Stabilisation of cascade and time-delay sampled-data systems

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Stabilisation des systèmes échantillonnés en cascade et avec retards

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Résumé étendu de la thèse

Ce manuscrit représente les mémoires de l'activité de recherche que j'ai développée pendant mon doctorat au sein du *Laboratoire des Signaux et Systèmes* (L2S) de l'*Université Paris-Saclay* et du *Dipartimento di Ingegneria Informatica, Automatica e Gestionale* (DIAG) de l'*Università degli Studi di Roma La Sapienza.*

Pendant cette période, des problèmes concernant la stabilisation de systèmes non linéaires sous échantillonnage et, au delà, des systèmes en temps discret en général ont été adressésral. Den plus, a été étudié l'effet de retards sur les entrées pour ces classes de systèmes.

En particulier, par système échantillonné (ou numérique), on se réfère à une dynamique évoluant continuellement dans le temps dont les entrées sont constante par morceaux pendant une durée fixée de temps (la période d'échantillonnage) et dont on ne capte les mesures que à certains instants de temps (les instants d'échantillonnage). Donc, ce type de système est caractérisé par des dynamiques à la fois discrètes et continues. Bien que la plupart des systèmes physiques entre dans cette catégorie, l'état de l'art s'systèmes numériques n'est pas satisfaisant dès lors qu'un ensemble compact de méthodologies constructives et générales pour la modélisation, l'analyse et la conception de commandes, n'est pas disponible.

Dans ce cadre, ce manuscrit se focalise sur la conception de lois de commande constantes par morceaux afin de stabiliser des systèmes non linéaires en temps continu et sous forme de cascades dont les données sont échantillonnées. Pour certaines classes de systèmes en cascade, on décrit un corps de méthodologies constructives pour la définition du contrôle échantillonné de telle sorte que, en boucle fermée, le système préserve les spécifications requises malgré la perte de certaines propriétés suite au processus d'échantillonnage. Finalement, la présence de retards dans les entrées est considérés en exploitant la forme en cascade induite par l'échantillonnage et du retard.

En particulier, l'approche développée se base sur le *système temps-discret équivalent* qui décrit les évolutions du système aux instants d'échantillonnage. Les difficultés principales abordées sont:

- la perte de structure par le modèle temps-discret équivalent ;
- la perte des propriétés de contrôle par le modèle temps-discret équivalent;
- la perte d'une structure géométrique caractérisant les évolutions du système aux instants d'échantillonnage;
- la non linéarité des équations par rapport à la commande.

On propose donc des lois de commande pour des systèmes non linéaires exhibant des formes en cascade soit *feedback*, soit *feedforward*, en termes de conception de type Lyapunov et de concepts d'*Immersion and Invariance* que l'on étend au contexte numériques. Supposant l'existence d'une stratégie de commande continue, on montre que dans beaucoup de situations, l'existence d'une commande échantillonnée stabilisante est assurée sous les mêmes hypothèses que celles du schéma continu. Pour cela, on exploite des concepts typiques des systèmes en temps discret tels, la forme de représentation différentielle et aux différences (F_0, G), la passivité moyenne par rapport à la commande, la passivité à partir d'une commande nominale. On développe des méthodologies itératives et constructives pour le calcul de la commande qui est décrite sous la forme d'une série formelle en puissances d'ordre croissant de δ, la période d'échantillonnage. Ainsi on peut définir des solutions approchées, faciles d'implantation, dont on étudie aussi les caractéristiques pour la préservation des propriétés de stabilisation en boucle fermée. Ensuite, on étudie les systèmes non linéaires en présence de retards sur les entrées et dans certains cas sur les variables d'état. Dans ce contexte, on montre l'effet positif de l'échantillonnage qui implicitement induit une forme en cascade dans un espace de dimension élargie mais finie (contrairement au cas continu). En exploitant cette forme, on propose plusieurs stratégies de commande numérique en mettant l'accent sur les simplifications que le contexte retards-échantillonnage apporte à cette classe de systèmes par rapport aux problèmes restant ouverts en temps-continu.

Enfin, on se concentre aussi sur certaines classes de dynamiques purement en temps-discret aussi ansi que d'un contexte retardé. En particulier, on étend aux systèmes en *feedforward* les méthodologies de type Lyapunov et utilisant la passivité qui sont largement populaires dans le domaine continu. On dépasse ainsi les problématiques typiquement liées aux systèmes temps-discret en proposant des techniques originales par rapport à celles développées pour le cas numérique.

Les contributions de la thèse sont ainsi résumées ici.

Conception de lois de commande pour système non linéaires échantillonnées

- Stabilisation à la *Immersion and Invariance* de dynamiques en forme *strict-feedback*;
- Stabilisation par *feedforwarding* de systèmes en cascade;
- Stabilisation des systèmes en présence des retards sur les entrées par prédiction;
- Stabilisation des systèmes en présence de retards sur les entrées par Immersion and Invariance;
- Stabilisation d'une classe de systèmes en présence des retards sur les états par Immersion and Invariance;
- Stabilisation des systèmes en présence des retards sur les entrées par réduction;

Conception de lois de commande pour système non linéaires temps-discret

- Stabilisation par *Immersion and Invariance* de dynamiques sous forme *strict-feedforward*;
- Stabilisation par *feedforwarding* en temps-discret;
- Stabilisation des systèmes en présence des retards sur les entrées par réduction;

Conception de lois de commande pour système non linéaires temps-continu

- Stabilisation de systèmes à déphasage non-minimal par inversion partielle de dynamique;
- Stabilisation des systèmes en présence des retards sur les entrées par réduction.

Sommario della tesi

Il presente lavoro di tesi è incentrato sulla stabilizzazione di sistemi non lineari in un contesto campionato. Nel contesto dell'era digitale, l'uso pervasivo di controllori e strumenti di sensoristica digitali definiscono un sistema eterogeneo caratterizzato da dinamiche sia a tempo continuo che discreto. Inoltre, tali dispositivi inducono necessariamente dei ritardi dovuti al tempo necessario alla definizione e alla trasmissione delle azioni di intervento e all'acquisizione delle misure utili ai fini controllistici. Questo fenomeno di digitalizzazione ha motivato, negli ultimi due decenni, un rinnovato interesse da parte della comunità scientifica nello studio dei sistemi digitali introducendo nuove sfide e domande di tipo metodologico e pratico.

Nonostante l'interesse, le difficoltà legate al contesto hanno fatto sì che spesso l'effetto del campionamento sia ignorato in fase di progettazione. La legge di controllo è quindi definita in un contesto idealmente continuo e poi implementata tramite i dispositivi digitali (emulazione). Le specifiche soddisfatte dal controllore ideale (tempo continuo) non saranno quindi preservato nel contesto reale (campionato) se non che per periodi di campionamento sufficientemente piccoli.

L'attività di dottorato si inquadra quindi in questo scenario nel tentativo di proporre un piccolo corpo di metodi che permettano lo studio e il controllo di sistemi a tempo campionato e, in seconda istanza, affetti da ritardi. In dettaglio, si considereranno di sistemi a tempo continuo a uscite campionate e il cui controllo è attuato tramite dispositivi di tenuta di ordine zero. Inoltre, lo studio sarà focalizzato nel caso in cui i dispositivi di campionamento e tenuta siano sincroni e caratterizzati da un periodo di campionamento costante nel tempo. l'obiettivo preposto è quindi quello di definire metodologie costruttive per il controllo di sistemi campionati che sfruttino sia le proprietà originali del sistema a tempo continuo che quelle indotte dal processo del campionamento. Inoltre, tali metodologie saranno poi estese per includere gli stati nei segnali di ingresso e negli stati del sistema stesso.

In questo spirito si studierà l'effetto del campionamento sul processo da controllare nel tentativo di sfruttarne la natura tempo continuo e delle proprietà nominali che questo soddisfa. In questo contesto, le difficoltà principali sono legate alla perdita di struttura del sistema, all'assenza di un apparato geometrico che permetta di descriverne le evoluzioni e la non linearità delle relazioni che definiscono le leggi di controllo. Per questi motivi, si introdurranno studieranno soluzioni approssimate che tengano conto della natura campionata del sistema e che, quindi, migliorino le prestazioni del sistema ad anello chiuso rispetto alle leggi di controllo emulate. Lo studio è effettuato in prima intenzione su classi di su sistemi in cascata (di tipo triangolare superiore e inferiore) proponendo strategie di controllo digitali stabilizzanti e che non richiedano ulteriori assunzioni se non quelle definite sul sistema ideale a tempo continuo. Le strategie di controllo si basano su tecniche di invarianza sotto

feedback, passivizzazione e stabilizzazione alla Lyapunov sfruttando il modo in cui il campionamento trasforma le cascate.

In seconda istanza, si studieranno sistemi a dati campionati e in presenza di ritardi dimostrando come il contesto digitale semplifichi notevolmente la progettazione rispetto la controparte puramente continuo a causa di una forma in cascata implicitamente indotta dal campionamento. Si proporranno quindi metodologie di controllo generali e costruttive che permettano la compensazione dei ritardi estendendo a questo contesto i concetti di Predizione, Immersion and Invariance e Riduzione (à la Artstein). Si considereranno ritardi sugli ingressi (anche di ampiezza diversa su ciascun canale) e, per classi di dinamiche, sugli stati. Inoltre, si sono considerati anche problemi di controllo per sistemi puramente a tempo discreto e a tempo continuo sia in contesto di mera stabilizzazione che in presenza di ritardi.

Nel caso di sistemi a tempo discreto si è dimostrato come un?opportuna combinazione di argomenti di passività e di stabilizzazione alla Lyapunov possano essere impiegati per la progettazione di controllori (limitati) per sistemi in forma triangolare superiore superando il problema dovuto alla generale non linearità delle relazioni che definiscono il controllo. Inoltre, si è dimostrato che queste tecniche sono applicabili in modo del tutto naturale alla stabilizzazione di sistemi a tempo discreto con ingresso ritardati.

Nel contesto di sistemi a tempo continuo, è stata proposta una nuova metodologia di stabilizzazione per sistemi non a minimo di fase tramite inversione parziale della dinamica con applicazione al caso della linearizzazione sotto feedback. Infine, nel contesto di sistemi ritardati, si è esteso il metodo di riduzione alla Artstein in un contesto puramente non lineare.

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Symbols

δ	The sampling period
τ	The delay length

Acronyms

GS	Globally Stable
GAS	Globally Asymptotically Stable
pGAS	Practically Globally Asymptotically Stable
S-GAS	Sampled-Data Globally Asymptotically Stable
LAS	Locally Asymptotically Stable
S-LAS	Sampled-Data Locally Asymptotically Stable
LES	Locally Exponentially Stable
S-LES	Sampled-Data Locally Exponentially Stable
GES	Globally Exponentially Stable
S-GES	Sampled-Data Globally Exponentially Stable
pGES	Practically Globally Exponentially Stable
ISS	Input-to-State Stable
ZSD	Zero-State Detectability
LHS	Left Hand Side
RHS	Right Hand Side
MASP	Maximum Allowable Sampling Period
PBC	Passivity-Based Control
u-PBC	u-Average Passivity-Based Control
I&I	Immersion and Invariance
ILM	Input-Lyapunov Matching
PISM	Partial Input-State Matching
ZOH	Zero-Order Holder

GHF Generalized Holding Function

PDE Partial Differential Equation

ODE Ordinary Differential Equation

List of Notations

$\mathbb{N}$ The set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$.

$\mathbb{N}_{>0}$ The set of natural numbers not including the 0.

$\mathbb{R}$ The set of real numbers.

$\mathbb{R}_{\geq 0}$ The set of non-negative real numbers.

$\mathbb{R}_{>0}$ The set of positive real numbers.

$\mathbb{C}$ The set of complex numbers.

$\mathbb{C}^+$ The set of complex numbers with positive real part (i.e., on the RHS of the complex plane).

$\mathbb{C}^-$ The set of complex numbers with negative real part (i.e., on the LHS of the complex plane).

$\mathbb{C}^0$ The set of complex numbers with zero real part (i.e., over the imaginary axis of the complex plane).

$\circ$ Composition of functions or operators (depending on the context).

$Mat_{\mathbb{R}}(n, m)$ the space of the $n \times m$ dimensional and real-valued matrices.

I the *Identity Matrix*.

Id the *Identity Function*.

$\mathbf{0}_{n\times m}$ the $n \times m$ real-valued matrix whose entries are all zeros.

$\mathbb{1}$ the vector whose entries are all ones.

$\sigma(A)$ the *Spectrum* of a matrix $A \in Mat_{\mathbb{R}}(n, n)$.

$\|\cdot\|$ The *Euclidean Norm*.

$\|x\|_{\mathcal{A}}$ The *distance from a closed set* as $\|x\|_{\mathcal{A}} = \inf_{z\in\dashv} \|x - z\|$.

$\mathcal{B}_r(c)$ a ball centered in c and of radius $r \in \mathbb{R}_{>0}$.

$\mathcal{K}$ the class of all strictly increasing continuous functions $\alpha : [0, a) \to [0, \infty)$ such that $\alpha(0) = 0$.

$\mathcal{K}_\infty$ the class of all $\mathcal{K}$ functions such that $a = \infty$ and $\lim_{s\to\infty} \alpha(s) = \infty$.

$\mathcal{L}$ the class of all strictly decreasing continuous functions $\gamma : [0, \infty) \to [0, \infty)$ so that $\lim_{r\to\infty} \gamma(0) = 0$.

$\mathcal{KL}$ the class of all continuous functions $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ such that for any fixed $\bar{r}$ then $\beta(\bar{r}, \cdot) \in \mathcal{K}$ and for any fixed $\bar{s}$ then $\beta(\cdot, \bar{s}) \in \mathcal{L}$.

$\mathcal{C}^\ell$ all continuous functions $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that its partial derivatives up to the ℓ^{th}-order exist and are continuous.

$\mathcal{C}^\infty$ the class of continuously differentiable (i.e., *smooth*) functions.

$O(\delta^p)$ all functions $R : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^m$ such that for any fixed $\delta > 0$ $R(x, \delta) = \delta^{p-1}\bar{R}(x, \delta)$ and there exists $\theta \in \mathcal{K}_\infty$ and $\delta^* > 0$ such that for each $\delta \leq \delta^*$ $\|\bar{R}(x, \delta)\| \leq \theta(\delta)$.

M_U^I the space of measurable and locally bounded functions $u : I \to U$ with $U \subseteq \mathbb{R}$ with $I \subset \mathbb{R}$.

M_U the space of measurable and locally bounded functions $u : \mathbb{R}_{\geq 0} \to U$ with $U \subseteq \mathbb{R}$.

$\nabla_{x_i}\lambda(x)$ the partial derivative of a smooth function $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the i^{th} component; i.e., $\nabla_{x_i}\lambda(x) = \frac{\partial\lambda}{\partial x_i}$.

$\nabla\lambda(x)$ $\lambda : \mathbb{R}^n \to \mathbb{R}^m$; i.e., $\nabla\lambda(x) = [\nabla_{x_1}\lambda(x) \ \ldots \ \nabla_{x_n}\lambda(x)]$.

$\mathcal{U}^\delta$ the set of piecewise constant functions over time intervals of length $\delta \in \mathbb{R}_{\geq 0}$ $\mathcal{U}^\delta = \{u \in M_U \ : \ u(t) = u(k\delta) \text{ for } t \in [k\delta, (k+1)\delta[\}$.

$\mathrm{L}_f\lambda(x)$ the *Lie Derivative* of a mapping $\lambda : \mathbb{R}^n \to \mathbb{R}$ along a vector field; i.e., $f : \mathbb{R}^n \to \mathbb{R}^n$ $\mathrm{L}_f\lambda(x) = \nabla\lambda(x)f(x)$.

$\mathrm{L}_f^\ell\lambda(x)$ the ℓ^{th}-order Lie Derivative $\mathrm{L}_f^\ell\lambda(x) = \mathrm{L}_f\mathrm{L}_f^{\ell-1}\lambda(x)$ and $\mathrm{L}_f^0\lambda(x) = \lambda(x)$.

$ad_f g$ the *Lie Bracket* among two vector fields f and g; i.e., $ad_f g = [f, g] = \nabla g f - \nabla f g$. $ad_f^\ell g$ the ℓ^{th}-order Lie Bracket $ad_f^\ell g = ad_f ad_f^{\ell-1} g$.

$e^{t\mathrm{L}_f}\mathrm{Id}$ the *Lie Exponential Operator* $e^{t\mathrm{L}_f}\mathrm{Id} = \mathrm{Id} + \sum_{i>0} \frac{t^i}{i!}\mathrm{L}_f^i\mathrm{Id}$.

$e^{t\mathrm{L}_f}x$ $e^{t\mathrm{L}_f}x = e^{t\mathrm{L}_f}\mathrm{Id}\big|_x$.

$e^{t\mathrm{L}_f}h(x)$ $e^{t\mathrm{L}_f}h(x) = h(e^{t\mathrm{L}_f}x)$.

$\Delta_k S(x)$ the *increment* of a function $S : \mathbb{R}^n \to \mathbb{R}$ along $x_{k+1} = F(x_k, u_k)$; i.e., $\Delta_k S(x) := S(x_{k+1}) - S(x_k)$.

List of Publications

Journal Articles

J.7 M. Mattioni, S. Monaco and D. Normand-Cyrot. *Forwarding stabilization in discrete time*. Submitted to Automatica. May, 2017.

J.6 M. Mattioni, S. Monaco and D. Normand-Cyrot. *Feedforwarding under sampling*. Submitted to IEEE Transactions on Automatic Control. September, 2017.

J.5 M. Mattioni, S. Monaco and D. Normand-Cyrot. *Reduction of discrete-time two-channel delayed systems*. IEEE Control System Letters (L-CSS). To Appear, 2018.

J.4 M. Mattioni, S. Monaco and D. Normand-Cyrot. *Nonlinear discrete-time systems with delayed control: a reduction*. Systems and Control Letters. Volume 114 (C). pp. 31-37. April, 2018.

J.3 M. Mattioni, S. Monaco and D. Normand-Cyrot. *Sampled-data reduction of nonlinear input-delayed dynamics*. IEEE Control System Letters (L-CSS). Volume 1 (1), pp. 116-121. July, 2017.

J.2 S. Monaco, D. Normand-Cyrot and M. Mattioni. *Sampled-data stabilization of nonlinear dynamics with input delays through Immersion and Invariance*. IEEE Transactions on Automatic Control, vol. 62 (5), pp. 2561-2567. May, 2017.

J.1 M. Mattioni, S. Monaco and D. Normand-Cyrot. *Immersion and Invariance stabilization of strict-feedback dynamics under sampling*. Automatica. Volume 76, pp. 78-86. February, 2017.

Book Chapters

B.1 S. Monaco, D. Normand-Cyrot and M. Mattioni. *Nonlinear Sampled-data stabilization with delays*. Accepted to *Delays and Interconnections: Methodology, Algorithms and Applications*, Series Advances on Delays and Dynamics at Springer. Editors: G. Valmorbida, R. Sipahi, I. Boussaada and A. Seuret.

Publications in Proceedings

C.8 M. Mattioni, S. Monaco and D. Normand-Cyrot. *Reduction-based stabilization of time-delay nonlinear dynamics*. Submitted to 57th IEEE Conference on Decision and Control (CDC). March, 2018.

C.7 M. Mattioni, S. Monaco and D. Normand-Cyrot. *Lyapunov stabilization of discrete-time feedforward dynamics.* 56th IEEE Conference on Decision and Control (CDC), pp. 4272-4277. Melbourne, Australia. 2017.

C.6 M. Mattioni, M. Hassan, S. Monaco and D. Normand-Cyrot. *On partially minimum phase systems and nonlinear sampled-data control.* 56th IEEE Conference on Decision and Control (CDC), pp. 6101-6106. Melbourne, Australia. 2017.

C.5 M. Mattioni, S. Monaco and D. Normand-Cyrot. *Further results on sampled-data stabilization of time-delay systems.* 20th IFAC World Congress. IFAC-PapersOnLine. Volume 50, Issue 1, pp. 14350–14355. Toulouse, France.

C.4 M. Mattioni, S. Monaco and D. Normand-Cyrot. *Sampled-data stabilization of feedforward dynamics with Lyapunov cross-term.* 55th IEEE Conference on Decision and Control (CDC). pp. 1322-1327. Las Vegas, USA. 2016.

C.3 S. Monaco, D. Normand-Cyrot and M. Mattioni. *Stabilisation of feedforward discrete-time dynamics through Immersion and Invariance.* American Control Conference (ACC). pp. 264-269. Boston, USA. 2016.

C.2 M. Mattioni, S. Monaco and D. Normand-Cyrot. *Sampled-data stabilization of a class of state-delayed nonlinear dynamics.* 54th IEEE Conference on Decision and Control (CDC). pp. 5695-5700. Osaka, Japan. 2015.

C.1 M. Mattioni, S. Monaco and D. Normand-Cyrot. *Digital Stabilization of strict-feedback dynamics through Immersion and Invariance.* 1st IFAC Conference on Modeling, Identification and Control of Nonlinear Systems (MICNON). Volume 48, Issue 11, pp. 1074-1079. St. Petersburg, Russia. 2015.

Communications in Conferences without Proceedings

W.3 M. Mattioni, S. Monaco and D. Normand-Cyrot. *Forwarding of sampled-data nonlinear systems. Automatica.it 2017.* September, 2017. Milan (Italy).

W.2 M. Mattioni, S. Monaco and D. Normand-Cyrot. *Stabilisation of input-delayed systems via Immersion and Invariance. Automatica.it 2016.* September, 2016. Rome (Italy).

W.1 S. Monaco, D. Normand-Cyrot and M. Mattioni. *Sampled-data stabilization of nonlinear dynamics with input delays through Immersion and Invariance.* 4th DelSys Workshop on Observers and Controllers for Complex Dynamical Systems. 25-27 November 2015. CentraleSupèlec. (France).

General Introduction

THIS manuscript represents a *mémoire* of the activity I have been carrying out during the last three years within' my PhD which has been developed in a joint program between the *Laboratoire des Signaux et Systèmes* (L2S) at *Université Paris-Saclay* and the *Dipartimento di Ingegneria Informatica e Automatica* (DIAG) at *Università degli Studi di Roma La Sapienza*. The mobility of my thesis has been partially funded by *Université Franco-Italienne/Università Italo-Francese* through the Vinci Grant 2016.

The goal of the research activity has been concerning issues arising with sampled-data control of nonlinear continuous-time systems and, beyond them, discrete and continuous-time systems at large. In doing so, we have faced problems also related to retarded systems in the sampled-data discrete-time and continuous-time scenarios.

In this context, this manuscript focuses toward the stabilizing design of sampled-data feedback laws for continuous-time systems admitting a cascade structure. In this sense, we provide constructive methodologies for the definition of the sampled-data feedbacks in the attempt of preserving the required specifications over the continuous-time systems despite the possible loss of the structure and properties due to the sampling process. Then, time-delay systems will serve as an applicative benchmark example as they indeed admit, under sampling, a cascade interconnection structure.

The general context of the thesis

Nonlinear system theory represents a powerful and essential tool for dealing with the innovations the world is constantly facing to. As a matter of fact, this *hidden technology* represents a fundamental instrument for describing and intervening on several situations spanning from different fields and disciplines. As time spends by, the need of developing new technologies has required a huge effort in establishing new and general methodologies properly serving those practical issues. In the control community, this has motivated a continuously evolving research line toward nonlinear systems since the early 70s when pioneering research by Lobry, Sussman, Jurdjevic, Krener, Bruni, Di Pillo, Koch, Ruberti, Brockett, Fliess, Kokotovic, Sontag, Isidori, Monaco and Sastry began to build up new mathematical frameworks based on the differential geometry and algebraic formalisms as in their first works in [90, 16, 177, 77, 17, 29, 76, 38, 174, 57, 58, 168]. Motivated by electro-mechanical applications, those works were mainly devoted to the case of continuous-time systems evolving continuously in time and being fed by continuous control signals. Then, research on these topics has gone on in the attempt to answer to the larger variety of problematics that control theory can answer to so spanning from engineering applications (e.g., robotics, mechatronics, telecommunications) to completely different scenarios such as biology, chemistry, medicine but even economy or politics (see, for instance, [150, 46, 138, 4, 41, 43]). To this end, new tools have been developed by introducing new formalisms and notions that have found a prolific applicability in this wide range of *complex systems*. Among these, hybrid, embedded or networked systems represent some of the most diffused paradigms trying to answer to these demands [48, 171, 13].

This thesis is contextualized in this framework within' the attempt of developing a small set of methodological tools to deal with the *digitalization* era we are currently living in with particular emphasis on sampled-data and discrete-time control systems. By sampled-data system, one describes a *computer-controlled* plant (or, in a more recent jargon, *cyber-physical system*) in the sense of continuously evolving physical dynamics whose informations are sporadically available over time and that are fed by piecewise constant input signals switching at certain time instants only. This framework is quite general as it models most of the practical situations one might encounter in the aforementioned scenarios. Indeed, every modern system is usually monitored and controlled through digital devices working at certain fixed frequencies. Accordingly, this hybrid nature of the system induces the need of developing ad hoc tools allowing to suitably describe and characterize the overall dynamics so to, then, perform control design to fulfill certain specifications.

In this sense, pioneering and important works have been proposed since the early 80s by Monaco and Normand-Cyrot in the attempt to fill the gap among the increasing amount of tools available for purely continuous-time systems and the almost on existing ones for nonlinear sampled-data dynamics with particular attention to purely discrete-time systems as well. Specifically to the sampling context, the following questions have been posed for the first time to the control community.

How to describe the behavior of a sampled-data system according to variations of the input signals?

What can one say about properties at large (e.g., structural or stability ones) of a given open-loop or closed-loop system under sampling?

How to preserve or recover the performances and the properties of the continuous-time control system?

What about the behavior of the closed-loop dynamics during the sampling period (i.e., when the control is kept constant and is not changing)?

Those difficulties give raise to increasing sources of complexity in the nonlinear setting. The works by Monaco and Normand-Cyrot have been focusing on both analysis and control design aspects by stressing on the effect of the sampling process over the continuous-time original dynamics both in terms of pros and cons. More in details, they have been introduced the so-called sampled-data equivalent model approach which is based on a discrete-time equivalent mathematical model describing the evolutions of the continuous-time states with respect to variations of the input. Accordingly, they have developed a body of instruments allowing to pursue analysis and design of sampled-data systems which take into account both the continuous and discrete time nature of the overall dynamics so giving a first push toward the gap bridging with existing methodologies in continuous time (e.g., [117, 118, 119, 120, 121, 32, 122, 123]). In this sense, when dealing with sampled-data dynamics several issues arise from the discrete-time intrinsic nature of the plant. With reference to control design, those problems are mainly related to the following aspects that are indeed shared with discrete-time dynamics:

- the discrete-time dynamics describing the sampled-data system does not preserve the same structure as the continuous-time model;
- the discrete-time dynamics describing the sampled-data system does not preserve the same properties the continuous-time model;
- the discrete-time dynamics describing the sampled-data system does not preserve the geometric-differential structure underlying the evolutions of the corresponding system;
- the discrete-time dynamics describing the sampled-data system is nonlinear in the control variable;
- the equations defining the control solutions are generally highly nonlinear.

After a transient silent period lasted almost twenty years, sampled-data systems have found a renewed interest by the control community throughout the last few decades motivated by the current technological developments. As a consequence, the aforementioned issues and questions have been re-addressed by several researchers in the field. Accordingly, an extended body of methodologies has been developed for sampled-data control so that the following rough classification can be deduced depending on the entry point one adopts for the definition of the control feedback.

Emulation-based design – The design is carried out over the continuous-time original system by neglecting the effect of sampling so that the overall feedback is directly implemented through Zero-Order-Hold (ZOH) devices. Such an approach is rather naive and requires no extra design effort though it does not keep into account the effect of sampling (both in the measures and the input signal) over the closed-loop system. Accordingly, the same performances as in the continuous-time case are not preserved as the sampling period increases so preventing from fulfilling the control specifications the ideal continuous-time feedback was design to satisfy. Several works are aimed at quantifying estimates of the sampling period preserving the performances of the continuous-time systems and, thus, the quality of emulation-based feedbacks as in the works by Carnevale, Hetel, Mazenc, Nesic Omran and many others (e.g., [143, 110, 147, 146, 157]) even in the case of aperiodic sampling.

Continuous-time redesign – Starting from the original continuous-time model, one first deduces a modified continuous-time plant taking into account the sampled-data nature of the system. The design is then carried out toward the definition of a continuous-time feedback law over this new model so to compensate the effect of sampling. Finally, the actual controller is implemented through ZOH devices and based on sampled measures. In this sense, rather than *discretizing* the plant one performs a sort of discretization over the continuous-time feedback so to compensate the effect of sampling and holding devices. The first results on this methodologies are due to Anderson, Comeau and Sastry [54, 72, 26, 11] though a more recent approach in this sense consists deducing the modified continuous-time system by looking at the sampled-data system as a continuous-time retarded system as developed, among others, by E. Fridman, Hetel, Richard (e.g., [47, 44, 45, 51]). Emulation-based control might be also considered as a trivialization of this family of approaches.

Direct discrete-time control – One first deduces a suitable discrete-time model describing the evolutions of the sampled-data system at each sampling instant. Then, the control design is performed over this new discrete-time model as if it were a completely discrete-time dynamics and, thus, discarding its original continuous-time nature. In this case, two sources of difficulties arise so that the design effort might be demanding. First, the discrete-time equivalent model might not be computable in closed form so that approximations are necessary. Then, one might face the lack of a suitable and sustained discrete-time design procedure to deal with the resulting system which might be general even if coming from continuous-time particular structures (as in the case of lower-triangular structures). Most of the works on this topics usually deal with the so-called Euler approximate models of the discrete-time equivalent model which indeed preserves the same mathematical structure as in continuous time as made in [160]. Accordingly, several investigations have been addressing the consistency property of the evolutions of the approximate discrete-time model with respect to the sampled evolutions of the continuous-time original one to quantify the confidence one might rely on it. Important works in this sense have been developed by Astolfi, Kokotovic, Laila, Nesic, Teel and co-workers (e.g., in [144, 140, 141, 84, 83]).

Indirect sampled-data control –The control is designed over a sampled-data equivalent model which is parametrized by the sampling period δ through a power series expansion. Accordingly, the definition of the feedback law is aimed at preserving, through piecewise constant control and at each sampling instant, the target performances of the continuous-time system under an ideal control designed in continuous-time. The feedback is thus deduced as the implicit solution to some nonlinear equality whose existence and uniqueness are generally ensured from the previous continuous-time design. The tools that are generally exploited make reference to the implicit function theorems and the formal series inversion so that the feedback is generally given as a series expansion in powers of δ around the continuous-time solution. Usually, exact solutions to the aforementioned equalities are hard to be computed so that only approximate feedback are implemented in practice by exploiting the general series expansion form of the solution. In these cases, attention should be directed toward the best trade off among computational efforts and required performances. A qualitative study of the properties yielded under approximate feedback have been developed in [179, 103, 86] with reference to practical and input-to-state stability arguments and multi-step consistency. Matching-based controllers belong to this family of design methods as developed by Monaco, Normand-Cyrot and co-workers (e.g., in [124, 134, 135]).

Direct sampled-data control – As in the indirect sampled-data control case, the control is designed over a sampled-data equivalent model which is parametrized by the sampling period δ through a power series expansion. Though, in this case, one first translates the continuous-time specifications to be fulfilled into the sampled-data context. Then, the exact sampled-data equivalent model is exploited to perform control design through discrete-time design strategies which take into account the original continuous-time nature of the sampled-data model and the corresponding properties. Thus, this design methodology stays in between direct-discrete time and indirect sampled-data control approaches usually yielding a constructive procedure for deducing the feedback controller both on the discrete and continuous-time nature of the dynamics by investigating on the ways the original properties (e.g., passivity) are transformed by the sampling process onto the discrete-time sampled-data equivalent model. Still, the feedbacks are inferred from highly nonlinear equalities for which a unique solution is guaranteed to exist based on the properties that the continuous-time original plant satisfies. Average passivity based controllers (with respect to the time or the control action) belong to this family as in the works by Monaco, Normand-Cyrot, Stramigliolì and Van der Schaft (e.g., [176, 27, 133]). In the linear case, signal lifting was suitably exploited to deduce infinite-dimensional sampled-data equivalent models which also allow to include informations on the inter-sampling behavior, as developed in several works by Yamamoto and co-workers [188, 189, 190] with special emphasis on the frequency domain.

This thesis is then contextualized in this framework as dealing with the design of sampled-data feedback laws making the origin an asymptotically stable equilibrium for a given continuous-time system through piecewise constant control actions. In doing so, the tools that we have been exploiting thought out the PhD thesis make reference to a suitable combination of direct and indirect sampled-data control strategies which are aimed at preserving, at the same time, the stabilizing properties of an ideal continuous-time feedback law and the properties the sampled-data equivalent model inherits from the original dynamics. We have focused on cascade dynamics as provided by the feedback and forward interconnection of suitable nonlinear systems in an attempt to deduce a constructive way of performing digital design regardless the possible loss the nested structure over the sampled-data equivalent model. In doing so, we have exploited Lyapunov, passivity and invariance arguments for enhancing the feedback control.

As a second step, we have suitably refined the proposed methodologies to deal with retarded nonlinear systems affected by a constant delay over the input channel. This is motivated by the implicit cascade structure induced by the combination of the effects of both the delay and the sampling process. As a matter of fact, it is now recognized that sampling notably solves several issues arising in the design of time-delay systems from two points of views: their cascade interpretation allows to carry out a rather simple and elegant design procedure as shown by Krstic, Karafyllis or other researchers in the field (e.g., [79, 80, 81, 67, 68]); sample-and-hold implementation of the stabilizing feedback through ZOH devices overcomes discontinuity issues one is usually stack with as studied by Pepe in several of his works [153, 158, 154, 155, 20, 156, 157].

In a parallel way, we have tried to go beyond sampling to deal with purely discrete-time systems by providing a general framework for addressing classes of cascade discrete-time dynamics possibly affected by time delays. The design methodologies we have proposed in this context suitably extend to the purely discrete-time context the Lyapunov and passivity-based design methods the continuous-time design extensively exploits. Thus, we have tried to overcome the difficulties arising from the general nonlinearities of the dynamics with respect to the control and the loss of a geometrical structure underlying the discrete-time evolutions. In doing so, we have provided a general framework for stabilizing cascade discrete-time dynamics at large. The discrete-time design methodologies we have proposed are not coinciding, in general, with their sampled-data counterpart practically and by nature.

Finally, some contributions for the stabilization of nonlinear continuous-time retarded systems are proposed based on the results in discrete time and under sampling.

Contributions of the PhD thesis

The contributions of this PhD thesis are summarized below by distinguishing among sampled-data and discrete-time systems and, finally, continuous-time dynamics.

Feedback design under sampling

I&I based stabilization of strict-feedback dynamics under sampling – Starting from the strict-feedback interconnection of nonlinear dynamics, we have shown that Immersion and Invariance (I&I) provides a powerful tool for establishing a constructive and general stabilizing procedure. Because the feedback nested structure is not preserved by sampling, a direct sampled-data design procedure has been proposed for deducing a suitable discrete-time target dynamics and the overall feedback making the corresponding manifold (which is not the same as in continuous time) attractive and invariant. I&I is generally applicable to sampled-data strict-feedforward systems even when backstepping-like procedures are not.

Sampled-data feedforwarding of nonlinear systems – Starting from a continuous-time systems admitting a feedforward structure, we have extended to the sampled-data context the well-known feedforwarding approach proposed by Kokotovic and co-workers in [169]. In doing so, we have provided an iterative procedure based, at each step, on the definition a suitable Lyapunov function and a corresponding passivating output (in an average sense) allowing to perform damping-like feedback control. In general, the general design we have proposed is less demanding that the discrete-time counterpart we have carried out as it takes into account the properties of the original continuous-time dynamics.

Sampled-data stabilization of input-delayed nonlinear systems via prediction– By assuming the ideal delay-free system stabilizable by smooth feedback, we have inferred a sampled-data prediction-based feedback which is fed by a discrete-time predictor dynamics evolving at each sampling instant. In doing so, we have overcome in a very natural way several numerical issues arising with the computability of general prediction-based controllers in continuous-time.

Sampled-data stabilization of input-delayed nonlinear systems via I&I – By rewriting the delay as an eventually non-entire multiple of the sampling period, we have first shown that the retarded system rewrites as an hybrid cascade interconnection of a continuous-time system with a linear discrete-time dynamics modeling the story of the input. By exploiting this structure, we have then proposed a direct sampled-data design approach based on I&I over the extended system. The ideal delay-free system defines the target dynamics so that the final feedback is aimed at making the corresponding manifold attractive and invariant. The I&I control for the retarded dynamics is shown to improve the prediction-based one through a feedback term over the prediction error.

Sampled-data stabilization of a class of state-delayed nonlinear systems via I&I – We have considered a retarded strict-feedback system affected by a constant delay over the interconnecting state fulfilling, in the delay-free continuous-time case, usual backstepping assumptions. Accordingly, we have inferred a suitable extended cascade sampled-data equivalent model for the retarded system over which we have deduced a stabilizing double-rate sampled-data feedback through I&I.

Sampled-data stabilization of input-delayed nonlinear systems via reduction – By rewriting the delay as suitable non-entire multiple of the sampling period, we have first inferred a new state (the reduction state) whose dynamics (the reduced dynamics) is free of delays and of the discrete-time type. Moreover, the inferred reduced dynamics is equivalent, at least in terms of stability, to the original retarded system. Then, we have shown that every feedback stabilizing the reduced dynamics achieves stabilization of the retarded continuous-time dynamics in turn. We have presented several ways of pursuing control design over the reduced model by exploiting some properties of the ideal delay-free system associated to the retarded one. This methodology extends prediction-based feedback and does not require any state augmentation.

Discrete-time control design

I&I-based stabilization of strict-feedforward dynamics in discrete time – When considering discrete-time strict-feedforward dynamics, we have shown how I&I can be profitably exploited as an alternative and less demanding stabilizing tool for this class of systems with respect to existing ones. As a very particular case, this design strategy applies to sampled-data system as well when pursuing a direct discrete-time design approach.

Forwarding of nonlinear dynamics in discrete time – When considering cascade nonlinear discrete-time dynamics, a unifying and constructive procedure for the stabilizing design was still missing. We have proposed a first attempt to achieve feedforwarding stabilization for nonlinear discrete-time cascade systems through a constructive and iterative procedure exploiting Lyapunov stability and u-average passivity arguments. Connections with the I&I procedure are detailed as well when specifying the proposed methodology to strict-feedforward systems. The case of a class of input-delayed dynamics provides an interesting case of application of the method (as an alternative to I&I) to also deduce a Lyapunov function for the extended closed-loop system. Specifying this approach to sampled-data systems might not be possible as the assumptions that are required do not take into account the continuous-time nature of the sampled-data dynamics. Moreover, whenever one might pursue a pure discrete-time forwarding over equivalent sampled-data dynamics, one would infer a feedback that is completely different from the one yielded by the ad-hoc sampled-data procedure.

Stabilization of input-delayed nonlinear systems via reduction – Analogously to the sampled-data case, we have first inferred a new state (the reduction state) whose dynamics (the reduced dynamics) is free of delays and equivalent, at least in terms of stability, to the original retarded system. Then, we have shown that every feedback stabilizing the reduced dynamics achieves stabilization of the retarded dynamics in turn. We have presented several ways of pursuing control design over the reduced model by exploiting some properties of the ideal delay-free system associated to the retarded one as passivity. This procedure is equivalent to the sampled-data one whenever, under sampling, the delay is an exact entire multiple of the sampling period. Whenever this does not hold, the sampled-data case needs to be dealt with the proposed ad hoc methodology.

Stabilization of continuous-time systems

As a minor activity within' the PhD we have also investigated on the control design of continuous-time systems.

Stabilization of input-delayed nonlinear systems via continuous-time reduction – Inspired by the sampled-data and discrete-time design methodologies based on reduction of retarded systems, we have extended the work by Arstein in 1982 [6] to nonlinear continuous-time dynamics in presence of a discrete delay over the input. The result is enforced by the input-affine structure of the dynamics and the general linearities in the control the involved mappings exhibit.

Stabilization of non-minimum phase systems through partial dynamic cancelation– We have investigated on the stabilization of non-minimum phase systems based on the definition of a new suitable output mappings defining a stable component of the original stable dynamics. The problem of achieving input-output feedback linearization is addressed and solved with stability despite the non-minimum phase property through partial dynamic cancelation.

Organization of the Manuscript

This manuscript is organized according three main parts distinguishing, respectively, among recalls on continuous-time and sampled-data systems and stabilization, sampled-data stabilization of cascade systems and sampled-data design for time-delay systems affected by a constant input delay.

Part I

In *Chapter 1*, preliminary and basic notions on continuous-time systems are given in the case of both autonomous and controlled dynamics with emphasis on Lyapunov stability and passivity.

In *Chapter 2*, the sampled-data framework we shall be dealing with is introduced together with the tools that will be extensively used throughout the manuscript. First, single-rate and multi-rate sampled-data equivalent models are given with emphasis on their approximations as truncations of the series expansion they are defined through. The notions of stability under sampled-data feedback are detailed. Then, some recalls on average passivity of sampled-data systems (in the discrete-time sense) are recalled with emphasis on the resulting average passivity-based design. Finally, a reminder on the Input-Lyapunov Matching approach is given.

Part II

In *Chapter 3*, we shall be dealing with the sampled-data stabilization of strict-feedback dynamics. For this purpose, first, the sampled-data equivalent model of such dynamics is introduced by putting in light the way the properties are destroyed and transformed through sampling. Then, a constructive procedure is proposed to construct (in an iterative manner) the sampled-data feedback through the Immersion and Invariance approach. This is made with reference to the double integrator-strict-feedback interconnection while the extension to the general multi-block cascade is sketched as a general procedure. It is shown that, as the sampling period decreases, one recovers the continuous-time feedback. Approximate control solutions are introduced and discussed as well.

In *Chapter 4*, the problem of stabilizing feedforward dynamics under sampling is addressed. To this end, the sampled-data equivalent model of such dynamics is inferred by emphasizing on the corresponding structure preservation. Then, with reference to a two block cascade, the design is pursued through an iterative procedure which is aimed at computing, first, a Lyapunov function for the dynamics under a partial feedback. The constructed Lyapunov function allows to deduce u-average passivity properties of the extended controlled dynamics so that stabilization is finally achieved through damping-like feedback over the average passivating output. The overall feedback recovers the continuous-time one as the sampling period decreases. The procedure is then extended to the general case through an iterative algorithm. When specified to strict-feedback systems, this approach implicitly recovers the one of making, at each step of the design, a suitable stable manifold attractive and invariant. Approximate control solutions are introduced and discussed as well.

Part III

In *Chapter 5*, time-delay systems are introduced by emphasizing on the implicit underlying structure both in the continuous-time and sampled-data scenarios. Then, continuous-time predictor-based feedbacks are recalled together with the corresponding issues they arise with. Finally, a sampled-data prediction-based controller is proposed by defining the predictor-dynamics as a discrete-time system evolving at each sampling instant. In this mixed time-delay and sampling scenario, some of the continuous-time problems find a natural solution as the one of the definition of approximate predictor-based controllers.

In *Chapter 6*, the extended cascade structure underlying the evolutions of the sampled-data retarded dynamics is exploited to deduce an I&I feedback over the corresponding discrete-time dynamics. By assuming stabilizability of the continuous-time ideal delay-free system, we first deduce a sampled-data stabilizing feedback over the delay-free sampled-data equivalent model through Input-Lyapunov Matching. Accordingly, I&I naturally intervenes under delay by setting the delay-free system under the stabilizing feedback as the target dynamics. The overall feedback can be split into two components: a mere prediction of the delay-free sampled-data feedback plus a feedback loop over the prediction error so enforcing robustness with respect to neglected and approximate higher order dynamics in power of δ. Moreover, in a very natural way, as the delay decreases, one recovers the delay-free feedback.

Some open perspectives and open works on these topics conclude the main body of the manuscript.

Appendix A finally contains all the works which were not included in the writing on this manuscript. The choice of omitting them was dictated by the will of focusing on sampled-data control design for cascade systems with emphasis on the following aspects:

- the preservation of the continuous-time properties over the sampled-data equivalent model;
- the preservation and translation of the continuous-time specifications over the sampled-data equivalent model and the corresponding feedback design;
- the involvement of cascade systems in the sampled-data stabilization of nonlinear retarded system.

Any chapter is compactly organized though a brief abstract stating the control problem we are addressing and, at the end, some concluding remarks and connections with the current state of the art on the same topics. For the sake of clarity, the case of the double integrator is dealt with to fix the main idea of the design procedure we provide with reference to approximate solutions. Then, a simple academic example is carried out to put in light computational aspects with respect to approximate solutions.

Part I

Basic tools for nonlinear systems: from continuous time to sampling

Chapter 1

Continuous-time nonlinear systems

In this chapter, some recalls and preliminary results will be given for continuous-time systems by emphasing on geometrical and physical properties that will be then exploited throughout the thesis together with basical notions regarding control design. The notions appearing hereinafter are recalled from [74, 8, 170, 149, 56]

In what follows, we are considering a continuous-time controlled system in the input-affine form

$$\dot{x} = f(x) + g(x)u \tag{1.1a}$$
$$y = h(x) \tag{1.1b}$$

with $x \in \mathbb{R}^n$ and $u \in M_U$, $y \in \mathbb{R}$ and where we are letting $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ be smooth mappings. We shall also assume that the dynamics (1.1a) is *forward complete*; i.e., its solutions exist at each time $t \geq 0$ for all $x_0 = x(0) \in \mathbb{R}^n$ and $u \in M_U$. We shall refer to the dynamics (1.1a) as time-invariant as they are not explicitly depending on time. Finally, we are saying that the couple $(x_e, 0) \in \mathbb{R}^n \times \mathbb{R}$ is an equilibrium of the system (1.1) if $f(x_e) = 0$ and $h(x_e) = 0$. Throughout the whole manuscript, as usual and without loss of generality, we are assuming that $x_e = 0$.

In the next few sections, we are first defining usual stability and geometrical properties [74, 56] of the uncontrolled dynamics associated to (1.1a) (i.e., when $u \equiv 0$). Then, some insights on the general properties (1.1) might verify are given together with control design tools.

1.1 Uncontrolled dynamical systems

In this part, we shall consider the uncontrolled system associated to (1.1a) for $u \equiv 0$; i.e.,

$$\dot{x} = f(x) \tag{1.2a}$$
$$y = h(x) \tag{1.2b}$$

with $f(0) = 0$, $h(0)$ and f being *locally Lipschitz* in the sense that it verifies the following inequality

$$\|f(x) - f(y)\| \leq L\|x - y\|, \quad L \in \mathbb{R}_{\geq 0} \tag{1.3}$$

for all x, y lying over a neighborhood of x_0.

1.1.1 Lyapunov Stability

First, let us define the stability properties of the origin of the dynamics (1.2a) and provide conditions allowing to deduce them based on Lyapunov functions and the corresponding extensions. When no confusion arises and unless differently specified, all the defined properties are meant to hold *locally.*

Definition 1.1. [Lyapunov stability] The equilibrium $x = 0$ of (1.2a) is

- *stable* if, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x_0\| < \delta$ implies $\|x(t)\| < \varepsilon$ for all $t \geq 0$;
- *unstable* if it is not stable;
- *attractive* if there exists a $\delta > 0$ such that $\|x_0\| < \delta$ implies $\lim_{t\to\infty} \|x(t)\| = 0$;
- *asymptotically stable* if it is stable and attractive;
- *exponentially stable* if it is asymptotically stable and there exist constant $c, \alpha > 0$ such that $\|x(t)\| \leq ce^{-\alpha t}\|x_0\|$.

If the above properties yield for all initial conditions $x_0 \in \mathbb{R}^n$, then they hold *globally.*

Remark 1.1. *When (1.2a) is linear (i.e., $f(x) = Ax$ with $A \in Mat_{\mathbb{R}}(n, n)$) stability of the origin implies stability of every other equilibrium $x_e \in \ker A$ and, thus, one might refer to stability of the system itself.*

Before stating the Lyapunov Theorem, let us introduce the concepts of positive and negative (semi-definiteness) functions and candidate Lyapunov function.

Definition 1.2. A function $V : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ such that $V(0) = 0$ is

- *positive semi-definite* if $V(x) \geq 0$ for all $x \in D$;
- *positive definite* if $V(x) > 0$ for all $x \in D/\{0\}$;
- *negative semi-definite* if $\mathcal{V}(x) := -V(x)$ is positive semi-definite;
- *negative definite* if $\mathcal{V}(x) := -V(x)$ is positive definite.

Definition 1.3. A function $V : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ such that $V(0) = 0$ is said to be *radially unbounded* if $\lim_{\|x\|\to\infty} V(x) = \infty$.

Remark 1.2. *The properties of positive/negative (semi-)definiteness apply to matrices* $P \in Mat_{\mathbb{R}}(n,n)$ *when specifying them to the corresponding quadratic form* $V(x) = x^\top P x$.

Definition 1.4. [Candidate Lyapunov Functions] We say that a continuously differentiable function $V : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ is a *candidate Lyapunov function* for (1.2a) if it is positive definite.

Remark 1.3. *Requiring the function* $V : D \to \mathbb{R}$ *to be continuously differentiable is not necessary and might be strongly weakened to embed much larger classes of functions as detailed in [15, 191]. The choice of assuming them continuously differentiable over the concerned domain is linked to the sampled-data context that will be discussed and investigated in the following.*

Definition 1.5. [Lyapunov functions] We say that a candidate Lyapunov function $V : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ is

- a (*weak*, [93]) *Lyapunov function* for (1.2a) if its derivative along (1.2a) is negative semidefinite, i.e., $\dot{V}(x) = \mathrm{L}_f V(x) \leq -W(x)$ with $W(x) \geq 0$;
- a *strict Lyapunov function* for (1.2a) if its derivative along (1.2a) is negative definite, i.e., $\dot{V}(x) = \mathrm{L}_f V(x) \leq -W(x)$ with $W(x) > 0$ for all $x \neq 0$.

Remark 1.4. *Given a candidate Lyapunov function one can always find class* $\mathcal{K}$ *functions* α_1 *and* α_2 *defined over* $[0, r)$ *such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \tag{1.4}$$

for all $x \in B_r(0) \subseteq D$ *and* $r > 0$. *If, moreover,* $D \equiv \mathbb{R}^n$*, then* $r = \infty$. *Finally, if* V *is radially unbounded then* α_1 *and* α_2 *can be chosen so to belong to class* $\mathcal{K}_\infty$.

Theorem 1.1 (Lyapunov criterion)**.** *Let* $x = 0$ *be an equilibrium of (1.2a) and* $D \subseteq \mathbb{R}^n$ *be a domain cointaining the origin. Let* $V : D \to \mathbb{R}^n$ *be a candidate Lyapunov function then, the following implications hold true:*

- *if* $V(x)$ *is a weak Lyapunov function, then the origin is stable;*
- *if* $V(x)$ *is a strict Lyapunov function, the origin is asymptotically stable;*
- *if* $V(x)$ *is a strict Lyapunov function and there exist constants* $a_i > 0$ $(i = 1, 2, 3)$ *such that*

$$a_1\|x\|^2 \leq V(x) \leq a_2\|x\|^2 \tag{1.5}$$
$$\dot{V}(x) \leq -a_3\|x\|^2. \tag{1.6}$$

the origin is exponentially stable.

Moreover, if $D \equiv \mathbb{R}^n$ *and* $V(x)$ *is radially unbounded, all the above properties hold globally.*

The above statement only provides sufficient conditions for proving stability. In addition, seeking for a Lyapunov function proving stability might be a tough task. Nevertheless, by exploiting some intrinsic (possibly physical) properties of the dynamics, one might construct a weak Lyapunov function and still hope for stronger properties to hold (e.g., GAS). For this purpose, the following statements are useful.

Theorem 1.2 (Barbashin and Krasovskii, 1959)**.** *Let $x = 0$ be an equilibrium for (1.2a) with weak Lyapunov function $V : D \to \mathbb{R}$ over a domain D. Let $S = \{x \in D \text{ s. t. } \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S other than the trivial $x(t) \equiv 0$. Then, the origin is asymptotically stable. If $D \equiv \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$ is radially unbounded, than the origin is globally asymptotically stable.*

A few months later the diffusion of the above statement, J.P. LaSalle got to prove its own *invariance principle* that can be seen as a generalization of the one by Krasovskii.

Definition 1.6. [LaSalle's Invariance Principle, 1960] Let $\Omega \subset D$ be a compact set that is positively invariant set [1] with respect to (1.2a). Let $V : D \to \mathbb{R}$ be a weak Lyapunov function such that $\dot{V}(x) \leq 0$ in Ω. Let $E = \{x \in \Omega \text{ s. t. } \dot{V}(x) = 0\}$ and M be the largest invariant set contained in E. Then, every solution starting in Ω approaches to M as $t \to \infty$.

When dealing with weak Lyapunov functions, the above theorems are of remarkable importance as they generally make the computation of a strict Lyapunov function unnecessary for studying asymptotic properties. Though, strict Lyapunov functions might still be needed in some case (e.g., for robustness issues). In that case, those results generally allow one to pursue *strictification* of the available weak Lyapunov function [93].

For completeness, the definition of boundedness of solutions of (1.2a) is recalled here.

Definition 1.7. [Boundedness of solutions] The solutions of (1.2a) are *bounded* if there exists a positive constant c such that for all $a \in]0, c[$, there is a $\beta = \beta(a) > 0$ such that $\|x_0\| \leq a$ implies $\|x(t)\| \leq \beta$ for all $t \geq 0$. If the above implication holds for every arbitrary $c > 0$ then the solutions of (1.2a) are *globally* bounded.

1.1.2 Stability through the linear approximation and the center manifold

Although Lyapunov theory represents an invaluable tool for studying stability properties one might also deduce local stability properties, under certain assumptions, by looking at the linear approximation of (1.2a) about the origin. This approach is sometimes referred to as *Lyapunov indirect method.*

[1] A set Ω is said to be *positively invariant* with respect to the dynamics (1.2a) if $x_0 \in M$ implies $x(t) \in M$ for all $t \geq 0$. If the above property holds for all $t \in \mathbb{R}$ then M is said to be *invariant.*

Theorem 1.3 (Lyapunov indirect method)**.** *Let $x = 0$ be an equilibrium of (1.2a) and let $A = \nabla f(0)$ be the dynamical matrix of its linear approximation around the origin. Then,*

- *the origin is asymptotically stable if $\sigma(A) \subset \mathbb{C}^-$;*
- *the origin is unstable if $\sigma(A) \cap \mathbb{C}^+ \neq \emptyset$.*

When A is critically stable (i.e., when all eigenvalues of A are either in the LHS of the complex plane or over the imaginary axis) nothing can be deduced from the linear approximation disregarding the geometric multiplicity of the 0 real part eigenvalues. In that case, one might proceed by studying the stability of the origin conditionally to the so-called *center manifold* [23]. For, let us consider a state partition $x = (x_1^\top \ x_2^\top)^\top$ (possibly after change of coordinates) so that (1.2a) rewrites as

$$\dot{x}_1 = A_1 x_1 + g_1(x_1, x_2) \tag{1.7a}$$
$$\dot{x}_2 = A_2 x_2 + g_2(x_1, x_2) \tag{1.7b}$$

with $\sigma(A_1) \subset \mathbb{C}^0$ and $\sigma(A_2) \subset \mathbb{C}^-$ and g_1, g_2 being at least twice differentiable and such that $\nabla g_i(0,0) = 0$ for $i = 1, 2$. Roughly speaking, based on reduction principles, one projects the original system onto an invariant manifold $\mathcal{C}$ whose tangent space at the origin coincides with the stable eigenspace associated with the critically stable eigenvalues of the matrix A_1 in (1.7a). Accordingly, one deduces a reduced dynamics describing the evolution of the system over $\mathcal{C}$ and with the property that the stability of the corresponding equilibrium reflects the one of the overall system (1.2) [23]. Specifically, the following results can be recalled.

Theorem 1.4 (Center Manifold)**.** *Let the dynamics (1.2) take the form (1.7) with $\sigma(A_1) \subset \mathbb{C}^0$, $\sigma(A_2) \subset \mathbb{C}^-$ and g_1, g_2 being at least twice differentiable. Then, there exist a constant $\delta > 0$ and a continuously differentiable function $\varphi(x_1)$ such that*

$$\varphi(0) = 0 \quad \textit{and} \quad \nabla_{x_1}\varphi(0) = 0 \tag{1.8}$$

defined for $\|x_1\| \leq \delta$ solution to the center manifold equation

$$\nabla_{x_1}\varphi(x_1)(A_1 x_1 + g_1(x_1, \varphi(x_1)) = A_2\varphi(x_1) + g_2(x_1, \varphi(x_1)). \tag{1.9}$$

Then, $x_2 = \varphi(x_1)$ defines the center manifold over which the trajectories are described by the reduced dynamics

$$\dot{x}_1 = A_1 x_1 + g_1(x_1, \varphi(x_1)). \tag{1.10}$$

Moreover, if the origin of the reduced dynamics is stable, unstable or asymptotically stable then, the origin of (1.2) is, respectively, stable, unstable or asymptotically stable.

A constructive procedure to deduce approximations of the reduced dynamics (1.10) over the center manifold have been proposed and exhaustively discussed by Carr in [23].

1.1.3 Tools on set stability and pGAS

In this part, we sketch definitions providing stability of the dynamics (1.1a) with respect to a given closed (but not necessarily compact) set $\mathcal{A}$ of $\mathbb{R}^n$ containing the origin.

Definition 1.8. [Global boundedness with respect to a set] The solutions of (1.1a) are said to be *globally bounded* with respect to $\mathcal{A}$ with respect to $\mathcal{A}$ if (1.1a) is forward complete and there exists a constant $c > 0$ such that for any $a \in]0, c[$ there exists a $\beta = \beta(a) > 0$ such that $\|x_0\| \leq a$ implies $\|x(t)\|_{\mathcal{A}} \leq \beta$ for any $t \geq 0$.

In the above definition, when $\mathcal{A} = \{0\}$ we recover *boundedness* of solutions as introduced in Definition 1.7. As the set $\mathcal{A}$ might be unbounded, trajectories might explode in finite time while $\|x(t)\|_{\mathcal{A}}$ remains bounded for any time. Forward completeness of (1.1a) prevents from this possibility. Also, forward completeness is not necessary whenever $\mathcal{A}$ is compact.

Definition 1.9. [GS of a set] Let (1.1a) be forward complete. The set $\mathcal{A}$ is said to be *globally stable* for (1.1) if solutions of (1.1) are globally bounded with respect to $\mathcal{A}$ and, for any constant $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x_0\| \leq \delta$ implies $\|x(t)\|_{\mathcal{A}} \leq \epsilon$ for any $t \geq 0$.

Definition 1.10. [Global Attractivity of a set] Let (1.1a) be forward complete. The set $\mathcal{A}$ is said to be *globally attractive* for (1.1) if, for any $r > 0$ and $\epsilon > 0$ there exists a positive time $T(r, \epsilon)$ such $\|x_0\| \leq r$ implies $\|x(t)\|_{\mathcal{A}} < \epsilon$ for any $t \geq T$.

Definition 1.11. [GAS of sets] Let (1.1a) be forward complete. The set $\mathcal{A}$ is said to be *globally asymptotically stable* for (1.1) if it is both globally stable and globally attractive.

This allows now the definition of pGAS of the origin of the system

$$\dot{x} = f(x, \theta) \tag{1.11}$$

where $x \in \mathbb{R}^n$ and $\theta \in \mathbb{R}^q$ a family of parameters. The following definitions are recalled from [?] where the corresponding local versions are also detailed.

Definition 1.12. [pGAS] Let $\Theta \subset \mathbb{R}^q$ be a set of parameters. The origin of the system (1.11) is said to be *practically globally asymptotically stable* on Θ if, for any $\delta > 0$, there exists $\theta^*(\delta) \in \Theta$ such that the ball $\mathcal{B}_\delta(0)$ is GAS for the system $\dot{x} = f(x, \theta^*)$.

In the next chapter, pGAS will be of paramount interest for systems under approximate sampled-data feedback. In that case, the sampling period will define the tuning parameter $\theta \in \mathbb{R}$ so proving that approximate sampled-data feedback will ensure convergence to a ball containing the origin whose radius will go to zero as the sampling period decreases.

1.2 Controlled dynamics: some properties and tools for control design

In this part, we shall define general properties for the controlled system (1.1) and some related tools for control design. First, some generalities are given with respect to what will be needed in the sequel of this manuscript.

1.2.1 Relative degree and Normal forms

In the following, we are defining the *relative degree* of (1.1) at $x_e = 0$ as $r \in \mathbb{N}$ such that $\mathrm{L}_g \mathrm{L}_f^k h(x) \equiv 0$ for $k = 0, 1 \dots, r-2$ and $\mathrm{L}_g \mathrm{L}_f^{r-1} h(x) \neq 0$ for any $x \in \mathcal{B}_\epsilon(0)$. Roughly speaking, it represents the differential delay acting over the influence of the input on the output. Whenever the above relations hold globally, the system (1.1) is said to possess a strong relative degree r at $x_e = 0$.

Whenever (1.1) possesses relative degree r at the origin, one can define a change of coordinates

$$\begin{pmatrix} z \\ \eta \end{pmatrix} = \begin{pmatrix} h(x) \\ \vdots \\ \mathrm{L}_f^{r-1} h(x) \\ \phi(x) \end{pmatrix} \quad \text{s. t.} \quad \nabla\phi(x) g(x) = 0 \tag{1.12}$$

with $z \in \mathbb{R}^r$ and $\eta \in \mathbb{R}^{n-r}$ so that the system rewrites in the so called *normal form*

$$\dot{z} = \hat{A}z + \hat{B}(a(z,\eta) + b(z,\eta)u) \tag{1.13}$$
$$\dot{\eta} = q(z,\eta) \tag{1.14}$$
$$y = \begin{pmatrix} 1 & \mathbf{0}_{1\times(n-1)} \end{pmatrix} \tag{1.15}$$

where $(\hat{A}, \hat{B})$ is in the companion Brunowski form while $a(z,\eta) = \mathrm{L}_f^r h(\phi^{-1}(z,\eta))$, $b(z,\eta) = \mathrm{L}_g \mathrm{L}^{r-1} h(\phi^{-1}(z,\eta))$ and $q(z,\eta) = \nabla\phi(\phi^{-1}(z,\eta)) f(\phi^{-1}(z,\eta))$. This representation is of paramount importance as it explicitly shows the so-called *zero-dynamics* evolving as

$$\dot{\eta} = q(0,\eta) \tag{1.16}$$

and defined as the residual internal dynamics of the system (1.1) when the output is identically equal to zero.

Definition 1.13. [Minimum-phase] Let a system (1.1) have relative degree $r \leq 0$. It is said to be:

- *weakly minimum-phase* if the origin of the zero-dynamics (1.16) is stable.
- *minimum-phase* if the origin of the zero-dynamics (1.16) is asymptotically stable.

The concept of zero-dynamics plays a crucial role in a lot of analysis, observation and control problems [56] even when dealing with sampled-data systems.

1.2.2 Passivity and PBC design

Passivity is a physical-inspired property that basically states that the input-output system (1.1) is dissipating energy. A large amount of studies have been developed in the control community in a purely input-output context [30] and in the state-space representation (e.g., [52, 186]).

Definition 1.14. The system (1.1) is *passive* if there exists a $\mathcal{C}^1$ positive semidefinite function $S : \mathbb{R}^n \to \mathbb{R}$ such that the following dissipation inequality holds

$$\dot{S}(x) \leq y^\top u. \tag{1.17}$$

The system is *lossless* if the dissipation relation holds as an equality; i.e., $\dot{S}(x) = y^\top u$. $S(x)$ is referred to as *storage function*.

By exploiting the above definition, one might deduce that whenever the storage function $S(x)$ is positive definite it can be exploited as a weak Lyapunov function to prove that the origin of the uncontrolled system associated to (1.1a) is stable.

Necessary and sufficient conditions for proving that a system is passive have been given by Kalman, Yacubovich and Popov stating that a system (1.1) is passive if and only if it has the KYP properties (see [19]); namely, there exists a $\mathcal{C}^1$ semi-definite function $S : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathrm{L}_f S(x) \leq 0 \quad \text{ and } \quad \mathrm{L}_g S(x) = h(x). \tag{1.18}$$

Remark 1.5. *Among many other consequences of these properties, one can immediately check that a necessary condition for passivity is for the system (1.1) to have relative degree $r = 1$ and being weakly minimum phase [19].*

Performing control design over passive systems generally results in very elegant and relatively simple feedback laws (e.g., [149, 183, 170]). As a matter of fact, by looking at the dissipation inequality (1.17), it is clear that, under certain circumstances, one might stabilize the system only through a damping output feedback of the form $u = -y$. This is the case if the following property holds.

Definition 1.15. [ZSD and ZSO] Let the system (1.1) be passive and consider the corresponding uncontrolled system (1.2). Let $\mathcal{Z} \subset \mathbb{R}^n$ be the largest invariant set contained into $\{x \in \mathbb{R}^n \text{ s. t. } h(x) = 0\}$. The system (1.1) is said to be *zero-state detectable* if the origin is globally asymptotically stable conditionally to $\mathcal{Z}$. Moreover, if $\mathcal{Z} \equiv \{0\}$, then (1.1) is said to be zero-state observable.

According to the above definition and by invoking the Barbashin-Krasovskii Theorem, any passive and ZSD system can be asymptotically stabilized by a proportional feedback $u = -y$. Also, note that whenever the above property holds true, one can deduce stability of the origin of (1.1) for $u \equiv 0$ even when the storage function is only positive semi-definite so relaxing the necessity of deducing a weak Lyapunov function,

Remark 1.6. *Even when dealing with more general control problems (e.g., regulation or tracking), passivity-based controllers admit rather simple structures as they come in general in the form of PI(D) feedback laws. Moreover, several works by Ortega and co-authors have been showing that passivity-based controls yield, in general, robustness in closed loop so qualifying for their application in a lot of practical situations and motivating the study toward the identification of outputs making a certain dynamics passive (e.g.,[5, 195, 165, 148]).*

1.2.3 Immersion and Invariance

Immersion and Invariance (I&I) was firstly introduced by Astolfi and Ortega in [9] as a very powerful and versatile tool for control of nonlinear systems. The application fields of this methodology span from purely control design to adaptive control and the design of observers [8].

I&I relies upon *reduction principles* (see [35, 92]) and upon the idea of forcing the trajectories of a given nonlinear system (1.1a) onto a lower-dimensional target manifold $\mathcal{M}$ where the dynamics are known to well behave (i.e., to be asymptotically stable). In other words, one seeks for a feedback making a lower dimensional stable manifold attractive and invariant. Several works have been proving that the applicability of such a methodology spans in a very large domain with huge impact on observer and feedback design and adaptive control (e.g., [163, 94, 53, 10, 55, 167]). Among these, it has been shown that I&I can be naturally applied to cascade systems as an alternative to backstepping and feedforwarding.

In the framework of control design, I&I stabilizability is recalled below.

Definition 1.16. [I&I stabilizability] The dynamics (1.1a) is said to be *I&I stabilizable* if, for some $p < n$, there exist mappings

$$\alpha : \mathbb{R}^p \to \mathbb{R}^p, \quad \pi : \mathbb{R}^p \to \mathbb{R}^n, \quad c : \mathbb{R}^p \to \mathbb{R} \tag{1.19}$$

$$\phi : \mathbb{R}^n \to \mathbb{R}^{n-p}, \quad \nu : \mathbb{R}^n \times \mathbb{R}^{n-p} \to \mathbb{R} \tag{1.20}$$

such that the following conditions hold true.

$i)$ *Target system* - The system

$$\dot{\xi} = \alpha(\xi) \tag{1.21}$$

with $\xi \in \mathbb{R}^p$ has a GAS equilibrium at the origin and $\pi(0) = 0$.

$ii)$ *Invariance condition* - For all $\xi \in \mathbb{R}^p$

$$\nabla\pi(\xi)\alpha(\xi) = f(\pi(\xi)) + g(\pi(\xi))c(\xi). \tag{1.22}$$

$iii)$ *Implicit manifold* - The following set identity holds

$$\{x \in \mathbb{R}^n \text{ s. t. } \phi(x) = 0\} \equiv \{x \in \mathbb{R}^n \text{ s. t. } x = \pi(\xi) \text{ for some } \xi \in \mathbb{R}^p\} \tag{1.23}$$

iv) Manifold attractivity and boundedness of trajectories - All trajectories of the system

$$\dot{z} = \nabla\phi(x)[f(x) + g(x)\nu(x,z)] \tag{1.24}$$
$$\dot{x} = f(x) + g(x)\nu(x,z) \tag{1.25}$$

are bounded with $\lim_{t\to\infty} z(t) = 0$, $\nu(\pi(\xi), 0) = c(\xi)$ and $z_0 = \phi(x_0)$.

It is evident that I&I stabilizability implies the existence of a feedback $u = \nu(x,z)$ so that the origin of the closed-loop system

$$\dot{x} = f(x) + g(x)\nu(x,z) \tag{1.26}$$

is GAS. Though, this feedback might not be unique together with the choice of the basis function $\phi(x)$ defining the stable target manifold $\mathcal{M} = \{x \in \mathbb{R}^n \text{ s. t. } \phi(x) = 0\}$.

Remark 1.7. *Condition* ***iv)*** *of the above definition can be weakened to prove asymptotic convergence of $x(t)$ to 0 by requiring that*

$$\lim_{t\to\infty} g(x(t))(\nu(x(t), z(t)) - \nu(x(t), 0)) = 0. \tag{1.27}$$

Remark 1.8. *Recently, I&I has been recast in the framework of contraction theory [91, 39] to provide a more systematic design tool [184]. In this sense, attractivity of the manifold is achieved by replacing attractivity of the manifold by virtual contraction of the off-the-manifold coordinate so to make the corresponding manifold horizontally contractive.*

In the next chapters we shall see that Immersion and Invariance represents a powerful tool even in the time-delay context when suitably settled in a sampled-data framework.

CHAPTER 2

Nonlinear systems under sampling

Contents

THE sampled-data framework is here introduced by deducing the so-called sampled-data equivalent model and by showing how properties of the original continuous-time system can be reformulated over the new discrete-time system. The preliminaries we are giving are based on several works on these topics by Monaco and Normand-Cyrot [120, 40, 125, 127, 179].

As already discussed, although a system might evolve continuously in time it is seldom subject to digitalization through physical devices that measure the output only at discrete time instants and apply the control input in the form of piecewise constant signals over the so-called sampling period. In this context, several analysis and control design tools have been proposed by leading the problem into other contexts such as the ones of hybrid, time-delay or purely discrete-time systems [49, 45, 125, 192, 51]. The approach we shall adopt throughout the manuscript relies upon the so-called *sampled-data equivalent model* [125] describing the evolutions of (1.1) at any sampling instants.

2.1 The equivalent sampled-data model

In what follows, we present sampled-data equivalent models of sampled-data systems based on the relation among the sampling periods of measures and outputs together with their advantages in control design. We assume control laws being implemented through zero-order-holders (ZOH) working at a given constant frequency.

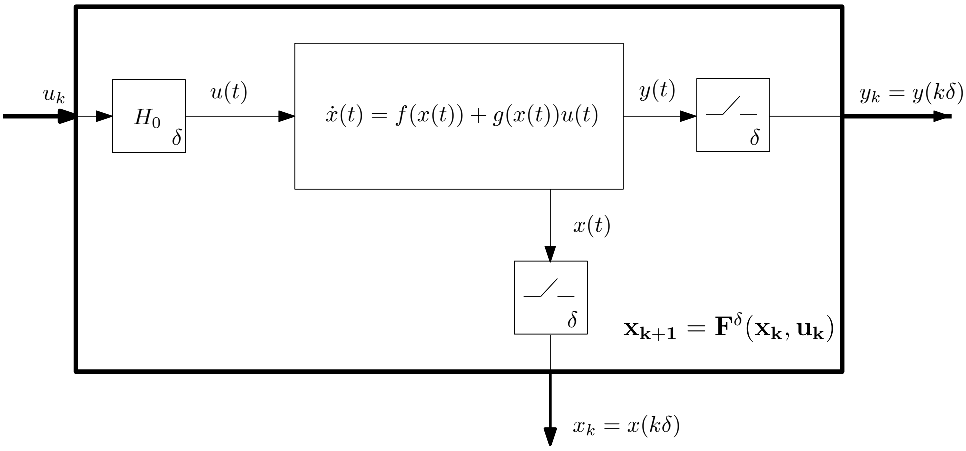


Figure 2.1: Single-rate sampling scheme

2.1.1 Single-rate sampling

Single-rate sampling describes the most common discretization scheme usually exploited in both theoretical and practical developments. In this context, as illustrated in Figure 2.3, one has what follows:

- the control signal is piecewise constant over time-intervals of fixed length $\delta > 0$, i.e., $u \in \mathcal{U}^\delta$ with

$$u(t) = u(k\delta) \quad \text{for all } t \in [k\delta, (k+1)\delta[\text{ and } k \geq 0; \tag{2.1}$$

- measures of the states and of the outputs are available only at the sampling instants $t = k\delta$, i.e.,

$$\begin{aligned} &x(t) = x(k\delta) \\ &y(t) = y(k\delta) = h(x(k\delta)) \quad \text{for all } t \in [k\delta, (k+1)\delta[\text{ and } k \geq 0. \end{aligned} \tag{2.2}$$

As a consequence, the evolutions of the continuous-time system (1.1) are described by the interval dynamics

$$\dot{x}(t) = f(x(t)) + g(x(t))u(k\delta), \quad t \in [k\delta, (k+1)\delta[\tag{2.3a}$$

$$y(t) = h(x(k\delta). \tag{2.3b}$$

One is now interested in describing the evolutions of the output and of the state with respect to jumps of control signal $u(t) = u(k\delta)$ at any sampling instant $t = k\delta$ with $k \geq 0$. For, let us integrate (2.3) over the sampling interval $[k\delta, (k+1)\delta[$ so getting

$$\begin{aligned} x((k+1)\delta) &= x(k\delta) + \int_{k\delta}^{(k+1)\delta} \big[f(x(s)) + g(x(s))u(k\delta)\big]\mathrm{d}s \\ y(k\delta) &= h(x(k\delta)). \end{aligned}$$

By invoking the Volterra and Taylor-like series expansions and through some nasty computations [118], one can verify that the above dynamics defines the so-called (*single-rate*) *sampled-data equivalent model* of (2.3) which is defined below.

Definition 2.1. [Sampled-data equivalent model] The discrete-time dynamics

$$x_{k+1} = F^\delta(x_k, u_k) \tag{2.4a}$$
$$y_k = h(x_k) \tag{2.4b}$$

with $x_k := x(k\delta)$, $u_k := u(k\delta)$, $y_k := y(k\delta)$ and

$$F^\delta(x_k, u_k) = e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)}\mathrm{Id}\big|_{x_k} \tag{2.5}$$

defines the (*single-rate*) *sampled-data equivalent model* of the sampled-data system (2.3) for any $t = k\delta$ and $k \geq 0$.

The mapping $F^\delta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is parametrized by δ and can be generally expressed as a formal series in powers of δ; namely, one gets[1]

$$F^\delta(x_k, u_k) = e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)}\mathrm{Id}\big|_{x_k} = x_k + \sum_{i>0} \frac{\delta^i}{i!}(\mathrm{L}_f + u_k \mathrm{L}_g)^i \mathrm{Id}\big|_{x_k}. \tag{2.6}$$

Remark 2.1. *Throughout this manuscript we shall not address the convergence of Lie series of the form (2.6). As it will be clear in the following, this is motivated by the necessity of computing approximate solutions (especially with respect to the control) by truncating the corresponding Lie series at any finite order δ^p with $p \geq 0$. For the interested reader, a deep investigation on the convergence of Lie series can be found in [50].*

Remark 2.2. *The expansions we shall be dealing with are deduced from series expansion of the mapping $F^\delta(x, u)$ in powers of δ as in (2.6). One other equivalent approach relies in expanding the above mapping in powers of u by exploiting the corresponding series expansion [130]; namely, one gets*

$$\begin{aligned} F^\delta(x, u) =& e^{\delta \mathrm{L}_f} x + u \int_0^\delta \gamma_1(s_1, \delta, x)\mathrm{d}s_1 \\ &+ \sum_{i\geq 2} u^i \int_0^\delta \int_0^{s_1} \cdots \int_0^{s_{i-1}} \gamma_i(s_1, \ldots, s_i, \delta, x)\mathrm{d}s_i \ldots \mathrm{d}s_1 \end{aligned}$$

with $\gamma_i(s_1, \ldots, s_i, \delta, x)$ denoting the Volterra kernels associated to (1.1a) and computed as

$$\gamma_i(s_1, \ldots, s_i, \delta, x) = e^{s_i \mathrm{L}_f}\mathrm{L}_g e^{(s_{i-1}-s_i)\mathrm{L}_f}\mathrm{L}_g \ldots \mathrm{L}_g e^{(\delta - s_1)\mathrm{L}_f} Id\big|_x.$$

[1]For the sake of brevity, in the following we might rewrite $e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)}\mathrm{Id}\big|_{x_k} = e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)}x\big|_{x_k}$. Analogously, given a smooth mapping $V : \mathbb{R}^n \to \mathbb{R}$ we might write, by exploiting the exchange theorem, $V(e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)}\mathrm{Id}\big|_{x_k}) = e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)}V(x)\big|_{x_k} = e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)}V(x_k)$.

2.1.1.1 On approximate sampled-data single-rate models and finite discretizability

Deducing a closed-form for sampled-data equivalent model (2.4) might be a tough task and generally depends on the capability of explicitly computing a closed-form solutions of the ODE (2.3a). Though, approximations can be easily carried out by truncating the series expansion (2.6) at any finite order $p \geq 0$ in δ as

$$F^{\delta[p]}(x_k, u_k) = x_k + \sum_{i=1}^{p} \frac{\delta^i}{i!}(\mathrm{L}_f + u_k \mathrm{L}_g)^i \mathrm{Id}\big|_{x_k}. \tag{2.7}$$

The above mapping describes the sampled-data evolutions of the original system (2.3) up to an error in $O(\delta^{p+1})$; namely, one gets

$$F^{\delta}(x_k, u_k) = F^{\delta[p]}(x_k, u_k) + O(\delta^{p+1}).$$

Accordingly, the following definition is given.

Definition 2.2. [Approximate sampled-data models] The discrete-time dynamics

$$x^p_{k+1} = F^{\delta[p]}(x^p_k, u_k) \tag{2.8}$$

with $F^{\delta[p]} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ as in (2.7) defines the p^{th}*-order approximate sampled-data equivalent model* of the system (2.3).

Remark 2.3. *By denoting x^p as the state of the approximate sampled-data system defined by (2.7) evolving as (2.8), one gets that, for any $k \geq 0$, $\|x_k - x^p_k\| \leq O(\delta^{p+1})$. This means that, as long as δ is sufficiently small, the trajectories of (2.8) will stay in a neighborhood of the ones of (2.4a). Whenever $u_k \equiv 0$, this implies that the properties of the origin of the uncontrolled (2.4a) yield* practically *over the approximate dynamics (2.8).*

Remark 2.4. *Whenever $p = 1$, one recovers the well-known Euler approximate sampled-data model of (2.3) as given by*

$$F^{\delta[1]}(x_k, u_k) = x_k + \delta(f(x_k) + g(x_k)u_k)$$

that is commonly used in the literature on sampled nonlinear systems.

Remark 2.5. *We have defined (2.8) as homogeneous approximations in δ^p of (2.4); namely, all the equations defining the sampled dynamics (2.4) are truncated at the same order in δ^p. This is motivated by the fact that approximations of (2.4) will not be computed, in general, for control design purposes as we shall specify in the sequel. For non-homogeneous approximations and their involvement in control design we refer to several works by Yuz and Goodwin (e.g., [192, 24]).*

As already commented on, the possibility of computing a closed-form sampled-data equivalent model depends on the capability of integrating (2.3a). In this sense, special classes of nonlinear systems admit a *finite* a finite series expansion (2.6) in powers of δ. We shall refer to this class of systems as *finitely discretizable* in the sense of the definition below.

Definition 2.3. [Finite discretizability] We say that the sampled-data system (2.3) under single-rate sampling is finitely discretizable if it admits a finite sampled-data equivalent model (2.4) in the form of a finite series expansion in powers of δ; namely,

$$F^{\delta}(x_k, u_k) = x_k + \sum_{i=1}^{q} \frac{\delta^i}{i!}(\mathrm{L}_f + u_k \mathrm{L}_g)^i \mathrm{Id}\big|_{x_k}.$$

Several classes of dynamics admit a finite sampled-data equivalent model such as the n^{th}-order integrator or systems admitting a chained-form. Equivalence or feedback equivalence of a nonlinear system (1.1a) to a finitely discretizable one has been firstly introduced [121] and then further investigated in [32, 33].

Remark 2.6. *When the system (1.1) is linear, the sampled-data equivalent model admits a closed-loop form that is not, in general, finite. Namely, if $f(x) = Ax$ and $g(x) = B$ with $A \in Mat_{\mathbb{R}}(n, n)$ and $B \in Mat_{\mathbb{R}}(n, 1)$ one gets*

$$F^{\delta}(x, u) = A^{\delta} x_k + B^{\delta} u_k$$

with

$$A^{\delta} = e^{A\delta} = I + \sum_{i>0} \frac{\delta^i}{i!} A^i, \quad B^{\delta} = \int_0^{\delta} e^{As} B \mathrm{d}s = B + \sum_{i>0} \frac{\delta^i}{i!} A^{i-1} B.$$

Remark 2.7. *We note that a linear system is finitely discretizable whenever there exists a $\bar{i} \in \mathbb{N}$ such that $A^i B = 0$ for any $i \geq \bar{i}$. It the nonlinear context, finitely discretizability has been proved to be depending on the Lie algebra induced by the vector fields f and g.*

2.1.1.2 An alternative state-space representation for sampled-data equivalent models

The sampled-data dynamics (2.4a) admits an equivalent state-space representation that was firstly introduced in [123] for purely discrete-time dynamics that is generally denoted as *Differential-Difference Representation* or (F_0, G)-form. Basically, one splits the dynamics (2.4a) into two difference-differential equations as

$$x^+ = F_0^{\delta}(x), \quad x^+ = x^+(0) \tag{2.9a}$$

$$\frac{\mathrm{d}x^+(u)}{\mathrm{d}u} = G^{\delta}(x^+(u), u) \tag{2.9b}$$

with

$$F_0^\delta(x) := F^\delta(x,0) = e^{\delta \mathrm{L}_f}\mathrm{Id}\big|_x, \quad G^\delta(x,u) = \int_0^\delta e^{-s\ ad_{f+ug}} g(x)\mathrm{d}s$$

and $G^\delta(x,u)$ being a complete vector field verifying $\nabla_u F^\delta(x,u) = G^\delta(F^\delta(x,u),u)$. More in details, $x^+(u)$ defines a curve over $\mathbb{R}^n$ parametrized by u and uniquely defined by the differential equation (2.9b) with initial condition (2.9a).

Both (2.4) and (2.9) are perfectly equivalent. For any triplet (k, x_k, u_k), by integrating (2.9b) over $[0, u_k[$ with initial condition (2.9a) at $x = x_k$, one gets $x_{k+1} = x^+(u_k)$ so recovering

$$F^\delta(x_k,u_k) = F_0^\delta(x_k) + \int_0^{u_k} G^\delta(x^+(v),v)\mathrm{d}v.$$

The (F_0, G)-form allows to settle analysis of nonlinear discrete-time systems in a very elegant geometrical framework and to deduce nice computational properties underlying the dynamics (e.g., [126, 130, 129]). Moreover, it allows to split the evolutions of a given smooth mapping $V : \mathbb{R}^n \to \mathbb{R}$ along the sampled trajectories of (2.4) as the contribution of the free and forced components; namely, for any $k \geq 0$, one gets

$$V(F^\delta(x_k,u_k)) = V(F_0^\delta(x_k)) + \int_0^{u_k} \mathrm{L}_{G^\delta(\cdot,v)} V(x^+(v))\mathrm{d}v.$$

Remark 2.8. *A different way of dealing with sampled-data systems was proposed by Yuz and Goodwin in [192] through the so-called δ-operator (where δ is not to confound with the sampling period) which is aimed at mimicking, in the sampled framework, the ordinary differential equations describing the variation in time of the continuous-time dynamics (1.1).*

2.1.1.3 Issues with single-rate sampling: the relative degree and the zero-dynamics

It is clear from (2.4a) that the input-affine structure of (1.1) is not preserved by sampling. As we shall show in the following chapters, sampling generally destroys the structure of a given nonlinear (or even linear) system and does not preserve, in general, other core properties of the control design such as the relative degree and, thus, the zero-dynamics. This will be illustrated through the following example.

Example 2.1. Consider the n^{th}-order integrator

$$\begin{aligned}\dot{x} &= \hat{A}x + \hat{B}u\\ y &= \hat{C}x\end{aligned}$$

with

$$\hat{A} = \begin{pmatrix} \mathbf{0}_{(n-1)\times 1} & \mathrm{I} \\ 0 & \mathbf{0}_{1\times(n-1)} \end{pmatrix}, \quad B = \begin{pmatrix} \mathbf{0}_{(n-1)\times 1} \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & \mathbf{0}_{1\times(n-1)} \end{pmatrix}$$

and transfer function $P(s) = \frac{1}{s^n}$ and relative degree $r = n$. Then, the finite sampled-data equivalent model is provided by

$$x_{k+1} = \begin{pmatrix} 1 & \delta & \dots & \frac{\delta^{n-1}}{(n-1)!} \\ 0 & 1 & \dots & \frac{\delta^{n-2}}{(n-2)!} \\ & & \ddots & \\ 0 & 0 & \dots & 1 \end{pmatrix} x_k + \begin{pmatrix} \frac{\delta^n}{(n)!} \\ \frac{\delta^{n-1}}{(n-1)!} \\ \vdots \\ \delta \end{pmatrix} u_k$$

with transfer function

$$\begin{aligned} G_d(z) &= \frac{\delta^n}{n!} \frac{B_d(z)}{(z-1)^n} \\ B_d(z) &= b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n \\ b_k &= \sum_{\ell=1}^{k} (-1)^{k-\ell} \ell^n \binom{n+1}{k-\ell} \end{aligned} \tag{2.10}$$

and, thus, possesses relative degree $r_d = 1$.

The n^{th}-order integrator clearly underlines that sampling induces $n-1$ new zeroes (the so-called *sampling zeroes*) defined as the roots of the polynomial $B_d(z)$ in (2.10) which are generally unstable. This study has been extended to sampled-data LTI systems with continuous-time relative degree $r \leq n$ in [12]. It was shown that

1. the relative degree r of the corresponding sampled-data equivalent model generally falls to $r_d = 1$;

2. $r-1$ zeros are induced by sampling coinciding, as $\delta \to 0$, with the zeros of the polynomial $B_d(z)$ in (2.10).

As a consequence, sampling induces a new $(r-1)$-dimensional zero-dynamics which is generally unstable as $r \geq 2$ [11] so not preserving, in general, the minimum-phase property of the original continuous-time plant.

This pathology extends to general nonlinear systems possessing relative degree $r > 2$. As a matter of fact, one can easily deduce that the relative-degree of the sampled-data equivalent model (2.4) falls to $r_d = 1$ independently of the continuous-time one; namely, by computing

$$\begin{aligned} y_{k+1} =& h(F^\delta(x_k, u_k)) = e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)} h(x_k) \\ =& h(x_k) + \sum_{i=1}^{n} \frac{\delta^i}{i!} \mathrm{L}_f^i h(x_k) + \frac{\delta^n}{n!} u_k \mathrm{L}_g \mathrm{L}_f^{i-1} h(x_k) + O(\delta^{n+1}) \end{aligned}$$

one gets that

$$\frac{\partial h(F^\delta(x, u))}{\partial u} = \frac{\delta^n}{n!} u \mathrm{L}_g \mathrm{L}_f^{i-1} h(x) + O(\delta^{n+1}) \neq 0.$$

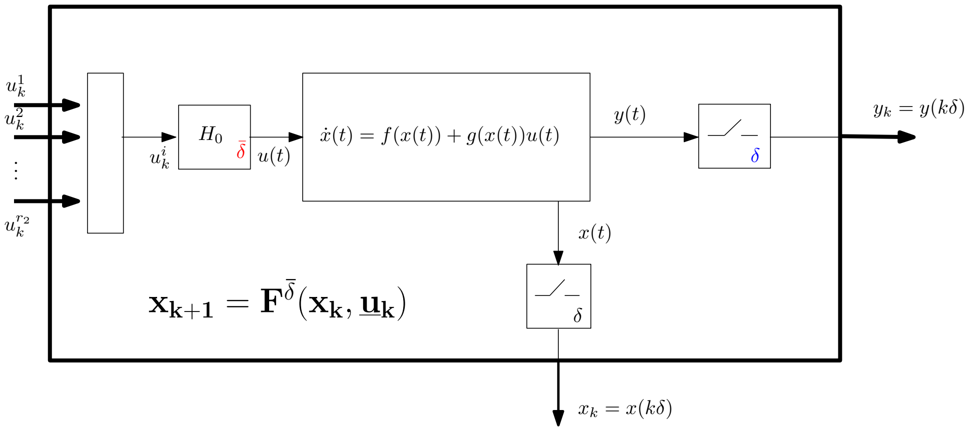


Figure 2.2: Multi-rate sampling scheme

As a consequence, it was proved in [119] that sampling induces a further zero-dynamics (the so-called *sampling zero-dynamics*) which is generally unstable as $r \geq 2$. Accordingly, the sampled-data system (2.4) is non-minimumphase as $r \geq 2$, independently of the original properties of the continuous-time original system (1.1).

This issue prevents from applying standard control techniques relying upon the zero-dynamics inversion (e.g., feedback linearization and output regulation [56]) that might indeed make the sampled-data single-rate system unstable in closed loop. Motivated by this, multi-rate-based control has been firstly introduced in [120] by proving that the relative degree is preserved under multi-rate sampling together with the original zero-dynamics and the minimum-phase property.

2.1.2 Multi-rate sampling

Multi-rate sampling generally refers to the situation in which the ZOH works at a faster frequency than the sampler device; roughly speaking, the input signal is "sampled" faster than the output as illustrated in Figure 2.2. More precisely, by defining δ as the sampling period of the output, one has:

- the control signal is piecewise constant over time interval of fixed length $\bar{\delta} = \frac{\delta}{m} > 0$ for some $m \in \mathbb{N}_{>0}$, i.e., for $i = 1, \ldots, m$

$$u(t) = u(k\delta + (i-1)\bar{\delta}) \qquad \text{for all } t \in [k\delta + (i-1)\bar{\delta}, k\delta + i\bar{\delta}[; \tag{2.11}$$

- measures of the states and of the outputs are available only at the sampling instants $t = k\delta$, i.e.,

$$\begin{aligned} x(t) &= x(k\delta) \\ y(t) &= y(k\delta) = h(x(k\delta)) \qquad \text{for all } t \in [k\delta, (k+1)\delta[\text{ and } k \geq 0. \end{aligned} \tag{2.12}$$

In this context, (1.1) reduces to the interval dynamics

$$\dot{x}(t) = f(x(t)) + g(x(t))u(k\delta + (i-1)\bar{\delta}), \quad t \in [k\delta + (i-1)\bar{\delta}, k\delta + i\bar{\delta}[\tag{2.13a}$$
$$y(t) = h(x(k\delta)). \tag{2.13b}$$

As in the single-rate context, the evolutions of the state at any sampling instant $t = k\delta$ are deduced by integrating (2.13) among two successive sampling instants so getting

$$\begin{aligned} x((k+1)\delta) =& x(k\delta) + \int_{k\delta}^{(k+1)\delta} f(x(s))\mathrm{d}s + \sum_{i=1}^{m} \int_{k\delta+(i-1)\bar{\delta}}^{k\delta+i\bar{\delta}} g(x(s))\mathrm{d}s u(k\delta + (i-1)\bar{\delta}) \\ y(k\delta) =& h(x(k\delta)). \end{aligned}$$

Through some nasty computations, one gets the *multi-rate sampled-data equivalent model* of (2.13) as defined below.

Definition 2.4. [Multi-rate sampled-data equivalent model of order m] The multi-input discrete-time dynamics

$$x_{k+1} = F^{\bar{\delta}}(x_k, \underline{u}_k) \tag{2.14a}$$
$$y_k = h(x_k) \tag{2.14b}$$

with $x_k := x(k\delta)$, $y_k = y(k\delta)$, $\underline{u}_k = (u_k^1, \ldots, u_k^m)^\top$ with $u_k^i := u(k\delta + (i-1)\bar{\delta})$ for $i = 1, \ldots, m$ and $F^{\bar{\delta}} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ as

$$\begin{aligned} F^{\bar{\delta}}(x_k, \underline{u}_k) =& F^{\bar{\delta}}(\cdot, u_k^m) \circ \cdots \circ F^{\bar{\delta}}(x_k, u_k^1) \\ =& e^{\bar{\delta}(\mathrm{L}_f + u_k^1 \mathrm{L}_g)} \circ \cdots \circ e^{\bar{\delta}(\mathrm{L}_f + u_k^m \mathrm{L}_g)} \mathrm{Id}\big|_{x_k}. \end{aligned}$$

defines the *multi-rate sampled-data equivalent model* of order m associated to (2.13) at any sampling instant $t = k\delta$, $k \geq 0$.

Again, as in the single-rate case, the sampled-data multi-rate model gets the form of an asymptotic series expansion in powers of $\bar{\delta}$ so that closed forms are hard to be computed. Nevertheless, one can define approximate sampled-data models along the lines of the single-rate case by truncating the corresponding series expansion at any finite order $p \geq 0$.

Remark 2.9. *The multi-rate sampled-data equivalent model (2.14) admits an equivalent (F_0, G) representation that is similar to the one presented for the single-rate case. Though, we are going to omit this as it will not be exploited in the rest of the manuscript. The interesting reader is referred to [130] for further details on this.*

Remark 2.10. *Multi-rate sampling increases the degrees of freedom in the control as the sampled evolutions of (2.13) along the sampling period are described by the multi-input dynamics (2.14a). This aspect makes multi-rate control qualified for its applications in a vaste domain of control problem such as, for example, motion planning or tracking [121].*

Remark 2.11. *When setting $u^1_k = \cdots = u^m_k$, one recovers the single-rate equivalent model (2.4).*

Remark 2.12. *When setting $u^i_k = \alpha_i u_k$ for $\alpha_i \in \mathbb{R}$ and $i = 1, \ldots, m$ and $u_k := u(k\delta)$, one recovers the sampling procedure through Generalized Holding Function (GHF, [59, 24]). In that case, the sampled-data equivalent model (2.14) reduces to a single-input dynamics.*

Remark 2.13. *As anticipated, multi-rate sampling preserves the minimum-phaseness of a given nonlinear system (1.1) without inducing any further zero-dynamics. This is clear by looking at the sampled-data equivalent model (2.14) and by considering the augmented output $\underline{y} = (h(x) \ \ldots \ \mathrm{L}_f^{r-1}h(x))^\top$ with respect to which the system has vector relative degree $r_d = (1 \ \ldots \ 1)$ [119].*

2.2 Stability and stabilization sampled-data nonlinear systems

In this part, we are defining properties of sampled-data nonlinear systems in terms of stability of equilibria and passivity. For the sake of clarity, we shall refer to systems under single-rate sampling as the extension to multi-rate is straightforward along the same lines.

To this end, the following definitions concerning stability of discrete-time systems are instrumental.

Definition 2.5. [Lyapunov stability in discrete time] The equilibrium $x = 0$ of the discrete-time system

$$x_{k+1} = F(x_k) \tag{2.15}$$

is said to be

- *stable* if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x_0\| < \delta$ implies $\|x_k\| < \varepsilon$ for any $k \geq 0$;
- *unstable* if it is not stable;
- *attractive* if there exists a $\delta > 0$ such that $\|x_0\| < \delta$ implies $\lim_{k\to\infty} \|x_k\| = 0$;
- *asymptotically stable* if it is stable and attractive;
- *exponentially stable* if it is asymptotically stable and there exist constant $c > 0$ and $\alpha \in (0, 1)$ such that $\|x_k\| \leq c\alpha^k \|x_0\|$.

If the above properties yield for any initial condition $x_0 \in \mathbb{R}^n$, then they hold *globally.*

Definition 2.6. [Candidate Lyapunov Functions in discrete time] We say that a continuous function $V : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ is a *candidate Lyapunov function* for (2.15) if it is positive definite.

Definition 2.7. [Lyapunov functions in discrete time] We say that a candidate Lyapunov function $V : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ is

- a (*weak*) *Lyapunov function* for (2.15) if its increment along (1.2a) is negative semidefinite, i.e., $V(F(x)) - V(x) \leq -W(x)$ with $W(x) \geq 0$;
- a *strict Lyapunov function* for (2.15) if its increment along (2.15) is negative definite, i.e., $V(F(x)) - V(x) \leq -W(x)$ with $W(x) > 0$ for any $x \neq 0$.

2.2.1 Stability under sampled-data feedback

First, we are going to define stability properties of the origin of the dynamics associated to (2.3) under sampled-data feedback in terms of the one of the sampled-data equivalent model (2.4a). The local or global version of the following properties will be specified later on depending on the context.

Definition 2.8. [Lyapunov stability under sampling] Let the dynamics (2.3a) possess an equilibrium at the origin and (2.4a) be its sampled-data equivalent model. Consider any feedback law $u_k = \gamma(x_k)$ with $\gamma(0) = 0$ and $\gamma : \mathbb{R}^n \to \mathbb{R}$. Let the corresponding closed-loop sampled-data equivalent model be provided by

$$x_{k+1} = F^{\delta}(x_k, \gamma(x_k)). \tag{2.16}$$

The origin of the closed-loop

$$\dot{x}(t) = f(x(t)) + g(x(t))\gamma(x_k), \quad t \in [k\delta, (k+1)\delta[\tag{2.17}$$

is said to be

- *sampled-data stable* (S-S) if it is stable for the sampled-data equivalent model (2.16);
- *sampled-data asymptotically stable* (S-AS) if it is asymptotically stable for the sampled-data equivalent model (2.16);
- *sampled-data exponentially stable* (S-ES) if it is exponentially stable for the sampled-data equivalent model (2.16).

The global versions of the above properties together with the other notions of stability can be carried out along the same lines by referring to the properties of the closed-loop sampled-data equivalent dynamics (2.16).

2.2.2 Passivity and Passivity-Based Control under sampling

The question we address in this part is about the definition of *passivity* of sampled-data system. Namely, given a continuous-time passive system (1.1) with storage function $S : \mathbb{R}^n \to \mathbb{R}_{\geq}$, we want to study the passivity properties of the sampled-data equivalent model (2.4) and the way they are affected by sampling. For the sake of convenience, we rewrite $h(x) = \mathrm{L}_g S(x)$ by exploiting the KYP properties (1.18); i.e., we consider the dynamics (1.1) as

$$\dot{x} = f(x) + g(x)u \tag{2.18a}$$
$$y = \mathrm{L}_g S(x) \tag{2.18b}$$

verifying the differential dissipation inequality

$$\dot{S}(x) \leq u\mathrm{L}_g S(x). \tag{2.19}$$

Those problems has been widely investigated in discrete time by proving that a necessary condition for passivity is for the discrete-time output mapping to be directly depending on u and thus the system to be causal (e.g., [122, 27, 176, 133]).

Under sampling, two notions of passivity arise by looking at (2.4) as coming from a continuous-time system or as a purely discrete-time system. Throughout the manuscript, we shall be make exhaustive use of the notion of u-Average passivity while we briefly recall the definition of δ-Average Passivity for the sake of completeness.

First, the definition of a passive discrete-time system is given.

Definition 2.9. [Passivity of discrete-time systems] Consider a nonlinear discrete-time system

$$x_{k+1} = F(x_k, u_k) \tag{2.20a}$$
$$y_k = H(x_k, u_k) \tag{2.20b}$$

with $x \in \mathbb{R}^n$ and y, $u \in \mathbb{R}$. We say that (2.20) is passive if there exists a positive semidefinite function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that, for any $k \geq 0$, the dissipativity inequality holds true

$$\Delta_k S(x) = S(x_{k+1}) - S(x_k) \leq y_k u_k.$$

Definition 2.10. [ZSD for discrete-time systems] The system (2.20) is said to be Zero-State Detectable (ZSD) if the origin is GAS conditionally to the largest invariant set $\mathcal{Z} \subset \{x \in \mathbb{R}^n \text{ s.t. } H(x, 0) = 0\}$. If, $\mathcal{Z} = \{0\}$, then (2.20) is said to be Zero-State observable (ZSO).

2.2.2.1 δ-Average Passivity

Assuming $u \in \mathcal{U}^\delta$ and the system (2.18) passive, we get that the dissipation inequality (2.19) rewrites as

$$\dot{S}(x(t)) \leq u_k \mathrm{L}_g S(x(t)).$$

By integrating now the above relation over $[k\delta, (k+1)\delta[$, the above inequality holds

$$S(x_{k+1}) - S(x_k) \leq u_k \int_{k\delta}^{(k+1)\delta} \mathrm{L}_g S(x(s)) \mathrm{d}s \tag{2.21}$$

so that passivity of the original system is not preserved under sampling (i.e., the sampled-data equivalent system (2.4) is not passive). Though, the above inequality suggests that the sampled-data dynamics (2.4a) is passive with respect to a new output that is deduced by averaging the continuous-time one (1.1b) over the sampling period and along the trajectories of (2.3a); i.e., the (2.4a) is passive with respect to

$$h^{\delta}_{\text{av}}(x_k, u_k) := \frac{1}{\delta}\int_{k\delta}^{(k+1)\delta} \mathrm{L}_g S(x(s))\mathrm{d}s \tag{2.22}$$

with

$$x(s) = e^{(s-k\delta)(\mathrm{L}_f + u_k \mathrm{L}_g)} Id\big|_{x_k}.$$

This leads to the notion of δ-*Average Passivity* (or *time-Average Passivity*) and to the following result which is recalled from [27, 99].

Theorem 2.1 (δ-average passivity)**.** *Let the continuous-time system (2.18) be passive with storage function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. Then, the sampled-data equivalent system (2.4) is δ-Average Passive; i.e., it verifies the dissipation inequality (2.21). Equivalently, the new system*

$$\begin{aligned} x_{k+1} &= F^{\delta}(x_k, u_k) \\ y^{\delta}_{av} &= h^{\delta}_{av}(x_k, u_k) \end{aligned}$$

is passive in the sense of definition (2.9).

Proof: By rewriting the increment of the storage function along (2.4a) we get

$$\begin{aligned} \Delta_k S(x) = &\int_{k\delta}^{(k+1)\delta} [\mathrm{L}_f S(x(s)) + u_k \mathrm{L}_g S(x(s))]\mathrm{d}s \\ &\leq u_k \mathrm{L}_g S(x(s))\mathrm{d}s := u_k h^{\delta}_{\text{av}}(x_k, u_k) \end{aligned}$$

so getting the result. ■

Remark 2.14. *We note that the new output (2.22) is explicitly depending on u*

Albeit this notion of passivity seems quite elegant and natural for sampled-data systems, it lacks of extendability to purely discrete-time systems and to general control purposes both in the possibility of defining Port-Hamiltonian-like representations [134] or in establishing some constructive procedure for passivity-based feedbacks (e.g., as in the case of cascade system). Contrarily to this case, the forthcoming discrete-time notion of u-average passivity has proven itself to serve for (some of) those purposes in a large variety of situations.

