I}, \Delta_d)$-partition. Montassier et Ochem [44] ont montré que cela implique que le problème de décider si un graphe de maille au moins 6 admet une $(\mathcal{I}, \Delta_d)$-partition est NP-complet pour tout $d \geq 1$. Ils ont également prouvé que le problème de décider si un graphe de maille au moins 7 a une $(\mathcal{I}, \Delta_2)$-partition est aussi NP-complet. Esperet, Montassier, Ochem et Pinlou [30] ont montré que le problème de décider si un graphe planaire de maille au moins 9 admet une $(\mathcal{I}, \Delta_1)$-partition est NP-complet.

Notons que si les théorèmes 25, 27 et 28 ne peuvent voir leurs résultats en terme de degré moyen maximum améliorés, leurs corollaires en terme de graphe planaire de grande maille le peuvent peut-être. En particulier, les questions suivantes restent ouvertes :

Question 30. *Tous les graphes planaires de maille au moins 7 admettent-ils une $(\mathcal{I}, \Delta_3)$-partition ?*

Question 31. *Tous les graphes planaires de maille au moins 10 admettent-ils une $(\mathcal{I}, \Delta_1)$-partition ?*

Une façon naturelle d'étendre les résultats précédents est de considérer les partitions de graphes peu denses en un ensemble indépendant (i.e. induisant un graphe vide) et un ensemble induisant une forêt de degré borné, c'est-à-dire une $(\mathcal{I}, \mathcal{F}_d)$-partition. Notons que si un graphe admet une $(\mathcal{I}, \mathcal{F}_d)$-partition, alors il admet une $(\mathcal{I}, \Delta_d)$-partition, et qu'une $(\mathcal{I}, \mathcal{F}_1)$-partition est identique à une $(\mathcal{I}, \Delta_1)$-partition. Donc les résultats précédents impliquent que :

- pour tout entier d, il existe un graphe planaire de maille au plus 6 qui n'admet pas de $(\mathcal{I}, \mathcal{F}_d)$-partition ;
- il existe un graphe planaire de maille au plus 7 qui n'admet pas de $(\mathcal{I}, \mathcal{F}_2)$-partition ;
- il existe un graphe planaire de maille au plus 9 qui n'admet pas de $(\mathcal{I}, \mathcal{F}_1)$-partition ;
- tout graphe planaire de maille au moins 11 admet une $(\mathcal{I}, \mathcal{F}_1)$-partition.

Nous prouvons les résultats suivants :

Theorème 32. *Soit M un nombre réel tel que $M < 3$. Soit $d \geq 0$ un entier et G un graphe tel que* $\mathrm{mad}(G) < M$. *Si $d \geq \frac{2}{3-M} - 2$, alors G admet une $(\mathcal{I}, \mathcal{F}_d)$-partition.*

Theorème 33. *Soit M un nombre réel tel que $\frac{8}{3} \leq M < 3$. Soit $d \geq 0$ un entier et G un graphe tel que* $\mathrm{mad}(G) < M$. *Si $d \geq \frac{1}{3-M}$, alors G admet une $(\mathcal{I}, \mathcal{F}_d)$-partition.*

On montre les deux théorèmes précédents en appliquant la méthode du déchargement. Certaines configurations peuvent être arbitrairement grandes, suite à la construction d'une structure que nous appelons *forêt légère*.

Par la formule d'Euler, on obtient le corollaire suivant :

Corollaire 34. *Soit G un graphe planaire de maille au moins g.*

1. *Si $g \geq 7$, alors G admet une $(\mathcal{I}, \mathcal{F}_5)$-partition.*
2. *Si $g \geq 8$, alors G admet une $(\mathcal{I}, \mathcal{F}_3)$-partition.*
3. *Si $g \geq 10$, alors G admet une $(\mathcal{I}, \mathcal{F}_2)$-partition.*

Les corollaires 34.2 et 34.3 viennent du théorème 33, alors que le corollaire 34.1 vient du théorème 32. Voir la table 2 pour un résumé des résultats sur les partitions des sommets de graphes planaires vus précédemment.

Pour autant que l'on sache, les graphes planaires de maille 7, 8 et 10 respectivement pourraient tous avoir une $(\mathcal{I}, \mathcal{F}_d)$-partition pour $d = 3$, $d = 2$ et $d = 1$ respectivement. Cependant, observons :

Remarque 35. *Pour tout entier d, il existe un graphe de degré moyen maximum strictement inférieur à $M = 3(1 - \frac{1}{8d})$, qui n'a pas de $(\mathcal{I}, \Delta_d)$-partition (et donc pas de $(\mathcal{I}, \mathcal{F}_d)$-partition).*

Comme $M = 3(1 - \frac{1}{8d})$ est équivalent à $d = \frac{3}{8(3-M)}$, cela montre que le Théorème 33 peut au plus être amélioré d'un facteur $\frac{3}{8}$.

Un graphe est un *chemin* si ses sommets peuvent être notés v_1, v_2, ..., v_n de telle manière que les arêtes sont les $v_i v_{i+1}$ pour i dans $\{1, 2, ..., n-1\}$. Les sommets v_1 et v_n sont les *extremités* du chemin. Un graphe G est *connexe* si pour toute paire de sommets de G, ces sommets sont les extremités d'un sous-graphe de G qui est un chemin. Dans un graphe G, les composantes connexes de G sont les sous-graphes connectés maximaux (pour la relation de sous-graphe) de G. Pour tout entier k, on note $\mathcal{O}_k$ la classe des graphes dont chaque composante connexe a au plus k sommets.

Esperet et Ochem [32] ont montré le théorème suivant :

Theorème 36 (Esperet et Ochem [32])**.** *Tout graphe planaire de maille au moins 6 admet une $(\mathcal{O}_{12}, \mathcal{O}_{12})$-partition.*

Borodin et Ivanova [12] donnent une partition pour les graphes planaires de maille au moins 7:

Theorème 37 (Borodin et Ivanova [12])**.** *Tout graphe planaire de maille au moins 7 admet une $(\mathcal{O}_3, \mathcal{O}_3)$-partition.*

Classes	Partitions	Références
Graphes planaires	$(\mathcal{I},\mathcal{I},\mathcal{I},\mathcal{I})$ $(\mathcal{I},\mathcal{F},\mathcal{F})$ $(\mathcal{F}_2,\mathcal{F}_2,\mathcal{F}_2)$ $(\mathcal{F},\mathcal{D}_2)$ $(\mathcal{I},\mathcal{D}_3)$	Le théorème des quatre couleurs [5, 6] Borodin [9] Poh [47] Thomassen [51] Thomassen [52]
Graphes planaires de maille 4	$(\mathcal{I},\mathcal{I},\mathcal{I})$ $(\mathcal{F},\mathcal{F})$ $(\mathcal{F}_5,\mathcal{F})$	Grötzsch [36] Folklore **Théorème 7**
Graphes planaires de maille 5	$(\mathcal{I},\mathcal{F})$	Borodin et Glebov [11]
Graphes planaires de maille 6	pas $(\mathcal{I},\Delta_d)$	Borodin et al. [13]
Graphes planaires de maille 7	pas $(\mathcal{I},\Delta_2)$ $(\mathcal{I},\Delta_4)$ $(\mathcal{I},\mathcal{F}_5)$	Montassier et Ochem [44] Borodin and Kostochka [15] **Corollaire 34**
Graphes planaires de maille 8	$(\mathcal{I},\Delta_2)$ $(\mathcal{I},\mathcal{F}_3)$	Borodin et Kostochka [15] **Corollaire 34**
Graphes planaires de maille 9	pas $(\mathcal{I},\Delta_1)$	Esperet et al.[30]
Graphes planaires de maille 10	$(\mathcal{I},\mathcal{F}_2)$	**Corollaire 34**
Graphes planaires de maille 11	$(\mathcal{I},\Delta_1)$	Kim, Kostochka et Zhu [42]

Table 2: Quelques partitions de graphes planaires.

Notons que $\mathcal{O}_2 = \Delta_1$, donc le théorème 28 peut être reformulé de la manière suivante :

Theorème 28 (Kim, Kostochka et Zhu [42], reformulé)**.** *Tout graphe sans triangles de degré moyen maximum au plus* $\frac{11}{9}$ *admet une* $(\mathcal{I},\mathcal{O}_2)$*-partition.*

Corollaire 29 (reformulé)**.** *Tout graphe planaire de maille au moins* 11 *admet une* $(\mathcal{I},\mathcal{O}_2)$*-partition.*

Le théorème suivant est aussi démontré grâce à la méthode du déchagement. Là encore, certaines configurations peuvent être arbitrairement grandes.

Theorème 38. *Tout graphe* G *tel que* $\mathrm{mad}(G) < \frac{5}{2}$ *admet une* $(\mathcal{I},\mathcal{O}_3)$*-partition.*

Avec la formule d'Euler, on obtient le résultat suivant :

Corollaire 39. *Tout graphe planaire de maille au moins* 10 *admet une* $(\mathcal{I},\mathcal{O}_3)$*-partition.*

Comme dans les graphes planaires de maille au moins 10, il n'y a pas de triangles, le corollaire 39 implique le corollaire 34.3, c'est à dire que tout graphe planaire de maille au moins 10 admet une $(\mathcal{I},\mathcal{F}_2)$-partition.

Pour étendre notre résultat aux $(\mathcal{I},\mathcal{O}_k)$-partitions pour d'autres valeurs de k nous prouvons le théorème suivant.

Theorème 40. *Soit* $k \geq 2$ *un entier. Tout graphe* G *tel que* $\mathrm{mad}(G) < \frac{8k}{3k+1} = \frac{8}{3}\left(1 - \frac{1}{3k+1}\right)$ *admet une* $(\mathcal{I},\mathcal{O}_k)$*-partition.*

Ce dernier théorème est prouvé à l'aide de la méthode du déchargement. Certaines configurations interdites peuvent être arbitrairement grandes (même à k fixé), et ces configurations ainsi que la procédure de déchargement reposent sur des constructions complexes dépendant directement de $(\mathcal{I},\mathcal{O}_k)$-partitions de sous-graphes du graphe considéré. Il est intéressant de noter que par conséquent, la preuve ne donne pas un algorithme en temps polynomial pour construire les partitions voulues.

Par conséquent, pour $k = 9$, on obtient que tout graphe tel que $\mathrm{mad}(G) < \frac{18}{7}$ admet une $(\mathcal{I},\mathcal{O}_9)$-partition, et donc, par la formule d'Euler, il suit :

Corollaire 41. *Tout graphe planaire de maille au moins* 9 *admet une* $(\mathcal{I}, \mathcal{O}_9)$*-partition.*

On peut également voir les graphes planaires de grandes mailles comme des graphes planaires avec un ensemble de cycles interdits. Il est alors naturel de chercher, pour diverses partitions, quels sont les ensembles S de cycles tels que les graphes planaires sans cycles dans S admettent de telles partitions.

Choi, Liu et Oum [38] ont prouvé qu'il existe deux ensembles minimaux pour l'inclusion S_1 et S_2 de cycles tels que pour $i \in \{1, 2\}$, tout graphe planaire sans cycles appartenant à S_i admet une $(\mathcal{I}, \Delta_d)$-partition pour une certaine constante d.

- S_1 est l'ensemble des cycles de taille impaire. Les graphes sans cycles dans S_1 sont exactement les graphes bipartis, i.e. les graphes qui admettent une $(\mathcal{I}, \mathcal{I})$-partition.
- S_2 est l'ensemble des cycles de taille 3, 4 et 6. Les graphes planaires sans cycles dans S_2 admettent une $(\mathcal{I}, \Delta_{45})$-partition.

Cela soulève la question suivante :

Question 42. *Quel est le plus petit entier* d *tel que tout graphe planaire sans cycles dans* S_2 *admet une* $(\mathcal{I}, \Delta_d)$*-partition.*

Notons qu'interdire les sous-graphes de S_2 comme sous-graphe ou comme sous-graphe induit est équivalent.

On prouve le théorème suivant à l'aide de la méthode du déchargement. On utilise des classes de configurations interdites arbitrairement grandes, bien que les règles de déchargement elles-mêmes soient locales.

Theorème 43. *Tout graphe sans cycles dans* S_2 *admet une* $(\mathcal{I}, \Delta_6)$*-partition.*

Notons que comme les graphes planaires de maille au moins 7 sont les graphes planaires sans cycles dans $S_2 \cup \{5\}$, le théorème précédent peut être comparé au Corollaire 26.

Corollaire 26 (Borodin et Kostochka [15], rappel)**.** *Tout graphe planaire de maille au moins* 7 *admet une* $(\mathcal{I}, \Delta_4)$*-partition, et tout graphe planaire de maille au moins* 8 *admet une* $(\mathcal{I}, \Delta_2)$*-partition.*

On prouve également le théorème suivant grâce à une réduction en temps polynomial depuis le problème de l'existence d'une $(\mathcal{I}, \Delta_1)$-partition pour les graphes planaires de maille au moins 9 [31].

Theorème 44. *Pour tout entier* $k \geq 1$*, soit tout graphe planaire sans cycles dans* S_2 *admet une* $(\mathcal{I}, \Delta_k)$*-partition, ou décider si un graphe planaire sans cycles dans* S_2 *admet une* $(\mathcal{I}, \Delta_k)$*-partition est un problème NP-complet.*

De plus, on construit un graphe planaire sans cycles dans S_2 qui n'admet pas de $(\mathcal{I}, \Delta_3)$-partition. Cela implique le corollaire suivant :

Corollaire 45. *Décider si un graphe planaire sans cycles dans* S_2 *admet une* $(\mathcal{I}, \Delta_3)$*-partition est un problème NP-complet.*

Summary

In this manuscript, we will talk about vertex partitions of sparse graphs. The study of vertex partitions of graphs originated with the Four Colour Theorem, that states that the vertex set of every planar graph can be partitioned into four sets such that no edge has both of its endpoints in one of those sets.

An *induced subgraph* of a graph G is a subgraph of G that is solely obtained by removing vertices (and their pendent edges) from G. An *induced forest* of a graph G is an induced subgraph of G that has no cycles. Albertson and Berman [3] conjectured that every planar graph has an induced forest containing at least $\frac{1}{2}$ of its vertices. One motivation for this conjecture was that it implies that every planar graph has an independent set on at least $\frac{1}{4}$ of its vertices, since every acyclic graph is bipartite. That last result is a corollary of the Four Colour Theorem, the proof of which was computer assisted, and an independent proof of that result would be appreciated.

Akiyama and Watanabe [1] on the one hand, and Albertson and Haas [2] on the other hand, independently conjectured that every bipartite planar graph has an induced forest on $\frac{5}{8}$ of its vertices. In an attempt to prove that conjecture, Alon, Mubayi, and Thomas [4] showed that every triangle-free planar graph admits an induced forest on at least $\frac{1}{2}$ of its vertices. That bound was later improved by Salavatipour [49], who showed that every triangle-free planar graph with n vertices and m edges admits an induced forest with at least $\frac{29n-6m}{32}$ vertices, and thus at least $\frac{17n+24}{32} \approx 0.531n$ vertices. We show that every triangle-free planar graph with n vertices and m edges admits an induced forest with at least $\frac{38n-7m}{44}$ vertices, and thus at least $\frac{6n+7}{11} \approx 0.545n$ vertices. The proof is presented in Section 2.3. See also the arXiv version [24].

An *induced linear forest* in a graph is an induced forest with maximum degree at most 2. Chappell conjectured that every planar graph with n vertices admits an induced linear forest with at least $\frac{4n}{9}$ vertices. Poh [47] proved that the vertices of any planar graph can be partitioned into three sets inducing linear forests, and thus that every planar graph admits an induced linear forest on at least $\frac{1}{3}$ of its vertices. On outerplanar graphs, Pelsmajer [45] proved that every outerplanar graph with n vertices admits an induced linear forest with at least $\frac{4n+2}{7}$ vertices, which is tight. We conjecture that every triangle-free planar graph has an induced linear forest on at least $\frac{1}{2}$ of its vertices, and prove that if this is true, it is tight. As a first step, we prove that every triangle-free planar graph of order n and size m admits an induced linear forest of order at least $\frac{9n-2m}{11} \geq \frac{5n+8}{11}$. The proof is presented in Section 2.4. See also the arXiv version [27].

The *girth* of a graph is the length of a smallest cycle in the graph. Shi and Xu [50] on the one hand, and Kelly and Liu [41] on the other hand, independently proved that every connected planar graph of girth at least 5, with n vertices and m edges has an induced forest on at least $\frac{8n-2m-2}{7}$ vertices, and thus at least $\frac{2n-2}{3}$ vertices. We conjecture that for all g, every planar graph with n vertices, m edges and girth at least g admits an induced forest on at least $n - \frac{m}{g}$ vertices. If that conjecture is true, it is the best possible, since a disjoint union of k cycles of length g has kg vertices and kg edges, and its largest induced forest has $k(g-1)$ vertices. There is a trivial bound of $n - \frac{2m}{g}$ vertices since removing $\frac{2m}{g}$ edges is enough to get a forest. We prove that such a graph admits an induced forest on at least $n - \frac{4m}{3g}$, improving on known lower bounds for $g \geq 8$. The proof is presented in Sections 2.5 and 2.6. Those results were published in Discrete Applied Mathematics [25].

The presence of a large induced forest in a graph G can be seen as a partition of the vertices of G into two sets: one that induces a forest and one whose size is bounded in terms of the number of vertices of G. One can think of other conditions on the second set of vertices, for instance that it induces an other graph

with nice properties (for example a second forest). That leads us to study vertex partitions.

We denote by $\mathcal{I}$ the class of empty graphs, which are graphs with no edges, and by $\mathcal{F}$ the class of forests, which are graphs with no cycles. For all d, we denote by $\mathcal{F}_d$ the class of forests with maximum degree at most d. For $\mathcal{C}_0$ and $\mathcal{C}_1$ two classes of graphs, a $(\mathcal{C}_0, \mathcal{C}_1)$-partition of a graph G is a partition of the vertices of G into two sets C_0 and C_1, such that for $i \in \{0, 1\}$, the graph induced by C_i in G is in $\mathcal{C}_i$.

It is easy to show that every triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F})$-partition, which corresponds to an $(\mathcal{F}, \mathcal{F}_d)$-partition where d goes to infinity. Borodin and Glebov [11] proved that every planar graph of girth at least 5 admits an $(\mathcal{I}, \mathcal{F})$-partition, that is an $(\mathcal{F}, \mathcal{F}_0)$-partition. We raise the question for all d whether every triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F}_d)$-partition. We show that the answer is yes for $d \geq 5$, and that if the answer is no for a given d, it is NP-difficult to decide whether a triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F}_d)$-partition. The proofs are presented in Sections 3.2 and 3.3. Those results were presented at Eurocomb 2015 and published in European Journal of Combinatorics [28].

For all d, let Δ_d be the class of graphs with maximum degree at most d. The *maximum average degree* of a graph is the maximum of the average degrees of its subgraphs. A lot of research has been done on $(\mathcal{I}, \Delta_d)$-partitions of sparse graphs. Borodin and Kostochka [15] showed that every planar graph of girth at least 7 (resp. 8) admits an $(\mathcal{I}, \Delta_4)$-partition (resp. an $(\mathcal{I}, \Delta_2)$-partition), and Kim, Kostochka, and Zhu [42] proved that every planar graph of girth at least 11 admits an $(\mathcal{I}, \Delta_1)$-partition (or, equivalently, admits an $(\mathcal{I}, \mathcal{F}_1)$-partition). We focused on $(\mathcal{I}, \mathcal{F}_d)$-partitions and showed that every planar graph of girth at least 7 (resp. 8, 10) admits an $(\mathcal{I}, \mathcal{F}_d)$-partition for $d = 5$ (resp. 3, 2). Our results, as well as those of Borodin and Kostochka [15] and Kim, Kostochka, and Zhu [42] are corollaries of more general results on graphs with low maximum average degree. The proofs are presented in Sections 3.4 and 3.5. See also the arXiv version [26].

For all k, let $\mathcal{O}_k$ be the class of graphs whose components have at most k vertices. We prove that every planar graph of girth at least 10 admits an $(\mathcal{I}, \mathcal{O}_3)$-partition. We also show that for all k, every graph G with maximum average degree less than $\frac{8k}{3k+1} = \frac{8}{3}\left(1 - \frac{1}{3k+1}\right)$ admits an $(\mathcal{I}, \mathcal{O}_k)$-partition, implying that every planar graph of girth at least 9 admits an $(\mathcal{I}, \mathcal{O}_9)$-partition. Again, those results are corollaries of results on graphs with low maximum average degree. The proofs are presented in Sections 4.2, 4.3, and 4.4.

The class $\mathcal{C}$ of (C_3, C_4, C_6)*-free planar graphs* is the class of planar graphs with no subgraph (or equivalently no induced subgraph) that is a cycle of size 3, 4 or 6. Choi, Liu, and Oum [38] showed that there are two minimal sets of cycles such that planar graphs with no cycles in one of those sets admit an $(\mathcal{I}, \Delta_d)$-partition for some d. Those sets are the set of all odd cycles on the one hand (graphs with no odd cycles are bipartite and thus have $(\mathcal{I}, \Delta_0)$-partitions), and the set of cycles with size 3, 4, and 6 on the other hand. They showed that graphs in $\mathcal{C}$ admit an $(\mathcal{I}, \Delta_{45})$-partition, and we show that such graphs have an $(\mathcal{I}, \Delta_6)$-partition. We also show that there exists a graph in $\mathcal{C}$ that has no $(\mathcal{I}, \Delta_3)$-partition. The proofs are presented in Sections 4.5 and 4.6. See also the arXiv version [29].

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Chapter 1

Introduction

Contents

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The study of colouring of planar graphs originated in 1852 with the famed Four Colour Conjecture made by Francis Guthrie. Allegedly, he was trying to colour a map of British counties such that no two counties sharing a border have the same colour, and noticed that he needed exactly four colours. He then conjectured that this result could be generalised to any map. He posed this problem to his younger brother Frederick Guthrie, who posed it to his professor Augustus De Morgan [54].

The first proof of this result was published by Kempe in 1879, but this proof was then disproved by Heawood in 1890. The conjecture then remained unproven until 1976, when it was proven by Appel, Haken and Koch [5, 6] with the help of computer computation. This proof was based on a method introduced in 1948 by Heesch, the discharging method.

When it was proven, the Four Colour Theorem was now presented in terms of graphs, using the fact that for any planar map, taking a vertex for each region of the map, and adding an edge between two vertices if the corresponding regions share a boundary leads to a planar graph.

Theorem 1.0.1 (Appel, Haken, and Koch [5, 6])**.** *Every planar graph admits a proper colouring with at most four colours.*

The work on the Four Colour Theorem initiated a lot of research on the colourings of graphs, especially planar graphs [40]. In this thesis, we will prove some results on those generalised colourings for sparse graphs.

1.1 Definitions and notation

A *graph* $G = (V, E)$ is a pair of sets, V and E, such that E is a set of unordered pairs of elements of V. The elements of V are called the *vertices* of G, and the elements of E are called the *edges* of G. The set of vertices of a graph G will be denoted by $V(G)$. The set of edges of a graph G will be denoted by $E(G)$. The *order* of a graph G, denoted by $|G|$, is $|V(G)|$, and the *size* of a graph G, denoted by $||G||$, is $|E(G)|$. See Figure 1.1.1 for an illustration of a graph of order 5 and size 7.

An edge $\{v, w\}$, denoted by vw or wv, is *incident* to the vertices v and w, and those vertices are called the *endpoints* of vw. Two vertices that are incident to the same edge are *adjacent*. The *neighbours* of a

vertex v in G are the vertices that are adjacent to v. If the graph G needs to be specified, we say that a neighbour of a vertex v in G is a *G-neighbour* of v. For a vertex v, the set of the neighbours of v, called the *open neighbourhood* of v, is denoted by $N(v)$, and $N[v] = N(v) \cup \{v\}$ is the *closed neighbourhood* of v. For all set $W \subset V(G)$, $N[W] = \bigcup_{v \in W} N[v]$ is the closed neighbourhood of W, and $N(W) = N[W] \setminus W$ is the open neighbourhood of W. For any set S of vertices, we say that any vertex in S is an *S-vertex*, and if a vertex in S is a neighbour of a vertex v, then we say it is an *S-neighbour* of v. For example, in the graph G_0 represented in Figure 1.1.1, $N(v_1) = \{v_0, v_2, v_4\}$, $N[v_1] = \{v_0, v_1, v_2, v_4\}$, $N(\{v_0, v_1\}) = \{v_2, v_3, v_4\}$, and $N[\{v_0, v_1\}] = \{v_0, v_1, v_2, v_3, v_4\}$.

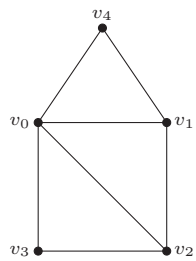

Figure 1.1.1: The graph G_0. The circles are vertices and the lines are edges.

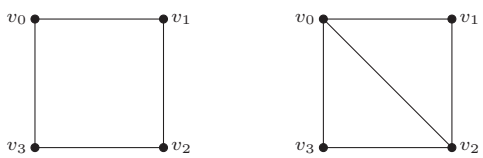

Figure 1.1.2: The graphs G_1 (left) and G_2 (right).

A *graph isomorphism* between two graphs G and H is a bijection f from $V(G)$ into $V(H)$ such that for all pair of vertices v and w in $V(G)$, $vw \in E(G)$ if and only if $f(v)f(w) \in E(H)$. If there exists an isomorphism between two graphs G and H, then G and H are *isomorphic*. A *copy* of a graph G is a graph that is isomorphic to G. Generally speaking, we will work up to isomorphism, saying that two graphs are equal if they are isomorphic.

A *subgraph* H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* H of a graph G is a subgraph of G such that $E(H)$ is the set of unordered pairs of vertices of H that are in $E(G)$. For a set $W \subseteq V(G)$ of vertices of a graph G, the subgraph of G *induced* by W, denoted by $G[W]$, is the induced subgraph of G with vertex set W. In Figure 1.1.2, the left graph (G_1) is a subgraph of G_0 (Figure 1.1.1), but it is not an induced subgraph of G_0. The one on the right (G_2) is an induced subgraph of G_0.

Let $G = (V, E)$ be a graph. For a set $W \subseteq V$, we denote by $G - W$ the graph $G[V \setminus W]$. For simplicity, for $v \in V$, we denote $G - \{v\}$ by $G - v$. For a set $F \subseteq E$, we denote by $G - F$ the graph $(V, E \setminus F)$. For simplicity, for $e \in E$, we denote $G - \{e\}$ by $G - e$. For F a set of unordered pairs of elements of V such that $F \cap E = \emptyset$, $G + F = (V, E \cup F)$. For simplicity, when F is a singleton, we denote $G + \{e\}$ by $G + e$. For example, $G_2 = G_0 - v_4$, and $G_1 = G_2 - v_0v_2$ (see Figures 1.1.1 and 1.1.2). The action of *subdividing* an edge $e = vw$ consists in considering a vertex $x \notin V$, and building the graph $(V \cup x, (E \setminus \{e\}) \cup \{vx, xw\})$.

The *degree* of a vertex v is the number of edges incident to v. The degree of a vertex v in a graph G is denoted by $d_G(v)$, or simply by $d(v)$ if there is no ambiguity on the graph G. For all $d \in \mathbb{N}$, a *d-vertex*, *d^+-vertex*, or *d^--vertex* denote a vertex with degree d, at least d, or at most d respectively. For all $d \in \mathbb{N}$ and all v in V, a *d-neighbour*, *d^+-neighbour*, or *d^--neighbour* of v denote a neighbour of v with degree d, at least d, or at most d respectively. The *maximum degree*, resp. *minimum degree*, of a graph is the maximum,

resp. minimum, of the degrees of the graph. A graph is *d-regular* for some integer d if all of its vertices are d-vertices, and a 3-regular graph is called a *cubic* graph. For example, in G_0 (Figure 1.1.1), the vertices v_3 and v_4 are 2-vertices, the vertices v_1 and v_2 are 3-vertices, and the vertex v_0 is a 4-vertex.

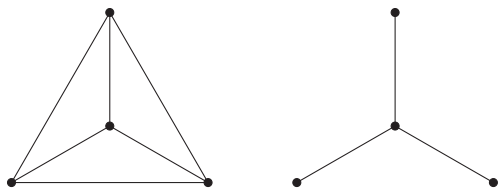

Figure 1.1.3: The graphs K_4 (left) and $K_{1,3}$ (right)

An *independent set* of a graph G is a set of vertices of G that are not adjacent to one another. A *clique* of a graph G is a set of vertices that are pairwise adjacent. A graph whose set of vertices is independent is called an *empty graph*, and a graph whose set of vertices is a clique is called a *complete graph*. In G_0 (Figure 1.1.1), the set $\{v_0, v_1, v_2\}$ is a clique, and $\{v_1, v_3\}$ is an independent set. For all k, the complete graph on k vertices is denoted by K_k. See Figure 1.1.3 (left) for an illustration of K_4. A *path* of *length* k, denoted by P_{k+1}, is the graph with $k+1$ vertices $\{v_0, v_1, ..., v_k\}$ such that for $i \in \{0, ..., k-1\}$, v_i is adjacent to v_{i+1}. The *cycle* of length k, or k-*cycle*, denoted C_k, is the graph with k vertices $\{v_0, v_1, ..., v_{k-1}\}$ such that for $i \in \{0, ..., k-2\}$, v_i is adjacent to v_{i+1}, and such that v_0 is adjacent to v_{k-1}. A k^+-*cycle* and a k^--*cycle* denote a cycle of length at least k and at most k respectively. The graph G_1 (Figure 1.1.2, left) is a 4-cycle.

A graph is *bipartite* if its set of vertices can be partitioned into two independent sets V_1 and V_2. Such a partition into two sets is called a *bipartition* of the graph, and the sets V_1 and V_2 are called the *partite sets*. A graph G is a *complete bipartite graph* if it has a bipartition into sets V_1 and V_2, such that for all $(v_1, v_2) \in V_1 \times V_2$, $v_1v_2 \in E(G)$. For all k_1 and k_2, the complete bipartite graph with partite sets of size k_1 and k_2 is denoted by K_{k_1,k_2}. For all k, the graph $K_{1,k}$ is called a *star*. See Figure 1.1.3 (right) for an illustration of $K_{1,3}$.

For two graphs G and H, a subgraph of G isomorphic to H is directly called an H of G. For example, a cycle of length three, or *triangle*, of a graph G is a subgraph of G isomorphic to C_3. Therefore, a graph G has no H, or is H-*free*, if it has no subgraph isomorphic to H. If a graph G has a cycle C on k vertices y_0, y_1, ..., y_{k-1} such that for all $i \in \{0, ..., k-2\}$, y_i is adjacent to y_{i+1} in C, and such that y_0 is adjacent to y_{k-1} in C, then we denote this cycle by $y_0y_1y_2...y_{k-1}$. For instance, the cycle $v_0v_1v_4$ is a triangle of G_0 (Figure 1.1.1). As another example, a path in a graph G is a subgraph of G isomorphic to a path. The *endvertices* of a path are the vertices of degree 1 in the path.

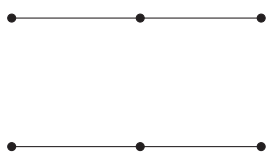

Figure 1.1.4: The graph $2P_3$.

A graph G is *connected* if any two vertices of G are the endvertices of a path. For example, G_0 (Figure 1.1.1) is connected, while the graph in Figure 1.1.4 is *disconnected*. Two vertices that are the endpoints of a path are said to be *connected* by this path. The *distance* between two vertices in a graph G is the length of a shortest path connecting these two vertices, if such a path exists. If two vertices are not connected by a path, then their distance is infinite. The distance between two vertices v and w in a graph G is denoted by $d_G(v, w)$, or simply by $d(v, w)$ if there is no ambiguity on the graph G. In a graph G, the *components* of a graph G are the maximal connected induced subgraphs of G. A *vertex cut* in a graph G is a set $S \subseteq V(G)$

such that $G - S$ is not connected. An *edge cut* in a graph G is a set $F \subseteq E(G)$ such that $G - F$ is not connected. For all $k \in \mathbb{N}$, a graph G is *k-connected* if $|G| \geq k+1$ and G has no vertex cut of size at most $k-1$. For all $k \in \mathbb{N}$, a graph G is *k-edge-connected* if it has no edge cut set of size at most $k-1$. In a connected graph, a vertex v such that $\{v\}$ is a vertex cut is a *cutvertex*, and an edge e such that $\{e\}$ is an edge cut is a *bridge*. In a graph G, a vertex cut S or an edge cut F *separates* two vertices if those vertices are in different components of $G - S$ or $G - F$ respectively. For example, the graph G_0 (Figure 1.1.1) is 2-connected and 2-edge-connected. The set $\{v_0, v_1\}$ is a vertex cut that separates v_2 and v_4, while $\{v_0v_3, v_2v_3\}$ is an edge cut that does not separate them. Note that cuts need not be minimal. For example, $\{v_0, v_1, v_3\}$ is also a vertex cut of G_0.

The *disjoint union* of two graphs G and H with disjoint sets of vertices is the graph $(V(G) \cup V(H), E(G) \cup E(H))$. If the sets of vertices are not actually disjoint, we take a copy of H such that they are disjoint. We denote by $G + H$ the disjoint union of the graphs G and H. For G a graph and for $n \in \mathbb{N}$, we define nG inductively by $0G = (\emptyset, \emptyset)$, and for all $n \in \mathbb{N}$, $(n+1)G = nG + G$. See Figure 1.1.4 for an illustration of the graph $2P_3$.

A *forest*, or *acyclic graph*, is a graph with no cycle. A *tree* is a connected forest. A *linear forest* is a forest with maximum degree at most two. Note that the components of a linear forest are paths. The *girth* of a graph G is the size of a smallest cycle in G. The girth of a forest is infinite. A *star forest* is a P_4-free forest. In other words, it is a graph whose components are stars. The graphs $K_{1,3}$ (Figure 1.1.3, right) and $2P_3$ (Figure 1.1.4) are forests, and even star forests. The graph $2P_3$ is also a linear forest. For $k \in \mathbb{N}$, a graph G is *k-degenerate* if every subgraph of G has a k-vertex. Forests are exactly 1-degenerate graphs.

We denote by $\mathcal{I}$ the class of empty graphs and by $\mathcal{F}$ the class of forests. For all $d \in \mathbb{N}$, we denote by Δ_d the class of graphs with maximum degree at most d, by $\mathcal{F}_d$ the class of forests with maximum degree at most d, by $\mathcal{D}_d$ the class of d-degenerate graphs, and by O_d the class of graphs whose components have order at most d.

A *colouring* or *vertex partition* of a graph G is a partition of $V(G)$ into several disjoint sets, which are called *colour classes*. A *proper colouring* of a graph G is a colouring of G such that every colour class is an independent set. An *acyclic colouring* of a graph G is a proper colouring of G such that the union of any two colour classes induces a forest.

For any classes of graphs $\mathcal{C}_1$, $\mathcal{C}_2$, ..., $\mathcal{C}_n$, a *$(\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n)$-partition* $(S_1, S_2, ..., S_n)$ of a graph G is a partition of the vertices of G into n sets S_1, ..., S_n such that for all $i \in \{1, ..., n\}$, $G[S_i] \in \mathcal{C}_i$. For example, $(\{v_0\}, \{v_1, v_2, v_3, v_4\})$ is an $(\mathcal{I}, \mathcal{F}_2)$-partition of G_0 (Figure 1.1.1). Note that an $(\mathcal{I}, \mathcal{I}, ..., \mathcal{I})$-partition is exactly a proper colouring.

On further figures, we will sometimes draw only parts of some graphs, and we will use the drawing conventions described in Figure 1.1.5 for all of our figures.

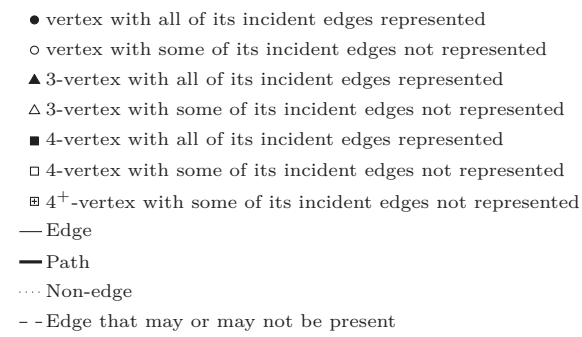

Figure 1.1.5: The drawing conventions for the figures.

A *directed graph* $G = (V, A)$ is a pair of sets, V and A, such that A is a set of ordered pairs of elements of V. The difference with graphs is that the elements of A, called *arcs*, are ordered pairs, contrary to the edges of a graph. In a directed graph, the arc uv corresponds to the pair (u, v) and is different from the arc

vu. In a directed graph, an arc uv is said to go from u towards v. For all vertex v, an arc that goes from v towards another vertex is an *outgoing arc* of v, and one that goes towards v is an *ingoing arc* of v. The *indegree* of a vertex v, denoted $d^-(v)$, is the number of ingoing arcs of v, and the *outdegree* of a vertex v, denoted $d^+(v)$, is the number of outgoing arcs of v.

A directed graph (V, A) is an *oriented graph* if for all arc uv in A, the arc vu does not belong to A. In a graph G, an *orientation* of G is a function which, with each edge uv of G, associates one of the corresponding ordered pairs, either (u, v) or (v, u). A graph together with an orientation yields an oriented graph. When we have a graph with an orientation, we will call the outgoing (resp. ingoing) arcs of a vertex in the corresponding oriented graph the *outgoing* (resp. *ingoing*) *edges* of the vertex.

A *rooted tree* is a tree where some specific vertex is called the *root* of the tree. With a rooted tree is associated a canonical orientation, where each edge is oriented from the endpoint farther from the root towards the endpoint closer to the root (in terms of distance in the graph). Such an orientation is called a bottom-up orientation. A rooted forest is a forest where every component is rooted. In a rooted forest, each vertex v that is not a root has one outgoing edge in that orientation, and the other endpoint of that edge is called the *father* of v. If a vertex w is the father of a vertex v, then v is a *son* of w.

A *multigraph* $G = (V, E)$ is a pair where V is a set of vertices and E is a multiset of unordered pairs of vertices and singletons of vertices. The elements of E are still called edges, and the singletons of E are called *loops*.

1.2 Sparse graphs

In this thesis, we will prove properties on some classes of graphs, mainly on sparse graphs. We will use several notions of sparsity, as opposed to density, throughout this thesis. Informally, a graph is dense if it has many edges, and sparse otherwise. The *average degree* of a graph G, noted $\mathrm{ad}(G)$, is the average of the degrees of the vertices of G, that is $\frac{2||G||}{|G|}$. One could take the average degree of a graph as a measure of its density, but that notion does not capture the local structure of the graph. For instance, a graph composed of a clique and a large independent set has a rather low average degree, though the clique itself is dense. We will need more local notions of density, so that a subgraph of a sparse graph is sparse.

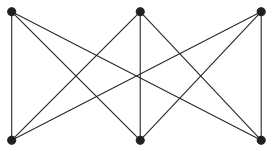

Figure 1.2.1: The graph $K_{3,3}$.

A *drawing* or *embedding* of a graph G is a bijection of the set $V(G)$ with a set of points in a surface Σ, and a bijection of the set $E(G)$ with a set of curves in the surface Σ, such that the image of an edge vw is a curve with v and w as its extremities. A graph is *planar* if it can be drawn into the plane such that no two edges intersect outside of their endpoints. This drawing is a *planar embedding* of the graph. A *plane graph* is the planar embedding of a planar graph. All the graphs represented in the figures of Section 1.1 are planar graphs, and their representations are planar embeddings. The graph $K_{3,3}$ (Figure 1.2.1) is not a planar graph.

In a plane graph, a *face* is a maximum connected surface that do not contain any edges and vertices. The edges and vertices that are in contact with some face ϕ of a plane graph G form a subgraph called the *boundary* of ϕ and denoted by $G[\phi]$. Two faces are *adjacent* if they share an edge in their boundary. The *degree* of a face ϕ, denoted by $d(\phi)$, is the number of edges that are in its boundary, counting twice the edges that are not in the boundary of any other face. A *d-face*, *d^+-face*, or *d^--face* denote a face with degree d, at least d, or at most d respectively. In a plane graph G, let $F(G)$ denote the set of faces of G. For example, the graph G_0 (Figure 1.1.1) has four faces, bounded by the cycles $v_0v_1v_2$, $v_0v_1v_4$, $v_0v_2v_3$, and $v_0v_3v_2v_1v_4$.

Note that in a plane graph, any cycle corresponds to a close curve, that thus separates the plane into two parts, the interior and the exterior. If a cycle is not in the boundary of a face, then it is a *separating cycle*. Note that if C is a separating cycle in a graph G, then $G - V(C)$ is disconnected. A cycle C *separates* two vertices if one of those two vertices is in the interior of C, and the other one is in the exterior of C. Note that if a cycle C separates two vertices, then those vertices are in different components of $G - V(C)$, i.e. $V(C)$ separates those vertices.

A planar graph is *outerplanar* if it admits a planar embedding where every vertex is on the boundary of the same face. For example, G_0 (Figure 1.1.1) is outerplanar, while K_4 (Figure 1.1.3, left) is not.

The *dual* of a plane graph G is a multigraph $\widehat{G}$ whose vertices are the faces of G and such that for all edge e of G, there is an edge between the two faces containing e in their boundaries (or the only face containing e in its boundary if there is only one such face).

For all $g \in \mathbb{N}$ with $g \geq 3$, let $\mathcal{P}_g$ denote the class of planar graphs with girth at least g. Note that $\mathcal{P}_3$ is the class of planar graphs and that $\mathcal{P}_4$ is the class of triangle-free planar graphs. The density of the graphs in $\mathcal{P}_g$ decreases, and thus the sparsity increases, as g augments. Indeed, the larger the girth, the closer the graph is to a tree, where the number of edges is the number of vertices minus one.

Another notion of graph density is the maximum average degree. The *maximum average degree* of a graph G, denoted by $\mathrm{mad}(G)$, is the maximum over all subgraphs H of G of $\mathrm{ad}(H)$. For example, the maximum average degree of G_0 (Figure 1.1.1) is $\frac{14}{5}$, and is equal to its average degree. The graph $G_0 + K_1$ has an average degree of $\frac{5}{2}$, but its maximum average degree is still $\frac{14}{5}$ since it contains G_0 as a subgraph.

Those two notions are not independent. Indeed, by Euler's formula, a connected plane graph of order $n \geq 1$, size m, and f faces verifies $n - m + f = 2$. Moreover, a planar graph of girth at least g has all of its faces of degree at least g, and thus $2m \geq fg$, as every edge is either in the boundary of two faces, or counted twice in the boundary of a face. The immediate consequence is that $\frac{2m}{n} < \frac{2g}{g-2}$ for every planar graph. As a subgraph of a planar graph is planar, we can deduce that for any planar graph G with girth at least g, $\mathrm{mad}(G) < \frac{2g}{g-2}$.

Note that we can obtain the girth or the girth plus one in quadratic time thanks to a breadth first search algorithm started from each vertex, stopping when the first cycle is obtained. The girth and the maximum average degree can also be computed in polynomial time [39, 46].

1.3 Discharging method

Most of the proofs in this thesis use the discharging method. For illustration, take the proofs of Theorems 3.1.4, 3.1.12, 4.1.4, or 4.1.9. Cranston and West [19] give an introduction to the discharging method, as well as some applications to graph colouring. Borodin [10] gives a survey of application of the discharging method to graph colourings.

The basic principle of the discharging method is rather simple. We consider a property that we want to prove, and a class of graphs, usually sparse, on which we want to prove the property. The proof is by contradiction, thus we assume that the property does not hold on every graph of the class. We consider an order on the graphs of the class, for example the number of vertices, and we pick a smallest counter-example G to the property in the class according to that order.

We then prove that some configurations, that we call *forbidden configurations* or *reducible configurations*, do not appear in the graph G. To prove this, first assume that the configuration appears in G. We then usually transform the graph G into a graph G' that is smaller than G in the order we chose previously. As G is minimum according to this order, G' verifies the property. Then, we prove that this implies the graph G also verifies the property, a contradiction.

Those forbidden configurations give us some local constraints on the structure of G. The remainder of the proof is to show that there is actually no graph in the considered class that verifies all of those constraints. This is done by first giving weight to the vertices, and sometimes to the faces of an embedding of the graph G, such that the total weight of the graph is negative. The next step is to move the weight from some vertices and faces to other vertices and faces, without changing the total weight of the graph, such that in the end, every vertex and every face has non-negative weight, leading to a contradiction. This part is called

the discharging procedure, and it gives its name to the discharging method.

1.4 Publications

Here is the list of publications and manuscripts of the author. First will be the works which are presented in this manuscript, and then the works that are not, but are available as an appendix.

1.4.1 Works presented in this manuscript

Large induced forests in triangle-free planar graphs

An *induced forest* in a graph is an induced subgraph (i.e. a subgraph obtained solely by removing vertices) with no cycles. Albertson and Berman [3] conjectured that every planar graph admits an induced forest on at least one half of its vertices. If true, that would imply that every planar graph admits an independent set on at least one fourth of its vertices, the only known proof of which relies on the Four Colour Theorem. Akiyama and Watanabe [1], and independently Albertson and Haas [2] conjectured that every bipartite planar graph admits an induced forest on at least five eights of its vertices. Salavatipour [49] proved that every triangle-free planar graph of order n and size m admits an induced forest of order at least $\frac{29n-6m}{32}$ and thus at least $\frac{17n+24}{32}$. We improve that bound by showing that a triangle-free planar graph of order n admits an induced forest of order at least $\frac{6n+7}{11}$.

The proof is presented in Section 2.3. See also the arXiv version [24].

Large induced linear forests in triangle-free planar graphs

An *induced linear forest* in a graph is an induced forest with maximum degree at most 2. Chappell conjectured that every planar graph admits an induced linear forest on at least four ninths of its vertices. Poh [47] proved that the vertices of any planar graph can be partitioned into three sets inducing linear forests, and thus that every planar graph admits an induced linear forest on at least one third of its vertices. We conjecture that every triangle-free planar graph has an induced linear forest on at least one half of its vertices, and prove that if this is true, it is tight. As a first step, we prove that every triangle-free planar graph of order n and size m admits an induced linear forest of order at least $\frac{9n-2m}{11} \geq \frac{5n+8}{11}$.

The proof is presented in Section 2.4. See also the arXiv version [27].

Large induced forests in planar graphs of large girth

The *girth* of a graph is the length of a smallest cycle in the graph. Shi and Xu [50], and Kelly and Liu [41] independently proved that every connected planar graph of girth at least 5, order n, and size m has an induced forest of order at least $\frac{8n-2m-2}{7}$, and thus at least $\frac{2n-2}{3}$. We conjecture that for all g, every planar graph with n vertices, m edges, and girth at least g admits an induced forest on at least $n - \frac{m}{g}$ vertices. That would be tight if true because of the union of cycles on g vertices. There is a trivial bound of $n - \frac{2m}{g}$ vertices since removing $\frac{2m}{g}$ edges is already enough to yield a forest. We prove that such a graph admits an induced forest on at least $n - \frac{4m}{3g}$, improving on known lower bounds for $g \geq 8$.

The proof is presented in Sections 2.5 and 2.6. Those results were published in Discrete Applied Mathematics [25].

$(\mathcal{F}, \mathcal{F}_d)$-partitions of triangle-free planar graphs

For all d, an $(\mathcal{F}, \mathcal{F}_d)$*-partition* of a graph is a partition of the vertices of the graph into two sets that are the vertex sets of an induced forest and an induced forest of maximum degree at most d respectively. It is easy to show that every triangle-free planar graph admits a partition of its vertices into two sets that are the vertex sets of induced forests, which corresponds to an $(\mathcal{F}, \mathcal{F}_d)$-partition where d goes to infinity. Borodin and Glebov [11] proved that every planar graph of girth at least 5 admits an $(\mathcal{F}, \mathcal{F}_0)$-partition. We raise

the question for all d whether every triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F}_d)$-partition. We show that the answer is yes for $d \geq 5$, and that if the answer is no for a given d, it is NP-difficult to decide whether a triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F}_d)$-partition.

The proofs are presented in Sections 3.2 and 3.3. Those results were presented at Eurocomb 2015 and published in European Journal of Combinatorics [28].

$(\mathcal{I}, \mathcal{F}_d)$-partitions of planar graphs of large girth

For all d, an $(\mathcal{I}, \mathcal{F}_d)$*-partition* of a graph is a partition of the vertices of the graph into two sets that are an independent set and the vertex set of an induced forest of maximum degree at most d respectively. For all d, an $(\mathcal{I}, \Delta_d)$*-partition* of a graph is a partition of the vertices of the graph into two sets that are an independent set and the vertex set of an induced subgraph (not necessarily acyclic) of maximum degree at most d respectively. The *maximum average degree* of a graph is the maximum of the average degrees of its subgraphs. A lot of research has been done on $(\mathcal{I}, \Delta_d)$-partitions of sparse graphs. Borodin and Kostochka [15] showed that every planar graph of girth at least 7 (resp. 8) admits an $(\mathcal{I}, \Delta_4)$-partition (resp. an$(\mathcal{I}, \Delta_2)$-partition), and Kim, Kostochka, and Zhu [42] proved that every planar graph of girth at least 11 admits an $(\mathcal{I}, \Delta_1)$-partition (or, equivalently, an admits an $(\mathcal{I}, \mathcal{F}_1)$-partition). We focus on $(\mathcal{I}, \mathcal{F}_d)$-partitions and show that every planar graph of girth at least 7 (resp. 8, 10) admits an $(\mathcal{I}, \mathcal{F}_d)$-partition for $d = 5$ (resp. 3, 2). Our results, as well as those of Borodin and Kostochka [15] and Kim, Kostochka, and Zhu [42] are corollaries of more general results on graphs with low maximum average degree.

The proofs are presented in Sections 3.4 and 3.5. See also the arXiv version [26].

$(\mathcal{I}, \mathcal{O}_k)$-partitions of planar graphs of large girth

For all k, an $(\mathcal{I}, \mathcal{O}_k)$*-partition* of a graph is a partition of the vertices of the graph into two sets that are an independent set and the vertex set of an induced subgraph whose components have at most k vertices. We prove that every planar graph of girth at least 10 admits an $(\mathcal{I}, \mathcal{O}_3)$-partition. We also show that for all k, every graph G with maximum average degree less than $\frac{8k}{3k+1} = \frac{8}{3}\left(1 - \frac{1}{3k+1}\right)$ admits an $(\mathcal{I}, \mathcal{O}_k)$-partition, implying that every planar graph of girth at least 9 admits an $(\mathcal{I}, \mathcal{O}_9)$-partition. Again, those results are corollaries of results on graphs with low maximum average degree.

The proofs are presented in Sections 4.2, 4.3, and 4.4.

$(\mathcal{I}, \Delta_d)$-partitions of (C_3, C_4, C_6)-free planar graphs

The class of (C_3, C_4, C_6)*-free planar graphs* is the class of planar graphs with no subgraphs (or equivalently no induced subgraphs) of size 3, 4 or 6. Choi, Liu, and Oum [38] showed that there are two minimal sets of cycles such that planar graphs with no cycles in one of those sets admit an $(\mathcal{I}, \Delta_d)$-partition for some d. Those sets are the set of all odd cycles on the one hand (graphs with no odd cycles are bipartite and thus have $(\mathcal{I}, \Delta_0)$-partitions), and the set of cycles with size 3, 4 and 6 on the other hand. They showed that graphs in $\mathcal{C}$ admit an $(\mathcal{I}, \Delta_{45})$-partition, and we prove that such graphs have an $(\mathcal{I}, \Delta_6)$-partition. We also show that there exists a graph in C that has no $(\mathcal{I}, \Delta_3)$-partition.

The proofs are presented in Sections 4.5 and 4.6. See also the arXiv version [29].

1.4.2 Works available as an appendix

Fractional triangle decomposition of dense graphs

A *triangle decomposition* of a graph is a partition of its edges into triangles. A *fractional triangle decomposition* of a graph is an assignment of a non-negative weight to each of its triangles such that the sum of the weights of the triangles containing any given edge is one. We prove that every graph on n vertices with minimum degree at least $0.9n$ has a fractional triangle decomposition. This improves a result of Garaschuk [35] that the same conclusion holds for graphs with minimum degree at least $0.956n$. Together with a result of

Barber, Kühn, Lo, and Osthus [7], this implies that for all $\epsilon > 0$, every large enough triangle divisible graph on n vertices with minimum degree at least $(0.9 + \epsilon)n$ admits a triangle decomposition.

Those results were published in Siam Journal on Discrete Mathematics [23]. See Appendix A.

List colouring of planar graphs

A colouring of a graph $G = (V, E)$ is a function $c : V \to \mathbb{N}^*$ such that $c(u) \neq c(v)$ for every $uv \in E$. A k-regular list assignment of G is a function L with domain V such that for every $u \in V$, $L(u)$ is a subset of $\mathbb{N}^*$ of size k. A colouring c of G respects a k-regular list assignment L of G if $c(u) \in L(u)$ for every $u \in V$. A graph G is k-choosable if for every k-regular list assignment L of G, there exists a colouring of G that respects L. We may also ask if for a given k-regular list assignment L of a given graph G, there exists a colouring of G that respects L. This yields the k-Regular List Colouring problem. For $k \in \{3, 4\}$ we determine a family of classes $\mathcal{G}$ of planar graphs, such that either k-Regular List Colouring is NP-complete for instances (G, L) with $G \in \mathcal{G}$, or every $G \in \mathcal{G}$ is k-choosable. By using known examples of non-3-choosable and non-4-choosable graphs, this enables us to classify the complexity of k-Regular List Colouring restricted to planar graphs, planar bipartite graphs, planar triangle-free graphs and to planar graphs with no 4-cycles and no 5-cycles. We also classify the complexity of k-Regular List Colouring and a number of related colouring problems for graphs with bounded maximum degree.

Those results were presented at IWOCA 2015 [20]. The full version is available on arXiv [21]. See Appendix B.

Colouring diamond-free graphs

The Colouring problem is that of deciding, given a graph G and an integer k, whether G admits a (proper) k-colouring. For all graphs H up to five vertices, we classify the computational complexity of Colouring for (diamond, H)-free graphs. Our proof is based on combining known results together with proving that the clique-width is bounded for (diamond, $P_1 + 2P_2$)-free graphs. Our technique for handling this case is to reduce the graph under consideration to a k-partite graph that has a very specific decomposition. As a by-product of this general technique we are also able to prove boundedness of clique-width for four other new classes of (H_1, H_2)-free graphs. As such, our work also continues a recent systematic study into the (un)boundedness of clique-width of (H_1, H_2)-free graphs, and our five new classes of bounded clique-width reduce the number of open cases from 13 to 8.

Those results were published in Journal of Computer an System Sciences [22]. See Appendix C.

Here are the references to the author's publications.

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[28] F. Dross, M. Montassier, and A. Pinlou. Partitioning a triangle-free planar graph into a forest and a forest of bounded degree. *European Journal of Combinatorics*, 66:81–94, 2017. Selected papers of EuroComb15.

[29] F. Dross and P. Ochem. Vertex partitions of (C_3, C_4, C_6)-free planar graphs. *arXiv1711.08710*, 2017.

Chapter 2

Large induced forests

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2.1 Introduction

One of the consequences of the Four Colour Theorem is that every planar graph admits an independent set on at least one fourth of its vertices. This result, of independent interest, was not known before the Four Colour Theorem, and actually no other proof of this result is known to date. An interesting other result that would imply this one is that every planar graph admits an induced forest on at least one half of its vertices. This was first conjectured by Albertson and Berman in 1979 [3] and has yet to be proven.

Conjecture 2.1.1 (Albertson and Berman [3])**.** *Every planar graph of order n admits an induced forest of order at least $\frac{n}{2}$.*

A notion that is perhaps more widely known is that of feedback vertex set. In a graph G, a *feedback vertex set* is a set of vertices that intersects all the cycles of G. Note that a set $S \subseteq V(G)$ is a feedback vertex set if and only if $G - S$ is a forest. Hence, an equivalent formulation of Conjecture 2.1.1 is that every planar graph of order n admits a feedback vertex set of size at most $\frac{n}{2}$.

Conjecture 2.1.1, if true, is best possible, as shown by the disjoint union of copies of the complete graph on four vertices. The best result known to date is that every planar graph admits an induced forest on at least two fifth of its vertices. This is a direct consequence of the following theorem by Borodin [9].

Theorem 2.1.2 (Borodin [9])**.** *Every planar graph admits an acyclic colouring with at most five colours.*

Corollary 2.1.3 (Borodin)**.** *Every planar graph of order n admits an induced forest of order at least $\frac{2n}{5}$.*

Conjecture 2.1.1, however, was proven, and even strengthened, on some smaller classes of graphs. On outerplanar graphs, Hosono [37] proved the following.

Theorem 2.1.4 (Hosono [37])**.** *Every outerplanar graph of order n admits an induced forest of order at least $\frac{2n}{3}$.*

This theorem is tight, as shown by the examples of Hosono [37]. Other results were deduced from results on acyclic colouring, for other classes of graphs. Fertin, Godard, and Raspaud [33] gave such results for several classes of graphs, stated in Table 2.1.

Family $\mathcal{F}$	Order of a largest induced forest: Lower bound	Upper bound
Planar	$\frac{2n}{5}$	$\lceil \frac{n}{2} \rceil$
Planar with girth 5, 6	$\frac{n}{2}$	$\frac{7n}{10} + 2$
Planar with girth ≥ 7	$\frac{2n}{3}$	$\frac{5n}{6} + 1$

Table 2.1: Bounds on the order of a largest induced forest for some families $\mathcal{F}$ of graphs [33].

On bipartite planar graphs, Akiyama and Watanabe [1], and Albertson and Haas [2] independently raised the same following conjecture.

Conjecture 2.1.5 (Akiyama and Watanabe [1], and Albertson and Haas [2])**.** *Every bipartite planar graph of order n admits an induced forest of order at least $\frac{5n}{8} = 0.625n$.*

As a first step towards Conjecture 2.1.5, Alon, Mubayi, and Thomas [4] proved the following.

Theorem 2.1.6 (Alon et al. [4])**.** *Every triangle-free graph of order n and size m admits an induced forest of order at least $n - \frac{m}{4}$.*

Corollary 2.1.7 (Alon et al. [4])**.** *Every triangle-free cubic graph of order n admits an induced forest of order at least $\frac{5n}{8}$.*

Corollary 2.1.8 (Alon et al. [4])**.** *Every triangle-free planar graph of order n admits an induced forest of order at least $\frac{n}{2}$.*

This latter lower bound was improved for $n \geq 1$ by Salavatipour [49].

Theorem 2.1.9 (Salavatipour [49])**.** *Every triangle-free planar graph of order n and size m admits an induced forest of order at least $\frac{29n-6m}{32}$ and thus at least $\frac{17n+24}{32} \approx 0.531n$.*

Relying on the methods of Kowalik, Lužar, and Škrekovski [43], we improve the previous result to prove the following in Section 2.3.

Theorem 2.1.10. *Every triangle-free planar graph of order n and size m admits an induced forest of order at least $\max\{\frac{38n-7m}{44}, n - \frac{m}{4}\}$, and thus at least $\frac{6n+7}{11} \approx 0.545n$.*

For girth 5, Kowalik, Lužar, and Škrekovski [43] made the following conjecture, and gave examples to show that it is tight.

Conjecture 2.1.11 (Kowalik et al. [43])**.** *Every planar graph of girth at least 5 and order n admits an induced forest of order at least $\frac{7n}{10}$.*

We used similar methods to that of Theorem 2.1.10 to prove that every planar graph with girth at least 5, order n, and size m admits an induced forest of order at least $n - \frac{5m}{23}$, and thus at least $\frac{44n+50}{69} \approx 0.638n$, but this result was improved independently by both Shi and Xu [50] on the one hand and by Kelly and Liu [41] on the other hand as follows:

Theorem 2.1.12 (Shi and Xu [50], Kelly and Liu [41])**.** *Every connected planar graph of girth at least* 5*, order* n*, and size* m *has an induced forest of order at least* $\frac{8n-2m-2}{7}$*, and thus at least* $\frac{2n-2}{3} \approx 0.667n$*.*

For higher girths, we make the following conjecture.

Conjecture 2.1.13. *For all* $g \geq 3$*, every planar graph of girth at least* g *with* n *vertices and* m *edges admits an induced forest of order at least* $n - \frac{m}{g}$*, and thus at least* $n - \frac{n-2}{g-2}$*.*

This conjecture is tight, as shown in Section 2.2. As a first step towards Conjecture 2.1.13, we prove the following in Section 2.6.

Theorem 2.1.14. *For all* $g \geq 3$*, every planar graph of girth at least* g *with* n *vertices and* m *edges admits an induced forest of order at least* $n - \frac{4m}{3g}$*, and thus at least* $n - \frac{4n}{3g-6}$*.*

In the proof of Theorem 2.1.14, we need the following theorem, that is of independent interest and will be proven in Section 2.5. Let $\mathcal{C}_{2,3^-}$ be the family of 2-connected graphs of maximum degree at most 3. Note that graphs in $\mathcal{C}_{2,3^-}$ do not need to be planar.

Theorem 2.1.15. *Every graph in* $\mathcal{C}_{2,3^-}$ *of order* n *has an induced forest of order at least* $\frac{2n-2}{3}$*.*

Conjecture 2.1.1 also led to some similar researches on induced linear forests. The following conjecture is due to Chappell (see Pelsmajer [45]).

Conjecture 2.1.16 (Chappell)**.** *Every planar graph of order* n *admits an induced linear forest of order at least* $\frac{4n}{9}$*.*

This conjecture is also tight. Pelsmajer [45] claims that Chappell found examples showing the tightness of Conjecture 2.1.16, citing personal correspondence with Chappell. The best known result to date is a consequence of the following theorem by Poh [47].

Theorem 2.1.17 (Poh [47])**.** *The set of vertices of every planar graph can be partitioned into three sets inducing linear forests.*

Corollary 2.1.18. *Every planar graph of order* n *admits an induced linear forest of order at least* $\frac{n}{3}$*.*

Again, the problem was solved for outerplanar graphs.

Theorem 2.1.19 (Pelsmajer [45])**.** *Every outerplanar graph of order* n *admits a linear forest of order at least* $\frac{4n+2}{7}$*.*

The previous theorem is tight, as shown by examples of Chappell (see Pelsmajer [45]).

Now, we focus on triangle-free planar graphs, and make the following conjecture:

Conjecture 2.1.20. *Every triangle-free planar graph of order* n *admits an induced linear forest of order at least* $\frac{n}{2}$*.*

Again, this conjecture is tight, as shown in Section 2.2. The first step towards proving Conjecture 2.1.20 is the following.

Theorem 2.1.21. *Every triangle-free planar graph of order* n *and size* m *admits an induced linear forest of order at least* $\frac{9n-2m}{11}$*, and thus at least* $\frac{5n+8}{11} \approx 0.455n$*.*

We will prove Theorem 2.1.21 in Section 2.4. As a generalisation of finding a largest induced linear forest, the problem of finding a largest induced forest with maximum degree at most d, with $d \geq 2$, was solved for graphs with treewidth at most k for all k by Chappell and Pelsmajer [16]. They made the following conjecture for planar graphs:

Conjecture 2.1.22 (Chappell and Pelsmajer [16]). *Let $d \geq 2$ be an integer. Every planar graph of order n admits an induced forest with maximum degree at most d of order at least $\frac{2dn}{4d+1}$.*

Note that for $d = 2$, this gives Conjecture 2.1.16, and for $d \to \infty$, this gives Conjecture 2.1.1. Chappell and Pelsmajer gave examples to show that their conjecture is tight. Up to the best knowledge of the author, no progress has been made towards Conjecture 2.1.22 besides the cases $d = 2$ and $d \to \infty$. Note though that bounds for $d = 2$ directly imply the same bounds for higher values of d, so the best result towards Conjecture 2.1.22 is Corollary 2.1.18.

2.2 Tightness of our conjectures

Let us first prove that Conjecture 2.1.13 is tight. To show this, we will prove that for all n and all g, there exists a graph G with ng vertices such that the maximum induced forest in G has order $ng - ||G||$. The example showing this is, for all $g \in \mathbb{N}$ and $n \in \mathbb{N}$, the graph nC_g. Indeed, nC_g is a planar graph of girth at least g, and an induced subgraph of nC_g is a forest if and only if it has at most $g - 1$ vertices in each copy of C_g.

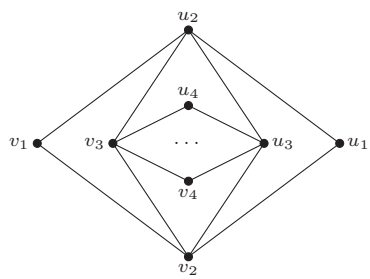

Figure 2.2.1: The graph G_k of Theorem 2.2.1.

The following Theorem proves that Conjecture 2.1.20 is tight.

Theorem 2.2.1. *For all integer $n \geq 2$, there exists a triangle-free planar graph of order n with an induced linear forest of order $\lceil \frac{n}{2} \rceil + 1$, and no induced linear forest of order more than $\lceil \frac{n}{2} \rceil + 1 \approx 0.545n$.*

Proof. Let us consider an integer $n \geq 2$, and let $k = \lfloor \frac{n}{2} \rfloor$. Let us define the graph G_k as follows: $G_k = (\bigcup_{1 \leq i \leq k} \{u_i, v_i\}, \bigcup_{1 \leq i \leq k-1} \{u_i u_{i+1}, u_i v_{i+1}, v_i u_{i+1}, v_i v_{i+1}\})$. If n is even, then the graph we consider is G_k, and otherwise, the graph we consider is $G_k + K_1$. It is clear that for any induced linear forest of G_k, there exists an induced linear forest of $G_k + K_1$ with one more vertex, that is the vertex of K_1. Therefore, we can just consider the case where n is even.

Let us first prove that there exists an induced forest with $k + 1$ vertices in G_k. If $k = 2j$ for some $j \geq 1$, then $G_k[\{u_1, v_1, u_3, v_3, ..., u_{2j-1}, v_{2j-1}, u_{2j}\}]$ is a linear forest on $2j + 1 = k + 1$ vertices. If $k = 2j + 1$ for some $j \geq 0$, then $G_k[\{u_1, v_1, u_3, v_3, ..., u_{2j+1}, v_{2j+1}\}]$ is a linear forest on $2j + 2 = k + 1$ vertices.

Now, consider an induced linear forest F of G_k, and prove that $|F| \leq k+1$. Let us prove this by induction on the value of $k \geq 1$.

- Suppose $k = 1$. As F has at most as many vertices as G_k, we have $|F| \leq 2 = k + 1$.
- Suppose $k = 2$. Since G_k is a cycle, F cannot be equal to G_k, and thus $|F| \leq 3 = k + 1$.
- Suppose $k = 3$. Suppose $|F| \geq 5$. At most one vertex of G_k is not in $V(F)$. This implies that for all $i \in \{1, 2, 3\}$, at least one vertex among u_i and v_i is in $V(F)$. Without loss of generality, u_1, u_2 and u_3 are in $V(F)$. Moreover, either $v_2 \in V(F)$, in which case $F[\{u_1, u_2, u_3, v_2\}]$ is a cycle of length 4, or $v_1 \in V(F)$, in which case $d_F(u_3) \geq 3$. In both cases, F is not a linear forest, a contradiction. Therefore $|F| \leq 4 = k + 1$.

- Suppose $k \geq 4$. By induction hypothesis, we can assume that every induced linear forest of G_{k-1}, G_{k-2}, and G_{k-3} respectively has at most k, $k-1$ and $k-2$ vertices, thus in particular, $|V(F) \cap V(G_{k-1})| \leq k$, $|V(F) \cap V(G_{k-2})| \leq k-1$ and $|V(F) \cap V(G_{k-3})| \leq k-2$. If at most one of v_k and u_k is in F, then $|F| \leq k+1$, thus we may assume that both v_k and u_k are in F. If both u_{k-1} and v_{k-1} are in F, then as $G_k[\{u_{k-1}, u_k, v_{k-1}, v_k\}]$ is a cycle, F is not a linear forest, a contradiction. Moreover, if none of u_{k-1} and v_{k-1} is in F, then $|F| \leq k-1+2 = k+1$. Therefore we can assume that exactly one of u_{k-1} and v_{k-1}, say u_{k-1}, is in F. If at least one of u_{k-2} and v_{k-2} is in F, then $d_F(u_{k-1}) \geq 3$, so F is not a linear forest, a contradiction. Therefore, none of u_{k-2} and v_{k-2} is in F, and thus $|F| \leq k-2+3 = k+1$.

That ends the proof of the theorem. □

2.3 Large induced forests in triangle-free planar graphs

Let us first recall the statement of Theorem 2.1.10.

Theorem 2.1.10 (recall)**.** *Every triangle-free planar graph of order n and size m admits an induced forest of order at least* $\max\{\frac{38n-7m}{44}, n - \frac{m}{4}\}$*, and thus at least* $\frac{6n+7}{11} \approx 0.545n$*.*

We first note that the values $\frac{38}{44}$ and $\frac{7}{44}$ in the statement of Theorem 2.1.10 are not arbitrary. Let a, b, and c be some real values such that every graph $G \in \mathcal{P}_4$ admits an induced forest on at least $a|G| + b||G|| + c$ vertices.

For any $G \in \mathcal{P}_4$ and for every $k \in \mathbb{N}$, the graph kG is in $\mathcal{P}_4$. Moreover, for all $n \in \mathbb{N}$, G admits an induced forest on n vertices if and only if kG admits an induced forest on kn vertices, by picking the same set of vertices on the different copies of G in kG. By linearity, this implies that we can take $c = 0$.

Let us now consider some graphs in $\mathcal{P}_4$. The cube, in particular, has 8 vertices, 12 edges, and the maximum order of an induced forest in the cube is 5. In particular, this implies that $8a + 12b \leq 5$. For the rest, we will use, in our proof, the fact that $3a + 10b \leq 1$, and that $a \leq 1$. These three inequalities, together with $a \geq 0$ and $b \leq 0$, define a polygon where the extremal points are $(1, -\frac{1}{4})$ and $(\frac{38}{44}, -\frac{7}{44})$. Note that $3a + 10b \leq 1$ is the inequality that would need to be relaxed in order to improve the theorem. In particular, if we could relax it to $b \leq 0$, we would get the point $(\frac{5}{8}, 0)$, and thus a proof of Conjecture 2.1.1.

Let us now proceed with the actual proof of Theorem 2.1.10. This proof is similar to the discharging method. The only difference is that we do not actually need a discharging procedure, and we can obtain the final contradiction from a simple counting argument. Let us assume that Theorem 2.1.10 is false, which means that there exists a triangle-free planar graph G with no induced forest on at least $\frac{38|G|-7||G||}{44}$ vertices. Consider such a graph G such that $|G|$ is as low as possible. We consider an arbitrary embedding of G in the plane. Let $n = |G|$, $m = ||G||$, and let f denote the number of faces of G.

Lemma 2.3.1. *The graph G is 2-edge-connected.*

Proof. Suppose G is not 2-edge-connected. If G is connected, this implies that G has a bridge e, and let $G' = G - e$. Otherwise, let $G' = G$. In both cases, the graph G' is not connected. Let C be a component of G'. Both C and $G' - V(C)$ are in $\mathcal{P}_4$ and have fewer vertices than G. Therefore they both verify the statement of Theorem 2.1.10, and there are two induced forests F_1 and F_2 in C and $G' - V(C)$ respectively such that $|F_1| \geq \frac{38|C|-7||C||}{44}$ and $|F_2| \geq \frac{38|G'-V(C)|-7||G'-V(C)||}{44}$. The graph F induced by $V(F_1) \cup V(F_2)$ in G is a forest, since it is the union of the two disjoint forests F_1 and F_2, plus at most one edge (e) connecting a vertex of F_1 and a vertex of F_2. Moreover, we have $|F| = |V(F_1) \cup V(F_2)| = |F_1| + |F_2| \geq \frac{38|C|-7||C||}{44} + \frac{38|G'-V(C)|-7||G'-V(C)||}{44} \geq \frac{38|C|+38|G'-V(C)|-7||C||-7||G'-V(C)||}{44} \geq \frac{38|G'|-7||G'||}{44}$. As $|G'| = |G|$ and $||G'|| \leq ||G||$, we have $|F| \geq \frac{38|G|-7||G||}{44}$. The induced forest F in G has order at least $\frac{38|G|-7||G||}{44}$, thus G verifies Theorem 2.1.10, a contradiction. □

Note that Lemma 2.3.1 implies in particular that every vertex in G has degree at least 2. Let us now prove that some configurations are forbidden in G.

Lemma 2.3.2. *Every vertex in G has degree at most 5.*

Proof. Let v be a vertex with degree at least 6. The graph $G-v$ has $n-1$ vertices and at most $m-6$ edges, and as it has fewer vertices than G, it has an induced forest F of order at least $\frac{38(n-1)-7(m-6)}{44} \geq \frac{38n-7m}{44}$. As F is an induced subgraph of $G-v$, that is an induced subgraph of G, F is an induced subgraph of G. Thus G verifies Theorem 2.1.10, a contradiction. □

We note that Lemma 2.3.2 can be reformulated as follows, and is thus indeed a forbidden configuration.

Lemma 2.3.2 (reformulated)**.** *The configuration where a vertex has degree at least* 6 *is forbidden in* G*.*

We will usually consider another graph G' that is smaller than G, that has a large induced forest by minimality of G. From this forest we will deduce that G also has a large induced forest, leading to a contradiction. Most of the lemmas also correspond to forbidden configurations, and use a similar approach. Therefore, we will use the following observation.