2.2.2.2 u-Average Passivity and u-Passivity-Based Control

When looking at the dynamics (2.4a) as a purely discrete-time one the notion of u-Average passivity arises as firstly introduced by Monaco and Normand-Cyrot in

[127]. Namely, starting from passivity of the continuous-time (2.18), one wants to deduce the existence of a new output mapping $Y^\delta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ with respect to which (2.4a) is u-Average passive.

In doing so, we consider the continuous-time storage function $S(x)$ verifying $\mathrm{L}_f S(x) \leq 0$ and compute its increment along the dynamics (2.4a); namely,

$$\Delta_k S(x) = S(F^\delta(x_k, u_k)) - S(x_k).$$

By exploiting the (F_0, G)-form (2.9) associated to (2.4a), one rewrites

$$\Delta_k S(x) = S(F_0^\delta(x_k)) - S(x_k) + \int_0^{u_k} \mathrm{L}_{G^\delta(\cdot,v)} S(x^+(v))\mathrm{d}v.$$

Since

$$\begin{aligned} S(F_0^\delta(x_k)) - S(x_k) =& e^{\delta \mathrm{L}_f} S(x_k) - S(x_k) \\ =& \int_{k\delta}^{(k+1)} \mathrm{L}_f S(x(s)\big|_{u=0})\mathrm{d}s \leq 0 \end{aligned}$$

we get the dissipation inequality

$$\Delta_k S(x) \leq \int_0^{u_k} \mathrm{L}_{G^\delta(\cdot,v)} S(x^+(v))\mathrm{d}v = u_k \int_0^1 \mathrm{L}_{G^\delta(\cdot,\theta u_k)} S(x^+(\theta u_k))\mathrm{d}\theta \tag{2.23}$$

so that (2.4a) is passive (in the sense of definition 2.9) with respect to the average of $\mathrm{L}_{G^\delta(\cdot,u)} S(x)$ over the control u. This leads to the result above recalled from [127].

Theorem 2.2 (u-Average passivity of sampled-data systems)**.** *Let the continuous -time system (2.18) be passive with storage function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. Then, the sampled-data equivalent dynamics (2.4a) with output $Y^\delta(x, u) = \mathrm{L}_{G^\delta(\cdot,u)} S(x)$ is* u-Average passive*; i.e., the sampled-data equivalent dynamics (2.4a) is passive with respect to the so-called* u-average output

$$Y_{av}^\delta(x, u) := \frac{1}{\delta u} \int_0^u \mathrm{L}_{G^\delta(\cdot,v)} S(x^+(v))\mathrm{d}v \tag{2.24}$$

and verifies, for any $k \geq 0$, the dissipative inequality

$$\Delta_k S(x) \leq \delta u_k Y_{av}^\delta(x_k, u_k).$$

Remark 2.15. *The u-average output (2.24) can be expressed as a series expansion in powers of δ around the continuous-time output $h(x) = \mathrm{L}_g h(x)$ as*

$$\begin{aligned} Y_{av}^\delta(x, u) =& h(x) + \frac{\delta}{2}((\mathrm{L}_f + u_k \mathrm{L}_g)h(x) + \mathrm{L}_g \mathrm{L}_f S(x)) + \frac{\delta^2}{6}((\mathrm{L}_f + u_k \mathrm{L}_g)^2 h(x) \\ &+ \mathrm{L}_f \mathrm{L}_g \mathrm{L}_f S(s) + u \mathrm{L}_g^2 \mathrm{L}_f S(x)) + O(\delta^2). \end{aligned}$$

Remark 2.16. *It is a matter of computations to verify that the u-average output (2.24) rewrites as*

$$Y^{\delta}_{av}(x,u) = \frac{1}{\delta u}\int_0^{\delta}\int_0^{u} \nabla_v \mathrm{L}_f S(F^s(x,v))\mathrm{d}v\mathrm{d}s + h^{\delta}_{av}(x,u)$$

where $h^{\delta}_{av}(x,u)$ denotes the δ-Average output (2.22). Accordingly, the two passivity notions come to coincide when the original continuous-time system (2.18) is lossless (i.e., when $\mathrm{L}_f S(x) \equiv 0$).

Based on u-Average passivity, one can now seek for a damping feedback over $Y^{\delta}_{\mathrm{av}}(x,u)$ making the origin of the sampled-data dynamics (2.4) globally asymptotically stable with

$$\Delta_k S(x) \leq -\delta\|Y^{\delta}_{\mathrm{av}}(x,u)\|^2 \leq 0.$$

It turns out that, if the original system (2.18) is ZSD, then the aforementioned control from the u-average output stabilizes the origin of the sampled-data equivalent (2.4a) in closed loop. For this purpose, the following intermediate result is first given.

Lemma 2.1. *Let the continuous-time system (2.18) be passive with storage function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and ZSD. Then, the sampled-data system*

$$x_{k+1} = F^{\delta}(x_k, u_k) \tag{2.25a}$$

$$Y^{\delta}(x_k, u_k) = \mathrm{L}_{G^{\delta}(\cdot,u_k)} S(x_k) \tag{2.25b}$$

with $G^{\delta}(x,u)$ as in (2.9) is ZSD; i.e., the origin of (2.25a) GAS conditionally to the largest invariant set $\mathcal{Z}_{\delta} \subset \{x \in \mathbb{R}^n$ s.t. $Y^{\delta}(x,0) = 0\}$. If, moreover, (2.18) is ZSO, then (2.25) is ZSO and, thus, $\mathcal{Z}_{\delta} \equiv \{0\}$.

Accordingly, one can state the following result which is recalled from [127].

Theorem 2.3 (u-PBC)**.** *Let the continuous-time system (2.18) be passive with positive definite storage function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and ZSD. Then, the feedback $u = u^{\delta}_{av}(x)$ defined as the solution to the* damping equality

$$u + KY^{\delta}_{av}(x,u) = 0, \quad K > 0 \tag{2.26}$$

makes the origin of (2.25a) GAS. Equivalently, the sampled-data feedback $u_k = u^{\delta}_{av}(x_k)$ makes the origin of (2.3) S-GAS.

Remark 2.17. *In [129], Monaco and Normand-Cyrot showed that u-average passivity-based controllers are inverse optimal with respect to a certain criterion that is indeed minimized in closed loop.*

Some computational aspects concerning the definition of the control law (2.26) are discussed now.

The stabilizing feedback (2.26) is implicitly defined by the damping equality (2.26) so that an exact solution might not be computed. Though, one can define approximate solutions still preserving global properties in closed loop. As an alternative, approximate solutions can be inferred so to guarantee *practical* stability properties while making the trajectories of the closed-loop (2.25) converge to a ball containing the origin whose radius descreases as δ^p does for some constant $p \geq 0$. In this sense, the above corollaries are given.

Corollary 2.1 (Existence and uniqueness of u-PBC solutions)**.** *Let the continuous time system (2.18) be passive with positive definite storage function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and ZSD. Then, there exists $\delta^* > 0$ so that, for any $\delta \in]0, \delta^*[$, the damping equality (2.26) admits a unique solution of the form*

$$u^{\delta}_{av}(x) = u_0(x) + \sum_{i>0} \frac{\delta^i}{(i+1)!} u_i(x) \tag{2.27}$$

with $u_0(x) = -\mathrm{L}_g S(x)$.

Solutions of the form (2.27) can be computed by setting up a recursive and constructive algorithm. After substituting (2.27) into (2.26) one solves, at each step i, a linear equation in the unknown $u_i(x)$ corresponding to the term with the i^{th}-power of δ. For the first terms, when setting for the sake of simplicity $K = 1$, one gets

$$u_0(x_k) = -h(x_k) \tag{2.28a}$$
$$u_1(x_k) = -\dot{h}(x_k) - \mathrm{L}_g \mathrm{L}_f S(x_k) \tag{2.28b}$$
$$u_2(x_k) = -\ddot{h}(x_k) + \frac{1}{2}\dot{h}(x_k)\mathrm{L}_g^2 S(x_k) - (\mathrm{L}_f \mathrm{L}_g + \mathrm{L}_g \mathrm{L}_f + h(x_k)\mathrm{L}_g^2)\mathrm{L}_f S(x_k) \tag{2.28c}$$

with $\dot{h}(x) = (\mathrm{L}_{f(\cdot)} - h(\cdot)\mathrm{L}_{g(\cdot)})h(x)$ and $\ddot{h}(x) = (\mathrm{L}_{f(\cdot)} - h(\cdot)\mathrm{L}_{g(\cdot)})^2 h(x)$. Thus, as $\delta \to 0$ one recovers the continuous-time solution.

Proposition 2.1 (Approximate solutions)**.** *Let the continuous time system (2.18) be passive with positive definite storage function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and ZSD. Then, then the p^{th}-order* approximate solution

$$u^{\delta[p]}_{av}(x) = u_0(x) + \sum_{i=0}^{p} \frac{\delta^i}{(i+1)!} u_i(x), \quad p \geq 0 \tag{2.29}$$

solves (2.26) in $O(\delta^{p+1})$. Moreover, the feedback $u_k = u^{\delta[p]}_{av}(x_k)$ makes the origin of (2.25a) pGAS. Equivalently, the sampled-data feedback $u_k = u^{\delta[p]}_{av}(x_k)$ makes the origin of (2.3) S-pGAS.

Remark 2.18. *We note that the 0-order approximate solution $u^{\delta[0]}_{av}(x_k) = -h(x_k)$ corresponds to the usual emulation-based feedback.*

Remark 2.19. *Several works have addressed the problem of exploiting the dissipation equality (2.19) to study properties of approximating Euler sampled-data models (2.8) when $p = 1$ in terms of passivity and with respect to emulation-based feedback [140, 141, 84].*

Remark 2.20. *The above proposition states that, in general, approximate solutions of the form (2.29) ensure convergence of the trajectories of the closed-loop system toward $B_{\delta^{p+1}}(0)$ whose radius decreases as δ does. Though, for certain values of δ, GAS of the origin might be still preserved. In the case of emulation-based controls, estimates of δ ensuring GAS of the origin of (2.3) in closed loop have been investigated in several works (e.g., [143, 110]).*

One other possible way of approximate (2.26) is by computing it over the free evolution of the system (2.25a) (i.e., along (2.9a)). More specifically, and setting $K = 1$, one computes

$$u + \int_0^1 \mathrm{L}_{G^\delta(\cdot,\theta u)} S(x^+(\theta u))\mathrm{d}\theta = 0$$

for $\theta = 0$ and gets

$$u_{\mathrm{ap}}(x) = -\mathrm{L}_{G^\delta(\cdot,0)} S(F_0^\delta(x))$$

solving (2.26) in $O(|u|^2)$. The above approximation is in general not suitable for stabilization purposes as it holds as u stay small. Though, it can be modified so to still guarantee global asymptotical stabilizing properties while ensuring the control to stay bounded. For, the following result is recalled from [113, 131] where a suitable dynamical gain is introduced to the approximate feedback $u_{\mathrm{ap}}(x)$.

Proposition 2.2 (A bounded approximate solution)**.** *Let the continuous time system (2.18) be passive with positive definite storage function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and ZSD. Then, for any $\mu > 0$, the feedback*

$$u_{av}^b(x) = -\lambda(x)\mathrm{L}_{G^\delta(\cdot,0)} S(F_0^\delta(x)) \tag{2.30}$$

with $|u_{av}^b(x)| \leq \mu$ for any $x \in \mathbb{R}^n$ and $\lambda : \mathbb{R}^n \to \mathbb{R}$ satisfying for any $x \in \mathbb{R}^n$

$$\lambda(x) \in]0, \frac{\mu}{(2\mu+1)(1+|\mathrm{L}_{G^\delta(\cdot,0)} S(F_0^\delta(x))|)} C(x)]$$
$$C(x) = \min_{|u|\leq\frac{1}{2}} \{1, \frac{|u|}{|Y_{av}^\delta(x,u) - \mathrm{L}_{G^\delta(\cdot,0)} S(F_0^\delta(x))|}\}$$

makes the origin of (2.25a) GAS. Equivalently, the sampled-data feedback $u_k = u_{av}^b(x)$ makes the origin of (2.3) S-GAS.

For completeness, we provide the notion of *u-Average Passivity from/around $\bar{u}$* as firstly introduced in our work [100].

Definition 2.11. [u-Average Passivity from/around $\bar{u}$] Let the continuous-time system (2.18) be passive with storage function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. Then, the sampled-data equivalent dynamics (2.4a) with output $Y^\delta(x,u) = \mathrm{L}_{G^\delta(\cdot,u)}S(x)$ is said to be *u-Average passive around* $\bar{u}$ if (2.4a) if, for any $k \geq 0$, the dissipative inequality

$$\Delta_k S(x) \leq \delta(u_k - \bar{u})Y^\delta_{\bar{u}}(x_k, u_k)$$

holds with

$$Y^\delta_{\bar{u}}(x,u) := \frac{1}{\delta(u-\bar{u})}\int_{\bar{u}}^{u} \mathrm{L}_{G^\delta(\cdot,v)}S(x^+(v))\mathrm{d}v. \tag{2.31}$$

Remark 2.21. *u-average passivity from $\bar{u}$ can be thought of as u-average passivity of (2.4) under the preliminary feedback transformation $u = \bar{u} + v$ so recovering, for $\bar{u} = 0$, the usual u-average passivity.*

Remark 2.22. *The notion of u-average passivity from $\bar{u}$ is strictly reminiscent of the concept of incremental passivity [152]. It defines incremental-like passivity of the overall system with respect to trajectories that are parametrized by different inputs u rather than time. Current work is addressing this aspect.*

Remark 2.23. *u-PBC extends to sampled-data systems which are u-Average Passive from $\bar{u}$ along the lines of Theorem 2.3.*

2.2.3 Input-Lyapunov Matching

The idea of matching the evolutions of the outputs of a given continuous-time system via piecewise constant control was firstly introduced in [117] within the context of feedback linearization. This strategy generally relies on the notion of formal series inversion (through the implicit function theorem [166]) and has been extended to larger problems of sampled-data feedback design through the idea of matching, at any sampling instants, the target behavior of a given nonlinear continuous-time system. As a target behavior one might be interested in (part of) the trajectories of the continuous-time system (e.g., as in feedback linearization [124]) or in the reproduction of a given smooth mapping testifying some properties of the closed loop one wants to preserve under sampled-data feedback. As an example of the latter case, one can think of matching the evolution of the Hamiltonian under a continuous-time optimal control (i.e., the continuous-time cost-value function [87]) so that the sampled-data closed-loop system preserves, at the sampling instants, optimality with respect to the functional minimized by continuous-time control.

Mathematically speaking, suppose that a nonlinear system (1.1a) admits an ideal continuous-time feedback $u(t) = \gamma(x(t))$ so that the origin of the closed-loop

$$\dot{x}(t) = f(x(t)) + g(x(t))\gamma(x(t)) \tag{2.32}$$

is GAS. The idea is to conceive a sampled-data feedback $u_k = \gamma^\delta(x_k)$ ensuring that the sampled-data closed-loop system

$$x_{k+1} = F^\delta(x_k, \gamma^\delta(x_k)) \tag{2.33}$$

reproduces, at any sampling instants $t = k\delta$ for $k \geq 0$, a target behavior of the ideal continuous-time system (2.32).

Input Lyapunov Matching (ILM) was firstly introduced in [180] and later refined in [179, 135]. The underlying idea consists in reproducing, at any sampling instant $t = k\delta$ for $k \geq 0$, the evolution of a (possibly strict) and radially unbounded Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ along (2.32) so to make the origin of (2.33) GAS. For this purpose, the following assumption is required.

Assumption 2.1 (Continuous-time smooth stabilizability)**.** *Let the origin of the continuous-time dynamics (1.1a) be globally asymptotically stabilized by a smooth feedback $u(t) = \gamma(x(t))$ with radially unbounded and strict-Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that*

$$\mathrm{L}_f V(x) + \gamma(x)\mathrm{L}_g V < 0 \quad \textit{and} \quad \mathrm{L}_g V(x) \neq 0 \tag{2.34}$$

for any $x \neq 0$.

Whenever this Assumption holds, one seeks for a feedback $u_k = \gamma^\delta(x_k)$ so that, at any sampling instant $t = k\delta$ and $k \geq 0$, the *ILM equality* holds

$$V(F^\delta(x_k, u_k)) - V(x_k) = \int_{k\delta}^{(k+1)\delta} \big[\mathrm{L}_{f(\cdot)} + \gamma(\cdot)\mathrm{L}_{g(\cdot)}\big] V(x(s))\mathrm{d}s. \tag{2.35}$$

The LHS and RHS of (2.35) define, respectively, the increment between to successive sampling instants of $V(x)$ over the sampled-data dynamics (2.4a) and the continuous-time one (2.32) when both initialized, at each step $k \geq 0$, as $x(k\delta) = x_k$. Because $\big[\mathrm{L}_{f(\cdot)} + \gamma(\cdot)\mathrm{L}_{g(\cdot)}\big] V(x(t)) < 0$ for any $x(t) \neq 0$ and $t \geq 0$, the feedback $u_k = \gamma^\delta(x_k)$ solution to (2.35) ensures GAS of (2.33) by matching. The following result can be thus recalled from [179].

Theorem 2.4 (Sampled-data stabilization under ILM)**.** *Let the dynamics (1.1a) verify Assumption 2.1 and let (2.4a) be its sampled-data equivalent model. Then, there exists $\delta^* > 0$ so that for any $\delta \in]0, \delta^*[$ the ILM equality (2.35) admits a unique solution $u_k = \gamma^\delta(x_k)$ in the form of an asymptotic series expansion in powers of δ around the continuous-time control $\gamma(x)$; i.e., it gets the form*

$$\gamma^\delta(x) = \gamma_0(x) + \sum_{i \geq 1} \frac{\delta^i}{(i+1)!}\gamma_i(x). \tag{2.36}$$

Moreover, the feedback $u_k = \gamma^\delta(x_k)$ makes the origin of the closed-loop (2.33) GAS. Equivalently, the sampled-data feedback $u_k = \gamma^\delta(x_k)$ makes the origin of (2.3) S-GAS in closed loop.

Remark 2.24. *Assumption 2.1 might be relaxed by requiring $V(x)$ to be a weak Lyapunov function. In this case, an ILM can be inferred and asymptotic stability of (2.33) follows invariance-like principles for discrete-time dynamics [99].*

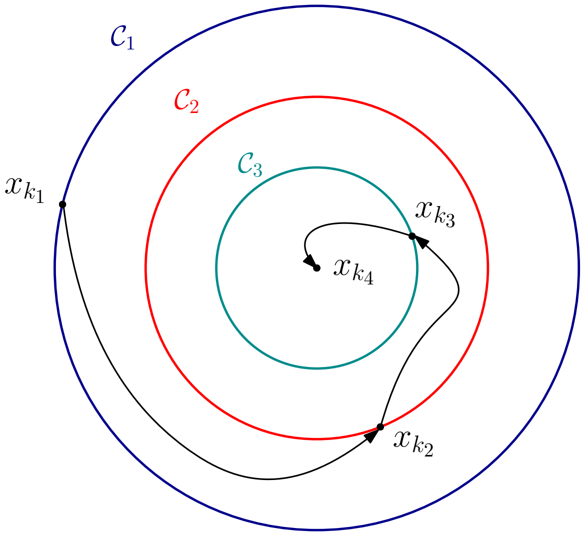


Figure 2.3: Trajectories of (2.33) and (2.3) under ILM with $x_{k_i} \in \mathcal{C}_i = \{x \in \mathbb{R}^n \text{ s.t. } V(x) = c_i, \quad c_i > 0\}$, $i = 1, 2, 3$ and $c_1 > c_2 > c_3$.

Remark 2.25. *When dealing with continuous-time passive systems (2.18), an ILM sampled-data feedback might be deduced from the continuous-time passivity-based controller if and only if (2.18) is ZSO. As a matter of fact, ZSD implies that* $L_g V(x) = 0$ *even for* $x \neq 0$ *so preventing from the inversion of (2.37).*

Remark 2.26. *At any sampling instant* $t = k\delta$*, the closed-loop trajectories of (2.33) jump onto descendent level sets of the function* $V(x)$ *which are approaching, at any* $t = k\delta$*, to the origin. As a drawback, the trajectories of* $V(x)$ *along the closed-loop (2.3a) and under the ILM feedback will be decreasing only at the sampling instants (and not for all time), albeit they stay, during the sampling period, in a neighborhood of the evolutions of* $V(x)$ *along (2.32).*

As in the u-PBC case, the feedback is implicitly defined by the nonlinear ILM equality (2.35) so that seeking for exact solutions might be tough. Still, each term of the series (2.36) can be computed through a recursive procedure by rewriting (2.35) as

$$e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)} V(x_k) - e^{\delta(\mathrm{L}_f + \gamma \mathrm{L}_g)} V(x_k) = 0. \tag{2.37}$$

By substituting (2.36) into (2.37), one needs to solve, at each step, a linear equation in the unknown $\gamma_i(x)$ by equating the terms in (2.37) with the same i^{th}-power of δ. For the first terms, one gets

$$\gamma_0(x_k) = \gamma(x_k) \tag{2.38a}$$
$$\gamma_1(x_k) = \dot{\gamma}(x_k) \tag{2.38b}$$
$$\gamma_2(x_k) = \ddot{\gamma}(x_k) + \frac{\mathrm{L}_{ad_f g} V(x_k)}{2\mathrm{L}_g V(x_k)} \dot{\gamma}(x_k) \tag{2.38c}$$

with $\dot{\gamma}(x) = (\mathrm{L}_{f(\cdot)} + \gamma(\cdot)\mathrm{L}_{f(\cdot)})\gamma(x)$ and $\ddot{\gamma}(x) = \dot{\gamma}(x) = (\mathrm{L}_{f(\cdot)} + \gamma(\cdot)\mathrm{L}_{f(\cdot)})^2\gamma(x)$ so that as $\delta \to 0$, $\gamma^\delta(x) \to \gamma(x)$ and recovers the continuous-time solution. Further computational aspects can be found in [179]. Again, only approximate solutions are implemented in practice so yielding pGAS of the origin in closed loop.

Proposition 2.3 (pGAS under approximate solutions)**.** *Let the continuous -time system (1.1) verify Assumption 2.1. Then, then the* p^{th}-order approximate solution

$$\gamma^{\delta[p]}(x) = \gamma_0(x) + \sum_{i=1}^{p} \frac{\delta^i}{(i+1)!}\gamma_i(x). \tag{2.39}$$

solves (2.35) in $O(\delta^{p+1})$*. Moreover, the feedback* $u_k = \gamma^{\delta[p]}(x_k)$ *makes the origin of (2.4a) pGAS. Equivalently, the sampled-data feedback* $u_k = \gamma^{\delta[p]}(x_k)$ *makes the origin of (2.3) S-pGAS.*

Remark 2.27. *As we shall show through simulations, considering first order approximate solutions of the form*

$$\gamma^{\delta[1]}(x) = \gamma(x) + \frac{\delta}{2}\dot{\gamma}(x).$$

significantly improves the performances in closed loop with respect to mere emulation schemes. Moreover, such an approximate solution is extremely easy to be computed and is shared by several other sampled-data feedback laws issued from both indirect and direct design strategies.

Remark 2.28. *Other than pGAS, approximate solutions to the ILM equality (2.35) guarantee a large variety of other properties in closed loop such as* one-step *consistency. For further details, the interested reader is referred to [179].*

Before concluding this chapter, it is worth to underline an important and common feature of the design methods we shall propose with emphasis on approximate control solutions for computational facilities. Starting from the exact sampled-data equivalent dynamics (2.4a), the approach we adopt is based on the following steps: first, we investigate the existence of a control solution to a given criterion over the exact sampled-data equivalent model; then, an approximate solution is computed and then implemented by approximating the criterion one wants the control to satisfy; finally, the properties yielded by those approximations in closed loop are studied with respect to the exact sampled-data model (2.4a). Hence, no approximation of the sampled-data equivalent model (2.4a) is needed throughout the design. This is different from the usual tendency in digital control. As a matter of fact, in this latter case, one first deduces a suitable approximate sampled-data equivalent model of the form (2.8) and then designs a feedback stabilizing the corresponding approximation of the model in closed loop (but, still, not the exact one (2.4a)) [160, 182]. Those two ways of performing digital control design do not commute, in general.

2.3 Conclusions and literature review

In this chapter, we have provided generalities and recalls on sampled-data nonlinear systems by emphasizing on the so-called sampled-data equivalent model approach. We have introduced the single and multi-rate equivalent sampled-data equivalent model to a given dynamics (1.1) and discussed on the issues and advantages arising in both cases. Accordingly, we have defined the notion of stability under sampling and provided several ways of carrying out the sampled-data feedback conception based on

- direct sampled-data average passivity for discrete time systems;
- indirect sampled-data Lyapunov-based design via Input-Lyapunov Matching.

Then, we have underlined the importance, in the sampled-data framework, of approximate feedback solutions as truncation of a given series expansion. Those controls are unavoidable in the actual implementation by stating the practical properties (in terms of stability) they usually yield when implemented.

Concerning sampled-data modeling of nonlinear systems, several other approaches are available other than the one we have discussed here. For example, Yuz and Goodwin have carried out different sampled-data equivalent models based on the so-called δ-operator where the sampled-data dynamics is described by its variation among two successive sampling instants in the attempt of mimicking the variation of the state evolutions with respect to time [192].

One other approach is based on looking at the effect of sampling as a time-delay so considering (2.3) as a time-delay dynamics of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t - \tau(t))$$

where $\tau(t) = t - k\delta$ and $\dot{\tau}(t) = 1$. Several works exploiting this representation have been carried out by Fridman, Hetel, Richard, Seuret and other researchers in the field also dealing with the case of aperiodic sampling sampling (e.g., [51, 45]). In that case, the sampled-data design might be developed within the time-delay frameworks.

One last way of looking at sampled-data systems is based on hybrid systems where the dynamics is described through set inclusions [137, 49] and in which both jumps and continuous flows might occur during the evolution in time. Accordingly, by considering an extended state space $\bar{x} = (x^\top, u, T)^\top$ one gets the dynamics

$$\begin{cases} \dot{x} = f(x) + g(x)u & \\ \dot{u} = 0 & \bar{x} \in \mathbb{R}^n \times \mathbb{R} \times [0, \delta] \\ \dot{T} = \alpha \in]0, 1] & \end{cases}$$

and

$$\begin{cases} x^+ = x & \\ u^+ = v & \bar{x} \in \mathbb{R}^n \times \mathbb{R} \times \{\delta\} \\ T^+ = 0 & \end{cases}$$

Concerning stability under approximate solutions, several works have been carried out in terms of preservation of global properties in closed loop or their practical or ISS characterization with reference to emulation-based controls. In this context, as an example, Mazenc [110] and Malisoff provide explicit bounds on the MASP by constructing a suitable Lyapunov-Krasovskii functional and by looking at sampling as a perturbing-like term even when considering actuator delays. Analysis within the framework of input time-delays has been performed by Karafyllis and Kravaris in [62] while some results for feedforward dynamics have been provided by Karafyllis and Krstic in [63]. Moreover, the Lyapunov-Razumikhin framework was also exploited to study stability of sampled-data systems when setting the sampling period as a design parameter which is adaptively changing over time as proposed by Fiter and co-authors in [37, 36].

Alternatively, other works are based on hybrid systems as the one by Carnevale, Teel and Nesic in [143] when inspired from networked control systems. Other than stability, step consistency properties of sampled-data controllers designed over approximate sampled-data models have been established by Nesic, Teel and Kokotovic in [144] to quantify the mismatch, at any sampling instants, of the real trajectories of the sampled-data systems with respect to its approximations. Other works exploiting dissipativity properties of sampled-data systems have been carried out by Omran, Hetel and co-workers to enhance local stability properties under emulation-based feedback and time-varying sampling (e.g., [147, 146]) for classes of nonlinear systems when considering the extended system

$$\begin{aligned} &\dot{x}(t) = f(x(t)) + g(x(t))\gamma(x(t)) + g(x(t))w(t) \\ &\nabla\gamma(x(t))\dot{x}(t) = y(t) \end{aligned}$$

for $t \in [t_k, t_{k+1}[$, $w(t) = -\int_{t_k}^{t_{k+1}} y(s)\mathrm{d}s$ and with $u = \gamma(x)$ being the continuous-time feedback. Finally, in [179] a characterization of general approximate sampled-data feedback laws computed over exact sampled-data models have been discussed in [179] in terms of ISS, one/multi-step consistency and practical stability.

Part II

Feedback design for cascade nonlinear systems under sampling

Chapter 3

Sampled I&I stabilization of strict-feedback dynamics

Contents

In this chapter, we shall focus on the stabilization of sampled-data cascade systems provided by the strict-feedback interconnection of nonlinear systems. In doing so, we are emphasizing on the way sampling affects both the structure of the interconnection (in terms of the corresponding sampled-data equivalent models) and, consequently, the properties of the overall dynamics. As far as stabilization is concerned we shall show that, although the feedback structure is not preserved under sampling, a constructive design can be still proposed via Immersion and Invariance. The contents of this chapter are based on part of the works in [97, 103].

M. Mattioni, S. Monaco and D. Normand-Cyrot. *Immersion and Invariance stabilization of strict-feedback dynamics under sampling.* Automatica. Volume 76, February 2017, pp. 78-86.

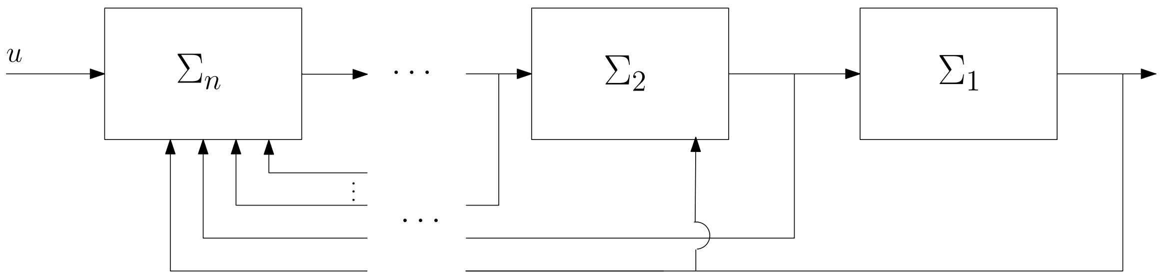


Figure 3.1: Feedback interconnection

M. Mattioni, S. Monaco and D. Normand-Cyrot. *Digital Stabilization of strict-feedback dynamics through Immersion and Invariance.* 1st IFAC Conference on Modeling, Identification and Control of Nonlinear Systems (MICNON). Volume 48, Issue 11, pp. 1074-1079. St. Petersburg, Russia. 2015.

3.1 Recalls on continuous-time strict-feedback systems

Strict-feedback systems are also referred to as *lower triangular structures* (Figure 3.1) and are described as follows

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \tag{3.1a}$$
$$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \tag{3.1b}$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, \dots, x_n) + g_n(x_1, \dots, x_n)u \tag{3.1c}$$

where $u \in \mathbb{R}$, $x_1 \in \mathbb{R}^p$ with $f_1(0) = 0$, $x_i \in \mathbb{R}$, $f_i(0, \dots, 0) = 0$ for $i = 2, \dots, n$ and, finally, $g_j(x_1, \dots, x_j) \neq 0$ for $j = 2, \dots, n$. Hereinafter, we are denoting $x = (x_1^\top, x_2, \dots, x_n)^\top$ and, for the sake of compactness

$$x^i = \begin{pmatrix} x_1 & \dots & x_i \end{pmatrix}^\top$$
$$f^i(x^i) = \begin{pmatrix} f^{i-1}(x^{i-1}) + x_i g^{i-1}(x^{i-1}) \\ f_i(x_1, \dots, x_i) \end{pmatrix}, \quad g^i(x^i) = \begin{pmatrix} \mathbf{0}_{(p+i-1)\times 1} \\ g_i(x_1, \dots, x_i) \end{pmatrix}$$

for $i = 2, \dots, n$ with $f^1(x^1) = f_1(x_1)$ and $g^1(x^1) = g_1(x_1)$.

The importance of dynamics admitting such a state-space representation relies upon the possibility of pursuing feedback design in an iterative and constructive fashion both in continuous and discrete time ([75, 61]). In the following parts, this aspect is illustrated over the elementary integrator-feedback interconnected cascade.

3.1.1 Backstepping design

Assume, for the time being, $n = 2$ and (3.1) specified as the simplest integrator-feedback interconnection; namely,

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \tag{3.2a}$$
$$\dot{x}_2 = u \tag{3.2b}$$

One first assumes x_2 to be a fictitious (or virtual) control input and seeks for a smooth feedback $x_2 = \gamma(x_1)$ so that the origin of the reduced system

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\gamma(x_1) \tag{3.3}$$

is GAS with a strict and radially unbounded function $W : \mathbb{R}^p \to \mathbb{R}$. Then, one defines an actual feedback $u = \nu(x)$ ensuring $z_2 := x_2 - \gamma(x_1) \to 0$ so achieving GAS of the origin of (3.2). This is made by considering the overall equivalent dynamics

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\gamma(x_1) + g_1(x_1)z_2 \tag{3.4a}$$
$$\dot{z}_2 = w \tag{3.4b}$$

under the feedback transformation

$$u = w + (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})\gamma(x_1)$$

where w is designed so to render $\mathcal{V}(x_1, z_2) = V(x_1) + \frac{1}{2}z_2^2$ a strict-Lyapunov function for the system in closed-loop. By computing

$$\dot{\mathcal{V}}(x_1, z_2) = (\mathrm{L}_{f_1} + \gamma\mathrm{L}_{g_1})V(x_1) + z_2\mathrm{L}_{g_1}V(x_1) + z_2 w$$

one gets that $w = -kz_2 - \mathrm{L}_g V(x_1)$ makes the origin of (3.4) GAS for any $k > 0$ so that the overall feedback rewrites as

$$u = -k(x_2 - \gamma(x_1)) - \mathrm{L}_{g_1}V(x_1) + (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})\gamma(x_1) \tag{3.5}$$

This extends to general strict-feedback structures (3.1) while exploiting the recurrent cascade interconnection. Basically, starting from smooth stabilizability of

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\gamma_1(x_1), \quad \gamma_1(x_1) = \gamma(x_1)$$

at each step i ($i = 1, \ldots, n-1$), one seeks for a virtual feedback $x_{i+1} = \gamma_i(x_1, \ldots, x_i)$ so to have, asymptotically, $z_{i+1} := x_i - \gamma_{i-1}(x_1, \ldots, x_{i-1}) \to 0$ until deducing the real feedback $u = \nu(x)$ stabilizing the origin of the overall system.

This procedure applies to a large class of practical systems (e.g., robot with flexible joints, spacecraft attitude dynamics) so that, when encountered in continuous time, one can rather easily deduce the stabilizing feedback. Though, when looking at (3.1) as a sampled-data system, things get complicated and a backstepping-like procedure cannot be generally carried out. With this in mind, in the following section, we'll recall how I&I applies to strict-feedback dynamics by claiming that this control technique is more suitable for extension to the sampled-data context.

3.1.2 I&I for continuous-time strict-feedback systems

Assuming $x_2 = \gamma(x_1)$ to be a stabilizing fictitious (or virtual) control for (3.3) with a strict and radially unbounded function $W : \mathbb{R}^p \to \mathbb{R}$ is enough to deduce I&I stabilizability of (3.2) in the sense of Definition 1.16. As a matter of fact, setting $\xi \in \mathbb{R}^p$, the origin of the target dynamics

$$\dot{\xi} = f_1(\xi) + g_1(\xi)\gamma(\xi) \tag{3.6}$$

is GAS and one can deduce an immersion mapping $\pi(\xi) = (\xi^\top\ \gamma(\xi))^\top$ and a feedback $c(\xi) = \dot{\gamma}(\xi)$. As a consequence, one can immediately set $z = x_2 - \gamma(x_1)$ as the off-the-manifold component so that *iii*) in Definition 1.16 holds. Thus, one can verify that *iv*) in Definition 1.16 holds under the feedback

$$\nu(x, z) = (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})\gamma(x_1) - K(x)z \tag{3.7}$$

with $K(x) > 0$ such that, for any smooth $\rho(x_1) > 0$ and $M > 0$, the following inequalities hold

$$(\mathrm{L}_{f_1} + \gamma\mathrm{L}_{g_1})W(x_1) + \frac{\|\mathrm{L}_{g_1}W\|^2}{\rho(x_1)} < 0 \text{ for any } \|x_1\| > M \tag{3.8a}$$

$$K(x) \geq \rho(x_1). \tag{3.8b}$$

Remark 3.1. *It is worth to note that the I&I feedback (3.7) gets a much more simple form than the one resulting from backstepping design (3.5).*

3.2 Strict-feedback systems under sampling

In this section, we are going to discuss on strict-feedback structures under sampling by focusing on the way this form is transformed by sampling onto the sampled-data equivalent model to the dynamics (3.1). For the sake of simplicity, we shall first provide the details for the simpler case (3.2).

3.2.1 The integrator-feedback interconnection case

For the sake of simplicity, consider again $n = 2$ and (3.2) and assume $u \in \mathcal{U}^\delta$ so that (3.2) rewrites as

$$\dot{x}_1(t) = f_1(x_1(t)) + g_1(x_1(t))x_2(t) \tag{3.9a}$$

$$\dot{x}_2(t) = u_k, \quad t \in [k\delta, (k+1)\delta[. \tag{3.9b}$$

Then, the sampled-data equivalent model of (3.2) gets the form

$$x_{1k+1} = F_1^\delta(x_{1k}, x_{2k}) + \frac{\delta^2}{2}u_k\mathcal{G}_1^\delta(x_{1k}, x_{2k}, u_k) \tag{3.10a}$$

$$x_{2k+1} = x_{2k} + \delta u_k \tag{3.10b}$$

with

$$
\begin{aligned}
F_1^\delta(x_1, x_2) &= e^{\delta \mathrm{L}_{f_1} + x_2 \mathrm{L}_{g_1}} x_1 && (3.11)\\
&= e^{\delta \mathrm{L}_{f_1}} x_1 + \int_0^{x_2} e^{\delta(\mathrm{L}_{f_1} + v\mathrm{L}_{g_1})} \mathcal{F}_1^\delta(x_1, v)\mathrm{d}v \\
\mathcal{F}_1^\delta(x_1, x_2) &= \int_0^\delta e^{-s ad_{f_1 + x_2 g_1}} g_1(x_1)\mathrm{d}s && (3.12)\\
\frac{\delta^2}{2} u \mathcal{G}_1^\delta(x_1, x_2, u) &= \int_0^u e^{\delta(\mathrm{L}_{f^2} + v\mathrm{L}_{g^2})} G_1^\delta(x_1, x_2, v)\mathrm{d}v && (3.13)\\
G_1^\delta(x, u) = G_1^\delta(x_1, x_2, u) &= e_1^\top \int_0^\delta e^{-s ad_{f^2 + uB_2}} B_2 \mathrm{d}s && (3.14)\\
B_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad e_1^\top &= \begin{pmatrix} 1 & 0 \end{pmatrix}.
\end{aligned}
$$

One can compute both (3.12) and (3.16) as series expansions in powers of δ, so getting

$$
F_1^\delta(x_1, x_2) = x_1 + \delta(\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})x_1 + \sum_{i>1} \frac{\delta^i}{i!} (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})^i x_1 \tag{3.15}
$$

$$
\frac{\delta^2}{2} u \mathcal{G}_1^\delta(x_1, x_2, u) = \frac{\delta^2}{2} \mathcal{G}_1^0(x_1, x_2, u) + \sum_{i>0} \frac{\delta^{i+2}}{(i+2)!} \mathcal{G}_1^i(x_1, x_2, u) \tag{3.16}
$$

with

$$
\begin{aligned}
\mathcal{G}_1^0(x_1, x_2, u) =& g_1(x_1)\\
\mathcal{G}_1^i(x_1, x_2, u) =& u\nabla_{x_2}\left[(\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})^{i-1} x_1 + \mathcal{G}_1^{i-1}(x_1, x_2, u)\right]\\
&+ (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})\mathcal{G}_1^{i-1}(x_1, x_2, u).
\end{aligned}
$$

It is clear from the above expressions that (3.10) does not preserve

- the input-affine structure of the continuous-time model;
- the feedback (and, thus, the cascade) structure so preventing from applying backstepping-like design strategies exploiting the nested structure.

Remark 3.2. *When considering the* 1^{st}*-order Euler approximate sampled-data equivalent deduced from (3.10) one obtains*

$$
\begin{aligned}
x_{1k+1}^{[1]} =& x_{1k}^{[1]} + \delta(f_1(x_{1k}^{[1]}) + x_{2k}^{[1]} g_1(x_{1k}^{[1]}))\\
x_{2k+1}^{[1]} =& x_{2k}^{[1]} + \delta u_k
\end{aligned}
$$

so preserving the strict-feedback structure. Accordingly, several works (e.g., [142, 160]) are aimed at performing control design over such an approximate sampled-data model through backstepping-like design methods. In [18] the authors design the controller over higher order approximate models.

Though the loss of the nested structure, it is worth to point out that the control u comes to directly affect the dynamics of the state x_1 in $O(\delta^r)$ where $r = 2$ is the relative degree of the dynamics (3.2) with respect to the dummy output $y = x_1$. Thus, the feedback structure is transformed by sampling into a scaled (in terms of δ) influence of the control over the dynamics of the state x_1.

3.2.2 The general case

Consider now the general feedback system (3.1). When $u \in \mathcal{U}^\delta$ and measures (of the state) are available only at the sampling instants $t = k\delta$, (3.1) rewrites, for $t \in [k\delta, (k+1)\delta[$, as

$$\dot{x}_1(t) = f_1(x_1(t)) + g_1(x_1(t))x_2(t) \tag{3.17a}$$
$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)) + g_2(x_1(t), x_2(t))x_3(t) \tag{3.17b}$$
$$\vdots$$
$$\dot{x}_n(t) = f_n(x_1(t), \ldots, x_n(t)) + g_n(x_1(t), \ldots, x_n(t))u_k. \tag{3.17c}$$

Accordingly, by exploiting Definition 2.1, the single-rate sampled-data equivalent model of (3.1) specifies as

$$x_{1k+1} = F_1^\delta(x_k) + \frac{\delta^n}{n!} u_k \mathcal{G}_1^\delta(x_k, u_k) \tag{3.18a}$$
$$x_{2k+1} = F_2^\delta(x_k) + \frac{\delta^{n-1}}{(n-1)!} u_k \mathcal{G}_2^\delta(x_k, u_k) \tag{3.18b}$$
$$\vdots$$
$$x_{nk+1} = F_n^\delta(x_k) + \delta u_k \mathcal{G}_n^\delta(x_k, u_k) \tag{3.18c}$$

where

$$\begin{aligned} F_i^\delta(x) =& e^{\delta(\mathrm{L}_{f^i} + x_{i+1}\mathrm{L}_{g^i})} x_i \\ =& F_{i,i}^\delta(x_1, \ldots, x_{i+1}) + \sum_{j=2}^{n-i} \frac{\delta^j}{j!} x_{i+j} F_{i,i+j-1}^\delta(x_1, \ldots, x_{i+j-1}) \\ =& x_i + \sum_{j=1}^{n-i} \frac{\delta^j}{j!} (\mathrm{L}_{f^{i+j-1}} + x_{i+j}\mathrm{L}_{g^{i+j-1}})^j x_i + \sum_{j>n-i} \frac{\delta^j}{j!} \mathrm{L}_{f^n}^j x_i \end{aligned} \tag{3.19}$$

$$\frac{\delta^{n-i+1}}{(n-i+1)!} u \mathcal{G}_i^\delta(x, u) = \int_0^u e^{\delta(\mathrm{L}_{f^n} + v\mathrm{L}_{g^n})} G_i^\delta(x, v) \mathrm{d}v \tag{3.20}$$

$$\tag{3.21}$$

$$G_i^\delta(x, u) = \int_0^\delta e^{-s ad_{f^n + ug^n}} g^n(x) \mathrm{d}s \tag{3.22}$$

and, letting $i = 1, \dots, n-1$

$$\begin{aligned}
&\lim_{\delta \to 0} F^{\delta}_{i,i}(x_1, \dots, x_{i+1}) = f_i(x_1, \dots, x_i) + x_{i+1} g_i(x_1, \dots, x_i) \\
&\lim_{\delta \to 0} F^{\delta}_{i,j}(x_1, \dots, x_{i+j-1}) = \prod_{\ell=i}^{j} g_\ell(x_1, \dots, x_\ell) \\
&\lim_{\delta \to 0} \mathcal{G}^{\delta}_{i}(x, u) = \prod_{\ell=i}^{n} g_\ell(x_1, \dots, x_\ell) \\
&\lim_{\delta \to 0} F^{\delta}_{n}(x) = f_n(x_1, \dots, x_n) \\
&\lim_{\delta \to 0} \mathcal{G}^{\delta}_{n}(x, u) = g_n(x_1, \dots, x_n).
\end{aligned}$$

As in the simpler case in (3.10), it is worth to remark that

- the input-affine structure is lost as any mapping $\mathcal{G}^{\delta}_{i}(\cdot, u)$ nonlinearly depends on u;
- the nested feedback structure is not preserved by the sampled-data equivalent model (3.18).

This loss prevents from applying backstepping-like methodologies under sampling. Though, again, the structure of the mapping defining (3.18) underlines that the feedback structure induces a scaling (in powers of δ) of the influence of the component x_i over the dynamics of x_j ($j < i-1$) in $O(\delta^{i-j})$; as an example, x_1 is influenced by the component x_3 in $O(\delta^2)$ while x_2 is affected by x_5 in $O(\delta^3)$. Moreover, the dynamics of any state x_i is explicitly depending by the control signal u in $O(\delta^{n-i+1})$ where $n - i$ corresponds to the relative degree of the continuous-time system (3.1) with respect to the dummy output $y_i = x_i$ for $i = 1, \dots, n$.

Remark 3.3. *For this class of systems, in [179], a multi-rate Input-Lyapunov Matching approach was introduced for designing feedback in a backstepping-like fashion through the reproduction of the closed-loop Lyapunov function deduced from the continuous-time design.*

3.2.3 Sampled-data I&I

Hereinafter, we are going to show how I&I can be recast into the framework of sampled-data design for cascade systems providing a constructive design procedure even when the continuous-time structure is lost and, thus, intuitive continuous-time-like methodologies fail in being applied.

As already commented in Chapter 2, when dealing with sampled-data feedback systems, properties might be ensured at any sampling instant $t = k\delta$ when $k \geq 0$. Accordingly, even when speaking about I&I under digital feedback, we are going to require that all the properties in Definition 1.16 hold at any sampling instant. In doing so, the following definition is instrumental.

Definition 3.1. [SD-I&I] The dynamics (1.1a) is said to be *SD-I&I stabilizable* if the sampled-data equivalent model (2.4a) is I&I stabilizable; i.e., if, for some $p < n$, there exist mappings

$$\alpha^\delta : \mathbb{R}^p \to \mathbb{R}^p, \quad \pi^\delta : \mathbb{R}^p \to \mathbb{R}^{p+n-1}, \quad c^\delta : \mathbb{R}^p \to \mathbb{R} \tag{3.23}$$

$$\phi^\delta : \mathbb{R}^{p+n-1} \to \mathbb{R}^{n-1}, \quad \nu^\delta : \mathbb{R}^{p+n-1} \times \mathbb{R}^{n-1} \to \mathbb{R} \tag{3.24}$$

such that the following conditions hold true.

i) *Target system* - The system

$$\xi_{k+1} = \alpha^\delta(\xi_k) \tag{3.25}$$

with $\xi \in \mathbb{R}^p$ has a GAS equilibrium at the origin and $\pi^\delta(0) = 0$.

ii) *Invariance condition* - For all $\xi \in \mathbb{R}^p$

$$\pi^\delta(\alpha^\delta(\xi)) = F^\delta(\pi^\delta(\xi), c^\delta(\xi)). \tag{3.26}$$

iii) *Implicit manifold* - The following set identity holds

$$\{x \in \mathbb{R}^n \text{ s. t. } \phi^\delta(x) = 0\} \equiv \{x \in \mathbb{R}^n \text{ s. t. } x = \pi^\delta(\xi) \text{ for some } \xi \in \mathbb{R}^p\} \tag{3.27}$$

iv) *Manifold attractivity and boundedness of trajectories* - All trajectories of the system

$$z_{k+1} = \phi^\delta(F^\delta(x_k, \nu^\delta(x_k, z_k))) \tag{3.28a}$$

$$x_{k+1} = F^\delta(x_k, \nu^\delta(x_k, z_k)) \tag{3.28b}$$

are bounded with $\lim_{k\to\infty} z_k = 0$, $\nu^\delta(\pi^\delta(\xi), 0) = c^\delta(\xi)$ and $z_0 = \phi^\delta(x_0)$.

Along the lines of Definition 3.1, the dynamics (1.1) is said to be *SD-I&I stabilizable under multi-rate sampling* (MR-I&I Stabilizable) if the multi-rate sampled-data equivalent model (2.14a) is I&I stabilizable.

Thus, when dealing with sampled-data systems, one seeks for a digital feedback $u_k = \nu^\delta(x_k, z_k)$ that makes a certain manifold $\mathcal{M}^\delta$ attractive and invariant, at the sampling instants, in the sense that trajectories of (2.3a) starting in $\mathcal{M}^\delta$ will stay in $\mathcal{M}^\delta$ at any $t = k\delta$ while staying close to $\mathcal{M}^\delta$ and bounded when $t \in]k\delta, (k+1)\delta[$. Accordingly, hereinafter we seek for an answer to the following question.

Does continuous-time I&I stabilizability of a continuous-time system (1.1a) imply I&I stabilizability of its sampled-data equivalent model (2.4) and, thus, SD-I&I stabilizability?

This is still an open question. Indeed, we shall provide answers for cascade systems as given by the strict-feedback interconnection of nonlinear systems. To this end, we are going to show how I&I stabilizability for the elementary integrator-feedback interconnection (3.2) can be enforced through single-rate sampling. Then, we shall extend the result to general strict-feedback systems (3.1) by showing how multi-rate sampling is necessary for preserving I&I stabilizability within' the digital context.

Hereinafter, the following assumption over the continuous-time plant (3.1a) is set as recurrent for strict-feedback dynamics.

Assumption 3.1 (Continuous-time stabilizability via fictitious feedback)**.** *The origin of (3.1a) (equivalently, (3.2a)) is globally asymptotically stabilizable by a smooth fictitious feedback $x_2(t) = \gamma(x_1(t))$ with $\gamma(0) = 0$ and a radially unbounded and strict Lyapunov function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ verifying, for any $x_1 \neq 0$*

$$(\mathrm{L}_{f_1(\cdot)} + \gamma(\cdot)\mathrm{L}_{g_1(\cdot)})V(x_1) < 0 \tag{3.29}$$

$$\mathrm{L}_{g_1} W(x_1) \neq 0. \tag{3.30}$$

As shown in Sections 3.1.1 and 3.1.2, Assumption 3.1 is sufficient to deduce I&I stabilizability of (3.1a) in continuous time with target dynamics

$$\dot{\xi} = f_1(\xi) + g_1(\xi)\gamma(\xi). \tag{3.31}$$

In what follows, we are going to study the involvement of the above target in the sampled-data I&I framework.

3.3 I&I for strict-feedback systems under sampling: the integrator-feedback interconnection case

Consider (3.2) and its sampled-data equivalent model (3.10). As one might expect, the loss of the sampled-data structure prevents from establishing I&I stabilizability of (3.10). As a matter of fact, choosing the sampled-data target dynamics as the discrete-time equivalent model of (3.31)

$$\xi_{k+1} = e^{\delta(\mathrm{L}_{f_1} + \gamma \mathrm{L}_{g_1})}\xi\big|_{\xi_k} \tag{3.32}$$

with $\xi_k := \xi(k\delta)$ is not even a suitable reduced dynamics for (3.10) in the sense of i) and ii) of Definition 3.1. Indeed, the fictitious feedback $x_2 = \gamma(x_1)$ acting over the continuous-time dynamics is allowed to be a fully continuous (and smooth) signal with possibly nonzero r^{th}-order derivative albeit, under sampling, x_2 is piecewise linear and provided by

$$x_2(t) = x_{2k} + (t - k\delta)u_k \quad t \in [k\delta, (k+1)\delta[.$$

Accordingly, one has $\frac{\mathrm{d}^i}{\mathrm{d}t^i} x_2(t) \equiv 0$ for any $i \geq 2$ and $t \in [k\delta, (k+1)\delta[$. Moreover, contrarily to (3.10a), the continuous-time target (3.31) is not explicitly influenced by the control $u = c(\xi)$. This motivates the necessity for the sampled-data redesign to fully address I&I stabilizability starting from the choice of a suitable target dynamics. Accordingly, we are going to propose a two steps design by

1. exhibiting a new discrete-time target dynamics evolving over a suitable stable manifold $\mathcal{M}^\delta$ so verifying $i)$ to $iii)$ in Definition 3.1;

2. designing a feedback ensuring convergence (and boundedness) of the trajectories of (3.10) onto $\mathcal{M}^\delta$ and, thus I&I stabilization.

In doing so, we aim at exploiting the particular structure that (3.10) inherits from the continuous-time strict-feedback form (3.2). Specifically, after exhibiting a suitable structure of the sampled-data target we introduce an ILM problem over it ensuring stability of its equilibrium and control-invariance of the consequent manifold $\mathcal{M}^\delta$. In doing so, the involved mappings and the invariant manifold as well will be parametrized by the sampling period δ so differing, in general, from their continuous-time counterparts.
Finally, the actual feedback is based on three different sampled-data design strategies that ensure convergence to $\mathcal{M}^\delta$ relying upon

1. direct discrete-time design by constructing the feedback so to stretch the trajectories of (3.2) onto $\mathcal{M}^\delta$ in exactly one step;

2. direct sampled-data design by constructing the feedback so to making a suitably deduced Lyapunov function bounded over the trajectories of (3.2) at any sampling instant;

3. indirect sampled-data design by tracking, at any sampling instant and under piecewise constant control, the off-the-manifold component associated to the continuous-time I&I design;

3.3.1 The double LTI integrator as a motivating example

Consider the double integrator

$$\dot{x}_1 = x_2 \tag{3.33a}$$
$$\dot{x}_2 = u \tag{3.33b}$$

clearly verifying Assumption 3.1 for

$$\gamma(x_1) = -x_1 \quad \text{and} \quad W(x_1) = \frac{1}{2}x_1^2. \tag{3.34}$$

Thus, I&I stabilizability in continuous-time can be directly inferred by setting the GES target dynamics as

$$\dot{\xi} = -\xi. \tag{3.35}$$

and $\pi(\xi) = (\xi \ -\xi)^\top$, $c(\xi) = \xi$. Assume now $u \in \mathcal{U}^\delta$ and compute the sampled-data equivalent model to (3.33) as

$$x_{1k+1} = x_{1k} + \delta x_{2k} + \frac{\delta^2}{2} u_k \tag{3.36a}$$

$$x_{2k+1} = x_{2k} + \delta u_k. \tag{3.36b}$$

It is clear that (3.36) does not preserve the feedback structure of (3.33) although the first term in (3.36a) where u appears is in $O(\delta^3)$. As a consequence, (3.36) is not I&I stabilizable with target dynamics deduced by discretizing the continuous-time (3.35) as

$$\xi_{k+1} = e^{-\delta} \xi_k. \tag{3.37}$$

and the same mappings in continuous time. As a matter of fact, the continuous-time choices $\pi(\xi) = (\xi \ -\xi)^\top$ and $c(\xi) = \xi$ do not verify the sampled-data invariance condition $ii)$ in Definition 3.1 which specifies here as

$$e^{-\delta}\xi = (1 - \delta + \frac{\delta^2}{2})\xi \tag{3.38a}$$

$$e^{-\delta}\xi = (1 - \delta)\xi. \tag{3.38b}$$

Accordingly, a total redesign of both the target system and of the consequent immersion mapping needs to be readdressed. To this end, in what follows, we are going to provide a schematic and constructive procedure for computing a new asymptotically stable target dynamics for the sampled-data equivalent model (3.36) together with new immersion mapping and on-the-manifold feedback fulfilling the invariance condition $ii)$ in Definition 3.1.

3.3.2 The design of the target dynamics under sampling

As we have previously commented on, starting from Assumption 3.1, the choice $\pi(\xi) = (\xi^\top \ \gamma(\xi))^\top$ and $c(\xi) = \dot{\gamma}(\xi)$ is not satisfying $i)$ to $iii)$ of Definition 3.1 about SD-I&I stabilizability although they do qualify for the continuous-time design. Consequently, a new target dynamics needs to be computed together with new immersion mapping $\pi^\delta(\cdot)$ and on-the-manifold control $c^\delta(\cdot)$ making the corresponding manifold invariant. In doing so, we shall show that it is necessary for all of the related mappings (and thus the manifold) to be explicitly smoothly depending on the sampling period δ.

Assume that the sampled-data immersion mapping $\pi^\delta : \mathbb{R}^p \to \mathbb{R}^{p+1}$ takes the form

$$x = \pi^\delta(\xi) = \begin{pmatrix} \xi \\ \gamma^\delta(\xi) \end{pmatrix} \tag{3.39}$$

with $\gamma^\delta : \mathbb{R}^p \to \mathbb{R}$, $\gamma^\delta(0) = 0$ and $\xi \in \mathbb{R}^p$ being the state of the sampled-data target dynamics we let possess the following structure

$$\xi_{k+1} = F_1^\delta(\xi_k, \gamma^\delta(\xi_k)) + \frac{\delta^2}{2!} c^\delta(\xi_k) \mathcal{G}_1^\delta(\xi_k, \gamma^\delta(\xi_k), c^\delta(\xi_k)) := \alpha^\delta(\xi_k) \tag{3.40}$$

with $F_1^\delta(\cdot, \cdot)$ and $\mathcal{G}_1^\delta(\cdot, \cdot, \cdot)$ as given in (3.12) and (3.16).

Thus, the problem is lead to the one of defining the pair $< \gamma^\delta(\xi), c^\delta(\xi) >$ so that (3.40) qualifies as a target dynamics for (3.10) with GAS equilibrium at the origin. Moreover, they need to verify the invariance condition *ii*) in Definition 3.1. The solution we propose is based on defining $< \gamma^\delta(\xi), c^\delta(\xi) >$ as the solution to an Input-Lyapunov Matching problem over the target dynamics (3.40) constrained to guarantee its invariance. Roughly speaking, under Assumption 3.1, we aim at constructing $\gamma^\delta(\cdot)$ so that, at any sampling instants $t = k\delta$, the evolutions of the Lyapunov function $W(\cdot)$ along the so-described sampled-data target (3.40) exactly reproduce the ones of $W(\cdot)$ along the continuous-time (3.6). At the same time, the feedback $c^\delta(\cdot)$ is computed to ensure the closed-loop invariance of (3.40) with respect to the original dynamics (3.10). This leads to the following result.

Proposition 3.1 (Existence and uniqueness of $< \gamma^\delta(\xi), c^\delta(\xi) >$). *Let the strict feedback dynamics (3.2) verify Assumption 3.1 and (3.10) be its sampled-data equivalent model. Consider $\alpha^\delta_{v_1,v_2} : \mathbb{R}^p \to \mathbb{R}^p$ as of the form*

$$\alpha^\delta_{v_1,v_2}(\xi) := F_1^\delta(\xi, v_1(\xi)) + \frac{\delta^2}{2!} v_2(\xi) \mathcal{G}_1^\delta(\xi, v_1(\xi), v_2(\xi)) \tag{3.41}$$

with $v_i : \mathbb{R}^p \to \mathbb{R}$ for $i = 1, 2$. Then, there exists $\delta^* > 0$ such that for any $\delta \in]0, \delta^*[$ the equalities

$$W(\alpha^\delta_{v_1,v_2}(\xi_k)) - W(\xi_k) = \int_{k\delta}^{(k+1)\delta} [\mathrm{L}_{f_1(\cdot)} + \gamma(\cdot)\mathrm{L}_{g_1(\cdot)}] W(\xi(s)) \mathrm{d}s \tag{3.42a}$$

$$v_1(\alpha^\delta_{v_1,v_2}(\xi_k)) = v_1(\xi_k) + \delta v_2(\xi_k) \tag{3.42b}$$

admit, for any $k \geq 0$, unique solutions $v_1(\xi) = \gamma^\delta(\xi)$ and $v_2(\xi) = c^\delta(\xi)$ of the form of series expansions in powers of δ; i.e.,

$$\gamma^\delta(\xi) = \gamma^0(\xi) + \sum_{i>0} \frac{\delta^i}{(i+1)!} \gamma^i(\xi) \tag{3.43a}$$

$$c^\delta(\xi) = c^0(\xi) + \sum_{i>0} \frac{\delta^i}{(i+1)!} c^i(\xi). \tag{3.43b}$$

Proof: The proof is constructive and is worked out by solving (3.42) via a bottom-up procedure. Assuming, first, $v_1 = \gamma^\delta(\xi)$ and $v_2 = c^\delta(\xi)$ of the form (3.43), one substitutes them into the corresponding values in (3.42) and then compares the terms with the same power of δ so getting an infinite number of linear equations to solve. The existence of a solution $(\gamma^i(\xi), c^i(\xi))$ to any equation corresponding to the term δ^{i+1} is ensured by Assumption 3.1 and the strict-feedback structure. More in details, by remarking that $\alpha^{\delta}_{\gamma^{\delta}(\xi),c^{\delta}(\xi)}(\xi)=\alpha^{\delta}(\xi)$ in (3.40) one rewrites (3.42) as

$$W(\alpha^{\delta}(\xi_k))-W(\xi_k)=\int_{k\delta}^{(k+1)\delta}[\mathrm{L}_{f1(\cdot)}+\gamma(\cdot)\mathrm{L}_{g_1(\cdot)}]W(\xi(s))\mathrm{d}s \tag{3.44a}$$

$$\gamma^{\delta}(\alpha^{\delta}(\xi_k))=\gamma^{\delta}(\xi_k)+\delta c^{\delta}(\xi_k). \tag{3.44b}$$

Accordingly, by exploiting the Lie exponential and carrying out some nasty computations, one obtains

$$\gamma^{\delta}(\alpha^{\delta}(\xi_k))=e^{\delta(\mathrm{L}_{f_1}+\gamma^{\delta}(\xi_k)\mathrm{L}_{g_1})}\gamma^{\delta}(\xi_k)+\int_0^{c^{\delta}(\xi_k)}\mathrm{L}_{G_1^{\delta}(\cdot,\gamma^{\delta}(\cdot),v)}\gamma^{\delta}(\alpha^{\delta}(\xi))\mathrm{d}v$$

By substituting the above expression into (3.44b) and denoting

$$\frac{\delta^2}{2}c^{\delta}(\xi)\Theta_1^{\delta}(\xi)=\int_0^{c^{\delta}(\xi_k)}\mathrm{L}_{G_1^{\delta}(\cdot,\gamma^{\delta}(\cdot),v)}\gamma^{\delta}(\alpha^{\delta}(\xi))\mathrm{d}v$$

let us define

$$\delta Q_2^{\delta}(\xi,\gamma^{\delta}(\xi_k),c^{\delta}(\xi_k))=e^{\delta(\mathrm{L}_{f_1}+\gamma^{\delta}(\xi_k)\mathrm{L}_{g_1})}\gamma^{\delta}(\xi_k)-\gamma^{\delta}(\xi_k)-\delta(1-\frac{\delta}{2}\Theta_1^{\delta}(\xi_k))c^{\delta}(\xi_k).$$

Similarly, one can define

$$\begin{aligned}\delta Q_1^{\delta}(\xi,\gamma^{\delta}(\xi_k),c^{\delta}(\xi_k))=&e^{\delta(\mathrm{L}_{f_1}+\gamma^{\delta}(\xi_k)\mathrm{L}_{g_1})}W(\xi_k)-e^{\delta(\mathrm{L}_{f_1(\cdot)}+\gamma(\cdot)\mathrm{L}_{g_1(\cdot)})}W(\xi_k)\\&+\frac{\delta^2}{2}\Theta_2^{\delta}(\xi)c^{\delta}(\xi)\end{aligned}$$

with

$$\frac{\delta^2}{2}c^{\delta}(\xi)\Theta_2^{\delta}(\xi)=\int_0^{c^{\delta}(\xi)}\mathrm{L}_{G_1^{\delta}(\cdot,,\gamma^{\delta}(\cdot),v)}W(\alpha^{\delta}(\xi))\mathrm{d}v.$$

As a consequence, (3.44) rewrites as a formal series equality in $<\gamma^{\delta}(\xi),c^{\delta}(\xi)>$; i.e.,

$$Q^{\delta}(\xi,\gamma^{\delta}(\xi),c^{\delta}(\xi))=\begin{pmatrix}Q_1^{\delta}(\xi,\gamma^{\delta}(\xi),c^{\delta}(\xi))\\Q_2^{\delta}(\xi,\gamma^{\delta}(\xi),c^{\delta}(\xi))\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{3.45}$$

with

$$Q_i^{\delta}(\xi,v_1,v_2)=Q_i^0(\xi,v_1,v_2)+\sum_{j>0}\frac{\delta^j}{(j+1)!}Q_i^j(\xi,v_1,v_2).$$

As a consequence, as $\delta\to 0$, one obtains

$$Q^{\delta}(\xi,\gamma^{\delta}(\xi),c^{\delta}(\xi))\to\begin{pmatrix}(\mathrm{L}_{f_1}+\gamma^0(\xi)\mathrm{L}_{g_1})W(\xi)-(\mathrm{L}_{f_1}+\gamma(\xi)\mathrm{L}_{g_1})W(\xi)\\(\mathrm{L}_{f_1}+\gamma^0\mathrm{L}_{g_1})\gamma^0(\xi)-c^0(\xi)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

that is solved by the continuous-time solution

$$\gamma^0(\xi) = \gamma(\xi), \quad \text{and} \quad c^0(\xi) = (\mathrm{L}_{f_1} + \gamma^0 \mathrm{L}_{g_1})\gamma^0(\xi) = c(\xi).$$

By invoking the Implicit Function Theorem [166], one gets that (3.45) admits unique solutions $v_1 = \gamma^\delta(\xi)$ and $v_2 = c^\delta(\xi)$ of the form (3.43) around the continuous-time solution $< \gamma(\xi), c(\xi) >$ as the matrix

$$\lim_{\delta \to 0} \begin{pmatrix} \nabla_{v_1} Q^\delta(\xi, v_1, v_2) & \nabla_{v_2} Q^\delta(\xi, v_1, v_2) \end{pmatrix} = \begin{pmatrix} L_{g_1} W(\xi) & 0 \\ * & 1 \end{pmatrix}$$

is full rank because of Assumption 3.1. This concludes the proof. ■

A **constructive procedure** for computing the solutions to (3.44) is discussed now. The couple $< \gamma^\delta(\xi), c^\delta(\xi) >$ is defined as the implicit solution to the nonlinear equality (3.44) and computing a closed form solution might not be possible in general. Though, because of sampling and the form the mappings account for, one can compute any term of the series expansions (3.43) through an iterative procedure equating the terms with the same power δ^i in (3.44) and solving, at any i^{th}-step, a couple of couple of linear equalities in the unknown $< \gamma^i(\xi), c^i(\xi) >$ and depending on the previously computed terms $< \gamma^j(\xi), c^j(\xi) >$ with $j < i$; namely, one solves $Q^i(\xi, \gamma^i(\xi), c^i(\xi)) = 0$ which is indeed linear in the unknowns. By carrying out some computations computations, one gets for the first terms

$$\begin{aligned} \gamma^0(\xi) &= \gamma(\xi), & c^0(\xi) &= (\mathrm{L}_{f_1} + \gamma \mathrm{L}_{g_1})\gamma(\xi) \\ \gamma^1(\xi) &= 0, & c^1(\xi) &= (\mathrm{L}_{f_1} + \gamma \mathrm{L}_{g_1})^2\gamma(\xi) + c^0(\xi)\mathrm{L}_{g_1}\gamma(\xi) \\ \gamma^2(\xi) &= \tfrac{1}{2}c^1(\xi) - c^0(\xi)\mathrm{L}_{g_1}\gamma(\xi), & \ldots \end{aligned}$$

Proposition 3.1 states that whenever Assumption 3.1 holds, one can solve an Input-Lyapunov Matching problem over the candidate target dynamics (3.40) while ensuring its feedback invariance. As a matter of fact, equation (3.42a) represents the ILM equality among the candidate sampled-data target (3.40) and the continuous-time one (3.31) while (3.42b) defines the invariance constraint the solutions need to fullfil. By construction, this implies that $< \gamma^\delta(\xi), c^\delta(\xi) >$ make (3.40) and (3.39), respectively, a target dynamics and an immersion mapping for (3.10) in the sense of Definition 3.1.

Lemma 3.1 (The target dynamics and immersion mapping)**.** *Let the strict feedback dynamics (3.2) verify Assumption 3.1 and (3.10) be its equivalent sampled-data equivalent model. Let $< \gamma^\delta(\xi), c^\delta(\xi) >$ be the unique solutions to (3.44). Then, the following holds:*

1. *the system (3.40) is a target dynamics for (3.2) with GAS equilibrium at the origin;*
2. *the immersion mapping (3.39) and on-the-manifold feedback $u = c^\delta(\xi)$ solution of (3.44) verify the invariance condition ii) in Definition 3.1;*

3. the associated stable manifold is implicitly described by

$$\mathcal{M}^\delta := \{x \in \mathbb{R}^n \ s.t.\ \phi^\delta(x) = x_2 - \gamma^\delta(x_1) = 0\}. \tag{3.47}$$

Remark 3.4. *We underline that both the sampled-data target dynamics (3.40) and the manifold $\mathcal{M}^\delta$ in (3.47) are, in general, parametrized by the sampling period and, hence, different from the continuous-time ones.*

Remark 3.5. *Whenever Assumption 3.1 holds with the trivial solution $\gamma(x_1) \equiv 0$, then one gets that (3.44) is satisfied by $\gamma^\delta(\xi) = 0$ and $c^\delta(\xi) = 0$ so getting that the sampled-data target dynamics (3.40) coincides with the sampled-data equivalent model of the continuous-time one (3.31) which reduces to*

$$\xi_{k+1} = e^{\delta \mathrm{L}_{f_1}} \xi\big|_{\xi_k}$$

intrinsically possessing a GAS equilibrium at the origin.

3.3.3 The design of the I&I feedback under sampling

Starting from the design the target proposed in the previous section, one defines the *off-the-manifold* component $z := \phi^\delta(x)$ with $z_0 = \phi^\delta(x_0)$ and computes the extended state trajectories as

$$z_{k+1} = z_k - [\gamma^\delta(x_{1k+1}) - \gamma^\delta(x_{1k})] + \delta u_k \tag{3.48a}$$

$$x_{1k+1} = F_1^\delta(x_{1k}, x_{2k}) + \frac{\delta^2}{2} u_k \mathcal{G}_1^\delta(x_{1k}, x_{2k}, u_k) \tag{3.48b}$$

$$x_{2k+1} = x_{2k} + \delta u_k. \tag{3.48c}$$

Remark 3.6. *We note that $\gamma^\delta(x_{1k+1})$ rewrites as*

$$\gamma^\delta(x_{1k+1}) = e^{\delta(\mathrm{L}_{f_1} + x_{2k}\mathrm{L}_{g_1})}\gamma^\delta(x_{1k}) + u_k \int_0^1 e^{\delta(\mathrm{L}_{f^2} + \theta u_k \mathrm{L}_{g^2})} \Big(\mathrm{L}_{G_1^\delta(\cdot, x_{2k}, \theta u_k)} \gamma^\delta(x_{1k})\Big) \mathrm{d}\theta$$

so emphasizing on the nonlinearity of the dynamics (3.48a) with respect to the control signal u. Nevertheless, we note that

$$\nabla_u \gamma^\delta(x_{k+1}) = 0 + O(\delta^2)$$

so that the first term of (3.48a) in which u appears nonlinearly is in $O(\delta^3)$.

Finally, one has to compute a feedback $u_k = \nu^\delta(x_k, z_k)$ ensuring that $z_k \to 0$ as $k \to \infty$ while guaranteeing boundedness of the extended trajectories (3.48) and $\nu^\delta(\pi^\delta(\xi), 0) = c^\delta(\xi)$. In the next section we will comment on several ways of defining the sampled-data feedback $u_k = \nu^\delta(x_k, z_k)$ so that (3.10) has a GAS equilibrium at the origin in closed loop. First, for the sake of simplicity, let us apply the feedback transformation

$$\delta u_k = \delta w_k + \gamma^\delta(x_{1k+1}) - \gamma^\delta(x_{1k}) \tag{3.49}$$

so that (3.48a) rewrites as a linear integrator

$$z_{k+1} = z_k + \delta w_k$$

and the further control action w_k needs to be chosen so that

- $z_k \to 0$ as $k \to \infty$;
- $w_k \to 0$ as $z_k \to 0$;
- the trajectories of

$$\begin{aligned} z_{k+1} =& z_k + \delta w_k && (3.50a)\\ x_{1k+1} =& F_1^\delta(x_{1k}, x_{2k}) \\ &+ \frac{\delta}{2}(\delta w_k + \Delta_k\gamma^\delta(x_1))\mathcal{G}_1^\delta(x_{1k}, x_{2k}, w_k + \frac{\Delta_k\gamma^\delta(x_1)}{\delta}) && (3.50b)\\ x_{2k+1} =& \gamma^\delta(x_{1k+1}) + z_k + \delta w_k && (3.50c) \end{aligned}$$

with

$$\Delta_k\gamma^\delta(x_1) = \gamma^\delta(x_{1k+1}) - \gamma^\delta(x_{1k})$$

stay bounded for any $k \geq 0$.

Remark 3.7. *The feedback transformation in (3.49) rewrites as*

$$\begin{aligned} \delta u =& \delta w + e^{\delta(\mathrm{L}_{f_1}+x_2\mathrm{L}_{f_2})}\gamma^\delta(x_1) - \gamma^\delta(x_1) + \\ & u\int_0^1 e^{\delta(\mathrm{L}_{f^2}+\theta u \mathrm{L}_{g^2})}\mathrm{L}_{G_1^\delta(\cdot,x_2,\theta u)}\gamma^\delta(x_1)\mathrm{d}\theta \end{aligned} \quad (3.51)$$

with

$$e^{\delta(\mathrm{L}_{f_1}+x_2\mathrm{L}_{g_1})}\gamma^\delta(x_1) - \gamma^\delta(x_1) = \sum_{i>0}\sum_{j\geq 0}\frac{\delta^{i+j}}{i!(j+1)!}(\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})\gamma^j(x_1) \quad (3.52a)$$

$$\begin{aligned} u\int_0^1 & e^{\delta(\mathrm{L}_{f^2}+\theta u \mathrm{L}_{g^2})}\mathrm{L}_{G_1^\delta(\cdot,x_2,\theta u)}\gamma^\delta(x_1)\mathrm{d}\theta = \\ & \sum_{i\geq 2}\sum_{j\geq 0}\frac{\delta^{i+j}}{i!(j+1)!}u(\mathrm{L}_{f^2} + u\mathrm{L}_{g^2})^{i-2}\mathrm{L}_{g^2}\mathrm{L}_{f^2}\gamma^j(x_1). \end{aligned} \quad (3.52b)$$

In what fallows, three design approaches are proposed for achieving I&I stabilization based on

1. *dead-beat control* where the control is design so to stretch z to 0 in exactly one time step so relying upon direct discrete-time design strategies;
2. *direct Lyapunov-based sampled-data design* where the design is carried out so to make the trajectories of the extended system (3.48) bounded (direct sampled-data design);
3. *Input-to-Partial State Matching* aimed at reproducing, at any sampling instants, the evolutions of the continuous-time off-the-manifold component $z = x_2 - \gamma(x_1)$ within the family of indirect sampled-data design strategies.

3.3.3.1 Dead-beat I&I feedback

The idea is to design the further control action so to stretch the trajectories of (3.10) onto $\mathcal{M}^\delta$ after exactly one sampling instant; namely, one needs to find w solution to the following equality

$$z_k + \delta w_k = 0 \tag{3.53}$$

so getting, $w_k = -\frac{z_k}{\delta}$. It is clear, that, the overall feedback solution to

$$\nu^\delta_{\rm db}(x_k, z_k) = -\frac{z_k}{\delta} + \frac{1}{\delta}\Delta_k \gamma^\delta(x_1) \tag{3.54}$$

will ensure that

1. $z_k = 0$ for $k \geq 1$;
2. $\nu^\delta_{\rm db}(\pi^\delta(\xi), 0) = c^\delta(\xi)$ as one recovers that it needs to solve the same invariance equation as in (3.42b).

Though, boundedness of the trajectories (3.48) will be ensured for δ "large enough"; as a matter of fact, as $\delta \to 0$ the feedback $\nu^\delta_{\rm db}(x,z)$ is not defined and induces a finite escape point in (3.48). This is a well-recognized problem in dead-beat control.

Thus, one gets the following result.

Theorem 3.1 (Existence and uniqueness of a dead-beat solution)**.** *Let (3.2) verify Assumption 3.1 and (3.10) be its equivalent sampled-data model. Assume $\gamma^\delta : \mathbb{R}^p \to \mathbb{R}$ be the solution to (3.44) and $z = x_1 - \gamma^\delta(x_1)$. Then, there exists δ^* so that for any $\delta \in]0, \delta^*[$, the equality*

$$\delta u_k = -z_k + \gamma^\delta(x_{1k+1}) - \gamma^\delta(x_{1k}) \tag{3.55}$$

admits a unique solution of the form $u_k = \nu^\delta_{db}(x_k, z_k)$

$$\nu^\delta_{db}(x,z) = \frac{1}{\delta}\nu^0_{db}(x,z) + \sum_{i>0} \frac{\delta^{i-1}}{(i-1)!}\nu^i_{db}(x,z). \tag{3.56}$$

Proof: The proof is constructive. To this end, exploiting Remark 3.6, we rewrite the equality (3.55) as

$$\mathcal{T}^\delta(x,z,u)u = -\, z + e^{\delta({\rm L}_{f_1} + x_2 {\rm L}_{f_2})}\gamma^\delta(x_1) - \gamma^\delta(x_1) \tag{3.57}$$

with

$$\mathcal{T}^\delta(x,z,u) = \delta - \int_0^1 e^{\delta({\rm L}_{f^2} + \theta u {\rm L}_{g^2})}\Big({\rm L}_{G_1^\delta(\cdot, x_2, \theta u)}\gamma^\delta(x_1)\Big){\rm d}\theta. \tag{3.58}$$

so that one needs to ensure that $\mathcal{T}^\delta(x,z,u)$ is invertible for any $(x,z,u) \in \mathbb{R}^{p+1} \times \mathbb{R} \times \mathbb{R}$. By rewriting

$$\mathcal{T}^\delta(x,z,u)u = \delta u - \sum_{i\geq 2}\sum_{j \geq 0} \frac{\delta^{i+j}}{i!(j+1)!} u({\rm L}_{f^2} + u{\rm L}_{g^2})^{i-2}{\rm L}_{g^2}{\rm L}_{f^2}\gamma^j(x_1)$$

we get the result by invoking, again, the Implicit Function Theorem as, for $\delta \to 0$, $\mathcal{T}^\delta(x,z,u)u \to u$ exhibiting a nonsingular Jacobian with respect to u. ∎

A **constructive procedure** to compute the solution to (3.55) is now given as it might not be an easy task in general. Though, as we have already discussed, one can implement an iterative procedure computing any term of the series expansion (3.56) by solving a set of linear equations in the unknown $\nu^i_{\mathrm{db}}(x,z)$. Namely, one rewrites (3.56) as

$$\begin{aligned}\delta u - &\sum_{i\geq 2}\sum_{j\geq 0}\frac{\delta^{i+j}}{i!(j+1)!}u(\mathrm{L}_{f^2}+u\mathrm{L}_{g^2})^{i-2}\mathrm{L}_{g^2}\mathrm{L}_{f^2}\gamma^j(x_1)\\ &= -z+\sum_{i>0}\sum_{j\geq 0}\frac{\delta^{i+j}}{i!(j+1)!}(\mathrm{L}_{f_1}+x_2\mathrm{L}_{g_1})\gamma^j(x_1)\end{aligned}$$

and substitutes in the above inequality $u=\nu^\delta_{\mathrm{db}}(x,z)$ as in (3.56). Accordingly, for the first terms, one gets the equality

$$\begin{aligned}&\nu^0_{\mathrm{db}}(x,z)+z=0\\ &\nu^1_{\mathrm{db}}(x,z)-\sum_{i\geq 1}\frac{(\nu^0_{\mathrm{db}}(x,z))^i}{i!}\mathrm{L}^i_{g^2}\mathrm{L}_{f^2}\gamma^0(x_1)-(\mathrm{L}_{f_1}+x_2\mathrm{L}_{g_1})\gamma^0(x_1)=0\\ &\dots\end{aligned}$$

which are solved by setting

$$\begin{aligned}&\nu^0_{\mathrm{db}}(x,z)=-z\\ &\nu^1_{\mathrm{db}}(x,z)=\sum_{i\geq 1}\frac{(-z)^i}{i!}\mathrm{L}^i_{g^2}\mathrm{L}_{f^2}\gamma^0(x_1)+(\mathrm{L}_{f_1}+x_2\mathrm{L}_{g_1})\gamma^0(x_1)\\ &\dots\end{aligned}$$

Accordingly, one can now set the following result.

Theorem 3.2 (I&I stabilization under dead-beat)**.** *Let (3.2) verify Assumption 3.1 and (3.10) be its equivalent sampled-data model. Assume $\gamma^\delta:\mathbb{R}^p\to\mathbb{R}$ be the solution to (3.44) and $z=x_1-\gamma^\delta(x_1)$. Then, the feedback $u_k=\nu^\delta_{db}(x_k,z_k)$ computed as the unique solution to (3.55) ensures I&I stabilization of (3.10) for $\delta>0$. Equivalently, the sampled-data feedback $u_k=\nu^\delta_{db}(x_k,z_k)$ achieves SD-I&I stabilization of (3.2) in closed-loop.*

Proof: The proof is straightforward from Lemma 3.1 and by noticing that, when $z_0=x_{20}-\gamma^\delta(x_{10})$, it ensures $z_k=0$ for $k\geq 1$ while guaranteeing boundedness of the trajecfories of the extended system (3.48) as δ stays away from 0. Moreover, when $z=0$ and $x=\pi^\delta(\xi)$, (3.55) reduces to the invariance equality (3.42b) so ensuring $\nu^\delta_{\mathrm{db}}(\pi^\delta(\xi),0)=c^\delta(\xi)$. ∎

Remark 3.8. *Practical stability properties of the closed-loop equilibrium of (3.10) under approximate dead-beat controllers can be deduced as recalled in Chapter 2.*

Remark 3.9. *The SD-I&I stabilizing feedback* $u_k = \nu^\delta_{\rm db}(x_k, z_k)$ *requires, in general, a huge control effort when* $\delta > 0$ *and the* z_0 *is large enough. For this reason, although the velocity of convergence of the trajectories of (3.10) onto* $\mathcal{M}^\delta$ *is rather fast, a dead-beat feedback might not be suitable for practical implementations.*

The dead-beat feedback $u_k = \nu^\delta_{\rm db}(x_k, z_k)$ is strictly model-based and well-known to lack of robustness. Moreover, it requires the knowledge of a closed-form sampled-data equivalent model (3.10) for achieving one step convergence onto $\mathcal{M}^\delta$. For this reason, we propose here two other alternatives qualifying for practical implementation.

3.3.3.2 I&I direct Lyapunov-based sampled-data design

In this section, we are providing a constructive way of defining the further control action w in (3.52a) based on Lyapunov arguments. Roughly speaking, we exploit a suitably defined Lyapunov function so to ensure convergence to the manifold $\mathcal{M}^\delta$ while preserving boundedness of the whole state trajectories (3.50). The design will be carried out over the increment of such a function over (3.50) in an exact context while underlining an alternative approximate procedure for computational purposes.

To this end, let us rewrite the dynamics (3.50) as

$$z_{k+1} = z_k + \delta w_k \tag{3.59a}$$
$$x_{1k+1} = \alpha^\delta(x_{1k}) + z_k F^\delta_z(x_{1k}, z_k) + w_k \tilde{\mathcal{G}}^\delta_1(x_{1k}, x_{2k}, w_k) \tag{3.59b}$$
$$x_{2k+1} = \gamma^\delta(x_{1k+1}) + z_k + \delta w_k \tag{3.59c}$$

with

$$\alpha^\delta(x_1) = F^\delta_1(x_1, \gamma^\delta(x_1)) + \frac{\delta}{2}\Delta_k\gamma^\delta(x_1)\mathcal{G}^\delta_1(x_{1k}, x_{2k}, \Delta_k\gamma^\delta(x_1)) \tag{3.60a}$$
$$F^\delta_z(x_1, z) = \int_0^1 e^{\delta({\rm L}_{f_1} + (\gamma^\delta(x_1) + z\theta){\rm L}_{g_1})}\Big(\mathcal{F}^\delta_1(x_1, \gamma^\delta(x_1) + z\theta)\Big){\rm d}\theta \tag{3.60b}$$
$$\tilde{\mathcal{G}}^\delta_1(x_1, x_2, w) = \int_0^1 e^{\delta({\rm L}_{f2} + (\Delta_k\gamma^\delta(x_1) + \theta w){\rm L}_{g2})}\Big(G^\delta_1(x, \Delta_k\gamma^\delta(x_1) + \theta w)\Big){\rm d}\theta \tag{3.60c}$$

and introduce the positive definite and radially unbounded function

$$V(x, z) = W(x_1) + \frac{1}{2}z^2. \tag{3.61}$$

Now, the following result can be stated.

Proposition 3.2. *Let (3.2) verify Assumption 3.1 and (3.10) be its equivalent sampled-data model. Let* $\delta K^\delta(x, z) \in]0, 1[$ *be such that, for a fixed* $M > 1$ *and smooth functions* $\rho^\delta_i(x_1, z) > 0$ *for* $i = 1, 2$, *for any* $\|x_1\| > M$

$$W(\alpha^\delta(x_1)) - W(x_1) + \frac{\|\Theta_1(x_1, z)\|^2}{2\rho^\delta_1(x_1, z)} + \frac{\|\Theta_2(x_1, z, w)\|^2}{2\rho^\delta_2(x_1, z)} < 0 \tag{3.62}$$

$$\frac{\rho^\delta_1(x, z)}{\delta(2 - \delta - \rho^\delta_2(x, z))} \le K^\delta(x, z) < \frac{1}{\delta} \tag{3.63}$$

with

$$\Theta_1(x_1, z) = \int_0^1 e^{\delta(\mathrm{L}_{f_1}+(\gamma^\delta(x_1)+z\theta)\mathrm{L}_{g_1})}\Big(\mathrm{L}_{\mathcal{F}_1^\delta(\cdot,\gamma^\delta(\cdot)+z\theta)}W(x_1)\Big)\mathrm{d}\theta$$
$$\Theta_2(x_1, z, w) = \int_0^1 e^{\delta(\mathrm{L}_{f^2}+(\Delta_k\gamma^\delta(x_1)+\theta w)\mathrm{L}_{g^2})}\Big(\mathrm{L}_{G_1^\delta(\cdot,x_2,\Delta_k\gamma^\delta(x_1)+\theta w)}W(x_1)\Big)\mathrm{d}\theta.$$

Then, the feedback $w = -K^\delta(x, z)z$ ensures I&I stabilization of (3.10) and, hence, SD-I&I stabilization of (3.2).

Proof: Because $\delta K^\delta(x, z) \in]0, 1[$ then $z_{k+1} = (1 - \delta K^\delta(x_k, z_k)z_k$ has a GAS equilibrium so ensuring $z_k \to 0$ as $k \to \infty$. Now, to prove boundedness, let us pick $M > 1$ such that the following inequality is verified for $\|x_1\| > M$

$$W(\alpha^\delta(x_1)) - W(x_1) < 0.$$

Introducing now the Lyapunov function (3.61) one computes

$$\begin{aligned}\Delta_k V(x, z) =& W(\alpha^\delta(x_1)) - W(x_1) + z\Theta_1(x, z) + w\Theta_2(x, z, w) + \delta w(z - \frac{\delta}{2}w)\\ \leq& W(\alpha^\delta(x_1)) - W(x_1) + \frac{\|\Theta_1(x_1, z)\|^2}{2\rho_1^\delta(x_1, z)} + \frac{\|\Theta_2(x_1, z, w)\|^2}{2\rho_2^\delta(x_1, z)}\\ &+ \frac{\rho_1^\delta(x, z)\|z\|^2}{2} + \frac{\rho_2^\delta(x, z)\|w\|^2}{2} + \delta w(z - \frac{\delta}{2}w).\end{aligned}$$

Substituting now $w = -K^\delta(x, z)z$ and exploiting (3.62) one obtains

$$\Delta_k V(x, z) \leq \Big(\frac{\rho_1^\delta(x, z)}{2} + \frac{\rho_2^\delta(x, z)K^\delta(x, z)}{2} - \delta(1 - \frac{\delta}{2})K^\delta(x, z)\Big)z^2$$

which is nonpositive under (3.63). ■

In general, conditions (3.62)-(3.63) are not easy to check as the concerned expressions do not admit a closed-form. Though, by letting $K^\delta(x, z)$ get the form

$$K^\delta(x, z) = \sum_{i\geq 0} \frac{\delta^i}{(i + 1)!}K^i(x, z) \quad \text{and} \quad K^i(x, z) > 0 \tag{3.64}$$

one can easily deduce bounds for any term $K^i(x, z)$ so to ensure the required properties. As a matter of fact, after computing

$$\begin{aligned}\Delta_k V(x, z) =& \delta(\mathrm{L}_{f_1} + \gamma^0\mathrm{L}_{g_1})W(x_1) + \frac{\delta^2}{2}(\mathrm{L}_{f_1} + \gamma^0\mathrm{L}_{g_1})^2W(x_1)+\\ &+ \delta z\mathrm{L}_{g_1}W(x_1) + \frac{\delta^2}{2}z\big[(\mathrm{L}_{f_1} + \gamma^0\mathrm{L}_{g_1})\mathrm{L}_{g_1}W(x_1) + \mathrm{L}_{g_1}(\mathrm{L}_{f_1} + \gamma^0\mathrm{L}_{g_1})W(x_1)\\ &+ z\mathrm{L}_{g_1}^2W(x_1)\big] - \frac{\delta^2}{2}K^0(x, z)z\mathrm{L}_{g^2}\mathrm{L}_{f^2}W(x_1)\\ &- \delta K^0(x, z)z^2 - \frac{\delta^2}{2}K^1(x, z)z^2 + \frac{\delta^2}{2}(K^0(x, z))^2z^2 + O(\delta^3).\end{aligned}$$

one equates the homogeneous terms in δ. Thus, for any smooth $\rho_1^0(x_1) > 0$, one gets

$$\begin{aligned}\mathcal{P}^0(x,z) =&(\mathrm{L}_{f_1} + \gamma^0 \mathrm{L}_{g_1})W(x_1) + z\mathrm{L}_{g_1}W(x_1) - K^0(x,z)z^2 \leq \\ &(\mathrm{L}_{f_1} + \gamma^0 \mathrm{L}_{g_1})W(x_1) + \frac{\|\mathrm{L}_{g_1}W(x_1)\|^2}{2\rho_1^0(x)} + \Big(\frac{\rho_1^0(x)}{2} - K^0(x,z)\Big)z^2\end{aligned}$$

which is nonnegative when $\rho_1^0(x_1)$ and $K^0(x,z)$ coincide with the continuous-time ones (3.8).

Then, the procedure goes on by equating the terms in δ^2 so deducing

$$\begin{aligned}\mathcal{P}^1(x,z) =&(\mathrm{L}_{f_1} + \gamma^0 \mathrm{L}_{g_1})^2 W(x_1) + z\big[(\mathrm{L}_{f_1} + \gamma^0 \mathrm{L}_{g_1})\mathrm{L}_{g_1}W(x_1) + \mathrm{L}_{g_1}(\mathrm{L}_{f_1} + \gamma^0 \mathrm{L}_{g_1})W(x_1) \\ &+ z\mathrm{L}_{g_1}^2 W(x_1)\big] - K^0(x,z)z\mathrm{L}_{g^2}\mathrm{L}_{f^2}W(x_1) - K^1(x,z)z^2 + (K^0(x,z))^2 z^2.\end{aligned}$$

For any smooth $\rho_1^1(x,z) > 0$ and $\rho_2^0(x,z) > 0$ one gets that

$$\begin{aligned}\mathcal{P}^1(x,z) \leq&\tilde{\mathcal{P}}^1(x,z) \\ \tilde{\mathcal{P}}^1(x,z) =&(\mathrm{L}_{f_1} + \gamma^0 \mathrm{L}_{g_1})^2 W(x_1) + \frac{1}{2\rho^1(x,z)}\|(\mathrm{L}_{f_1} + \gamma^0 \mathrm{L}_{g_1})\mathrm{L}_{g_1}W(x_1) \\ &+ \mathrm{L}_{g_1}(\mathrm{L}_{f_1} + \gamma^0 \mathrm{L}_{g_1})W(x_1) + z\mathrm{L}_{g_1}^2 W(x_1)\|^2 + \frac{\|\mathrm{L}_{g^2}\mathrm{L}_{f^2}W(x_1)\|^2}{2\rho_2^0(x,z)} \\ &+ \Big(\frac{\rho_1^1(x,z)}{2} + \frac{\rho_2^0(x,z)+1}{2}(K^0(x,z))^2 - K^1(x,z)\Big)z^2 \leq 0.\end{aligned}$$

Accordingly, one sets the gains so to satisfy, as $\|x_1\| > M$

$$\begin{aligned}&\tilde{\mathcal{P}}^1(x,0) < 0 \\ &K^1(x,z) \geq \frac{\rho_1^1(x,z)}{2} + \frac{\rho_2^0(x,z)+1}{2}(K^0(x,z))^2.\end{aligned}$$

This procedure applies for any term $K^i(x,z)$ in (3.64) which indeed admits a closed-form expression. Finally, the existence of the complete I&I feedback is stated by the following theorem.

Theorem 3.3 (Existence of a Lyapunov-based SD-I&I feedback). *Let (3.2) verify Assumption 3.1 and (3.10) be its equivalent sampled-data model. Assume $\gamma^\delta : \mathbb{R}^p \to \mathbb{R}$ be the solution to (3.44) and $z = x_1 - \gamma^\delta(x_1)$. Then, there exists δ^* such that for any $\delta \in]0, \delta^*[$, the equality*

$$\delta u_k = -\delta K^\delta(x_k, z_k)z_k + \gamma^\delta(x_{1k+1}) - \gamma^\delta(x_{1k}) \tag{3.65}$$

admits a unique solution of the form $u = \nu_\ell^\delta(x,z)$

$$\nu_\ell^\delta(x,z) = \nu_\ell^0(x,z) + \sum_{i>0} \frac{\delta^i}{(i+1)!}\nu_\ell^i(x,z) \tag{3.66}$$

around the continuous-time solution $\nu_\ell^0(x,z) = \nu(x,z)$ in (3.7).

Proof: The proof is omitted as it follows the line of the proof of Proposition 3.1. After substituting (3.73) into (3.65) and rewriting the deduced expression as a formal series equality, one gets the result by invoking the Implicit Function Theorem. ∎

A **constructive procedure** for computing the solutions to (3.65) is now commented. As in commented for the other feedback laws, a closed-form solution for (3.65) cannot be deduced easily. Though, any term $\nu_\ell^i(x,z)$ of (3.73) can be deduced by implementing a suitable iterative algorithm solving, at any step, a linear equality. For the first terms, one gets

$$\begin{aligned}
\nu_\ell^0(x,z) =& -K^0(x,z)z + (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})\gamma^0(x_1)\\
\nu_\ell^1(x,z) =& -K^1(x,z)z + \nu_\ell^0(x,z)\mathrm{L}_{g^2}\mathrm{L}_{f^2}\gamma^0(x_1) + (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})^2\gamma^0(x_1)
\end{aligned}$$

so that, as $\delta \to 0$, one recovers the continuous-time solution (3.7).

For the sake of implementation, only approximate solutions to (3.65) can be computed in practice, as truncation of (3.73) at any finite order δ^p for $p \geq 0$; i.e., the p^{th}-order approximate feedback is defined as

$$\nu_\ell^{\delta[p]}(x,z) = \nu_\ell^0(x,z) + \sum_{i=1}^{p} \frac{\delta^i}{(i+1)!}\nu_\ell^i(x,z). \tag{3.67}$$

Remark 3.10. *Practical stability properties of the closed-loop equilibrium of (3.10) under approximate solutions (3.67) can be deduced as recalled in Chapter 2.*

Remark 3.11. *As $z = 0$, $x = \pi^\delta(\xi)$ and, hence, $\nu_\ell^\delta(x,z) = c^\delta(\xi)$ as (3.65) reduces to the invariance equality (3.42b).*

Concluding, the feedback $u = \nu_\ell^\delta(x,z)$ is based over a complete sampled-data design in the sense that it is constructed so to guarantee that the exact sampled-data equivalent model (3.10) fulfills the specifications required by the I&I procedure in the discrete-time sense. Although the above results guarantee the existence of the so-defined sampled-data feedback, checking for the required properties might be tough. For this purpose, in the next section, we propose a way of defining the I&I feedback that completely and properly exploits the continuous-time design while ensuring the required properties as a direct implication.

3.3.3.3 Sampled-data I&I based Input-to-State Matching

The third approach we propose is based on indirect design methods and belongs to the matching-based strategies. In particular, we are going to introduce a *Partial Input-to-State Matching* (PISM) problem in which we define the residual component of the control w in (3.49) so to ensure that, in closed-loop, the dynamics (3.50) matches, at any sampling instants, part of the state evolution of the continuous-time dynamics in closed-loop.

To this end, let us consider again the extended continuous-time dynamics under the I&I feedback (3.7)

$$\dot{z} = -K(x)z \tag{3.68a}$$
$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \tag{3.68b}$$
$$\dot{x}_2 = -K(x)z + (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})\gamma(x_1) \tag{3.68c}$$

with $z = \phi(x) = x_2 - \gamma(x)$.

We aim at defining the feedback w_k so to ensure matching, for any $t = k\delta$ for $k \geq 0$, of the sampled-data off-the-manifold component $z = \phi^\delta(x) = x_2 - \gamma^\delta(x_1)$ evolving as in (3.50a) with the continuous-time one $z = \phi(x) = x_2 - \gamma(x_1)$ evolving as in (3.68a). For, let us rewrite (3.50) compactly as

$$z_{k+1} = z_k + \delta w_k \tag{3.69a}$$
$$x_{k+1} = F^\delta(x_k, w_k + \frac{1}{\delta}\Delta_k\gamma^\delta(x_1)) \tag{3.69b}$$

Theorem 3.4 (Existence and uniqueness of a solution to PISM)**.** *Let (3.2) verify Assumption 3.1 and (3.10) be its equivalent sampled-data model. Assume $\gamma^\delta : \mathbb{R}^p \to \mathbb{R}$ be the solution to (3.44) and $z = x_2 - \gamma^\delta(x_1)$. Consider the ideal closed-loop dynamics (3.68) under the feedback (3.7). Then, there exists δ^* such that for any $\delta \in]0, \delta^*[$, the PISM equality*

$$\phi^\delta(F^\delta(x_k, w_k + \frac{1}{\delta}\Delta_k\gamma^\delta(x_1))) - \phi^\delta(x_k) = \int_{k\delta}^{(k+1)\delta} \dot{\phi}(x(s))\mathrm{d}s \tag{3.70}$$

with $x_k = x(k\delta)$ and $z_k = z(k\delta)$ for any $k \geq 0$, admits a unique solution of the form $w = w^\delta(x, z)$

$$w^\delta(x, z) = w^0(x, z) + \sum_{i>0} \frac{\delta^i}{(i+1)!} w^i(x, z) \tag{3.71}$$

with $w^\delta(x, 0) = 0$. As a consequence, the overall feedback solving (3.49) is provided by $u = \nu_P^\delta(x, z)$ computed as the unique solution to

$$\delta u = \delta w^\delta(x, z) + \Delta_k\gamma^\delta(x_1) \tag{3.72}$$

of the form

$$\nu_P^\delta(x, z) = \nu_P^0(x, z) + \sum_{i>0} \frac{\delta^i}{(i+1)!} \nu_P^i(x, z). \tag{3.73}$$

around the continuous-time solution $\nu_P^0(x, z) = \nu(x, z)$ as in (3.7).

Proof: The proof is constructive and consists of two parts: first, we are showing that a unique solution of the form (3.71) to (3.70) exists; then, we'll show that the overall feedback $u = \nu_{\mathrm{P}}^\delta(x, z)$ exists as the unique solution to (3.72).

As far as the first part is concerned, we first expand

$$\begin{aligned}\int_{k\delta}^{(k+1)\delta}\dot{\phi}(x(s))\mathrm{d}s =& e^{\delta(\mathrm{L}_{f^2}+(K(x)z+\dot{\gamma}(x_1))\mathrm{L}_{g_2})}\phi(x_k)-\phi(x_k)\\ \phi^\delta(F^\delta(x_k,w_k+\frac{1}{\delta}\Delta_k\gamma^\delta(x_1)))-\phi^\delta(x_k) =& e^{\delta(\mathrm{L}_{f^2}+(w_k+\frac{1}{\delta}\Delta_k\gamma(x_1))\mathrm{L}_{g_2})}\phi^\delta(x_k)-\phi^\delta(x_k)\\ =&\delta w_k\end{aligned}$$

so getting the formal series equality

$$\delta Q^\delta(x_k,z_k,w) = \delta w - e^{\delta(\mathrm{L}_{f^2}+(K(x)z+\dot{\gamma}(x_1))\mathrm{L}_{g_2})}\phi(x_k)+\phi(x_k)=0$$

with

$$Q^\delta(x,z,w)=Q^0(x,z,w)+\sum_{i\geq 1}\frac{\delta^i}{i!}Q^i(x,z,w) \tag{3.74}$$

and

$$Q^0(x,z,w)=w+K(x)z.$$

Accordingly, as $\delta\to 0$ the above equality is solved by setting

$$w=-K(x)z.$$

Thus, by applying the Implicit Function Theorem one gets that a unique solution to (3.70) exists and admits the form (3.71) with $w^0(x,z)=-K(x)z$ as the $\nabla_w Q^0(x,z,w)=1$ is full-rank.