Observation 2.3.3. *Let* α*,* β*,* γ *be integers satisfying* $\alpha \geq 1$*,* $\beta \geq 0$ *and* $\gamma \geq 0$*.*

Let $G' \in \mathcal{P}_4$ *be a graph with* $|G'| = n - \alpha$ *and* $||G'|| \leq m - \beta$*.*

By minimality of G*, the graph* G' *admits an induced forest of order at least* $\frac{38(n-\alpha)-7(m-\beta)}{44} = \frac{38n-7m}{44} - \frac{38\alpha-7\beta}{44}$*.*

Given an induced forest F' *of* G' *of order at least* $\frac{38n-7m}{44} - \frac{38\alpha-7\beta}{44}$*, if there is an induced forest* F *of* G *of order* $|F| \geq |F'| + \gamma$*, then as* $|F| < \frac{38n-7m}{44}$*, we have* $\frac{38\alpha-7\beta}{44} > \gamma$*.*

Now, the proof of Lemma 2.3.2 just becomes:

Proof. Let v be a vertex with degree at least 6. The graph $G' = G - v$ has $n-1$ vertices and at most $m-6$ edges, and any induced forest F' of $G-v$ is an induced forest $F = F'$ of G. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (1, 6, 0)$, we have $\frac{38}{44} - 6\frac{7}{44} > 0$, a contradiction. □

We are now ready to prove the next series of lemmas. The aim here is to prove that G has no 2^--vertex, that no 3-vertex is in the boundary of a 4-face in G, and that every 4-face has at least two 4^+-vertices in its boundary. This will go through a series of lemmas on the structure of 4-faces, 5-faces, and 4-cycles in G.

Lemma 2.3.4. *Let* v *be a* 3*-vertex with a* 4^+*-neighbour* w *in* G*. The two neighbours of* v *distinct from* w *have a common neighbour distinct from* v*.*

Proof. Let x and y be the two neighbours of v distinct from w. Suppose by contradiction that they do not have a common neighbour besides v. Let $G' = G + xy - \{v, w\}$. As x and y do not have a common neighbour in G', $G' \in \mathcal{P}_4$. Let F' be an induced forest of G'. The graph $G[V(F') \cup \{v\}]$ is a forest, since it is equal to F' with at most an edge subdivided, or with an additional 1^--vertex. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (2, 5, 1)$, we have $2\frac{38}{44} - 5\frac{7}{44} > 1$, a contradiction. □

Lemma 2.3.5. *There is no* 2*-vertex with a* 4^+*-neighbour in* G*.*

Proof. Let v be a 2-vertex with a 4^+-neighbour w in G. Let $G' = G - \{v, w\}$. For all induced forest F' of G', the graph $G[V(F') \cup \{v\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (2, 5, 1)$, we have $2\frac{38}{44} - 5\frac{7}{44} > 1$, a contradiction. □

Lemma 2.3.6. *There is no* 3*-vertex with two* 2*-neighbours in* G*.*

Proof. Let v be a 3-vertex with two 2-neighbours x and y. Let $G' = G - \{v, x, y\}$. For all induced forest F' of G', the graph $G[V(F') \cup \{x, y\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$, we have $3\frac{38}{44} - 5\frac{7}{44} > 2$, a contradiction. □

Lemma 2.3.7. *There is no* 2*-vertex in* G*.*

Proof. Assume G has a 2-vertex.

Let us first prove that G has at least one 3^+-vertex. If that is not the case, then G is a collection of disjoint cycles, and in particular $n = m$. To get an induced forest, we can get a feedback vertex set with at most one vertex per cycle. As the cycles have length at least 4, this means that there is an induced forest of order at least $\frac{3n}{4} \geq \frac{31n}{44} = \frac{38n-7m}{44}$.

Let v be a 2-vertex in G with a 3^+-neighbour, that exists since G is connected and not 2-regular. By Lemma 2.3.5, the two neighbours x and y of v are 3^--vertices.

Suppose one of x and y, say x, is a 2-vertex. Let $G' = G - \{v, x, y\}$. For all induced forest F' of G', the graph $G[V(F') \cup \{v, x\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$, we have $3\frac{38}{44} - 5\frac{7}{44} > 2$, a contradiction, so both x and y are 3-vertices.

We will now consider different cases depending on the number of common neighbour of x and y.

- Suppose x and y have no common neighbour besides v. Let $G' = G - v + xy$. For all induced forest F' of G', the graph $G[V(F') \cup \{v\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (1, 1, 1)$, we have $\frac{38}{44} - \frac{7}{44} > 1$, a contradiction.

- Suppose x and y have exactly two common neighbours v and w. Let u be the neighbour of x distinct from v and w. Let $G' = G - \{u, v, w, x, y\}$. By Lemma 2.3.6, since v is a 2-vertex, u and w are 3^+-vertices, and since $G \in \mathcal{P}_4$, u and w are not adjacent, so $||G'|| \leq m - 9$. For all induced forest F' of G', the graph $G[V(F') \cup \{v, x, y\}]$ is a forest. By observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (5, 9, 3)$, we have $5\frac{38}{44} - 9\frac{7}{44} > 3$, a contradiction.

- Now x and y have three common neighbours, u, v and w. Suppose one of those, say u, is a 4^+-vertex. Let $G' = G - \{u, v, w, x, y\}$. Again, u and w are not adjacent and w is a 3^+-vertex, so $||G'|| \leq m - 9$. For all induced forest F' of G', the graph $G[V(F') \cup \{v, x, y\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (5, 9, 3)$, we have $5\frac{38}{44} - 9\frac{7}{44} > 3$, a contradiction.

  Therefore u and w are 3-vertices. Let z be the third neighbour of u. Let $G' = G - \{u, v, w, x, y, z\}$. We have $||G'|| \leq m - 8$. For all induced forest F' of G', the graph $G[V(F') \cup \{u, v, w, x\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (6, 8, 4)$, we have $6\frac{38}{44} - 8\frac{7}{44} > 4$, a contradiction.

This ends the proof of the lemma. □

Lemma 2.3.8. *Let $C = v_0v_1v_2v_3$ be a 4-cycle of G. If v_0 and v_2 are 3-vertices in G, then v_1 and v_3 are 3-vertices too.*

In particular, there is no 4-cycle with exactly three 3-vertices in G.

Proof. Suppose v_1 and v_3 are 4^+-vertices. Let $G' = G - V(C)$. Since $G \in \mathcal{P}_4$, v_0 is not adjacent to v_2, and v_1 is not adjacent to v_3. Therefore, $||G'|| \leq m - 10$. For all induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_2\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (4, 10, 2)$, we have $4\frac{38}{44} - 10\frac{7}{44} > 2$, a contradiction.

We now assume now that v_0, v_1, and v_2 are 3-vertices and that v_3 is a 4^+-vertex. Let u_0, u_1, and u_2 be the third neighbours of v_0, v_1, and v_2 respectively. Suppose that two of the u_i's are equal. This implies that $u_0 = u_2$. Let $G' = G - V(C) - \{u_0\}$. Since u_0 is not adjacent to v_3, we have $||G'|| \leq m - 10$. For all induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1, v_2\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (5, 10, 3)$, we have $5\frac{38}{44} - 10\frac{7}{44} > 3$, a contradiction.

Therefore all of the u_i's are distinct. By Lemma 2.3.4, since the 3-vertex v_0 is adjacent to the 4-vertex v_3, the two vertices u_0 and v_1 have a common neighbour, which cannot be v_2 since $u_0 \neq u_2$, and is thus u_1. Therefore, $u_0u_1 \in E(G)$, and similarly, $u_1u_2 \in E(G)$. Assume u_0 or u_2, say u_0, has at most one neighbour $w \notin \{v_0, v_1, v_2, v_3, u_0, u_1, u_2\}$. Let $G' = G - V(C) - \{u_0, u_1, u_2\}$. We have $||G'|| \leq m - 13$. For all induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1, v_2, u_0\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (7, 13, 4)$, we have $7\frac{38}{44} - 13\frac{7}{44} > 4$, a contradiction.

Therefore u_0 and u_2 both have at least two neighbours that are not in $\{v_0, v_1, v_2, v_3, u_0, u_1, u_2\}$. Let $G' = G - V(C) - \{u_0, u_2\}$. We have $||G'|| \leq m - 15$. For all induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1, v_2\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (6, 15, 3)$, we have $6\frac{38}{44} - 15\frac{7}{44} > 3$, a contradiction. □

Lemma 2.3.9. *Let $C = v_0v_1v_2v_3$ be a 4-cycle of G. If v_0 is a 3-vertex and v_2 is a 4-vertex, then the two neighbours of v_2 distinct from v_1 and v_3 in G are either both in the interior of C, or both in the exterior of C.*

Proof. Suppose one of the neighbours of v_2 is in the interior of C, and another one is in the exterior of C. In particular, any path containing two neighbours of v_2 contains a vertex of C. The vertices v_1 and v_3 are not both 3-vertices by Lemma 2.3.8. Say v_1 is a 4^+-vertex. Let $G' = G - V(C)$. We have $||G'|| \leq m - 10$. For all induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_2\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (4, 10, 2)$, we have $4\frac{38}{44} - 10\frac{7}{44} > 2$, a contradiction. □

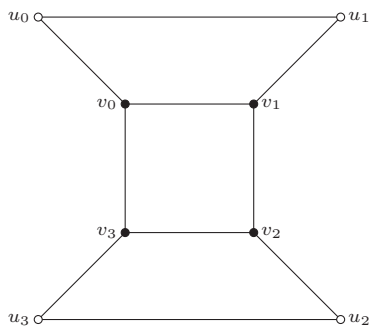

Figure 2.3.1: The graph G in Lemma 2.3.10

Lemma 2.3.10. *There is no 4-face in G with four 3-vertices in its boundary.*

Proof. Let $C = v_0v_1v_2v_3$ be a cycle bounding a 4-face. For $i \in \{0, 1, 2, 3\}$, let u_i be the neighbour of v_i that is not in $V(C)$. In this proof, we consider the indices of he u_i's and v_i's modulo 4. As $G \in \mathcal{P}_4$, if two of the u_i's are adjacent, they correspond to some u_i and u_{i+2}, say u_0 and u_2. In the cycle $v_0v_1v_2u_0$, the vertices v_0 and v_2 are 3-vertices, thus u_0 is a 3-vertex by Lemma 2.3.8. The cycle $v_0v_1v_2u_0$ separates the plane into two parts, one containing u_1 and the other u_3. Therefore, the graph $G - V(C) - u_0$ is disconnected, and there are at most three edges that connect $G[V(C) \cup \{u_0\}]$ to the rest of the graph. Therefore, there is a component of $G - V(C) - u_0$ that has at most one edge connecting it to the rest of the graph, and that edge is a bridge, contradicting Lemma 2.3.1.

Therefore all of the u_i's are distinct. We now consider the question of the presence or not of the edges u_iu_{i+1}. Assume $u_iu_{i+1} \notin E(G)$ and $u_{i+1}u_{i+2} \notin E(G)$ for some $i \in \{0, 1, 2, 3\}$, say for $i = 0$. If $u_0u_2 \in E(G)$, then one of u_0u_3 an u_2u_3 is not in $E(G)$, as otherwise G would have a triangle, and, by planarity of G, $u_1u_3 \notin E(G)$. If $u_0u_3 \notin E(G)$, then $\{u_3u_0, u_0u_1, u_3u_1\} \cap E(G) = \emptyset$. If $u_2u_3 \notin E(G)$, then $\{u_1u_2, u_2u_3, u_1u_3\} \cap E(G) = \emptyset$. Lastly, if $u_0u_2 \notin E(G)$, then $\{u_0u_1, u_1u_2, u_0u_2\} \cap E(G) = \emptyset$. In all cases, $u_iu_{i+1} \notin E(G)$ and $u_{i+1}u_{i+2} \notin E(G)$ and $u_iu_{i+2} \notin E(G)$ for some $i \in \{0, 1, 2, 3\}$. Therefore up to permutation of the indices, we can assume that $\{u_0u_1, u_1u_2, u_0u_2\} \cap E(G) = \emptyset$. Let $G' = G - \{v_0, v_1, v_2, v_3\} + (\{x\}, \emptyset) + \{xu_0, xu_1, xu_2\}$. As $\{u_0u_1, u_1u_2, u_0u_2\} \cap E(G) = \emptyset$, $G \in \mathcal{P}_4$. We have $||G'|| \leq m - 5$. Let F' be an induced forest G'. If $x \in V(F')$, then the graph $G[(V(F') \setminus \{x\}) \cup \{v_0, v_1, v_2\}]$ is a forest, and otherwise, the graph $G[V(F') \cup \{v_0, v_2\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (3, 5, 1)$, we have $3\frac{38}{44} - 5\frac{7}{44} > 1$, a contradiction.

Therefore for all $i \in \{0, 1, 2, 3\}$, either $u_iu_{i+1} \in E(G)$, or $u_{i+1}u_{i+2} \in E(G)$. In particular, there must be an i, say $i = 0$, such that $u_iu_{i+1} \in E(G)$ and $u_{i+2}u_{i+3} \in E(G)$. Let $G^* = G - C$. We have $|G^*| = |G| - 4$ and $||G^*|| = ||G|| - 8$. See Figure 2.3.1 for a representation of the graph G.

Let us now count, for each of the u_i's, its neighbours that are not in $A = \{v_0, v_1, v_2, v_3, u_0, u_1, u_2, u_3\}$.

- Suppose one of the u_i's, say u_0, has no $(V(G) \setminus A)$-neighbours, and that another one, say u_{i_0}, has at most one $(V(G) \setminus A)$-neighbour. Let $G' = G^* - \{u_0, u_1, u_2, u_3\}$. We have $||G'|| \leq m - 12$. For any induced forest F' of G', the graph $G[V(F') \cup \{v_1, v_2, v_3, u_0, u_{i_0}\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (8, 12, 5)$, we have $8\frac{38}{44} - 12\frac{7}{44} > 5$, a contradiction.

- Suppose one of the u_i's, u_0 say, has at most one $(V(G) \setminus A)$-neighbour, and each of the other u_i's has at least one $(V(G) \setminus A)$-neighbour. Since G is triangle-free, u_0 cannot be adjacent both to u_2 and to u_3. Let $i_0 \in \{2, 3\}$ be such that $u_0 u_{i_0} \notin E(G)$. Let $G' = G^* - \{u_{i_0+1}, u_{i_0+2}, u_{i_0+3}\}$ (we remove all the vertices of A except u_{i_0}). Let us now count the number of edges in G^* that are incident to a vertex in $\{u_{i_0+1}, u_{i_0+2}, u_{i_0+3}\}$. There are the two edges u_0u_1 and u_2u_3, plus one edge since u_0 has degree at least 3, plus, for all $i \in \{1, 2, 3\} \setminus \{i_0\}$, one edge for the $(V(G) \setminus A)$-neighbour of u_i. Therefore there are at least five edges in G^* incident to a vertex in $\{u_{i_0+1}, u_{i_0+2}, u_{i_0+3}\}$, and thus $||G'|| \leq ||G|| - 13$. Let F' be an induced forest of G'. The graph $G[(V(F')] \cup \{v_1, v_2, v_3, u_0\}]$ is a forest, since there is not path between u_0 and u_{i_0} in $G[\{v_1, v_2, v_3, u_0, u_{i_0}\}]$. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (7, 13, 4)$, we have $7\frac{38}{44} - 13\frac{7}{44} > 4$, a contradiction.

- Therefore each of the u_i's has at least two $(V(G) \setminus A)$-neighbours. Let $G' = G^* - \{u_0, u_2\}$. We have $||G'|| \leq m - 14$. For any induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1, v_2\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (6, 14, 3)$, we have $6\frac{38}{44} - 14\frac{7}{44} > 3$, a contradiction.

That completes the proof of the lemma. □

Lemma 2.3.11. *There is no separating 4-cycle with four 3-vertices in G.*

Proof. Let $C = v_0v_1v_2v_3$ be such a cycle. For $i \in \{0, 1, 2, 3\}$, let u_i be the neighbour of v_i that is not in $V(C)$. In this proof, we consider the indices of he u_i's and v_i's modulo 4. By Lemma 2.3.1, the graph G is 2-connected. Therefore all of the u_i's are distinct, two of the u_i's are inside of C, and the other two are outside of C. There is an $i \in \{0, 1, 2, 3\}$, say $i = 0$, such that u_{i+1} is inside of C and u_{i+2} is outside of C. Let $G' = G - C - u_0$. We have $||G'|| \leq m - 10$. For any induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1, v_2\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (5, 10, 3)$, we have $5\frac{38}{44} - 10\frac{7}{44} > 3$, a contradiction. □

Lemma 2.3.12. *There is no 3-vertex with a 5-neighbour in G.*

Proof. Let v be a 3-vertex and u a 5-neighbour of v. Let w and x be the two remaining neighbours of v. Suppose first that one of w and x, say w, is a 4^+-vertex. Let $G' = G - \{u, v, w\}$. We have $||G'|| \leq m - 10$. For any induced forest F' of G', the graph $G[V(F') \cup \{v\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (3, 10, 1)$, we have $3\frac{38}{44} - 10\frac{7}{44} > 1$, a contradiction. Therefore w and x are 3-vertices. By Lemma 2.3.4, w and x have a common neighbour y distinct from v. By Lemma 2.3.8, y has degree 3, which contradicts either Lemma 2.3.10 or Lemma 2.3.11. □

Lemma 2.3.13. *There is no separating 4-cycle with at least two 3-vertices in G.*

Proof. Let $C = v_0v_1v_2v_3$ be such a cycle. By Lemmas 2.3.8 and 2.3.11, C has exactly two 3-vertices, and by Lemmas 2.3.7, 2.3.8 and 2.3.12, the two 3-vertices are adjacent and the two remaining vertices are 4-vertices. We can assume that the two 3-vertices are v_0 and v_1. Let u_0 be the third neighbour of v_0, and u_1 be the third neighbour of v_1.

Suppose $u_0v_2 \in E(G)$ or $u_1v_3 \in E(G)$, say $u_0v_2 \in E(G)$. Let u_2 be the fourth neighbour of v_2. By Lemma 2.3.9 in C, u_0 and u_2 are either both in the interior of C or both in the exterior of C. Say that the embedding of G in the plane is such that they are both on the exterior of C. Either u_2 is in the interior of the cycles $v_0v_1v_2u_0$ and $v_0v_3v_2u_0$, or it is in the exterior of those cycles. Moreover, if v_3 is in the interior of $v_0v_1v_2u_0$, then v_1 is in the exterior of $v_0v_3v_2u_0$, and reciprocally. Thus one of $v_0v_1v_2u_0$ and $v_0v_3v_2u_0$ contradicts Lemma 2.3.9. Therefore $u_0v_2 \notin E(G)$ and $u_1v_3 \notin E(G)$.

By Lemma 2.3.4, $u_0u_1 \in E(G)$. Therefore u_0 and u_1 are either both in the exterior of C or both in the interior of C. Say they are both in the exterior of C. By Lemmas 2.3.7, 2.3.8, 2.3.10, 2.3.11 and 2.3.12, u_0 and u_1 are 4-vertices. By Lemma 2.3.9 in C, the two neighbours of v_2 distinct from v_1 and v_3 are either both in the interior of C or both in the exterior of C, and the same holds for the two neighbours of v_3 distinct from v_2 and v_0. At least one of v_2 or v_3, say v_2, has two neighbours inside of C, otherwise the cycle is not separating. Let $G' = G - \{v_0, v_1, v_3, u_1\}$. We have $||G'|| \leq m - 10$. For any induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (4, 10, 2)$, we have $4\frac{38}{44} - 10\frac{7}{44} > 2$, a contradiction. □

Lemma 2.3.14. *There is no 4-face with exactly two 3-vertices in its boundary.*

Proof. Let $C = v_0v_1v_2v_3$ be a cycle bounding such a face. By Lemmas 2.3.7 and 2.3.8 the two 3-vertices are adjacent. Say v_0 and v_1 are 3-vertices. By Lemmas 2.3.7 and 2.3.12, v_2 and v_3 are 4-vertices. Let u_0 and u_1 be the third neighbours of v_0 and v_1 respectively. By Lemma 2.3.4, since v_0 is a 3-vertex adjacent to a 4-vertex v_3, its two remaining neighbours, which are v_1 and u_0, have a common neighbour. Similarly, by Lemma 2.3.4 applied to v_1 and v_2, v_0 and u_1 have a common neighbour. Since by planarity, u_0v_2 and u_1v_3 are not both in $E(G)$, we have $u_0u_1 \in E(G)$. By Lemma 2.3.13, $v_0v_1u_1u_0$ cannot be a separating cycle, so it is the boundary of some 4-face. If both u_0 and u_1 have degree 3, we have a contradiction to Lemma 2.3.10. If one has degree 3 and the other has degree at least 4, we have a contradiction to Lemma 2.3.8. Finally, by Lemma 2.3.12, u_0 and u_1 are 4-vertices.

If v_2 is adjacent to u_0, then $u_0v_0v_1v_2$ is a separating 4-cycle with two 3-vertices, contradicting Lemma 2.3.13. Hence v_2 and u_0 are not adjacent. Similarly, v_3 and u_1 are not adjacent. If u_0 and v_2 have a common neighbour and u_1 and v_3 have a common neighbour, then by planarity this common neighbour must be the same vertex, say w, and u_0u_1w is a triangle in G, a contradiction. By symmetry, assume that u_0 and v_2 do not have a common neighbour. Let $G' = G + u_0v_2 - \{v_0, v_1, v_3, u_1\}$. We have $||G'|| \leq m - 10$. For any induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1\}]$ is isomorphic to the graph F' with the edge u_0v_2 subdivided twice, hence it is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (4, 10, 2)$, we have $4\frac{38}{44} - 10\frac{7}{44} > 2$, a contradiction. □

Lemma 2.3.15. *There is no 4-cycle with at least two 3-vertices in G.*

Proof. It follows from Lemmas 2.3.8, 2.3.10, 2.3.13 and 2.3.14. □

Lemma 2.3.16. *There is no 4-face with exactly one 3-vertex in its boundary.*

Proof. Let $C = v_0v_1v_2v_3$ be a cycle bounding such a face, where v_0 is the 3-vertex. By Lemma 2.3.7, v_1, v_2 and v_3 are 4^+-vertices. By Lemma 2.3.12, v_1 and v_3 are 4-vertices. Let u_0 be the third neighbour of v_0. Vertex u_0 is different from v_2 and non-adjacent to v_1 and v_3 since G is triangle-free.

Let us first assume that u_0 is adjacent to v_2. By Lemmas 2.3.7, 2.3.12 and 2.3.15, u_0 is a 4-vertex. Assume v_2 has degree 5. Let $G' = G - \{u_0, v_0, v_2\}$. We have $||G'|| \leq m - 10$. For any induced forest F' of G', the graph $G[V(F') \cup \{v_0\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (4, 10, 2)$, we have $4\frac{38}{44} - 10\frac{7}{44} > 2$, a contradiction. Hence v_2 has degree 4. Then either $v_0v_1v_2u_0$ or $v_0v_3v_2u_0$ contradicts Lemma 2.3.9.

Thus u_0 is non-adjacent to v_2. By Lemma 2.3.4, v_1 and u_0 have a common neighbour distinct form v_0, call it u_1. It is distinct from all the vertices we defined previously. By Lemma 2.3.15 applied to $v_0v_1u_1u_0$, u_0 and u_1 are 4^+-vertices. By Lemma 2.3.12, u_0 is a 4-vertex.

Suppose u_1 is adjacent to v_3. As C bounds a face, the neighbour of v_1 distinct from v_0, v_2 and u_1, say w_1, is not in the interior of C. The cycle $v_0v_1u_1v_3$ separates u_0 and v_2. Suppose first that $v_0v_1u_1v_3$ does not separate u_0 and w_1. Then $v_0v_1u_1u_0$ separates v_3 and w_1. Let $G' = G - \{v_0, v_1, v_2, v_3, u_0, u_1\}$. We have $||G'|| \leq m - 14$. For any induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1, v_3\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (6, 14, 3)$, we have $6\frac{38}{44} - 14\frac{7}{44} > 3$, a contradiction. Therefore $v_0v_1u_1v_3$ separates u_0 and w_1. Suppose u_1 has degree 5. Let $H' = G - \{v_0, v_3, u_1\}$. We have $||G'|| \leq m - 14$. For any induced forest F' of G', the graph $G[V(F') \cup \{v_0\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (3, 10, 1)$, we have $3\frac{38}{44} - 10\frac{7}{44} > 1$, a contradiction. Thus u_1 has degree 4. Then $v_0v_1u_1v_3$, $v_0u_0u_1v_3$ or $v_0v_1u_1u_0$ contradicts Lemma 2.3.9.

So u_1 is not adjacent to v_3. As $u_1v_3 \notin E(G)$ and $u_0v_2 \notin E(G)$, by Lemma 2.3.4, v_3 and u_0 have a common neighbour distinct from v_0, say u_3. By what precedes and by symmetry, it is of degree at least 4 and non-adjacent to v_0, v_1, v_2 and u_1 (it has a role similar to that of u_1, and is non-adjacent to u_1 because of the girth assumption). See Figure 2.3.2 for a reminder of the structure of $G[\{v_0, v_1, v_2, v_3, u_0, u_1, u_3\}]$. Vertex v_0 is a 3-vertex, v_1, v_3 and u_0 are 4-vertices, and v_2, u_1 and u_3 are 4^+-vertices. Recall that $u_1v_3 \notin E(G)$, $u_3v_1 \notin E(G)$ and $u_0v_2 \notin E(G)$.

In the following we will no longer use the information that C bounds a face. In particular, there is a symmetry between the vertices v_1, v_3, and u_0 and between the vertices v_2, u_1, and u_3. Let w_0, w_1, and

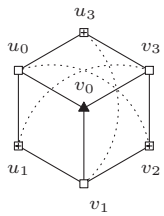

Figure 2.3.2: Graph $G[\{v_0, v_1, v_2, v_3, u_0, u_1, u_3\}]$.

w_3 be the fourth neighbours of u_0, v_1, and v_3 respectively. Since G is triangle-free, w_0 is not adjacent to u_1 nor to u_3. Suppose w_0 is adjacent to v_1 or to v_3, say to v_1. Then since G is triangle-free, w_0 is not adjacent to v_2. However, it may be adjacent to v_3. By Lemma 2.3.15 applied to $v_0v_1w_0u_0$, w_0 is a 4^+-vertex. Let $G' = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0\}$. We have $||G'|| \leq m - 19$. For any induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1, v_3, u_0\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (8, 19, 4)$, we have $8\frac{38}{44} - 19\frac{7}{44} > 4$, a contradiction. Hence, w_0 is not adjacent to v_1 nor to v_3, thus it is distinct from w_1 and w_3. Similarly, w_1 and w_3 are distinct.

Suppose $w_0v_2 \in E(G)$. Assume that C separates w_1 and w_3, or that it does not separate w_1 and w_3 nor w_0 and w_1. Then either C or $v_0v_1v_2w_0u_0$ separates w_1 and w_3. Let $G' = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0\}$. We have $||G'|| \leq m - 19$. For any induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1, v_3, u_0\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (8, 19, 4)$, we have $8\frac{38}{44} - 19\frac{7}{44} > 4$, a contradiction. Thus C does not separate w_1 and w_3 but separates w_1 and w_0. Let $G' = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_3\}$. We have $||G'|| \leq m - 19$. For any induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1, v_3, u_0\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (8, 19, 4)$, we have $8\frac{38}{44} - 19\frac{7}{44} > 4$, a contradiction. So $w_0v_2 \notin E(G)$, and similarly $w_1u_3 \notin E(G)$ and $w_3u_1 \notin E(G)$.

Thus the only edges that may or may not exist between the vertices we defined are w_0w_1, w_0w_3 and w_1w_3. See Figure 2.3.3 for a reminder of the edges and vertices we know to this point. Vertex v_0 has degree 3, v_1, v_3 and u_0 are 4-vertices and v_2, u_1 and u_3 are 4^+-vertices. Vertices v_0, v_1, v_3 and u_0 have all their incident edges represented in Figure 2.3.3.

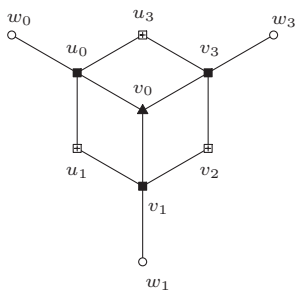

Figure 2.3.3: Vertices v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1, and w_3.

Suppose $w_0w_1 \notin E(G)$, $w_0w_3 \notin E(G)$, and $w_1w_3 \notin E(G)$. Let x be a vertex that is not in $V(G)$, and $G' = G + (\{x\}, \emptyset) + \{xw_0, xw_1, xw_3\} - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3\}$. The graph G' is in $\mathcal{P}_4$ and $||G'|| \leq m - 15$. Let F' be any induced forest of G'. If $x \in V(F')$, then $G[V(F') \cup \{v_0, v_1, v_3, u_0\} \setminus \{x\}]$ is a forest, and if $x \notin V(F')$, then $G[V(F') \cup \{v_1, v_3, u_0\}$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (6, 15, 3)$, we have $6\frac{38}{44} - 15\frac{7}{44} > 3$, a contradiction. Thus there is at least one edge in G among w_0w_1, w_0w_3 and w_1w_3. Moreover, since G is triangle free, there are at most two of these edges in G. We can assume that $w_0w_1 \notin E(G)$ and $w_0w_3 \in E(G)$.

Let us now prove some claims that we will use later :

(a) Suppose that either w_0 and w_1 are 4^+-vertices, or one is a 3-vertex, the other is a 4^+-vertex, and v_2, u_1 or u_3 have degree 5. Let $G' = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1\}$. We have $||G'|| \leq m - 24$. For any induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1, v_3, u_0\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (9, 24, 4)$, we have $9\frac{38}{44} - 24\frac{7}{44} > 4$, a contradiction.

(b) Suppose w_0 or w_3, say w_{i_0}, is a 3-vertex and either one of the w_i is a 4^+-vertex, or $w_1w_3 \notin E(G)$. Let $G' = G - \{v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1, w_3\}$. We have $||G'|| \leq m - 23$. For any induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1, v_3, u_0, w_{i_0}\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (10, 23, 5)$, we have $10\frac{38}{44} - 23\frac{7}{44} > 5$, a contradiction.

(c) Suppose w_0 and w_3 are 3-vertices and $w_1w_3 \in E(G)$. Let $G' = G - \{v_0, v_1, v_3, u_0, u_1, u_3, w_0, w_1, w_3\}$. We have $||G'|| \leq m - 23$. Since $w_1w_3 \in E(G)$ and $w_0w_3 \in E(G)$, the cycle $v_0v_1w_1w_3v_3$ separates v_2 from w_0 in G. Hence, for any induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1, u_0, w_0, w_3\}]$ is a forest.By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (9, 19, 5)$, we have $9\frac{38}{44} - 19\frac{7}{44} > 5$, a contradiction.

If $w_1w_3 \in E$, then by (b) and (c), both w_0 and w_3 are 4^+-vertices, and by symmetry w_1 is also a 4^+-vertex, which contradicts (a). Hence $w_1w_3 \notin E(G)$.

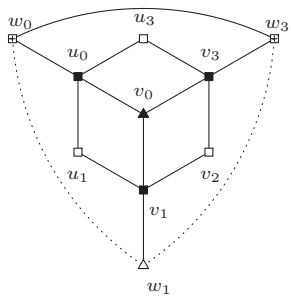

Figure 2.3.4: Vertices v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1, and w_3.

Therefore by (b), w_0 and w_3 are 4^+-vertices. By (a), w_1 is a 3-vertex, and v_2, u_1, and u_3 are 4-vertices (see Figure 2.3.4). Let y_0 and y_1 the two neighbours of w_1 distinct from v_1. By Lemma 2.3.4 they have a common neighbour distinct from w_1, say t. So by Lemmas 2.3.12 and 2.3.15 in the cycle $w_1y_0ty_1$, the vertices y_0 and y_1 have degree 4. By Lemma 2.3.4 each of y_0 and y_1 has a common neighbour with v_1 distinct from w_1, and thus is adjacent either to v_2 or to u_1. If they are both adjacent to the same one, say to v_2, then either $v_2v_1w_1y_0$ or $v_2v_1w_1y_1$ is a 4-cycle with a 3-vertex (w_1) not adjacent to a 4-vertex (v_2) that has both a neighbour inside and the neighbour outside of is, contradicting Lemma 2.3.8. Therefore y_0 and y_1 are not both adjacent to u_1 or both adjacent to v_2. We can assume that y_0 is adjacent to v_2 and y_1 is adjacent to u_1. See Figure 2.3.5 for a reminder of the edges and vertices we know to this point. At this point we know that v_0, v_1, v_2, v_3, u_0, u_1, w_1, y_0, and y_1 are distinct and do not share an edge that is not represented in Figure 2.3.5.

Let z be the neighbour of v_2 distinct from v_1, v_3, and y_0. The only edges with vertices among v_0, v_1, v_2, v_3, u_0, u_1, w_1, y_0, y_1, and z that may or may not be in G are zy_1 and zu_1, and as G is triangle-free, there is at most one of those edges. Let $G' = G - \{v_0, v_1, v_2, v_3, u_0, u_1, w_1, y_0, y_1, z\}$. We have $||G'|| \leq m - 23$. For any induced forest F' of G', the graph $G[V(F') \cup \{v_0, v_1, u_0, w_0, w_3\}]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (10, 23, 5)$, we have $10\frac{38}{44} - 23\frac{7}{44} > 5$, a contradiction. □

Lemma 2.3.17. *There is no 5-face with five 3-vertices in its boundary.*

Proof. Let $C = v_0v_1v_2v_3v_4$ be the cycle bounding such a face, and u_0, u_1, u_2, u_3, and u_4 be the third neighbours of v_0, v_1, v_2, v_3 and v_4 respectively. The u_i's are all distinct since the graph is triangle-free and by Lemma 2.3.15. We will consider the indices of the u_i's and v_i's modulo 5. By Lemma 2.3.15, for all

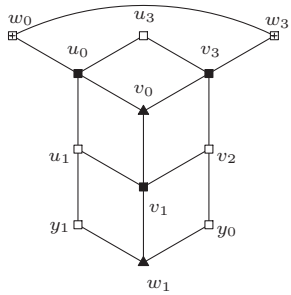

Figure 2.3.5: Vertices v_0, v_1, v_2, v_3, u_0, u_1, u_3, w_0, w_1, w_3, y_0, and y_1.

i, $u_iu_{i+1} \notin E(G)$. Let x and y be two vertices that are not in $V(G)$, and let $G' = G + (\{x, y\}, \{xy\}) + \{xu_0, xu_1, yu_2, yu_3\} - C$. We have $||G'|| \leq m - 5$. Let F' be an induced forest of G'. Let S be defined as the set of vertices that are in $\{v_0, v_3\}$, plus v_1 if $x \in V(F')$, and plus v_3 if $y \in V(F')$. The graph $G[V(F') \setminus \{x, y\} \cup S]$ is a forest. By Observation 2.3.3 applied to $(\alpha, \beta, \gamma) = (3, 5, 2)$, we have $3\frac{38}{44} - 5\frac{7}{44} > 2$, a contradiction. □

Lemma 2.3.18. *There is no 3-vertex with a 3-neighbour and a 4-neighbour in G.*

Proof. Let v be a 3-vertex, u a 3-neighbour of v, and w a 4-neighbour of v. Let x be the third neighbour of v. By Lemma 2.3.4, x and u have a common neighbour distinct from v, which contradicts Lemma 2.3.15. □

We now have all the forbidden configurations we need, expressed in terms of lemmas on the local structure of G. All that is left to do is to prove that no graph in $\mathcal{P}_4$ verifies all of those lemmas. In a standard discharging proof, this is done with the help of a discharging procedure. Here, however, the discharging procedure is so simple that we can just do a double counting on the number of 4^+-vertices in G.

For every face ϕ of G, let $c_{4^+}(\phi)$ be the number of 4^+-vertices in the boundary of ϕ. Let f be the number of faces of G, and for every $3 \leq d \leq 5$ and every $4 \leq l$, let f_l be the number of l-faces and n_d the number of d-vertices in G.

Each 4-vertex is in the boundary of at most four faces, and each 5-vertex is in the boundary of at most five faces. Therefore the sum of the $c_{4^+}(\phi)$ over all the 4-faces and 5-faces is $\sum_{\phi, 4 \leq d(\phi) \leq 5} c_{4^+}(\phi) \leq 4n_4 + 5n_5$. From Lemmas 2.3.12, 2.3.17, and 2.3.18 we can deduce that for each 5-face ϕ we have $c_{4^+}(\phi) \geq 2$. Moreover, by Lemmas 2.3.15 and 2.3.16, for each 4-face ϕ, we have $c_{4^+}(\phi) \geq 4$. Thus $\sum_{\phi, d(\phi)=4} c_{4^+}(\phi) + \sum_{\phi, d(\phi)=5} c_{4^+}(\phi) \geq 4f_4 + 2f_5$. Thus it follows that:

$$4n_4 + 5n_5 \geq 4f_4 + 2f_5$$

By Euler's formula, we have:

$$\begin{aligned}
-12 &= 6m - 6n - 6f \\
&= 2 \sum_{v \in V(G)} d(v) + \sum_{\phi \in F(G)} d(\phi) - 6n - 6f \\
&= \sum_{d \geq 3} (2d - 6) n_d + \sum_{l \geq 4} (l - 6) f_l \\
&\geq 2n_4 + 4n_5 - 2f_4 - f_5 \\
&\geq 0
\end{aligned}$$

That contradiction completes the proof of Theorem 2.1.10.

2.4 Large induced linear forests in triangle-free planar graphs

Let us first recall the statement of Theorem 2.1.21.

Theorem 2.1.21 (recall)**.** *Every triangle-free planar graph of order n and size m admits an induced linear forest of order at least $\frac{9n-2m}{11}$, and thus at least $\frac{5n+8}{11} \approx 0.455n$.*

This proof is similar to the proof of Theorem 2.1.10 (see Section 2.3). We will use the same reasoning: we will first prove a series of lemmas on the structure of a minimal counter-example, and then obtain a contradiction by a double counting argument.

Let $G = (V, E)$ be a counter-example to Theorem 2.1.21 with minimum order. Let $n = |G|$ and $m = ||G||$. We use a similar observation to Observation 2.3.3, adapted to the framework of linear forests.

Observation 2.4.1. *Let α, β, γ be integers satisfying $\alpha \geq 1$, $\beta \geq 0$ and $\gamma \geq 0$.*

Let $G' \in \mathcal{P}_4$ be a graph with $|G'| = n - \alpha$ and $||G'|| \leq m - \beta$.

By minimality of G, the graph G' admits an induced linear forest of order at least $\frac{9(n-\alpha)-2(m-\beta)}{11} = \frac{9n-2m}{11} - \frac{9\alpha-2\beta}{11}$.

Given an induced linear forest F' of G' of order at least $\frac{9n-2m}{11} - \frac{9\alpha-2\beta}{11}$, if there is an induced forest F of G of order $|F| \geq |F'| + \gamma$, then as $|F| < \frac{9n-2m}{11}$, we have $\frac{9\alpha-2\beta}{11} > \gamma$.

As in the previous section, we will use Observation 2.4.1 with bounded values for α, β, and γ. However, we would like to extend the method to enable us to remove possibly large portions of the graph with an induced linear forest that is not too much connected to the rest of the graph. To do this, we will use Observations 2.4.2 and 2.4.3.

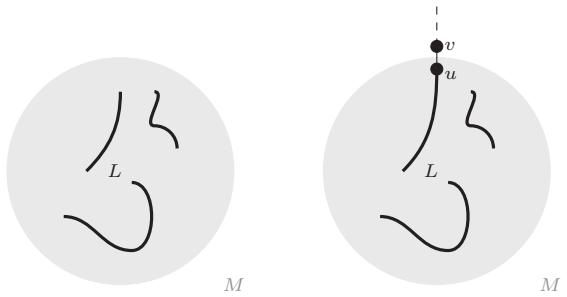

Figure 2.4.1: The situation in Observation 2.4.2 (left) and Observation 2.4.3 (right).

Observation 2.4.2. *Suppose $L \subset V(G)$ induces a linear forest in G, and M is a set of vertices such that $M \cap L = \emptyset$ and $M \supseteq N(L)$. Let $G' = G - M - L$. See Figure 2.4.1 (left) for an illustration.*

By minimality of G, G' admits a linear forest F' with $|F'| \geq \frac{9}{11}|G'| - \frac{2}{11}||G'||$. Observe that $F = G[V(F') \cup L]$ is an induced linear forest of G. As G is a counter-example to Theorem 2.1.21, we have $|F| < \frac{9}{11}|G| - \frac{2}{11}||G||$. Therefore $|L| = |F| - |F'| < \frac{9}{11}(|M| + |L|) - \frac{2}{11}(||G|| - ||G'||)$.

Observation 2.4.3. *Suppose $L \subset V(G)$ induces a linear forest in G. Suppose there is a set of vertices M and two vertices $u \in L$ and v such that $M \cap L = \emptyset$, $N(L) \setminus M = \{v\}$, and $N(v) \cap L = \{u\}$. Let $G' = G - M - L$. Suppose v is a 1^--vertex in G' and u is a 1^--vertex in $G[L]$. See Figure 2.4.1 (right) for an illustration.*

By minimality of G, G' admits a linear forest F' with $|F'| \geq \frac{9}{11}|G'| - \frac{2}{11}||G'||$. Observe that $F = G[V(F') \cup L]$ is an induced linear forest of G. As G is a counter-example to Theorem 2.1.21, we have $|F| < \frac{9}{11}|G| - \frac{2}{11}||G||$. Therefore $|L| = |F| - |F'| < \frac{9}{11}(|M| + |L|) - \frac{2}{11}(||G|| - ||G'||)$.

Note that Observations 2.4.2 and 2.4.3 will be used to obtain a contradiction, similarly to Observation 2.4.1. Unlike in the previous section, we will have several observations and lemmas of this type, thus

we do not expect our reader to read them every time they are used, and will directly express the implied inequalities in the proofs where those observations and lemmas are used.

Now we want to prove some structural properties of G, so that we can later show that the counter-example G does not exist, and thus that Theorem 2.1.21 is true. First note that G is connected, otherwise one of its components would be a smaller counter-example to Theorem 2.1.21. Then note that every vertex of G has degree at most 4. Otherwise, by considering a vertex of degree at least 5 and by applying Observation 2.4.1 to $H^* = G - v$ with $(\alpha, \beta, \gamma) = (1, 5, 0)$ and $F = F^*$, we have $\frac{9}{11} - 5\frac{2}{11} > 0$, a contradiction

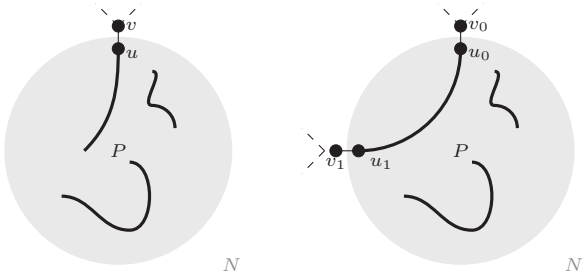

Figure 2.4.2: A simple chain (left) and a double chain (right).

For the rest of the procedure, we will need to define some notions that will appear in the remainder of this section. Those notions, that we will call *chain* and *double chain*, will be used to simplify the rest of the proof, by regrouping similar arguments. We note that all of the forbidden configurations could be obtained by doing a direct analysis of a finite number of cases on bounded structures. This analysis would however be dull and redundant, and is rendered unnecessary by the analysis of chains and double chains.

Let us define the notion of a *chain* (or *simple chain*) of G which is a quadruplet $C = (P, N, u, v)$ such that:

- $P \subset V(G)$, $N \subset V(G) \setminus P$, $u \in P$, and $v \in V(G) \setminus (N \cup P)$;
- $G[P]$ is a linear forest;
- u is a 1^--vertex of $G[P]$, and $N(v) \cap P = \{u\}$;
- $N(P) \subset N \cup \{v\}$ in G;
- v is a 2^--vertex in $G - (N \cup P)$.

See Figure 2.4.2 (left) for an illustration. We will use the following notation for a chain $C = (P, N, u, v)$ of G:

- $|C| = |P| + |N|$;
- $G - C = G - (N \cup P)$;
- $d(C)$ is the degree of v in $G - C$ (thus $d(C) \leq 2$);
- $||C|| = ||G|| - ||G - C||$.

The following lemma imposes a condition on the chains. It will be used to derive a contradiction whenever a chain does not meet this condition.

Lemma 2.4.4. *For every chain $C = (P, N, u, v)$ of G, we have $|P| < \frac{9}{11}|C| - \frac{2}{11}(||C|| - \frac{1}{2})$.*

Proof. Let us consider by contradiction a chain $C = (P, N, u, v)$ such that $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - \frac{1}{2})$ maximizing $|C|$.

- Suppose $d(C) = 0$. Let $G' = (G-C)-v$. The set $P \cup \{v\}$ induces a linear forest, and its neighbourhood is a subset of N. By applying Observation 2.4.2 to $L = P \cup \{v\}$ and $M = N$, we have $|P| + 1 < \frac{9}{11}(|C| + 1) - \frac{2}{11}||C||$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - \frac{1}{2})$, it follows that $1 < \frac{9}{11} - \frac{2}{11}\frac{1}{2}$, a contradiction.
- Suppose $d(C) = 1$. The set P induces a linear forest, and its neighbourhood is a subset of $N \cup \{v\}$. Furthermore, $N(v) \cap P = \{u\}$, $N(u) \cap (G - C) = \{v\}$, and u and v are 1-vertices in P and $G - C$ respectively. By applying Observation 2.4.3 to $L = P$ and $M = N$, we have $|P| < \frac{9}{11}|C| - \frac{2}{11}||C||$, thus $|P| < \frac{9}{11}|C| - \frac{2}{11}(||C|| - \frac{1}{2})$, a contradiction.
- Suppose $d(C) = 2$. Let w_0 and w_1 be the neighbours of v in $G - C$.
  - Suppose one of the w_i's, say w_0, has degree 1 in $G - C$. Let $G' = (G - C) - \{v, w_0, w_1\}$. The set $P \cup \{v, w_0\}$ induces a linear forest and its neighbourhood is a subset of $N \cup \{w_1\}$. By applying Observation 2.4.2 to $L = P \cup \{v, w_0\}$ and $M = N \cup \{w_1\}$, we have $|P|+2 < \frac{9}{11}(|C|+3) - \frac{2}{11}(||C||+2)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - \frac{1}{2})$, it follows that $2 < \frac{9}{11}3 - \frac{2}{11}\frac{5}{2}$, a contradiction.
  - Suppose the w_i's are 2-vertices in $G - C$. Observe that they are not adjacent since G is triangle-free. The set $P \cup \{v\}$ induces a linear forest, and its neighbourhood is a subset of $N \cup \{w_0, w_1\}$. Furthermore, $N(w_1) \cap (P \cup \{v\}) = \{v\}$, $N(v) \cap ((G - C) - \{v, w_0\}) = \{w_1\}$, and v and w_1 are 1-vertices in $P \cup \{v\}$ and $G - C - \{v, w_0\}$ respectively. By applying Observation 2.4.3 to $L = P \cup \{v\}$ and $M = N \cup \{w_0\}$, we have $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(||C|| + 3)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - \frac{1}{2})$, it follows that $1 < \frac{9}{11}2 - \frac{2}{11}\frac{7}{2}$, a contradiction.
  - Suppose the w_i's both have degree 4 in $G - C$. Again they are not adjacent since G is triangle-free. The set $P \cup \{v\}$ induces a linear forest, and its neighbourhood is a subset of $N \cup \{w_1, w_0\}$. By applying Observation 2.4.2 to $L = P \cup \{v\}$ and $M = N \cup \{w_0, w_1\}$, we have $|P| + 1 < \frac{9}{11}(|C| + 3) - \frac{2}{11}(||C|| + 8)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - \frac{1}{2})$, it follows that $1 < \frac{9}{11}3 - \frac{2}{11}\frac{17}{2}$, a contradiction.
  - Suppose one of the w_i's, say w_0, is a 3^--vertex in $G - C$ and the other one is a 3^+-vertex in $G - C$. Let $C' = (P \cup \{v\}, N \cup \{w_1\}, v, w_0)$. Then C' is a chain of G, and by maximality of $|C|$, we have $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(||C|| + \frac{7}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - \frac{1}{2})$, it follows that $1 < \frac{9}{11}2 - \frac{2}{11}4$, a contradiction. □

Let us now define a new notion quite similar to the notion of chain. A *double chain* of G is a sextuplet $C = (P, N, u_0, u_1, v_0, v_1)$, so that:

- $P \subset V(G)$, $N \subset V(G) \setminus P$, $u_0 \in P$, $u_1 \in P$, $v_0 \in V(G) \setminus (N \cup P)$ and $v_1 \in V(G) \setminus (N \cup P)$;
- $v_0 \neq v_1$;
- $G[P]$ is a linear forest;
- u_0 and u_1 are 1^--vertices of $G[P]$ if they are distinct, a 0-vertex of $G[P]$ if they are equal, and for $i \in \{0, 1\}$, $N(v_i) \cap P = \{u_i\}$;
- $N(P) \subset N \cup \{v_0\} \cup \{v_1\}$;
- v_0 and v_1 are 2^--vertices in $G - (N \cup P)$.

See Figure 2.4.2 (right) for an illustration. We will use the following notation for a double chain $C = (P, N, u_0, u_1, v_0, v_1)$ of G :

- $|C| = |P| + |N|$;
- $G - C = G - (N \cup P)$;
- $d_0(C)$ is the degree of v_0 in $G - C$ (thus $d_0(C) \leq 2$);

- $d_1(C)$ is the degree of v_1 in $G - C$ (thus $d_1(C) \leq 2$);
- $||C|| = ||G|| - ||G - C||$.

A double chain $C = (P, N, u_0, u_1, v_0, v_1)$ of G such that v_0 and v_1 are on different components of $G - C$ is called a *separating double chain* of G. We will use the following two lemmas to get a contradiction when double chains do not verify certain conditions.

Lemma 2.4.5. *For every double chain $C = (P, N, u_0, u_1, v_0, v_1)$ of G, $|P| < \frac{9}{11}|C| - \frac{2}{11}(||C|| - 3)$.*

Proof. Let us consider by contradiction a double chain $C = (P, N, u_0, u_1, v_0, v_1)$ such that $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 3)$ maximizing $|C|$.

Suppose first that v_0 and v_1 are not adjacent.

- Suppose $d_0(C) = 0$. Then $(P \cup \{v_0\}, N, u_1, v_1)$ is a simple chain of G. By Lemma 2.4.4, $|P| + 1 < \frac{9}{11}(|C| + 1) - \frac{2}{11}(||C|| - \frac{1}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 3)$, we have $1 < \frac{9}{11} - \frac{5}{2}\frac{2}{11}$, a contradiction.
- Suppose $d_0(C) = 1$. Let w be the neighbour of v_0 in $G - C$. Then $(P \cup \{v_0\}, N \cup \{w\}, u_1, v_1)$ is a simple chain of G. By Lemma 2.4.4, $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(||C|| + \frac{1}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 3)$, we have $1 < \frac{9}{11}2 - \frac{2}{11}\frac{7}{2}$, a contradiction.
- Suppose $d_0(C) = 2$. Let w_0 and w_1 be the neighbours of v_0 in $G - C$.
  - Suppose one of the w_i's, say w_0, has degree 1 in $G - C$. Then $(P \cup \{v_0, w_0\}, N \cup \{w_1\}, u_1, v_1)$ is a simple chain of G. By Lemma 2.4.4, $|P| + 2 < \frac{9}{11}(|C| + 3) - \frac{2}{11}(||C|| + \frac{3}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 3)$, we have $2 < \frac{9}{11}3 - \frac{2}{11}\frac{9}{2}$, a contradiction.
  - Suppose that the w_i's both have degree 2 in $G - C$. Note that they are not adjacent since G is triangle-free. They may, however, be adjacent to v_1.

    Suppose both of the w_i's are adjacent to v_1. The set $P \cup \{v_0, v_1\}$ induces a linear forest in G, and its neighbourhood is a subset of $N \cup \{w_0, w_1\}$. By applying Observation 2.4.2 to $L = P \cup \{v_0, v_1\}$ and $M = N \cup \{w_0, w_1\}$, we have $|P| + 2 < \frac{9}{11}(|C| + 4) - \frac{2}{11}(||C|| + 4)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 3)$, it follows that $2 < \frac{9}{11}4 - \frac{2}{11}7$, a contradiction.

    Therefore one of the w_i's, say w_0, is not adjacent to v_1. Let x be the neighbour of w_0 in $G - C$ distinct from v_0. Suppose x has degree 4 in $G - C - \{v_0, w_1\}$. Now $(P \cup \{v_0, w_0\}, N \cup \{w_1, x\}, u_1, v_1)$ is a chain of G. By Lemma 2.4.4, $|P| + 2 < \frac{9}{11}(|C| + 4) - \frac{2}{11}(||C|| + \frac{13}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 3)$, we have $2 < \frac{9}{11}4 - \frac{2}{11}\frac{19}{2}$, a contradiction. Therefore x is a 3^--vertex in $G - C - \{v_0, w_1\}$. Then $(P \cup \{v_0, w_0\}, N \cup \{w_1\}, w_0, u_1, x, v_1)$ is a double chain of G, so by maximality of $|C|$, $|P| + 2 < \frac{9}{11}(|C| + 3) - \frac{2}{11}(||C|| + 4 - 3)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 3)$, we have $2 < \frac{9}{11}3 - \frac{2}{11}4$, a contradiction.
  - Suppose that the w_i's are 4-vertices in $G - C$. Now $(P \cup \{v_0\}, N \cup \{w_0, w_1\}, u_1, v_1)$ is a chain of G. By Lemma 2.4.4, $|P| + 1 < \frac{9}{11}(|C| + 3) - \frac{2}{11}(||C|| + \frac{15}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 3)$, we have $1 < \frac{9}{11}3 - \frac{2}{11}\frac{21}{2}$, a contradiction.
  - Suppose one of the w_i's, say w_0, is a 3^--vertex in $G - C$ and the other one is a 3^+-vertex in $G - C$. Then $(P \cup \{v_0\}, N \cup \{w_1\}, v_0, u_1, w_0, v_1)$ is a double chain of G. By maximality of $|C|$, $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(||C|| + 4 - 3)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 3)$, we have $1 < \frac{9}{11}2 - \frac{2}{11}4$, a contradiction.

Now v_0 and v_1 are adjacent.

- Suppose $d_0(C) = 1$ or $d_1(C) = 1$. Say $d_0(C) = 1$. The set $P \cup \{v_0\}$ induces a linear forest, and its neighbourhood is a subset of $N \cup \{v_1\}$. By applying Observation 2.4.2 to $L = P \cup \{v_0\}$ and $M = N \cup \{v_1\}$, we have $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(||C|| + 1)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 3)$, it follows that $1 < \frac{9}{11}2 - \frac{2}{11}4$, a contradiction.

- Now $d_0(C) = 2$ and $d_1(C) = 2$. Let w be the neighbour of v_0 in $(G - C) - v_1$.

  Suppose w is a 2^--vertex in $(G-C)-v_1$. The set $P\cup\{v_0\}$ induces a linear forest, and its neighbourhood is a subset of $N \cup \{v_1, w\}$. Furthermore, $N(w) \cap (P \cup \{v_0\}) = \{v_0\}$, $N(v_0) \cap (G - C) - \{v_0, v_1\} = \{w\}$, and v_0 and w are 1^--vertices in $G[P \cup \{v_0\}]$ and $(G - C) - \{v_0, v_1\}$ respectively. By applying Observation 2.4.3 to $L = P \cup \{v_0\}$ and $M = N \cup \{v_1\}$, we have $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(||C|| + 3)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 3)$, it follows that $1 < \frac{9}{11}2 - \frac{2}{11}6$, a contradiction.

  Now w is a 3^+-vertex in $(G - C) - v_1$. The set $P \cup \{v_0\}$ induces a linear forest, and its neighbourhood is a subset of $N \cup \{v_1, w\}$. By applying Observation 2.4.2 to $L = P \cup \{v_0\}$ and $M = N \cup \{w\}$, we have $|P| + 1 < \frac{9}{11}(|C| + 3) - \frac{2}{11}(||C|| + 5)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 3)$, it follows that $1 < \frac{9}{11}3 - \frac{2}{11}8$, a contradiction. □

Lemma 2.4.6. *For every separating double chain $C = (P, N, u_0, u_1, v_0, v_1)$ of G, $|P| < \frac{9}{11}|C| - \frac{2}{11}(||C|| - 1)$.*

Proof. Let us consider by contradiction a separating double chain $C = (P, N, u_0, u_1, v_0, v_1)$ such that $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 1)$ maximizing $|C|$.

- Suppose $d_0(C) = 0$. Then $(P \cup \{v_0\}, N, u_1, v_1)$ is a simple chain of G. By Lemma 2.4.4, $|P| + 1 < \frac{9}{11}(|C| + 1) - \frac{2}{11}(||C|| - \frac{1}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 1)$, we have $1 < \frac{9}{11} - \frac{2}{11}\frac{1}{2}$, a contradiction.

- Suppose $d_0(C) = 1$. Let w be the neighbour of v_0 in $G - C$. Suppose w is a 4^+-vertex in $G - C$. Then $(P\cup\{v_0\}, N\cup\{w\}, u_1, v_1)$ is a simple chain of G. By Lemma 2.4.4, $|P|+1 < \frac{9}{11}(|C|+2)-\frac{2}{11}(||C||+\frac{7}{2})$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 1)$, we have $1 < \frac{9}{11}2 - \frac{2}{11}\frac{9}{2}$, a contradiction.

  Now w is a 3^--vertex in $G - C$. Let $C' = (P \cup \{v_0\}, N, v_0, u_1, w, v_1)$. Then C' is a separating double chain of G, and by maximality of $|C|$, $|P|+1 < \frac{9}{11}(|C|+1) - \frac{2}{11}(||C||)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 1)$, we have $1 < \frac{9}{11} - \frac{2}{11}$, a contradiction.

- Suppose $d_0(C) = 2$. Let w_0 and w_1 be the neighbours of v_0 in $G - C$.

  - Suppose one of the w_i's, say w_0, has degree 1 in $G-C$. We have a simple chain $(P\cup\{v_0, w_0\}, N\cup\{w_1\}, u_1, v_1)$. By Lemma 2.4.4, $|P|+2 < \frac{9}{11}(|C|+3)-\frac{2}{11}(||C||+\frac{3}{2})$. As $|P| \geq \frac{9}{11}|C|-\frac{2}{11}(||C||-1)$, we have $2 < \frac{9}{11}3 - \frac{2}{11}\frac{5}{2}$, a contradiction.
  - Suppose the w_i's are both 2-vertices in $G-C$. Note that they are not adjacent since G is triangle-free. Let x be the second neighbour of w_0 in $G - C$. Suppose x is a 4-vertex in $G - C - \{w_1\}$. Then $(P \cup \{v_0, w_0\}, N \cup \{w_1, x\}, u_1, v_1)$ is a simple chain of G. By Lemma 2.4.4, $|P| + 2 < \frac{9}{11}(|C|+4)-\frac{2}{11}(||C||+\frac{13}{2})$. As $|P| \geq \frac{9}{11}|C|-\frac{2}{11}(||C||-1)$, we have $2 < \frac{9}{11}4-\frac{2}{11}\frac{15}{2}$, a contradiction. Now x is a 3^--vertex in $G-C-\{w_1\}$, so $(P\cup\{v_0, w_0\}, N\cup\{w_1\}, w_0, u_1, x, v_1)$ is a separating double chain of G. By maximality of $|C|$, $|P|+2 < \frac{9}{11}(|C|+3)-\frac{2}{11}(||C||+3)$. As $|P| \geq \frac{9}{11}|C|-\frac{2}{11}(||C||-1)$, we have $2 < \frac{9}{11}3 - \frac{2}{11}4$, a contradiction.
  - Suppose the w_i's are 4-vertices in $G - C$. Again they are not adjacent since G is triangle-free. We have a simple chain $(P \cup \{v_0\}, N \cup \{w_0, w_1\}, u_1, v_1)$. By Lemma 2.4.4, $|P| + 1 < \frac{9}{11}(|C|+3)-\frac{2}{11}(||C||+\frac{15}{2})$. As $|P| \geq \frac{9}{11}|C|-\frac{2}{11}(||C||-1)$, we have $1 < \frac{9}{11}3-\frac{2}{11}\frac{17}{2}$, a contradiction.
  - Suppose one of the w_i's, say w_0, is a 3^--vertex in $G - C$ and the other one is a 3^+-vertex in $G - C$. Then $(P \cup \{v_0\}, N \cup \{w_1\}, v_0, u_1, w_0, v_1)$ is a separating double chain. By maximality of $|C|$, $|P| + 1 < \frac{9}{11}(|C| + 2) - \frac{2}{11}(||C|| + 3)$. As $|P| \geq \frac{9}{11}|C| - \frac{2}{11}(||C|| - 1)$, we have $1 < \frac{9}{11}2 - \frac{2}{11}4$, a contradiction. □

Now we have all the lemmas we need to deal with simple and double chains. Note that Lemmas 2.4.4-2.4.6 could also have been expressed in terms of forbidden configurations. Those configurations could be arbitrarily large. Let us now prove some lemmas on the structure of G. Again, these lemmas can be seen as forbidden configurations, but all of these configurations are bounded. All of these could be proven directly, but the proof would be longer and a lot more redundant.

Lemma 2.4.7. *Graph G has no 2^--vertex.*

Proof. As G is connected, if it has a 0-vertex, then G is the graph with one vertex and it satisfies Theorem 2.1.21, a contradiction.

By contradiction, suppose $u \in V$ is a 1-vertex. Let v be the neighbour of u. If v is a 3^--vertex in G, then $(\{u\}, \emptyset, u, v)$ is a chain of G, thus by Lemma 2.4.4, $1 < \frac{9}{11} - \frac{2}{11}(1 - \frac{1}{2})$, a contradiction. Now v is a 4-vertex. Let $G' = G - \{u, v\}$. We have $||G'|| = m - 4$. Let F' be an induced linear forest of G'. The graph $F = G[V(F') \cup \{u\}]$ is a linear forest. By applying Observation 2.4.1 to $(\alpha, \beta, \gamma) = (2, 4, 1)$, $1 < \frac{9}{11}2 - \frac{2}{11}4$, a contradiction.

Therefore G has no 1^--vertex. Suppose now that $u \in V$ is a 2-vertex. Let v_0 and v_1 be the two neighbours of u.

Suppose v_0 or v_1, say v_1, is a 3^--vertex. We have a simple chain $(\{u\}, \{v_0\}, u, v_1)$. By Lemma 2.4.4, $1 < \frac{9}{11}2 - \frac{2}{11}(4 - \frac{1}{2})$, a contradiction.

Now v_0 and v_1 are 4-vertices. Let $G' = G - \{u, v_0, v_1\}$. We have $||G'|| = m - 8$. Let F' be an induced linear forest of G'. The graph $F = G[V(F') \cup \{u\}]$ is a linear forest. By applying Observation 2.4.1 to $(\alpha, \beta, \gamma) = (3, 8, 1)$, $1 < \frac{9}{11}3 - \frac{2}{11}8$, a contradiction. □

Lemma 2.4.8. *Graph G has no 3-vertex with a 3-neighbour and two 4-neighbours.*

Proof. By contradiction, suppose G has a 3-vertex u, adjacent to a 3-vertex v and two 4-vertices w_0 and w_1. We have a simple chain $(\{u\}, \{w_0, w_1\}, u, v)$. By Lemma 2.4.4, $1 < \frac{9}{11}3 - \frac{2}{11}\frac{17}{2}$, a contradiction. □

Lemma 2.4.9. *Graph G has no 3-vertex with two 3-neighbours and a 4-neighbour.*

Proof. Let u be a 3-vertex with two 3-neighbours v_0 and v_1, and a 4-neighbour w. Let x_0 and x_1 be the two neighbours of v_0 distinct from u. Note that x_0 and x_1 are 3^+-vertices in G by Lemma 2.4.7, and thus 1^+-vertices in $G' = G - \{u, w, v_0\}$ since they are not adjacent to u. Let $G' = G - \{u, w, v_0\}$.

Suppose that x_0 and x_1 are 2^+-vertices in G', or that one is a 3-vertex and the other a 1^+-vertex. We have a simple chain $(\{u, v_0\}, \{x_0, x_1, w\}, u, v_1)$. By Lemma 2.4.4, $2 < \frac{9}{11}5 - \frac{2}{11}\frac{23}{2}$, a contradiction.

Suppose one of the x_i's, say x_0, is a 2^+-vertex in G', and the other one is a 1-vertex in G'. We have a double chain $(\{u, v_0\}, \{w, x_0\}, u, v_0, v_1, x_1)$. By Lemma 2.4.5, $2 < \frac{9}{11}4 - \frac{2}{11}7$, a contradiction.

Now the x_i's are 1-vertices in G'. By Lemma 2.4.7, the x_i's are 3-vertices in G, and are both adjacent to w. By planarity of G, one of the x_i's, say x_0, is not adjacent to v_1. Let y be the neighbour of x_0 in G'. By Lemmas 2.4.7 and 2.4.8, y is a 3-vertex in G. We have a simple chain $(\{u, v_0, x_0\}, \{w, v_1, x_1\}, x_0, y)$. By Lemma 2.4.4, $3 < \frac{9}{11}6 - \frac{2}{11}\frac{21}{2}$, a contradiction. □

Lemma 2.4.10. *Graph G has no two adjacent 3-vertices in G.*

Proof. By Lemma 2.4.7, every vertex in G has degree 3 or 4. By Lemmas 2.4.8 and 2.4.9, there is no 3-vertex with a 3-neighbour and a 4-neighbour in G. Suppose by contradiction that there are two adjacent 3-vertices in G. Then as G is connected, G only has 3-vertices.

Suppose there is a 4-cycle $v_0v_1v_2v_3$ in G. For all i, let u_i be the third neighbour of v_i. Since G has no triangle, the only vertices among the u_i's and v_i's that may not be distinct are u_0 and u_2 on the one hand, and u_1 and u_3 on the other hand. Suppose $u_0 = u_2$ and $u_1 = u_3$. In particular, one of those vertices is in the interior of $v_0v_1v_2v_3$ and the other one is not. Let $G' = G - \{v_0, v_1, v_2, v_3\}$. We have $||G'|| = m - 8$. Let F' be an induced linear forest of G'. As u_0 and u_1 are 1-vertices in G', separated by $v_0v_1v_2v_3$ in G, the graph $F = G[V(F') \cup \{v_0, v_1\}]$ is a linear forest. By applying Observation 2.4.1 to $(\alpha, \beta, \gamma) = (4, 8, 2)$, we have $2 < \frac{9}{11}4 - \frac{2}{11}8$, a contradiction. Now say u_0 and u_2 are distinct. We have a double chain $(\{v_0, v_1, v_2\}, \{v_3, u_1\}, v_0, v_2, u_0, u_2)$. By Lemma 2.4.5, $3 < \frac{9}{11}5 - \frac{2}{11}6$, a contradiction.

Therefore there is no 4-cycle in G. Suppose there is a 5-cycle $v_0v_1v_2v_3v_4$ in G. For all i, let u_i be the third neighbour of v_i. Now all the u_i's are distinct, otherwise there is a 4-cycle and we fall into the previous case. We have a double chain $(\{v_0, v_1, v_2, v_3\}, \{v_4, u_1, u_2\}, v_0, v_3, u_0, u_3)$. By Lemma 2.4.5, $4 < \frac{9}{11}7 - \frac{2}{11}11$, a contradiction.

Now G is a 3-regular planar graph with girth at least 6, which contradicts Euler's formula. □

Lemma 2.4.11. *There is no 4-cycle with at least two 3-vertices in G.*

Proof. By contradiction, suppose there is such a 4-cycle $v_0v_1v_2v_3$. By Lemmas 2.4.7 and 2.4.10, this cycle has exactly two 3-vertices and two 4-vertices, and the two 3-vertices are not adjacent. Say v_0 and v_2 are 3-vertices, and v_1 and v_3 are 4-vertices. Let u_0 and u_2 be the third neighbours of v_0 and v_2 respectively. By Lemma 2.4.10, u_0 and u_2 are 4-vertices.

Suppose that v_0 and v_2 have three neighbours in common, v_1, v_3, and $u = u_0 = u_2$. Let $G' = G - \{v_0, v_1, v_2, v_3, u\}$. Let F' be an induced linear forest of G'. The graph $F = G[V(F') \cup \{v_0, v_2\}]$ is a linear forest. By applying Observation 2.4.1 to $(\alpha, \beta, \gamma) = (5, 12, 2)$, $2 < \frac{9}{11}5 - \frac{2}{11}12$, a contradiction.

Now u_0 and u_2 are distinct. Suppose that $u_0u_2 \in E(G)$. We have a chain $(\{v_0, v_2\}, \{v_1, v_3, u_2\}, v_0, u_0)$. By Lemma 2.4.4, $2 < \frac{9}{11}5 - \frac{2}{11}\frac{25}{2}$, a contradiction.

Now $u_0u_2 \notin E(G)$. Let $G' = G - \{v_0, v_1, v_2, v_3, u_0, u_2\}$. Let F' be an induced linear forest of G'. The graph $F = G[V(F') \cup \{v_0, v_2\}]$ is a linear forest. By applying Observation 2.4.1 to $(\alpha, \beta, \gamma) = (6, 16, 2)$, $2 < \frac{9}{11}6 - \frac{2}{11}16$, a contradiction. □

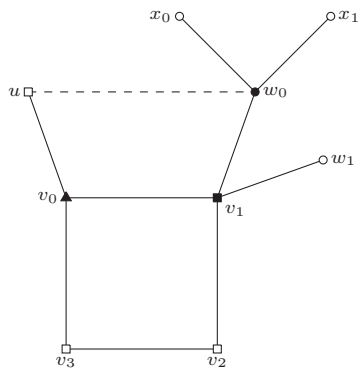

Figure 2.4.3: The vertices of Lemma 2.4.12. The dashed line is an edge that may or may not be present.

Lemma 2.4.12. *There is no 4-face with exactly one 3-vertex in its boundary in G.*

Proof. By contradiction, suppose there is a 4-face bounded by a cycle $v_0v_1v_2v_3$, such that v_0 is a 3-vertex and the other v_i's are 4-vertices. Let u be the third neighbour of v_0. Note that u is a 4-vertex by Lemma 2.4.10.

Suppose first that $vv_2 \in E(G)$. By planarity of G, v_1 and v_3 are in different components of $G - \{v_0, u, v_2\}$. Therefore $(\{v_0\}, \{u, v_2\}, v_0, v_0, v_1, v_3)$ is a separating double chain of G. By Lemma 2.4.6, $1 < \frac{9}{11}3 - \frac{2}{11}8$, a contradiction.

Now $uv_2 \notin E(G)$. Let w_0 and w_1 be the neighbours of v_1 distinct from v_0 and v_2.

- Suppose w_0 and w_1 are adjacent to v_3. Let $G' = G - \{v_0, v_1, v_2, v_3, v, w_0, w_1\}$. By Lemma 2.4.7, the w_i's are 3^+-vertices, and since G is triangle-free, they cannot be adjacent to v_2. Moreover, by planarity, one of the w_i's at least is not adjacent to u. This implies that $||G'|| \leq m - 15$. Let F' be an induced linear forest of G'. The graph $F = G[V(F') \cup \{v_0, v_1, v_3\}]$ is a linear forest. By applying Observation 2.4.1 to $(\alpha, \beta, \gamma) = (7, 15, 3)$, $3 < \frac{9}{11}7 - \frac{2}{11}15$, a contradiction.

- Suppose that one of the w_i's, say w_0, is adjacent to v_3 and that the other one (w_1) is not adjacent to v_3. Let w_2 be the neighbour of v_3 distinct from v_0, v_2, and w_0.

  - Suppose w_1 or w_2, say w_1, is a 3-vertex in $G - \{v_0, v_1, v_2, v_3, u, w_0\}$. Suppose w_2 is a 3-vertex in $G - \{v_0, v_1, v_2, v_3, u, w_0, w_1\}$ (note that this implies that $w_1w_2 \notin E(G)$). Let $G' = G - \{v_0, v_1, v_2, v_3, u, w_0, w_1, w_2\}$. We have $||G'|| \leq m - 20$. Let F' be an induced linear forest of G'. The graph $F = G[V(F') \cup \{v_0, v_1, v_3\}]$ is a linear forest. By applying Observation 2.4.1 to $(\alpha, \beta, \gamma) = (8, 20, 3)$, $3 < \frac{9}{11}8 - \frac{2}{11}20$, a contradiction.

Now w_2 is a 2^--vertex in $G-\{v_0, v_1, v_2, v_3, u, w_0, w_1\}$, and thus $(\{v_0, v_1, v_3\}, \{v_2, u, w_0, w_1\}, v_3, w_2)$ is a chain of G. By Lemma 2.4.4, $3 < \frac{9}{11}7 - \frac{2}{11}(17 - \frac{1}{2})$, a contradiction.

- Suppose w_1 and w_2 are 2^--vertex in $G - \{v_0, v_1, v_2, v_3, u, w_0\}$. Then we have a double chain $(\{v_0, v_1, v_3\}, \{v_2, u, w_0\}, v_1, v_3, w_1, w_2)$. By Lemma 2.4.5, $3 < \frac{9}{11}6 - \frac{2}{11}11$, a contradiction.

- Suppose the w_i's are not adjacent to v_3. Let us prove by contradiction that the w_i's are 2^--vertices in $G - \{v_0, v_1, v_2, u\}$.

  - Suppose the w_i's are 3-vertices in $G-\{v_0, v_1, v_2, u\}$. The quadruplet $(\{v_0, v_1\}, \{v_2, u, w_0, w_1\}, v_0, v_3)$ forms a chain. By Lemma 2.4.4, $2 < \frac{9}{11}6 - \frac{2}{11}(18 - \frac{1}{2})$, a contradiction.
  - Now one of the w_i's, say w_0, is a 2^--vertex in $G - \{v_0, v_1, v_2, u\}$. Suppose w_1 is a 3-vertex in $G - \{v_0, v_1, v_2, u\}$. Then we have the following double chain: $(\{v_0, v_1\}, \{v_2, u, w_1\}, v_0, v_1, v_3, w_0)$. By Lemma 2.4.5, $2 < \frac{9}{11}5 - \frac{2}{11}12$, a contradiction.