As far as the second part about (3.72) is concerned, one proceeds in the same way as before by rewriting (3.72) as a series expansion in power of δ which is solved, as $\delta\to 0$, by

$$u=w^0(x,z)+(\mathrm{L}_{f_1}+x_2\mathrm{L}_{g_1})\gamma^0(x_1)$$

so that the unique solution is provided by the series expansion (3.73) around

$$\nu^0_{\mathrm{P}}(x,z)=w^0(x,z)+(\mathrm{L}_{f_1}+x_2\mathrm{L}_{g_1})\gamma^0(x_1).$$

■

A **constructive procedure** for computing the solutions to (3.70) and (3.72) is now discussed. As usual in the framework we propose, the control is implicitly defined by the nonlinear equalities (3.70) and (3.72) so that exact forms are tough to be found. Though, a closed-form expression for any term of the series expansions (3.71) and (3.73) can be deduced through a constructive procedure solving a brunch of linear equalities in the corresponding unknowns. As far as (3.70) is concerned, one gets

$$
\begin{aligned}
&w^0(x,z) + K(x)z = 0 \implies w^0(x,z) = -K(x)z\\
&w^1(x,z) - z(K(x))^2 + z\dot{K}(x) = 0 \implies w^1(x,z) = z(K(x))^2 - z\dot{K}(x)\\
&w^2(x,z) + z(K(x))^3 - 3zK(x)\dot{K}(x) + z\ddot{K}(x) = 0\\
&\qquad\qquad \implies w^2(x,z) = -z(K(x))^3 + 3zK(x)\dot{K}(x) - z\ddot{K}(x)\\
&\dots
\end{aligned}
$$

with $\dot{K}(x) = (\mathrm{L}_{f^2} + \nu(x,z)\mathrm{L}_{g^2})K(x)$ and $\ddot{K}(x) = (\mathrm{L}_{f^2} + \nu(x,z)\mathrm{L}_{g^2})^2 K(x)$. Accordingly, one computes the any term of (3.73) by solving

$$
\begin{aligned}
&\nu_{\mathrm{P}}^0(x,z) - w^0(x,z) - (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})\gamma^0(x_1) = 0\\
&\nu_{\mathrm{P}}^1(x,z) - w^1(x,z) - \nu_{\mathrm{P}}^0(x,z)\mathrm{L}_{g^2}\mathrm{L}_{f^2}\gamma^0(x_1) - (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})^2\gamma^0(x_1) = 0\\
&\nu_{\mathrm{P}}^2(x,z) - w^2(x,z) - \frac{3}{2}\nu_{\mathrm{P}}^1(x,z)\mathrm{L}_{g^2}\mathrm{L}_{f^2}\gamma^0(x_1) - \nu_{\mathrm{P}}^0(x,z)(\mathrm{L}_{f_1} + \nu_{\mathrm{P}}^0(x,z)\mathrm{L}_{g^2})\mathrm{L}_{g^2}\mathrm{L}_{f^2}\gamma^0(x_1)\\
&\quad - (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})^3\gamma^0(x_1) - (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})\gamma^2(x_1) = 0\\
&\dots
\end{aligned}
$$

so getting

$$
\begin{aligned}
\nu_{\mathrm{P}}^0(x,z) =& w^0(x,z) + (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})\gamma^0(x_1)\\
\nu_{\mathrm{P}}^1(x,z) =& w^1(x,z) + \nu_{\mathrm{P}}^0(x,z)\mathrm{L}_{g^2}\mathrm{L}_{f^2}\gamma^0(x_1) + (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})^2\gamma^0(x_1)\\
\nu_{\mathrm{P}}^2(x,z) =& w^2(x,z) + \frac{3}{2}\nu_{\mathrm{P}}^1(x,z)\mathrm{L}_{g^2}\mathrm{L}_{f^2}\gamma^0(x_1) + \nu_{\mathrm{P}}^0(x,z)(\mathrm{L}_{f_1} + \nu_{\mathrm{P}}^0(x,z)\mathrm{L}_{g^2})\mathrm{L}_{g^2}\mathrm{L}_{f^2}\gamma^0(x_1)\\
&+ (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})^3\gamma^0(x_1) + (\mathrm{L}_{f_1} + x_2\mathrm{L}_{g_1})\gamma^2(x_1)\\
&\dots.
\end{aligned}
$$

Remark 3.12. *As $\delta \to 0$ one recovers the continuous-time solution (3.7); i.e., $\lim_{\delta\to 0}\nu_P^\delta(x,z) = \nu(x,z) = -K(x) + \dot{\gamma}(x_1)$. Moreover, it is a matter of computations to verify that $\nu_P^1(x,z) = \dot{\nu}(x,z) = (\mathrm{L}_{f^2} + \nu(x,z)\mathrm{L}_{g^2})\nu(x,z)$.*

Because of matching, $z_k \to 0$ as $k \to \infty$ while ensuring the properties of the closed-loop continuous-time system (3.68) and, in particular, boundedness of the trajectories. Moreover, because $w^\delta(x,0) = 0$, one recovers that, as $z = 0$, (3.72) reduces to (3.44) and, hence, $\nu_{\mathrm{P}}^\delta(\pi^\delta(x,z),0) = c^\delta(\xi)$. Accordingly, the following result can be stated.

Theorem 3.5 (I&I stabilization under PISM)**.** *Let (3.2) verify Assumption 3.1 and (3.10) be its equivalent sampled-data model. Assume $\gamma^\delta : \mathbb{R}^p \to \mathbb{R}$ be the solution to (3.44) and $z = x_1 - \gamma^\delta(x_1)$. Then, the feedback $u_k = \nu_P^\delta(x_k, z_k)$ computed as the unique solution to (3.72) ensures I&I stabilization of (3.10). Equivalently, the sampled-data feedback $u_k = \nu_P^\delta(x_k, z_k)$ achieves SD-I&I stabilization of (3.2) in closed-loop.*

For the sake of implementation, only approximate solutions to (3.72) can be computed in practice, as truncation of (3.73) at any finite order δ^p for $p \geq 0$; i.e., the p^{th}-order approximate feedback is defined as

$$\nu_{\mathrm{P}}^{\delta[p]}(x, z) = \nu_{\mathrm{P}}^{0}(x, z) + \sum_{i=1}^{p} \frac{\delta^i}{(i+1)!} \nu_{\mathrm{P}}^{i}(x, z). \tag{3.75}$$

Remark 3.13. *Practical stability properties of the closed-loop equilibrium of (3.10) under approximate solutions (3.75) can be deduced as recalled in Chapter 2.*

Concluding, among the three feedback laws we have been proposing, the PISM-based design completely exploits the continuous-time design and prevents from carrying out some tedious analysis (as in the case of the design proposed in Section 3.3.3.2) either over the exact sampled-data equivalent model (through dead-beat) or for each term defining the control for guaranteeing the required specifications under sampling in the discrete-time sense of Definition 3.1. As a matter of fact, through matching, the sampled-data equivalent model inherits, in closed loop, all of the properties yielded by the continuous-time design.

3.3.4 On approximate controllers

As shown in the previous part, any stabilizing sampled-data I&I feedback is described as the implicit solution of a formal series equality. In this section, we are providing a small insight on approximate controllers by focusing on the PISM-based one defined by (3.72). The extension to the other cases discussed in Section 3.3.3.1-3.3.3.2 is straightforward along these lines.

The feedback $u_k = \nu_{\mathrm{P}}^{\delta}(x_k, z_k)$ is described by an asymptotic series expansion around the continuous-time solution $\nu_{\mathrm{P}}^{0}(x, z) = \nu(x, z)$ in the form of (3.73). GAS of the closed-loop equilibrium of (3.10) under this control implies the existence of a $\mathcal{KL}$ function β such that for each $k \geq 0$ and any initial condition x_0

$$\|x_k\| \leq \beta(\|x_0\|, k). \tag{3.76}$$

Nevertheless, implementation issues arise when considering that only approximations of the controller can be computed. To this end, define the *p-th order approximate* feedback as in (3.75).

The stability property of the closed-loop system under such a controller is stated below.

Proposition 3.3. *Consider (3.10) with stabilizing feedback $u_k = \nu_P^{\delta}(x_k, z_k)$ solution to (3.72). Then the approximated controller (3.75) of order p makes the equilibrium practically globally asymptotically stable in $\Theta(\bar{\delta}) = \{O(\bar{\delta}^{p+2}) \ : \ \delta \in]0, \delta^*[\}$.*

Proof: Denote by x_{k+1} and x_{k+1}^p the states of (3.10) under, respectively, the exact and approximate controllers from the same initial condition at $t = k\delta$. Then, at each instant $t = (k+1)\delta$, they coincide up to an error in $O(\delta^{p+2})$. In virtue of (3.76) we can write, for all $k \geq 0$

$$\begin{aligned}\|x^p_{k+1}\| &\leq \|x_{k+1}\| + \|x_{k+1} - x^p_{k+1}\| \\ &\leq \beta(\|x_{k+1}\|, k) + \delta^{p+1} R(\delta, x_k)\end{aligned}$$

where R is a $\mathcal{K}_\infty$ function defined as the sum of the norms of the remaining terms of the dynamics (3.10). One concludes that the trajectories of the system converge to $B_{\delta^{p+2}}(0)$. ■

3.4 I&I for strict-feedback systems under sampling: extensions

In this section, we shall extend the proposed design methodology to more general strict-feedback dynamics by first assuming a nonlinear dynamics in the last component of (3.2). Then, the case of generally feedback interconnected systems (3.1) will be sketched as well.

3.4.1 A nonlinear dynamics on the last component

Up to now we considered a strict-feedback dynamics in which the last component of the cascade is an integrator. Now, we assume a more general strict-feedback dynamics over $\mathbb{R}^{p+1}$

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u_1\end{aligned}$$

with $g_2 \neq 0$ for any $x \in \mathbb{R}^n$. Assuming $u_1 \in \mathcal{U}^\delta$ one gets the sampled-data equivalent model

$$x_{1k+1} = F^\delta_1(x_k) + \frac{\delta^2}{2!} u_{1k} \mathcal{G}^\delta_1(x_k, u_{1k}) \tag{3.77a}$$

$$x_{2k+1} = F^\delta_2(x_k) + \delta u_{1k} \mathcal{G}^\delta_2(x_k, u_{1k}). \tag{3.77b}$$

In that case, one applies the feedback transformation defined by

$$x_2 + \delta u = F^\delta_2(x) + \delta u_1 \mathcal{G}^\delta_2(x, u_1)$$

so recovering (3.10). Thus, the design proceeds now as discussed in Sections 3.3.3.1, 3.3.3.2 and 3.3.3.3 by computing the I&I feedback $u = \nu^\delta_j(x, z)$ ($j = \mathrm{db}, \ell, \mathrm{P}$). Then, once the I&I design is over, the actual feedback $u_1 = u^\delta_j(x, z)$ is provided by the solution to the following equality for $j = \mathrm{db}, \ell, \mathrm{P}$

$$x_2 + \delta \nu^\delta_j(x, z) = F^\delta_2(x_k) + \delta u^\delta_j(x, z) \mathcal{G}^\delta_2(x_k, u^\delta_j(x, z))$$

which always admits a solution of the form

$$u_j^\delta(x,z) = u_j^0(x,z) + \sum_{i\geq 1} \frac{\delta^i}{(i+1)!} u_j^i(x,z)$$

by virtue of the Implicit Function Theorem and the cascade structure. Any term of the above expansion can be computed through an iterative procedure so obtaining, for the first term

$$u^0(x,z) = \frac{\nu_j^0(x,z) - f_2(x_1,x_2) - \dot{\gamma}(x)}{g_2(x_1,x_2)} \text{ for } j = \mathrm{db}, \ell, \mathrm{P}.$$

3.4.2 I&I for multiple feedback interconnected dynamics under sampling

Here, we are providing a sketch on the extension the results discussed in Section 3.3 to general dynamics (3.1) composed by the strict-feedback interconnection of n-blocks. Again, we shall show how do deduce a suitable δ-dependent sampled-data target dynamics whose equilibrium is GAS through the definition suitable mappings $< \gamma_1^\delta(\xi), \dots, \gamma_{n-1}^\delta(\xi), c^\delta(\xi) >$ which also identify the stable manifold $\mathcal{M}^\delta$. Then, we shall give an idea on how to achieve SD-I&I stabilization in closed-loop through multi-rate sampling of order n.

3.4.2.1 The choice of the target dynamics

As in the integrator-feedback interconnection case, whenever Assumption 3.1 holds, the choice of the sampled-data target dynamics (3.32) deduced from (3.31) is not satisfactory because:

- the feedback structure is lost and, thus, the x_1-sampled dynamics (3.18a) is influenced by all the successive x_i states with $i = 3, \dots, n$ (other than x_2) and the control itself;
- any x_i $(i = 2, \dots, n)$ is no longer a general smooth signal as constrained by the piecewise constant nature of the control u.

Accordingly, these motivations induce the necessity of defining a new sampled-data target dynamics, a suitable immersion mapping and, finally, a piecewise constant on-the-manifold control ensuring its invariance.

Accordingly, let as assume, for $\xi \in \mathbb{R}^p$ the immersion mapping

$$x = \pi^\delta(\xi) = \begin{pmatrix} \xi \\ \gamma_1^\delta(\xi) \\ \vdots \\ \gamma_{n-1}^\delta(\xi) \end{pmatrix} \tag{3.78}$$

and the target sampled-data dynamics of the form

$$\begin{aligned}\xi_{k+1} =& F^{\delta}_{1,1}(\xi,\gamma^{\delta}_1(\xi)) + \sum_{j=2}^{n-1}\frac{\delta^j}{j!}\gamma^{\delta}_j(\xi)F^{\delta}_{1,j}(\xi,\gamma^{\delta}_1(\xi),\dots,\gamma^{\delta}_{j-1}(\xi))\\ &+\frac{\delta^n}{n!}c^{\delta}(\xi)\mathcal{G}^{\delta}_1(\xi,\gamma^{\delta}_1(\xi),\dots,\gamma^{\delta}_n(\xi),c^{\delta}(\xi)) := \alpha^{\delta}(\xi)\end{aligned} \tag{3.79}$$

while the mappings $F_{1,i}(\cdot,\dots,\cdot)$ and $\mathcal{G}^{\delta}_1(\cdot,\vdots,\cdot)$ are given in (3.19) and (3.20). Accordingly, the problem consists in defining $< \gamma^{\delta}_1(\xi),\dots,\gamma^{\delta}_{n-1}(\xi),c^{\delta}(\xi) >$ so that (3.79) qualifies as a target dynamics for (3.18) in the sense that it verifies the following properties:

- it has a GAS equilibrium at the origin;
- it is feedback invariant with respect to (3.18).

To this end, by exploiting Assumption 3.1, we introduce an ILM problem over the continuous-time target (3.31) which is also aimed, in turn, to guarantee invariance of the sampled-data target (3.79).

Accordingly, one gets the following result.

Proposition 3.4 (Existence and uniqueness of $< \gamma^{\delta}_1(\xi),\dots,\gamma^{\delta}_{n-1}(\xi),c^{\delta}(\xi) >$)**.** *Let the strict feedback dynamics (3.1) verify Assumption 3.1 and (3.18) be its equivalent sampled-data equivalent model. Consider $\alpha^{\delta}_{v_1,\dots,v_n}:\mathbb{R}^p\to\mathbb{R}^p$ as of the form*

$$\alpha^{\delta}_{v_1,\dots,v_n}(\xi) := F^{\delta}_{1,1}(\xi,v_1(\xi)) + \sum_{j=2}^{n-1}\frac{\delta^j}{j!}v_j(\xi)F^{\delta}_{1,j}(\xi,v_1(\xi),\dots,v_{j-1}(\xi)) \tag{3.80}$$

$$+\frac{\delta^n}{n!}v_n(\xi)\mathcal{G}^{\delta}_1(\xi,v_1(\xi),\dots,v_n(\xi)) \tag{3.81}$$

with $v_i:\mathbb{R}^p\to\mathbb{R}$ for $i=1,\dots,n$. Then, there exists $\delta^*>0$ such that for any $\delta\in]0,\delta^*[$ the equalities

$$W(\alpha^{\delta}_{v_1,\dots,v_n}(\xi_k))) - W(\xi_k) = \int_{k\delta}^{(k+1)\delta}[\mathrm{L}_{f_1(\cdot)}+\gamma(\cdot)\mathrm{L}_{g_1(\cdot)}]W(\xi(s))\mathrm{d}s \tag{3.82a}$$

$$\begin{aligned}v_i(\alpha^{\delta}_{v_1,\dots,v_n}(\xi_k)) =\ & F^{\delta}_{i,i}(\xi,v_1(\xi)\dots,v_i(\xi))\\ &+\sum_{j=2}^{n-i}\frac{\delta^j}{j!}v_{i+j-1}(\xi)F^{\delta}_{i,i+j-1}(\xi,v_1(\xi)\dots,v_{i+j-2}(\xi))\\ &+\frac{\delta^{n-i+1}}{(n-i+1)!}v_n(\xi)\mathcal{G}^{\delta}_i(\xi,v_1(\xi)\dots,v_n(\xi))\end{aligned} \tag{3.82b}$$

for $i=1,\dots,n$ admit, for any $k\geq 0$, unique solutions $v_i(\xi)=\gamma^{\delta}_i(\xi)$ and $v_n(\xi)=$

$c^\delta(\xi)$ of the form of series expansions in powers of δ; i.e.,

$$\gamma_i^\delta(\xi) = \gamma_i^0(\xi, \gamma_1^0(\xi), \ldots, \gamma_{i-1}^0(\xi)) + \sum_{i>0} \frac{\delta^j}{(j+1)!}\gamma_i^j(\xi) \tag{3.83a}$$

$$c^\delta(\xi) = c^0(\xi) + \sum_{j>0} \frac{\delta^j}{(j+1)!} c^j(\xi). \tag{3.83b}$$

for $i = 1, \ldots, n$ around the continuous-time solution.

The proof of the above results follows the lines of the one of Proposition 3.1 by invoking the Implicit Function Theorem. Any term of the series expansion (3.83) can be easily deduced through an iterative procedure that solves, at any step, a set of linear equation in the corresponding unknowns. For the first terms, one gets,

$$\gamma_1^0(\xi) = \gamma(\xi), \quad \gamma_1^j(\xi) = 0 \text{ for } j = 1, \ldots, n$$

$$\gamma_i^0(\xi, \gamma_1^0(\xi), \ldots, \gamma_{i-1}^0(\xi)) = \frac{(\mathrm{L}_{f^{i-1}} + \gamma_{i-1}^0 \mathrm{L}_{g^{i-1}})\gamma_i^0 - f_i(\xi, \gamma_1^0(\xi), \ldots, \gamma_{i-1}^0(\xi))}{g_i(\xi, \gamma_1^0(\xi), \ldots, \gamma_{i-1}^0(\xi))}$$

$$\gamma_i^j(\xi) = 0$$

$$c^0(\xi) = \frac{(\mathrm{L}_{f^{n-1}} + \gamma_{n-1}^0 \mathrm{L}_{g^{n-1}})\gamma_{n-1}^0 - f_n(\xi, \gamma_1^0(\xi), \ldots, \gamma_{n-1}^0(\xi))}{g_n(\xi, \gamma_1^0(\xi), \ldots, \gamma_{n-1}^0(\xi))}$$

for $j = 1, \ldots, n - i + 1$ and $i = 2, \ldots, n$.

Accordingly, once $< \gamma_1^\delta(\xi), \ldots, \gamma_{n-1}^\delta(\xi), c^\delta(\xi) >$ are computed as above, one gets the following result.

Lemma 3.2 (The target dynamics and immersion mapping)**.** *Let the strict feedback dynamics (3.1) verify Assumption 3.1 and (3.18) be its equivalent sampled-data equivalent model. Let $< \gamma_1^\delta(\xi), \ldots, \gamma_{n-1}^\delta(\xi), c^\delta(\xi) >$ be the unique solutions to (3.82). Then, the following holds.*

1. *the system (3.79) is a target dynamics for (3.1) with GAS equilibrium at the origin;*
2. *the immersion mapping (3.78) and on-the-manifold feedback $u = c^\delta(\xi)$ solution of (3.82) verify the invariance condition ii) in Definition 3.1.*

Remark 3.14. *Contrarily to the case of the interconnected-feedback cascade (3.1), even when Assumption 3.1 is verified by the trivial solution $\gamma(x_1) = 0$, the sampled-data target dynamics and immersion mappings do not preserve the same structure as (3.32); namely, even in this case, the continuous-time target dynamics (3.31) does not allow to straightforwardly deduce a sampled-data target.*

The following Lemma is then useful for showing the existence of a mapping $\phi^\delta(\cdot) : \mathbb{R}^{p+n-1} \to \mathbb{R}^{n-1}$ implicitly defining the stable manifold.

Lemma 3.3. *Let the strict feedback dynamics (3.1) verify Assumption 3.1 and (3.18) be its equivalent sampled-data equivalent model. Let* $<\gamma_1^\delta(\xi),\ldots,\gamma_{n-1}^\delta(\xi),c^\delta(\xi)>$ *be the unique solutions to (3.82). Then, there exists* $\pi_{inv}^\delta(\cdot)$ *defined the inverse function of (3.78) verifying* $\pi_{inv}^\delta(\pi^\delta(\xi))=\xi$*. Accordingly, the associated stable manifold is implicitly described by*

$$\mathcal{M}^\delta := \{x\in\mathbb{R}^n \ s.t.\ \phi^\delta(x)=\mathbf{0}_{(n-1)\times 1}\}. \tag{3.85}$$

with the mapping

$$\phi^\delta(x)=\begin{pmatrix}\phi_1^\delta(x)\\ \vdots\\ \phi_{n-1}^\delta(x)\end{pmatrix}:=\begin{pmatrix}x_2-\gamma_1^\delta(\pi_{inv}^\delta(x))\\ \vdots\\ x_n-\gamma_{n-1}^\delta(\pi_{inv}^\delta(x))\end{pmatrix}. \tag{3.86}$$

Remark 3.15. *We note that the mapping (3.86) rewrites as a series expansion in powers of* δ *with, for* $i=1,\ldots,n-1$

$$\phi_i^\delta(x)=x_{i+1}-\gamma_i^0(x_1,x_2,\ldots,x_i)+O(\delta^{n-i+1}).$$

3.4.2.2 SD-I&I Stabilization through multi-rate sampling

According to Lemma (3.2) one defines now the off-the-manifold component $z=\phi^\delta(x)$ and $z=\mathrm{col}(z_1,\ldots,z_{n-1})$ evolving as

$$z_{k+1}=e^{\delta(\mathrm{L}_{f^n}+u_k\mathrm{L}_{g^n})}\phi^\delta(x_k).$$

Accordingly, the control objective is reduced to the one of designing a control action $u_k=\nu^\delta(x_k,z_k)$ with $\nu^\delta(\pi^\delta(\xi),0)=c^\delta(\xi)$ ensuring $z_k\to 0$ as $k\to$ while making the extended trajectories of

$$z_{k+1}=e^{\delta(\mathrm{L}_{f^n}+u_k\mathrm{L}_{g^n})}\phi^\delta(x_k) \tag{3.87a}$$

$$x_{k+1}=F^\delta(x_k,u_k) \tag{3.87b}$$

bounded with $F^\delta(\cdot,u)$ denoting, in a compact way, the stack of the state dynamics (3.18). For this purpose, single-rate sampling is not enough. As a matter of fact, it has been proven in several contributions by Monaco and Normand-Cyrot, that single-rate sampling does not preserve invariance of an m-dimensional manifold when $m>1$ (e.g., in [120]). In our case, $\mathcal{M}^\delta$ is of dimension $m=n-1$ so that a multi-rate feedback strategy of order $n-1$ is necessary for guarateeing I&I stabilization. Accordingly, the following result is given.

Theorem 3.6 (I&I stabilizability under multi-rate sampling)**.** *Let the strict feedback dynamics (3.1) verify Assumption 3.1. Let* $<\gamma_1^\delta(\xi),\ldots,\gamma_{n-1}^\delta(\xi),c^\delta(\xi)>$ *be the unique solutions to (3.82). Then, (3.1) is MR-SD I&I stabilizable; namely, the* *multi-rate sampled-data equivalent model of order $m = n-1$*

$$x_{1k+1} = F_1^{m\bar{\delta}}(x_k) + \frac{\bar{\delta}^n}{n!}\Omega_1 \underline{u}_k \mathcal{G}_1^{\bar{\delta}}(x_k, \underline{u}_k) \tag{3.88a}$$

$$\vdots \tag{3.88b}$$

$$x_{nk+1} = F_1^{m\bar{\delta}}(x_k) + \frac{\bar{\delta}^n}{n!}\Omega_n \underline{u}_k \mathcal{G}_n^{\bar{\delta}}(x_k, \underline{u}_k) \tag{3.88c}$$

with $\bar{\delta} = \frac{\delta}{n-1}$

$$\underline{u} = \begin{pmatrix} u^1 & \dots & u^{n-1} \end{pmatrix}^\top$$
$$\Omega_i = \begin{pmatrix} i^{n-i} - (i-1)^{n-i} & (i-1)^{n-i} - (i-2)^{n-i} & \dots & 1 \end{pmatrix}, \quad i = 1, \dots, n$$

is I&I stabilizable in the sense of Definition 3.1.

The proof of the result is straightforward because, as $\bar{\delta} < \delta < \delta^*$, then $< \gamma_1^\delta(\xi), \dots, \gamma_{n-1}^\delta(\xi), c^\delta(\xi) >$ will still define suitable target dynamics and immersion mappings for the multi-rate sampled-data model. Accordingly, one computes the multi-rate sampled-data equivalent model associated to (3.87) as provided by

$$z_{k+1} = e^{\bar{\delta}(\mathrm{L}_{f^n} + u_k^1 \mathrm{L}_{g^n})} \circ \dots \circ e^{\bar{\delta}(\mathrm{L}_{f^n} + u_k^{n-1} \mathrm{L}_{g^n})} \phi^{\bar{\delta}}(x_k) \tag{3.89a}$$

$$x_{1k+1} = F_1^{m\bar{\delta}}(x_k) + \frac{\bar{\delta}^n}{n!}\Omega_1 \underline{u}_k \mathcal{G}_1^{\bar{\delta}}(x_k, \underline{u}_k) \tag{3.89b}$$

$$\vdots \tag{3.89c}$$

$$x_{nk+1} = F_1^{m\bar{\delta}}(x_k) + \frac{\bar{\delta}^n}{n!}\Omega_n \underline{u}_k \mathcal{G}_n^{\bar{\delta}}(x_k, \underline{u}_k). \tag{3.89d}$$

One sets the feedback transformation

$$\underline{u} = \underline{w} + \beta^\delta(x, z, \underline{u})$$

transforming (3.89a) into a linear dynamics

$$z_{k+1} = A^{\bar{\delta}} z_k + B^{\bar{\delta}} \underline{w}_k$$

with

$$A^{\bar{\delta}} = \begin{pmatrix} 1 & (n-1)\bar{\delta} & \dots & \frac{((n-1)\bar{\delta})^{n-1}}{(n-1)!} \\ 0 & 1 & \dots & \frac{((n-1)\bar{\delta})^{n-2}}{(n-2)!} \\ & & \ddots & \\ 0 & 0 & \dots & 1 \end{pmatrix} \qquad B^{\bar{\delta}} = \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_n \end{pmatrix}.$$

One computes the further control action w so to guarantee I&I stabilization of (3.88) along the lines of the methodologies proposed in Sections 3.3.3.1, 3.3.3.2 and 3.3.3.3.

3.5 Some illustrating examples

In this section, two examples are discussed to illustrate the proposed methodology. First, the case of a double integrator will serve to show exact computation of the solutions to the involved equalities. Then, an academic example of the nonlinear type will be exploited to emphasize on computational aspects related to approximate solutions.

3.5.1 The double LTI integrator (cont'd)

Consider, again, the double integrator (3.33) with sampled-data equivalent model (3.36) verifying Assumption 3.1 with

$$\gamma(x_1) = -x_1 \quad \text{and} \quad W(x_1) = \frac{1}{2}x_1^2. \tag{3.90}$$

Set the continuous-time target dynamics as

$$\dot{\xi} = -\xi \tag{3.91}$$

with sampled-data equivalent model

$$\xi_{k+1} = e^{-\delta}\xi_k. \tag{3.92}$$

According to the procedure we have been proposing, the I&I procedure consists of the following two steps.

3.5.1.1 Step 1: the choice of the target dynamics

Accordingly, the I&I design under sampling first consists in defining $< \gamma^\delta(\xi), c^\delta(\xi) >$ so to satisfy conditions $i)$ and $ii)$ in Definition 3.1. To this end, setting

$$\gamma^\delta(\xi) = \Pi^\delta\xi \quad \text{and} \quad c^\delta(\xi) = F^\delta\xi \tag{3.93}$$

we assume the sampled-data target dynamics as

$$\xi_{k+1} = (1 + \delta\Pi^\delta + \frac{\delta^2}{2}F^\delta)\xi_k. \tag{3.94}$$

According to Lemma 3.1, equations (3.44) specify as

$$(1 + \delta\Pi^\delta + \frac{\delta^2}{2}F^\delta)^2 = e^{-2\delta} \tag{3.95a}$$

$$\Pi^\delta(1 + \delta\Pi^\delta + \frac{\delta^2}{2}F^\delta) = \Pi^\delta + \delta F^\delta \tag{3.95b}$$

Accordingly, we seek for the unique solution $< \Pi^\delta, F^\delta >$ which is smooth in δ. By solving (3.95b) in F^δ we easily get

$$F^\delta = \frac{(\Pi^\delta)^2}{1 - \frac{\delta}{2}\Pi^\delta}. \tag{3.96}$$

Solving (3.95a) in Π^δ we get the pair of solutions

$$1 + \delta\Pi^\delta + \frac{\delta^2}{2}F^\delta - e^{-\delta} = 0 \tag{3.97}$$

$$1 + \delta\Pi^\delta + \frac{\delta^2}{2}F^\delta + e^{-\delta} = 0 \tag{3.98}$$

By substituting (3.96) into both (3.97)-(3.98) we get the following solutions

$$\Pi^\delta = -\frac{2(1-e^{-\delta})}{\delta(1+e^{-\delta})} \tag{3.99}$$

$$\Pi^\delta = -\frac{2(1+e^{-\delta})}{\delta(1-e^{-\delta})} \tag{3.100}$$

By noticing that (3.100) is not defined for $\delta = 0$, we get that the only admissible solution in the sense of Proposition 3.1 is given by (3.99). Accordingly, substituting (3.99) into (3.96) one gets

$$F^\delta = \frac{2(1-e^{-\delta})^2}{\delta^2(1+e^{-\delta})}. \tag{3.101}$$

Substituting (3.99)-(3.101) into (3.94) one gets that the target dynamics coincides, as a particular case, with (3.92). Accordingly, condition *ii*) of Definition (3.1) holds setting

$$x = \begin{pmatrix} 1 \\ \Pi^\delta \end{pmatrix} \xi \quad \text{and} \quad c^\delta(\xi) = F^\delta \xi. \tag{3.102}$$

Remark 3.16. *In the case of the double integrator, the ILM-invariance-based design yields a sampled-data target dynamics that coincides, at any sampling instant, with the continuous-time one. Though, the sampled-data immersion mapping and feedback making the corresponding set invariant are not the same as in continuous time and come to be parametrized by the sampling period δ.*

As a consequence, condition *iii*) of Definition (3.1) is satisfied by setting

$$z = \Phi^\delta x = x_2 - \Pi^\delta x_1, \quad \Phi^\delta = \begin{pmatrix} -\Pi^\delta & 1 \end{pmatrix} \tag{3.103}$$

and with

$$\begin{pmatrix} 1 \\ \Pi^\delta \end{pmatrix} \in \ker\{\Phi^\delta\}$$

so consequently defining the off-the-set component.

3.5.1.2 Step 2: the I&I feedback design

Setting the off-the-set component as (3.103), one gets the extended dynamics

$$\begin{aligned} z_{k+1} =& z_k + \delta(1 - \frac{\delta}{2}\Pi^\delta)u_k - \delta\Pi^\delta x_{2k} \\ x_{1k+1} =& x_{1k} + \delta x_{2k} + \frac{\delta^2}{2}u_k \\ x_{2k+1} =& x_{2k} + \delta u_k \end{aligned}$$

so that one seeks for a feedback ensuring condition iv) of Definition (3.1). To this end, we apply the feedback transformation (3.49) which specifies to this context as

$$u = \frac{\Pi^\delta}{1 - \frac{\delta}{2}\Pi^\delta}(w + x_2) \tag{3.105}$$

so yielding

$$z_{k+1} = z_k + \delta w_k \tag{3.106a}$$

$$x_{1k+1} = e^{-\delta}x_{1k} + \delta z_k + \frac{\delta^2}{2}\frac{\Pi^\delta}{1 - \frac{\delta}{2}\Pi^\delta}w_k \tag{3.106b}$$

$$x_{2k+1} = x_{2k} + \frac{\delta\Pi^\delta}{1 - \frac{\delta}{2}\Pi^\delta}(w_k + x_{2k}). \tag{3.106c}$$

According to Sections 3.3.3.1, 3.3.3.2 and 3.3.3.3, the remaining component of the feedback can be set as follows:

- As a **dead-beat** feedback $w_k = -\frac{1}{\delta}z_k$ so getting for any $k \geq 1$

$$\begin{aligned} z_{k+1} =& 0 \\ x_{1k+1} =& e^{-\delta}x_{1k} \\ x_{2k+1} =& \Pi^\delta x_{1k}. \end{aligned}$$

- As a **direct Lyapunov-based** sampled-data control $w_k = -Kz_k$ with $\delta K \in]0, 1[$ as the solution to

$$\frac{\delta}{2}K^2\Big(\rho_2^\delta + \delta^2 - \frac{\delta}{4}(1 - e^{-\delta})\Big) + \delta\rho_1^\delta + \delta^2 < 0$$

with ρ_1^δ, $\rho_2^\delta > 0$ such that

$$\rho_1^\delta > \delta\frac{e^{-\delta}}{1 - e^{-\delta}}, \quad \rho_2^\delta > \delta\frac{e^{-\delta}}{1 + e^{-\delta}}.$$

- As the **PISM** feedback $w_k = \frac{e^{-K_c\delta} - 1}{\delta}z_k$ with $K_c > \frac{1}{2}$.

3.5.2 A simple academic example

Here we are going to apply the proposed methodology to stabilize via sampled-data I&I the following dynamics

$$\begin{aligned}\dot{x}_1 =& x_1^2 + x_2\\ \dot{x}_2 =& x_3\\ \dot{x}_3 =& u\end{aligned} \tag{3.108}$$

with $x = col(x_1, x_2, x_3) \in \mathbb{R}^3$ and $u \in \mathbb{R}$.

Continuous-time I&I design

Consider (3.108), then by setting $\gamma(x_1) = -x_1 - x_1^2$, the evolutions of x_1 when $x_2 = \gamma(x_1)$ possess a GAS equilibrium with $W(x_1) = \frac{1}{2}x_1^2$. Then one sets the target dynamics over $\mathbb{R}$ as

$$\dot{\xi} = \alpha(\xi) = \xi^2 + \gamma(\xi) = -\xi. \tag{3.109}$$

At this point, one has to look for a mapping $\pi(\cdot) : \mathbb{R} \to \mathbb{R}^3$ in the form $\pi(\xi) = col(\pi_1(\xi), \pi_2(\xi), \pi_3(\xi))$ and control $c(\cdot) : \mathbb{R} \to \mathbb{R}$ solutions to

$$\nabla\pi_1(\xi)\alpha(\xi) = -\pi_1^2(\xi) + \pi_2(\xi) \tag{3.110a}$$
$$\nabla\pi_2(\xi)\alpha(\xi) = \pi_3(\xi) \tag{3.110b}$$
$$\nabla\pi_3(\xi)\alpha(\xi) = c(\xi). \tag{3.110c}$$

Substituting $\alpha(\xi) = \xi^2 + \gamma(\xi) = -\xi$ in the latter equations, one gets that

$$\begin{aligned}\pi_1(\xi) =& \xi =\\ \pi_2(\xi) =& \gamma(\xi) = -\xi - \xi^2\\ \pi_3(\xi) =& \dot{\gamma}(\xi) = \nabla\gamma(\xi)\alpha(\xi) = \xi + 2\xi^2\\ c(\xi) =& \ddot{\gamma}(\xi) = \nabla\pi_3(\xi)\alpha(\xi) = -\xi - 4\xi^2.\end{aligned} \tag{3.111}$$

In order to solve ensure I&I stabilization in the sense of Definition 1.16, one sets $x = \pi(\xi)$ and, consequently, $z = \phi(x)$ where

$$\begin{aligned}z_1 =& \phi_1(x_1, x_2) = x_2 - \gamma(x_1) = x_2 + x_1 + x_1^2\\ z_2 =& \phi_2(x_1, x_2, x_3) = x_3 - \dot{\gamma}(x_1) = x_3 + (1 + 2x_1)(x_1^2 + x_2).\end{aligned} \tag{3.112}$$

Consider now the overall dynamics over $\mathbb{R}^5$

$$\begin{aligned}\dot{x}_1 =& -x_1 + z_1\\ \dot{x}_2 =& x_3\\ \dot{x}_3 =& u\\ \dot{z}_1 =& z_2\\ \dot{z}_2 =& u - \ddot{\gamma}(x_1).\end{aligned} \tag{3.113}$$

The problem consists in finding a feedback $u = \nu(x, z)$ such that $\lim_{t\to\infty} z(t) = 0$ with boundedness of the trajectories of the dynamics (3.113). For this purpose one show keep in mind that the mapping $col(x_1, z) \mapsto col(x_1, x_2, x_3)$ is well-defined through $\phi(x)$. Hence, the problem is simplified by considering only the partial dynamics

$$\begin{aligned}\dot{x}_1 =& - x_1 + z_1\\ \dot{z}_1 =& z_2\\ \dot{z}_2 =& u - \ddot{\gamma}(x_1).\end{aligned}$$

At this point one sets $u = -z_1 - z_2 + \ddot{\gamma}(x_1)$ making the closed-loop dynamics

$$\begin{aligned}\dot{x}_1 =& - x_1 + z_1\\ \dot{z}_1 =& z_2\\ \dot{z}_2 =& - z_1 - z_2.\end{aligned}$$

globally asymptotically stable. Hence, Boundedness of trajectories and $\lim_{t\to\infty} z(t) = 0$ are ensured. Hence the closed-loop dynamics

$$\begin{aligned}\dot{x}_1 =& x_1^2 + x_2\\ \dot{x}_2 =& x_3\\ \dot{x}_3 =& x_1 + x_2 + x_3 + x_1^2.\end{aligned}$$

has a GAS equilibrium at the origin.

Sampled-data design

Assuming $u \in \mathcal{U}^\delta$, the approximated single-rate sampled-equivalent model associated to (3.108) is provided by

$$\begin{aligned}x_{1k+1} =& x_1 + \delta(x_1^2 + x_2) + \frac{\delta^2}{2}(2x_1^3 + 2x_1x_2 + x_3) + \frac{\delta^3}{3!}[2(3x_1^2 + x_2)(x_1^2 + x_2)+\\ & 6x_1x_3 + u] + O(\delta^4)\\ x_{2k+1} =& x_2 + \delta x_3 + \frac{\delta^2}{2!}u\\ x_{3k+1} =& x_3 + \delta u.\end{aligned} \tag{3.114}$$

As a result, one has that the strict-feedback form is lost when considering the sampled-data equivalent dynamics (3.114).

One has that the approximate multi-rate sampled-data dynamics of order $r = 2$ associated to (3.108) is given by

$$\begin{aligned}x_{1k+1} =& x_1 + 2\bar{\delta}(x_1^2 + x_2) + \bar{\delta}^2(4x_1^3 + 4x_2x_1 + 2x_3) + \frac{\bar{\delta}^3}{3}[24x_1^4 + 32x_1^2x_2+\\ & 8x_3x_1 + 8x_2^2 + \frac{1}{2}(7u^1 + u^2)] + O(\delta^4)\\ x_{2k+1} =& x_2 + 2\bar{\delta}x_3 + \frac{\bar{\delta}^2}{2!}(3u^1 + u^2)\\ x_{3k+1} =& x_3 + \bar{\delta}(u^1 + u^2)\end{aligned} \tag{3.115}$$

with $u_k^1 = u(k\delta)$ and $u_k^2 = u(k\delta + \frac{\delta}{2})$.

3.5.2.1 Step 1: the choice of the target dynamics

One sets the sampled-data target dynamics as

$$\xi_{k+1} = \xi_k + \delta(\xi_k^2 + \gamma_1^\delta(\xi_k)) + \frac{\delta^2}{2}(\xi_k^3 + \xi_k\gamma_1^\delta(\xi_k) + \gamma_2^\delta(\xi_k)) + O(\delta^3). \tag{3.116}$$

Hence, one introduces the triplet $< \gamma_1^\delta(\xi), \gamma_2^\delta(\xi), c^\delta(\xi) >$ of the form

$$\begin{aligned}
\gamma_1^\delta(\xi) =& \gamma_1^0(\xi) + \frac{\delta}{2}\gamma_1^1(\xi) + O(\delta^2)\\
\gamma_2^\delta(\xi) =& \gamma_2^0(\xi) + \frac{\delta}{2}\gamma_2^1(\xi) + O(\delta^2)\\
c^\delta(\xi) =& c_0(\xi) + \frac{\delta}{2}c_1(\xi) + O(\delta^2).
\end{aligned}$$

and rewrites the ILM and invariance conditions (3.82) as

$$\begin{aligned}
&\xi^3 + \xi\gamma_1^0(\xi) + \frac{\delta}{2}(\xi\gamma_1^1(\xi) + \xi(\xi^3 + \xi\gamma_1^0(\xi) + \gamma_2^0(\xi)) + (\xi^2 + \gamma_1^0)^2) = -\xi^2 + \delta\xi^2\\
&\nabla\gamma_1^0(\xi)(\xi^2 + \gamma_1^0(\xi)) + \frac{\delta}{2}\big(\nabla\gamma_1^0(\xi)\gamma_1^1(\xi) + \nabla\gamma_1^1(\xi)(\xi^3 + \xi\gamma_1^0(\xi) + \gamma_2^0(\xi))\\
&+ \nabla^2\gamma_1^0(\xi)(\xi^2 + \gamma_1^0(\xi))^2\big) = \gamma_2^0(\xi) + \frac{\delta}{2}(\gamma_2^1(\xi) + c_0(\xi))\\
&\nabla\gamma_2^0(\xi)(\xi^2 + \gamma_1^0(\xi)) + \frac{\delta}{2}\big(\nabla\gamma_2^0(\xi)\gamma_1^1(\xi) + \nabla\gamma_2^1(\xi)(\xi^3 + \xi\gamma_1^0(\xi) + \gamma_2^0(\xi))\\
&+ \nabla^2\gamma_2^0(\xi)(\xi^2 + \gamma_1^0(\xi))^2\big) = c_0(\xi) + \frac{\delta}{2}c_1(\xi).
\end{aligned}$$

By equating the terms with the same power of δ, one can find the expressions for any term composing $(\gamma_1^\delta(\cdot), \gamma_2^\delta(\cdot), c^\delta(\cdot))$. More in details, for the terms in δ^0 one has

$$\begin{aligned}
\xi^3 + \xi\gamma_1^0(\xi) = -\xi^2 \quad &\Longrightarrow \quad \gamma_1^0(\xi) = -\xi - \xi^2\\
\nabla\gamma_1^0(\xi)(\xi^2 + \gamma_1^0(\xi)) = \gamma_2^0(\xi) \quad &\Longrightarrow \quad \gamma_2^0(\xi) = 2\xi + \xi^2\\
\nabla\gamma_2^0(\xi)(\xi^2 + \gamma_1^0(\xi)) = c_0(\xi) \quad &\Longrightarrow \quad c_0(\xi) = -4\xi^2 - \xi.
\end{aligned}$$

Proceeding in this way for the terms in δ^1, one gets

$$\begin{aligned}
&\xi\gamma_1^1(\xi) + \xi(\xi^3 + \xi\gamma_1^0(\xi) + \gamma_2^0(\xi)) + (\xi^2 + \gamma_1^0)^2 = 2\xi^2\\
&\Longrightarrow \gamma_1^1(\xi) = 0\\
&\nabla\gamma_1^0(\xi)\gamma_1^1(\xi) + \nabla\gamma_1^1(\xi)(\xi^3 + \xi\gamma_1^0(\xi) + \gamma_2^0(\xi)) + \nabla^2\gamma_1^0(\xi)(\xi^2 + \gamma_1^0(\xi))^2 = \gamma_2^1(\xi) + c_0(\xi)\\
&\Longrightarrow \gamma_2^1(\xi) = 0\\
&\nabla\gamma_2^0(\xi)\gamma_1^1(\xi) + \nabla\gamma_2^1(\xi)(\xi^3 + \xi\gamma_1^0(\xi) + \gamma_2^0(\xi)) + \nabla^2\gamma_2^0(\xi)(\xi^2 + \gamma_1^0(\xi))^2 = c_1(\xi)\\
&\Longrightarrow c_1(\xi) = 32\xi^6 + 32\xi^5 - 24\xi^4 - 12\xi^3.
\end{aligned}$$

Finally, one gets

$$\begin{aligned}\gamma_1^\delta(\xi) =& -\xi - \xi^2 + O(\delta^2)\\ \gamma_2^\delta(\xi) =& \xi + 2\xi^2 + O(\delta^2)\\ c^\delta(\xi) =& -4\xi^2 - \xi + \delta(16\xi^6 + 16\xi^5 - 12\xi^4 - 6\xi^3) + O(\delta^2)\end{aligned}$$

so defining

$$\begin{aligned}z_1 =& x_2 + x_1 + x_1^2 + O(\delta^2)\\ z_2 =& x_3 + (1 + 2x_1)(z_1 - x_1) + O(\delta^2).\end{aligned}$$

3.5.2.2 Step 2: the sampled-data I&I feedback via PISM

Suppose now we want to solve the problem by means of the PISM approach. The approximated sampled-data extended dynamic of order 2 is provided as

$$\begin{aligned}x_{1k+1} =& x_1 + 2\bar\delta(x_1^2 + x_2) + \bar\delta^2(4x_1^3 + 4x_2x_1 + 2x_3) + \frac{\bar\delta^3}{3}[24x_1^4 + 32x_1^2x_2 +\\ & 8x_3x_1 + 8x_2^2 + \frac{1}{2}(7u^1 + u^2)] + O(\bar\delta^3)\\ x_{2k+1} =& x_2 + 2\bar\delta x_3 + \frac{\bar\delta^2}{2!}(3u^1 + u^2)\\ x_{3k+1} =& x_3 + \bar\delta(u^1 + u^2)\\ z_{1k+1} =& z_1 + 2\bar\delta z_2 + \frac{\bar\delta^2}{2}(3u^1 + u^2 + x_1 - z_1 + z_2 - 6x_1z_1 + 2x_1z_2 + 4x_1^2 + 2z_1^2) +\\ & \frac{\bar\delta^3}{3!}(7u^1 + u^2 + 14u^1x_1 + 2u^2x_1 + 48x_1z_1 - 32x_1z_2 + 48z_1z_2 + 32x_1z_1^2 -\\ & 96x_1^2z_1 + 32x_1^2z_2 - 16x_1^2 + 64x_1^3 - 32z_1^2) + O(\bar\delta^4)\\ z_{2k+1} =& z_2 + \bar\delta(u^1 + u^2 + 2x_1 - 2z_1 + 2z_2 - 12x_1z_1 + 4x_1z_2 + 8x_1^2 + 4z_1^2) +\\ & \bar\delta^2(\frac{3u^1}{2} + \frac{u^2}{2} + 3u^1x_1 + u^2x_1 + 12x_1z_1 - 8x_1z_2 + 12z_1z_2 + 8x_1z_1^2 -\\ & 24x_1^2z_1 + 8x_1^2z_2 - 4x_1^2 + 16x_1^3 - 8z_1^2) + O(\bar\delta^3)\end{aligned} \tag{3.117}$$

with $\delta = 2\bar\delta$.

Remark 3.17. *Note that the approximation is not homogeneous in the sense the dynamical equations are not approximated at the same order of $\bar\delta$. This is due in order to report only the necessary terms characterizing the 1^{st}-order approximate PISM controller we shall use in the following.*

We denote by z^c the continuous-time closed-loop z-dynamics in (3.5.2). Its sampled-data equivalent model is provided as

$$\begin{aligned}z^c_{1k+1} =& z^c_{1k} + 2\bar\delta z^c_{1k} - 2\bar\delta^2(z^c_{1k} + z^c_{2k}) + \frac{4}{3}\bar\delta z^c_{1k} + O(\delta^4)\\ z^c_{2k+1} =& z^c_{2k} - 2\bar\delta(z^c_{1k} + z^c_{2k}) + 2\bar\delta^2 z^c_{1k} + O(\delta^3)\end{aligned} \tag{3.118}$$

According to the previous result we want to compute the 1^{st}-order approximate PISM controller $\underline{u} = \underline{\nu}_P^{\delta[1]}(x,z)$ as

$$\underline{\nu}_P^{\delta[1]}(x,z) = \begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix} + \frac{\bar{\delta}}{2}\begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix}. \tag{3.119}$$

At this point, one substitutes the latter expression in the z dynamics in (3.117). The PISM equality is provided by equating both the right-hand sides of (3.118) and (3.117).

$$\begin{aligned}
&z_1 + 2\bar{\delta}z_2 + \frac{\bar{\delta}^2}{2}(3u_1^0 + u_2^0 + x_1 - z_1 + z_2 - 6x_1z_1 + 2x_1z_2 + 4x_1^2 + 2z_1^2) + \\
&\frac{\bar{\delta}^3}{4}(3u_1^1 + u_2^1) + \dots = z_1^c + 2\bar{\delta}z_2^c - 2\bar{\delta}^2(z_1^c + z_2^c) + \frac{4}{3}\bar{\delta}z_1^c + \dots \\
&z_2 + \bar{\delta}(u_1^0 + u_2^0 + 2x_1 - 2z_1 + 2z_2 - 12x_1z_1 + 4x_1z_2 + 8x_1^2 + 4z_1^2) + \frac{\bar{\delta}^2}{2}(u_1^1 + u_2^1) = \\
&+ \dots z_2^c - 2\bar{\delta}(z_1^c + z_2^c) + 2\bar{\delta}^2 z_1^c + \dots
\end{aligned} \tag{3.120}$$

Now, by setting $z^c(k\delta) = z_k$ in (3.120), the expression for (3.119) is found by equating in the so defined equality the terms at the same power of $\bar{\delta}$. At each step, an algebraic set of linear equations in the unknowns u_{ij} $(i,j = 1,2)$ has to be solved. More in details, for u_{i0} one gets

$$\begin{cases} 3\mathbf{u_1^0} + \mathbf{u_2^0} + x_1 - z_1 + z_2 - 6x_1z_1 + 2x_1z_2 + 4x_1^2 + 2z_1^2) = 4z_1 + 4z_2 \\ \mathbf{u_1^0} + \mathbf{u_2^0} + 2x_1 - 2z_1 + 2z_2 - 12x_1z_1 + 4x_1z_2 + 8x_1^2 + 4z_1^2 = -2z_1 + -2z_2 \end{cases} \tag{3.121}$$

which is solved by

$$u_1^0 = u_2^0 = z_1 - x_1 - z_2 - k_1z_1 - k_2z_2 + 6x_1z_1 - 2x_1z_2 - 4x_1^2 - 2z_1^2$$

Similarly for u_{i1} one gets the algebraic system

$$\begin{cases} \frac{1}{4}(3\mathbf{u_1^1} + \mathbf{u_2^1}) + \frac{1}{3!}(7u_1^0 + u_2 + 14u_1^0x_1 + 2u_2^0x_1 + 48x_1z_1 - 32x_1z_2 + 48z_1z_2 + \\ \qquad 32x_1z_1^2 - 96x_1^2z_1 + 32x_1^2z_2 - 16x_1^2 + 64x_1^3 - 32z_1^2) = \frac{4}{3}z_1 \\ \frac{1}{2}(\mathbf{u_1^1} + \mathbf{u_2^1}) + \frac{3u_1^0}{2} + \frac{u_2^0}{2} + 3u_1^0x_1 + u_2^0x_1 + 12x_1z_1 - 8x_1z_2 + 12z_1z_2 + 8x_1z_1^2 - \\ \qquad 24x_1^2z_1 + 8x_1^2z_2 - 4x_1^2 + 16x_1^3 - 8z_1^2 = 2z_1 \end{cases} \tag{3.122}$$

whose solution is

$$\begin{aligned}
u_1^1 = &\frac{u_2^0}{3} - \frac{5u_1^0}{3} - \frac{4k_1z_2}{3} - \frac{10u_1^0x_1}{3} + \frac{2u_2^0x_1}{3} - 8x_1z_1 + \frac{16x_1z_2}{3} - 8z_1z_2 + \\
&\frac{4k_2^2z_2}{3} - \frac{16x_1z_1^2}{3} + 16x_1^2z_1 - \frac{16x_1^2z_2}{3} + \frac{8x_1^2}{3} - \frac{32x_1^3}{3} + \frac{16z_1^2}{3} + \frac{4k_1k_2z_1}{3} \\
u_2^1 = &\frac{80x_1z_2}{3} - \frac{7u_2^0}{3} - \frac{20k_1z_2}{3} - \frac{26u_1^0x_1}{3} - \frac{14u_2^0x_1}{3} - 40x_1z_1 - \frac{13u_1^0}{3} - 40z_1z_2 + \\
&\frac{20k_2^2z_2}{3} - \frac{80x_1z_1^2}{3} + 80x_1^2z_1 - \frac{80x_1^2z_2}{3} + \frac{40x_1^2}{3} - \frac{160x_1^3}{3} + \frac{80z_1^2}{3} + \frac{20k_1k_2z_1}{3}.
\end{aligned}$$

3.5.2.3 Simulations

Simulations are referred the dynamics discussed in the academic example developed in Section 3.5.2. In particular, we implement and compare closed-loop performances under three sample-data I&I controllers:

- the control $u_{de}(x_k) = u_c(x(k\delta))$ implemented by emulation of the continuous-time one;
- 1^{st}-order approximate PISM controller;
- 1^{st}-order approximate dead-beat controller.

In the two latter cases, the mappings $< \gamma_1^\delta(\xi), \gamma_2^\delta(\xi), c^\delta(\xi) >$ is defined in $O(\delta^2)$; thus the implicit manifold is the same as in the continuous-time design (which is reported in pointed red). The continuous-time closed-loop behavior and control are also reported since the PISM law is partially based on it. Simulations are performed for different values of the sampling period δ (0.1, 0.3 and 0.5 seconds). We focus on two aspects: first we want to verify if invariance is guaranteed under the three approximate feedbacks as the sampling period δ increases; secondly, we shall care of its attractivity as δ increases.

Concerning invariance, the initial state was set as $x = col(2, -5, -10)$ with zero initial displacement from the corresponding surface (i.e., $z = col(0, 0)$). Figures 3.2 and 3.3 depict the results of the simulations in this sense. Figure 3.2 underlines that, although δ is quite small, the emulation-based feedback does not guarantee invariance of the corresponding surface. This is even clearer when considering Figure 3.3. Hence, the emulation-based controller is no longer stabilizing the closed-loop equilibrium in the I&I sense. Contrarily, the feedback laws relying upon sampled-data redesign (even if approximate) preserve invariance of the target even for increasing values of the sampling period. Also, simulations confirm that invariance under sampling is guaranteed at any sampling instant $t = k\delta$. The control effort is acceptable in all of the simulations for each controller. We recall that when the trajectories of the system lie in the surface $c^\delta(\cdot)$ is acting. Hence, one has not to wonder if the deadbeat amplitude of the control is acceptable as $u^\delta_{\mathrm{db}}(\pi^\delta(\xi), 0) = c^\delta(\xi)$ so that no inversion by δ is needed in the definition of $c^\delta(\xi)$.

Concerning attractivity, simulations have been carried out when setting $z_0 = col(0.5, 0.5)$ as initial displacement from the stable surface. As one may expect the deadbeat approach leads to the best performance in terms of convergence. Indeed, it stretches the closed-loop trajectories onto the target manifold in exactly one step, at least as δ small enough. As one is applying an approximate solution, when the sampling period increases such a property is lost but convergence to the surface is still guaranteed over a finite number of sampling instants. As we already pointed out while developing theory, the price to pay is given in terms of control effort making such solution not applicable in practice. Nevertheless, when δ increases the

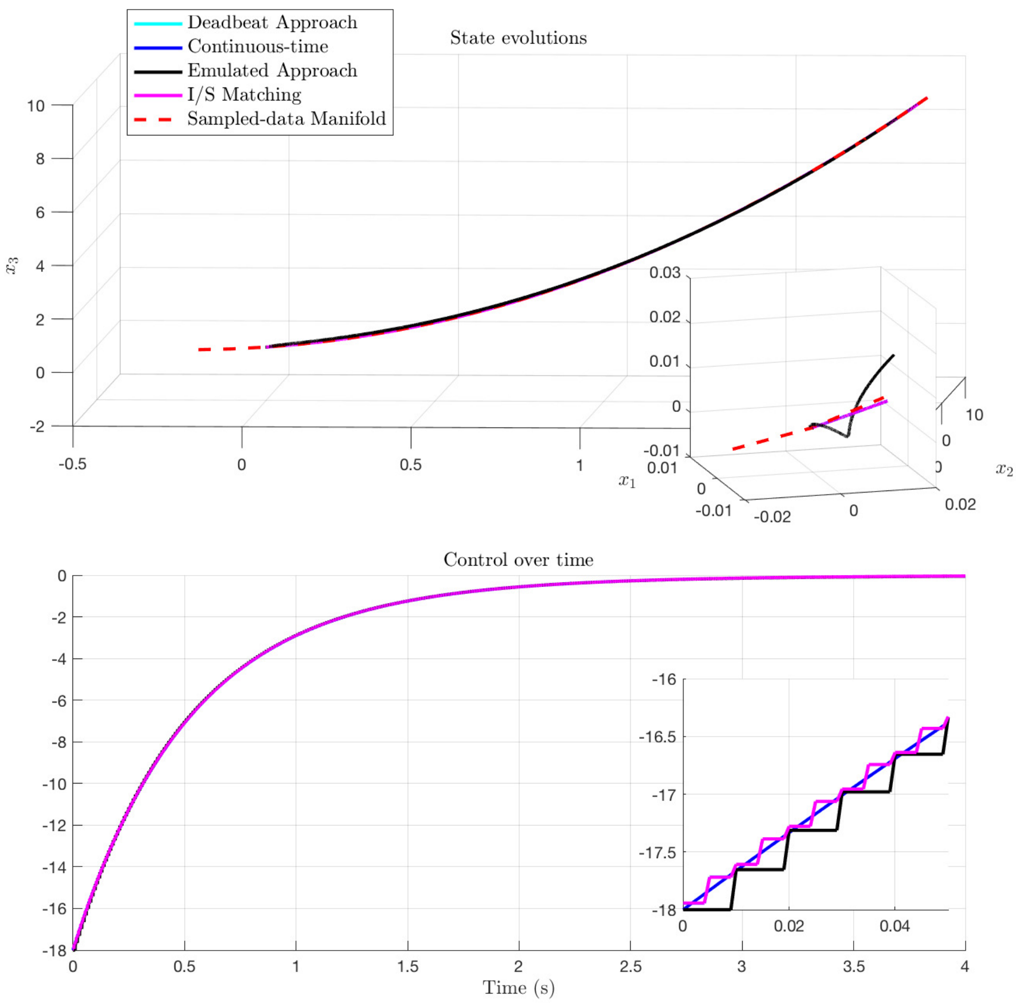


Figure 3.2: Invariance of I&I sampled-data controllers with $\delta = 0.01s$.

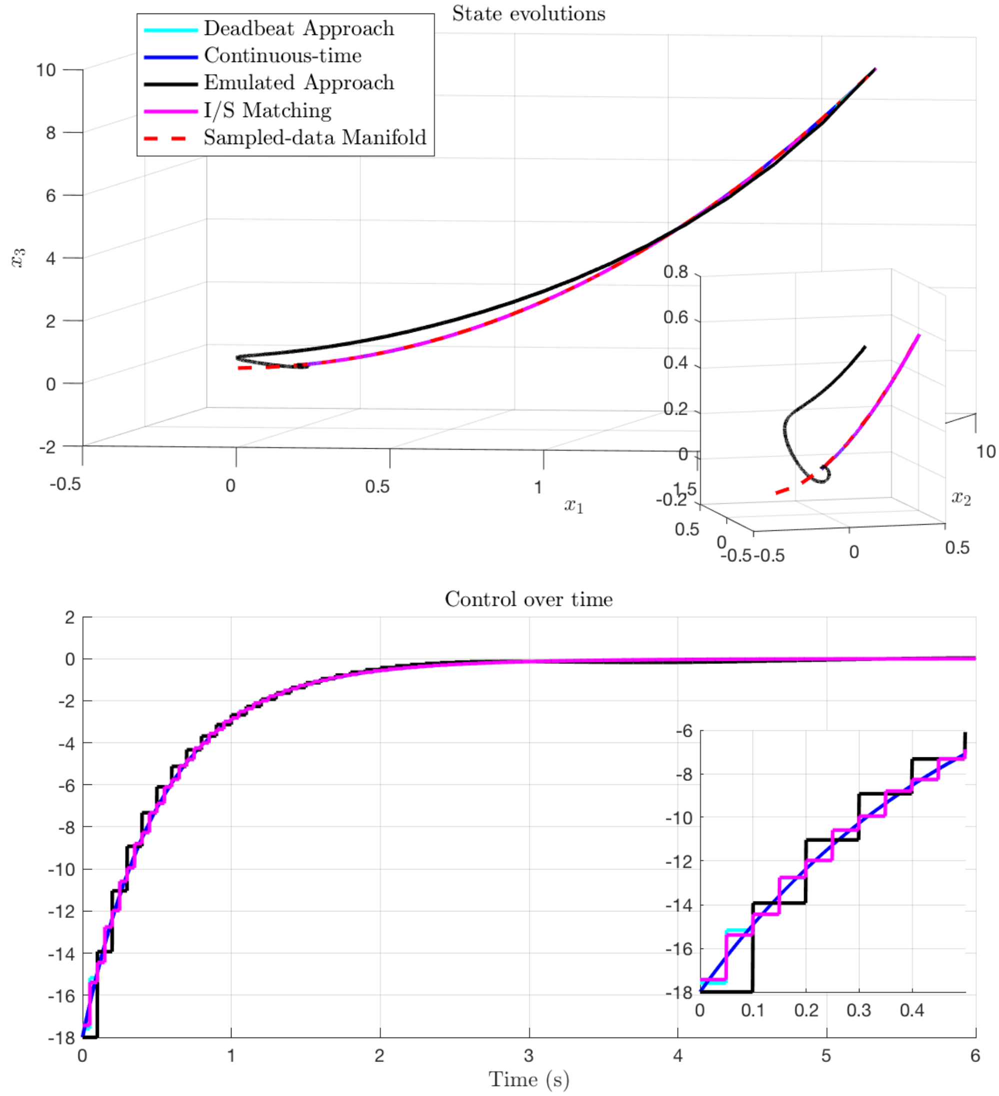


Figure 3.3: Invariance of I&I sampled-data controllers with $\delta = 0.1s$.

control effort significantly decreases making such a control law implementable when the sampling period is large enough.
Concerning the PISM solution, simulations confirm that the sampled-data convergence to the manifold follows the continuous-time one. Such a property ensures smoothness of the closed-loop evolutions contrarily to the case of the emulation-based and deadbeat feedback closed-loop dynamics. The control effort is limited and closed to the continuous-time one.
Finally, it is evident from the plots that the emulated controller performances degrade for even small values of δ yielding instability of the equilibrium for $\delta = 0.5$ s. On the other side, a sampled-data I&I design (even if in an approximate scenario) guarantees stability in closed-loop even as δ increases and the emulation-based feedback fails.

3.6 Conclusions and literature review

In this chapter, we have provided a first extension of the Immersion and Invariance design tool to sampled-data nonlinear systems admitting a strict-feedback structure. The results are based on the works in [97, 103]. We have shown that, regardless the loss of the feedback form under sampling, I&I provides a constructive ways of deducing a sampled-data feedback stabilizing the origin of the overall dynamics. Basically, starting from backstepping-like assumptions (over the continuous-time "ideal" system) on part of the dynamics, we have exhibited a new sampled-data dynamics defining the target together with suitable immersion mappings and on-the-manifold control. This is achieved by introducing an Input-Lyapunov Matching Problem over the continuous-time target dynamics coupled with invariance conditions. All of the concerned mappings (and the manifold itself) come to be parametrized by the sampling period δ and recover the continuous-time counterparts only as $\delta \to 0$. Finally, with reference to the elementary integrator-feedback interconnection, we have deduced three way of designing the sampled-data I&I feedback based on

- direct discrete-time design through the so-called dead-beat control;
- direct sampled-data design through the sampled-data Lyapunov-based control;
- indirect sampled-data design through the definition of a suitable Partial Input-to-State Matching Problem.

The extension to the case of higher order cascade is sketched as well while further details are given in [103].

As far as stabilization of strict-feedback structure under sampling is concerned, several works have been proposed to extend backstepping-like design procedures to the sampled-data context based on the Euler or higher order approximate models by Burlion, Postoyan, Nesic and other researchers in the field [182, 18, 160, 142]. All

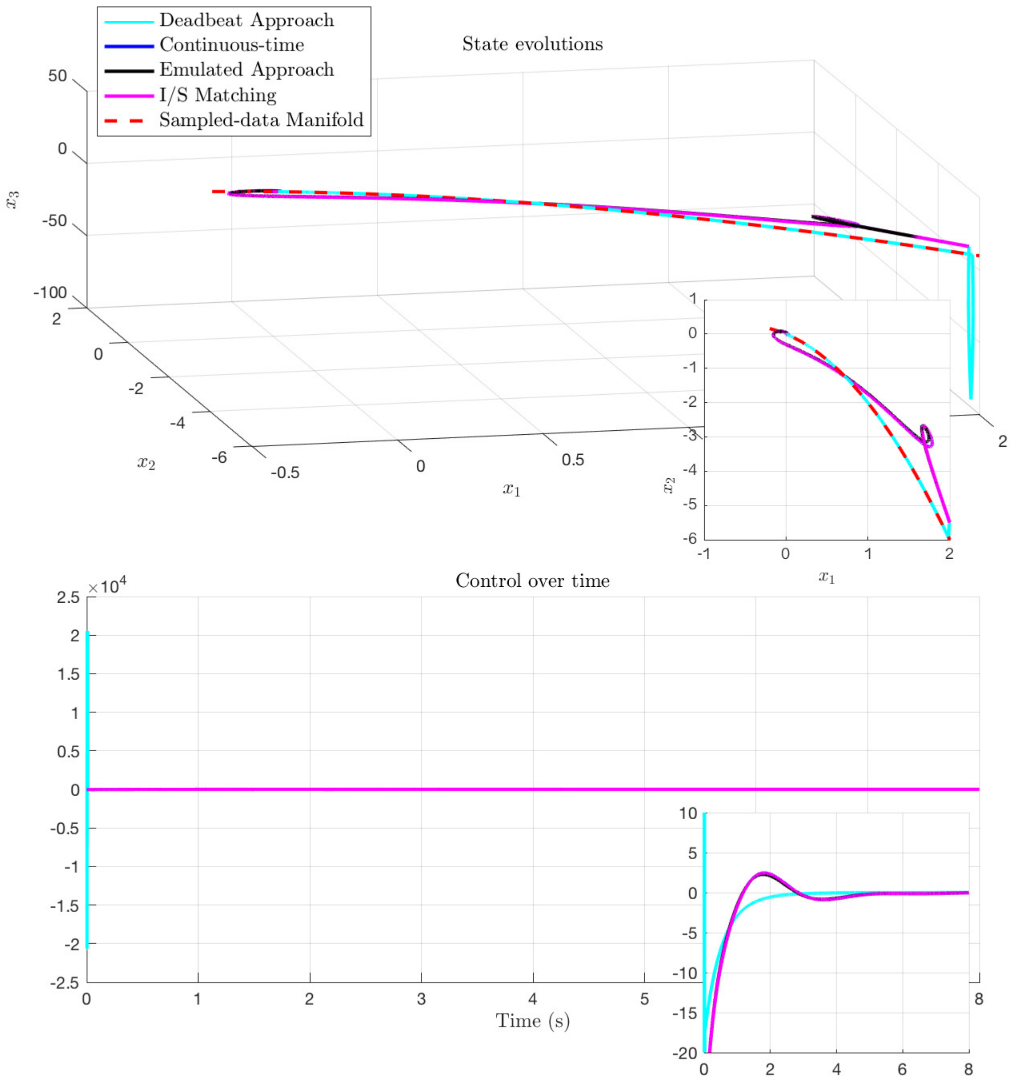


Figure 3.4: Manifold attractivity under I&I sampled-data controllers with $\delta = 0.01s$ and zooming on the first step control amplitude.

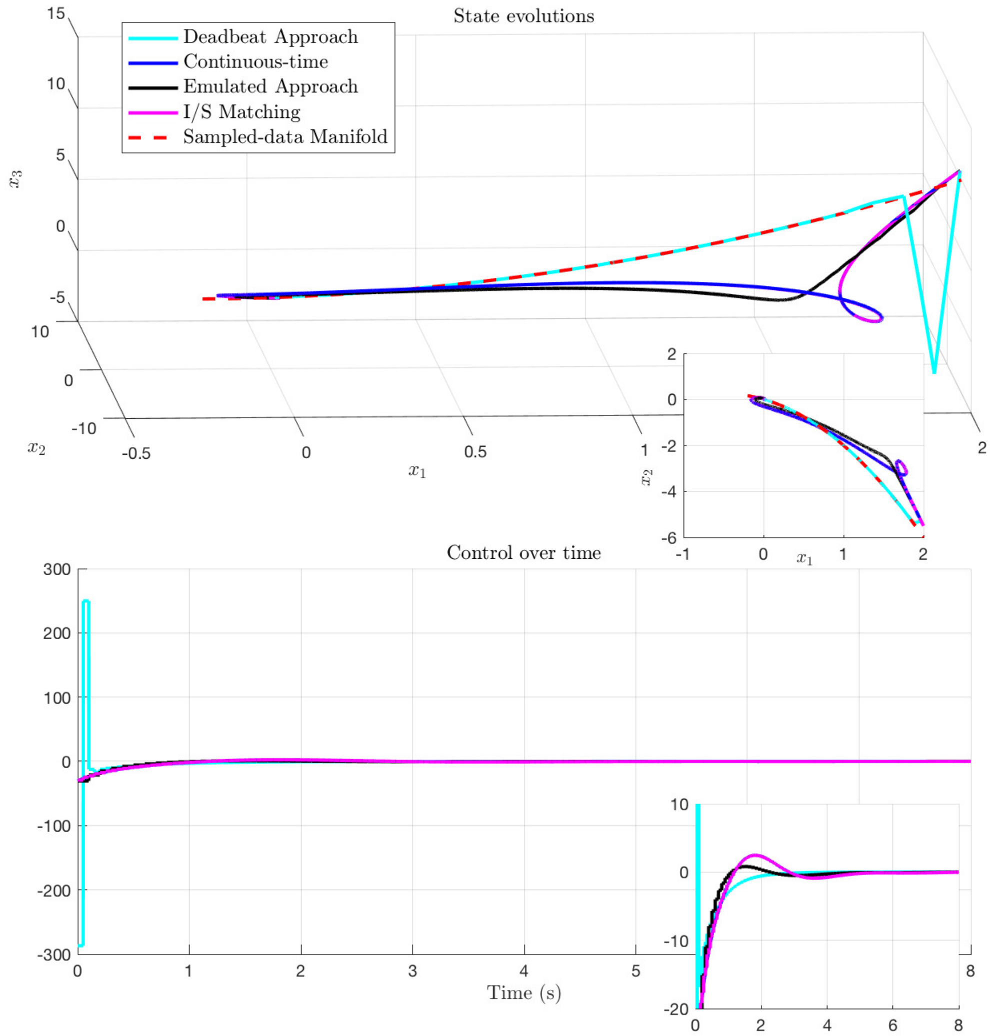


Figure 3.5: Manifold attractivity under I&I sampled-data controllers with $\delta = 0.1s$.

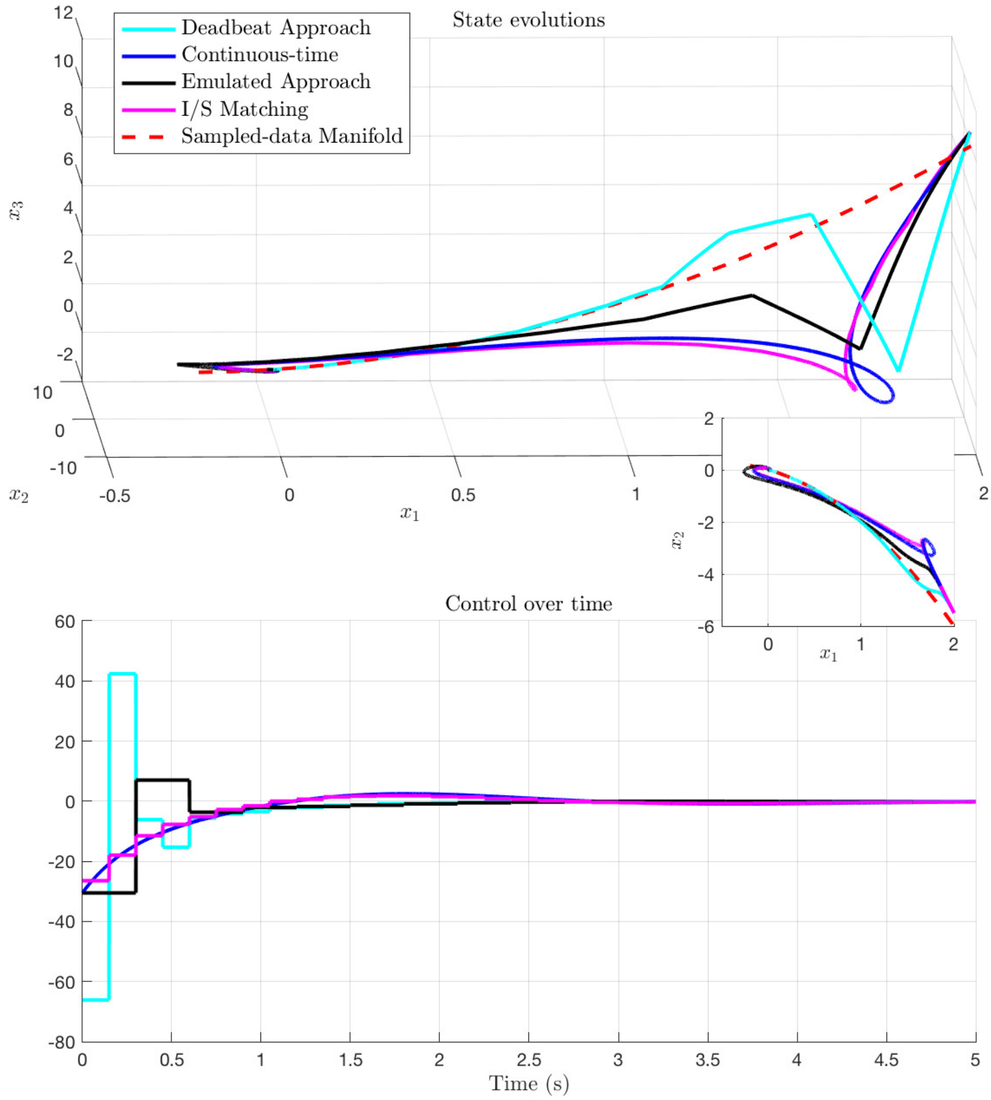


Figure 3.6: Manifold attractivity under I&I sampled-data controllers with $\delta = 0.3s$.

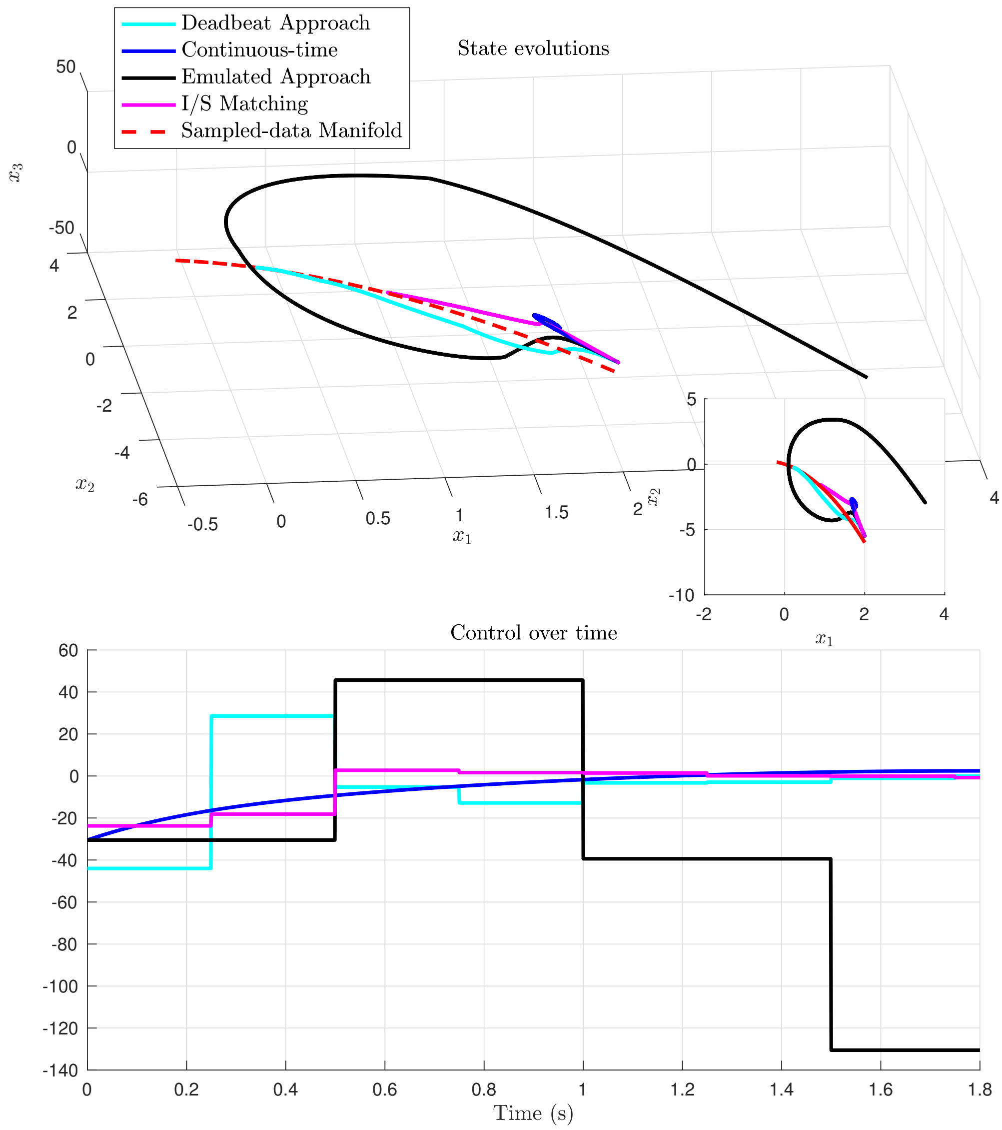


Figure 3.7: Attractivity under I&I sampled-data controllers with $\delta = 0.5s$

of these methodologies are generally trajectory-based as a Lyapunov-based methodology is hard to be developed under sampling because of the nonlinearity (in the control variable) of the involved mappings. In [179], Tanasa et al. propose a new way of performing backstepping under sampling through a multi-rate design based on a generalized Input-Lyapunov Matching problem.

Concerning I&I, its first extension to discrete-time systems in strict-feedforward form (issued from sampling) is due to Yalcin and Astolfi in [187] in the context of adaptive control and where the design is carried out based on the Euler approximate sampled-data equivalent model of (3.1). A similar work on the same topic has been recently proposed by Franco in [42]. In terms of stabilization, a preliminary work concerning Immersion and Invariance stabilization for sampled-data dynamics has been proposed Mattei et al. in [95] where a stronger assumption than Assumption 3.1 is set. This work has been then applied by the same authors for the robust digital attitude stabilization of a rigid spacecraft in [96]. In [131] we have extended I&I to sampled-data feedforward systems by exploiting discrete-time average passivity-based arguments and providing a sampled-data I&I bounded feedback. Recently, we have also shown how I&I can be profitably exploited, together with sampling, to deal with time-delay systems with delays acting over the input [132, 102] and, for classes of nonlinear systems, over the state [98].

Chapter 4

Sampled-data feedforwarding

Contents

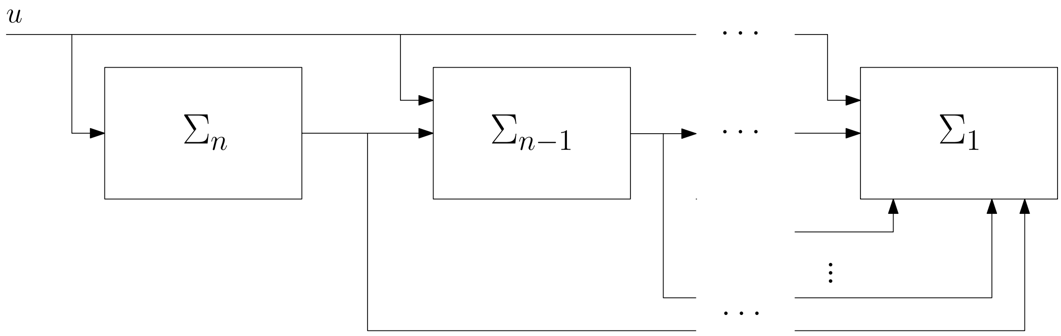


Figure 4.1: Feedforward interconnection

THIS chapter concerns stabilization of feedforward multiple cascade dynamics under sampling. First, we discuss about feedforward dynamics under sampling by emphasizing on the fact that their structure is preserved by the corresponding sampled-data equivalent model and how the concerned passivity properties are transformed. Then, we show that u-average passivity concepts and Lyapunov methods can be profitably exploited to provide a systematic sampled-data design procedure. The proposed iterative stabilizing technique is reminiscent of continuous-time feedforwarding and can be applied under the same assumptions as those set over the continuous-time cascade dynamics. The final sampled feedback is carried out through a three steps procedure that involves passivation and stabilization in the u-average sense.

The results of this chapter are based on [99, 100].

M. Mattioni, S. Monaco and D. Normand-Cyrot. *Sampled-data stabilization of feedforward dynamics with Lyapunov cross-term.* 55th IEEE Conference on Decision and Control (CDC). pp. 1322-1327. Las Vegas, USA.

M. Mattioni, S. Monaco and D. Normand-Cyrot. *Feedforwarding under sampling.* Submitted to IEEE Transactions on Automatic Control. September, 2017.

4.1 Feedforward dynamics

As illustrated in Figure 4.1, feedforward systems exhibit an *upper triangular structure* of the form

$$\dot{x}_n = f_n(x_n) + \varphi_n(x_1, \dots, x_n) + g_n(x_1, \dots, x_n)u \tag{4.1a}$$

$$\vdots$$

$$\dot{x}_2 = f_2(x_2) + \varphi_2(x_1, x_2) + g_2(x_1, x_2)u \tag{4.1b}$$

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u \tag{4.1c}$$

where $x_i \in \mathbb{R}^{n_i}$, $f_i(0) = 0$, $\varphi_i(0, \dots, 0) = 0$, $u \in \mathbb{R}^n$. For the sake of compactness, we might denote $x^\top := (x_1^\top, \dots, x_n^\top)$ $x^{i\top} = (x_1^\top, \dots, x_i^\top)$ and

$$f^i(x^i) = \begin{pmatrix} f_i(x_i) \\ \vdots \\ f_2(x_2) \\ f_1(x_1) \end{pmatrix}, \quad \varphi^i(x^i) = \begin{pmatrix} \varphi_i(x_1, \dots, x_i) \\ \vdots \\ \varphi_2(x_1, x_2) \\ 0 \end{pmatrix} \text{ and } g^i(x^i) = \begin{pmatrix} g_i(x_1, \dots, x_i) \\ \vdots \\ g_2(x_1, x_2) \\ g_1(x_1) \end{pmatrix}.$$

Feedforward structures are very fascinating from both the points of view of analysis and control design because of their upper nested interconnection and their involvement in theoretical and practical situations such as the *driving-driven system decomposition* discussed in [19] or in the case of input-output linearization of input-affine systems [56].

A particular class of feedforward systems is represented by the so-called *strict-feedforward* forms occurring when, for any $i = 1, \dots, n$

$$\begin{aligned} &f_i(x_i) = F_i x_i, \quad F_i \in Mat_{\mathbb{R}}(n_i, n_i) \\ &\nabla_{x_i} \varphi_i(x_1, \dots, x_i) = 0 \\ &\nabla_{x_i} g_i(x_1, \dots, x_i) = 0. \end{aligned}$$

Thus, a strict-feedforward system is described by

$$\dot{x}_n = F_n x_n + \varphi_n(x_1, \dots, x_{n-1}) + g_n(x_1, \dots, x_{n-1})u \tag{4.2a}$$
$$\vdots$$
$$\dot{x}_2 = F_2 x_2 + \varphi_2(x_1) + g_2(x_1)u \tag{4.2b}$$
$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u. \tag{4.2c}$$

Remark 4.1. *A geometric characterization of continuous-time feedforward forms has been provided by Astolfi and Kaliora in [7] and by Tall and Respondek in [178] providing necessary and conditions for an input-affine dynamic (1.1a) to be feedback equivalent to a feedforward structure. Similar results in discrete time have been proposed by Moog and Kotta in [136] for strict-feedforward systems.*

Because of their nested upper triangular structure, feedforward dynamics have attracted the interest of a lot of researches in automatic control. As a matter of fact, it allows to establish constructive bottom-up procedure that may arise from different properties (e.g., passivity) one wants to exploit and enforce on the overall system [115, 151, 78, 169, 106, 161].

4.1.1 Feedforward systems and forwarding

Let us briefly recall the main idea of the forwarding-based control when applied to the elementary feedforwarding two-block interconnection; namely, let (4.1) specify as

$$\dot{x}_2 = f_2(x_2) + \varphi_2(z, x_1) + g_2(x_1, x_2)u \tag{4.3a}$$
$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u \tag{4.3b}$$

and possess an equilibrium at the origin. The following standing standard feedforwarding assumptions [170] will hold from now on.

Assumption 4.1 (Linear growth). *The functions $\varphi_2(x_1, x_2)$ and $g_2(x_1, x_2)$ satisfy the* linear growth *property with respect to the state x_2*[1]*.*

Assumption 4.2 (GS of the decoupled continuous-time x_2-dynamics). *The origin of $\dot{x}_2 = f_2(x_2)$ is globally stable (GS), with radially unbounded and locally quadratic Lyapunov function $W(x_2)$ so that $\mathrm{L}_{f_2}W(x_2) \le 0$ for all $x_2 \in \mathbb{R}^{n_2}$. Moreover, there exist real constants c and M such that, for $\|x_2\| > M$,*

$$\|\nabla W(x_2)\|\|x_2\| \le cW(x_2).$$

Assumption 4.3 (GS of the decoupled continuous-time x_1-dynamics). *$\dot{x}_1 = f_1(x_1)$ is globally stable (GS), with radially unbounded and locally quadratic Lyapunov function $U(x_1)$ such that $\mathrm{L}_{f_1}U(x_1) \le 0$ for any x_1.*

The above assumptions allow to deduce the following result.

Theorem 4.1 (Continuous-time feedforwarding, [170]). *Let the cascade dynamics (4.3) verify Assumptions 4.1 to 4.3 and the sub-dynamics (4.3b) with output $y_0 = \mathrm{L}_{g_1}U(x_1)$ be Zero State Detectable (ZSD). Let the pair $(\nabla f^2(0),\ g^2(0))$ be stabilizable. Then:*

(i) *the function*

$$\Psi(x_1, x_2) = \int_0^\infty \mathrm{L}_{\varphi_2(\cdot, x_1(s)) - g_2(\cdot, x_1(s))\mathrm{L}_{g_1}U(x_1(s))} W(x_2(s))\mathrm{d}s \tag{4.4}$$

evaluated along the solutions of the closed loop dynamics

$$\begin{aligned}\dot{x}_2 &= f_2(x_2) + \varphi_2(z, x_1) - g_2(x_1, x_2)\mathrm{L}_{g_1}U(x_1)\\ \dot{x}_1 &= f_1(x_1) - g_1(x_1)\mathrm{L}_{g_1}U(x_1)\end{aligned} \tag{4.5}$$

qualifies as a cross term in the construction of a radially unbounded Lyapunov function

$$V(x) = U(x_1) + \Psi(x_1, x_2) + W(x_2) \tag{4.6}$$

verifying

$$(\mathrm{L}_{f^2} - \mathrm{L}_{g_1}U(x_1)\mathrm{L}_{g^2})V(x) \le -\|\mathrm{L}_{g_1}U(x_1)\|^2 \tag{4.7}$$

[1] A function $\omega : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is said to satisfy a *linear growth property* with respect to the first variable if there exist functions $\gamma_1(\cdot), \gamma_2(\cdot) \in \mathcal{K}$ differentiable at $x_1 = 0$, such that $\|\omega(x_1, x_2)\| \le \gamma_1(\|x_1\|)\|x_2\| + \gamma_2(\|x_1\|)$.

(ii) *the dynamics (4.3) with output $y = \mathrm{L}_{g^2} V(x)$ is passive with storage function (4.6);*

(iii) *the control law $u = -\mathrm{L}_{g^2} V(x)$ achieves global asymptotic stability of the closed-loop equilibrium of (4.3). If the Jacobian linearization of (4.3) is stabilizable, such a feedback ensures LES of the equilibrium.*

Under Assumptions 4.1 to 4.3 there exists a damping feedback $u_0 = -\mathrm{L}_{g_1} U(x_1)$ ensuring GAS and LES of the equilibrium of (4.3b) and GS of the overall (4.3). Accordingly, one deduces the existence of a cross term of the form (4.4) satisfying the partial derivative equation

$$\dot{\Psi}(x_1, x_2) = -\mathrm{L}_{\varphi_2(x_1,x_2) - g_2(x_1,x_2)\mathrm{L}_{g_1} U(x_1)} W(x_2) \tag{4.8}$$

along the trajectories of (4.5). Hence, the Lyapunov function (4.6) is non-increasing along the closed-loop dynamics (4.5) so verifying (4.7)

Assumption 4.3 could be modified to require the asymptotic stabilizability of (4.3b) through any smooth feedback $u = k(x_1)$ without affecting the forthcoming results. Though, it is here assumed for the sake of uniformity in the iterative procedure.

4.1.2 Feedforward dynamics under sampling

Under sampling, feedforward systems are of paramount interest as, contrarily to feedback structures, their nested interconnection form is preserved by sampling. This allows to carry out analysis and control of sampled-data feedforward dynamics in a constructive and iterative way and they can be used as a benchmark example for comparing different sampled-data control strategies arising from several frameworks (e.g., direct/indirect digital design or direct discrete-time design).

Assume now $u \in \mathcal{U}^\delta$ and measures of the state being available only at the sampling instants. Then, (4.1) rewrites, for any $k \geq 0$, as for $t \in [k\delta, (k+1)\delta[$

$$\dot{x}_n(t) = f_n(x_n(t)) + \varphi_n(x_1(t), \ldots, x_n(t)) + g_n(x_1(t), \ldots, x_n(t))u_k \tag{4.9a}$$

$$\vdots$$

$$\dot{x}_2(t) = f_2(x_2(t)) + \varphi_2(x_1(t), x_2(t)) + g_2(x_1(t), x_2(t))u_k \tag{4.9b}$$

$$\dot{x}_1(t) = f_1(x_1(t)) + g_1(x_1(t))u_k. \tag{4.9c}$$

By integrating (4.9) one as in Definition 2.1 one can infer the following result.

Lemma 4.1. *The equivalent sampled-data model to (4.1) preserves the feedforward structure; i.e., it gets the cascade form*

$$x_{nk+1} = f_n^\delta(x_{nk}) + \varphi_n^\delta(x_{1k}, \ldots, x_{nk}) + g_n^\delta(x_{1k}, \ldots, x_{nk}, u_k) \tag{4.10a}$$

$$\vdots$$

$$x_{2k+1} = f_2^\delta(x_{2k}) + \varphi_2^\delta(x_{1k}, x_{2k}) + g_2^\delta(x_{1k}, x_{2k}, u_k) \tag{4.10b}$$

$$x_{1k+1} = f_1^\delta(x_{1k}) + g_1^\delta(x_{1k}, u_k) \tag{4.10c}$$

with $g_i^\delta(x_1,\dots,x_i,0)=0$ *and*

$$\begin{aligned}
f_i^\delta(x_i) =& e^{\delta \mathrm{L}_{f_i}} x_i = x_i + \sum_{j>0} \frac{\delta^j}{j!}\mathrm{L}^j_{f_i} x_i\\
\varphi_i^\delta(x_1,\dots,x_n) =& e^{\delta \mathrm{L}_{f^i+\varphi^i}} x_i - e^{\delta \mathrm{L}_{f_i}} x_i\\
=& \delta\varphi_i(x_1,\dots,x_n) + \sum_{j>1}\frac{\delta^j}{j!}[\mathrm{L}^{j-1}_{f^i+\varphi^i}\mathrm{L}_{f_i+\varphi_i} - \mathrm{L}^j_{f_i}]x_i\\
g_i^\delta(x_1,\dots,x_i,u) =& \int_0^u G_i^\delta(x_1^+(v),\dots,x_i^+(v),v)\mathrm{d}v\\
G_i^\delta(x_1,\dots,x_i,u) =& \int_0^\delta e^{-s ad_{f^i+\varphi^i+ug^i}} g_i(x_1,\dots,x_i)\mathrm{d}s
\end{aligned}$$

with $x_i^+(u) = f_i^\delta(x_i)+\varphi_i^\delta(x_1,\dots,x_i)+g_i^\delta(x_1,\dots,x_i,u)$ *for* $i=1,\dots,n$.

Remark 4.2. *Albeit the feedforward form is preserved under sampling, the input-affine structure of the equations is not.*

As a consequence, the equivalent (F_0, G) representation of (4.10) exhibits a feedforward cascade interconnection; namely, it gets the form

$$\begin{aligned}
x_i^+ =& f_i^\delta(x_i) + \varphi_i^\delta(x_1,\dots,x_i), \quad x_i^+ = x_i^+(0) && \text{(4.11a)}\\
x_1^+ =& f_1^\delta(x_1), \quad x_1^+ = x_1^+(0) && \text{(4.11b)}\\
\frac{\mathrm{d}x_i^+(u)}{\mathrm{d}u} =& G_i^\delta(x_1^+(u),\dots,x_i^+(u),u) && \text{(4.11c)}\\
\frac{\mathrm{d}x_1^+(u)}{\mathrm{d}u} =& G_1^\delta(x_1^+(u),u) && \text{(4.11d)}
\end{aligned}$$

for $i=2,\dots,n$ and with

$$G_i^\delta(x_1,\dots,x_i,u) = \nabla_u g_i^\delta(x_1,\dots,x_i,u)\big|_{x^i = e^{-\delta(\mathrm{L}_{f^i}+\mathrm{L}_{\varphi^i}+u\mathrm{L}_{g^i})}x^i}.$$

For the sake of compactness, we introduce the following notation

$$F_0^{i,\delta}(x^i) = \begin{pmatrix} f_i^\delta(x_i)+\varphi_i^\delta(x_1,\dots,x_i)\\ F_0^{i-1,\delta}(x^{i-1})\end{pmatrix}, \quad G^{i,\delta}(x^i,u) = \begin{pmatrix} G_i^\delta(x_1,\dots,x_i,u)\\ G^{i-1,\delta}(x^{i-1},u)\end{pmatrix}$$
$$g^{i,\delta}(x^i,u) = \begin{pmatrix} g_i^\delta(x_1,\dots,x_i,u)\\ g^{i-1,\delta}(x^{i-1},u)\end{pmatrix}$$

with

$$F_0^{1,\delta}(x^1) = f_1^\delta(x_1), \quad G^{1,\delta}(x^1,u) = G_1^\delta(x_1,u), \quad g^{1,\delta}(x^1,u) = g_1^\delta(x_1,u).$$

Remark 4.3. *For computational facilities, it is worth pointing out that the sampled-data equivalent model of the feedforward system (4.1) can be iteratively computed through a bottom-up procedure. As a matter of fact, one can start by computing the sampled-data equivalent model to (4.9c) and then proceed by defining the equivalent one to (4.9c) depending on the previous one and so on until (4.9a).*

4.1.2.1 Strict-feedforward systems under sampling

Assume again $u \in \mathcal{U}^\delta$ and measures of the state being available only at the sampling instants. Then, the strict-feedforward system (4.2) rewrites, for any $k \geq 0$, as for $t \in [k\delta, (k+1)\delta[$ Assume again that

$$\dot{x}_n(t) = F_n x_n(t) + \varphi_n(x_1(t), \ldots, x_{n-1}(t)) + g_n(x_1(t), \ldots, x_{n-1}(t))u_k \tag{4.12a}$$
$$\vdots$$
$$\dot{x}_2(t) = F_2 x_2(t) + \varphi_2(x_1(t)) + g_2(x_1(t))u_k \tag{4.12b}$$
$$\dot{x}_1(t) = f_1(x_1(t)) + g_1(x_1(t))u_k. \tag{4.12c}$$

By specifying to (4.12) the content of Definition 2.1 one can infer the following result.

Lemma 4.2. *The equivalent sampled-data model to (4.2) preserves the* strict-*feedfoward structure; i.e., it gets the cascade form*

$$x_{nk+1} = F_n^\delta x_{nk} + \varphi_n^\delta(x_{1k}, \ldots, x_{n-1k}) + g_n^\delta(x_{1k}, \ldots, x_{n-1k}, u_k) \tag{4.13a}$$
$$\vdots$$
$$x_{2k+1} = F_2^\delta x_{1k} + \varphi_2^\delta(x_{1k}) + g_2^\delta(x_{1k}, u_k) \tag{4.13b}$$
$$x_{1k+1} = f_1^\delta(x_{1k}) + g_1^\delta(x_{1k}, u_k) \tag{4.13c}$$

with $g_i^\delta(x_1, \ldots, x_i, 0) = 0$ *and*

$$F_i^\delta = e^{\delta F_i} \qquad \text{such that}$$
$$\varphi_i^\delta(x_1, \ldots, x_{i-1}) = \int_0^\delta e^{sF_i} e^{s\mathrm{L}_{f^i+\varphi^i}} \varphi_i(x_1, \ldots, x_{i-1}) \mathrm{d}s$$
$$g_i^\delta(x_1, \ldots, x_{i-1}, u) = \int_0^u G_i^\delta(x_1^+(v), \ldots, x_{i-1}^+(v), v) \mathrm{d}v$$
$$G_i^\delta(x_1, \ldots, x_{i-1}, u) = \int_0^\delta e^{-sad_{f^i+\varphi^i+ug^i}} g_i(x_1, \ldots, x_{i-1}) \mathrm{d}s.$$

It is straightforward to verify that the equivalent (F_0, G) representation of (4.13) exhibits a strict-feedforward cascade interconnection as well; namely, it gets the form

$$x_i^+ = F_i^\delta x_i + \varphi_i^\delta(x_1, \ldots, x_{i-1}), \quad x_i^+ = x_i^+(0) \tag{4.14a}$$
$$x_1^+ = f_1^\delta(x_1), \quad x_1^+ = x_1^+(0) \tag{4.14b}$$
$$\frac{\mathrm{d}x_i^+(u)}{\mathrm{d}u} = G_i^\delta(x_1^+(u), \ldots, x_{i-1}^+(u), u) \tag{4.14c}$$
$$\frac{\mathrm{d}x_1^+(u)}{\mathrm{d}u} = G_1^\delta(x_1^+(u), u) \tag{4.14d}$$

for $i = 2, \ldots, n$ and

$$G_i^\delta(x_1, \ldots, x_{i-1}, u) = \nabla_u g_i^\delta(x_1, \ldots, x_{i-1}, u)\big|_{x^{i-1} = e^{-\delta(\mathrm{L}_{f^{i-1}} + \mathrm{L}_{\varphi^{i-1}} + u\mathrm{L}_{g^{i-1}})} x^{i-1}}.$$

When considering strict-feedforward structures, we are going to introduce a constructive I&I feedback exploiting u-average passivity and u-average passivation of each component of the cascade so extending the results in [128, 131]. Then, this design method is enlarged to general feedforward systems by exploiting Lyapunov-based design and by underlying on connections to the I&I-inspired results.

4.2 Feedforwarding stabilization under sampling

First, we are going to specify the result to the case of a two-block feedforward interconnection structure of the form (4.3). Then, the extension to (4.1) is sketched through an iterative algorithm.

4.2.1 The two-block feedforwarding interconnection case

Consider the case of the two-block feedforwarding interconnection case (4.3) so that, letting $u \in \mathcal{U}^\delta$ one gets the interval dynamics as for $t \in [k\delta, (k+1)\delta[$

$$\dot{x}_2(t) = f_2(x_2(t)) + \varphi_2(x_1(t), x_2(t)) + g_2(x_1(t), x_2(t))u_k \tag{4.15a}$$
$$\dot{x}_1(t) = f_1(x_1(t)) + g_1(x_1(t))u_k. \tag{4.15b}$$

and, hence, the sampled-data equivalent model described as

$$x_{2k+1} = f_2^\delta(x_{2k}) + \varphi_2^\delta(x_{1k}, x_{2k}) + g_2^\delta(x_{1k}, x_{2k}, u_k) \tag{4.16a}$$
$$x_{1k+1} = f_1^\delta(x_{1k}) + g_1^\delta(x_{1k}, u_k) \tag{4.16b}$$

or, equivalently, in the (F_0, G) form as

$$x_2^+ = f_2^\delta(x_2) + \varphi_2^\delta(x_1, x_2), \quad x_2^+ = x_2^+(0) \tag{4.17a}$$
$$x_1^+ = f_1^\delta(x_1), \quad x_1^+ = x_1^+(0) \tag{4.17b}$$
$$\frac{\mathrm{d}x_2^+(u)}{\mathrm{d}u} = G_2^\delta(x_1^+(u), x_2^+(u), u) \tag{4.17c}$$
$$\frac{\mathrm{d}x_1^+(u)}{\mathrm{d}u} = G_1^\delta(x_1^+(u), u). \tag{4.17d}$$

Given (4.3) verifying the assumptions set in Theorem 4.1, the following three items will be proven for its sampled-data equivalent model (4.16):

1. there exists a feedback $u_0^\delta(x_1)$ ensuring GAS and LES of the equilibrium of the x_1-dynamics (4.16b) via u-average passivity arguments;
2. a new and explicitly δ-dependent Lyapunov function $V^\delta(\cdot) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}_{>0}$ can be constructed for the augmented dynamics (4.16) through the definition of a δ-depending cross-term;

3. there exists a passivating output mapping $Y^\delta(x_1, x_2, u)$ so that (4.16) is u-average passive from $u_0^\delta(x, u)$ and ZSD with storage function $V^\delta(x_1, x_2)$; accordingly, one can construct a sampled-data feedback ensuring GAS and LES of the equilibrium of the complete cascade (4.16).

Remark 4.4. *All mappings involved in the passivation-based design and Lyapunov analysis are, in general, different from the continuous-time ones although no further hypotheses than the continuous-time ones are needed to ensure their existence.*

Remark 4.5. *Although the feedforwarding procedure might seem similar to the continuous -time one, it is important to underline that it exploits different notions as, for example, average passivation and takes into account the hybrid nature of the sampled-data system (4.15).*

4.2.1.1 Stabilization of the x_1-dynamics

As pointed out in Theorem 4.1, Assumption 4.3 allows to deduce passivity of the continuous-time system (4.1) with respect to the output

$$y_0 = \mathrm{L}_{g_1} U(x_1) \tag{4.18}$$

and storage function $U(x_1)$. Accordingly, by invoking Theorem 2.3 one gets the following result establishing the initial step of the sampled-data design procedure.