  Now the w_i's are 2^--vertices in $G - \{v_0, v_1, v_2, u\}$. The w_i's are 3-vertices or 4-vertices in G, they are not adjacent to v_0 and v_2 since G is triangle-free, and by Lemma 2.4.11, if for some i, w_i is adjacent to u, then w_i is a 4-vertex. Therefore each of the w_i's is either a 3-vertex non-adjacent to u or a 4-vertex adjacent to u, and thus the w_i's are 2-vertices in $G - \{v_0, v_1, v_2, u\}$.

  Let x_0 and x_1 be the two neighbours of w_0 in $G - \{v_0, v_1, v_2, v_3, u\}$. See Figure 2.4.3 for a reminder of the vertices considered. Let d be the sum of the degrees of x_0 and x_1 in $G - \{v_0, v_1, v_2, u, w_0, w_1\}$.

  - Suppose $d \geq 4$. We have a simple chain $(\{v_0, v_1, w_0\}, \{v_2, u, w_1, x_0, x_1\}, v_0, v_3)$. By Lemma 2.4.4, $3 < \frac{9}{11}8 - \frac{2}{11}(20 - \frac{1}{2})$, a contradiction.
  - Suppose $d \leq 3$. Suppose one of the x_i, say x_0, is a 0-vertex in $G - \{v_0, v_1, v_2, u, w_0, w_1\}$. We have a simple chain $(\{v_0, v_1, w_0, x_0\}, \{v_2, u, w_1, x_1\}, v_0, v_3)$. By Lemma 2.4.4, $4 < \frac{9}{11}8 - \frac{2}{11}(16 - \frac{1}{2})$, a contradiction.

    Suppose one of the x_i, say x_0, is a 1-vertex in $G - \{v_0, v_1, v_2, u, w_0, w_1\}$ and the other one (x_1) is a 1-vertex or a 2-vertex in $G - \{v_0, v_1, v_2, u, w_0, w_1\}$.

    Let us prove by contradiction that x_0 is not adjacent to v_3. Suppose x_0 is adjacent to v_3.

    Suppose $x_0w_1 \in E(G)$. By Lemma 2.4.11, at least one of the w_i's, w_0 say, is a 4-vertex, and thus is adjacent to u. By planarity, w_1 is not adjacent to u, and thus w_1 is a 3-vertex in G. In this case, x_0 is adjacent to w_0, w_1 and v_3, but not to any other vertex since it is a 1-vertex in $G - \{v_0, v_1, v_2, u, w_0, w_1\}$ and G is triangle-free. Then x_0 and w_1 are adjacent 3-vertices in G, which contradicts Lemma 2.4.10.

    Now $x_0w_1 \notin E(G)$. We have a simple chain $(\{v_0, v_1, x_0\}, \{v_2, v_3, u, w_0\}, v_1, w_1)$. By Lemma 2.4.4, $3 < \frac{9}{11}7 - \frac{2}{11}(16 - \frac{1}{2})$, a contradiction.

    Therefore x_0 is not adjacent to v_3. Let y be the neighbour of x_0 in $G - \{v_0, v_1, v_2, u, w_0, w_1\}$. Suppose y is a 3-vertex in $G - \{v_0, v_1, v_2, u, w_0, w_1, x_0, x_1\}$. Now the following quadruplet is a simple chain: $(\{v_0, v_1, w_0, x_0\}, \{v_2, u, w_1, x_1, y\}, v_0, v_3)$. By Lemma 2.4.4, $4 < \frac{9}{11}9 - \frac{2}{11}(21 - \frac{1}{2})$, a contradiction. Therefore y is a 2^--vertex in $G - \{v_0, v_1, v_2, u, w_0, w_1, x_0, x_1\}$. Then we have a double chain $(\{v_0, v_1, w_0, x_0\}, \{v_2, u, w_1, x_1\}, v_0, x_0, v_3, y)$. By Lemma 2.4.5, $4 < \frac{9}{11}8 - \frac{2}{11}15$, a contradiction. □

For every face ϕ of G, let $c_4(\phi)$ be the number of 4-vertices in the boundary of ϕ. Let f be the number of faces of G, and for every $3 \leq d \leq 4$ and every $4 \leq l$, let f_l be the number of l-faces and n_d the number of d-vertices in G.

Each 4-vertex is in the boundary of at most four faces. Therefore the sum of the $c_4(\phi)$ over all the 4-faces and 5-faces is $\sum_{\phi, 4 \leq d(\phi) \leq 5} c_4(\phi) \leq 4n_4$. Now, by Lemmas 2.4.7, 2.4.11, and 2.4.12, every 4-face of G has only 4-vertices in its boundary, so for each 4-face ϕ, $c_4(\phi) = 4$. By Lemma 2.4.10, every 5-face of G has at least three 4-vertices in its boundary, so for each 5-face ϕ, $c_4(\phi) \geq 3 \geq 2$.

Thus $\sum_{\phi, d(\phi)=4} c_4(\phi) + \sum_{\phi, d(\phi)=5} c_4(\phi) \geq 4f_4 + 2f_5$. It follows that:

$$4n_4 \geq 4f_4 + 2f_5$$

By Euler's formula, we have:

$$\begin{aligned}
-12 &= 6m - 6n - 6f \\
&= 2\sum_{v\in V(G)} d(v) + \sum_{\phi\in F(G)} d(\phi) - 6n - 6f \\
&= \sum_{d\geq 3}(2d-6)n_d + \sum_{l\geq 4}(l-6)f_l \\
&\geq 2n_4 - 2f_4 - f_5 \\
&\geq 0
\end{aligned}$$

That contradiction completes the proof of Theorem 2.1.21.

2.5 Large induced forests in 2-connected graphs with maximum degree at most 3

Let us first recall the statement of Theorem 2.1.15.

Theorem 2.1.15 (recall)**.** *Every graph in $\mathcal{C}_{2,3^-}$ of order n admits an induced forest of order at least $\frac{2n-2}{3}$.*

The following lemma is folklore, but we include a proof for the sake of completeness.

Lemma 2.5.1. *For every $k \in \{1,2,3\}$, a graph with maximum degree at most 3 is k-connected if and only if it is k-edge-connected.*

Proof. Let G be a graph with maximum degree at most 3. One can easily check that the result holds for the complete graphs on at most four vertices.

Suppose now that G is not complete. Let C_v be a vertex cut of G and C_e be a edge cut of G, both of minimum size. If we show that $|C_v| = |C_e|$, then the lemma holds.

Let V_1 and V_2 be the vertex sets of the two components of $G - C_e$. We have $V_1 \cup V_2 = V(G)$. By minimality of $|C_e|$, every edge of C_e has an endpoint in V_1 and the other one in V_2. Suppose every vertex of V_1 is adjacent to every vertex of V_2 in G. We have $|C_e| = |V_1||V_2| \geq |V_1| + |V_2| - 1 = |V(G)| - 1$. Moreover, for any vertex in G, the set of the edges incident to this vertex is an edge cut of G. Therefore, since G is not complete, by minimality of C_e, $|C_e| \leq |V(G)| - 2$, a contradiction. Therefore there are two vertices $v_1 \in V_1$ and $v_2 \in V_2$ such that $v_1v_2 \notin E(G)$. Let $C'_v = \{x \neq v_1 | \exists y \in V_2, xy \in C_e\} \cup \{y | v_1y \in C_e\}$. Note that $|C'_v| = |\{x \neq v_1 | \exists y \in V_2, xy \in C_e\}| + |\{y | v_1y \in C_e\}| \leq |C_e|$. For each edge in C_e, one of the endpoints of this edge is in C'_v. As neither v_1 nor v_2 is in C'_v, C'_v separates v_1 from v_2 in G. Therefore $|C_v| \leq |C'_v|$, and thus $|C_v| \leq |C_e|$.

Let W_1 and W_2 be the vertex sets of two components of $G - C_v$. Let $x \in C_v$. Since x has degree at most 3, x has at most one neighbour in W_1 or at most one neighbour in W_2, and it has at least one neighbour in W_1 and one in W_2 by minimality of C_v. Let y_x be the neighbour of x that is in W_1 if there is only one neighbour of x in W_1, and the neighbour of x in W_2 otherwise, and $e_x = xy_x$. Observe that this defines a unique edge e_x for every $x \in C_v$. Let $C'_e = \{e_x | x \in C_v\}$. Assume C'_e does not separate W_1 and W_2. There are $v_1 \in W_1$ and $v_2 \in W_2$ such that there is a path P from v_1 to v_2 in $H - C'_e$. Let us consider v_1 and v_2 such that P has minimal length. Then there are w_1 and w_2 in C_v such that $v_1w_1 \in E(P)$ and $v_2w_2 \in E(P)$. If $w_1 = w_2$, then either $v_1w_1 \in C'_e$ or $v_2w_2 \in C'_e$, a contradiction. If $w_1 \neq w_2$, then w_1 has a neighbour in $V(G) \setminus (W_1 \cup W_2)$, so it has only one neighbour in W_1, that is v_1, so $v_1w_1 \in C'_e$, a contradiction. Therefore C'_e separates W_1 and W_2. We have $|C'_e| = |C_v|$, thus $|C_e| \leq |C_v|$. Finally, since $|C_v| \leq |C_e|$, $|C_v| = |C_e|$. $\square$

Let us consider G a counter-example to Theorem 2.1.15 of minimum order, and let $n = |V| \geq 3$ be the order of G. Let us prove some lemmas on the structure of G.

Lemma 2.5.2. *Graph G is cubic (i.e. every vertex in G has degree 3).*

Proof. Suppose there is a 2^--vertex v in G. As G is 2-connected, v is a 2-vertex. Let u and w be the two neighbours of v in G. Suppose $uw \notin E(G)$. Let $G' = G - v + uw$. Since u and w have degree at least 2 in G (since G is 2-connected), $|V(G')| \geq 3$. Then G' is in $\mathcal{C}_{2,3^-}$ since G is. By minimality of G, G' has a an induced linear forest F' of order $|F'| \geq \frac{2(n-1)-2}{3}$. The graph $F = G[V(F) \cup \{v\}]$ is a forest of order $|F| = |F'|+1 \geq \frac{2n-1}{3} \geq \frac{2n-2}{3}$ vertices, a contradiction. Therefore $uw \in E(G)$. If both u and w are 2-vertices, then $G = C_3$, and G admits an induced forest of order $2 \geq \frac{2n-2}{3} = \frac{4}{3}$, a contradiction. If one of u and w has degree 2 and the other one has degree 3, then G is not 2-connected, a contradiction. Therefore u and w are 3-vertices. Note that more generally, we proved that there are no two adjacent vertices of degree 2 in G.

Let u' and w' be the third neighbours of u and w respectively. If $u' = w'$, then $V(G) = \{u, v, w, u'\}$ (since G is 2-connected), and G admits an induced forest of order $3 \geq \frac{2n-2}{3} = 2$ ($\{v, w, u'\}$ for example), a contradiction. Thus u' and w' are distinct. Suppose $u'w' \in E(G)$. Let $G' = G - \{u, v, w\}$. If $|V(G')| < 3$, then u' and w' are adjacent 2-vertices in G and we fall into a previous case. Therefore $|V(G')| \geq 3$. The graph G' is in $\mathcal{C}_{2,3^-}$ since G is. By minimality of G, G' has an induced forest F' of order $|F'| \geq \frac{2(n-3)-2}{3}$. The graph $F = G[V(F') \cup \{v, w\}]$ is a forest of order $|F| \geq \frac{2n-8}{3} + 2 = \frac{2n-2}{3}$, a contradiction. Therefore $u'w' \notin E(G)$. Let $G' = G - \{u, v, w\} + u'w'$. The graph G' is in $\mathcal{C}_{2,3^-}$ since G is. By minimality of G, G' has an induced forest F' of size $|F'| \leq \frac{n-3+2}{3}$. The graph $F = G[F' \cup \{v, w\}]$ is a forest of order $|F| \geq \frac{2n-6-2}{3} + 2 = \frac{2n-2}{3}$, a contradiction. □

In the rest of this section, we will use the fact that G is cubic without referring explicitly to Lemma 2.5.2.

Lemma 2.5.3. *No two triangles share an edge in G.*

Proof. Assume that there are two triangles $v_0v_1v_2$ and v_0v_1u sharing an edge v_0v_1 in G. If $v_2u \in E(G)$, then $G = K_4$ (since G is connected), and G has an induced forest of order $2 = \frac{2n-2}{3}$. Therefore $v_2u \notin E(G)$. Let w be the neighbour of v_2 in G distinct from v_0 and v_1. Observe that $uw \notin E(G)$, since G is cubic and 2-connected. Let $G' = G - \{v_0, v_1, v_2\} + uw$. The graph G' is in $\mathcal{C}_{2,3^-}$ since G is. By minimality of G, G' has an induced forest F' of order $|F'| \geq \frac{2(n-3)-2}{3}$. The graph $F = G[V(F') \cup \{v_0, v_2\}]$ is a forest of order $|F| \geq \frac{2(n-3)-2}{3} + 2 = \frac{2n-2}{3}$, a contradiction. □

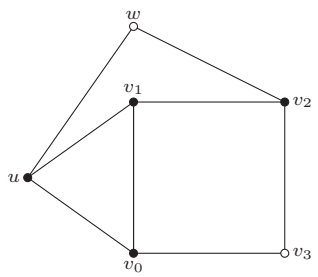

Figure 2.5.1: The configuration of Lemma 2.5.4.

Lemma 2.5.4. *There is no triangle that shares an edge with a 4-cycle in G.*

Proof. By Lemma 2.5.3, there is no triangle that shares two edges with a 4-cycle in G. Assume that there are a triangle v_0v_1u and a 4-cycle $v_0v_1v_2v_3$ that share the edge v_0v_1. Let w be the neighbour of u distinct from v_0 and v_1. By Lemma 2.5.3, $w \notin \{v_2, v_3\}$.

Suppose first that w adjacent to v_2. See Figure 2.5.4 for an illustration of those vertices. If $v_3w \in E(G)$, then $V(G) = \{v_0, v_1, v_2, v_3, u, w\}$ (since G is connected), and $G[\{v_0, v_1, v_2, w\}]$ is a forest of order $4 \geq \frac{2n-2}{3} = \frac{10}{3}$ a contradiction. Therefore $v_3w \notin E(G)$. Let $G' = G - \{v_0, v_1, u\} + v_3w$. Graph G' is in $\mathcal{C}_{2,3^-}$ since G is. By minimality of G, G' admits an induced forest F' of order $|F'| \geq \frac{2(n-3)-2}{3}$. The graph $F = G[V(F') \cup \{v_0, u\}]$ is a forest of order $|F| \geq \frac{2(n-3)-2}{3} + 2 = \frac{2n-2}{3}$, a contradiction.

Therefore w is not adjacent to v_2. Let $G' = G - \{v_0, v_1, u\} + wv_2$. Graph G' is in $\mathcal{C}_{2,3^-}$. By minimality of G, G' admits an induced forest F' of order $|F'| \geq \frac{2(n-3)-2}{3}$. The graph $F = G[F' \cup \{u, v_0\}]$ is a forest of order $|F| \geq \frac{2(n-3)-2}{3} + 2 = \frac{2n-2}{3}$, a contradiction. □

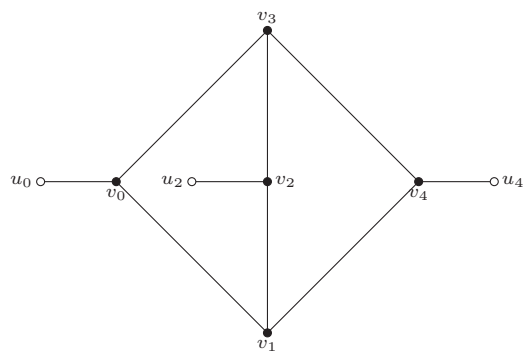

Figure 2.5.2: The configuration of Lemma 2.5.5.

Lemma 2.5.5. *There are no two 4-cycles that share two edges in G.*

Proof. Let $v_0v_1v_2v_3$ and $v_1v_2v_3v_4$ be two 4-cycles of G. Let u_0, u_2 and u_4 be the third neighbours of v_0, v_2 and v_4 respectively. By Lemma 2.5.3, they are distinct from the vertices defined previously. If $u_0 = u_2 = u_4$, then $G = K_{3,3}$ admits an induced forest of order $4 \geq \frac{12-2}{3} = \frac{2n-2}{3}$ (for example $G[\{v_0, v_2, v_3, v_4\}]$), a contradiction.

Suppose $u_0 \neq u_2 \neq u_4 \neq u_0$. See Figure 2.5.2 for an illustration of the configuration. Let $G' = G - \{v_0, v_1, v_2, v_4\} + \{v_3u_0, v_3u_2, v_3u_4\}$. If G' is not 2-connected, then without loss of generality v_4 separates u_0 and u_2 in G', and thus v_0 separates u_0 and u_2 in G, a contradiction. Therefore G' is in $\mathcal{C}_{2,3^-}$. By minimality of G, G' admits an induced forest F' of order at least $\frac{2(n-4)-2}{3}$. The graph $F = G[V(F') \cup \{v_0, v_2, v_4\}]$ is a forest of order $|F| = |F'| + 3 \geq \frac{2n-2}{3}$, a contradiction.

Thus the u_i's are not all equal and not all distinct, thus we can assume that $u_0 = u_2 \neq u_4$. Let w be the neighbour of $u_0 = u_2$ distinct from v_0 and v_2. Observe that w is distinct from u_4 since G is cubic and 2-connected. Let $G' = G - \{v_0, v_1, v_2, v_3, v_4, u_0\}$ if $u_4w \in E(G)$ and $G' = G - \{v_0, v_1, v_2, v_3, v_4, u_0\} + u_4w$ otherwise. Graph G' is in $\mathcal{C}_{2,3^-}$ since G is. By minimality of G, G' admits an induced forest F' of order $|F'| \geq \frac{2(n-6)-2}{3}$. The graph $F = G[V(F) \cup \{v_0, v_1, v_2, v_4\}$ is a forest of order $|F| = |F'| + 4 \geq \frac{2n-12-2}{3} + 4 = \frac{2n-2}{3}$, a contradiction. □

Lemma 2.5.6. *Graph G is 3-connected.*

Proof. Suppose by contradiction that G is not 3-connected. By Lemma 2.5.2, $|V(G)| \geq 4$. Since $G \in \mathcal{C}_{2,3^-}$, G is 2-connected. By Lemma 2.5.1, G is 2-edge-connected but not 3-edge-connected. Let $\{e, f\}$ be an edge cut of G that induces two componentss V_1 and V_2 such that $|V_1|$ is minimum.

We will now prove the two following properties:

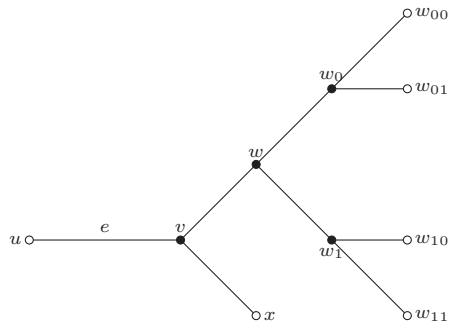

Figure 2.5.3: The vertices of Lemma 2.5.6. The vertices w_{00}, w_{01}, w_{10}, and w_{11} may coincide.

- P_e: The deletion of any edge in $G[V_1]$ preserves the 2-edge connectivity of G.

  By contradiction, suppose there is an edge $e' \in G[V_1]$ such that $G - e'$ is not 2-edge-connected (but connected since G is 2-edge-connected). Let v be the endpoint of e in V_1, and w be the endpoint of f in V_1. Suppose e' is a bridge in $G[V_1]$. Then at least one of the components of $G[V_1] - e'$ does not contain w. It is also a component of $G - \{e, e'\}$, contradicting the minimality of V_1. Therefore there is a path from one endpoint of e' to the other in $G[V_1]$. Let f' be a bridge of $G - e'$. By what precedes, $f' \in E(G[V_1])$. As $\{f, f'\}$ is an edge cut of G, it is also an edge cut of $G[V_1]$. Note that v and w have a path connecting them that does not contain any edge of $G[V_1]$. Therefore, they are in the same component of $G \setminus \{f, f'\}$. The other component of $G \setminus \{f, f'\}$ then contradicts the minimality of $|V_1|$.

- P_v: For every vertex v in V_1 with no V_2-neighbour, $H - v$ is 2-edge-connected, and thus 2-connected by Lemma 2.5.1.

  Suppose there is a vertex $v \in V_1$ that is not incident to an edge of $\{e, f\}$ such that $H - v$ is not 2-edge-connected. Let f' be a cut edge of $H - v$. As vertex v has degree 3, there is an edge e' incident to v such that $H - \{e', f'\}$ is disconnected. As v is not incident to an edge of $\{e, f\}$, e' is an edge of $G[V_1]$, a contradiction with P_e.

For the rest of the proof, you can refer to Figure 2.5.3 for a reminder of the vertices considered. Let $v \in V_1$ and $u \in V_2$ be the endpoints of e. Let w and x be the two neighbours of v distinct from u. Vertices w and x are in V_1, otherwise either $f = vw$ or $f = vx$, say $f = vw$, and vx is a bridge of G, a contradiction.

Let us show that $wx \notin E(G)$. By contradiction assume that $wx \in E(G)$. Let w' be the neighbour of w distinct from v and x, and x' be the neighbour of x distinct from v and w. By Lemmas 2.5.3 and 2.5.4, w', x' and u are distinct and pairwise not adjacent. Moreover, if $w' \notin V_1$ or $x' \notin V_1$, say $w' \notin V_1$, then $f = ww'$, and thus xx' is a bridge of G, a contradiction. Hence v, w, x, w' and x' are all in V_1, and thus, by P_v, $G - w$ is 2-connected. Let $G' = G - \{v, w, x\} + ux'$. Graph G' is in $\mathcal{C}_{2,3^-}$. By minimality of G, G' admits an induced forest F' of order $|F'| \geq \frac{2(n-3)-2}{3}$. The graph $F = G[V(F') \cup \{v, x\}$ is a forest of order $|F'| + 2 \geq \frac{2n-6-2}{3} + 2 = \frac{2n-2}{3}$, a contradiction. Therefore $wx \notin E(G)$.

Let w_0 and w_1 be the two neighbours of w distinct from v. If w_0 or w_1 is in V_2, say $w_0 \in V_2$, then $ww_0 = f$, and $\{vx, ww_1\}$ is an edge cut of G, contradicting the minimality of $|V_1|$. Therefore w_0 and w_1 are in V_1.

Let us show that $w_0w_1 \notin E(G)$. By contradiction assume that $w_0w_1 \in E(G)$. Let w_0' be the neighbour of w_0 distinct from w and w_1, and w_1' be the neighbour of w_1 distinct from w and w_0. By Lemmas 2.5.3 and 2.5.4, w_0' and w_1' are distinct and not adjacent. Vertices v, w, w_0 and w_1 are all in V_1, thus, by P_v, $G - w$ is 2-connected. Let $G' = G - \{w, w_0, w_1\} + w_0'w_1'$. Graph G' is in $\mathcal{C}_{2,3^-}$. By minimality of G, G' admits an induced forest F' of order $|F'| \geq \frac{2(n-3)-2}{3}$. The graph $F = G[V(F') \cup \{w_0, w_1\}]$ is a forest of order $|F'| + 2 \geq \frac{2n-6-2}{3} + 2 = \frac{2n-2}{3}$, a contradiction.

Let w_{00} and w_{01} be the two neighbours of w_0 distinct from w. Let us show that $w_{00}w_{01} \notin E(G)$. By contradiction assume that $w_{00}w_{01} \in E(G)$. By Lemma 2.5.4, w_{00} and w_{01} are distinct from x. Let w'_{00} be the neighbour of w_{00} distinct from w_0 and w_{01}, and w'_{01} be the neighbour of w_{01} distinct from w_0 and w_{00}. By Lemmas 2.5.3 and 2.5.4, w'_{00} and w'_{01} are distinct and not adjacent. Suppose w_{00} or w_{01} is in V_2, say $w_{00} \in V_2$. Then $w_0w_{00} = f$, and e, f is not an edge cut of H (since $w_0w_{00}w_{01}$ is a triangle), a contradiction. Therefore w, w_0, w_{00} and w_{01} are in V_1, and thus, by P_v, $G - w_0$ is 2-connected. Let $G' = G - \{w_0, w_{00}, w_{01}\} + w'_{00}w'_{01}$. Graph G' is in $\mathcal{C}_{2,3^-}$. By minimality of G, G' admits an induced forest F' of order $|F'| \geq \frac{2(n-3)-2}{3}$. The graph $G[V(F') \cup \{w_{00}, w_{01}\}$ is a forest of order $|F'| + 2 \leq \frac{2n-6-2}{3} + 2 = \frac{2n-2}{3}$, a contradiction.

Let w_{10} and w_{11} be the two neighbours of w_1 distinct from w. By symmetry, $w_{10}w_{11} \notin E(G)$. Suppose $\{w_{00}, w_{01}\} = \{w_{10}, w_{11}\}$; say $w_{00} = w_{10}$ and $w_{01} = w_{11}$. Lemma 2.5.5 leads to a contradiction. Therefore the pairs $\{w_{00}, w_{01}\}$ and $\{w_{10}, w_{11}\}$ are not equal. As v, w, w_0 and w_1 are in V_1, by P_v, $G - w$ is 2-connected. Let $G' = G - \{w, w_0, w_1\} + \{w_{00}w_{01}, w_{10}w_{11}\}$. Graph G' is in $\mathcal{C}_{2,3^-}$. By minimality of G, G' admits an induced forest F' of order $|F'| \geq \frac{2(n-3)-2}{3}$. The graph $G[V(F) \cup \{w_0, w_1\}]$ is a forest of order $|F'| + 2 \geq \frac{2n-6-2}{3} + 2 = \frac{2n-2}{3}$, a contradiction, which completes the proof. □

Lemma 2.5.7. *There is no triangle in G.*

Proof. Suppose there is a triangle $v_0v_1v_2$ in G. Let u_0, u_1, and u_2 be the third neighbours of v_0, v_1 and v_2 respectively. By Lemmas 2.5.3 and 2.5.4, u_0, u_1 and u_2 are distinct and non-adjacent. Let $G' = G - \{v_0, v_1, v_2\} + u_1u_2$. Observe that by Lemma 2.5.6, $G - v_0$ is 2-connected. Therefore G' is in $\mathcal{C}_{2,3^-}$. By minimality of G, G' admits an induced forest F' of order $|F'| \geq \frac{2(n-3)-2}{3}$. The graph $G[V(F') \cup \{v_1, v_2\}]$ is a forest of order $|F'| + 2 \geq \frac{2n-6-2}{3} + 2 \geq \frac{2n-2}{3}$, a contradiction. □

Let v be a vertex of G, and x and y be two neighbours of v. They are not adjacent by Lemma 2.5.7. Let x_0, x_1, y_0, and y_1 the two other neighbours of x and y respectively. Vertices x_0 and x_1 are not adjacent by Lemma 2.5.7, and similarly y_0 and y_1 are not adjacent. The pairs $\{x_0, x_1\}$ and $\{y_0, y_1\}$ are distinct by Lemma 2.5.5. Let $G' = G - \{v, x, y\} + \{x_0x_1, y_0y_1\}$. By Lemma 2.5.6, $G - v$ is 2-connected, so G' is in $\mathcal{C}_{2,3^-}$. By minimality of G, G' admits an induced forest F' of order $|F'| \geq \frac{2(n-3)-2}{3}$. The graph $G[V(F') \cup \{x, y\}]$ is a forest of order $|F'| + 2 \geq \frac{2n-6-2}{3} + 2 \geq \frac{2n-2}{3}$, a contradiction. That completes the proof of Theorem 2.1.15.

2.6 Large induced forests in planar graphs of large girth

Let us first recall the statement of Theorem 2.1.14.

Theorem 2.1.14 (recall)**.** *For all $g \geq 3$, every planar graph of girth at least g with n vertices and m edges admits an induced forest of order $n - \frac{4m}{3g}$, and thus at least $n - \frac{4n}{3g-6}$.*

Let $g \geq 3$ be a fixed integer. For G a planar graph, $\omega : E(G) \to \mathbb{N}$ a weight function, and $F \subseteq E(G)$, we denote $\sum_{e \in F} \omega(e)$ by $\omega(F)$, and $\sum_{e \in E(G)} \omega(e)$ by $\omega(G)$. We will prove the following claim:

Claim 2.6.1. *Let G be a planar graph, and $\omega : E(G) \to \mathbb{N}$ a weight function such that for each cycle C of G, $\omega(C) \geq g$. There exists an induced forest F of G of order at least $|G| - \frac{4\omega(G)}{3g}$.*

Observe that fixing ω constant equal to 1 in Claim 2.6.1 yields Theorem 2.1.14.

In the following, we will generally reason on 2-connected plane graphs. A particularity of those graphs is that every edge is always in the boundary of exactly two faces, and that the boundary of every face is a cycle.

Let G be a 2-connected plane graph. Three faces f_0, f_1, and f_2 of G are said to be *mergeable* if:

1. there exists a vertex v that is in the boundary of f_0, f_1, and f_2.

2. w.l.o.g. f_0 and f_1 (resp f_1 and f_2) have at least one common edge in their boundary.

Given three mergeable faces f_0, f_1, and f_2, the *merge* of f_0, f_1, and f_2 consists in removing the edges belonging to the boundary of two faces among f_0, f_1, and f_2 as well as the vertices that end up being isolated. The common vertex v of f_0, f_1, and f_2 is called the *crucial* vertex of the merge. A merge is *nice* if the sum of the weights of the edges removed is at least $\frac{3g}{4}$. Observe that a merge cannot decrease the minimum weight of a cycle in G, since we only delete vertices and edges. See Figure 2.6.1 for an example of the merge of three faces.

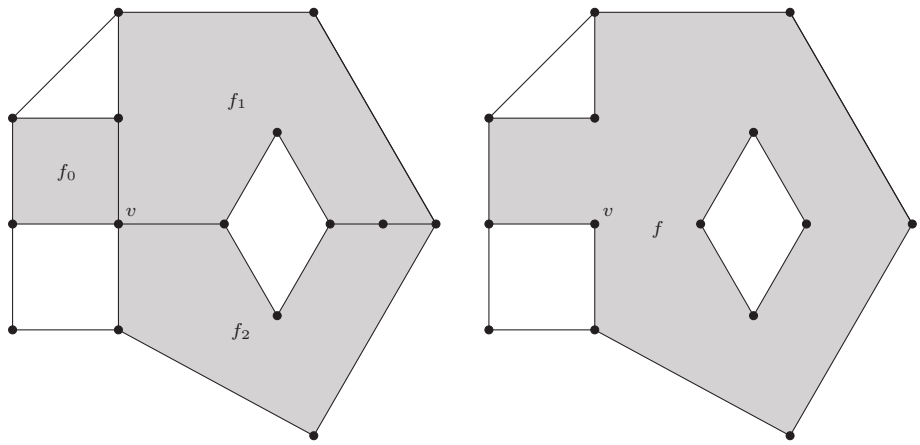

Figure 2.6.1: The merge of faces f_0, f_1, and f_2 into f with crucial vertex v.

Lemma 2.6.2. *Let G be a 2-connected plane graph, and let G' be a graph obtained from G by applying a merge with the crucial vertex v. If F' is an induced forest of G', then $F = G[(V(F') \cup (V(G) \setminus V(G')) \setminus \{v\}]$ is a forest.*

Proof. Let C be a cycle of G that contains an edge $e \in E(G) \setminus E(G')$. Edge e is in the boundary of two of the faces that are merged, say ϕ_0 and ϕ_1. Cycle C separates ϕ_0 and ϕ_1 (i.e. one of the ϕ_i's is in the interior of C and the other one is in the exterior of C). Therefore C contains all the vertices of that are both in the boundary of ϕ_0 and in that of ϕ_1. In particular, it contains v.

Therefore each cycle of G either is entirely contained in G', or contains v. Thus as F' is a forest, F is also a forest. □

Lemma 2.6.3. *Let G be a 2-connected plane graph, and G' be obtained from G by applying a nice merge. If graph G' satisfies Claim 2.6.1, then graph G also satisfies Claim 2.6.1.*

Proof. Let v be the crucial vertex of the merge. We have $\omega(G') \leq \omega(G) - \frac{3g}{4}$. Since G' verifies Claim 2.6.1, there exists an induced forest F' of G' such that $|F'| \geq |G'| - \frac{4\omega(G')}{3g} \geq |G'| - \frac{4\omega(G)}{3g} + 1$. Then $F = G[(V(F') \cup (V(G) \setminus V(G')) \setminus \{v\}]$ is a forest (by Lemma 2.6.2), and $|F| \geq |F'| + |G| - |G'| - 1 \geq |G| - \frac{4\omega(G)}{3g}$, which completes the proof. □

Let us assume by contradiction that there are couples (G, ω) that do not satisfy Claim 2.6.1. Among all counter-examples (G, ω) to Claim 2.6.1 minimizing $\omega(G)$, we consider such a couple (G, ω) minimizing $\sum_{v \in V(G)} (\max\{0.5, d(v) - 2.5\})$.

Lemma 2.6.4. *Graph G is 2-connected.*

Proof. By contradiction, assume G is not 2-connected. Graph G has at least 2 vertices, otherwise it would satisfy Claim 2.6.1. Let S be a minimal vertex cut of G. We have $|S| \leq 1$. Let V_1 and V_2 be non-empty sets of vertices separated by S.

Let $\omega_1 = \omega(G[V_1 \cup S])$ and $\omega_2 = \omega(G[V_2 \cup S])$. By minimality of (G, ω), there exist F_1 and F_2 which are induced forests of $G[V_1 \cup S]$ and $G[V_2 \cup S]$ respectively, such that $|F_1| \geq |V_1| + |S| - \frac{4\omega_1}{3g}$ and $|F_2| \geq |V_2| + |S| - \frac{4\omega_2}{3g}$. Now $G[V(F_1) \cap V(F_2)]$ is an induced forest of G, and $|V(F_1) \cap V(F_2)| \geq |F_1| + |F_2| - |S| \geq |V_1| + |V_2| + |S| - \frac{4\omega_1}{3g} - \frac{4\omega_2}{3g} = \frac{4\omega(G)}{3g}$. Thus G satisfies Claim 2.6.1, a contradiction. □

Lemma 2.6.5. *No nice merge can be done in G.*

Proof. It follows from Lemma 2.6.3 and the minimality of (G, ω). □

Lemma 2.6.6. *Every face in G has at least three 3^+-vertices in its boundary.*

Proof. Let us assume that there is a face ϕ in G with at most two 3^+-vertices in its boundary. Face ϕ is adjacent to at most two other faces in G. Suppose ϕ is adjacent to exactly one face, say ϕ'. As G is 2-connected by Lemma 2.6.4, $G[\phi]$ and $G[\phi']$ are cycles. As ϕ is adjacent only to ϕ', $E(G[\phi]) \subseteq E(G[\phi'])$, and thus $G[\phi] = G[\phi']$. So two faces of G have exactly the same boundary, and so the graph G has only two faces. In particular, G is a cycle, thus it satisfies Claim 2.6.1, a contradiction.

Thus ϕ is adjacent to exactly two other faces, say ϕ_0 and ϕ_1. Then $E(G[\phi]) \subseteq E(G[\phi_0]) \cup E(G[\phi_1])$, and $E(G[\phi]) \cap E(G[\phi_0]) \neq \emptyset \neq E(G[\phi]) \cap E(G[\phi_1])$. As $G[\phi]$ is a cycle, and in particular is connected, there is a vertex v in $V(G[\phi])$ incident to an edge in $E(G[\phi]) \cap E(G[\phi_0])$ and to an edge in $E(G[\phi]) \cap E(G[\phi_1])$. Merging the faces ϕ, ϕ_0 and ϕ_1 with crucial vertex v is nice, since we remove all the edges of $G[\phi]$ and $\omega(\phi) \geq g \geq \frac{3g}{4}$. This contradicts Lemma 2.6.5. □

Lemma 2.6.7. *There is no 4^+-vertex in G.*

Proof. Suppose v is a d-vertex in G with $d \geq 4$. Let u_0, ..., u_{d-1} be the neighbours of v. Let w and w' be fresh vertices (i.e. vertices that are not in $V(G)$). Let $G' = G - v + (\{w, w'\}, \emptyset) + \{wu_0, wu_1, ww', w'u_2, ..., w'u_{d-1}\}$, $\omega(wu_0) = \omega(vu_0)$, $\omega(wu_1) = \omega(vu_1)$, $\omega(w'u_2) = \omega(vu_2)$, ..., $\omega(w'u_{d-1}) = \omega(vu_{d-1})$, and $\omega(ww') = 0$. See Figure 2.6.2 for an illustration of this construction. Clearly, $\omega(G') = \omega(G)$. As we removed a d-vertex, added a 3-vertex and a $(d-1)$-vertex, and did not change the degree of the other vertices, $\sum_{v \in V(G')}(\max\{0.5, d(v) - 2.5\}) = \sum_{v \in V(G)}(\max\{0.5, d(v) - 2.5\}) - 0.5$.

For any cycle C' of G', there is a cycle in G that has the same weight, so $\omega(C') \geq g$.

By minimality of (G, ω), let F' be an induced forest of G' with $|F'| \geq |G'| - \frac{4\omega(G')}{3}$. For any cycle C of G there is a cycle C' of G' such that $C = C'$ or $V(C) = (V(C') \setminus \{w, w'\}) \cup \{v\}$. If $w \notin V(F')$ or $w' \notin V(F')$, then let $F = G[V(F') \setminus \{w, w'\} \cup (V(G) \setminus (V(G') \cup \{v\}))$ and otherwise let $F = G[V(F') \setminus \{w, w'\} \cup (V(G) \setminus V(G'))$. Then $|F| \geq |F'| - 1 \geq |G'| - \frac{4\omega(G')}{3} - 1 = |G| - \frac{4\omega(G)}{3}$, and F is an induced forest of G, a contradiction. □

Lemma 2.6.8. *Every cycle has at least three 3-vertices in G.*

Proof. Let C be a cycle of G. By Lemma 2.6.7, every vertex in $V(C)$ has degree at most 3. Suppose C is a separating cycle. By Lemma 2.6.4, graph G is 2-connected, so at least two vertices of $V(C)$ have a neighbour in the interior of C, and at least two vertices of $V(C)$ have a neighbour in the exterior of C. Therefore C has at least four 3-vertices. Now if C is not separating, then it is the boundary of a face, and Lemma 2.6.6 concludes the proof. □

Lemma 2.6.9. *Graph G is cubic (i.e. 3-regular).*

Proof. Suppose v is a 2^--vertex in G. Vertex v has degree 2 by Lemma 2.6.4. Let u and w be the two neighbours of v. By Lemma 2.6.8, $uw \notin E(G)$.

Let $G' = G - v + uw$ and $\omega(uw) = \omega(uv) + \omega(vw)$. See Figure 2.6.3 for an illustration of this construction. Clearly, $\omega(G') = \omega(G)$. As we removed a 2-vertex and did not change the degree of the other vertices, $\sum_{v \in V(G')}(\max\{0.5, d(v) - 2.5\}) = \sum_{v \in V(G)}(\max\{0.5, d(v) - 2.5\}) - 0.5$.

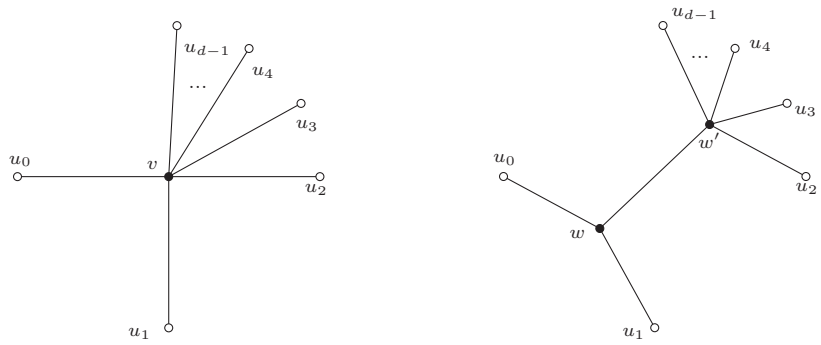

Figure 2.6.2: The construction of Lemma 2.6.7.

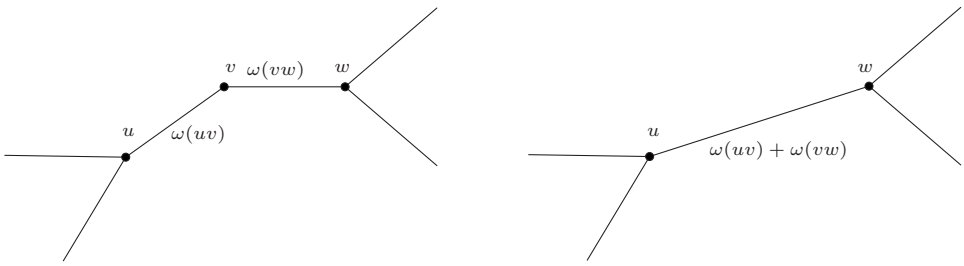

Figure 2.6.3: The construction of Lemma 2.6.9.

Let C' be any cycle of G'. If $uw \notin E(C')$, then C' is a cycle of G, and so $\omega(C') \geq g$. Otherwise, $C = C' - uw + (\{v\}, \emptyset) + \{uv, vw\}$ is a cycle of G, and $\omega(C) = \omega(C')$, so $\omega(C') \geq g$.

For any cycle C of G there is a cycle C' of G' that contains all the vertices of $V(C) \setminus \{v\}$. By minimality of (G, ω), let F' be an induced forest of G' with $|F'| \geq |G'| - \frac{4\omega(G')}{3} = |G| - 1 - \frac{4\omega(G)}{3}$. The graph $G[V(F') \cup \{v\}]$ is a forest of order $|F'| + 1 \geq |G| - \frac{4\omega(G)}{3}$, a contradiction. $\square$

By Lemmas 2.6.4 and 2.6.9, the graph G is a 2-connected cubic graph. By Theorem 2.1.15, G admits an induced forest of order at least $\frac{2|V(G)|-2}{3}$. Let us denote by n the order of G, by m the size of G and by f the number of faces of G.

By Euler's formula, we have $n - m + f = 2$. We have $3n = 2m$ as G is cubic. Therefore, $f = 2 + m - n = 2 + \frac{n}{2}$, i.e. $n = 2(f - 2)$. Therefore G has an induced forest F of order $|F| \geq \frac{n-(2f-4)-2}{3} \geq \frac{n-2f}{3}$. As each face has weight at least g, we have $gf \leq 2\omega(G)$, so $|F| \geq n - \frac{4\omega(G)}{3g}$, a contradiction, completing the proof of Theorem 2.1.14.

2.7 Conclusion

We thus proved lower bounds on the order of the largest induced forest in triangle-free planar graphs and in planar graphs of large girth, as well as on the order of the largest induced linear forest in triangle-free planar graphs. Namely, we recall our three main theorems.

Theorem 2.1.10 (recall)**.** *Every triangle-free planar graph of order n and size m admits an induced forest of order at least* $\max\{\frac{38n-7m}{44}, n-\frac{m}{4}\}$*, and thus at least* $\frac{6n+7}{11} \approx 0.545n$*.*

Theorem 2.1.14 (recall)**.** *For all $g \geq 3$, every planar graph of girth at least g with n vertices and m edges admits an induced forest of order at least $n - \frac{4m}{3g}$, and thus at least $n - \frac{4n}{3g-6}$.*

Theorem 2.1.21 (recall)**.** *Every triangle-free planar graph of order n and size m admits an induced linear forest of order at least $\frac{9n-2m}{11}$, and thus at least $\frac{5n+8}{11} \approx 0.455n$.*

We note that those bounds have some consequences on other bounds. For example, consider a forest F, pick a vertex r in F as the root, and for $i \in \{0,1,2\}$, let $S_i = \{v \in V(F), d(r,v) \equiv i[3]\}$. The graph induced by the union of any two of the S_i's is a star forest. Therefore, every forest of order n has an induced star forest of order $\frac{2n}{3}$. Consequently, our results on induced forests directly extend to star forest. Thus we have the following two corollaries, which are the best know results to date on induced star forests in these classes of graphs:

Corollary 2.7.1. *Every triangle-free planar graph of order n and size m admits an induced star forest of order at least* $\max\{\frac{38n-7m}{66}, \frac{2n}{3}-\frac{m}{6}\}$*, and thus at least* $\frac{12n+14}{33}$*.*

Corollary 2.7.2. *For all $g \geq 3$, every planar graph of girth at least g with n vertices and m edges admits an induced star forest of order at least $\frac{2n}{3} - \frac{8m}{9g}$, and thus at least $\frac{2n}{3} - \frac{8m}{9g-18}$.*

We note that every forest also has an independent set on at least half of its vertices (since they are bipartite), and has an induced linear forest on at least three fourth of its vertices, and these bounds are tight. However, those do not improve existing bounds for independent sets, nor does Theorem 2.1.10 imply Theorem 2.1.21.

Chapter 3

Partition into independent sets, forests, and forests of bounded degree

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3.1 Introduction

We recall that we denote by $\mathcal{I}$ the class of empty graphs and by $\mathcal{F}$ the class of forests. For all $d \in \mathbb{N}$, we denote by Δ_d the class of graphs with maximum degree at most d, by $\mathcal{F}_d$ the class of forests with maximum degree at most d, and by $\mathcal{D}_d$ the class of d-degenerate graphs.

The study of vertex partitions of planar graphs originated from the Four Colour Theorem, which can be restated this way:

Theorem 1.0.1 (restated)**.** *Every planar graph admits an $(\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I})$-partition.*

Many of the theorems in the literature can also be restated in terms of vertex partitions. We recall the theorem of Poh that we previously stated for one of its corollaries.

Theorem 2.1.17 (recall)**.** *Every planar graph admits an $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_2)$-partition.*

Moreover a consequence of Theorem 2.1.2 of Borodin [9] (every planar graph admits an acyclic colouring with at most five colours) is that every planar graph admits an $(\mathcal{I}, \mathcal{F}, \mathcal{F})$-partition.

Thomassen launched a study on the $(\mathcal{D}_i, \mathcal{D}_j)$-partitions of planar graphs. He proved that every planar graph admits an $(\mathcal{F}, \mathcal{D}_2)$-partition [51], and an $(\mathcal{I}, \mathcal{D}_3)$-partition [52] (recall that $\mathcal{F} = \mathcal{D}_1$ and $\mathcal{I} = \mathcal{D}_0$). However, there are planar graphs that do not admit any $(\mathcal{F}, \mathcal{F})$-partition [17]. Wegner [53] even showed that there are planar graphs that do not admit any $(\mathcal{I}, \mathcal{I}, \mathcal{F})$-partition. Borodin and Glebov [11] proved the following.

Theorem 3.1.1 (Borodin and Glebov [11])**.** *Every planar graph of girth at least 5 admits an $(\mathcal{I}, \mathcal{F})$-partition.*

Raspaud and Wang [48] proved that every planar graph with no triangles at distance at most 2 (and thus in particular every triangle-free planar graph) admits an $(\mathcal{F}, \mathcal{F})$-partition.

The result that every triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F})$-partition is folklore, and can be proven very easily. Indeed, if we consider a minimum counter-example (in terms of the number of vertices) to this result, then this counter-example does not have any 3^--vertex (just put it in a forest that contains at most one neighbour of the 3-vertex). A direct application of Euler's formula leads to a contradiction (triangle-free planar graphs must contain at least one 3^--vertex).

However, it is not known whether every triangle-free planar graph admits an $(\mathcal{I}, \mathcal{F})$-partition. We pose the following questions:

Question 3.1.2. *Does every triangle-free planar graph admit an $(\mathcal{I}, \mathcal{F})$-partition?*

Question 3.1.3. *More generally, what is the lowest d such that every triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F}_d)$-partition?*

Note that proving $d = 0$ in Question 3.1.3 would prove Question 3.1.2. We prove the following:

Theorem 3.1.4. *Every triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F}_5)$-partition.*

Theorem 3.1.4 will be proven in Section 3.2. It implies that $d \leq 5$ in Question 3.1.3. Our proof uses the discharging method. It is constructive and immediately yields an algorithm for finding an $(\mathcal{F}, \mathcal{F}_5)$-partition of a triangle-free planar graph in quadratic time.

Montassier and Ochem [44] give an example showing that not every triangle-free planar graph can be partitioned into two graphs of bounded degree (which shows that Theorem 3.1.4 is tight in some sense).

In Section 3.3, we show that if for some d, there exists a triangle-free planar graph that does not admit an $(\mathcal{F}, \mathcal{F}_d)$-partition, then deciding whether a triangle-free planar graph admits such a partition is an NP-complete problem. That is, if the answer to Question 3.1.3 is some $k > 0$, then for all $0 \leq d < k$, deciding whether a triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F}_d)$-partition is NP-complete. We prove this by reduction to Planar 3-Sat.

Borodin and Kostochka [15] showed the following:

Theorem 3.1.5 (Borodin and Kostochka [15])**.** *For all $j \geq 0$ and $k \geq 2j+2$, every graph G with* $\mathrm{mad}(G) < 2\left(2 - \frac{k+2}{(j+2)(k+1)}\right)$ *admits a (Δ_j, Δ_k)-partition.*

In particular, Theorem 3.1.5 implies that every graph G with $\mathrm{mad}(G) < \frac{8}{3}$ admits an $(\mathcal{I}, \Delta_2)$-partition, and every graph G with $\mathrm{mad}(G) < \frac{14}{5}$ admits an $(\mathcal{I}, \Delta_4)$-partition. With Euler's formula, this yields the following corollary.

Corollary 3.1.6. *Every planar graph with girth at least 7 admits an $(\mathcal{I}, \Delta_4)$-partition, and every planar graph with girth at least 8 admits an $(\mathcal{I}, \Delta_2)$-partition.*

Borodin and Kostochka [14] proved the following theorem.

Theorem 3.1.7 (Borodin and Kostochka [14])**.** *Every graph G with* $\mathrm{mad}(G) < \frac{12}{5}$ *admits an $(\mathcal{I}, \Delta_1)$-partition.*

Theorem 3.1.7 implies that every planar graph with girth at least 12 admits an $(\mathcal{I}, \Delta_1)$-partition. This last result was improved by Kim, Kostochka, and Zhu [42]:

Theorem 3.1.8 (Kim, Kostochka, and Zhu [42])**.** *Every triangle-free graph with maximum average degree less than $\frac{11}{9}$ admits an $(\mathcal{I}, \Delta_1)$-partition.*

Corollary 3.1.9. *Every planar graph with girth at least 11 admits an $(\mathcal{I}, \Delta_1)$-partition.*

In contrast to this, Borodin, Ivanova, Montassier, Ochem, and Raspaud [13] proved that for every d, there exists a planar graph of girth 6 that admits no $(\mathcal{I}, \Delta_d)$-partition. Montassier and Ochem [44] showed that this implies that deciding if a planar graph of girth 6 admits an $(\mathcal{I}, \Delta_d)$-partition is an NP-complete problem for all $d \geq 1$, and they proved that deciding if a planar graph of girth 7 has an $(\mathcal{I}, \Delta_2)$-partition is NP-complete. Esperet, Montassier, Ochem, and Pinlou [30] showed that deciding if a planar graph of girth 9 has an $(\mathcal{I}, \Delta_1)$-partition is NP-complete.

Note that although Theorems 3.1.5, 3.1.7, and 3.1.8 are tight, their corollaries in terms of planar graphs are not necessarily so. In particular, the following questions remain open:

Question 3.1.10. *Does every planar graph of girth* 7 *admit an* $(\mathcal{I}, \Delta_3)$*-partition ?*

Question 3.1.11. *Does every planar graph of girth* 10 *admit an* $(\mathcal{I}, \Delta_1)$*-partition ?*

A natural way to extend the previous results is to consider partitions of sparse graphs into an independent set and a set inducing a forest of bounded degree, that is $(\mathcal{I}, \mathcal{F}_d)$-partitions. Note that if a graph admits an $(\mathcal{I}, \mathcal{F}_d)$-partition, then it admits an $(\mathcal{I}, \Delta_d)$-partition, and that an $(\mathcal{I}, \mathcal{F}_1)$-partition is the same as an $(\mathcal{I}, \Delta_1)$-partition. Therefore the previous results imply that:

- for every d, there exists a planar graph of girth 6 that admits no $(\mathcal{I}, \mathcal{F}_d)$-partition;
- there exists a planar graph of girth at least 7 that admits no $(\mathcal{I}, \mathcal{F}_2)$-partition;
- there exists a planar graph of girth at least 9 that admits no $(\mathcal{I}, \mathcal{F}_1)$-partition;
- every planar graph with girth at least 11 admits an $(\mathcal{I}, \mathcal{F}_1)$-partition.

We will prove the following theorems in Sections 3.4 and 3.5 respectively.

Theorem 3.1.12. *Let M be a real number such that $M < 3$. Let $d \geq 0$ be an integer and let G be a graph with* $\mathrm{mad}(G) < M$*. If $d \geq \frac{2}{3-M} - 2$, then G admits an $(\mathcal{I}, \mathcal{F}_d)$-partition.*

Theorem 3.1.13. *Let M be a real number such that $\frac{8}{3} \leq M < 3$. Let $d \geq 0$ be an integer and let G be a graph with* $\mathrm{mad}(G) < M$*. If $d \geq \frac{1}{3-M}$, then G admits an $(\mathcal{I}, \mathcal{F}_d)$-partition.*

By a direct application of Euler's formula, every planar graph with girth at least g has maximum average degree less than $\frac{2g}{g-2}$. That yields the following corollary:

Corollary 3.1.14. *Let G be a planar graph with girth at least g.*

1. *If $g \geq 7$, then G admits an $(\mathcal{I}, \mathcal{F}_5)$-partition.*
2. *If $g \geq 8$, then G admits an $(\mathcal{I}, \mathcal{F}_3)$-partition.*
3. *If $g \geq 10$, then G admits an $(\mathcal{I}, \mathcal{F}_2)$-partition.*

Corollaries 3.1.14.2 and 3.1.14.3 are obtained from Theorem 3.1.13, whereas Corollary 3.1.14.1 is obtained from Theorem 3.1.12. See Table 3.1 for an overview of the results on vertex partitions of planar graphs presented above.

Recently, Chen, Yu, and Wang [18], improved Theorems 3.1.12 and 3.1.13, as well as the case $j = 0$ of Theorem 3.1.5 by proving the following:

Theorem 3.1.15 (Chen, Yu, and Wang [18])**.** *For all integer $d \geq 2$, every graph with maximum average degree less than $2 + \frac{d}{d+1}$ has an $(\mathcal{I}, \mathcal{F}_d)$-partition.*

Classes	Vertex-partitions	References
Planar graphs	$(\mathcal{I},\mathcal{I},\mathcal{I},\mathcal{I})$ $(\mathcal{I},\mathcal{F},\mathcal{F})$ $(\mathcal{F}_2,\mathcal{F}_2,\mathcal{F}_2)$ $(\mathcal{F},\mathcal{D}_2)$ $(\mathcal{I},\mathcal{D}_3)$	The Four Color Theorem [5, 6] Borodin [9] Poh [47] Thomassen [51] Thomassen [52]
Planar graphs with girth 4	$(\mathcal{I},\mathcal{I},\mathcal{I})$ $(\mathcal{F},\mathcal{F})$ $(\mathcal{F}_5,\mathcal{F})$	Grötzsch [36] Folklore **Theorem 3.1.4**
Planar graphs with girth 5	$(\mathcal{I},\mathcal{F})$	Borodin and Glebov [11]
Planar graphs with girth 6	no $(\mathcal{I},\Delta_d)$	Borodin et al. [13]
Planar graphs with girth 7	no $(\mathcal{I},\Delta_2)$ $(\mathcal{I},\Delta_4)$ $(\mathcal{I},\mathcal{F}_5)$	Montassier and Ochem [44] Borodin and Kostochka [15] **Corollary 3.1.14**
Planar graphs with girth 8	$(\mathcal{I},\Delta_2)$ $(\mathcal{I},\mathcal{F}_3)$	Borodin and Kostochka [15] **Corollary 3.1.14**
Planar graphs with girth 9	no $(\mathcal{I},\Delta_1)$	Esperet et al.[30]
Planar graphs with girth 10	$(\mathcal{I},\mathcal{F}_2)$	**Corollary 3.1.14**
Planar graphs with girth 11	$(\mathcal{I},\Delta_1)$	Kim, Kostochka and Zhu [42]

Table 3.1: Known results on planar graphs.

With Euler's formula, that implies the following:

Corollary 3.1.16. *Every planar graph with girth at least 8 (resp. 7) has an $(\mathcal{I},\mathcal{F}_4)$-partition (resp. an $(\mathcal{I},\mathcal{F}_2)$-partition).*

in terms of tightness, it is not known whether every planar graph of girth 7, 8 or 10 admits an $(\mathcal{I},\mathcal{F}_d)$-partition for $d=3$, $d=2$, and $d=1$ respectively. However, we note that Borodin, Ivanova, Montassier, Ochem, and Raspaud [13] constructed, for all d, a planar graph of girth 6 with $16d+14$ vertices that has no $(\mathcal{I},\Delta_d)$-partition (and thus in particular no $(\mathcal{I},\mathcal{F}_d)$-partition). Let us denote this graph by G_d. By Euler's formula, for every planar graph with girth at least g, n vertices, m edges, and at least one cycle, we have $\frac{m}{n-2}\leq\frac{g}{g-2}$. Moreover, if a graph with n vertices, m edges has no cycle, then $\frac{m}{n-2}\leq 1\leq\frac{g}{g-2}$ for all $g\geq 3$. Therefore for every planar graph of girth at least 6 with n vertices and m edges, we have $\frac{2m}{n}\leq 3\cdot\frac{n-2}{n}$. In particular, this is true for every subgraph of G_d. Thus, as $\frac{n-2}{n}$ increases when $n\geq 2$ increases, if we denote the number of vertices of G_d by n, then $\mathrm{mad}(G_d)\leq 3\cdot\frac{n-2}{n}=3-\frac{6}{n}=3-\frac{3}{8d+7}<3-\frac{3}{8d}$. That shows the following claim.

Claim 3.1.17. *For all integer d, there exists a graph with maximum average degree less than M, with $d=\frac{3}{8(3-M)}$, that admits no $(\mathcal{I},\Delta_d)$-partition (and thus no $(\mathcal{I},\mathcal{F}_d)$-partition).*

That shows that Theorem 3.1.13 is tight up to a multiplicative factor of $\frac{3}{8}$.

3.2 $(\mathcal{F},\mathcal{F}_5)$-partition of triangle-free planar graphs

This proof will make full use of the discharging method, and can be a good example of this kind of proof. Contrary to the proofs in the previous chapter, we will actually use a non-trivial discharging procedure in addition to the reducible configurations.

Let us first recall the statement of Theorem 3.1.4.

Theorem 3.1.4 (recall)**.** *Every triangle-free planar graph admits an $(\mathcal{F},\mathcal{F}_5)$-partition.*

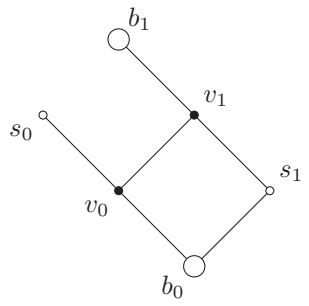

Figure 3.2.1: The forbidden configuration of Lemma 3.2.5. The big vertices are represented with big circles, and the small vertices with small circles. The filled circles represent vertices whose incident edges are all represented.

We prove Theorem 3.1.4 by contradiction. Let $G = (V, E)$ be a counter-example to Theorem 3.1.4 with minimum order.

Graph G is connected, otherwise at least one of its components is a counter-example to Theorem 3.1.4, contradicting the minimality of G.

Let us consider any planar embedding of G. Let us prove a series of lemmas on the structure of G, that correspond to forbidden configurations in G.

Lemma 3.2.1. *There is no 2^--vertex in G.*

Proof. Suppose there is a 2^--vertex v in G. By minimality of G, $G - v$ admits an $(\mathcal{F}, \mathcal{F}_5)$-partition (F, D). If v is a 1^--vertex, then $G[F \cup \{v\}] \in \mathcal{F}$. Suppose v is a 2-vertex. If both of its neighbours are in F, then $G[D \cup \{v\}] \in \mathcal{F}_5$. Otherwise, $G[F \cup \{v\}] \in \mathcal{F}$. In all cases, one can obtain an $(\mathcal{F}, \mathcal{F}_5)$-partition of G, a contradiction. □

In this section, a *big vertex* is an 8^+-vertex, and a *small vertex* is a 7^--vertex. By extension, a *big neighbour* of a vertex v is a big vertex that is adjacent to v, and a *small neighbour* of a vertex v is a small vertex that is adjacent to v.

Lemma 3.2.2. *Every 3-vertex in G has at least one big neighbour.*

Proof. Suppose there is a 3-vertex v in G that has three small neighbours. By minimality of G, $G - v$ admits an $(\mathcal{F}, \mathcal{F}_5)$-partition (F, D). If v has at least two D-neighbours, then $G[F \cup \{v\}] \in \mathcal{F}$. If v has no D-neighbour, then $G[D \cup \{v\}] \in \mathcal{F}_5$. Suppose v has exactly one D-neighbour u. If u has at most one F-neighbour, then $G[F \cup \{u\}] \in \mathcal{F}$, and $G[D \backslash \{u\} \cup \{v\}] \in \mathcal{F}_5$. Otherwise, since u is small, at most four of the neighbours of u are in D, thus $G[D \cup \{v\}] \in \mathcal{F}_5$. In all cases, one can obtain an $(\mathcal{F}, \mathcal{F}_5)$-partition of G, a contradiction. □

Lemma 3.2.3. *Every 4-vertex or 5-vertex in G has at least one 4^+-neighbour.*

Proof. Suppose there is a 4-vertex or 5-vertex v in G that has no 4^+-neighbour. Let the u_i's be the neighbours of v, for $i \in \{0, ..., 3\}$ or $i \in \{0, ..., 4\}$. Let $G' = G - v - \bigcup_i \{u_i\}$. By minimality of G, G' admits an $(\mathcal{F}, \mathcal{F}_5)$-partition (F, D). Add v to D, and for all i, add u_i to D if its two neighbours distinct from v are in F, and add u_i to F otherwise. Vertex v has at most five D-neighbours, and each of the u_i's that is in D has one D-neighbour. Each of the u_i that is in F has at most one F-neighbour. We have an $(\mathcal{F}, \mathcal{F}_5)$-partition of G, a contradiction. □

We will need the following observation in the next two lemmas.

Observation 3.2.4. *Let $v_0v_1v_2v_3$ be the boundary of a face of G, u_0 be a neighbour of v_0 distinct from v_1 and v_3, and u_1 be a neighbour of v_1 distinct from v_0 and v_2. Either u_0 and v_2 are at distance at least 3, or u_1 and v_3 are at distance at least 3.*

By contradiction, suppose that u_0 and v_2 are at distance at most two, and that u_1 and v_3 are at distance at most two. Since G is triangle-free, a shortest path from u_0 to v_2 (resp. from u_1 to v_3) does not contain any of the u_i and v_i except for its extremities. Then by planarity there exists a vertex w adjacent to u_0, v_2, u_1, and v_3. In particular v_2v_3w is a triangle, a contradiction.

Lemma 3.2.5. *The following configuration does not occur in G: two adjacent 3-vertices v_0 and v_1 such that for all $i \in \{0,1\}$, v_i has a big neighbour b_i and a small neighbour s_i distinct from the v_i's, and such that $v_0v_1s_1b_0$ bounds a face of G.*

Proof. Suppose such a configuration exists in G. See Figure 3.2.1 for an illustration of this configuration. Observe that all the vertices defined in the statement are distinct (since G is triangle-free). By Observation 3.2.4, either b_0 and b_1 are at distance at least 3, or s_0 and s_1 are at distance at least 3. For the remainder of the proof, we no longer need the fact that $b_0s_1 \in E(G)$ or that $v_0v_1s_1b_0$ bounds a face. We forget this assumption, and only remember that either b_0 and b_1 are at distance at least 3, or s_0 and s_1 are at distance at least 3. This provides some symmetry in the graph.

Let $G_0 = G - \{v_0, v_1\} + b_0b_1$ and $G_1 = G - \{v_0, v_1\} + s_0s_1$. By what precedes, either G_0 or G_1 is triangle-free, thus there exists a j such that G_j is a triangle-free planar graph. By minimality of G, G_j admits an $(\mathcal{F}, \mathcal{F}_5)$-partition (F, D).

Let us first prove that if we do not have b_0 and b_1 in D, and s_0 and s_1 in F, then the conditions $G[F] \in \mathcal{F}$ and $G[D] \in \mathcal{F}_5$ lead to a contradiction. We will see that we can always extend the $(\mathcal{F}, \mathcal{F}_5)$-partition of G_j to G.

- If at least three of the b_i and s_i are in D, then $G[F \cup \{v_0, v_1\}] \in \mathcal{F}$.
- If all of the b_i's and s_i's are in F, then $G[D \cup \{v_0, v_1\}] \in \mathcal{F}_5$.
- Suppose now that exactly three of the b_i's and s_i's are in F. W.l.o.g., $b_0 \in D$ or $s_0 \in D$. We have $G[F \cup \{v_0\}] \in \mathcal{F}$ and $G[D \cup \{v_1\}] \in \mathcal{F}_5$.
- Suppose now that exactly two of the b_i's and s_i's are in F. If b_0 and s_0 are in F (resp. b_1 and s_1 are in F), then $G[D \cup \{v_0\}] \in \mathcal{F}_5$ and $G[F \cup \{v_1\}] \in \mathcal{F}$ (resp. $G[F \cup \{v_0\}] \in \mathcal{F}$ and $G[D \cup \{v_1\}] \in \mathcal{F}_5$).

  Now since we assumed that b_0 and b_1 are not both in D, one of them, say b_0, is in F, and by what precedes, $s_0 \in D$. If s_0 has at most one G-neighbour in F, then $G[F \cup \{s_0\}] \in \mathcal{F}$, we can replace F by $F \cup \{s_0\}$ and D by $D \backslash \{s_0\}$, and we fall into a previous case. We can thus assume that s_0 has at least two of its G-neighbours in F, and thus it has at most four of its G-neighbours in D. Therefore $G[D \cup \{v_0\}] \in \mathcal{F}_5$, and $G[F \cup \{v_1\}] \in \mathcal{F}$.

In all cases, G has an $(\mathcal{F}, \mathcal{F}_5)$-partition, a contradiction.

Remains the case where b_0 and b_1 are in D, and s_0 and s_1 are in F. In the case where we added the edge b_0b_1 (i.e. the case $j = 0$), we have $G[D \cup \{v_0, v_1\}] \in \mathcal{F}_5$, since $G[D \cup \{v_0, v_1\}]$ is equal to $G_0[D]$ where an edge is subdivided twice. Similarily, in the case where we added the edge s_0s_1 (i.e. the case $j = 1$), we have $G[F \cup \{v_0, v_1\}] \in \mathcal{F}$, since $G[F \cup \{v_0, v_1\}]$ is equal to $G_0[F]$ where an edge is subdivided twice. Again, G has an $(\mathcal{F}, \mathcal{F}_5)$-partition, a contradiction. □

Lemma 3.2.6. *The following configuration does not occur in G: a 3-vertex v_0 adjacent to a 4-vertex v_1 such that v_0 has a big neighbour b and a small neighbour s_0 distinct from v_1, and v_1 has three small neighbours s_1, w_0, and w_1 distinct from v_0 such that $v_0v_1s_1b$ bounds a face of G and s_1 has degree 3.*

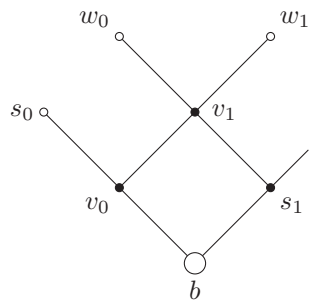

Figure 3.2.2: The forbidden configuration of Lemma 3.2.6.

Proof. Suppose such a configuration exists in G. See Figure 3.2.2 for an illustration of this configuration. Observe that all the vertices defined in the statement are distinct (since G is triangle-free). By Observation 3.2.4, either b and w_0 are at distance at least 3, or s_0 and s_1 are at distance at least 3. Let $G_0 = G - \{v_0, v_1\} + bw_0$ and $G_1 = G - \{v_0, v_1\} + s_0s_1$. By what precedes, either G_0 or G_1 is triangle-free, thus there exists a j such that G_j is a triangle-free planar graph. By minimality of G, G_j has an $(\mathcal{F}, \mathcal{F}_5)$-partition (F, D).

Let us first prove that except in the case where $\{b, w_0, w_1\} \subset D$ and $\{s_0, s_1\} \subset F$, the conditions $G[F] \in \mathcal{F}$ and $G[D] \in \mathcal{F}_5$ lead to a contradiction. We will see that we can always extend the $(\mathcal{F}, \mathcal{F}_5)$-partition of G_j to G.

If at least four among the w_i's, the s_i's, and b are in D, then $G[F \cup \{v_0, v_1\}] \in \mathcal{F}$.

Suppose now that at most three among the w_i's, the s_i's, and b are in D. Suppose $x \in \{b, s_0, s_1, w_0, w_1\}$ is in D. If x has at most one G-neighbour in F, then $G[F \cup \{x\}] \in \mathcal{F}$, and we could consider $F \cup \{x\}$ instead of F and $D \backslash \{x\}$ instead of D. Note that this cannot lead to the case we excluded ($\{b, w_0, w_1\} \subset D$ and $\{s_0, s_1\} \subset F$) unless at least four among the w_i's, the s_i's, and b are in D. Thus we can assume that for any x among the w_i's and s_i's such that $x \in D$, x has at most four G-neighbours in D, and thus adding one neighbour of x in D cannot cause x to have at least six D-neighbours. We consider two cases according to b:

- Suppose $b \in F$. If at least three of the w_i's and s_i's are in F, then $G[D \cup \{v_0, v_1\}] \in \mathcal{F}_5$.

  If at least two among the w_i's and s_1 are in D, then $G[F \cup \{v_1\}] \in \mathcal{F}$ and $G[D \cup \{v_0\}] \in \mathcal{F}_5$. Else, at least two among the w_i's and s_1 are in F. If $s_0 \in F$, then $G[D \cup \{v_0, v_1\}] \in \mathcal{F}_5$, and if $s_0 \in D$, then $G[D \cup \{v_1\}] \in \mathcal{F}_5$ and $G[F \cup \{v_0\}] \in \mathcal{F}$.

- Suppose now that $b \in D$. As s_1 has degree 3, it has at most one G-neighbour in F, and thus as previously, if $s_1 \in D$ then we can consider $F \cup \{s_1\}$ instead of F and $D \backslash \{s_1\}$ instead of D. Again, this cannot lead to the case we excluded ($\{b, w_0, w_1\} \subset D$ and $\{s_0, s_1\} \subset F$) unless at least four among the w_i's, the s_i's, and b are in D. Therefore we can assume that $s_1 \in F$. The w_i's are not both in D (otherwise we fall into the case we excluded). We have $G[D \cup \{v_1\}] \in \mathcal{F}_5$ and $G[F \cup \{v_0\}] \in \mathcal{F}$.

In all cases, G has an $(\mathcal{F}, \mathcal{F}_5)$-partition, a contradiction.

Remains the case $\{b, w_0, w_1\} \subset D$ and $\{s_0, s_1\} \subset F$. In the case where we added the edge bw_0 (i.e. the case $j = 0$), b has at most five G_0-neighbours in D, and thus at most four G-neighbours in D, so $G[D \cup \{v_0\}] \in \mathcal{F}_5$, and $G[F \cup \{v_1\}] \in \mathcal{F}$. In the case where we added the edge s_0s_1 (i.e. the case $j = 1$), we have $G[F \cup \{v_0, v_1\}] \in \mathcal{F}$, since $G[F \cup \{v_0, v_1\}]$ is equal to $G_0[F]$ where an edge is subdivided twice. Again, G has an $(\mathcal{F}, \mathcal{F}_5)$-partition, a contradiction. □

We define a specific configuration:

Configuration 3.2.7. *Two 4-faces bounded by $b_0v_0v_1w_0$ and $v_0v_1v_2v_3$, such that b_0 is a big vertex, v_0 and w_0 are 3-vertices, v_1 is a 4-vertex, v_2 and v_3 are small vertices, and the neighbour of v_1 distinct from v_0, w_0, and v_2, say b_1, is a big vertex. See Figure 3.2.3 for an illustration of this configuration.*

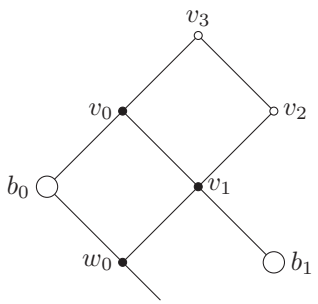

Figure 3.2.3: Configuration 3.2.7.

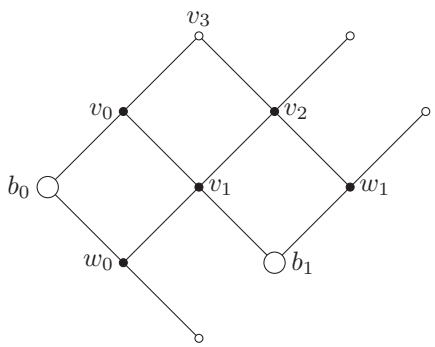

Figure 3.2.4: The forbidden configuration of Lemma 3.2.8.