Lemma 4.3. *Let $(\nabla f_1(0), g_1(0))$ be stabilizable and verify Assumption 4.3. Moreover, assume it is ZSD with respect to the output (4.18). Then, the following holds:*

- *the sampled-data equivalent model (4.16b) is u-average passive and ZSD with respect to the output*
$$Y_0(x_1, u) = \mathrm{L}_{G_1^\delta(\cdot,u)} U(x_1); \tag{4.19}$$
- *there exists $\delta^* > 0$ such that, for any $\delta \in]0, \delta^*[$, there exists a unique solution $u = u_0^\delta(x_1)$ to the damping equality*
$$u + Y_{0,av}^\delta(x_1, u) = 0 \quad \textit{with} \quad Y_{0,av}^\delta(x_1, u) = \frac{1}{\delta u}\int_0^u \mathrm{L}_{G_1^\delta(\cdot,v)} U(x_1) \tag{4.20}$$
of the form
$$u_0^\delta(x_1) = u_0^0(x_1) + \sum_{i>0} \frac{\delta^i}{(i+1)!} u_0^i(x_1) \tag{4.21}$$
with $u_0^0(x_1) = u_0(x_1) = -\mathrm{L}_{g_1} U(x_1)$;
- *the feedback $u = u_0^\delta(x_1)$ solution to (4.20) ensures GAS and LES of the equilibrium the origin of (4.16b) in closed-loop;*
- *the sampled-data feedback $u_k = u_0^\delta(x_{1k})$ solution to (4.20) ensures S-GAS and S-LES of the equilibrium the origin of (4.3b) in closed-loop.*

Remark 4.6. *The terms of the series expansion (4.21) coincide with the ones reported in (2.28) as the feedback solving (4.20) is a classical u-average passivity-based control.*

4.2.1.2 A sampled-data Lyapunov function

Whenever Assumption 4.3 holds true, feedback $u = u_0^\delta(x_1)$ solution to (4.20) ensures GAS and LES of the closed-loop equilibrium of (4.16b). Moreover, as Assumptions 4.1 and 4.2 are met as well, we also get that the origin of the sampled-data closed-loop dynamics

$$x_{2k+1} = f_2^\delta(x_{2k}) + \varphi_2^\delta(x_{1k}, x_{2k}) + g_2^\delta(x_{1k}, x_{2k}, u_0^\delta(x_{1k})) \tag{4.22a}$$
$$x_{1k+1} = f_1^\delta(x_{1k}) + g_1^\delta(x_{1k}, u_0^\delta(x_{1k})) \tag{4.22b}$$

is GS and, thus, the one of the interval dynamics for $t \in [k\delta, (k+1)\delta[$ as

$$\dot{x}_2(t) = f_2(x_2(t)) + \varphi_2(x_1(t), x_2(t)) + g_2(x_1(t), x_2(t)) u_0^\delta(x_{1k}) \tag{4.23a}$$
$$\dot{x}_1(t) = f_1(x_1(t)) + g_1(x_1(t)) u_0^\delta(x_{1k}). \tag{4.23b}$$

is S-GS. We would like to deduce a suitable radially unbounded weak Lyapunov Function $V^\delta(x) = V^\delta(x_1, x_2) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ verifying, along the trajectories of (4.22)

$$\Delta_k V^\delta(x) \leq -\Delta_k U(x_1) \leq 0, \quad \text{for any } x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \tag{4.24}$$

To this purpose, let us assume the aforementioned Lyapunov function of the form

$$V^\delta(x) = W(x_2) + U(x_1) + \Psi^\delta(x_1, x_2) \tag{4.25}$$

where $\Psi^\delta(x) = \Psi^\delta(x_1, x_2) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ defines the *cross-term* to be constructed and, possibly, smoothly parametrized by δ. Accordingly, let us compute the increment of (4.25) along the trajectories of (4.22) as

$$\Delta_k V^\delta(x) = \Delta_k W(x_2) + \Delta_k U(x_1) + \Delta_k \Psi^\delta(x) \tag{4.26}$$

with

$$\Delta_k U(x_1) \leq -\delta \|Y_{0,\mathrm{av}}^\delta(x_1, u_0^\delta(x_1))\|^2 \leq 0 \tag{4.27}$$
$$\Delta_k W(x_2) = W(f_2^\delta(x_{2k}) + \varphi_2^\delta(x_{1k}, x_{2k}) + g_2^\delta(x_{1k}, x_{2k}, u_0^\delta(x_{1k})) - W(x_{2k}). \tag{4.28}$$

Accordingly, one rewrites (4.28) through the integral form

$$\begin{aligned}
\Delta_k W(x_2) = &\int_{k\delta}^{(k+1)\delta} (\mathrm{L}_{f_2(\cdot)+\varphi_2(x_1(s),\cdot)} + u_0^\delta(x_{1k}) \mathrm{L}_{g_2(x_1(s),\cdot)}) W(x_2(s)) \mathrm{d}s \\
= &\int_{k\delta}^{(k+1)\delta} \mathrm{L}_{f_2(\cdot)} W(x_2(s)) \mathrm{d}s \\
&+ \int_{k\delta}^{(k+1)\delta} \mathrm{L}_{\varphi_2(x_1(s),\cdot)+u_0^\delta(x_{1k}) g_2(x_1(s),\cdot)} W(x_2(s)) \mathrm{d}s
\end{aligned}$$

with

$$x_1(s) = e^{(s-k\delta)(\mathrm{L}_{f_1} + u_0^\delta(x_{1k}) \mathrm{L}_{g_1})} x_1 \big|_{x_{1k}}, \quad x_2(s) = e^{(s-k\delta)(\mathrm{L}_{f^2} + u_0^\delta(x_{1k}) \mathrm{L}_{g^2})} x_2 \big|_{x_{1k}, x_{2k}}.$$

Because of Assumption 4.2 $\mathrm{L}_{f_2}W(\cdot) \leq 0$ and one deduces that

$$\int_{k\delta}^{(k+1)\delta} \mathrm{L}_{f_2(\cdot)}W(x_2(s))\mathrm{d}s \leq 0$$

so obtaining the inequality

$$\Delta_k W(x_2) \leq \int_{k\delta}^{(k+1)\delta} \mathrm{L}_{\varphi_2(x_1(s),\cdot)+u_0^\delta(x_{1k})g_2(x_1(s),\cdot)}W(x_2(s))\mathrm{d}s$$

whose RHS is with indefinite sign. By applying the above inequality in (4.26), one gets

$$\Delta_k V^\delta(x) \leq \Delta_k U(x_1) + \Delta_k \Psi^\delta(x) + \int_{k\delta}^{(k+1)\delta} \mathrm{L}_{\varphi_2(x_1(s),\cdot)+u_0^\delta(x_{1k})g_2(x_1(s),\cdot)}W(x_2(s))\mathrm{d}s \tag{4.29}$$

so that a candidate cross-term $\Psi^\delta(x)$ is the one verifying the equality

$$\Delta_k \Psi^\delta(x) = -\int_{k\delta}^{(k+1)\delta} \mathrm{L}_{\varphi_2(x_1(s),\cdot)+u_0^\delta(x_{1k})g_2(x_1(s),\cdot)}W(x_2(s))\mathrm{d}s \tag{4.30}$$

so ensuring, in turn, the required (4.24). Accordingly, the following result can be deduced stating that a solution to (4.30) exists under the same Assumptions as in continuous time.

Proposition 4.1. *Let the continuous-time feedforward dynamics (4.3) verify Assumptions 4.1 to 4.3 and the hypotheses of Lemma 4.3. Consider the sampled-data equivalent model (4.16) and assume $u = u_0^\delta(x_1)$ as the solution to (4.20). Then, the origin of (4.22) is GS and (4.30) admits a solution $\Psi^\delta(\cdot) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ of the form*

$$\Psi^\delta(x) = \Psi^\delta(x_1, x_2) = \sum_{\ell=0}^{\infty} \int_{\ell\delta}^{(\ell+1)\delta} \mathrm{L}_{\varphi_2(x_1(s),\cdot)+u_0^\delta(x_{1\ell})g_2(x_1(s),\cdot)}W(x_2(s))\mathrm{d}s \tag{4.31}$$

with

$$x_1(s) = e^{(s-\ell\delta)(\mathrm{L}_{f_1}+u_0^\delta(x_{1\ell})\mathrm{L}_{g_1})}x_1\big|_{x_{1\ell}}, \; x_2(s) = e^{(s-\ell\delta)(\mathrm{L}_{f^2}+u_0^\delta(x_{1\ell})\mathrm{L}_{g^2})}x_2\big|_{x_{1\ell},x_{2\ell}} \tag{4.32}$$

that is continuous. Furthermore, $V^\delta : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ defined in (4.25) is a radially unbounded and weak Lyapunov function for (4.22) and, equivalently, for (4.23) at any sampling instant $t = k\delta$, $k \geq 0$.

Proof: The proof of the existence of the sampled-data cross term follows the lines of the continuous-time case [170]. It is achieved by showing that the x_2-dynamics remains bounded for $k \geq 0$ and that the terms in the sum in the right hand side of (4.31) are positive and bounded. As far as boundedness of x_2 is concerned, from Assumption 4.1 one computes

$$\Delta_k W(x_2) \leq \int_{k\delta}^{(k+1)\delta} \|\nabla_{x_2}W(x_2(s))\|(\gamma_0(\|x_1(s)\|) + \gamma_1(\|x_1(s)\|)\|x_2(s)\|)\mathrm{d}s$$

for some $\mathcal{K}$-functions $\gamma_0(\cdot)$ and $\gamma_1(\cdot)$. Moreover, because of LES of (4.22b), there exists a $\mathcal{K}$-function $\gamma(\cdot)$ upper-bounding both $\gamma_1(\cdot)$ and $\gamma_0(\cdot)$; this yields

$$\Delta_k W(x_2) \leq \int_{k\delta}^{(k+1)\delta} \|\nabla_{x_2} W(x_2(s))\|(\gamma(\|x_1(s)\|) + \gamma(\|x_1(s)\|)\|x_2(s)\|)\mathrm{d}s.$$

From now on, let us assume that $\|x_2(s)\| > 1$ in the sampling interval (the result is straightforward if not). Accordingly, one gets

$$\Delta_k W(x_2) \leq c \int_{k\delta}^{(k+1)\delta} \gamma(\|x_1(s)\|)W(x_2(s))\mathrm{d}s. \tag{4.33}$$

By construction of $u_0^\delta(\cdot)$ in Lemma 4.3 one has $\Delta_k U(x_1) \leq -\delta\|Y_{0,\mathrm{av}}^\delta(x_1, u_0^\delta(x_1))\|^2$ ensuring GAS and LES of the closed-loop equilibrium of (4.22b); namely, when x_1 is sufficiently close to the origin

$$\|x_{1k+1}\| \leq e^{-\alpha\delta k}\gamma_{x_1}(\|x_1\|)$$

for some $\mathcal{K}$-function $\gamma_{x_1}(\cdot)$, constant $\alpha > 0$ and initial condition $x_{10} = x_1$. As a consequence, by rewriting (4.33) as

$$\int_{k\delta}^{(k+1)\delta} \dot{W}(x_2(s))\mathrm{d}s \leq c \int_{k\delta}^{(k+1)\delta} \gamma(\|x_1(s)\|)W(x_2(s))\mathrm{d}s.$$

and exploiting LES of the closed-loop equilibrium of (4.22b), one gets

$$W(x_{2k+1}) \leq e^{c\delta(e^{-\alpha\delta}-1)\gamma(\|x_{1k}\|)}W(x_{2k})$$

so proving $\Delta_k W(x_2) \leq 0$ and thus GS of the equilibrium of (4.22).

Since $W(x_2)$ is radially unbounded, one has that x_2 and $\|\nabla_{x_2} W\|$ stay bounded for $k \geq 0$. Thus, one deduces for a suitable $\lambda^\delta \in (0,1)$

$$\int_{k\delta}^{(k+1)\delta} \mathrm{L}_{\varphi_2(x_1(s),\cdot)+u_0^\delta(x_{1k})g_2(x_1(s),\cdot)} W(x_2(s))\mathrm{d}s \leq \delta\gamma_1(\|(x_1,x_2)\|)\lambda^{k\delta}$$

so concluding that

$$\int_{k\delta}^{(k+1)\delta} \mathrm{L}_{\varphi_2(x_1(s),\cdot)+u_0^\delta(x_{1k})g_2(x_1(s),\cdot)} W(x_2(s))\mathrm{d}s$$

is summable for $k \geq 0$ and that (4.31) exists and is bounded for all bounded (x_1, x_2). The remaining part of the proof follows the lines of the continuous-time one [170] *mutatis mutandis* as developed in [104] in discrete time. ■

Remark 4.7. *It is a matter of computations to verify that $\Psi^\delta(x_1, 0) = 0$ by construction and because of (4.30).*

Remark 4.8. *As we shall discuss, the cross-term (4.31) is smoothly parametrized by the sampling period δ so implying that, in general, it does not coincide with the continuous-time one (4.4). This is mainly motivated by the impact of the piecewise-constant nature of the feedback $u_k = u_0^\delta(x_{1k})$ over the continuous-time dynamics (4.22) whose trajectories do not coincide, hence, with the ones of (4.3) under the continuous-time control $u = u_0(x_1)$.*

Remark 4.9. *The construction of the cross-term might be carried out by considering the sampled-data equivalent model (4.22) under $u_k = u_0^\delta(x_{1k})$ as a purely discrete-time system. Namely, one would look for a Lyapunov function*

$$V_d(x_1, x_2) = U(x_1) + W(x_2) + \Psi_d(x_1, x_2)$$

where the new cross-term should be chosen to satisfy the equality

$$\begin{aligned}\Delta_k \Psi_d(x)\big|_{u=u_0^\delta(x_1)} =& - W\big(f_1^\delta(x_1) + \varphi_2^\delta(x_1,x_2) + g_1^\delta(x_1,x_2,u_0^\delta(x_1))\big)\\ &+ W(f_1^\delta(x_1)).\end{aligned} \tag{4.34}$$

The above equality is in general different and more conservative than (4.30) and its solvability requires further assumptions than the continuous-time ones (see [104] for further details). As a matter of fact, (4.34) does not take into account the continuous-time nature of the plant and the properties of the original vector fields defining its dynamics.

Remark 4.10. *By exploiting the (F_0, G) representation (4.17), (4.29) rewrites as*

$$\begin{aligned}\Delta_k V^\delta(x) =& V^\delta(F_0^{2,\delta}(x)) - V^\delta(x) + \int_0^{u_0^\delta(x_1)} \mathrm{L}_{G^{2,\delta}(\cdot,v)} V^\delta(x^+(v))\mathrm{d}v\\ \le& -\delta\|Y_{0,av}^\delta(x_1, u_0^\delta(x_1))\|^2\end{aligned}$$

when setting

$$F_0^{2,\delta}(x) = \begin{pmatrix} f_2^\delta(x_2) + \varphi_2^\delta(x_1,x_2)\\ f_1^\delta(x_1)\end{pmatrix}, \quad G^{2,\delta}(x,u) = \begin{pmatrix} G_2^\delta(x_1,x_2,u)\\ G_1^\delta(x_1,u)\end{pmatrix}. \tag{4.35}$$

It is worth to underline that the equality the sampled-data cross-term $\Psi^\delta(x)$ needs to verify (4.30) can be rewritten as

$$\begin{aligned}&\int_{k\delta}^{(k+1)\delta} \mathrm{L}_{f^2 + u_0^\delta(x_{1k})\mathrm{L}_{g^2}} \Psi^\delta(x(s))\mathrm{d}s\\ &= -\int_{k\delta}^{(k+1)\delta} \mathrm{L}_{\varphi_2(x_1(s),\cdot) + u_0^\delta(x_{1k}) g_2(x_1(s),\cdot)} W(x_2(s))\mathrm{d}s\end{aligned} \tag{4.36}$$

so extending to the sampled-data context the partial differential equality in (4.8).

The cross-term requires the computation of (4.31) over an infinite horizon and along the future trajectories of the system (4.23). To this end, for computing an exact solution of the form (4.31), the exact sampled-data equivalent model (4.22) needs to be computed in a closed form. Though, this is seldom the case.

Some constructive aspects about the computation of the sampled-data cross-term starting from the continuous-time one $\Psi(x)$ deduced from $u = u_0(x_1)$ in Lemma 4.3 are given below.

Consider the continuous-time cross-term (4.4) verifying, for any $x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ the equality (4.8) and rewrite it as

$$\mathrm{L}_{f^2}\Psi(x) + \mathrm{L}_{\varphi(x_1,\cdot)}W(x_2) = -u_0(x_1)\big(\mathrm{L}_{g^2}\Psi(x) + \mathrm{L}_{g_2}W(x_2)\big). \tag{4.37}$$

By adding to both sides of (4.37) the term

$$u_0^\delta(x_{1k})\big(\mathrm{L}_{g^2}\Psi(x(t)) + \mathrm{L}_{g_2}W(x_2(t))\big)$$

one gets, for $t \in [k\delta, (k+1)\delta[$ and $k \geq 0$,

$$\begin{aligned} &\mathrm{L}_{f^2+u_0^\delta(x_{1k})g^2}\Psi(x(t)) + \mathrm{L}_{\varphi(x_1(t),\cdot)+u_0^\delta(x_{1k})g_2}W(x_2(t)) \\ &= -\big(u_0(x_1(t)) - u_0^\delta(x_{1k})\big)\big(\mathrm{L}_{g^2}\Psi(x(t)) + \mathrm{L}_{g_2}W(x_2(t))\big). \end{aligned} \tag{4.38}$$

By integrating (4.38) over $t \in [k\delta, (k+1)\delta[$ and along the trajectories of the sampled-data interval system (4.23), we get

$$\begin{aligned} &\Delta_k\Psi(x) - \Delta_k\Psi^\delta(x) \\ &= -\int_{k\delta}^{(k+1)\delta} \big(u_0(x_1(s)) - u_0^\delta(x_{1k})\big)\big(\mathrm{L}_{g^2}\Psi(x(s)) + \mathrm{L}_{g_2}W(x_2(s))\big)\mathrm{d}s \end{aligned} \tag{4.39}$$

with the sampled-data $x_i(s)$ as in (4.32) for $i = 1, 2$ and $\Delta_k\Psi(x)$ being the increment of the continuous-time cross-term (4.4) along (4.22).

Remark 4.11. *Equality (4.39) emphasizes on the fact that the continuous-time cross-term does not solve, in general, the sampled-data equation (4.30) as the following equality*

$$\int_{k\delta}^{(k+1)\delta} u_0(x_1(s))\big(\mathrm{L}_{g^2}\Psi(x(s)) + \mathrm{L}_{g_2}W(x_2(s))\big)\mathrm{d}s =$$
$$u_0^\delta(x_{1k})\int_{k\delta}^{(k+1)\delta} \big(\mathrm{L}_{g^2}\Psi(x(s)) + \mathrm{L}_{g_2}W(x_2(s))\big)\mathrm{d}s$$

is not verified in general.

Remark 4.12. *Whenever the continuous-time (4.3b) has a GAS and LES equilibrium at the origin for $u_0(x_1) = 0$, then the sampled-data equivalent model (4.16b) has a GAS and LES equilibrium at the origin under the trivial sampled-data feedback $u^\delta(x_1) = 0$. In that case, as a consequence and as equality (4.39) underlines, the continuous-time (4.4) is a cross-term for the sampled-data system (4.22). This is also due to the fact that, for $u_0^\delta(x_1) = 0$ and $u_0(x_1) = 0$, the trajectories of (4.3) coincide, at any sampling instants $t = k\delta$, with the ones of (4.22).*

Equality (4.39) puts in light that one can transform the problem of computing an exact solution to the integral-differential equality (4.30) into the one of solving an infinite number of partial-differential equations (PDE). To this purpose, set the sampled-data cross-term of the form

$$\Psi^{\delta}(x) = \Psi^{0}(x) + \sum_{i>0} \frac{\delta^{i}}{(i+1)!}\Psi^{i}(x). \tag{4.40}$$

By substituting (4.40) and (4.21) into (4.39) and equating the homogeneous terms with δ^i in the corresponding expansion, one will deduce any $\Psi_i(x)$ in (4.40) as the solution of the corresponding PDE. For the first terms, one shall get the corresponding PDEs

$$\begin{aligned}
\mathrm{L}_{f^2+u_0^0(x_{1k})g^2}\Psi^0(x) =& -\mathrm{L}_{\varphi_2(x_1,\cdot)+u_0^0(x_{1k})g_2}W(x_2)\\
\mathrm{L}_{f^2+u_0^0(x_{1k})g^2}\Psi^1(x) =& -\big(u_0^1(x_{1k}) - \mathrm{L}_{f^2+u_0^0(x_{1k})g^2}u_0(x_1)\big)\mathrm{L}_{g^2}\big(W(x_2)+\Psi(x)\big)\\
\mathrm{L}_{f^2+u_0^0(x_{1k})g^2}\Psi^2(x) =& -\big(u_0^2(x_{1k}) - \frac{3}{2}u_0^1(x_{1k})\mathrm{L}_{g^2}u_0(x_1)\\
&- \mathrm{L}^2_{f^2+u_0^0(x_{1k})g^2}u_0(x_1)\big)\mathrm{L}_{g^2}\big(W(x_2)+\Psi(x)\big) - \big(\frac{3}{2}u_0^1(x_{1k})\\
&- \mathrm{L}_{f^2+u_0^0(x_{1k})g^2}u_0(x_1)\big)\mathrm{L}_{f^2+u_0^0(x_{1k})g^2}\mathrm{L}_{g^2}\big(W(x_2)+\Psi(x)\big)\\
&- \frac{3}{2}u_0^1(x_{1k})\mathrm{L}_{g^2}\Psi^1(x) - \frac{3}{2}\mathrm{L}^2_{f^2+u_0^0(x_{1k})g^2}\Psi^1(x)
\end{aligned}$$

so getting that $\Psi^0(x)$ coincides with the continuous-time solution (4.4) and, thus, $\Psi^0(x) = \Psi(x)$. Note that in the PDEs above, the terms appearing with a k-dependency need to be considered as constant when integrating.

Remark 4.13. *By virtue of the above expressions and the one of the term $u_0^1(x_1)$ given in (2.28), one gets $\Psi^1(x) \equiv 0$ whenever the continuous-time system (4.3b) with output $y_0 = \mathrm{L}_{g_1}U(x_1)$ is lossless (i.e., $\mathrm{L}_{f_1}U(x_1) \equiv 0$).*

These computational aspects enlighten the impact of the piecewise constant nature of the feedback $u_k = u_0^{\delta}(x_{1k})$ over the redefinition of the cross-term for the sampled-data dynamics with respect to the continuous-time one (4.4) deduced from the continuous-time feedback $u_0(x_1)$. Moreover, the formal-series expansion form of the cross-term (4.40) will be useful for deducing approximate solutions of the overall feedback as shown in the next section.

4.2.1.3 Sampled-data u-average passivity feedforwarding

Once a Lyapunov function $V^{\delta} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ of the form (4.25) for the sampled-data closed-loop system (4.22) has been exhibited, one has to design the overall feedback $u = u_1^{\delta}(x)$ ensuring GAS of the origin of (4.16). To this end, we shall exploit u-average passivity arguments and make extensive use of the (F_0, G)-representation of the sampled-data dynamics (4.16) as described in (4.17).

First, we prove the following result.

Lemma 4.4. *Let the continuous-time dynamics (4.3) verify Assumptions 4.1 to 4.3 and the hypotheses of Lemma 4.3. Assume the sampled-data feedback $u = u_0^\delta(x_1)$ as the solution to (4.20) and consider the Lyapunov function $V^\delta : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ as in (4.25) with cross-term $\Psi^\delta : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ solution to (4.30). Then, the sampled-data system (4.16) with output*

$$Y_1^\delta(x, u) = \mathrm{L}_{G^{2,\delta}(\cdot,u)} V^\delta(x) \tag{4.41}$$

is u-average passive from $u_0^\delta(x_1)$ with storage function $V^\delta(x)$ as in (4.25) and averaged output

$$Y^\delta_{1,u_0^\delta(x_1)}(x, u) = \frac{1}{\delta(u - u_0^\delta(x_1))} \int_{u_0^\delta(x_1)}^{u} \mathrm{L}_{G^{2,\delta}(\cdot,v)} V^\delta(x^+(v))\mathrm{d}v. \tag{4.42}$$

Proof: To prove u-average passivity from $u_0^\delta(x_1)$ of the dynamics (4.16) with output (4.41), we compute

$$\begin{aligned}
\Delta_k V^\delta(x) =& V^\delta(F_0^{2,\delta}(x_k)) - V^\delta(x_k) + \int_0^{u_k} \mathrm{L}_{G^{2,\delta}(\cdot,v)} V^\delta(x^+(v))\mathrm{d}v \\
=& V^\delta(F_0^{2,\delta}(x_k)) - V^\delta(x_k) + \int_0^{u_0^\delta(x_{1k})} \mathrm{L}_{G^{2,\delta}(\cdot,v)} V^\delta(x^+(v))\mathrm{d}v \\
&+ \int_{u_0^\delta(x_{1k})}^{u_k} \mathrm{L}_{G^{2,\delta}(\cdot,v)} V^\delta(x^+(v))\mathrm{d}v \\
\leq& \delta(u_k - u_0^\delta(x_{1k})) Y^\delta_{1,u_0^\delta(x_1)}(x, u)
\end{aligned}$$

so getting the result by virtue of Proposition 4.1 and Remark 4.10. ■

Remark 4.14. *When $x_2 \equiv 0$, the output (4.41) rewrites as*

$$\begin{aligned}
Y_1^\delta(x_1, 0, u) =& \mathrm{L}_{G^{2,\delta}(x_1,0,u)} \Big(U(x_1) + W(0) + \Psi^\delta(x_1, 0) \Big) \\
=& \mathrm{L}_{G_1^\delta(\cdot,u)} U(x_1) \\
=& Y_0^\delta(x_1, u)
\end{aligned}$$

so recovering the output (4.19).

Remark 4.15. *It is a matter of computations to verify that the averaged output (4.42) rewrites as*

$$\delta Y^\delta_{1,u_0^\delta(x_1)}(x, u) = \int_0^1 \mathrm{L}_{G^{2,\delta}(\cdot,(\theta u + (1-\theta)u_0^\delta(x_1)))} V^\delta(x^+(\theta u + (1-\theta)u_0^\delta(x_1)))\mathrm{d}\theta$$

so getting, when $x_2 \equiv 0$

$$\delta Y^\delta_{1,u_0^\delta(x_1)}(x_1, 0, u) = \int_0^1 \mathrm{L}_{G_1^\delta(\cdot,(\theta u + (1-\theta)u_0^\delta(x_1)))} U(x_1^+(\theta u + (1-\theta)u_0^\delta(x_1)))\mathrm{d}\theta.$$

By construction of $u_0^\delta(x_1)$ and $V^\delta(x)$ in, respectively, Lemma 4.3 and Proposition 4.1, one upper-bounds the increment of $V^\delta(x)$ along (4.16) as

$$\Delta_k V^\delta(x) \leq -\delta\|Y_{0,\mathrm{av}}^\delta(x_{1k}, u_0^\delta(x_{1k}))\|^2 + \delta(u_k - u_0^\delta(x_{1k}))Y_{1,u_0^\delta(x_1)}^\delta(x_k, u_k). \tag{4.43}$$

Accordingly, because $u_0^\delta(x_1)$ solves (4.20), the inequality (4.43) rewrites as

$$\Delta_k V^\delta(x) \leq -\delta\|Y_{0,\mathrm{av}}^\delta(x_{1k}, u_0^\delta(x_{1k}))\|^2 + \delta(u_k + Y_{0,\mathrm{av}}^\delta(x_{1k}, u_0^\delta(x_{1k}))Y_{1,u_0^\delta(x_1)}^\delta(x_k, u_k)$$

so finally getting, when exploiting the Young inequality[2],

$$\begin{aligned}\Delta_k V^\delta(x) \leq &-\frac{\delta}{2}\|Y_{0,\mathrm{av}}^\delta(x_{1k}, u_0^\delta(x_{1k}))\|^2 + \frac{\delta}{2}\|Y_{1,u_0^\delta(x_1)}^\delta(x_k, u_k)\|^2 \\ &+ \delta u_k Y_{1,u_0^\delta(x_1)}^\delta(x_k, u_k).\end{aligned} \tag{4.44}$$

This shows that any feedback making $\Delta_k V^\delta(x) \leq 0$ ensures GAS of the origin of (4.16) provided ZSD with respect to the output (4.41). In this sense, a suitable choice for the control law is provided by the feedback given by the solution, if any, to the following equality

$$u + Y_{1,u_0^\delta(x_1)}^\delta(x, u) = 0. \tag{4.45}$$

One can now state the main result related to stability properties of the origin of (4.16) under the feedback $u = u_1^\delta(x)$ solution to (4.45).

Theorem 4.2 (Stabilizing forwarding through u-average passivity)**.** *Let the continuous-time feedforward dynamics (4.3) verify Assumptions 4.1 to 4.3 and the hypotheses of Lemma 4.3. Consider the sampled-data equivalent model (4.16) and assume $u = u_0^\delta(x_1)$ as the solution to (4.20). Consider the Lyapunov function $V^\delta : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ as in (4.25) with cross-term $\Psi^\delta : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ solution to (4.30). Then, the following statements are equivalent:*

1. *the feedback $u = u_1^\delta(x)$ solution to (4.45) ensures GAS and LES of the zero equilibrium of (4.16) in closed loop;*
2. *the sampled-data feedback $u_k = u_1^\delta(x_k)$ solution to (4.45) ensures S-GAS and S-LES of the zero equilibrium of (4.3) in closed loop.*

Proof: By exploiting (4.45) into (4.44) one immediately gets

$$\Delta_k V^\delta(x) \leq -\frac{\delta}{2}\|Y_{0,\mathrm{av}}^\delta(x_{1k}, u_0^\delta(x_{1k}))\|^2 - \frac{\delta}{2}\|Y_{1,u_0^\delta(x_1)}^\delta(x_k, u_1^\delta(x_k))\|^2.$$

Accordingly, GAS of the origin the system (4.16) under $u = u_1^\delta(x)$ follows if it is GAS conditionally to the largest invariant set $\mathcal{Z}_1^\delta$ contained

$$\begin{aligned}&\{x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \text{ s.t. } \|Y_{0,\mathrm{av}}^\delta(x_1, 0))\| \equiv \|Y_{1,u_0^\delta(x_1)}^\delta(x, 0)\| \equiv 0\} \\ &\equiv \{x_1 \in \mathbb{R}^{n_1} \text{ s.t. } \|Y_0^\delta(f_1^\delta(x_1), 0))\| \equiv 0\}.\end{aligned}$$

[2]For any $a, b \geq 0$ then for any $\lambda > 0$, $ab \leq \frac{a^2}{2\lambda} + \lambda\frac{b^2}{2}$.

This implies that $\mathcal{Z}_1^\delta$ coincides with the largest invariant set contained into the one induced by $u_0^\delta(x_1)$ in Lemma 4.3. Accordingly, ZSD (in the discrete-time sense) of (4.16b) with output (4.19) is enough to ensure ZSD of (4.16) with output (4.41) so getting the GAS of the origin of (4.16) under $u = u_1^\delta(x)$ in (4.45). LES follows from stabilizability of the continuous-time system (4.3) in first approximation implying the one of (4.16) [175]. S-GAS and S-LES of the equilibrium the origin of (4.3) in closed loop follow. ■

Remark 4.16. *As an alternative design strategy, one might apply Input-Lyapunov Matching. In this sense, one defines the sampled-data feedback $u_k = \gamma^\delta(x_k)$ so to ensure matching, at any sampling instant, of the continuous-time Lyapunov function (4.6) under the continuous-time feedback $u(t) = -\mathrm{L}_{g^2}V(x(t))$. Nonetheless, for ILM to be applicable to sampled-data stabilization of the feedforward (4.3), it is necessary for the continuous-time system with output $y = \mathrm{L}_{g^2}V(x)$ to be zero-state observable so to verify Assumption 2.1. Hence, ILM requires a stronger assumption than the feedforwarding design we are proposing where zero-state-detectability of the partial dynamics (4.3b) with output $y_0 = \mathrm{L}_{g_1}U(x_1)$ is needed.*

Remark 4.17. *By virtue of Remarks 4.7 and 4.15, when $x_2 = 0$, (4.45) rewrites as*

$$u_1^\delta(x_1,0) + \frac{1}{\delta}\int_0^1 \mathrm{L}_{G_1^\delta(\cdot,(\theta u_1^\delta(x_1,0)+(1-\theta)u_0^\delta(x_1)))}U(x_1^+(\theta u_1^\delta(x_1,0)+(1-\theta)u_0^\delta(x_1)))\mathrm{d}\theta = 0$$

so recovering, as expected, $u_1^\delta(x_1,0) = u_0^\delta(x_1)$.

The following result states the existence of a unique solution to (4.45). For this purpose, by exploiting (4.40), we first rewrite (4.25) as

$$\mathrm{V}^\delta(x) = V(x) + \sum_{i>0}\frac{\delta^i}{(i+1)!}\Psi^i(x) \tag{4.46}$$

with $V(x) := U(x_1) + W(x_2) + \Psi(x)$ being the continuous-time solution (4.6)

Theorem 4.3 (Existence and uniqueness of a control solution)**.** *Let the continuous -time feedforward system (4.3) verify Assumptions 4.1 to 4.3 and the hypotheses of Lemma 4.3. Assume $u = u_0^\delta(x_1)$ as the solution to (4.20) and consider the Lyapunov function $V^\delta : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ as in (4.25) with cross-term $\Psi^\delta : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \to \mathbb{R}$ solution to (4.30). Then, there exists $\delta^* > 0$ such that, for any $\delta \in]0,\delta^*[$, there exists a unique solution $u = u_1^\delta(x)$ to the damping equality (4.45) of the form*

$$u_1^\delta(x) = u_1^0(x) + \sum_{i>0}\frac{\delta^i}{(i+1)!}u_1^i(x) > 0 \tag{4.47}$$

with $u_1^0(x) = u_0(x_1) = -\mathrm{L}_{g^2}V(x)$.

Proof: The proof, again, is constructive. First, we substitute (4.21) and (4.46) into (4.45). Then, we rewrite the corresponding equation as a formal series equality in powers of δ of the form

$$Q^{\delta}(x,u) = Q^0(x,u) + \sum_{i>0} \frac{\delta}{(i+1)!} Q^i(x,u)$$

with

$$Q^0(x,u) = u + \mathrm{L}_{g^2} V(x).$$

Accordingly, by invoking the Implicit Function Theorem, one deduces the result. ■

For computational facilities, one can easily deduce a closed-form expression for any term $u_1^i(x)$ in (4.47) by running an iterative algorithm solving, at any step, a linear equality in the corresponding unknown. For the first terms and by discarding the state dependencies, one gets

$$\begin{aligned}
u_1^0(x) =& -\mathrm{L}_{g^2} V(x) && (4.48a)\\
u_1^1(x) =& -\mathrm{L}_{f^2+u_1^0(x)g^2}\mathrm{L}_{g^2} V(x) - \mathrm{L}_{g^2}\Psi^1(x) && (4.48b)\\
& - \mathrm{L}_{g^2}\mathrm{L}_{f^2} V(x) - u_0^0(x)\mathrm{L}_{g^2}^2 V(x) &&\\
& - \mathrm{L}_{g^2}\Psi^1(x) + \mathrm{L}_{f^2+u_1^0(x)g^2} u_1^0(x) - \mathrm{L}_{g^2}\mathrm{L}_{f^2+u_0^0 g^2} V(x) &&\\
u_1^2(x) =& -\mathrm{L}_{g^2}\Psi^2(x) + \mathrm{L}^2_{f^2+u_1^0(x)g^2} u_1^0(x) - \frac{1}{2}(u_1^1(x) + 3u_0^1(x))\mathrm{L}_{g^2}^2 V(x) &&\\
& - \frac{3}{2}\mathrm{L}_{f^2+u_0^0(x)g^2}\mathrm{L}_{g^2}\Psi^1(x) - \frac{3}{2}\mathrm{L}_{g^2}\mathrm{L}_{f^2+u_1^0(x)g^2}\Psi^1(x) && (4.48c)\\
& - \mathrm{L}_{f^2+u_0^0(x)g^2}\mathrm{L}_{g^2}\mathrm{L}_{f^2+u_1^0(x)g^2} V(x) - \mathrm{L}_{g^2}\mathrm{L}^2_{f^2+u_0^0(x)g^2} V(x) &&\\
& \quad - (u_1^0(x) - u_0^0(x))\mathrm{L}_{g^2}\Big(\mathrm{L}_{g^2}\mathrm{L}_{f^2+u_1^0(x)g^2} + \mathrm{L}_{f^2+u_0^0(x)g^2}\mathrm{L}_{g^2}\Big) V(x) && (4.48d)
\end{aligned}$$

so verifying $Q^i(x,u) = 0$ for $i = 0, 1, 2$.

As usual in sampling, the possibility of computing exact solutions to the involved equalities is sporadic so that approximate feedback solutions are usually implemented in practice as truncation of the series expansion (4.47) defining the final feedback at any finite order p. As a matter of fact, those approximate solutions

$$u_1^{\delta[p]}(x) = u_1^0(x) + \sum_{i=1}^{p} \frac{\delta^i}{(i+1)!} u_1^i(x) \tag{4.49}$$

will embed approximate solutions to the other concerned equalities (4.20) and (4.30) and will guarantee practical properties in closed loop only.

In what follows, we are going to provide an iterative procedure generalizing the proposed solution to multiple feedforward-interconnected dynamics (4.1).

4.2.2 The case of multiple feedforwarding interconnection

Considering now the feedforward dynamics (4.1) as

$$\begin{aligned}\dot{x}_n =& f_n(x_n) + \varphi_n(x_1, \dots, x_n) + g_n(x_1, \dots, x_n)u\\ &\vdots\\ \dot{x}_2 =& f_2(x_2) + \varphi_2(x_1, x_2) + g_2(x_1, x_2)u\\ \dot{x}_1 =& f_1(x_1) + g_1(x_1)u\end{aligned}$$

Assumptions 4.1 and 4.2 will extend as follows.

Assumption 4.4 (Linear growth)**.** *For any $i = 2, \dots, n$, the functions $\varphi_i(x_1, \dots, x_i)$ and $g_i(x_1, \dots, x_i)$ satisfy the* linear growth *property with respect to the state x_i.*

Assumption 4.5 (GS of the decoupled continuous-time x_i-dynamics)**.** *For any $i = 2, \dots, n$, the origin of $\dot{x}_i = f_i(x_i)$ is globally stable, with radially unbounded and locally quadratic Lyapunov function $W_i(x_i)$ so that $\mathrm{L}_{f_i} W_i(x_i) \le 0$ for all $x_i \in \mathbb{R}^{n_i}$. Moreover, there exist real constants c_i and M_i such that, for $\|x_i\| > M_i$,*

$$\|\nabla W_i(x_i)\| \|x_i\| \le c_i W(x_i).$$

Now, assume that the pair $(\nabla f^n(0), g^n(0))$ is stabilizable and that (4.1c) verifies Assumption 4.3 being also ZSD with respect to the output (4.18). Thus, when $u \in \mathcal{U}^\delta$, S-GAS and S-LES of the closed-loop equilibrium can be achieved by an iterative procedure over the sampled-data equivalent model (4.10) passivating (in the u-average sense), at each step i, the partial x^i-dynamics with respect to a suitably defined output $Y_i^\delta(x^i, u)$ deduced from the construction of a construced Lyapunov function $V_i^\delta(x^i)$.

Step 0: The dynamics (4.10c) is u-average passive with respect to the output $Y_0^\delta(x^1, u) = \mathrm{L}_{G^{1,\delta}(\cdot,u)} U(x^1)$ and the feedback $u = u_0^\delta(x^1)$ solution to the damping equality

$$u = -\frac{1}{\delta u} \int_0^u \mathrm{L}_{G^{1,\delta}(\cdot,v)} U(x^{1+}(v)) \mathrm{d}v$$

ensures GAS and LES of the equilibrium of (4.10c).

Step 1: Denoting $V_0^\delta(x^1) = U(x_1)$ and $x^1 = x_1$, for any $t \in [k\delta, (k+1)\delta[$, the dynamics

$$\dot{x}_2(t) = f_2(x_2(t)) + \varphi_2(x_1(t), x_2(t)) + g_2(x_1(t), x_2(t)) u_0^\delta(x_k^1) \tag{4.51a}$$
$$\dot{x}^1(t) = f^1(x^1(t)) + g^1(x^1(t)) u_0^\delta(x_k^1) \tag{4.51b}$$

has a S-GS equilibrium at the origin (Proposition 4.1). Then, its sampled-data equivalent model

$$x_{2k+1} = f_2^\delta(x_{2k}) + \varphi_2^\delta(x_{1k}, x_{2k}) + g_2^\delta(x_{1k}, x_{2k}, u_k) \tag{4.52a}$$
$$x_{k+1}^1 = F_0^{1,\delta}(x_k^1) + g^{1,\delta}(x_k^1, u_k) \tag{4.52b}$$

compactly rewritten as

$$x^2_{k+1} = f^{2,\delta}(x^2_k) + g^{2,\delta}(x^2_k, u_k) \tag{4.53}$$

is u-average passive from $u^\delta_0(x^1)$ and ZSD with respect to the output $Y^\delta_1(x^2, u) = \mathrm{L}_{G^{2,\delta}(\cdot,u)} V^\delta_1(x^2)$ where

$$V^\delta_1(x^2) = V_0(x^1) + W_2(x_2) + \Psi^\delta_2(x^2)$$

and

$$\Psi^\delta_2(x^2) = \sum_{\ell=0}^{\infty} \int_{\ell\delta}^{(\ell+1)\delta} \Big(\mathrm{L}_{\varphi_2(x^1(s),\cdot)} + u^\delta_0(x^1_\ell)\mathrm{L}_{g_2(x^1(s),\cdot)} W_2(x_2(s))\Big)\mathrm{d}s$$

$$x^1(s) = e^{\delta(\mathrm{L}_{f^1} + u^\delta_0(x^1_\ell)\mathrm{L}_{g^1})} x^1\big|_{x^1_\ell} \quad \text{and} \quad x_2(s) = e^{\delta(\mathrm{L}_{f_2} + \varphi_2 + u^\delta_0(x^1_\ell)\mathrm{L}_{g_2})} x_2\big|_{x^1_\ell, x_{2\ell}}.$$

Accordingly, the feedback $u = u^\delta_1(x^2)$ solution to

$$u = -\frac{1}{\delta(u - u^\delta_0(x^1))} \int_{u^\delta_0(x^1)}^{u} \mathrm{L}_{G^{2,\delta}(\cdot,v)} V^\delta_1(x^{2+}(v))\mathrm{d}v$$

makes the origin of (4.53) GAS and LES.

...

Step i: For any $t \in [k\delta, (k+1)\delta[$, the dynamics

$$\dot{x}_{i+1}(t) = f_{i+1}(x_i(t)) + \varphi_i(x^i(t), x_{i+1}(t)) + g_i(x^i(t), x_{i+1}(t))u^\delta_i(x^i_k) \tag{4.54a}$$

$$\dot{x}^i(t) = f^i(x^i(t)) + g^i(x^i(t))u^\delta_i(x^i_k) \tag{4.54b}$$

has a S-GS equilibrium at the origin as a result of the previous iteration. Then, its sampled-data equivalent model

$$x_{i+1k+1} = f^\delta_{i+1}(x_{i+1k}) + \varphi^\delta_{i+1}(x^i_k, x_{i+1k}) + g^\delta_{i+1}(x^i_k, x_{i+1k}, u_k) \tag{4.55a}$$

$$x^i_{k+1} = F^{i,\delta}_0(x^i_k) + g^{i,\delta}(x^i_k, u_k) \tag{4.55b}$$

compactly rewritten as

$$x^{i+1}_{k+1} = F^{i+1,\delta}_0(x^{i+1}_k) + g^{i+1,\delta}(x^{i+1}_k, u_k) \tag{4.56}$$

is u-average passive from $u^\delta_{i-1}(x^i)$ and ZSD with respect to the output $Y^\delta_i(x^{i+1}, u) = \mathrm{L}_{G^{i+1,\delta}(\cdot,u)} V^\delta_i(x^{i+1})$ where

$$V^\delta_i(x^{i+1}) = V_{i-1}(x^i) + W_{i+1}(x_{i+1}) + \Psi^\delta_{i+1}(x^{i+1})$$

and

$$\Psi^\delta_{i+1}(x^{i+1}) = \sum_{\ell=0}^{\infty} \int_{\ell\delta}^{(\ell+1)\delta} \Big(\mathrm{L}_{\varphi_{i+1}(x^i(s),\cdot)} + u^\delta_{i-1}(x^i_\ell)\mathrm{L}_{g_{i+1}(x^i(s),\cdot)}\Big) W(x_{i+1}(s))\mathrm{d}s$$

$$x^i(s) = e^{\delta(\mathrm{L}_{f^i} + u^\delta_{i-1}(x^i_\ell)\mathrm{L}_{g^i})} x^i\big|_{x^i_\ell}$$

$$x_{i+1}(s) = e^{\delta(\mathrm{L}_{f_{i+1}} + \varphi_{i+1} + u^\delta_{i-1}(x^i_\ell)\mathrm{L}_{g_{i+1}})} x_{i+1}\big|_{x^{i+1}_\ell}.$$

Accordingly, the feedback $u = u_i^\delta(x^{i+1})$ solution to

$$u = -\frac{1}{\delta(u - u_{i-1}^\delta(x^i))} \int_{u_{i-1}^\delta(x^i)}^{u} \mathrm{L}_{G^{i+1,\delta}(\cdot,v)} V_i^\delta(x^{i+1+}(v))\mathrm{d}v$$

makes the origin of (4.56) GAS and LES.

By applying the procedure for $i = 1, \ldots, n-1$, one gets the following result generalizing Theorem 4.2.

Theorem 4.4 (Forwarding sampled-data stabilization)**.** *Let the continuous-time feedforward system (4.1) verify Assumptions 4.4 and 4.5 with the pair $(\nabla f^n(0), g^n(0))$ being stabilizable. Assume(4.1c) verifies Assumption 4.3 and be ZSD with output $y_0 = L_{g_1}U(x_1)$. Then, the sampled-data equivalent model (4.10) is u-average passive from $u_{n-2}^\delta(x^{n-1})$ with respect to the output*

$$Y_{n-1}^\delta(x^n, u) = \mathrm{L}_{G^{n,\delta}(\cdot,u)} V_{n-1}^\delta(x^n)$$

and storage function

$$V_{n-1}^\delta(x^n) = U(x_1) + \sum_{i=2}^{n} \big(W_i(x_i) + \Psi_i^\delta(x^i)\big).$$

Moreover the control $u = u_{n-1}^\delta(x^n)$ computed as the unique solution to the damping equality

$$u = -\frac{1}{\delta(u - u_{n-2}^\delta(x^{n-1}))} \int_{u_{n-2}^\delta(x^{n-1})}^{u} \mathrm{L}_{G^{n,\delta}(\cdot,v)} V_{n-1}^\delta(x^{n+}(v))\mathrm{d}v$$

makes the equilibrium of (4.1) S-GAS and S-LES.

Remark 4.18. *As in the case of the simple double cascade interconnection 4.3, any $u_i^\delta(x^{i+1}) = u_i^\delta(x_1, \ldots, x_{i+1})$ verifies $u_i^\delta(x_1, \ldots, x_i, 0) = u_{i-1}^\delta(x_1, \ldots, x_i)$ for any $i = 1, \ldots, n-1$. Moreover, as $\delta \to 0$, one recovers all the continuous-time solutions.*

4.3 The strict-feedforward case

The procedure we have presented so far relies upon the construction of a suitable sampled-data cross-term characterizing a weak Lyapunov-function for a partial feedback dynamics. The corresponding Lyapunov function is then exploited as a storage function with respect to the natural passivating output for the overall system so allowing stabilization through u-average passivity arguments. This design is reminiscent of the continuous-time case although it appears to be quite far from the available design strategies in discrete time [113, 128, 106] which are mainly devoted to strict-feedforward-like forms as in (4.13). When dealing with classes of such structures, the design methodologies are essentially based on the definition of a coordinate transformation that makes the concerned sub-dynamics (usually assumed scalar) driftless (and, as a consequence, decoupled from the lower one) when a partial feedback is applied. Accordingly, the stabilizing feedback is deduced with respect to the system in the new coordinates based on different arguments.

In this part, we are specifying the results in Section 4.2 for nonlinear dynamics admitting a strict-feedforward cascade structure of the form (4.2) with $i = 2$ so discussing on the way the cross-term based feedforwarding we have been proposing implicitly recovers (and extends) the usual aforementioned procedure. Namely, we shall consider the system

$$\dot{x}_2 = F_2 x_2 + \varphi_2(x_1) + g_2(x_1)u \tag{4.57a}$$

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)u. \tag{4.57b}$$

We shall assume F_2 being skew-symmetric (i.e, $F_2^\top + F_2 = 0$) and, hence, regular with all of its eigenvalues over the imaginary axis and with unitary geometric multiplicity. Accordingly, Assumption 4.2 is trivially verified with $W(x_2) = x_2^\top x_2$. Moreover, because $\nabla_{x_2} g_2 = 0$ and $\nabla_{x_2} \varphi_2 = 0$ Assumption 4.1 naturally holds too. Finally, we shall also let (4.57b) verify Assumption 4.3.

Now, assuming $u \in \mathcal{U}^\delta$, we detail the sampled-data equivalent model (4.13) as

$$x_{2k+1} = F_2^\delta x_{1k} + \varphi_2^\delta(x_{1k}) + g_2^\delta(x_{1k}, u_k) \tag{4.58a}$$

$$x_{1k+1} = f_1^\delta(x_{1k}) + g_1^\delta(x_{1k}, u_k) \tag{4.58b}$$

and, in the (F_0, G)-representation as

$$x_2^+ = F_2^\delta x_2 + \varphi_2^\delta(x_2), \quad x_2^+ = x_2^+(0) \tag{4.59a}$$

$$x_1^+ = f_1^\delta(x_1), \quad x_1^+ = x_1^+(0) \tag{4.59b}$$

$$\frac{\mathrm{d}x_2^+(u)}{\mathrm{d}u} = G_2^\delta(x_1^+(u), u) \tag{4.59c}$$

$$\frac{\mathrm{d}x_1^+(u)}{\mathrm{d}u} = G_1^\delta(x_1^+(u), u) \tag{4.59d}$$

with $F_2^\delta = e^{\delta F_2}$ verifying $(F_2^\delta)^\top F_2^\delta = \mathrm{I}$ and, thus, being regular and possessing all the eigenvalues over the unit circle.

The lower dynamics (4.57b) is not impacted by the strict-feedforward form in the sense that it coincides with the first block of the general feedforward structure (4.3b). As a consequence, Lemma 4.3 still applies so yielding GAS and LES of the corresponding equilibrium under the feedback $u = u_0^\delta(x_1)$ solution to the damping equality (4.20). In what follows, we shall specialize the construction of the Lyapunov function (4.25) and of the sampled-data forwarding feedback (4.45) to the strict-feedforward dynamics (4.58) by emphasizing on the geometric properties they yield.

4.3.1 The cross-term and a decoupling state transformation

Consider now the closed-loop interval dynamics

$$\dot{x}_2 = F_2 x_2 + \varphi_2(x_1) + g_2(x_1) u_0^\delta(x_{1k}) \tag{4.60a}$$
$$\dot{x}_1 = f_1(x_1) + g_1(x_1) u_0^\delta(x_{1k}). \tag{4.60b}$$

and, the equivalent sampled-data one (4.58) under the feedback $u_k = u_0^\delta(x_1)$ which is provided by

$$x_{2k+1} = F_2^\delta x_{1k} + \varphi_2^\delta(x_{1k}) + g_2^\delta(x_{1k}, u_0^\delta(x_{1k})) \tag{4.61a}$$
$$x_{1k+1} = f_1^\delta(x_{1k}) + g_1^\delta(x_{1k}, u_0^\delta(x_{1k})) \tag{4.61b}$$

and consider the problem of deducing a Lyapunov function (4.25). To this end, the cross-term $\Psi^\delta(x) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ needs to solve (4.30) which specifies to this context as

$$\begin{aligned}
\Delta_k \Psi^\delta(x) =& -2 \int_{k\delta}^{(k+1)\delta} x_2^\top(s)[\varphi_2(x_1(s)) + u_0^\delta(x_{1k}) g_2(x_1(s))] \mathrm{d}s \\
=& -2 \int_{k\delta}^{(k+1)\delta} x_2^\top(s)[\dot{x}_2(s) - F_2 x_2(s)] \mathrm{d}s \\
=& -x_{2k+1}^\top x_{2k+1} + x_{2k}^\top x_{2k}.
\end{aligned} \tag{4.62}$$

Accordingly, by substituting (4.62) into (4.31) one gets

$$\Psi^\delta(x) = \sum_{\ell=0}^{\infty} (x_{2\ell+1}^\top x_{2\ell+1} - x_{2\ell}^\top x_{2\ell}) = \lim_{\ell\to\infty} x_{2\ell}^\top x_{2\ell} - x_{20}^\top x_{20}$$

so obtaining, when setting

$$\begin{aligned}
x_{2\ell} =& e^{\ell\delta F_2} x_2 + \sum_{i=0}^{\ell-1} e^{(\ell-i-1)\delta F_2} \int_0^\delta e^{(\delta-s)F} \big(\varphi_2(x_1(s)) + u_0^\delta(x_{1i}) g_2(x_1(s))\big) \mathrm{d}s \\
x_2 =& x_{20}
\end{aligned}$$

$$\Psi^\delta(x) = \lim_{\ell\to\infty} x_{2\ell}^\top x_{2\ell} - x_2^\top x_2 \tag{4.63}$$

and, thus,

$$V^\delta(x) = U(x_1) + \lim_{\ell\to\infty} x_{2\ell}^\top x_{2\ell} \tag{4.64}$$

Remark 4.19. *We emphasize the fact that the construction of the cross-term for strict-feedforward dynamics still applies albeit* $\lim_{\ell\to\infty} x_{2\ell}$ *might not exist. Indeed, in that case, the cross-term is defined through* $\lim_{\ell\to\infty} x_{2\ell}^\top x_{2\ell}$ *whose existence is guaranteed.*

Remark 4.20. *In the strict-feedforward case, the equation the sampled-data cross-term verifies recovers the one arising in (4.34) when looking at (4.13) as a purely discrete-time system.*

Other than studying the stability properties of the origin of (4.61) via Lyapunov functions, one can notice that, by construction, (4.61) exhibits a stable manifold $\mathcal{M}^\delta$ over which the trajectories are described by (4.61b). Moreover, such a manifold is implicitly defined as

$$\mathcal{M}^\delta = \{x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \text{ s.t. } \phi^\delta(x_1) = 0\} \tag{4.65}$$

where $\phi^\delta : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is a smooth mapping solving the invariance condition

$$\phi^\delta(f_1^\delta(x_1) + g_1^\delta(x_1, u_0^\delta(x_1))) = F_2^\delta \phi^\delta(x_1) + \varphi_2^\delta(x_1) + g_2^\delta(x_1, u_0^\delta(x_1)). \tag{4.66}$$

It is a matter of computations to verify that the above equality is solved by setting

$$\begin{aligned}\phi^\delta(x_1) &= -\sum_{\ell=\ell_0}^{\infty} e^{(\ell_0-\ell-1)\delta F_2} \int_0^\delta e^{(\delta-s)F}\big(\varphi_2(x_1(s)) + u_0^\delta(x_{1\ell})g_2(x_1(s))\big)\mathrm{d}s \\ &= -\sum_{\ell=\ell_0}^{\infty} e^{(\ell_0-\ell-1)\delta F_2}\varphi^\delta(x_{1\ell})\end{aligned} \tag{4.67}$$

with $\varphi^\delta(x_1) = \varphi_2^\delta(x_1) + g_2^\delta(x_1, u_0^\delta(x_1))$. Accordingly, introducing now the coordinate transformation

$$\zeta = x_2 - \phi^\delta(x_1) \tag{4.68}$$
$$= x_2 + \sum_{\ell=\ell_0}^{\infty} e^{(\ell_0-\ell-1)\delta F_2} \int_0^\delta e^{(\delta-s)F}(\varphi_2(x_1(s)) + u_0^\delta(x_{1\ell})g_2(x_1(s))\mathrm{d}s \tag{4.69}$$

the closed-loop system (4.61) rewrites as

$$\zeta_{k+1} = F_2^\delta \zeta_k \tag{4.70a}$$
$$x_{1k+1} = f_1^\delta(x_{1k}) + g_1^\delta(x_{1k}, u_0^\delta(x_{1k})) \tag{4.70b}$$

and thus is provided by two decoupled dynamics. By decoupling, one can directly infer that the overall system has a GS equilibrium with Lyapunov function

$$\mathcal{V}(\zeta, x_1) = U(x_1) + W_2(\zeta) = U(x_1) + \zeta^\top \zeta. \tag{4.71}$$

A natural arising question is now concerned with the relation among the Lyapunov function (4.64) deduced from the cross-term (4.63) and the one (4.71) deduced from the decoupling coordinate transformation (4.68). The answer is given through the proposition below.

Proposition 4.2. *Consider the strict-feedback dynamics (4.57) and its sampled-data equivalent model. Assume the hypotheses of Lemma 4.3. Then, the following relation holds*

$$V(x) = \mathcal{V}(x_2 - \phi^\delta(x_1), x_1) \tag{4.72}$$
$$\Psi^\delta(x) = (x_2 - \phi^\delta(x_1))^\top (x_2 - \phi^\delta(x_1)) - x_2^\top x_2. \tag{4.73}$$

Proof: The proof is straightforward from computing $(x_2 - \phi^\delta(x_1))^\top (x_2 - \phi^\delta(x_1))$ as

$$\begin{aligned}
&(x_2 + \sum_{i=0}^{\infty} e^{-(i+1)\delta F_2} \varphi^\delta(x_{1i}))^\top (x_2 + \sum_{i=0}^{\infty} e^{-(i+1)\delta F_2} \varphi^\delta(x_{1i})) \\
&= (x_2 + \sum_{i=0}^{\infty} e^{-(i+1)\delta F_2} \varphi^\delta(x_{1i}))^\top (e^{\ell\delta F_2})^\top e^{\ell\delta F_2} (x_2 + \sum_{i=0}^{\infty} e^{-(i+1)\delta F_2} \varphi^\delta(x_{1i})) \\
&= \| x_{2\ell} + \sum_{i=0}^{\infty} e^{(i-i-1)\delta F_2} \varphi^\delta(x_{1i}) - \sum_{i=0}^{i-1} e^{(i-i-1)\delta F_2} \varphi^\delta(x_{1i}) \|^2
\end{aligned}$$

with $\varphi^\delta(x_1) = \varphi_2^\delta(x_1) + g_2^\delta(x_1, u_0^\delta(x_1))$. Thus, by letting $\ell \to \infty$ in the above equality, one gets the result. ■

Along these lines, those results apply to other classes of feedforward dynamics of the form (4.3) as discussed below.

At first, consider the case of the feedforward dynamics(4.3) under Assumption 4.1 and verifying $\mathrm{L}_{f_2} W(x_2) \equiv 0$. In this case, equality (4.30) reduces to

$$\Delta_k \Psi^\delta(x_1, x_2) = -\Delta_k W(x_2) = -\int_{k\delta}^{(k+1)\delta} \big(\mathrm{L}_{\varphi_2(x_1(s),\cdot) + g_2(x_1(s),\cdot))}\big) W(x_2(s)) \mathrm{d}s$$

so analogously getting $\Psi(x_1, x_2) = \lim_{k\to\infty} W(x_{2k}) - W(x_2)$ with $(x_2, x_1) = (x_{20}, x_{10})$.

Moreover, if $f_2(x_2) \equiv 0$, one deduces

$$x_{2\infty}(x_1, x_2) := \lim_{k\to\infty} x_{2k} = x_2 + \sum_{i=0}^{\infty} \int_{i\delta}^{(i+1)\delta} (\varphi_2 + u_0^\delta(x_{1i}) g_2)(x_2(s), x_1(s)) \mathrm{d}s$$

and

$$\lim_{k\to\infty} W(x_{2k}) = W(x_{2\infty}(x_1, x_2)).$$

In this latter case, the mapping $(x_1, x_2) \to (x_{2\infty}(x_1, x_2), x_1)$ defines the local change of coordinates provided by

$$\nabla_{x_2} x_{2\infty}(x_1, x_2) = \mathrm{I} + \sum_{i=0}^{\infty} \int_{i\delta}^{(i+1)\delta} \nabla_{x_2} \big[\varphi_2 + u_0^\delta(x_{1i}) g_2)(x_1, x_2)\big] (x_1(s), x_2(s)) \mathrm{d}s$$

whose integral vanishes for $x_1 = 0$. From the above relation it is clear that the above of coordinate transformation is globally defined in case of strict-feedforward structures.

4.3.2 Feedforwarding and I&I

Based on the above considerations, one can now discuss about the geometric properties yielded by the final feedback discussed in Theorem 4.2. To this end, applying the coordinate transformation (4.68) to the original system (4.57) and exploiting the (F_0, G) representation, one gets

$$\zeta_{k+1} = x_{2k+1} - \phi^\delta(x_{1k+1})$$

with

$$\begin{aligned}\phi^\delta(x_{1k+1}) =& \phi^\delta(f_1^\delta(x_{1k})) + \int_0^{u_k} \mathrm{L}_{G_1^\delta(\cdot,v)}\phi^\delta(x_1^+(v))\mathrm{d}v \\ =& \underbrace{\phi^\delta(f_1^\delta(x_{1k})) + \int_0^{u_0^\delta(x_{1k})} \mathrm{L}_{G_1^\delta(\cdot,v)}\phi^\delta(x_1^+(v))\mathrm{d}v}_{=\phi^\delta(f_1^\delta(x_{1k})+g^\delta(x_{1k},u_0^\delta(x_{1k})))} + \int_{u_0^\delta(x_{1k})}^{u_k} \mathrm{L}_{G_1^\delta(\cdot,v)}\phi^\delta(x_1^+(v))\mathrm{d}v.\end{aligned}$$

Similarly, one rewrites

$$\begin{aligned}x_{2k+1} =& F_2^\delta x_{2k} + \varphi_2^\delta(x_{1k}) + g_2^\delta(x_{1k}, u_k) \\ =& F_2^\delta x_{2k} + \varphi_2^\delta(x_{1k}) + \int_0^{u_k} G_2^\delta(x_1^+(v))\mathrm{d}v \\ =& F_2^\delta x_{2k} + \varphi_2^\delta(x_{1k}) + \underbrace{\int_0^{u_0^\delta(x_{1k})} G_2^\delta(x_1^+(v))\mathrm{d}v}_{=g_2^\delta(x_{1k},u_0^\delta(x_{1k}))} + \int_{u_0^\delta(x_{1k})}^{u_k} G_2^\delta(x_1^+(v))\mathrm{d}v.\end{aligned}$$

As a consequence, by exploiting (4.66), one gets

$$\zeta_{k+1} = F_2^\delta \zeta_k + \int_{u_0^\delta(x_{1k})}^{u_k} G_\zeta(x_1^+(v), v)\mathrm{d}v \tag{4.74a}$$

$$x_{1k+1} = f_1^\delta(x_{1k}) + g^\delta(x_{1k}, u_k) \tag{4.74b}$$

with

$$G_\zeta(x_1, u) := G_2^\delta(x_1, u) - \mathrm{L}_{G_1^\delta(\cdot,u)}\phi^\delta(x_1).$$

By resettling Theorem 4.2 into the (ζ, x_1)-coordinates, one exploits the new form of the Lyapunov function (4.71) to rewrite the output (4.41) as

$$\begin{aligned}\mathcal{Y}_1^\delta(\zeta, x_1, u) =& Y_1^\delta(\zeta + \phi^\delta(x_1), x_1, u) \\ =& \mathrm{L}_{G_1^\delta(\cdot,u)}U(x_1) + 2\zeta^\top G_\zeta(x_1, u)\end{aligned}$$

and, thus, deduce the stabilizing feedback $u = \mathbf{u}_1^\delta(\zeta, x_1) = u_1^\delta(\zeta + \phi^\delta(x_1), x_1)$ defined by (4.45).

Remark 4.21. *Those new coordinates emphasize on the fact that the u-average passivity based feedback solving (4.45) is implicitly making the manifold $\mathcal{M}^\delta$ in (4.65) attractive and invariant. In this sense, the sampled-data feedforwarding design we propose recovers Immersion and Invariance, when specified to strict-feedforward dynamics (4.57). As a matter of fact, one can ensure that, in the I&I jargon, (4.61b) represents the target dynamics evolving over $\mathcal{M}^\delta$ while*

$$\pi^\delta(x_1) = \begin{pmatrix} \phi^\delta(x_1) \\ x_1 \end{pmatrix}, \quad c^\delta(x_1) = u_0^\delta(x_1)$$

with (4.68) define the immersion mapping and the control making $\mathcal{M}^\delta$ feedback invariant.

Those considerations extend to multi-block strict-feedforward dynamics (4.2). In this context, at each step i, one looks for a coordinate change

$$\zeta_i = x_{i+1} - \phi_i^\delta(\zeta_1, \ldots, \zeta_{i-1}, x_1)$$

that makes the corresponding dynamics decoupled under the feedback

$$u = \mathbf{u}_{i-1}^\delta(\zeta_1, \ldots, \zeta_{i-1}, x_1).$$

Accordingly, at each step, one makes the manifold

$$\mathcal{M}_i^\delta = \{(\zeta_1 \ldots, \zeta_{i-1}, x_1) \in \mathbb{R}^{n_i} \times \cdots \times \mathbb{R}^{n_1} \text{ s.t. } x_{i+1} - \phi_i^\delta(\zeta_1, \ldots, \zeta_{i-1}, x_1) \equiv 0\}$$

invariant and attractive for the corresponding augmented cascade. Accordingly, this procedure extends our previous work in [131].

4.4 Some illustrating examples

In this sections, several examples will be developed to illustrate the proposed feedforwarding strategy. First, the case of the LTI double integrator will be exploited for illustrating the exact computations that are required. Then, a nonlinear dynamics in feedforward form will be exploited to focus on approximations of the feedback controller.

4.4.1 The double integrator

Consider again the case of the double integrator

$$\dot{x}_2 = x_1 \tag{4.75a}$$
$$\dot{x}_1 = u \tag{4.75b}$$

which verifies Assumptions 4.1 to 4.3 with $U(x_1) = \frac{1}{2}x_1^2$ and $W(x_2) = \frac{1}{2}x_2^2$. Then, one computes the corresponding sampled-data equivalent as

$$x_{2k+1} = x_{2k} + \delta x_{1k} + \frac{\delta^2}{2} u_k \tag{4.76a}$$
$$x_{1k+1} = x_{1k} + \delta u_k. \tag{4.76b}$$

or, equivalently, as

$$\begin{aligned}
x_2^+ &= x_2 + \delta x_1 \\
x_1^+ &= x_1 \\
\frac{\partial x_2^+(u)}{\partial u} &= \frac{\delta^2}{2} \\
\frac{\partial x_1^+(u)}{\partial u} &= \delta.
\end{aligned}$$

The feedforwarding stabilizing design proceeds through the following steps.

4.4.1.1 Step 0: stabilization of the partial dynamics

Consider the partial dynamics (4.76b). Then, it can be easily shown that it is u-average passive with respect to the output

$$Y_0^\delta(x_1) = x_1$$

and u-average output

$$Y_{0,\text{av}}^\delta(x_1) = x_1 + \frac{\delta}{2}u$$

Accordingly, one computes the feedback $u = u_0^\delta(x_1)$ as solution to the damping equality

$$u = -x_1 - \frac{\delta}{2}u$$

which is hence provided by

$$u_0^\delta(x_1) = -\frac{2}{2+\delta}x_1.$$

Accordingly, the dynamics

$$x_{2k+1} = \frac{2-\delta}{2+\delta}x_{1k}$$

has a GES equilibrium at the origin.

4.4.1.2 Step 1: computation of the cross-term

Consider now the closed-loop interval dynamics under $u_0^\delta(x_1) = -\frac{2}{2+\delta}x_1$ for $t \in [k\delta, (k+1)\delta[$

$$\begin{aligned}
\dot{x}_2(t) &= x_1(t) \\
\dot{x}_1(t) &= -\frac{2}{2+\delta}x_{1k}.
\end{aligned}$$

and its sampled-data equivalent model

$$\begin{aligned} x_{2k+1} =& x_{2k} + \frac{2\delta}{2+\delta} x_{1k} \\ x_{1k+1} =& \frac{2-\delta}{2+\delta} x_{1k} \end{aligned} \tag{4.79}$$

which, because of the triangular structure, is GS. Accordingly, the goal is now to compute a Lyapunov function for (4.84) through the definition of a cross-term; i.e., we look for

$$V_0(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \Psi^\delta(x)$$

where the cross-term $\Psi^\delta(x)$ satisfies

$$\Delta_k \Psi^\delta(x) = -\frac{\delta}{2+\delta} x_1 \Big(2x_2 + \frac{\delta}{2+\delta} x_1. \Big)$$

Accordingly, by specifying (4.31) and (4.63) to this case we get

$$\begin{aligned} \Psi^\delta(x) =& \frac{1}{2} \lim_{\ell \to \infty} x_{2\ell}^2 - \frac{1}{2} x_2^2 \\ =& \frac{1}{2}(x_1 + x_2)^2 - \frac{1}{2} x_2^2 \end{aligned}$$

and the corresponding Lyapunov function

$$V_0(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_1 + x_2)^2$$

verifying

$$\Delta_k V_0(x) = -\frac{4\delta}{(2+\delta)^2} x_1^2.$$

4.4.1.3 Step 2: u-average PBC from $u_0^\delta(x_1)$

Consider the sampled-data dynamics (4.76). By construction, it is u-average passive from $u_0^\delta(x_1) = -\frac{2}{2+\delta}x_1$ with storage function $V_0(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_1+x_2)^2$ and output

$$Y_1^\delta(x) = (1 + \frac{\delta}{2})x_2 + (2 + \frac{\delta}{2})x_1.$$

As a matter of fact, one gets

$$\Delta_k V_0(x) = -\frac{4\delta}{(2+\delta)^2} + \delta \int_{u_0^\delta(x_1)}^{u} \Big((1 + \frac{\delta}{2}) x_2^+(v) + (2 + \frac{\delta}{2}) x_1^+(v)) \Big) \mathrm{d}v.$$

with

$$\begin{aligned} x_2^+(u) &= x_2 + \delta x_1 + \frac{\delta^2}{2} u \\ x_1^+(u) &= x_1 + \delta u. \end{aligned}$$

Accordingly, through easy computations, one gets

$$\Delta_k V_0(x) \leq (1+\frac{\delta}{2})u\Big(x_2 + (1+\delta)x_1 + (1+\frac{\delta}{2})(u - u_0^\delta(x_1))\Big)$$

so achieving

$$\Delta_k V_0(x) \leq 0$$

when setting $u = u_1^\delta(x)$ as the solution to the damping equality

$$u = -(1+\frac{\delta}{2})x_2 - (1+\delta)(1+\frac{\delta}{2})x_1 - (1+\frac{\delta}{2})^2(u - u_0^\delta(x_1)).$$

As a consequence, the stabilizing feedback is provided by

$$\begin{aligned} u_1^\delta(x) =& -\frac{1}{(\delta+2)^2}x_2 - u_0^\delta(x_1) \\ =& -\frac{1}{(\delta+2)^2}x_2 - \frac{2}{\delta+2}x_1 \end{aligned}$$

with the property that $u_1^\delta(x_1, 0) = u_0^\delta(x_1)$.

4.4.2 The case of a feedforward dynamics

Consider the simple cascade dynamics

$$\dot{x}_2 = x_2 x_1 \tag{4.80}$$
$$\dot{x}_1 = u. \tag{4.81}$$

The system (4.80) admits an exactly computable sampled-data equivalent model

$$\begin{aligned} x_{2k+1} =& e^{\delta(x_{1k} + \frac{\delta}{2}u_k)}x_{2k} \\ x_{1k+1} =& x_{1k} + \delta u_k \end{aligned} \tag{4.82}$$

which clearly preserves the feedforward structure with nonlinear dependency in u_k. The equivalent (F_0^δ, G^δ) representation takes the form

$$\begin{aligned} x_2^+ &= e^{\delta x_1}x_2; & \frac{\partial x_2^+(u)}{\partial u} &= \frac{\delta^2}{2}e^{-\delta(x_1^+(u) - \frac{\delta}{2}u)}x_2^+(u) \\ x_1^+ &= x_1; & \frac{\partial x_1^+(u)}{\partial u} &= \delta \end{aligned}$$

and, thus,

$$G_1^\delta(x_1, u) = \delta, \quad G_2^\delta(x_1, x_2, u) = \frac{\delta^2}{2}e^{-\delta(x_1 - \frac{\delta}{2}u)}x_2.$$

One verifies that (4.80) satisfies Assumptions 4.1 to 4.3 with $W(x_2) = \frac{1}{2}x_2^2$ and $U(x_1) = \frac{1}{2}x_1^2$. We illustrate below how to design and compute the sampled-data stabilizing feedback.

The **Step 0** of the procedure coincides with the one developed in Section 4.4.1.1 so that the feedback

$$u_0^\delta(x_1) = -\frac{2}{2+\delta}x_1$$

makes

$$x_{1k+1} = \frac{2-\delta}{2+\delta}x_{1k}$$

globally exponentially stable.

4.4.2.1 Step 1: computation of the cross-term

Consider now the closed-loop interval dynamics under $u_0^\delta(x_1) = -\frac{2}{2+\delta}x_1$ for $t \in [k\delta, (k+1)\delta[$

$$\begin{aligned}\dot{x}_2(t) =& x_2(t)x_1(t)\\ \dot{x}_1(t) =& -\frac{2}{2+\delta}x_{1k}.\end{aligned}$$

and its sampled-data equivalent model

$$\begin{aligned}x_{2k+1} =& e^{\frac{2}{\delta+2}x_{1k}}x_{2k}\\ x_{1k+1} =& \frac{2-\delta}{2+\delta}x_{1k}\end{aligned} \tag{4.84}$$

possessing a globally stable equilibrium at the origin. Then, we aim at constructing a Lyapunov Function

$$V_0(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \Psi^\delta(x)$$

where the cross-term verifies

$$\begin{aligned}\Delta_k\Psi^\delta(x) =& \int_{k\delta}^{(k+1)\delta} x_1(s)x_2^2(s)\mathrm{d}s\\ =& \int_0^\delta \Big(1 - \frac{2s}{2+\delta}\Big)e^{2s(1-\frac{2s}{2+\delta})x_{1k}}\mathrm{d}s x_{1k}x_{2k}^2\\ =& \Big(e^{\frac{4\delta}{2+\delta}x_{1k}} - 1\Big)x_{2k}.\end{aligned}$$

Accordingly, exploiting that

$$\begin{aligned}x_{1k} &= \Big(\frac{2-\delta}{2+\delta}\Big)^k x_1\\ x_{2k} &= e^{\big(1-(\frac{2-\delta}{2+\delta})^k\big)x_1}x_2\end{aligned}$$

the relation (4.31) reduces to

$$\Psi^\delta(x) = \frac{1}{2}(e^{2x_1} - 1)x_2^2$$

so obtaining

$$V_0(x) = \frac{1}{2}x_1^2 + \frac{1}{2}e^{2x_1}x_2^2$$

verifying

$$\Delta_k V_0(x) = -\frac{4\delta}{(2+\delta)^2}x_1^2.$$

4.4.2.2 Step 2: u-average PBC from $u_0^\delta(x_1)$

Consider the sampled-data dynamics (4.82). By construction, it is u-average passive from $u_0^\delta(x_1) = -\frac{2}{2+\delta}x_1$ with storage function $V_0(x) = \frac{1}{2}x_1^2 + \frac{1}{2}e^{2x_1}x_2^2$ and output

$$Y_1^\delta(x,u) = x_1 + e^{2x_1}(1 + \frac{\delta}{2}e^{-\delta(x_1 - \frac{\delta}{2}u)})x_2^2.$$

As a matter of fact, one obtains

$$\Delta_k V_0(x) = -\frac{4\delta}{(2+\delta)^2}x_1^2 + \int_{u_0^\delta(x_1)}^{u} \Big(x_1^+(v) + e^{2x_1^+(v)}(1 + \frac{\delta}{2}e^{-\delta(x_1^+(v) - \frac{\delta}{2}v)})(x_2^+(v))^2\Big)\mathrm{d}v$$

with

$$\begin{aligned} x_1^+(u) =& x_1 + \delta u \\ x_2^+(u) =& e^{\delta(x_1 + \frac{\delta}{2}u)}x_2. \end{aligned}$$

Accordingly, one gets

$$\begin{aligned} \Delta_k V_0(x) \le& (u - u_0^\delta(x_1))\Big(\frac{2}{2+\delta}x_1 + \frac{\delta}{2}u + e^{\frac{2(2+\delta^2)}{2+\delta}x_1}\frac{e^{3\delta(u-u_0^\delta(x_1))} - 1}{3\delta(u - u_0^\delta(x_1))}x_2^2 \\ &+ e^{\frac{-4(\delta^2-\delta+1)}{2+\delta}x_1}\frac{\frac{5}{2}\delta^2(u - u_0^\delta(x_1)) - 1}{5\delta(u - u_0^\delta(x_1))}x_2^2\Big). \end{aligned}$$

Accordingly, the stabilizing feedback is given by the unique solution $u = u_1^\delta(x)$ to the damping equality

$$\begin{aligned} (1 + \frac{\delta}{2})u =& -\frac{2}{2+\delta}x_1 - e^{\frac{2(2+\delta^2)}{2+\delta}x_1}\frac{e^{3\delta(u-u_0^\delta(x_1))} - 1}{3\delta(u - u_0^\delta(x_1))}x_2^2 \\ &- e^{\frac{-4(\delta^2-\delta+1)}{2+\delta}x_1}\frac{e^{\frac{5}{2}\delta^2(u-u_0^\delta(x_1))} - 1}{5\delta(u - u_0^\delta(x_1))}x_2^2. \end{aligned}$$

Remark 4.22. *The damping equality above underlines that, as $x_2 = 0$, one recovers $u_1^\delta(x_1, 0) = u_0^\delta(x_1)$.*

Computing an exact solution to the damping equality is tough in general. So, by exploiting the series representation (4.47) and the corresponding expressions (4.48) one gets that an approximate solution is given by

$$u_1^{\delta[1]}(x) = -x_1 - e^{2x_1}x_2^2 + \delta(e^{4x_1}x_2^4 + 2x_1).$$

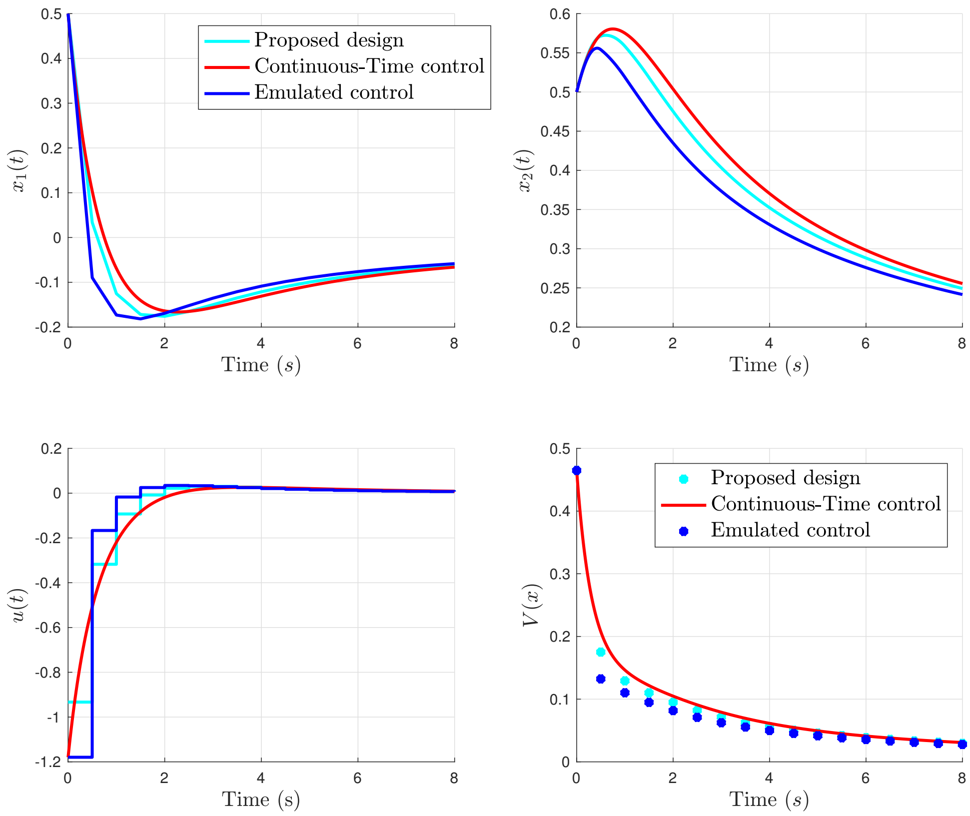


Figure 4.2: $\delta = 0.5$ s

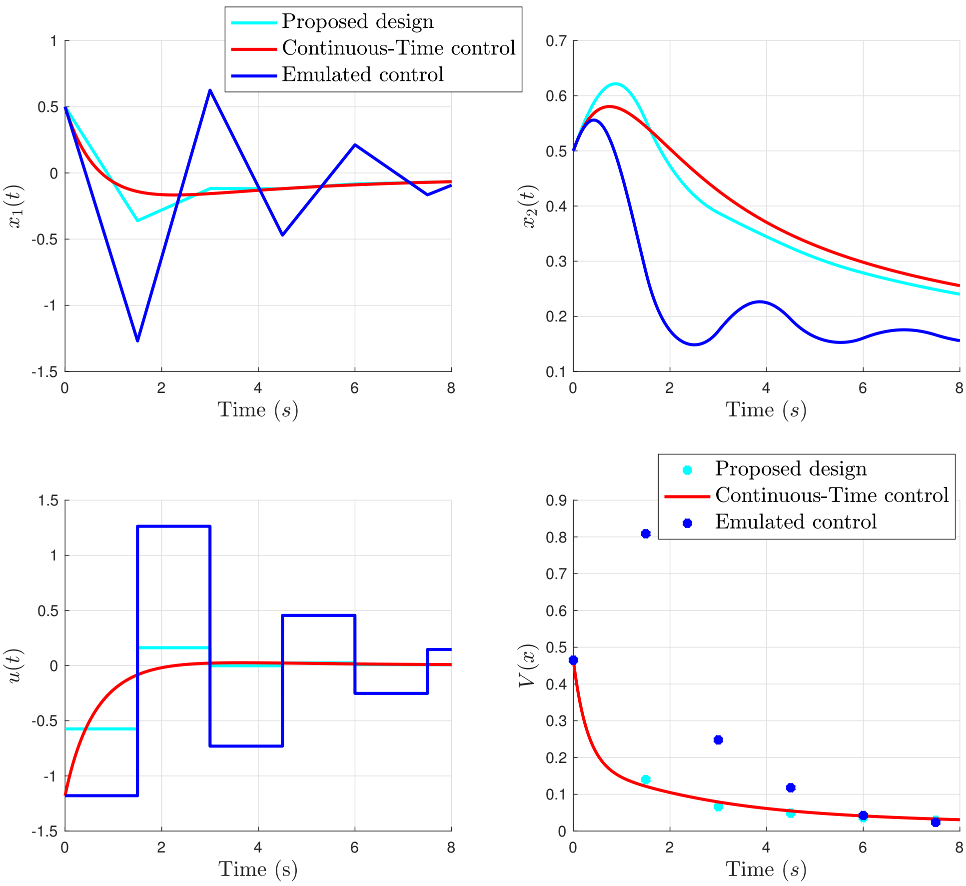


Figure 4.3: $\delta = 1.5$ s

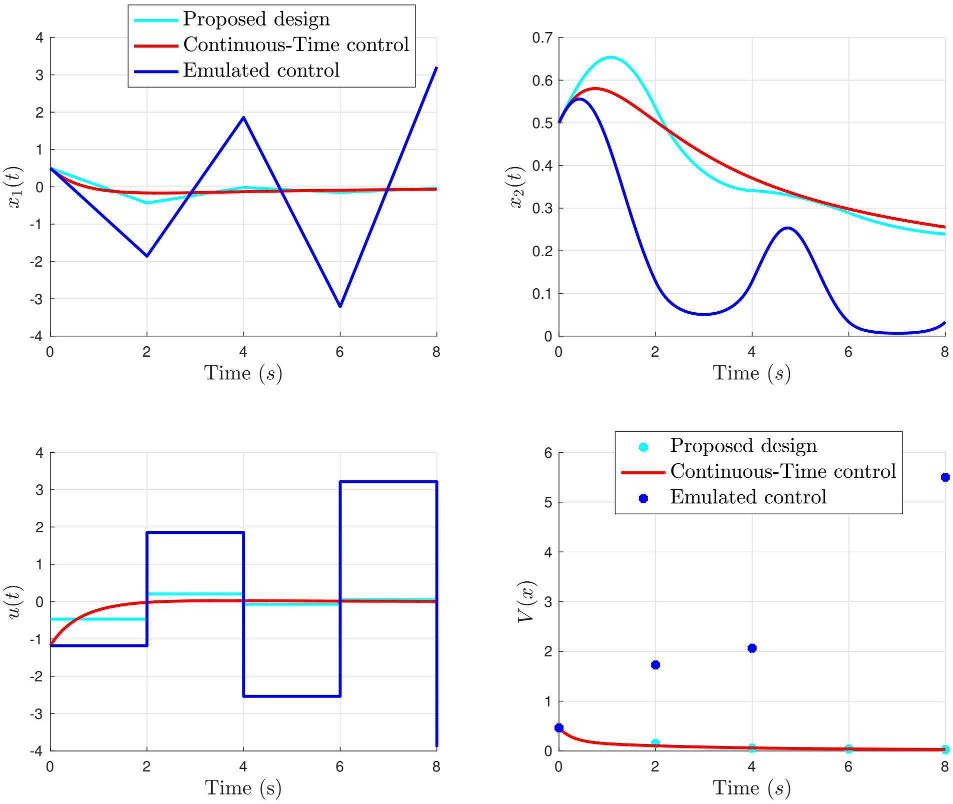


Figure 4.4: $\delta = 2$ s

4.4.2.3 Some simulations

The proposed control strategy is compared through simulations with the continuous-time one and the so-called emulated control (i.e., when implementing the continuous feedback by means of sample-and-hold devices).

Simulations are depicted in Figures 4.2, 4.3 and 4.4 for initial condition $x_0 = (0.5, 0.5)$. They clearly show that, as the sampling period increases, the proposed control strategy achieves very good performances (with smooth trajectories) especially when the emulated one degrades or even fails (Figures 4.4). This empirically proves the efficiency of the sampled-data direct design when compared to mere sample-and-hold implementation [34, 194] of the continuous-time feedback. Moreover, contrarily to the emulated feedback, the evolutions of the Lyapunov function along the trajectories under the proposed sampled-data feedback are decreasing even when δ significantly increases. More in detail, the continuous-time $V_0(x) = \frac{1}{2}x_1^2 + \frac{1}{2}e^{2x_1}x_2^2$ is no longer a Lyapunov function for the closed-loop system under the emulated feedback. Finally, the simulated results underline the nested nature of the feedback in the sense of Remark 4.17; namely, first one drives z to zero so recovering the integrator x_1-dynamics $x_{1k+1} = x_{1k} + \delta u_k$ evolving according to the feedback computed at the first step of the design. In this sense, when the emulated feedback is implemented (i.e., $u_0(x_1) = -x_1$) and $z \equiv 0$ the reduced linear x_1-dynamics is clearly unstable for higher values of δ as Figure 4.4 clearly underlines; in this same scenario, the dynamics under the proposed feedback $u = u_1^\delta(x)$ still shows good stablizing performances. Finally, in order to emphasize on the hierarchical structure of the overall feedback $u = u_1^\delta(x_1)$ (Remark 4.17), Figures 4.5 and 4.6 depict the results of the simulation when setting $x_{20} = 0$ and $x_{10} = 0.5$. The plots underline that, in this situation and when $u = u^{[1]\delta}(x_1)$, the x_2-dynamics in closed-loop stays in the equilibrium while x_1 evolves as the controlled sampled-data integrator $\dot{x}_1(t) = -\frac{2}{2+\delta}x_{1k}$. Moreover, when the emulated feedback $u = -x_1 - e^{2x_1}x_2^2$ is applied and $x_2 = 0$, the closed-loop sampled-data x_1-dynamics is given by $x_{1k+1} = (1-\delta)x_{1k}$ which is critically stable as $\delta = 2$ and unstable as $\delta > 2$ as Figure 4.6 testifies.

4.5 Conclusions and literature review

Feedforward structures are of impressive importance in nonlinear control as they allow both constructive design of stabilizing controllers and as they cover a wild range of real processes. A geometric characterization of feedback equivalence of nonlinear (both continuous and discrete-time) systems to feedforward forms has been given in several works by Astolfi, Moog and Respondeck in [7, 136, 178].

As far as control design is concerned, most of the works on these topics address the case of continuous-time strict-feedforward systems where the design is shown to be based upon coordinate transformations successively integrating the lower states of the system as shown by several authors as Mazenc, Praly, Respondek, Ortega

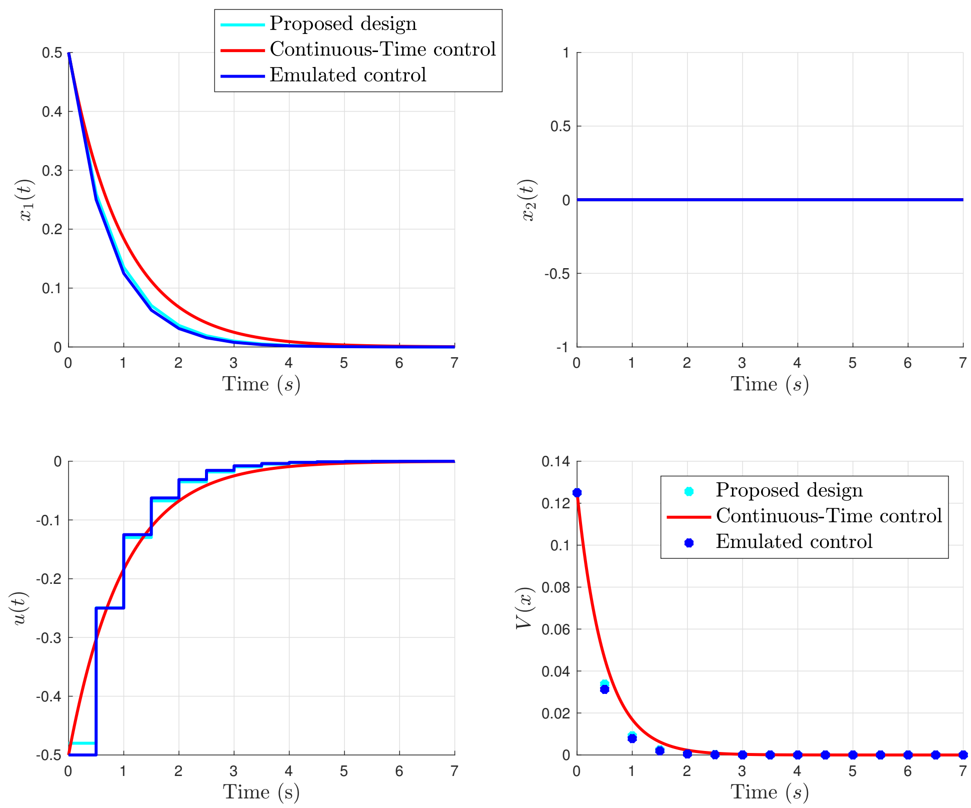


Figure 4.5: $\delta = 0.5$ s and $x_{20} = 0$

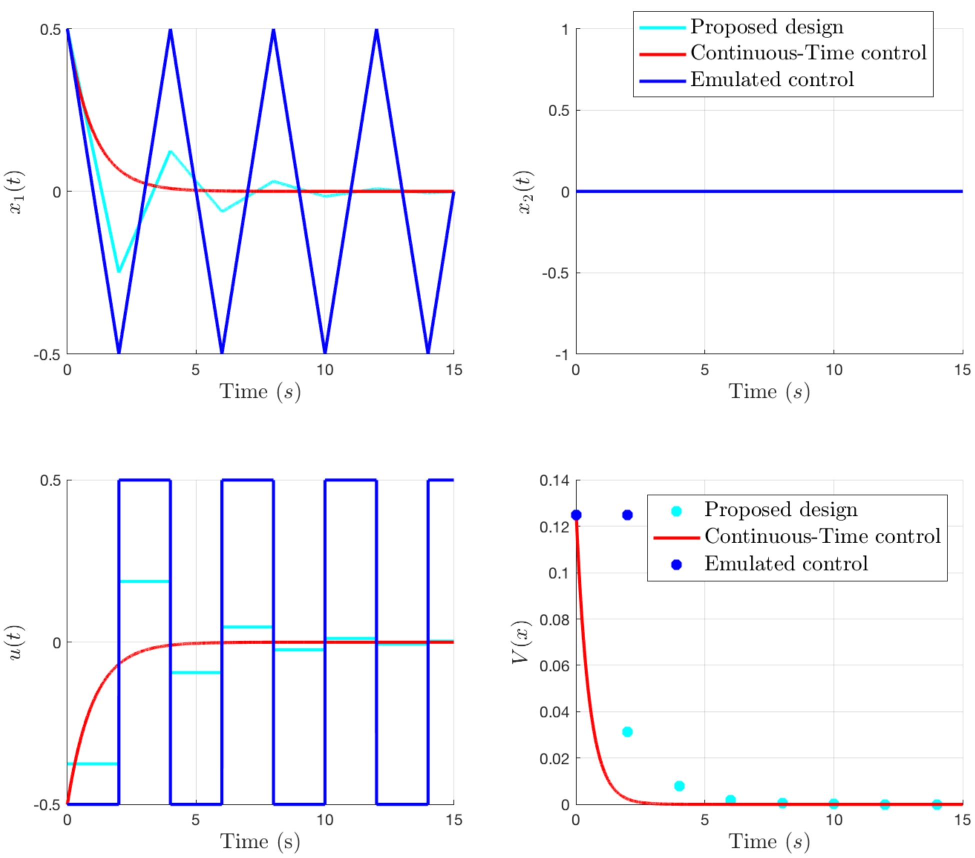


Figure 4.6: $\delta = 2$ s and $x_{20} = 0$

and Krstic [115, 178, 161, 78]. When dealing with continuous-time feedforward systems, this procedure has been generalized by Sepulchre et al. in [169] where the construction of the cross-term was firstly introduced. Then, several alternative solutions have been proposed as in [181, 196, 3, 25, 194, 162, 193, 89, 34, 88].

When dealing with discrete-time systems, most works are devoted to strict-feedforward structure and based upon the definition of a coordinate transformation successively integrating the concerned states. Moreover, we have recently presented a new feedforwarding procedure for general and purely discrete-time feedforward systems so extending the one that are applicable to strict-feedforward structures [104, 101]. Though, in discrete time, the design needs to face the nonlinearities of the concerned equations in the control variable and the apparent loss of a geometrical structure. Accordingly, several methods have been proposed to overcome these problem through approximate solutions and optimization algorithms that might easily work online [113, 104, 128].

A challenging perspective is provided by sampled-data feedforward cascade systems as, contrarily to feedback forms, they indeed preserve the interested nested structure. Accordingly, the design is made less demanding than in discrete time both in the required assumptions and in the computation of the solutions to the concerned equalities. As a matter of fact, a constructive and nested design can still be pursued while approximate solutions can be easily deduced by exploiting the δ parametrization characterizing all of the concerned equations. Moreover, approximate feedbacks can be computed at any required order as a trade off among computational issues and the required preservation of the continuous-time specifications in closed-loop.

In this sense, we have proposed a constructive and unifying procedure to preserve feedforwarding stabilization under sampling of nonlinear dynamics admitting the structure (4.3) by exploiting the same assumptions as in continuous time and without requiring any further one. To this end, after showing the preservation of the structure under sampling, the design we have carried out is composed of three steps:

1. starting from (4.16b), we have exhibited a new δ-dependent output with respect to which the system is u-average passive so deducing a stabilizing controller for the lower dimensional dynamics;

2. we have constructed a new weak Lyapunov function (4.25) for the corresponding closed-loop system (4.22) based on the definition of a new cross-term (4.31) which is smoothly parametrized by δ and takes into account the piecewise constant nature of the control signal over the interval dynamics (4.23);

3. we have defined the final stabilizing controller for the augmented dynamics (4.16) by exploiting u-average passivity arguments arising with the Lyapunov function constructed for the preliminary closed-loop system (4.22).

All the solutions we have been exhibiting recover the continuous-time ones as $\delta \to 0$. Then, we have proposed an iterative procedure stabilizing the multiple feedforward cascade interconnection (4.1) under sampling. A geometric interpretation of the design procedure together with connections to I&I have been given with reference to strict-feedforward systems by showing that, implicitly, the overall feedback is aimed at making a suitably defined manifold attractive and invariant. In this sense, this procedure extends the one we previously proposed for strict-feedforward structure in [131].

Comparison of the proposed methodologies with other sampled-data design methodologies that are still applicable to feedforward systems have been discussed in our work [98] with reference to both sampled-data direct or indirect redesign strategies (e.g., Input-Lyapunov Matching or δ-average passivity arguments).

As we shall sketch in the following, this procedure also applies for the stabilization of cascade systems deduced from sampled-data time-delay systems so providing an alternative tool for the design in this scenario.

Part III

Time-delay systems under sampling

Chapter 5

Time-delay systems as cascades

Contents

The seminal works by Smith [173] and Artstein [6] have inspired a research toward time-delay systems as an unavoidable paradigm in control theory because of their involvement in a lot of practical situations [172, 80, 71]. Investigations have been addressed to the study of the effects of time delay in a control system emphasizing on drawback and also, unexpectedly, advantages. As an example, it has been shown that introducing a delay over the control system might make a non stabilizable (or not controllable) system stabilizable (or controllable) as shown, among others, by Fridman and Niculescu [145, 45] or in the works by Califano and Moog (e.g., [21, 22]) where the notion of *extended Lie Bracket* has been introduced for time-delay systems.

In this chapter, we are going to emphasize on the cascade structure lying behind nonlinear systems when affected by a constant and fixed time-delay over the input channel and, thus, how the tools developed for cascade nonlinear systems can be profitably exploited in this new framework. To this end, we shall start from continuous-time systems by underlying some unavoidable numerical pathologies and difficulties which might be encountered, even for linear systems, in the feedback design. Then, the impact of the sampling framework over time-delay systems is discussed by showing how it positively intervenes to overcome some of the issues arising in a completely continuous-time scenario. The case of predictor-based feedback design will serve as a benchmark and comparative example in this sense when assuming, under sampling, that the length of the delay is a suitable multiple of the sampling period.

The contents of this chapter are mainly based on [71] for continuous-time systems. The results for the design of sampled-data stabilizing controllers are based on

M. Mattioni, S. Monaco and D. Normand-Cyrot. *Further results on sampled-data stabilization of time-delay systems.* 20th IFAC World Congress. IFAC-PapersOnLine. Volume 50, Issue 1, pp. 14350-14355. Toulouse, France.

S. Monaco, D. Normand-Cyrot and M. Mattioni. *Nonlinear Sampled-data stabilization with delays.* Accepted to *Delays and Interconnections: Methodology, Algorithms and Applications*, Series Advances on Delays and Dynamics at Springer. Editors: G. Valmorbida, R. Sipahi, I. Boussaada and A. Seuret.

5.1 Recalls on time-delay continuous-time systems

In this section, we shall provide a sketch on available tools for prediction-based stabilization of continuous-time nonlinear dynamics affected by a *discrete* input delay. We are going to consider nonlinear retarded systems of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t-\tau) \tag{5.1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and f and g assumed smooth vector fields. We are considering the delay $\tau > 0$ to be constant and a-priori known and we shall refer to (1.1) as the *delay-free dynamics* associated to (5.1) when computed for $\tau = 0$. We assume $u : [-\tau, \infty) \to \mathbb{R}$ belongs to $M_U^{[-\tau,\infty)}$. We'll be assuming (1.1) forward complete so implying that, as $u \in M_U^{[-\tau,\infty)}$, the retarded system (5.1) is forward complete, too [81]. For any $t \geq 0$ and $\theta \in [-\tau, 0]$, $u_t := u_t(\theta) = u(t+\theta)$ and $x_t := x_t(\theta) = u(t+\theta)$ respectively denote the story of u and x of (5.1) over the time window $[t-\tau, t]$.

5.1.1 Continuous-time input-delayed dynamics as cascades

In several contributions (e.g., [79, 80, 81, 82]), Krstic and co-workers have been showing that, because of their intrinsic infinity, one can represent (5.1) as a dynamics governed by both a ODE and PDE components; namely, by introducing the spatial variable $r \in [0, \tau]$ and the new state

$$v(t, r) := u(t - \tau + r)$$

one gets a transport PDE describing the infinite dimensional input dynamics as

$$\nabla_t v(t, r) = \nabla_r v(t, r).$$

Thus, one rewrites the plant (5.1) as the following (feedback) cascade interconnection

$$\begin{aligned} \dot{x}(t) =& f(x(t)) + g(x(t))v(t, 0) && (5.2a) \\ \nabla_t v(t, r) =& \nabla_r v(t, r) && (5.2b) \\ v(t, \tau) =& u(t) && (5.2c) \end{aligned}$$

so clearly enlightening the implicit cascade structure the time-delay system (5.1) lays on. Thus, the problem of stabilizing the origin of (5.1) is led to the one of stabilizing (5.2) through boundary control. In this sense, the equivalent PDE representation (5.2) has allowed to deduce nice proofs and alternative ways of defining predictor-based feedback laws. Though, huge practical and computational issues still remain unsolved in this case because of the difficulties arising with (5.2), even in the LTI case.

Remark 5.1. *An easier (but approximate) cascade representation of the dynamics (5.1) is given by exploiting the first-order* Padé approximation *of the delay-transfer function; i.e.,*

$$e^{-\tau s} \approx \frac{1 - \frac{\tau s}{2}}{1 + \frac{\tau s}{2}}.$$

Accordingly, as τ is small enough, (5.1) is equivalent to the (feedforward) cascade dynamics

$$\begin{aligned} \dot{x}(t) =& f(x(t)) - g(x(t))v(t) + \frac{4}{\tau} g(x(t))u(t) && (5.3a) \\ \dot{v}(t) =& -\frac{2}{\tau} v(t) + u(t). && (5.3b) \end{aligned}$$

5.1.2 Predictors for continuous-time dynamics

The easiest way of stabilizing an input-delayed system is based on compensating the effect of the delay over (5.1). To this end, one computes the prediction of the future state trajectories τ units of time ahead by introducing the predictor state as the functional

$$\zeta(t) = x(t+\tau), \quad t \geq \tau. \tag{5.4}$$

Accordingly, one gets the predictor dynamics as a mere copy of the delay-free (1.1) as given by

$$\dot{\zeta}(t) = f(\zeta(t)) + g(\zeta(t))u(t) \tag{5.5}$$

so that any feedback

$$u(t) = k(\zeta(t)) \tag{5.6}$$

ensuring stability of the origin of (5.5) in closed-loop will ensure stability of the one of (5.1) as well provided that (5.4) is suitably initialized. By nature, the control (5.6) is typically computed over the ideal delay-free system (1.1) and is assumed to be a-priori known.

Thus, predictor-based feedback can be applied to nonlinear time-delay systems in a straightforward manner. Though, in this kind of design, several problems arise:

1. first of all, suitably initializing (5.4) is not a trivial task as, usually, the initial condition defining (5.1) is defined over an infinite dimensional function space depending on the story of x and of u over the time window $[-\tau, 0]$;

2. computing a closed-loop form for the mapping (5.4) might not be possible as it requires closed-form solutions for (5.1) to be available although this is hardly the case in dynamical systems;

3. usually, an infinite dimensional buffer needs to be available for storing all the past values of the state and control inputs over the time window $[t-\tau, t]$ with $t \geq 0$.

As far as the initial condition problem is concerned, several works have been proposing the introduction of estimates of $x(t+\tau)$ governed by exponentially convergence observers. Albeit the impact of this approach, very few results are available in the general context as they mainly refer to linear or some very restrictive classes of nonlinear systems (e.g., [14]).
The problem of exactly computing the future trajectories for (5.1) is quite tough although solvability has been proven for classes of nonlinear systems which are equivalent, up to a coordinate transformation, to the following cases:

- LTI dynamics of the form

  $$\dot{x}(t) = Ax(t) + Bu(t-\tau)$$

  for which one gets

  $$\zeta(t) = e^{A\tau}x(t) + \int_{t-\tau}^{t} e^{(t-s)A}Bu(s)\mathrm{d}s;$$

- bilinear systems

$$\dot{x}(t) = Ax(t) + (Nx(t) + B)u(t - \tau)$$

with predictor mapping provided by

$$\zeta(t) = e^{A\tau + N\int_{t-\tau}^{t} u(s)\mathrm{d}s}x(t) + \int_{t-\tau}^{t} e^{(t-s)A + N\int_{t-s-\tau}^{t-s} u(\ell-\tau)\mathrm{d}\ell}Bu(s)\mathrm{d}s;$$

- strict-feedforward dynamics as in the special case

$$\begin{aligned}
\dot{x}_n(t) =& f_n(x_1(t), \ldots, x_{n-1}(t)) + g_n(x_1(t), \ldots, x_{n-1}(t))u(t - \tau) \\
&\vdots \\
\dot{x}_2(t) =& f_2(x_1(t)) + g_2(x_1(t))u(t - \tau) \\
\dot{x}_1(t) =& Ax_1(t) + Bu(t - \tau)
\end{aligned}$$

so getting

$$\begin{aligned}
\zeta_n(t) =& x_n(t) + \int_{t-\tau}^{t} [f_n(\zeta_1(s), \ldots, \zeta_{n-1}(s)) + g_n(\zeta_1(s), \ldots, \zeta_{n-1}(s))u(s)]\mathrm{d}s \\
&\vdots \\
\zeta_2(t) =& x_2(t) + \int_{t-\tau}^{t} [f_2(\zeta_1(s)) + g_2(\zeta_1(s))u(s)]\mathrm{d}s \\
\zeta_1(t) =& e^{A\tau}x_1(t) + \int_{t-\tau}^{t} e^{(t-s)A}Bu(s)\mathrm{d}s.
\end{aligned}$$

For other classes of dynamics that do not admit a closed-form form of the predictor mapping, one can apply approximations by exploiting the properties of (5.1). As discussed in [71], approximate predictors can be based on three main families of approaches mainly exploiting

- numerical approximation schemes (e.g., [67]);
- nested approximation strategies applying whenever (5.1) is globally Lipschitz (e.g., [60, 66]);
- suitably defined dynamical systems for generating the approximate predictor (e.g., [1, 70]).

Though, generally, those problems still require an infinite dimensional buffer for storing required values of the control and state and, more importantly, only embed a few among the many situations one might encounter. Moreover, by nature, any prediction-based feedback $u(t) = \gamma(\zeta(t))$ comes to be the implicit solution to some functional-integral equality as $\zeta(t)$ is implicitly defined by the past story of the control itself.

Thus, one other possible way of deducing non-exact numerical predictors is based on sampling the states and the control input fast enough (with respect to the length of the delay) to approximately compute the predictor mapping through numerical integration. Basically, by assuming δ small enough, one gets the closed-loop predictor dynamics

$$\begin{aligned}\dot{\tilde{\zeta}}(t) &= f(\tilde{\zeta}(t)) + g(\tilde{\zeta}(t))k(\tilde{\zeta}(k\delta)), \quad t \in [k\delta, (k+1)\delta[\\ \tilde{\zeta}(k\delta) &:= \Phi(x(k\delta), u(k\delta), \ldots, u((k-N)\delta))\end{aligned}$$

where $N \in \mathbb{N}$ is a suitable integer to be specified. In that case, a trade off is needed among the required performances, the relation among the delay and the sampling period and the computational issues that might arise in the approximation scheme. Moreover, we underline that this approach does not rely on the sampled-data framework we have been adopted. As a matter of fact, the control is here only sampled while it still assumed to be a fully continuous-time signal (and, thus, not piecewise constant) affecting (5.1).

In the next few parts, we shall show how settling the problem in a completely sampled-data framework notably simplifies the design and allows to deduce predictor-based mappings (even in an approximate sense) which might work in general for large delays and for a huge variety of dynamical systems.

5.2 Cascade forms for input-delayed dynamics under sampling

Assume now that (5.1) is a sampled-data system in the usual sense; i.e., the following holds:

- measures of the states are available only at the sampling instants $t = k\delta$ for $k \geq 0$;
- the control is piecewise constant over time-intervals of length δ.

In this context, one can consider assume the case of *non-entire* delay whenever τ is a non-entire multiple of the sampling period δ; i.e.,

$$\tau = N\delta + \sigma \tag{5.7}$$

for some known constants $N \in \mathbb{N}$ and $\sigma \in [0, \delta[$. We are saying that (5.1) is affected by an *entire delay* whenever $\sigma = 0$ in (5.7). Accordingly, we are now providing a cascade representation for the sampled-data time-delay system (5.1) by distinguishing among the two cases by also exhibiting a suitable sampled-data equivalent model for the equivalent dynamics.

5.2.1 The case of an entire delay

Assume (5.1) is a sampled-data system affected by an entire delay

$$\tau = N\delta \tag{5.8}$$

for some known $N \in \mathbb{N}$. Then, because of the piecewise constant nature of the control signal $u : [-\tau, 0] \cup \mathbb{R}_{\geq 0}$, one can describe the corresponding story as a discrete-time LTI dynamics of the form

$$v_{k+1} = \hat{A} v_k + \hat{B} u_k \tag{5.9}$$

where $v \in \mathbb{R}^N$ with

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \quad \text{and} \quad v_{ik} := u((k - N + i - 1)\delta), \quad i = 1, \ldots, N \tag{5.10}$$

and $(\hat{A}, \hat{B})$ being in the Brunowskii form

$$\hat{A} = \begin{pmatrix} \mathbf{0}_{(N-1)\times 1} & \mathrm{I} \\ 0 & \mathbf{0}_{1\times(N-1)} \end{pmatrix} \quad \text{and} \quad \hat{B} = \begin{pmatrix} \mathbf{0}_{(N-1)\times 1} \\ 1 \end{pmatrix}. \tag{5.11}$$

Accordingly, (5.1) rewrites, over an extended state-space, as the *hybrid dynamics*

$$\dot{x}(t) = f(x(t)) + g(x(t)) v_{1k}, \quad t \in [k\delta, (k+1)\delta[\tag{5.12a}$$

$$v_{k+1} = \hat{A} v_k + \hat{B} u_k \tag{5.12b}$$

or, more explicitly, as

$$\begin{aligned} \dot{x}(t) =& f(x(t)) + g(x(t)) v_{1k}, \quad t \in [k\delta, (k+1)\delta[\\ v_{1k+1} =& v_{2k} \\ &\vdots \\ v_{Nk+1} =& u_k. \end{aligned}$$

Thus, the time-delay dynamics (5.1) rewrites as the strict-feedback cascade interconnection of the interval continuous-time dynamics (5.12a) with the LTI discrete-time one (5.12b).

By integrating (5.12a) over $[k\delta, (k+1)\delta[$ with initial condition $x_k = x(k\delta)$, one gets the extended and finite-dimensional sampled-data equivalent model of (5.1) as the strict-feedback form below

$$x_{k+1} = F^\delta(x_k, v_{1k}) \tag{5.14a}$$

$$v_{k+1} = \hat{A} v_k + \hat{B} u_k \tag{5.14b}$$

with

$$F^\delta(x, v_1) = e^{\delta(\mathrm{L}_f + v_1 \mathrm{L}_g)} x.$$

Remark 5.2. *As $\tau = 0$ and, thus, $N = 0$, (5.14a) recovers the delay-free sampled-data equivalent model (2.4a).*

Remark 5.3. *We underline that, whenever $\dot{x} = f(x)$ verifies Assumption 4.1 (i.e., it possesses a GS equilibrium with a weak Lyapunov function $W(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$), the hybrid dynamics (5.12) exhibits a feedforward-like structure with the origin of (5.12b) being GES. Accordingly, one might apply the design procedure proposed in Chapter 4 to deduce a stabilizing procedure over the corresponding sampled-data equivalent model (5.14) that indeed preserves the required form.*

5.2.2 The case of a non-entire delay

Assume (5.1) is a sampled-data system affected by an entire delay (5.7) for some known $N \in \mathbb{N}$ and $\sigma \in [0\,\delta[$. Then, because of the piecewise constant nature of the control signal $u : [-\tau, 0] \cup \mathbb{R}_{\geq 0}$, one can describe the corresponding story as a discrete-time LTI dynamics of the form

$$\begin{aligned} v_{0k+1} =& v_{1k} \\ v_{k+1} =& \hat{A}v_k + \hat{B}u_k \end{aligned}$$

where $v \in \mathbb{R}^N$ and $(\hat{A}, \hat{B})$ being in the Brunowskii form as in (5.10) and

$$v_{0k} = u((k - N - 1)\delta).$$

Accordingly, (5.1) rewrites, over an extended state-space, as the *hybrid dynamics*

$$\dot{x}(t) = f(x(t)) + g(x(t))v_{0k}, \quad t \in [k\delta, k\delta + \sigma[\tag{5.15a}$$
$$\dot{x}(t) = f(x(t)) + g(x(t))v_{1k}, \quad t \in [k\delta + \sigma, (k+1)\delta[\tag{5.15b}$$
$$v_{0k+1} = v_{1k} \tag{5.15c}$$
$$v_{k+1} = \hat{A}v_k + \hat{B}u_k. \tag{5.15d}$$

By integrating (5.15a)-(5.15b) over $[k\delta, (k+1)\delta[$ with initial condition $x_k = x(k\delta)$, one gets the extended and finite-dimensional sampled-data equivalent model of (5.1) as the strict-feedback form below

$$x_{k+1} = F^{\delta}(\sigma, x_k, v_{0k}, v_{1k}) \tag{5.16a}$$
$$v_{0k+1} = v_{1k} \tag{5.16b}$$
$$v_{k+1} = \hat{A}v_k + \hat{B}u_k \tag{5.16c}$$

with

$$F^{\delta}(\sigma, x, v_0, v_1) = F^{\delta-\sigma}(\cdot, v_1) \circ F^{\sigma}(x, v_0) = e^{\sigma(\mathrm{L}_f + v_0\mathrm{L}_g)} \circ e^{(\delta-\sigma)(\mathrm{L}_f + v_1\mathrm{L}_g)}x$$

and

$$x(k\delta + \sigma) = F^{\sigma}(x_k, v_{0k}).$$

Thus, again, the time-delay dynamics (5.1) under sampling rewrites as the strict-feedback cascade interconnection of (5.16a)-(5.16b) with the LTI dynamics (5.16c).

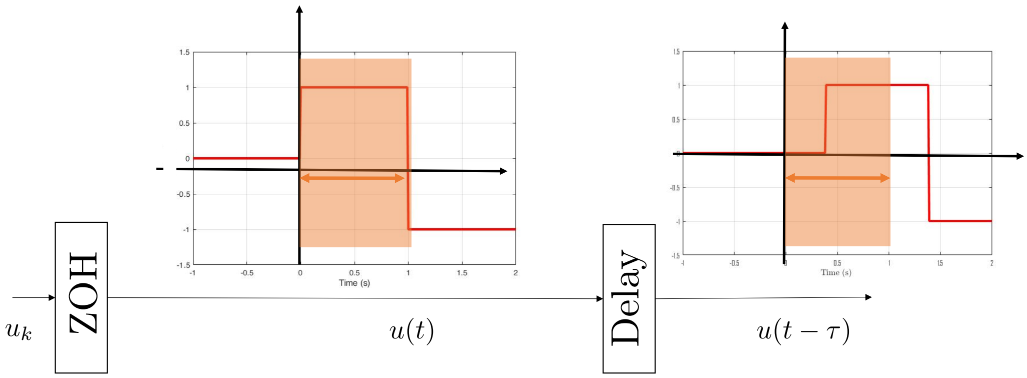


Figure 5.1: Effect of non-entire delays over sampled-data systems with $\tau = \sigma$ and $\delta = 1$ s and $u(t) = 0$ for $t \in [-\tau, 0[$.

As a drawback, it is clear from (5.16a), that the discrete nature of the time-delay acting over the continuous-time system (5.1) is mapped, through sampling, to a distributed delay acting over its sampled-data equivalent model (5.16). As a matter of fact, in the time-delay domain, one gets

$$x_{k+1} = F^{\delta}(\sigma, x_k, u_{k-N-1}, u_{k-N}).$$

This implies that, through out any sampling period $[k\delta, (k+1)\delta[$ the continuous-time system (5.1) is affected by two retarded values of the control. Still, the action of the two retarded signals is not simultaneous as (5.1) is affected by u_{k-N-1} and u_{k-N} over, respectively, $t \in [k\delta, k\delta + \sigma[$ and $t \in [k\delta + \sigma, (k+1)\delta[$. This is illustrated in Figure 5.1 when setting $N = 0$ and $\delta = 1$ second.

Remark 5.4. *When $\sigma = 0$, one recovers the extended sampled-data equivalent model (5.14). Moreover, as $\tau = 0$ and, thus, both $N = 0$ and $\sigma = 0$, (5.16a) recovers the delay-free sampled-data equivalent model (2.4a).*

In the next section, we are proposing a way of computing prediction-based stabilization schemes for stabilizing the origin of (5.1) based on its sampled-data equivalent model. Specifically, we shall exhibit a discrete-time predictor (i.e., evolving as a discrete-time dynamics) and a corresponding sampled-data stabilizer for (5.1). In doing so, we shall comment on approximate predictor-dynamics and, especially, approximate sampled-data prediction-based feedback laws. In the next chapter, then, the cascade structure will be exploited to deduce an Immersion and Invariance improved feedback. Other control laws exploiting the feedback or feedforward structure of the extended model (5.14) might be designed as well.

5.3 Prediction-based stabilization under sampling

In this section, assuming continuous-time stabilizability of the delay-free continuous -time system (1.1a) associated to (5.1), we are going to deduce a sampled-data stabilizing feedback for the origin of (5.1) based on prediction. In doing so, we shall proceed in two steps by:

1. constructing a sampled-data feedback $u_k = \gamma^\delta(x_k)$ over the delay-free system (1.1a) making its origin S-GAS;
2. exhibiting a discrete-time prediction dynamics for (5.1) and a corresponding feedback making the origin of the retarded system (5.1) S-GAS in closed-loop.

To this end, let us introduce this standing assumption over the continuous-time delay-free system (1.1a) associated to (5.1).

Assumption 5.1 (Continuous-time delay-free smooth stabilizability)**.** *The delay-free system (1.1a) associated to (5.1) verifies Assumption 2.1; i.e., the origin of the continuous-time dynamics (1.1a) is globally asymptotically stabilized by a smooth feedback $u(t) = \gamma(x(t))$ with radially unbounded and strict-Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that*

$$\mathrm{L}_f V(x) + \gamma(x)\mathrm{L}_g V < 0 \quad \text{and} \quad \mathrm{L}_g V(x) \neq 0 \tag{5.17}$$

for any $x \neq 0$.

Remark 5.5. *Assumption 5.1 might be weakened or modified so to require the sampled-data stabilizability of (1.1a) through other arguments including, as an example, u-average passivity or other design methodologies that are applicable over the sampled-data equivalent model of the delay-free system (1.1a).*

As a straightforward consequence of Assumption 5.1, one can construct a sampled-data feedback stabilizing the origin of (2.4a) through Input-Lyapunov Matching by applying Theorem 2.4 and compute $u_k = \gamma^\delta(x_k)$ as the solution to the ILM equality

$$V(F^\delta(x_k, u_k)) - V(x_k) = \int_{k\delta}^{(k+1)\delta} \left[\mathrm{L}_{f(\cdot)} + \gamma(\cdot)\mathrm{L}_{g(\cdot)}\right] V(x(s))\mathrm{d}s \tag{5.18}$$

with $x(s) = e^{s(\mathrm{L}_{f(\cdot)}+\gamma(\cdot)\mathrm{L}_{g(\cdot)})}x\big|_{x_k}$. Accordingly, the sampled-data feedback $u_k = \gamma^\delta(x_k)$ ensures S-GAS of the origin of the delay-free (1.1a) associated to (5.1) as $\tau = 0$.

Remark 5.6. *Assumption 5.1 requires the existence of a Lyapunov-based continuous-time feedback for the delay-free system associated to (5.1). This requirement is generally much weaker than the ones usually introduced for prediction-based stabilization of sampled-data time-delay systems. As a matter of fact, assumptions for prediction-based stabilization of sampled-data system might require the existence of*

- *a feedback ensuring GAS of a certain (usually the Euler) approximate discrete-time equivalent model associated to the delay-free (2.4a);*
- *a sampled-data feedback ensuring GAS of (1.1a) for δ small enough as in the methodologies lying within' the continuous-time redesign framework.*

For further details on this, the reader is referred to [64] where the case of delayed measurements is addressed as well.

Now, based on Assumption 5.1, we shall now first provide the prediction-based feedback for (5.1) in case of an entire delay (5.8) acting over (5.1).

5.3.1 The case of an entire delay

The stabilizing procedure we propose here is aimed at compensating the effect of the delay acting over (5.1) at any sampling instant $t = k\delta$, $k \geq 0$. For this purpose, assuming the case of an entire delay (5.8), we define the discrete-time predictor over (5.14) as

$$\zeta_k = x(k\delta + \tau) = x_{k+N}$$

with initial condition $x_k := x(k\delta)$ so taking the explicit *exact* form

$$\begin{aligned} \zeta_k =& F^\delta(\cdot, v_{Nk}) \circ \cdots \circ F^\delta(x_k, v_{1k}) \\ =& e^{\delta(\mathrm{L}_f + v_{1k}\mathrm{L}_g)} \circ \cdots \circ e^{\delta(\mathrm{L}_f + v_{Nk}\mathrm{L}_g)} x\big|_{x_k} \end{aligned} \tag{5.19}$$

with $v_{i0} = u_{-N+i-1}$ for $i = 1, \ldots, N$. Accordingly, one computes the predictor dynamics as

$$\zeta_{k+1} = F^\delta(\zeta_k, u_k) \tag{5.20}$$

which is a mere copy of the delay-free sampled-data equivalent (2.4a) when computed N steps ahead in time. As a consequence, one immediately gets the following result. As far as stabilization is concerned, any feedback ensuring sampled-data stabilization of the delay-free system (2.4a) ensures, when suitably implemented over the prediction state ζ_k, stabilization of the prediction dynamics (5.20) and, hence, of (5.14). This is stated by the following result.

Theorem 5.1 (Stability under entire delay via prediction)**.** *Let (5.1) verify Assumption 5.1 and be affected by an entire delay (5.8). Consider its sampled-data equivalent model (5.14) and let $\gamma^\delta : \mathbb{R}^n \to \mathbb{R}$ be the unique solution to the ILM equality (5.18). Then, the feedback $u_k = \gamma^\delta(\zeta_k)$ with ζ_k as in (5.19) ensures GAS of the origin of (5.14) and, thus, S-GAS of (5.1).*

The above theorem states that the existence of a sampled-data feedback stabilizing the retarded system (5.1) can be directly inferred whenever a delay-free smooth stabilizing feedback exists for the continuous time delay-free system (1.1a).

Remark 5.7. *It is a matter of computation to verify that, when implemented over ζ, the ILM-based feedback solution to (5.18) ensures N-steps ahead matching, under both sampling and delay, of the continuous-time and delay-free Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. Namely, it solves the following retarded version of the ILM equality*

$$V(x_{k+N+1}) - V(x_{k+N}) = \int_{k\delta}^{(k+1)\delta} \left[\mathrm{L}_{f(\cdot)} + \gamma(\cdot)\mathrm{L}_{g(\cdot)}\right] V(x(s))\mathrm{d}s \tag{5.21}$$

with

$$x(s) = e^{(s-k\delta)(\mathrm{L}_{f(\cdot)}+\gamma(\cdot)\mathrm{L}_{g(\cdot)})} x\big|_{x_k}$$

so implying, in turn, GAS of the origin of (5.14) and, thus, S-GAS of (5.1) with

$$\begin{aligned} V(x_{k+N+1}) =& V(\cdot) \circ F^\delta(\cdot, u_k) \circ F^\delta(\cdot, v_{Nk}) \circ \cdots \circ F^\delta(x_k, v_{1k}) \\ =& e^{\delta(\mathrm{L}_f + v_{1k}\mathrm{L}_g)} \circ \cdots \circ e^{\delta(\mathrm{L}_f + v_{Nk}\mathrm{L}_g)} \circ e^{\delta(\mathrm{L}_f + u_k\mathrm{L}_g)} V(x_k). \end{aligned}$$

Remark 5.8. *Because of the matching property, the Lyapunov function*

$$\mathcal{V}(x, v) := e^{\delta(\mathrm{L}_f + v_1\mathrm{L}_g)} \circ \cdots \circ e^{\delta(\mathrm{L}_f + v_N\mathrm{L}_g)} V(x)$$

is a strict and radially unbounded Lyapunov function for the extended dynamics (5.14) under the feedback $u_k = \gamma^\delta(\zeta_k)$ with ζ as in (5.19).

As a drawback, the prediction-based feedback presented in Theorem 5.1 generally works in open loop with respect to possible prediction errors. As a consequence, those kind of control laws suffer, in general, from robustness with respect to prediction errors that unavoidably arise from possible approximations of the delay-free feedback $u = \gamma^\delta(x)$ or from neglected higher order (in δ) dynamics in the sampled-data equivalent model (5.14). This issue is of paramount importance in sampled-data control and cannot be discarded. To this end, we shall propose a possible solution based on I&I in the next chapter.

In what follows, some computational aspects are discussed.

5.3.1.1 Computational aspects and approximate solutions

Contrarily to the work in [64], we compute a sampled-data prediction-based feedback which is aimed at compensating the effect of the delay at the sampling instants only. As a consequence, the following differences hold as well:

1. the predictor evolves according to a discrete-time dynamics at any sampling instants;

2. only a finite N-dimensional buffer is required to store the N past values of the control signal v_i (for $i = 1, \dots, N$) which are indeed used to compute the stabilizing feedback.

These aspects take into account the piecewise constant nature of the control signal acting over (5.1) and underline that a continuous (i.e., at any time instant $t \geq 0$) prediction of the future states is unnecessary. Moreover, the feedback $u_k = \gamma^\delta(\zeta_k)$ is no longer implicitly defined as ζ_k only depends on a finite number of past values of the control signal itself.

Still, the problem of exactly computing (5.19) remains although similar considerations holding for classical sampled-data systems hold true so that approximations are naturally introduced in this extended sampled-data retarded framework. As a matter of fact, one might compute approximations of the predictor (5.19) or of its corresponding dynamics as developed in Chapter 2 through their series expansion in powers of δ and τ. Though, as typical of the sampled-data design approach we have been presenting, we are rather interested in approximate solutions of the prediction-based feedback $u_k = \gamma^\delta(\zeta_k)$ in both powers of δ and τ. In this scenario, one has two deal with two coupled sources of approximations:

- approximations of $\gamma^\delta(\cdot)$ as solution to the delay-free ILM problem in (5.18) in powers of δ;

- approximations of $\gamma^\delta(\cdot)$ as solution to the time-delay ILM problem (5.21) through approximations of the prediction state ζ in (5.19) in both powers of δ and τ.

In this sense, assuming any a-priori computed approximate solution $\gamma^{\delta[p]}(x)$ to the delay-free ILM problem, we are going to exhibit approximate based-solutions to the retarded (5.21) and, thus, approximate sampled-data prediction-based feedbacks.

To this end and for the sake of clarity, we shall first rewrite the predictor-feedback in terms of the state (x, v) as

$$\gamma^\delta(\zeta) = K^\delta(\tau, x, v). \tag{5.22}$$

By substituting $\delta = \frac{\tau}{N}$ in (5.19) and exploiting the Lie-exponential form, one gets

$$K^\delta(\tau, x, v) = e^{\frac{\tau}{N}(\mathrm{L}_f + v_1 \mathrm{L}_g)} \circ \dots \circ e^{\frac{\tau}{N}(\mathrm{L}_f + v_N \mathrm{L}_g)} \gamma^\delta(x) \tag{5.23}$$

so showing that (5.22) itself can be expressed as a power series expansion in powers of τ as

$$K^\delta(\tau, x, v) = K_0^\delta(x, v) + \sum_{i>0} \frac{\tau^i}{N^i i!} K_i^\delta(x, v) \tag{5.24}$$

with any i^{th}-order term still parametrized by the sampling period δ and provided by

$$\begin{aligned}
K_0^\delta(x,v) =& \gamma^\delta(x), \quad K_1^\delta(x,v) = \sum_{i=1}^{N}(\mathrm{L}_f + v_i\mathrm{L}_g)\gamma^\delta(x)\\
K_2^\delta(x,v) =& \sum_{i=1}^{N}(\mathrm{L}_f + v_i\mathrm{L}_g)^2\gamma^\delta(x) + 2!\sum_{i_1=1}^{N}\sum_{i_2=i_1+1}^{N}(\mathrm{L}_f + v_{i_1}\mathrm{L}_g)(\mathrm{L}_f + v_{i_2}\mathrm{L}_g)\gamma^\delta(x)\\
K_i^\delta(x,v) =& \sum_{i_1=1}^{N}\cdots\sum_{i_i=i_{i-1}}^{N} a_i(i_1,\ldots,i_i)(\mathrm{L}_f + v_{i_1}\mathrm{L}_g)\ldots(\mathrm{L}_f + v_{i_i}\mathrm{L}_g)\gamma^\delta(x)
\end{aligned} \tag{5.25}$$

for computable positive coefficients $a_i(i_1,\ldots,i_i) \geq 0$.

Remark 5.9. *The series expansion (5.23) of the predictor-based feedback (5.22) underlines that, as $\tau \to 0$, one recovers the delay-free ILM-based control in Theorem 2.4; i.e., $K^\delta(0,x,v) = \gamma^\delta(x)$.*

Recalling that the mapping $\gamma^\delta(\cdot)$ rewrites itself as the series expansion (2.36) in powers of δ, one can easily expand any term $K_i^\delta(x,v)$ in (5.25) as

$$\begin{aligned}
K_0^\delta(x,v) =& \gamma^0(x) + \sum_{j\geq 1}\frac{\delta^j}{(j+1)!}\gamma^j(x)\\
K_1^\delta(x,v) =& \sum_{i=1}^{N}\sum_{j\geq 0}\frac{\delta^j}{(j+1)!}(\mathrm{L}_f + v_i\mathrm{L}_g)\gamma^j(x)\\
&\ldots .
\end{aligned}$$

Accordingly, one can deduce approximate prediction-based feedbacks as truncations of (5.24) at any fixed order $q > 0$ in τ and under delay-free ILM approximate solutions (2.39).

Definition 5.1. Given an approximate p^{th}-order solution $\gamma^{\delta[p]}(x)$ of the form (2.39), we define the $[p,q]^{th}$*-order approximate solution* to the retarded ILM equality (5.21) as

$$K^{\delta[p,q]}(\tau,x,v) = K_0^{\delta[p]}(x,v) + \sum_{i=1}^{q}\frac{\tau^i}{N^i i!}K_i^{\delta[p]}(x,v) \tag{5.26}$$

with

$$\begin{aligned}
K_0^{\delta[p]}(x,v) =& \gamma^{\delta[p]}(x), \quad K_1^{\delta[p]}(x,v) = \sum_{i=1}^{N}(\mathrm{L}_f + v_i\mathrm{L}_g)\gamma^{\delta[p]}(x)\\
K_2^{\delta[p]}(x,v) =& \sum_{i=1}^{N}(\mathrm{L}_f + v_i\mathrm{L}_g)^2\gamma^\delta(x) + 2!\sum_{i_1=1}^{N}\sum_{i_2=i_1+1}^{N}(\mathrm{L}_f + v_{i_1}\mathrm{L}_g)(\mathrm{L}_f + v_{i_2}\mathrm{L}_g)\gamma^{\delta[p]}(x)\\
K_i^{\delta[p]}(x,v) =& \sum_{i_1=1}^{N}\cdots\sum_{i_i=i_{i-1}}^{N} a_i(i_1,\ldots,i_i)(\mathrm{L}_f + v_{i_1}\mathrm{L}_g)\ldots(\mathrm{L}_f + v_{i_i}\mathrm{L}_g)\gamma^{\delta[p]}(x).
\end{aligned}$$

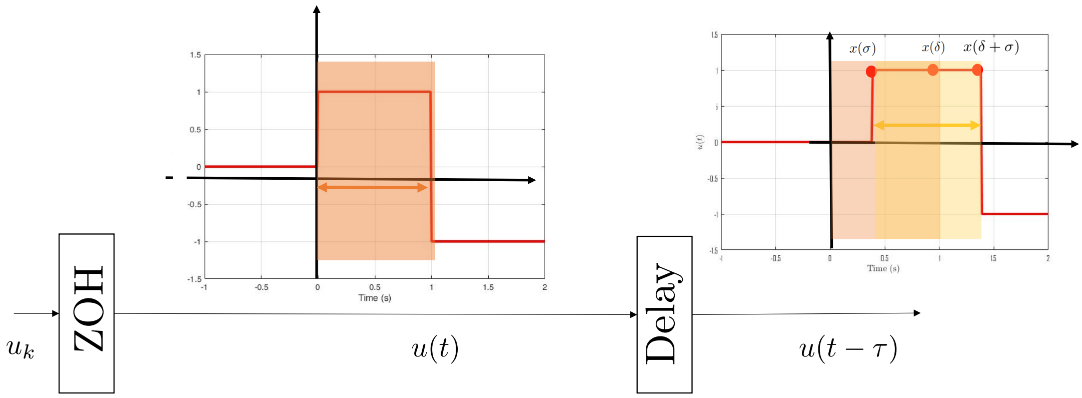


Figure 5.2: Effect of non-entire delays over sampled-data systems with $\tau = \sigma$ and $\delta = 1$ s and $u(t) = 0$ for $t \in [-\tau, 0[$.

Along the lines of the results in the delay-free case, one can verify that the approximate sampled-data feedback $u_k = K^{\delta[p,q]}(\tau, x_k, v_k)$ ensures pGAS of the origin of (5.14) and, thus, S-pGAS of the one time-delay system (5.1); namely, all the solutions of (5.1) will converge, in closed-loop, to a ball centered at the origin and with radius in $O(\delta^p(1 + \tau^q))$.

Remark 5.10. *Closed or finitely computable forms for the predictor (5.19) are available whenever the sampled-data equivalent model of the delay-free system associated to (5.1) admits a closed-form sampled-data equivalent model or is finitely discretizable in the sense of Definition 2.3.*

5.3.2 The case of non-entire delay

When dealing with non-entire delays of the form $\tau = N\delta + \sigma$ as in (5.7), things get complicated as the sampled-data equivalent model (5.16) computed over two successive sampling instants gets to be distributively influenced by the delayed control signal. Accordingly, naively setting

$$\zeta_k := x(k\delta + \tau)$$

with initial condition $x_k := x(k\delta)$ provides the predictor dynamics

$$\zeta_{k+1} = F^{\delta}(\sigma, \zeta_k, v_{Nk}, u_k)$$

with $v_{Nk} = u_{k-1}$. Thus, the above choice does not define a predictor as its corresponding dynamics is not free of delays and, in particular, a copy of the delay-free system (2.4a). As already commented on, this is due to the fact that throughout any sampling interval $[k\delta, (k+1)\delta[$, two retarded values of the control signal are affecting the retarded system (5.1). Though, by virtually shifting the point of view by σ units of time, one might notice that, throughout any interval $[k\delta + \sigma, (k+1)\delta + \sigma[$ this pathology is over and only the retarded signal u_{k-N} is acting over the corresponding plant (5.1). This fact is illustrated in Figure 5.2. This corresponds to first moving the initial state of the sampled-data dynamics (5.16) by σ units of time as

$$\tilde{x}_k : x(k\delta) \mapsto x(k\delta + \sigma) \tag{5.27}$$

$$\tilde{x}_k = F^{\sigma}(x_k, v_{0k}) \tag{5.28}$$

with $v_{0k} = u_{k-N-1}$ and evolving, over time-intervals of length δ and initial condition $\tilde{x}_k := x(k\delta + \sigma)$, as

$$\tilde{x}_{k+1} = F^{\delta}(\tilde{x}_k, v_{1k}) \tag{5.29}$$

which is indeed affected by a discrete delay $v_{1k} := u_{k-N}$. Hence, one can deduce the over all predictor by computing the shifted trajectories (5.29) N-steps ahead so getting with initial condition $\tilde{x}_k := x(k\delta + \sigma)$; namely, one sets

$$\zeta_k = \tilde{x}_{k+N} = x((k+N)\delta + \sigma)$$

explicitly provided by

$$\begin{aligned} \zeta_k =& F^{\delta}(\cdot, v_{Nk}) \circ \cdots \circ F^{\delta}(\tilde{x}_k, v_{1k}) \\ =& e^{\delta(\mathrm{L}_f + v_{1k}\mathrm{L}_g)} \circ \cdots \circ e^{\delta(\mathrm{L}_f + v_{Nk}\mathrm{L}_g)} x\big|_{\tilde{x}_k} \end{aligned} \tag{5.30}$$

and evolving as

$$\zeta_{k+1} = F^{\delta}(\zeta_k, u_k) \tag{5.31}$$

which is indeed a copy of the delay-free sampled-data equivalent model (2.4a) computed over (5.30).

Because measures of the state are provided only at the sampling instants $t = k\delta$, $\tilde{x}_k = x(k\delta + \sigma)$ is not available, in general, and needs to be predicted by using the expression (5.27). As a consequence, (5.27) represents a preliminary predictor compensating the non-entire part of the delay. As a consequence, the overall predictor rewrites, in terms of the available measures of the state $x_k = x(k\delta)$, as

$$\begin{aligned} \zeta_k =& F^{\delta}(\cdot, v_{Nk}) \circ \cdots \circ F^{\delta}(\cdot, v_{1k}) \circ F^{\sigma}(x_k, v_{0k}) \\ =& e^{\sigma(\mathrm{L}_f + v_{0k}\mathrm{L}_g)} \circ e^{\delta(\mathrm{L}_f + v_{1k}\mathrm{L}_g)} \circ \cdots \circ e^{\delta(\mathrm{L}_f + v_{Nk}\mathrm{L}_g)} x\big|_{x_k}. \end{aligned}$$

Remark 5.11. *Closed or finitely computable forms for the predictor (5.30) are available whenever the sampled-data equivalent model of the delay-free system associated to (5.1) admits a closed-form sampled-data equivalent model or is finitely* *discretizable in the sense of Definition 2.3; namely, whenever the Lie exponential* $e^{\delta(\mathrm{L}_f+u_k\mathrm{L}_g)}Id$ *is admits a closed-form or rather a finite number of terms in powers of* δ.

It follows, again, that any sampled-data feedback stabilizing the origin of delay-free sampled-data equivalent model (2.4a) stabilizes the one of predictor dynamics when computed over (5.30) and, thus, the origin of the retarded (5.16a). The following statement summarizes the result by underlying, once again, that smooth stabilizability of the delay-free continuous-time system (2.4a) associated to (5.1) is enough to deduce a stabilizing sampled-data predictor feedback for (5.1).

Theorem 5.2 (Stability under non-entire delay via prediction)**.** *Let the retarded (5.1) verify Assumption 5.1 and be affected by an entire delay (5.7). Consider its sampled-data equivalent model (5.16) and let* $\gamma^\delta : \mathbb{R}^n \to \mathbb{R}$ *be the unique solution to the ILM equality (5.18). Then, the feedback* $u_k = \gamma^\delta(\zeta_k)$ *with* ζ_k *as in (5.30) ensures GAS of the origin of (5.16) and, thus, GAS of (5.1) at any time-instant* $t = k\delta + \sigma$ *and* $k \geq 0$.

The following remarks hold true even in the non-entire case.

Remark 5.12. *It is a matter of computation to verify that, when implemented over* ζ, *the ILM-based feedback solution to (5.18) ensures* $N+1$*-steps ahead matching, under both sampling and delay, of the continuous-time and delay-free Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. *Namely, it solves the following retarded version of the ILM equality*

$$V(\tilde{x}_{k+N+1}) - V(\tilde{x}_{k+N})) = \int_{k\delta+\sigma}^{(k+1)\delta+\sigma} \big[\mathrm{L}_{f(\cdot)} + \gamma(\cdot)\mathrm{L}_{g(\cdot)}\big] V(x(s))\mathrm{d}s \tag{5.32}$$

with

$$x(s) = e^{(s-k\delta-\sigma)(\mathrm{L}_{f(\cdot)}+\gamma(\cdot)\mathrm{L}_{g(\cdot)})}x\big|_{\tilde{x}_k}$$

so implying, in turn, GAS of the origin of (5.14) and, thus, S-GAS of (5.1) with

$$\begin{aligned} V(\tilde{x}_{k+N+1}) =& V(\cdot) \circ F^\delta(\cdot, u_k) \circ F^\delta(\cdot, v_{Nk}) \circ \cdots \circ F^\delta(\cdot, v_{1k}) \circ F^\sigma(x_k, v_{0k}) \\ =& e^{\sigma(\mathrm{L}_f+v_{0k}\mathrm{L}_g)} \circ e^{\delta(\mathrm{L}_f+v_{1k}\mathrm{L}_g)} \circ \cdots \circ e^{\delta(\mathrm{L}_f+v_{Nk}\mathrm{L}_g)} \circ e^{\delta(\mathrm{L}_f+u_k\mathrm{L}_g)} V(x_k). \end{aligned}$$

Remark 5.13. *Because of matching, the Lyapunov function*

$$\mathcal{V}(x, v_0, v) := e^{\sigma(\mathrm{L}_f+v_0\mathrm{L}_g)} \circ e^{\delta(\mathrm{L}_f+v_1\mathrm{L}_g)} \circ \cdots \circ e^{\delta(\mathrm{L}_f+v_N\mathrm{L}_g)} V(x)$$

is a strict and radially unbounded Lyapunov function for the extended dynamics (5.16) under the feedback $u_k = \gamma^\delta(\zeta_k)$ *with* ζ *as in (5.30).*

Remark 5.14. *Because of the non-entire nature of the delay (5.7), all properties yielded under the prediction-based controller in Theorem (5.2) are shifted to the time instants* $t = k\delta + \sigma$ *rather than at the sampling instants* $t = k\delta$. *This is implicitly* *motivated by the shifting operation we have introduced to define the predictor dynamics (5.31) describing the evolutions of the future trajectories of the system among two inter-sampling instants $t_1 = k\delta + \sigma$ and $t_2 = k\delta + \delta + \sigma$. Accordingly, the effect of the delay is compensated at any time instant $t = k\delta + \sigma$ with $k \geq 0$.*

As in the case of entire delays, the prediction-based feedback works in open loop in terms of possible prediction errors at any time instants. An Immersion and Invariance redesign will be introduced in the next chapter for covering that case as well.

Some computational details are given below together with the case of LTI dynamics as an illustrative example.

5.3.2.1 Some computational aspects

The same considerations underlined in Section 5.3.1.1 hold true in the case of prediction-based feedbacks for non-entire input-delayed dynamics (5.1). As a matter of fact, the prediction still evolves as a discrete-time dynamics so underlining that computing the future trajectories of (5.1) for any $t \geq 0$ is unnecessary. Moreover, the feedback $u_k = \gamma^\delta(\zeta_k)$ is still explicitly defined and requires a finite N-dimensional buffer for storing the most N-recent values of the control signal u_k.

As in the entire case, approximate solutions in terms of δ and τ can be naturally defined by first rewriting

$$\begin{aligned} K^\delta(\tilde{\tau}, \sigma, x, v_0, v) =&\gamma^\delta(\zeta) \\ =&e^{\sigma(\mathrm{L}_f + v_0\mathrm{L}_g)} \circ e^{\frac{\tilde{\tau}}{N}(\mathrm{L}_f + v_1\mathrm{L}_g)} \circ \cdots \circ e^{\frac{\tilde{\tau}}{N}(\mathrm{L}_f + v_N\mathrm{L}_g)}\gamma^\delta(x) \end{aligned} \tag{5.33}$$

with $\tilde{\tau} = \tau - \sigma$. This implies that (5.33) itself can be expressed as a power series expansion in powers of $\tilde{\tau}$ and σ as

$$K^\delta(\tilde{\tau}, \sigma, x, v_0, v) = K_0^\delta(x, v_0, v) + \sum_{\substack{i,\, j\, \geq\, 0 \\ i\, +\, j\, >\, 0}} \frac{\tilde{\tau}^i \sigma^j}{N^i i! j!} K_{i+j}^\delta(x, v_0, v) \tag{5.34}$$

with any i^{th}-order term still parametrized by the sampling period δ and provided by

$$\begin{aligned} K_0^\delta(x, v_0, v) =&\gamma^\delta(x), \quad K_1^\delta(x, v_0, v) = \sum_{i=0}^{N}(\mathrm{L}_f + v_i\mathrm{L}_g)\gamma^\delta(x) \\ K_2^\delta(x, v_0, v) =& \sum_{i=0}^{N}(\mathrm{L}_f + v_i\mathrm{L}_g)^2\gamma^\delta(x) + 2! \sum_{i_1=0}^{N}\sum_{i_2=i_1+1}^{N}(\mathrm{L}_f + v_{i_1}\mathrm{L}_g)(\mathrm{L}_f + v_{i_2}\mathrm{L}_g)\gamma^\delta(x) \\ K_i^\delta(x, v_0, v) =& \sum_{i_1=0}^{N} \cdots \sum_{i_i=i_{i-1}}^{N} a_i(i_1, \dots, i_i)(\mathrm{L}_f + v_{i_1}\mathrm{L}_g)\dots(\mathrm{L}_f + v_{i_i}\mathrm{L}_g)\gamma^\delta(x) \end{aligned} \tag{5.35}$$

for suitable positive coefficients $a_i(i_1, \dots, i_i) \geq 0$.

Remark 5.15. *The series expansion (5.34) of the predictor-based feedback (5.33) underlines that, as* $\tau \to 0$ *(namely,* $\sigma \to 0$ *and* $\tilde{\tau} \to 0$*), one recovers the delay-free ILM-based control in Theorem 2.4; i.e.,* $K^{\delta}(0, 0, x, v_0, v) = \gamma^{\delta}(x)$.

Accordingly, one can deduce approximate prediction-based feedbacks as truncations of (5.34) at any fixed order $q > 0$ in $\tilde{\tau}$ and σ and under delay-free ILM approximate solutions (2.39).

Definition 5.2. Given an approximate p^{th}-order solution $\gamma^{\delta[p]}(x)$ of the form (2.39), we define the $[p, q]^{th}$*-order approximate solution* to the retarded ILM equality (5.21) as

$$K^{\delta[p,q]}(\tau, x, v) = K_0^{\delta[p]}(x, v) + \sum_{\substack{i,\, j\, =\, 0\\ i\, +\, j\, >\, 0}}^{i+j=q} \frac{\tilde{\tau}^i \sigma^j}{N^i i! j!} K_{i+j}^{\delta[p]}(x, v_0, v) \tag{5.36}$$

with

$$K_0^{\delta[p]}(x, v_0, v) = \gamma^{\delta}(x), \quad K_1^{\delta[p]}(x, v_0, v) = \sum_{i=0}^{N} (\mathrm{L}_f + v_i \mathrm{L}_g)\gamma^{\delta}(x)$$

As in the entire case, the approximate sampled-data feedback $u_k = K^{\delta[p,q]}(\tau, x_k, v_{0k}, v_k)$ ensures pGAS of the origin of (5.16) and, thus, S-pGAS of the one time-delay system (5.1); namely, all the solutions of (5.1) will converge, in closed-loop, to a ball centered at the origin and with radius in $O(\delta^p(1 + \tau^q))$.

5.3.2.2 The linear case as an example

Consider, as a case study, the LTI system

$$\dot{x}(t) = Ax(t) + Bu(t - \tau) \tag{5.37}$$

where $x \in \mathbb{R}^n$ with $u \in \mathcal{U}^{\delta}$ and τ being a non-entire delay of the form (5.7). In that case, Assumption 5.1 reformulates as follows:

Assumption 5.2 (Linear continuous-time delay-free stabilizability)**.** *When* $\tau = 0$ *the couple* (A, B) *is stabilizable and the continuous-time feedback* $u = Fx$ *stabilizes in closed-loop with Lyapunov function* $V(x) = x^{\top}Qx$*,* $Q > 0$ *such that* $(A+BF)^{\top}Q + Q(A + BF) < 0$ *and* QB *is full rank.*

In the LTI case, the sampled-data equivalent dynamics are exactly computable.

As well known, the LTI nature of (5.37) is preserved under sampling so that the extended sampled-data equivalent model gets the form

$$x_{k+1} = A^{\delta}x_k + A^{\delta-\sigma}B^{\sigma}v_{0k} + B^{\delta-\sigma}v_{1k} \tag{5.38a}$$
$$v_{0k+1} = v_{1k} \tag{5.38b}$$
$$v_{k+1} = \hat{A}v_k + \hat{B}u_k \tag{5.38c}$$

where $x_k = x(k\delta)$ for $k \geq 0$ and

$$A^\delta = e^{\delta A}, \quad B^\sigma = \int_0^\sigma e^{sA} B \mathrm{d}s.$$

From the above definitions, it is straightforward to verify that

$$A^{\delta-\sigma} B^\sigma + B^{\delta-\sigma} = A^\sigma B^{\delta-\sigma} + B^\sigma = B^\delta.$$

Accordingly, one gets the following result by detailing Theorem 5.2 to the present case.

Corollary 5.1. *Consider (5.37) under Assumption 5.2 and let F^δ be computed as the solution to the ILM equality*

$$(A^\delta + B^\delta F^\delta)^\top Q (A^\delta + B^\delta F^\delta) = e^{(A+BF)^\top \delta} Q e^{(A+BF)\delta}. \tag{5.39}$$

Then, the predictor-based feedback

$$u = F^\delta \zeta \tag{5.40}$$

with

$$\tilde{x}_k := x(k\delta + \sigma) = A^\sigma x_k + B^\sigma v_{0k} \tag{5.41}$$

$$\zeta_k := A^{N\delta} \tilde{x}_k + \sum_{i=k}^{k+N-1} A^{(k+N-1-i)\delta} B^\delta u_{i-N} \tag{5.42}$$

asymptotically stabilizes (5.38) As a consequence, (5.40) asymptotically stabilizes (5.37) at the time instants $t = k\delta + \sigma$, $k \geq 0$.

5.4 An example

Consider the case of the van der Pol oscillator typically exploited in the context of predictor-based control [80, 81, 69] whose dynamics is provided by

$$\begin{aligned} \dot{x}_2(t) &= x_1(t) - x_1^2(t) u(t-\tau) \\ \dot{x}_1(t) &= u(t-\tau). \end{aligned} \tag{5.43}$$

5.4.1 Delay-free design

Consider the delay-free system associated to (5.43)

$$\begin{aligned} \dot{x}_2(t) &= x_1(t) - x_1^2(t) u(t) \\ \dot{x}_1(t) &= u(t) \end{aligned} \tag{5.44}$$

which verifies Assumption 5.1 with

$$u = \gamma(x) = -x_1(1 + \frac{x_1^2}{6}) - \frac{1}{2} x_2, \quad V(x) = \frac{1}{2} x_1^2 + \frac{1}{2}(x_2 + x_1 + \frac{x_1^3}{3})^2.$$

As a consequence, one considers the *finite* sampled-data equivalent model associated to the delay-free dynamics (5.44) provided by

$$\begin{aligned} x_{2k+1} =& x_{2k} + \delta x_{1k}(1 - x_{1k}u_k) + \frac{\delta^2}{2}u_k(1 - 2x_{1k}u_k) - \frac{\delta^3}{3}u_k^3 \\ x_{1k+1} =& x_{1k} + \delta u_k. \end{aligned} \tag{5.45}$$

As a consequence, one can computes the stabilizing feedback $u_k = \gamma^\delta(x_k)$ as the solution to the ILM equality (5.18) so getting the approximate solution

$$\gamma^{\delta[1]}(x) = \gamma^0(x) + \frac{\delta}{2}\gamma^1(x) \tag{5.46}$$

with

$$\begin{aligned} \gamma^0(x) =& - x_1(1 + \frac{x_1^2}{6}) - \frac{1}{2}x_2 \\ \gamma^1(x) =& - \gamma^0(x)(1 + \frac{x_1^2}{2}) - \frac{1}{2}x_1(1 - x_1\gamma^0(x)) \\ =& \frac{x_1}{2}(1 + \frac{x_1^2}{3}) + \frac{x_2}{2}. \end{aligned}$$

Remark 5.16. *Albeit (5.44) is finitely discretizable (i.e., its sampled-data equivalent model (5.45) possesses a finite number of terms in powers of δ), computing an exact solution to the ILM equality (5.18) is still not possible and only approximate feedback laws can be carried out.*

5.4.2 Design of the prediction-based feedback

Consider now the retarded system (5.43) and assume $u \in \mathcal{U}^\delta$ and the case of non-entire delay $\tau = \delta + \sigma$ with $\sigma \in [0, \delta[$ and $N = 1$. Then, the extended hybrid dynamics (5.15) specifies as the interval dynamics

$$\begin{aligned} \dot{x}_2(t) =& x_1(t) - x_1^2(t)v_{0k} \\ \dot{x}_1(t) =& v_{0k} \\ v_{0k+1} =& v_{1k} \\ v_{1k+1} =& u_k \end{aligned} \tag{5.47}$$

for $t \in [k\delta, k\delta + \sigma[$ and

$$\begin{aligned} \dot{x}_2(t) =& x_1(t) - x_1^2(t)v_{1k} \\ \dot{x}_1(t) =& v_{1k} \\ v_{0k+1} =& v_{1k} \\ v_{1k+1} =& u_k \end{aligned} \tag{5.48}$$

for $t \in [k\delta + \sigma, (k+1)\delta[$ and with

$$\begin{aligned} v_0 =& u((k-1)\delta - \sigma) = u_{k-2} \\ v_1 =& u((k-1)\delta) = u_{k-1}. \end{aligned}$$

Accordingly, one computes the extended sampled-data equivalent model to (5.43) as

$$
\begin{aligned}
x_{2k+1} =& x_{2k} + \sigma x_{1k}(1 - x_{1k}v_{0k}) + \frac{\sigma^2}{2}v_{0k}(1 - 2x_{1k}v_{0k}) - \frac{\sigma^3}{3}v_{0k}^3 \\
&+ (\delta - \sigma)(x_{1k} + \sigma v_{0k})(1 - v_{1k}(x_{1k} + \sigma v_{0k})) \\
&+ \frac{(\delta - \sigma)^2}{2}v_{1k}(1 - 2v_{1k}(x_{1k} + \sigma v_{0k})) - \frac{(\delta - \sigma)^3}{3}v_{1k}^3 \\
x_{1k+1} =& x_{1k} + \sigma v_{0k} + (\delta - \sigma)v_{1k} \\
v_{0k+1} =& v_{1k} \\
v_{1k+1} =& u_k
\end{aligned}
\tag{5.49}
$$

which is clearly (nonlinearly) affected by both u_{k-1} and u_{k-2} as long as $\sigma > 0$. Accordingly, following the lines of Section 5.2.2 one first construct the non-entire part of the predictor (5.27) as

$$
\tilde{x}_k = \begin{pmatrix} \tilde{x}_{2k} \\ \tilde{x}_{1k} \end{pmatrix} = \begin{pmatrix} x_2(k\delta + \sigma) \\ x_1(k\delta + \sigma) \end{pmatrix}
\tag{5.50}
$$

with

$$
\begin{aligned}
\tilde{x}_2 =& x_2 + \sigma x_1(1 - x_1v_0) + \frac{\sigma^2}{2}v_0(1 - 2x_1v_0) - \frac{\sigma^3}{3}v_0^3 \\
\tilde{x}_1 =& x_1 + \sigma v_0
\end{aligned}
$$

and evolving as

$$
\begin{aligned}
\tilde{x}_{2k+1} =& \tilde{x}_{2k} + \delta\tilde{x}_{1k}(1 - \tilde{x}_{1k}v_{1k}) + \frac{\delta^2}{2}v_{1k}(1 - 2\tilde{x}_{1k}v_{1k}) - \frac{\delta^3}{3}v_{1k}^3 \\
\tilde{x}_{1k+1} =& \tilde{x}_{1k} + \delta v_{1k}.
\end{aligned}
$$

Thus, the predictor mapping (5.30) is given by

$$
\begin{aligned}
\zeta_2 =& \tilde{x}_2 + \delta\tilde{x}_1(1 - \tilde{x}_1v_1) + \frac{\delta^2}{2}v_1(1 - 2\tilde{x}_1v_1) - \frac{\delta^3}{3}v_1^3 \\
\zeta_1 =& \tilde{x}_1 + \delta v_1.
\end{aligned}
\tag{5.51}
$$

or, in terms of the original state (x_2, x_1) as

$$
\begin{aligned}
\zeta_2 =& x_2 + \sigma x_1(1 - x_1v_0) + \frac{\sigma^2}{2}v_0(1 - 2x_1v_0) - \frac{\sigma^3}{3}v_0^3 \\
&+ \delta(x_1 + \sigma v_0)(1 - v_1(x_1 + \sigma v_0)) + \frac{\delta^2}{2}v_1(1 - 2v_1(x_1 + \sigma v_0)) - \frac{\delta^3}{3}v_1^3 \\
\zeta_1 =& x_1 + \sigma v_0 + \delta v_1.
\end{aligned}
\tag{5.52}
$$

Accordingly, the stabilizing prediction-based feedback in Theorem 5.2 specifies as

$$
\gamma^{\delta[1]}(\zeta) = \gamma^0(\zeta) + \frac{\delta}{2}\gamma^1(\zeta)
\tag{5.53}
$$

with

$$\begin{aligned}\gamma^0(\zeta) =& -\zeta_1(1+\frac{\zeta_1^2}{6}) - \frac{1}{2}\zeta_2\\ \gamma^1(\zeta) =& -\gamma^0(\zeta)(1+\frac{\zeta_1^2}{2}) - \frac{1}{2}\zeta_1(1-\zeta_1\gamma^0(\zeta))\\ =& \frac{\zeta_1}{2}(1+\frac{\zeta_1^2}{3}) + \frac{\zeta_2}{2}.\end{aligned}$$

Remark 5.17. *Note that, because (5.44) is finitely discretizable, the predictor mapping admits the exact expression (5.52). Accordingly, the feedback (5.53) completely compensates the effect of the delay although it ensures practical properties in the amplitude of the sampling period δ.*

Remark 5.18. *Whenever the delay is entire (i.e., $\tau = \delta$ with $\sigma = 0$), (5.52) recovers the case described in (5.19) detailing as*

$$\begin{aligned}\zeta_2 =& x_2 + \tau x_1(1-x_1v_1) + \frac{\tau^2}{2}v_1(1-2x_1v_1) - \frac{\tau^3}{3}v_1^3\\ \zeta_1 =& x_1 + \tau v_1\end{aligned} \tag{5.54}$$

which, in turn, gives $\zeta = x$ as $\tau \to 0$.

Simulations over this example are reported to Chapter 6 so to compare the results with respect to the I&I approach.

5.5 Conclusions and Literature review

In this chapter, we have settled the problem of stabilizing nonlinear input-delayed dynamics of the form (5.1) under sampling. We have considered the case of discrete input delays by rewriting the delay τ as a non-entire multiple of the sampling period δ, which is always possible. Accordingly, we have emphasized on the cascade structure implicitly describing the dynamics (5.1) and enforced on the simplification arising in the sampled-data context with respect to the fully continuous counterpart. Moreover, we have shown that whenever the delay-free system associated to (5.1) is smoothly stabilizable, one can always infer a computable sampled-data feedback for (5.1) based on prediction and Input-Lyapunov Matching. We have proposed a predictor compensating the delay at any time instant $t = k\delta + \sigma$ with $k \geq 0$ and evolving as a suitable discrete-time dynamics defined as the copy of the discrete-time equivalent model of the delay-free system associated to (5.1). Then, we have shown that, within the sampled-data scenario, approximate predictor-based feedback are naturally defined as truncations of the series expansion in powers of τ, σ and δ defining the feedback at any finite order.

In the framework of continuous-time time-delay systems, a lot of works have been proposed throughout the last decades. As far as prediction-based control is concerned, the very first result goes back to 1959 when the Smith's predictor [173] was introduced for input delayed linear stable systems. Then, it was later improved by several other works as the one by Wanatabe, Ito [185] also to deal with unstable linear plants. Successively, extensions to more general cases have been studied as well by considering nonlinear plants as made by Kristic and co-workers via the definition of suitable Lyapunov-Krasovskii functionals [81] to deal with robustness issues as well [65].Then, predictors for larger variety of situations have been proposed by embedding time-varying and distributed delays for both time invariant or time-invariant systems as proposed, among many others, by Mazenc, Malisoff, Niculescu and Pepe [111, 112, 109, 20]. Sequential subpredictors have been investigated by Najafi and co-workers in [139] for linear systems with a long input delays and extended to classes of time-varying systems by Polyakov and co-workers [159], Léchappé [85] and Mazenc and Malisoff [108].

When dealing with sampled-data systems different prediction-based schemes have been proposed based on the scenario one faces to. As an example, in the context of sampled-data measurements (and continuous-time control signal) extensive works have been carried out by Fridman, Krstic or Mazenc and co-authors in [107, 64, 2] through the definition of interval observers exploiting the continuous-time predictor-dynamics. When dealing with fully sampled-data systems (both in inputs and outputs), most prediction-based methodologies have been addressing the problem of computing continuous-time predictors over the emulation-based delay-free feedback as, for example, in the work by Mazenc and Normand-Cyrot in [114]. Though, even in this case, important works have been underlining the impact of sampling for dealing with numerical approximations and robustness issues as underlined by Karafyllis and collaborators [67, 68, 69] and in a larger variety of problems concerned with time-delay systems as pointed out by many works by Pepe [155, 157, 156]

As an alternative to prediction, reduction and descriptor-based methods have been proposed for both continuous-time and sampled-data retarded dynamics [164, 45, 105]. In this case, those methodologies give further degrees of freedom in the design that is no longer constrained to the one carried out over the idea delay-free system.

In the next Chapter, we shall show that Immersion and Invariance represents a natural tool for the stabilization of retarded systems under sampling.

CHAPTER 6

Sampled-data I&I for time-delay systems

Contents

In this chapter, we are going to extend Immersion and Invariance to deal with sampled-data time delay systems affected by a constant and known delay over the input. We shall show that, exploiting the cascade dynamics underlying the sampled-data dynamics, one can naturally deduce an I&I problem admitting a solution which can be rewritten as the prediction based feedback proposed in Chapter 5 plus a robustiying component over the prediction error. To this end, we shall assume the existence of a continuous-time smooth feedback for the origin of the delay-free system and infer an ILM-based feedback stabilizing the origin of the delay-free dynamics. Accordingly, I&I naturally intervenes by setting as a target the sampled-data equivalent model of the delay-free model under the ILM-based control.

We shall first present the result for the case of an entire delay $\tau = N\delta$ for $N \in \mathbb{N}$ and then extend it to the case of a non-entire delay $\tau = N\delta + \sigma$ with $N \in \mathbb{N}$ and $\sigma \in [0, \delta[$.

The results of this section are based on [132, 102].

M. Mattioni, S. Monaco and D. Normand-Cyrot. *Further results on sampled-data stabilization of time-delay systems.* 20th IFAC World Congress. IFAC-PapersOnLine. Volume 50, Issue 1, pp. 14350-14355. Toulouse, France.

S. Monaco, D. Normand-Cyrot and M. Mattioni. *Sampled-data stabilization of non-linear dynamics with input delays through Immersion and Invariance.* IEEE Transactions on Automatic Control, vol. 62 (5), pp. 2561-2567, May 2017.

S. Monaco, D. Normand-Cyrot and M. Mattioni. *Nonlinear Sampled-data stabilization with delays.* Accepted to *Delays and Interconnections: Methodology, Algorithms and Applications*, Series Advances on Delays and Dynamics at Springer. Editors: G. Valmorbida, R. Sipahi, I. Boussaada and A. Seuret.

6.1 I&I stabilization under entire input delays

Consider the sampled-data time-delay system (5.1) and assume an entire input-delay (5.8) so that it rewrites in the form of an hybrid cascade representation provided by (5.12). Accordingly, its sampled-data equivalent model rewrites as

$$x_{k+1} = F^{\delta}(x_k, v_{1k}) \tag{6.1a}$$

$$v_{k+1} = \hat{A} v_k + \hat{B} u_k \tag{6.1b}$$

with

$$F^{\delta}(x, v_1) = e^{\delta(\mathrm{L}_f + v_1 \mathrm{L}_g)} x, \quad v = \begin{pmatrix} v_1 & \dots & v_N \end{pmatrix}^{\top}$$
$$\hat{A} = \begin{pmatrix} \mathbf{0}_{(N-1)\times 1} & \mathrm{I} \\ 0 & \mathbf{0}_{1\times(N-1)} \end{pmatrix} \quad \text{and} \quad \hat{B} = \begin{pmatrix} \mathbf{0}_{(N-1)\times 1} \\ 1 \end{pmatrix}.$$

Then, assuming (5.1) verifies Assumption 5.1 as in Chapter 5, one gets the equilibrium of the dynamics

$$x_{k+1} = F^{\delta}(x_k, \gamma^{\delta}(x_k)) \tag{6.2}$$

is GAS whenever $\gamma^{\delta} : \mathbb{R}^n \to \mathbb{R}$ is computed as the solution to the Input-Lyapunov Matching equality (5.18). Accordingly, I&I naturally comes into play by exploiting the cascade structure of (6.1). Indeed, the idea we want to enforce relies on defining the closed-loop delay-free equivalent model (6.2) as the target dynamics. Thus, the design is reduced to making the manifold describing the evolutions of the delay-free (6.2) attractive and invariant under feedback.

Remark 6.1. *Under Assumption 5.1 and because the sampled-data equivalent model (6.1) exhibits a strict-feedback dynamics, one might apply the same arguments presented in Chapter 3 to conclude on the corresponding I&I stabilizability. This is indeed the case although we underline that (6.1) is not a fully sampled-data system as it* *is given by the interconnection of (6.1a) with a completely discrete-time dynamics through the completely discrete-time coupling state v_1. Accordingly, I&I stabilizability can be still deduced although it does not follows the same lines as in Chapter 3 which was rather dealing with fully sampled-data systems and the coupling state arising from sampling of continuously evolving ones.*

We shall now prove, step by step, the following result.

Theorem 6.1 (I&I for retarded systems)**.** *Let the retarded dynamics (5.1) verify Assumption 5.1 and $\gamma^\delta : \mathbb{R}^n \to \mathbb{R}$ be the solution to the Input-Lyapunov Matching equality (5.18). Then, (5.1) is SD-I&I stabilizable with sampled-data target dynamics*

$$\xi_{k+1} = F^\delta(\xi_k, \gamma^\delta(\xi_k)) \tag{6.3}$$

with $\xi \in \mathbb{R}^n$. Equivalently, the extended sampled-data equivalent model (6.1) to (5.1) is I&I stabilizable with target dynamics (6.3).

For showing I&I stabilizability, one has to show that I&I stabilizability under sampling holds in the sense of Definition 3.1. Accordingly, one has that (6.3) defines a target dynamics for (6.1) as it possesses a GAS equilibrium at the origin by construction of $\gamma^\delta(\cdot)$ as in Theorem 2.4. Then, the invariance condition naturally holds by setting

$$\pi^\delta(\xi_k) = \begin{pmatrix} \xi_k \\ \gamma^\delta(\xi_k) \\ \gamma^\delta(\xi_{k+1}) \\ \vdots \\ \gamma^\delta(\xi_{k+N-1}) \end{pmatrix}, \quad c^\delta(\xi_k) = \gamma^\delta(\xi_{k+N}) \tag{6.4}$$

with

$$\begin{aligned} \gamma^\delta(\xi_{k+i}) :=& \gamma^\delta(\cdot) \circ \underbrace{F^\delta(\cdot, \gamma^\delta(\cdot)) \circ \cdots \circ F^\delta(\xi_k, \gamma^\delta(\xi_k))}_{i \text{ times}} \\ =& e^{\delta(\mathrm{L}_f + \gamma^\delta(\xi_k)\mathrm{L}_g)} \circ \cdots \circ e^{\delta(\mathrm{L}_f + \gamma^\delta(\xi_{k+i-1})\mathrm{L}_g)} \gamma^\delta(\xi)\big|_{\xi_k} \end{aligned}$$

for $i = 0, \ldots, N$. Then, one implicitly defines the manifold $\mathcal{M}^\delta$

$$\mathcal{M}^\delta = \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^N \text{ s.t. } \phi^\delta(x, v) = 0\} \tag{6.5}$$

with

$$\phi^\delta(x, v) = \begin{pmatrix} \phi_1^\delta(x, v) \\ \vdots \\ \phi_N^\delta(x, v) \end{pmatrix} = \begin{pmatrix} v_1 - \gamma^\delta(x_k) \\ \vdots \\ v_N - \gamma^\delta(x_{k+N-1}) \end{pmatrix} \tag{6.6}$$

and, for $i = 0, \ldots, N$

$$\begin{aligned}\gamma^\delta(x_{k+i}) :=& \gamma^\delta(\cdot) \circ \underbrace{F^\delta(\cdot, v_{ik}) \circ \cdots \circ F^\delta(x_k, v_{1k})}_{i \text{ times}}\\ =& e^{\delta(\mathrm{L}_f + v_{1k}\mathrm{L}_g)} \circ \cdots \circ e^{\delta(\mathrm{L}_f + v_{ik}\mathrm{L}_g)} \gamma^\delta(x)\big|_{x_k}.\end{aligned}$$

Consequently, one defines the off-the-manifold component as

$$z = \begin{pmatrix} z_1 & \ldots & z_N \end{pmatrix}^\top \quad z_i = \phi_i^\delta(x, v) = v_i - \gamma^\delta(x_{k+i-1}) \tag{6.7}$$

and evolving as

$$z_{k+1} = \hat{A} z_k + \hat{B}(u_k - \gamma^\delta(x_{k+N})) \tag{6.8}$$

for initial condition $z_0 = \phi^\delta(x_0, v_0)$.
Accordingly, any feedback $u = \nu^\delta(\tau, x, v, z)$ ensuring that $z_k \to 0$ as $k \to \infty$ with $\nu^\delta(\tau, \pi^\delta(\xi), 0) = \gamma^\delta(\xi_{k+N})$ and boundedness of the trajectories of the extended system

$$z_{k+1} = \hat{A} z_k + \hat{B}(u_k - \gamma^\delta(x_{k+N})) \tag{6.9a}$$
$$x_{k+1} = F^\delta(x_k, v_{1k}) \tag{6.9b}$$
$$v_{k+1} = \hat{A} v_k + \hat{B} u_k \tag{6.9c}$$

makes the origin a GAS equilibrium for (6.1).

Remark 6.2. *We underline that, the feedback $u_k = c^\delta(\xi_k)$ in (6.4) making the stable manifold $\mathcal{M}^\delta$ in (6.5) coincides with the prediction-based control as defined in Theorem 5.1.*

Remark 6.3. *Theorem 6.1 shows that the problem of stabilizing the origin of (5.1) under sampling and entire delay admits a solution whenever there exists a continuous-time smooth feedback stabilizing the origin of the delay-free system associated to (5.1).*

In the following, a possible choice of the I&I stabilizing feedback will be studied together with some computational aspects for the case $N = 1$.

6.1.1 The I&I stabilizing feedback

We are now defining the I&I feedback ensuring *iv*) of Definition 3.1. To this purpose, we notice that the dynamics (6.1) rewrites in compact form as

$$x^e_{k+1} = F^\delta_e(x^e_k) + B_e u_k \tag{6.10}$$

with

$$x^e = \begin{pmatrix} x \\ v \end{pmatrix}, \quad F^\delta_e(x^e) = \begin{pmatrix} F^\delta(x, v) \\ \hat{A} v \end{pmatrix}, \quad B_e = \begin{pmatrix} \mathbf{0}_{n\times 1} \\ \hat{B} \end{pmatrix}.$$

Accordingly, the condition iv) in Definition 3.1 relaxes to requiring that the following limit condition holds

$$\lim_{k\to\infty} B_e(\nu^\delta(\tau, x_k, v_k, z_k) - \nu^\delta(\tau, x_k, v_k, 0)) = 0. \tag{6.11}$$

To this end, because of linearity of the controlled part of (6.10) one gets the following result.

Proposition 6.1. *Let the retarded dynamics (5.1) verify Assumption 5.1 and $\gamma^\delta : \mathbb{R}^n \to \mathbb{R}$ be the solution to the Input-Lyapunov Matching equality (5.18). Consider the extended system (6.9) with $z = \phi^\delta(x, v)$ as in (6.7). Then, the feedback $u = \nu^\delta(\tau, x, v, z)$*

$$\nu^\delta(\tau, x, v, z) = Lz + \gamma^\delta(x_{k+N}) \tag{6.12}$$

with

$$\gamma^\delta(x_{k+N}) := e^{\frac{\tau}{N}(\mathrm{L}_f + (z_{1k}+\gamma^\delta(x_k))\mathrm{L}_g)} \circ \dots \circ e^{\frac{\tau}{N}(\mathrm{L}_f + (z_{Nk}+\gamma^\delta(x_{k+N-1}))\mathrm{L}_g)} \gamma^\delta(x)\big|_{x_k}$$

and L such that $\hat{A}+\hat{B}L$ is Schur (i.e., possessing all eigenvalues within' the open unit circle) ensures I&I stabilization of (6.9) and, thus, SD-I&I of (5.1) in closed-loop.

Proof: The proof is straightforward as one gets, in closed-loop, that the z dynamics in (6.9a) rewrites as the GES linear dynamics

$$z_{k+1} = (\hat{A} + \hat{B}L)z_k.$$

Accordingly, (6.11) rewrites as

$$\lim_{k\to\infty} B_e(Lz_k + e^{\frac{\tau}{N}(\mathrm{L}_f + (z_{1k}+\gamma^\delta(x_k))\mathrm{L}_g)} \circ \dots \circ e^{\frac{\tau}{N}(\mathrm{L}_f + (z_{Nk}+\gamma^\delta(x_{k+N-1}))\mathrm{L}_g)} \gamma^\delta(x)\big|_{x_k}$$
$$- e^{\frac{\tau}{N}(\mathrm{L}_f + \gamma^\delta(x_k)\mathrm{L}_g)} \circ \dots \circ e^{\frac{\tau}{N}(\mathrm{L}_f + \gamma^\delta(x_{k+N-1})\mathrm{L}_g)} \gamma^\delta(x)\big|_{x_k}) = 0$$

so getting the result. ■

Remark 6.4. *Whenever one sets $L = \mathbf{0}_{1\times N}$, one gets dead beat control of z to 0 in exactly N sampling instants.*

Remark 6.5. *Proposition 6.1 underlines that, the extended dynamics (6.9) with dummy output $y = z_1$ is minimum-phase with zero-dynamics corresponding to the dynamics (6.2) which is indeed free of delays.*

6.1.2 Some computational aspects for $N = 1$

We are now providing some computational aspects when $N = 1$ in (5.8). In that case, (6.1) rewrites as

$$x_{k+1} = F^\delta(x_k, v_k) \tag{6.13a}$$
$$v_{k+1} = u_k \tag{6.13b}$$

so that Theorem (6.1) holds with target dynamics (6.3) and

$$\pi^\delta(\xi) = \begin{pmatrix} \xi \\ \gamma^\delta(\xi) \end{pmatrix}, \quad c^\delta(\xi) = \gamma^\delta(F^\delta(\xi, \gamma^\delta(\xi)))$$

and $z = v - \gamma^\delta(x)$. Accordingly, (6.9) specifies as

$$z_{k+1} = u_k - \gamma^\delta(F^\delta(x_k, v_k))) \tag{6.14a}$$

$$x_{k+1} = F^\delta(x_k, v_k) \tag{6.14b}$$

$$v_{k+1} = u_k \tag{6.14c}$$

Accordingly, by exploiting the (F_0, G) representation (2.9) associated with the delay-free (2.4a) one can rewrite

$$F^\delta(x, v) = F_0^\delta(x) + \int_0^v G^\delta(x^+(w), w)\mathrm{d}w$$

with $F_0^\delta(x) = F^\delta(x, 0)$ and, thus, when rewriting $v = z + \gamma^\delta(x)$ as

$$\begin{aligned} F^\delta(x, z + \gamma^\delta(x)) =& F_0^\delta(x) + \int_0^{\gamma^\delta(x)} G^\delta(x^+(w), w)\mathrm{d}w \\ &+ \int_0^z G^\delta(x^+(\gamma^\delta(x) + w), \gamma^\delta(x) + w)\mathrm{d}w \\ =& F^\delta(x, \gamma^\delta(x)) + z\int_0^1 G^\delta(x^+(\gamma^\delta(x) + \theta z), \gamma^\delta(x) + \theta z)\mathrm{d}z \end{aligned}$$

where $F^\delta(x, \gamma^\delta(x))$ possess a GAS equilibrium at the origin by construction. As a consequence, the I&I feedback aimed at forcing $z_k \to 0$ rewrites, in this case, as

$$\nu^\delta(\tau, x, v, z) = \underbrace{\gamma^\delta(F^\tau(x, \gamma^\delta(x)))}_{=\gamma^\delta(\zeta)} + L^\delta(x, z)z \tag{6.15}$$

with dynamical gain

$$L^\delta(x, z) = L + \int_0^1 G^\delta(x, \gamma^\delta(x) + \theta z)\mathrm{d}z. \tag{6.16}$$

The form (6.15) underlines that the I&I feedback is composed of two contributions:

- $\gamma^\delta(F^\delta(x, \gamma^\delta(x))) = \gamma^\delta(\zeta)$ and $\zeta = F^\delta(x, v)$ as in (5.19) coinciding with the predictor-based feedback presented in Chapter 5;
- a feedback term $L^\delta(x, z)z$ over the off-the-manifold component z representing the prediction-error over the feedback dynamics at any sampling instants.

Accordingly, the I&I procedure improves the prediction-based control by including a feedback term (through a dynamical gain) over the prediction error. Thus, as $z \to 0$, the I&I control recovers the prediction-based feedback $u = \nu^\delta(\tau, \pi^\delta(\xi), 0) = c^\delta(\xi) =$

$\gamma^\delta(F^\delta(\xi, \gamma^\delta(\xi))$. This aspect is of paramount importance in the sampled-data framework as it implicitly robustifies with respect to variations of δ, approximations of the feedback control and possibly higher order discarded dynamics in the sampled-data model (6.1).

Now, it is a matter of computation to verify that (6.15) rewrites as a series expansion in powers of τ as

$$\nu^\delta(\tau, x, v, z) = \nu^\delta(x, z) + \sum_{i>0} \frac{\tau^i}{i!} \nu_i^\delta(x, \gamma^\delta(x)) \tag{6.17}$$

with

$$\begin{aligned}
\nu_0^\delta(x, z) =& \gamma^\delta(x) + L^\delta(x, z)z \\
\nu_1^\delta(x, v) =& (\mathrm{L}_f + \gamma^\delta(x)\mathrm{L}_g)\gamma^\delta(x) \\
\nu_i^\delta(x, v) =& \Big((\mathrm{L}_f + v\mathrm{L}_g)^i \gamma^\delta(x)\Big)_{v=\gamma^\delta(x)} \\
& \dots
\end{aligned}$$

and

$$L^\delta(x, z) = L_0(x, z) + \sum_{i>0} \frac{\delta^i}{i!} L_i(x, z) \tag{6.18}$$

with, for the first terms

$$\begin{aligned}
L_0(x, z) =& L \\
L_1(x, z) =& \mathrm{L}_g x \\
L_2(x, z) =& (\mathrm{L}_f\mathrm{L}_g + \mathrm{L}_g\mathrm{L}_f)x + (z + 2\gamma_0(x))\mathrm{L}_g^2 x \\
& \vdots \, .
\end{aligned}$$

Remark 6.6. *The series expansion (6.17) enlightens that, when $\tau = 0$ (and, thus $z = 0$) one recovers the delay-free feedback $u = \gamma^\delta(x)$.*

Thus, the feedback (6.15) admits the form of a series expansion in powers of δ and τ. Still, no exact solutions can be computed in practice. Though, approximations can be easily carried out as depending on

- approximations of $\gamma^\delta : \mathrm{R}^n \to \mathbb{R}$ as solution to the ILM equality (5.18) in the sense of Definition 2.39;
- approximations of the predictor component $\gamma^\delta(F^\tau(x, \gamma^\delta(x)))$ in terms of τ as in Definition 5.1;
- approximations of the gain $L^\delta(x, z)$ in powers of δ.

Accordingly, the following definition is given by extending the one proposed in Definition 5.1.

Definition 6.1. Given an approximate p^{th}-order solution $\gamma^{\delta[p]}(x)$ of the form (2.39), we define the $[p,q]^{th}$*-order approximate solution* to I&I feedback (6.15) as

$$\nu^{\delta[p,q]}(\tau,x,v) = \gamma^{\delta[p]}(x) + L^{\delta[q]}(x,z)z + \sum_{i=1}^{p} \frac{\tau^i}{Ni!}\nu_i^{\delta}(x,\gamma^{\delta[p]}(x)) \tag{6.19}$$

with

$$L^{\delta[q]}(x,z) = L_0(x,z) + \sum_{i=1}^{q} \frac{\delta^i}{i!}L_i(x,z).$$

Similarly to the other scenarios we have been dealing with, one can conclude that the approximate sampled-data feedback $u = \nu^{\delta[p,q]}(\tau,x,v)$ ensures pGAS of the origin of (6.1) and, thus, S-pGAS of the one time-delay system (5.1); namely, all the solutions of (5.1) will converge, in closed-loop, to a ball centered at the origin and with radius in $O(\delta^p(1+\tau^q))$.

Remark 6.7. *Those computations and commentaries easily extend to the case of $N > 1$ along the lines of the case $N = 1$.*

6.2 I&I stabilization under non-entire input delays

In this section, the I&I design for the sampled-data retarded system (5.1) is extended to the case of non-entire delays $\tau = N\delta + \sigma$ as in (5.7).

6.2.1 The choice of the target dynamics

Consider the sampled-data time-delay systems (5.1) and consider now a non-entire input-delay (5.7) so that it rewrites in the form of an hybrid cascade representation provided by (5.15). Accordingly, its sampled-data equivalent model rewrites as

$$x_{k+1} = F^{\delta}(\sigma, x_k, v_{0k}, v_{1k}) \tag{6.20a}$$
$$v_{0k+1} = v_{1k} \tag{6.20b}$$
$$v_{k+1} = \hat{A}v_k + \hat{B}u_k \tag{6.20c}$$

with

$$F^{\delta}(\sigma,x,v_0,v_1) = F^{\delta-\sigma}(\cdot,v_1)\circ F^{\sigma}(x,v_0) = e^{\sigma(\mathrm{L}_f+v_0\mathrm{L}_g)}e^{(\delta-\sigma)(\mathrm{L}_f+v_1\mathrm{L}_g)}x$$
$$v = \begin{pmatrix} v_1 & \dots & v_N \end{pmatrix}^{\top}$$
$$\hat{A} = \begin{pmatrix} \mathbf{0}_{(N-1)\times 1} & \mathrm{I} \\ 0 & \mathbf{0}_{1\times(N-1)} \end{pmatrix} \quad \text{and} \quad \hat{B} = \begin{pmatrix} \mathbf{0}_{(N-1)\times 1} \\ 1 \end{pmatrix}.$$

Then, assuming (5.1) verifies Assumption 5.1, one gets the equilibrium of the dynamics (6.2) is GAS whenever $\gamma^{\delta} : \mathbb{R}^n \to \mathbb{R}$ is computed as the solution to the Input-Lyapunov Matching equality (5.18). Thus, as in the case of $\sigma = 0$, I&I

naturally comes into play by exploiting the cascade structure of (6.20). Though, this time we are going to exploit the cascade interconnection of the augmented system (6.20a)-(6.20b) with the LTI dynamics (6.20c). Indeed, the idea we want to enforce relies on defining target dynamics through an extended version of the closed-loop delay-free equivalent model (6.2) keeping into account the non entire part of the delay.

We shall now prove, step by step, the following result.

Theorem 6.2 (I&I design for retarded systems). *Let the retarded dynamics (5.1) verify Assumption 5.1 and $\gamma^\delta : \mathbb{R}^n \to \mathbb{R}$ be the solution to the Input-Lyapunov Matching equality (5.18). Then, (5.1) is I&I stabilizable under sampling at the time instants $t = k\delta + \sigma$ and sampled-data target dynamics*

$$\xi_{1k+1} = F^\delta(\xi_{1k}, \gamma^\delta(\xi_{1k})) \tag{6.21a}$$

$$\xi_{2k+1} = \gamma^\delta(\xi_{1k}) \tag{6.21b}$$

with $\xi = (\xi_1^\top \ \xi_2)^\top \in \mathbb{R}^n \times \mathbb{R}$. Equivalently, the extended sampled-data equivalent model (6.20) to (5.1) is I&I stabilizable with target dynamics (6.21).

For showing I&I stabilizability, one has to show that I&I stabilizability under sampling holds in the sense of Definition 3.1.

Accordingly, one has that (6.21a) possesses a GAS equilibrium at the origin by construction of $\gamma^\delta(\cdot)$ as in Theorem 2.4 as it coincides with the closed-loop delay-free system (6.2). Moreover, by nature of the feedforward cascade interconnection, the extended target (6.21) possesses a GAS equilibrium at the origin so verifying condition i) of Definition 3.1.

The I&I invariance condition holds by setting

$$\pi^\delta(\xi_k) = \begin{pmatrix} F^{-\sigma}(\xi_{1k}, \xi_{2k}) \\ \xi_{2k} \\ \gamma^\delta(\xi_{1k+1}) \\ \vdots \\ \gamma^\delta(\xi_{1k+N-1}) \end{pmatrix}, \quad c^\delta(\xi_k) = \gamma^\delta(\xi_{1k+N}) \tag{6.22}$$

with

$$\begin{aligned} \gamma^\delta(\xi_{k+i}) :=& \gamma^\delta(\cdot) \circ \underbrace{F^\delta(\cdot, \gamma^\delta(\cdot)) \circ \cdots \circ F^\delta(\xi_k, \gamma^\delta(\xi_k))}_{i \text{ times}} \\ =& e^{\delta(\mathrm{L}_f + \gamma^\delta(\xi_k)\mathrm{L}_g)} \circ \cdots \circ e^{\delta(\mathrm{L}_f + \gamma^\delta(\xi_{k+i-1})\mathrm{L}_g)} \gamma^\delta(\xi)\big|_{\xi_k} \end{aligned}$$

for $i = 0, \ldots, N$. Then, one implicitly defines the manifold $\mathcal{M}^\delta_\sigma$ as

$$\mathcal{M}^\delta_\sigma = \{(x, v_0, v) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^N \text{ s.t. } \phi^\delta(\sigma, x, v_0, v) = 0\} \tag{6.23}$$

with

$$\phi^\delta(\sigma, x, v_0, v) = \begin{pmatrix} \phi_1^\delta(\sigma, x, v_0, v) \\ \vdots \\ \phi_N^\delta(\sigma, x, v_0, v) \end{pmatrix} = \begin{pmatrix} v_1 - \gamma^\delta(F^\sigma(x, v_0)) \\ \vdots \\ v_N - \gamma^\delta(F^\sigma(x_{k+N-1}, v_{N-1})) \end{pmatrix} \tag{6.24}$$

and, for $i = 0, \dots, N$

$$\begin{aligned}\gamma^\delta(F^\sigma(x_{k+i}, v_i)) :=& \gamma^\delta(\cdot) \circ \underbrace{F^\delta(\cdot, v_i) \circ \cdots \circ F^\delta(\circ, v_1)}_{i \text{ times}} \circ F^\sigma(x_k, v_0) \\ =& e^{\sigma(\mathrm{L}_f + v_0 \mathrm{L}_g)} \circ e^{\delta(\mathrm{L}_f + v_1 \mathrm{L}_g)} \circ \cdots \circ e^{\delta(\mathrm{L}_f + v_i \mathrm{L}_g)} \gamma^\delta(x)\big|_{x_k}.\end{aligned}$$

Consequently, one defines the off-the-manifold component as

$$z = \begin{pmatrix} z_1 & \dots & z_N \end{pmatrix}^\top \quad z_i = \phi_i^\delta(\sigma, x, v_0, v) = v_i - \gamma^\delta(F^\sigma(x_{k+i-1}, v_{i-1})) \tag{6.25}$$

and $z_0 = \phi^\delta(x_0, v_{00}, v_0)$ evolving as

$$z_{k+1} = \hat{A} z_k + \hat{B}(u_k - \gamma^\delta(F^\sigma(x_{k+N}, v_{Nk}))). \tag{6.26}$$

Accordingly, any feedback $u = \nu^\delta(\tau, x, v_0, v, z)$ ensuring that $z_k \to 0$ as $k \to \infty$ with $\nu^\delta(\tau, \pi^\delta(\xi), 0) = \gamma^\delta(\xi_{1k+N})$ and boundedness of the trajectories of the extended system

$$z_{k+1} = \hat{A} z_k + \hat{B}(u_k - \gamma^\delta(F^\sigma(x_{k+N}, v_{Nk}))) \tag{6.27a}$$
$$x_{k+1} = F^\delta(\sigma, x_k, v_{0k}, v_{1k}) \tag{6.27b}$$
$$v_{0k} = v_{1k} \tag{6.27c}$$
$$v_{k+1} = \hat{A} v_k + \hat{B} u_k \tag{6.27d}$$

makes the origin a GAS equilibrium for (6.20).

Remark 6.8. *As in the case of predictors for retarded systems under non-entire delay (5.7), the I&I feedback first performs a time shifting of the state from $t = k\delta$ to $t = k\delta + \sigma$; as a matter of fact, one gets*

$$\tilde{x}_k = x(k\delta + \sigma) = F^\sigma(x_k, v_{0k}) \tag{6.28}$$

so that, even in this case, one needs a preliminary prediction over the time window $[k\delta, k\delta + \sigma[$ defined by the non-entire size of the delay and from the available measure of the state at $t = k\delta$ for $k \geq 0$. This is enlighten by the immersion mapping (6.22). Accordingly, all the properties of the sampled-data I&I feedback will be yielded, over the continuous-time system, at the time instants $t = k\delta + \sigma$ for $k \geq 0$.

Remark 6.9. *We underline that, the feedback $u_k = c^\delta(\xi_k)$ in (6.22) making the stable manifold $\mathcal{M}_\sigma^\delta$ in (6.23) coincides with the prediction-based control as defined in Theorem 5.2.*

Remark 6.10. *As $\sigma \to 0$, one recovers the I&I procedure developed for the case of entire delays (5.8).*

Remark 6.11. *Theorem 6.2 shows that the problem of stabilizing the origin of (5.1) under sampling and non-entire delay still admits a solution whenever there exists a continuous-time smooth feedback stabilizing the origin of the delay-free system associated to (5.1).*

6.2.2 The I&I stabilizing feedback

For designing the I&I feedback ensuring *iv*) of Definition 3.1, we first rewrite (6.20) as

$$x^e_{k+1} = F^\delta_e(x^e_k) + B_e u_k \tag{6.29}$$

with

$$x^e = \begin{pmatrix} x \\ v_0 \\ v \end{pmatrix}, \quad F^\delta_e(x^e) = \begin{pmatrix} F^\delta(\sigma, x, v) \\ v_0 \\ \hat{A}v \end{pmatrix}, \quad B_e = \begin{pmatrix} \mathbf{0}_{n\times 1} \\ 0 \\ \hat{B} \end{pmatrix}.$$

Accordingly, the condition *iv*) in Definition 3.1 relaxes to requiring that the following limit condition holds

$$\lim_{k\to\infty} B_e(\nu^\delta(\tau, x_k, v_{0k}, v_k, z_k) - \nu^\delta(\tau, x_k, v_{0k}, v_k, 0)) = 0. \tag{6.30}$$

To this end, because of linearity of the controlled part of (6.29) one gets the following result.

Proposition 6.2. *Let the retarded dynamics (5.1) verify Assumption 5.1 and $\gamma^\delta : \mathbb{R}^n \to \mathbb{R}$ be the solution to the Input-Lyapunov Matching equality (5.18). Consider the extended system (6.27) with $z = \phi^\delta(\sigma, x, v_0, v)$ as in (6.25). Then, the feedback $u = \nu^\delta(\tau, x, v, z)$*

$$\nu^\delta(\tau, x, v, z) = Lz + \gamma^\delta(F^\sigma(x_{k+N}, v_{Nk})) \tag{6.31}$$

with $\tilde{\tau} = \tau - \sigma$ and

$$\begin{aligned}\gamma^\delta(F^\sigma(x_{k+N}, v_{Nk})) =& e^{\sigma(\mathrm{L}_f + v_{0k}\mathrm{L}_g)} \circ e^{\frac{\tilde{\tau}}{N}(\mathrm{L}_f + (z_{1k}+\gamma^\delta(\cdot)))\mathrm{L}_g)} \circ \dots \\ &\circ e^{\frac{\tilde{\tau}}{N}(\mathrm{L}_f + (z_{Nk}+\gamma^\delta(\cdot))\mathrm{L}_g)}\gamma^\delta(x_k)\end{aligned}$$

and L such that $\hat{A} + \hat{B}L$ is Schur (i.e., possessing all eigenvalues within' the open unit circle) ensures I&I stabilization of (6.9) and, thus, I&I under sampling of (5.1) in closed-loop at any sampling instant $t = k\delta + \sigma$.

Proof: One gets, in closed-loop, that the z dynamics in (6.27a) rewrites as the GES linear dynamics

$$z_{k+1} = (\hat{A} + \hat{B}L)z_k.$$

Accordingly, (6.30) rewrites as

$$\begin{aligned}
\lim_{k\to\infty} B_e(Lz_k &+ e^{\sigma(\mathrm{L}_f+v_{0k}\mathrm{L}_g)} \circ e^{\frac{\bar{\tau}}{N}(\mathrm{L}_f+(z_{1k}+\gamma^\delta(\cdot)))\mathrm{L}_g)} \circ \dots \\
&\circ e^{\frac{\bar{\tau}}{N}(\mathrm{L}_f+(z_{Nk}+\gamma^\delta(\cdot))\mathrm{L}_g)}\gamma^\delta(x_k) \\
&- e^{\sigma(\mathrm{L}_f+v_{0k}\mathrm{L}_g)} \circ e^{\frac{\bar{\tau}}{N}(\mathrm{L}_f+\gamma^\delta(\cdot)\mathrm{L}_g)} \circ \dots \circ e^{\frac{\bar{\tau}}{N}(\mathrm{L}_f+\gamma^\delta(\cdot)\mathrm{L}_g)}\gamma^\delta(x_k)) = 0
\end{aligned}$$

so getting the result. ■

Remark 6.12. *Whenever one sets $L = \mathbf{0}_{1\times N}$, one gets dead beat convergence of z to 0 in exactly N+1 sampling instants.*

Remark 6.13. *Proposition 6.2 underlines that, the extended dynamics (6.27) with dummy output $y = z_1 = v_1 - \gamma^\delta(F^\sigma(x, v_0))$ is minimum-phase with zero-dynamics corresponding to the dynamics (6.2) which is indeed free of delays.*

Along the lines of Section 6.1.2, computational facilities can be carried out to define approximate solutions. What we underline is that, again, one can rewrite the I&I feedback as

$$\nu^\delta(\tau, x, v, z, \zeta) = L^\delta(x, v_0, z)z + \gamma^\delta(\zeta_k) \tag{6.32}$$

where $\zeta_k = x(k\delta + \tau)$ is the prediction state as defined in (5.30). Accordingly, the I&I feedback rewrites as composed of two terms:

- the mere prediction-based feedback $\gamma^\delta(\zeta_k)$;
- $L^\delta(x, v_0, z)z$ which can be interpreted as a feedback loop over the prediction error at any time-step;

Though, two main pathologies still remain even in this case:

1. a preliminary prediction of the state $x(k\delta + \sigma)$ at the time instant $t = k\delta + \sigma$ from the measure $x(k\delta)$ is necessary;
2. no feedback term is introduced by the sampled-data procedure over the prediction error of the non entire component of the state (i.e., on $x(k\delta + \sigma)$);
3. the properties yielded by the sampled-data I&I feedback hold for (5.1) in closed loop at the time instants $t = k\delta + \sigma$.

6.3 An example

Consider, again, the case of the van der Pol oscillator (5.43). When $\tau = 0$, the delay-free system (5.44) verifies Assumption 5.1 with

$$u = \gamma(x) = -x_1(1 + \frac{x_1^2}{6}) - \frac{1}{2}x_2, \quad V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1 + \frac{x_1^3}{3})^2.$$

Accordingly, the origin of the delay-free sampled-data equivalent model (5.45) is made stable by the ILM-based feedback $u_k = \gamma^{\delta[1]}(x_k)$ given in (5.46).

When assuming $u \in \mathcal{U}^\delta$ and a non-entire delay $\tau = \delta + \sigma$ for $\sigma \in [0, \delta[$, the sampled-data extended equivalent dynamics to (5.43) is provided by

$$\begin{aligned}
x_{2k+1} =& x_{2k} + \sigma x_{1k}(1 - x_{1k}v_{0k}) + \frac{\sigma^2}{2}v_{0k}(1 - 2x_{1k}v_{0k}) - \frac{\sigma^3}{3}v_{0k}^3 \\
&+ (\delta - \sigma)(x_{1k} + \sigma v_{0k})(1 - v_{1k}(x_{1k} + \sigma v_{0k})) \\
&+ \frac{(\delta-\sigma)^2}{2}v_{1k}(1 - 2v_{1k}(x_{1k} + \sigma v_{0k})) - \frac{(\delta-\sigma)^3}{3}v_{1k}^3 \\
x_{1k+1} =& x_{1k} + \sigma v_{0k} + (\delta - \sigma)v_{1k} \\
v_{0k+1} =& v_{1k} \\
v_{1k+1} =& u_k
\end{aligned} \tag{6.33}$$

when setting

$$\begin{aligned}
v_0 =& u((k-1)\delta - \sigma) = u_{k-2} \\
v_1 =& u((k-1)\delta) = u_{k-1}.
\end{aligned}$$

Accordingly, the I&I procedure proceeds as follows.

The **target dynamics** is given by

$$\begin{aligned}
\xi^2_{1k+1} =& \xi^2_{1k} + \delta\xi^1_{1k}(1 - \xi^1_{1k}\gamma^{\delta[1]}(\xi_{1k})) \\
&+ \frac{\delta^2}{2}\gamma^{\delta[1]}(\xi_{1k})(1 - 2\xi^1_{1k}\gamma^{\delta[1]}(\xi_{1k})) - \frac{\delta^3}{3}\gamma^{\delta[1]}(\xi_{1k})^3 \\
\xi^1_{1k+1} =& \xi^1_{1k} + \delta\gamma^{\delta[1]}(\xi_{1k}) \\
\xi_{2k+1} =& \gamma^{\delta[1]}(\xi_{1k})
\end{aligned} \tag{6.34}$$

with $\xi^1 = \begin{pmatrix} \xi_1^1 & \xi_1^2 \end{pmatrix}$.

Accordingly, the **immersion mapping** is given by

$$\pi^\delta(\xi) = \begin{pmatrix} \xi_1^2 - \sigma\xi_1^1(1 - \xi_1^1\xi_2) + \frac{\sigma^2}{2}\xi_2(1 - 2\xi_1^1\xi_2) + \frac{\sigma^3}{3}\xi_2^3 \\ \xi_1^1 - \sigma\xi_2 \\ \xi_2 \\ \gamma^{\delta[1]}(\xi_{1k}) \end{pmatrix}$$

while the **on-the-manifold feedback** takes the form $c^\delta(\xi) = \gamma^{\delta[1]}(\xi_{1k+1})$.

Thus, the manifold $\mathcal{M}^\delta_\sigma$ is implicitly defined by

$$\mathcal{M}^\delta_\sigma = \{(x, v_0, v_1) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \text{ s.t. } v_1 - \gamma^\delta(\tilde{x}) = 0\}$$

with $\tilde{x} = (\tilde{x}_2\ \tilde{x}_1)^\top$ as in (5.50) with

$$\begin{aligned}\tilde{x}_2 =& x_2 + \sigma x_1(1 - x_1 v_0) + \frac{\sigma^2}{2} v_0(1 - 2x_1 v_0) - \frac{\sigma^3}{3} v_0^3\\ \tilde{x}_1 =& x_1 + \sigma v_0.\end{aligned}$$

Accordingly, by defining the **off-the-manifold component** as

$$z = v_1 - \gamma^{\delta[1]}(\tilde{x})$$

one gets the dynamics

$$z_{k+1} = u_k - \gamma^{\delta[1]}(\zeta_k)$$

with ζ_k being the predictor computed in (5.52) and provided by

$$\begin{aligned}\zeta_2 =& x_2 + \sigma x_1(1 - x_1 v_0) + \frac{\sigma^2}{2} v_0(1 - 2x_1 v_0) - \frac{\sigma^3}{3} v_0^3\\ &+ \delta(x_1 + \sigma v_0)(1 - v_1(x_1 + \sigma v_0)) + \frac{\delta^2}{2} v_1(1 - 2v_1(x_1 + \sigma v_0)) - \frac{\delta^3}{3} v_1^3\\ \zeta_1 =& x_1 + \sigma v_0 + \delta v_1.\end{aligned}$$

As a result, the feedback

$$u = \ell z + \gamma^{\delta[1]}(\zeta) \tag{6.35}$$

ensures I&I stabilization of (6.33) in closed loop, and thus, SD-I&I stabilization of (5.43) whenever $\ell \in]0, 1[$.

6.3.1 Simulations

In this section, we are providing the simulation results of the prediction-based and I&I feedback laws designed over the van der Pol oscillator (5.43) under several values of the delay and of the sampling period. More in details, we are plotting

- the closed-loop trajectories of the delay-free sampled-data system (5.44) in closed loop under the ILM-based approximate feedback (5.46);
- the closed-loop trajectories of the retarded system (5.43) under sampled-data predictor-based feedback (5.53);
- the closed-loop trajectories of the retarded system (5.43) under sampled-data I&I feedback (6.35).

In doing so, we are considering three main scenarios:

1. the delay affecting (5.43) is of the entire-type (i.e., $\tau = \delta$ with $\sigma = 0$, Figures 6.1, 6.2 and 6.3);

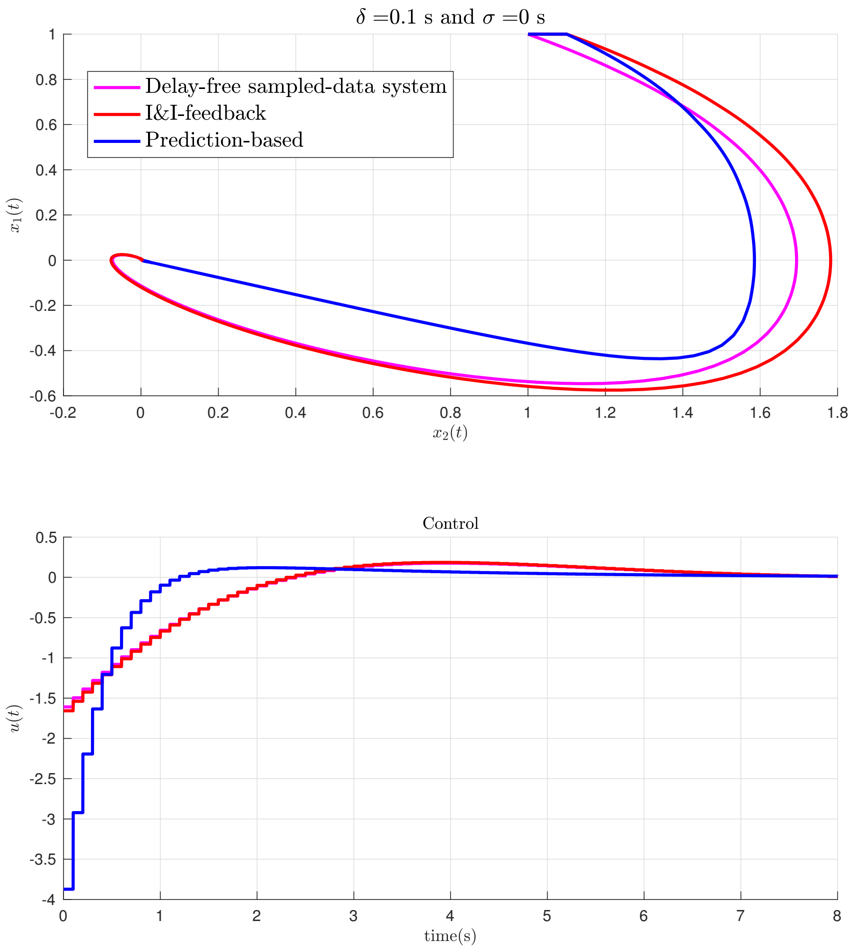


Figure 6.1: The van der Pol oscillator under prediction-based and I&I feedback laws and entire delay $\tau = \delta$.

2. the delay affecting (5.43) is of the non-entire-type (i.e., $\tau = \delta$ with $\sigma > 0$) and the design of the feedback laws (5.53) and (6.35) is carried out so to compensate its effect (Figures 6.4, 6.5 and 6.6);

3. the delay affecting (5.43) is of the non-entire-type (i.e., $\tau = \delta$ with $\sigma > 0$) but the design discards the non-entire part of the delay (i.e., the feedback laws (5.53) and (6.35) are computed for $\sigma = 0$, Figures 6.7, 6.8 and 6.9).