Lemma 3.2.8. *The following configuration is forbidden: Configuration 3.2.7 with the added condition that there is a 4-face $b_1v_1v_2w_1$ with w_1 a 3-vertex, v_2 a 4-vertex, and the fourth neighbour of v_2, the third neighbour of w_1, and the third neighbour of w_0 are small vertices.*

Proof. Suppose such a configuration exists in G. See Figure 3.2.4 for an illustration of this configuration. Observe that all the vertices named in the statement are distinct since G is triangle-free and w_1 is a small vertex whereas b_0 is a big one.

Let us prove that either b_0 and b_1 are at distance at least 3, or w_0 and w_1, and w_0 and v_3 are at distance at least 3. By contradiction, suppose that b_0 and b_1 are at distance at most two, and that either w_0 and w_1 are at distance at most two, or w_0 and v_3 are at distance at most 2. Since G is triangle-free, a shortest path from b_0 to b_1, from w_0 to w_1 or from w_0 to v_3 does not go through any of the vertices named in the statement. Then by planarity there exists a vertex w adjacent to b_0, b_1, w_0 and either w_1 or v_3. In particular b_0w_0w is a triangle, a contradiction.

Let $G_0 = G - \{v_0, v_1\} + b_0b_1$ and $G_1 = G - \{v_0, v_1\} + w_0w_1 + w_0v_3$. By what precedes, either G_0 or G_1 is triangle-free, thus there exists a j such that G_j is a triangle-free planar graph. By minimality of G, G_j has an $(\mathcal{F}, \mathcal{F}_5)$-partition (F, D).

Let s_0 be the third neighbour of w_0, s_1 be the third neighbour of w_1 and s_2 be the fourth neighbour of v_2. They are all small vertices, but there may be some that are equal between themselves, or equal to some vertices named in the statement of the lemma. However, if one of the s_i is in $\{v_0, v_1, v_2, w_0, w_1\}$, then this s_i is a 4^--vertex in G (and in particular it has at most 4 D-neighbours in G).

Suppose first that b_0 and b_1 are both in D.

1. Suppose w_0 is in D. Here we only consider (F, D) as an $(\mathcal{F}, \mathcal{F}_5)$-partition of $G - \{v_0, v_1\}$.

   If v_3 is also in D, then adding v_0 and v_1 to F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. Suppose v_3 is in F. We show now that we can assume that v_2 is in D. By contradiction, suppose v_2 is in F. We remove v_2 from F.

   Observe that we can assume that v_2 has no G-neighbour in D with five G-neighbours in D. Indeed, suppose v_2 has a G-neighbour in D with five G-neighbours in D. This G-neighbour is a 5^+-vertex, so it is s_2. Moreover, s_2 is not equal to v_3 (because v_3 is in F), and is not equal to any of the other vertices named in the statement (because of the degree conditions). As s_2 is a small D-vertex, has at least five G-neighbours in D and is adjacent to v_2 that is neither in F nor in D, s_2 has at most one F-neighbour. Therefore we can put s_2 in F.

   Observe that we can assume that v_2 has at most one G-neighbour in D. Suppose v_2 has two G-neighbours in D. These G-neighbours must be s_2 and w_1. Vertex w_1 has at most one F-neighbour, that is s_1, so we can put w_1 in F.

   Now v_2 has at most one G-neighbour in D, and no G-neighbour of v_2 in D has five G-neighbours in D, so we can put v_2 in D. Therefore we can always assume that v_2 is in D. Note that we do not need to change where s_2 is in the partition if it is equal to one of the vertices named in the statement. Adding v_0 and v_1 to F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.

2. Suppose w_0 is in F, v_3 is in D, and w_1 is in D. If s_2 is in D, then putting v_0, v_1 and v_2 in F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. Suppose s_2 is in F. We put v_0, v_1 and w_1 in F, and v_2 in D. If this increases the number of G-neighbours of v_3 in D above five, then since v_3 is small, v_3 has at most one F-neighbour, which is v_0, and we put v_3 in F. This leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.

3. Suppose w_0 is in F, v_3 is in D and w_1 is in F. Suppose s_2 is in F. We put v_0 and v_1 in F, and v_2 in D. If this increases the number of G-neighbours of v_3 in D above five, then since v_3 is small, v_3 has at most one F-neighbour, which is v_0, and we put v_3 in F. This leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. Suppose s_2 is in D. If v_2 is not in F, we may put it in F, since it has only one G_j-neighbour in F, that is w_1. Therefore we can assume that v_2 is in F. If $j = 0$, then b_1 has at most four G-neighbours in D (since it has at most 5 such G_0-neighbours), so adding v_0 to F and v_1 to D leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. If $j = 1$, then adding v_0 and v_1 to F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.

4. Suppose w_0 is in F and v_3 is in F. Suppose $j = 0$. The vertex b_0 has at most 4 G-neighbours in D (since it has at most 5 such G_0-neighbours), so we can add v_0 to D. If v_2 is in D, then adding v_1 to F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. If v_2 is in F, then adding v_1 to D makes $G[D]$ equal to $G_0[D]$ with an edge subdivided twice, and this leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. Suppose $j = 1$. Here we only consider (F, D) as an $(\mathcal{F}, \mathcal{F}_5)$-partition of $G - \{v_0, v_1\} + w_0v_3$. As in 1, we can suppose, up to changing where s_2 and w_1 are in the partition, that v_2 is in D. Note that if s_2 is equal to one of the vertices named in the statement, we do not need to move s_2 in the partition. Adding v_0 and v_1 to F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.

Now we may assume that at least one of b_0 and b_1 is in F. From now on we only consider (F, D) as an $(\mathcal{F}, \mathcal{F}_5)$-partition of $G - \{v_0, v_1\}$.

- Suppose b_0 is in F and b_1 is in D. In that case we put v_0 and w_0 in D, and v_1 in F. Adding v_0 in D (resp. w_0 in D) may violate the degree condition of $G[D]$; however, if it happens, one can put v_3 (resp. s_0) in F. In any case, we obtain an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.
- Suppose b_0 is in D and b_1 is in F. If at least one of w_0 and v_2 is in F, then adding v_0 in F and v_1 in D leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. Assume w_0 and v_2 are both in D. If v_3 is in D, then adding v_0 and v_1 in F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. Assume v_3 is in F. We consider three cases:
  - Suppose s_2 and w_1 are in F. Adding v_0 in F and v_1 in D leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.
  - Suppose s_2 is in F and w_1 is in D. If s_1 is in D, then we can put w_1 in F and we fall into the previous case. If s_1 is in F, then adding v_0 in F and v_1 in D leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.
  - Suppose s_2 is in D. If s_1 is in D and has five G-neighbours in D distinct from w_1, then as s_1 is small, it is distinct from all the vertices named in the statement, and we can put it in F. Therefore we can put w_1 in D and v_2 in F. We fall into a previous case (at least one of w_0 and v_2 is in F).
- Suppose b_0 and b_1 are in F. If s_0 is in D and has five G-neighbours in D distinct from w_0, then as s_0 is small, it is distinct from all the vertices named in the statement aside from v_3, and we can put it in F. Therefore we can put w_0 in D. We consider the following cases:
  - If v_2 and v_3 are in F, then adding v_0 and v_1 to D leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.
  - If v_2 is in F and v_3 is in D, then adding v_0 to F and v_1 to D leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.
  - If v_2 is in D and v_3 is in F, then adding v_0 to D and v_1 to F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.
  - If v_2 and v_3 are in D, then adding v_0 to D and v_1 to F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. Adding v_0 to D may violate the degree condition of $G[D]$, but in that case we can put v_3 in F.

□

We have proven all of the forbidden configurations we need, and we are now ready for the discharging procedure. Initially, for all j, every j-vertex v has a charge equal to $c_0(v) = j - 4$, and every j-face ϕ has a charge equal to $c_0(\phi) = j - 4$. Let n denote the number of vertices of G, m be the number of edges of G, and f the number of faces of G. By Euler's formula, we have:

$$\begin{aligned}
-8 &= 4m - 4n - 4f \\
&= \sum_{v \in V(G)} d(v) + \sum_{\phi \in F(G)} d(\phi) - 4n - 4f \\
&= \sum_{v \in V(G)} (d(v) - 4) + \sum_{\phi \in F(G)} (d(\phi) - 4) \\
&= \sum_{v \in V(G)} c_0(v)) + \sum_{\phi \in F(G)} e(c_0(\phi))
\end{aligned}$$

Therefore the total charge is negative (equal to -8). Observe that, since G is triangle-free, every face has a non-negative initial charge, and by Lemma 3.2.1, the vertices that have negative initial charges are exactly the 3-vertices of G, and they have an initial charge of -1. The following discharging procedure distributes the weight of the vertices in five steps, and is a bit more complex than classic discharging procedures.

Discharging procedure

- *Step 1*: Every big vertex gives $\frac{1}{2}$ to each of its small neighbours. Furthermore, for every 4-face bounded by a cycle $uvwx$ where u and v are big, and w and x are small, v gives $\frac{1}{4}$ to x (and u gives $\frac{1}{4}$ to w).
- *Step 2*: Consider a 4-vertex v such that there is no instance of Configuration 3.2.7 where $v = v_1$. Vertex v gives $\frac{1}{4}$ to each of its small neighbours that are consecutive (as neighbours of v) to exactly one big vertex, and $\frac{1}{2}$ to each of its small neighbours that are consecutive (as neighbours of v) to two big vertices.

  Consider a 4-vertex v corresponding to v_1 in an instance of Configuration 3.2.7. We use the notation of Configuration 3.2.7. If w_0 has two big neighbours, then v_1 gives $\frac{1}{4}$ to v_0 and $\frac{1}{4}$ to v_2. Otherwise, it gives $\frac{1}{4}$ to w_0 and $\frac{1}{4}$ to v_0.

  Every small 5^+-vertex that has a big neighbour gives $\frac{1}{4}$ to each of its small neighbours, and an additional $\frac{1}{4}$ to each of its small neighbours that is consecutive (as neighbours of v) to at least one big vertex. Every small 5^+-vertex that has no big neighbour gives $\frac{1}{4}$ to each of its 3-neighbours.
- *Step 3*: For every 4-face bounded by a cycle $uvwx$, with u a big vertex, v a 3-vertex, w a 4-vertex, and x a small vertex such that x gave charge to w in Step 2, w gives $\frac{1}{4}$ to v.
- *Step 4*: Every 5^+-face that has a big vertex in its boundary gives $\frac{1}{4}$ to each of the small vertices in its boundary. Every 5^+-face that has no big vertex in its boundary gives $\frac{1}{5}$ to each of the vertices in its boundary.
- *Step 5*: For every 4-face bounded by a cycle $uvwx$, with u a big vertex, v a 3-vertex, w a 4-vertex and x a 3-vertex such that the other face that has vw in its boundary is a 5^+-face, w gives $\frac{1}{5}$ to v.

For every vertex or face x of G, for every $i \in \{1, 2, 3, 4, 5\}$, let $c_i(x)$ be the charge of x at the end of Step i. Observe that during the procedure, no charge is created and no charge disappears; hence the total charge is kept fixed.

We now prove that every vertex and every face has a non-negative charge at the end of the procedure. That leads to the following contradiction:

$$0 \leq \sum_{x \in V(G) \cup F(G)} c_5(x) = \sum_{x \in V(G) \cup F(G)} c_0(x) = -8$$

Lemma 3.2.9. *Every face has non-negative charge at the end of the procedure.*

Proof. At the beginning of the procedure, for every j-face ϕ we have $c_0(\phi) = j - 4 \geq 0$ (as $j \geq 4$). The procedure does not involve 4-faces. Hence if $j = 4$, then $c_5(\phi) = c_0(\phi) = 0$. If $j = 5$, then ϕ gives at most four times $\frac{1}{4}$ if it is incident to a big vertex and at most five times $\frac{1}{5}$ otherwise in Step 4. It follows that $c_5(\phi) \geq 0$. If $j \geq 6$, then ϕ can give $\frac{1}{3}$ to each of its incident vertices (and so $\frac{1}{4}$ or $\frac{1}{5}$) during Step 4, and $c_5(\phi) \geq j - 4 - \frac{j}{3} \geq 0$. □

Lemma 3.2.10. *A 4^+-vertex never has negative charge.*

Proof. Consider a j-vertex z with $j \geq 4$. At the beginning, $c_0(z) = j - 4 \geq 0$. We will show that $c_i(z) \geq 0$ for $i = 1, ..., 5$.

- Suppose z is a big vertex. Such a vertex only loses charge in Step 1. Since $j \geq 8$, we have $c_0(z) \geq \frac{j}{2}$. In Step 1, vertex z loses $\frac{1}{2}$ for each of its small neighbours, and at most $\frac{1}{2}$ for each of its big neighbours. Therefore it has more charge than what it gives, and thus it keeps a non-negative charge.

- Suppose z is a small 5^+-vertex. It does not lose charge in Steps 1, 3, 4 and 5.

  Suppose z has a big neighbour. It has at most $j-1$ small neighbours, and it has charge at least $\frac{1}{4}(j-1)$ at the beginning of the procedure, since $j \geq 5$. Moreover, it receives $\frac{1}{2}$ from each of its big neighbours in Step 1. Therefore it does not give more charge that it has in Step 2.

  Suppose now that z has no big neighbour. If z is a 5-vertex, then by Lemma 3.2.3, it has at most four 3-neighbours, and $c_2(z) \geq 1 - 4 \cdot \frac{1}{4} \geq 0$. If z is a 6^+-vertex, then $c_2(z) \geq j - 4 - j \cdot \frac{1}{4} \geq 0$.

- Suppose z is a 4-vertex. It does not lose charge in Steps 1 and 4. Suppose z gives charge in Step 2. Consider first that z does not correspond to v_1 in an instance of Configuration 3.2.7. If z is adjacent to a small vertex that is consecutive (as a neighbour of z) to two big neighbours, then z gives at most twice $\frac{1}{2}$ in Step 2 and received twice $\frac{1}{2}$ in Step 1; hence $c_2(z) \geq 0$. Otherwise, z gives at most twice $\frac{1}{4}$ in Step 2, and received at least once $\frac{1}{2}$ in Step 1; hence $c_2(z) \geq 0$. Let us now consider the case where z corresponds to v_1 in an instance of Configuration 3.2.7. The vertex z has a big neighbour that gave $\frac{1}{2}$ to z in Step 1, and z gives $\frac{1}{4}$ to two of its neighbours in Step 2. Therefore z received in Step 1 at least as much charge as what it gives in Step 2.

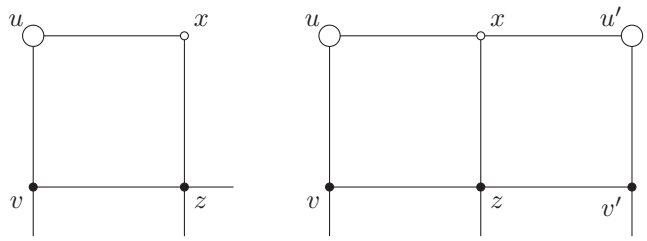

Figure 3.2.5: Some configurations that appear in Lemma 3.2.10.

  Suppose z gives charge in Step 3. There is a 4-face bounded by a cycle $uvzx$ with u a big vertex, v a 3-vertex, and x a small vertex such that x gave charge to z in Step 2. Suppose z is consecutive to exactly one big vertex (as neighbours of x). The vertex x gave at least $\frac{1}{4}$ to z in Step 2, and there is exactly one such face with the same z and x (i.e. there is no pair (u', v') distinct from (u, v) that verifies the properties we stated for (u, v))(see Figure 3.2.5, left). Therefore z can give $\frac{1}{4}$ to v in Step 3. Suppose z is consecutive to exactly two big vertices (as neighbours of x). The vertex x gave $\frac{1}{2}$ to z in Step 2, and there are at most two such faces with the same z and x (i.e. there is at most one pair (u', v') distinct from (u, v) that verifies the properties we stated for (u, v)) (see Figure 3.2.5, right). Therefore z can give $\frac{1}{4}$ to each of the corresponding v's in Step 3. Therefore z received in Step 2 at least as much charge as what it gives in Step 3.

  Suppose z gives charge in Step 5. There is a 4-face bounded by a cycle $uvzx$, with u a big vertex, v a 3-vertex, and x a 3-vertex such that the other face, say ϕ, that has vz in its boundary is a 5^+-face. Vertex z received at least $\frac{1}{5}$ from ϕ in Step 4, and it gives $\frac{1}{5}$ to v. We would lack charge only if there was another 4-face bounded by a cycle $u'v'zx'$, such that vz and zv' are on the boundary of ϕ, u' is a big vertex, and x' and v' are 3-vertices. But then z would have four 3-neighbours, contradicting Lemma 3.2.3. Therefore z received in Step 4 at least as much charge as what it gives in Step 5.

In all cases, z never has negative charge. □

Lemma 3.2.11. *At the end of the procedure, every* 3*-vertex has non-negative charge.*

Proof. Let z be a 3-vertex. It never loses charge in the procedure, so we only need to prove that it received at least 1 over the whole procedure. Assume by contradiction that it received less than that.

By Lemma 3.2.2, vertex z has at least one big neighbour b. Let x_0 and x_1 be its two other neighbours. Vertex b gives $\frac{1}{2}$ to z in Step 1, so z only needs to receive $\frac{1}{2}$ from x_0, x_1, and its surrounding faces. In

particular, if one of the x_i's is a big vertex, then it gives $\frac{1}{2}$ to z in Step 1, and z receives all the charge it needs, a contradiction. Therefore x_0 and x_1 are small vertices.

Let ϕ be the face that contains x_0z and zx_1 in its boundary, ϕ_0 be the face that contains x_0z and zb in its boundary and ϕ_1 the face that contains x_1z and zb in its boundary. Let y_0 and y_1 be such that bzx_0y_0 and bzx_1y_1 are 4-paths that are in the boundaries of ϕ_0 and ϕ_1 respectively. Let us count the charge that x_0, y_0, and ϕ_0 give to z plus half the charge that ϕ gives to z. If we show that this sum is at least $\frac{1}{4}$, then by symmetry we will know that z received at least $\frac{1}{2}$ from x_0, x_1, y_0, y_1, and the faces ϕ, ϕ_0, and ϕ_1, and that leads to a contradiction.

Observe that ϕ_0 is a 4-face. If it was a 5^+-face, then since it has the big vertex b in its boundary, it would gives $\frac{1}{4}$ to z in Step 4, a contradiction.

Observe that y_0 is a small vertex. If y_0 was a big vertex, then y_0 would give $\frac{1}{4}$ to z in Step 1, a contradiction. See Figure 3.2.6 for a representation of the vertices we know.

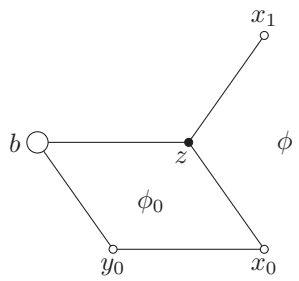

Figure 3.2.6: The face ϕ_0 and the vertex x_1.

Observe that x_0 has degree 4. Indeed, suppose x_0 is a 5^+-vertex. It gives at least $\frac{1}{4}$ to z in Step 2, a contradiction. Suppose x_0 is a 3-vertex. Then x_0 has a big neighbour by Lemma 3.2.2, and it cannot be y_0. That contradicts Lemma 3.2.5.

Let a and a' be the neighbours of x_0 distinct from z and y_0, such that a is consecutive to z(as a neighbour of x_0). Suppose a is a big vertex. If x_0 does not correspond to v_1 in an instance of Configuration 3.2.7, then x_0 gives $\frac{1}{4}$ to z in Step 2. If x_0 corresponds to v_1 in an instance of Configuration 3.2.7, then z corresponds to w_0 and is not adjacent to two big vertices, so x_0 also gives $\frac{1}{4}$ to z in Step 2. Therefore a is a small vertex.

Observe that y_0 is a 4^+-vertex. Indeed, suppose y_0 is a 3-vertex. By Lemma 3.2.6, there is at least one big vertex in $\{a, a'\}$, which has to be a'. If ϕ is a 4-face, then x_0 corresponds to v_1 in an instance of Configuration 3.2.7, and it gives $\frac{1}{4}$ to z in Step 2. Therefore ϕ is a 5^+-face, and it gives at least $\frac{1}{5}$ to z in Step 4, and x_0 gives $\frac{1}{5}$ to z in Step 5. As $\frac{1}{10} + \frac{1}{5} \geq \frac{1}{4}$, this leads to a contradiction.

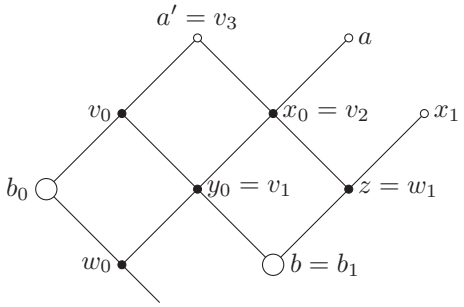

Figure 3.2.7: The case in Lemma 3.2.11 where y_0 corresponds to v_1 in Configuration 3.2.7.

Suppose first that y_0 corresponds to v_1 in an instance of Configuration 3.2.7. See Figure 3.2.7 for an illustration of the vertices we know, and of the correspondence with vertices of Configuration 3.2.7. By

Lemma 3.2.8, the third neighbour of w_0 is big. Therefore y_0 gives $\frac{1}{4}$ to x_0 in Step 2. It follow that x_0 gives $\frac{1}{4}$ to z in Step 3, a contradiction.

Now y_0 does not correspond to v_1 in an instance of Configuration 3.2.7. Vertex y_0 gives $\frac{1}{4}$ to x_0 in Step 2, since x_0 is a neighbour of y_0 consecutive (as a neighbour of y_0) to a big neighbour. Therefore x_0 gives $\frac{1}{4}$ to z in Step 3, a contradiction. □

Lemmas 3.2.9–3.2.11 conclude the proof of Theorem 3.1.4.

3.3 Complexity of finding an $(\mathcal{F}, \mathcal{F}_d)$-partition

By Theorem 3.1.4, there exists a smallest integer $d_0 \leq 5$ such that every triangle-free planar graph has an $(\mathcal{F}, \mathcal{F}_{d_0})$-partition. For all $d \geq d_0$, every triangle-free planar graph has an $(\mathcal{F}, \mathcal{F}_d)$-partition. Let us assume that $d_0 \geq 1$.

In this section, for a fixed integer d we consider the complexity of the following problem P_d: given a triangle-free planar graph G, does G have an $(\mathcal{F}, \mathcal{F}_d)$-partition? This can be answered positively in constant time for $d \geq d_0$. However, we prove the following:

Theorem 3.3.1. *For $d < d_0$, the problem P_d is NP-complete.*

The problem is clearly in NP, since checking that a graph is acyclic and/or has maximum degree at most d can be done in polynomial time. Let us show that the problem is NP-hard.

The general idea is first to build some gadgets that can force where some vertices are in the partition. Then we will use these gadgets to make some polynomial reductions and prove that our problem is NP-hard.

Let G be a counter-example to the property that every triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F}_d)$ partition. We consider such a counter-example G with minimum number of vertices, and with minimum number of edges among the counter-examples with minimum number of vertices. Let $e = uv$ be an edge of G, and $G' = G - e$. By minimality of G, G' admits an $(\mathcal{F}, \mathcal{F}_d)$-partition. In such a partition (F, D), u and v are either both in F or both in D, and if they are in F, then there is a path from u to v in $G'[F]$ (otherwise it would be an $(\mathcal{F}, \mathcal{F}_d)$-partition of G). Observe that in G', u and v are at distance at least 3, since G is triangle-free. We call a copy of G' an anti-edge uv.

We want to make a gadget H with a vertex x that admits an $(\mathcal{F}, \mathcal{F}_d)$-partition, and such that x is in F for all $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of H.

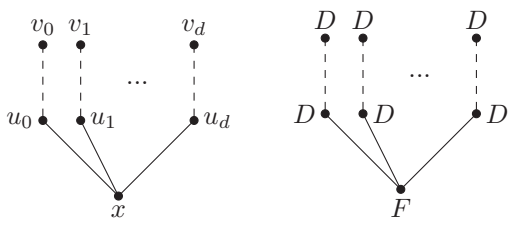

Figure 3.3.1: The gadget H in Case 1, and an $(\mathcal{F}, \mathcal{F}_d)$-partition. Dashed lines are anti-edges.

We construct H as follows:

1. Suppose for all $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of G', u and v are in D. See Figure 3.3.1 for an illustration of the construction of H and an $(\mathcal{F}, \mathcal{F}_d)$-partition of H in this case. Take $d + 1$ copies of G', called G'_0, ..., G'_d, and add a new vertex x adjacent to each instance of u in those copies. Consider an $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of G'. This leads to an $(\mathcal{F}, \mathcal{F}_d)$-partition (F_i, D_i) of each G_i, and $(\bigcup_i F_i \cup \{x\}, \bigcup_i D_i)$ is an $(\mathcal{F}, \mathcal{F}_d)$-partition of H.

   Let us now prove that for any $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of H, x belongs to F. For any $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of H, if $x \in D$, then there exists a u_i that is in F, so the corresponding G'_i admits an $(\mathcal{F}, \mathcal{F}_d)$-partition with $u_i \in F$, a contradiction.

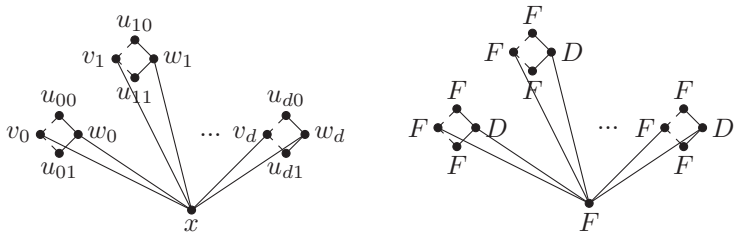

Figure 3.3.2: The gadget H in Case 2, and an $(\mathcal{F}, \mathcal{F}_d)$-partition.

2. Suppose there exists an $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of G' such that u and v are in F. See Figure 3.3.2 for an illustration of the construction of H and an $(\mathcal{F}, \mathcal{F}_d)$-partition of H in this case. We construct H as follows. Consider a vertex x. We add new vertices $v_0, ..., v_d$ and $w_0, ..., w_d$ to the graph, adjacent to x. Then for $0 \leq i \leq d$ and $0 \leq j \leq 1$, we add a new vertex u_{ij}, the anti-edge $v_i u_{ij}$, and the edge $u_{ij} w_i$.

   Graph H admits an $(\mathcal{F}, \mathcal{F}_d)$-partition. Indeed, consider an $(\mathcal{F}, \mathcal{F}_d)$-partition of G' with u and v in F, and apply it to every anti-edge of H (as before, we take the union of the F_i and the union of the D_i). Then the v_i and u_{ij} are all in F. Add all the w_i to D. Add x to F. We then have an $(\mathcal{F}, \mathcal{F}_d)$-partition of H.

   Let us now prove that for any $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of H, x belongs to F. For any $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of H, if $x \in D$, then there exists an i such that v_i and w_i are in F, thus u_{i0} and u_{i1} are in F, so there is a cycle in $H[F]$, a contradiction.

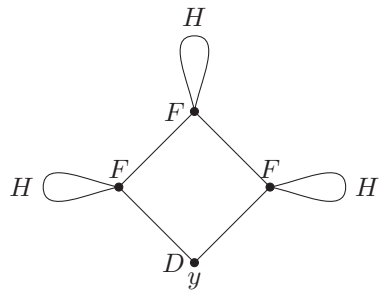

Figure 3.3.3: The gadget H' with an $(\mathcal{F}, \mathcal{F}_d)$-partition.

Observe that we can make a gadget H' with a vertex y that admits an $(\mathcal{F}, \mathcal{F}_d)$-partition, and such that y is in D for all $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of H' (see Figure 3.3.3): we take three copies of H, and make a 4-cycle with the corresponding copies of x and a new vertex y. Taking an $(\mathcal{F}, \mathcal{F}_d)$-partition of H for each copy of H, and adding y to D leads to an $(\mathcal{F}, \mathcal{F}_d)$-partition of H'. Conversely, in an $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of H', all the copies of x are in F, so y is in D.

We will first make a reduction from the problem Planar 3-sat to P_0, and then from P_0 to P_d with $d < d_0$.

First reduction: from Planar 3-sat to P_0

Here we will use the gadget H for $d = 0$.

Consider an instance I of Planar 3-sat. The instance I is a boolean formula in conjunctive normal form, associated with a planar graph G_I. For each clause C of I with variables x, y and z, we make a

4-cycle $K_C = x_C y_C z_C a_C$. For each variable x that appears k_x times in the formula, we make the following gadget G_x: a path $p_{x,0}...p_{x,2k_x-1}$, and for all $i \in [0, 2k_x - 2]$, we add two adjacent vertices, $q_{x,i}$ and $r_{x,i+1}$, adjacent to $p_{x,i}$ and $p_{x,i+1}$ respectively (see Figure 3.3.5). We then add a copy of H for each clause C such that a_C corresponds to the vertex x of H, and a copy of H for each $q_{x,i}$ and each $r_{x,i}$ such that $q_{x,i}$ and $r_{x,i}$ respectively correspond to the vertex x of H. Then for every clause C and every variable x that appears in C, we add an edge from x_C to a $p_{x,i}$, with an even i if the literal associated with x in C is a positive literal and an odd i otherwise, such that no two x_C are adjacent to the same $p_{x,i}$ (see Figure 3.3.4). It is possible to do so without breaking planarity, since the graph G_I is planar. We call G'_I the graph we obtain.

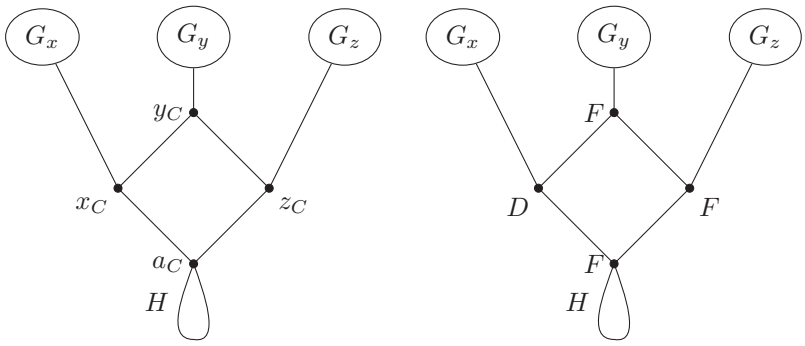

Figure 3.3.4: The cycle K_C of a clause C with variables x, y and z, and an $(\mathcal{F}, \mathcal{F}_d)$-partition in the case where variable x satisfies the clause.

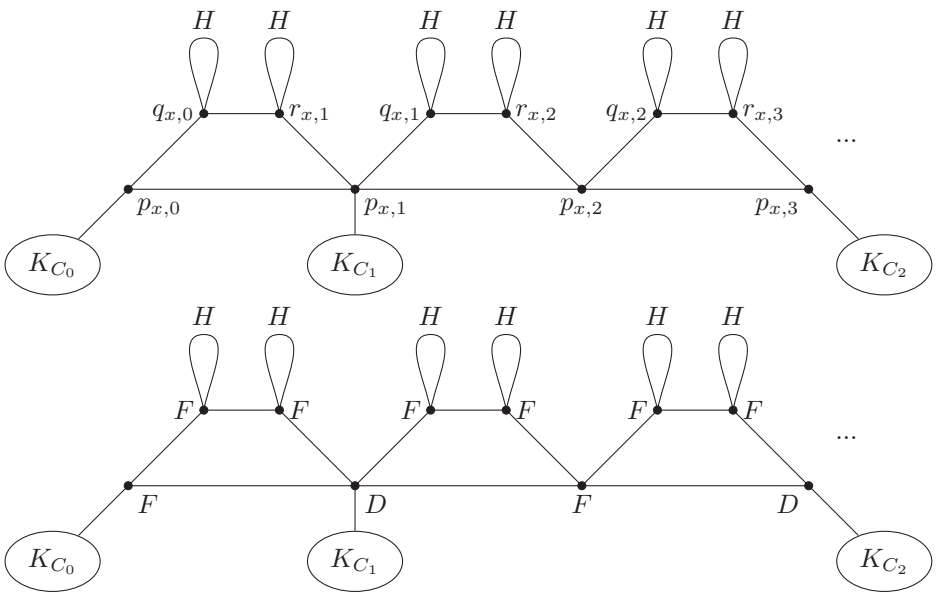

Figure 3.3.5: The gadget G_x for a variable x, with an $(\mathcal{F}, \mathcal{F}_d)$-partition that corresponds to the assignation of x to true. Here the literal associated with x in C_0 is positive, and that associated with x in C_1 and C_2 is negative.

Suppose I is satisfiable, and let us consider an assignation σ of the variables that satisfies I. Let us make an $(\mathcal{F}, \mathcal{F}_0)$-partition of G'_I. We first take an $(\mathcal{F}, \mathcal{F}_0)$-partition for each copy of H. All the a_C, $q_{x,i}$ and $r_{x,i}$ are in F. For each variable x, if $\sigma(x) = 1$, then we put all the $p_{x,2i}$ in F and the $p_{x,2i+1}$ in D, else we put all

the $p_{x,2i}$ in D and the $p_{x,2i+1}$ in F. Then for each clause C, we choose a variable x of C that satisfies the clause (i.e. x is true if the literal associated with x in C is a positive literal, and false otherwise), we put x_C in D and for the two other variables of C, we put the corresponding y_C in F.

All the vertices are in F or in D. Let v be a G'_I-vertex in D. If v is in a copy of H, then it has no D-neighbour. If v is a x_C, then the three other vertices of K_C are in F. If v is a $p_{x,i}$, then $p_{x,i+1}$ and $p_{x,i-1}$ are in F if they exist, and all the q_j and r_j are in F. Suppose there are two $G'_I[F]$-neighbours in D. One is a x_C and the other is a $p_{x,i}$ (with the same x). Then by construction the variable x satisfies clause C (i.e. x is true if the literal associated with x in C is a positive literal, and false otherwise). If x is associated with a positive literal in clause C, then $\sigma(x) = 1$ and i is even, thus $p_{x,i}$ is in F, a contradiction. If x is associated with a negative literal in clause C, then $\sigma(x) = 0$ and i is odd, thus $p_{x,i}$ is in F, a contradiction. Graph $G'_I[F]$ has no cycle: there is no cycle in the copies of H with every vertex in F; for each clause C, K_C has a vertex in D, and for each $i \in [0, 2k_x - 2]$, $p_{x,2i}$ or $p_{x,2i+1}$ is in D. Therefore (F, D) is an $(\mathcal{F}, \mathcal{F}_0)$-partition of G'_I.

Suppose now that there is an $(\mathcal{F}, \mathcal{F}_0)$-partition (F, D) of G'_I. All the a_C, the $q_{x,i}$ and the $r_{x,i}$ are in F. For all variable x and all $i \in [0, 2k_x - 2]$, either $p_{x,i} \in F$ and $p_{x,i+1} \in D$, or $p_{x,i} \in D$ and $p_{x,i+1} \in F$. Therefore for all x, either all the $p_{x,i}$ are in F for i even and in D for i odd, or all the $p_{x,i}$ are in D for i even and in F for i odd. Let σ be the assignation of the variables x such that $\sigma(x) = 1$ if $p_{x,0}$ is in F, and $\sigma(x) = 0$ otherwise. Let C be a clause of I. At least one of the x_C is in D (otherwise K_C is a cycle with every vertex in F), and it is adjacent to a $p_{x,i}$ with i even if x is positive and i odd if x is negative in C. This $p_{x,i}$ is in F, so if x is positive in C, then $\sigma(x) = 1$, else $\sigma(x) = 0$. Therefore σ satisfies clause C, and this is true for all C, so σ satisfies I.

It is easy to see that the reduction is polynomial, and that G'_I is a triangle-free planar graph. Thus this is a polynomial reduction from Planar 3-sat to P_0.

Second reduction: from P_0 to P_d with $d < d_0$

Consider an instance I of P_0. For each vertex v in I, add d copies of H', such that the corresponding copies of y are adjacent to v. We call I_d the resulting graph.

Suppose I admits an $(\mathcal{F}, \mathcal{F}_0)$-partition. Consider an $(\mathcal{F}, \mathcal{F}_d)$-partition of H'. Apply it to every copy of H' we made in I_d. Complete it with an $(\mathcal{F}, \mathcal{F}_0)$-partition of I. The obtained partition is an $(\mathcal{F}, \mathcal{F}_d)$-partition of I_d.

Suppose now that I_d admits an $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D). In each copy of H', we have $y \in D$, so each vertex in I has exactly d $(I_d - V(I))$-neighbours in D and no $(I_d - V(I))$-neighbours in F. Therefore $(F \cap V(I), D \cap V(I))$ is an $(\mathcal{F}, \mathcal{F}_0)$-partition of I.

It is easy to see that the reduction is polynomial, and that I_d is a triangle-free planar graph. Thus this is a polynomial reduction from P_0 to P_d.

3.4 $(\mathcal{I}, \mathcal{F}_d)$-partition of graphs with maximum average degree less than to 3

Let us first recall the statement of Theorem 3.1.12.

Theorem 3.1.12 (recall)**.** *Let M be a real number such that $M < 3$. Let $d \geq 0$ be an integer and let G be a graph with* $\mathrm{mad}(G) < M$*. If $d \geq \frac{2}{3-M} - 2$, then G admits an $(\mathcal{I}, \mathcal{F}_d)$-partition.*

This section and the following one will use the same kind of proof, the second one being more involved. In particular, some of the lemmas of this section will be used in the following one. This is again a good example of an application of the discharging method. Additionally, besides finite configurations that appear in previous sections, here we will use some families of configurations that can be arbitrarily large. Because of this, the proof does not immediately yield a polynomial time algorithm for finding an $(\mathcal{I}, \mathcal{F}_d)$-partition. We will see, however, that there is still a polynomial time algorithm.

Let $M < 3$, and let d be an integer such that $d \geq \frac{2}{3-M} - 2$. Let us first define a notion that will enable us to deal with 2-vertices. Let us call a *good* d-partition of a graph G a partition (I, F) of the vertices of G such that I is an independent set of G, $G[F]$ is a graph with maximum degree at most d, and every cycle in $G[F]$ contains a 2-vertex of G. Note that for any graph G, if G admits a good d-partition, then G admits an $(\mathcal{I}, \mathcal{F}_d)$-partition: while there is a vertex v with degree 2 in G that is in F and has two neighbours in F, move v from F to I. Therefore Theorem 3.1.12 is implied by the following lemma:

Lemma 3.4.1. *Every graph G with $\mathrm{mad}(G) < M$ has a good d-partition.*

For the sake of contradiction, assume that Lemma 3.4.1 is false. Let G be a counter-example to Lemma 3.4.1 with minimum order.

A $(d+1)^-$-vertex of G is a *small* vertex, and a $(d+2)^+$-vertex of G is a *big* vertex. Let v be a vertex of G. A neighbour of v that is a big vertex is a *big neighbour* of v, and a neighbour of v that is a small vertex is a *small neighbour* of v. We start by proving some lemmas on the structure of G. Specifically, we prove that some configurations are reducible, and thus cannot occur in G.

Lemma 3.4.2. *There is no 1^--vertex in G.*

Proof. Assume there is a 1^--vertex v in G. The graph $G - v$ has one fewer vertex than G, and thus, by minimality of G, admits a good d-partition (I, F). If v has no neighbours in I, then we can add it to I. Otherwise, it has no neighbours in F, and we can add it to F. In both cases, that leads to a good d-partition of G, a contradiction. □

Lemma 3.4.3. *Every 2-vertex of G has at least one big neighbour.*

Proof. Assume v is a 2-vertex adjacent to two small vertices, u and w. The graph $G - v$ has one fewer vertex than G, and thus, by minimality of G, admits a good d-partition (I, F). If u and w are both in F, then we can put v in I, and if they are both in I, then we can put v in F. Therefore without loss of generality, we can assume that $u \in I$ and $w \in F$. If w has no neighbours in I, then we can put it in I, and put v in F. Therefore we can assume that w has at least one neighbour in I and thus at most $d - 1$ neighbours in F (since w is a small vertex in G). Then v has at most one F-neighbour, and this neighbour w has at most $d - 1$ neighbours in $G[F]$, thus we can add v to F. In every case, this leads to a good d-partition of G, a contradiction. □

We are going to build a structure on the graph G that we will call the *light forest*, and that will enable us to state some more lemmas. To define this light forest, we need several notions, that we are now going to define. A 2-vertex is a *light 2-vertex* if it is adjacent to a small vertex, and it is a *heavy 2-vertex* otherwise. Note that by Lemma 3.4.3, each 2-vertex has at most one small neighbour.

Lemma 3.4.4. *Let B be a set of small 3^+-vertices such that $G[B]$ is a tree. There exists a 3^+-vertex $v \notin B$ that is adjacent to a vertex of B.*

Proof. Assume the lemma is false, that is every vertex that is not in B but has a B-neighbour is a 2-vertex in G. By minimality of G, $G - B$ admits a good d-partition (I, F). For every vertex v in B, successively, we put v in I if v has no I-neighbours and we put it in F otherwise. Note that this way a vertex that we add to F has at most d neighbours that are not in I, and we cannot make any cycle in $G[F]$ that does not go through a 2-vertex of G, since $G[B]$ is a tree. Thus we have a good d-partition of G, a contradiction. □

Let B be a (maximal) set of small 3^+-vertices of G such that:

(a) $G[B]$ is a tree,

(b) there is only one edge that links a vertex of B to a 3^+-vertex u outside of B,

(c) u is a big vertex.

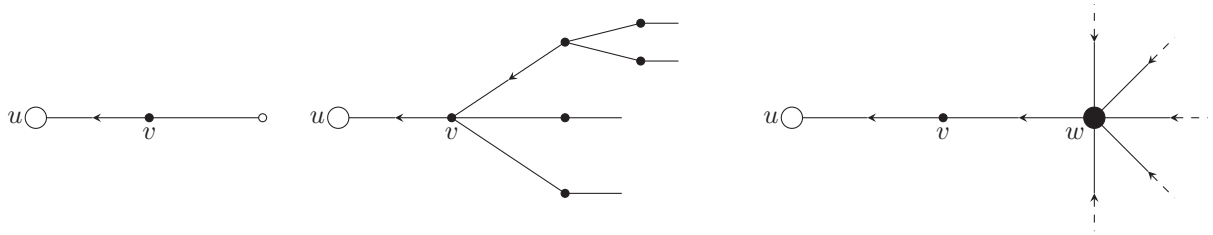

Figure 3.4.1: The construction of the light forest L. The big vertices are represented with big circles, and the small vertices with small circles. The filled circles represent vertices whose incident edges are all represented. The dashed lines are the continuation of the light forest. The arrows point from son to father in L.

We call B a *bud* with father u.

We will now build the light forest inductively on the graph G. We notice that although the light forest can be arbitrarily large, finding it can be done in polynomial time due to its inductive nature (just follow the following three steps, each iteration of a step adding at least one vertex to the light forest).

We note that at this point, the light forest is not necessarily unique, since we can choose the order in which the vertices are treated. We do not need the construction to be uniquely defined. We just build one instance of the light forest, and we will always use the same one.

We start with $L = (\emptyset, \emptyset)$, the graph with no vertices and no edges.

1. While there is a light 2-vertex that is not in L, do the following. Pick a light 2-vertex v, and let u be the big neighbour of v (that exists by Lemma 3.4.3). Add to L the vertex v, the edge uv, and the vertex u (if it is not already in L). Also set that u is the *father* of v (and v is a *son* of u). See Figure 3.4.1, left. Note that by doing this, we obtain a star forest with only big vertices and light 2-vertices. Also note that the set of the big vertices and the set of the light 2-vertices are independent sets in L (but not necessarily in G).

2. While there are buds whose vertices are not all included in L, do the following. Pick a bud B. Let u be the father of B, and let v be the vertex of B adjacent to u. Add $G[B]$ to L, as well as the edge uv, and the vertex u (if it is not already in L). The vertex u is the father of v, and the father/son relationship in B is that of the tree $G[B]$ rooted at v. See Figure 3.4.1, middle. We recall that the buds do not share vertices. Since we add the vertices of the buds to L all in one go, in the end all the vertices of the buds are in L. Note that each iteration of this step always adds to L a tree with at most one vertex (u) that was already in L. Hence, L is still a forest. Moreover, it is still a rooted forest since the tree we add has the orientation of a tree rooted at u.

3. While, for some k, there exists a big k-vertex $w \in L$ that has $k-1$ sons in L and a 2-neighbour v that is not in L, do the following. Let u be the neighbour of v distinct from w. Note that v is a heavy 2-vertex (since it was not added to L in Step 1), therefore u is a big vertex. Add to L the vertex v, the edges uv and vw, and the vertex u (if it is not already in L). We set that v is the father of w, and that u is the father of v. See Figure 3.4.1, right. Note that this operation just takes a root of L, adds a vertex as a father of this root and another vertex as a father of that new vertex. Therefore, L is still a rooted forest. Also note that each of the set of the big vertices and the set of the 2-vertices remains independent in L.

As noticed previously, L is a rooted forest. We say that a vertex v is a *descendant* of a vertex $u \neq v$ in L if there are vertices $v_0 = v$, v_1, ..., $v_k = u$ in L, such that for $i \in \{0, 1, ..., k-1\}$, v_{i+1} is the father of v_i in L. Consider a vertex v in L. If v is a heavy 2-vertex, then v was added in Step 3, and its two incident edges were added at the same time. If v is a big vertex and is not the root of its component in L, then the father of v was added in Step 3, thus all the neighbours of v distinct from its father are its sons in L. If v is a small 3^+-vertex in L, then v was added in Step 2 and thus is in a bud. Therefore, a vertex v in L is

incident to an edge that is not in L only if either v is a big vertex and the root of its component in L, or v is a light 2-vertex, or v is in a bud. The *pending vertices* of L are the vertices that are not in L but are adjacent to a light 2-vertex. Note that the pending vertices are small.

Now we would like to prove that the light forest respects some properties, by proving that some configurations are reducible. To prove this, we will need to add some whole parts of the light forest to a good d-partition. Let us develop some ways to do that.

Let B be a bud with father u. Let $S \subseteq V(G) \setminus (B \cup \{u\})$ and let (I, F) be a good d-partition of $S \cup \{u\}$ such that u either is in I or has at most $d-1$ neighbours in F. We show that we can extend the good d-partition to $S \cup \{u\} \cup B$. We proceed as follows: for every vertex $v \in B$ successively, we add v to I if it has no neighbours in I or to F otherwise. The vertices in I clearly form an independent set. Moreover, $G[F]$ has maximum degree at most d and every cycle of $G[F]$ goes through a 2-vertex by construction of a bud. This leads to a good d-partition of $S \cup B \cup \{u\}$. We call that process *colouring the bud B*.

Let v be a 2-vertex of L, u its father and D_v the set of the descendants of v. Let (I, F) be a good d-partition of $S \subseteq V(G) \setminus (D_v \cup \{u, v\})$. We show that we can extend the good d-partition to $S \cup D_v \cup \{v\}$. We proceed as follows:

Step 1. We add every big vertex of D_v to I. We can do this, since big vertices form an independent set in L by construction, and as we noted previously, big vertices that are in L and are not the root of their component (in particular big vertices that are in D_v) have no incident edge outside of L.

Step 2. While there is a pending vertex $w \in S$ that has no neighbours in I, we add one such vertex to I.

Step 3. We add every 2-vertex of D_v and v to F. We can do that, since the 2-vertices of D_v form a stable set in L. Moreover, Step 2 ensures that the maximum degree of $G[F]$ is at most d.

Step 4. Finally, we colour every bud. We can do that, since the father of every bud whose vertices are in D_v has been put in I in Step 1.

This leads to a good d-partition of $S \cup D_v \cup \{v\}$. We call that process *descending v*.

We will now use the previous construction to prove the following lemma on the structure of the Light forest.

Lemma 3.4.5. *For all k, there is no big k-vertex in G that is in L and has k sons in L.*

Proof. Let u be a big k-vertex that has k sons in L. Note that this implies that u is the root of its component in L. Let C be the component of u in L. Let $H = G - V(C)$. The graph H has fewer vertices than G and thus, by minimality of G, H admits a good d-partition (I, F). Let N be the set of the 2-neighbours of u. We descend every vertex of N. Note that this implies that every vertex of N is put in F. We add u to I. Then we colour every bud of father u. That leads to a good d-partition of G, a contradiction. □

Discharging procedure

Let $\epsilon = 3 - M$. Recall that $d \geq \frac{2}{3-M} - 2 = \frac{2}{\epsilon} - 2$, therefore $\epsilon \geq \frac{2}{d+2} > 0$. For all k, we assign to each k-vertex a charge equal to $k - M = k - 3 + \epsilon$. Note that since M is bigger than the average degree of G, the sum of the charges of the vertices is negative. The initial charge of each 3^+-vertex is at least ϵ, and thus is positive.

For every big vertex v, v gives charge $1 - \epsilon$ to each of its 2-neighbours that are its sons in L, does not give anything to its father in L (if it has one), and gives $\frac{1-\epsilon}{2}$ to its other 2-neighbours.

Lemma 3.4.6. *Every vertex has non-negative charge at the end of the procedure.*

Proof. The small 3^+-vertices start with a non-negative charge, and do not give or receive charge throughout the procedure, thus they have non-negative charge at the end of the procedure.

Every 2-vertex is either in L, in which case it receives $1 - \epsilon$ from its father in L, or is not in L and is a heavy 2-vertex, in which case it receives $\frac{1-\epsilon}{2}$ from each of its neighbours. As 2-vertices have charge $\epsilon - 1$ at the start of the procedure, and as they receive $1 - \epsilon$, they have charge 0 at the end of the procedure.

Let v be a big k-vertex. By Lemma 3.4.5, v has at most $k-1$ sons in L. Moreover, by construction of L, if v has $k-1$ sons in L, then either its neighbour that is not its son in L is a 3^+-vertex, or it is the father of v in L (and in both cases v does not give charge to this vertex). Therefore v gives charge amounting to at most $(k-1)(1-\epsilon)$. Since its initial charge is $k-3+\epsilon$, in the end it has at least $k-3+\epsilon-(k-1)(1-\epsilon) = k\epsilon-2$. Since every big vertex has degree at least $d+2 \geq \frac{2}{\epsilon}$, the final charge of each big vertex is at least $\frac{2}{\epsilon}\epsilon-2 = 2-2 = 0$. □

By Lemma 3.4.6, every vertex has non-negative charge at the end of the procedure, thus the sum of the charges at the end of the procedure is non-negative. Since no charge was created nor removed, this is a contradiction with the fact that the initial sum of the charges is negative. That ends the proof of Lemma 3.4.1.

3.5 $(\mathcal{I}, \mathcal{F}_d)$-partition of graphs with maximum average at least $\frac{8}{3}$ and less than 3

Let us first recall the statement of Theorem 3.1.13.

Theorem 3.1.13 (recall)**.** *Let M be a real number such that $\frac{8}{3} \leq M < 3$. Let $d \geq 0$ be an integer and let G be a graph with* $\mathrm{mad}(G) < M$*. If $d \geq \frac{1}{3-M}$, then G admits an $(\mathcal{I}, \mathcal{F}_d)$-partition.*

This proof is quite similar to the proof of Theorem 3.1.12 above, but is a bit more involved. Let $\frac{8}{3} \leq M < 3$, and let d be an integer such that $d \geq \frac{1}{3-M}$. We define good d-partitions as in Section 3.4. Theorem 3.1.13 is implied by the following lemma:

Lemma 3.5.1. *Every graph G with* $\mathrm{mad}(G) < M$ *has a good d-partition.*

For the sake of contradiction, assume that Lemma 3.5.1 is false. Let G be a counter-example to Lemma 3.5.1 with minimum order.

We take the same definitions as before. Lemmas 3.4.2–3.4.5 of the previous section still hold in this setting.

In this section, we will not only consider components of the light forest independently, we will consider several of those components together to get some more charge. To explain our method, let us first consider a small example.

Assume that in G there is a cycle of length 6 $v_1v_2v_3v_4v_5v_6$ of vertices that are all in $V(L)$, such that

- v_1 and v_4 are $(d+2)$-vertices, and each of them either has $d+1$ sons in L or has d sons in L. If one of them has d sons in L, then all of its neighbours are 2-vertices.
- v_2 and v_6 are 2-vertices of G that are sons of v_1.
- v_3 and v_5 are 2-vertices of G that are sons of v_4.

Note that this corresponds to an instance where the weights of the v_i's are exactly 0 at the end of the discharging procedure of the previous section. Let $H = G - (D_{v_1} \cup D_{v_4} \cup \{v_1, v_4\})$. As H has fewer vertices than G, it has a good d-partition (I, F). We descend every son of v_1 and every son of v_4. This implies that every son of v_1 and v_4 is in F.

If v_1 and v_4 have no neighbours in I, then put them both in I. If one of them, say v_1, has a neighbour in I and the other one has no neighbour in I, then put v_1 in F, v_2 and v_6 in I, and v_4 in I. If v_1 and v_4 both have a neighbour in I, then put them both in F, and put v_2 and v_5 in I. In all cases, each of v_1 and v_4 that is in F has at least two neighbours in I, and thus at most d neighbours that are not in I. We colour every bud of father u. That leads to a good d-partition of G, a contradiction.

In what follows, we are going to generalise the previous reasoning to more general cases, where the big vertices considered may be farther away than v_1 and v_4, and may have degree more than $d+2$. To do this, we will consider a subgraph of G induced by vertices that belong to the light forest, together with an orientation of this subgraph that will enable us to distribute the charges. Frank and Gyárfás [34] prove the following theorem:

Theorem 3.5.2 (Frank and Gyárfás [34]). *Let $H = (V, E)$ be a graph, and let $\omega : V \to \mathbb{N}$. There exists an orientation such that $\forall v \in V, d^+(v) \geq \omega(v)$ if and only if for all $X \subset V$, $\omega(X) \leq |\{uv \in E, u \in X\}|$.*

Given $H = (V, E)$ and $\omega : V \to \mathbb{N}$, a *good ω-orientation* of H is an orientation of H such that $\forall v \in V, d^+(v) \geq \omega(v)$. We prove some additional lemmas.

Lemma 3.5.3. *Let $H = (V, E)$ be a graph on $n \geq 1$ vertices and m edges. Let $\omega : V \to \mathbb{N}$ such that $\omega(V) \leq m$. There exists a subgraph S of H with at least one vertex such that S admits a good ω-orientation.*

Proof. For a graph I and a set $X \subset V(I)$, let $e_I(X) = |\{uv \in E(I), u \in X\}|$. If $\omega(X) \leq e_I(X)$, we say that X is *good* in I.

If every subset of V is good in H, then by Theorem 3.5.2, we have a good ω-orientation of H. Therefore we may assume that there is a subset of V that is not good in H. Let X be a maximum subset of V that is not good in H. Let $Y = V - X$, and let $H' = H[Y]$. Note that V is good in H, since $\omega(V) \leq m$, so $Y \neq \emptyset$.

If every subset of Y is good in H', then by Theorem 3.5.2, we have a good ω-orientation of H'. Therefore there is a $Z \subset Y$ such that Z is not good in H', i.e. $\omega(Z) > e_{H'}(Z)$. As X is not good in H, we also have $\omega(X) > e_H(X)$. Therefore we have $\omega(X \cup Z) = \omega(X) + \omega(Z) > e_H(X) + e_{H'}(Z) = |\{uv \in E(H), u \in X\}| + |\{uv \in E(H'), u \in Z\}| = |\{uv \in E(H), u \in (X \cup Z)\}| = e_H(X \cup Z)$. Therefore $X \cup Z$ is not good in H, which contradicts the maximality of X. □

We recall that L is the light forest of G.

Lemma 3.5.4. *Let U be a non-empty subset of $V(L)$ with no small 3^+-vertices. Let $H = G[U]$ (i.e. the subgraph of G induced by the 2-vertices and the big vertices of $U \subseteq L$). Suppose:*

1. *There is an orientation of the edges of H such that every 2-vertex in H has at least one out-going edge, and for all $i \geq 1$, every big $(d + i + 1)$-vertex in G has at least i out-going edges.*
2. *There are no 1^--vertices in H.*

Then H contains an edge that is not in L and that is incident to a big vertex.

The graph H of Lemma 3.5.4 is as follows: it is composed by subtrees of L plus some additional edges (that do not belong to L). Such edges are edges between light 2-vertices, and maybe edges between roots of trees of L. The aim of Lemma 3.5.4 is to prove the existence of such latter edges.

The orientation in Lemma 3.5.4 does not correspond to the orientation defined by the father/son relation in the light forest. This orientation will allow us to extend a partial partition (I, F): consider a big $(d+i+1)$-vertex v being in F. Vertex v must have at least $i + 1$ neighbours in I. The orientation will point towards i sons of v that will be added to I. Moreover we will see that v will have one extra neighbour in I: either its father in L, or a neighbour outside L.

Proof of Lemma 3.5.4. Assume the lemma is false: every edge of H that is not in L is between two 2-vertices. Let R_0 be the set of the vertices of H that are not the descendants in L of a vertex of H. In particular, R_0 contains the roots of L that are in H, plus big vertices that have no ancestor in U. Note that R_0 contains only big vertices ; otherwise, H would contain 1-vertices. Moreover, $H - R_0$ has at least one vertex, otherwise U would contain only big vertices, there would be an edge between two big vertices, and this edge could not be in L. Let S be the set of the vertices that are not in H, but are descendants of vertices of H.

By minimality of G, the graph $G - (V(H - R_0) \cup S)$ admits a good d-partition (I, F). While there is a vertex $v \in R_0$ that is in F and has no neighbours in I, we put v in I. Now we can assume that every vertex in $R_0 \cap F$ has a neighbour in I. Let $R = R_0$ (in the following we describe a procedure that modifies R but we need to refer to vertices of R_0). While there is a vertex in R, do the following:

- Suppose u is in I. We descend every 2-vertex with father u (by the procedure every 2-vertex is added to F) and colour every bud with father u. This leads to a good d-partition of u and all its descendants.

- Suppose u is in F. We remove u from R. For every 2-vertex v in H with father u such that the edge uv is oriented from u to v (according to the orientation defined in the statement of the lemma), we first add v to I, and then add the son of v to F and R. By hypothesis, if u is a $(d+i+1)$-vertex, then it has at least i outgoing edges. These edges lead to sons of u:
  - either $u \in R_0$, thus u has no ancestor in H by construction and all its neighbours in H are its sons ; moreover it has a neighbour outside H that is in I (by construction).
  - or, $u \in R \setminus R_0$, this means that u was added to R during the procedure, this implies that his father, say w, is a 2-vertex added to I and the edge wv is oriented from w to v (as every 2-vertex has an out-going edge by hypothesis). It follows that all the out-going neighbours of u are sons of u.

  It follows that u has at least $i+1$ neighbours in I, and so all other neighbours can be added to F without violating the degree condition on F. Now we descend every 2-vertex $v \notin H$ with father u, and every 2-vertex $v \in H$ with father u such that the edge uv is oriented from v to u, and colour every bud with father u. The only problem that could occur is when two adjacent light 2-vertices ℓ and ℓ' are added to I: in that case, since ℓ and ℓ' were added to I, the edge that links ℓ (resp. ℓ') to its father is towards ℓ (resp. ℓ'); it follows that one of ℓ, ℓ' has no out-going edges, contradicting the hypothesis.

In all cases, that leads to a good d-partition of G, a contradiction. □

Lemma 3.5.5. *Let U be a non-empty subset of $V(L)$ with no small 3^+-vertices. Let $H = G[U]$ (i.e. the subgraph of G induced by the 2-vertices and the big vertices of $U \subseteq L$). Suppose that H has no edge linking two roots of two components of L. Let us denote by $n_2^G(H)$ the number of vertices of H that are 2-vertices in G. Then,*

$$|E(H)| < n_2^G(H) + \sum_{big\ v \in H} (d_G(v) - d - 1)\,.$$

Proof. By contradiction, suppose there is such an H such that $|E(H)| \geq n_2^G(H) + \sum_{big\ v\in H}(d_G(v) - d - 1)$. Let us define a weight function $\omega : V(H) \to \mathbb{N}$ such that, for every 2-vertex u, $\omega(u) = 1$ and, for every $(d+i+1)$-vertex v with $i \geq 1$ in $\mathbb{N}$, $\omega(v) = i$. By hypothesis, $|E(H)| \geq \sum_{v\in V(H)} \omega(v)$. By Lemma 3.5.3, H contains a non empty subgraph I that has a good ω-orientation.

Suppose that I has a vertex of degree 1, say v, and let u be the neighbour of v in I. As $\omega(v) \geq 1$, the only edge incident to v goes from v to u. It follows that, for all $w \neq v$, w has the same number of outgoing edges in I and in $I - \{v\}$. Hence $I - \{v\}$ is a subgraph of H with at least one vertex (it contains u) and has a good ω-orientation. By successively removing vertices of degree 1 from I, we can assume that I has no 1-vertices.

By Lemma 3.5.4, I has an edge e that is not in L and is incident to a big vertex. As no light 2-vertex is adjacent to a big vertex besides its father, edge e has to link the roots of two connected components of L, contradicting the hypothesis. □

Let $\widehat{L}$ be the graph induced by $V(L)$, where we remove every edge that links the roots of two components of L and we remove every bud. An *internal* 2-vertex is a 2-vertex in $\widehat{L}$ that has its two neighbours in $\widehat{L}$. By applying Lemma 3.5.5 to $\widehat{L}$, we can bound the number of internal 2-vertices in $\widehat{L}$. We obtain the following lemma:

Lemma 3.5.6. *The number of internal 2-vertices is at most $2\sum_{big\ v\in H}(d_G(v) - d - 1)$.*

Proof. Let $\widehat{L}'$ be the graph $\widehat{L}$ where every 2-vertex is removed in the following way: if v is an internal 2-vertex with neighbours u and w, then we remove v and add an edge from u to w, and we iterate. For the 2-vertices that are not internal, we just remove them. Note that $\widehat{L}'$ may have multiple edges and even loops. As for each 2-vertex that was removed, exactly one edge was removed, the number of edges in $\widehat{L}'$ is at most $\sum_{big\ v\in H}(d_G(v) - d - 1)$. By Lemma 3.4.3, every edge of $\widehat{L}'$ corresponds to at most two internal 2-vertices. Therefore there are at most $2\sum_{big\ v\in H}(d_G(v) - d - 1)$ internal 2-vertices. □

Discharging procedure

Let $\epsilon = 3 - M$ (recall that $\frac{8}{3} \leq M \leq 3$). Recall that $d \geq \frac{1}{3-M} = \frac{1}{\epsilon}$, therefore $\epsilon \geq \frac{1}{d} > 0$.

We assign to each k-vertex a charge equal to $k - M = k - 3 + \epsilon$. Note that since M is bigger than the average degree of G, the sum of the charges of the vertices is negative.

Every 3^+-vertex has a charge of at least $\epsilon > 0$. Therefore every vertex that has a negative charge is a 2-vertex and has charge $\epsilon - 1$. We will redistribute the weight from the 3^+-vertices to the 2-vertices, in order to obtain a non-negative weight on each vertex, by the following three steps:

- *Step 1*: Let S be a maximal set of small 3^+-vertices such that $G[S]$ is connected. Let S_2 be a set of 2-vertices that have exactly one (by Lemma 3.4.3) neighbour in S. Note that since $\epsilon \leq 1$, every k-vertex in S has charge at least $(k-2)\epsilon$. The vertices in S give ϵ to each of the vertices in S_2.

  Suppose that the total charge of S becomes negative. This implies that the number of vertices in S_2 is more than $\sum_{v \in S}(d(v) - 2)$. Therefore there are at most $|S| - 1$ edges in $G[S]$. Since $G[S]$ is connected, this implies that $G[S]$ is a tree. Now by Lemma 3.4.4, there is at least one big vertex outside of S that has a neighbour in S. Note that if there are at least two of these vertices, or if one of them has at least two neighbours in S, then one can observe that $G[S]$ has at most $|S| - 2$ edges, contradicting the connectivity of $G[S]$. Therefore S is a bud. In this case, S ends up with a charge of at least $-\epsilon$. We will, in Step 2, make sure that every son of a big vertex in L receives at least $1 - 2\epsilon \geq \epsilon$ (since $\epsilon \leq \frac{1}{3}$) from its father, and this will ensure that every bud ends up with a non-negative charge.

  We do this step for every maximal set S of small 3^+-vertices such that $G[S]$ is connected successively. Note that such sets are distinct.

- *Step 2*: For every big vertex v, v gives $1 - 2\epsilon$ to each of its sons, does not give anything to its father (if it has one), and gives $\frac{1-\epsilon}{2}$ to its other 2-neighbours. Additionally, every big k-vertex gives $2(k-d-1)\epsilon$ to a common pot.

- *Step 3*: The common pot gives ϵ to every internal 2-vertex.

Lemma 3.5.7. *Every vertex has non-negative charge at the end of the procedure.*

Proof. Note that by what precedes every small 3^+-vertex v has non-negative charge.

Every 2-vertex that is not in L receives $\frac{1-\epsilon}{2}$ from each of its neighbours. Every light 2-vertex that is not adjacent to a 2-vertex (i.e. every 2-vertex of L that is not an internal 2-vertex) receives $1 - 2\epsilon$ from its father and ϵ from its other neighbour (which is a small 3^+-vertex). Every internal 2-vertex receives $1 - 2\epsilon$ from its father and ϵ from the common pot. Therefore every 2-vertex has charge 0 at the end of the procedure.

Let us prove that every big vertex has non-negative charge at the end of the procedure. Let v be a big k-vertex. Let $c(v)$ be the initial charge of v, and $c'(v)$ be the final charge of v. Suppose by contradiction that $c'(v) < 0$. By Lemma 3.4.5, vertex v has at most $k - 1$ sons. Moreover, if v has $k - 1$ sons, then its last neighbour is either the father of v, or a 3^+-vertex (by construction of L). Recall that $\epsilon \leq \frac{1}{3}$, therefore $1 - 2\epsilon \geq \frac{1-2\epsilon}{2}$. If v has $k - 1$ sons, then v gives $1 - 2\epsilon$ to each of its $k - 1$ sons, and $2(k - d - 1)\epsilon$ to the common pot, therefore $c'(v) = c(v) - (1 - 2\epsilon)(k - 1) + 2(k - d - 1)\epsilon$, and thus as $c'(v) < 0$, we have $c(v) < (1 - 2\epsilon)(k - 2) + 1 - \epsilon + 2(k - d - 1)\epsilon$. If v has $k - 2$ sons, then it gives $1 - 2\epsilon$ to each of its $k - 2$ sons, and may give at most $\frac{1-\epsilon}{2}$ to its two other neighbours and $2(k - d - 1)\epsilon$ to the common pot, therefore $c(v) < (1 - 2\epsilon)(k - 2) + 1 - \epsilon + 2(k - d - 1)\epsilon$. If we decrease the number of sons of v further than $k - 2$, we will still have $c(v) < (1 - 2\epsilon)(k - 2) + 1 - \epsilon + 2(k - d - 1)\epsilon$.

Thus if $c(v) \geq (1 - 2\epsilon)(k - 2) + 1 - \epsilon + 2(k - d - 1)\epsilon$, we get a contradiction. Recall that $c(v)$ is equal to $k - 3 + \epsilon$. Therefore we only need to prove that $k - 3 + \epsilon \geq (1 - 2\epsilon)(k - 2) + 1 - \epsilon + 2(k - d - 1)\epsilon$, which is equivalent to $d \geq \frac{1}{\epsilon}$. □

Let us prove that the common pot also has non-negative charge at the end of the procedure. It receives charge $\sum_{v \text{ big}} 2(d(v) - d - 1)\epsilon$. By Lemma 3.5.6, this charge is at least ϵ times the number of internal 2-vertices. The common pot gives ϵ to each internal 2-vertex, therefore it has non-negative charge at the end of the procedure.

By Lemma 3.5.7, every vertex has non-negative charge at the end of the procedure, thus the sum of the charges at the end of the procedure is non-negative. Since no charge was created nor removed, and since the common pot also has non-negative charge, this is a contradiction with the fact that the initial sum of the charges is negative. That ends the proof of Theorem 3.1.13.

3.6 Conclusion

We proved some results on vertex partitions of sparse graphs into independent sets, forests, and forests of bounded degree. Namely, we proved the following results.

Theorem 3.1.4 (recall)**.** *Every triangle-free planar graph admits an* $(\mathcal{F}, \mathcal{F}_5)$*-partition.*

Theorem 3.1.12 (recall)**.** *Let* M *be a real number such that* $M < 3$*. Let* $d \geq 0$ *be an integer and let* G *be a graph with* $\mathrm{mad}(G) < M$*. If* $d \geq \frac{2}{3-M} - 2$*, then* G *admits an* $(\mathcal{I}, \mathcal{F}_d)$*-partition.*

Theorem 3.1.13 (recall)**.** *Let* M *be a real number such that* $\frac{8}{3} \leq M < 3$*. Let* $d \geq 0$ *be an integer and let* G *be a graph with* $\mathrm{mad}(G) < M$*. If* $d \geq \frac{1}{3-M}$*, then* G *admits an* $(\mathcal{I}, \mathcal{F}_d)$*-partition.*

The proofs of the previous theorems yield polynomial time algorithms to find these partitions. It is clear for Theorem 3.1.4, since all of the reducible configurations have bounded size. For Theorems 3.1.12 and 3.1.13, it is less obvious since some configurations can be arbitrarily large. However, as argued in Section 3.4, one can construct these configurations in polynomial time, and all the reductions can also be done in polynomial time.

In contrast to that, we proved that if for some d, there is a triangle-free planar graph with no $(\mathcal{F}, \mathcal{F}_d)$-partition, then it is an NP-complete problem to find such a partition in a triangle-free planar graph.

In the following chapter, we will consider some other partitions. First we will consider partitions into sets inducing graphs with bounded components. We note that putting a lower bound on the girth of a graph is the same as forbidding all cycles of small length. We will consider partitions of graphs with some other sets of forbidden cycles.

Chapter 4

Partitions into independent sets and graphs of bounded degrees/with bounded components

4.1 Introduction

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In the previous chapter, we treated on the one hand partitions into a forest and a forest of bounded degree, and on the other hand partitions into an independent set and a forest of bounded degree. In this chapter, we will introduce some new partitions. First, we will treat partitions where one of the sets induces a graph with bounded components.

We recall that $\mathcal{I}$ denotes the family of empty graphs and that for all k, $\mathcal{O}_k$ denotes the family of graphs whose components have at most k vertices, and Δ_k denotes the family of graphs with maximum degree at most k.

Esperet and Ochem [32] proved the following:

Theorem 4.1.1 (Esperet and Ochem [32])**.** *Every planar graph with girth at least* 6 *admits an* $(\mathcal{O}_{12}, \mathcal{O}_{12})$*-partition.*

Borodin and Ivanova [12] showed a similar result:

Theorem 4.1.2 (Borodin and Ivanova [12])**.** *Every planar graph with girth at least* 7 *admits an* $(\mathcal{O}_3, \mathcal{O}_3)$*-partition.*

Note that $\mathcal{O}_2 = \Delta_1$, therefore Theorem 3.1.8 can be restated as follows:

Theorem 3.1.8 (Kim, Kostochka, and Zhu [42], restated)**.** *Every triangle-free graph with maximum average degree at most* $\frac{11}{9}$ *admits an* $(\mathcal{I}, \mathcal{O}_2)$*-partition.*

Corollary 4.1.3. *Every planar graph with girth at least* 11 *admits an* $(\mathcal{I}, \mathcal{O}_2)$*-partition.*

We prove the following theorem in Section 4.2:

Theorem 4.1.4. *Every graph G with* $\mathrm{mad}(G) < \frac{5}{2}$ *admits an* $(\mathcal{I}, \mathcal{O}_3)$*-partition.*

Euler's formula yields the following corollary:

Corollary 4.1.5. *Every planar graph with girth at least* 10 *admits an* $(\mathcal{I}, \mathcal{O}_3)$*-partition.*

Note that graphs with girth at least 10 do not contain triangles, so Corollary 4.1.5 implies Corollary 3.1.14.3, that is that every planar graph with girth at least 10 admits an $(\mathcal{I}, \mathcal{F}_2)$-partition.

To extend our result to $(\mathcal{I}, \mathcal{O}_k)$-partitions for other values of k, we prove the following theorem in Section 4.4, after giving the rough ideas of the proof in Section 4.3:

Theorem 4.1.6. *Let $k \geq 2$ be an integer. Every graph G with* $\mathrm{mad}(G) < \frac{8k}{3k+1} = \frac{8}{3}\left(1 - \frac{1}{3k+1}\right)$ *admits an* $(\mathcal{I}, \mathcal{O}_k)$*-partition.*

Consequently, for $k = 9$, we get that every graph with $\mathrm{mad}(G) < \frac{18}{7}$ admits an $(\mathcal{I}, \mathcal{O}_9)$-partition, and thus, as by Euler's formula, every planar graph of girth 9 has a maximum average degree of less than $\frac{2\times 9}{9-2}$, we get the following:

Corollary 4.1.7. *Every planar graph with girth at least* 9 *admits an* $(\mathcal{I}, \mathcal{O}_9)$*-partition.*

Another quite interesting way to see planar graphs of large girth is as planar graphs with some forbidden set of (induced or not) cycles. A natural question is then what sets of forbidden cycle lengths imply some partitions on planar graphs.

Choi, Liu, and Oum [38] proved that there exist exactly two minimal sets of forbidden cycle lengths such that every planar graph admits an $(\mathcal{I}, \Delta_d)$-partition for some constant d:

- Planar graphs without odd cycles are bipartite, i.e. admit an $(\mathcal{I}, \mathcal{I})$-partition.
- Planar graphs without cycles of length 3, 4, or 6 admit an $(\mathcal{I}, \Delta_{45})$-partition.

This raises the following question:

Question 4.1.8. *What is the smallest integer d such that every planar graph without cycles of length* 3*,* 4*, or* 6 *admits an* $(\mathcal{I}, \Delta_d)$*-partition.*

Notice that forbidding cycles of length 3, 4, or 6 as subgraphs or induced subgraphs result in the same graph class. We denote by $\mathcal{C}$ the class of (C_3, C_4, C_6)-free planar graphs.

We will prove Theorems 4.1.9 and 4.1.10 in Sections 4.5 and 4.6 respectively.

Theorem 4.1.9. *Every graph in $\mathcal{C}$ admits an* $(\mathcal{I}, \Delta_6)$*-partition.*

We note that since planar graphs of girth at least 7 are exactly C_5-free graphs of $\mathcal{C}$, Theorem 4.1.9 can be compared with Corollary 3.1.6.

Corollary 3.1.6 (Borodin and Kostochka [15], recall)**.** *Every planar graph with girth at least* 7 *admits an* $(\mathcal{I}, \Delta_4)$*-partition, and every planar graph with girth at least* 8 *admits* $(\mathcal{I}, \Delta_2)$*-partition.*

We also worked on the complexity of the related decision problem:

Theorem 4.1.10. *For every $k \geq 1$, either every graph in $\mathcal{C}$ admits an $(\mathcal{I}, \Delta_k)$-partition, or deciding whether a graph in $\mathcal{C}$ admits an $(\mathcal{I}, \Delta_k)$-partition is an NP-complete problem.*

In addition, we construct a graph in $\mathcal{C}$ that does not admit an $(\mathcal{I}, \Delta_3)$-partition in Section 4.6. This graph and Theorem 4.1.10 imply the following.

Corollary 4.1.11. *Deciding whether a graph in $\mathcal{C}$ admits an $(\mathcal{I}, \Delta_3)$-partition is an NP-complete problem.*

4.2 $(\mathcal{I}, \mathcal{O}_3)$-partition of graphs maximum average degree less than $\frac{5}{2}$

Let us first recall the statement of Theorem 4.1.4.

Theorem 4.1.4 (recall)**.** *Every graph G with* $\mathrm{mad}(G) < \frac{5}{2}$ *admits an $(\mathcal{I}, \mathcal{O}_3)$-partition.*

In this proof, we will use a similar method to that of Sections 3.4 and 3.5. This gives an insight that this method can be applied to several problems. As our discharging procedure is different, we have a different definition for the light forest.

In our $(\mathcal{I}, \mathcal{O}_3)$-partitions, we will usually call the colour sets I and O respectively. Note that I and O will indifferently be considered as colours or as the sets of vertices with those colours. We say that an O-vertex is *saturated* if it has either two O-neighbours, or an O-neighbour u that has two O-neighbours. We say that an O-vertex is *partially saturated* if it has an O-neighbour and is not saturated. At last, we say that an O-vertex is *unsaturated* if it has no O-neighbour. Note that O-vertices that are not saturated are not necessarily unsaturated, as they can also be partially saturated.

We will prove Theorem 4.1.4 via the discharging method. Let $G = (V, E)$ be a counter-example to Theorem 4.1.4 with as few vertices as possible.

The graph G is connected, since otherwise at least one of its components would be a counter-example to Theorem 4.1.4 with fewer vertices than G.

Lemma 4.2.1. *Every vertex in G has degree at least 2.*

Proof. Let v be a 1^--vertex in G. Since G is connected, if v has degree 0, then $V = \{v\}$ and G admits an $(\mathcal{I}, \mathcal{O}_3)$-partition, a contradiction. Therefore v has degree 1. The graph $G - v$ has fewer vertices than G. Thus it admits an $(\mathcal{I}, \mathcal{O}_3)$-partition, which can be extended to an $(\mathcal{I}, \mathcal{O}_3)$-partition of G by giving to v a colour distinct from that of its neighbour. That is a contradiction. □

Lemma 4.2.2. *Every 3^--vertex in G has at least one 3^+-neighbour.*

Proof. Let v be a 3^--vertex such that every neighbour of v is a 2^--vertex (and thus a 2-vertex by Lemma 4.2.1). Let $G' = G - N[v]$. The graph G' has fewer vertices than G, therefore it admits an $(\mathcal{I}, \mathcal{O}_3)$-partition. For every vertex in $N(v)$ that has a neighbour in G', we can colour that vertex with the colour distinct from that of its neighbour that is in G'. If two vertices of $N(v)$ are adjacent, we can colour one with colour O and one with colour I. If all the vertices in $N(v)$ have colour O, then we can colour v with colour I. Otherwise, v has at most two neighbours in G that are coloured O, and each of them has no neighbours in G that are coloured O. Therefore we can give v colour O, and we obtain an $(\mathcal{I}, \mathcal{O}_3)$-partition of G, a contradiction. □

A 2-vertex that has a 2-neighbour is called a *light 2-vertex*, and a 2-vertex that does not have a 2-neighbour is a *heavy 2-vertex*. Let us build a rooted forest in G, that we will call the *light forest* L, using the following rules.

1. For all light 2-vertex v, let u be the 3^+-neighbour of v (that exists by Lemma 4.2.2), we add v and u (if it is not already in L) to the light forest, as well as the edge uv. We set that u is the *father* of v.

2. Whenever there is a 4-vertex w that is the father of three vertices in L, such that w has a 2-neighbour v that is not in L, do the following. Let u be the neighbour of v distinct from w. Since v is not in L, v is a heavy 2-vertex, and thus u is a 3^+-vertex. Add the vertices u (if it is not already in L) and v to L, as well as the edges uv and vw. The vertices u and v are the fathers of v and w respectively.

   Whenever there is a 3-vertex w that is the father of a vertex in L and has a 2-neighbour v that is not in L, do the following. Let u be the neighbour of v distinct from w. Since v is not in L, v is a heavy 2-vertex, and thus u is a 3^+-vertex. Add the vertices u (if it is not already in L) and v to L, as well as the edges uv and vw. The vertices u and v are the fathers of v and w respectively.

A vertex u is a *son* of a vertex v if v is the father of u. By construction, every son of a 3^+-vertex is a 2-vertex, and every son of a 2-vertex is a 3^+-vertex.

Note that, by Lemma 4.2.2, every 3-vertex has a 3^+-neighbour, thus every 3-vertex that has a son can have at most one father (its sons and fathers being 2-vertices). Hence, every vertex has at most one father. Moreover, there are no cycles, thus that construction leads to a rooted forest. We say that a vertex v *descends* from a vertex u if there are vertices $w_1 = u$, w_2, ..., w_{k-1}, $w_k = v$ for some $k \geq 2$, such that for all $1 \leq i \leq k-1$, w_i is the father of w_{i+1}.

Lemma 4.2.3. *Let v be a 3^+-vertex of L that has a father u and let D be the set of vertices that descend from v in L. Let H be an induced subgraph of $G - (D \cup \{u, v\})$. We can find an $(\mathcal{I}, \mathcal{O}_3)$-partition of $G[V(H) \cup D \cup \{v\}]$ such that, if v is coloured O, then it is not saturated.*

Proof. Since H has fewer vertices than G, the graph H admits an $(\mathcal{I}, \mathcal{O}_3)$-partition.

We will prove the lemma by induction. We assume that, for every 3^+-vertex x that descends from v, we can colour x and all of its descendants such that, if x is coloured O, then it is not saturated.

Note that by construction of the light forest, since v is a 3^+-vertex that has a father, v is either a 4-vertex with three sons or a 3-vertex with one son.

Suppose v is a 4-vertex. Vertex v has three sons, say v_1, v_2 and v_3. Note that they are 2-vertices by construction. For all i in $\{1, 2, 3\}$, let w_i be the neighbour of v_i distinct from v. Let $i \in \{1, 2, 3\}$. Assume first that v_i is a light 2-vertex, that is that w_i is a 2-vertex. Let x be the neighbour of w_i distinct from v_i. If w_i and x are both in O, then we can recolour w_i to I. Therefore we can assume that, if w_i is coloured O, then it is unsaturated. Assume now that v_i is not a leaf of L. By induction, we can colour w_i and its descendants such that, if w_i is coloured O, then it is not saturated. Now, for all i, we can colour v_i with colour O. Finally, we can colour v with colour I.

Suppose now that v is a 3-vertex with son v_1 (which is a 2-vertex). Let v_2 be the third neighbour of v, and w be the neighbour of v_1 distinct from v. If v_1 is a heavy 2-vertex, then we colour w and its descendant such that if w is in O, then it is not saturated. If v_1 is a light 2-vertex, then we can also assume as above that, if w is in O, then it is not saturated (otherwise we can recolour w with I). If v_2 is in I, then we colour v_1 with the colour distinct from that of w, and we colour v with colour O. Note that in that case v is not saturated. If v_2 is not in I, then we can colour v_1 with colour O and v with colour I.

That ends the proof of the lemma. □

Lemma 4.2.4. *Every 3-vertex of G has at most one son in L.*

Proof. Let v be a 3-vertex with at least two sons, v_1 and v_2, in L. Let w_1 and w_2 be the neighbours of v_1 and v_2 respectively distinct from v, and let v_3 be the neighbour of v distinct from v_1 and v_2. Note that by Lemma 4.2.2, v_3 is a 3^+-vertex. Let D be the set of the descendants of v. The graph $G - (D \cup \{v\})$ has fewer vertices than G. Therefore it admits an $(\mathcal{I}, \mathcal{O}_3)$-partition. For $i \in \{1, 2\}$, if v_i is a light 2-vertex, then we can assume, up to recolouring w_i, that if w_i is in O, then it is unsaturated. For $i \in \{1, 2\}$, if v_i is a heavy 2-vertex, then w_i is a 3^+-vertex, and by Lemma 4.2.3, we can colour w_i and all of its descendants such that, if w_i is in O, then it is not saturated.

If v_3 is in I, then, for all $i \in \{1, 2\}$, we give to v_i the colour distinct from that of w_i, and since v has at most two O-neighbours and each of them is unsaturated, we can give colour O to v. If v_3 is in O, then we put v_1 and v_2 in O and v in I. In both cases, that leads to an $(\mathcal{I}, \mathcal{O}_3)$-partition of G, a contradiction. □

Lemma 4.2.5. *Every 4-vertex of G has at most three sons in L.*

Proof. Let v be a 4-vertex with four sons, v_1, v_2, v_3, and v_4, in L. Let w_1, w_2, w_3, and w_4 be the neighbours of v_1, v_2, v_3, and v_4 respectively, distinct from v. Let D be the set of the descendants of v. The graph $G - (D \cup \{v\})$ has fewer vertices than G. Therefore it admits an $(\mathcal{I}, \mathcal{O}_3)$-partition. For $i \in \{1, 2, 3, 4\}$, if v_i is a light 2-vertex, then we can assume up to recolouring w_i that, if w_i is in O, then it is unsaturated. For $i \in \{1, 2, 3, 4\}$, if v_i is a heavy 2-vertex, then w_i is a 3^+-vertex, and by Lemma 4.2.3, we can colour w_i and all of its descendants such that, if w_i is in O, then it is not saturated. We can put all of the v_i's in O, and v in I. That leads to an $(\mathcal{I}, \mathcal{O}_3)$-partition of G, a contradiction. □

Now all the structural lemmas we need have been proven, and we can move on to the discharging procedure. We give an initial charge to each vertex of degree d equal to $d - \frac{5}{2}$. Since $\mathrm{mad}(G) < \frac{5}{2}$, the sum of the initial charges in the graph is negative.

Discharging procedure

- *Step 1*: Every 3^+-vertex of L gives $\frac{1}{2}$ to each of its sons.
- *Step 2*: Every 3^+-vertex gives $\frac{1}{4}$ to each of its 2-neighbours that are not its neighbours in L.

Throughout that procedure, no charge is created and no charge is deleted. It follows that the sum of the final charges of the vertices is negative. Let us now prove that every vertex has a non-negative final charge. That contradiction completes the proof of Theorem 4.1.4.

Lemma 4.2.6. *Every vertex has a non-negative final charge.*

Proof. Each 2-vertex has initial charge equal to $-\frac{1}{2}$. Every 2-vertex that is in L receives $\frac{1}{2}$ from its father in Step 1 (recall that every 2-vertex in L has a father in L that is a 3^+-vertex). Every 2-vertex that is not in L has two 3^+-neighbours by construction of L, and thus receives $\frac{1}{4}$ from each of its two neighbours in Step 2. Thus every 2-vertex has a non-negative final charge.

Let v be a 3-vertex. Vertex v has initial charge equal to $\frac{1}{2}$. By Lemma 4.2.2, v has at most two 2-neighbours. Thus, if v is not in L, then it gives no charge in Step 1 and at most twice $\frac{1}{4}$ in Step 2. Suppose v is in L. By Lemma 4.2.4, v has at most one son in L, and by construction, if it has one son and another 2-neighbour u, then u is the father of v. Therefore either v has at most one 2-neighbour, or it has two 2-neighbours and one of its 2-neighbours is its father in L. Consequently, if v is in L, then it gives $\frac{1}{2}$ to its son in Step 1 and gives nothing in Step 2. In all cases, v has non-negative final charge.

Let v be a 4-vertex. Vertex v has initial charge equal to $\frac{3}{2}$. By Lemma 4.2.5, v has at most three sons in L. If v has at most three 2-neighbours, then it gives charge amounting to at most $3 \times \frac{1}{2} = \frac{3}{2}$ in Step 1 and Step 2. Suppose v has three sons in L and has another 2-neighbour u. By Lemma 4.2.5, v cannot have four sons, therefore u is not the son of v. Hence, if u is not the father of v, then u is not in L, and we would have applied Step 2 of the construction of the light forest. Thus u is the father of v, and v gives $3 \times \frac{1}{2} = \frac{3}{2}$ in Step 1 and nothing in Step 2. The remaining case is when v has at most two sons in L. In that case, it gives charge amounting to at most $2 \times \frac{1}{2} + 2 \times \frac{1}{4} = \frac{3}{2}$ in Step 1 and Step 2. In all cases, v has a non-negative final charge.

For all $d \in \mathbb{N}$, $d \geq 5$, every d-vertex has initial charge equal to $d - \frac{5}{2} \geq \frac{d}{2}$, and gives charge amounting to at most $\frac{d}{2}$ over Step 1 and Step 2. Therefore every 5^+-vertex has non-negative final charge. □

4.3 Ideas for $(\mathcal{I}, \mathcal{O}_k)$-partition of graphs with bounded maximum average degree

Let us first recall the statement of Theorem 4.1.6.

Theorem 4.1.6 (recall)**.** *Let $k \geq 2$ be an integer. Every graph G with* $\mathrm{mad}(G) < \frac{8k}{3k+1} = \frac{8}{3}\left(1 - \frac{1}{3k+1}\right)$ *admits an $(\mathcal{I}, \mathcal{O}_k)$-partition.*

In our $(\mathcal{I}, \mathcal{O}_k)$-partitions, we will usually call the colour sets I and O respectively. Sets I and O will indifferently be used to call the vertex sets of the partition and their respective colours.

Let us give some intuition on how our reasoning proceeds. We are going to use the discharging method.

We first consider a counter-example G to Theorem 4.1.6 with as few vertices as possible.

Lemma 4.3.1. *The graph G is connected and its minimum degree is at least 2.*

Proof. If G is not connected, then one of its components is a smaller counter-example to Theorem 4.1.6, contradicting the minimality of G.

If G contains a 0-vertex, then G is reduced to that vertex and admits an $(\mathcal{I}, \mathcal{O}_k)$-partition.

Let v be a 1-vertex of G. The graph $G - v$ has fewer vertices than G, therefore it admits an $(\mathcal{I}, \mathcal{O}_k)$-partition. If the neighbour of v is coloured I, then we put v in O, and otherwise we put v in I. That leads to an $(\mathcal{I}, \mathcal{O}_k)$-partition of G, and thus to a contradiction. □

We would like to prove that some more configurations are reducible. Consider a 2-vertex v. The graph $G-v$ has fewer vertices than G, and thus admits an $(\mathcal{I}, \mathcal{O}_k)$-partition. Since this partition cannot be extended to G, one of the neighbours of v is in I, and the other is in O, and its component in $G[O]$ has exactly k vertices. We add v to O. This gives us a *near $(\mathcal{I}, \mathcal{O}_k)$-partition* of G, where $G[I]$ is an independent set and every component of $G[O]$ has at most k vertices, except for one component which has exactly $k+1$ vertices. Now, every configuration in which the near $(\mathcal{I}, \mathcal{O}_k)$-partition can be transformed into an $(\mathcal{I}, \mathcal{O}_k)$-partition of G is reducible.

There are some reducible configurations that can be directly deduced from that. Mainly, from the fact that each vertex in the component of v in $G[O]$ (recall that v is now in O in the near $(\mathcal{I}, \mathcal{O}_k)$-partition) must have an I-neighbour. This implies in particular that if $k \geq 2$, then the O-neighbour of v must be a 3^+-vertex (it has an I-neighbour and an O-neighbour distinct from v), and thus every 2-vertex in G has a 3^+-neighbour.

Let w be the O-neighbour of v, and x be an I-neighbour of w. Suppose w has degree 3 and x has degree 2 (those are the minimum degrees these vertices can have). Then the neighbour z of x distinct from w is in a component of $G[O]$ with exactly k vertices, otherwise we could exchange the colours of w and x and obtain an $(\mathcal{I}, \mathcal{O}_k)$-partition of G. Let us exchange the colours of w and e. We obtain a new near $(\mathcal{I}, \mathcal{O}_k)$-partition and we can apply the same reasoning as before for vertex z. That way, we can derive arbitrarily many forbidden configurations. Our aim is to give a scheme for these configurations so that we can actually use them in a discharging procedure.

The initial repartition of the charges will be mainly on the vertices of high degree (we can give each vertex v a weight of $d(v) - \mathrm{mad}(G)$). Therefore with the idea that the higher the degree, the higher the charge, we can start by reasoning with vertices whose degree is as small as possible. By applying that rule, we can assume that $w_0 = w$ has degree 3 with an I-neighbour $x_0 = x$ of degree 2 and an O-neighbour w_1 distinct from v, that w_1 has degree 3 and has an I-neighbour x_1 of degree 2 and an O-neighbour w_2... until w_{k-1}, that actually has degree 2 and has an I-neighbour x_{k-1}. The same reasoning can then be done taking any of the x_i's instead of v, and thus for all i, x_i has a similar structure to the one described above where it takes the place of v and its neighbour distinct from w_i takes the place of w. That is, if we put w_i in I and x_i in O, then we obtain a partition where the component of x_i in $G[O]$ has more than k vertices, and we can reason on that new partition.

If every vertex v has an initial weight of $d(v) - \mathrm{mad}(G)$, and assuming that $\mathrm{mad}(G) < 3$, only the 2-vertices have a negative initial weight. In the previous configuration, one would like the vertices w_0, w_1, ..., w_{k-2} to give all their weight (that is $(k-1)(3 - \mathrm{mad}(G))$) to v and w_{k-1}, such that v and w_{k-1} receive the weight they need (that is $\mathrm{mad}(G) - 2$ each). The x_i's then receive the weight they need from their neighbour distinct from the w_i's. We do that inductively, obtaining layers of vertices as in Figure 4.3.1. That gives us a local discharging procedure on some vertices of the graph, including the vertex v, such that the final weight of those vertices in non-negative. The idea is then to do that same reasoning around each 2-vertex.

There are several problems with what is described above.

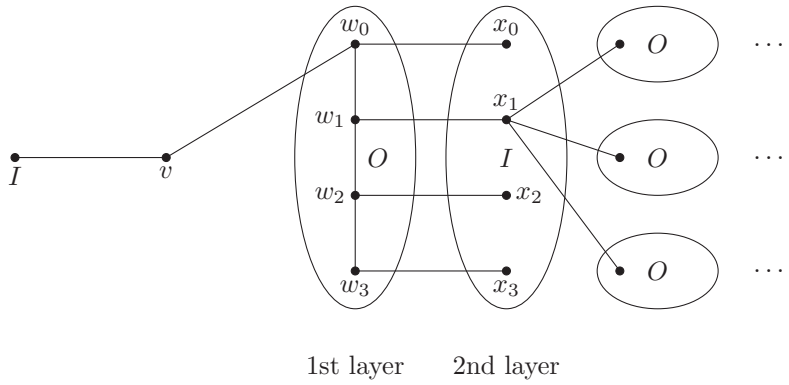

Figure 4.3.1: The configuration considered around a 2-vertex v.

Problems

1. We just considered the extremal case where the w_i's form a path, but they could form any other graph. We need to check that in case the components of $G[O]$ are not actually paths we can still do our weight distribution.

2. The x_i's do not need to be 2-vertices. Actually, we have a similar weight distribution if they are for example 3-vertices inside a path of $G[O]$ instead of 2-vertices at the end of that path. This implies that the discharging procedure described above cannot easily be described in terms of the structure of the graph, and is heavily dependant on the near $(\mathcal{I}, \mathcal{O}_k)$-partition of G we considered. Since that partition depends in particular on the vertex v we started with, we will need to make sure that the discharging procedures devised around each 2-vertex are compatible with one another.

3. The structure can sometimes not be that easy to determine. Starting from the second layer, the components of $G[O]$ can become arbitrarily large. For example, in the previous reasoning, the vertex x_1 could have degree 3 and have two O-neighbours besides w_1, each being in a component of $G[O]$ with k vertices. That would lead, by changing the colours of x_1 and w_1, to a partition where the component of x_1 in $G[O]$ has a lot more than $k+1$ vertices (here $2k+1$ vertices). Then some of these vertices can be recoloured to O without there being a contradiction. Therefore it can be complicated to find which vertices can take the place of the x_i's in the previous reasoning and give the weight needed.

4. The x_i's do not need to be all distinct. That could be dealt with if it was only on one layer, but there can actually be an I-vertex that is adjacent to vertices of several layers, which makes things a lot more complicated. Actually, we could not completely remove this issue, and we had to revise our discharging procedure so that in that case, the two O-neighbours of the I-vertex considered each give it the necessary weight, which produced a worse bound than what we hoped to get.

The first of those problems is quite easy to handle. The idea is that if instead of a path, the graph considered is any tree with the same number of vertices, then the number of edges remains the same, and thus the sum of the degrees of the vertices also remains the same. Therefore, we actually have the same weight if the component of v in $G[O]$ is any tree instead of a path, and we have a larger weight if it is not a tree.

Let us move on to the second of those problems, that is the fact that different local discharging procedures may interfere. We will always insure that vertices only give weight to their neighbours, and give at most $\mathrm{mad}(G)-2$ to their neighbours (which is the weight needed by the 2-vertices). In each of the local discharging procedure, we describe some discharging that apply among vertices of a specific set S, in such a way that vertices in S end up with non-negative weight and give $\mathrm{mad}(G)-2$ to their neighbours outside of S. That

way, we can just choose one of the two overlapping local discharging procedures, and say that it applies in priority over the other ones. For the vertices of the local discharging procedure that has priority, they end up with a non-negative weight since the discharging that is applied is that of that procedure. For the vertices of the other local discharging procedure, they get at least as much weight from each of their neighbours as in that local discharging procedure, so they also end up with a non-negative weight.

For the third problem, we move on to more complex aspects of the reasoning. In large components of $G[O]$, we need to know, for each vertex, whether that vertex can give weight or if it can be recoloured to I. If when every vertex that can be put in I is put in I, the remaining component of $G[O]$ still contains at least $k+1$ vertices, then there is (locally) enough weight to do the discharging procedure, and otherwise the recolouring is (locally) an $(\mathcal{I}, \mathcal{O}_k)$-partition. But to know if one of those vertices can be put in I, we need to put it in I and check if we can make the colouring consistent in the next layer. To do that, we build a recursive procedure that chooses, at each layer, whether to recolour or ensure the weight is enough for the discharging procedure. We will describe that procedure (actually, two procedures because of the changes between the first layer and the following ones) in Section 4.4, and we will give a series of lemmas on this procedure. It can be noted that this reasoning cannot actually be done layer by layer (breadth first), and actually needs to be done, in each layer, completely for one vertex before getting to the next one (depth first). That is why we do not consider layers in our recursive procedure, and instead consider the local layer associated with each vertex (that will be called the *cluster set* of that vertex).

For the fourth problem, we did not find a way to recolour the vertices when an I-vertex is adjacent to vertices in different layers. The resulting change in the discharging procedure quite worsens our results. If the w_i's did not need to give any weight to the x_i's, as described above the problems, we would only need to ensure that $(k-1)(3-\mathrm{mad}(G)) > 2(\mathrm{mad}(G)-2)$, i.e. $\mathrm{mad}(G) < \frac{3k+1}{k+1}$. This would imply, for $k=3$, Theorem 4.1.4, and for $k \to \infty$, $\mathrm{mad}(G) \to 3$. Our actual bound of $\mathrm{mad}(G) < \frac{8k}{3k+1}$ in Theorem 4.4 (the calculations are done in Section 4.4) is quite worse, but still gives the best known bounds to date for $k \geq 9$.

4.4 $(\mathcal{I}, \mathcal{O}_k)$-partition of graphs with bounded maximum average degree

Let us first recall the statement of Theorem 4.1.6.

Theorem 4.1.6 (recall)**.** *Let $k \geq 2$ be an integer. Every graph G with $\mathrm{mad}(G) < \frac{8k}{3k+1} = \frac{8}{3}\left(1 - \frac{1}{3k+1}\right)$ admits an $(\mathcal{I}, \mathcal{O}_k)$-partition.*

Let us consider a counter-example G to Theorem 4.1.6 with as few vertices as possible. In this section, unless we specify otherwise, the degrees of the vertices we consider are the degrees in G. That is, when we speak for example about a k-vertex, we mean a vertex of degree k in G. Our discharging procedure consists in giving weights to vertices so that the total sum of the weights is negative and then in moving these weights according to some rules from vertex to vertex so that all the vertices eventually have non-negative weights (due to some structural properties of G). That contradiction completes the proof.

Given an $(\mathcal{I}, \mathcal{O}_k)$-partition of a graph, we say that a vertex v with colour O is *saturated* if the component in $G[O]$ containing v is of order k.

The following lemma was proven in the previous section:

Lemma 4.3.1 (recall)**.** *The graph G is connected and its minimum degree is at least 2.*

The key point of the proof is the use of two procedures, a main one (Procedure 4.4.13) that calls a secondary inductive one (Procedure 4.4.1). Procedure 4.4.13 starts with a colouring that is a near $(\mathcal{I}, \mathcal{O}_k)$-partition of G. From that partition, the procedures try to deduce an $(\mathcal{I}, \mathcal{O}_k)$-partition of G, leading to a contradiction when it is possible. When this is not possible, the procedures build a structure that will later enable us to apply a discharging procedure on G, leading to a contradiction.

The vertices of G will be *marked* up to twice. At the start the vertices are unmarked. We will mark the vertices up to twice and may unmark some of them. The principle is that vertices are marked once when we

first encounter them, and a second time when we actually treat them. In the end, each vertex will be either twice marked or unmarked (i.e. the state of being marked once is only temporary), and the twice marked vertices will be able to give weight to their neighbours that are unmarked. For the rest, the weight will move between twice marked vertices. For all $i \in \{0, 1, 2\}$, let V^i (resp I^i, O^i) be the set of vertices of $V(G)$ (resp. I, O) marked i times.

In Procedures 4.4.1 and 4.4.13, for some of the vertices $v \in V$, we will define the *cluster set* $S(v)$ of v to be a set of vertices such that $v \notin S(v)$, $S(v) \cup \{v\}$ induces a connected graph. Given a vertex v, an element of $S(v)$ is a *subordinate* of v. We say that v *supervises* the elements of $S(v)$. We will also sometimes define some cluster sets with no supervisor. For some of the vertices $v \in V$, we will also define a *giving edge* $e(v)$ incident to v. Finally, some edges of E may become *neutral edges*. All these notions are useful to describe our discharging process. Some ideas to keep in mind are the following:

- We will specify the discharging procedure in each cluster set.
- Only twice marked vertices will give weight.
- A twice marked vertex will give less weight to its adjacent subordinates than to its unmarked neighbours.
- A giving edge will link two twice marked vertices. A giving edge $e(v) = vw$ of v means that w will be able to give weight to v as if v is unmarked (at some point of the procedures, w is twice marked and v is not marked yet, and thus w could give weight to v).
- Neutral edges are edges between vertices that do not give weight to each other.

Procedures 4.4.1 and 4.4.13 try to recolour some of the vertices of V. When we recolour some vertices, we remember the colour they had, such that we can revert all the recolourings that have been done since the start of the procedure. When we say that we revert all the recolourings that have been done since the start of the procedure, or that we reset the colourings, we mean since the start of the call of the procedure that we are currently treating. This is important since the procedure calls itself recursively. For example, suppose we make a call A to the procedure, and that call makes a call B to the procedure. If we say in the code of A before the call to B that we reverse all the recolourings that have been done, then we undo all the recolourings done since the start of call A. If we say this in the code of B, then we only reverse what has been recoloured since the start of call B. Lastly, if we say this in the code of A after the call to B, then we reverse all the changes that were done since the beginning of call A, including those that were done in call B.

Similarly, we will also reverse all the markings that have been done since the beginning of the procedure. When we do so, we also reset the cluster sets and the giving and neutral edges as they were at the start of the procedure.

Procedure 4.4.1 supposes that all the vertices of G are coloured with colours O and I, such that no two adjacent vertices have colour I (but no assumption is made on vertices with colour O). Considering an O^2-vertex v, the aim of Procedure 4.4.1 is either to recolour v with colour I, or to get a cluster set or a neutral edge for v.

The sets W, C, and C' in Procedure 4.4.1 are not sets defined for the whole procedure, but local sets defined at each call of the procedure.

Let us present now Procedure 4.4.1 that is applied to an O^2-vertex v:

Procedure 4.4.1. *Let W be the set of I-neighbours of v.*

1. *If a vertex w of W is twice marked, then we set $S(v) = \emptyset$, we set vw as a neutral edge, and we end the procedure.*

Now every vertex of W is marked up to once.

2. *If a vertex w of W has an O^2-neighbour u distinct from v, then we set $S(v) = \{w\}$ and $e(w) = wu$, we mark w twice, and we end the procedure.*

3. *If a vertex w of W has an O^1-neighbour u, then we set $S(v) = S(u) = \{w\}$, we mark u a second time, we mark w twice, and we end the procedure.*

4. *All the vertices of W only have O^0-neighbours. We recolour v with I and every vertex of W with O^0. Let X be the set of the vertices of W that are in a component of $G[O]$ having at least $k+1$ vertices. Let C be the set of vertices of the components of $G[O^0]$ containing (at least) a vertex of X.*

   While there exists a vertex in $C \setminus X$ that has all its neighbours in O, put that vertex in I, and redefine X and C as before (but with respect with the new colouring of the graph).

   Mark once all the 3^+-vertices of $C \setminus X$, and twice the vertices of X.

   Now while there is a vertex $c \in C \setminus X$ marked once, do the following:

   (a) *Mark c a second time and apply Procedure 4.4.1 to the vertex c.*

   (b) *If c is in I, then we undo all the markings that have been done since the start of the procedure, reset the cluster sets and giving and neutral edges as they were at the start of the procedure, redefine X and C as before (but with respect to the new colouring of the graph), mark once all the 3^+-vertices of $C \setminus X$ and twice all the elements of X.*

5. *If X is empty at the beginning of (4) or becomes empty during (4).b, then we stop the procedure (in that case, we succeed to recolour v with I and to manage all the resulting conflicts).*

   Now, set $S(v) = \emptyset$.

   For every component C' of $G[C]$ with at least $k+1$ vertices, we add to $S(v)$ the 3^+-vertices of C', plus the 2-vertices of $C' \cap X$.

   For every component C' of $G[C]$ with at most k vertices such that a 3^+-vertex x in C' is adjacent to a vertex y that was already twice marked at the start of the procedure, we add to $S(v)$ the 3^+-vertices of C', plus the 2-vertices of $C' \cap X$, and set $e(x) = xy$ for one such x.

   We undo all the recolourings that have been done since the start of the procedure.

We now prove a series of properties of Procedure 4.4.1.

Lemma 4.4.2. *In a given call to Procedure 4.4.1, the only vertices that become unmarked are vertices that have been marked during the procedure.*

Proof. Every time we undo the markings that have been done in the procedure, the marking state resets to what it was at the start of the procedure. Since we never unmark a vertex aside from those resets, no vertex that was marked at least once at the start of the procedure becomes unmarked in the procedure. □

By *single call*, we mean a call without considering the inner recursive calls. When we say that a change (of the colouring or the markings, for example) happens *directly*, we mean it does not happen during a reset of the colouring or the markings of the vertices. The following lemma is straightforward.

Lemma 4.4.3. *In a single call to Procedure 4.4.1 applied to vertex v, the only time a vertex is recoloured from O to I is by undoing some recolourings (end of Step 5), except for v and some vertices in O^0 with no neighbours in I during Step 4.*

Lemma 4.4.4. *At the end of a call to Procedure 4.4.1 applied to vertex v, either all the vertices have the same colour as before the call, or they have all the same markings, subordinates, giving edges and neutral edges as before the call. Moreover, at the end of Procedure 4.4.1, v is in O if and only if all the vertices have the same colour as before the call.*

Proof. In Steps 1, 2, and 3, there are no recursive calls and no recolourings, therefore if the procedure stops in one of those, then all the vertices have the same colour as before the call. In those cases, the vertex v is still in O. If Step 4 is reached, then Step 5 is also reached, and either all the recolourings are undone (end of Step 5), or all the markings, subordinates and giving edges are reset (Step 4.b) or unchanged (the case where X is empty). As v is given colour I in Step 4, if the colourings are not undone, then v is in I. If all the recolourings are undone, then v is back in O. That ends the proof of the lemma. □

In the following, we will generally ignore what was done then undone. In particular, by Lemma 4.4.4, we can assume that every call to Procedure 4.4.1 either does not change the colouring or does not change the markings, subordinates, and giving and neutral edges of the vertices.

The following two lemmas ensure that the procedure keeps the colouring at least as good as it was.

Lemma 4.4.5. *If no two vertices of I are adjacent before a call to Procedure 4.4.1, then no two vertices of I are ever adjacent during the call.*

Proof. By contradiction, let v and w be two adjacent vertices that are not both in I and that become adjacent I-vertices during a call A to Procedure 4.4.1. Let us consider the first time v and w are both in I. This happens after one of v and w, say v, was recoloured from O to I. That recolouring cannot be the result of a reset of the colourings, since v and w were never in I together before. Thus v was recoloured to I while its neighbour w was in I. By Lemma 4.4.3, since v has a neighbour in I, this implies that v was recoloured as the vertex for which some call B to Procedure 4.4.1 was applied. In call B, v was recoloured from O to I in Step 4, and w, that was in W at this point as an I-neighbour of v, was recoloured from I to O at the same time. Therefore v and w were never in I at the same time. □

Lemma 4.4.6. *After a call to Procedure 4.4.1, the vertices of all the components of $G[O]$ with more than k vertices were in O at the start of the call.*

Proof. Consider a call to Procedure 4.4.1 for which the lemma is not true. We assume by induction that the recursive calls verify the lemma. By Lemma 4.4.4, we can assume that we are in a call for a vertex v which is in I at the end of the call (otherwise the colouring is the same as before the call). This implies that in Step 5, the set X was empty, and thus that all the elements of W were not in components of $G[O]$ with more than k vertices. The only time when vertices are put from I to O in Procedure 4.4.1 is when we recolour the vertices of W in the beginning of Step 4. That proves the lemma. □

The next two lemmas prove that Procedure 4.4.1 terminates.

Lemma 4.4.7. *In Step 4 of Procedure 4.4.1, every time the set C changes, it ends up with fewer vertices than before.*

Proof. Let A be the current call to Procedure 4.4.1. Assume that the set C of call A has changed after a call A' to Procedure 4.4.1 for some vertex $c \in C$. By the definition of C in Step 4b and by Lemma 4.4.4, this implies that c has changed colour from O to I. The vertex c is no longer in $G[O^0]$, thus it is removed from C.

Let us prove that no new vertex was added to C. Consider a vertex that is in C after a change of C and was not in C the previous time C was defined. Note that the state of the markings just before the change of C is the same as at the start of call A, either because they did not change or because they were reset. Also note that the only time a vertex may have been recoloured from I to O is in a recursive call called by A. As the elements of X correspond to vertices of W that are in components of C with at least $k+1$ vertices, by Lemma 4.4.6 no new vertex is put in X. Furthermore, the elements of X that remained in X do not have any new vertex in their component of $G[O]$, and thus, since the markings are the same, no new vertex in their component of $G[O^0]$ either. Therefore C does not have any new vertex. This ends the proof of the lemma. □

Lemma 4.4.8. *Procedure 4.4.1 terminates, and at the end of the procedure, every vertex that is marked once was marked once at the start of the procedure.*

Proof. Procedure 4.4.1 calls itself recursively. To make sure that it terminates, we only need to make sure on the one hand that such calls cannot be infinitely nested, and on the other hand that each of these calls only makes a finite number of calls to the procedure. First, note that every time we call Procedure 4.4.1 recursively, at least one additional vertex is twice marked (the vertex for which the new call is applied). Therefore there cannot be infinitely nested inductive iterations.

Now we proceed by induction. Assume that for a given call A to the procedure, each recursive call stops while not marking once any vertex that was not marked once. If the procedure stops in Step 1, 2 or 3 then

clearly no new vertex is marked once. Now we assume that Step 4 is reached. Let n_1 be the number of vertices in C, and let n_2 be the number of vertices that are marked once in C.

By Lemma 4.4.7, every time C changes, n_1 decreases strictly. When it does not change during an iteration of Step 4b, then at least one vertex of C that was marked once is now twice marked, thus n_2 decreases strictly. At each iteration of Steps 4a and 4b, the lexicographical order (n_1, n_2) decreases strictly, therefore there are finitely many such iterations. Moreover, whenever the set C changes, all the markings made during the procedure are cancelled, and in the end, all the vertices that have been marked once (only vertices of C, by induction hypothesis), are now twice marked. In Step 5, there are no new vertices that are marked once, and the procedure stops. □

Lemma 4.4.9. *Throughout the procedure, no vertex is ever in I^1.*

Proof. Let v be the first vertex that is put in I^1. We consider the call A where v was first in I^1. As we never alter the colourings and markings at the same time, either v was put from O to I while marked once, or it became marked once while in I.

Suppose v was put from O to I while marked once. If call A was applied for vertex v, then v was twice marked at the start of call A, and by Lemmas 4.4.2 and 4.4.8 v remained twice marked throughout call A, a contradiction. Therefore call A was not applied for vertex v, and thus by Lemma 4.4.3, the recolouring from O to I of v was done during a reset of the recolouring, and thus at the end of Step 5 for call A. Therefore v was marked once at the end of call A. This implies by Lemma 4.4.8 that v was already marked once at the start of call A. Moreover, since v was recoloured from O to I during the reset of the colouring in Step 5, it was in I at the start of call A. Therefore v was in I^1 at the start of call A, a contradiction.

Suppose v became marked once while in I. The only vertices that we mark once directly are vertices of C, that are in O. Therefore v became marked once due to a reset of the markings, thus v was marked once at the start of call A, thus it was in O^1. By Lemma 4.4.2, v never became unmarked until the end of call A. By Lemma 4.4.3, the first time v was put in I, v was the vertex that a certain call B of Procedure 4.4.1 was applied for. But this implies that when the call B was applied, v was in some set C, and thus had been unmarked at some point since the start of call A, which leads to a contradiction. □

Lemma 4.4.10. *Every time Procedure 4.4.1 changes the cluster set of a vertex, if this change is not undone afterwards during the procedure, then that vertex either is the vertex we are applying the procedure for, or was not twice marked at the start of the procedure and is twice marked at the end of the procedure.*

Proof. We consider a call A to the procedure. Let v be a vertex whose cluster set is changed during call A or one of its recursive calls. If v is the vertex some call $B \neq A$ is applied for, then it became twice marked just before call B, since calls to Procedure 4.4.1 are always made for vertices that have just become twice marked. By Lemmas 4.4.2 and 4.4.8, since we only mark once vertices that were previously unmarked, v was not twice marked at the start of call A and it remains twice marked until the end of call A. The only time the procedure changes the cluster set of another vertex is when u was marked once in Step 2, in which case the vertex u is also marked a second time. Similarly, by Lemmas 4.4.2 and 4.4.8, and since we only mark once vertices that were previously unmarked, u was not twice marked in the beginning of the procedure. As the cluster sets and the markings are always reset together, the lemma follows. □

Note that this implies that as long as we only apply Procedure 4.4.1 to vertices that just became twice marked (as we do in Step 4a), we will never set twice the cluster set of a vertex.

Let us define a property Π:

Property (Π)**.** *No 3^+-vertex in O^0 has an O^1-neighbour, and no 2-vertex in O^0 has a neighbour in O^1 and the other neighbour in O.*

That property is ensured by the fact that when we put vertices in O^1, all the vertices of the component in O^0 are put there together, except from some 2-vertices that have an I-neighbour.

Lemma 4.4.11. *If Π is true before a call A to Procedure 4.4.1 until the end of call A, every time Π becomes false, it is restored before any recursive call is applied, and before the procedure ends.*

Proof. We consider the first time Π is not true at the start or the end of some call to Procedure 4.4.1. We consider the event that caused Π to fail. By induction, we suppose that every recursive call preserves Π. Let us consider the change that caused the property Π to be broken, and prove that Π is restored immediately.

Suppose the change was direct, i.e. not because of some reset. When we mark a vertex u once, which only happens in Step 4, we do so for all the vertices in the component of u in $G[O^0]$ except some 2-vertices that have an I-neighbour, and thus Π is maintained. Moreover, by Lemma 4.4.2, no vertex directly becomes unmarked. Therefore the change was a change in the colouring. Hence some vertex w was put from I to O. Moreover, as no vertex that is marked once ever changes colour directly, w was unmarked when it changed colour. Therefore either w has a neighbour u in O^1, or it has a 2-neighbour u in O^0 that has a neighbour x in O^1. Vertex w is put in O as a vertex of W in Step 4 of some call B to Procedure 4.4.1. If call B was applied for vertex u, then u is put in I when w is put in O, and the property is not broken. Therefore call B was not applied for u. Then u is not in O^1, otherwise we would have stopped call B in Step 2. Thus u is a 2-vertex in O^0 incident to w with two O-neighbours, and thus it is put in I in Step 4 of call B.

Now we suppose the considered change is some reset. Suppose first that the reset was a reset of the colouring. This happens only at the end of Step 5, therefore at the end of the procedure, and the colouring is reset to what it was at the start of the procedure. By Lemmas 4.4.2 and 4.4.8, no new vertex is marked once or zero times at the end of the procedure compared to the start, so the property was already broken at the start of the current call to the procedure, a contradiction.

Thus the reset was a reset of the markings, subordinates and giving edges. Then some vertex w must have been put from I to O between the start of the procedure and this reset was such that w has an O-neighbour u that either was in O^1 at the start of the procedure, or is a 2-vertex that was in O^0 at the start of the procedure, such that the other neighbour x of u was in O at the start of the procedure. Since the resets are well parenthesised, we may assume that the recolouring of w was not the consequence of a reset, i.e. was done directly. We are now interested in the state of the procedure at the time when w is put from I to O. Note that this recolouring must happen in Step 4 for some call B, and that the marking in the beginning of Step 4 is still the same as at the start of call B. Also note that, unless u is a 2-vertex with its two neighbours in O, u cannot be in O^0 by Lemma 4.4.2. If call B was applied for vertex u, then u is put in I when w is put in O, and the property is not broken. Therefore call B was not applied for u. Then u is not in O^1 nor O^2, otherwise we would have stopped call B in Step 2. Thus u is a 2-vertex in O^0 incident to w with two O-neighbours, and thus it is put in I in Step 4 of call B. □

The following lemma implies that Procedure 4.4.1 either manages to recolour vertex v with colour I, or gives it a neutral edge or a cluster set.

Lemma 4.4.12. *Let A be a call to Procedure 4.4.1. Suppose that before a call A to Procedure 4.4.1,* Π *is true. After call A, if the vertex v for which the call was applied is in O, then either v is incident to a neutral edge, or $S(v) \neq \emptyset$.*

Proof. If Procedure 4.4.1 stops in Step 1, then v is incident to a neutral edge. If Procedure 4.4.1 stops in Step 2 or Step 3, then $S(v)$ is not empty. If the procedure reached Step 4, then it reached Step 5. We can assume that the vertex v is in O at the end of the procedure, and thus by Lemma 4.4.4 that all the recolourings have been undone.

All that remains to prove is that $S(v) \neq \emptyset$. We know that X is not empty in Step 5, otherwise v would be in I. Let x be a vertex of X. By Lemma 4.4.9, since x was in I at the start of the procedure (as a member of W), it was not marked once. Therefore $x \in C$. By definition of X, the component K of x in $G[O]$ has at least $k+1$-vertices. By Lemma 4.4.6, since K still has at least $k+1$ vertices the last time X is redefined, K did not gain any new vertex since the beginning of the recursive calls. Let C' be the component of x in $G[C]$. If C' has at least $k+1$ vertices, then $x \in S(v)$. Otherwise, by definition of X, there is a vertex y in C' that has a neighbour z that was in O^1 or O^2 last time X was defined. By Lemma 4.4.11, the graph G verified Π just before we last marked once all the vertices of $C \setminus X$ in Step 4, and thus the last time X was defined. Therefore z was in O^2 the last time X was defined, and thus also the first time C was defined. The vertex y is not in W (otherwise we would have stopped in Step 2), therefore it is not a 2-vertex (otherwise we would have put it in I as a 2-vertex in C with two O-neighbours). Therefore $x \in S(v)$. □

Let us now define a new procedure, applied on the graph G with no colouring, but where some vertices may be twice marked. The aim of that new procedure is, for some unmarked 2-vertex v, to make one of its neighbours become twice marked.

We start with every vertex of G unmarked. While there is a 2-vertex v that is not twice marked and has no neighbour that is twice marked, we apply the following procedure.

Procedure 4.4.13. *1. The graph $G - v$ has fewer vertices than G, therefore it admits an $(\mathcal{I}, \mathcal{O}_k)$-partition. The vertex v has a neighbour u in I and a neighbour w in O that is saturated, otherwise we can colour v with either I or O, which leads to a contradiction. Give colour O to v (note that this leads to a component of order $k+1$ in $G[O]$).*

2. Let us consider the component C of v in $G[O^0]$. Mark once every 3^+-vertex of C. For every once marked vertex x of C, we mark x a second time and we apply Procedure 4.4.1 for x. Note that there is no 2-vertex in C with both neighbours in O, as otherwise we could put such a vertex in I and get a colouring of G. We define a cluster set S as the set of 3^+-vertices in C. The set S is defined with no supervisor, and if there is an edge from a vertex x of S to a vertex y that was twice marked at the beginning of Procedure 4.4.13, then let $e(x) = xy$ for one such x.

We are now ready to prove some more lemmas, to ensure that the graph obtained after all the calls to Procedure 4.4.13 have been applied verifies the properties we will need in the discharging procedure.

Lemma 4.4.14. *In Procedure 4.4.13, no vertex of C is ever put in I, and at the end of each call to Procedure 4.4.1, all the recolourings have been undone.*

Proof. By Lemma 4.4.6, the calls to Procedure 4.4.1 never add any new vertex to the component of v in $G[O]$, otherwise this component ends up with at most k vertices, and by Lemmas 4.4.5 and 4.4.6, we get an $(\mathcal{I}, \mathcal{O}_k)$-partition of G, a contradiction.

Procedure 4.4.13 cannot lead to recolouring to I any of the vertices of C. Indeed, by Lemmas 4.4.5 and 4.4.6, this would lead to an $(\mathcal{I}, \mathcal{O}_k)$-partition of G. Thus by Lemma 4.4.4, in every application of Procedure 4.4.1, all the recolourings have been undone. □

Lemma 4.4.15. *After any number of calls to Procedure 4.4.13, no vertex is marked once.*

Proof. The property is true initially, when every vertex is unmarked. Suppose that no vertex is marked once at the beginning of some call to Procedure 4.4.13. We mark each 3^+-vertex in C once. Afterwards, for each vertex $x \in C$ that was marked once, we mark it twice and apply Procedure 4.4.1, that does not mark once any new vertex by Lemma 4.4.8. Therefore at the end of the call, no vertex is marked once. The lemma follows by induction. □

Lemma 4.4.16. *After any number of calls to Procedure 4.4.13, every twice marked vertex is either adjacent to one of its supervisors, or has a neutral edge, or has a non-empty cluster set.*

Proof. Let w be a twice marked vertex. Initially, no vertex is twice marked. Therefore w became twice marked at some point, and this change was not reset afterwards. Let us consider the moment w became twice marked. If w is twice marked in Step 2 or 3 of Procedure 4.4.1, then either it is adjacent to one of it supervisor, or it is given a subordinate. If w is twice marked in Step 4 of Procedure 4.4.1 as a vertex of W, it later becomes adjacent to its supervisor v in Step 5.

Suppose v became twice marked in Procedure 4.4.13 or in Step 4a of Procedure 4.4.1. Then some call A to Procedure 4.4.1 is applied for this vertex right after. Therefore, by Lemma 4.4.12, since Π is true at the start of each call to Procedure 4.4.1, at the end of call A, either v has a neutral edge, or it has a non-empty cluster set, or it is in I. If it is in I, then by Lemma 4.4.14, v became twice marked in Step 4a of Procedure 4.4.1, and thus the marking of v is reset in Step 4b, a contradiction. Therefore v has a neutral edge or a non-empty cluster set. That ends the proof of the lemma. □

Lemma 4.4.17. *No unmarked vertex has a cluster set or a giving edge.*

Proof. This is true in the beginning as no vertex has a cluster set or a giving edge. Every time a vertex is given a cluster set or a giving edge in Procedures 4.4.1 and 4.4.13, this vertex is twice marked. We always reset the cluster sets, the giving edge and the markings together. Finally, no twice marked vertex directly becomes unmarked or marked once. □

Lemma 4.4.18. *Every vertex belongs to at most two cluster sets. Moreover if a vertex belongs to two cluster sets, then that vertex has no giving edge and the two sets contain only that vertex.*

Proof. Let v be a vertex in two cluster sets S and T. When S was set, v was twice marked. Indeed, every vertex that is put in a cluster set is twice marked. As the cluster sets are reset together with the markings, and no twice marked vertex directly becomes unmarked or marked once, this implies that v remains twice marked afterwards.

Let us prove that v was unmarked at the start of the call to Procedure 4.4.1 or 4.4.13 in which S is set. If S is set in Steps 1, 2 or 3 of Procedure 4.4.1, this is trivial. If S is set in Step 5 of Procedure 4.4.1, then the last time C was set in Step 4, v was put in C, and therefore v was in O^0. Furthermore, at that point the set of the markings was the same as it was at the start of the procedure. Therefore if S is set in Procedure 4.4.1, then v was unmarked at the start of the procedure. If S is set in Procedure 4.4.13, then v is in C, and thus was unmarked at the start of the procedure.

Therefore v was unmarked at the start of the call to Procedure 4.4.1 or 4.4.13 in which S was set, and was twice marked at the end of that call. The same holds for T. Therefore S and T were set in the same call to Procedure 4.4.1 or 4.4.13. The only time this can happen is in Step 2 of procedure 4.4.1, and this implies that $S = T = \{v\}$, and v has no giving edge by Lemma 4.4.17. By what precedes, if v is also in a third cluster set U, then U was also set in the same call to Procedure 4.4.1 or 4.4.13 as S and T, which is impossible. □

Note that the previous proof also implies the following lemma.

Lemma 4.4.19. *Every vertex that is in a cluster set is twice marked.*

Lemma 4.4.20. *After any number of calls to Procedure 4.4.13, every twice marked vertex is in a cluster set.*

Proof. In Procedures 4.4.1 and 4.4.13, whenever a vertex that was in V^2 is put in $V^0 \cup V^1$, it happens because of a reset of the markings. Therefore, since the resets are well nested (they correspond to the nested calls to Procedures 4.4.1 and 4.4.13), the resets of the markings can never put a vertex from $V^0 \cup V^1$ to V^2.

When a vertex w that was unmarked becomes twice marked in Procedure 4.4.1 or 4.4.13, it either happens in Steps 2 or 3 of Procedure 4.4.1, in which case w is put in $S(v)$, or it happens in Step 4 of Procedure 4.4.1, in which case w is in C. Besides that, whenever a vertex w becomes twice marked, w was marked once, so it was in C for some call A to Procedure 4.4.1 or Procedure 4.4.13. Hence every time a vertex w becomes twice marked, either it is in a cluster set or it is in C for some call A to Procedures 4.4.1 or 4.4.13, in which case, when call A stops, either the vertex w is still in C, in which case it is put in a cluster set, or it is no longer in C and the markings were reset since w became twice marked. Since the cluster sets and the markings are reset at the same time, this proves the lemma. □

Lemma 4.4.21. *After any number of calls to Procedure 4.4.13, if a vertex x has a giving edge $e(x) = xw$, then x and w are twice marked, they are not in the same cluster set, neither of them is the subordinate of the other one, and we do not have $e(w) = wx$.*

Proof. Whenever a giving edge xw is given to a vertex x in a call A to Procedure 4.4.1 or 4.4.13, w was already twice marked at the start of call A, and x was unmarked at the start of call A. This implies that we do not have $e(w) = wx$ unless the considered edge is reset. Moreover, x is put in the cluster set defined at the end of call A, and w is not since it was already twice marked at the start of call A. This implies that x and w are not put in the same cluster set. Moreover, by Lemmas 4.4.17 and 4.4.19, x and w were not the subordinates of one another before the start of call A. Furthermore, no vertex that was twice marked before the start of a procedure is ever put in a cluster set or given a subordinate in that procedure unless it has just

become twice marked and Procedure 4.4.13 is applied to it. Therefore x and w were not the subordinates of one another at the end of call A. After call A, by Lemma 4.4.17, x and w are both twice marked, and thus neither of them is given new subordinates. As the cluster sets and the giving edges are always reset together, that proves the lemma. □

Lemma 4.4.22. *A neutral edge is incident to two twice marked vertices. A neutral edge is not incident to a vertex and one of its subordinates. A neutral edge is not incident to two vertices belonging to a same cluster set. A neutral edge is not a giving edge.*

Proof. Let us consider a neutral edge vw. The only time it can have been set as a neutral edge is in Step 1 of a call A to Procedure 4.4.1 for vertex v. The cluster set of v is empty, therefore w is not a subordinate of v. Moreover, w is a vertex in I^2 when A is called in some call B to Procedure 4.4.1 or 4.4.13, and v became twice marked just before A was called. The vertex v is in C for call B, and will be put in the set S containing all the element of C (unless C is redefined, in which case the giving edge vw is reset). Let us consider the time when call A is applied. When the set C (for call B) was defined, it contained only elements of O, and if after some call A' called by B before A, the colouring had changed, then by Lemma 4.4.4 the set C would have been redefined. Therefore when A is applied, all the elements of C for call B are in O, therefore w is not in that set C. Therefore v and w are not in the same cluster set. Additionally, when the giving edges are defined in B, they are between vertices in O, so vw is not defined as a giving edge in B. Before call A, the vertex v is not twice marked, and after call A, the vertices v and w are both twice marked, thus vw is never defined as a giving edge. Moreover, if B is a call to Procedure 4.4.1, then v is not in the set W of call B, therefore v is not adjacent to its supervisor if it has one, and thus v is not a subordinate of w. As the markings, the cluster sets, the giving edges and the neutral edges are reset together, that ends the proof of the lemma. □

Lemma 4.4.23. *After any number of calls to Procedure 4.4.13, every non-empty cluster set S satisfies one of the following:*

- *S is a singleton $\{v\}$ with two supervisors ; moreover v has no subordinates and no giving edges.*
- *For each component K of $G[S]$, either K has a vertex with a giving edge, or $\sum_{x\in K}(d(x)-2) \geq k-1$.*

Proof. The lemma is true initially as there are no cluster sets.

Let us consider a set S that does not verify the lemma. The set S is defined in some call A to Procedure 4.4.1 or 4.4.13. If S is defined in Step 1, 2, or 3 of Procedure 4.4.1, then the lemma is trivially verified for S. We can assume that A is either a call to Procedure 4.4.13, or that Step 5 is reached. We consider the set C as it is last defined in call A, and if A is a call to Procedure 4.4.13, then let $X = \emptyset$. Let K be the component of S defined from a component C' of $G[C]$ (K is defined from C' by removing some vertices that have degree 1 in C', and degree 2 in G). By construction, either C' contains a 3-vertex with a giving edge, or $|C'| \geq k+1$. Suppose $|C'| \geq k+1$. For every vertex u of $C' \setminus X$, since u was not put in I in Step 4, u has a neighbour that is not in C (that was in I when C was defined), and thus not in C'. For every vertex u of X, u is adjacent to its supervisor, which is not in C'. Therefore $\sum_{x\in K}(d(x)-2) = \sum_{x\in C'}(d(x)-2) = \sum_{x\in C'}(d_{C'}(x)-1)$, where $d_{C'}(x)$ is the number of neighbours of x in C'. The graph C' being connected, $\sum_{x\in C'}(d_{C'}(x)-1) \geq |C'|-2 \geq k-1$. Therefore, the set S verifies the lemma. □

Lemma 4.4.24. *Let v be a 2-vertex in V^2. The vertex v is adjacent to all of its supervisors.*

Proof. Procedure 4.4.13 does not mark twice a 2-vertex, except in its calls to Procedure 4.4.1. Moreover, every time a vertex becomes marked once, that vertex is a 3^+-vertex, thus only 3^+-vertices are ever marked once.

In Procedure 4.4.1, if a 2-vertex becomes twice marked in Step 2 or 3, then it is adjacent to its supervisor(s), and if a 2-vertex becomes twice marked in Step 4, then it belongs to $X \subseteq W$ and thus is also adjacent to its supervisor. No vertex becomes twice marked anywhere else in Procedure 4.4.1. As the cluster sets and the markings are reset together, that proves the lemma. □

Discharging procedure

Let $M = \frac{8k}{3k+1}$. Every vertex v starts with a weight equal to $d(v) - M$. As M is larger than the average degree of G, the sum of the weights of the vertices is negative. We will move some weights from vertices to other vertices, without introducing or removing weights, so that every vertex has non-negative weight at the end of the procedure. That contradiction will complete the proof.

In the discharging procedure, the weights will move via the edges (i.e. the weights will always move from one vertex to one of its neighbours). We apply the following four steps.

- *Step 1*: Every vertex w with two supervisors receives $\frac{M-2}{2}$ from each of its supervisors.
- *Step 2*: Every vertex, that is twice marked, gives $M - 2$ to each of its unmarked neighbours.
- *Step 3*: Every vertex v that has a giving edge, say $e(v) = vw$, receives $M - 2$ from w.
- *Step 4*: For every component C of the graph induced by a cluster set S, we do the following. Let T be a spanning tree of C. For each pair of adjacent vertices u and v, let n_{Tuv} be the sum, for all vertex w of the component of v in $T - u$, of $d(w) - 2$. For each vertex u of T and each neighbour v of u in T, the vertex u gives $M - 2 - n_{Tuv}(4 - \frac{3M}{2})$ to v if this value is positive, unless v is in the same component of $T - u$ as a vertex that has a giving edge (in that case u does not give weight to v).

Lemma 4.4.25. *No unmarked vertex gives weight to another vertex.*

Proof. By contradiction, suppose there is an unmarked vertex v that gives weight to another vertex u.

By Lemma 4.4.17, v has no subordinate, so it does not give weight in Step 1 of the discharging procedure.

Since v is unmarked, v does not give weight in Step 2.

Suppose v gives weight to u in Step 3. Then $e(u) = uv$. Every time a giving edge is set in Procedures 4.4.1 and 4.4.13, its other endpoint is twice marked. Moreover the marking and giving edges are always reset together. That implies v is twice marked, a contradiction.

Suppose v gives weight in Step 4. Then v is in a cluster set and thus v is twice marked by Lemma 4.4.19, a contradiction. □

Lemma 4.4.26. *Every unmarked vertex has non-negative weight at the end of the discharging procedure.*

Proof. We applied Procedure 4.4.13 until every 2-vertex is either twice marked or has a neighbour that is twice marked. Each unmarked vertex either has degree at least $3 > M$ or has initial weight $2 - M$ and receives at least $M - 2$ in Step 2 of the discharging procedure. Furthermore, each of these vertices does not give weight by Lemma 4.4.25. That ends the proof of the lemma. □

Lemma 4.4.27. *No weight is given through a neutral edge.*

Proof. This is a direct consequence of Lemma 4.4.22. □

Lemma 4.4.28. *A subordinate does not give weight to its supervisor. A supervisor does not give weight to its subordinates except maybe in Step 1.*

Proof. Suppose v is the supervisor of a vertex w. By Lemmas 4.4.15, 4.4.17, and 4.4.19, v and w are twice marked and thus do not give weight to each other in Step 2. By Lemma 4.4.21, they do not give weight to each other in Step 3. By Lemma 4.4.18, v and w cannot be both contained in a same cluster set ; hence they do not give weight to each other in Step 4.

Now, if w gives weight to v in Step 1, then v is in two cluster sets. Then, by Lemma 4.4.18, v has no giving edge and is alone in its cluster sets, therefore by Lemma 4.4.23, w has no subordinates, a contradiction. Thus w does not give weight to v in Step 1. □

Lemma 4.4.29. *Every vertex v gives weight to each of its neighbours w at most once in the discharging procedure. Moreover, if v gives weight to w, then w does not give weight to v.*

Proof. Let v and w be two adjacent vertices. Suppose that v gives weight to w at some point in the discharging procedure. By Lemmas 4.4.15 and 4.4.25, v is twice marked.

Suppose first that v gives weight to w in Step 1. By Lemma 4.4.28, w does not give weight to v and v does not give weight to w in Steps 2-4.

Suppose now that v gives weight to w in Step 2. As w is then unmarked, it does not give weight by Lemma 4.4.25. By Lemma 4.4.17 (resp. Lemma 4.4.19), v does not give weight to w in Step 3 (resp. Step 4).

Suppose that v gives weight to w in Step 3. By Lemma 4.4.21, v and w are in distinct cluster sets, thus they do not give weight to each other in Step 4. Also by Lemma 4.4.21, we do not have $e(v) = vw$, thus w does not give weight to v in Step 3.

Lastly, suppose for contradiction that v gives weight to w and w gives weight to v in Step 4. Then v and w are in the same (unique by Lemma 4.4.18) cluster set S and in the same component C of $G[S]$. Let T be the tree that was chosen in Step 4 of the discharging procedure for C.

Observe that no vertex in the component of v in $T - w$ nor in the component of w in $T - v$ has a giving edge, and thus that no vertex of C has a giving edge. Also observe that $n_{Tvw} + n_{Twv} = \sum_{x \in C}(d(x) - 2) \geq k - 1$ by Lemma 4.4.23. Since v and w both give weight, we have $M - 2 - n_{Twv}(4 - \frac{3M}{2}) > 0$ and $M - 2 - n_{Tvw}(4 - \frac{3M}{2}) > 0$. Therefore $2M - 4 - (k-1)(4 - \frac{3M}{2}) > 0$, i.e. $M > \frac{8k}{3k+1}$, a contradiction. That ends the proof of the lemma. □

Lemma 4.4.30. *Every vertex that is twice marked has non-negative weight at the end of the discharging procedure.*

Proof. Let v be a d-vertex that is twice marked ($d \geq 2$). By Lemma 4.4.20, v is in a cluster set S. By Lemma 4.4.23,

- either S is the singleton $\{v\}$ with two supervisors, say u and w, and no giving edge,
- or the component of $G[S]$ containing v, say C, has a vertex having a giving edge or satisfies $\sum_{x \in C}(d(x) - 2) \geq k - 1$.

Consider the former case. The vertices u and w both give $\frac{M-2}{2}$ to v in Step 1. The vertex v gives nothing to u and w by Lemma 4.4.29, and at most $(d(v) - 2)(M - 2) \leq d(v) - 2$ to its other neighbours. It follows that v has weight at least equal to $d(v) - M + 2\frac{M-2}{2} - (d(v) - 2) = 0$.

Consider the latter case. Let T be defined for C in Step 4 of the discharging procedure. By Lemma 4.4.16, v is either adjacent to one of its supervisors, or is incident to a neutral edge, or has a non-empty cluster set. In each case, v is adjacent to a vertex w that is either the supervisor of v, or the other endpoint of a neutral edge, or one of the subordinates of v. By Lemmas 4.4.18 and 4.4.22, w does not belong to S. By Lemmas 4.4.27 and 4.4.28, v gives weight at most $\frac{M-2}{2}$ to w, and it does not give weight to w if w is the supervisor of v. Let v_1, ..., v_{d-1} be the neighbours of v distinct from w. For the v_i's that are not the neighbours of v in T, n_{Tvv_i} is not defined and we set $n_{Tvv_i} = 0$, and v gives at most $M - 2 = M - 2 - n_{Tvv_i}(4 - \frac{3M}{2})$ to v_i.

Suppose that at least two of the v_i's give weight to v or are in a component of $T - v$ containing a vertex that has a giving edge. Those two vertices do not receive weight from v by Lemma 4.4.29 and Step 4. As v has initial weight equal to $d - M$ and gives at most $M - 2$ to each of its neighbours distinct from w, the final weight of v is at least $d - M - \frac{M-2}{2} - (d-3)(M-2) = (d-3)(3-M) + 4 - \frac{3M}{2} \geq 0$ since $M < \frac{8}{3}$.

Suppose one of the v_i's, say v_1, gives weight to v (by Step 3 or 4) or is into a component of $T - v$ containing a vertex that has a giving edge. By Lemma 4.4.29 and Step 4, v does not give weight to v_1. If $e(v) = vv_1$, then v receives $M - 2$ from v_1. In that case, n_{Tvv_i} is not defined and we set $n_{Tv_1v} = 0$. So by Step 3 or 4, v receives at least $M - 2 - n_{Tv_1v}(4 - \frac{3M}{2}) = \frac{4-M}{2} - (n_{Tv_1v} + 1)(4 - \frac{3M}{2})$ from v_1. Moreover v gives at most $\frac{4-M}{2} - (n_{Tvv_i} + 1)(4 - \frac{3M}{2})$ to each of the other v_i's. Observe that $\sum_{i>1}(n_{Tvv_i} + 1) \geq n_{Tv_1v}$ (since v has one neighbour outside from the v_i's). At last, v gives at most $\frac{M-2}{2}$ to its last neighbour w. Overall, v receives at least $\frac{4-M}{2} - (n_{Tv_1v} + 1)(4 - \frac{3M}{2})$ and gives at most $(d-2)\frac{4-M}{2} - n_{Tv_1v}(4 - \frac{3M}{2}) + \frac{M-2}{2}$, and initially it has a weight equal to $d - M$. Therefore, the final weight of v is at least $d - M + \frac{4-M}{2} - (n_{Tv_1v} + 1)(4 - \frac{3M}{2}) - (d-2)\frac{4-M}{2} + n_{Tv_1v}(4 - \frac{3M}{2}) - \frac{M-2}{2} = d - M - (d-3)\frac{4-M}{2} - (3 - M) = (d-3)\frac{M-2}{2}$. Hence, if $d \geq 3$, then the final weight of v is non-negative. If $d = 2$, then v is adjacent to its supervisor by

Lemma 4.4.24 and v gives nothing to its supervisor. Therefore the final weight of v is at least $\frac{M-2}{2}$ more than what we computed above, and thus is non negative.

Suppose now that none of the v_i's give weight to v and that none of them are into a component of $T - v$ containing a vertex having a giving edge. That implies that no vertex of C has a giving edge, $\sum_{x \in C}(d(x) - 2) \geq k - 1$ by Lemma 4.4.23, and thus $\sum_i n_{Tvv_i} \geq k - d + 1$. For all i, v gives at most $M - 2 - n_{Tvv_i}(4 - \frac{3M}{2})$ to v_i. Moreover, v gives at most $\frac{M-2}{2}$ to its last neighbour w. Overall, v gives at most $(d-1)(M-2)-(k-d+1)(4-\frac{3M}{2})+\frac{M-2}{2}$, and initially it has a weight equal to $d-M$. Therefore the final weight of v is at least $d-M-(d-1)(M-2)+(k-d+1)(4-\frac{3M}{2})-\frac{M-2}{2}$. As $M = \frac{8k}{3k+1}$, we have $k(4-\frac{3M}{2}) = \frac{M}{2}$, therefore the final weight of v is at least $d - M - (d-1)(M-2) + \frac{M}{2} - (d-1)(4-\frac{3M}{2}) - \frac{M-2}{2} = (d-3)\frac{M-2}{2}$. As in the previous paragraph, the final weight of v is at least 0 if $d \geq 3$. If $d = 2$, then v is adjacent to its supervisor by Lemma 4.4.24, so v does not give weight to its supervisor, and so its final weight is at least $(d-2)\frac{M-2}{2} = 0$. That ends the proof of Lemma 4.4.30. □

By Lemmas 4.4.15, 4.4.26, and 4.4.30, every vertex has non-negative weight at the end of the procedure. As the sum of the weights of the vertices is negative, that leads to a contradiction. Therefore the counter-example G to Theorem 4.1.6 does not exist and that ends the proof of Theorem 4.1.6.

4.5 $(\mathcal{I}, \Delta_6)$-partition of planar graphs with no C_3, C_4 or C_6

Recall that $\mathcal{C}$ denotes the set of planar graphs with no cycles of length 3, 4 or 6. We are going to prove the following theorem.

Theorem 4.1.9 (recall)**.** *Every graph in $\mathcal{C}$ admits an $(\mathcal{I}, \Delta_6)$-partition.*

Given an $(\mathcal{I}, \Delta_6)$-partition of a graph, we call the respective colour sets I and Δ, and we similarly call the corresponding colours I and Δ.

The proof will be using the discharging method. Let us define the partial order $\preceq$. Let $n_3(G)$ be the number of 3^+-vertices in G. For any two graphs G_1 and G_2, we have $G_1 \prec G_2$ if and only if one of the following conditions holds:

- $n_3(G_1) < n_3(G_2)$;
- $n_3(G_1) = n_3(G_2)$ and $|V(G_1)| < |V(G_2)|$.

Note that the partial order $\preceq$ is well-defined and is a partial linear extension of the subgraph poset.

We suppose for contradiction that G is a graph in $\mathcal{C}$ that has no $(\mathcal{I}, \Delta_6)$-partition and is minimal according to $\preceq$. We consider an embedding of G in the plane. Let n denote the number of vertices, m the number of edges, and f the number of faces of G. In this section, a 7^--vertex will be called a *small vertex* and an 8^+-vertex will be called a *big vertex*. A *small neighbour* (resp. *big neighbour*) of a vertex v is a neighbour of v that is a small vertex (resp. a big vertex).

Let us first prove some results on the structure of G, and then we will prove that G cannot exist, thus proving the theorem.

Lemma 4.5.1. *G is connected.*

Proof. If G is not connected, then every component of G is smaller than G and thus admits an $(\mathcal{I}, \Delta_6)$-partition. The union of these $(\mathcal{I}, \Delta_6)$-partitions gives an $(\mathcal{I}, \Delta_6)$-partition of G, a contradiction. □

Lemma 4.5.2. *G has no 1^--vertex.*

Proof. Since G is connected and the graph with one vertex is not a counter-example, G has no 0-vertex. Let v be a 1-vertex and w be the neighbour of v. The graph $G - v$ admits an $(\mathcal{I}, \Delta_6)$-partition since $G - v \prec G$. We get an $(\mathcal{I}, \Delta_6)$-partition G by assigning to v the colour distinct from the colour of w, a contradiction. □

Lemma 4.5.3. *Every small vertex of G has a big neighbour.*

Proof. Let v be a small vertex with no big neighbours. The graph $G - v$ admits an $(\mathcal{I}, \Delta_6)$-partition since $G - v \prec G$. If there is a neighbour w of v with no I-neighbour, then we colour w with I. Thus, we can assume that every neighbour of v that is coloured Δ has an I-neighbour in $G - v$, and thus at most five Δ-neighbours in $G - v$. Also, we can assume that v has at least one I-neighbour, since otherwise v can be coloured I. Thus, v has at most six Δ-neighbours and v can be coloured Δ, a contradiction. □

Lemma 4.5.4. *Every small 3^+-vertex has two big neighbours.*

Proof. Suppose for contradiction that G contains a d-vertex v such that $3 \leq d \leq 7$ and such that v has at most one big neighbour. By Lemma 4.5.3, v has exactly one big neighbour w. Let $w_1, \ldots, w_{d-1}$ be the other neighbours of v. Let H be the graph obtained from $G - v$ by adding $d - 1$ 2-vertices $v_1, \ldots, v_{d-1}$, such that for every $i \in \{1, d-1\}$, v_i is adjacent to w and w_i.

Notice that $H \prec G$ since $n_3(H) = n_3(G) - 1$. Moreover, every cycle of length ℓ in H is associated with a cycle of length ℓ or $\ell - 2$ in G. Therefore $H \in \mathcal{C}$, so H has an $(\mathcal{I}, \Delta_6)$-partition.

If w is coloured I, then every v_i is coloured Δ, and colouring v with Δ leads to an $(\mathcal{I}, \Delta_6)$-partition of G, a contradiction. Therefore w is coloured Δ.

If at least one of the v_i's is coloured Δ, then w has at most five Δ-neighbours in $G - v$. We then give colour Δ to v, and every w_i that has all of its neighbours in Δ is put in I. That leads to an $(\mathcal{I}, \Delta_6)$-partition of G. Otherwise, every v_i is coloured I, every w_i is coloured Δ, and w is coloured Δ. Thus we can colour v with I to obtain an $(\mathcal{I}, \Delta_6)$-partition of G, a contradiction. □

Lemma 4.5.5. *No 3-vertex is adjacent to a 2-vertex.*

Proof. Let w be a 3-vertex adjacent to a 2-vertex v, let x_1 and x_2 be the other two neighbours of w, and let u be the other neighbour of v. Let H be the graph obtained from $G - \{v, w\}$ by adding five 2-vertices v_1, v_2, w_1, w_2, and x which form the 8-cycle $uv_1w_1x_1xx_2w_2v_2$. It is easy to check that H is in $\mathcal{C}$. By Lemmas 4.5.3 and 4.5.4, u, x_1, and x_2 are big vertices in G and thus are 9^+-vertices in H. Since w is in G but not in H, $n_3(H) = n_3(G) - 1$, so $H \prec G$. Therefore H has an $(\mathcal{I}, \Delta_6)$-partition.