All simulations are assuming the initial state $x_0 = (1\ 1)^\top$ and are carried out for different values of the sampling period and of the non-entire part of the delay σ while N is constant and fixed as $N = 1$. The control signal is assumed as $u(t) = 0$ for $t \in [-\tau, 0[$

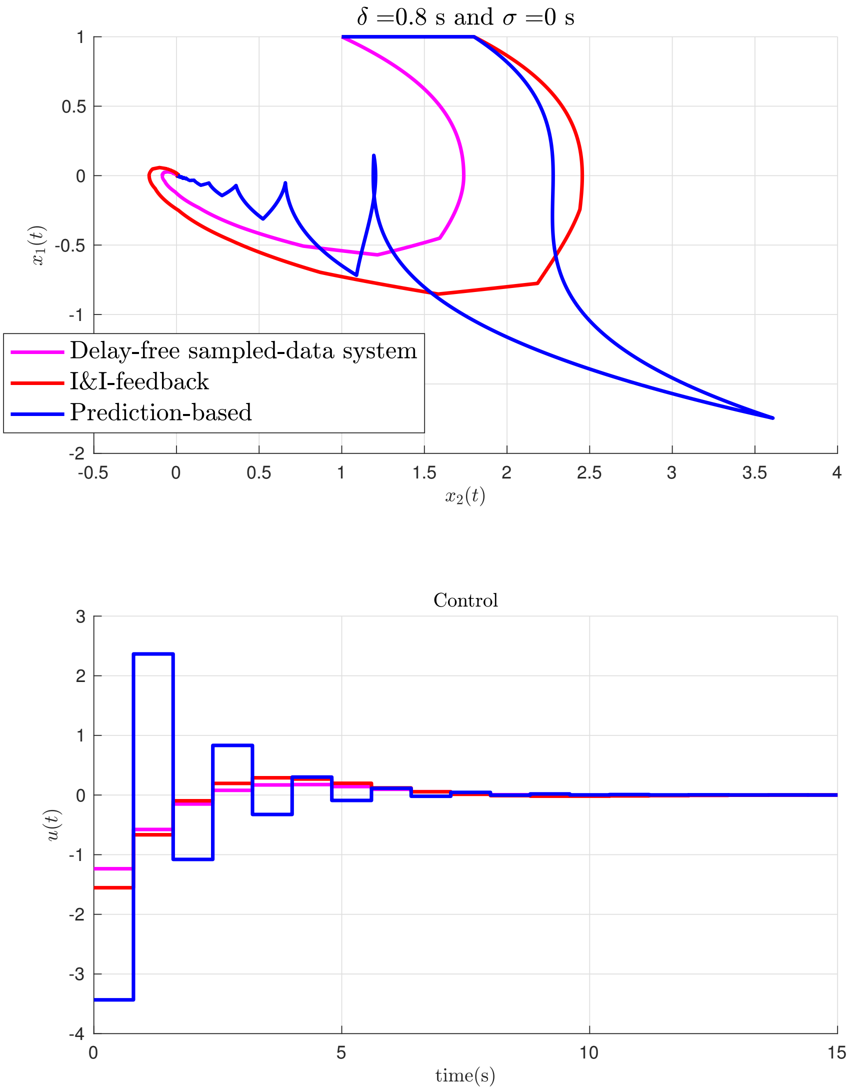


Figure 6.2: The van der Pol oscillator under prediction-based and I&I feedback laws and entire delay $\tau = \delta$.

Whenever the design is aimed at perfectly compensating the effect of the delay, it is evident that both control strategies yield good performances as δ and σ are small enough as shown in Figures 6.1 and 6.4. Though, as they both increase, the prediction-based feedback (5.53) causes degradation of the closed-loop (Figures 6.2 and 6.5) until failing in stabilizing the closed-loop origin as testified by Figures 6.3 and 6.6. These drawbacks are mainly due to the fact that the delay-free feedback (5.46) is computed as the first-order approximate solution to the ILM equality (5.18)

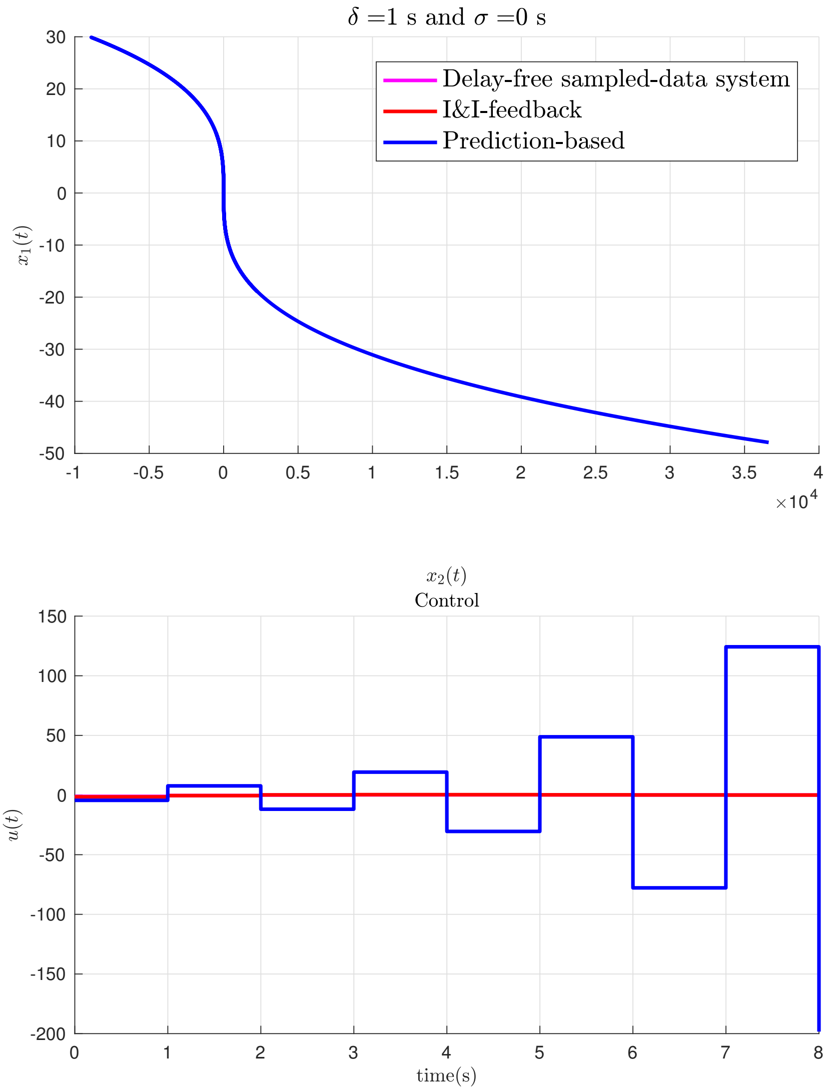


Figure 6.3: The van der Pol oscillator under prediction-based and I&I feedback laws and entire delay $\tau = \delta$.

so that the prediction-based feedback does not keep into account the corresponding discarded terms in $O(\delta^3)$. Contrarily to that case, the I&I controller (6.35) yields satisfying performances even in the case of increasing values of δ and σ being successful in achieving stability (with promising performances both in terms of smoothness of the trajectories and control effort) even when the predictor fails. Whenever σ is fully compensated by both the prediction and I&I feedbacks, the behavior of the corresponding closed-loop system is similar to the one resulting when $\sigma = 0$.

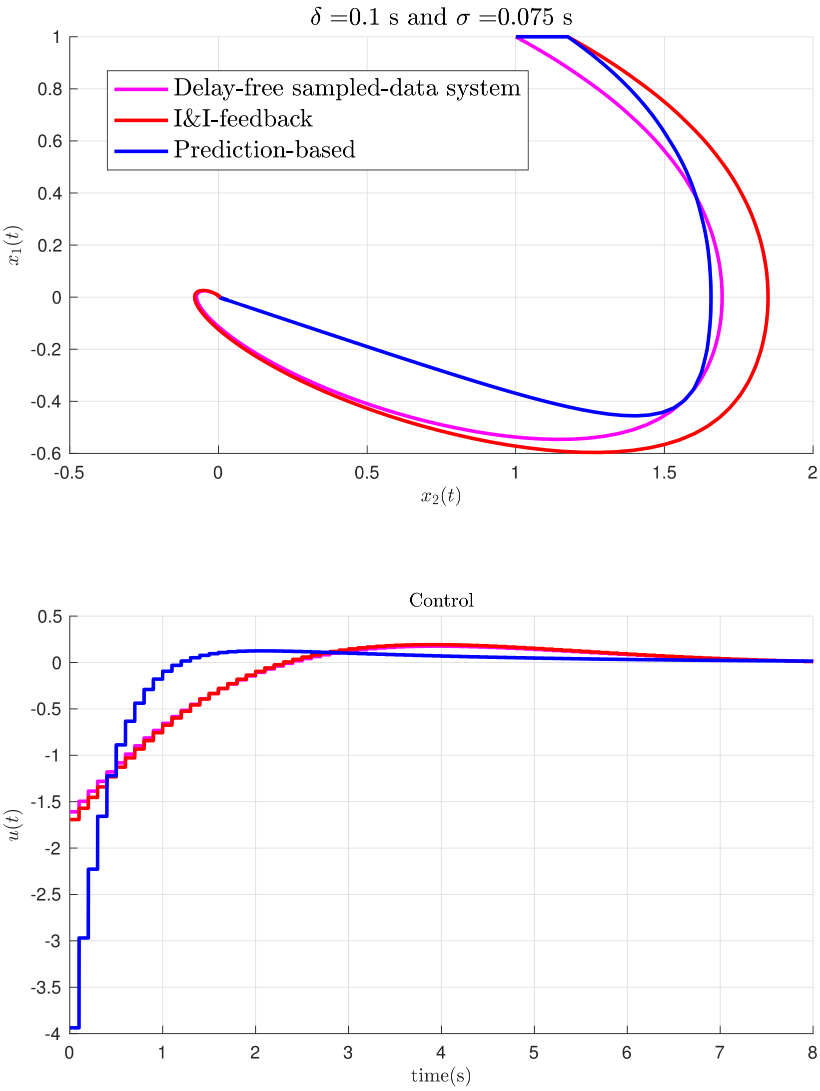


Figure 6.4: The van der Pol oscillator under prediction-based and I&I feedback laws and non-entire delay $\tau = \delta + \sigma$.

When the delay is assumed non-entire (i.e., $\tau = \delta + \sigma$) but the feedback laws are designed and implemented when discarding the effect of σ (thus, for $\tau = \delta$), the degradation of the performances yielded by the prediction-feedback (5.53) computed for $\sigma = 0$ is anticipated with respect to increasing values of δ and σ. As a matter of fact, although for small δ the results are unchanged with respect to the nominal case (Figure 6.7, as δ increases, the intrinsic non-robustness of the predictor-based feedback is evident by yielding degrading performances as in Figure 6.8 until failing

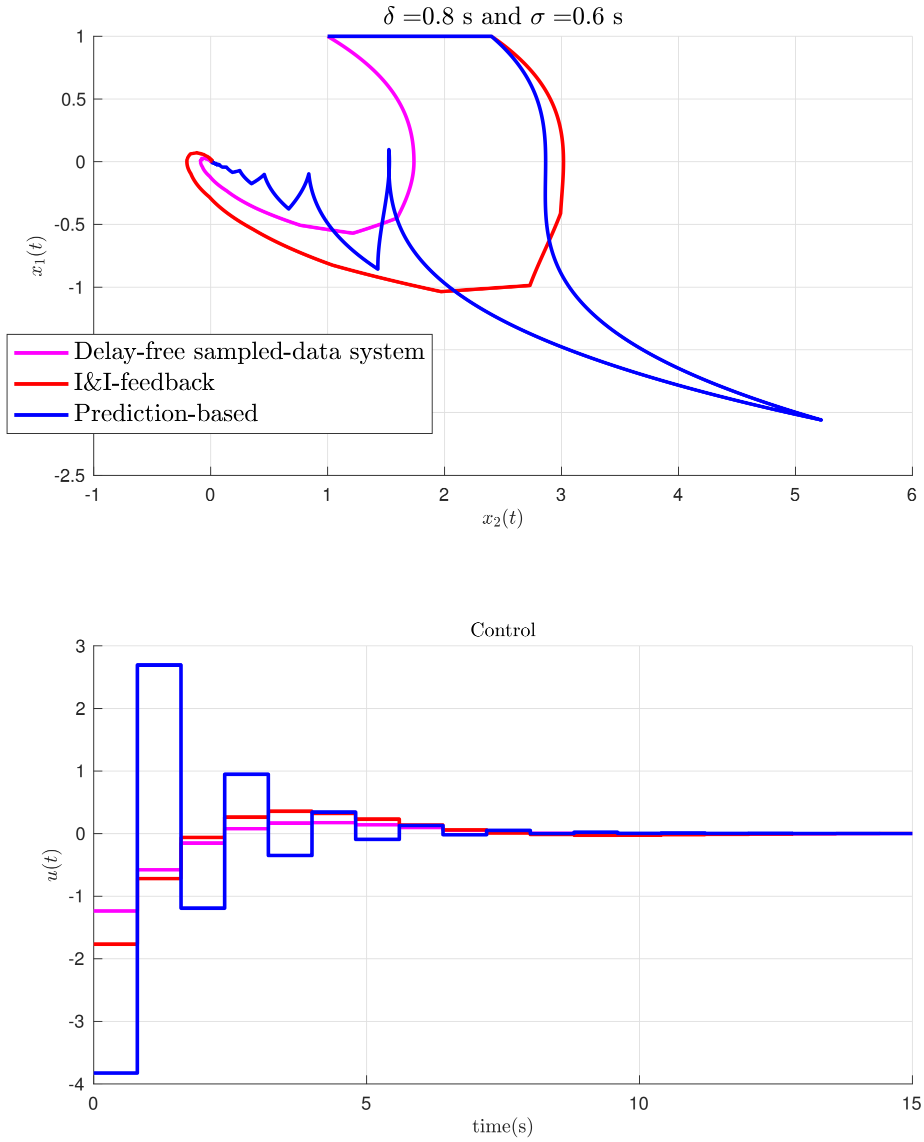
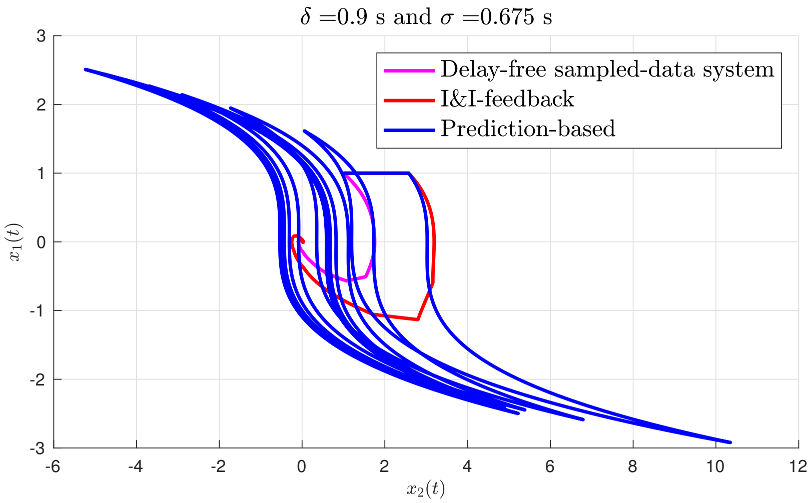


Figure 6.5: The van der Pol oscillator under prediction-based and I&I feedback laws and non-entire delay $\tau = \delta + \sigma$.

in ensuring stability as in Figure 6.9 where $\delta = 0.4$ seconds. On the other side, the I&I feedback law (5.53) yields improved performances being able to guarantee closed-loop stability even as δ and σ significantly increase.



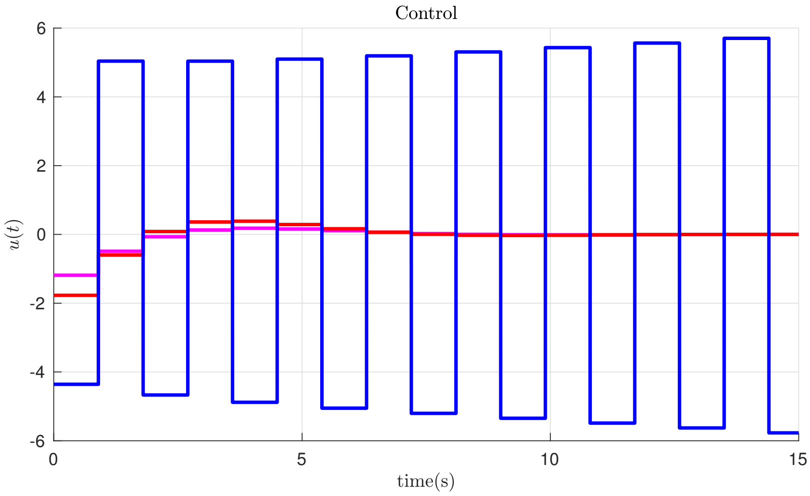


Figure 6.6: The van der Pol oscillator under prediction-based and I&I feedback laws and non-entire delay $\tau = \delta + \sigma$.

6.4 Conclusions and literature review

In this Chapter, we have proposed a new application of Immersion and Invariance for the stabilization of retarded systems under sampling affected by a constant input delay. To this end, we have exploited the finite-dimensional cascade structure describing the sampled-data equivalent model of the retarded system (5.1) in both the entire and non-entire scenarios. Smooth stabilizability of the delay-free continuous-time system (1.1a) associated to (5.1) is required. Accordingly, the I&I procedure

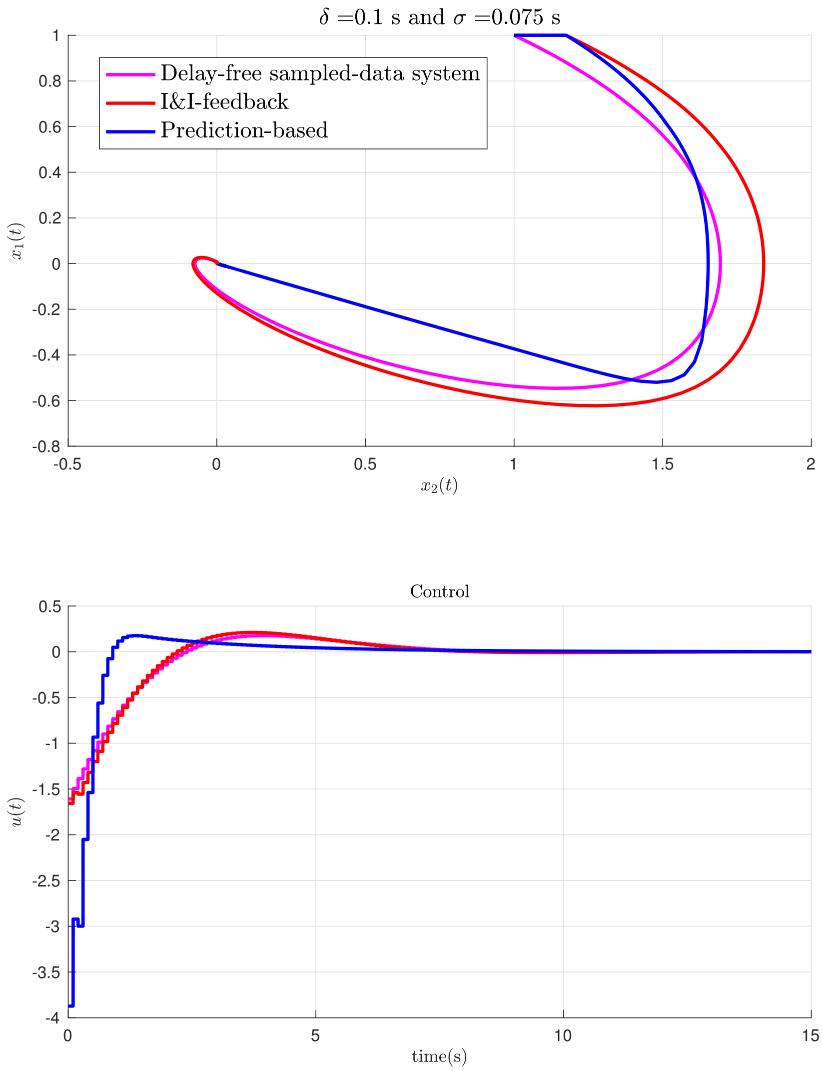


Figure 6.7: The van der Pol oscillator under non-entire delay $\tau = \delta + \sigma$ when discarding the effect of σ in the design of the prediction-based and I&I feedback laws.

we have proposed relies upon two main steps:

1. design an ILM-based feedback making the origin a GAS equilibrium for the delay-free sampled-data equivalent model (2.4a);
2. design an I&I sampled-data feedback over the extended cascade dynamics (6.1) aimed at stretching all the trajectories onto the manifold associated to the closed-loop delay-free dynamics (6.2) which, hence, define the target.

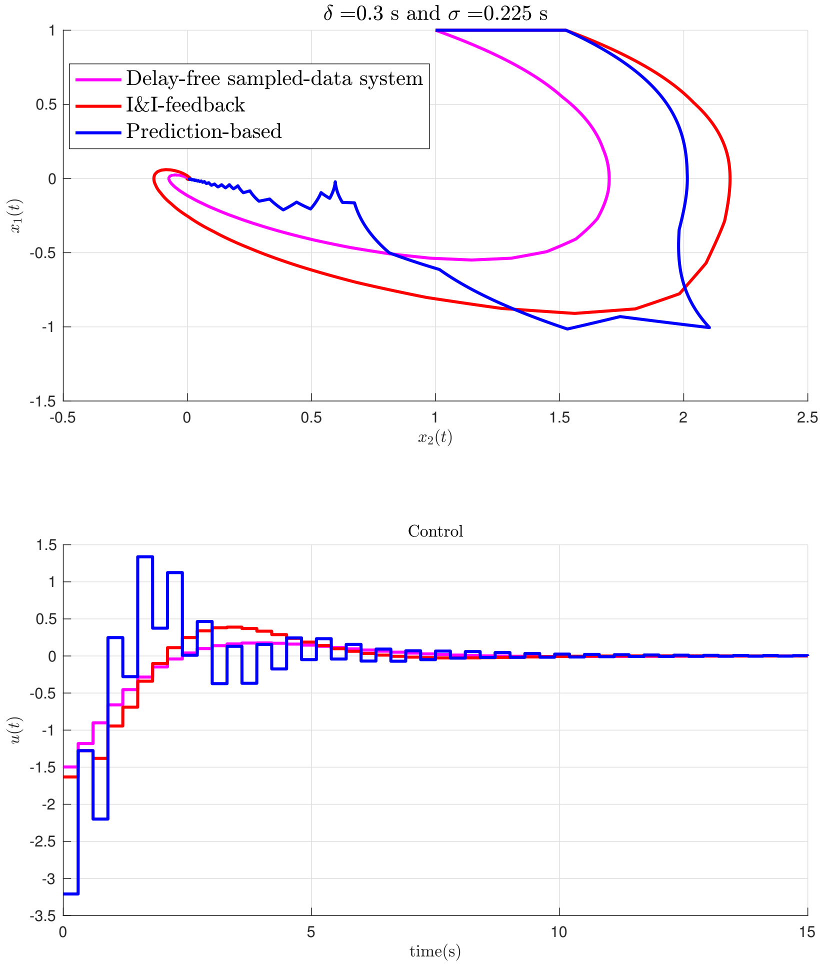


Figure 6.8: The van der Pol oscillator under non-entire delay $\tau = \delta + \sigma$ when discarding the effect of σ in the design of the prediction-based and I&I feedback laws.

Accordingly, in a very natural manner, the sampled-data I&I control law is aimed at stretching all the trajectories of the retarded system over a manifold identified by the controlled delay-free sampled-data dynamics. Moreover, the I&I control adds a new feedback loop to the standard prediction-based one which can be interpreted as a feedback over the feedback prediction error. This term implicitly enforces robustness with respect to approximations of the control law and discarded higher order dynamics (in powers of δ) in defining the predictor mapping. Though, when dealing with non-entire delays (5.7) still an open loop preliminary prediction is required to compensate the non-entire component of the delayed evolutions (i.e., $x(k\delta + \sigma)$). As a consequence, all the closed-loop properties are ensured at the

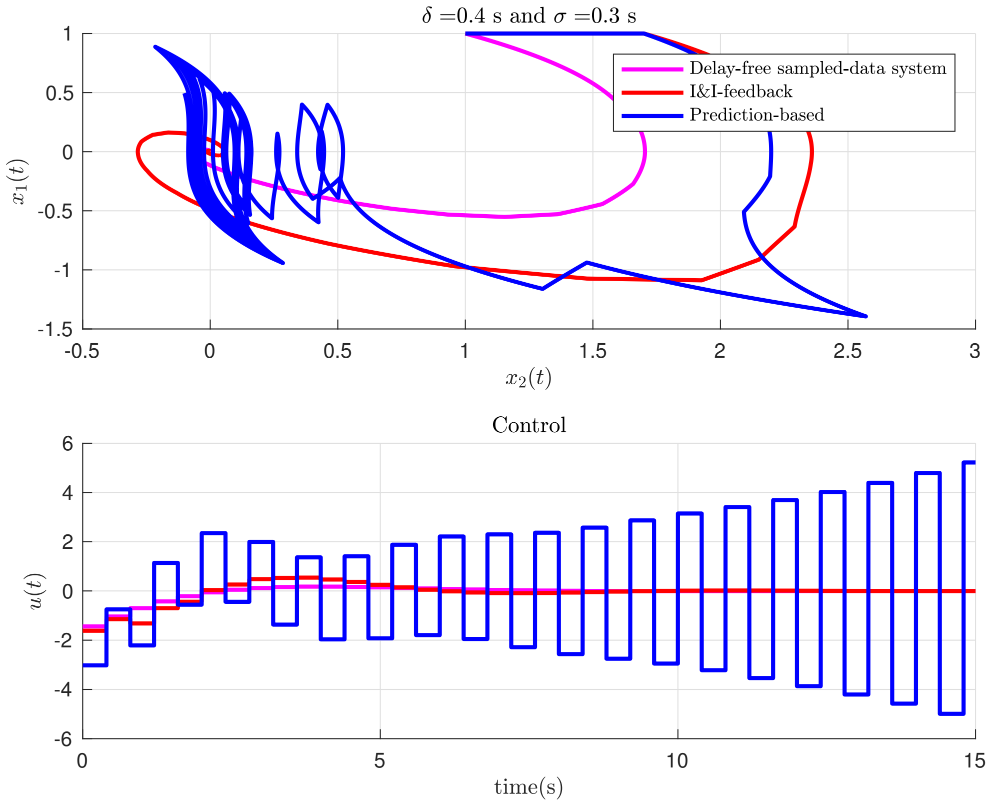


Figure 6.9: The van der Pol oscillator under non-entire delay $\tau = \delta + \sigma$ when discarding the effect of σ in the design of the prediction-based and I&I feedback laws.

shifted time instants $t = k\delta + \sigma$.

At the best of our knowledge, there is no available result on the literature about time-delay systems exploiting Immersion and Invariance for the stabilizing feedback design within' a completely sampled-data framework. A few results for discrete-time systems are available to exploit the cascade structure of retarded discrete-time dynamics for numerical robustification of the naive prediction-based feedback [67]. Those methods usually, as shown, do not apply to the case of sampled-data systems affected by non-entire delays as they do not keep into account the distributed nature of the retarded sampled-data equivalent model. Moreover, those strategies usually suffer from applicability in reality as solvability and optimization of the involved equations usually need to be done offline as they generally demand for high computational effort.

We have recently proposed a new solution to overcome the issues still remaining in the I&I retarded framework by enlarging the point of view of predictors to general reduction-based methods. Basically, we exhibit a new state whose dynamics (the reduced dynamics) is free of delays and equivalent (at least as far as stabilizability is concerned) to the original retarded system. The reduced dynamics is not coinciding with the delay-free model associated to the retarded system although they share the same drift. We have shown that this new method do not require any prediction of the inter-sampling state $x(k\delta + \sigma)$ as it only exploits $x(k\delta)$ and the most recent $N + 1$ values of the control signal. For further details on this, the reader is referred to [105].

Conclusions and Perspectives

In this manuscript, we have focused on cascade systems under sampling by providing a set of constructive procedures for designing sampled-data feedback laws achieving stabilization of the original continuous-time systems.

We have considered classes of nonlinear systems whose states are measured sporadically in time (at any sampling instant) and whose control inputs are assumed piecewise constant over time intervals of length δ, the sampling period. To this end, the design has been carried out over the sampled-data equivalent model describing the evolutions of the given system at any sampling instants $t = k\delta$ with $k \geq 0$ by emphasizing on the way the continuous-time properties (e.g., passivity) are transformed by the sampling process. Accordingly, the procedures we have proposed are aimed at preserving the ideal continuous-time design under no further assumptions involving the sampled-data nature of the system. Both direct and indirect sampled-data design methodologies have been exploited such as Input-Lyapunov Matching or u-average passivity-based design. The final controllers have been defined through nonlinear equalities which we have proved to admit a unique solution defining the feedback. As exact solutions to these equalities are tough to compute, approximate solutions and the corresponding feedback laws have been defined by discussing on the properties they yield in closed-loop. Moreover, as δ falls to zero, all the designed feedback laws and constructed mappings recover the ideal continuous-time ones. Simple academic examples have been developed to easily enlighten the computational aspects that are encountered. The extension of the proposed methodologies to the multi-input case follows these lines.

Starting from nonlinear systems exhibiting strict-feedforward structure, we have shown that a sampled-data contextualization of the I&I procedure yields a constructive way of designing the feedback under sampling despite the sampled-data equivalent model does not preserve the strict-feedback structure. To this end, we have proposed a two step procedure involving Input-Lyapunov Matching for the design of the target dynamics and multi-rate-based design for stretching the trajectories of the overall system onto the corresponding stable manifold.

Then, when considering the case of feedforward dynamics, we have proposed an iterative and constructive procedure involving u-average passivity and Lyapunov arguments. Namely, at each step, one constructs a weak Lyapunov function for the partial globally stable sampled-data equivalent dynamics so allowing to deduce a suitable average passivating output. Accordingly, the feedback is deduced by solving the damping equality over the average passivating output.

Finally, we have shown how sampling positively affects nonlinear retarded dynamics in presence of a fixed and known time-delay over the input signal. By rewriting the delay as a non-entire multiple of the sampling period (which is always possible), we have shown that the retarded sampled-data system admits an explicit and finite dimensional cascade representation that we have exploited for the design. First, we have deduced prediction-based feedback aimed at compensating the effect of the delay acting over. Then, an I&I redesign has been performed by taking advantage of the extended cascade structure so to stretching the trajectories of the retarded system onto the manifold identified by the delay-free closed-loop dynamics under an ideal stabilizing feedback.

Accordingly, a wide range of perspectives is still open on these topics. A few of them are cited below.

As far as general sampled-data systems at large are concerned, an imminent problem to face is the one arising from approximate feedback solutions and a quantitative and precise study on their stabilizing performances with respect to the amplitude of the sampling period δ. As already discussed, although some results are available for classical emulation-based feedback (i.e., zero-order approximate solutions), a deep investigation on the enhancement of adding correcting terms (i.e., increasing the order of the approximation) is still missing.

Throughout the manuscript, we have been assuming measures of the states to be available at any sampling instants. Although this is a recurrent assumption, it is not much realistic so that the investigation of the proposed designed strategies under output feedback and (possibly) state observers deserves particular attention. In this sense, interesting works on the preservation of sampled-data stabilizing feedback under emulated observers are due, among many others, to Di Fernando and Pepe in [31] and Khalil in [28, 73].

The proposed methodologies should be extended to cover the case of sampled-data systems under aperiodic or non-uniform sampling process as a new trend in the community (e.g., in [37, 36, 51, 147, 146]). The extension to this scenario is straightforward whenever the amplitude of each sampling period is apriori known; as a matter of fact, for any given a sampling sequence $t_0 < t_1 < \cdots < t_\ell$ with $t_\ell \to \infty$, one can set the sampled-data feedback as the series expansion in powers of $\delta_k := t_{k+1} - t_k$ as

$$u^{\delta_k}(x) = u^0(x) + \sum_{i>0} \frac{\delta_k^i}{(i+1)!} u^i(x).$$

Whenever those feedbacks are exactly computable and each δ_k is known, global asymptotic stability in closed loop is guaranteed turning out to be practical when introducing approximations. This is mainly due to the fact that (direct and indirect) sampled-data design strategies are carried out to deduce the feedback. Still, deducing uniformity of those properties might not be trivial. In addition, whenever the length of any δ_k is unknown (possibly due to uncertainties), a suitable stability analysis needs to be carried out to enhance robustness properties.

Concerning I&I design, the problem of preserving I&I stabilization under sampling of generally nonlinear systems and in terms of different control specifications (e.g., tracking) is of extreme interest. This problem is more and more important due to the increasing amount of applications and extensions I&I has been recently attracting in continuous time. Accordingly, a general procedure to perform sampled-data I&I redesign, starting from continuous time, might be intriguing and useful for implementational issues arising from practical applications.

Concerning feedforwarding design, an alternative way of computing the sampled-data Lyapunov functions is under current investigation. As a matter of fact, in the design we have proposed, the construction of the cross-term defining the Lyapunov function requires explicit integration of the coupling terms in the Lyapunov increment over each sampling period and along the system trajectories. Accordingly, we would like to weaken this demand by investigating other possibilities. It should be noted, that the procedure proposed by Mazenc in [116] through composite Lyapunov functions does not apply in the sampled-data context. As a matter of fact, composite Lyapunov functions are strong from the continuous-time input-affine structure that is indeed not at all preserved by sampling.

As far as time-delay systems, we are currently working on quantifying the improvements of the I&I design approach with respect to uncertainties in the length of the time-delay and higher order discarded terms in powers of δ. In doing so, we are also trying to weaken the assumption of having τ a-priori known and constant so enlarging the proposed methodology to wider classes of retarded systems. Moreover, opening this class of controllers to the case of nonlinear dynamics affected by distributed delays offers an intriguing perspective as well. In this sense, new methods combining I&I and reduction arguments through a Lyapunov-Krasovskii characterization seem to provide a promising framework to deal with those problems.

In any case, research on sampled-data systems is constantly fed by a huge amount of new challenges that go far beyond the issues addressed in this manuscript. Those challenges arise from both practical and theoretical problems which deserve particular attention and new tools as in the case of space manipulators where actuators and sensors generally work at different sampling frequencies (that cannot be assumed similar) so preventing from applying emulation-based feedback laws.

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Appendix

Appendix A

Other works

In this Appendix, the works that have not been included throughout the manuscript are included. The contributions of these papers are concerned with are sketched below.

- A constructive procedure for the **stabilization of discrete-time cascade dynamics** admitting a discrete-time feedforward structure.

  In this work, conditions for systems arising from the cascade interconnection of stable systems are given, Then, those conditions are exploited for enhancing a constructive and iterative design for stabilizing the origin of the augmented system in closed-loop. Although the tools we are using are similar to the ones presented in Chapter 4, the design is different for the following aspects: here, the dynamics are completely discrete time while, in Chapter 4, we rather study the problem of preserving forwarding design under sampling starting from the continuous-time scenario; the assumptions we require each component of the system to fullfill are stronger, in general, than the ones we require in the sampled-data setting; the final feedback controllers are not the same as applying the discrete-time procedure to the case in Chapter 4 yields a different feedback that is, in general, more conservative. A particular case is provided by strict-feedforward dynamics for which both the discrete-time and sampled-data design methodologies provide the same feedback law. This class of system is exploited to derive an alternative I&I stabilizing control.

- **The sampled-data I&I stabilization of strict-feedback dynamics with delay over the interconnecting state**.

  In this work, we consider strict-feedback dynamics affected by a time-delay over the interconnecting state component. Starting from backstepping-like assumptions over the delay-free system associated to the first component of the cascade, we show that one can solve an I&I problem over an extended system so to ensure stabilization of the retarded dynamics. The final feedback relies upon multi-rate strategy where one component is aimed at stabilizing the delay-free component of the system, while the remaining ones are devoted to compensating the effect of the delay.

- **Reduction of retarded continuous-time, discrete-time and sampled-data systems** where an alternative tool for stabilizing design is proposed based on the definition of *reduction*.

One seeks for a new state whose dynamics (the reduced dynamics) is delay-free and equivalent, at least in term of stability of the equilibrium, to the original retarded one. Accordingly, we show that any feedback asymptotically stabilizing the origin of the reduced dynamics makes the origin of the retarded system asymptotically stable as well. Several ways of designing the controller based on the properties in the delay-free case are exploited (i.e., passivity) by underlying on how the reduced dynamics preserved and/or transforms them. Under sampling, whenever the system is affected by entire delay, one can proceed through a direct design over the sampled-data equivalent model by exploiting the discrete-time tools. When the delay is of the non-entire type, one needs to exploit the sampled-data nature of the plant and cannot consider the sampled-data equivalent model as a pure discrete-time system. In that case, the reduction-based feedback preserves all the properties at the sampling instants and does not require any kind of prediction of the future trajectories. This methodology has been also shown to be applicable for stabilizing retarded dynamics affected by multi-channel input delays.

- **Stabilization of non-minimum phase systems through partial dynamic cancelation**

  In this work, we consider nonlinear non-minimum phase systems with hyperbolic equilibria. Accordingly, by exploiting a suitable partition of the polynomial identifying the zeros of the linear approximation at the origin, we exhibit a new output which is locally identifying to the stable component of the unstable original dynamics. This output rewrites as the solution of a PDE defined by the actual output of the system and the coefficients of the unstable component of the zero polynomial. This relation is exploited to achieve feedback linearization of the original dynamics while preserving stability of the internal components.

Forwarding stabilization in discrete time ⋆

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Abstract

This paper presents a feedforwarding stabilizing design for nonlinear discrete-time cascade systems. The procedure is constructive and iterative and exploits Lyapunov stability and average passivity arguments. The case of input delayed dynamics provides an interesting case of application of the method. An academic simulated example illustrates the performances.

Key words: asymptotic stabilization; discrete-time systems; application of nonlinear analysis and design; systems with time-delays.

1 Introduction

Over the last decades, nonlinear control theory has found a prolific interest with particular devotion towards continuous-time systems (e.g., Khalil (2002), Isidori (1995), Sepulchre et al. (1997)). In this framework, a lot of challenging problems have been finding several solutions exploiting, as an example, the differential geometry lying behind and the properties of the differential mathematical model describing the evolutions. As a parallel field of investigation, discrete-time systems have been attracting a growing attention in the control community especially for their recent involvement in sampled-data, networked or hybrid control. Though similar problems as in continuous time can be settled and important improvements have been made throughout the years (e.g., Monaco and Normand-Cyrot (1986), Wei and Byrnes (1994), Mazenc and Nijmeijer (1998), Nešić et al. (1999), Kazakos and Tsinias (1994), Jiang and Wang (2001), Navarro-LóPez and Fossas-Colet (2004) and Kazantzis (2004)), several difficulties and open questions still remain unsolved. Those are essentially concerned with the intricate geometric structure of the discrete-time recurrent model which is even nonlinear in the control variables. This unavoidably requires the resolution of highly nonlinear algebraic equations that define the control feedback.

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Those issues have been representing obstacles in extending ideas and methodologies that are well-known and elegant in continuous time. Among these obstacles, one can include even the primitive concept of passivity and the consequent passivity-based and Lyapunov-based designs that are fundamental when addressing stabilization of cascaded dynamics (see Lin and Gong (2003), Chiang et al. (2010), Lin and Pongvuthithum (2002), Jankovic (2006)). Several works by the authors are aimed at bridging this gap. In this sense, the differential-difference state space representation (or, (F_0,G)-form) was introduced in Monaco and Normand-Cyrot (1997) as an alternative to difference state-space representation for providing a differential geometric flow interpretation to the input to state evolutions. It was then profitably exploited to define a notion of u-average passivity Monaco and Normand-Cyrot (2011) so extending the concept of passivity to systems without direct input-output link too, a well known request in discrete time.
The present paper formulates in this framework stabilization of discrete-time cascade systems exhibiting an upper-triangular (or feedforward) mathematical model. Several studies and design methodologies in the literature are concerned with these cascade forms, with particular emphasis on a class of strict-feedforward structures. In Aranda-Bricaire and Moog (2004) and Marquez-Martinez and Moog (2004), the authors investigate equivalence to feedforward dynamics, up to coordinates change and preliminary feedback. In Mazenc and Nijmeijer (1998), the design is carried out through bounded control and then extended to the presence of disturbances in Ahmed-Ali et al. (1999). In Monaco and Normand-Cyrot (2013), a stabilizing procedure for dynamics in strict feedforward-form is developed through the computation of successive coordinates change making each

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successive sub-dynamics driftless and passive. In Monaco et al. (2016), forwarding is revisited in the Immersion and Invariance context so relaxing the a-priori knowledge of a Lyapunov function for the initial step of the design.
Stabilizing discrete-time systems in feedforward form remains challenging, because these structures are recovered in the formulation of many control problems. The aim of this work is to provide a discrete-time design approach which represents the discrete-time counterpart of continuous-time forwarding (see Sepulchre et al. (1997)). This is achieved making use of a constructive and iterative stabilizing procedure by suitably exploiting the notion of u-average passivity. In particular, by preliminarily considering the feedforward interconnection of two dynamics, we show that global asymptotic stabilization of the lower subsystem plus global stability of the upper decoupled dynamics is enough for exhibiting a u-average passivity based controller ensuring global asymptotic stability of the interconnected system. This can be achieved whenever suitable growth assumptions are verified by the coupling nonlinearities. Then, the design is extended to multiple cascade interconnection by establishing an iterative procedure aimed at stabilizing, at each step, a lower dimensional augmented cascade through average passivation. Specifically, at each step, a Lyapunov function for the partial cascade is computed through the definition of a suitable cross-term so proving average passivity of the concerned block with respect to the induced passivating $\mathrm{L}_g V$-like output. At the end of the procedure, an actual feedback for the whole system is constructive through damping control over the augmented averaged output. Additionally, we show an original application to stabilization of discrete-time dynamics affected by input delays. This is of peculiar interest when considering discrete-time dynamics issued from the sampling of retarded continuous-time systems so proposing computable stabilizing procedures. Moreover, arguing that robustness and optimality performances of the cascade design can be proved in continuous time (Sepulchre et al. (1997)), interesting perspectives are opened, as discussed in the present paper with reference to Linear Time Invariant (LTI) cascade dynamics.
More in general, any discrete-time forwarding has an immediate application into the sampled-data context, since the feedforward structure is preserved through sampling. However, taking advantage of the continuous-time original system one might deduce a less conservative sampled-data forwarding strategy which stays in-between the continuous and discrete-time scenarios. In this sense, a work on the sampling of continuous-time feedforward design has been proposed by Mattioni et al. (2017a) where the difference among the two approaches are discussed as well.
A preliminary contribution was provided by Mattioni et al. (2017b) when assuming part of the cascade globally asymptotically stable already so immediately implying u-average passivity of the augmented system. Here, those assumptions are weakened and the notion of Output-Feedback-Passivity with respect to the average output is exploited to carry out the design in this extended framework.
The paper is organized as follows. Preliminaries on discrete-time dynamics and average passivity are in Section 2. Section 3 is devoted to the computation of a Lyapunov function for uncontrolled feedforward dynamics. Constructive aspects are discussed when referring to strict-feedforward structures and other related particular forms. Section 4 states the main results. The stabilizing strategy is first described for the restricted two-block case and then extended to multi block thanks to the notion of average passivation around a nominal value. Further analysis is performed in the case of strict-feedforward dynamics to underline how known results based on invariance under suitable coordinates change are recovered. Section 5 explains how the stabilization of systems affected by input delays can be formulated as the stabilization of a particular feedforward cascade. Specifying the study on linear time invariant cascade dynamics in Section 6, one puts in light some further optimality properties of the design. Section 7 develops some computations over a simulated example while conclusions are carried out in Section 8

Notations and basic assumptions: All mappings and vector fields are assumed smooth in their arguments. Given a mapping $H : \mathbb{R}^n \to \mathbb{R}^m$ with $H(x_1,\dots,x_n)$ we define $\nabla_{x_i} H = \frac{\partial H}{\partial x_i}$ and $\nabla H = (\nabla_{x_1} H \dots \nabla_{x_n} H)$. Accordingly, $\nabla_{x_i} H(\bar{x}) = \nabla_{x_i} H(x)\big|_{x=\bar{x}}$ and, equivalently, $\nabla_x H(\bar{x}) = \nabla_x H(x)\big|_{x=\bar{x}}$.
Given a vector field G over $\mathbb{R}^n$ and a scalar function $V : \mathbb{R}^n \to \mathbb{R}$, we define the Lie derivative of V along G as $\mathrm{L}_{G(\cdot)} V(x) = \nabla V(x) G(x)$. A function $\rho : [0,\infty) \to [0,\infty)$ is said of class $\mathscr{K}$ if its continuous, strictly increasing and $\rho(0) = 0$. It is said of class $\mathscr{K}_\infty$ if it is $\mathscr{K}$ and it is unbounded. Given a mapping $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $F^{-1}(\cdot,u)$ denotes the inverse function verifying $F(F^{-1}(x,u),u) = x$. The symbol "$\circ$" denotes the composition of functions.

2 Preliminaries on discrete-time systems

Given a nonlinear discrete-time single-input dynamics Σ_D described as usual in the form of a map

$$x_{k+1} = F(x_k, u_k) \tag{1}$$

where $F(\cdot,u) : \mathbb{R}^n \to \mathbb{R}^n$ is smooth for all $u \in \mathbb{R}$ and smoothly parameterized by the control variable u, it has been proposed in Monaco and Normand-Cyrot (1997), to rewrite Σ_D in the form of two coupled differential and difference equations

$$x^+ = F_0(x) \tag{2a}$$

$$\frac{\mathrm{d}x^+(u)}{\mathrm{d}u} = G(x^+(u), u) \tag{2b}$$

with $F_0(\cdot) = F(\cdot, 0)$ and $G(\cdot,u) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfying the equality

$$G(F(x,u),u) := \nabla_u F(x,u). \tag{3}$$

In equations (2), $x^+(u)$ denotes any curve over $\mathbb{R}^n$, parameterized by u. It is a matter of computations to verify that, for any given pair (x_k, u_k) for which a solution exists, the

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integration of the differential equation (2b) over $u \in [0, u_k[$ with initial condition $x_k^+(0) = F_0(x_k)$ gives

$$x_k^+(u_k) = x_k^+(0) + \int_0^{u_k} G(x_k^+(v), v)\mathrm{d}v$$

so recovering $x_k^+(u_k) = F(x_k, u_k)$. This is straightforward from (3) when computing the Taylor expansion of the map $F(x,u)$ around $u = 0$ so obtaining

$$F(x,u) = F_0(x) + \int_0^u \nabla_v F(x,v)\mathrm{d}v.$$

Conversely, a given $\mathbb{R}^n$-valued smooth map $F(\cdot,u)$ can be split in the form (2) whenever there exists a complete vector field $G(\cdot,u)$ over $\mathbb{R}^n$, parameterized by u satisfying (3). The existence and uniqueness of $G(\cdot,u)$ are ensured by the invertibility of the mapping $F(x,0)$ in (1) with respect to x; thus, one uniquely defines $G(x,u)$, for u sufficiently small, as $G(x,u) := \nabla_u F(F^{-1}(x,u),u)$.
Consequently, given any smooth function $S(\cdot) : \mathbb{R}^n \to \mathbb{R}$, its variation with respect to u around $S(F(x,0))$ can be rewritten as

$$S(F(x,u)) - S(F(x,0)) = \int_0^u \mathrm{L}_{G(\cdot,v)} S(x^+(v))\mathrm{d}v \qquad (4)$$

where $\mathrm{L}_{G(\cdot,v)} S(\cdot)$ indicates the Lie derivative of $S(\cdot)$ along the vector field $G(\cdot,v)$; i.e. $\mathrm{L}_{G(\cdot,v)} S(x) := \nabla_x S(x) G(x,v)$.

Remark 2.1 *The representation (2) can be extended along the same lines to the multi-input case (see Monaco and Normand-Cyrot (2011) for further details).*

In the sequel, $\Sigma_D(H)$ will denote either the dynamics (1) with invertible drift term $F_0(\cdot)$ or its (F_0, G) representation with output mapping $H(\cdot) : \mathbb{R}^n \to \mathbb{R}$. Without loss of generality, it will be assumed that $\Sigma_D(H)$ possesses an equilibrium at $x = 0$; i.e. $F_0(0) = 0$ and $H(0) = 0$.

2.1 u-average passivity

The notion of *u-average passivity* has been introduced in discrete time by Monaco and Normand-Cyrot (2011) to overpass the necessity of a direct input-output link when referring to a more usual passivity notion.

Definition 2.1 (*u*-average passivity) *$\Sigma_D(H)$ is said to be u-*average passive *(or* average passive*) if it is passive in the usual sense with respect to the u-*average output

$$H^{av}(x,u) := \frac{1}{u}\int_0^u H(x^+(v))\mathrm{d}v \qquad (5)$$

$H^{av}(x,0) = H(x^+(0)) = H(F_0(x))$; i.e, there exists a positive semi-definite storage function $S : \mathbb{R}^n \to \mathbb{R}$ such that, for $k \geq 0$

$$S(x_{k+1}) - S(x_k) \leq H^{av}(x_k,u_k)u_k. \qquad (6)$$

Remark 2.2 *The u-average output rewrites as*

$$H^{av}(x,u) := \int_0^1 H(x^+(su))\mathrm{d}s = \int_0^1 H(F(x,su))\mathrm{d}s.$$

Remark 2.3 *A necessary condition for u-average passivity of $\Sigma_D(H)$ is provided by*

$$\mathrm{L}_{G(\cdot,0)} H(F_0(x)) = \nabla_u (H(F(x,u))\big|_{u=0} > 0 \qquad (7)$$

in a neighborhood of the origin. As a consequence, this requires $\Sigma_D(H)$ to possess a relative degree $r = 1$ at the origin (see Monaco and Normand-Cyrot (2011)).

Neglecting the k dependencies and exploiting the (F_0, G) representation, the passivity inequality (6) rewrites as

$$S(F_0(x)) - S(x) + \int_0^u \mathrm{L}_{G(\cdot,v)} S(x^+(v))\mathrm{d}v \leq \int_0^u H(x^+(v))\mathrm{d}v \qquad (8)$$

with by definition $\int_0^u H(x^+(v))\mathrm{d}v = H^{\mathrm{av}}(x,u)u$.

Remark 2.4 *When the output map itself depends on u, one defines analogously $H^{av}(x,u) := \frac{1}{u}\int_0^u H(x^+(v),v)\mathrm{d}v$ with $H(\cdot,u) : \mathbb{R}^n \to \mathbb{R}$, smoothly parameterized by u.*

More in general, one can define u-average passivity from some nominal control value $\bar{u}$ as follows.

Definition 2.2 (*u*-average passivity from $\bar{u}$) *$\Sigma_D(H)$ is u-*average passive from $\bar{u}$ *if there exists a positive semi-definite storage function $S : \mathbb{R}^n \to \mathbb{R}$ such that, for any $k \geq 0$*

$$S(x_{k+1}) - S(x_k) \leq H_{\bar{u}}^{av}(x_k,u_k)(u_k - \bar{u}) \qquad (9)$$

with

$$H_{\bar{u}}^{av}(x,u) = \frac{1}{u - \bar{u}}\int_{\bar{u}}^u H(x^+(v),v)\mathrm{d}v. \qquad (10)$$

Remark 2.5 *u-average passivity from $\bar{u}$ can be understood as u-average passivity of the closed loop dynamics under preliminary feedback $\bar{u}$ because*

$$\int_{\bar{u}}^u H(x^+(v),v)\mathrm{d}v = \int_0^{u-\bar{u}} H(x^+(\bar{u}+v),\bar{u}+v)\mathrm{d}v$$
$$= (u-\bar{u})\int_0^1 H(x^+((1-s)\bar{u}+su),(1-s)\bar{u}+su)\mathrm{d}s.$$

Remark 2.6 *As $\bar{u} = 0$, one recovers the classical u-average passivity definition.*

Remark 2.7 *u-average passivity from $\bar{u}$ is strictly reminiscent of the notion of incremental passivity (see Pavlov and Marconi (2008)). It defines incremental-like passivity of the overall system with respect to trajectories that are*

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parametrized by different inputs u rather than time. Moreover, contrarily to incremental passivity, u-average passivity from $\bar{u}$ is referred to the influence of the incremental-like input $\Delta u = u - \bar{u}$ over the same output trajectories.

In the sequel, the following definition is given for characterizing an excess of passivity (in the average sense).

Definition 2.3 (u-OFP(ρ)) *$\Sigma_D(H)$ is said to be u-*average output feedback passive *with $\rho \in \mathbb{R}$ (u-OFP(ρ)), if it is output-feedback passive in the classical sense with respect to the u-average output (5); i.e.,there exists a storage function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that, for any $k \geq 0$*

$$S(x_{k+1}) - S(x_k) \leq H^{av}(x_k, u_k)u_k - \rho(H^{av}(x_k, u_k))^2. \quad (11)$$

The notion of $(u - \bar{u})$*-average output feedback passivity* can be deduced through the same lines.

Definition 2.4 ($(u-\bar{u})$-OFP(ρ)) *$\Sigma_D(H)$ is said to be $(u - \bar{u})$-*average output feedback passive *with $\rho \in \mathbb{R}$ (u-OFP(ρ)), if it is output-feedback passive in the classical sense with respect to the $(u - \bar{u})$-average output (10); i.e.,there exists a storage function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that, for any $k \geq 0$*

$$S(x_{k+1}) - S(x_k) \leq H^{av}_{\bar{u}}(x_k, u_k)u_k - \rho(H^{av}_{\bar{u}}(x_k, u_k))^2.$$

2.2 u-average passivity based controller

On these bases, stabilizing u-average passivity based controller (u-AvPBC) can be deduced (Monaco and Normand-Cyrot (2011)). For, the notion of zero state detectability is instrumental.

Definition 2.5 *Consider the discrete-time system $\Sigma_D(H)$. For $u = 0$, let $\mathscr{Z} \subset \mathbb{R}^n$ be the largest positively invariant set contained in $\{x \in \mathbb{R}^n \mid y = H(x) = 0\}$. We say that $\Sigma_D(H)$ is Zero-State-Detectable (ZSD) if $x = 0$ is asymptotically stable conditionally to $\mathscr{Z}$.*

We underline for completeness that the Zero State Detectability requirement makes reference to the real system output $H(\cdot)$.
The following result extends the celebrated negative output feedback to the discrete-time context via the notion of u-average passivity.

Theorem 2.1 (Monaco and Normand-Cyrot (2011)) *Let $\Sigma_D(H)$ be u-average passive with positive storage function $S : \mathbb{R}^n \to \mathbb{R}$ and be ZSD. Then, any feedback $u = \gamma(x)$ solving the algebraic equation*

$$u + KH^{av}(x, u) = 0, \quad K > 0 \quad (12)$$

achieves global asymptotic stability of the origin of $\Sigma_D(H)$.

The existence of a solution to (12) for u sufficiently small is guaranteed by the condition

$$1 + \frac{K}{2}\mathrm{L}_{G(\cdot,0)}H(F_0(x)) > 0 \quad (13)$$

which is ensured, locally, by average passivity yielding indeed (7). However, computing a closed-form solution requires the inversion of the corresponding series expansion in u deduced from (12) (see Monaco and Normand-Cyrot (1997, 2011)). In practice, only approximate solutions can be computed by solving such algebraic equality up to a certain degree of approximation in u so yielding local properties of the closed-loop equilibrium. Nevertheless, a bounded approximate solution is deduced from the first order approximation of (12) while still guaranteeing global properties.

2.3 A computable bounded solution

Solving the equality (12) in $O(u^2)$, one easily computes

$$\begin{aligned} u^{[1]}_{ap}(x) &= -L(x)H(F_0(x)) \\ L(x) &= \frac{K}{1 + \frac{K}{2}\mathrm{L}_{G(\cdot,0)}H(F_0(x))} \end{aligned} \quad (14)$$

with $K > 0$. The approximate feedback (14) results to be a negative feedback on the output computed one step ahead over free evolution (i.e., $H(F_0(x))$). It can be proved that such a solution is bounded and still guarantees global asymptotic stability in closed loop for a suitably tuned gain $K(x)$. The following result is recalled from Mazenc and Nijmeijer (1998); Monaco et al. (2016).

Theorem 2.2 *Let $\Sigma_D(H)$ be u-average passive with positive storage function $S(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and be ZSD. Then, for any real $\mu > 0$, the feedback $u_b(x) = -\lambda(x)H(F_0(x))$ with $\lambda(\cdot)$ satisfying*

$$0 < \lambda(x) \leq \frac{\mu}{(2\mu+1)(1+\left|\frac{K}{2}\mathrm{L}_{G(\cdot,0)}H(F_0(x))\right|)} min\{1, C\} \quad (15)$$

where $K > 0$ and

$$C = min_{|u|\leq\frac{1}{2}}\left\{\frac{1}{[\int_0^1 H(F(x, su))\mathrm{d}s]^2}\right\} \quad (16)$$

is bounded (i.e., $\|u_b(x_k)\| < \mu$ for any $x_k \in \mathbb{R}^n$) and ensures global asymptotic stability of the origin of $\Sigma_D(H)$.

3 Lyapunov cross term for cascade dynamics

Consider the elementary feedforward uncontrolled dynamics

$$\Sigma_0 : \quad \begin{cases} z_{k+1} = & f(z_k) + \varphi(z_k, \xi_k) \\ \xi_{k+1} = & a(\xi_k) \end{cases}$$

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where $\xi \in \mathbb{R}^{n_\xi}$, $z \in \mathbb{R}^{n_z}$ and $u \in \mathbb{R}$; f, φ and a are assumed smooth functions in their arguments with $\varphi(z,0) = 0$. Let Σ_0 possess an equilibrium at the origin. In the sequel, for the sake of brevity, we might refer to the properties of the equilibria of the system as properties of the corresponding dynamics. The following standing assumptions are set.

A.1 $z_{k+1} = f(z_k)$ is Globally Stable - GS - with $\mathscr{K}_\infty$ Lyapunov function $W(z)$;
A.2 $\xi_{k+1} = a(\xi_k)$ is Globally Asymptotically Stable - GAS - and Locally Exponentially Stable - LES - with a C^2 and $\mathscr{K}_\infty$ Lyapunov function $U(\xi)$;
A.3 $\varphi(z,\xi)$ satisfies the *linear growth assumption*; i.e. there exist two class $\mathscr{K}$-functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ such that

$$||\varphi(z,\xi)|| \le \gamma_1(||\xi||)||z|| + \gamma_2(||\xi||);$$

A.4 $W(z)$ is C^2 and verifies what follows:
- given any $s(\cdot) : \mathbb{R}^{n_z} \to: \mathbb{R}^{n_z}$ and $d(\cdot,\cdot)\mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_z}$

$$|W(s(z)+d(z,\xi)) - W(s(z))| \le \Big|\nabla W(s(z))d(z,\xi)\Big|;$$

- there exist $c, M \in \mathbb{R}_{>0}$ such that for $\|z\| > M$, $\|\nabla W(f(z))\|\|z\| \le cW(f(z))$.

For concluding GS of the origin of Σ_0, Assumptions A.1 and A.2 are not enough because of the coupling term $\varphi(z,\xi)$ which might grow unboundedly albeit ξ converges to zero exponentially fast. To this end, we show how assumptions A.3 and A.4 enable us to deduce GS of Σ_0 and, furthermore, build a Lyapunov function $V_0 : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ for Σ_0. Thus, assume it of the form

$$V_0(z,\xi) = W(z) + U(\xi) + \Psi(z,\xi). \tag{17}$$

The additional cross-term $\Psi : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ is properly defined to ensure the semi-negativity of the increment $\Delta_k V_0(z,\xi) := V(z_{k+1},\xi_{k+1}) - V(z_k,\xi_k)$ along Σ_0. More precisely, $\Psi(z,\xi)$ is chosen so to get rid of all the coupling terms with indefinite sign $\Delta_k V_0(z,\xi)$; i.e., it has to satisfy the equality

$$\Delta_k \Psi(z,\xi) = -W(f(z_k) + \varphi(z_k,\xi_k)) + W(f(z_k)). \tag{18}$$

A solution to (18) is provided by the infinite sum

$$\Psi(z,\xi) = \sum_{k=0}^{\infty} \big[W(f(z_k) + \varphi(z_k,\xi_k)) - W(f(z_k))\big] \tag{19}$$

computed along the trajectories $(z_k,\xi_k) = (z_k(z,\xi),\xi_k(\xi))$ of Σ_0 starting at $(z_0,\xi_0) = (z,\xi)$. With such a choice, one gets that the Lyapunov function is not increasing along the trajectories of Σ_0; i.e., $\Delta_k V_0(z,\xi) \le \Delta_k U(\xi) \le 0$.
The existence of a solution is guaranteed by the Theorem below.

Theorem 3.1 *Consider Σ_0 under Assumptions **A.1** to **A.4**, then:*

(i) *there exists a continuous function $\Psi : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ solution of (18);*
(ii) *the function $V_0 : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ in (17) is positive-definite and radially unbounded.*

Proof: A complete proof is reported in the Appendix. $\triangleleft$

Remark 3.1 *It must be noted that an alternative approach for constructing a Lyapunov function for the system Σ_0 consists in defining the so-called composite Lyapunov function as developed by Mazenc and Praly (1996) in continuous time. The extension of this methodology to the discrete-time scenario is not straightforward as notable difficulties arise from the composition (rather than differentiation) of non-linear functions defining the increment of a given Lyapunov over the discrete-time Σ_0.*

A particular situation arises when Σ_0 exhibits the so-called strict-feedforward structure. In the next section approach, we shall discuss and emphasize on an interesting interpretation of the cross-term approach when detailed to this class of feedforward dynamics.

3.1 The case of strict-feedforward dynamics

Let the *strict-feedforward dynamics*

$$\Sigma_{20} : \begin{cases} z_{k+1} = & Fz_k + \varphi(\xi_k) \\ \xi_{k+1} = & a(\xi_k) \end{cases}$$

where $\varphi(0) = 0$ and the matrix F satisfies $F^\top F = I$ (all the eigenvalues are on the unit circle and with unitary geometric multiplicity). In this case, Assumption **A.1** is satisfied with $W(z) = z^\top z$ and **A.4** follows.
Specifying (18) for Σ_{20} one gets that $\Psi(\cdot)$ must satisfy the equality

$$\Delta_k \Psi(z,\xi) = -2z_k^\top F^\top \varphi(\xi_k) - \varphi^\top(\xi_k)\varphi(\xi_k). \tag{20}$$

Because in this case $\Delta_k \Psi(z,\xi) = -\Delta_k W(z)$, a solution to (20) is given by

$$\begin{aligned}\Psi(z,\xi) =& \sum_{k=0}^{\infty} \big[z_{k+1}^\top(z,\xi)z_{k+1}(z,\xi) - z_k^\top(z,\xi)z_k(z,\xi)\big] \\ =& (z_k^\top(z,\xi)z_k(z,\xi))_\infty - z^\top z\end{aligned}$$

with $(z_k^\top(z,\xi)z_k(z,\xi))_\infty = \lim_{k\to\infty} z_k^\top(z,\xi)z_k(z,\xi)$ so getting according to (17), a candidate Lyapunov function for Σ_{20} of the form

$$V_0(z,\xi) = U(\xi) + (z_k^\top(z,\xi)z_k(z,\xi))_\infty. \tag{21}$$

Other than studying the stability properties of Σ_{20} through Lyapunov functions and the definition of a cross-term, one

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might notice that, by nature, Σ_{20} possesses a stable set $\mathscr{S}$ over which the trajectories are described by

$$\xi_{k+1} = a(\xi_k).$$

$\mathscr{S}$ is implicitly defined by

$$\mathscr{S} = \{(z,\xi) \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \text{ s.t. } \phi(\xi) = 0\} \tag{22}$$

where $\phi : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_z}$ is a smooth mapping verifying

$$\phi(a(\xi)) = F\phi(\xi) + \varphi(\xi) \tag{23}$$

provided by

$$\phi(\xi) = -\sum_{\ell=k_0}^{\infty} F^{k_0-1-\ell}\varphi(\xi_\ell) \tag{24}$$

with $\xi = \xi_\ell$. Introducing now the coordinates transformation

$$\zeta = z - \phi(\xi) = z + \sum_{\ell=k_0}^{\infty} F^{k_0-1-\ell}\varphi(\xi_\ell) \tag{25}$$

one gets that Σ_{20} rewrites as the decoupled dynamics

$$\zeta_{k+1} = F\zeta_k \tag{26a}$$
$$\xi_{k+1} = a(\xi_k) \tag{26b}$$

possessing a globally stable equilibrium at the origin. A Lyapunov function for the decoupled dynamics (26) is then

$$\tilde{V}_0(\zeta,\xi) = U(\xi) + \zeta^\top\zeta. \tag{27}$$

Such a Lyapunov function comes to coincide, up to a coordinates change, with the one computed through cross-term in (21). This fact provides an interesting interpretation to the cross-term (19) as stated in the following result.

Proposition 3.1 *Let the strict-feedforward dynamics satisfy Assumption **A.1**. Then, the Lyapunov function (21) deduced from (19) and (27) computed through (25) coincide, up to a coordinates transformation; namely,*

$$V_0(z,\xi) = \tilde{V}_0(z - \phi(\xi), \xi).$$

As a consequence, the cross-term takes the form

$$\Psi(z,\xi) = (z-\phi(\xi))^\top(z-\phi(\xi)) - z^\top z. \tag{28}$$

Proof: First, rewrite $\zeta^\top\zeta$ for $k_0 = 0$ as

$$(z + \sum_{\ell=0}^{\infty} F^{-1-\ell}\varphi(\xi_\ell))^\top (F^k)^\top F^k (z + \sum_{\ell=0}^{\infty} F^{-1-\ell}\varphi(\xi_\ell))$$
$$= \|z_k(z,\xi) + \sum_{\ell=0}^{\infty} F^{k-\ell-1}\varphi(\xi_\ell) - \sum_{\ell=0}^{k-1} F^{k-\ell-1}\varphi(\xi_\ell)\|^2$$

because $(F^k)^\top F^k = I$. Letting $k \to \infty$, one gets

$$\zeta^\top\zeta = (z_k^\top(z,\xi))(z_k(z,\xi))_\infty.$$

Accordingly, setting $\Psi(z,\xi) = (z-\phi(\xi))^\top(z-\phi(\xi)) - z^\top z$ one easily recovers that the cross term verifies (20) due to the invariance equality (23). $\triangleleft$ It is important to note that the cross-term in (21) depends on $\lim_{k\to\infty}\|z_k(z,\xi)\|^2$ which always exists, for strict-feedforward structures, albeit $\lim_{k\to\infty} z_k(z,\xi)$ does not (but for the particular case of $F = 1$ and $n_z = 1$).

Remark 3.2 *The discussion we have developed in this section relates the existence of a cross-term of the form (19) with the existence of an invariant set $\mathscr{S}$ as in (22) for the strict-feedback dynamics Σ_{20} and a coordinates transformation (25) decoupling the subsystems dynamics. In this special case, this is a consequence of the non-resonance condition among the eigenvalues of both F and $\nabla a(0)$ describing Σ_{20}.*

3.2 Some further particular cases

Some particular cases are examined below.

Let Σ_0 verify **A.1** with Lyapunov function $W(z)$ such that $W(f(z)) - W(z) = 0, \forall z \in \mathbb{R}^{n_z}$. Then, (18) specializes as

$$\Delta_k\Psi(z,\xi) = -W(f(z_k) + \varphi(z_k,\xi_k)) + W(z_k) = -\Delta_k W(z)$$

and the cross-term takes the form

$$\Psi(z,\xi) = \sum_{k=0}^{\infty}\left[W(z_{k+1}) - W(z_k)\right] = W_\infty(z,\xi) - W(z)$$

with $W_\infty(z,\xi) := \lim_{k\to\infty} W(z_k(z,\xi))$. Thus, one gets

$$V_0(z,\xi) = U(\xi) + W_\infty(z,\xi).$$

If, moreover, $f(z) = z$ in Σ_0, one computes

$$z_\infty(z,\xi) = z + \sum_{\ell=0}^{\infty}\varphi(z_\ell,\xi_\ell)$$

and thus $W_\infty(z,\xi) = W(z_\infty(z,\xi))$. Accordingly, the mapping $(z,\xi) \mapsto (z_\infty,\xi)$ defines a *local* coordinates change since

$$\nabla_z z_\infty(z,\xi) = I + \sum_{\ell=0}^{\infty}\nabla_z\varphi(z_\ell,\xi_\ell)$$

and the sum vanishes at $\xi = 0$. When the connection term $\varphi(\xi,z)$ does not depend on z, the above coordinates change is globally defined as one recovers a strict-feedforward form. Let $\varphi(\xi)$ in Σ_0 be a finite polynomial of degree p. Then, the cross-term takes a polynomial form of degree $2p$; the following example illustrates the case.

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Example: Let the second order dynamics with coupling term of degree 2

$$z_{k+1} = z_k + \frac{3}{4}\xi_k^2, \quad \xi_{k+1} = \frac{1}{2}\xi_k.$$

It clearly verifies Assumptions **A.1** to **A.4** with $W(z) = z^2$ and $U(\xi) = \xi^2$. Setting the cross-term in the form of a polynomial of degree 4, $\Psi(z,\xi) = a_1 z\xi^2 + a_2\xi^4$, one easily specialises (20) as

$$\frac{a_1}{2}(z + \frac{3}{4}\xi^2)\xi^2 + \frac{a_2}{16}\xi^4 - a_1 z\xi^2 - a_2\xi^4 = \\ \frac{1}{16}\xi^4 + \frac{1}{2}\xi^2(z + \frac{3}{4}\xi^2) - \xi^4 - 2z\xi^2$$

so computing $a_1 = 2$ and $a_2 = 1$. Applying the result of the previous section, one easily gets the complete Lyapunov function $V_0(z,\xi) = z^2 + \xi^2 + 2z\xi^2 + \xi^4$ or, alternatively, the decoupling coordinates transformation with $\zeta = z + \xi^2$.

4 Feedforwarding stabilization

The previous arguments are used in the sequel to achieve stabilization of controlled feedforward dynamics of the form

$$\Sigma_e : \begin{cases} z^n_{k+1} = & f_n(z^n) + \varphi_n(z^1,\dots,z^n,\xi) + g_n(z^1,\dots,z^n,\xi,u) \\ & \vdots \\ z^1_{k+1} = & f_1(z^1) + \varphi_1(z^1,\xi) + g_1(z^1,\xi,u) \\ \xi_{k+1} = & a(\xi) + b(\xi,u) \end{cases}$$

with $\xi \in \mathbb{R}^{n_\xi}$ and $z^i \in \mathbb{R}^{n_{z^i}}$ $(i = 1,\dots,n)$, $u \in \mathbb{R}$; moreover,

$$g_i(0,\dots,0,z^i,0,0) = g_i(z^1,\dots,z^i,\xi,0) = 0 \\ \varphi_i(0,\dots,0,z^i,0,0) = \varphi_i(z^1,\dots,z^i,\xi,0) = 0$$

and $b(\xi,0) = 0$. Finally, it is assumed that any $f_i(\cdot) + \varphi_i(\cdot)$ is invertible with respect to the corresponding z^i $(i = 1,\dots,n)$ and $a(\cdot)$ invertible with respect to ξ.

The results will be first discussed with reference to the two block cascade and then generalized to Σ_e. Basically, it will be shown that, whenever Σ_e verifies, for $u = 0$, Assumptions **A.1**, **A.3**, **A.4** and a relaxed version of **A.2**, one can deduce an average passivity-based feedback achieving stabilization in closed-loop. When specified to the lower two block cascade, these arguments are then iteratively applied to a new augmented cascade including at each step a new upper block. Up to the authors' knowledge, the final result provides an original and new unifying and general feedforwarding stabilizing design for discrete-time systems. The typical difficulties related to the necessity to compute the control law as the implicit solutions to nonlinear algebraic equations still remain as usual in discrete-time. Though, average passivity concepts reveal their efficiency in describing passivating output maps for the dynamics.

4.1 *The two block controlled cascade*

Let the augmented two-blocks feedforward cascade

$$\Sigma_1 : \begin{cases} z_{k+1} = f(z_k) + \varphi(z_k,\xi_k) + g(z_k,\xi_k,u_k) & \text{(29a)} \\ \xi_{k+1} = a(\xi_k) + b(\xi_k,u_k) & \text{(29b)} \end{cases}$$

defined on $\mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi}$ with $u \in \mathbb{R}$, $g(z,\xi,0) = b(\xi,0) = 0$, which recovers Σ_0 when setting $u = 0$. let $(0,0)$ be an equilibrium. The following assumptions are set.

A.5 The mapping $g(z,\xi,u)$ satisfies the *linear growth assumption* in z for any (ξ,u).

According to Section 2, the existence of vector fields $G(\cdot,\cdot,u) : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_z}$ and $B(\cdot,u) : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_\xi}$ is guaranteed by assumption. Moreover, they satisfy

$$\nabla_u g(z,\xi,u) = G(z^+(u),\xi^+(u),u);\ \nabla_u b(\xi,u) = B(\xi^+(u),u)$$

or equivalently

$$g(z,\xi,u) = \int_0^u G(z^+(v),\xi^+(v),v)\mathrm{d}v \\ b(\xi,u) = \int_0^u B(\xi^+(v),v)\mathrm{d}v$$

with for all u

$$z^+(u) = f(z) + \varphi(z,\xi) + g(z,\xi,u) \\ \xi^+(u) = a(\xi) + b(\xi,u).$$

When necessary, one writes in a compact form $x = \mathrm{col}(z,\xi)$, $\bar{F}_0(x) = \mathrm{col}(f(z) + \varphi(z,\xi), a(\xi))$ and $\bar{G}(x,u) = \mathrm{col}(G(z,\xi,u), B(\xi,u))$.

4.1.1 *Average passivation*

A preliminary result is recalled from Mattioni et al. (2017b).

Proposition 4.1 *Let Σ_1 verify the assumptions **A.1** to **A.4** when $u \equiv 0$. Then, it is u-average passive with respect to the output mapping $H(z,\xi,u) = \mathrm{L}_{\bar{G}(\cdot,u)}V_0(z,\xi)$ and storage function (17); i.e.*

$$V_0(z_{k+1},\xi_{k+1}) - V_0(z_k,\xi_k) \le H^{av}(z_k,\xi_k,u_k)u_k \tag{30}$$

with

$$H^{av}(z,\xi,u) := \frac{1}{u}\int_0^u \mathrm{L}_{\bar{G}(\cdot,v)}V_0(z^+(v),\xi^+(v))\mathrm{d}v. \tag{31}$$

The result is an immediate consequence of the construction performed in Section 3 which provides a Lyapunov function $V_0(z,\xi)$ for the associated uncontrolled dynamics. As a consequence, over Σ_1 this implies

$$\begin{aligned}\Delta_k V_0 &= V_0(F_0(x_k) - V_0(x_k) + \int_0^{u_k} \mathrm{L}_{\bar{G}(\cdot,v)} V_0(z_k^+(v), \xi_k^+(v))\mathrm{d}v \\ &\leq \int_0^{u_k} \mathrm{L}_{\bar{G}(\cdot,v)} V_0(z_k^+(v), \xi_k^+(v))\mathrm{d}v = H^{\mathrm{av}}(z_k, \xi_k, u_k)u_k.\end{aligned}$$

For completeness, we note that by definition in (31)

$$\begin{aligned}H(z,\xi,u) &= \mathrm{L}_{\bar{G}(\cdot,v)} V_0(z,\xi) \\ &= \nabla_z V_0(z,\xi)G(z,\xi,u) + \nabla_\xi V_0(z,\xi)B(\xi,u) \\ H^{\mathrm{av}}(z,\xi,u) &= \frac{1}{u}\int_0^u \nabla_v V_0(z^+(v), \xi^+(v))\mathrm{d}v.\end{aligned}$$

4.1.2 uOFP(ρ) and uPBC

Assumption **A.2** is here relaxed as follows.

AR.2 The ξ-dynamics (29b) with output $Y_0(\xi,u) = \mathrm{L}_{B(\cdot,\xi)}U(\xi)$ is uOFP(-$\frac{1}{2}$) with radially unbounded storage function $U(\xi)$; i.e. for all $k \geq 0$

$$U(x_{k+1}) - U(x_k) \leq Y_0^{\mathrm{av}}(\xi_k,u_k)u_k + \frac{1}{2}(Y_0^{\mathrm{av}}(\xi_k,u_k))^2 \quad (32)$$

with by definition

$$Y_0^{\mathrm{av}}(\xi,u) := \frac{1}{u}\int_0^u \mathrm{L}_{B(\cdot,v)}U(\xi^+(v))\mathrm{d}v = \frac{1}{u}\int_0^u \nabla_v U(\xi^+(v))\mathrm{d}v.$$

According to Theorem 2.1, the following Lemma is straightforward.

Lemma 4.1 *Let the subdynamics (29b) verify **AR.2** and be ZSD with output $Y_0(\xi,0) = \mathrm{L}_{B(\cdot,0)}U(\xi)$. Then the control $u_0 = u_0(\xi)$ solution to*

$$u_0 = -Y_0^{av}(\xi, u_0) \quad (33)$$

makes the closed-loop equilibrium of the ξ-dynamics GAS. Moreover, if the linearization of (29b) at the origin is stabilizable, then u_0 achieves LES of the closed-loop equilibrium.

The following result is a straightforward consequence of Theorem 3.1.

Lemma 4.2 *Let Σ_1 verify Assumptions **A.1**, **AR.2**, **A.3**, **A.4** and **A.5** and let the linearization of (29b) at $\xi = 0$ be stabilizable. Then the cross-term $\Psi(\cdot,\cdot) : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ provided by*

$$\Psi(z,\xi) = \sum_{k=0}^{\infty} W(z_k^+(u_0(\xi_k)) - W(f(z_k))$$

computed along the trajectories of

$$\tilde{\Sigma}_1 : \begin{cases} z_{k+1} = & f(z_k) + \varphi(z_k,\xi_k) + g(z_k, z_k, u_0(\xi_k)) \\ \xi_{k+1} = & a(\xi_k) + b(\xi_k, u_0(\xi_k)) \end{cases}$$

exists and $V_0(z,\xi) = U(\xi) + \Psi(z,\xi) + W(z)$ is a Lyapunov function for $\tilde{\Sigma}_1$.

The result below is deduced from Theorem 2.1 and Lemmas 4.1 and 4.2. It shows that the partial state feedback $u_0(\xi)$ enables to conclude $(u - u_0(\xi))$-OFP$(-\frac{1}{2})$ of Σ_1 with storage function V_0 so recovering assumption **AR.2** stated on Σ_1.

Theorem 4.1 *Let Σ_1 verify Assumptions **A.1**, **A.3**, **A.4** and **A.5** and let (29b) verify **AR.2** and be ZSD with output $Y_0(\xi,0) = \mathrm{L}_{B(\cdot,0)}U(\xi)$. Let $u_0(\xi)$ be the solution to (33). Then, the following holds:*

(i) *Σ_1 is $(u - u_0(\xi))$-OFP(-$\frac{1}{2}$) with respect to the output*

$$Y_1(z,\xi,u) = \mathrm{L}_{\bar{G}(\cdot,u)} V_0(z,\xi) \quad (34)$$

and radially unbounded storage function $V_0(z,\xi)$ defined in (17);

(ii) *the feedback $u_1(z,\xi)$ solution of*

$$\begin{aligned}u_1 &= -Y_{u_0(\xi)}^{av}(z,\xi,u_1) \quad (35)\\ &= -\frac{1}{(u_1 - u_0(\xi))}\int_{u_0(\xi)}^{u_1} \mathrm{L}_{\bar{G}(\cdot,v)} V_0(z^+(v), \xi^+(v), v)\mathrm{d}v\end{aligned}$$

achieves GAS of the origin of Σ_1 in closed lopp;

(iii) *if the linearization of Σ_1 at the origin is stabilizable, then (35) yields LES of the closed-loop equilibrium.*

Proof: When $u = u_0(\xi)$, Lemmas 4.1 and 4.2 imply

$$\begin{aligned}\Delta_k V_0(z,\xi)\big|_{u=u_0(\xi_k)} &\leq \Delta_k U(\xi)\big|_{u=u_0(\xi_k)} \\ &\leq -\frac{1}{2}(Y_0^{\mathrm{av}}(\xi_k, u_0(\xi_k)))^2.\end{aligned}$$

Computing $\Delta_k V_0(z,\xi) = V_0(z_{k+1},\xi_{k+1}) - V_0(z_k,\xi_k)$ along Σ_1 one gets

$$\begin{aligned}\Delta_k V_0(z,\xi) &= U(a(\xi)) - U(\xi) + \int_0^u \mathrm{L}_{B(\cdot,v)}U(\xi^+(v))\mathrm{d}v \\ &+ W(f(z) + \varphi(z,\xi)) - W(z) + \int_0^u \mathrm{L}_{G(\cdot,\xi^+(v),v)}W(z^+(v))\mathrm{d}v \\ &+ \Psi(F(z,\xi)) - \Psi(z,\xi) + \int_0^u \mathrm{L}_{\bar{G}(\cdot,v)}\Psi(z^+(v),\xi^+(v))\mathrm{d}v.\end{aligned}$$

Exploiting now the properties of the cross-term $\Psi(\cdot)$ com-

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puted for $u = u_0(\xi)$, one verifies that

$$\begin{aligned}
\Delta_k V_0(z,\xi) &= \Delta_k V_0(z,\xi)\big|_{u=u_0(\xi)} + \int_{u_0(\xi)}^{u} \mathrm{L}_{\bar{G}(\cdot,v)} V_0(z^+(v),\xi^+(v))\mathrm{d}v \\
&\le -\frac{1}{2}|Y_0^{\mathrm{av}}(\xi,u_0(\xi))|^2 + \int_{u_0(\xi)}^{u} \mathrm{L}_{\bar{G}(\cdot,v)} V_0(z^+(v),\xi^+(v))\mathrm{d}v \\
&= -\frac{1}{2}|Y_0^{\mathrm{av}}(\xi,u_0(\xi))|^2 + (u-u_0(\xi))Y_{u_0(\xi)}^{\mathrm{av}}(z,\xi,u) \\
&= -\frac{1}{2}(Y_{u_0(\xi)}^{\mathrm{av}}(z,\xi,u) - Y_0^{\mathrm{av}}(z,\xi,u_0(\xi)))^2 \\
&\quad + \frac{1}{2}(Y_{u_0(\xi)}^{\mathrm{av}}(z,\xi,u))^2 + uY_{u_0(\xi)}^{\mathrm{av}}(z,\xi,u)
\end{aligned}$$

so implying $(u-u_0(\xi))$-OFP($-\frac{1}{2}$) with respect to the dummy output $Y_1(z,\xi,u) = \mathrm{L}_{\bar{G}(\cdot,u)} V_0(z,\xi)$. Consequently, the solution to the implicit equality (35) describes the damping controller ensuring

$$\begin{aligned}
\Delta_k V_0(z,\xi) \le &-\frac{1}{2}(Y_{u_0(\xi)}^{\mathrm{av}}(z,\xi,u) - Y_0^{\mathrm{av}}(z,\xi,u_0(\xi)))^2 \\
&-\frac{1}{2}(Y_{u_0(\xi)}^{\mathrm{av}}(z,\xi,u))^2 \le 0.
\end{aligned}$$

Accordingly, GAS under $u = u_1(z,\xi)$ as in (35) follows if the equilibrium of Σ_2 is GAS conditionally to the largest invariant set contained into

$$\begin{aligned}
&\{(z,\xi) \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \text{ s.t. } (Y_{u_0(\xi)}^{\mathrm{av}}(z,\xi,0) - Y_0^{\mathrm{av}}(z,\xi,u_0(\xi)))^2 \\
&\quad + (Y_{u_0(\xi)}^{\mathrm{av}}(z,\xi,0))^2 = 0\} \equiv \\
&\{(z,\xi) \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \text{ s.t. } Y_{u_0(\xi)}^{\mathrm{av}}(z,\xi,0) = Y_0^{\mathrm{av}}(z,\xi,u_0(\xi)), \\
&\quad Y_{u_0(\xi)}^{\mathrm{av}}(z,\xi,0) = 0\} \equiv \\
&\{(z,\xi) \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \text{ s.t. } Y_0^{\mathrm{av}}(z,\xi,0) = 0\}
\end{aligned}$$

so getting that ZSD of (29b) with respect to $Y_0(\xi,0) = \mathrm{L}_{B(\cdot,0)} U(\xi)$ ensures the result. LES follows when the dynamics is stabilizable in first approximation. $\triangleleft$

4.2 Extended feedforward structure-n blocks

The proposed procedure extends to the n-blocks feedforward dynamics Σ_e under the same assumptions **A.1**, **A.3**, **A.4**, **A.5** reformulated for each sub-dynamics $j = 1,\dots,n$ in a straightforward manner. Moreover, the ξ-dynamics is required to verify Assumption **AR.2**.

Basically, If the linearization of Σ_e at the origin is stabilizable, then GAS and LES of the closed-loop equilibrium can be achieved by extending the here presented strategy in a bottom-up way. The consequent procedure is aimed at exploiting OFP-like properties that are implicitly ensured, at each step i, with respect to the corresponding output Y_i. For the sake of compactness, we introduce the following notations.

$$\begin{aligned}
&\mathbf{z} = \mathrm{col}(z^n,\dots,z^1), \\
&G(\mathbf{z}^+(u),u) = \mathrm{col}(G_n(\cdot),\dots,G_1(\cdot),B(\cdot))
\end{aligned}$$

with any $G_i(\cdot,u)$ being such that

$$\nabla_u g_i(z^1,\dots,z^i,\xi,u) = G_i(z^{1+}(u),\dots,z^{i+}(u),\xi^+(u),u)$$

Step 0: Initialize with

$$\begin{aligned}
&Y_0(\xi,u) = \mathrm{L}_{B(\cdot,u)} U(\xi^+(u)) \\
&u_0 = -\frac{1}{u_0}\int_0^{u_0} \mathrm{L}_{B(\cdot,v)} U(\xi^+(v))\mathrm{d}v
\end{aligned}$$

ensuring GAS and LES of the ξ-dynamics.

Step 1: Set

$$\begin{aligned}
&V_0(z^1,\xi) = W_0(z^1) + \Psi_0(z^1,\xi) + U(\xi) \\
&\Psi_0(z^1,\xi) = \sum_{k=0}^{\infty}\left[W_0(f_1(z_k^1) + \tilde{\varphi}_1(z_k^1,\xi_k)) - W_0(f_1(z_k^1)\right] \\
&\tilde{\varphi}_1(z_1,\xi) = \varphi_1(z_1,\xi) + \int_0^{u_0} \mathrm{L}_{G(\cdot,v)} V_0(\mathbf{z}^+(v),\xi^+(v))\mathrm{d}v \\
&Y_1(z^1,\xi,u) = \mathrm{L}_{G(\cdot,u)} V_0(\mathbf{z},\xi) \\
&u_1 = -\frac{1}{u_1-u_0}\int_{u_0}^{u_1} Y_1(z^{1+}(v),\xi^+(v),v)\mathrm{d}v.
\end{aligned}$$

Now the design can be reported to the case $n = 2$. For, one sets at each step i, $\bar{\xi}_i = col(z^{i-1},\dots,z,\xi)$ that clearly verifies **AR.2** by construction. More in detail, one proceeds as follows.

Step i: Define

$$\begin{aligned}
V_{i-1}(\cdot) &= W_{i-1}(z^i) + \Psi_{i-1}(z^1,\dots,z^i,\xi) \\
&\quad + V_{i-2}(z^1,\dots,z^{i-1},\xi) \\
\Psi_{i-1}(\cdot) &= \sum_{k=0}^{\infty}\left[W_{i-1}(f_i(z_k^1) + \tilde{\varphi}_i(z_k^1,\dots,z_k^i,\xi_k))\right. \\
&\quad \left. - W_{i-1}(f_1(z_k^1)\right] \\
\tilde{\varphi}_i(\cdot) &= \varphi_i(z^1,\dots,z^i,\xi) + \\
&\quad \int_0^{u_{i-1}} \mathrm{L}_{G(\cdot,v)} V_{i-1}(\mathbf{z}^+(v),\xi^+(v))\mathrm{d}v \\
Y_i(z^1,\dots,z^i,\xi,u) &= \mathrm{L}_{G(\cdot,u)} V_{i-1}(\mathbf{z},\xi) \\
u_i &= -\frac{1}{u_i-u_{i-1}}\int_{u_{i-1}}^{u_i} Y_i(z^{1+}(v),\dots,z^{i+}(v),\xi^+(v),v)\mathrm{d}v
\end{aligned}$$

where the sum is evaluated along the trajectories of Σ_e from the initial state $(z^i,\dots,z^1,\xi)$ and under the feedback u_{i-1}.

Accordingly, by applying this procedure n times one gets the following result.

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Theorem 4.2 *Let any sub-dynamics z^j of Σ_e verify Assumptions **A.1**, **A.3**, **A.4**, **A.5**. Moreover, the ξ-dynamics is required to verify Assumption **AR.2** and to be ZSD with respect to the output $Y_0(\xi,u) = \mathrm{L}_{B(\cdot,u)}U(\xi)$. Then, if Σ_e is stabilizable in first approximation, the control control $u = u_n(\mathbf{z},\xi)$ computed as the implicit solution of*

$$u_n = -\frac{1}{u_n - u_{n-1}}\int_{u_{n-1}}^{u_n} \mathrm{L}_{G(\cdot,v)}V_{n-1}(\mathbf{z}^+(v),\xi^+(v))\mathrm{d}v.$$

with

$$V_{n-1}(\mathbf{z},\xi) = U(\xi) + \sum_{i=1}^{n}\left[W_{i-1}(z^i) + \Psi_{i-1}(z^1,\ldots,z^i,\xi)\right]$$

makes the closed-look origin of Σ_e GAS and LES.

4.3 The case of strict-feedforward dynamics

Consider now the augmented strict-feedforward dynamics

$$\Sigma_2: \begin{cases} z_{k+1} = & Fz_k + \varphi(\xi_k) + g(\xi_k,u_k) \\ \xi_{k+1} = & a(\xi_k) + b(\xi_k,u_k) \end{cases}$$

with F satisfying $F^\top F = I$ and the dynamics $\xi_{k+1} = a(\xi_k)$ invertible. Moreover, one verifies by definition that

$$\begin{aligned}\nabla_u g(\xi,u) &= G(a(\xi)+b(\xi,u),u);\\ \nabla_u b(\xi,u) &= B(a(\xi)+b(\xi,u),u).\end{aligned}$$

As previously noted for Σ_0, **A.1** and **A.4** are verified by setting $W(z) = z^\top z$, while **A.3** and **A.5** relax to requiring that $\|\varphi(\xi)\| \leq \gamma_1(\|\xi\|)$ and $\|g(\xi,u)\| \leq \gamma_2(\|(\xi,u)\|)$ for some $\mathscr{K}$ functions $\gamma_i(\cdot)$ $(i=1,2)$. Assuming now **AR.2** and stabilizability of the ξ-system at the origin, the control $u = u_0(\xi)$ can be constructed so to make the equilibrium of the ξ-dynamics GAS and LES. Consequently, Lemma 4.2 applies and one can find a cross-term $\Psi(z,\xi)$ solution to

$$\Delta_k\Psi(z,\xi) = -W(Fz_k + \varphi(\xi_k) + g(\xi_k,u_0(\xi_k))) + W(z_k)$$

along the trajectories of

$$\tilde{\Sigma}_2 \begin{cases} z_{k+1} = & Fz_k + \varphi(\xi_k) + g(\xi_k,u_0(\xi_k)) \\ \xi_{k+1} = & a(\xi_k) + b(\xi_k,u_0(\xi_k)). \end{cases}$$

As discussed before, under the preliminary feedback $u_0(\xi)$, one computes the coordinates change $\zeta = z - \tilde{\phi}(\xi)$ as

$$\tilde{\phi}(\xi) = -\sum_{\ell=k_0}^{\infty} F^{k_0-1-\ell}\tilde{\varphi}(\xi_\ell).$$

In the new coordinates, one gets the decoupled system

$$\zeta_{k+1} = F\zeta_k, \quad \xi_{k+1} = \tilde{a}(\xi_k)$$

with Lyapunov function $\tilde{V}_0(\zeta,\xi) = U(\xi) + \zeta^\top\zeta$ which coincides, up to a coordinate transformation, with $V_0(z,\xi)$. Thus, the problem of stabilizing Σ_2 via the cross-term can be re-addressed into the one of stabilizing Σ_2 in the new coordinates by exploiting the preliminary design $u_0(\xi) = U_0$. Thus, in the (ζ,ξ) coordinates one has that Σ_2 gets the form

$$\begin{aligned}\zeta_{k+1} &= F\zeta_k + \int_{u_0(\xi_k)}^{u_k} G_\zeta(\xi^+(v),v)\mathrm{d}v\\ \xi_{k+1} &= a(\xi_k) + \int_0^{u_0(\xi_k)} B(\xi^+(v),v)\mathrm{d}v + \int_{u_0(\xi_k)}^{u_k} B(\xi^+(v),v)\mathrm{d}v\end{aligned}$$

where

$$G_\zeta(\xi^+(u),u) = G(\xi^+(u),u) - \mathrm{L}_{B(\cdot,u)}\phi(\xi^+(u)).$$

Hence, Theorem 4.1 holds with output

$$Y_1(\zeta,\xi,u) = \mathrm{L}_{\bar{G}_\zeta(\cdot,u)}\tilde{V}_0(\zeta,\xi) \tag{36}$$

and stabilizing feedback $u = u_1(z,\xi)$ solution of

$$u = -\frac{1}{(u-u_0(\xi))}\int_{u(\xi)}^{u} \mathrm{L}_{G_\zeta(\cdot,v)}\tilde{V}_0(\zeta^+(v),\xi^+(v),v)\mathrm{d}v. \tag{37}$$

Remark 4.1 *When $F = I$ and $n_z = 1$, the coordinates change $\zeta = z - \tilde{\phi}(\xi)$ makes the ζ-dynamics driftless once the preliminary control $u_0(\xi)$ has been applied. Accordingly, one recovers the result in Monaco and Normand-Cyrot (2013) proposed when assuming directly in Σ_2, $\xi_{k+1} = u_k$.*

Remark 4.2 *In Monaco et al. (2016), the stabilization problem of strict-feedforward systems is set in the framework of Immersion and Invariance (I&I, Astolfi and Ortega (2003)) when $n_z = 1$. Assuming **AR.2**, a stable set over which the closed loop ξ-dynamics evolves is exhibited. The design aims at stretching the off-stable set components ζ to zero while ensuring boundedness of the full state trajectories. Moreover, I&I is less demanding since the knowledge of a Lyapunov function $U(\xi)$ for the ξ-system is not necessary. On the other hand, the presented cross term approach covers a wider range of cases.*

Example: Consider the discrete-time system in strict-feedforward form (skipping the $\big|_k$ index into the right hand side)

$$\begin{aligned}z_{k+1} &= z + \xi + u(\frac{1}{2} - \xi^2) - u^2\xi - \frac{1}{3}u^3\\ \xi_{k+1} &= \xi + u\end{aligned}$$

or equivalently

$$\begin{aligned}&z^+ = z + \xi; \quad \xi^+ = \xi\\ &\frac{\partial z^+(u)}{\partial u} = \frac{1}{2} - (\xi^+(u))^2; \quad \frac{\partial \xi^+(u)}{\partial u} = 1\end{aligned}$$

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which verifies Assumption **A.1** with $W(z) = \frac{1}{2}z^2$. Assumption **AR.2** holds so that one computes the preliminary feedback $u_0(\xi) = -\frac{2}{3}\xi$ with $U(\xi) = \frac{1}{2}\xi^2$. The cross term $\Psi(z,\xi) = \frac{1}{2}(z+\xi+\frac{\xi^3}{3})^2 - \frac{1}{2}z^2$ verifies $\Delta_k V_0(z,\xi) = \Delta_k U(\xi) = -\frac{4}{9}\xi_k^2$. Finally, the u-average output and the consequent control are provided by

$$H(z,\xi,u) = 4\xi + \frac{3}{2}z + \frac{13}{8}u + \frac{1}{2}\xi^3$$
$$u_1(z,\xi) = -\frac{4}{7}z - \frac{32}{21}\xi - \frac{4}{21}\xi^3.$$

Remark 4.3 *The procedure in Section 4.2 specifies to multi-block strict feedforward dynamics along the same lines. At each step, one looks for a coordinates change $\zeta^i = z^i - \phi_i(\zeta^1,\dots,\zeta^{i-1},\xi)$ that decouples the corresponding dynamics in the new coordinates when $u = u_{i-1}(z^1,\dots,z^{i-1},\xi)$. As a matter of fact, at each step, one makes the set $\zeta_i = 0$ globally asymptotically stable for the augmented cascade. Furthermore, such a set is made invariant by the control $u_i(z^1,\dots,z^i,\xi)$ which also makes it attractive and achieves GAS of the augmented cascade.*

5 Stabilization of delayed dynamics

When considering discrete-time retarded dynamics affected by input delays, the feedforward structure is naturally recovered when introducing a suitably defined extended system. This yields an interesting application of the stablizing procedure previously discussed. As a matter of fact, a u-average passivity based controller for the retarded system is here proposed as an alternative to predictor-based or reduction strategies (see, for example, Fridman (2014), Monaco et al. (2017), Karafyllis et al. (2016)).
Consider the nonlinear discrete-time retarded system

$$\Sigma_{del}: \quad z_{k+1} = f(z_k) + \varphi(z_k, u_{k-N})$$

with $z \in \mathbb{R}^n$, $N \in \mathbb{N}$ and equilibrium at $z = 0$ satisfying, mutatis mutandis, **A.1**, **A.3** and **A.4**.
Making use of the usual representation of delayed systems over the extended state space $\mathbb{R}^n \times \mathbb{R}^N$ (see Monaco and Normand-Cyrot (2015), Karafyllis and Krstic (2013)), one gets

$$\bar{\Sigma}_{del}: \begin{cases} z_{k+1} = f(z_k) + \varphi(z_k, \xi_{1k}) \\ \xi_{k+1} = A\xi_k + Bu_k \end{cases}$$

with $\xi = col(\xi_1,\dots,\xi_N)$ and

$$A = \begin{pmatrix} 0_{(N-1)\times 1} & I_{N-1} \\ 0_{1\times 1} & 0_{1\times(N-1)} \end{pmatrix}; \quad B = \begin{pmatrix} 0_{(N-1)\times 1} \\ 1 \end{pmatrix}.$$

$\bar{\Sigma}_{del}$ clearly exhibits the feedforward form of Σ_1 with in particular $g(z,\xi,u) = 0$ and linear ξ-dynamics so that it can be immediately verified that Assumptions **A.1**, **A.2**, **A.3** and **A.4** hold for the extended system $\bar{\Sigma}_{del}$.

Before proceeding on the stabilization of the retarded system, let us note that the delay free system ($N = 0$) satisfying Assumption **A.1**

$$\Sigma_{free}: \quad z_{k+1} = f(z_k) + \varphi(z_k, u_k)$$

admits a stabilizing average-based feedback of the form

$$\begin{aligned} u_f &= -\frac{1}{u_f}\int_0^{u_f} \mathrm{L}_{G_\varphi(\cdot,v)} W(f(z) + \varphi(z,v))\mathrm{d}v \\ &= -\int_0^1 \mathrm{L}_{G_\varphi(\cdot,su_f)} W(f(z) + \varphi(z,su_f))\mathrm{d}s \end{aligned} \tag{38}$$

whenever it is ZSD with respect to the passive output

$$Y_f(z,u) = \mathrm{L}_{G_\varphi(z,u)} W(z) \tag{39}$$

with $G_\varphi(f(z) + \varphi(z,u), u) = \nabla_u \varphi(z,u)$.
The same arguments can be used to stabilize the delayed system on the basis of u-average passivity of $\bar{\Sigma}_{del}$ with respect to a Lyapunov function constructed along the lines of Section 4. More in detail, one sets

$$V_0(z,\xi) = \frac{1}{2}\xi^\top \xi + \Psi(z,\xi) + W(z)$$

where the cross-term $\Psi(z,\xi)$ has to verify

$$\Delta_k \Psi(z,\xi) = -W(f(z_k) + \varphi(z_k, \xi_{1k})) + W(f(z_k)) \tag{40}$$

along the trajectories of $\bar{\Sigma}_{del}$ when $u = 0$. Thus, one gets

$$\Psi(z,\xi) = \sum_{\ell=0}^{\infty} [W(f(z_\ell) + \varphi(z_\ell, \xi_{1\ell})) + W(f(z_\ell))].$$

Now, by recalling that $\varphi(\cdot, 0) = 0$ and that, for $u = 0$, $\xi_{1\ell} = 0$ for $\ell \geq N$, one gets that the above expression is finitely provided by

$$\begin{aligned} \Psi(z,\xi) &= \sum_{\ell=0}^{N-1} [W(f(z_\ell) + \varphi(z_\ell, \xi_{1\ell})) + W(f(z_\ell))] \\ &= \sum_{\ell=0}^{N-1} \int_0^{\xi_{\ell+1,0}} \mathrm{L}_{G_\varphi(\cdot,v)} W(f(z_\ell) + \varphi(z_\ell, v))\mathrm{d}v \end{aligned}$$

where $G_\varphi(f(z) + \varphi(z,\xi_1), \xi_1) = \nabla_{\xi_1}\varphi(z,\xi_1)$ and $\xi_{\ell 0} = \xi_\ell$ for $\ell = 1,\dots,N$.

Accordingly, the following result can be stated specifying Theorem (4.1) to this case.

Corollary 5.1 *Under Assumptions **A.1**, **A.3** and **A.4**, $\bar{\Sigma}_{del}$ is average passive with respect to the output*

$$Y_1(z_k, \xi_k) = \xi_{Nk} + \mathrm{L}_{G_\varphi(\cdot,\xi_{Nk})} W(z_{k+N}) \tag{41}$$

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with, for $i = 1, \dots, N$

$$z_{k+i} = \big(f(\cdot) + \varphi(\cdot, \xi_{ik})\big) \circ \cdots \circ \big(f(z_k) + \varphi(z_k, \xi_{1k})\big).$$

Then, the feedback solution to the implicit equality

$$u_r = -\frac{2}{3} \int_0^1 \mathrm{L}_{G_\varphi(\cdot, su_r)} W(f(z_{k+N}) + \varphi(z_{k+N}, su_r))\mathrm{d}s \quad (42)$$

globally asymptotically stabilizes the closed-loop system Σ_{del} whenever Σ_{free} is ZSD with respect to (39).

Remark 5.1 *The feedback solution (42) is proportional to prediction-based feedback computed over (38) through the gain $k = \frac{2}{3}$ which is introduced by the feedforwarding approach over the extended system Σ_{del}. Accordingly, applying this approach to the time-delay framework restitutes the prediction approach that is indeed enriched by a control-Lyapunov function constructed through the cross-term in a very natural and constructive way.*

6 Linear systems as a case study

In the sequel the feedforward procedure is specified for stabilizable Linear Time Invariant (LTI) systems of the form

$$\Sigma_2^L: \quad \begin{cases} z_{k+1} = & Fz_k + \Gamma\xi_k + Gu_k \\ \xi_{k+1} = & A\xi_k + Bu_k \end{cases}$$

verifying Assumption **AR.2** with $U(\xi) = \xi^\top P_0 \xi$ ($P_0 > 0$) while the remaining Assumptions clearly hold. F, G, A, B are matrices of appropriate dimensions defining Σ_2^L.

It is a matter of computations to verify that the preliminary output mapping and feedback verifying Lemma 4.1 are

$$Y_0(\xi) = 2B^\top P_0 \xi \quad (43)$$
$$u_0 = -K_0\xi \quad \text{with} \quad K_0 = 2(1 + B^\top P_0 B)^{-1} B^\top P_0 A \quad (44)$$

under which exponential stability of the ξ-subsystem is achieved provided that the couple $(A, B^\top P_0)$ is detectable.

Computing now the cross-term $\Psi(z, \xi)$ as the solution to

$$\Delta_k \Psi(z, \xi) = -2z_k^\top F^\top \tilde{\Gamma} \xi_k \quad \text{with} \quad \tilde{\Gamma} = \Gamma - GK_0 \quad (45)$$

one gets

$$\Psi(z, \xi) = \begin{pmatrix} z^\top & \xi^\top \end{pmatrix} \begin{pmatrix} \mathbf{0} & \Psi_{12} \\ \Psi_{12}^\top & \Psi_{22} \end{pmatrix} \begin{pmatrix} z \\ \xi \end{pmatrix} \quad (46)$$

with

$$\Psi_{12} = \sum_{\ell = k_0}^{\infty} F^{k_0 - \ell - 1} F^\top \tilde{\Gamma} (A - BK_0)^{\ell - k_0}$$
$$\Psi_{22} = \Psi_{12}^\top \Psi_{12}.$$

Denoting now

$$P_1 = \begin{pmatrix} I & \Psi_{12} \\ \Psi_{12}^\top & P_0 + \Psi_{22} \end{pmatrix}, \ \bar{A} = \begin{pmatrix} F & \Gamma \\ 0 & A \end{pmatrix} \text{ and } \bar{B} = \begin{pmatrix} G \\ B \end{pmatrix}$$

Theorem 4.1 specifies as follows.

Corollary 6.1 *Let the strict-feedforward linear system Σ_2^L verify **AR.2** with $U(\xi) = \xi^\top P_0 \xi$ and suppose it is stabilizable. Then, the following holds.*

- *Σ_2^L is u-average passive with respect to the output*

  $$Y_1(z, \xi) = 2\bar{B}^\top P_1 \begin{pmatrix} z \\ \xi \end{pmatrix}$$

  and storage function

  $$V_0(z, \xi) = \begin{pmatrix} z^\top & \xi^\top \end{pmatrix} P_1 \begin{pmatrix} z \\ \xi \end{pmatrix};$$

- *the feedback*

  $$u_1 = -2\Big(1 + \bar{B}^\top P_1 \bar{B}\Big)^{-1} \bar{B}^\top P_1 A_c \begin{pmatrix} z \\ \xi \end{pmatrix} \quad (47)$$

  with $A_c = \bar{A} - \frac{\bar{B}}{2}\begin{pmatrix} K_0 & 0 \end{pmatrix}$ achieves asymptotic stability of the equilibrium provided Σ_2^L with output $Y_1(z, \xi)$ is detectable.

Remark 6.1 *In the LTI case, at each step of the procedure, one computes a feedback that is optimal with respect to a given linear quadratic index. As a matter of fact, at the initial step, the linear feedback*

$$u_0^* = -K_0^* \xi \quad \textit{with} \quad K_0^* = \Big(1 + B^\top P_0 B\Big)^{-1} B^\top P_0 A \quad (48)$$

optimally asymptotically stabilizes the ξ-system with cost functional

$$J_0(\xi_0) = \sum_{k=0}^{\infty} \left[\xi_k^\top Q_0 \xi_k + u_k^\top u_k\right]$$
$$Q_0 = -(A - BK_0^*)^\top P_0 (A - BK_0^*) + P_0 + K_0^{*\top} K_0^*.$$

Then, at the fist step, the feedback

$$u_1^* = -K_1^* \begin{pmatrix} z \\ \xi \end{pmatrix} \quad \textit{with} \quad K_1^* = \Big(1 + \bar{B}^\top P_0 \bar{B}\Big)^{-1} \bar{B}^\top P_0 A_c^*$$

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with $A_c^* = \bar{A} - \frac{\bar{B}}{2}\begin{pmatrix} K_0^* & 0 \end{pmatrix}$*, optimally asymptotically stabilizes the closed-loop system with cost functional*

$$J_1(z_0,\xi_0) = \sum_{k=0}^{\infty}\left[\begin{pmatrix} z_k^\top & \xi_k^\top \end{pmatrix} Q_1 \begin{pmatrix} z_k \\ \xi_k \end{pmatrix} + u_k^\top u_k\right]$$
$$Q_1 = -(A_c^* - \bar{B}K_1^*)^\top P_1 (A_c^* - \bar{B}K_1^*) + P_1 + K_1^{*\top} K_1^*.$$

Similar considerations are far from being understood in the nonlinear case.

7 A simulated example

Let us apply the results in Section 4 to the dynamics in feedforward form

$$z_{k+1} = e^{\xi_k + \frac{u_k}{2}} z_k; \quad \xi_{k+1} = \xi_k + u_k$$

also described as

$$z^+ = e^{\xi} z; \quad \frac{\partial z^+(u)}{\partial u} = \frac{1}{2} z^+(u)$$
$$\xi^+ = \xi; \quad \frac{\partial \xi^+(u)}{\partial u} = 1.$$

The standing assumptions are verified with $U(\xi) = \frac{1}{2}\xi^2$, $W(z) = \frac{1}{2}z^2$. Accordingly to the step 0 of the forwarding procedure, one computes over the ξ-dynamics

$$Y_0(\xi) = \xi, \quad u_0(\xi) = -\frac{2}{3}\xi.$$

As a consequence, because $f(z) = z$, one has that the cross term can be directly computed as in Section 3.2 so getting

$$\Psi(z,\xi) = \frac{1}{2}(e^{2\xi} - 1)z^2.$$

The corresponding Lyapunov function is $V_0(z,\xi) = \frac{1}{2}\xi^2 + \frac{1}{2}z^2$. Thus, the overall system is u average passive from u_0 with respect to the output $Y_1(z,\xi) = \xi + \frac{3}{2}e^{2\xi}z^2$ verifying $Y_1(0,\xi) = Y_0(\xi)$ and corresponding feedback $u = u_1$ solution to $u_1(z,\xi,u) = 0$ with

$$u_1(z,\xi,u) = u + \xi + \frac{1}{2}(u + u_0) + \frac{1}{2}e^{4\xi}\frac{e^{3u} - e^{3u_0}}{u - u_0}z^2.$$

Because the above equation is hard to be solved, we consider the approximate bounded feedback recalled in Section 2.3 that is provided by

$$\bar{u}_1 = -\lambda(z,\xi)u_1(z,\xi,0)$$
$$u_1(z,\xi,0) = \frac{2}{3}\xi + \frac{3}{2}e^{4\xi}z^2$$

with $\|\bar{u}_1\| \le \mu$, $\lambda(z,\xi) \in]0, C(z,\xi)[$ and

$$C(z,\xi) = \frac{\mu}{(2\mu+1)(1+|u_1(z,\xi,0)|)} S(z,\xi)$$
$$S(z,\xi) = \min_{|u|\le 1/2}\left\{1, \frac{|u|}{|u_1(z,\xi,u) - u_1(z,\xi,0)|}\right\}.$$

For the sake of completeness, some simulations are depicted in Figure 1 by implementing the bounded control and simulating the trajectories of the closed-loop system for several initial states. Specifically, we fix $\mu = 0.5$ and test the behave of the closed-loop dynamics under the bounded feedback $\bar{u}_1 = -\lambda(z,\xi)u_1(z,\xi,0)$. The results of the simulations (Figure 1) correspond to different initial conditions varying over $[0,10]$ and put in light the effectiveness of the proposed methodology complemented with the aforementioned bounded feedback. In thick black, the evolutions which correspond to the initial condition $(z_0,\xi_0) = (8,8)$ are taken as a sample. We only underline that

- regardless the initial conditions, the bound over the control is satisfied and $|u_k| < 0.5$ for any $k \ge 0$;
- the bound over the final feedback is conservative in this case as it turns out that, for any initial condition, $|u_k| \le 0.25 < 0.5$ for $k \ge 0$;
- for any initial condition, the behavior of the closed-loop system is similar although the velocity of convergence to the equilibrium decreases as the initial distance from zero increases.

8 Conclusions

This paper describes a constructive forwarding design for discrete-time cascade dynamics. The design is iterative and involves, at each step, average-passivation and the construction of a Lyapunov function. In case of strict-feedforward dynamics, the proposed strategy recovers the one developed in the literature through successive coordinate changes and average passivation. The strategy is then applied to stabilize time-delay systems affected by a constant and known input delay. The case of LTI systems is carried out as an illustrative example. A simulated academic example over a nonlinear system illustrates the design computations.

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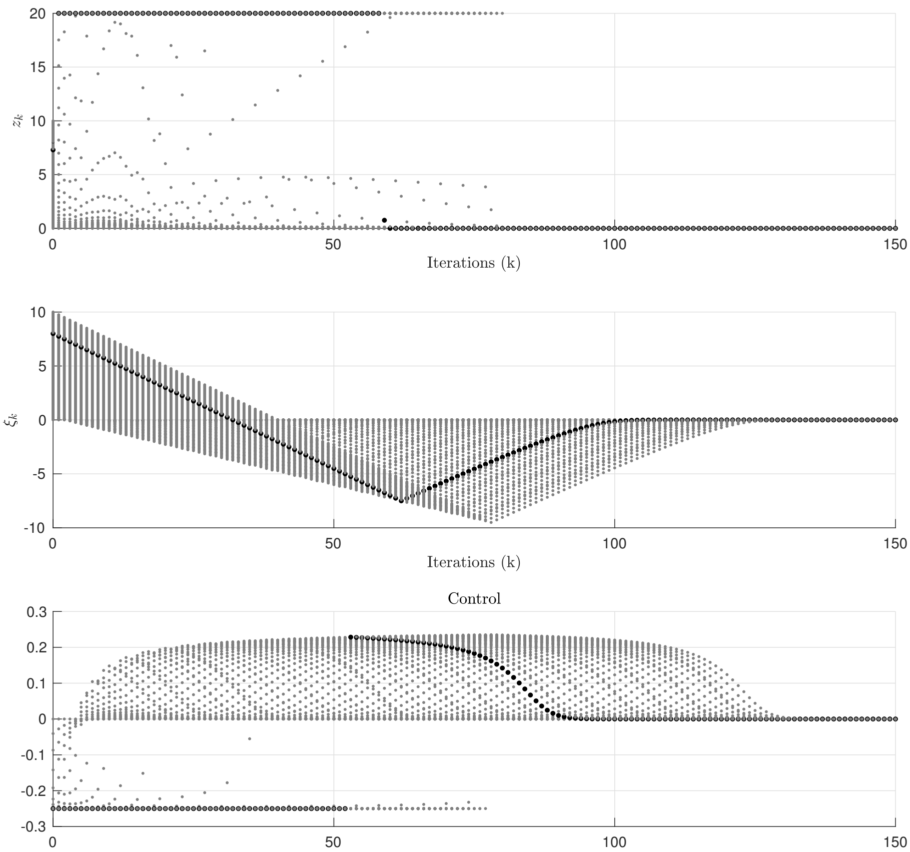

Fig. 1. $\mu = 0.5$ and $\xi,\ z_0 \in [0, 10]$

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A Proof of Theorem 3.1

The proof starts by showing **(i)**. To this purpose, since the equilibrium of $\xi_{k+1} = a(\xi_k)$ is LES, we can write that for a real constant $\alpha \in (0,1)$ and function $\gamma(\cdot) \in \mathscr{K}$, then $\gamma(\|\xi_k\|) \leq \alpha^k \gamma(\|\xi\|)$ for any $k \geq 0$. To prove the result we need to prove that $W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z))$ is summable. To this end, we first prove that z_k is bounded for any $k \geq 0$. For, by exploiting **A.4**, we compute

$$\begin{aligned}W(z_{k+1}) =&W(f(z_k)+\varphi(z_k,\xi_k))\\ \leq&W(z_k)+\|\nabla W(f(z_k))\|\|\varphi(z_k,\xi_k)\|.\end{aligned}$$

Using now **A.3** we have

$$\begin{aligned}W(z_{k+1}) \leq&W(z_k)+\|\nabla W(f(z_k))\|\big(\gamma_1(\|\xi_k\|)\|z_k\|+\gamma_2(\|\xi_k\|)\big)\\ \leq&W(z_k)+\|\nabla W(f(z_k))\|\alpha^k\big(\gamma_1(\|\xi\|)\|z_k\|+\gamma_2(\|\xi\|)\big)\end{aligned}$$

where $\xi = \xi_0$ and the latter bound exploits the LES property of $\xi_{k+1} = a(\xi_k)$. Accordingly, now, because $\gamma_i(\|\xi\|)$ for $i = 1,2$ are constant, one can find $\gamma(\cdot) \in \mathscr{K}$ such that

$$\begin{aligned}W(z_{k+1}) \leq&W(z_k)+\|\nabla W(f(z_k))\|\gamma(\|\xi\|)\alpha^k\big(1+\|z_k\|\big)\\ \leq&W(z_k)+2\gamma(\|\xi\|)\alpha^k\|\nabla W(f(z_k))\|\|z_k\|.\end{aligned}$$

Applying now **A.1** and **A.4** and assuming $\|z_k\| > \max\{1,M\}$

$$W(z_{k+1}) \leq (1+\mathbf{c}_1(\|\xi\|)\alpha^k)W(z_k)$$

with constant $\mathbf{c}_1(\|\xi\|) = 2c\gamma(\|\xi\|)$ implying that, as $k \to \infty$, $W(z_{k+1}) = W(z_k)$ and, thus, boundedness of $W(z_k)$ for any $k \geq 0$. Because $W(\cdot)$ is assumed radially unbounded, boundedness of $W(z_k)$ implies the one of $\|z_k\|$.
Accordingly, considering now $W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k))$ and exploiting the above bound, one gets

$$W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k)) \leq \mathbf{c}_1(\|\xi\|)\alpha^k W(z_k) \tag{A.1}$$

Because $W(z_k)$ and $\|z_k\|$ are bounded for any time $k \geq 0$, one gets that there exists a constant $\mathbf{c_2}(\|(z,\xi)\|)$ depending on the initial state (z,ξ) such that

$$W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k)) \leq \mathbf{c}_2(\|(z,\xi)\|)\alpha^k \tag{A.2}$$

so getting that $W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k))$ is summable over $[0,\infty)$ and (19) exists and is bounded for all (z,ξ).
Continuity of (19) comes from the fact that it is the composition and the sum of continuous-functions on $[0,\infty)$.
As far as **(ii)** is concerned, positive definiteness of V_0 is ob-

tained by exploiting the radial unboundedness of $W(z)$.

$$W(z_k) = W(z) + \sum_{t=0}^{k-1} \left[W(f(z_t) + \varphi(z_t, \xi_t)) - W(z_t)\right] =$$
$$W(z) + \sum_{t=0}^{k-1} \left[W(f(z_t) + \varphi(z_t, \xi_t)) - W(f(z_t))\right] +$$
$$\sum_{t=0}^{k-1} \left[W(f(z_t)) - W(z_t)\right]$$

where the term $W(f(z_t)) - W(z_t)$ is non-increasing for any $t \geq 0$. By substracting both sides of the last equality by $W(f(z_t)) - W(z_t)$ and taking the limit for $k \to \infty$ one gets

$$W_\infty(z) - \sum_{t=0}^{\infty} \left[W(f(z_t)) - W(z_t)\right] =$$
$$W(z) + \sum_{t=0}^{\infty} \left[W(f(z_t) + \varphi(z_t, \xi_t)) - W(f(z_t))\right]$$

where $W_\infty(z) = \lim_{k\to\infty} W(z_k)$ and $\Psi(z, \xi) = \sum_{t=0}^{\infty} \left[W(f(z_t) + \varphi(z_t, \xi_t)) - W(f(z_t))\right]$. Hence, one gets that $V_0(z, \xi)$ rewrites

$$V_0(z, \xi) = W_\infty(z) - \sum_{t=0}^{\infty} \left[W(f(z_t)) - W(z_t)\right] + U(\xi) \geq 0. \tag{A.3}$$

From the radially unboundedness of W and U one has that if $V_0(z, \xi) = 0$ then $\xi = 0$. By construction, $V_0(z, 0) = W(z)$ so concluding that $V_0(z, \xi) = 0$ implies $(z, \xi) = (0, 0)$. According to the last inequality this proves that V_0 is positive-definite.
To prove its radial unboundedness we first point out that from (A.3) it follows that $V_0(z, \xi) \to \infty$ as $\|\xi\| \to \infty$ for any z. Hence, one has to show that

$$\lim_{\|z\|\to+\infty} \left[W_\infty(z) - \sum_{t=0}^{\infty} \left(W(f(z_t)) - W(z_t)\right)\right] = +\infty. \tag{A.4}$$

This will be achieved by lowerbounding (A.4) by means of a radially unbounded function deduced from $W(z)$. For, consider $C = \mathbf{c}(\|\xi\|)$ in (A.1). Accordingly, for any $k \geq 0$ we write

$$|W(f(z_k) + \varphi(z_k, \xi_k)) - W(f(z_k))| \leq$$
$$\|\frac{\partial W}{\partial z}\|(C|\alpha|^k + C|\alpha|^k \|z_k\|).$$

It follows that

$$W(f(z_k) + \varphi(z_k, \xi_k)) - W(f(z_k)) \geq$$
$$-|W(f(z_k) + \varphi(z_k, \xi_k)) - W(f(z_k))| \geq$$
$$\geq -2\|\frac{\partial W}{\partial z}\|C|\alpha|^k\|z_k\| - C(1 - \|z_k\|)\|\frac{\partial W}{\partial z}\||\alpha|^k.$$

When $1 - \|z_k\| > 0$ the term $-C(1 - \|z_k\|)\frac{\partial W}{\partial z}\||\alpha|^k$ can be discarded without affecting the inequality. On the other hand, when $1 - \|z_k\| \leq 0$, it is bounded by $K_2|\alpha|^k$ so that

$$W(f(z_k) + \varphi(z_k, \xi_k)) - W(f(z_k)) \geq$$
$$-2\|\frac{\partial W}{\partial z}\|C|\alpha|^k\|z_k\| - K_2|\alpha|^k.$$

Using **A.4** we obtain

$$W(f(z_k) + \varphi(z_k, \xi_k)) - W(z_k) \geq \tag{A.5}$$
$$\begin{cases} -K|\alpha|^k W(z_k) - K_2|\alpha|^k + W(f(z_k)) - W(z_k), & \|z\| > r \\ -K_1|\alpha|^k W(z_k) - K_2|\alpha|^k + W(f(z_k)) - W(z_k), & \|z\| \leq r \end{cases}$$

with $r \geq 1$ and real K, K_1, K_2.

$\|z\| > r$ and $k \in [0, t)$

$$W(z_k) \geq \phi(k, 0)W(z) + \sum_{t=0}^{k-1} \phi(k-1, t)\left[-K_2|\alpha_1|^t + W(f(z_t)) - W(z_t)\right]$$

$\|z\| \leq r$ and $k \in [0, t)$

$$W(z_k) \geq W(z) + \sum_{t=0}^{k-1} \left[-K_1|\alpha|^t - K_2|\alpha|^t + W(f(z_t)) - W(z_t)\right]$$

with $\phi(k, t) = \prod_{j=t}^{k}(1 - K|\alpha|^j)$. Accordingly, by mixing both the bounds, one gets

$$W(z_k) \geq \phi(k, 0)W(z) + \sum_{t=0}^{k-1}(-K_1|\alpha|^t - K_2|\alpha|^t + W(f(z_t)) - W(z_t))$$

so that for all $k \geq 0$, $\phi(k, 0)$ admits a lower bound K_3 and

$$W(z_k) \geq K_3 W(z) + \sum_{t=0}^{k-1} \left[W(f(z_t)) - W(z_t)\right] + r_k$$

with $r_k := \sum_{t=0}^{k-1} \left[-K_1|\alpha|^t - K_2|\alpha|^t\right]$ which converges to a bounded solution r^* over $[0, \infty)$. So, taking the limit when $k \to \infty$ one obtains

$$W_\infty(z, \xi) - \sum_{t=0}^{k-1} \left[W(f(z_t)) - W(z_t)\right] \geq K_3 W(z) + r^*.$$

It is clear that r^* and K_3 may depend on ξ but are independent of z so that (A.4) holds.
Accordingly, by construction $V_0(z_{k+1}, \xi_{k+1}) - V_0(z_k, \xi_k)) = W(f(z_k)) - W(z_k) + U(a(\xi_k)) - U(\xi_k) \leq 0$ so concluding the proof.

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Stabilization of feedforward discrete-time dynamics through immersion and invariance

Salvatore Monaco, Dorothée Normand-Cyrot and Mattia Mattioni

***Abstract*— The paper deals with the problem of stabilizing discrete-time feedforward dynamics through Immersion and Invariance. Closed loop stabilization of the equilibrium is achieved making use of a passivity-based controller combined with a domination argument. A simulated example illustrates the performances.**

***Index Terms*— Nonlinear output feedback; Stability of nonlinear systems; Lyapunov methods**

I. INTRODUCTION

Forwarding is a stabilizing approach developed in continuous time (see [1], [2]) for dynamics admitting a particular cascaded (or triangular) structure. It provides a systematic bottom up recursive Lyapunov-based design procedure which can be interpreted as the dual of the celebrated back-stepping one [3]: instead of assuming a state component as a virtual control and controlling through forwarding, stabilization is achieved by iteratively adding a state component which "integrates" the other ones. Such an approach has been developed in discrete time in [4] to stabilize classes of nonlinear dynamics under Lyapunov-based bounded feedback.
In the present work we consider discrete-time strict-feedforward dynamics of the form

$$\begin{aligned} x_{jk+1} &= x_{jk} + F_j(x_{1k}, x_{2k}, \dots, x_{j-1k}, u_k); \quad j = 2, \dots, n \\ x_{1k+1} &= F_1(x_{1k}, u_k) \end{aligned} \tag{1}$$

where $x_1 \in \mathbb{R}^p$ and the $x_j \in \mathbb{R}$ for $j = 2, \dots, n$ and $u \in U \subseteq \mathbb{R}^m$, $m = 1$. The proposed control strategy extends the design introduced in [5] for systems in strict-feedforward form (when setting $x_{1k+1} = u_k$, $m = 1$). Here, the general non linear x_1-dynamics is known to be stabilizable under a suitable state feedback but the knowledge of a control Lyapunov Function is not assumed. The problem is presently set in the context of Immersion and Invariance - I&I. I&I was proposed in [6]-[7] for stabilizing continuous-time systems and reformulated in [8]-[9] in discrete-time; the overall design results to be less demanding in such a context.

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With reference to (1), a preliminary controller ensuring global asymptotic stability - GAS - of the origin of the x_1-dynamics is computed so defining the target system. Then the design is completed for ensuring attractiveness of a certain invariant set associated with the target x_1-dynamics. Boundedness of the state trajectories guarantees global asymptotic stabilization of the closed-loop equilibrium. Attractiveness of the invariant set is achieved thanks to passivity arguments and negative output feedback as proposed in [10]. As usual in a discrete-time context, the negative output feedback is only implicitly defined; for this reason, domination arguments as developed in [11]-[4] are used to provide an explicit bounded solution.

The paper is organized as follows. Some preliminaries on discrete-time state space representations and I&I stabilizability are in Section II. A brief motivating discussion is in Section III. The proposed control design is developed for the elementary feedforward two block dynamics in Section IV. An example concludes the paper in Section V.

II. RECALLS AND PRELIMINARIES

A. The Differential Difference Representation

Following [12], nonlinear discrete-time dynamics in the form of a map $x_{k+1} = \mathscr{F}(x_k, u_k)$ with $\mathscr{F}(\cdot, u)$ a $\mathbb{R}^n$-valued smooth map, smootly parameterized by $u \in U$, can be represented as a couple of a difference and a differential equation

$$x^+ = \mathscr{F}_0(x) \tag{2}$$

$$\frac{\partial x^+(u)}{\partial u} = \mathscr{G}(x^+(u), u) \tag{3}$$

where $x^+(u)$ represents a curve in $\mathbb{R}^n$ parameterized by u, $\mathscr{F}_0(x) := \mathscr{F}(x, 0) := x^+(0)$, defines the initial condition of the differential equation (3). $\mathscr{G}(\cdot, u)$ on $\mathbb{R}^n$, parameterized by u, is computed to satisfy the equality

$$\mathscr{G}(\mathscr{F}(x,u), u) = \frac{\partial \mathscr{F}(x,u)}{\partial u}.$$

Given $\mathscr{F}(\cdot, u)$, the existence of $\mathscr{G}(\cdot, u)$ is ensured by the reversibility of $\mathscr{F}_0(\cdot)$ as a function of x, so uniquely defining and computing $\mathscr{G}(\cdot, u)$ for u sufficiently small as

$$\mathscr{G}(x,u) := \frac{\partial \mathscr{F}(\cdot, u)}{\partial u}\Big|_{x = \mathscr{F}^{-1}(x,u)} \tag{4}$$

where $\mathscr{F}^{-1}(x,u)$ denotes the reverse function, i.e., satisfying $\mathscr{F}(\mathscr{F}^{-1}(x,u), u) = x$.

For any x, completeness of the vector field $\mathscr{G}(\cdot, u)$ for all $u \in U$ ensures integrability of (3) so recovering the usual representation in the form of a map

$$x^+(u) = \mathscr{F}(x,u) = x^+(0) + \int_0^u \mathscr{G}(x^+(v),v)\mathrm{d}v$$

with $x^+(0) := \mathscr{F}_0(x)$. Consequently, given any smooth function $\mathscr{H} : \mathbb{R}^n \to \mathbb{R}$, its variation with respect to u around $\mathscr{H}(x^+(0))$ admits the integral form representation

$$\mathscr{H}(\mathscr{F}(x,u)) - \mathscr{H}(\mathscr{F}_0(x)) = \int_0^u \mathrm{L}_{\mathscr{G}(\cdot,v)}\mathscr{H}(x^+(v))\mathrm{d}v$$

where $\mathrm{L}_{\mathscr{G}(\cdot,\cdot)}\mathscr{H}(\cdot)$ represents the Lie derivative of $\mathscr{H}(\cdot)$ along the vector field $\mathscr{G}(\cdot,\cdot)$; i.e.

$$\mathrm{L}_{\mathscr{G}(\cdot,v)}\mathscr{H}(x^+(v)) = \Big[\frac{\partial \mathscr{H}(\cdot)}{\partial x}\mathscr{G}(\cdot,v)\Big]_{x^+(v)}.$$

(2) and (3) define the $(\mathscr{F}_0,\mathscr{G})$ representation of a discrete-time dynamics.

B. $(\mathscr{F}_0,\mathscr{G})$ representation of feedforward dynamics

According to these definitions it is a matter of computations to verify that any nonlinear feedforward dynamics of the form (1), with $F_1(x,u)$ reversible in x_1 for all $u \in U$, admits the following $(\mathscr{F}_0,\mathscr{G})$ representation

$$\begin{aligned} x_j^+ &= x_j + F_{j0}(x_{j-1},\dots,x_1); \quad j=2,\dots,n \\ x_1^+ &= F_{10}(x_1) \qquad (5) \\ \frac{\partial x_j^+(u)}{\partial u} &= G_j(x_{j-1}^+(u),\dots,x_1^+(u),u); \quad j=2,\dots,n \\ \frac{\partial x_1^+(u)}{\partial u} &= G_1(x_1^+(u),u) \qquad (6) \end{aligned}$$

with

$$\mathscr{F}_0(x) = col[x_n + F_{n0}(x_{n-1},\dots,x_1),\dots,x_2+F_{20}(x_1),F_{10}(x_1)]$$

$\mathscr{F}_0(0) = 0$ and $\mathscr{G}(\cdot,u) := col[G_n(\cdot,u),\dots,G_1(\cdot,u)]$.

For, it is sufficient to verify that reversibility of $F_1(x_1,u)$ in x_1 is sufficient to imply reversibility of $\mathscr{F}_0(x)$ in x. More precisely, the reverse dynamics can be iteratively computed so getting

$$\begin{aligned} x_1 &= F_1^{-1}(x_1^+(u),u) \\ x_2 &= x_2^+(u) - F_2(x_1,u) = x_2^+(u) - F_2\big(F_1^{-1}(x_1^+(u),u),u\big) \\ x_3 &= x_3^+(u) - F_3\Big(x_2^+(u) - \\ & \quad F_2(F_1^{-1}(x_1^+(u),u),u),F_1^{-1}(x_1^+(u),u),u\Big) \\ \cdots & \quad . \end{aligned}$$

Then, according to (4), one computes for $j=2,\dots,n$, the control vector fields $G_j(x_{j-1}^+(u),\dots,x_1^+(u),u)$ as

$$\begin{aligned} G_j(x_{j-1},\dots,x_1,u) &= \frac{\partial F_j(x_{j-1},\dots,x_1,u)}{\partial u}\Big|_{x=\mathscr{F}^{-1}(x,u)} \\ G_1(x_1,u) &= \frac{\partial F_1(x_1,u)}{\partial u}\Big|_{x_1=F_1^{-1}(x_1,u)} \end{aligned}$$

which maintain the required triangular form.

In the sequel, the design is instrumentally developed with reference to strict-feedforward dynamics which admit the $(\mathscr{F}_0,\mathscr{G})$ representation (5,6), but the solution can be applied to a discrete-time system of the form (1).

C. The discrete-time I&I stabilizability conditions

Following [6], let us preliminarily formulate I&I stabilization for generally nonlinear difference equations [9].

Theorem 2.1: Consider the nonlinear discrete-time dynamics

$$x_{k+1} = \mathscr{F}(x_k,u_k) \qquad (7)$$

with state $x \in \mathbb{R}^n$, control $u \in \mathbb{R}$ and let $x^* \in \mathbb{R}^n$ the equilibrium to be stabilized. Let $p < n$ and assume that we can find mappings

$$\begin{aligned} \alpha(\cdot) &: \mathbb{R}^p \to \mathbb{R}^p; \quad \pi(\cdot) : \mathbb{R}^p \to \mathbb{R}^n \\ \phi(\cdot) &: \mathbb{R}^n \to \mathbb{R}^{n-p}; \quad \psi(\cdot,\cdot) : \mathbb{R}^{n\times(n-p)} \to \mathbb{R} \end{aligned}$$

such that the following hold.

H$_{1d}$ (*Target System*) - The dynamics with state $\xi \in \mathbb{R}^p$

$$\xi_{k+1} = \alpha(\xi_k) \qquad (8)$$

has a globally asymptotically stable equilibrium at $\xi^* \in \mathbb{R}^p$ and $x^* = \pi(\xi^*)$.

H$_{2d}$ (*Immersion and invariance condition*) - For all $\xi \in \mathbb{R}^p$, there exists $c(\cdot) : \mathbb{R}^p \to \mathbb{R}$ such that

$$\mathscr{F}(\pi(\xi),c(\xi)) = \pi(\alpha(\xi)). \qquad (9)$$

H$_{3d}$ (*Implicit manifold*) - The following identity between sets holds

$$\{x \in \mathbb{R}^n | \phi(x) = 0\} = \{x \in \mathbb{R}^n | x = \pi(\xi) \text{ for } \xi \in \mathbb{R}^p\} \qquad (10)$$

H$_{4d}$ (*Manifold attractivity and trajectory boundedness*) - All the trajectories of the system

$$x_{k+1} = \mathscr{F}(x_k,\psi(x_k,z_k)) \qquad (11a)$$

$$z_{k+1} = \phi(\mathscr{F}(x_k,\psi(x_k,z_k))) \qquad (11b)$$

with $z \in \mathbb{R}^{n-p}$, $z_0 = \phi(x_0)$, are bounded for all $k \geq 0$ and satisfy $\lim_{k\to\infty} z_k = 0$ and $\psi(\cdot,0)\Big|_{x=\pi(\xi)} = c(\xi)$.

Then, x^* is a globally asymptotically stable equilibrium of the closed loop dynamics $x_{k+1} = \mathscr{F}(x_k,\psi(x_k,\phi(x_k)))$.

Definition 2.1: The discrete-time nonlinear dynamics (7) is said to be I&I-stabilizable with target dynamics $\xi_{k+1} = \alpha(\xi_k)$ when **H**$_{1d}$-**H**$_{2d}$-**H**$_{3d}$-**H**$_{4d}$ in Theorem 2.1 are satisfied.

We show in this paper that I&I stabilization of feedforward dynamics reduces the control design to the iterative stabilization of suitably defined one-dimensional dynamics.

III. A MOTIVATING DISCUSSION

To briefly discuss the idea behind forwarding, let the elementary discrete-time cascade-connected dynamics

$$y_{k+1} = y_k + H(x_k); \quad x_{k+1} = F(x_k)$$

where $x \in \mathbb{R}^p$, y is scalar, H and F are continuous functions and the origin of the x-subsystem is assumed asymptotically stable; i.e. there exists a positive definite Lyapunov function $V(x)$ such that $V(x_{k+1}) - V(x_k) < 0$ for all $x_k \neq 0$. The existence of an invariant stable set described by the graph of a function $y = \Phi(x)$, implies the stability of the cascade.

In fact, if such a function $\Phi(x)$ (with $\Phi(0) = 0$) exists, it satisfies the invariance implication

$$(y_0, x_0) \in \Omega = \{(y, x) \text{s.t.} y = \Phi(x)\} \Rightarrow (y_k, x_k) \in \Omega;\ \forall k > 0$$

for any (y_k, x_k) solutions to the cascade system. $\Phi(\cdot)$ can be computed by solving the equality

$$\Phi(F(x)) - \Phi(x) = H(x) \tag{12}$$

with initial condition $\Phi(0) = 0$. On these bases, it is easily verified that a Lyapunov function for the overall system is given by $W(x,y) = V(x) + \frac{1}{2}(y - \Phi(x))^2$ with first order increment $\Delta_k W = W(x_{k+1}, y_{k+1}) - W(x_k, y_k)$ negative definite equal to $\Delta_k V = V(F(x_k)) - V(x_k)$.

Forwarding relies on this basic idea to stabilize controlled cascade-connected dynamics of the form

$$y_{k+1} = y_k + H(x_k), \quad x_{k+1} = F(x_k, u_k).$$

In this case, according to the (F_0, G) representation of the x-dynamics

$$x_k^+(u_k) := F(x_k, u_k) = F_0(x_k) + \int_0^{u_k} \mathrm{L}_{G(\cdot,v)}(x^+(v))\mathrm{d}v$$

with $F_0(x) := F(x, 0)$, one has

$$\begin{aligned} \Delta_k W &= V(F(x_k)) - V(x_k) + \int_0^{u_k} \mathrm{L}_{G(\cdot,v)} V(x^+(v))\mathrm{d}v \\ &+ \frac{1}{2}\Big(\int_0^{u_k} \mathrm{L}_{G(\cdot,v)}\Phi(x^+(v))\mathrm{d}v\Big)^2 \\ &- (y_k - \Phi(x_k)) \int_0^{u_k} \mathrm{L}_{G(\cdot,v)}\Phi(x^+(v))\mathrm{d}v. \end{aligned} \tag{13}$$

It clearly comes out that choosing u_k to render negative $\Delta_k W$ in (13) is a difficult task since it involves solving an implicit inequality in u_k.

An equivalent solution, which recalls the structure of the continuous-time one, can be obtained by rewriting $\Delta_k W$ as

$$\begin{aligned} \Delta_k W &= V(F(x_k)) - V(x_k) + u_k \int_0^1 \mathrm{L}_{G(\cdot,su_k)} V(x^+(su_k))\mathrm{d}s \\ &+ \frac{u_k^2}{2}\Big(\int_0^1 \mathrm{L}_{G(\cdot,su_k)}\Phi(x^+(su_k))\mathrm{d}s\Big)^2 \\ &- u_k(y_k - \Phi(x_k)) \int_0^1 \mathrm{L}_{G(\cdot,su_k)}\Phi(x^+(su_k))\mathrm{d}s \end{aligned}$$

and solving the implicit equality below

$$u = -\Gamma^{-1}\int_0^1 [\mathrm{L}_{G(\cdot,su)} V\Big|_{x^+(su)} - (y - \Phi(x))\mathrm{L}_{G(\cdot,su)}\Phi\Big|_{x^+(su)}]\mathrm{d}s \tag{14}$$

with positive gain function

$$\Gamma = [1 + \frac{[\int_0^1 \mathrm{L}_{G(\cdot,su)}\Phi|_{x^+(su)}\mathrm{d}s]^2}{2}]. \tag{15}$$

Remark 3.1: The control solution (14) recalls the continuous-time stabilizing control

$$u = -[\mathrm{L}_g V(x) - (y - \phi(x))\mathrm{L}_g \phi(x)] \tag{16}$$

which ensures global asymptotic stabilization of the system

$$\dot{y} = h(x), \quad \dot{x} = f(x) + ug(x) \tag{17}$$

when the origin of the x-subsystem is assumed asymptotically stable (there exists a positive definite Lyapunov function $V(x)$ such that $\mathrm{L}_f V < 0$ for all $x \neq 0$) and when the function $\phi(x)$ is computed to satisfy $\mathrm{L}_f \phi = h$ with $\phi(0) = 0$.

It will be shown in the sequel that the concept of I&I stabilization combined with a domination argument makes the design constructive in discrete-time too.

IV. I&I FORWARDING STABILIZATION

Consider the following elementary feedforward dynamics over $\mathbb{R}^{p+1}$

$$x_{2k+1} = x_{2k} + F_2(x_{1k}, u_k), \quad x_{1k+1} = F_1(x_{1k}, u_k) \tag{18}$$

and assume that:

A1) - the origin of the x_1-subsystem is asymptotically stable, i.e. there exists a positive definite Lyapunov function $V(\cdot) : \mathbb{R}^p \to \mathbb{R}$ such that $V(F_1(x_{1k}, 0)) - V(x_{1k}) < 0$ for all $x_{1k} \neq 0$;

A2) - there exists a function $\Phi_1(x_1)$ with $\Phi_1(0) = 0$ satisfying the following equality

$$\Phi_1(F_{10}(x_1)) - \Phi_1(x_1) = F_{20}(x_1) \tag{19}$$

with $F_{10}(x_1) = F_1(x_1, 0)$ and $F_{20}(x_1) = F_{20}(x_1, 0)$.

The (F_0, G) representation of (18) takes the form:

$$\begin{aligned} x_2^+ &= x_2 + F_{20}(x_1), \quad \frac{\partial x_2^+(u)}{\partial u} = G_2(x_1^+(u), u) \\ x_1^+ &= F_{10}(x_1), \quad \frac{\partial x_1^+(u)}{\partial u} = G_1(x_1^+(u), u) \end{aligned} \tag{20}$$

with

$$G_1(F_1(x_1, u), u) = F'_{1u}(x, u);\ G_2(F_1(x_1, u), u) = F'_{2u}(x_1, u).$$

Setting $z_2 = x_2 - \Phi_1(x_1)$ and

$$G_z(\cdot, u) \ := \ G_2(\cdot, u) - \mathrm{L}_{G_1(\cdot,u)}\Phi_1(\cdot) \tag{21}$$

with $\mathrm{L}_{G_1(\cdot,u)}\Phi_1(F_1(x_1, u)) := \frac{\partial \Phi_1}{\partial x_1}(F_1(x_1, u))F'_{1u}(x_1, u)$, the following result holds true.

Proposition 4.1: - Given the discrete-time feedforward dynamics (18) satisfying **A1) A2)**, then the equilibrium is I&I stabilizable with target dynamics $\xi_{k+1} = F_{10}(\xi_{1k})$.

Proof: For, it is sufficient to show that the $\mathbf{H}_{1d}$ to $\mathbf{H}_{3d}$ conditions in Theorem 2.1 are satisfied. First, setting $x_1 = \xi$, the target dynamics is defined by the x_1-dynamics in free evolution because of assumption **A1)** (i.e. $\xi_{k+1} = F_{10}(\xi_{1k})$). Then, the immersion mapping $\pi(\xi)$ is defined from $\mathbb{R}^p$ to $\mathbb{R}^{p+1}$ as $\pi(\xi) = col(\Phi_1(\xi), \xi)$ so immediately verifying the invariance condition (9) because of assumption **A2)** with $c(\xi) = 0$. Setting now $z_2 = x_2 - \Phi_1(x_1)$, one expresses with (21) the z_2-dynamics, driftless by construction of Φ_1; i.e.

$$\begin{aligned} z_2^+ &= z_2 \\ \frac{\partial z_2^+(u)}{\partial u} &= G_z(x_1^+(u), u). \end{aligned} \tag{22}$$

By construction, the p-dimensional set described by $z_2 = 0$ is invariant and the I&I design reduces to find a control that makes such set attractive while guaranteeing boundedness of the closed loop trajectories. ■

The control design is discussed below.

A. I&I dead-beat stabilizing control

A first dead-beat solution can be computed by solving in u_k for all z_{2k}, the implicit equality $z_{2k+1} = 0$; i.e.

$$z_{2k} + u_k \int_0^1 \Big(G_2(\cdot, su_k) - \mathrm{L}_{G_1(\cdot,su_k)}\Phi_1 \Big)(x_1^+(su_k))\mathrm{d}s = 0$$

so bringing in one step z_{2k+1} to zero (equivalently x_{2k+1} to $\Phi(x_{1k+1})$). Then $u_{k+i} = 0$ for $i \geq 1$ guarantees $z_{2k+i+1} = 0$ so that the trajectory lays on the stable set.

B. I&I negative output feedback

It is shown in this section that the concept of u-average passivity, introduced by the authors in [10], can be fruitfully used to get asymptotic stabilization of the origin of the z_2-dynamics. With this in mind, let us associate to (20) in the coordinates (x_1, z_2) the output

$$H(x_1, z_2, u) = \mathrm{L}_{G_z(x_1,u)} V_z(z_2) \tag{23}$$

ant its average [10]

$$H_{av}^+(x_1, z_2, u) = \frac{1}{u} \int_0^u H(x_1^+(v), z_2^+(v), v)\mathrm{d}v.$$

The next result is an immediate consequence of the driftless property of (22).

Proposition 4.2: The feedforward dynamics (18) satisfying **A1)-A2)** with output (23) is u-average lossless with storage function $V_z(z_2) = \frac{1}{2}z_2^2$; i.e. it satisfies

$$V_z(z_{2k+1}) - V_z(z_{2k}) = \int_0^{u_k} H(x_1^+(v), z_2^+(v), v)\mathrm{d}v.$$

In the present case, one easily computes with $G_z(\cdot, u)$ in (21)

$$\begin{aligned} H(x_1^+(u), z_2^+(u), u) &= z_2^+(u) G_z(x_1^+(u), u) \\ H_{av}^+(x_1, z_2, u) &= \int_0^1 z_2^+(su) G_z(x_1^+(su), su)\mathrm{d}s. \end{aligned}$$

From [10], any feedback law making (23) negative renders the origin of the z_2-dynamics GAS, provided the so defined output $H(x_1, z_2, u)$ is zero state detectable - ZSD -; i.e.

ZSD: no solution of the uncontrolled dynamics (18) can stay in the set $\{(x_1, z_2) \in \mathbb{R}^{p+1}$ s.t $H_{av}^+(x_1, z_2, 0) = z_2 G_z(F_{10}(x_1), 0)\}$ other than solutions converging asymptotically to the zero equilibrium.

Proposition 4.3 (I&I Negative u-average output feedback): Given the feedforward dynamics (18) satisfying **A1) A2)** with output mapping $z_2^+(u) G_z(x_1^+(u), u)$ assumed zero state detectable then, for all $(x_1, z_2) \in \mathbb{R}^{p+1}$, the control law u solution of the algebraic equation

$$u = -\varepsilon(x_1) z_2 \Big[1 + \frac{1}{2}\big(\int_0^1 G_z(x_1^+(su), su)\mathrm{d}s\big)^2\Big]^{-1} \int_0^1 G_z(x_1^+(su), su)\mathrm{d}s \tag{24}$$

with sufficiently small $\varepsilon(x_1) > 0$, ensures global asymptotic stabilization of the origin of (18).

Proof: From the lossless equality (23) rewritten as

$$V_z(z_{2k+1}) - V_z(z_{2k}) = u_k H_{av}^+(x_{1k}, z_{2k}, u_k) \tag{25}$$

and according to Proposition 4.2, the control law solution of the algebraic equality

$$u = -\varepsilon(x_1) H_{av}^+(x_1, z_2, u) =$$

$$-\varepsilon(x_1) \Big[z_2 \int_0^1 G_z(x_1^+(su), su)\mathrm{d}s - \frac{u}{2}\big[\int_0^1 G_z(x_1^+(su), su)\mathrm{d}s\big]^2 \Big]$$

with sufficiently small $\varepsilon(x_1) > 0$ achieves in closed loop

$$V_z(z_{2k+1}) - V_z(z_{2k}) = -\varepsilon(x_1)[H_{av}^+(x_{1k}, z_{2k}, u_k)]^2 \leq 0.$$

Asymptotic stability to the origin of the z_2-dynamics follows under the requested zero state detectability condition of the mapping $z_2 G_z(F_{10}(x_1), 0)$. Boundedness of the trajectories is ensured by sufficiently small $\varepsilon(x_1)$. Then, I&I stabilization of the origin of the feedforward dynamics (18) follows. ■

C. A constructive bounded solution

The solution proposed in (24) is implicitly defined and thus cannot be exactly computed in general. Setting $s = 0$ in the right hand side of (24), one gets a computable approximation of the solution in the form

$$u = -\varepsilon(x_1) z_2 [1 + \frac{1}{2}[G_z(F_{10}(x_1), 0)]^2]^{-1} G_z(F_{10}(x_1), 0) \tag{26}$$

with $G_z(F_{10}(x_1), 0) = G_2(F_{10}(x_1), 0) - \mathrm{L}_{G^1(\cdot,0)}\Phi_1(F_{10}(x_1))$. Lemma 4.1 below, recalled from [11], [4] is used to provide a bounded feedback of the form (26) with suitably chosen gain.

Lemma 4.1: [11] Let $K(\xi, u)$ be a continuous function. For any strictly positive real number $\bar{u}$, there exists a function $\lambda(\xi)$, as smooth as $K(\xi, u)$ is, such that if $u(\xi) = -\lambda(\xi) K(\xi, u)$ then, for all ξ, we have

$$|u(\xi)| \leq \bar{u} \text{ and } u(\xi) K(\xi, u) \leq \frac{1}{2}\lambda(\xi)|K(\xi, 0)|^2.$$

Moreover, if $K(\xi, u)$ is C^1, then λ is strictly positive on any compact set.

On the basis of Proposition 4.3 and Lemma 4.1 we can now prove the following result

Theorem 4.1: Bounded I&I negative u-average output feedback - Given the feedforward dynamics (18) satisfying **A1)-A2)** then, for any bound $\mu > 0$, the feedback

$$u(x_1, z_2) = -\varepsilon(x_1)\lambda(x_1, z_2) z_2 G_z(F_{10}(x_1)) \tag{27}$$

where $\lambda(x_1, z_2)$ is any function that satisfies

$$0 < \varepsilon(x_1)\lambda(x_1, z_2) \leq \frac{\mu}{(2\mu + 1)(1 + |z_2 G_z(F_{10}(x_1))|)} min\{1, C\} \tag{28}$$

with

$$C := min_{|u| \leq \frac{1}{2}} \{ \frac{2}{[\int_0^1 G_z(x_1^+(su), su)\mathrm{d}s]^2} \} \tag{29}$$

and $\varepsilon(x_1) > 0$ sufficiently small, ensures global asymptotic stabilization of the origin of (18) provided the set $\{x_1 \in \mathbb{R}^p$ s.t. $G_z(F_{10}(x_1)) = 0\}$ is made of isolated points.

Proof: From Lemma 4.1, the feedback law (27) with

$$C := min_{|u| \leq \frac{1}{2}} \{ \frac{|u|}{|\int_0^1 z_2^+(su) G_z(x_1^+(su), su)\mathrm{d}s - z_2 G_z(F_{10}(x_1))|} \}$$

rewritten as in (29) is bounded and guarantees negativity of $uH_{av}^{+}(x_1,z_2,u)$; i.e. for all (x_1,z_2), one has $|u(x_1,z_2)| \leq \mu$ and

$$\int_0^u z_2^{+}(v)G_z(x_1^{+}(v),v)\mathrm{d}v \leq -\frac{1}{2}\varepsilon(x_1)\lambda(x_1,z_2)|z_2G_z(F_{10}(x_1))|^2.$$

When $z_2 = 0$, $u(x_1,0) = 0$. Boundedness of the whole state trajectories follows from Proposition 4.3 with sufficiently small $\varepsilon(x_1) > 0$. ■

Remark 4.1: Theorem 4.1 can be used repeatedly to deduce that the feedforward system (1) is globally asymptotically stabilizable under **A1)** and successive conditions of the type **A2)**. At each step of the design, the closed loop $col(x_1,z_2)$-dynamics defines a new stable $\bar{x}_1$-dynamics in $\mathbb{R}^{p+1}$ which defines a new target dynamics and one adds an integrating variable $z_3 = x_3 - \bar{\Phi}(\bar{x}_1)$ which satisfies the invariance condition $\bar{\Phi}(\bar{F}_{10}(\bar{x}_1)) = \bar{F}_{30}(\bar{x}_1)$. This is repeated for the n blocks.

Remark 4.2: The choice of the gain $\lambda(x_1,z_2)$ in (27) can be made (within the suitable interval) according to several strategies. In particular, it can be chosen so as to obtain robustness of the closed-loop system under the nominal control with respect to parameters uncertainty and external disturbances. In this case, suitable Lipschitz-like assumptions should be introduced for the mappings F_i $(i = 1,2)$.

Remark 4.3: The same approach can be pursued when relaxing assumption **A1)** to the existence of a preliminary stabilizing feedback $c(x_1)$ for the p-dimensional x_1-dynamics. Then, the same result holds when substituting the dynamics $F_{10}(x_1)$ with $F_1(x_1,c(x_1))$ and $F_{20}(x_1)$ with $F_2(x_1,c(x_1))$.

Remark 4.4: Consider the continuous-time cascade

$$\dot{x}_2 = f_2(x_1) + ug_2(x_1); \quad \dot{x}_1 = f_1(x_1) + ug_1(x_1) \tag{30}$$

with the origin of the x_1-subsystem asymptotically stable (there exists a positive definite Lyapunov function $V(x_1)$ such that $\mathrm{L}_{f_1}V(x_1) < 0$ for all $x_1 \neq 0$) and assume the existence of the function $\phi_1(x_1)$ solving $\mathrm{L}_{f_1}\phi_1 = f_2$ with $\phi_1(0) = 0$. Under piecewise constant control over time intervals of length δ, the sampled-data equivalent model takes the form (2)

$$x_{2k+1} = x_{2k} + F_2^{\delta}(x_{1k},u_k); \quad x_{1k+1} = F_1^{\delta}(x_{1k},u_k)$$

with

$$\begin{aligned} F_2^{\delta}(x_{1k},u_k) &= \int_0^{\delta} e^{\tau(f_1+u_kg_1)}(f_2+u_kg_2)(x_1)\mathrm{d}\tau\Big|_{x_{1k}} \\ F_1^{\delta}(x_{1k},u_k) &= e^{\delta(f_1+u_kg_1)}x_1\Big|_{x_{1k}}. \end{aligned}$$

It is a matter of computations to show that the two conditions **A1)-A2)** hold with $\Phi_1(\cdot) = \phi_1(\cdot)$; which proves the existence of a piecewise constant solution from sampled state measures.

V. Example

Let the academic example on the plane

$$\dot{x}_2(t) = x_1(t) + x_1^3(t); \quad \dot{x}_1(t) = -x_1(t) + u(t). \tag{31}$$

It is easily verified that continuous-time I&I stabilization is achieved by the control $u = -Kz_2(1+x_1^2)$ with $K > 0$, $z_2 = x_2 - \phi_1(x_1)$ and $\phi_1(x_1) = -x_1 - \frac{1}{3}x_1^3$. Furthermore, the dynamics is zero state detectable with respect to the "output" map $z_2(1+x_1^2)$. Consider now the sampled-data equivalent to (31) which is exactly computable and given by

$$\begin{aligned} x_{2k+1} =& x_{2k} + (1-e^{-\delta})x_{1k} + x_{1k}^3\int_0^{\delta} e^{-3\tau}d\tau \\ &+ u_k[e^{-\delta} + \delta - 1 + 3x_{1k}^2\int_0^{\delta} e^{-2\tau}(1-e^{-\tau})d\tau] \\ &+ 3x_{1k}u_k^2\int_0^{\delta} e^{-\tau}(1-e^{-\tau})^2 d\tau + u_k^3\int_0^{\delta}(1-e^{-\tau})^3 d\tau \\ x_{1k+1} =& e^{-\delta}x_{1k} + (1-e^{-\delta})u_k. \end{aligned} \tag{32}$$

The origin is still GAS for the discrete-time x_1-dynamics. Hence, one defines the discrete-time target as $\xi_{k+1} = e^{-\delta}\xi_k$. Then, setting $z_2 = x_2 - \phi_1(x_1)$ with the same $\phi_1(x_1)$ as in the continuous-time case, one verifies that $z_{2k+1} = z_{2k}$ under $u_k \equiv 0$. Consequently, the immersion mapping is described, as in the continuous-time case, by $\pi(\xi) = \mathrm{col}(-\xi - \frac{1}{3}\xi^3,\ \xi)$. The z_2-dynamics rewrites in the (F_0, G) form as

$$\begin{aligned} &z_2^{+} = z_2 \\ &\frac{\partial z_2^{+}(u)}{\partial u} = \delta + 3[e^{\delta}x_1^{+}(u) + (1-e^{\delta})u]^2\int_0^{\delta} e^{-2\tau}(1-e^{-\tau})d\tau \\ &+ [e^{\delta}x_1^{+}(u) + (1-e^{\delta})u]^2 e^{-2\delta}(1-e^{-\delta}) \\ &+ 6u[\delta e^{\delta}x_1^{+}(u) + (1-e^{\delta})u][\int_0^{\delta} e^{-\tau}(1-e^{-\tau})^2 d\tau + \frac{e^{-\delta}(1-e^{-\delta})}{3}] \\ &+ 3u^2[\int_0^{\delta}(1-e^{-\tau})^3 d\tau + \frac{(1-e^{-\delta})^3}{3}] \end{aligned} \tag{33}$$

with $z^{+}(u) = z_{2k+1}$ and $x_1^{+}(u) = x_{1k+1}$. The problem results in defining a digital I&I control law which makes $z_2 \to 0$ as $k \to \infty$ preserving boundedness of the whole state trajectories (32)-(33). According to Section IV.2, one defines the control Lyapunov function $V(z_2) = z_2^2$ and the output $H(x_1,z_2,u) = L_{G_z(x_1,u)}V_z(z_2)$. It is a matter of computation to verify that zero-state detectability is inherited from the continuous-time one. At this point, one notices that in order to find $u = -\int_0^1 H(x_1^{+}(su), z_2^{+}(su), su)ds$ one has to solve a fifth degree equation in u, which is hard. Hence, one looks for the bounded I&I negative output feedback defined as in Theorem 4.1. In particular, one computes

$$G_z(F_0(x_1),0) = \delta + 3x_1^2\int_0^{\delta} e^{-2\tau}(1-e^{-\tau})d\tau + x_1^2 e^{-2\delta}(1-e^{-\delta})$$

and sets $u = -\lambda(x_1,z_2)\varepsilon(x_1)z_2G_z(F_0(x_1),0)$ choosing any $\mu > 0$ and $\varepsilon(x_1)\lambda(x_1,z_2)$ so as to satisfy Theorem 4.1.

A. Simulations

Simulations are referred to the continuous-time I&I feedback (red); its emulation (dotted red) which corresponds to hold constant over the sampling time the continuous-time controller; the dead beat approach (blue) and the bounded I&I output feedback (dotted blue). The invariant set is plotted in black. The bounded feedback is implemented by choosing $\lambda(\cdot)$ as the upper-bound of (28). Simulations are reported for increasing values of the sampling period

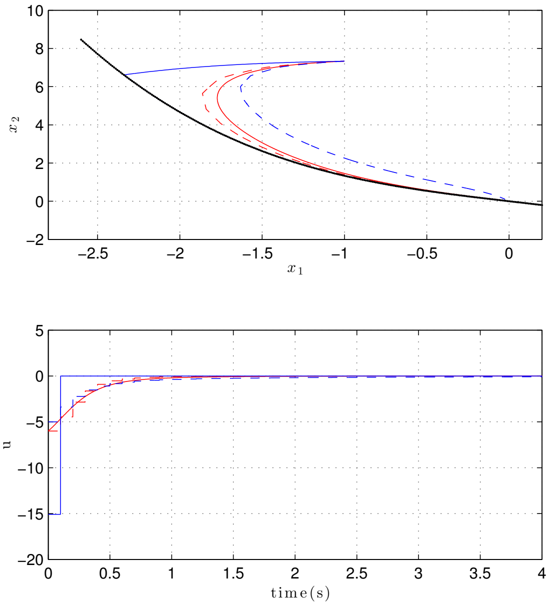

Fig. 1. $\delta = 0.1$ seconds.

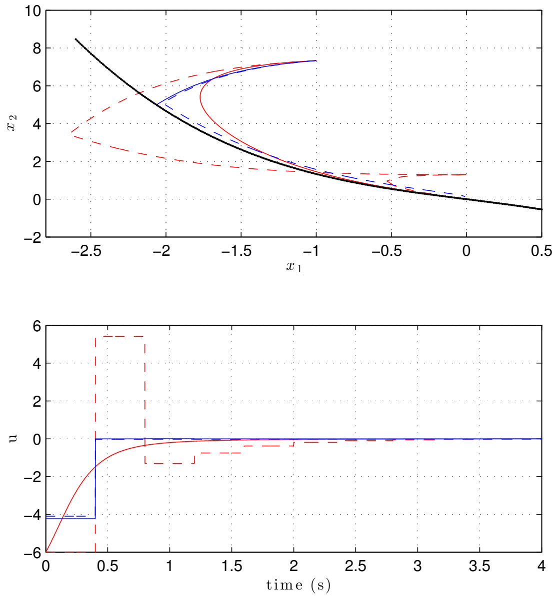

Fig. 2. $\delta = 0.4$ seconds.

δ (0.1, 0.4 and 0.6 seconds). We can see that while the proposed controllers (dead-beat and average-passivity based) succeed in preserving the I&I properties and the stability of the closed-loop equilibrium, the emulated-based control yields degradate performances for $\delta = 0.4s$ and instability for $\delta = 0.6s$. We also note that, as the sampling period increases, the sampled-data controllers require a lower control effort than the continuous-time and emulated-based ones.

VI. Conclusions

With respect to usual forwarding, the proposed I&I approach enables us to relax the knowledge of the x_1-control

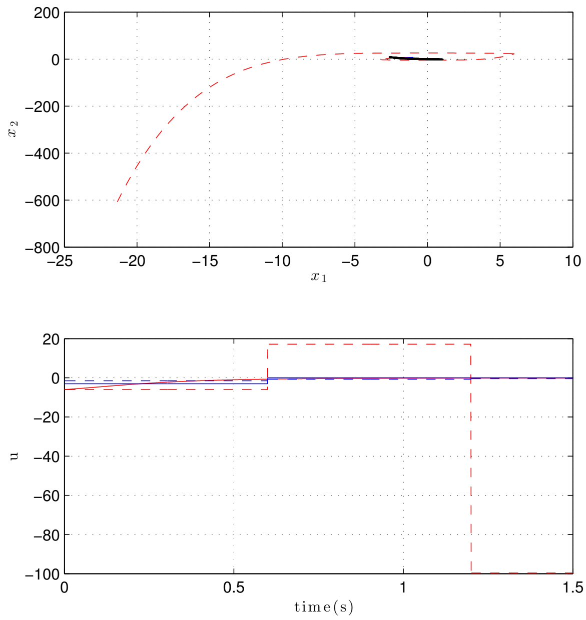

Fig. 3. $\delta = 0.6$ seconds.

Lyapunov function V. Secondly, the bounded design relaxes the necessity to solve an implicit algebraic equality thanks to a suitable dynamic gain. Regarding sampled-data dynamics, it results that I&I stabilization of continuous-time feedforward dynamics implies I&I stabilization of its equivalent sampled-data dynamics in the case of two interconnected blocks.

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Lyapunov stabilization of discrete-time feedforward dynamics

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Abstract— **The paper discusses stabilization of nonlinear discrete-time dynamics in feedforward form. First it is shown how to define a Lyapunov function for the uncontrolled dynamics via the construction of a suitable cross-term. Then, stabilization is achieved in terms of u-average passivity. Several constructive cases are analyzed.**

Index Terms— **Lyapunov Methods; Stability of nonlinear systems; Algebraic/geometric methods**

I. Introduction

Nonlinear discrete-time control theory has been attracting a growing interest in the control community because of its impact into the sampled-data, or more generally hybrid context. Although important works bridge the gap between the continuous-time and discrete-time domains through different methodologies (e. g., [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]), hard difficulties still represent obstacles in extending results that are well-known and elegant in continuous time. These are essentially concerned with the generic nonlinearity in the control variable of the dynamics and the difficulty to settle the geometric structure underlying the evolutions.

As a first attempt to characterize accessibility properties of nonlinear discrete-time dynamics, an alternative differential-difference state-space representation (or (F_0,G)-form) was introduced in [11]. In this context, a discrete-time dynamics over $\mathbb{R}^n$ is described by two coupled differential-difference equations as

$$x^+ = F(x), \quad x^+ := x^+(0) \tag{1a}$$

$$\frac{\partial x^+(u)}{\partial u} = G(x^+(u),u). \tag{1b}$$

Denoting by $x^+(u)$ a curve in $\mathbb{R}^n$ parametrized by $u \in \mathbb{R}$, (1a) models the free evolution described by a smooth mapping $F(\cdot)$ while (1b) models the variational effect of the control by a vector field $G(\cdot,u)$, parameterized by u and assumed complete. Further exploiting this differential geometric framework, structural properties (e.g., invariance, decoupling [12]) have been characterized up to introducing the concept of u-average passivity [13]. This latter notion enables to relax the necessity of a direct throughput as usually required when defining passivity for discrete-time systems. Recently, u-average passivity based feedback design (or control Lyapunov design at large) has been introduced in [14] and is exploited in the present paper with reference to stabilization of cascade dynamics.

More precisely, asymptotic stabilization of cascade discrete-time dynamics exhibiting an upper-triangular (or feedforward) form is addressed. Discrete-time forwarding design was firstly addressed in [15] via the construction of a bounded solution to a suitable control-dependent inequality. Arguing so, the difficulty of solving the nonlinear algebraic equation which implicitly defines the feedback solution is overcome. In [16], a discrete-time forwarding design is proposed by exploiting the framework of Immersion and Invariance so relaxing the a-priori knowledge of a Lyapunov function for the first part of the cascade dynamics. In the present paper, we propose a two steps procedure based on control Lyapunov design and feedback average passivation so reminding of the continuous-time forwarding technique ([17], [18]). Preliminarily considering a two block cascade dynamics with nonlinear coupling mapping, a Lyapunov function is firstly constructed for the uncontrolled stable system via the computation of a suitable cross-term. Then, asymptotic stabilization is achieved in terms of u-average passivity. Constructive solutions are discussed based on specifications of the interconnection term. As a particular case, one recovers the case of dynamics in strict-feedforward form studied in [19] where the construction of a cross term reduces to the one of a coordinates transformation rendering the overall dynamics driftless. Finally, it is shown how similar cascade connected forms are recovered when representing input-delayed dynamics through dynamical extension. It follows that the proposed forwarding design procedure may represent an original control Lyapunov design for discrete-time input delayed dynamics.

The paper is organized as follows: in Section II, the existence of a cross-term is proven for the uncontrolled dynamics. It is employed in while in Section III for stabilizing feedforward dynamics through u-average passivity. In Section IV, case studies specifying the connection term structure are discussed. In Section V conclusions are set.

II. Lyapunov cross term for cascade dynamics

Consider a two block cascade dynamics of the form

$$z_{k+1} = f(z_k) + \varphi(z_k,\xi_k), \quad \xi_{k+1} = a(\xi_k). \tag{2}$$

with $\xi \in \mathbb{R}^{n_\xi}$, $z \in \mathbb{R}^{n_z}$; f, φ and a are continuous functions in their arguments and $(z,\xi) = (0,0)$ is an equilibrium state. We note that the dynamics (2) is uncontrolled with nonlinear connecting map $\varphi(z,\xi)$. The following standing assumptions are introduced.

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A.1 $z_{k+1} = f(z_k)$ has a Globally Stable (GS) equilibrium at the origin with continuously differentiable, positive definite, radially unbounded Lyapunov function $W : \mathbb{R}^{n_z} \to \mathbb{R}_{\geq 0}$ such that $W(f(z)) - W(z) \leq 0$;

A.2 $\xi_{k+1} = a(\xi_k)$ has a Globally Asymptotically Stable (GAS) and Locally Exponentially Stable (LES) equilibrium at the origin with continuously differentiable, positive definite, radially unbounded Lyapunov function $U : \mathbb{R}^{n_\xi} \to \mathbb{R}_{\geq 0}$ such that $U(a(\xi)) - U(\xi) < 0$ for $\xi \neq 0$.

Assumptions **A.1** and **A.2** are not enough to deduce GS of the origin for the complete cascade. For this purpose, further assumptions are needed.

A.3 the function $\varphi(z,\xi)$ satisfies the *linear growth assumption*; i.e. there exist $\mathscr{K}$-functions[1] $\gamma_1(\cdot), \gamma_2(\cdot)$ such that

$$\|\varphi(z,\xi)\| \leq \gamma_1(\|\xi\|)\|z\| + \gamma_2(\|\xi\|);$$

A.4 the function $W(z)$ verifies :

- given any $s(\cdot) : \mathbb{R}^{n_z} \to: \mathbb{R}^{n_z}$ and $d(\cdot,\cdot) : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_z}$

$$|W(s(z) + d(z,\xi)) - W(s(z))| \leq \left|\frac{\partial W}{\partial z} d(z,\xi)\right|;$$

- there exist $c, M \in \mathbb{R}_{>0}$ such that for $\|z\| > M$

$$\|\frac{\partial W}{\partial z}\|\|z\| \leq cW(z).$$

The above assumptions imply the possibility of constructing a Lyapunov function $V_0(\cdot)$ for the complete dynamics starting from the respective ones $W(\cdot)$ and $U(\cdot)$. Setting

$$V_0(z,\xi) = W(z) + U(\xi) + \Psi(z,\xi) \tag{3}$$

we aim at defining an additional continuous cross term $\Psi(z,\xi) : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ to dominate the part with not definite sign when computing the difference

$$\Delta_k V_0(z,\xi) = V_0(z_{k+1},\xi_{k+1}) - V_0(z_k,\xi_k).$$

It is a matter of computations to verify that

$$\begin{aligned}\Delta_k V_0(z,\xi) =& \Delta_k U(\xi) + W(f(z_k)) - W(z_k) \\ &+ W(f(z_k) + \varphi(z_k,\xi_k)) - W(f(z_k)) + \Delta_k \Psi(z,\xi)\end{aligned}$$

with $\Delta_k U(\xi) < 0$ and $W(f(z_k)) - W(z_k) \leq 0$. It turns out that, for ensuring $\Delta_k V_0(z,\xi) \leq 0$, the cross term $\Psi(z,\xi)$ can be chosen to satisfy

$$\Delta_k \Psi(z,\xi) = -W(f(z_k) + \varphi(z_k,\xi_k)) + W(f(z_k)) \tag{4}$$

where the right hand side represents the part in $\Delta_k V_0$ whose sign is not definite. As a consequence, $\Psi(z,\xi)$ is defined as

$$\Psi(z,\xi) = \sum_{k=0}^{\infty} W(f(z_k) + \varphi(z_k,\xi_k)) - W(f(z_k)) \tag{5}$$

along the trajectories $(z_k,\xi_k) = (\tilde{z}(k,z,\xi), \tilde{\xi}(k,\xi))$ of (2) starting at $(z_0,\xi_0) = (z,\xi)$. The stability of the whole system follows from the existence of such function V_0.

Theorem 2.1: Under assumptions **A.1** to **A.4**

(i) $\Psi : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ exists and is continuous;

(ii) $V_0 : \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \to \mathbb{R}$ in (3) is positive-definite and radially unbounded.

As a consequence the origin is a GS equilibrium of (2).

A. Some particular cases

Some constructive cases are discussed below in relation with the connection term $\varphi(z,\xi)$.

1) Strict-feedforward dynamics: Consider strict-feedforward dynamics described by

$$z_{k+1} = Fz_k + \varphi(\xi_k), \quad \xi_{k+1} = a(\xi_k) \tag{6}$$

with $\varphi(0) = 0$ and $F^\top F = I$. Assumption **A.1** is satisfied with $W(z) = z^\top z$ and **A.4** is obviated. Specifying (4) for (6), one gets that the cross term $\Psi(z,\xi)$ must satisfy

$$\Delta_k \Psi(z,\xi) = -2z_k^\top F^\top \varphi(\xi_k) - \varphi^\top(\xi_k)\varphi(\xi_k). \tag{7}$$

As a consequence $\Delta_k \Psi(z,\xi) = -\Delta_k W(z)$ and, according to (5), one sets

$$\begin{aligned}\Psi(z,\xi) =& \sum_{k=0}^{\infty} \left[z_{k+1}^\top(z,\xi) z_{k+1}(z,\xi) - z_k^\top(z,\xi) z_k(z,\xi)\right] \\ =& (z_k^\top(z,\xi) z_k(z,\xi))_\infty - z^\top z\end{aligned}$$

where $(z_k^\top(z,\xi) z_k(z,\xi))_\infty = \lim_{k\to\infty} z_k^\top(z,\xi) z_k(z,\xi)$ and $z_k(z,\xi)$ denotes the z-trajectory at time k starting at (z,ξ). According to (3), a Lyapunov function for (6) is thus

$$V_0(z,\xi) = U(\xi) + (z_k^\top(z,\xi) z_k(z,\xi))_\infty. \tag{8}$$

More in detail, the dynamics (6) possess two invariant sets: a stable set where the evolutions are described by $\xi_{k+1} = a(\xi_k)$; a center set where the evolutions are described by $z_{k+1} = Fz_k$. It is a matter of computations to verify that the projection of the trajectories of (6) onto the center set are described by the map

$$\phi(\xi) = -\sum_{\tau=k_0}^{\infty} F^{k_0-1-\tau}\varphi(\xi_\tau) \tag{9}$$

verifying the invariance equality

$$\phi(\xi_{k+1}) = F\phi(\xi_k) + \varphi(\xi_k). \tag{10}$$

Thus, under the coordinates change $\zeta = z - \phi(\xi)$, (6) is transformed into the decoupled dynamics

$$\zeta_{k+1} = F\zeta_k, \quad \xi_{k+1} = a(\xi_k). \tag{11}$$

Hence, a Lyapunov function for the cascade is given by $\tilde{V}_0(\zeta,\xi) = U(\xi) + \zeta^\top\zeta$. Exploiting the strict-feedforward form, one easily verifies that the two Lyapunov functions $\tilde{V}_0$ and $\tilde{V}$ coincide up to a coordinates change.

Proposition 2.1: Consider the strict-feedforward dynamics (6). Then, one has $V_0(z,\xi) = \tilde{V}_0(z + \phi(\xi),\xi)$ with $\phi(\xi) : \mathbb{R}^{n_\xi} \to \mathbb{R}^{n_z}$ described in (9). As a consequence, the cross-term takes the form

$$\Psi(z,\xi) = (\zeta - \phi(\xi))^\top(\zeta - \phi(\xi)) - z^\top z. \tag{12}$$

[1] A function ρ is said of class $\mathscr{K}$ if its continuous, strictly increasing and $\rho(0) = 0$. It is said of class $\mathscr{K}_\infty$ if it is $\mathscr{K}$ and it is unbounded.

Proof: First, rewrite $\zeta^\top\zeta$ for $k_0=0$ as

$$(z+\sum_{\tau=0}^{\infty}F^{-1-\tau}\varphi(\xi_\tau))^\top(F^k)^\top F^k(z+\sum_{\tau=0}^{\infty}F^{-1-\tau}\varphi(\xi_\tau))$$
$$=\|z_k(z,\xi)+\sum_{\tau=0}^{\infty}F^{k-\tau-1}\varphi(\xi_\tau)-\sum_{\tau=0}^{k-1}F^{k-\tau-1}\varphi(\xi_\tau)\|^2.$$

Because $(F^k)^\top F^k=I$ and

$$z_k(z,\xi)=F^{k-k_0}z+\sum_{\tau=k_0}^{k-1}F^{k-\tau-1}\varphi(\xi_\tau)$$

then, letting $k\to\infty$, one gets

$$\zeta^\top\zeta=(z_k^\top(z,\xi))(z_k(z,\xi))_\infty.$$

Setting $\Psi(z,\xi)=(z-\phi(\xi))^\top(z-\phi(\xi))-z^\top z$, the cross term verifies (7) because of (10). ■

Remark 2.1: The cross-term in (8) depends on $\|z_k(z,\xi)\|^2$ that admits a limit for $k\to\infty$. This is not so in general for the solution $z_k(z,\xi)$, except in the particular case of $n_z=1$. $V_0(z,\xi)$ can be thus computed even if a decoupling change of coordinates does not exist.

2) $W(f(z))\equiv W(z)$: Here, (4) specifies as

$$\Delta_k\Psi(z,\xi)=-W(f(z_k)+\varphi(z_k,\xi_k))+W(z_k)=-\Delta_kW(z)$$

so that the cross term takes the form

$$\Psi(z,\xi)=\sum_{k=0}^{\infty}\left[W(z_{k+1})-W(z_k)\right]=W_\infty(z,\xi)-W(z)$$

with $W_\infty(z,\xi):=\lim_{k\to\infty}W(z_k(z,\xi))$. Consequently, one gets

$$V_0(z,\xi)=U(\xi)+W_\infty(z,\xi).$$

3) $f(z)=z$: In such a case, one computes

$$z_\infty(z,\xi)=z+\lim_{N\to\infty}\sum_{k=0}^{N}\varphi(z_k,\xi_k)$$

and thus $W_\infty(z,\xi)=W(z_\infty(z,\xi))$. Accordingly, the map $(z,\xi)\mapsto(z_\infty,\xi)$ defines a *local* coordinates change since

$$\frac{\partial z_\infty}{\partial z}=I+\lim_{N\to\infty}\sum_{k=0}^{N}\frac{\partial\varphi}{\partial z}(z_k,\xi_k)$$

and the sum vanishes at $\xi=0$. When the connection term $\varphi(\xi,z)$ does not depend on z, the above coordinates change is globally defined as one recovers a strict-feedforward form.

4) Particular structures of $\varphi(\xi)$: When the connection function $\varphi(\xi)$ is a finite polynomial of degree p, the cross term is quadratic of degree $2p$; the following example illustrates the case.

Example: Given

$$z_{k+1}=z_k+\frac{3}{4}\xi_k^2,\quad \xi_{k+1}=\frac{1}{2}\xi_k.$$

which verifies Assumptions **A.1** to **A.4** with $U(\xi)=\xi^2$ and $W(z)=z^2$. Assuming the connection term $\varphi(\cdot)$ to be a finite polynomial of degree 2, we set the cross term as a polynomial of degree 4, $\Psi(z,\xi)=a_1z\xi^2+a_2\xi^4$. Accordingly, one computes $a_1,a_2\in\mathbb{R}$ to solve (7) that specialises as

$$\begin{aligned}&\frac{a_1}{2}(z+\frac{3}{4}\xi^2)\xi^2+\frac{a_2}{16}\xi^4-a_1z\xi^2-a_2\xi^4\\&=\frac{1}{16}\xi^4+\frac{1}{2}\xi^2(z+\frac{3}{4}\xi^2)-\xi^4-2z\xi^2.\end{aligned}$$

III. Stabilization of extended cascade dynamics

The so built Lyapunov function $V_0(z,\xi)$ is now exploited to show u-average passivity of the extended controlled cascade and to compute the corresponding stabilizing feedback. Without loss of generality, the problem is set in the (F_0,G) formalism (1).

A. Feedforward dynamics

Consider the two block controlled feedforward dynamics

$$z^+=f(z)+\varphi(z,\xi),\quad z^+:=z^+(0)\tag{13a}$$
$$\frac{\partial z^+(u)}{\partial u}=G_z(z^+(u),\xi^+(u),u)\tag{13b}$$
$$\xi^+=a(\xi),\quad \xi^+:=\xi^+(0)\tag{13c}$$
$$\frac{\partial\xi^+(u)}{\partial u}=B_\xi(\xi^+(u),u)\tag{13d}$$

with uncontrolled part defined in (2) and controlled vector fields $G_z(\cdot,\cdot,u)$ and $B_\xi(\cdot,u)$. In a more compact way, one writes over $\mathbb{R}^{n_z}\times\mathbb{R}^{n_\xi}$

$$x^+=F(x),\quad ,\frac{\partial x^+(u)}{\partial u}=G(x^+(u),u),\quad x^+:=x^+(0)$$

with $x=col(z,\xi)$, $F(x)=col(f(z)+\varphi(z,\xi),a(\xi))$ and $G(x^+(u),u)=col(G_z(z^+(u),B_\xi(\xi^+(u)),u)$.

For any triplet (z_k,ξ_k,u_k), by integrating (13b)-(13d) over $[0,u_k[$ with initial condition (13a)-(13c), one recovers a feedforward dynamics in the form of a map

$$\begin{aligned}z_{k+1}&=f(z_k)+\varphi(z_k,\xi_k)+g(z_k,\xi_k,u_k)\\ \xi_{k+1}&=a(\xi_k)+b(\xi_k,u_k)\end{aligned}$$

where $(z_{k+1},\xi_{k+1})=(z^+(u_k),\xi^+(u_k))$ and

$$\begin{aligned}\frac{\partial g(z,\xi,u)}{\partial u}&:=G_z(z^+(u),\xi^+(u),u)\\ \frac{\partial b(\xi,u)}{\partial u}&:=B_\xi(\xi^+(u),u).\end{aligned}$$

Property 3.1: Given any C^1-function $S:\mathbb{R}^{n_z}\times\mathbb{R}^{n_\xi}\to\mathbb{R}$, one can rewrite

$$S(x_{k+1})=S(F(x_k))+\int_0^{u_k}\mathrm{L}_{G(\cdot,v)}S(x^+(v))\mathrm{d}v$$

where $\mathrm{L}_{G(\cdot,v)}S(x)$, denotes the usual Lie derivative of the function $S(\cdot)$ along $G(\cdot,v)$; i.e., $\mathrm{L}_{G(\cdot,v)}S(x):=\frac{\partial S}{\partial x}G(x,v)$. Furthermore, one has

$$\int_0^{u_k}\mathrm{L}_{G(\cdot,v)}S(x^+(v))\mathrm{d}v=u_k\int_0^1\mathrm{L}_{G(\cdot,\theta u_k)}S(x^+(\theta u_k))\mathrm{d}\theta.$$

B. u-average passivity and PBC design

GAS of the equilibrium can now be achieved through u-average passivity-based control as introduced in [14]. The following definitions are recalled.

Definition 3.1 (u-average passivity): The dynamics (13), with output $y = H(x,u)$ is u-average passive with positive definite storage function $S(\cdot)$ if the following inequality holds for any $u \in \mathbb{R}$

$$S(x^+(u)) - S(x) \leq \int_0^u H(x^+(v),v)\mathrm{d}v. \tag{14}$$

Definition 3.2 (ZSD): Given (13) with output $H(x,u)$, let $Z \subset \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi}$ be the largest positively invariant set contained in $\{x \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_\xi} \mid H(x,0) = 0\}$. (13) is Zero-State-Detectable (ZSD) if $x = 0$ is asymptotically stable conditionally to Z.

Theorem 3.1: Consider (13) under **A.1** to **A.4**, then:

- (13) is u-average passive with respect to the output
$$H(z,\xi,u) = \mathrm{L}_{G(\cdot,u)}V_0(z,\xi) \tag{15}$$
and storage function $V_0(z,\xi)$;
- if, furthermore, (13) with output $H(z,\xi,0)$ is ZSD, the feedback u_d solving the equality
$$u_d = -\frac{1}{u_d}\int_0^{u_d} \mathrm{L}_{G(\cdot,v)}V_0(z^+(v),\xi^+(v),v)\mathrm{d}v \tag{16}$$
achieves GAS of the equilibrium $(z,\xi) = (0,0)$;
- if the linear approximation of (13) is stabilizable then (16) ensures LES of the closed-loop.

Proof: Computing $\Delta_k V_0(z,\xi) = V_0(z_{k+1},\xi_{k+1}) - V_0(z_k,\xi_k)$ along the dynamics (13) one gets (dropping the k-index in the right hand side)

$$\begin{aligned}\Delta_k V_0(z,\xi) &= U(a(\xi)) - U(\xi) + \int_0^u \mathrm{L}_{B_\xi(\cdot,v)}U(\xi^+(v))dv\\ &+W(f(z)+\varphi(z,\xi)) - W(z) + \int_0^u \mathrm{L}_{G_z(\cdot,\xi^+(v),v)}W(z^+(v))dv\\ &+\Psi(F(z,\xi)) - \Psi(z,\xi) + \int_0^u \mathrm{L}_{G(\cdot,v)}\Psi(z^+(v),\xi^+(v))dv.\end{aligned}$$

By construction of $\Psi(\cdot)$ for $u = 0$, one concludes u-average passivity with respect to the dummy output $H(\cdot,u) = \mathrm{L}_{G(\cdot,u)}V_0$ and storage function $V_0(\cdot)$; i.e.

$$\Delta_k V_0(z,\xi) \leq \int_0^u \mathrm{L}_{G(\cdot,v)}V_0(z^+(v),\xi^+(v))dv. \tag{17}$$

Accordingly, the control u solution to (16) achieves GAS of the equilibrium whenever (13) is ZSD with respect to $H(\cdot,0)$. LES follows from u-average passivity plus the stabilizability of the linear approximation of (13) at the origin. ■

Remark 3.1: The damping controller u_d solution of the equality (16) can equivalently be rewritten as the solution of

$$u_d = -\int_0^1 \mathrm{L}_{G(\cdot,\theta u_d)}V_0(x^+(\theta u_d))\mathrm{d}\theta. \tag{18}$$

To avoid the difficult problem of solving implicit equalities, approximate solutions can be computed. In [16], the authors provide an explicit and exactly computable expression of the feedback u which preserves u-average passivity and stability. The consequent solution is bounded by a positive constant $\mu \in \mathbb{R}$ and is defined as

$$u_{dap}(x) = -K(x)\mathrm{L}_{G(\cdot,0)}V_0(x^+(0))$$

for a suitable gain $K(\cdot) > 0$.

Example: Consider the discrete-time cascade dynamics

$$\begin{aligned}&z^+ = z+\xi, \quad \xi^+ = \xi\\ &\frac{\partial z^+(u)}{\partial u} = \frac{1}{2} - (\xi^+(u))^2, \quad \frac{\partial \xi^+(u)}{\partial u} = 1\end{aligned}$$

or, equivalently,

$$z_{k+1} = z+\xi+u(\frac{1}{2}-\xi^2) - u^2\xi - \frac{1}{3}u^3, \quad \xi_{k+1} = \xi+u$$

which verifies Assumption **A.1** with $W(z) = \frac{1}{2}z^2$ and Assumption **A.2** with preliminary feedback $u = -\frac{2}{3}\xi$ and $U(\xi) = \frac{1}{2}\xi^2$. The cross term $\Psi(z,\xi) = \frac{1}{2}(z+\xi+\frac{\xi^3}{3})^2 - \frac{1}{2}z^2$ verifies $\Delta_k V_0(z,\xi) = \Delta_k U(\xi) = -\frac{4}{9}\xi_k^2$. Finally, the u-average output and the consequent control are provided by

$$\begin{aligned}H(z,\xi,u) &= 4\xi + \frac{3}{2}z + \frac{13}{8}u + \frac{1}{2}\xi^3\\ u &= -\frac{4}{7}z - \frac{32}{21}\xi - \frac{4}{21}\xi^3.\end{aligned}$$

IV. Some cases of study

A. The case of strict-feedforward dynamics

Consider the controlled strict-feedforward dynamics

$$z^+ = Fz + \varphi(\xi), \quad \frac{\partial z^+(u)}{\partial u} = G(\xi^+(u),u) \tag{19a}$$

$$\xi^+ = a(\xi), \quad \frac{\partial \xi^+(u)}{\partial u} = B(\xi^+(u),u) \tag{19b}$$

or equivalently

$$z_{k+1} = Fz_k + \varphi(\xi_k) + g(\xi_k,u_k), \quad \xi_{k+1} = a(\xi_k) + b(\xi_k,u_k)$$

with uncontrolled part (6) and by definition

$$\begin{aligned}g(\xi_k,u_k) &:= \int_0^{u_k} G(\xi^+(v),v)\mathrm{d}v\\ b(\xi_k,u_k) &:= \int_0^{u_k} B(\xi^+(v),v)\mathrm{d}v\end{aligned}$$

with $g(\cdot,0) = 0$ and $b(\cdot,0)$. As already detailed, when $u \equiv 0$, the coordinates change $\zeta = z - \phi(\xi)$ in (9) transforms the system into a decoupled dynamics of the form (11). Specyfying to (19), one gets

$$\zeta^+ = F\zeta_k, \quad \frac{\partial \zeta^+(u)}{\partial u} = G_\zeta(\xi^+(u),u) \tag{20a}$$

$$\xi^+ = a(\xi_k), \quad \frac{\partial \xi^+(u)}{\partial u} = B(\xi^+(v),v)\mathrm{d}v \tag{20b}$$

where

$$G_\zeta(\xi^+(u),u) = G(\xi^+(u),u) - \mathrm{L}_{B(\cdot,u)}\phi(\xi^+(u)).$$

As a consequence, Theorem 3.1 holds with output

$$Y_1(\zeta,\xi,u) = \mathrm{L}_{G_\zeta(\cdot,u)}\tilde{V}_0(\zeta,\xi). \tag{21}$$

Remark 4.1: When $F = I$ and $n_z = 1$, the coordinates change $\zeta = z - \phi(\xi)$ makes the ζ-dynamics driftless. Accordingly, one recovers the result in [19] proposed when assuming directly in (19), $\xi_{k+1} = u_k$ and $n_z = 1$.

Remark 4.2: In [16], the strict-feedforward stabilization is set in the Immersion and Invariance (I&I) framework, [20] when $n_z = 1$. Assuming **A.2**, a stable set over which the closed loop ξ-dynamics evolves is exhibited. The design aims at driving the off-stable set state components ζ to zero while ensuring boundedness of the full state trajectories. I&I is less demanding since the knowledge of a Lyapunov function $U(\xi)$ for the ξ-system is not necessary. On the other hand, the cross term approach covers a wider range of cases.

B. Stabilization of input-delayed dynamics

The result is now applied to design u-average passivity-based controllers for discrete-time dynamics affected by input delay of the form

$$z_{k+1} = f(z_k) + \varphi(z_k, u_{k-1}). \tag{22}$$

Setting the usual extension $\xi_k = u_{k-1}$, (22) rewrites as

$$z_{k+1} = f(z_k) + \varphi(z_k, \xi_k), \quad \xi_{k+1} = u_k \tag{23}$$

that clearly takes the form of (13) with $g(z, \xi, u) = 0$ and $a(\xi) = 0$. Assuming GS the origin of the dynamics $z_{k+1} = f(z_k)$ with C^1 and radially unbounded Lyapunov function $W(z)$ and setting $U(\xi) = \xi^2$, the Lyapunov function $V_0(z, \xi)$ for (23) takes the form $V_0(z, \xi) = \xi^2 + W(z) + \Psi(z, \xi)$ with cross term solution of

$$\Delta_k \Psi(z, \xi)\big|_{u\equiv 0} = -W(f(z) + \varphi(z, \xi)) + W(f(z)).$$

Under the assumptions in Theorem 3.1, one specifies the output map $H_{del}(z, \xi) = \frac{\partial V_0}{\partial \xi}(z, \xi)$ with respect to which (23) is u-average passive so satisfying the inequality

$$V_0(f(z) + \varphi(z, \xi), u) - V_0(z, \xi) \leq \int_0^u \frac{\partial V_0}{\partial \xi}(f(z) + \varphi(z, \xi), v)\mathrm{d}v.$$

Accordingly, the control u_{del} solution of the equality

$$u_{del} = -\frac{1}{u_{del}} \int_0^{u_{del}} \frac{\partial V_0}{\partial \xi}(f(z) + \varphi(z, \xi), v)\mathrm{d}v$$

stabilizes the equilibrium provided the ZSD property holds.

This comment can be generalized to multiple input delays and to a z-dynamics explicitly depending on u as well. This is of peculiar interest when the problem of stabilizing a continuous-time time-delay system is set in the sampled-data context and reformulated as a discrete-time stabilizing one over an extended state space [21].

V. CONCLUSIONS

Stabilization of discrete-time dynamics in feedforward form via Lyapunov-based and passivity-based methodologies has been addressed. The study is detailed for the case of two interconnected dynamics by constructing a Lyapunov function through the definition of a suitable cross-term. When considering dynamics issued from sampling, a similar approach has been developed in [22], taking advantage of the primitive continuous-time properties. Work is progressing regarding multi-block cascade dynamics and analyzing the variety of control problems involving these structures.

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APPENDIX

Let us first prove **(i)**. Being the equilibrium of the dynamics $\xi_{k+1} = a(\xi_k)$ LES, there exist a real constant $|\alpha| < 1$ and a function $\gamma(\cdot) \in \mathscr{K}$ so that $\tilde{\xi}(s,\xi) \leq \gamma(\|\xi\|)|\alpha|^s$ for any $s \geq 0$. Then, because of Assumption **A.4**, the following inequality holds

$$W(f(z_k)+\varphi(z_k,\xi_k)) - W(z_k) \leq \tag{24}$$

$$W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k)) \leq \Big|\frac{\partial W}{\partial z}\varphi(z_k,\xi_k)\Big| \leq$$

$$\|\frac{\partial W}{\partial z}\|(\gamma(\|\xi\|))|\alpha|^k + \gamma(\|\xi\|))|\alpha|^k\|z_k\|) \leq c\gamma(\|\xi\|)|\alpha|^k W(z_k).$$

Accordingly, $W(z)$ is not decreasing along the trajectories of (2) and $\|z_k\|$ and $\|\frac{\partial W}{\partial z}(z_k)\|$ are bounded on $[0,\infty)$ (because $W(z)$ is radially unbounded). Consequently, one can write

$$W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k)) \leq \gamma_1(\|(z,\xi)\|)\alpha^k \tag{25}$$

so getting that $W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k))$ is summable over $[0,\infty)$ and (5) exists and is bounded for all (z,ξ).

Continuity of $\Psi(\cdot)$ in (5) comes from the fact that it is the composition and the sum of continuous functions on $[0,\infty)$.

As far as **(ii)** is concerned, positive definiteness of $V_0(\cdot)$ is obtained by exploiting the radial unboundedness of $W(z)$.

$$\begin{aligned} W(z_k) &= W(z) + \sum_{t=0}^{k-1}\big[W(f(z_t)+\varphi(z_t,\xi_t)) - W(z_t)\big] \\ &= W(z) + \sum_{t=0}^{k-1}\big[W(f(z_t)+\varphi(z_t,\xi_t)) - W(f(z_t))\big] \\ &+ \sum_{t=0}^{k-1}\big[W(f(z_t)) - W(z_t)\big] \end{aligned}$$

where the term $W(f(z_t)) - W(z_t)$ is non-increasing for any $t \geq 0$. By substracting both sides of the last equality by $W(f(z_t)) - W(z_t)$ and taking the limit for $k \to \infty$ one gets

$$\begin{aligned} &W_\infty(z) - \sum_{t=0}^{\infty}\big[W(f(z_t)) - W(z_t)\big] = \\ &W(z) + \sum_{t=0}^{\infty}\big[W(f(z_t)+\varphi(z_t,\xi_t)) - W(f(z_t))\big] \end{aligned}$$

where $W_\infty(z) = \lim_{k\to\infty} W(z_k)$ and $\Psi(z,\xi) = \sum_{t=0}^{\infty}\big[W(f(z_t)+\varphi(z_t,\xi_t)) - W(f(z_t))\big]$. Hence, one gets that $V_0(z,\xi)$ rewrites as

$$V_0(z,\xi) = W_\infty(z) - \sum_{t=0}^{\infty}\big[W(f(z_t)) - W(z_t)\big] + U(\xi) \geq 0. \tag{26}$$

From the radially unboundedness of $W(\cdot)$ and $U(\cdot)$ one has that if $V_0(z,\xi) = 0$ then $\xi = 0$. By construction, $V_0(z,0) = W(z)$ so concluding that $V_0(z,\xi) = 0$ implies $(z,\xi) = (0,0)$. This last inequality proves that $V_0(\cdot)$ is positive-definite.

To prove its radial unboundedness we first point out that $V_0(z,\xi) \to \infty$ as $\|\xi\| \to \infty$ for any z because of (26). Hence, one has to show that

$$\lim_{\|z\|\to+\infty}\Big[W_\infty(z) - \sum_{t=0}^{\infty}\big(W(f(z_t)) - W(z_t)\big)\Big] = +\infty. \tag{27}$$

This is achieved by lower bounding (27) by means of a radially unbounded function deduced from $W(z)$. For, fix ξ so that $\gamma(\|\xi\|)$ in (24) is a constant C. Accordingly, for any $k \geq 0$, we write

$$\begin{aligned} &|W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k))| \leq \\ &\|\frac{\partial W}{\partial z}\|(C|\alpha|^k + C|\alpha|^k\|z_k\|). \end{aligned}$$

It follows that

$$\begin{aligned} &W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k)) \geq \\ &-|W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k))| \geq \\ &-2\|\frac{\partial W}{\partial z}\|C|\alpha|^k\|z_k\| - C(1-\|z_k\|)\|\frac{\partial W}{\partial z}\||\alpha|^k. \end{aligned}$$

When $1-\|z_k\| > 0$, the term $-C(1-\|z_k\|)\frac{\partial W}{\partial z}\||\alpha|^k$ can be discarded without affecting the inequality. On the other hand, when $1-\|z_k\| \leq 0$, it is bounded by $K_2|\alpha|^k$ so that

$$\begin{aligned} &W(f(z_k)+\varphi(z_k,\xi_k)) - W(f(z_k)) \geq \\ &-2\|\frac{\partial W}{\partial z}\|C|\alpha|^k\|z_k\| - K_2|\alpha|^k. \end{aligned}$$

Using **A.4** we obtain

$$W(f(z_k)+\varphi(z_k,\xi_k)) - W(z_k) \geq \tag{28}$$

$$\begin{cases} -K|\alpha|^k W(z_k) - K_2|\alpha|^k + W(f(z_k)) - W(z_k), & \|z\| > r \\ -K_1|\alpha|^k W(z_k) - K_2|\alpha|^k + W(f(z_k)) - W(z_k), & \|z\| \leq r \end{cases}$$

with $r \geq 1$ and real K, K_1, K_2.From (28) one gets the following lower bounds on $W(z_k)$.

When $\|z\| > r$ and $k \in [0,t)$

$$\begin{aligned} W(z_k) \geq \phi(k,0)W(z) + \sum_{t=0}^{k-1}\phi(k-1,t)\big[-K_2|\alpha_1|^t + \\ W(f(z_t)) - W(z_t)\big] \end{aligned}$$

When $\|z\| \leq r$ and $k \in [0,t)$

$$W(z_k) \geq W(z) + \sum_{t=0}^{k-1}\big[-K_1|\alpha|^t - K_2|\alpha|^t + W(f(z_t)) - W(z_t)\big]$$

with $\phi(k,t) = \prod_{j=t}^{k}(1-K|\alpha|^j)$. Accordingly, by mixing both the bounds, one gets

$$W(z_k) \geq \phi(k,0)W(z) + \sum_{t=0}^{k-1}(-K_1|\alpha|^t - K_2|\alpha|^t + W(f(z_t)) - W(z_t))$$

so that for all $k \geq 0$, $\phi(k,0)$ admits a lower bound K_3 and

$$W(z_k) \geq K_3 W(z) + \sum_{t=0}^{k-1}\big[W(f(z_t)) - W(z_t)\big] + r_k$$

with $r_k := \sum_{t=0}^{k-1}\big[-K_1|\alpha|^t - K_2|\alpha|^t\big]$ converging to a bounded solution r^* over $[0,\infty)$. Taking the limit as $k \to \infty$, one obtains

$$W_\infty(z,\xi) - \sum_{t=0}^{k-1}\big[W(f(z_t)) - W(z_t)\big] \geq K_3 W(z) + r^*.$$

We note that r^* and K_3 may depend on ξ but are independent of z so that (27) holds. Finally, by construction $V_0(z_{k+1},\xi_{k+1}) - V_0(z_k,\xi_k)) = W(f(z_k)) - W(z_k) + U(a(\xi_k)) - U(\xi_k) \leq 0$ so concluding the proof.

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Sampled-data stabilisation of a class of state-delayed nonlinear dynamics

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Abstract— **The paper deals with the stabilisation of strict-feedback dynamics with a delay on the last component of the state. It is shown that the Immersion and Invariance approach provides a natural framework for solving the problem. An accademic simulated example is provided.**

Index Terms— **Nonlinear stabilisation, Systems with delays, Nonlinear sampled-data systems.**

I. INTRODUCTION

Stabilisation under Immersion and Invariance - I&I -, proposed in [1] for continuous-time dynamics, has been the object of several investigations in the last decade. Several extensions and applicatisve results have been developed which identify a recognized control approach ([2], [3], [4]). It was extended to the discrete-time domain in [5] in relation with adaptive control in presence of parameter uncertainties.

More recently, in [6], it has been shown that the I&I approach provides a natural framework to deal with sampled-data stabilisation of input-delayed dynamics; while in [7] it has been fruitfully applied to design sampled-data controllers for dynamics which exhibit specific structures such as strict-feedback forms. Exploiting sampling to control systems with delayed inputs is a well known practice which has found renewed interest in the current literature ([8], [9], [10], [11], [12]). The present work follows these lines.

In this paper, the stabilization of a strict-feedback dynamics with delays on the last connecting state is addressed. More precisely and for simplicity we consider dynamics with one cascade of the form

$$\dot{x}_1(t) = f(x_1(t)) + g(x_1(t))x_2(t-\tau), \quad \dot{x}_2(t) = u(t) \qquad (1)$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}$, $u \in U \subseteq \mathbb{R}$, f and g are smooth vector fields on $\mathbb{R}^n$, i. e. C^∞, and τ denotes a delay acting on x_2, the connecting state.

The problem is set in the digital context assuming that the measures of the state are available at the sampling instants $t = k\delta$, $k \geq 0$ and the control is maintained constant over time intervals of length δ. The sampling period δ is chosen so that $\tau = N\delta$ for a positive $N \in \mathbb{N}^+$.

The idea developed in the sequel starts by noting that under a simple coordinates change, the delayed dynamics admits a higher but finite dimensional sampled-data equivalent model over which stabilization is reformulated in the I&I context with target given by the sampling of the delay-free dynamics. Then the design of the controller is achieved by driving the dynamics to the invariant manifold with boundedness of all the extended state trajectories.

The proposed solution combines two previous contributions of the authors:

- the sampled-data I&I stabilizer discussed in [6] which naturally identifies the target with the delay-free dynamics;
- the direct sampled-data I&I stabilizing in [7] to define the immersion mapping and feedback which render invariant the target manifold.

We note that the same type of state delays on connected dynamics was studied in [13] according to a continuous-time backstepping procedure.

This paper is organized as follows: in *Section II* the class of system under study is defined and some preliminary results are given; in *Section III* the main result is given and specified in the particular case of $\delta = \tau$ in *Section IV*; an academic example is discussed with some simulations in *Section V*.

II. PROBLEM SETTLEMENT AND PRELIMINARY RESULTS

We summarize in the following the recurrent assumptions:

A) the sampling period δ, small enough, is a multiple of the delay τ, i.e. $\tau = N\delta$ for a suitable $N \in \mathbb{N}^+$;
B) the input $u(t)$ is set constant over time intervals of length δ; namely, $u(t) = u_k$ $t \in [k\delta, (k+1)\delta[$;
C) the delay free x_1-dynamics of (1) is smoothly stabilizable trough a *fictitious* continuous-time controller $x_2 = \gamma(x_1)$ and a control Lyapunov function, $W : \mathbb{R}^n \to \mathbb{R}$, is assumed known (see [14]);

$$[L_f + \gamma L_g]W(x_1) < 0 \quad \forall x_1 \in \mathbb{R}^n/\{0\}.$$

Accordingly, assuming the I&I framework [1], one defines for the delay-free dynamics:

- the target dynamics $\dot{\xi} = f(\xi) + g(\xi)\gamma(\xi)$;
- the immersion map $\pi(\xi) = \begin{pmatrix} \xi' & \gamma(\xi) \end{pmatrix}'$;
- the implicit manifold $z = \phi(x) = x_2 - \gamma(x_1)$, with $z(0) = x_2(0) - \gamma(x_1(0))$;
- the *on-the-manifold* control law $\varphi(\xi) = \dot{\gamma}(\xi)$ which renders invariant the manifold.

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Finally, the control law which makes the manifold attractive with boundedness of the trajectories of the full dynamics

$$\begin{aligned}\dot{x}_1 &= f(x_1)+g(x_1)[z+\gamma(x_1)]\\ \dot{x}_2 &= \psi(x,z)\\ \dot{z} &= \psi(x,z)-\dot{\gamma}(x_1)\end{aligned}$$

is set as $\psi(x,z)=\dot{\gamma}(x_1)-K(x)z$ with suitably chosen gain function $K(x)$.

A. The extended hybrid representation

Consider the continuous-time dynamics (1) and set $x_3(t)=x_2(t-\tau)$ so moving the delay into the input variable

$$\dot{x}_1(t)=f(x_1(t))+g(x_1(t))x_3(t), \quad \dot{x}_3(t)=u(t-\tau) \tag{2}$$

so that the approach proposed in [6] can be used. Setting $\tau=N\delta$ and under Assumption A, the *hybrid extended dynamics* over $\mathbb{R}^{n+1+N}$ is defined for $t\in[k\delta,(k+1)\delta[$ as:

$$\begin{aligned}&\dot{x}_1(t)=f(x_1(t))+g(x_1(t))x_3(t)\\ &\dot{x}_3(t)=v_k^1, \quad v_{k+1}^1=v_k^2, \quad \ldots \quad v_{k+1}^N=u_k.\end{aligned} \tag{3}$$

It results that the control design problem can be set on the sampled-data equivalent of (3), which is finite dimensional dynamics, with state extension of order N, strictly related to the delay length.

B. Sampled-data delay free I&I stabilization

Following [7], Assumption C provides sufficient conditions for the existence of an I&I sampled-data controller preserving GAS of the equilibrium when $\tau=0$. Setting $\tau=0$, one defines the equivalent sampled-data dynamics of (1) through integration over the time interval $[k\delta,(k+1)\delta[;k\geq 0$, as in [15]. It is provided in the form of a map parameterized by δ:

$$\begin{aligned}x_{1k+1}&=F^{\delta}(x_{1k},x_{2k})+\frac{\delta^2}{2!}u_kG^{\delta}(x_{1k},x_{2k},u_k)\\ x_{2k+1}&=x_{2k}+\delta u_k\end{aligned} \tag{4}$$

when $x_k=x(t)\big|_{t=k\delta}$. The following proposition summarises the results in [7], where a complete proof is given.