Suppose that v_1 and v_2 are both coloured I. Then w_1, w_2, and u are coloured Δ. We colour v with I and w with Δ. The number of Δ-neighbours of x_1 (resp. x_2) in G is at most the number of Δ-neighbours of x_1 (resp. x_2) in H. Thus we have an $(\mathcal{I}, \Delta_6)$-partition of G, a contradiction. Now we assume without loss of generality that v_1 is coloured Δ. We colour w with the colour of x and we colour v with Δ. The number of Δ-neighbours of u (resp. x_1, x_2) in G is at most the number of Δ-neighbours of u (resp. x_1, x_2) in H. Thus we have an $(\mathcal{I}, \Delta_6)$-partition of G, a contradiction. □

A *special face* is a 5-face with three 2-vertices and two non-adjacent big vertices. See Figure 4.5.1, left. A *semi-special face* is a 5-face with, in this order, a big vertex, a 3-vertex, another big vertex, and two 2-vertices. A *special configuration* is three semi-special faces sharing a common 3-vertex. See Figure 4.5.1, right. We say *special structure* to speak indifferently about a special face or a special configuration.

Let us define a hypergraph $\widehat{G}$ whose vertices are the big vertices of G and the hyperedges correspond to the sets of big vertices contained in the same special structure. For every vertex v of $\widehat{G}$, let $\hat{d}(v)$ denote the degree of v in $\widehat{G}$, that is the number of hyperedges containing v.

Lemma 4.5.6. *Let α be a special structure, with the notation of Figure 4.5.1, and consider an $(\mathcal{I}, \Delta_6)$-partition of α.*

We can change the colour of the x_i's, y_i's, and u such that the v_i's have no more neighbours with colour Δ than before, and for all i, if v_i has colour Δ, then v_i has an I-neighbour.

Proof. If all of the v_i's are in I, then there is nothing to do. If they are all coloured Δ, then we give colour I to u. If one of the v_i's, say v_0, is coloured I and another one, say v_1, is coloured Δ, then u and x_0 are coloured Δ and we can give colour I to y_0. Moreover, if α is a special configuration and v_2 is coloured Δ, then y_2 is coloured Δ and we can give colour I to x_2. □

Lemma 4.5.7. *For every vertex v in $\widehat{G}$, $d(v) - \hat{d}(v) \geq 7$.*

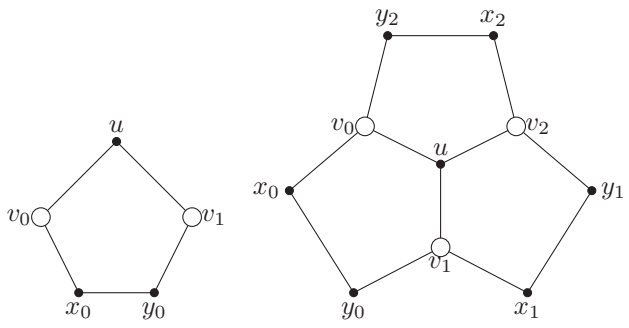

Figure 4.5.1: A special face (left) and a special configuration (right).

Proof. Let v be a vertex that does not verify the lemma, i.e. such that $d(v) - \hat{d}(v) \leq 6$. As v is a big vertex in G, $\hat{d}(v) \geq 2$. Let α be a special structure incident to v in $\widehat{G}$. We use the notation of Figure 4.5.1, with say $v = v_0$. The graph $G - x_0$ is smaller than G, thus it admits an $(\mathcal{I}, \Delta_6)$-partition. As G does not admit such a colouring, v_0 has colour Δ and y_0 has colour I. By Lemma 4.5.6, we can assume that v has an I-neighbour in each of its special structures distinct from α. Since y_0 has colour I, v_1 is coloured Δ. If α is a special face, or if v_2 has colour Δ, then we can colour u with I. If α is a special configuration and v_2 has colour I, then x_2 has colour Δ and we can colour y_2 with I. In both cases, v has at least $\hat{d}(v)$ I-neighbours, thus even if we give colour Δ to x_0, v has at most $d(v) - \hat{d}(v) \leq 6$ neighbours in Δ, thus G has an $(\mathcal{I}, \Delta_6)$-partition, a contradiction. □

Lemma 4.5.8. *Every component of $\widehat{G}$ has at least one vertex v such that $d(v) - \hat{d}(v) \geq 8$.*

Proof. Suppose the lemma is false, and let C be a component of $\widehat{G}$ that does not verify the lemma. If C has only one vertex, then this vertex is a big vertex, which verifies $d(v) - \hat{d}(v) \geq 8$ (as $\hat{d}(v) = 0$.) Therefore C has at least one hyperedge, which corresponds to a special structure α of G. By Lemma 4.5.7, every vertex of C verifies $d(v) - \hat{d}(v) = 7$. We use the notation of Figure 4.5.1. The graph $G - \{x_0, y_0\}$ is smaller than G, thus it admits an $(\mathcal{I}, \Delta_6)$-partition. Since G admits no $(\mathcal{I}, \Delta_6)$-partition, v_0 and v_1 are coloured Δ. If α is a special configuration and v_2 is coloured I, then x_2 and y_1 are coloured Δ and we can colour y_2 and x_1 with I. Otherwise, we can colour u with I. Note that v_0 and v_1 both have six Δ-neighbours. Moreover, by Lemma 4.5.6, we can assume that each of the v_i's that is in Δ has at least one I-neighbour in each of its special structures other than α.

If one of the either v_0 or v_1, say v_0, has an additional neighbour with colour I, it verifies $d(v_0) - \hat{d}(v_0) \geq 8$, a contradiction. In particular each of v_0 and v_1 has at most one I-neighbour in α. Suppose α is a special configuration and u is in I. Then x_2 and y_1 are also in I and v_2 has six Δ-neighbours, otherwise one of y_2 and x_1 could be put in I and one of v_0 and v_1 would have at most five Δ-neighbours, a contradiction. Thus, for every v_i, either v_i is coloured I or v_i has no I-neighbour outside of its special structures and at most one I-neighbour in each special structure other than α.

We uncolour u and all the x_i's and y_i's, and let H be equal to G where u, the x_i's and the y_i's are removed. By symmetry, we only consider the vertex v_0. The following procedure either gives colour I to v_0 or makes it so that it has two I-neighbours in one of its special structures:

- For each special structure β containing v_0 and completely contained in H, we use the notation of Figure 4.5.1, keeping the same vertex for v_0, but changing the other ones for the vertices in β, and do the following:
  - By Lemma 4.5.6, we can assume that every v_i coloured Δ has an I-neighbour in each of its special structures that are completely contained in H.

- Suppose that one of the big vertices of β distinct from v_0, say v_1, has two I-neighbours in a special structure distinct from β or an I-neighbour that is not in one of its special structures. As $d(v_1) - \hat{d}(v_1) = 7$, v_1 has at most five Δ-neighbours outside of β if β is a special face, and at most four Δ-neighbours outside of β if β is a special configuration. We colour y_0 with colour Δ and x_0 with colour I. If v_2 exists and is coloured I, then we can colour y_2 with I, and otherwise we can colour u with I. We end the procedure.
- We uncolour the small vertices of β and remove them from H.
- For every big vertex $w \neq v_0$ in β coloured Δ, we apply the procedure with w instead of v_0. Now w is coloured I or has two I-neighbours in the same special structure.
- We add back to H the small vertices of β. If v_0 is coloured I, then we give colour Δ to those that have an I-neighbour, we give colour I to the other ones, and we end the procedure. Now v_0 is coloured Δ. If β is a special face and v_1 is coloured Δ, or if β is a special configuration and v_1 and v_2 are coloured Δ, then we colour u and x_0 with I, we colour the other 2-vertices with Δ, and we end the procedure. Suppose β is a special configuration, either v_1 or v_2, say v_1, has colour Δ, and the other one has colour I. We give colour I to x_0, x_1, and y_2, and colour Δ to u, y_0, y_1, and x_1, and we end the procedure. Now all of the v_i's distinct from v_0 have colour I. We colour x_0 and y_2 (if it exists) with I and we colour the other small vertices in β with colour Δ.

- If we reach this point, then in each special structure containing v_0 and completely contained in H, all of the big vertices distinct from v_0 have colour I. We give colour Δ to all of the neighbours of v_0, and colour I to v_0.

Let us prove that the previous procedure terminates. It always calls itself iteratively on a graph with fewer vertices, thus the number of nested iterations is bounded by the order of the initial graph. Furthermore, each iteration of the procedure only does a bounded number of calls to the procedure (at most two). That proves that the procedure terminates.

In the end, if one of the v_i's is coloured Δ, then it has at most five Δ-neighbours outside of α if α is a special face, and at most four Δ-neighbours outside of α if α is a special structure. If every v_i is coloured Δ, then colour u with colour I and the other small vertices of α with colour Δ. Otherwise, give colour Δ to u, and do the following:

- If every v_i is coloured I, then colour all of the x_i's and y_i's with colour Δ.
- If α is a special face and one of the v_i's, say v_0, is coloured I while the other one is coloured Δ, then give colour Δ to x_0 and I to y_0.
- If α is a special structure, then give colour Δ to the y_i's, and for all $i \in \{0, 1, 2\}$, if v_i is coloured Δ, then give colour I to x_i, and if v_i is coloured I, then give colour Δ to x_i.

In all cases, we get an $(\mathcal{I}, \Delta_6)$-partition of G, a contradiction. □

For each component C of $\widehat{G}$, we choose a vertex v in C such that $d(v) - \hat{d}(v) \geq 8$ (by Lemma 4.5.8) as the *root* of C. We set that the vertex v *sponsors* all of its hyperedges. Then while there is a hyperedge α in C that has no sponsor, we choose a vertex in one such hyperedge that is also in a hyperedge that is already sponsored, and we set it to sponsor α. We can always do that until all of the hyperedges are sponsored, since C is connected. That way, every vertex in C other than v is in at least one hyperedge it does not sponsor. As the hyperedges of $\widehat{G}$ correspond exactly to the special structures of G, we say that a vertex sponsors one of its special structures in G if and only if it sponsors the corresponding hyperedge in $\widehat{G}$.

Discharging procedure

We are now going to give some weight on the vertices and faces of the graph. Initially, for all integer $d \geq 2$, every d-vertex has weight $d - 4$, and every d-face has weight $d - 4$. Thus every face and every 4^+-vertex has non-negative initial weight.

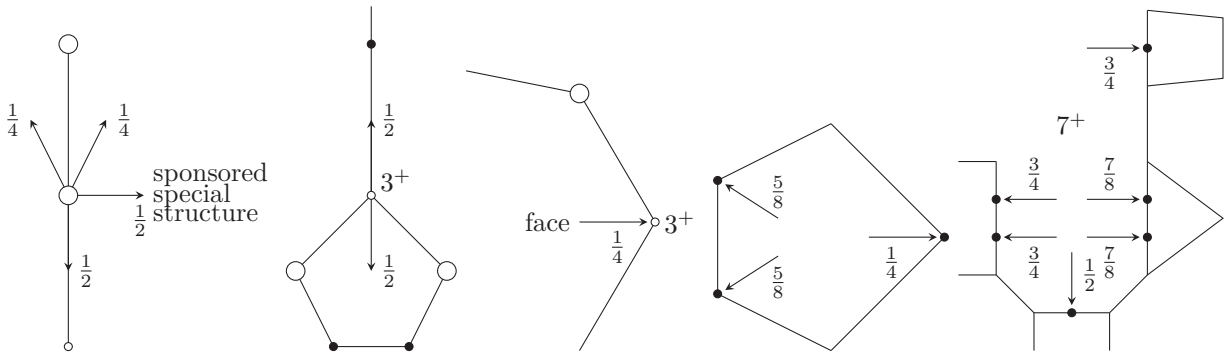

Figure 4.5.2: The discharging steps. From left to right, Steps 1 to 5.

We are now ready to give the discharging procedure. See Figure 4.5.2 for an illustration of the different steps.

- *Step 1*: Every big vertex gives weight $\frac{1}{2}$ to each of its small neighbours, $\frac{1}{2}$ to each special face it sponsors, and an additional $\frac{1}{2}$ to the 3-vertex of each special configuration it sponsors. Additionally, for every edge vw where v and w are big vertices, v and w each give $\frac{1}{4}$ to each of the faces containing the edge vw, and $\frac{1}{4}$ more to the face containing vw if there is only one face containing vw.

- *Step 2*: For each small 3^+-vertex v in G, v gives $\frac{1}{2}$ to each of its 2-neighbours. Moreover, it gives $\frac{1}{2}$ to each of its incident 5-faces where it is adjacent to two big vertices and where there are two 2-vertices.

- *Step 3*: Every time a small 3^+-vertex v appears in the boundary of a face f, if (at least) one of the two neighbours of v that appear consecutively to v in the boundary of G is a big vertex, f gives $\frac{1}{4}$ to v.

- *Step 4*: Each 5-face gives $\frac{1}{4}$ to each of its 2-vertices with no 2-neighbour and $\frac{5}{8}$ to its 2-vertices with a 2-neighbour.

- *Step 5*: Each 7^+-face gives $\frac{3}{4}$ to each of its 2-vertices that belong to a 5-face and have no 2-neighbours, $\frac{7}{8}$ to each of its 2-vertices that belong to a 5-face and have a 2-neighbour, $\frac{1}{2}$ to each of its 2-vertices that do not belong to a 5-face and have no 2-neighbours, and $\frac{3}{4}$ to each of its 2-vertices that do not belong to a 5-face and have a 2-neighbour.

Let ω be the initial weight distribution, and let ω' be the final weight distribution (after the discharging procedure).

Lemma 4.5.9. *Every vertex v verifies $\omega'(v) \geq 0$.*

Proof. Let v be a vertex of degree d. We have $\omega(v) = d - 4$.

- Suppose first that $d \geq 8$. The vertex v gives $\frac{1}{2}$ to each of its small neighbours and twice $\frac{1}{4}$ for each of its big neighbours in Step 1, for a total of $\frac{d}{2}$. As $d \geq 8$, we have $\omega(v) = d - 4 \geq \frac{d}{2}$. Therefore if v sponsors no special structure, then $\omega'(v) = d - 4 - \frac{d}{2} \geq 0$.

  Suppose v sponsors a special structure. If v sponsors all of its special structures, then v is the root of its component in $\widehat{G}$, thus $d - \hat{d}(v) \geq 8$, and we have $\omega'(v) = d - 4 - \frac{\hat{d}(v)}{2} - \frac{d}{2} = d - \hat{d}(v) - 4 - \frac{d - \hat{d}(v)}{2} \geq 0$. If v does not sponsor all of its special structures, then $d - \hat{d}(v) \geq 7$, and $\omega'(v) = d - 4 - \frac{\hat{d}(v) - 1}{2} - \frac{d}{2} = d - \hat{d}(v) - \frac{7}{2} - \frac{d - \hat{d}(v)}{2} \geq 0$.

- Suppose now that $4 \leq d < 8$. By Lemma 4.5.4, v has at least two big neighbours. The vertex v only gives weight in Step 2. Moreover, it gives at most $\frac{1}{2}$ to each of its 2-neighbours plus $\frac{1}{2}$ for each pair of consecutive big vertices in Step 2. If v has only big neighbours, then it receives $\frac{d}{2}$ in Step 1, and gives at most $\frac{d}{2}$ in Step 2, so $\omega'(v) \geq \omega(v) = d - 4 \geq 0$. Suppose v has at least one small neighbour. Let $d' \geq 2$ be the number of big neighbours of v. The vertex v receives $\frac{d'}{2}$ in Step 1. It gives at most $\frac{d-d'}{2}$ to the 2-vertices and at most $\frac{d'-1}{2}$ to the faces for a total of at most $\frac{d-d'}{2} + \frac{d'-1}{2} = \frac{d}{2} - \frac{1}{2}$ in Step 2. It receives at least $\frac{d'}{4}$ in Step 3. We have $\omega'(v) \geq d - 4 - \frac{d}{2} + 3\frac{d'}{4} + \frac{1}{2} \geq 0$, since $d' \geq 2$ and $d \geq 4$.

- Suppose that $d = 3$. By Lemma 4.5.4, v has at least two big neighbours, and by Lemma 4.5.5, v has no 2-neighbours. If v has exactly two big neighbours, then it receives 1 in Step 1, gives $\frac{1}{2}$ in Step 2, and receives $\frac{3}{4}$ in Step 3, therefore $\omega'(v) \geq \frac{1}{4} > 0$. If v has three big neighbours and is not in a special configuration, then v receives $\frac{3}{2}$ in Step 1 and an additional $\frac{3}{4}$ in Step 3, and it gives at most 1 in Step 2 to its semi-special faces. If v is in a special configuration, then it receives 2 in Step 1 an additional $\frac{3}{4}$ in step 3, and it gives at most $\frac{3}{2}$ in Step 2. Therefore if v has three big neighbours, then $\omega'(v) \geq \frac{1}{4} > 0$.

- Suppose that $d = 2$. Note that v cannot be in two 5-faces since $G \in \mathcal{C}$ (otherwise there would be a C_6 in G).

  - If v is in a 5-face and adjacent to another 2-vertex, then it receives $\frac{1}{2}$ from its big neighbour in Step 1, $\frac{5}{8}$ from its 5-face in Step 4, and $\frac{7}{8}$ from its other face in Step 5.
  - If v is in a 5-face and adjacent to no other 2-vertex, then it receives 1 from its 3^+-neighbours in Steps 1 and 2, $\frac{1}{4}$ from its 5-face in Step 4, and $\frac{3}{4}$ from its other face in Step 5.
  - If v is not in a 5-face and is adjacent to another 2-vertex, then it receives $\frac{1}{2}$ from its big neighbour in Step 1, and $\frac{3}{2}$ from its faces in Step 5.
  - If v is not in a 5-face and adjacent to no other 2-vertex, then it receives 1 from its 3^+-neighbours in Steps 1 and 2, and 1 from its faces in Step 5.

  In all cases, v receives 2 over the procedure, and thus $\omega'(v) = 2 - 4 + 2 = 0$.

□

Lemma 4.5.10. *Every face α satisfies $\omega'(\alpha) \geq 0$.*

Proof. Let α be a face of degree d. We have $\omega(\alpha) = d - 4$. When we consider the vertices in the boundary of α, we will assume that they are all distinct. If that is not the case, i.e. if one vertex appears several times in the boundary of α, we can treat all of these appearances independently. Let S be the set of small vertices in the boundary of α that appear consecutively to a big vertex in this boundary. Note that one big vertex in this boundary is consecutive to at most two vertices in S, so S contains at most two thirds of the vertices of the boundary. Note that the only vertices that α gives weight to are vertices of S, in steps 3 and 4 (recall that a 2-vertex always has a big neighbour).

- Suppose $d = 5$. Since $\frac{2}{3} \cdot 5 < 4$, there are at most three vertices in S. If α is a special face, then it receives $\frac{1}{2}$ in Step 1 and gives $\frac{1}{4} + 2 \cdot \frac{5}{8} = \frac{3}{2}$ in Step 4.

  If α has no two consecutive 2-vertices, then it gives at most $\frac{1}{4}$ to vertices in S over Steps 3 and 4.

  If α has two consecutive 2-vertices and its three other vertices are big vertices, then it receives 1 in Step 1 and gives at most $2 \cdot \frac{5}{8} = \frac{5}{4} \leq 2$ overall.

  The only remaining case is when α has, in this consecutive order, two 2-vertices, a big vertex, a small 3^+-vertices, and another big vertex. In that case, α receives $\frac{1}{2}$ in Step 2, and gives $2 \cdot \frac{5}{8} + \frac{1}{4} = \frac{3}{2}$ over Steps 3 and 4.

  In all cases, $\omega'(\alpha) \geq 1 - 1 = 0$.

- Suppose $d = 7$. Note that if there are two adjacent 2-vertices in α, then these two vertices are not in a 5-face, otherwise there would be a cycle of length 6 in G. The face α has an initial charge of 3, gives at most $\frac{3}{4}$ to vertices of S, and there are at most $4 = \lfloor \frac{2}{3} \cdot 7 \rfloor$ of those vertices. Therefore $\omega'(\alpha) \geq 3 - 4 \cdot \frac{3}{4} = 0$.

- Suppose $d = 8$. Note that at most one pair of adjacent 2-vertices in α is also in a 5-face, otherwise there would be a cycle of length 6 in G. The face α has an initial charge of 4 and gives at most $\frac{7}{8}$ to vertices in S. There can be at most $5 = \lfloor \frac{2}{3} \cdot 8 \rfloor$ of these vertices, and at most two are given $\frac{7}{8}$, the other being given at most $\frac{3}{4}$. Therefore $\omega'(\alpha) \geq 4 - 2 \cdot \frac{7}{8} - 3 \cdot \frac{3}{4} = 0$.

- Suppose $d = 9$. Note that at most two pairs of adjacent 2-vertices in α are also in a 5-face, otherwise there would be a cycle of length 6 in G. The face α has an initial charge of 5, and gives at most $\frac{7}{8}$ to vertices in S. There can be at most $6 = \lfloor \frac{2}{3} \cdot 9 \rfloor$ of these vertices, at most four are given $\frac{7}{8}$, and the others are given at most $\frac{3}{4}$. Therefore $\omega'(\alpha) \geq 5 - 4 \cdot \frac{7}{8} - 2 \cdot \frac{3}{4} = 0$.

- Suppose $d \geq 10$. The face α has an initial charge of $d - 4$ and gives at most $\frac{7}{8}$ to vertices in S. There are at least $\lceil \frac{10}{3} \rceil = 4$ vertices in the boundary of α that are not in S. There can be at most $d - 4$ vertices in S, therefore $\omega'(\alpha) \geq d - 4 - (d - 4) \cdot \frac{7}{8} > 0$.

□

By Euler's formula, since G is connected by Lemma 4.5.1 and has at least one vertex, $n + f - m = 2$. The initial weight of the graph is $\sum_{v \in V(G)} \omega(v) + \sum_{\alpha \in F(G)} \omega(\alpha) = \sum_{v \in V(G)} (d(v) - 4) + \sum_{\alpha \in F(G)} (d(\alpha) - 4) = \sum_{v \in V(G)} d(v) + \sum_{\alpha \in F(G)} d(\alpha) - 4n - 4f = 4m - 4n - 4f = -8 < 0$. Therefore the initial weight of the graph is negative, thus the final weight of the graph is negative. Since by Lemmas 4.5.9 and 4.5.10, the final weight of every face and every vertex is non-negative, we get a contradiction. This completes the proof of Theorem 4.1.9.

4.6 NP-hardness of finding $(\mathcal{I}, \Delta_k)$-partitions

Let $k \geq 3$ be a fixed integer. Suppose that there exists a graph in $\mathcal{C}$ that has no $(\mathcal{I}, \Delta_k)$-partition. We consider such a graph H_k that is minimal according to $\preceq$ (see Section 4.5, page 103). By adapting the proofs of Lemmas 4.5.1, 4.5.2, and 4.5.3, we obtain that the minimum degree of H_k is at least two and every $(k+1)^-$-vertex in H_k is adjacent to a $(k+2)^+$-vertex. Suppose for contradiction that H_k contains no 2-vertex. We consider the discharging procedure such that the initial charge of every vertex is equal to its degree and every 5^+-vertex gives $\frac{1}{3}$ to every adjacent 3-vertex. Then the final charge of a 3-vertex is at least $3 + \frac{1}{3} = \frac{10}{3}$, the final charge of a d-vertex with $d \geq k + 2$ is at least $d - d \times \frac{1}{3} = \frac{2}{3}d \geq 2(k+2)/3 \geq \frac{10}{3}$, and the final charge of every remaining vertex is at least $4 > \frac{10}{3}$. This implies that the maximum average degree of H_k is at least $\frac{10}{3}$, which is a contradiction since H_k is a planar graph with girth at least 5. Thus, H_k contains a 2-vertex v adjacent to the vertices u_1 and u_5.

By minimality of H_k, $H_k - v$ has an $(\mathcal{I}, \Delta_k)$-partition, every $(\mathcal{I}, \Delta_k)$-partition of $H_k - v$ is such that u_1 and u_5 get distinct colours, and the vertex in $\{u_1, u_5\}$ that is coloured Δ has exactly k Δ-neighbours.

Consider the graph H'_k obtained from $H_k - v$ by adding three 2-vertices u_2, u_3, and u_4 which form a path $u_1u_2u_3u_4u_5$. Notice that an $(\mathcal{I}, \Delta_k)$-partition of $H_k - v$ can be adapted into an $(\mathcal{I}, \Delta_k)$-partition of H'_k and that every $(\mathcal{I}, \Delta_k)$-partition of H'_k is such that u_3 is coloured Δ and is adjacent to exactly one vertex coloured Δ. It is easy to see that H'_k is in $\mathcal{C}$.

We are ready to prove that deciding whether a graph in $\mathcal{C}$ admits an $(\mathcal{I}, \Delta_k)$-partition is NP-complete. The reduction is from the NP-complete problem of deciding whether a planar graph with girth at least 9 admits an $(\mathcal{I}, \Delta_1)$-partition [31]. Given in instance G of this problem, we construct a graph $G' \in \mathcal{C}$, as follows. For every vertex v in G, we add $k - 1$ copies of H'_k and we add an edge between v and the vertex u_3 of each these copies. Notice that G' is in $\mathcal{C}$ since G' is planar and every cycle of length at most 8 is contained in a copy of H'_k which is in $\mathcal{C}$. Notice that an $(\mathcal{I}, \Delta_1)$-partition of G can be extended to an $(\mathcal{I}, \Delta_k)$-partition of G'.

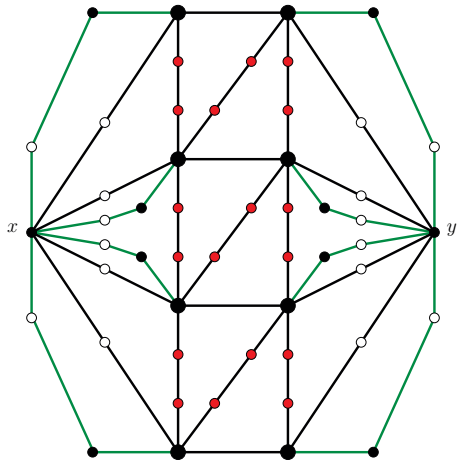

Figure 4.6.1: The gadget $F_{x,y}$.

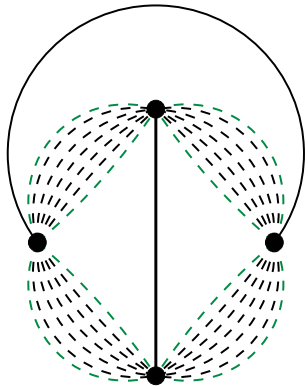

Figure 4.6.2: The graph in $\mathcal{C}$ with no $(\mathcal{I}, \Delta_3)$-partition.

Conversely, an $(\mathcal{I}, \Delta_k)$-partition of G' induces an $(\mathcal{I}, \Delta_1)$-partition of G. So G admits an $(\mathcal{I}, \Delta_1)$-partition if and only if G' admits an $(\mathcal{I}, \Delta_k)$-partition.

We thus proved that for all $k \geq 3$, if there is a graph in $\mathcal{C}$ that has no $(\mathcal{I}, \Delta_k)$-partition, then the problem of deciding whether a graph in $\mathcal{C}$ has an $(\mathcal{I}, \Delta_k)$-partition is NP-complete. Let us now build a graph in $\mathcal{C}$ that has no $(\mathcal{I}, \Delta_3)$-partition.

Consider the graph $F_{x,y}$ depicted in Figure 4.6.1. Suppose for contradiction that $F_{x,y}$ admits an $(\mathcal{I}, \Delta_3)$-partition such that all the neighbours of x and y are coloured I (the white vertices in the picture). Then the neighbours of those white vertices are coloured Δ. We consider the eight big vertices. Each of them is coloured Δ and is adjacent to two vertices coloured Δ. For every pair of adjacent red vertices, at least one of them is coloured Δ. Notice that every red vertex is adjacent to a big vertex. Since there are nine pairs of adjacent red vertices, there exists a big vertex that is adjacent to at least two red vertices coloured Δ. This big vertex is thus adjacent to four vertices coloured Δ, which is a contradiction.

In the graph depicted in Figure 4.6.2, every dashed line represents a copy of $F_{x,y}$ whose extremities are x and y. Suppose for contradiction that this (C_3, C_4, C_6)-free planar graph admits an $(\mathcal{I}, \Delta_3)$-partition. Each of the two drawn edges has at least one extremity coloured Δ. Thus, there exist two vertices u and v coloured Δ that are linked by seven copies of $F_{x,y}$. Since at most three neighbours of u and at most three neighbours of v can be coloured Δ, one of these seven copies of $F_{x,y}$ is such that all the neighbours of x and y are coloured I. This contradiction proves Theorem 4.1.10.

Following the proof above, we see that if we remove the green parts in Figures 4.6.1 and 4.6.2, we obtain a planar graph with girth 7 that has no $(\mathcal{I}, \Delta_2)$-partition. A graph with such properties is already known [44], but this new graph is smaller (184 vertices instead of 1304) and the proof of non-$(\mathcal{I}, \Delta_2)$-partition is simpler.

4.7 Conclusion

We proved some results on the vertex partitions of sparse graphs into an independent set and a graph with bounded components:

Theorem 4.1.4 (recall)**.** *Every graph G with* $\mathrm{mad}(G) < \frac{5}{2}$ *admits an $(\mathcal{I}, \mathcal{O}_3)$-partition.*

Theorem 4.1.6 (recall)**.** *Let $k \geq 2$ be an integer. Every graph G with* $\mathrm{mad}(G) < \frac{8k}{3k+1} = \frac{8}{3}\left(1 - \frac{1}{3k+1}\right)$ *admits an $(\mathcal{I}, \mathcal{O}_k)$-partition.*

Additionally, we focused on $(\mathcal{I}, \Delta_6)$-partition in the class $\mathcal{C}$ of (C_3, C_4, C_6)-free planar graphs:

Theorem 4.1.9 (recall)**.** *Every graph in $\mathcal{C}$ admits an $(\mathcal{I}, \Delta_6)$-partition.*

Theorem 4.1.10 (recall)**.** *For every $k \geq 1$, either every graph in $\mathcal{C}$ admits an $(\mathcal{I}, \Delta_k)$-partition, or deciding whether a graph in $\mathcal{C}$ admits an $(\mathcal{I}, \Delta_k)$-partition is NP-complete.*

We additionally exhibited a graph in C that has no $(\mathcal{I}, \Delta_3)$-partition.

On the algorithmic side, the proof of Theorem 4.1.4 easily yields a polynomial time algorithm to find an $(\mathcal{I}, \mathcal{O}_3)$-partition of planar graph, as the light forest can be found in polynomial time quite readily thanks to its inductive construction.

For the proof of Theorem 4.1.9, the special structures can be identified in polynomial time, and thus the multigraph $\widehat{G}$ can be computed in polynomial time. However, the procedure presented in Lemma 4.5.8 is not necessarily polynomial. To get a polynomial algorithm, we would need to do the reductions in polynomial time, which is not the case. However, if one can obtain a polynomial version of the procedure of Lemma 4.5.8, then the resulting algorithm would be polynomial.

In the proof of Theorem 4.1.6, Procedures 4.4.1 and 4.4.13 are not polynomial. Moreover, each application of Procedure 4.4.13 requires the computation of an $(\mathcal{I}, \mathcal{O}_k)$-partition of a graph with one fewer vertex than the graph considered, and there can be several such applications. Although that proof uses the discharging method, the computations to obtain the forbidden configurations are so complicated that the current proof does not yield a polynomial algorithm.

To sum up, the proofs of Theorems 4.1.6 and 4.1.9, although constructive, rely on complex constructions, and thus the algorithms obtained by simply translating the proofs are not efficient, and if one wants an algorithm to compute the desired partitions, we would rather advise using the naive algorithm of trying every possible partition, especially in the case of Theorem 4.1.6.

Chapter 5

Conclusion

In this thesis, we saw a series of results, first on large induced (linear) forests on planar graphs of large girth, then on vertex partitions of sparse graphs into forests, forests of bounded degree, independent sets, graphs of bounded degrees, and graphs with bounded components. All our results can be seen as a partition of the vertices of the graphs of some class into two sets with some specific properties.

Most of our proofs were based on the discharging method. The first proofs (Sections 2.3 and 2.4) relied on basic counting arguments, later proofs (Section 3.2) relied on standard discharging arguments, while some other ones relied on global discharging (Sections 3.4, 3.5, and 4.2). The last discharging proofs (Sections 4.4 and 4.5) used some complex constructions to build the forbidden configurations.

These methods enabled us to prove some new theorems on the partition of some classes of sparse graphs, and can probably be applied to other such partitions. Among classes we did not study, there are the classes of graphs with bounded degeneracy, the classes of graphs with bounded diameter... We could also study partitions of vertices into more than two sets.

We note that some of our results are for graphs with bounded maximum average degree and imply results on planar graphs of large girth, and that others are directly proven for planar graphs of large girth. Some proofs in the literature [8] were first made for planar graphs, and then generalised for graphs of bounded maximum average degree, sometimes even strengthened such that the bound on the maximum average degree is better than what we need for the bound on planar graphs. In these cases, the generalisation is of course worthwile. However, sometimes, as in the case of the Four Colour Theorem, it is possible to get better bounds by making full use of the planarity of the graphs studied. It is thus also interesting to directly study planar graphs of large girth to get better bounds (although in less general cases) than those obtained for graphs of bounded maximum average degree. For instance, it would be interesting to try and prove that for some integer k, planar graphs of girth at least 8 admit an $(\mathcal{I}, \mathcal{O}_k)$-partition, which would improve Theorem 4.1.6 in the specific case of planar graphs.

The idea of Choi, Liu, and Oum [38] of restricting not only the girth of the considered graphs, but forbidding any set of cycles, can also be extended to some other partitions (they do it for some of these partitions), and it can be interesting to study them.

Another interesting future work would be to try and apply our methods on edge partition instead of vertex partition. Note that edge partitions can be seen as vertex partition of line graphs. However, since line graphs of planar graphs are not planar in general, and as line graphs of graphs with bounded maximum average degree do not have bounded maximum average degree (they can have arbitrary large cliques), our results do not directly apply.

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Appendix A

Fractional triangle decompositions in graphs with large minimum degree

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Fractional triangle decompositions in graphs with large minimum degree

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Abstract

A triangle decomposition of a graph is a partition of its edges into triangles. A fractional triangle decomposition of a graph is an assignment of a non-negative weight to each of its triangles such that the sum of the weights of the triangles containing any given edge is one. We prove that every graph graph on n vertices with minimum degree at least $0.9n$ has a fractional triangle decomposition. This improves a result of Garaschuk that the same conclusion holds for graphs with minimum degree at least $0.956n$. Together with a recent result of Barber, Kühn, Lo and Osthus, this implies that for all $\epsilon > 0$, every large enough triangle divisible graph on n vertices with minimum degree at least $(0.9 + \epsilon)n$ admits a triangle decomposition.

1 Introduction

Decomposition and packing problems are central and classical problems in combinatorics, and, in particular, in design theory. Kirkman's theorem [8] from the middle of 19th century gives a necessary and sufficient condition on the existence of a Steiner triple system with a certain number of elements. In the language of graph theory, Kirkman's result asserts that every complete graph with an odd number of vertices and a number of edges divisible by three can be decomposed into triangles. Note that if a graph can be decomposed into triangles, then its vertex degrees are even and the total number of edges is divisible by three. Barber, Kühn, Lo and Osthus [2] showed that the same conclusion is true for large graphs satisfying these divisibility conditions if their minimum degree is not too far from the number of their vertices. In this short paper, we study the fractional variant of the problem and we use it, together with a result of Barber, Kühn, Lo and Osthus [2], to improve the best known bound.

Let us fix the terminology we are going to use. A *graph* is a pair of sets (V, E) such that elements of E are unordered pairs of elements of V. The elements of V are called *vertices* and the elements of E are called *edges*. We denote by uv (or vu) the edge with vertices u and v. We denote by $|G|$ the number of vertices of G. Two distinct vertices contained in the same edge are said to be *adjacent* or to be *neighbours*. Two edges that share a vertex are said to be *adjacent*. The *degree* of a vertex v is equal to the number

of neighbours of v. Let $\gcd(G)$ denote the greatest common divisor of the degrees of the vertices of G.

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a bijection b from V_1 to V_2 such that uv is an edge of G_1 if and only if $b(u)\,b(v)$ is an edge of G_2 for every two vertices u and v of G_1. The *complete graph* K_k is the graph with k vertices all mutually adjacent. The graph K_3 is also called a *triangle*. A graph $G_1 = (V_1, E_1)$ is a *subgraph* of $G_2 = (V_2, E_2)$ if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. The subgraphs of G_2 isomorphic to G_1 will be referred to as copies of G_1.

Let H be a graph. An *H-decomposition* of a graph G is a set of subgraphs of G isomorphic to H that are edge disjoint such that each edge of G is contained in one of them. A graph is *H-decomposable* if it admits an H-decomposition. A K_3-decomposition is also called a *triangle decomposition* and a graph is *triangle decomposable* if it is K_3-decomposable. A graph G is *H-divisible* if $\gcd(G)$ is a multiple of $\gcd(H)$ and the number of edges of G is a multiple of the number of edges of H. It is easy to see that every H-decomposable graph is H-divisible. However, the converse is not true. For example a cycle on six vertices is K_3-divisible, but not K_3 decomposable. As noted previously, Kirkman [8] proved that every K_3-divisible complete graph is K_3-decomposable. The fact that, for all H, every H-divisible complete graph is H-decomposable remained an open problem for over one hundred years before it was solved by Wilson [10].

The first generalisation to graphs that are near complete is due to Gustavsson [6]. He proved that for every graph H, there exist $n_0(H)$ and $\epsilon(H)$ such that every H-divisible graph with $n \geq n_0(H)$ vertices and minimum degree at least $(1 - \epsilon(H))n$ is H-decomposable. This has been generalised to hypergraphs in a recent result of Keevash [7]. The best that is known to date for a general graph H is due to Barber, Kühn, Lo and Osthus [2], who gave a way to turn a fractional decomposition into an integral one. They proved the following: Let F be a graph with minimum degree $\delta(F)$ and let H be a graph with chromatic number $\chi(H)$ and $e(H)$ edges. Let $C := \min\{9\chi(H)^2(\chi(H)-1)^2/2, 10^4\chi(H)^{3/2}\}$ and let $t := \max\{C, 6e(H)\}$. Then for all $\epsilon > 0$, there exists an n_0 such that every H-divisible graph G on $n \geq n_0$ vertices with $\delta(G) \geq (1 - 1/t + \epsilon)n$ has an H-decomposition. For some particular classes of graphs, the exact asymptotic minimum degree threshold is known [2][11].

A *fractional H-decomposition* of a graph G is an assignment of non-negative weights to the copies of H in G such that for an edge e, the sum of the weights of the copies of H that contain e is equal to one. A graph is *fractionally H-decomposable* if it admits a fractional H-decomposition. A graph can be fractionally H-decomposable without being H-divisible. A fractional K_3-decomposition is also called a *fractional triangle decomposition* and a graph is *fractionally triangle decomposable* if it is fractionally K_3-decomposable. For all $r \geq 2$, Yuster [12] proved that every graph on n vertices with minimum degree at least $\left(1 - \frac{1}{9r^{10}}\right)n$ is fractionally K_r-decomposable, and Dukes [3][4] proved that the same result holds for sufficiently large graphs on n vertices with minimum degree at least $\left(1 - \frac{2}{9r^2(r-1)^2}\right)n$. Our paper already led to further research: Barber, Kühn, Lo, Montgomery and Osthus [1] proved, using our method, that every graph on $n \geq 10^4 r^3$ vertices with minimum degree at least $(1 - 1/10^4 r^{3/2})n$ has a fractional K_r-decomposition.

In this paper we will focus on triangle decompositions of graphs with large minimum degree. The following conjecture is due to Nash-Williams [9]:

Conjecture 1 (Nash-Williams [9])**.** *Let G be a K_3-divisible graph with n vertices and minimum degree at least $\frac{3}{4}n$. If n is large enough, then G is K_3-decomposable.*

The best result towards a proof of Conjecture 1 is due to the combination of results of Garaschuk [5] and Barber, Kühn, Lo and Osthus [2].

Theorem 2 (Garaschuk [5], Barber, Kühn, Lo and Osthus [2])**.** *There exists an n_0 such that every K_3-divisible graph G on $n \geq n_0$ vertices with minimum degree at least $0.956n$ is K_3-decomposable.*

The proof of Theorem 2 relies on a result on fractional K_3-decomposability, which we now state. The following appears as a conjecture in [5]. Note that for K_3-divisible graphs, this is a consequence of Conjecture 1.

Conjecture 3 (Garaschuk [5])**.** *Let G be a graph with n vertices and minimum degree at least $\frac{3}{4}n$. If n is large enough, then G is fractionally K_3-decomposable.*

The best known result towards proving Conjecture 1 was established by Garaschuk [5].

Theorem 4 (Garaschuk [5])**.** *Let G be a graph with n vertices and minimum degree at least $0.956n$. The graph G admits a fractional triangle decomposition.*

In this paper we use a different method to prove the following.

Theorem 5. *Every graph with n vertices and minimum degree at least $\frac{9}{10}n$ admits a fractional triangle decomposition.*

In [2], a particular case of Theorem 11.1 and Lemma 12.3 imply the following.

Theorem 6 (Barber, Kühn, Lo and Osthus [2])**.** *Suppose there exist n_0 and δ such that every graph on $n \geq n_0$ vertices with minimum degree at least δn is fractionally K_3-decomposable. For all $\epsilon > 0$, there exist n_1 such that every K_3-divisible graph on $n \geq n_1$ vertices with minimum degree at least $\left(\max\left(\delta, \frac{3}{4}\right) + \epsilon\right) n$ vertices is K_3-decomposable.*

Together with Theorem 6, our result improves Theorem 2.

Theorem 7. *Let $\epsilon > 0$. There exists an n_0 such that every K_3-divisible graph on $n \geq n_0$ vertices with minimum degree at least $(\frac{9}{10} + \epsilon)n$ is K_3-decomposable.*

2 Proof of Theorem 5

Let $\delta = \frac{1}{10}$. Fix a graph G with n vertices and minimum degree at least $(1-\delta)\,n$. Suppose the graph G has at least one triangle with three vertices of degree at least $(1-\delta)\,n+2$. Let G' be the graph G where the edges of one such triangle are removed. Observe that G' has minimum degree at least $(1-\delta)\,n$ and that if G' has a fractional triangle decomposition, then G has one too. By doing this operation several times, we can assume that G has no triangle with three vertices of degree at least $(1-\delta)\,n+2$. Let m be the number of edges of G.

Initially, we give the same weight w_Δ to every triangle such that the sum of the weights of the triangles is equal to $\frac{m}{3}$. We will modify the weights of the triangles to obtain a fractional triangle decomposition. We will do so in a way that the total sum of the weights is preserved.

We define the weight of an edge e to be the sum of the weights of the triangles that contain e. Given H a copy of K_4 in G, and two non-adjacent edges e_1 and e_2 in H, let

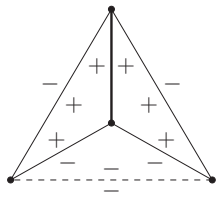

Figure 1: By removing some weight w from the two triangles containing the thick edge and adding w to the two triangle containing the dashed edge, we remove $2w$ from the dashed edge and add $2w$ to the thick edge.

us call $(H, \{e_1, e_2\})$ a *rooted* K_4 of G. We will use the following procedure to modify the weights of the edges of a rooted K_4 of G:

Let $(H, \{e_1, e_2\})$ be a rooted K_4 of G. By removing a weight w from the two triangles of H that contain e_1 and adding the same weight w to each of the other two triangles (i.e. those that contain e_2), we transfer a weight of $2w$ from e_1 to e_2. The weights of all the other edges of the graph remain unchanged (see Figure 1).

To prevent the weight of any triangle from becoming negative, we have to restrict how much weight we can transfer using the procedure above. If for some w we use the procedure to transmit a weight of $2w$ from an edge to another one, then any triangle's weight is lowered by at most w for triangles that are in the K_4, and does not change for other triangles. Moreover, since every triangle contains a vertex with degree at most $(1-\delta)n+1$, any triangle is in at most $(1-\delta)n-1$ copies of K_4, and thus in at most $3(1-\delta)n-3$ rooted K_4 (since for each K_4 there are three possible choices for the pair of edges). Since each triangle has an initial weight of w_Δ, if each rooted K_4 containing that triangle is used to transfer weight of at most $\frac{2w_\Delta}{3(1-\delta)n-3}$ between its labelled edges, its final weight will be non-negative.

For each edge e, let T_e be the number of triangles of G containing e. We express redistributing the weights as a flow problem in an auxiliary graph, which is denoted by $\widehat{G}$. The vertices of $\widehat{G}$ are the edges of G, plus two special vertices, called the *supersource* and the *supersink*. Two edges of G are adjacent as vertices in $\widehat{G}$ if they form a pair in a rooted K_4. The edge between them is set to have the capacity $c = \frac{2w_\Delta}{3(1-\delta)n-3}$. Let E_c be the set of these edges. If $T_e w_\Delta > 1$, then the vertex of $\widehat{G}$ corresponding to e is joined to the supersource and the capacity of the corresponding edge of $\widehat{G}$ is $T_e w_\Delta - 1$. Likewise, if $T_e w_\Delta < 1$, then the vertex of $\widehat{G}$ corresponding to e is joined to the supersink and the capacity of the corresponding edge is $1 - T_e w_\Delta$. The vertices of G adjacent to the supersource are referred to as *sources* and those adjacent to the supersink as *sinks*. Let

$$M = \sum_{e \text{ source}} (T_e w_\Delta - 1) = \sum_{e \text{ sink}} (1 - T_e w_\Delta)\,.$$

We will show that $\widehat{G}$ has a flow of value M from the supersource to the supersink.

If $\widehat{G}$ does not have a flow of value M, then it has a vertex cut (A_0, B_0) such that the supersource is contained in A_0, the supersink in B_0 and the sum of the capacities of the edges from A_0 to B_0 is less than M. Let A be the edges of G corresponding to the vertices of A_0 and B the edges corresponding to the vertices of B_0. Note that $|A| = |A_0| - 1$ and $|B| = |B_0| - 1$. Finally, let $k = |A|$, and observe that $|B| = m - k$.

Let T_A and T_B be the average T_e for e in A and in B respectively. Let $e = uv$ be an edge of G. Let W_e be the set of the vertices w such that uvw is a triangle. By the definition of T_e, $|W_e| = T_e$. Each vertex of W_e is non-adjacent to at most δn vertices of G, and thus is non-adjacent to at most δn vertices of W_e. So each vertex of W_e is adjacent to at least $T_e - \delta n$ vertices of W_e. Therefore e is in at least $\frac{T_e(T_e-\delta n)}{2}$ distinct copies of K_4.

Let e be a vertex of A. It is adjacent to at least $\frac{T_e(T_e-\delta n)}{2} - k$ vertices of B. Therefore the cut contains at least

$$\sum_{e\in A}\left(\frac{T_e\left(T_e-\delta n\right)}{2} - k\right)$$

edges of E_c. Similarly, it contains at least

$$\sum_{e\in B}\left(\frac{T_e\left(T_e-\delta n\right)}{2} - (m-k)\right)$$

edges of E_c. Moreover, for each source e that is in B and each sink e that is in A, the cut contains the edge between e and the supersource or the supersink. Recall that the capacities of the edges of E_1 is $c = \frac{2w_\Delta}{3(1-\delta)n}$. Therefore the sum of the capacities of the edges of $\widehat{G}$ is at least

$$\sum_{e\in A}\left(\frac{T_e\left(T_e-\delta n\right)}{2} - k\right)c + \sum_{e \text{ source}\in B}\left(T_e w_\Delta - 1\right) + \sum_{e \text{ sink}\in A}\left(1 - T_e w_\Delta\right).$$

At the same time, it is also at least

$$\sum_{e\in B}\left(\frac{T_e\left(T_e-\delta n\right)}{2} - (m-k)\right)c + \sum_{e \text{ source}\in B}\left(T_e w_\Delta - 1\right) + \sum_{e \text{ sink}\in A}\left(1 - T_e w_\Delta\right).$$

Since the sum of the capacities of the edges in the considered cut is less than M, we get that

$$\sum_{e\in A}\left(\frac{T_e\left(T_e-\delta n\right)}{2} - k\right)c + \sum_{e \text{ source}\in B}\left(T_e w_\Delta - 1\right) + \sum_{e \text{ sink}\in A}\left(1 - T_e w_\Delta\right) < M \tag{1}$$

and

$$\sum_{e\in B}\left(\frac{T_e\left(T_e-\delta n\right)}{2} - (m-k)\right)c + \sum_{e \text{ source}\in B}\left(T_e w_\Delta - 1\right) + \sum_{e \text{ sink}\in A}\left(1 - T_e w_\Delta\right) < M. \tag{2}$$

The inequalities (1) and (2) can be rewritten using that

$$M = \sum_{e \text{ source}}\left(T_e w_\Delta - 1\right)$$

and

$$M = \sum_{e \text{ sink}}\left(1 - T_e w_\Delta\right)$$

respectively as follows.

$$\sum_{e\in A}\left(T_e\left(T_e-\delta n\right) - 2k\right)c - 2\sum_{e\in A}\left(T_e w_\Delta - 1\right) < 0 \tag{3}$$

$$\sum_{e\in B}\left(T_e\left(T_e-\delta n\right) - 2(m-k)\right)c - 2\sum_{e\in B}\left(1 - T_e w_\Delta\right) < 0 \tag{4}$$

Since the summand is a convex function of T_e, we obtain the following.

$$(T_A(T_A - \delta n) - 2k)c - 2(T_A w_\Delta - 1) < 0 \tag{5}$$

$$(T_B(T_B - \delta n) - 2(m-k))c - 2(1 - T_B w_\Delta) < 0 \tag{6}$$

The inequality (5) implies that

$$T_A(T_A - \delta n) + \frac{2}{c}(1 - T_A w_\Delta) < 2k. \tag{7}$$

The inequality (6) implies that

$$2k < 2m - T_B(T_B - \delta n) + \frac{2}{c}(1 - T_B w_\Delta). \tag{8}$$

We now combine the inequalities (7) and (8) and we substitute $c = \frac{2w_\Delta}{3(1-\delta)n-3}$ to get the following.

$$T_A(T_A - \delta n) - (3n(1-\delta) - 3)T_A < 2m - T_B(T_B - \delta n) - (3n(1-\delta) - 3)T_B \tag{9}$$

Let e be an edge of G. Each end-vertex of e is non-adjacent to at most δn vertices of G. Hence, the edge e is contained in at least $n - 2\delta n$ triangles. Since e cannot be contained in more than n triangles, we get that $n - 2\delta n \leq T_e \leq n$. Consequently, we have $n - 2\delta n \leq T_A, T_B \leq n$.

A standard analytic argument shows that the left hand side of (9) is minimized when $T_A = n$ and the right hand side is maximized when $T_B = n - 2\delta n$. Consequently, it must hold that

$$n(n-\delta n) - 3(1-\delta)n^2 + 3n < 2m - (n-2\delta n)(n-3\delta n) - 3n(1-\delta)(n-2\delta n) + 3n - 6\delta n. \tag{10}$$

Therefore $(2 - 12\delta + 12\delta^2)n^2 < 2m - 6\delta n$.

Note that we proved that in G there is no triangle with three vertices of degree at least $(1-\delta)n + 2$. Let V_b be the set of vertices of degree at least $(1-\delta)n + 2$ in G, and let n_b be the number of vertices in V_b. Let us prove that $n_b \leq 2\delta n - 4$. Assume by contradiction that $n_b \geq 2\delta n - 3$. Since every vertex in V_b has at most $\delta n - 3$ non-neighbours in $V(G)$, every vertex in V_b has at most $\delta n - 3$ non-neighbours in V_b, and thus has at least $n_b - \delta n + 2$ neighbours in V_b. The graph induced by V_b is triangle-free, has $n_b \geq 2\delta n - 3$ vertices, and has minimum degree at least $n_b - \delta n + 2$. A triangle-free graph on k vertices has at most $\frac{k^2}{4}$ edges, thus it has minimum degree at most $\frac{k}{2}$. This implies that $n_b - \delta n + 2 \leq n_b/2$, i.e. $n_b/2 \leq \delta n - 2$, and thus $\delta n - 1.5 \leq \delta n - 2$, a contradiction. Therefore there are at most $2\delta n - 4$ vertices in V_b. We have

$$2m \leq (2\delta n - 4)n + ((1-2\delta)n + 4)((1-\delta)n + 1) = (1 - \delta - 2\delta^2)n^2 + 4 + n - 6\delta n \tag{11}$$

and thus

$$(2 - 12\delta + 12\delta^2)n^2 < (1 - \delta - 2\delta^2)n^2 + 4 + n - 12\delta n. \tag{12}$$

Assume $n < 20$. Since G has minimum degree at least $(1-\delta)n = 0.9n$, the graph G is a complete graph. Then giving the same weight to every triangle leads to a fractional triangle decomposition of G.

Therefore we can assume that $n \geq 20$, and thus $4 + n - 12\delta n \leq 0$. We get that $(1 - 10\delta)(1 - \delta) = 1 - 11\delta + 10\delta^2 < 0$. Since $\delta = 0.1$, this leads to a contradiction. Therefore, there must exist a flow of value M in $\widehat{G}$, and hence, as described previously, the weights of the triangles can be adjusted in such a way that these weights now form a fractional decomposition of G. This finishes the proof of Theorem 5.

3 Conclusion

In this paper we proved that every graph on n vertices with minimum degree at least $\frac{9}{10}n$ is fractionally triangle decomposable. Together with a result of Barber, Kühn, Lo and Osthus [2], this implies that, for all $\epsilon > 0$, there exists a constant n_0 such that every triangle divisible graph on $n \geq n_0$ vertices with minimum degree at least $(0.9 + \epsilon)\, n$ is triangle decomposable.

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Appendix B

Filling the Complexity Gaps for Colouring Planar and Bounded Degree Graphs

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Filling the Complexity Gaps for Colouring Planar and Bounded Degree Graphs*

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Abstract. A colouring of a graph $G = (V, E)$ is a function $c : V \to \{1, 2, \dots\}$ such that $c(u) \neq c(v)$ for every $uv \in E$. A k-regular list assignment of G is a function L with domain V such that for every $u \in V$, $L(u)$ is a subset of $\{1, 2, \dots\}$ of size k. A colouring c of G respects a k-regular list assignment L of G if $c(u) \in L(u)$ for every $u \in V$. A graph G is k-choosable if for every k-regular list assignment L of G, there exists a colouring of G that respects L. We may also ask if for a given k-regular list assignment L of a given graph G, there exists a colouring of G that respects L. This yields the k-Regular List Colouring problem. For $k \in \{3, 4\}$ we determine a family of classes $\mathcal{G}$ of planar graphs, such that either k-Regular List Colouring is NP-complete for instances (G, L) with $G \in \mathcal{G}$, or every $G \in \mathcal{G}$ is k-choosable. By using known examples of non-3-choosable and non-4-choosable graphs, this enables us to classify the complexity of k-Regular List Colouring restricted to planar graphs, planar bipartite graphs, planar triangle-free graphs and to planar graphs with no 4-cycles and no 5-cycles. We also classify the complexity of k-Regular List Colouring and a number of related colouring problems for graphs with bounded maximum degree.

Keywords: list colouring, choosability, planar graphs, maximum degree

1 Introduction

A *colouring* of a graph is a labelling of its vertices such that adjacent vertices do not have the same label. We call these labels *colours*. Graph colouring problems are central to the study of combinatorial algorithms and they have many theoretical and practical applications. A typical problem asks whether a colouring exists

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under certain constraints, or how difficult it is to find such a colouring. For example, in the List Colouring problem, a graph is given where each vertex has a list of colours and one wants to know if the vertices can be coloured using only colours in their lists. The Choosability problem asks whether such list colourings are guaranteed to exist whenever all the lists have a certain size k (if so, then the graph is said to be k-choosable). In fact, an enormous variety of colouring problems can be defined and there is now a vast collection of literature on this subject. For longer introductions to the type of problems we consider we refer to two recent surveys [10, 21].

In this paper, we are concerned with the computational complexity of colouring problems (we give formal definitions of these problems in Section 2). For many such problems, the complexity is well understood in the case where we allow every graph as input, so it is natural to consider problems with restricted inputs in order to increase our understanding of their hardness. Our two main objectives are related to this aim. They are:

1. to exploit structural results in order to obtain complexity results;
2. to fill a number of complexity gaps in order to obtain complete complexity classifications.

The graph classes we consider are the classes of planar graphs and of graphs with bounded maximum degree. As such our paper consists of two main parts.

In the first (Section 3), we study planar graphs and consider a natural variant of the List Colouring problem, closely related to the Choosability problem. As we will discuss later in more detail, there exist many results for the latter problem restricted to planar graphs (see for example [1,8,14,23,25,26,28–31,35–37,39–41]). Most of these results are structural. They either show that every graph of some subclass $\mathcal{G}$ of planar graphs is k-choosable for some small value of k, or construct a concrete example of a non-k-choosable graph that belongs to $\mathcal{G}$. When not every graph in $\mathcal{G}$ is k-choosable, it is natural to ask if the vertices of a given graph from $\mathcal{G}$ can be coloured in polynomial time using only colours from their lists for some given list assignment L, which assigns a list of size k to each vertex. For $k \in \{3, 4\}$, we prove two general theorems, which give us a family of subclasses $\mathcal{G}$ of planar graphs, such that either this problem is NP-complete for instances (G, L) with $G \in \mathcal{G}$ or every graph in $\mathcal{G}$ is k-choosable. We will then combine known structural choosability results with these theorems to obtain NP-hardness results for this problem on planar graphs and on a number of subclasses of planar graphs. This enables us to fill some complexity gaps in order to obtain a complete classification of the computational complexity of the problem for these graph classes.

Some of the known results we use in the first part of our paper are for planar graphs with bounded degree. In the second part of the paper (Section 4), we combine these and other old and new results to fill some more complexity gaps and obtain complete complexity classifications for a number of colouring problems on graphs with bounded maximum degree.

We first introduce necessary definitions and terminology in Section 2.

2 Preliminaries

A *colouring* of a graph $G = (V, E)$ is a function $c : V \to \{1, 2, \ldots\}$ such that $c(u) \neq c(v)$ whenever $uv \in E$. We say that $c(u)$ is the *colour* of u. For a positive integer k, if $1 \leq c(u) \leq k$ for all $u \in V$, then c is a *k-colouring* of G. We say that G is *k-colourable* if a k-colouring of G exists. The Colouring problem is to decide whether a graph G is k-colourable for some given integer k. If k is fixed, that is, not part of the input, we obtain the k-Colouring problem.

A *list assignment* of a graph $G = (V, E)$ is a function L with domain V such that for each vertex $u \in V$, $L(u)$ is a subset of $\{1, 2, \ldots\}$. This set is called the *list* of *admissible* colours for u. If $L(u) \subseteq \{1, \ldots, k\}$ for each $u \in V$, then L is a *k-list assignment*. The *size* of a list assignment L is the maximum list size $|L(u)|$ over all vertices $u \in V$. A colouring c *respects* L if $c(u) \in L(u)$ for all $u \in V$. Given a graph G with a k-list assignment L, the List Colouring problem is to decide whether G has a colouring that respects L. If k is fixed, then we have the List k-Colouring problem. Fixing the size of L to be at most ℓ gives the ℓ-List Colouring problem. We say that a list assignment L of a graph $G = (V, E)$ is *ℓ-regular* if, for all $u \in V$, $L(u)$ contains exactly ℓ colours. This gives us the following problem, which is one focus of this paper. It is defined for each integer $\ell \geq 1$ (note that ℓ is *fixed*; that is, ℓ is not part of the input).

ℓ-Regular List Colouring *Instance:* a graph G with an ℓ-regular list assignment L. *Question:* does G have a colouring that respects L?

A *k-precolouring* of a graph $G = (V, E)$ is a function $c_W : W \to \{1, 2, \ldots, k\}$ for some subset $W \subseteq V$. A k-colouring c of G is an *extension* of a k-precolouring c_W of G if $c(v) = c_W(v)$ for each $v \in W$. Given a graph G with a precolouring c_W, the Precolouring Extension problem is to decide whether G has a k-colouring that extends c_W. If k is fixed, we obtain the k-Precolouring Extension problem. The relationships amongst the problems introduced are shown in Fig. 1.

For an integer $\ell \geq 1$, a graph $G = (V, E)$ is *ℓ-choosable* if, for every ℓ-regular list assignment L of G, there exists a colouring that respects L. The corresponding decision problem is the Choosability problem. If ℓ is fixed, we obtain the ℓ-Choosability problem.

We emphasize that ℓ-Regular List Colouring and ℓ-Choosability are two fundamentally different problems. For the former we must decide whether there exists a colouring that respects a *particular* ℓ-regular list assignment. For the latter we must decide whether or not *every* ℓ-regular list assignment has a colouring that respects it. As we will see later, this difference also becomes clear from a complexity point of view: for some graph classes ℓ-Regular List Colouring is computationally easier than ℓ-Choosability, whereas, perhaps more surprisingly, for other graph classes, the reverse holds.

For two vertex-disjoint graphs G and H, we let $G + H$ denote the disjoint union $(V(G) \cup V(H), E(G) \cup E(H))$, and kG denote the disjoint union of k copies of G. If G is a graph containing an edge e or a vertex v then $G - e$ and $G - v$

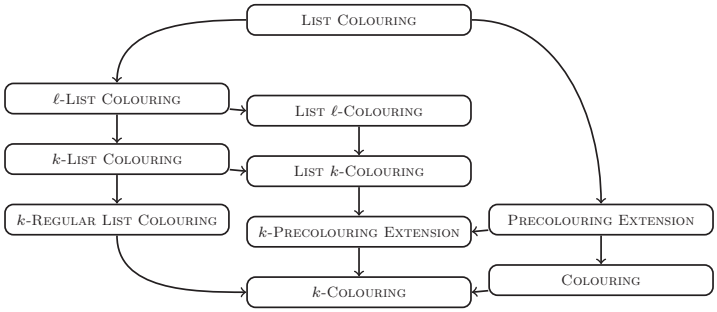

Fig. 1. Relationships between Colouring and its variants. An arrow from one problem to another indicates that the latter is (equivalent to) a special case of the former; k and ℓ are any two arbitrary integers for which $\ell \geq k$. For instance, k-Colouring is a special case of k-Regular List Colouring. This can be seen by giving the list $L(u) = \{1, \ldots, k\}$ to each vertex u in an instance graph of Colouring. We also observe that ℓ-Regular List Colouring and k-Regular List Colouring are not comparable for any $k \neq \ell$.

denote the graphs obtained from G by deleting e or v, respectively. If G' is a subgraph of G then $G - G'$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(G')$.

We let C_n, K_n and P_n denote the cycle, complete graph and path on n vertices, respectively. A wheel is a cycle with an extra vertex added that is adjacent to all other vertices. The wheel on n vertices is denoted W_n; note that $W_4 = K_4$. A graph on at least three vertices is 2-*connected* if it is connected and there is no vertex whose removal disconnects it. A *block* of a graph is a maximal subgraph that is connected and cannot be disconnected by the removal of one vertex (so a block is either 2-connected, a K_2 or an isolated vertex). A *block graph* is a connected graph in which every block is a complete graph. A *Gallai tree* is a connected graph in which every block is a complete graph or a cycle. We say that B is a *leaf-block* of a connected graph G if B contains exactly one cut vertex u of G and $B \setminus u$ is a component of $G - u$.

For a set of graphs $\mathcal{H}$, a graph G is $\mathcal{H}$-*free* if G contains no induced subgraph isomorphic to a graph in $\mathcal{H}$, whereas G is $\mathcal{H}$-*subgraph-free* if it contains no subgraph isomorphic to a graph in $\mathcal{H}$. The *girth* of a graph is the length of its shortest cycle.

3 Planar Graphs

3.1 Known Results for Planar Graphs

We start with a classical result observed by Erdős et al. [17] and Vizing [33].

Theorem 1 ([17, 33]). 2-List Colouring *is polynomial-time solvable.*

Garey et al. proved the following result, which is in contrast to the fact that every planar graph is 4-colourable by the Four Colour Theorem [2].

Theorem 2 ([19]). 3-Colouring *is* NP*-complete for planar graphs of maximum degree* 4.

Next we present results found by several authors on the existence of k-choosable graphs for various graph classes.

Theorem 3. *The following statements hold for k-choosability:*

(i) Every planar graph is 5*-choosable* [30].
(ii) There exists a planar graph that is not 4*-choosable* [35].
(iii) Every planar triangle-free graph is 4*-choosable* [25].
(iv) Every planar graph with no 4*-cycles is* 4*-choosable* [26].
(v) There exists a planar triangle-free graph that is not 3*-choosable* [36].
(vi) There exists a planar graph with no 4*-cycles, no* 5*-cycles and no intersecting triangles that is not* 3*-choosable* [29].
(vii) Every planar bipartite graph is 3*-choosable* [1].

We note that smaller examples of graphs than were used in the original proofs have been found for Theorems 3.(ii) [23], 3.(v) [28] and 3.(vi) [41] and that Theorem 3.(vi) strengthens a result of Voigt [37]. We recall that Thomassen [31] first showed that every planar graph of girth at least 5 is 3-choosable, and that a number of results have since been obtained on 3-choosability of planar graphs in which certain cycles are forbidden; see, for example, [8, 14, 39, 40].

We will also use the following result of Chlebík and Chlebíková.

Theorem 4 ([9]). List Colouring *is* NP*-complete for* 3*-regular planar bipartite graphs that have a list assignment in which each list is one of* $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$ *and all the neighbours of each vertex with three colours in its list have two colours in their lists.*

3.2 New Results for Planar Graphs

Theorems 1–3 have a number of immediate consequences for the complexity of ℓ-Regular List Colouring when restricted to planar graphs. For instance, Theorem 2 implies that 3-Regular List Colouring is NP-complete for planar graphs, whereas Theorem 3.(i) shows that 5-Regular List Colouring is polynomial-time solvable on this graph class. In fact, we notice a complexity jump from being NP-complete to being trivial (that is, the answer is always yes) when ℓ changes from 3 to 5. It is a natural question to determine the complexity for the missing case $\ell = 4$.

In this section we settle this missing case and also present a number of new hardness results for ℓ-Regular List Colouring restricted to various subclasses of planar graphs. At the end of this section we show how to combine the known

results with our new ones to obtain a number of dichotomy results (Corollaries 2–5). We deduce some of our new results from two more general theorems, namely Theorems 5 and 7, which we state below, but which we prove in Sections 5.1 and 5.2, respectively.

Theorem 5. *Let $\mathcal{H}$ be a finite set of 2-connected planar graphs. Then* 4-Regular List Colouring *is* NP*-complete for planar $\mathcal{H}$-subgraph-free graphs if there exists a planar $\mathcal{H}$-subgraph-free graph that is not 4-choosable.*

Note that the class of $\mathcal{H}$-subgraph-free graphs is contained in the class of $\mathcal{H}$-free graphs. Hence, whenever a problem is NP-complete for $\mathcal{H}$-subgraph-free graphs, it is also NP-complete for $\mathcal{H}$-free graphs.

An alternative formulation of Theorem 5 is that, for every finite set $\mathcal{H}$ of 2-connected planar graphs, either every pair (G, L), where G is a planar $\mathcal{H}$-subgraph-free graph, is a yes-instance of 4-Regular List Colouring, or 4-Regular List Colouring is NP-complete for planar $\mathcal{H}$-subgraph-free graphs. Results of the same flavour were proved for 3-Colouring, Acyclic 3-Colouring, $(1, 0)$-Colouring and C_{2p+1}-Colouring restricted to classes of planar graphs by Esperet et al. [18]. To give a more recent example of such a result, Dross, Montassier and Pinlou [12] proved that either every triangle-free planar graph is near-bipartite – that is, can be decomposed into an independent set and a forest – or recognizing near-bipartite triangle-free planar graphs is NP-complete. By using their construction and the existence of a planar graph that is not near-bipartite (for instance K_4), Bonamy et al. [3, see arXiv version] observed that the problem of recognizing near-bipartite graphs is NP-complete on planar graphs.

We will exploit the power of Theorem 5 by combining it with Theorem 3. For instance, combining Theorem 5 with Theorem 3.(ii) yields that 4-Regular List Colouring is NP-complete for planar graphs. We are not aware of any paper proving this result although it seems to have been known: the result is mentioned by Thomassen [32] without proof, and Dvořák and Thomas [15] mistakenly attribute it to [23]. We can sharpen this result as follows. By the Four Colour Theorem [2], every planar graph is 4-colourable, that is, has a colouring respecting the list assignment that assigns list of colours $\{1, 2, 3, 4\}$ to every vertex of the graph. In contrast, Voigt and Wirth [38] showed the existence of a planar graph that does not allow a colouring respecting some specific list assignment L, in which each list contains four distinct colours from $\{1, 2, 3, 4, 5\}$. We use their example to prove the following result in Section 5.1.

Theorem 6. 4-Regular List Colouring *is* NP*-complete for planar graphs even if every list contains four colours from* $\{1, 2, 3, 4, 5\}$.

As we will see later, the $k = 4$ case was the only one for which the complexity of k-Regular List Colouring for planar graphs was not settled.

Theorem 5 has further applications and can also be combined with other results from the literature. For instance, consider the non-4-choosable planar graph H from the proof of Theorem 1.7 in [23]. It can be observed that H is W_p-subgraph-free for all $p \geq 8$. Wheels are 2-connected and planar. Hence,

if $\mathcal{H}$ is any finite set of wheels on at least eight vertices, then 4-Regular List Colouring is NP-complete for planar $\mathcal{H}$-subgraph-free graphs.

Our basic idea for proving Theorem 5 is similar to the proof technique used in [12, 18]. We pick a minimal counterexample H with list assignment L. We select an "appropriate" edge $e = uv$ and consider the graph $F' = F - e$. We reduce from an appropriate colouring problem restricted to planar graphs and use copies of F' as a gadget to ensure that we can enforce a regular list assignment. The proof of the next theorem also uses this idea.

Theorem 7. *Let $\mathcal{H}$ be a finite set of 2-connected planar graphs. Then 3-Regular List Colouring is NP-complete for planar $\mathcal{H}$-subgraph-free graphs if there exists a planar $\mathcal{H}$-subgraph-free graph that is not 3-choosable.*

Just like Theorem 5, Theorem 7 has a number of applications. For instance, if we let $\mathcal{H} = \{K_3\}$ then Theorem 7, combined with Theorem 3.(v), leads to the following result (Dvořák and Kawarabayashi also briefly mentioned how to obtain this result in their paper [13] but do not provide a full proof).

Corollary 1. *3-Regular List Colouring is NP-complete for planar triangle-free graphs.*

Theorem 7 can also be used for other classes of graphs. For example, let $\mathcal{H}$ be a finite set of graphs, each of which includes a 2-connected graph on at least five vertices as a subgraph. Let $\mathcal{I}$ be the set of these 2-connected graphs. The graph K_4 is a planar $\mathcal{I}$-subgraph-free graph that is not 3-choosable (since it is not 3-colourable). Therefore, Theorem 7 implies that 3-Regular List Colouring is NP-complete for planar $\mathcal{H}$-subgraph-free graphs. We can obtain more hardness results by taking some other planar graph that is not 3-choosable, such as a wheel on an even number of vertices. Also, if we let $\mathcal{H} = \{C_4, C_5\}$ we can use Theorem 7 by combining it with Theorem 3.(vi) to find that 3-Regular List Colouring is NP-complete for planar graphs with no 4-cycles and no 5-cycles. We strengthen this result as follows (see Section 5.3 for the proof).

Theorem 8. *3-Regular List Colouring is NP-complete for planar graphs with no 4-cycles, no 5-cycles and no intersecting triangles.*

Theorem 6, Corollary 1 and Theorem 8 can be seen as strengthenings of Theorems 3.(ii), 3.(v) and 3.(vi), respectively. Moreover, they complement Theorem 2, which implies that 3-List Colouring is NP-complete for planar graphs, and a result of Kratochvíl [24] that, for planar bipartite graphs, 3-Precolouring Extension is NP-complete. Theorem 6 and Corollary 1 also complement results of Gutner [23] who showed that 3-Choosability and 4-Choosability are Π_2^p-complete for planar triangle-free graphs and planar graphs, respectively. However, we emphasize that, for special graph classes, it is not necessarily the case that ℓ-Choosability is computationally harder than ℓ-Regular List Colouring. For instance, contrast the fact that Choosability is polynomial-time solvable on $3P_1$-free graphs [20] with our next result, which we prove in Section 5.4.

Theorem 9. 3-Regular List Colouring *is* NP*-complete for* $(3P_1, P_1 + P_2)$*-free graphs.*

Our new results, combined with known results, close a number of complexity gaps for the ℓ-Regular List Colouring problem. Combining Theorem 6 with Theorems 1, 2 and 3.(i) gives us Corollary 2. Combining Theorem 8 with Theorems 1 and 3.(iv) gives us Corollary 3. Combining Corollary 1 with Theorems 1 and 3.(iii) gives us Corollary 4, whereas Theorems 1 and 3.(vii) imply Corollary 5.

Corollary 2. *Let* ℓ *be a positive integer. Then* ℓ-Regular List Colouring*, restricted to planar graphs, is* NP*-complete if* $\ell \in \{3, 4\}$ *and polynomial-time solvable otherwise.*

Corollary 3. *Let* ℓ *be a positive integer. Then* ℓ-Regular List Colouring*, restricted to planar graphs with no* 4*-cycles and no* 5*-cycles and no intersecting triangles, is* NP*-complete if* $\ell = 3$ *and polynomial-time solvable otherwise (even if we allow intersecting triangles and* 5*-cycles).*

Corollary 4. *Let* ℓ *be a positive integer. Then* ℓ-Regular List Colouring*, restricted to planar triangle-free graphs, is* NP*-complete if* $\ell = 3$ *and polynomial-time solvable otherwise.*

Corollary 5. *Let* ℓ *be a positive integer. Then* ℓ-Regular List Colouring*, restricted to planar bipartite graphs, is polynomial-time solvable.*

4 Bounded Degree Graphs

4.1 Known Results for Bounded Degree Graphs

First we present a result of Kratochvíl and Tuza [25].

Theorem 10 ([25]). List Colouring *is polynomial-time solvable on graphs of maximum degree at most* 2*.*

Brooks' Theorem [7] states that every graph G with maximum degree d has a d-colouring unless G is a complete graph or a cycle with an odd number of vertices. The next result of Vizing [34] generalizes Brooks' Theorem to list colourings.

Theorem 11 ([34]). *Let* d *be a positive integer. Let* $G = (V, E)$ *be a connected graph of maximum degree at most* d *and let* L *be a* d*-regular list assignment for* G*. If* G *is not a cycle or a complete graph then* G *has a colouring that respects* L*.*

And we need another result of Chlebík and Chlebíková [9].

Theorem 12 ([9]). Precolouring Extension *is polynomial-time solvable on graphs of maximum degree* 3*.*

4.2 New Results for Bounded Degree Graphs

In Section 5.5 we prove the following result by making a connection to Gallai trees (which is a standard approach) and using a result of Bonomo, Durán and Marenco [4] on List Colouring for block graphs.

Theorem 13. *Let k be a positive integer. Then k-Precolouring Extension is polynomial-time solvable for graphs of maximum degree at most k.*

We have the following two classifications. The first one is an observation obtained by combining only previously known results, whereas the second one also makes use of our new result.

Corollary 6. *Let d be a positive integer. The following two statements hold for graphs of maximum degree at most d.*

(i) List Colouring *is* NP*-complete if $d \geq 3$ and polynomial-time solvable if $d \leq 2$.*
(ii) Precolouring Extension *and* Colouring *are* NP*-complete if $d \geq 4$ and polynomial-time solvable if $d \leq 3$.*

Proof. We first consider (i). If $d \geq 3$, we use Theorem 4. If $d \leq 2$, we use Theorem 10. We now consider (ii). If $d \geq 4$, we use Theorem 2. If $d \leq 3$, we use Theorem 12. □

Corollary 7. *Let d and k be two positive integers. The following two statements hold for graphs of maximum degree at most d.*

(i) k-List Colouring *and* List k-Colouring *are* NP*-complete if $k \geq 3$ and $d \geq 3$ and polynomial-time solvable otherwise.*
(ii) k-Regular List Colouring *and* k-Precolouring Extension *are* NP*-complete if $k \geq 3$ and $d \geq k+1$ and polynomial-time solvable otherwise.*

Proof. We first consider (i). If $k \geq 3$ and $d \geq 3$, we use Theorem 4. If $k \leq 2$ or $d \leq 2$, we use Theorems 1 or 10, respectively.

We now consider (ii). We start with the hardness cases and so let $k \geq 3$ and $d \geq k+1$.

First consider k-Precolouring Extension. Theorem 2 implies that 3-Colouring is NP-complete for graphs of maximum degree at most d for all $d \geq 4$. The $k = 3$ case follows immediately from this result. Suppose $k \geq 4$ and $d \geq k+1$. Consider a graph G of maximum degree 4. For each vertex v, we add $k-3$ new vertices $x_1^v, \ldots, x_{k-3}^v$ and edges $vx_1^v, \ldots, vx_{k-3}^v$. Let G' be the resulting graph. Note that G' has maximum degree at most $4 + k - 3 = k + 1 \leq d$. We define a precolouring c on the newly added vertices by assigning colour $i + 3$ to each x_i^v. Then G' has a k-colouring extending c if and only if G has a 3-colouring.

Now consider k-Regular List Colouring. The $k = 3$ case follows immediately from Theorem 2. Suppose $k \geq 4$ and $d \geq k+1$. Consider a graph G of maximum degree 4. We define the list $L(v) = \{1, \ldots, k\}$ for each vertex $v \in V(G)$. For each vertex v, we add $k-3$ new vertices $x_1^v, \ldots, x_{k-3}^v$ and edges

$vx_1^v, \ldots, vx_{k-3}^v$. We define the list $L(x_i^v) = \{i, k+1, k+2, \ldots, 2k-1\}$ for each x_i^v. For each vertex x_i^v, we also add k new vertices $w_1(x_i^v), \ldots, w_k(x_i^v)$ and edges such that $x_i^v, w_1(x_i^v), \ldots, w_k(x_i^v)$ form a clique (on $k+1$ vertices). We define the list $L(w_j(x_i^v)) = \{k+1, \ldots, 2k\}$ for each $w_j(x_i^v)$. Let G' be the resulting graph. Note that G' has maximum degree at most $k+1$ and that the resulting list assignment L is a k-regular list assignment of G'. Then G' has a k-colouring respecting L if and only if G has a 3-colouring.

We continue with the polynomial-time solvable cases. If $k \leq 2$, the result follows from Theorem 1. Suppose that $k \geq 3$ and $d \leq k$. Then the result for k-Regular List Colouring follows from Theorems 10 and 11 and the result for k-Precolouring Extension follows from Theorem 13. □

Note that Corollary 7 does not contain a dichotomy for k-Colouring restricted to graphs of maximum degree at most d. A full classification of this problem is open, but a number of results are known. Molloy and Reed [27] classified the complexity for all pairs (k, d) for sufficiently large d. Emden-Weinert et al. [16] proved that k-Colouring is NP-complete for graphs of maximum degree at most $k + \lceil\sqrt{k}\rceil - 1$. It follows from Brooks' Theorem [7] that for every integer $k \geq 1$, k-Colouring is polynomial-time solvable for graphs of maximum degree k. Combining this observation with the result of [16] means that the smallest open case is when $k = 5$ and $d = 6$.

5 Proofs

In Section 5.1 we prove Theorems 5 and 6. In Section 5.2 we prove Theorem 7, whereas in Section 5.3 we prove Theorem 8, and in Section 5.4 we prove Theorem 9. Finally, in Section 5.5 we prove Theorem 13.

5.1 The Proofs of Theorems 5 and 6

We need an additional result.

Theorem 14. *For every integer $p \geq 3$, 3-List Colouring is NP-complete for planar graphs of girth at least p that have a list assignment in which each list is one of $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$.*

Proof. By Theorem 4, List Colouring is NP-complete for 3-regular planar bipartite graphs that have a list assignment in which each list is one of $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$ and all the neighbours of each vertex with three colours in its list have two colours in their lists. We modify the hardness construction as follows. Note that for each edge at least one of the incident vertices has a list of size 2. We replace each edge by a path on an odd number of edges in such a way that the girth of the graph obtained is at least p. The new vertices on the path are all given the same list of size 2, identical to the list on one or other of the end-vertices. It is readily seen that these modifications do not affect whether or not the graph can be coloured. □

We are now ready to prove Theorem 5, which we restate below.

Theorem 5 (restated). *Let $\mathcal{H}$ be a finite set of 2-connected planar graphs. Then* 4-Regular List Colouring *is* NP*-complete for planar $\mathcal{H}$-subgraph-free graphs if there exists a planar $\mathcal{H}$-subgraph-free graph that is not 4-choosable.*

Proof. The problem is readily seen to be in NP. Let F be a planar $\mathcal{H}$-subgraph-free graph with a 4-regular list assignment L such that F has no colouring respecting L. We may assume that F is minimal (with respect to the subgraph relation). In particular, this means that F is connected. Let r be the length of a longest cycle in any graph of $\mathcal{H}$. We reduce from the problem of 3-List Colouring restricted to planar graphs of girth at least $r+1$ in which each vertex has list $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ or $\{1,2,3\}$. This problem is NP-complete by Theorem 14. Let a graph G and list assignment L_G be an instance of this problem. We will construct a planar $\mathcal{H}$-subgraph-free graph G' with a 4-regular list assignment L' such that G has a colouring that respects L_G if and only if G' has a colouring that respects L'.

If every pair of adjacent vertices in F has the same list, then the problem of finding a colouring that respects L is just the problem of finding a 4-colouring which, by the Four Colour Theorem [2], we know is possible. Thus we may assume that, on the contrary, there is an edge $e = uv$ such that $|L(u) \cap L(v)| \leq 3$. Let $F' = F - e$. Then, by minimality, F' has at least one colouring respecting L, and moreover, for any colouring of F' that respects L, u and v are coloured alike (otherwise we would have a colouring of F that respects L). Let T be the set of possible colours that can be used on u and v in colourings of F' that respect L and let $t = |T|$. As $T \subseteq L(u) \cap L(v)$, we have $1 \leq t \leq 3$. Up to renaming the colours in L, we can build copies of F' with 4-regular list assignments such that

(i) the set T is any given list of colours of size t, and
(ii) the vertex corresponding to u has any given list of 4 colours containing T.

We will implicitly make use of this several times in the remainder of the proof.

We say that a vertex w in G is a *bivertex* or *trivertex* if $|L_G(w)|$ is 2 or 3, respectively. We construct a planar $\mathcal{H}$-subgraph-free graph G' and 4-regular list assignment L' as follows.

First suppose that $t = 1$. For each bivertex w in G, we do as follows. We add two copies of F' to G, which we label $F_1(w)$ and $F_2(w)$. The vertex in $F_i(w)$ corresponding to u is labelled u_i^w for $i \in \{1,2\}$ and we set $U^w = \{u_1^w, u_2^w\}$. We add the edges wu_1^w and wu_2^w. We give list assignments to the vertices of $F_1(w)$ and $F_2(w)$ such that $T = \{4\}$ for F_1 and $T = \{5\}$ for F_2. We let $L'(w) = L_G(w) \cup \{4,5\}$. For each trivertex w in G, we do as follows. We add one copy of F' to G, which we label $F_1(w)$. The vertex in $F_1(w)$ corresponding to u is labelled u_1^w and we set $U^w = \{u_1^w\}$. We add the edge wu_1^w. We give list assignments to vertices of $F_1(w)$ such that $T = \{4\}$ for F_1. We let $L'(w) = L_G(w) \cup \{4\}$. This completes the construction of G' and L' when $t = 1$.

Now suppose that $t = 2$. Let $s = r$ if r is even and $s = r+1$ if r is odd (so s is even in both cases). For each bivertex w in G, we do as follows. We

add a copy of F' to G, which we label $F_1(w)$, and identify the vertex in $F_1(w)$ corresponding to u with w. We give list assignments to vertices of $F_1(w)$ such that $T = L_G(w)$ and $L'(w) = L_G(w) \cup \{4, 5\}$. For each trivertex w in G, we do as follows. We add s copies of F' to G which we label $F_i(w)$, $1 \leq i \leq s$. The vertex in $F_i(w)$ corresponding to u is labelled u_i^w. Let $U^w = \{u_i^w \mid 1 \leq i \leq s\}$. Add edges such that the union of w and U^w induces a cycle on $s+1$ vertices. For all $1 \leq i \leq s$, we give list assignments to vertices of $F_i(w)$ such that $T = \{4, 5\}$. We let $L'(w) = \{1, 2, 3, 4\}$. This completes the construction of G' and L' when $t = 2$.

Now suppose that $t = 3$. For each bivertex w in G, we do as follows. We add two copies of F' to G which we label $F_1(w)$ and $F_2(w)$, such that for $i \in \{1, 2\}$, the vertex in $F_i(w)$ corresponding to u is identified with w. We give list assignments to vertices of $F_1(w)$ and $F_2(w)$ such that $T = L_G(w) \cup \{4\}$ for $F_1(w)$, $T = L_G(w) \cup \{5\}$ for $F_2(w)$ and $L'(w) = L_G(w) \cup \{4, 5\}$. For each trivertex w in G, we do as follows. We add a copy of F' to G which we label $F_1(w)$, such that the vertex in $F_1(w)$ corresponding to u is identified with w. We give list assignments to the vertices of $F_1(w)$ such that $T = \{1, 2, 3\}$ and $L'(w) = \{1, 2, 3, 4\}$. This completes the construction of G' and L' when $t = 3$.

Note that G' is planar. Suppose that there is a subgraph H in G' that is isomorphic to a graph of $\mathcal{H}$. Since F is $\mathcal{H}$-subgraph-free, and since F' is obtained from F by removing one edge, F' is also $\mathcal{H}$-subgraph-free. Therefore for all w, H is not fully contained in any $F_i(w)$. Since H is 2-connected and since for all w only one vertex of any $F_i(w)$ has a neighbour outside of $F_i(w)$, we find that H has at most one vertex in each $F_i(w)$. In particular, H cannot contain any vertex of any $F_i(w)$ in which the vertex corresponding to u has been attached to w (as opposed to being identified with w); this includes the case when the union of w and U^w induces a cycle on $s + 1$ vertices. Hence we have found that H is a subgraph of G, which contradicts the fact that G has girth at least $r + 1$. Therefore G' is $\mathcal{H}$-subgraph-free.

Note that in any colouring of G' that respects L', each copy of F' must be coloured such that the vertices corresponding to u and v have the same colour, which must be one of the colours from the corresponding set T. If $t = 1$ and w is a trivertex, this means that the unique neighbour of w in U^w must be coloured with colour 4, so w cannot be coloured with colour 4. Similarly, if $t = 1$ and w is a bivertex or $t = 2$ and w is a trivertex then the two neighbours of w in U^w must be coloured with colours 4 and 5, so w cannot be coloured with colours 4 or 5. If $t = 2$ and w is a bivertex or $t = 3$ and w is a trivertex then w belongs to a copy of F' with $T = L_G(w)$, so w cannot have colour 4 or 5. If $t = 3$ and w is a bivertex then w belongs to two copies of F', one with $T = L_G(w) \cup \{4\}$ and one with $T = L_G(w) \cup \{5\}$. Therefore, w must be coloured with a colour from the intersection of these two sets, that is it must be coloured with a colour from $L_G(w)$. Therefore none of the vertices of G can be coloured 4 or 5. Thus the problem of finding a colouring of G' that respects L' is equivalent to the problem of finding a colouring of G that respects L_G. This completes the proof. □

We will now prove Theorem 6.

Theorem 6 (restated). 4-Regular List Colouring *is* NP*-complete for planar graphs even if every list contains four colours from* $\{1, 2, 3, 4, 5\}$.

Proof. Recall that there exists a planar graph F^* with a 4-regular list assignment L in which each list $L(u)$ contains four colours from $\{1, 2, 3, 4, 5\}$ such that F^* has no colouring respecting L [38]. We may assume without loss of generality that F^* is minimal (with respect to the subgraph relation) and use F^* as the graph F in the proof of Theorem 5. This means that we give each vertex of the graph G' in the proof of Theorem 5 a list of colours from $\{1, 2, 3, 4, 5\}$. The result follows. □

5.2 The Proof of Theorem 7

Theorem 7 (restated). *Let* $\mathcal{H}$ *be a finite set of* 2*-connected planar graphs. Then* 3-Regular List Colouring *is* NP*-complete for planar* $\mathcal{H}$*-subgraph-free graphs if there exists a planar* $\mathcal{H}$*-subgraph-free graph that is not* 3*-choosable.*

Proof. The problem is readily seen to be in NP. Every graph in $\mathcal{H}$ is 2-connected, and therefore contains a cycle. Let r be the length of a longest cycle in any graph of $\mathcal{H}$. By assumption, there exists a planar graph F and 3-regular list assignment L such that F is $\mathcal{H}$-subgraph-free and has no colouring respecting L. We may assume that F is minimal by removing edges and vertices until any further removal would give a graph with a colouring respecting L. In particular, we note that F is connected.

We distinguish two cases.

Case 1. $L(v)$ is the same for every vertex v in F.
Then we may assume without loss of generality that $L(v) = \{1, 2, 3\}$ for all v. We reduce from 3-Colouring which is NP-complete even for planar graphs by Theorem 2. Let G be a planar graph. We will construct a planar $\mathcal{H}$-subgraph-free graph G' as follows.

Let $e = uv$ be an edge of F. Let $F' = F - e$. Then, by minimality, F' has at least one colouring respecting L, which must be a 3-colouring as every list consists of the colours $1, 2, 3$. For every 3-colouring c of F', it holds that $c(u) = c(v)$ (otherwise c would be a colouring of F that respects L). Moreover, since we can permute the colours, there is such a colouring c that colours u (and thus v) with colour i for each $i \in \{1, 2, 3\}$. Note that in F' the vertices u and v must be at distance at least 2 from each other.

Let $s = \lceil \frac{r}{6} \rceil$. Assume that the vertices of G are ordered. For each edge $xy \in E(G)$ with $x < y$, we do the following:

(i) delete xy;
(ii) add s copies of F' labelled $F_1(xy), \ldots, F_s(xy)$ and, for $1 \leq i \leq s$, let u_i^{xy} and v_i^{xy} be the vertices in $F_i(xy)$ corresponding to u and v;
(iii) identify x with u_1^{xy} and, for $1 \leq i \leq s - 1$, identify v_i^{xy} with u_{i+1}^{xy};
(iv) add an edge from v_s^{xy} to y.

Let G' be the obtained graph and note that G' is planar. Every cycle in G' that is not contained in a copy of F' has length at least $3(2s+1) \geq r+1$, since it corresponds to a cycle in G and in which every edge has been replaced by s successive copies of F' plus an edge.

Suppose that there is a subgraph H in G' that is isomorphic to a graph of $\mathcal{H}$. Since F is $\mathcal{H}$-subgraph-free, and since F' is obtained from F by removing one edge, F' must also be $\mathcal{H}$-subgraph-free. Therefore H is not fully contained in a copy of F'. Since H is 2-connected, this implies that there is a cycle in H that is not fully contained in a copy of F'. By definition, this cycle has length at most r, a contradiction.

Suppose the graph G' has a 3-colouring c. For each copy of F', the vertices corresponding to u and v must be coloured the same. For all edges xy with $x < y$ in G, there is a vertex in G' coloured the same as x in G that is adjacent to y in G', so $c(x) \neq c(y)$. Therefore c restricted to $V(G)$ is a 3-colouring of G.

On the other hand, suppose the graph G has a 3-colouring c. We can extend this 3-colouring to G' by doing the following: for all edges xy with $x < y$ in G, colour every $F_i(xy)$ in such a way that the vertex corresponding to u and the vertex corresponding to v have colour $c(x)$. This leads to a 3-colouring of G'.

Case 2. F contains two vertices u and v with $L(u) \neq L(v)$.
As F is connected, we assume without loss of generality that u and v are adjacent; let $e = uv$.

We reduce from the problem of 3-List Colouring restricted to planar graphs of girth at least $r+1$ in which each vertex has list $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ or $\{1,2,3\}$. This problem is NP-complete by Theorem 14. Let a graph G and list assignment L_G be an instance of this problem. We will construct a planar $\mathcal{H}$-subgraph-free graph G' with a 3-regular list assignment L' such that G has a colouring that respects L_G if and only if G' has a colouring that respects L'.

We define $F' = F - e$. Then, by minimality, F' has at least one colouring respecting L, and moreover, for any colouring of F' that respects L, u and v are coloured alike (otherwise we would have a colouring of F that respects L). Let T be the set of possible colours that can be used on u and v in colourings of F' that respect L and let $t = |T|$. As $T \subseteq L(u) \cap L(v)$, we have $1 \leq t \leq 2$. Let us assume, without loss of generality, that $T \subseteq \{4,5\}$ and that $4 \in T$.

We say that a vertex w in G is a bivertex or trivertex if $|L(w)|$ is 2 or 3, respectively. We construct a planar $\mathcal{H}$-free graph G'.

First suppose that $t = 1$. For each bivertex w in G, we do as follows. We add a copy of F' to G which we label $F(w)$. The vertex in $F(w)$ corresponding to u is labelled u^w and we set $U^w = \{u^w\}$. We add the edge wu^w. This completes the construction of G' when $t = 1$.

Now suppose that $t = 2$. Let $s = r$ if r is even and $s = r+1$ if r is odd (so s is even in both cases). For each bivertex w in G, we add s copies of F' to G which we label $F_i(w)$, $1 \leq i \leq s$. The vertex in $F_i(w)$ corresponding to u is labelled u_i^w. Let $U^w = \{u_i^w \mid 1 \leq i \leq s\}$. Add edges such that, for each bivertex w in G, the union of w and U^w induces a cycle on $s+1$ vertices. This completes the construction of G' when $t = 2$.

Note that G' is planar, since it is made of planar graphs (G and copies of F') connected in a way that does not obstruct planarity. Suppose that there is a subgraph H in G' that is isomorphic to a graph of $\mathcal{H}$. Since F is $\mathcal{H}$-subgraph-free, and since F' is obtained from F by removing one edge, F' is also $\mathcal{H}$-subgraph-free. Therefore for all w, H is not fully contained in $F(w)$. Since H is 2-connected and since for all w, only one vertex of $F(w)$ has a neighbour outside of $F(w)$, we find that H has at most one vertex in each $F(w)$. This means that H is a subgraph of G, which contradicts the fact that G has girth at least $r+1$. Therefore G' is $\mathcal{H}$-subgraph-free.

Now we define a list assignment L'. We give the vertices of each copy of F' the same lists as their corresponding vertices in F, and for each bivertex w in G, we define $L'(w) = L_G(w) \cup \{4\}$, and for each trivertex w in G, we define $L'(w) = L_G(w)$. This gives us the 3-regular list assignment L' of G'.

The graph $G' - G$ has a colouring that respects (the restriction of) L' and we notice that in such a colouring each copy of F' must be coloured in such a way that, for each bivertex w in G, the t vertices of U^w that are adjacent to w are coloured with the t colours of T. So one of the neighbours of w in U^w must be coloured 4. Thus the problem of finding a colouring of G' that respects L' is equivalent to the problem of finding a colouring of G that respects L_G. The proof is complete. □

5.3 The Proof of Theorem 8

Theorem 8 is not quite implied by Theorem 7. However, we can adapt the proof of Theorem 7 to prove Theorem 8, which we restate below.

Theorem 8 (restated). 3-Regular List Colouring *is* NP*-complete for planar graphs with no* 4*-cycles, no* 5*-cycles and no intersecting triangles.*

Proof. Let B be the graph on five vertices with two triangles sharing exactly one vertex (this graph is known as the *butterfly*). Consider the previous proof with $\mathcal{H} = \{C_4, C_5, B\}$. Note that the only problem is that B is not 2-connected.

Consider the example in [29] of a graph with no 4-cycle, no 5-cycle and no intersecting triangles that is not 3-choosable. Let I be this example, with L a 3-regular list assignment such that there is no colouring of I respecting L. We remove edges and vertices from I until any further removal would give a graph with a colouring respecting L. This leads to a connected graph F. There are no four vertices in I with the same list inducing a connected subgraph, so there are no such four vertices in F. Therefore in F there is an edge uv such that $L(u) \neq L(v)$ (otherwise F would have at most three vertices, and thus would be 3-choosable).

Therefore we can skip Case 1 and directly adapt the proof in Case 2. Note that the only thing to prove is that in this case G' does not contain a subgraph isomorphic to B. Suppose H is such a subgraph. Since F' is B-subgraph-free, H cannot be fully contained in $F(w)$ for any w. Since no vertex is in two different $F(w)$ and no vertex of $F(w)$ has two adjacent neighbours outside $F(w)$,

this implies that there is a triangle in G, which is impossible since G has girth at least 6. □

5.4 The Proof of Theorem 9

The proof is obtained by a modification of the NP-hardness construction of 3-List Colouring for $(3P_1, P_1 + P_2)$-free graphs from [22]. Recall that we included this result in our paper to illustrate that k-Choosability and k-Regular List Colouring can have different complexities when restricted to special graph classes. Indeed, since Choosability is polynomial-time solvable on $3P_1$-free graphs [20], Theorem 9 shows that k-Choosability may even be easier than k-Regular List Colouring.

Theorem 9 (restated). 3-Regular List Colouring *is* NP*-complete for* $(3P_1, P_1 + P_2)$*-free graphs.*

Proof. The problem is readily seen to belong to NP. Golovach et al. [20] showed that 3-List Colouring is NP-complete for $(3P_1, P_1 + P_2)$-free graphs in which every vertex has a list of size 2 or 3. Let G be such an instance. We add three new vertices s, t, u to G. We make s, t, u, v adjacent to each other and to each original vertex of G. This results in a $(3P_1, P_1 + P_2)$-free graph G'. We take three new colours $1, 2, 3$ and set $L(s) = L(t) = L(u) = \{1, 2, 3\}$. This forces colour 1 to be used to colour one of s, t or u. Then all that remains is to add colour 1 to the list of every vertex of G that has a list of size 2. □

5.5 The Proof of Theorem 13

We need some additional results. We begin with a theorem of Bonomo, Durán and Marenco.

Theorem 15 ([4]). List Colouring *is polynomial-time solvable on block graphs.*

By generalizing their proof we extend this result to classes of graphs where List Colouring is polynomial-time solvable on the blocks of graphs in the class.

Theorem 16. *Let $\mathcal{G}$ be a class of graphs and let $\mathcal{G}_B$ be the class of graphs that appear as blocks of graphs in $\mathcal{G}$. If* List Colouring *is polynomial-time solvable on $\mathcal{G}_B$ then it is polynomial-time solvable on $\mathcal{G}$.*

Proof. Let G be a graph in $\mathcal{G}$ that, together with a list assignment L, forms an instance of List Colouring. We may assume G is connected. If $G \notin \mathcal{G}_B$ then consider a cut-vertex u in a leaf-block B (such B and u exist). Let L_B be the restriction of L to $V(B) \setminus \{u\}$. For each colour $i \in L(u)$, we do as follows. We remove i from the list of every neighbour of u in B and check whether B admits a colouring that respects L_B. Note that we can do this in polynomial time, as B

is in $\mathcal{G}_B$. If so then we put i in a set A_u for vertex u; otherwise we do not do this. If, after considering each colour in $L(u)$, we find that $A_u = \emptyset$ then we return no. Otherwise we define a new list assignment L' for the subgraph G' of G induced by $V(G) \setminus (B \setminus \{u\})$ by setting $L'(u) = A(u)$ and $L'(v) = L(v)$ if $v \in V(G) \setminus B$. Note that G has a colouring that respects L if and only if G' has a colouring that respects L'. We continue with the pair G', L'. We do this exhaustively until we obtain in polynomial time a graph in $\mathcal{G}_B$. Since List Colouring is polynomial-time solvable in $\mathcal{G}_B$, this completes the proof. □

Theorem 16, combined with Theorem 10, leads to the following generalization of Theorem 15.

Corollary 8. List Colouring *is polynomial-time solvable on Gallai trees.*

We also state the following theorem, proved independently by Borodin and Erdős et al.

Theorem 17 ([5, 6, 17]). *Let $G = (V, E)$ be a connected graph with a list assignment L such that $|L(u)| = \deg(u)$ for all $u \in V$. If G is not a Gallai tree then G has a colouring respecting L.*

We now restate and prove Theorem 13.

Theorem 13 (restated). *Let k be a positive integer. Then k-*Precolouring Extension *is polynomial-time solvable for graphs of maximum degree at most k.*

Proof. Let $G = (V, E)$, together with a k-precolouring c_W defined on a subset $W \subseteq V$, be an instance of k-Precolouring Extension. We let G' be the subgraph of G induced by $V \setminus W$. For each $v \in V(G')$, we set $L(v) = \{1, \ldots, k\} \setminus \{c_W(u) \mid u \in W \cap N(v)\}$. Observe that G has a k-colouring extending c_W if and only if G' has a colouring that respects L. Hence we may consider G' instead.

Note that, for every vertex $v \in V(G')$, the colours that are removed from $\{1, \ldots, k\}$ to obtain $L(v)$ are exactly the colours of those neighbours of v in G that are not in G'. This implies, together with the assumption that G has maximum degree at most k, that every $v \in V(G')$ has at most $|L(v)|$ neighbours in G'.

We now apply the following procedure on G' exhaustively. If a vertex v has fewer than $|L(v)|$ neighbours then remove v from G'. In the end we obtain a graph G^* with the property that $L(u) = \deg_{G^*}(u)$ for all $u \in V(G^*)$. Moreover, G' has a colouring that respects L if and only if G^* has a colouring that respects the restriction of L to $V(G^*)$. Hence we may consider G^* instead. We consider every connected component C of G^* in turn. If C is a Gallai tree then we apply Corollary 8. Otherwise we apply Theorem 17. □

6 Conclusions

As well as filling the complexity gaps of a number of colouring problems for graphs with bounded maximum degree, we have given several dichotomies for the

k-Regular List Colouring problem restricted to subclasses of planar graphs. In particular we showed NP-hardness of the cases $k = 3$ and $k = 4$ restricted to planar $\mathcal{H}$-subgraph-free graphs for several sets $\mathcal{H}$ of 2-connected planar graphs. Our method implies that for such sets $\mathcal{H}$ it suffices to find a counterexample to 3-choosability or to 4-choosability, respectively. It is a natural to ask whether we can determine the complexity of 3-Regular List Colouring and 4-Regular List Colouring for any class of planar $\mathcal{H}$-subgraph-free graphs. However, we point out that even when restricting $\mathcal{H}$ to be a finite set of 2-connected planar graphs, this would be very hard (and beyond the scope of this paper) as it would require solving several long-standing conjectures in the literature. For example, when $\mathcal{H} = \{C_4, C_5, C_6\}$, Montassier [28] conjectured that every planar $\mathcal{H}$-subgraph-free graph is 3-choosable.

A drawback of our method is that we need the set of graphs $\mathcal{H}$ to be 2-connected. If we forbid a set $\mathcal{H}$ of graphs that are not 2-connected, the distinction between polynomial-time solvable and NP-complete cases is not clear, and both cases may occur even if we forbid only one graph. We illustrate this below with an example.

Example. Let $\mathcal{H}$ contain only the star $K_{1,r}$ for some $r \geq 2$. Note that $K_{1,r}$-subgraph-free graphs are exactly those graphs that have maximum degree at most $r - 1$. Hence, if $r = 3$, then 3-Regular List Colouring is polynomial-time solvable due to Theorem 10. However, there exist larger values of r for which the problem is NP-complete. In order to see this we adapt the proof of Theorem 7. The hardness reductions in this proof multiply the maximum degree of our instances by some constant d that is at most the maximum degree of the no-instance F. By Theorems 2 and 4, the problems we reduce from are NP-complete even for graphs with maximum degree at most 4. Hence, we have proven the following: if $\mathcal{H}$ is a finite set of 2-connected planar graphs and F is a non-3-choosable planar $\mathcal{H}$-subgraph-free graph with maximum degree d, then 3-Regular List Colouring is NP-complete on planar $\mathcal{H}$-subgraph-free graphs with maximum degree at most $4d$. We can take $F = K_4$ to deduce that 3-Regular List Colouring is NP-complete on planar $K_{1,13}$-subgraph-free graphs.

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Appendix C

Coloring Diamond-free Graphs

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Colouring Diamond-free Graphs*

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Abstract. The Colouring problem is that of deciding, given a graph G and an integer k, whether G admits a (proper) k-colouring. For all graphs H up to five vertices, we classify the computational complexity of Colouring for (diamond, H)-free graphs. Our proof is based on combining known results together with proving that the clique-width is bounded for (diamond, P_1+2P_2)-free graphs. Our technique for handling this case is to reduce the graph under consideration to a k-partite graph that has a very specific decomposition. As a by-product of this general technique we are also able to prove boundedness of clique-width for four other new classes of (H_1, H_2)-free graphs. As such, our work also continues a recent systematic study into the (un)boundedness of clique-width of (H_1, H_2)-free graphs, and our five new classes of bounded clique-width reduce the number of open cases from 13 to 8.

1 Introduction

The Colouring problem is that of testing whether a given graph can be coloured with at most k colours for some given integer k, such that any two adjacent vertices receive different colours. The complexity of Colouring is fully understood for general graphs: it is NP-complete even if $k = 3$ [35]. Therefore it is natural to study its complexity when the input is restricted. A classic result in this area is due to Grötschel, Lovász, and Schrijver [26], who proved that Colouring is polynomial-time solvable for perfect graphs.

As surveyed in [14,20,25,43], Colouring has been well studied for hereditary graph classes, that is, classes that can be defined by a family $\mathcal{H}$ of forbidden induced subgraphs. For a family $\mathcal{H}$ consisting of one single forbidden induced subgraph H, the complexity of Colouring is completely classified: the problem is polynomial-time solvable if H is an induced subgraph of P_4 or $P_1 + P_3$ and NP-complete otherwise [34]. Hence, many papers (e.g. [13,18,29,34,37,40,41,45]) have considered the complexity of Colouring for bigenic hereditary graph classes, that is, graph classes defined by families $\mathcal{H}$ consisting of two forbidden graphs H_1 and H_2; such classes of graphs are also called (H_1, H_2)-free. This classification is far from complete (see [25] for the state of art). In fact there are still an infinite number of open cases, including cases where both H_1 and H_2 are small. For instance, Lozin and Malyshev [37] determined the computational complexity of Colouring for (H_1, H_2)-free graphs for all graphs H_1 and H_2 up to four vertices except when $(H_1, H_2) \in \{(K_{1,3}, 4P_1), (K_{1,3}, 2P_1 + P_2), (C_4, 4P_1)\}$ (we refer to Section 2 for notation and terminology).

The *diamond* is the graph $\overline{2P_1 + P_2}$, that is, the graph obtained from the complete graph on four vertices by removing an edge. Diamond-free graphs are well studied in the

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literature. For instance, Tucker [46] gave an $O(kn^2)$ time algorithm for Colouring for perfect diamond-free graphs. It is also known that that Colouring is polynomial-time solvable for diamond-free graphs that contain no induced cycle of even length [32] as well as for diamond-free graphs that contain no induced cycle of length at least 5 [8]. Diamond-free graphs also played an important role in proving that the class of P_6-free graphs contains 24 minimal obstructions for 4-Colouring [15] (that is, the Colouring problem for $k = 4$).

1.1 Our Main Result

In this paper we focus on Colouring for (diamond, H)-free graphs where H is a graph on at most five vertices. It is known that Colouring is NP-complete for (diamond, H)-free graphs when H contains a cycle or a claw [34] and polynomial-time solvable for $H = sP_1 + P_2$ ($s \geq 0$) [18], $H = 2P_1 + P_3$ [5], $H = P_1 + P_4$ [11], $H = P_2 + P_3$ [19] and $H = P_5$ [1]. Hence, the only graph H on five vertices that remains is $H = P_1 + 2P_2$, for which we prove polynomial-time solvability in this paper. This leads to the following result.

Theorem 1. *Let H be a graph on at most five vertices. Then* Colouring *is polynomial-time solvable for (diamond, H)-free graphs if H is a linear forest and* NP*-complete otherwise.*

To solve the case $H = P_1 + 2P_2$, one could try to reduce to a subclass of diamond-free graphs, for which Colouring is polynomial-time solvable, such as the aforementioned results of [8,32,46]. This would require us to deal with the presence of small cycles up to C_7, which may not be straightforward. Instead we aim to identify tractability from an underlying property: we show that the class of (diamond, $P_1 + 2P_2$)-free graphs has bounded clique-width. This approach has several advantages and will lead to a number of additional results, as we will discuss in the remainder of Section 1.

Clique-width is a graph decomposition that can be constructed via vertex labels and four specific graph operations, which ensure that vertices labelled alike will always keep the same label and thus behave identically. The clique-width of a graph G is the minimum number of different labels needed to construct G using these four operations (we refer to Section 2 for a precise definition). A graph class $\mathcal{G}$ has *bounded* clique-width if there exists a constant c such that every graph from $\mathcal{G}$ has clique-width at most c.

Clique-width is a well-studied graph parameter (see, for instance, the surveys [27,31]). An important reason for the popularity of clique-width is that a number of classes of NP-complete problems, such as those that are definable in Monadic Second Order Logic using quantifiers on vertices but not on edges, become polynomial-time solvable on any graph class $\mathcal{G}$ of bounded clique-width (this follows from combining results from [16,23,33,44] with a result from [42]). The Colouring problem is one of the best-known NP-complete problems that is solvable in polynomial time on graph classes of bounded clique-width [33]; another well-known example of such a problem is Hamilton Path [23].

1.2 Methodology

The key technique for proving that (diamond, P_1+2P_2)-free graphs have bounded clique-width is the use of a certain graph decomposition of k-partite graphs. We obtain this decomposition by generalizing the so-called canonical decomposition of bipartite graphs, which decomposes a bipartite graph into two smaller bipartite graphs such that edges between these two smaller bipartite graphs behave in a very restricted way. Fouquet,

Giakoumakis and Vanherpe [24] introduced this decomposition and characterized exactly those bipartite graphs that can recursively be canonically decomposed into graphs isomorphic to K_1. Such bipartite graphs are said to be totally decomposable by canonical decomposition. We say that k-partite graphs are totally k-decomposable if they can be, according to our generalized definition, recursively k-decomposed into graphs isomorphic to K_1. We show that totally k-decomposable graphs have clique-width at most $2k$. We prove this result in Section 3, where we also give a formal definition of canonical decomposition, along with our generalization.

Our goal is to transform (diamond, $P_1 + 2P_2$)-free graphs into graphs in some class for which we already know that the clique-width is bounded. Besides the class of totally k-decomposable graphs, we will also reduce to other known graph classes of bounded clique-width, such as the class of (diamond, $P_2 + P_3$)-free graphs [19] and certain classes of H-free bipartite graphs [21]. Of course, our transformations must not change the clique-width by "too much". We ensure this by using certain graph operations (described in Section 2) that are known to preserve (un)boundedness of clique-width [31,38].

1.3 Consequences for Clique-Width

There are numerous papers (as listed in, for instance, [22,27,31]) that determine the (un)boundedness of the clique-width or variants of it (see e.g. [4,28]) of special graph classes. Due to the complex nature of clique-width, proofs of these results are often long and technical, and there are still many open cases. In particular, gaps exist in a number of dichotomies on the (un)boundedness of clique-width for graph classes defined by one or more forbidden induced subgraphs. As such our paper also continues a line of research [5,6,19,21,22] in which we focus on these gaps in a *systematic* way. It is known [22] that the class of H-free graphs has bounded clique-width if and only if H is an induced subgraph of P_4. Over the years many partial results [2,7,9,10,11,12,20,39] on the (un)boundedness of clique-width have appeared for classes of (H_1, H_2)-free graphs, but until recently [22] it was not even known whether the number of missing cases was bounded. Combining these older results with recent progress [5,18,19,22] reduced the number of open cases to 13 (up to an equivalence relation) [22].

As a by-product of our general methodology, we are able not only to settle the case $(H_1, H_2) = (\text{diamond}, P_1 + 2P_2)$, but in fact we solve **five** of the remaining 13 open cases by proving that the class of (H_1, H_2)-free graphs has bounded clique-width if

1–4: $H_1 = K_3$ and $H_2 \in \{P_1 + 2P_2, P_1 + P_2 + P_3, P_1 + P_5, S_{1,2,2}\}$ or

5: $H_1 = \text{diamond}$ and $H_2 = P_1 + 2P_2$.

The above graphs are displayed in Fig. 1. Note that the $(K_3, P_1 + 2P_2)$-free graph case is properly contained in all four of the other cases. These four other newly solved cases are pairwise incomparable. In Section 4 we use our key technique on totally k-decomposable graphs to find a number of sufficient conditions for a graph class to have bounded clique-width. We use these conditions in Section 5 to prove Results 1–4 and we then prove Result 5 (which relies on Result 1) in Section 6.

Updating the classification (see [22]) with our five new results gives the following theorem. Here, $\mathcal{S}$ is the class of graphs each connected component of which is either a subdivided claw or a path, and we write $H \subseteq_i G$ if H is an induced subgraph of G; see Section 2 for notation that we have not formally defined yet.

K_3 diamond $P_1 + 2P_2$ $P_1 + P_2 + P_3$ $P_1 + P_5$ $S_{1,2,2}$

Fig. 1. The forbidden graphs considered in this paper.

Theorem 2. *Let $\mathcal{G}$ be a class of graphs defined by two forbidden induced subgraphs. Then:*

(i) $\mathcal{G}$ has bounded clique-width if it is equivalent[1] *to a class of (H_1, H_2)-free graphs such that one of the following holds:*
1. *H_1 or $H_2 \subseteq_i P_4$;*
2. *$H_1 = sP_1$ and $H_2 = K_t$ for some s, t;*
3. *$H_1 \subseteq_i P_1 + P_3$ and $\overline{H_2} \subseteq_i K_{1,3} + 3P_1,\ K_{1,3} + P_2,\ P_1 + P_2 + P_3,\ P_1 + P_5,\ P_1 + S_{1,1,2},\ P_6,\ S_{1,1,3}$ or $S_{1,2,2}$;*
4. *$H_1 \subseteq_i 2P_1 + P_2$ and $\overline{H_2} \subseteq_i P_1 + 2P_2,\ 2P_1 + P_3,\ 3P_1 + P_2$ or $P_2 + P_3$;*
5. *$H_1 \subseteq_i P_1 + P_4$ and $\overline{H_2} \subseteq_i P_1 + P_4$ or P_5;*
6. *$H_1 \subseteq_i 4P_1$ and $\overline{H_2} \subseteq_i 2P_1 + P_3$;*
7. *$H_1, \overline{H_2} \subseteq_i K_{1,3}$.*

(ii) $\mathcal{G}$ has unbounded clique-width if it is equivalent to a class of (H_1, H_2)-free graphs such that one of the following holds:
1. *$H_1 \notin \mathcal{S}$ and $H_2 \notin \mathcal{S}$;*
2. *$\overline{H_1} \notin \mathcal{S}$ and $\overline{H_2} \notin \mathcal{S}$;*
3. *$H_1 \supseteq_i K_{1,3}$ or $2P_2$ and $\overline{H_2} \supseteq_i 4P_1$ or $2P_2$;*
4. *$H_1 \supseteq_i 2P_1 + P_2$ and $\overline{H_2} \supseteq_i K_{1,3},\ 5P_1,\ P_2 + P_4$ or P_6;*
5. *$H_1 \supseteq_i 3P_1$ and $\overline{H_2} \supseteq_i 2P_1 + 2P_2,\ 2P_1 + P_4,\ 4P_1 + P_2,\ 3P_2$ or $2P_3$;*
6. *$H_1 \supseteq_i 4P_1$ and $\overline{H_2} \supseteq_i P_1 + P_4$ or $3P_1 + P_2$.*

1.4 Future Work

Naturally we would like to extend Theorem 1 and solve the following open problem.

Open Problem 1. *What is the computational complexity of the* Colouring *problem for (diamond, H)-free graphs when H is a graph on at least six vertices?*

Solving Open Problem 1 is highly non-trivial. It is known that 4-Colouring is NP-complete for (C_3, P_{22})-free graphs [30]. Hence, the polynomial-time results in Theorem 1 cannot be extended to all linear forests. The first open case to consider would be $H = P_6$, for which only partial results are known. Indeed, Colouring is polynomial-time solvable for (C_3, P_6)-free graphs [9], but its complexity is unknown for (C_3, P_7)-free graphs (on a side note, a recent result for the latter graph class is that 3-Colouring is polynomial-time solvable [3]).

[1] Given four graphs H_1, H_2, H_3, H_4, the class of (H_1, H_2)-free graphs and the class of (H_3, H_4)-free graphs are *equivalent* if the unordered pair H_3, H_4 can be obtained from the unordered pair H_1, H_2 by some combination of the operations (i) complementing both graphs in the pair and (ii) if one of the graphs in the pair is K_3, replacing it with $\overline{P_1 + P_3}$ or vice versa. If two classes are equivalent, then one of them has bounded clique-width if and only if the other one does (see [22]).

We observe that boundedness of the clique-width of (diamond, $P_1 + 2P_2$)-free graphs implies boundedness of the clique-width of $(2P_1 + P_2, \overline{P_1 + 2P_2})$-free graphs (recall that the diamond is the complement of the graph $2P_1 + P_2$). Hence our results imply that Colouring can also be solved in polynomial time for graphs in this class. After incorporating the consequences of our new results and this additional observation, there are 13 classes of (H_1, H_2)-free graphs for which Colouring could potentially still be solved in polynomial time by showing that their clique-width is bounded (see also [25]):

Open Problem 2. *Is* Colouring *polynomial-time solvable for* (H_1, H_2)*-free graphs when:*

1. $\overline{H_1} \in \{3P_1, P_1 + P_3\}$ *and* $H_2 \in \{P_1 + S_{1,1,3}, S_{1,2,3}\}$*;*
2. $H_1 = 2P_1 + P_2$ *and* $\overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + P_5\}$*;*
3. $H_1 =$ *diamond and* $H_2 \in \{P_1 + P_2 + P_3, P_1 + P_5\}$*;*
4. $H_1 = P_1 + P_4$ *and* $\overline{H_2} \in \{P_1 + 2P_2, P_2 + P_3\}$*;*
5. $\overline{H_1} = P_1 + P_4$ *and* $H_2 \in \{P_1 + 2P_2, P_2 + P_3\}$*;*
6. $H_1 = \overline{H_2} = 2P_1 + P_3$.

As mentioned in Section 1.3, after updating the list of remaining open cases for clique-width from [22], we find that eight non-equivalent open cases remain for clique-width. These are the following cases.

Open Problem 3. *Does the class of* (H_1, H_2)*-free graphs have bounded or unbounded clique-width when:*

1. $H_1 = 3P_1$ *and* $\overline{H_2} \in \{P_1 + S_{1,1,3}, P_2 + P_4, S_{1,2,3}\}$*;*
2. $H_1 = 2P_1 + P_2$ *and* $\overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + P_5\}$*;*
3. $H_1 = P_1 + P_4$ *and* $\overline{H_2} \in \{P_1 + 2P_2, P_2 + P_3\}$ *or*
4. $H_1 = \overline{H_2} = 2P_1 + P_3$.

Bonomo, Grippo, Milanič and Safe [4] determined all pairs of connected graphs H_1, H_2 for which the class of (H_1, H_2)-free graphs has power-bounded clique-width. In order to compare their result with our results for clique-width, we would only need to solve the single open case $(H_1, H_2) = (K_3, S_{1,2,3})$, which is equivalent to the (open) case $(H_1, H_2) = (3P_1, \overline{S_{1,2,3}})$ mentioned in Open Problem 3. This follows because our new result for the case $(H_1, H_2) = (K_3, S_{1,2,2})$ has reduced the number of open cases (H_1, H_2) with H_1, H_2 both connected from two to one.

2 Preliminaries

Throughout our paper we only consider finite, undirected graphs without multiple edges or self-loops. Below we define further graph terminology.

The *disjoint union* $(V(G) \cup V(H), E(G) \cup E(H))$ of two vertex-disjoint graphs G and H is denoted by $G + H$ and the disjoint union of r copies of a graph G is denoted by rG. The *complement* of a graph G, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and an edge between two distinct vertices if and only if these vertices are not adjacent in G. For a subset $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of G *induced* by S, which has vertex set S and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. If $S = \{s_1, \ldots, s_r\}$ then, to simplify notation, we may also write $G[s_1, \ldots, s_r]$ instead of $G[\{s_1, \ldots, s_r\}]$. We use $G \setminus S$ to denote the graph obtained from G by deleting every vertex in S, i.e. $G \setminus S = G[V(G) \setminus S]$. We write $H \subseteq_i G$ to indicate that H is an induced subgraph of G.

The graphs $C_r, K_r, K_{1,r-1}$ and P_r denote the cycle, complete graph, star and path on r vertices, respectively. The graph $K_{1,3}$ is also called the *claw*. The graph $S_{h,i,j}$, for $1 \leq h \leq i \leq j$, denotes the *subdivided claw*, that is, the tree that has only one vertex x of degree 3 and exactly three leaves, which are of distance h, i and j from x, respectively. Observe that $S_{1,1,1} = K_{1,3}$. The graph $S_{1,2,2}$ is also known as the E, since it can be drawn like a capital letter E (see Fig. 1). Recall that the graph $\overline{2P_1 + P_2}$ is known as the *diamond*. The graphs K_3 and $\overline{P_1 + 2P_2}$ are also known as the *triangle* and the 5-vertex *wheel*, respectively. For a set of graphs $\{H_1, \ldots, H_p\}$, a graph G is $(H_1, \ldots, H_p)$*-free* if it has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$; if $p = 1$, we may write H_1-free instead of (H_1)-free.

Let X be a set of vertices in a graph $G = (V, E)$. A vertex $y \in V \setminus X$ is *complete* to X if it is adjacent to every vertex of X and *anti-complete* to X if it is non-adjacent to every vertex of X. Similarly, a set of vertices $Y \subseteq V \setminus X$ is *complete* (*anti-complete*) to X if every vertex in Y is complete (anti-complete) to X. A vertex y or a set Y is *trivial* to X if it is either complete or anti-complete to X. Note that if Y contains both vertices complete to X and vertices not complete to X, we may have a situation in which every vertex in Y is trivial to X, but Y itself is not trivial to X.

For a graph $G = (V, E)$, the set $N(u) = \{v \in V \mid uv \in E\}$ denotes the neighbourhood of $u \in V$. Let X and Y be disjoint sets of vertices in a graph $G = (V, E)$. If every vertex of X has at most one neighbour in Y and vice versa then we say that the edges between X and Y form a *matching*. If every vertex of X has exactly one neighbour in Y and vice versa then we say that the edges between X and Y form a *perfect matching*.

A graph is k*-partite* if its vertex set can be partitioned into k independent sets (some of which may be empty). A graph is *bipartite* if it is 2-partite. A graph is *complete bipartite* if its vertex set can be partitioned into two independent sets that are complete to each other. For integers $r, s \geq 0$, the *biclique* $K_{r,s}$ is the complete bipartite graph with sets in the partition of size r and s respectively. The *bipartite complement* of a bipartite graph G with bipartition (X, Y) is the graph obtained from G by replacing every edge from a vertex in X to a vertex in Y by a non-edge and vice versa.

Clique-Width. The *clique-width* of a graph G, denoted $\mathrm{cw}(G)$, is the minimum number of labels needed to construct G by using the following four operations:

1. creating a new graph consisting of a single vertex v with label i;
2. taking the disjoint union of two labelled graphs G_1 and G_2;
3. joining each vertex with label i to each vertex with label j $(i \neq j)$;
4. renaming label i to j.

An algebraic term that represents such a construction of G and uses at most k labels is said to be a k*-expression* of G (i.e. the clique-width of G is the minimum k for which G has a k-expression). Recall that a class of graphs $\mathcal{G}$ has bounded clique-width if there is a constant c such that the clique-width of every graph in $\mathcal{G}$ is at most c; otherwise the clique-width of $\mathcal{G}$ is *unbounded*.

Let G be a graph. We define the following operations. For an induced subgraph $G' \subseteq_i G$, the *subgraph complementation* operation (acting on G with respect to G') replaces every edge present in G' by a non-edge, and vice versa. Similarly, for two disjoint vertex subsets S and T in G, the *bipartite complementation* operation with respect to S and T acts on G by replacing every edge with one end-vertex in S and the other one in T by a non-edge and vice versa.

We now state some useful facts about how the above operations (and some other ones) influence the clique-width of a graph. We will use these facts throughout the paper. Let $k \geq 0$ be a constant and let γ be some graph operation. We say that a graph class $\mathcal{G}'$ is (k, γ)*-obtained* from a graph class $\mathcal{G}$ if the following two conditions hold:

(i) every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing γ at most k times, and
(ii) for every $G \in \mathcal{G}$ there exists at least one graph in $\mathcal{G}'$ obtained from G by performing γ at most k times.

We say that γ *preserves* boundedness of clique-width if for any finite constant k and any graph class $\mathcal{G}$, any graph class $\mathcal{G}'$ that is (k, γ)-obtained from $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width.

Fact 1. Vertex deletion preserves boundedness of clique-width [38].

Fact 2. Subgraph complementation preserves boundedness of clique-width [31].

Fact 3. Bipartite complementation preserves boundedness of clique-width [31].

The following lemma is easy to show.

Lemma 1. *The clique-width of a graph of maximum degree at most* 2 *is at most* 4.

Two vertices are *false twins* if they have the same neighbourhood (note that such vertices must be non-adjacent). The following lemma follows immediately from the definition of clique-width.

Lemma 2. *If a vertex x in a graph G has a false twin then* $\mathrm{cw}(G) = \mathrm{cw}(G \setminus \{x\})$.

We will also make use of the following two results.

Lemma 3 ([19]). *The class of* $(\textit{diamond}, P_2 + P_3)$*-free graphs has bounded clique-width.*

Lemma 4 ([21]). *Let H be a graph. The class of H-free bipartite graphs has bounded clique-width if and only if*

- $H = sP_1$ *for some* $s \geq 1$*;*
- $H \subseteq_i K_{1,3} + 3P_1$*;*
- $H \subseteq_i K_{1,3} + P_2$*;*
- $H \subseteq_i P_1 + S_{1,1,3}$ *or*
- $H \subseteq_i S_{1,2,3}$.

In some of our proofs we will use the fact that $S_{1,2,3}$-free bipartite graphs have bounded clique-width, which follows from Lemma 4. Alternatively we could have used the result of Lozin [36], who showed that $S_{1,2,3}$-free bipartite graphs have clique-width at most 5.

3 Totally k-Decomposable Graphs

In this section we describe our key technique, which is based on a decomposition of bipartite graphs introduced by Fouquet, Giakoumakis and Vanherpe [24], which is defined as follows.

Let G be a bipartite graph with a vertex bipartition (V_1, V_2). A 2-*decomposition of* G with respect to (V_1, V_2) consists of two *non-empty* graphs $G[V_1' \cup V_2']$ and $G[V_1'' \cup V_2'']$ such that:

(i) for $i \in \{1, 2\}$, $V_i' \cup V_i'' = V_i$ and $V_i' \cap V_i'' = \emptyset$;
(ii) V_1' is either complete or anti-complete to V_2'' in G;
(iii) V_2' is either complete or anti-complete to V_1'' in G.

Note that $V_1' \cup V_1''$ and $V_2' \cup V_2''$ are independent sets in G and that the last two conditions imply that each of $G[V_1' \cup V_2'']$ and $G[V_1'' \cup V_2']$ is either an independent set or a biclique. Observe that we do not impose restrictions on the bipartite graphs $G' = G[V_1' \cup V_2']$ and $G'' = G[V_1'' \cup V_2'']$. If G has a 2-decomposition G', G'' with respect to some bipartition, we say that G can be 2-*decomposed* into G' and G''. A graph G is *totally decomposable by canonical decomposition* if it can be recursively 2-decomposed into graphs isomorphic to K_1. Note that if G has a 2-decomposition G', G'' with respect to some bipartition (V_1, V_2), this does not force us to decompose G' and G'' with respect to a sub-partition of (V_1, V_2). As we will see, this distinction does not make a difference for bipartite graphs, but it will become an issue when we extend the notion to k-partite graphs when $k \geq 3$.

Fouquet, Giakoumakis and Vanherpe proved the following characterization, which we will need for our proofs (see Fig. 2 for pictures of P_7 and $S_{1,2,3}$).

Lemma 5 ([24]). *A bipartite graph is totally decomposable by canonical decomposition if and only if it is $(P_7, S_{1,2,3})$-free.*

Fig. 2. The forbidden graphs from Lemma 5.

For our purposes we need to generalize the notion of totally decomposable bipartite graphs to k-partite graphs for $k \geq 2$, and we will also need to partially classify graphs with this modified notion, in effect generalizing Lemma 5.

Let G be a k-partite graph with a *fixed* vertex k-partition $(V_1, \ldots, V_k)$. A *k-decomposition of* G with respect to the partition $(V_1, \ldots, V_k)$ consists of two non-empty graphs, each with their own partition: $G' = G[V_1' \cup \cdots \cup V_k']$ with partition $(V_1', V_2', \ldots, V_k')$ and $G'' = G[V_1'' \cup \cdots \cup V_k'']$ with partition $(V_1'', V_2'', \ldots, V_k'')$, such that:

(i) for $i \in \{1, \ldots, k\}$, $V_i' \cup V_i'' = V_i$ and $V_i' \cap V_i'' = \emptyset$;
(ii) for all $i, j \in \{1, \ldots, k\}$, V_i' is either complete or anti-complete to V_j'' in G.

Note that the last condition holds for $i = j$ by definition, since $V_i = V_i' \cup V_i''$ is an independent set in G. Also note that in the above definition, $(V_1', V_2', \ldots, V_k')$ and $(V_1'', V_2'', \ldots, V_k'')$ are sub-partitions of $(V_1, V_2, \ldots, V_k)$, in the sense that $V_i' = V_i \cap V(G')$ and $V_i'' = V_i \cap V(G)$ for $i \in \{1, \ldots, k\}$, so the original partition on G uniquely specifies the partitions on G' and G''.

If a graph G with a fixed k-partition has a k-decomposition with respect to this partition into two graphs G' and G'' (with their associated sub-partitions), we say that G can be *k-decomposed* into G' and G'' (with each of these subgraphs getting the appropriate sub-partition). We say that G is *totally k-decomposable* with respect to some fixed partition if G can be recursively k-decomposed *with respect to this fixed partition* into graphs isomorphic to K_1. Note that by definition, if a graph H appears in a total k-decomposition of G with respect to some fixed partition $(V_1, \ldots, V_k)$, then the k-partition $(V_1^H, V_2^H, \ldots, V_k^H)$ of H used to partition H satisfies $V_i^H = V_i \cap V(H)$ for $i = 1, \ldots, k$. This property will be necessary for us to be able to use inductive arguments "safely."

To compare graphs that are totally decomposable by canonical decomposition and graphs that are totally 2-decomposable, we observe that every connected bipartite graph G has a unique bipartition (up to isomorphism and swapping the two independent sets in the bipartition). Also, if G is totally decomposable by canonical decomposition, then this decomposition can recursively be done component-wise. Hence, in each step of the recursion, we may decompose with respect to an arbitrary bipartition of the graph under consideration. This means that the definitions of total canonical decomposability and total 2-decomposability are equivalent. However, for $k > 2$, a connected graph can have multiple k-partitions, even up to isomorphism and permuting the independent sets of the partition. Therefore, unlike for $k = 2$, we need to fix the partition of the subgraphs G' and G'' in the definition of total k-decomposability.

As mentioned, for our proofs we need to generalize Lemma 5. It seems difficult to give a full characterization of totally k-decomposable graphs for $k \geq 3$. However, the following lemma is sufficient for our purposes.

Lemma 6. *A 3-partite graph G is totally 3-decomposable with respect to a 3-partition (V_1, V_2, V_3) if the following two conditions are both satisfied:*

- *$G[V_1 \cup V_2], G[V_1 \cup V_3]$ and $G[V_2 \cup V_3]$ are all $(P_7, S_{1,2,3})$-free, and*
- *for every $v_1 \in V_1$, every $v_2 \in V_2$ and every $v_3 \in V_3$, the graph $G[v_1, v_2, v_3]$ is isomorphic neither to K_3 nor to $3P_1$.*

Proof. Let G be a 3-partite graph with a 3-partition (V_1, V_2, V_3) such that both conditions are satisfied. Note that any induced subgraph H of G (with partition $(V(H) \cap V_1, V(H) \cap V_2, V(H) \cap V_3)$) also satisfies the hypotheses of the lemma. This enables us to apply induction. It is therefore sufficient to show that G has a 3-decomposition with respect to the given 3-partition.

If V_1 is empty then G is a $(P_7, S_{1,2,3})$-free bipartite graph and is therefore totally 2-decomposable with respect to the partition (V_2, V_3) by Lemma 5 (and is thus totally 3-decomposable with respect to the partition (V_1, V_2, V_3)). By symmetry, we may therefore assume that every set V_i is non-empty.

Now $G[V_1, V_2]$ is a bipartite $(P_7, S_{1,2,3})$-free graph, so by Lemma 5, $G[V_1 \cup V_2]$ is totally 2-decomposable. Since V_1 and V_2 are both non-empty, it follows that V_1 can be partitioned into two sets V_1' and V_1'' and V_2 can be partitioned into two sets V_2' and V_2'', such that V_1' is either complete or anti-complete to V_2'', and V_2' is either complete or anti-complete to V_1''. Since the graphs $G[V_1' \cup V_2']$ and $G[V_1'' \cup V_2'']$ in this decomposition must be non-empty, it follows that $V_1' \cup V_2'$ and $V_1'' \cup V_2''$ must be non-empty. Since for $i \in \{1, 2\}$ we know that $V_i = V_i' \cup V_i''$ is non-empty, at least one of V_i' and V_i'' is non-empty. Hence, combining these two observations, we may assume without loss of generality that V_1' and V_2'' are non-empty. Assume that these sets are maximal, that is, no vertex of V_1'' (respectively V_2') can be moved to V_1' (respectively V_2''). Note that V_1'' or V_2' may be empty.

We will prove that we can partition V_3 into sets V_3' and V_3'', such that for all $i, j \in \{1, 2, 3\}$, V_i' is complete or anti-complete to V_j''. Note that we already know that V_1' (respectively V_2') is complete or anti-complete to V_2'' (respectively V_1''). Also note that for $i \in \{1, 2, 3\}$, V_i' is automatically anti-complete to V_i'', since V_i is an independent set.

First suppose that V_1' is complete to V_2''. If a vertex of V_3 has a neighbour in both V_1' and V_2'' then these three vertices would form a forbidden K_3, so every vertex in V_3 is anti-complete to V_1' or V_2''. Let V_3' be the set of vertices in V_3 that are anti-complete to V_2'' and let $V_3'' = V_3 \setminus V_3'$. Note that V_3'' must be anti-complete to V_1'. Suppose, for contradiction, that $z \in V_3'$ has a non-neighbour $v \in V_1''$. Since V_1' is maximal, v must have a non-neighbour $w \in V_2''$. This means that $G[v, w, z]$ is a $3P_1$.

This contradiction means that V_1'' is complete to V_3'. Similarly, V_2' is complete to V_3''. Therefore $G[V_1' \cup V_2' \cup V_3']$ and $G[V_1'' \cup V_2'' \cup V_3'']$ form the required 3-decomposition of G.

Now suppose that V_1' is anti-complete to V_2''. If a vertex of V_3 has a non-neighbour in both V_1' and V_2'' then these three vertices would induce a forbidden $3P_1$, so every vertex in V_3 is complete to V_1' or V_2''. Let V_3' be the set of vertices in V_3 that are complete to V_2'' and let $V_3'' = V_3 \setminus V_3'$. Note that V_3'' must be complete to V_1'. By using similar arguments to those in the previous case, we find that V_1'' is anti-complete to V_3' and V_2' is anti-complete to V_3''. Hence, $G[V_1' \cup V_2' \cup V_3']$ and $G[V_1'' \cup V_2'' \cup V_3'']$ form the required 3-decomposition of G. This completes the proof. □

We also need the following lemma.

Lemma 7. *Let G be a k-partite graph with vertex partition $(V_1, \ldots, V_k)$. If G is totally k-decomposable with respect to this partition, then the clique-width of G is at most $2k$. Moreover, there is a $2k$-expression for G that assigns, for $i \in \{1, \ldots, k\}$, label i to every vertex of V_i.*

Proof. We prove the lemma by induction on the number of vertices. If G contains only one vertex then the lemma holds trivially. Suppose that the lemma is true for all k-partite graphs H on at most n vertices and for all k-partitions $(V_1^H, \ldots, V_2^H)$ with respect to which H is totally k-decomposable. Let G be a graph on $n+1$ vertices that is totally k-decomposable with respect to a vertex partition $(V_1, \ldots, V_k)$. Then, we can partition every set V_i into two sets V_i' and V_i'' in such a way that each set V_i' is either complete or anti-complete to each set V_j'' for all $i, j \in \{1, \ldots, k\}$ and $G' = G[V_1' \cup \ldots \cup V_k']$ and $G'' = G[V_1'' \cup \ldots \cup V_k'']$ are totally k-decomposable with respect to the partitions $(V_1', \ldots, V_k')$ and $(V_1'', \ldots, V_k'')$, respectively.

As both G' and G'' are smaller graphs that G, we can apply the induction hypothesis. Hence, we can find a $2k$-expression that constructs G' such that the vertices in each set V_i' have label i for $i \in \{1, \ldots, k\}$. Similarly, we can find a $2k$-expression that constructs G'' such that the vertices in each set V_j'' have label $k+j$ for $j \in \{1, \ldots, k\}$. We take the disjoint union of these two constructions. Next, for $i, j \in \{1, \ldots, k\}$, we join the vertices with label i to the vertices with label $k+j$ if and only if V_i' is complete to V_j'' in G. Finally, for $i \in \{1, \ldots, k\}$, we relabel the vertices with label $k+i$ to have label i. This completes the proof of the lemma. □

4 Sufficient Conditions for $(K_3, S_{1,2,3})$-free Graphs

We observe that the classes of (K_3, P_1+2P_2)-free, $(K_3, P_1+P_2+P_3)$-free, (K_3, P_1+P_5)-free and $(K_3, S_{1,2,2})$-free graphs are all subclasses of the class of $(K_3, S_{1,2,3})$-free graphs. In order to prove that each of the four subclasses has bounded clique-width, we investigate, in this section, sufficient conditions for a subclass of $(K_3, S_{1,2,3})$-free graphs to be of bounded clique-width. We present these conditions in Corollary 1 and Lemma 9. Corollary 1 follows from a structural result (Lemma 8), which we prove first. The proof of Lemma 9 uses the results from the previous section. We will not use Corollary 1 and Lemma 9 directly when proving that the class of (diamond, $P_1 + 2P_2$)-free graphs has bounded clique-width. However, our proof of that result does rely on these two results indirectly, as it depends on the $(K_3, P_1 + 2P_2)$-free case.

Lemma 8. *Let G be a connected $(K_3, C_5, S_{1,2,3})$-free graph that does not contain a pair of false twins. Then G is either bipartite or an induced cycle.*

Proof. Let G be a connected $(K_3, C_5, S_{1,2,3})$-free graph that does not contain a pair of false twins. We may assume that G is not bipartite, otherwise we are done. We know that G is (C_3, C_5)-free (since $C_3 = K_3$). We may therefore assume that G contains an induced odd cycle C on k vertices, say $v_1 - v_2 - \cdots - v_k - v_1$, where $k \geq 7$. Assume that C is an odd cycle of minimum length in G.

Suppose that not every vertex of G is in C. Since G is connected, we may assume that there is a vertex v not in C that has a neighbour in C. Suppose v is adjacent to precisely one vertex of C. If v is adjacent to v_3, but has no other neighbours on C then $G[v_3, v, v_2, v_1, v_4, v_5, v_6]$ is an $S_{1,2,3}$, a contradiction. By symmetry, it follows that v must be adjacent to at least two vertices of C. Note that since G is K_3-free, no vertex outside of C can be adjacent to two consecutive vertices of C.

Suppose that v is adjacent to v_1 and v_i and non-adjacent to $v_2, \ldots, v_{i-1}$ for some even i with $i \leq k-2$. Then $G[v, v_1, v_2, \ldots, v_i]$ would be an odd cycle on less than k vertices, contradicting the minimality of k. By a parity argument, since C is an odd cycle, it follows that v must be adjacent to precisely two vertices of C, which must be at distance 2 away from each other on the cycle.

Let V_i be the set of vertices outside of C that are adjacent to v_{i-1} and v_{i+1} (subscripts interpreted modulo k) and let U be the set of vertices that have no neighbour in C. Suppose, for contradiction, that U is non-empty. Since G is connected, without loss of generality there is a vertex $u \in U$ that has a neighbour $v \in V_1$. Then $G[v_2, v_1, v, u, v_3, v_4, v_5]$ is an $S_{1,2,3}$, a contradiction. We conclude that U must be empty.

Now since G is K_3-free, for every i the set V_i is anti-complete to the set V_{i+2}. Moreover, if i and j are such that the vertices v_i and v_j are at distance more than 2 on the cycle, then V_i and V_j must be anti-complete, as otherwise there would be a smaller odd cycle than C in G, which would contradict the minimality of k.

Note that every set V_i is independent in G, since G is K_3-free. If a vertex $x_1 \in V_1$ is non-adjacent to a vertex $x_2 \in V_2$ then $G[v_3, x_2, v_2, x_1, v_4, v_5, v_6]$ is an $S_{1,2,3}$, a contradiction. Therefore a vertex $x_i \in V_i$ is adjacent to a vertex $x_j \in V_j$ if and only if v_i and v_j are consecutive vertices of C. In other words, for every i, every vertex in V_i is a false twin of v_i. Therefore every set V_i must be empty, so G is an induced odd cycle. This completes the proof. □

We immediately get the following corollary, which implies that the four triangle-free cases in our new results hold when the graph class under consideration is in addition C_5-free.

Corollary 1. *The class of $(K_3, C_5, S_{1,2,3})$-free graphs has bounded clique-width.*

Proof. Let G be a $(K_3, C_5, S_{1,2,3})$-free graph. If G contains a pair of false twins then by Lemma 2 we may delete one of them. By Lemma 8, every component of the resulting graph is either a bipartite graph or an induced cycle. In the first case, such a component is an $S_{1,2,3}$-free bipartite graph, so it has bounded clique-width by Lemma 4. In the second case, the component has clique-width at most 4 by Lemma 1. The corollary follows. □

In our second lemma we state a number of sufficient conditions for a subclass of $(K_3, S_{1,2,3})$-free graphs to be of bounded clique-width when C_5 is no longer a forbidden induced subgraph. To prove it we will need Lemmas 6 and 7.

Lemma 9. *Let $\mathcal{G}$ be the subclass of $(K_3, S_{1,2,3})$-free graphs for which the vertices in each graph $G \in \mathcal{G}$ can be partitioned into ten independent sets $V_1, \ldots, V_5, W_1, \ldots, W_5$, such that the following seven conditions hold (we interpret subscripts modulo 5):*

(i) for all i, V_i is anti-complete to $V_{i-2} \cup V_{i+2} \cup W_{i-1} \cup W_{i+1}$;

(ii) for all i, W_i is complete to $W_{i-1} \cup W_{i+1}$;
(iii) for all i, every vertex of V_i is trivial to at least one of the sets V_{i+1} and V_{i-1};
(iv) for all i, every vertex in V_i is trivial to W_i;
(v) for all i, W_i is trivial to W_{i-2} and to W_{i+2};
(vi) for all i, j, the graphs induced by $V_i \cup V_j$ and $V_i \cup W_j$ are P_7-free;
(vii) for all i, there are no three vertices $v \in V_i$, $w \in V_{i+1}$ and $x \in W_{i+3}$ such that v, w and x are pairwise non-adjacent.

Then $\mathcal{G}$ has bounded clique-width.

Proof. Let G be a $(K_3, S_{1,2,3})$-free graph with such a partition that satisfies Conditions (i)–(vii) of the lemma. Note that for all i, every vertex $v \in V_i$ is trivial to $V_{i-2}, V_{i+2}, W_{i-1}, W_{i+1}, W_i$ and either trivial to V_{i-1} or trivial to V_{i+1}. Therefore a vertex $v \in V_i$ can only be non-trivial to W_{i-2}, W_{i+2} and at most one of V_{i-1} and V_{i+1}. Likewise, every vertex $w \in W_i$ is trivial to $W_{i-1}, W_{i+1}, W_{i-2}, W_{i+2}, V_{i-1}$ and V_{i+1}. Therefore, a vertex $w \in W_i$ can only be non-trivial to V_i, V_{i-2} and V_{i+2} (and every vertex in V_i is trivial to W_i).

For $i \in \{1, \ldots, 5\}$, let W_i' be the set of vertices in W_i that are non-trivial to both V_{i-2} and V_{i+2}, let V_i' be the set of vertices in V_i that are non-trivial to both V_{i+1} and W_{i-2} and let V_i'' be the set of vertices in V_i that are non-trivial to both V_{i-1} and W_{i+2}. Note that $V_i' \cap V_i'' = \emptyset$ by Condition (iii).

We say that an edge is *irrelevant* if one of its end-vertices is in a set V_i, V_i', V_i'', W_i or W_i', and its other end-vertex is complete to this set, otherwise we say that the edge is *relevant*. We will now show that for $i \in \{1, \ldots, 5\}$, the graph $G[V_i' \cup V_{i+1}'' \cup W_{i-2}']$ can be separated from the rest of G by using a bounded number of bipartite complementations. To do this, we first prove the following claim.

Claim 1. *If $u \in V_i' \cup V_{i+1}'' \cup W_{i-2}'$ and $v \notin V_i' \cup V_{i+1}'' \cup W_{i-2}'$ are adjacent then uv is an irrelevant edge.*
We split the proof of Claim 1 into the following cases.

Case 1: $u \in V_i'$.
Since u is in V_i, v must be in $V_{i-1} \cup V_{i+1} \cup W_{i-2} \cup W_{i+2}$, otherwise uv would be irrelevant by Condition (i) or (iv). We consider the possible cases for v.

Case 1a: $v \in V_{i-1}$.
Since u is in V_i', it is non-trivial to V_{i+1}, so by Condition (iii), u is trivial to V_{i-1}. Therefore uv is irrelevant.

Case 1b: $v \in V_{i+1}$.
Suppose, for contradiction, that v is complete to W_{i-2}. Let $w \in W_{i-2}$ be a neighbour of u (such a vertex w exists, since u is non-trivial to W_{i-2}). Then $G[u, v, w]$ is a K_3, a contradiction, so v cannot be complete to W_{i-2}. Now suppose, for contradiction that v is anti-complete to W_{i-2}. We may assume that v has a non-neighbour $u' \in V_i'$, otherwise v would be trivial to V_i', in which case uv would be irrelevant. Since $u' \in V_i'$, u' is non-trivial to W_{i-2}, so it must have a non-neighbour $w \in W_{i-2}$. Then, since v is anti-complete to W_{i-2}, it follows that $G[u', v, w]$ is a $3P_1$, contradicting Condition (vii). We may therefore assume that v is non-trivial to W_{i-2}. We know that $v \notin V_{i+1}''$. Therefore v must be trivial to V_i, so uv is irrelevant.

Case 1c: $v \in W_{i-2}$.
Reasoning as in the previous case, we find that v cannot be complete or anti-complete to V_{i+1}. Hence, as $v \notin W_{i-2}'$, v must be trivial to V_i, so uv is irrelevant.

Case 1d: $v \in W_{i+2}$.
Since u is non-trivial to W_{i-2} (by definition of V_i'), there is a vertex $w \in W_{i-2}$ that is

adjacent to u. By Condition (ii), w is adjacent to v. Therefore $G[u,v,w]$ is a K_3. This contradiction implies that $v \notin W_{i+2}$. This completes Case 1.

Now assume that $u \notin V_i'$. Then, by symmetry, $u \notin V_{i+1}''$. This means that the following case holds.

Case 2: $u \in W_{i-2}'$.
We argue similarly to Case 1b. We may assume that v is non-trivial to W_{i-2}', otherwise uv would be irrelevant. By Conditions (i), (ii), (iv) and (v), it follows that $v \in V_i \cup V_{i+1}$. Without loss of generality assume that $v \in V_i$. Since $v \notin V_i'$ and v is non-trivial to W_{i-2}, it follows that v is trivial to V_{i+1}. If v is complete to V_{i+1} then since u is non-trivial to V_{i+1}, there must be a vertex $w \in V_{i+1}$ adjacent to u, in which case $G[u,v,w]$ is a K_3, a contradiction. Therefore v must be anti-complete to V_{i+1}. Since v is non-trivial to W_{i-2}', there must be a vertex $u' \in W_{i-2}'$ that is non-adjacent to v. Since $u' \in W_{i-2}'$, u' must have a non-neighbour $w \in V_{i+1}$. Then $G[u',v,w]$ is a $3P_1$, contradicting Condition (vii). This completes Case 2.

We conclude that, if $u \in V_i' \cup V_{i+1}'' \cup W_{i-2}'$ and $v \notin V_i' \cup V_{i+1}'' \cup W_{i-2}'$ are adjacent, then uv is an irrelevant edge. Hence we have proven Claim 1.

By Claim 1 we find that if $u \in V_i' \cup V_{i+1}'' \cup W_{i-2}'$ and $v \notin V_i' \cup V_{i+1}'' \cup W_{i-2}'$ are adjacent then u or v is complete to some set V_j, V_j', V_j'', W_j or W_j' that contains v or u, respectively. By applying a bounded number of bipartite complements (which we may do by Fact 3), we can separate $G[V_i' \cup V_{i+1}'' \cup W_{i-2}']$ from the rest of G. By Conditions (vi) and (vii) and the fact that G is $(K_3, S_{1,2,3})$-free, Lemmas 6 and 7 imply that $G[V_i' \cup V_{i+1}'' \cup W_{i-2}']$ has clique-width at most 6. Repeating this argument for each i, we may assume that $V_i' \cup V_{i+1}'' \cup W_{i-2}' = \emptyset$ for every i.

For $i \in \{1,\ldots,5\}$ let V_i^* be the set of vertices in V_i that are either non-trivial to V_{i+1} or non-trivial to W_{i+2} and let V_i^{**} be the set of the remaining vertices in V_i. For $i \in \{1,\ldots,5\}$, let W_i^* be the set of vertices that are non-trivial to V_{i+2} and let W_i^{**} be the set of the remaining vertices in W_i.

We claim that every vertex in V_i that is non-trivial to V_{i-1} or that is non-trivial to W_{i-2} is in V_i^{**}. Indeed, if $v \in V_i$ is non-trivial to V_{i-1} then by Condition (iii), v is trivial to V_{i+1} and since V_i'' is empty, v must be trivial to W_{i+2}. If $v \in V_i$ is non-trivial to W_{i-2} then v must be trivial to V_{i+1} since V_i' is empty. Moreover, in this case v must also be trivial to W_{i+2}, otherwise, by Condition (ii) the vertex v, together with a neighbour of v in each of W_{i+2} and W_{i-2}, would induce a K_3 in G. It follows that every vertex in V_i that is non-trivial to V_{i-1} or that is non-trivial to W_{i-2} is indeed in V_i^{**}. Similarly, for all i, since W_i' is empty, every vertex in W_i that is non-trivial to V_{i-2} is in W_i^{**}.

We say that an edge uv is *insignificant* if u or v is in some set V_i^*, V_i^{**}, W_i^* or W_i^{**} and the other vertex is trivial to this set; all other edges are said to be *significant*. We prove the following claim.

Claim 2. *If $u \in W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**}$ and $v \notin W_i^* \cup V_{i+2}^{**} \cup V_{i+1}^* \cup W_{i-2}^{**}$ are adjacent then the edge uv is insignificant.*
To prove this claim suppose, for contradiction, that uv is a significant edge. We split the proof into two cases.

Case 1: $u \in W_i$.
We will show that $v \in V_{i+2}^{**}$ or $v \in V_{i-2}^*$ if $u \in W_i^*$ or $u \in W_i^{**}$, respectively. By Conditions (i), (ii), (iv) and (v) we know that u is trivial to V_{i-1}, V_{i+1}, W_{i-1}, W_{i+1}, W_{i-2} and W_{i+2}, and that every vertex of V_i is trivial to W_i. Furthermore, u is trivial to $W_i^{**} \setminus \{u\}$ since W_i is independent. Therefore $v \in V_{i-2} \cup V_{i+2}$. Note that v is non-trivial to W_i (by choice of v). If $u \in W_i^*$ then u must be trivial to V_{i-2}, since W_i' is

empty. Therefore $v \in V_{i+2}$. Now if $v \in V^*_{i+2}$ then v is non-trivial to V_{i-2} or non-trivial to W_{i-1}. In the first case v is non-trivial to both V_{i-2} and W_i, contradicting the fact that V'_{i+2} is empty. In the second case v has a neighbour $w \in W_{i-1}$. By Condition (ii), w is adjacent to u, so $G[u, v, w]$ is a K_3. This contradiction implies that if $u \in W^*_i$ then $v \in V^{**}_{i+2}$, contradicting the choice of v. Now suppose $u \in W^{**}_i$. Then u is trivial to V_{i+2}, so $v \in V_{i-2}$. If $v \in V^{**}_{i-2}$ then v is trivial W_i (by definition of V^{**}_{i-2}). Therefore if $u \in W^{**}_i$ then $v \in V^*_{i-2}$, contradicting the choice of v.

We conclude that for every $i \in \{1, \ldots, 5\}$ the vertex u is not in W_i. Similarly, we may assume $v \notin W_i$. This means that the following case holds.

Case 2: *$u \in V_i$, $v \in V_j$ for some i, j.*
Then $i \neq j$, since V_i is an independent set. By Condition (i), $j \notin \{i-2, i+2\}$. Without loss of generality, we may therefore assume that $j = i+1$. If $u \in V^{**}_i$ then u is trivial to V_{i+1}, so we may assume that $u \in V^*_i$. If $v \in V^*_{i+1}$ then v is non-trivial to V_{i+2}, so by Condition (iii) v is trivial to V_i, contradicting the fact that uv is significant. Therefore $v \in V^{**}_{i+1}$, contradicting the choice of v.

We conclude that if for some i, $u \in W^*_i \cup V^{**}_{i+2} \cup V^*_{i+1} \cup W^{**}_{i-2}$ and $v \notin W^*_i \cup V^{**}_{i+2} \cup V^*_{i+1} \cup W^{**}_{i-2}$ are adjacent then the edge uv is insignificant. Hence we have proven Claim 2.

Note that $W^*_i, V^{**}_{i+2}, V^*_{i+1}$ and W^{**}_{i-2} are independent sets. By Condition (i), W^*_i is anti-complete to V^*_{i+1} and V^{**}_{i+2} is anti-complete to W^{**}_{i-2}. Therefore $W^*_i \cup V^*_{i+1}$ and $V^{**}_{i+2} \cup W^{**}_{i-2}$ are independent sets. Thus $G[W^*_i \cup V^{**}_{i+2} \cup V^*_{i+1} \cup W^{**}_{i-2}]$ is an $S_{1,2,3}$-free bipartite graph, which has bounded clique-width by Lemma 4. Applying a bounded number of bipartite complementations (which we may do by Fact 3), we can separate $G[W^*_i \cup V^{**}_{i+2} \cup V^*_{i+1} \cup W^{**}_{i-2}]$ from the rest of the graph. We may thus assume that $W^*_i \cup V^{**}_{i+2} \cup V^*_{i+1} \cup W^{**}_{i-2} = \emptyset$. Repeating this process for each i we obtain the empty graph. This completes the proof. □

5 The Four Triangle-free Cases

We can now give the following result, which also implies the $(K_3, P_1 + 2P_2)$-free case.

Theorem 3. *For $H \in \{P_1 + P_5, S_{1,2,2}, P_1 + P_2 + P_3\}$, the class of (K_3, H)-free graphs has bounded clique-width.*

The proofs for all three cases are broadly similar. We will prove the $H = P_1 + P_2 + P_3$ case separately, as it is a little more involved than the other two cases.

5.1 Proof of the $H = P_1 + P_5$ and $H = S_{1,2,2}$ Cases.

Proof. Let $H \in \{P_1 + P_5, S_{1,2,2}\}$ and consider a (K_3, H)-free graph G. We may assume that G is connected.

By Corollary 1, we may assume that G contains an induced cycle on five vertices, say $C = v_1 - v_2 - \cdots - v_5 - v_1$. Again, we will interpret subscripts on vertices and vertex sets modulo 5.

Since G is K_3-free, no vertex v is adjacent to two consecutive vertices of the cycle. Therefore every vertex of G has either zero, one or two neighbours on the cycle and if it has two neighbours then they must be non-consecutive vertices of the cycle.

We partition the vertices of G that are not on C as follows:

- U: the set of vertices adjacent to no vertices of C,
- W_i: the set of vertices whose unique neighbour in C is v_i and

– V_i: the set of vertices adjacent to v_{i-1} and v_{i+1}.

In the remainder of the proof we will show how to modify the graph using operations that preserve boundedness of clique-width, such that in the resulting graph the set U is empty and the partition $V_1, \ldots, V_5, W_1, \ldots, W_5$ satisfies Conditions (i)–(vii) of Lemma 9. In order to do this we prove a number of claims.

The first two claims follow immediately from the fact that G is K_3-free.

Claim 1. *For all i, V_i and W_i are independent sets.*
Claim 2. *For all i, V_i is anti-complete to $V_{i-2} \cup V_{i+2} \cup W_{i-1} \cup W_{i+1}$.*

Claim 3. *We may assume that U is empty.*
We prove Claim 3 as follows. First consider the case where $H = S_{1,2,2}$ and suppose, for contradiction, that U is not empty. Since G is connected there must be a vertex $u \in U$ that is adjacent to a vertex $v \notin U$ that has a neighbour on the cycle C. Without loss of generality, we may assume that $v \in V_1 \cup W_2$, in which case v is adjacent to v_2 and non-adjacent to v_1, v_3 and v_4. Now $G[v_2, v_1, v_3, v_4, v, u]$ is an $S_{1,2,2}$. This contradiction means that $U = \emptyset$ if $H = S_{1,2,2}$.

Now consider the case where $H = P_1 + P_5$ and suppose that U is non-empty. Suppose, for contradiction, that there are two vertices $u, u' \in U$ that do not have the same neighbourhood in some set V_i or W_i. Without loss of generality, assume $v \in V_1 \cup W_2$ is adjacent to u, but not u'. Note that v is adjacent to v_2, but non-adjacent to v_1, v_3 and v_4. Then $G[v_4, u', u, v, v_2, v_1]$ is a $P_1 + P_5$ if u and u' are adjacent and $G[u', u, v, v_2, v_3, v_4]$ is a $P_1 + P_5$ if they are not. This contradiction means that every vertex in U has the same neighbourhood in every set V_i and every set W_i. Since G is connected there must be a vertex v in some V_i or W_i that is adjacent to every vertex of U. Since G is K_3-free, U must therefore be an independent set. Applying a bipartite complementation (which we may do by Fact 3) between U and the vertices adjacent to the vertices of U disconnects U from the rest of the graph. Since $G[U]$ is independent, it has clique-width at most 1. We may therefore assume that U is empty.

Claim 4. *For all i, W_i is complete to $W_{i-1} \cup W_{i+1}$.*
Suppose, for contradiction, that $v \in W_1$ has a non-neighbour $w \in W_2$. Then $G[w, v, v_1, v_5, v_4, v_3]$ is a $P_1 + P_5$ and $G[v_1, v, v_2, w, v_5, v_4]$ is an $S_{1,2,2}$. This contradiction proves the claim.

See Fig. 3 for an illustration of the graph G.

Claim 5. *For all i, every vertex of V_i is trivial to at least one of the sets V_{i+1} and V_{i-1}.*
Suppose, for contradiction that the claim is false. Without loss of generality, there is a vertex $v \in V_2$ with non-neighbours $u \in V_1$ and $w \in V_3$. By Claim 2, u and w must be non-adjacent. Then $G[v_5, u, v_1, v, v_4, w]$ is an $S_{1,2,2}$ and $G[u, v_1, v, v_3, v_4, w]$ is a $P_1 + P_5$. This contradiction completes the proof of the claim.

Claim 6. *For all i, every vertex in V_i is trivial to W_i.*
Suppose, for contradiction, that the claim is false. Without loss of generality, we may assume there are vertices $v \in V_1$ and $w, w' \in W_1$ such that v is adjacent to w, but not to w'. Then $G[v_2, v, v_1, w', v_3, v_4]$ is an $S_{1,2,2}$ and $G[w', w, v, v_2, v_3, v_4]$ is a $P_1 + P_5$. This contradiction completes the proof of the claim.

Claim 7. *For all i, W_i is trivial to W_{i-2} and to W_{i+2}.*
Suppose, for contradiction, that this does not hold. Without loss of generality, assume $v \in W_1$ is adjacent to $w \in W_3$ and non-adjacent to $w' \in W_3$. Then $G[v_1, v_2, v_5, v_4, v, w]$ is an $S_{1,2,2}$ and $G[w', w, v, v_1, v_5, v_4]$ is a $P_1 + P_5$. This contradiction proves the claim.

Claim 8. *For all i, j, the graphs induced by $V_i \cup V_j$ and $V_i \cup W_j$ are P_7-free.*
Note that $P_1 + P_5$ is an induced subgraph of P_7. Therefore if $H = P_1 + P_5$ then the

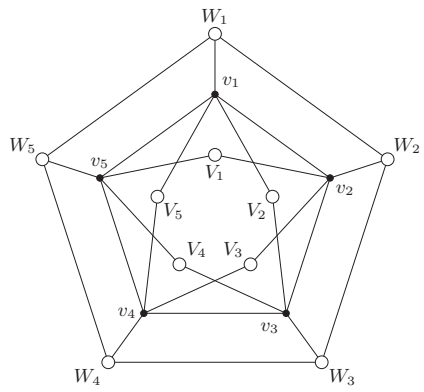

Fig. 3. The graph G. The black points are the vertices of the cycle C. The circles are (possibly empty) independent sets of vertices and the lines are complete bipartite graphs. Note that G may contain additional edges that are not represented in this figure.

claim follows immediately. Now suppose $H = S_{1,2,2}$. Without loss of generality, we may assume $i = 1$. Suppose that $G[V_1 \cup V_j]$ or $G[V_1 \cup W_j]$ contains an induced P_7, for some i, j. By Claims 1, 2 and 6 and symmetry, we may assume that $G[V_1 \cup V_2]$ or $G[V_1 \cup W_3]$ contains this P_7. This P_7 contains an induced subgraph isomorphic to $2P_2$, say on vertices v, v', w, w'. Then $G[v_5, v_4, v, v', w, w']$ is an $S_{1,2,2}$. This contradiction completes the proof of the claim.

Claim 9. *For all i, there are no three vertices $v \in V_i$, $w \in V_{i+1}$ and $x \in W_{i+3}$ such that v, w and x are pairwise non-adjacent.*
Suppose, for contradiction that such pairwise non-adjacent vertices exist, say with $v \in V_1, w \in V_2$ and $x \in W_4$. Then $G[v_4, x, v_3, w, v_5, v]$ is an $S_{1,2,2}$ and $G[x, v_3, w, v_1, v_5, v]$ is a $P_1 + P_5$. This contradiction completes the proof of the claim.

We now consider the graph obtained G' from G by removing the five vertices of C. Claims 1 and 3 show that we may assume $V_1, \ldots, V_5, W_1, \ldots, W_5$ are independent sets that form a partition of the vertex set of G'. Claims 2 and 4–9 correspond to the seven conditions of Lemma 9. Therefore G' has bounded clique-width. By Fact 1, G also has bounded clique-width. This completes the proof. □

5.2 Proof of the $H = P_1 + P_2 + P_3$ Case.

Proof. Consider a $(K_3, P_1+P_2+P_3)$-free graph G. We may assume that G is connected.

By Corollary 1, we may assume that G contains an induced cycle on five vertices, say $C = v_1 - v_2 - \cdots - v_5 - v_1$. Again, we will interpret subscripts on vertices and vertex sets modulo 5.

Since G is K_3-free, no vertex v is adjacent to two consecutive vertices of C. Therefore every vertex of G has either zero, one or two neighbours on C and if it has two neighbours then they must be non-consecutive vertices of C.

We partition the vertices of G that are not on C as follows:

- U: the set of vertices adjacent to no vertices of C,
- W_i: the set of vertices whose unique neighbour in C is v_i and

– V_i: the set of vertices adjacent to v_{i-1} and v_{i+1}.

In the remainder of the proof we will show how to modify the graph using operations that preserve boundedness of clique-width, such that in the resulting graph the set U is empty and the partition $V_1, \ldots, V_5, W_1, \ldots, W_5$ satisfies Conditions (i)–(vii) of Lemma 9.

The first two claims follow immediately from the fact that G is K_3-free.

Claim 1. *For all i, V_i and W_i are independent sets.*
Claim 2. *For all i, V_i is anti-complete to $V_{i-2} \cup V_{i+2} \cup W_{i-1} \cup W_{i+1}$.*

Claim 3. *We may assume that U is empty.*
In order to proof Claim 3, we first suppose that there are two adjacent vertices $u, u' \in U$. Since G is connected, we may assume without loss of generality that u is adjacent to some vertex $v \in V_1 \cup W_2$. Then u' must be non-adjacent to v, otherwise $G[u, u', v]$ would be a K_3. Note that v is adjacent to v_2, but not to v_1, v_3 or v_4. Now $G[v_1, v_3, v_4, u', u, v]$ is a $P_1 + P_2 + P_3$. This contradiction implies that U must be an independent set.

Now suppose, for contradiction, that a vertex $u \in U$ has two neighbours in some set $V_i \cup W_{i+1}$. Without loss of generality assume that u is adjacent to $v, v' \in V_1 \cup W_2$. Note that v and v' are adjacent to v_2, but not adjacent to v_1, v_3 and v_4. Now $G[v_1, v_3, v_4, v, u, v']$ is a $P_1 + P_2 + P_3$. This contradiction implies that every vertex of U has at most one neighbour in $V_i \cup W_{i+1}$ for each i. In particular, this means that every vertex of U has degree at most 5. Therefore, if $u \in U$ then we delete $\{u\} \cup N(u)$ (a set of at most 6 vertices). This gives us a $(K_3, P_2 + P_3)$-free graph, which has bounded clique-width by Lemma 3. By Fact 1, we may therefore assume that U is empty, that is, we have proven Claim 3.

We say that a set V_i or W_i is *large* if it contains at least two vertices and *small* if it contains exactly one vertex. If any set V_i is not large then by Fact 1 we may assume that it is empty. (Later in the proof, we may delete vertices from some sets V_i or W_i. In doing so, some sets that were previously large may become small. If this happens, we will simply repeat the argument. We will only do this a bounded number of times, so boundedness of clique-width will be preserved.)

Claim 4. *For all i, W_i is complete to $W_{i-1} \cup W_{i+1}$.*
Suppose, for contradiction, that $v \in W_1$ has a non-neighbour $w \in W_2$. Since W_2 is non-empty, it must be large, so it must contain a vertex w' distinct from w. Then $G[w, v_3, v_4, v_1, v, w']$ is a $P_1 + P_2 + P_3$ if v and w' are adjacent and $G[v, v_4, v_5, w, v_2, w']$ is a $P_1 + P_2 + P_3$ if they are not. This contradiction completes the proof of Claim 4.

Claim 5. *For all i, every vertex of V_i is trivial to at least one of the sets V_{i+1} and V_{i-1}.*
Suppose, for contradiction that the claim is false. Without loss of generality, there is a vertex $v \in V_2$ with non-neighbours $u \in V_1$ and $w \in V_3$ and neighbour $u' \in V_1$. By Claim 2, w and must be non-adjacent to both u and u'. Then $G[u, v_4, w, v_1, v, u']$ is a $P_1 + P_2 + P_3$. This contradiction completes the proof of Claim 5.

Claim 6. *For all i, every vertex in V_i is trivial to W_i.*
In fact we will prove a stronger statement, namely that for all i, V_i is trivial to W_i. Suppose, for contradiction, that this is not the case. Without loss of generality, assume that V_1 is not trivial to W_1. First suppose that there are vertices $w \in W_1$ and $v, v' \in V_1$ such that w is adjacent to v, but not to v'. Then $G[v', v_3, v_4, v_1, w, v]$ is a $P_1 + P_2 + P_3$. Therefore every vertex in W_1 must be trivial to V_1. Since we assumed that V_1 is not trivial to W_1, there must therefore be vertices $v \in V_1$ and $w, w' \in W_1$ such that v is adjacent to w, but not to w'. Since V_1 is non-empty, it must be large, so there must be another vertex $v' \in V_1$. Since every vertex of W_1 is trivial to V_1, v' must be

adjacent to w and non-adjacent to w'. Then $G[w', v_3, v_4, v, w, v']$ is a $P_1 + P_2 + P_3$. This contradiction completes the proof of Claim 6.

Claim 7. *We may assume that for all i, W_i is anti-complete to W_{i-2} and to W_{i+2}.*
We start by showing that the edges between W_i and W_{i+2} form a matching. Indeed, suppose for contradiction that there is a vertex $v \in W_1$ with two neighbours $w, w' \in W_3$. Then $G[v_2, v_4, v_5, w, v, w']$ is a $P_1 + P_2 + P_3$, a contradiction. By symmetry, no vertex of W_3 has two neighbours in W_1. We conclude that the edges between W_i and W_{i+2} form a matching.

Let W_1' be the set of vertices in W_1 that have a neighbour in W_3. Similarly, let W_3'' be the set of vertices in W_3 that have a neighbour in W_1. Note that $|W_1'| = |W_3''|$ since the edges between W_1' and W_3'' form a perfect matching. We will show that every vertex of $G \setminus (W_1' \cup W_3'')$ is trivial to W_1' and W_3''. This follows immediately if $|W_1'| = |W_3''| = 1$.

Assume $|W_1'| = |W_3''| \geq 2$. Suppose there is a vertex $w \in V(G) \setminus (W_1' \cup W_3'')$ that is non-trivial to W_1'. Then we may choose $u, u' \in W_1'$ and $v, v' \in W_3''$ such that u is adjacent to v and w, but non-adjacent to v' while u' is adjacent to v', but non-adjacent to v and w. Since w is non-trivial to W_1, it cannot be in W_1 (by Claim 1), $V_2 \cup V_5$ (by Claim 2), $W_2 \cup W_5$ (by Claim 4), V_1 (by Claim 6) or W_3 (since we assumed $w \notin W_3''$). Furthermore, $w \notin C$ by definition of W_1. Therefore $w \in V_4 \cup W_4 \cup V_3$. By Claims 2, 4 and 6 respectively, we conclude that w is trivial to W_3. Since u is adjacent to v and w, it follows that w must be non-adjacent to v, otherwise $G[u, v, w]$ would be a K_3, a contradiction. Therefore w must be anti-complete to W_3. If $w \in V_3 \cup W_4$, let $z = v_5$ and otherwise (if $w \in V_4$) let $z = v_4$. Then z is non-adjacent to u, u', v, v' and w. Now $G[z, u', v', v, u, w]$ is a $P_1 + P_2 + P_3$, a contradiction. Therefore every vertex in $V(G) \setminus (W_1' \cup W_3'')$ is trivial to W_1'. By symmetry, every vertex in $V(G) \setminus (W_1' \cup W_3'')$ is trivial to W_3''.

Therefore, by applying a bipartite complementation (which we may do by Fact 3) between W_1' and the vertices in $V(G) \setminus W_3''$ that are complete to W_1' and another bipartite complementation between W_3'' and the vertices in $V(G) \setminus W_1'$ that are complete to W_3'', we separate $G[W_1' \cup W_3'']$ from the rest of the graph. Since $G[W_1' \cup W_3'']$ is a perfect matching, it has clique-width at most 2. We may therefore assume that $W_1' \cup W_3''$ is empty i.e. that W_1 is anti-complete to W_3. Repeating this argument for each $i \in \{1, \ldots, 5\}$, we show that we may assume that W_i is anti-complete to W_{i-2} for every i. This completes the proof of Claim 7.

Note that when applying Claim 7 we may delete vertices in some sets W_i, which may cause some large sets to become small. In this case, as stated earlier, we may simply delete the small sets as before. Thus we may assume that every set W_i is either large or empty.

Claim 8. *For all i, j, the graphs induced by $V_i \cup V_j$ and $V_i \cup W_j$ are P_7-free.*
Suppose, for contradiction, that the claim is false. Then there is an i and a j such that $G[V_i \cup V_j]$ or $G[V_i \cup W_j]$ contains an induced P_7, say on vertices $u_1, \ldots, u_7$. There must be a vertex $v_k \in C$ that is non-adjacent to every vertex of $V_i \cup V_j$ or $V_i \cup W_j$, respectively (since every vertex not in C has at most two neighbours in C). Then $G[v_k, u_1, u_2, u_4, u_5, u_6]$ is a $P_1 + P_2 + P_3$, a contradiction. This completes the proof of Claim 8.

Claim 9. *For all i, if there are vertices $v \in V_i$, $w \in V_{i+1}$ and $x \in W_{i+3}$ such that v, w and x are pairwise non-adjacent then G has bounded clique-width.*
Suppose that such pairwise non-adjacent vertices exist, say with $v \in V_1, w \in V_2$ and $x \in W_4$. We start by showing that $V_3 \cup V_4 \cup V_5 \cup W_1 \cup W_2 \cup W_3 \cup W_5$ is empty.

First suppose there is a vertex $y \in V_3$. Then y is non-adjacent to v and x by Claim 2. Then $G[x, v, v_5, v_3, w, y]$ or $G[v, v_1, w, x, v_4, y]$ is a $P_1 + P_2 + P_3$ if y is adjacent

or non-adjacent to w, respectively. This contradiction implies that V_3 is empty. By symmetry V_5 is also empty.

Next, suppose there is a vertex $y \in V_4$. Then y is non-adjacent to v and w by Claim 2. Then $G[v, v_1, w, v_4, x, y]$ or $G[y, v_4, x, v_2, v_1, w]$ is a $P_1+P_2+P_3$ if y is adjacent or non-adjacent to x, respectively. This contradiction implies that V_4 is empty.

Next, suppose there is a vertex $y \in W_1$. Then y is non-adjacent to w and x by Claims 2 and 7, respectively. Then $G[w, x, v_4, v_2, v, y]$ or $G[v, x, v_4, w, v_1, y]$ is a $P_1 + P_2 + P_3$ if y is adjacent or non-adjacent to v, respectively. This contradiction implies that W_1 is empty. By symmetry W_2 is also empty.

Finally, suppose that W_3 is not empty. Then W_3 must be large, so it contains two vertices, say y and y'. Then y and y' are each non-adjacent to w and adjacent to x by Claims 2 and 4, respectively. If y is non-adjacent to v then $G[v, v_1, w, v_4, x, y]$ would be a $P_1 + P_2 + P_3$, a contradiction. Therefore y is adjacent to v, and similarly y' is adjacent to v. Now $G[v_4, v_1, w, y, v, y']$ is a $P_1 + P_2 + P_3$. This contradiction implies that W_3 is empty. By symmetry, we may assume that W_5 is also empty.

The above means that $V_3 \cup V_4 \cup V_5 \cup W_1 \cup W_2 \cup W_3 \cup W_5$ is indeed empty, so $V(G) = V_1 \cup V_2 \cup W_4 \cup V(C)$.

Let V_1' and V_1'' be the set of vertices in V_1 that are anti-complete or complete to $\{w, x\}$, respectively. Let V_2' and V_2'' be the set of vertices in V_2 that are anti-complete or complete to $\{v, x\}$, respectively. Let W_4' and W_4'' be the set of vertices in W_4 that are anti-complete or complete to $\{v, w\}$, respectively. Observe that $v \in V_1', w \in V_2'$ and $x \in W_4'$. We will show that $V_1', V_1'', V_2', V_2'', W_4'$ and W_4'' form a partition of $V(G) \setminus V(C)$.

Suppose, for contradiction, that there is a vertex $v' \in V_1$ with exactly one neighbour in $\{w, x\}$. Then $G[v, v_4, x, v', w, v_1]$ or $G[v, v_1, w, v_4, x, v']$ is a $P_1 + P_2 + P_3$ if this neighbour is w or x, respectively. Therefore every vertex of V_1 is in $V_1' \cup V_1''$. Similarly, every vertex of V_2 is in $V_2' \cup V_2''$.

Suppose, for contradiction, that there is a vertex $x' \in W_4$ with exactly one neighbour in $\{v, w\}$. Without loss of generality, suppose that x' is adjacent to v, but not to w. Then $G[x, w, v_3, v_5, v, x']$ is a $P_1+P_2+P_3$. Therefore every vertex of W_4 is in $W_4' \cup W_4''$. Thus every vertex of $V(G) \setminus V(C)$ is in $V_1' \cup V_1'' \cup V_2' \cup V_2'' \cup V_4' \cup V_4''$.

Observe that the remarks made above for v, w and x also hold if one of these is replaced by a vertex of V_1', V_2' or W_4', respectively. Indeed, suppose $v' \in V_1' \setminus \{v\}$, then every vertex of W_4 must be either complete or anti-complete to $\{v', w\}$. Since the vertices of W_4' are non-adjacent to w, but the vertices of W_4'' are adjacent to w, it follows that W_4' is anti-complete to $\{v', w\}$ and that W_4'' is complete to $\{v', w\}$. Therefore W_4' is anti-complete to V_1', and W_4'' is complete to V_1'. Since G is K_3-free and every vertex of $V_1'' \cup V_2''$ is adjacent to x, it follows that V_1'' is anti-complete to V_2''. Similarly, we conclude that V_1', V_2' and W_4' are pairwise anti-complete, V_1'', V_2'' and W_4'' are pairwise anti-complete and for every pair of sets $S \in \{V_1', V_2', W_4'\}$ and $T \in \{V_1'', V_2'', W_4''\}$ such that $(S, T) \notin \{(V_1', V_1''), (V_2', V_2''), (W_4', W_4'')\}$, S and T are complete to each-other.

Now if we delete the vertices of C (which we may do by Fact 1) and apply bipartite complementations between $V_1'\&V_2''$, $V_1'\&W_4''$, $V_2'\&V_1''$, $V_2'\&W_4''$, $W_4'\&V_1''$ and $W_4'\&V_2''$, we obtain an edgeless graph, which therefore has clique-width at most 1. By Fact 3, it follows that G has bounded clique-width. This completes the proof of Claim 9.

We now consider the graph G' obtained from G by removing the five vertices of C. Claims 1 and 3 show that we may assume $V_1, \ldots, V_5, W_1, \ldots, W_5$ are independent sets that form a partition of the vertex set of G'. Claims 2 and 4–9 correspond to the seven conditions of Lemma 9. Therefore G' has bounded clique-width. By Fact 1, G also has bounded clique-width. This completes the proof. □

6 The Diamond-free Case

In this section, we prove that (diamond, $P_1 + 2P_2$)-free graphs have bounded clique-width. In order to do this, we first need to prove the following two lemmas.

Lemma 10. *The class of disconnected (diamond, $P_1 + 2P_2$)-free graphs has bounded clique-width.*

Proof. If G is a disconnected (diamond, $P_1 + 2P_2$)-free graph then it contains at least two components. Therefore every component of G must be (diamond, $2P_2$)-free and thus has bounded clique-width by Lemma 3. We conclude that G has bounded clique-width.

Lemma 11. *The class of (diamond, $P_1 + 2P_2$)-free graphs that contain a K_4 has bounded clique-width.*

Proof. Let G be a (diamond, $P_1 + 2P_2$)-free graph containing an induced K_4. By Lemma 10, we may assume that G is connected. Let K be a maximum clique of G and note that $|K| \geq 4$. We may assume that G contains vertices outside K, otherwise G is a clique on at least four vertices, in which case it has clique-width 2.

Suppose there is a vertex v in G that is not in K, but has at least two neighbours $x, y \in K$. By maximality of K, there must be a vertex $z \in K$ that is not adjacent to v. However this means that $G[x, y, v, z]$ is a diamond, a contradiction. Therefore every vertex not in K has at most one neighbour in K.

Choose $v_1, v_2, v_3, v_4 \in K$ arbitrarily. For $i \in \{1, 2, 3, 4\}$, let V_i be the set of vertices not in K whose unique neighbour in K is v_i. Let U be the set of vertices not in K that do not have a neighbour in $\{v_1, v_2, v_3, v_4\}$. Note that vertices of U may have neighbours in $K \setminus \{v_1, v_2, v_3, v_4\}$.

Claim 1. *For $i, j \in \{1, 2, 3, 4\}$, $G[U \cup V_i \cup V_j]$ must be $(P_1 + P_2)$-free.*
Indeed, if $G[U \cup V_1 \cup V_2]$ contains an induced $P_1 + P_2$ on vertices y_1, y_2, y_3, say, then $G[y_1, y_2, y_3, v_3, v_4]$ is a $P_1 + 2P_2$, a contradiction. The claim follows by symmetry.

Claim 2. *For $i \in \{1, 2, 3, 4\}$, we may assume $G[V_i]$ is either a clique on at most two vertices or an independent set.*
If $G[V_1]$ contains an induced P_3 on vertices y_1, y_2, y_3, say, then $G[v_1, y_2, y_1, y_3]$ is a diamond, a contradiction. Therefore $G[V_1]$ is a disjoint union of cliques. Claim 1 implies that $G[V_1]$ is either a clique, or else every clique in $G[V_1]$ contains at most one vertex i.e. V_1 is an independent set.

Suppose, for contradiction, that V_1 is a clique on at least three vertices. We will show that the clique-width of G is bounded in this case. First suppose, for contradiction, that there is a vertex $u \in U \cup V_2 \cup V_3 \cup V_4$. Since $G[\{u\} \cup V_1]$ is $(P_1 + P_2)$-free by Claim 1, u must be adjacent to all but at most one vertex of V_1. Let $x, y \in V_1$ be neighbours of u. Then $G[x, y, u, v_1]$ is a diamond, a contradiction. We conclude that $U \cup V_2 \cup V_3 \cup V_4 = \emptyset$, so $V(G) = K \cup V_1$. Deleting v_1 we obtain a disconnected (diamond, $P_1 + 2P_2$)-free graph, which has bounded clique-width by Lemma 10. Therefore G has bounded clique-width by Fact 1. Therefore if V_1 is a clique then it contains at most two vertices. The claim follows by symmetry.

Claim 3. *For distinct $i, j \in \{1, 2, 3, 4\}$, if V_i is an independent set then every vertex of V_j is either complete or anti-complete to V_i.*
Indeed, this follows directly from Claim 1, which states that $G[V_i \cup V_j]$ is $(P_1 + P_2)$-free. (Note that if V_j is a clique then it may contain a vertex that is complete to V_i and another that is anti-complete to V_i.)

Claim 4. *We may assume U contains at least three vertices.*
Suppose that U has at most two vertices. By Fact 1 and Claim 2, we may remove every

vertex of U and every vertex of V_i for those V_i that are cliques. After this, by Claim 2, every set V_i will either be empty or an independent set. Furthermore, for distinct $i, j \in \{1, 2, 3, 4\}$, by Claim 3, every vertex of V_i is trivial to V_j and vice versa, so V_i is complete or anti-complete to V_j. By Fact 3, we may apply a bipartite complementation between V_i and V_j if they are complete. By Fact 1, we may delete v_1, v_2, v_3, v_4. We obtain a graph that is the disjoint union of a clique and at most four independent sets and therefore has clique-width at most 2. It follows that the graph G must also have had bounded clique-width. We may therefore assume that U contains at least three vertices. This completes the proof of the claim.

We now consider a number of cases:

Case 1: *Every vertex of K has at most one neighbour outside of K.*
By Fact 2, we may remove all the edges connecting pairs of vertices in K. Let G' be the resulting graph and note that in G', every vertex of K has at most one neighbour. Then $\mathrm{cw}(G') \leq \mathrm{cw}(G' \setminus K) + 1$. (Given a k-expression for $G' \setminus K$, whenever we create a vertex v that has a neighbour w in K, we immediately create w with a special new label $*$, take the disjoint union and join v to w by an edge. For any vertices in K with no neighbours outside of K, we simply add them with label $*$ at the end of the process. This will give a $(k+1)$-expression for G'.) Now $G' \setminus K = G \setminus K$. Since V_1 contains at most one vertex, by Fact 1, it is sufficient to show that $G \setminus (V_1 \cup K)$ has bounded clique-width. However, $G \setminus (V_1 \cup K)$ is $(\text{diamond}, 2P_2)$-free, since if it contained an induced $2P_2$ then this, together with v_1 would induce a $P_1 + 2P_2$ in G. Therefore $G \setminus (V_1 \cup K)$ has bounded clique-width by Lemma 3 and therefore G also has bounded clique-width. This completes the proof of this case.

We may now assume that at least one vertex of K has at least two neighbours outside of K.

Case 2: *Exactly one vertex of K has neighbours outside K.*
Suppose that v_1 is the only vertex of K that has neighbours outside of K (at least one vertex of K has a neighbour outside of K since G is connected and not a clique). Now $G \setminus \{v_1\}$ is a disconnected $(\text{diamond}, P_1 + 2P_2)$-free graph, so it has bounded clique-width by Lemma 10. By Fact 1, G also has bounded clique-width. This completes the proof of this case.

We may now assume that at least two vertices of K have neighbours outside of K. Without loss of generality, we may therefore assume that the following case holds.

Case 3: *V_1 contains at least two vertices and V_2 contains at least one vertex.*
Fix $x, y, z \in V_1 \cup V_2$, with two of these vertices in V_1 and one in V_2. If these vertices are pairwise adjacent then $G[x, y, v_1, z]$ would be a diamond, a contradiction. We may therefore assume that x and y are non-adjacent. Now every vertex of $v \in U$ is either complete or anti-complete to $\{x, y\}$, otherwise $G[v, x, y]$ would be a $P_1 + P_2$ in $G[U \cup V_1 \cup V_2]$, which would contradict Claim 1.

Suppose $u, v \in U$. If u and v are adjacent then they cannot both be complete to $\{x, y\}$, otherwise $G[u, v, x, y]$ would be a diamond and they cannot both be anti-complete to $\{x, y\}$, otherwise $G[x, u, v]$ would be a $P_1 + P_2$ in $G[U \cup V_1 \cup V_2]$, which would contradict Claim 1. Therefore if u and v are adjacent then one of them is complete to $\{x, y\}$ and the other is anti-complete to $\{x, y\}$. If u and v are non-adjacent then they must either both be complete to $\{x, y\}$ or both be anti-complete to $\{x, y\}$. Indeed, suppose for contradiction that u is complete to $\{x, y\}$ and v is anti-complete to $\{x, y\}$. Then $G[v, u, x]$ would be an induced $P_1 + P_2$ in $G[U \cup V_1 \cup V_2]$, which would contradict Claim 1. The above holds for every pair of vertices $u, v \in U$. This implies that $G[U]$ is a complete bipartite graph with one of the sets in the bipartition consisting

of the vertices complete to $\{x, y\}$ and the other consisting of the vertices anti-complete to $\{x, y\}$. (Note that one of the parts of the complete bipartite graph $G[U]$ may be empty, as we allow the case where U is an independent set.)

Note that the arguments in the above paragraph only used the facts that $G[U \cup V_1 \cup V_2]$ is $(P_1 + P_2, \text{diamond})$-free and that $V_1 \cup V_2$ contains two non-adjacent vertices. Let U_1 and U_2 be the independent sets that form the bipartition of U. Note that since U contains at least three vertices (by Claim 4), we may assume without loss of generality that U_1 contains at least two vertices. If U_2 contains exactly one vertex, by Fact 1, we may delete it. (Note that this may cause U to contain only two vertices, rather than at least three, however this does not affect our later arguments.) We may therefore assume that U_2 is either empty or contains at least two vertices. Repeating the argument in the previous paragraph with the roles of U and $V_1 \cup V_2$ reversed, we find that $G[V_1 \cup V_2]$ is a complete bipartite graph, with one side of the bipartition complete to U_1 and the other anti-complete to U_1 and if U_2 is non-empty then one side of the bipartition is complete to U_2 and the other is anti-complete to U_2. Similarly, for each pair of distinct $i, j \in \{1, 2, 3, 4\}$, the same argument shows that $G[V_i \cup V_j]$ is also a complete bipartite graph with a similar bipartition.

We now proceed as follows: if V_i is a clique for some i then it contains at most two vertices, in which case we delete them and make V_i empty. For every pair of distinct $i, j \in \{1, 2, 3, 4\}$ (V_i or V_j may be empty) $G[V_i \cup V_j]$ must then be an independent set, in which case we do nothing, or a complete bipartite graph with bipartition (V_i, V_j), in which case we apply a bipartite complementation between V_i and V_j. Now every set V_i is either complete or anti-complete to U_1 and complete or anti-complete to U_2. Applying at most $4 \times 2 = 8$ bipartite complementations, we can remove all edges between $V_1 \cup \cdots \cup V_4$ and U. Next, we apply a bipartite complementation between U_1 and U_2. Finally, we apply a complementation to the clique K. Let G' be the resulting graph and note that $G'[V_1 \cup \cdots \cup V_4 \cup U]$ and $G'[K]$ are independent sets and that in G' every vertex in $V_1 \cup \cdots \cup V_4 \cup U$ has at most one neighbour in K. Therefore G' is a disjoint union of stars, and so has clique-width at most 2. By Facts 1, 2 and 3, it follows that G also has bounded clique-width. This completes proof for this case and therefore completes the proof of the lemma. □

To prove the main result of this section, we will need an additional notion. Let G be a graph. For each set T that induces a triangle in G, let U^T be the set of vertices in G that have no neighbour in T. Let $\mathcal{U} = \{u \in U^T \mid T \text{ induces a triangle in } G\}$. We say that the graph G is *basic* if we can partition the vertices of $G \setminus \mathcal{U}$ into three sets V_1, V_2, V_3 and also into sets $T^1, W_1, T^2, W_2, \ldots, T^p, W_p$ for some p such that the following properties hold:

(i) No triangle in G contains a vertex of $\mathcal{U}$.
(ii) For every triangle T, the set U^T is independent and there is a vertex $x \in V(T)$ such that $N(x) = N(u) \cup (V(T) \setminus \{x\})$ for all $u \in U^T$.
(iii) V_1, V_2 and V_3 are independent.
(iv) $\{G[T^1], \ldots, G[T^p]\}$ is the set of all induced triangles in G and each of them has exactly one vertex in each of V_1, V_2 and V_3.
(v) $G[W_i]$ is $(P_1 + 2P_2)$-free and does not contain an induced $3P_1$ with one vertex in each of V_1, V_2 and V_3.
(vi) If $i < j$ and $k + 1 \not\equiv \ell \pmod 3$ then:
  1. $T^i \cap V_k$ is anti-complete to $T^j \cap V_\ell$,
  2. $T^i \cap V_k$ is anti-complete to $W_j \cap V_\ell$,
  3. $W_i \cap V_k$ is anti-complete to $T^j \cap V_\ell$ and
  4. $W_i \cap V_k$ is anti-complete to $W_j \cap V_\ell$,

(vii) If $i < j$ and $k + 1 \equiv \ell \pmod 3$ then:
  1. $T^i \cap V_k$ is complete to $T^j \cap V_\ell$,
  2. $T^i \cap V_k$ is complete to $W_j \cap V_\ell$ and
  3. $W_i \cap V_k$ is complete to $T^j \cap V_\ell$.

(viii) If $i + 1 < j$ and $k + 1 \equiv \ell \pmod 3$ then:
  1. $W_i \cap V_k$ is complete to $W_j \cap V_\ell$.

(ix) If $i + 1 = j$ and $k + 1 \equiv \ell \pmod 3$ then:
  1. $W_i \cap V_k$ is either complete or anti-complete to $W_j \cap V_\ell$.

(x) If $i = j$ and $k + 1 \equiv \ell \pmod 3$ then:
  1. $T^i \cap V_k$ is complete to $W_j \cap V_\ell$.

(xi) If $i = j$ and $k + 1 \not\equiv \ell \pmod 3$ then:
  1. $T^i \cap V_k$ is anti-complete to $W_j \cap V_\ell$.

Next, we show that basic graphs have bounded clique-width.

Lemma 12. *If G is a basic graph then it has clique-width at most* 9.

Proof. Let G be a graph with vertices partitioned into sets as above. This means that we have sets of vertices $T^1, W_1, T^2, W_2, \ldots, T^p, W_p$ in order, such that if X and Y are sets in this order with X coming before Y then $X \cap V_k$ is complete to $Y \cap V_\ell$ if $k + 1 \equiv \ell \pmod 3$ and anti-complete otherwise in all cases except where $X = W_i, Y = W_{i+1}$ for some i, in which case $X \cap V_k$ may either be complete or anti-complete to $Y \cap V_{k+1}$. Also recall that U^{T^i} is an independent set for every i and there is a vertex $x \in T^i$ such that every vertex of U^{T^i} has the same neighbourhood in $G \setminus T^i$ as x.

Note that $W_i \subseteq V_1 \cup V_2 \cup V_3$. Then $G[W_i]$ is a 3-partite graph with 3-partition $(W_i \cap V_1, W_i \cap V_2, W_i \cap V_3)$. Furthermore, $G[W_i]$ is K_3-free, and contains no induced $3P_1$ with exactly one vertex in each V_j. Since $G[W_i]$ is $(P_1 + 2P_2)$-free it must therefore be $(P_7, S_{1,2,3})$-free. Therefore, by Lemma 6, the graph $G[W_i]$ is totally 3-decomposable with respect to this partition. By Lemma 7, we can construct $G[W_i]$ using at most six labels such that the resulting labelled graph has all vertices in W_i labelled with label i for $i \in \{1, 2, 3\}$.

We are now ready to describe how to construct G. We do this by constructing $G[T^i \cup U^{T^i}]$ then $G[W_i]$ for each $i \in \{1, \ldots, p\}$ in turn and adding it to the graph. More formally, we start with the empty graph, then for $i = 1, \ldots, p$ in turn, we do the following:

1. Let $\{x_1^i, x_2^i, x_3^i\} = T^i$, where $x_j^i \in V_j$ for $j \in \{1, 2, 3\}$. Add vertices x_1^i, x_2^i and x_3^i with labels $4, 5$ and 6, respectively, then add edges between vertices labelled $4\&5, 5\&6$ and $4\&6$.
2. If U^{T^i} is non-empty then the vertices in this set have the same neighbourhood in $G \setminus T^i$ as x_1^i, x_2^i or x_3^i. Add the vertices of U^{T^i} with label $4, 5$ or 6, respectively.
3. Add edges between vertices labelled $1\&5, 2\&6$ and $3\&4$.
4. Relabel vertices labelled $4, 5$ or 6 to have labels $1, 2$ or 3, respectively.
5. Construct $G[W_i]$ with vertices labelled $4, 5$ or 6, if they are in V_1, V_2 or V_3, respectively.
6. Add edges between vertices labelled $1\&5, 2\&6$ and $3\&4$.
7. If $i > 1$ then add edges between vertices labelled $4\&9, 5\&7$ and $6\&8$ if $V_k \cap W_i$ is complete to $V_{k-1} \cap W_{i-1}$ for $k = 1, 2, 3$, respectively.
8. Relabel vertices labelled $7, 8$ or 9 to have labels $1, 2$ or 3, respectively.
9. Relabel vertices labelled $4, 5$ or 6 to have labels $7, 8$ or 9, respectively.

Note that at the end of any iteration of the above procedure, the vertices of W_i will have labels in $\{7, 8, 9\}$ and all other constructed vertices will have labels in $\{1, 2, 3\}$.

This construction builds a copy of G using at most nine labels. Thus G has clique-width at most 9. This concludes the proof of the lemma. □

We are now ready to prove our main theorem of this section. To do so, we show that if a graph G is (diamond, $P_1 + 2P_2$)-free then either we can show that G has bounded clique-width directly (possibly by applying some graph operations that do not change the clique-width the graph by "too much") or else the (unmodified) graph G is itself basic (in which case it has clique-width at most 9).

Theorem 4. *The class of (diamond, $P_1 + 2P_2$)-free graphs has bounded clique-width.*

Proof. Let G be a (diamond, $P_1 + 2P_2$)-free graph. By Lemma 10, we may assume that G is connected. By Theorem 3, we may assume that G contains an induced K_3. By Lemma 11, we may assume that G is K_4-free.

Let T be an arbitrary induced triangle (i.e. K_3) in G with vertices v_1^T, v_2^T and v_3^T. Since G is (diamond, K_4)-free, every vertex not in T has at most one neighbour in T. For $i \in \{1, 2, 3\}$ let V_i^T be the set of vertices not in T whose unique neighbour in T is v_i^T and let U^T be the set of vertices that have no neighbour in T. We will now prove a series of claims. More formally, we will show that if the conditions of any of these claims are not satisfied, then either we obtain a contradiction or we can directly prove that G has bounded clique-width, in which case we are done.

Claim 1. *For every triangle T, the sets V_1^T, V_2^T and V_3^T each contain at least three vertices.*
If for some i the set V_i^T contains at most two vertices then v_i^T has at most four neighbours in G. If we delete every vertex in $N(v_i^T)$, then v_i^T has no neighbours in the resulting graph. Therefore either G has at most five vertices (in which case it has clique-width at most 5), or $G \setminus N(v_i^T)$ is a disconnected (diamond, $P_1 + 2P_2$)-free graph, so it has bounded clique-width by Lemma 10. By Fact 1, it follows that G has bounded clique-width. This completes the proof of the claim.

Claim 2. *For every triangle T, the sets V_1^T, V_2^T and V_3^T are independent.*
Suppose, for contradiction, that V_1^T is not an independent set. Since G is K_4-free and every vertex of V_1^T is adjacent to v_1^T, it follows that $G[V_1^T]$ is K_3-free. Since V_1^T contains at least three vertices by Claim 1, there must be vertices $x, y, z \in V_1^T$ such that x is adjacent to y, but not to z. Then $G[v_1^T, y, x, z]$ is a diamond if y and z are adjacent and $G[z, x, y, v_2^T, v_3^T]$ is a $P_1 + 2P_2$ if they are not. This contradiction implies that V_1^T is an independent set. The claim follows by symmetry.

Claim 3. *Every pair of triangles in G is vertex-disjoint.*
Consider a triangle T with vertex v_1^T. The neighbourhood of v_1^T is $V_1^T \cup \{v_2^T, v_3^T\}$. Now V_1^T is independent by Claim 2 and anti-complete to $\{v_2^T, v_3^T\}$ by definition. Therefore, if a triangle in G contains v_1^T then it must also contain v_2^T and v_3^T. In other words, v_1^T is contained in only one triangle in G, namely T. The claim follows by symmetry.

Claim 4. *For every triangle T, the set U^T is independent.*
By Claim 1, we can choose $x, y \in V_1^T$ and by Claim 2, x must be non-adjacent to y. If a vertex $u \in U^T$ is adjacent to x, but not to y then $G[y, u, x, v_2^T, v_3^T]$ is a $P_1 + 2P_2$, a contradiction. Therefore every vertex of U^T is either complete or anti-complete to $\{x, y\}$. Suppose $u, v \in U^T$. First suppose u and v are non-adjacent. If x is adjacent to u but v is not, then $G[v, u, x, v_2^T, v_3^T]$ is a $P_1 + 2P_2$, a contradiction. Therefore if $u, v \in U^T$ are non-adjacent, then $\{u, v\}$ is either complete or anti-complete to

$\{x, y\}$. Now suppose u and v are adjacent. Then $G[u, v, x, y]$ is a diamond if $\{u, v\}$ is complete to $\{x, y\}$ and $G[x, u, v, v_2^T, v_3^T]$ is a $P_1 + 2P_2$ if $\{u, v\}$ is anti-complete to $\{x, y\}$. Therefore if $u, v \in U^T$ are adjacent then exactly one of them is complete to $\{x, y\}$ and the other is anti-complete to $\{x, y\}$. This means that $G[U^T]$ is a complete bipartite graph, with partition classes U_1^T and U_2^T, say, and furthermore, one of U_1^T and U_2^T is complete to V_1^T and the other is anti-complete to V_1^T. Similarly, this holds with the same partition (U_1^T, U_2^T) if we replace V_1^T by V_2^T or V_3^T. Thus every vertex of U_1^T (respectively U_2^T) has the same neighbourhood in $V_1^T \cup V_2^T \cup V_3^T$.

Suppose that V_i^T and V_j^T are both complete to U_k^T for some $i, j \in \{1, 2, 3\}$ with $i \neq j$ and some $k \in \{1, 2\}$ and that U_k^T contains at least two vertices, say u and v. If $x \in V_i^T$ and $y \in V_j^T$ are adjacent, then $G[x, y, u, v]$ is a diamond, a contradiction. Therefore V_i^T is anti-complete to V_j^T.

Suppose that U_1^T and U_2^T each contain at least one vertex, say u and v, respectively. We will show that in this case the clique-width of G is bounded. Suppose, for contradiction, that $G \setminus (T \cup \{u, v\})$ contains an induced K_3, say with vertex set T'. Since $G[U^T]$ is a complete bipartite graph with bipartition (U_1^T, U_2^T) and no vertex of a set V_i^T can have neighbours in both U_1^T and U_2^T, at most one vertex of T' can be in U^T. Suppose that U_1^T contains at least two vertices (so $U_1^T \setminus \{u\}$ is non-empty) and that U_1^T is complete to V_i^T and V_j^T for some $i \neq j$ (in which case U_2^T is anti-complete to V_i^T and V_j^T). Then V_i^T and V_j^T must be anti-complete. We conclude that in this case no vertex of U_1^T can belong to T'. No vertex of U_2^T can belong to T' either, since vertices in U_2^T can only have neighbours in U_1^T and in V_k^T where $k \notin \{i, j\}$ (if U_1^T is anti-complete to V_k^T). Furthermore, since V_i^T is anti-complete to V_j^T, and V_1^T, V_2^T, V_3^T are independent (by Claim 2), there is no induced K_3 in $G[V_1^T \cup V_2^T \cup V_3^T]$. Thus T' cannot exist, a contradiction.

The above means that if such a triangle T' does exist and a set U_i^T contains at least two vertices, then U_i^T must be anti-complete to at least two distinct sets V_j^T and V_k^T (in which case U_i^T cannot contain a vertex of T'). Since T' consists of vertices of $G \setminus (T \cup \{u, v\})$, this means that no vertex of U^T is in T' (if U_i^T contains a single vertex for some i then by definition T' does not include it). By Claim 2, it follows that T' must consist of vertices $x \in V_1^T, y \in V_2^T$ and $z \in V_3^T$. Since each set V_i^T is anti-complete to exactly one of U_1^T and U_2^T, we may assume without loss of generality that U_1^T (and therefore u) is complete to both V_1^T and V_2^T. Now $G[x, y, z, u]$ is a K_4 or diamond if u and z are adjacent or non-adjacent, respectively. This contradiction means that $G \setminus (T \cup \{u, v\})$ must in fact be K_3-free. Since $G \setminus (T \cup \{u, v\})$ is a $(K_3, P_1 + 2P_2)$-free graph, it has bounded clique-width by Theorem 3. By Fact 1, we conclude that G also has bounded clique-width. We may therefore assume that either U_1^T or U_2^T is empty. It follows that U^T is an independent set. This completes the proof of the claim.

Claim 5. *For every triangle T, there is a vertex $x \in V(T)$ such that $N(x) = N(u) \cup (V(T) \setminus \{x\})$ for all $u \in U^T$.*

By the previous claim, we may assume that U^T is independent. Note that by the same arguments as for the previous claim, for all $i \in \{1, 2, 3\}$, U^T is trivial to V_i^T. Suppose $u \in U^T$. By the same arguments as for the previous claim, U^T must be anti-complete to at least two distinct sets V_i^T and V_j^T, otherwise $G \setminus (T \cup \{u\})$ would be K_3-free and the clique-width of G would be bounded as before. Since G is connected, it follows that U^T must be complete to at least one set V_i^T. Therefore U^T must be complete to exactly one set V_i^T. It follows that $N(v_i^T) = V_i^T \cup (V(T) \setminus \{v_i^T\}) = N(u) \cup (V(T) \setminus \{v_i^T\})$ for all $u \in U^T$. This completes the proof of the claim.

Claim 6. *No triangle in G contains a vertex of $\mathcal{U}$.*

If $u \in \mathcal{U}$ then $u \in U_T$ for some triangle T. By the previous claim, the neighbourhood

of every vertex of U^T is V_i^T, for some i. Since V_i^T is an independent set, the claim follows immediately.

Claim 7. *If T and T' are distinct triangles in G then the edges between them form an induced matching.*
Suppose T and T' are distinct triangles in G. By Claim 3, T and T' must be vertex-disjoint. By Claim 6, it follows that every vertex of T' is in $V_1^T \cup V_2^T \cup V_3^T$, so every vertex of T' has exactly one neighbour in T. By Claim 2, for $i \in \{1,2,3\}$, the set V_i^T is an independent set, so it can contain at most one vertex of T'. Therefore T' has exactly one vertex in each of V_1^T, V_2^T and V_3^T. By definition of V_i^T, this means that every vertex of T' has a different neighbour in T. The claim follows.

Claim 8. *For every triangle T and for every pair of distinct $i, j \in \{1,2,3\}$, $G[V_i^T \cup V_j^T]$ is $2P_2$-free.*
Suppose, for contradiction, that $G[V_1^T \cup V_2^T]$ contains an induced $2P_2$. Then this $2P_2$, together with the vertex v_3^T would induce a $P_1 + 2P_2$ in G. The claim follows by symmetry.

Claim 9. *For every triangle T, there is no induced $3P_1$ in G with one vertex in each of V_1^T, V_2^T and V_3^T.*
Suppose that there are three vertices $x \in V_1^T, y \in V_2^T$ and $z \in V_3^T$ that are pairwise non-adjacent. We will show that in this case G has bounded clique-width. Suppose $u \in U^T$. By Claim 5, u has exactly one neighbour in $\{x, y, z\}$. Without loss of generality, assume that u is adjacent to x. Then $G[z, u, x, y, v_2^T]$ is a $P_1 + 2P_2$, a contradiction. We may therefore assume that U^T is empty. If there is a vertex $x' \in V_1^T \setminus \{x\}$ that is adjacent to y, but not to z then $G[x, x', y, v_3^T, z]$ is a $P_1 + 2P_2$ in G. This contradiction means that every vertex of V_1^T is either complete or anti-complete to $\{y, z\}$. Similarly, every vertex of V_2^T is either complete or anti-complete to $\{x, z\}$ and every vertex of V_3^T is either complete or anti-complete to $\{x, y\}$. Note that the above holds for any three pairwise non-adjacent vertices in V_1^T, V_2^T and V_3^T, respectively.

Let $V_1'^T$ and $V_1''^T$ be the sets of vertices in V_1^T that are anti-complete or complete to $\{y, z\}$, respectively. Let $V_2'^T$ and $V_2''^T$ be the sets of vertices in V_2^T that are anti-complete or complete to $\{x, z\}$, respectively. Let $V_3'^T$ and $V_3''^T$ be the sets of vertices in V_3^T that are anti-complete or complete to $\{x, y\}$, respectively. Note that $x \in V_1'^T, y \in V_2'^T$ and $z \in V_3'^T$.

Suppose $x' \in V_1'^T$ and $y' \in V_2'^T$. Since x' is non-adjacent to y and to z, it follows that $G[x', y, z]$ is a $3P_1$. Since y' is non-adjacent to z, it must therefore be anti-complete to $\{x', z\}$. In particular, this means that if $i, j \in \{1,2,3\}$ are distinct then $V_i'^T$ is anti-complete to $V_j'^T$.

Suppose $x' \in V_1'^T$ and $y' \in V_2''^T$. Since x' is non-adjacent to y and to z, it follows that $G[x', y, z]$ is a $3P_1$. Since y' is adjacent to z, it must therefore be complete to $\{x', z\}$. In particular, this means that if $i, j \in \{1,2,3\}$ are distinct then $V_i'^T$ is complete to $V_j''^T$.

Note that for all $i \in \{1,2,3\}$, $V_i'^T$ is anti-complete to $V_i''^T$, since V_i^T is an independent set.

Suppose $x' \in V_1''^T$ and $y' \in V_2''^T$. If x' and y' are non-adjacent then $G[x', y, x, y']$ is a $2P_2$ in $G[V_1^T \cup V_2^T]$, which would contradict Claim 8. This means that if $i, j \in \{1,2,3\}$ are distinct then $V_i''^T$ is complete to $V_j''^T$.

We now proceed as follows: from G, we delete the three vertices of T. We then apply a bipartite complementation between every pair of sets $V_i'^T$ and $V_j''^T$ and every pair of distinct sets $V_i''^T$ and $V_j''^T$ (a total of nine bipartite complementations). After doing this, we obtain an edge-less graph, which therefore has clique-width at most 1. By

Facts 1 and 3, it follows that G must also have bounded clique-width. This completes the proof of the claim.

Claim 10. *G contains at least three vertex-disjoint triangles.*
Suppose, for contradiction, that the claim is false. Then G contains at most two vertex-disjoint triangles, in which case, we can delete at most six vertices to obtain a $(K_3, P_1 + 2P_2)$-free graph, which has bounded clique-width by Theorem 3. By Fact 1, G also has bounded clique-width. This completes the proof of the claim.

We will now assume that the above claims are satisfied and show that this implies that G is basic. We arbitrarily fix a triangle T^1 with vertices $v_1^{T^1}$, $v_2^{T^1}$ and $v_3^{T^1}$. To simplify notation, set $v_i = v_i^{T^1}$ for $i \in \{1, 2, 3\}$. Recall that by Claim 6, no K_3 in G has a vertex in $\mathcal{U}$. By Claim 2, it follows that every K_3 in G apart from T^1 has exactly one vertex in each of $V_1^{T^1} \setminus \mathcal{U}$, $V_2^{T^1} \setminus \mathcal{U}$ and $V_3^{T^1} \setminus \mathcal{U}$. We now set $V_1 = (V_1^{T^1} \cup \{v_2\}) \setminus \mathcal{U}$, $V_2 = (V_2^{T^1} \cup \{v_3\}) \setminus \mathcal{U}$ and $V_3 = (V_3^{T^1} \cup \{v_1\}) \setminus \mathcal{U}$.

Claim 11. *V_1, V_2 and V_3 are independent.*
The vertices in $V_i^{T^1}$ are exactly the vertices outside T^1 whose unique neighbour in T^1 is v_i. The claim follows by Claim 2.

By Claim 3 any two triangles in G must be vertex-disjoint. By Claim 7, the edges between any two triangles in G form a perfect matching. Let $T^x = \{x_1, x_2, x_3\}$ and $T^y = \{y_1, y_2, y_3\}$ be two distinct triangles in G with $x_i, y_i \in V_i$ for $i \in \{1, 2, 3\}$. By Claim 11, x_i is non-adjacent to y_i for $i \in \{1, 2, 3\}$. This means that the set of edges between T^x and T^y is either $\{x_1y_2, x_2y_3, x_3y_1\}$ or $\{x_1y_3, x_2y_1, x_3y_2\}$. We say that $T^x < T^y$ holds in the first case and $T^y < T^x$ holds in the second. Note that exactly one of these statements holds for any two distinct triangles in G. Furthermore, note that if T^x is a triangle other than T^1 then the definition of the sets V_i implies that $T^1 < T^x$.

We show that the relation $<$ is transitive. Suppose, for contradiction, that this is not the case. Then there must be three pairwise distinct triangles in G, say $T^x = \{x_1, x_2, x_3\}, T^y = \{y_1, y_2, y_3\}$ and $T^z = \{z_1, z_2, z_3\}$, where $x_i, y_i, z_i \in V_i^T$ for $i \in \{1, 2, 3\}$, with $T^x < T^y$, $T^y < T^z$ and $T^z < T^x$. Then x_1 is adjacent to y_2, y_2 is adjacent to z_3 and z_3 is adjacent to x_1. Therefore $G[x_1, y_2, z_3]$ is a K_3 which shares exactly one vertex with T^x, which would contradict Claim 3. Therefore $<$ is a transitive, anti-symmetric relation on the triangles in G. We may now order the triangles in G, say $T^1 < T^2 < \cdots < T^p$ for some p. By Claim 10, it follows that $p \geq 3$. We now conclude the following:

Claim 12. *$\{G[T^1], \ldots, G[T^p]\}$ is the set of all induced triangles in G and each of them has exactly one vertex in each of V_1, V_2 and V_3.*
Claim 13. *If $i < j$ and $k + 1 \not\equiv \ell \pmod 3$ then $T^i \cap V_k$ is anti-complete to $T^j \cap V_\ell$.*
Claim 14. *If $i < j$ and $k + 1 \equiv \ell \pmod 3$ then $T^i \cap V_k$ is complete to $T^j \cap V_\ell$.*

Consider a vertex x that is not in any induced triangle in G. If $x \notin \mathcal{U}$ then $x \in V_1 \cup V_2 \cup V_3$ and x must have exactly one neighbour in every triangle in G. Let W be the set of vertices that are not in any triangle in G and have exactly one neighbour in every induced triangle in G.

We extend the relation $<$ as follows: suppose $T = \{x_1, x_2, x_3\}$ is an induced triangle in G with $x_1 \in V_1, x_2 \in V_2$ and $x_3 \in V_3$ and suppose $w \in W$. Then w is a vertex in V_i for some $i \in \{1, 2, 3\}$. By Claim 11, w is not adjacent to x_i. Since $w \in W$, w must be adjacent to exactly one vertex of T. We say that $x < T$ holds if x is adjacent to x_{i+1} and $T < x$ if x is adjacent to x_{i-1} (we interpret indices modulo 3).

Let $w \in W$ and let T and T' be triangles in G such that $w < T$ and $T < T'$. We will show that $w < T'$. Say $T = \{x_1, x_2, x_3\}$ and $T' = \{y_1, y_2, y_3\}$, where $x_i, y_i \in V_i$ for

$i \in \{1,2,3\}$. Without loss of generality, assume $w \in V_1$. Since $w < T$, w is adjacent to x_2. Since $T < T'$, x_2 is adjacent to y_3. Since $w \in V_1$, w is non-adjacent to y_1. Now w cannot be adjacent to y_3, otherwise $G[w, x_2, y_3]$ would be a triangle that is not vertex-disjoint from T, which would contradict Claim 3. Since $w \in W$, it must have a neighbour in T', so w must therefore be adjacent to y_2. It follows that $w < T'$. Similarly, if $T < T'$ and $T' < w$ then $T < w$ and if $T < w$ and $w < T'$ then $T < T'$.

This means that we can now partition W into sets $W_1, \ldots, W_p$ where W_i contains the vertices $x \in W$ such that $T^j < x$ for $j \leq i$ and $x < T^j$ for $j > i$. (Note that $T^1 < w$ for all $w \in W$, by construction.) We immediately conclude the following:

Claim 15. *If $i < j$ and $k+1 \not\equiv \ell \pmod 3$ then $T^i \cap V_k$ is anti-complete to $W_j \cap V_\ell$.*
Claim 16. *If $i = j$ and $k+1 \not\equiv \ell \pmod 3$ then $T^i \cap V_k$ is anti-complete to $W_j \cap V_\ell$.*
Claim 17. *If $i < j$ and $k+1 \not\equiv \ell \pmod 3$ then $W_i \cap V_k$ is anti-complete to $T^j \cap V_\ell$.*
Claim 18. *If $i < j$ and $k+1 \equiv \ell \pmod 3$ then $T^i \cap V_k$ is complete to $W_j \cap V_\ell$.*
Claim 19. *If $i = j$ and $k+1 \equiv \ell \pmod 3$ then $T^i \cap V_k$ is complete to $W_j \cap V_\ell$.*
Claim 20. *If $i < j$ and $k+1 \equiv \ell \pmod 3$ then $W_i \cap V_k$ is complete to $T^j \cap V_\ell$.*

We also prove the following claim:

Claim 21. *$G[W_i]$ is $(P_1 + 2P_2)$-free and does not contain an induced $3P_1$ with one vertex in each of V_1, V_2 and V_3.*
Since G is $(P_1 + 2P_2)$-free, it follows that $G[W_i]$ is also $(P_1 + 2P_2)$-free. Since the vertices of W_i do not belong to any triangle of G and do not belong to $\mathcal{U}$, it follows that $W_i \subseteq V_1^{T^1} \cup V_2^{T^1} \cup V_3^{T^1}$. The claim then follows by Claim 9.

It remains to analyse the edges between the sets $W_1, \ldots, W_p$.

Claim 22. *If $i < j$ and $k+1 \not\equiv \ell \pmod 3$ then $W_i \cap V_k$ is anti-complete to $W_j \cap V_\ell$.*
Let $i, j \in \{1, \ldots, p\}$ be such that $i < j$. Let $T^j = \{x_1, x_2, x_3\}$ with $x_k \in V_k$ for $k \in \{1,2,3\}$. Note that if $x \in W_i$ and $y \in W_j$ then $x < T^j$ and $T^j < y$. Now $W_i \cap V_k$ is anti-complete to $W_j \cap V_k$ for $k \in \{1,2,3\}$, since V_k is an independent set by Claim 11. Suppose $x \in W_i \cap V_1$ and $y \in W_j \cap V_3$. Then x and y are both adjacent to x_2. Therefore x and y cannot be adjacent, otherwise $G[x_2, x, y]$ would be a triangle which is not vertex-disjoint from T^j, which would contradict Claim 3. By symmetry we conclude that $W_i \cap V_k$ is anti-complete to $W_j \cap V_{k+2}$ for $k \in \{1,2,3\}$ (interpreting subscripts modulo 3). This completes the proof of the claim.

The edges between $W_i \cap V_k$ and $W_j \cap V_{k+1}$ for $k \in \{1,2,3\}$ are more complicated, as shown in the following two claims:

Claim 23. *If $i+1 < j$ and $k+1 \equiv \ell \pmod 3$ then $W_i \cap V_k$ is complete to $W_j \cap V_\ell$.*
Let $i, j \in \{1, \ldots, p\}$ be such that $i+1 < j$. Suppose, for contradiction, that $x \in W_i \cap V_1$ and $y \in W_j \cap V_2$ are non-adjacent. Since $i+2 \leq j$ we find that $x < T^{j-1}, x < T^j, T^{j-1} < y$ and $T^j < y$. Let $T^j = \{x_1, x_2, x_3\}$ with $x_k \in V_k$ for $k \in \{1,2,3\}$. Let $T^{j-1} = \{y_1, y_2, y_3\}$, where $y_k \in V_k$ for $k \in \{1,2,3\}$. Then x is adjacent to y_2, but non-adjacent to x_1, while y is adjacent to x_1, but non-adjacent to y_2. Since $T^{j-1} < T^j$ it follows that y_2 is non-adjacent to x_1. Since $T^1 < x, y$, the vertex v_3 must be non-adjacent to x and y (recall that $v_3 = v_3^{T^1}$ and that this vertex has no neighbours in V_1 or V_2 apart from v_1 and v_2). Now $G[v_3, x, y_2, x_1, y]$ is a $P_1 + 2P_2$, a contradiction. By symmetry this completes the proof of the claim.

Claim 24. *If $i+1 = j$ and $k+1 \equiv \ell \pmod 3$ then $W_i \cap V_k$ is either complete or anti-complete to $W_j \cap V_\ell$.*
Let $i, j \in \{1, \ldots, p\}$ with $i+1 = j$. Let $T^j = \{x_1, x_2, x_3\}$ with $x_k \in V_k$ for $k \in \{1,2,3\}$. Assume, for contradiction, that the vertex sets $W_i \cap V_k$ and $W_j \cap V_{k+1}$ are not trivial to each-other for some $k \in \{1,2,3\}$. Without loss of generality, we may assume that there

is a vertex x with a neighbour y and a non-neighbour y' such that either $x \in W_i \cap V_1$ and $y, y' \in W_j \cap V_2$ or $y, y' \in W_i \cap V_1$ and $x \in W_j \cap V_2$. Note that x_3 is non-adjacent to x, y and y'. Since $T^1 < x, y, y'$, the vertex v_3 must be non-adjacent to x and y (recall that $v_3 = v_3^T$ and that this vertex has no neighbours in V_1 or V_2 apart from v_1 and v_2). Now $G[y', x, y, v_3, x_3]$ is a $P_1 + 2P_2$, a contradiction. By symmetry this completes the proof of the claim.

The claims proved above imply all the necessary properties for G to be basic. Indeed, Claim 6 implies Property (i) and Claims 4 and 5 imply Property (ii). Claims 11, 12, 21, 13, 15, 17, 22, 14, 18, 20, 23, 24, 19 and 16 imply Properties (iii), (iv), (v), (vi).1, (vi).2, (vi).3, (vi).4, (vii).1, (vii).2, (vii).3, (viii).1, (ix).1, (x).1, (xi).1 and respectively. Therefore G is basic, so it has bounded clique-width by Lemma 12. This completes the proof. □

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