Proposition 2.1: Consider the nonlinear continuous-time dynamics in (1) under Assumptions A, B and C in the delay free case (i.e., $\tau=0$). Then, its sampled-data equivalent dynamics (4) is I&I stabilizable with target dynamics

$$\xi_{k+1}=F^{\delta}(\xi_k,\gamma^{\delta}(\xi_k))+\frac{\delta^2}{2!}\varphi^{\delta}(\xi_k)G^{\delta}(\xi_k,\gamma^{\delta}(\xi_k),\varphi^{\delta}(\xi_k)). \tag{5}$$

The mappings $\gamma^{\delta}(\cdot)$ and $\varphi^{\delta}(\cdot)$ are solutions of the two equalities:

$$W(\xi_{k+1}) = W(\xi_k)+\int_{k\delta}^{(k+1)\delta}L_{f+g\gamma}W(\xi(\tau))d\tau \tag{6}$$

$$\gamma^{\delta}(\xi_{k+1}) = \gamma^{\delta}(\xi_k)+\delta\varphi^{\delta}(\xi_k). \tag{7}$$

We note that the mappings $\gamma^{\delta}(\cdot)$ and $\varphi^{\delta}(\cdot)$ are defined by their asymptotic series expansions in powers of δ as follows

$$\gamma^{\delta}(\xi_k)=\gamma_0(\xi_k)+\sum_{i\geq 0}\frac{\delta^i}{(i+1)!}\gamma_i(\xi_k)$$

$$\varphi^{\delta}(\xi_k)=\varphi_0(\xi_k)+\sum_{i\geq 0}\frac{\delta^i}{(i+1)!}\varphi_i(\xi_k).$$

Accordingly, both the immersion mapping $\pi^{\delta}(\xi)=\begin{pmatrix}\xi' & \gamma^{\delta}(\xi)\end{pmatrix}'$ and the implicit manifold characterisation $\phi^{\delta}(x)=x_2-\gamma^{\delta}(x_1)$ are parameterized by the sampling period δ. Setting $\delta=0$, one recovers the continuous-time solutions $(\pi(\cdot),\phi(\cdot),\varphi(\cdot))$.

We note that the equality (6) ensures Input Lyapunov Matching - ILM - at the sampling instants (see [15], [16]) of the closed loop behavior of the function $W(\cdot)$ on the target dynamics (5).This guarantees that the equilibrium of (5) is GAS. On the other hand, equality (7) guarantees the invariance of the manifold. Accordingly, it is implicitly defined as $\phi^{\delta}(x)=0$. On these bases, the I&I stabilizing sampled-data feedback $u=\psi^{\delta}(x,z)$ is designed to drive z to zero while preserving boundedness of the complete state trajectories

$$\begin{aligned}&x_{1k+1}=F^{\delta}(x_{1k},x_{2k})+\frac{\delta^2}{2!}u_kG^{\delta}(x_{1k},x_{2k},u_k)\\ &x_{2k+1}=x_{2k}+\delta u_k, \quad z_{k+1}=z_k+\delta u_k-\gamma^{\delta}(x_{1k+1})+\gamma^{\delta}(x_{1k}).\end{aligned}$$

It follows that the equilibrium of the closed-loop x dynamics is GAS in the delay free case.

III. Main result

Consider the continuous-time dynamics (2) (or, equivalently, (1)) and its hybrid representation (3) over $\mathbb{R}^{n+1+N}$ when $\tau=N\delta$. Its sampled-data equivalent dynamics is described as

$$\begin{aligned}x_{1k+1}&=F^{\delta}(x_{1k},x_{3k})+\frac{\delta^2}{2!}v_k^1G^{\delta}(x_{1k},x_{3k},v_k^1)\\ x_{3k+1}&=x_3+\delta v_k^1, \; v_{k+1}^1=v_k^2 \quad \ldots \quad v_{k+1}^N=u_k\end{aligned} \tag{8}$$

or, in a more compact way, as $x_{k+1}^e=\bar{F}^{\delta}(x_k^e,u_k)$ with $x^e=col(x_1', \; x_3, \; v^1, \; \ldots, \; v^N)'\in\mathbb{R}^{n+1+N}$. In [6], the authors define the GAS sampled-data I&I target dynamics as the closed-loop dynamics (4) under the delay-free feedback $\psi^{\delta}(\cdot,\cdot)$, as defined in Proposition 2.1. Hence, the attractive manifold is the one where the delay on the input is recovered. An alternative approach is stated by the following result.

Theorem 3.1: Consider the input-affine continuous-time dynamics in (1) with state delay $\tau=N\delta$ under Assumptions A, B and C. Let the extended sampled-data dynamics (8) with equilibrium $x_*^e=col(x_*', \; 0_{N\times 1}')$, then it is I&I stabilizable with target dynamics 5 and $\gamma^{\delta}(\cdot),\varphi^{\delta}(\cdot):\mathbb{R}^n\to\mathbb{R}$ defined as in Proposition 2.1.

Proof: To prove the thesis, one has to show that the conditions in Theorem 2.2 in [6] are verified. For this purpose, suppose $\gamma^{\delta}(\cdot)$ and $\varphi^{\delta}(\cdot)$ defined according to Proposition 2.1 as solutions to the I-LM problem in (6-7) with control Lyapunov function $W:\mathbb{R}^n\to\mathbb{R}^+$. Consequently, one can define the extended immersion mapping $\bar{\pi}^{\delta}:\mathbb{R}^n\to\mathbb{R}^{n+1+N}$

$$\bar{\pi}^{\delta}(\xi_k)=\begin{pmatrix}\xi_k' & \gamma^{\delta}(\xi_k) & \varphi^{\delta}(\xi_k) & \ldots & \varphi^{\delta}(\xi_{k+N-1})\end{pmatrix}' \tag{9}$$

and extended mapping $\bar{\phi}^{\delta} : \mathbb{R}^{n+N+1} \to \mathbb{R}^{N+1}$ as

$$\begin{aligned} z_{1k} &= \bar{\phi}_1^{\delta}(x_k, v_k) = x_{3k} - \gamma^{\delta}(x_{1k}) \\ z_{2k} &= \bar{\phi}_2^{\delta}(x_k, v_k) = v_k^1 - \varphi^{\delta}(x_{1k}) \\ &\dots \\ z_{N+1,K} &= \bar{\phi}_{N+1}^{\delta}(x_k, v_k) = v_k^N - \varphi^{\delta}(x_{1,k+N-1}) \end{aligned} \tag{10}$$

where $v = col(v^1, \dots, v^N)'$.

By construction, the three instrumental condition for I&I stabilization are satisfied (see Theorem 2.2 in [6]). More in detail, the target dynamics $\xi_{k+1} = \alpha^{\delta}(\xi)$ as in (5) has a GAS equilibrium $\xi_* \in \mathbb{R}^p$. Then, the *Immersion Condition* is satisfied by the choices (9)-(10) with $z_0 = \bar{\phi}(x_0, v_0)$. On these bases, it is straightforward that the sampled-data feedback $u_k = \bar{\psi}^{\delta}(x_k^e, z_k)$ designed in order to bring z to zero and make all the trajectories of the dynamics

$$\begin{aligned} z_{1k+1} =& z_{1k} + \delta[z_{2k} - \varphi^{\delta}(x_{1k})] + \gamma^{\delta}(x_{1k}) - \gamma^{\delta}(x_{1k+1}) \\ z_{2k+1} =& z_{3k} \quad \dots \quad z_{Nk+1} = z_{N+1k} \\ z_{N+1k+1} =& \bar{\psi}^{\delta}(x_k^e, z_k) - \varphi^{\delta}(x_{1k+N}) \\ x_{k+1}^e =& \bar{F}^{\delta}(x_k^e, \bar{\psi}^{\delta}(x_k^e, z_k)). \end{aligned} \tag{11}$$

bounded, globally asymptotically stabilizes the equilibrium of (1). ■

Remark 3.1: When the system dynamics reaches the invariant manifold, the feedback reduces to $c^{\delta}(\cdot)$ corresponding to the delay-free stabilizing feedback $\varphi^{\delta}(\cdot)$ in Proposition 2.1. When $N = 0$, the delay-free case, one recovers $c^{\delta}(\xi) = \varphi^{\delta}(\xi)$.

Remark 3.2: For a given τ, the pair $(N, \ \delta)$ has to be chosen as a trade off between computational effort and required performances on the closed-loop system.

1) On the definition of the sampled-data control law: Theorem 3.1 states sufficient conditions for the existence of a I&I stabilizing controller $u_k = \bar{\psi}^{\delta}(x_k, v_k, z_k)$. In this section, we describe a multirate design strategy of order equal to the dimension of the off the manifold state component z. For, let us introduce the $(N+1)$-order multirate sampled-data dynamics associated with (11) when the I&I controller $\bar{\psi}^{i\bar{\delta}}(\cdot,\cdot)$ is denoted as u_k^i

$$\begin{aligned} z_{1k+1} =& z_{1k} + \bar{\delta} \sum_{i=2}^{N+1} [z_{ik} + \varphi^{\bar{\delta}}(x_{1k+\frac{i-2}{N+1}})] + \\ & \gamma^{\bar{\delta}}(x_{1k}) - \gamma^{\bar{\delta}}(x_{1k+1}) + \bar{\delta} u_k^1 \\ z_{2k+1} =& u_k^2 - \varphi^{\bar{\delta}}(x_{1k+1}) \quad \dots \quad z_{Nk+1} = u_k^N - \varphi^{\bar{\delta}}(x_{1k+\frac{2N-1}{N+1}}) \\ z_{N+1k+1} =& u_k^{N+1} - \varphi^{\bar{\delta}}(x_{1k+\frac{2N}{N+1}}) \\ x_{1k+1} =& F_1^{\bar{\delta}}(x_k^e, u_k^1) \quad x_{3k+1} = x_{3k} + \bar{\delta}[u_k^1 + \sum_{i=1}^{N} v_k^i] \\ v_{k+1}^1 =& u_k^2 \quad \dots \quad v_{k+1}^N = u_k^{N+1} \end{aligned} \tag{12}$$

in which the control $u(t)$ is maintained constant at values u_k^i over intervals of length $\bar{\delta} = \frac{\delta}{N+1}$ for all $t \in [k\delta + (i-1)\bar{\delta}, k\delta + i\bar{\delta}[$, $i = \frac{1}{N+1}$.

Remark 3.3: The prediction steps required with a single rate strategy is N; the multirate strategy requires, at most, $\frac{2N}{N+1}$ prediction steps.

The hypotheses of Theorem 2.2 in [6] are naturally preserved under the multirate controller. Though, an accurate rewriting of the immersion condition may be useful to point out that the so-defined sampled-data controller preserves manifold invariance under multirate-sampling. In particular, by defining $c^{i,\bar{\delta}}(\xi) = c^{\bar{\delta}}(\xi_{k+\frac{i}{N+1}})$ $(i = 1, \dots, N+1)$, one has that $\forall \xi \in \mathbb{R}^n$

$$\begin{aligned} &\xi_{k+1} = \alpha^{\bar{\delta}}(\xi_k) \\ &\gamma^{\bar{\delta}}(\xi_{k+1}) = \gamma^{\bar{\delta}}(\xi_k) + \bar{\delta} \sum_{i=2}^{N+1} \varphi^{\bar{\delta}}(\xi_{k+\frac{i-2}{N+1}}) + \bar{\delta} c^{1,\bar{\delta}}(\xi_k) \\ &c^{2,\bar{\delta}}(\xi_k) = \varphi^{\bar{\delta}}(x_{1k+1}) \quad \dots \quad c^{N,\bar{\delta}}(\xi_k) = \varphi^{\bar{\delta}}(x_{1k+\frac{2N-1}{N+1}}) \\ &c^{N+1,\bar{\delta}}(\xi_k) = \varphi^{\bar{\delta}}(x_{1k+\frac{2N}{N+1}}). \end{aligned} \tag{13}$$

Finally, one can see that the I&I stabilisation is achieved by the $(N+1)$-rate control u defined as

$$\begin{aligned} \bar{\delta} u_k^1 =& -\bar{\delta}\Gamma_1 z_{1k} - \gamma^{\bar{\delta}}(x_{1k}) + \gamma^{\bar{\delta}}(x_{1k+1}) - \\ & \bar{\delta} \sum_{i=2}^{N+1} [z_{ik} + \varphi^{\bar{\delta}}(x_{1k+\frac{i-2}{N+1}})] \\ u_k^2 =& -\Gamma_2 z_{2k} + \varphi^{\bar{\delta}}(x_{1k+1}) \\ &\dots \\ u_k^N =& -\Gamma_N z_{Nk} + \varphi^{\bar{\delta}}(x_{1k+\frac{2N-1}{N+1}}) \\ u_k^{N+1} =& -\Gamma_{N+1} z_{N+1k} + \varphi^{\bar{\delta}}(x_{1k+\frac{2N}{N+1}}) \end{aligned} \tag{14}$$

with suitably defined gains Γ_i $(i = 1, ..., N+1)$. More in detail, when such a controller is applied, one has that all trajectories of (12) are bounded for all $k \geq 0$ with

$$\lim_{k\to\infty} z_k = 0 \quad \bar{\psi}_k^{i\bar{\delta}}(\bar{\pi}^{\bar{\delta}}(\xi), \mathbf{0}) = c^{i\bar{\delta}}(\xi)$$

for $i = 1, \dots, N+1$.

Without loss of generality, the proof of the existence of such a solution is reported for the particular case of $\tau = \delta$.

IV. THE CASE $\tau = \delta$

Let us discuss more in detail the design of the feedback $\bar{\psi}^{\delta}(x^e, z)$ in the single-rate case in which $\tau = \delta$. In such a case, Theorem 3.1 specifies as follows.

Proposition 4.1: Consider the continuous-time dynamics (1) satisfying Assumptions A, B and C with state delay $\tau = \delta$. Let the extended dynamics on $\mathbb{R}^{n+2}$ be

$$\begin{aligned} x_{1k+1} &= F^{\delta}(x_{1k}, x_{3k}) + \frac{\delta^2}{2!} v_k G^{\delta}(x_{1k}, x_{3k}, v_k) \\ x_{3k+1} &= x_3 + \delta v_k, \qquad v_{k+1} = u_k. \end{aligned} \tag{15}$$

Then it is I&I stabilizable with target dynamics (5) whose equilibrium is made GAS with suitable choice of γ^{δ}, φ^{δ} : $\mathbb{R}^n \to \mathbb{R}$.

Proof: The proof proceeds in the same way as in the one of Theorem 3.1, so it will be omitted. ■

A. On the design of the sampled-data stabilizer

In this section, a possible choice of the controller which satisfies the condition on *Manifold invariance and attractivity with trajectory boundedness* is proposed. When $\tau = \delta$, the double rate sampled-data equivalent model of the hybrid dynamics (3) over time intervals of length $\delta = 2\bar{\delta}$ is defined as in (12) with

$$\bar{\psi}^{1\bar{\delta}}(x_{1k}, v_k, z_k) = \bar{\psi}^{\bar{\delta}}(x_{1k}, z_k)$$

and

$$\bar{\psi}^{2\bar{\delta}}(x_{1k}, v_k, z_k) = \bar{\psi}^{\bar{\delta}}(x_{1k+\frac{1}{2}}, z_{k+\frac{1}{2}}).$$

Setting

$$\begin{aligned}\bar{\delta}\bar{\psi}^{1\bar{\delta}}(x_{1k}, v_k, z_k) =& \bar{\delta}\Gamma_1 z_{1k} + \gamma^{\bar{\delta}}(x_{1k+1}) \qquad (16a)\\ &- \gamma^{\bar{\delta}}(x_{1k}) - \bar{\delta}[z_{2k} + \varphi^{\bar{\delta}}(x_{1k})]\\ \bar{\psi}^{2\bar{\delta}}(x_{1k}, v_k, z_k) =& \Gamma_2 z_{2k} + \varphi^{\bar{\delta}}(x_{1k+1}) \qquad (16b)\end{aligned}$$

the reduced z-dynamics is

$$z_{1k+1} = [1 + \bar{\delta}\Gamma_1]z_{1k}, \quad z_{2k+1} = [1 + \Gamma_2]z_{2k}.$$

The existence of such a controller is proved in the following Proposition.

Proposition 4.2: Given the sampled-data dynamics in (15) verifying Theorem 3.1, then there exists a double-rate control ensuring, at each step, I&I stabilisation of the dynamics in (15).

Proof: Denoting by $\bar{\psi}^{1\bar{\delta}} = u_k^1$. The proof consists in verifying that there exist solutions in the form

$$\bar{\psi}^{j\bar{\delta}}(x_{1k}, v_k, z_k) = \bar{\psi}_0^j(x_{1k}, v_k, z_k) + \sum_{i\geq 1} \bar{\delta}^i \bar{\psi}_i^j(x_{1k}, v_k, z_k) \qquad (17)$$

for $j = 1, 2$ to equalities (16a) and (16b).

The existence of a solution to (16b) is guaranteed since the right-hand side of the equality does not depend on $\bar{\psi}^{2\bar{\delta}}$ itself; hence, a series inversion is needed in order to compute the resulting controller. For, one rewrites $\gamma^{\bar{\delta}}(x_{1k+1})$ as the sum of two component:

$$\gamma^{\bar{\delta}}(x_{1k+1}) = \bar{\gamma}_1^{\bar{\delta}}(x_{1k}, z_k, 0) + \bar{\delta}u_k^1 \bar{\gamma}_2^{\bar{\delta}}(x_{1k}, z_k, u_k^1)$$

where

$$\bar{\gamma}_1^{\bar{\delta}}(x_{1k}, z_k, 0) = \gamma^{\bar{\delta}}(\tilde{F}^{\bar{\delta}}(x_{1k}, z_{1k} + \gamma^{\bar{\delta}}(x_{1k}), z_{2k} + \varphi^{\bar{\delta}}(x_{1k})))$$

does not depend on the control while

$$\begin{aligned}&\bar{\gamma}_2^{\bar{\delta}}(x_{1k}, z_k, u_k) =\\ &\frac{\delta}{2}\sum_{i\geq 1}\frac{\partial^i \gamma^{\bar{\delta}}}{\partial x_1^i}\Big|_{x_1=\tilde{F}^{\bar{\delta}}}[\tilde{G}(x_{1k}, z_{1k} + \gamma^{\bar{\delta}}(x_{1k}), z_{2k} + \varphi^{\bar{\delta}}(x_{1k}), u_k^1)]^i\end{aligned}$$

is control dependent. One can now rewrite the equality among formal series in (16a) as

$$\begin{aligned}&\bar{\delta}S(\bar{\delta}, x_{1k}, z_k, u_k^1) = \bar{\delta}u_k^1[1 - \bar{\gamma}_2^{\bar{\delta}}(x_{1k}, z_k, u_k)] - \bar{\delta}\Gamma_1(x_k^e)z_{1k} -\\ &\bar{\gamma}_1^{\bar{\delta}}(x_{1k}, z_k, 0) + \gamma^{\bar{\delta}}(x_{1k}) + \bar{\delta}[z_{2k} + \varphi^{\bar{\delta}}(x_{1k})] = 0.\end{aligned}$$

The existence of a solution is proved by means of the Implicit Function Theorem. Indeed, for $\delta = 0$ one has

$$\begin{aligned}&S(0, x_{1k}, z_k, u_k^1) = \psi_0^1(x_{1k}, z_k) - \Gamma_1(x_k^e)z_{1k} + 2\frac{\partial \gamma_0}{\partial x_1}\Big|_{x_{1k}}\{f(x_1)\\ &+ g(x_1)[z_{1k} + \gamma_0(x_1)]\} + [z_{2k} + \varphi_0(x_1)] = 0\end{aligned}$$

where $\gamma_0(\cdot)$ and $\varphi_0(\cdot)$ are defined according to Proposition 2.1. Such an equality is solved if

$$\begin{aligned}\psi_0^1(x_{1k}, z_k) = \Gamma_1(x_k^e)z_{1k} - 2\frac{\partial \gamma_0}{\partial x_1}\Big|_{x_{1k}}\{f(x_{1k}) +\\ g(x_{1k})[z_{1k} + \gamma_0(x_{1k})]\} - [z_{2k} + \varphi_0(x_{1k})]\end{aligned}$$

i.e., the controller defined on the double-rate Euler sampled-data model of (3). Since the partial derivative

$$\frac{\partial S(\bar{\delta}, u)}{\partial u}\Big|_{\bar{\delta}=0} = 1$$

is non-zero for any (x_1, z), one can conclude that there exists, for $\bar{\delta}$ small enough, a control $u = \bar{\psi}^{1\bar{\delta}}(x, v, z)$ in a neighbourhood of $\bar{\psi}_0^1(x, v, z)$ such that

$$S(\bar{\delta}, \rho(\bar{\delta})) = 0 \iff u = \bar{\psi}^{1\bar{\delta}}(x, v, z) = \rho(\bar{\delta}),$$

where ρ is the formal inversion $\rho(\bar{\delta}) = S^{-1}(\bar{\delta}, \rho(\bar{\delta}))$. Such a solution can be defined as an asymptotic series of $\bar{\delta}$ in the form (17) with $\bar{\psi}_0^1(\cdot,\cdot) = \rho(0)$. The I&I stabilisation is guaranteed since the invariance of the multi-rate controller is verified at the inter-sampling and sampling instants as in (13). Hence, Theorem 3.1 is satisfied for a suitable choice of $\bar{\delta}\Gamma_1$ and Γ_2 (not necessarily static) in order to have boundedness of the whole state trajectories in 12, with $N = 1$. At this point, the choice of $\bar{\delta}\Gamma_1$ and Γ_2 can be performed by means of a control Lyapunov function defined as $V^{\delta}(x, v, z) = W(x_1) + \sum_{i=1}^{2} z_i^2$. ■

B. Some constructive aspects

In this part, some constructive aspects are sketched for the computation of the solution in an approximate context, [17]. More in detail, considering (12), with $N = 1$, one gets in $O(|z|^2)$ the approximation below

$$\begin{aligned}x_{1k+1} =& \tilde{F}^{\bar{\delta}}(x_{1k}, \gamma^{\bar{\delta}}(x_{1k}), \varphi^{\bar{\delta}}(x_{1k})) + P_1^{\bar{\delta}}(x_{1k}, z_{1k}, z_{2k})z_{1k} +\\ &P_2^{\bar{\delta}}(x_{1k}, z_{1k}, z_{2k})z_{2k} + \frac{\bar{\delta}^2}{2}\bar{\psi}^{1\bar{\delta}}(x_k, v_k, z_k)\\ &\tilde{G}(x_{1k}, z_{1k} + \gamma^{\bar{\delta}}(x_{1k}), z_{2k} + \varphi^{\bar{\delta}}(x_{1k}), \bar{\psi}^{1\bar{\delta}}(x_k, v_k, z_k))\end{aligned}$$

with, discarding the dependence on the state and the control,

$$P_1^{\bar{\delta}} = \frac{\partial \tilde{F}^{\bar{\delta}}}{\partial x_3}\Big|_{x_3=\gamma^{\delta}(x_1)} \quad P_2^{\bar{\delta}} = \frac{\partial \tilde{F}^{\bar{\delta}}}{\partial v}\Big|_{v=\varphi^{\delta}(x_1)}.$$

Accordingly, one can write the Taylor expansion of $\gamma^{\bar{\delta}}(x_{1k+1})$ and $\varphi^{\bar{\delta}}(x_{1k+1})$ in $O(|z|^2)$ as

$$\begin{aligned}\gamma^{\bar{\delta}}(x_{1k+1}) =& \gamma^{\bar{\delta}}(\tilde{F}^{\bar{\delta}}) + \frac{\partial \gamma^{\bar{\delta}}}{\partial x_1}\Big|_{\tilde{F}^{\bar{\delta}}}[P_1^{\bar{\delta}} z_{1k} + \\ & P_2^{\bar{\delta}} z_{2k} + \frac{\bar{\delta}^2}{2}\bar{\psi}^{1\bar{\delta}}\tilde{G}^{\bar{\delta}}] + O(|z|^2) + O(|\bar{\psi}^{1\bar{\delta}}|^2)\\ \varphi^{\bar{\delta}}(x_{1k+1}) =& \varphi^{\bar{\delta}}(\tilde{F}^{\bar{\delta}}) + \frac{\partial \varphi^{\bar{\delta}}}{\partial x_1}\Big|_{\tilde{F}^{\bar{\delta}}}[P_1^{\bar{\delta}} z_{1k} + \\ & P_2^{\bar{\delta}} z_{2k} + \frac{\bar{\delta}^2}{2}\bar{\psi}^{1\bar{\delta}}\tilde{G}^{\bar{\delta}}] + O(|z|^2) + O(|\bar{\psi}^{1\bar{\delta}}|^2).\end{aligned} \tag{18}$$

One can now define the controls $\bar{\psi}^{1\bar{\delta}}$ and $\bar{\psi}^{2\bar{\delta}}$ by their asymptotic series expansions with respect to $\bar{\delta}$ truncated at the p-th order; namely,

$$\bar{\psi}^{j\bar{\delta},[p_j]}(x_1,v,z) = \bar{\psi}_0^j(x_1,v,z) + \sum_{i\geq 1}^{p} \bar{\delta}^i \bar{\psi}_i^j(x_1,v,z) \tag{19}$$

for $j=1,2$. Substituting (19) and (18) into (16a) and (16b), under suitable boundedness assumptions on $\Gamma_j(x_k,v_k)$, $j=1,2$, for the corresponding approximated dynamics in $O(\bar{\delta}^p)$ and $O(|z|^2)$, one has that $lim_{k\to\infty} z_k = 0$ with manifold invariance and boundedness of the approximated state trajectories. This implies that the computed feedback at least locally stabilizes the delayed continuous time dynamics in (1).

V. Example

Let us consider the system in strict-feedback form

$$\dot{x}_1(t) = x_1^2(t) + x_2(t-\tau), \quad \dot{x}_2(t) = u(t). \tag{20}$$

A. Continuous-time design - the delay free case

Let us consider $\tau = 0$. In the continuous time case, one has that the I&I control law which makes the origin globally asymptotically stable is

$$u_c(x) = -\Gamma_c z + \dot{\gamma}_c(x_1) \quad \varphi_c(x) = \dot{\gamma}_c(x_1) \quad K > a > 1$$

with $\Gamma_c = 2$ and $\gamma_c(x_1) = -x_1^2 - x_1$. The immersion mapping and invariant manifold are defined as in the proof of Proposition 2.1. The target dynamics is $\dot{\xi} = -\xi$.

B. Sampled-data design - the delay free case

Once again, suppose $\tau = 0$ and introduce the sampled-data equivalent model associated to (20) as below

$$\begin{aligned} x_{1k+1} &= x_{1k} + \delta(x_{1k}^2 + x_{2k}) + \delta^2 x_{1k}(x_{1k}^2 + x_{2k}) + \tfrac{\delta^2}{2!}u_k + O(\delta^3)\\ x_{2k+1} &= x_{2k} + \delta u_k. \end{aligned}$$

In this case, the resulting target dynamics is GAS by setting

$$\begin{aligned} &\gamma_0(\xi_k) = -\xi_k - \xi_k^2 \quad \gamma_2(\xi_k) = 2\xi_k^3\\ &\varphi_0(\xi_k) = \xi_k + 2\xi_k^2 \quad \varphi_1(\xi_k) = -2\xi_k - 8\xi_k^2 - 4\xi_k^3 \end{aligned}$$

where γ_0, γ_1, and c_0 are the terms of $\gamma^{\delta,[2]}$ and $\varphi^{\delta,[1]}$ which are defined according to Proposition 2.1. The final second-order approximated sampled-data I&I control law is provided by $\delta\psi_D(x_{1k},x_{2k},z_k) = -\Gamma_D z_k + \gamma^{\delta}(x_{1k+1}) - \gamma^{\delta}(x_{1k})$,where $\gamma^{\delta}(x_{1k+1})$ is computed as its Taylor extension around x_{1k} truncated at the second-order.

C. Sampled-data design - the case of $\tau = \delta$

According to Section II-A, one introduces $x_3(t) = x_2(t-\tau)$ and the sampled-equivalent extended dynamics associated to (20) as

$$\begin{aligned} x_{1k+1} &= x_{1k} + \delta(x_{1k}^2 + x_{3k}) + \delta^2 x_{1k}(x_{1k}^2 + x_{3k}) + \frac{\delta^2}{2!}v_k + O(\delta^3)\\ x_{3k+1} &= x_{3k} + \delta v_k \quad v_{k+1} = u_k. \end{aligned} \tag{21}$$

According to the sampled-data delay-free design, one introduces the target dynamics, immersion mapping and off-manifold component as in Proposition 4.1 and the problem results in finding $\bar{\psi}^{\delta}(x,v,z)$ such that $\lim_{k\to\infty} z_k = 0$ and $\bar{\psi}^{\delta}(\bar{\pi}^{\delta}(\xi),0) = c^{\delta}(\xi)$, with boundedness of the trajectories of the system with state $(z,\ x,\ v)$. Accordingly to Section V-A, one can define sampled-data controller by considering the double-rate sampled-equivalent model. Setting

$$\begin{aligned} \frac{\delta}{2}\bar{\psi}^{1\delta,[2]}(x_{1k},v_k,z_k) =& \frac{\delta}{2}\Gamma_1 z_{1k} + \gamma^{\delta}(x_{1k+1}) - \\ & \gamma^{\delta}(x_{1k}) - \frac{\delta}{2}[z_{2k} + \varphi^{\delta}(x_{1k})]\\ \bar{\psi}^{2\delta,[2]}(x_{1k},v_k,z_k) =& \Gamma_2 z_{2k} + \varphi^{\delta}(x_{1k+1}) \end{aligned} \tag{22}$$

one ensures stability of the closed-loop sampled-data input-delayed dynamics.

D. Simulations

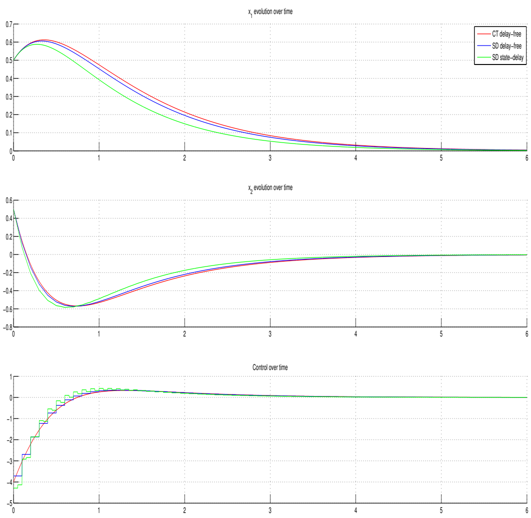

Fig. 1. $\delta = 0.1$ s and $x_0 = (0.5, 0.5)^T$

Simulations are carried out on the example in Section V for different sampling periods δ. The control law is defined according to an I&I double-rate design when $\tau = \delta$ with gains $\Gamma_1,\ \Gamma_2 = 1$. All the simulations are performed for the initial condition $x = \begin{pmatrix}0.5 & 0.5\end{pmatrix}^T$. The control approach presented in this paper is compared with the continuous-time and sampled-data ones (respectively in [1] and in [7]), when the former ones are applied to the delay-free dynamics. In general it can be pointed out that the so-defined feedback leads to good performances even with respect to the delay-free case. This is achieved since the control law is not explicitly designed in order to predict the delayed-state, but to stabilize the dynamics with no information on the delay-free controller. As a matter of fact, the proposed controller directly stabilizes the delayed-dynamics by leading it to the invariant manifold where the implicit prediction aim is fulfilled. Promising performances are obtained when δ increases with still limited control efforts and reasonable smoothness of the trajectories.

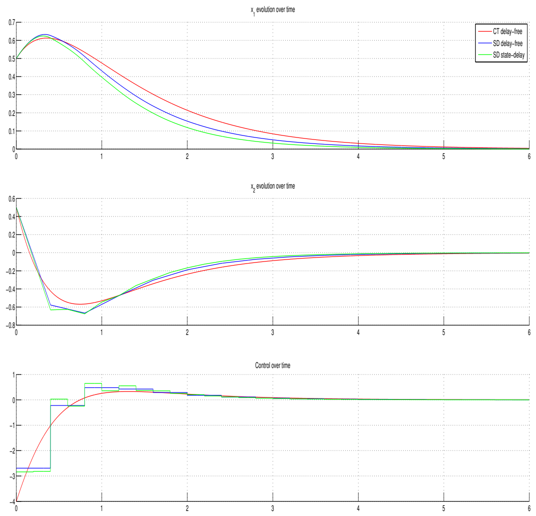

Fig. 2. $\delta = 0.4$ s and $x_0 = (0.5, 0.5)^T$

VI. Conclusions

In this paper a multi-rate sampled-data I&I controller is proposed for a special class of dynamics in which one state is affected by delays. The performances are shown through simulations on an academic example. The proposed approach can be extended to the case of dynamics with delayed interconnection (e.g. through the state component x_2) taking advantage of possible intrinsic properties of the sampled-data equivalent models [18].

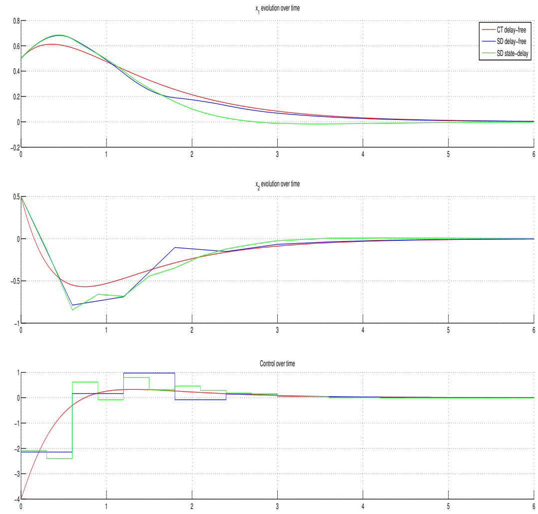

Fig. 3. $\delta = 0.6$ s and $x_0 = (0.5, 0.5)^T$

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Reduction-based stabilization of time-delay nonlinear dynamics

Mattia Mattioni, Salvatore Monaco and Dorothée Normand-Cyrot

Abstract— **This paper represents a first attempt toward an alternative way of computing reduction-based feedback *à la Arstein* for input-delayed systems. For, we first exhibit a new reduction state evolving as the reduced dynamics which is free of delays. Then, the feedback design is carried out by enforcing passivity-based arguments in the reduction time-delay scenario. The case of strict-feedforward dynamics serves as a case study to discuss in details the computational advantages. A simulated exampled highlights the performances.**

Index Terms— **Predictive control for nonlinear systems, Delay systems, Lyapunov methods**

I. Introduction

Time-delay systems have been deeply investigated throughout the last decades. As far as prediction-based control is concerned, the very first result goes back to 1959 when the Smith's predictor [1] was introduced for input delayed linear stable systems. Then, it was later improved by several other works as [2] also to deal with unstable linear plants. Successively, extensions to more general cases have been studied as well by considering nonlinear plants via the definition of suitable Lyapunov-Krasovskii functionals to deal with robustness issues as well [3], [4].Then, predictors for larger variety of situations have been proposed by embedding time-varying and distributed delays for both time invariant or time-invariant systems as proposed, among many others, in [5], [6], [7]. Sequential subpredictors have been investigated in [8] for linear systems with a long input delays and extended to classes of time-varying systems in [9].

As an alternative to prediction-based control, reduction-based methods have been firstly introduced by Arstein in 1982 [10] for linear time-invariant systems. More recently, this results has been reformulated in an extended nonlinear and time-varying context by Mazenc an Malisoff in several of their works [11], [12], [13].

The aim of this work is to provide an alternative way of designing reduction-based feedback for input-affine retarded dynamics affected by a discrete delay τ over the input. To this end, we first exhibit a new state whose dynamics (the reduced dynamics) is free of delays and equivalent, in terms of stability, to the original delayed system. Then, we prove that any stabilizing feedback computed over this new delay-free dynamics achieves stabilization of the original system as well. The new reduced dynamics is not a copy of the delay-free dynamics associated to the retarded system when $\tau = 0$. Indeed, the reduced dynamics preserves the same drift (i.e., the free evolution of the retarded system) as the retarded dynamics but is transformed in the forced component through a control vector field which is explicitly parameterized by the delay. Consequently, the design over the reduced dynamics can be pursued by exploiting the properties of the uncontrolled retarded system which are indeed preserved by reduction. In this scenario, passivity-based arguments naturally extend to reduction-based feedback. This work extends to the continuous-time framework our previous contributions for discrete-time and sampled-data dynamics [14], [15].

The paper is organized as follows. In Section II, the reduction state is described and the reduced dynamics is inferred. In Section III, reduction-based design is proposed through passivity and passivation arguments when proposing negative output damping over the reduced model. The result is specified to strict-feedforward system as a case study in Section IV for which exact computations can be carried out. This results in extending the feedforwarding design to the time-delay scenario through reduction. In Section V an academic simulated is carried out while conclusions and perspective are in Section VI.

Notations and assumptions: We say that a system $\dot{x} = f(x,u)$ (with $x \in \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^p$) is forward complete if for every $x_0 \in \mathbb{R}^n$ and $u\mathbb{R}$ the solution $x(t)$ of such system with $x(0) = x_0 \in \mathbb{R}^n$ exists for all $t \geq 0$. Vector fields and mappings are assumed smooth. Given a vector field f, L_f denotes the Lie derivative operator, $\mathrm{L}_f = \sum_{i=1}^n f_i(\cdot)\nabla_{x_i}$ with $\nabla_{x_i} := \frac{\partial}{\partial x_i}$ while $\nabla = (\nabla_{x_1}, \dots, \nabla_{x_n})$. Given two vector fields f and g, $ad_f g = [f,g]$ and iteratively $ad_f^i g = [f, ad_f^{i-1} g]$. The Lie exponent operator is denoted as $e^{\mathrm{L}_f} Id$ and defined as $e^{\mathrm{L}_f} := \mathrm{I} + \sum_{i\geq 1} \frac{\mathrm{L}_f^i}{i!}$. Given two vector fields f, g on $\mathbb{R}^n$, their Lie bracket is defined as $ad_f g := [f,g] := [\mathrm{L}_f, \mathrm{L}_g] := \mathrm{L}_f \circ \mathrm{L}_g - \mathrm{L}_g \circ \mathrm{L}_f$, and in an iterative way, $ad_f^i g := [f, ad_f^{i-1} g]$, with $ad_f^0 g := g$. Given two vector fields f, g and a constant $\tau \in \mathbb{R}$, the *transport* operator is defined as $e^{\tau ad_f} g(x) = e^{\tau \mathrm{L}_f} g(e^{-\tau \mathrm{L}_f} x)$.

II. Stabilization of time-delay systems: from prediction to reduction

Let the continuous-time dynamics

$$\dot{x}(t) = f(x(t)) + u(t-\tau) g(x(t)) \tag{1}$$

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with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ possess an equilibrium at the origin. We shall denote the dynamics inferred from (1) when $\tau = 0$ as the *delay-free* dynamics

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t). \tag{2}$$

which we assume forward complete so implying that (1) is forward complete as well [3]. In the following, we are going to define a stabilizing feedback based on reduction. Namely, we shift the problem of stabilizing the origin of (1) onto a new dynamics which is free of delays but equivalent, in terms of stability, to the original retarded (1). Accordingly, any feedback stabilizing the deduced dynamics will ensure stabilization of (1) as well.

A. The predictor-based feedback

Setting

$$\zeta(t) = x(t+\tau) = x(t) + \int_{t-\tau}^{t} (f + u(s)g)(\zeta(s))\mathrm{d}s. \tag{3}$$

one immediately verifies that the predictor dynamics recovers the delay free dynamics

$$\dot{\zeta}(t) = f(\zeta(t)) + u(t)g(\zeta(t)) \tag{4}$$

which is a copy of the delay-free dynamics (2) as proposed in several works (e.g.,[3]). As a straightforward consequence, any feedback $u = k(x)$ making the delay-free (2) globally asymptotically stable (GAS) in closed loop will ensure stability of the predictor-dynamics (4) and, by construction, of the retarded dynamics (1). The main obstruction to prediction-based feedbacks relies upon the fact that the resulting feedback is just a copy of the one computed computed over the delay-free (2) with any possible redesign taking into account the delay acting over (1). In the following, starting from prediction, we propose a new feedback based on the definition of a new state whose dynamics preserves the same free evolution as the delay-free (2) but is changed into the forced component.

B. The reduction-based feedback

In this section, we extend the notion of *reduction* (or, *reduction state*), as firstly introduced by Arstein in the linear case [10], to nonlinear continuous-time dynamics. Basically, we define a new state $\eta(t) = r(\tau, x(t), u_{[t-\tau,t[})$ whose dynamics is free of delays but equivalent, at least as far as stability is concerned, to the original retarded dynamics (1). To this end, we define

$$\eta(t) = T(\tau)(\zeta(t)) = T(\tau)(x(t+\tau)) \tag{5}$$

with the causal operator

$$T(\tau)(x) = e^{-\tau \mathrm{L}_f}(x) = x + \sum_{i>0} (-1)^i \frac{\tau^i}{i!} \mathrm{L}_f^i x$$

as a candidate reduction state. Accordingly, by exploiting the transport operator, (5) evolves as the *reduced dynamics*

$$\dot{\eta}(t) = f(\eta(t)) + u(t)e^{\tau \mathrm{ad}_f} g(\eta(t)) \tag{6}$$

with

$$g^\tau(\eta) = e^{\tau \mathrm{ad}_f} g(\eta) = g(\eta) + \sum_{i>0} \frac{\tau^i}{i!} ad_f^i g(\eta).$$

Then, one gets the following result.

Theorem 2.1: Consider the retarded system (1) affected by a discrete delay $\tau > 0$. Let the reduction state (5) evolve as the reduced dynamics (6). Then, any feedback $u = \alpha(\eta)$ making the origin of (6) GAS in closed loop makes the origin of (1) GAS as well; namely, the extended system

$$\dot{\eta}(t) = f(\eta(t)) + e^{\tau \mathrm{ad}_f} g(\eta(t))\alpha(\eta(t)) \tag{7a}$$
$$\dot{x}(t) = f(x(t)) + g(x(t))\alpha(\eta(t-\tau)) \tag{7b}$$

possesses a GAS equilibrium at the origin.

Proof: The proof is straightforward by noticing that $\eta(t-\tau) = e^{-\tau \mathrm{L}_f} x(t)$ so obtaining

$$\alpha(\eta(t-\tau)) = \alpha(e^{-\tau \mathrm{L}_f} x(t)).$$

The closed-loop (7b) rewrites as

$$\dot{x}(t) = f(x(t)) + g(x(t))\alpha(e^{-\tau \mathrm{L}_f} x(t))$$

so that, introducing the coordinates change $\bar{x}(t) = e^{\tau \mathrm{L}_f} x(t)$ one obtains

$$\dot{\bar{x}}(t) = f(\bar{x}(t)) + e^{\tau \mathrm{L}_f} g(e^{-\tau \mathrm{L}_f} \bar{x}(t))\alpha(\bar{x}(t)).$$

By using the Baker-Campbell-Hausdorff formula [16] one gets that

$$e^{\tau \mathrm{L}_f} g(e^{-\tau \mathrm{L}_f} \bar{x}(t)) = e^{\tau ad_f} g(\bar{x}(t)) = g^\tau(\bar{x}(t))$$

and, thus,

$$\dot{\bar{x}}(t) = f(\bar{x}(t)) + e^{\tau ad_f} g(\bar{x}(t))\alpha(\bar{x}(t))$$

possessing a GAS equilibrium at the origin as coinciding with (7a). This concludes the proof. ■

Remark 2.1: As $u \equiv 0$, one gets for the prediction-state that $x(t+\tau) = e^{\tau \mathrm{L}_f} x(t)$ so implying that

$$\eta(t) = e^{-\tau \mathrm{L}_f}(e^{\tau \mathrm{L}_f})x(t) = x(t).$$

Thus, as the control effect vanishes, the reduction coincides with the current state at time t. This is different from the case of the prediction that goes on predicting the future trajectories of the system even when the control (the delay acts through) is set to zero. Moreover, whenever $u_{[-\tau,0[} = 0$ one gets that $\eta(0) = x(0)$ so solving the typical issues arising with the predictor-based control involving the choice of the initial state.

Remark 2.2: The transformed control vector field $g^\tau(\cdot) = e^{\tau \mathrm{ad}_f} g(\cdot)$ is τ-dependent and recovers $g^\tau(\cdot) = g(\cdot)$ as $\tau \to 0$. Moreover, the controlled vector field of the reduced dynamics (6) differs from the one of the retarded (1) by a term which corresponds to a projection of the control vector field $g(\cdot)$ backward in time through the free evolution.

Remark 2.3: Whenever (1) is driftless ($f(x) \equiv 0$) the reduction (5) coincides with the prediction (3) as $e^{-\tau \mathrm{L}_f} x = x$.

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The above result states that any feedback stabilizing the reduced dynamics achieves stabilization of the original retarded system (1). This opens a wide range of possibility to the feedback design which is no longer limited to the delay-free design as in case of prediction. To this end, one might exploit the properties of the delay-free system (2) in free evolution as they are indeed preserved though reduction. In the following, the case of passivity-based design will be carried out over the reduced dynamics (6) by exploiting passivity of the delay-free (2). First, some computational aspects are given.

C. Some computational issues

The main obstruction which remains in this reduction-based control is linked to the computation of $\eta(t)$ which requires the integration of the implicit equation

$$\eta(t) = e^{-\tau \mathrm{L}_f}(x(t)) + \int_{t-\tau}^{t} (f + u(s)e^{\tau \mathrm{ad}_f} g)(\eta(s))\mathrm{d}s. \quad (8)$$

However, the η and x time trajectories differ from control dependent terms only so that they coincide whenever $u \equiv 0$. This is the consequence of the definition of the reduction η in 5 which aims at compensating the effect of the delay acting over (1) only in the controlled evolutions which are indeed explicitly affected by τ. It follows that the computation of $\eta(t)$ can be worked out through truncation of the associated Volterra series expansion. As far as the first Volterra kernel is concerned, one gets

$$\eta(t) = x(t) + \int_{t-\tau}^{t} e^{s\mathrm{ad}_f} g^{\tau}(x(t))u(s)\mathrm{d}s + O(u^2) \quad (9)$$

where $O(u^2)$ contains higher order kernels of order greater or equal to 2 in the control variable.

For computational purposes, sampled-data implementation schemes for the reduction (5) might be considered. As a matter of fact, if one assumes a finite number of samples of the past history of the control over $[t-\tau, t[$ available, (5) can be computed through numerical approximations by exploiting the results proposed in [15], [14] for sampled-data systems and, thus, overcoming computational issues.

III. REDUCTION PASSIVITY-BASED CONTROL

Based on the preservation of the free evolution of (1) under reduction (5), we are now proposing a reduced passivity-based control for the retarded system (1) over the reduced model (6). To this purpose, the following assumption over the delay-free dynamics (2) is instrumental.

Assumption 3.1: There exists a positive-definite and C^1 (i.e., once differentiable with continuous derivative) function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that $V(0) = 0$ and $\mathrm{L}_{f(\cdot)}V(x) \leq 0$ for any $x \in \mathbb{R}^n$.

Under Assumption 3.1, the following implications hold.

- the delay-free system (2) with output $h(x) = \mathrm{L}_g V(x)$, is passive, with storage function $V(x)$;
- the feedback $u(x) = -\mathrm{L}_g V(x)$ makes the origin GAS for (2) if the delay-free system (2) with output $h(x) = \mathrm{L}_g V(x)$ is Zero State Detectable[1].

Accordingly, the following result hold true for the reduced dynamics.

Theorem 3.1: Let the retarded dynamics (1) satisfy Assumption 3.1. Consider the reduction (5) evolving as the reduced dynamics (6). Then, the following holds true:

1) the reduced dynamics with output $h^{\tau}(x) = \mathrm{L}_{g^{\tau}}V(\eta)$ is passive;
2) if (6) with $h^{\tau}(x) = \mathrm{L}_{g^{\tau}}V(\eta)$ is ZSD, then the feedback
$$u(\eta) = -\mathrm{L}_{g^{\tau}}V(\eta) \quad (10)$$
makes the origin a GAS equilibrium for the reduced dynamics (6) and, thus, for (1).

Proof: As far as passivity is concerned, by exploiting Assumption 3.1, one computes over (6)

$$\dot{V}(\eta) = \mathrm{L}_f V(\eta) + u\mathrm{L}_{g^{\tau}}V(\eta) \leq h^{\tau}(x)u$$

so getting the result. Accordingly, whenever (6) with $h^{\tau}(x) = \mathrm{L}_{g^{\tau}}V(\eta)$ is ZSD, the feedback (10) makes the reduced dynamics (6) GAS. From Theorem 2.1, one gets that (10) makes the retarded system GAS as well. ■

The reduction passivity-based feedback (10) is parameterized by the delay τ through the vector field $g^{\tau}(\cdot) = e^{\tau ad_f} g(\cdot)$. As a consequence, it rewrites as

$$\begin{aligned} u(\eta) &= -\mathrm{L}_{g^{\tau}}V(\eta) = -\nabla V(\eta) g^{\tau}(\eta) \\ &= -\nabla V(\eta)\Big(g(\eta) + \sum_{i>0} \frac{\tau^i}{i!} ad_f^i g(\eta)\Big) \end{aligned}$$

so underlining that as $\tau \to 0$, and because $\eta \to 0$, one recovers the delay-free passivity-based feedback over (2). Such a form naturally introduces approximations of the reduction-based feedback (10) as truncation of the aforementioned series expression at any finite power of τ; namely, one defines for some $p \in \mathbb{N}$

$$u^{[p]}(\eta) = -\nabla V(\eta)\Big(g(\eta) + \sum_{i=1}^{p} \frac{\tau^i}{i!} ad_f^i g(\eta)\Big).$$

Of course, those solutions will ensure stability of (1) in closed-loop only under suitable limits in the length of the delay τ with respect to the approximation order p.

Remark 3.1: Considering again the dynamics (1), the previous approach can be pursued when assuming the delay free dynamics (2) with output map $y = h(x)$ passive; namely, there exists a definite positive, C^1 function $S(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ such that $\dot{S}(x) \leq uh(x)$ over the delay-free trajectories (2). From the Kalman-Yakubovich-Popov properties [17] (i.e. $\mathrm{L}_f S \leq 0$ and $h(\cdot) = \mathrm{L}_g S(\cdot)$), the result in Theorem 3.1 still holds when assuming passivity of the delay-free dynamics (2) with output map $y = h(x(t))$. In that case, the stabilizing feedback is given by $u(\eta) = -\mathrm{L}_{g^{\tau}}S(\eta)$ with $\mathrm{L}_g S(\cdot) = h(\cdot)$.

[1]Consider the dynamics (2) with output $y = h(x)$. Let $u \equiv 0$ and $\mathscr{X} \in \mathbb{R}^n$ be the largest positively invariant set contained in $\{x \in \mathbb{R}^n \text{ s. t. } h(x) = 0\}$. We say that (2) with output $y = h(x)$ is zero state detectable (ZSD) if $x = 0$ is asymptotically stable conditionally to $\mathscr{X}$.

Remark 3.2: If the reduction-based controller $u(\eta) = -\mathrm{L}_{g^\tau}V(\eta)$ achieves GAS of the origin of the reduced dynamics (6), then it also solves a global optimal stabilization problem over the reduction (6) with cost functional

$$J = \int_0^\infty (l(\eta(t)) + \frac{u^2(t)}{2})\mathrm{d}t \tag{11}$$

with $l(\eta)$ as

$$l(\eta) = -\mathrm{L}_f V(\eta) + \frac{1}{2}(\mathrm{L}_{g^\tau}V(\eta))^T \mathrm{L}_{g^\tau}V(\eta) \geq 0 \tag{12}$$

and optimal value function $V(\eta)$.

IV. STRICT-FEEDFORWARD SYSTEMS AS A CASE STUDY

Consider the case of a strict-feedforward dynamics [18]

$$\dot{x}_1(t) = Fx_1(t) + \varphi(x_2(t)) + g(x_2(t))u(t-\tau) \tag{13a}$$
$$\dot{x}_2(t) = Ax_2(t) + Bu(t-\tau) \tag{13b}$$

with $u \in \mathbb{R}$, $x_i \in \mathbb{R}^{n_i}$ for $i = 1,2$ possessing an equilibrium at the origin and verifying standard feedforwarding conditions

- F.1 A is Hurwitz with positive definite matrix $P \succ 0$ such that $A^\top P + PA \prec 0$
- F.2 F is skew-symmetric; i.e., $F^\top + F = 0$.

It is well known that, whenever $\tau = 0$, one can stabilize (13) via an iterative forwarding procedure consisting in defining a decoupling change of coordinate for the delay-free dynamics deduced from (13) when $u \equiv 0$ and then perform passivity-based control [19]. In what follows, we show that this procedure extends to retarded dynamics (13) by suitably exploiting the proposed reduction-based design. Moreover, in that case, the reduction (5) and the reduced dynamics (6) are finitely computable because of the strict-feedforward interconnection. For the sake of brevity, we rewrite (13) in a compact way as (1) when setting $x = \mathrm{col}(x_1,\ x_2)^\top$, $f(x) = \mathrm{col}(Fx_1 + \varphi(x_2),\ Ax_2)$ and $g(x) = \mathrm{col}(g(x_2),\ B)$.

A. Reduction of strict-feedforward systems

For detailing (5) to (13), one first describes

$$e^{-\tau \mathrm{L}_f}x(t) = \begin{pmatrix} e^{-\tau \mathrm{L}_f}x_1(t) \\ e^{-\tau \mathrm{L}_f}x_2(t) \end{pmatrix}$$

with

$$\begin{aligned} e^{-\tau \mathrm{L}_f}x_1(t) &= e^{-F\tau}x_1(t) - \int_{t-\tau}^{t} e^{F(t-\tau-\ell)}\varphi(e^{A(\ell-t)}x_2(t))\mathrm{d}\ell \\ e^{-\tau \mathrm{L}_f}x_2(t) &= e^{-A\tau}x_2(t). \end{aligned}$$

Accordingly, setting $\eta = \mathrm{col}(\eta_1, \eta_2)$ one gets the reductio variables

$$\begin{aligned} \eta_1(t) &= e^{-F\tau}x_1(t+\tau) - \int_{t-\tau}^{t} e^{F(t-\tau-\ell)}\varphi(e^{A(\ell-t)}x_2(t+\tau))\mathrm{d}\ell \\ \eta_2(t) &= e^{-\tau \mathrm{L}_f}x_2(t+\tau) \end{aligned} \tag{15}$$

with

$$\begin{aligned} x_1(t+\tau) &= e^{F\tau}x_1(t) + \int_{t-\tau}^{t} e^{F(t-\ell)}\varphi(x_2(\ell+\tau))\mathrm{d}\ell \\ &\quad + \int_{t-\tau}^{t} e^{F(t-\ell)}g(x_2(\ell+\tau))u(\ell)\mathrm{d}\ell \\ x_2(t+\tau) &= e^{A\tau}x_2(t) + \int_{t-\tau}^{t} e^{A(t-\tau-\ell)}Bu(\ell)\mathrm{d}\ell. \end{aligned}$$

By differentiating (15) with respect to time and exploiting the relation $x_2(t+\tau) = e^{A\tau}\eta_2(t)$, one gets the reduced dynamics (6) specified as

$$\dot{\eta}_1(t) = F\eta_1(t) + \varphi(\eta_2(t)) + g_1^\tau(\eta_2(t))u(t) \tag{16a}$$
$$\dot{\eta}_2(t) = A\eta_2(t) + e^{-\tau A}Bu(t) \tag{16b}$$

with

$$\begin{aligned} g_1^\tau(\eta_2(t)) &= e^{-F\tau}g(e^{A\tau}\eta_2(t)) \\ &\quad - \int_{t-\tau}^{t} e^{F(t-\tau-\ell)}\nabla\varphi(e^{A(\ell-t+\tau)}\eta_2(t))e^{A(\ell-t)}B\mathrm{d}\ell. \end{aligned}$$

It is clear from (16) that reduction preserves the strict-feedforward structure of (13). Moreover, as (16) possesses the same free evolution as (13), the reduced dynamics still verifies Assumption F.1 and F.2. For this reason, we can now stabilize the retarded dynamics (13) via reduction-based feedforwarding so extending the methodology proposed in [19] to the time-delay case.

B. Reduction-based feedforwarding

When $u \equiv 0$, the uncontrolled reduced dynamics

$$\dot{\eta}_1(t) = F\eta_1(t) + \varphi(\eta_2(t)), \quad \dot{\eta}_2(t) = \quad A\eta_2(t)$$

exhibits an invariant manifold where the trajectories are described by the globally exponentially stable (GES) dynamics

$$\dot{\eta}_2(t) = A\eta_2(t).$$

Such a manifold is implicitly defined as $\mathscr{M} = \{\eta \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \text{ s.t. } \eta_1 = \phi(\eta_2)\}$ where the smooth mapping $\phi : \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that $\phi(0) = 0$ and given by

$$\phi(\eta_2) = -\int_t^\infty e^{-F(\ell-t)}\varphi(e^{A\ell}\eta_2)\mathrm{d}\ell \tag{18}$$

also verifying the invariance condition

$$\nabla_{\eta_2}\phi(\eta_2)A\eta_2 = F\phi(\eta_2) + \varphi(\eta_2).$$

Thus, by applying to (16) the coordinates transformation

$$\bar{\eta}_1 = \eta_1 - \phi(\eta_2)$$

the reduced model rewrites as

$$\dot{\bar{\eta}}_1(t) = F\bar{\eta}_1(t) + g_1^\tau(\eta_2(t))u(t) \tag{19a}$$
$$\dot{\eta}_2(t) = A\eta_2(t) + e^{-\tau A}Bu(t) \tag{19b}$$

exhibiting a decoupling structure for $u \equiv 0$. Accordingly, by Assumption F.1 and F.2, the reduced dynamics (19) in free evolution (i.e., computed for $u \equiv 0$) possesses a globally stable equilibrium at the origin with Lyapunov function

$$\mathscr{V}(\bar{\eta}_1, \eta_2) = \frac{1}{2}(\bar{\eta}_1^\top \bar{\eta}_1 + \eta_2^\top P\eta_2) \tag{20}$$

verifying by assumption

$$\dot{\mathscr{V}}(\bar{\eta}_1,\eta_2)\big|_{u=0} = \eta_2^\top(PA + A^\top P)\eta_2 \le 0.$$

Accordingly, computing now the derivative of the Lyapunov (20) along the reduced dynamics (19) one gets that

$$\begin{aligned}\dot{\mathscr{V}}(\bar{\eta}_1,\eta_2) =&\eta_2^\top(PA + A^\top P) + (\bar{\eta}_1^\top g_1^\tau(\eta_2) + \eta_2^\top P e^{-\tau A}B)u\\ \le&(\bar{\eta}_1^\top g_1^\tau(\eta_2) + \eta_2^\top P e^{-\tau A}B)u.\end{aligned}$$

Hence, Assumption 3.1 is recovered so concluding that the reduced dynamics (19) is passive with output

$$\begin{aligned}y =&\mathrm{L}_{g^\tau(\eta_2)}\mathscr{V}(\bar{\eta}_1,\eta_2)\\ =&(g_1^\tau(\eta_2))^\top\bar{\eta}_1 + B^\top e^{-A^\top\tau}P\eta_2\end{aligned} \tag{21}$$

and storage function (20). As a straightforward application of Theorem 3.1, the reduction passivity-based feedback

$$u = -(g_1^\tau(\eta_2))^\top\bar{\eta}_1 - B^\top e^{-A^\top\tau}P\eta_2 \tag{22}$$

makes the closed-loop origin of the retarded dynamics (13) GAS if the reduced dynamics (19) with output (21) is ZSD.

Remark 4.1: In the original reduction coordinates, the stabilizing feedback rewrites as

$$u = -(g_1^\tau(\eta_2))^\top(\eta_1 - \phi(\eta_2) - B^\top e^{-A^\top\tau}P\eta_2 \tag{23}$$

yielding the origin a GAS equilibrium for the reduced dynamics (16) with weak Lyapunov function

$$V(\eta) = \mathscr{V}(\eta_1 - \phi(\eta_2),\eta_2). \tag{24}$$

Remark 4.2: By rewriting the Lyapunov function (23) and the feedback (23) in the original x-coordinates and over the closed-loop retarded system (13), one deduces a Lyapunov-Krasovskii functional that might be useful for further redesign (e.g., aimed at robustifying in closed-loop) [20], [21].

Remark 4.3: Assumption F.1 can be weakened to requiring A being critically stable with a positive definite matrix $P \succ 0$ such that $A^\top P + PA \preceq 0$. In that case, the construction of the reduction (5) proceeds as in Section IV-A so deducing the reduced dynamics (16). Still, a reduction-based preliminary stabilizing feedback $u(t) = G\eta_2(t) + v(t)$ over the partial reduced dynamics (16b) is needed so to ensure $A + e^{-\tau A}BG$ Hurwitz. Then, one can directly apply the procedure in Section IV-B to the modified reduced dynamics

$$\begin{aligned}\dot{\eta}_1(t) =&F\eta_1(t) + \tilde{\varphi}(\eta_2(t)) + g_1^\tau(\eta_2(t))v(t)\\ \dot{\eta}_2(t) =&\tilde{A}\eta_2(t) + e^{-\tau A}Bv(t)\end{aligned}$$

with $\tilde{\varphi}(\eta_2) = \varphi(\eta_2) + g_1^\tau(\eta_2)G\eta_2$ and $\tilde{A} = A + e^{-\tau A}BG$.

Remark 4.4: The reduction-based feedforwarding strategies extends, along these lines, to more general strict-feedforwarding structures where (13b) is assumed a general input-affine forward complete dynamics of the form

$$\dot{x}_2(t) = a(x_2(t)) + b(x_2(t))u(t-\tau).$$

Remark 4.5: The application of this reduction-based design to strict-feedforward structures can be seen as an alternative to the work in [3] within' the framework of prediction and to the one in [22] where time-varying coordinate transformations and Lyapunov-Krasovskii functional are iteratively constructed.

V. An academic simulated example

Consider the feedforward dynamics

$$\dot{x}_1(t) = x_2(t) - x_2(t)u(t-\tau), \quad \dot{x}_2(t) = \quad -x_2(t) + u(t-\tau)$$

clearly verifying Assumptions F.1 and F.2. In this case, the reduction (15) gets the form

$$\begin{aligned}\eta_1(t) =&x_1(t+\tau) + (1-e^\tau)x_2(t+\tau)\\ =&x_1(t) + \int_{t-\tau}^t x_2(\ell+\tau)\mathrm{d}\ell\\ &+ \int_{t-\tau}^t e^{-(t-\ell)}x_2(\ell+\tau)u(\ell)\mathrm{d}\ell - \eta_2(t)\\ \eta_2(t) =&e^\tau x_2(t+\tau) = x_2(t) + \int_{t-\tau}^t e^{-(t-\tau-\ell)}u(\ell)\mathrm{d}\ell\end{aligned}$$

and evolves as the reduced dynamics

$$\begin{aligned}\dot{\eta}_1(t) =&\eta_2(t) - (e^{-\tau}\eta_2(t) - 1 + e^\tau)u(t)\\ \dot{\eta}_2(t) =&-\eta_2(t) + e^\tau u(t).\end{aligned} \tag{26}$$

As far as the reduction-based stabilizing design is concerned, one follows the lines of Section IV-B by deducing the mapping (18) as $\bar{\eta}_1 = \eta_1 + \eta_2$ and thus the feedback

$$u = (\eta_1 + \eta_2)(e^{-\tau}\eta_2 - 1) - e^\tau\eta_2 \tag{27}$$

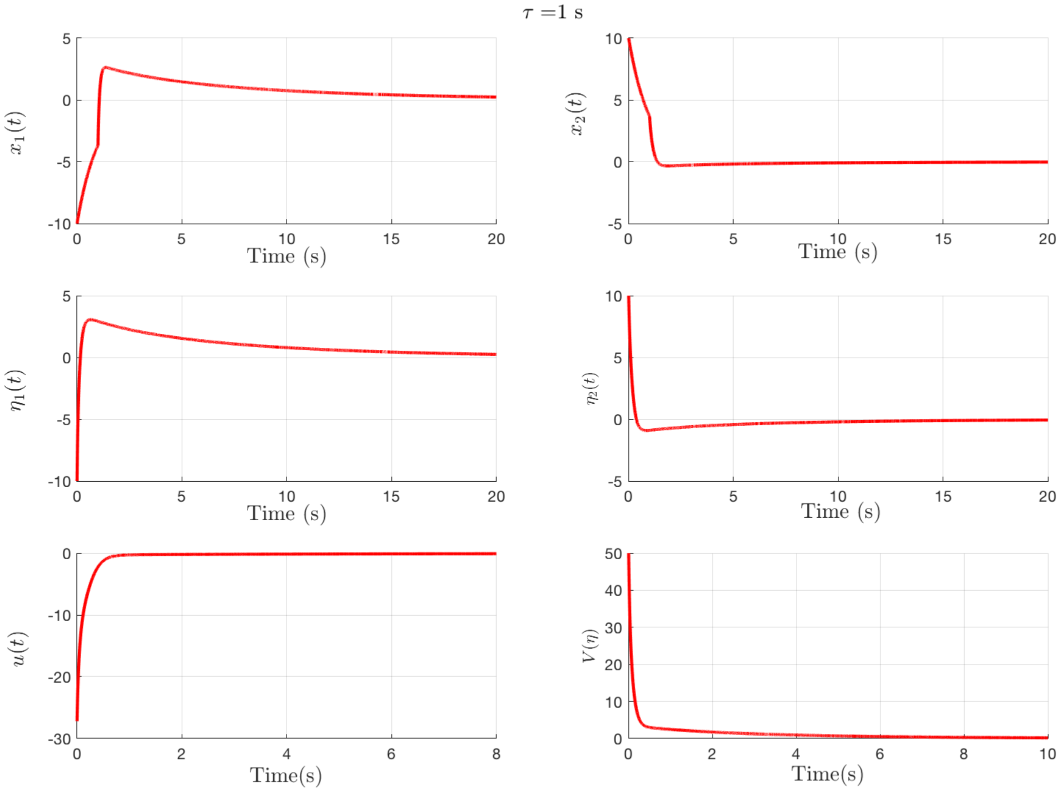
with Lyapunov function

$$V(\eta) = \frac{1}{2}(\eta_2 + \eta_1)^2 + \frac{1}{2}\eta_2^2. \tag{28}$$

Several simulations have been performed over the aforementioned example for initial condition $x_0 = (-10, 10)$ and increasing values of the time-delay τ. We set $u(t) = 0$ as $t \in [-\tau, 0[$ so that the reduction initial condition is given by $\eta_0 = x_0$. We simulated the reduction-passivity based feedback (27) computed over the reduced dynamics (26). A sample result is depicted in Figures V where the evolutions of the reduced dynamics are reported together with the one of the storage function (28). Simulations testify the efficiency of the reduction passivity-based feedback while yielding good performances even as the delay length increases.

VI. Conclusions

This work represents a first attempt to providing an alternative way to perform reduction-based design for input-delayed dynamics so that a wide range of perspective is opened. Among these, a further study of the proposed methodology with emphasis on reduction passivity-based control at large deserves paramount attention. Moreover, a deeper investigation on computational issues arising from the difficulties in the exact computation of the reduced dynamics is ongoing for general input-affine dynamics. In this sense, complementing the proposed design with Lyapunov-Krasovskii arguments should provide prolific tools to address those problems starting with general strict-feedforward retarded systems. A comparative analysis with respect to existing reduction strategies are under investigation.



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Nonlinear discrete-time systems with delayed control: A reduction

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ABSTRACT

In this work, the notion of reduction is introduced for discrete-time nonlinear input-delayed systems. The retarded dynamics is reduced to a new system which is free of delays and equivalent (in terms of stabilizability) to the original one. Different stabilizing strategies are proposed over the reduced model. Connections with existing predictor-based methods are discussed. The methodology is also worked out over particular classes of time-delay systems as sampled-data dynamics affected by an entire input delay.

1. Introduction

The seminal works by Smith [1] and Artstein [2] have inspired a research toward time-delay systems as an unavoidable paradigm in control theory because of their involvement in a lot of practical situations. Investigations have been addressed to the study of the effects of time delay in a control system emphasizing on drawback and also, unexpectedly, advantages. As an example, it has been shown that introducing a delay over the control system might make a non stabilizable (or not controllable) system stabilizable (or controllable) as shown, among others, in [3] or [4]. Furthermore, the huge developments in classical (non-delayed) nonlinear control motivated several important works devoted to extend those well-known results to time-delay systems (e.g., [3,5–8] and references therein). Nevertheless, a lot of questions still remain unanswered in the case of both continuous and discrete-time dynamics.

In this paper, the focus is set toward time-delay discrete-time systems which have proven themselves to be of extreme interest for several reasons [9–12]. Among them, a well-known motivation is provided by the fact that retarded discrete-time systems are finite dimensional so enabling one to restate the design problem over an extended and delay-free state–space model. That is even more interesting when the discrete-time retarded system is issued from the sampling of dynamics affected by input delays [13].

This paper addresses the stabilization of discrete-time nonlinear dynamics affected by input-delay. In this context, several works were carried out, especially in the linear context, by employing descriptor (mostly for linear systems, [3]) or prediction based feedback [14]. As this latter technique usually lacks in robustness, it was recently improved through Immersion and Invariance in [15]. Though, the aforementioned strategy is still hard to extend to larger classes of time-delay systems. Inspired by the work by Artstein [2], we aim at extending the reduction model approach to the discrete-time nonlinear context. Roughly speaking, given a nonlinear discrete-time dynamics affected by a N step input delay, we seek for a model which is delay-free and equivalent to the original retarded system at least as far as stabilizability is concerned. In doing so, we provide an explicit way of computing such a reduction and we prove that any feedback stabilizing its corresponding dynamics also achieves stabilization of the retarded dynamics. Then, we present several ways of designing control by exploiting the properties of the original delay-free system (i.e., the retarded system computed for $N = 0$) such as smooth stabilizability (in the Lyapunov sense) and u-average passivity (in the sense of [16]). Connections to predictor-based feedback laws are established and commented. The cases of Linear Time Invariant (LTI) and input-affine-like dynamics are illustrated as cases study as well as the case of sampled-data systems affected by the so-called entire delay [17,18].

The paper is organized as follows: the problem is formulated in Section 2 and general recalls on discrete-time delay-free systems are provided in Section 3; the definition of the reduction and its stabilizing properties with respect to the original retarded dynamics are in Section 4; the control design is addressed in Section 5 while some case studies are discussed in Section 6; conclusions and perspectives end the paper in Section 7.

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Notations and definitions: $\mathbb{N}$ and $\mathbb{R}$ denote, respectively, the set of natural and real numbers including the 0. For any $u^j \in \mathbb{R}$ and $j = 1, \dots, m$ and $w^i \in \mathbb{R}$ for a fixed $i \leq m$, we denote $\mathbf{w}^i = (u^1, \dots, u^{i-1}, w^i, 0, \dots, 0) \in \mathbb{R}^m$. $u_{[k-N,k[}$ denotes the story of u over time window $[k-N, k[$ (i.e., $u_{[k-N,k[} = \{u_{k-N}, \dots, u_{k-1}\}$). All the functions and vector fields defining the dynamics are assumed smooth over the respective definition spaces. Id and I denote the identity function and matrix respectively. Given a vector field f, L_f denotes the Lie derivative operator, $\mathrm{L}_f = \sum_{i=1}^n f_i(\cdot)\nabla_{x_i}$ with $\nabla_{x_i} := \frac{\partial}{\partial x_i}$. Given two vector fields f and g, $ad_f g = \mathrm{L}_f \circ \mathrm{L}_g Id - \mathrm{L}_g \circ \mathrm{L}_f Id = [f, g]$ and iteratively $ad_f^i g = [f, ad_f^{i-1} g]$. $e^{\mathrm{L}_f} Id$ (or $e^f Id$, when no confusion arises) denotes the associated Lie series operator, $e^{\mathrm{L}_f} := \mathrm{I} + \sum_{i\geq 1} \frac{\mathrm{L}_f^i}{i!}$. Given any smooth function $h : \mathbb{R}^n \to \mathbb{R}$ then $e^{\mathrm{L}_f} h(x) = h(e^{\mathrm{L}_f} Id\big|_x)$.

2. Problem statement

In this paper, we address the problem of stabilizing via reduction *discrete-time dynamics with discrete input delays* of the form

$$x_{k+1} = F(x_k, u_{k-N}) \tag{1}$$

with $N \in \mathbb{N}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $F(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and the origin as equilibrium to be stabilized. The approach consists in defining a *reduction variable* (or simply reduction) whose dynamics (the *reduced dynamics*) is delay-free and of the same dimension as the original retarded system. Moreover, the stabilizability properties of the reduced model are equivalent to those of the original system; namely, any feedback stabilizing the reduce model ensures stabilization of the retarded dynamics as well.

3. Recalls on discrete-time systems

In the following, we refer to

$$\Sigma_d : x_{k+1} = F(x_k, u_k) \tag{2}$$

as the *delay-free dynamics* associated to (1) when $N = 0$.

3.1. The differential-difference (or generically (F_0, G)) representation

As proposed in [19], (2) can be equivalently represented by two coupled difference and differential equations whenever the drift term dynamics $F(\cdot, 0) := F_0(\cdot)$ admits an inverse.[1] More in detail, assuming $m = 1$, Σ_d described as a map by (2) can be equivalently represented in the (F_0, G)-form below

$$x^+ = F_0(x), \quad x^+ := x^+(0) \tag{3a}$$

$$\frac{\partial x^+(u)}{\partial u} = G(x^+(u), u) \tag{3b}$$

where $x^+(u)$ denotes a curve parametrized by u over $\mathbb{R}^n$ and $G(\cdot, u) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfies[2] $G(x, u) := \nabla_u F(x, u)\big|_{x=F^{-1}(x,u)}$. It is a matter of computations to verify that for any pair (x, u), the map $F(\cdot, u)$ can be recovered by integrating (3b) over $[0, u[$ with initial condition fixed by (3a) as $x^+ = F_0(x)$. One gets

$$F(x, u) = x^+(u) = F_0(x) + \int_0^u G(x^+(v), v)\mathrm{d}v \tag{4}$$

and thus $x^+(u_k) = x_{k+1} = F(x_k, u_k)$ for any pair (x_k, u_k).

Remark 3.1. Invertibility of $F_0(\cdot)$ guarantees the existence of $G(\cdot, u)$ and integrability of (3b) with well defined solution (4) for u sufficiently close to zero. Invertibility of $F_0(\cdot)$ can be relaxed to require the existence of a nominal control value $\bar{u} \in \mathbb{R}$ for which $F(\cdot, \bar{u})$ admits an inverse. In such a case, integrability of (3b) between $\bar{u}$ and u is still guaranteed for u in a neighborhood of $\bar{u}$.

In the *multi-input case* ($m > 1$), one defines analogously the (F_0, G)-form with $G(x, u) = (G^1(x, u), \dots, G^m(x, u))$ and $G^i(\cdot, u) := \nabla_{u^i} F(x, u)\big|_{x=F^{-1}(x,u)}$ for $i = (1, \dots, m)$ by setting

$$x^+ = F_0(x), \quad x^+ := x^+(0) \tag{5a}$$

$$\frac{\partial x^+(u)}{\partial u^1} = G^1(x^+(u), u) \tag{5b}$$

$$\dots \tag{5c}$$

$$\frac{\partial x^+(u)}{\partial u^m} = G^m(x^+(u), u). \tag{5d}$$

The family of controlled vector fields $(G^j(\cdot, u))_{j=1,\dots,m}$ verifies by definition the so-called *compatibility conditions* that guarantee integrability of the so built system of partial derivatives (see [19]). In the multi-input case, (4) generalizes as

$$F(x, u) = F_0(x) + \sum_{i=1}^m \int_0^{u^i} G^i(x^+(\mathbf{w}^i), \mathbf{w}^i)\mathrm{d}w^i \tag{6}$$

with $\mathbf{w}^i = (u^1, \dots, u^{i-1}, w^i, 0, \dots, 0)$.

As discussed through several contributions (e.g., [20,21]), the $(G^j(\cdot, u))_{j=1,\dots,m}$ provide a differential geometric apparatus to analyze and formulate in an elegant way the properties of nonlinear discrete-time dynamics and their associated flows. Some of the aspects that are instrumental in the present context are recalled below when $m = 1$ with intuitive extension to $m > 1$.

At first, given $G(\cdot, u)$, one defines $Ad_{F_0} G(\cdot, u)$ as its *transport along the drift term* $F_0(\cdot)$ as (see [19,21])

$$Ad_{F_0} G(x, u) := [\nabla_x F_0(x)]_{F_0^{-1}(x)} G(F_0^{-1}(x), u). \tag{7}$$

Iteratively, one sets $Ad_{F_0}^i G(x, u) := Ad_{F_0} \circ Ad_{F_0}^{i-1} G(x, u)$ with $Ad_{F_0}^0 G(x, u) := G(x, u)$.

Given any smooth mapping $S(\cdot) : \mathbb{R}^n \to \mathbb{R}$, a useful outcome of the (F_0, G)-representation is to split the evolution of $S(\cdot)$ along the dynamics (2) into the free (or uncontrolled) and forced contributions; namely, one writes

$$S(F(x, u)) = S(F_0(x)) + \int_0^u \mathrm{L}_{G(\cdot,v)} S(x^+(v))\mathrm{d}v. \tag{8}$$

This is useful in the definition of u-average passivity that is recalled below [16].

3.2. u-average passivity and stabilization

The notion of *u-average passivity* has been introduced in discrete time in [16]. First, consider the case of a single-input system (i.e., when $m = 1$).

Definition 3.1. Σ_d with $u \in \mathbb{R}$ and output $H(\cdot)$ is *u-average passive* (or *average passive*) if there exists a positive semi definite function $S(\cdot) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, the storage function, such that for any pair (x_k, u_k), $k \geq 0$, one verifies the inequality

$$S(F(x_k, u_k)) - S(x_k) \leq H_{av}(x_k, u_k)u_k \tag{9}$$

where $H_{av}(x, u)$ denotes the *u-average output mapping* associated with $H(x)$; i.e.

$$H_{av}(x, u) := \frac{1}{u} \int_0^u H(x^+(v))\mathrm{d}v$$

with $H_{av}(x, 0) = H(x^+(0)) = H(F_0(x))$.

[1] There exists $F_0^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ such that $F_0^{-1}(\cdot) \circ F_0(x) = F_0(\cdot) \circ F_0^{-1}(x) = x$.

[2] Given a smooth mapping $F(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $F^{-1}(x, u)$ denotes the inverse of F with respect to x; i.e., $F(F^{-1}(x, u), u) = F^{-1}(F(x, u), u) = x$.

According to (8), the dissipativity inequality (9) rewrites as

$$S(F_0(x_k)) - S(x_k) + \int_0^{u_k} \mathrm{L}_{G(\cdot,v)}S(x^+(v))\mathrm{d}v \le \int_0^{u_k} H(x^+(v))\mathrm{d}v \quad (10)$$

with by definition $\int_0^{u_k} H(x^+(v))\mathrm{d}v := H_{av}(x_k, u_k)u_k$.

Remark 3.2. u-average passivity can be generalized to systems with direct input–output link $H(\cdot, u)$ smoothly parametrized by u thus setting

$$H_{av}(x, u) := \frac{1}{u}\int_0^u H(x^+(v), v)\mathrm{d}v$$

with $H_{av}(x, 0) = H(x^+(0), 0) = H(F_0(x), 0)$.

We recall the notion of zero-state detectability.

Definition 3.2. Let Σ_d with output $H(\cdot, u)$ and let $\mathscr{Z} \subset \mathbb{R}^n$ be the largest positively invariant set contained in $\{x \in \mathbb{R}^n \mid H(x, 0) = 0\}$. We say that Σ_d is Zero-State-Detectable (ZSD) if $x = 0$ is asymptotically stable conditionally to $\mathscr{Z}$.

The following result extends u-average passivity to the case $m > 1$.

Theorem 3.1. *Consider Σ_d with $m \geq 1$ and assume the existence of a positive definite storage function $S(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that $S(F_0(x)) - S(x) \leq 0$. Then,*

(i) *Σ_d with output $H(x, u) = (\mathrm{L}_{G(\cdot,u)}S(x))^\top$ is u-average passive; i.e., the dissipativity inequality holds*

$$S(x^+(u)) - S(x) \le H_{av}^\top(x, u)u = \sum_{i=1}^m H_{av}^i(x, u)u^i$$

with

$$H_{av}^i(x, u) = \frac{1}{u^i}\int_0^{u^i} \mathrm{L}_{G^i(\cdot,\mathbf{w}^i)}S(x^+(\mathbf{w}^i))\mathrm{d}w^i. \quad (11)$$

(ii) *If Σ_d with output $H(x, u)$ is ZSD, then any feedback $u = \gamma(x)$ solving*

$$u + KH_{av}(x, u) = 0, \quad \text{with } K > 0 \quad (12)$$

achieves global asymptotic stabilization of the equilibrium of Σ_d.

Accordingly, the feedback satisfying (12) is an u-average passivity based controller (uAvPBC) that we shall refer to as the *negative u-average output feedback* for discrete-time systems in the form of (3) with output $H(\cdot, u)$.

Remark 3.3. The feedback $u = \gamma(x)$ is defined as the implicit solution to the nonlinear equality (12) which is hard to solve in practice. Nevertheless, an approximate and bounded solution still yielding GAS of the closed-loop equilibrium was proposed in [22,23] and takes the form $\gamma(x) = -\hat{K}(x)H_{av}(x, 0)$ with a suitable dynamical gain $\hat{K}(x) > 0$ and $H_{av}(x, 0) = H(F_0(x), 0)$.

4. Reduction of time-delay systems

Considering now the input-delayed dynamics (1) with invertible drift $F_0(x)$, we show how to recast the problem of stabilizing (1) into the one of stabilizing a delay-free dynamics of the form $\eta_{k+1} = F_r(N, \eta_k, u_k) : \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ for a suitably defined reduction variable $\eta_k := r(x_k, u_{[k-N,k[}) : \mathbb{R}^n \times \{\mathbb{R}^m\}^N \to \mathbb{R}^n$.

Theorem 4.1. *Consider the dynamics (1) with invertible drift term $F_0(\cdot)$. Then,*

$$\eta_k = F_0^{-N}(\cdot) \circ F^N(x_k, u_{[k-N,k[}) \quad (13)$$

with

$$F^N(x_k, u_{[k-N,k[}) = F^{N-1}(\cdot, u_{[k-N+1,k[}) \circ F(x_k, u_{k-N})$$
$$= F(\cdot, u_{k-1}) \circ \cdots \circ F(x_k, u_{k-N})$$
$$F^1(x_k, u_{[k-1,k[}) = F(x_k, u_{k-1})$$

is a reduction for (1) evolving according to the reduced model

$$\eta_{k+1} = F_r(N, \eta_k, u_k) \quad (14)$$

with $F_r(N, \eta, u) := F_0^{-N}(\cdot) \circ F(\cdot, u) \circ F_0^N(\eta)$ or, equivalently

$$\eta^+ = F_0(\eta), \quad \eta^+ := \eta^+(0) \quad (15a)$$

$$\frac{\partial \eta^+(u)}{\partial u} = Ad_{F_0^{-N}}G(\eta^+(u), u). \quad (15b)$$

Proof. In order to show the result, one computes

$$\eta_{k+1} = F_0^{-N}(\cdot) \circ F^N(\cdot, u_{[k-N+1,k+1[}) \circ F(x_k, u_{k-N}) \quad (16)$$
$$= F_0^{-N}(\cdot) \circ F(\cdot, u_k) \circ F^N(x_k, u_{[k-N,k[}).$$

By rewriting $x_k = F^{-N}(\cdot, u_{[k-N,k[}) \circ F_0^N(\eta_k)$ and substituting it into (16) one gets the result. ◁

The representation (15) of the reduced model emphasizes its geometric structure: the free evolution (15a) is unchanged while the forced component (15b) (which is actually affected by the delay) is transported backward along the drift dynamics composed N times. As a result, the reduced system is delay free over the input but explicitly parametrized by the delay N.

Now, the problem of stabilizing the retarded system (1) is recast into the one of stabilizing the equilibrium reduced dynamics (14).

Theorem 4.2. *Consider the dynamics (1) with $F_0(\cdot)$ invertible and reduced model (14). Then, any feedback $u = \alpha(\eta_k)$ such that $\alpha(0) = 0$ ensuring GAS of the equilibrium of (14) achieves GAS of the equilibrium of (1) in closed-loop. Furthermore, if $\eta_k = 0$ for $k \geq \bar{k}$, then $x_k = 0$ for $k \geq \bar{k} + N + 1$ and, thus, converges to zero in finite time.*

Proof. Introduce the auxiliary state $v_{ik} = u_{k-N+1-i}$ so that $v_{ik+1} = v_{i+1k}$ for $i = 1, \ldots, N$. Because $F_0(x)$ is invertible, $F(x, u)$ is locally invertible so that one can introduce the cascade system

$$x_{k+1} = F_1(\eta_k, v_k)$$
$$v_{k+1} = A_0 v_k + B_0 u_k$$
$$\eta_{k+1} = F_r(N, \eta_k, u_k)$$

with $v_k = (v_{1k}, \ldots, v_{Nk})^\top$, $v_{jk} = \mathrm{col}(v_{jk}^1, \ldots, v_{jk}^m)$ for $i, j = 1, \ldots, N$ and

$$F_1(\eta_k, v_k) = F^{-(N-1)}(\cdot, u_{[k-N+1,k-1[}) \circ F_0^N(\eta_k)\big|_{u_{k-N+1-i}=v_{ik}}$$

$$A_0 = \begin{pmatrix} \mathbf{0}_{m(N-1)\times m} & D \\ \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m(N-1)} \end{pmatrix}$$

$$D = \mathrm{diag}\{I_{m\times m}, \ldots, I_{m\times m}\}, \quad B_0 = \begin{pmatrix} \mathbf{0}^\top & I_{m\times m} \end{pmatrix}^\top.$$

By exploiting the strict feedforward structure [24], one gets the result. ◁

We note that methodologies involving a suitable dynamical state extension over the delayed inputs transform the system into an equivalent one where the effect of the delay is explicitly hidden [3,13,14]. As a matter of fact, the corresponding augmented dynamics is free of delays in the control but also in the mapping characterizing the evolutions. The corresponding design then leads to stabilizing the extended dynamics which is apparently free of any delay. In the case of reduction, the controlled component of the reduced dynamics is explicitly parametrized by N. This explicit dependence might be exploited to directly infer control laws which take advantage of the properties of the uncontrolled systems (e.g., passivity) while possibly guaranteeing robustness with respect to variations of N within a fixed range of values. Moreover, the reduced dynamics preserves the same dimension as the original retarded system.

Remark 4.1. Since here the problem of stabilizing the origin of (1) is addressed, the reduction is directly computed over the x-dynamics. However, this method extends to a larger variety of control problems that can be turned into the one of stabilizing the origin of a suitably defined dynamics (e.g., tracking, regulation). In those cases, one deduces the reduction over the dynamics defining the control objective. As an example, in the case of tracking of a reference signal r, one should compute the reduction over the error dynamics $\varepsilon_k = x_k - r_k$ and then stabilize the origin of the reduced error model.

Remark 4.2. Assuming, for the sake of brevity, $u \in \mathbb{R}$, the story of the control is governed by the LTI dynamics

$$v_{k+1} = \begin{bmatrix} \mathbf{0} & I \\ 0 & \mathbf{0}^\top \end{bmatrix} v_k + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} u_k$$

with $v_k = (u_{k-N}, \dots, u_{k-1})^\top$ that is globally finitely stable (i.e., with all eigenvalues in 0) in free evolution so that $v_k \to 0$ as long as $u_k \to 0$ (in exactly N steps). As a consequence, in the context of stabilization of the origin, unstable controllers are naturally excluded. This comment applies to general stabilization problems.

5. Control design

In the following we present two stabilizing feedback strategies over the reduced model designed upon the notion of Discrete Input-Lyapunov Matching (D-ILM) and u-average Passivity (u-AvPB) respectively. A comparison with a purely Prediction-Based (PB) strategy is discussed.

5.1. Stabilization via D-ILM

The following standing assumption is set.

Assumption 5.1. The delay-free dynamics (2) (equivalently (3)) is smoothly stabilizable; i.e., there exist a smooth feedback $u_k = \gamma(x_k) : \mathbb{R}^n \to \mathbb{R}^m$ and radially unbounded and positive definite Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that $\Delta_k V(x_k) := V(F(x_k, \gamma(x_k))) - V(x_k) < 0$ and $\text{rank}\{\mathrm{L}_{G(\cdot,0)} V(F_0(x_k))\} = 1$ whenever $x_k \neq 0$.

In the delay free case $N = 0$, $\eta \equiv x$ and thus $F_r(0, \eta, u) = F(x, u)$ so that the feedback $\gamma(\cdot)$ defined in Assumption 5.1 is clearly stabilizing for the reduced dynamics so that $\eta_{k+1} = F_r(0, \eta_k, \gamma(\eta_k))$ has a GAS equilibrium at the origin with strictly decreasing Lyapunov function $V(\eta)$. Thus, for a generic $N > 0$, the idea is to look for the stabilizing control $u_k = L_r(N, \eta_k)$ which satisfies the Input Lyapunov Matching equality at any time instant; i.e. $\forall \eta_k$, $k \geq 0$, u_k is such that

$$V(F_r(N, \eta_k, u_k)) - V(\eta_k) = V(F_r(0, \eta_k, \gamma(\eta_k))) - V(\eta_k)$$

which simplifies as

$$V(F_r(N, \eta_k, u_k)) - V(F_r(0, \eta_k, \gamma(\eta_k))) = 0, \quad \forall k \geq 0. \tag{17}$$

Theorem 5.1. *Consider the retarded dynamics (1) and the corresponding reduced model (14). If the delay-free dynamics (2) verifies Assumption 5.1, then the feedback $u_k = L_r(N, \eta_k)$ computed as the solution to the D-ILM equality (17) ensures GAS of the equilibrium of (14). As a consequence, the aforementioned feedback globally asymptotically stabilizes the closed-loop equilibrium of (1).*

Proof. From Assumption 5.1, one has $V(F_r(0, \eta_k, \gamma(\eta_k))) - V(\eta_k) < 0$, $\forall k \geq 0$ and thus, because of matching $V(F_r(N, \eta_k, L_r(N, \eta_k))) - V(\eta_k) < 0$, $\forall k \geq 0$. Thus, GAS of the equilibrium of (14) follows by construction while stability of (1) comes from direct application of Theorem 4.2. Existence of a solution to the above equality is ensured by the fact that $\text{rank}\{\mathrm{L}_{G(\cdot,0)} V(F_0(x))\} = 1$ for any $x \neq 0$ as it implies $\text{rank}\{\mathrm{L}_{\bar{G}_N(\cdot,0)} V(F_0(\eta))\} = 1$ for $\eta \neq 0$. $\triangleleft$

Remark 5.1. The rank condition in Assumption 5.1 is necessary for proving (through suitably invoking the implicit function theorem) the existence of a solution to the implicit equality (17) that rewrites as a former series expansion in powers of N. Such a series is invertible if the rank condition holds for $N = 0$ and $u = 0$. In that case, the solution takes the form of an asymptotic series expansion in powers of N around the delay-free solution $u_k = \gamma(\eta_k)$ (i.e., $u = \gamma(\eta) + \sum_{i>0} \frac{N^j}{(j+1)!} \gamma^j(\eta)$).

When $u \in \mathbb{R}$, it is a matter of computation to rewrite (17) as

$$\int_0^u \mathrm{L}_{\bar{G}_N(\cdot,v)} V(F_r(N, \eta, v)) \mathrm{d}v = \int_0^{\gamma(\eta)} \mathrm{L}_{G(\cdot,v)} V(F(\eta, v)) \mathrm{d}v \tag{18}$$

so specifying the control $u = L_r(N, \eta)$ as

$$L_r(N, \eta) = K(N, \eta, u)\gamma(\eta) \tag{19}$$

with $K(N, \eta, u)$ solution of the implicit equality

$$K(N, \eta, u) = \left(\int_0^1 \mathrm{L}_{\bar{G}_N(\cdot,su)} V(F_r(N, \eta, su)) \mathrm{d}s \right)^{-1} \times \int_0^1 \mathrm{L}_{G(\cdot,s\gamma(\eta))} V(F(\eta, s\gamma(\eta))) \mathrm{d}s.$$

It is important to note that one recovers the delay-free feedback when setting in the above equation $N = 0$ so getting $K(0, \eta, \gamma(\eta)) = 1$. This can be easily extended to the multi-input case along the same lines

Remark 5.2. The D-ILM equality (17) is approximately solved[3] in $O(|u|^2)$ by setting $u_{app} = K(N, \eta, 0)\gamma(\eta)$.

5.2. Stabilization via average passivity

In this part, the following assumption is set.

Assumption 5.2. Considering the delay-free dynamics (2) (equivalently (3)), there exists a proper and positive definitive $S(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that $S(F_0(x)) - S(x) \leq 0$.

As a consequence, the following result can be proven.

Theorem 5.2. *Consider the retarded dynamics (1) with invertible $F_0(\cdot)$ and the corresponding reduced model (14). If the delay-free dynamics (2) verifies Assumption 5.2, then the following holds true:*

(i) *The reduced model (14) is u-average passive with output*

$$H(N, \eta, u) = (\mathrm{L}_{\bar{G}_N(\cdot,u)} S(\eta))^\top.$$

(ii) *If the reduced model (14) is ZSD with respect to $H(N, \cdot, u) = (\mathrm{L}_{\bar{G}_N(\cdot,u)} S(\cdot))^\top$ then, the feedback solution to*

$$u + KH_{av}(N, \eta, u) = 0, \quad K > 0 \tag{20}$$

[3] A function $R(\eta, u) = O(|u|^p)$ is said of order $|u|^p$; $p \geq 1$ if whenever it is defined it can be written as $R(x, |u|) = |u|^{p-1}\tilde{R}(x, |u|)$ and there exist a function $\theta \in \mathscr{K}_\infty$ and $|u|^* > 0$ s. t. $\forall |u| \leq |u|^*$, $|\tilde{R}(x, |u|)| \leq \theta(|u|)$.

with $H_{av}(N,\eta,u) = (H^1_{av}(N,\eta,u)\dots H^m_{av}(N,\eta,u))^T$ *and, for* $i = 1,\dots,m$

$$H^i_{av}(N,\eta,u) = \frac{1}{u^i}\int_0^{u^i} \mathrm{L}_{\bar{G}^i_N(\cdot,\mathbf{w}^i)} S(\eta^+(\mathbf{w}^i))dw^i$$

ensures GAS of the closed-loop equilibrium of (14) and, hence, of (1).

Proof. Item (i) directly follows from $S(F_0(\eta)) - S(\eta) < 0$ since by definition

$$\begin{aligned} S(\eta_{k+1}) - S(\eta_k) = &\ S(F_0(\eta_k)) - S(\eta_k) \\ &+ \sum_{i=1}^m \int_0^{u^i} \mathrm{L}_{\bar{G}^i_N(\cdot,\mathbf{w}^i)} S(\eta^+(\mathbf{w}^i))dw^i \\ \le &\ H^\top_{av}(N,\eta_k,u_k)u_k. \end{aligned}$$

Concerning (ii), from ZSD of the reduced dynamics with output $(\mathrm{L}_{\bar{G}_N(\cdot,0)}S(\cdot))^\top$ the result follows from Theorem 3.1. ◁

When $u \in \mathbb{R}$, the solution $u = L_{av}(N,\eta)$ solution to (20) rewrites as

$$L_{av}(N,\eta) = -\int_0^1 \mathrm{L}_{\bar{G}_N(\cdot,sL_{av}(N,\eta))} V(F_r(N,\eta,sL_{av}(N,\eta)))ds \tag{21}$$

which recovers when $N = 0$ the delay-free solution

$$L_{av}(N,\eta) = -\int_0^1 \mathrm{L}_{G(\cdot,sL_{av}(N,\eta))} V(F(\eta,sL_{av}(N,\eta)))ds. \tag{22}$$

Remark 5.3. Approximate and bounded solutions to (17)–(20) can be explicitly computed by exploiting the result in [22,23] as pointed out in Remark 3.3.

An academic example: Consider the retarded dynamics

$$x_{1k+1} = e^{\frac{2}{3}x_{2k}+\frac{1}{2}u_{k-N}}x_{1k}, \quad x_{2k+1} = \frac{1}{3}x_{2k} + u_{k-N}. \tag{23}$$

It is a matter of computations to verify that when $N = 0$, the delay-free system is described by the (F_0, G) representation

$$x_1^+ = e^{\frac{2}{3}x_2}x_1, \quad \frac{\partial x_1^+(u)}{\partial u} = \frac{1}{2}x_1^+(u), \quad x_2^+ = \frac{1}{3}x_2, \quad \frac{\partial x_2^+(u)}{\partial u} = 1$$

and verifies Assumption 5.2 with storage function $S(x) = \frac{1}{2}(e^{2x_2}x_1^2 + x_2^2)$ so that $S(F_0(x)) - S(x) = -\frac{4}{9}x_2^2 \le 0$.

Assume now $N = 1$ and define the reduction variable $\eta = (\eta_1,\ \eta_2)^\top$ as

$$\eta_{1k} = e^{-\frac{3}{2}u_{k-1}}x_{1k}, \quad \eta_{2k} = x_{2k} + 3u_{k-1}$$

evolving according to the dynamics

$$\begin{aligned} \eta_1^+ &= e^{\frac{2}{3}\eta_2}\eta_1, & \frac{\partial \eta_1^+(u)}{\partial u} &= -\frac{3}{2}\eta_1^+(u), \\ \eta_2^+ &= \frac{1}{3}\eta_2, & \frac{\partial \eta_2^+(u)}{\partial u} &= 3. \end{aligned} \tag{24}$$

As a direct consequence of Theorem 5.2, (24) is u-average passive with respect to output $H(1,\eta) = \frac{3}{2}e^{2\eta_2}\eta_1^2 + 3\eta_2$ and average

$$H_{av}(1,\eta,u) = \eta_2 + \frac{9}{2}u + \frac{1}{2}e^{2\eta_2}\eta_1^2\frac{e^{3u}-1}{u}$$

that are computed through the same storage function $S(\cdot)$ as in the delay-free case. Thus, the feedback $u = L_{av}(N,\eta)$ solution to the implicit equality

$$u = \varphi(\eta,u) = -\frac{2}{11}\eta_2 - \frac{1}{11}e^{2\eta_2}\eta_1^2\frac{e^{3u}-1}{u} \tag{25}$$

ensures GAS of the closed-loop equilibrium. The equality (25) is highly nonlinear in the control u and is approximatively solved in $O(|u|^2)$ by

$$u_{app}(\eta) = \lim_{u\to 0}\varphi(\eta,u) = -\frac{2}{11}\eta_2 - \frac{3}{11}e^{2\eta_2}\eta_1^2.$$

Though, the above solution only ensures stability of the equilibrium as long as u is bounded and closed to 0. To overcome this issue, according to Remark 3.3, we compute an approximate solution to (25) which is bounded and globally asymptotically stabilizing. It takes the form

$$\bar{u}(\eta) = \hat{K}(\eta)u_{app}(\eta)$$

with $|\bar{u}(\eta)| \le \mu$ for any fixed positive $\mu \in \mathbb{R}$ and $\hat{K}(\eta) \in]0, C(\eta)[$ where the mappings

$$C(\eta_k) = \frac{\mu}{(1+2\mu)(1+|u_{app}(\eta_k)|)}S(\eta),$$

$$S(\eta_k) = \min_{|u|\le\frac{1}{2}}\left\{1,\ \frac{|u|}{|\varphi(\eta_k,u) - u_{app}(\eta_k)|}\right\}$$

are computed at any time instant $k \ge 0$.

5.2.1. Reduction vs. prediction

Assumption 5.1 guarantees also the existence of a prediction-based feedback that stabilizes the origin of (1) in closed-loop. As a matter of fact, defining as usual the prediction state $z_k = F^N(x_k, u_{[k-N,k[})$, one gets the predictor dynamics

$$z_{k+1} = F(z_k, u_k) \tag{26}$$

which coincides with the delay free one. As a consequence, applying the feedback $\gamma(\cdot)$ in Assumption 5.1 over z (i.e., setting $u_k = \gamma(z_k)$) ensures stabilization of the predictor dynamics and, thus, of the retarded system (1). It turns out, that the above prediction-based feedback can be interpreted a particular case of reduction-based system because $z_k = F_0^N(\eta_k)$ so that $\gamma(z_k)$ rewrites in terms of reduction as $u_k = \gamma(F_0^N(\eta_k))$. Accordingly, the existence of a stabilizing prediction-based feedback for input delayed dynamics (1) implies the existence a reduction-based one.

Remark 5.4. We note that by construction the prediction-based feedback $\gamma(z) = \gamma(F^N(x_k, u_{[k-N,k[}))$ ensures Input-Lyapunov Matching of the closed loop delay free dynamics with N step delays (i.e., at step $k+N$) while the reduction-based feedback proposed in Section 5.1 guarantees Input-Lyapunov Matching of the closed loop delay free dynamics without any delay (i.e., at step k).

Remark 5.5. Because of the mere compensation purpose, the prediction-based feedback $u = \gamma(z_k)$ lacks in robustness with respect to prediction error and uncertainty over the delay length. This issue was discussed in [15] in the context of Immersion and Invariance (I&I) by also exploiting a suitable dynamical extension that makes (1) delay-free. Roughly speaking, the I&I feedback $u_k = \gamma(z_k) - \mathrm{L}(z_k)e_k$ adds a proportional term over the prediction error for a suitable dynamic gain $\mathrm{L}(\cdot)$ and prediction error $e_k = \mathrm{col}\{e^1,\dots,e^N\}$ with $e^i_k = v^i_k - \gamma(x_{k+i-1})$ and $v^i_k = u_{k-N+i-1}$ for $i = 1,\dots,N$.

6. Case studies

6.1. LTI systems

Consider the case of linear time-invariant (LTI) systems of the form

$$x_{k+1} = Ax_k + Bu_{k-N} \tag{27}$$

then, (13) specifies as

$$\eta_k = x_k + \sum_{j=k-N}^{k-1} A^{k-1-N-j} B u_j \tag{28}$$

and evolves according to

$$\eta_{k+1} = A\eta_k + A^{-N} B u_k \tag{29}$$

so that controllability of $(A, A^{-N}B)$ is enough to ensure the existence of a reduction-based feedback. For, the following result is proven.

Proposition 6.1. *Consider the LTI system (27) and let (28) be a reduction with model (29). Then, (29) is controllable if and only if*

(i) *the couple (A, B) is controllable;*
(ii) *A has no zero eigenvalue.*

As a consequence, any feedback $u = Lx$ ensuring that $A + A^{-N}BL$ is Schur asymptotically stabilizes (27).

Proof. In order to show the result, one has to prove that the above conditions are necessary and sufficient to guarantee that the matrix

$$\mathscr{R}_N = \left(A^{-N}B \vdots A^{-N+1}B \vdots \quad \dots \quad \vdots A^{n-1-N}B\right) \tag{30}$$

is full-rank n. For, we rewrite $\mathscr{R}_N = A^{-N}\mathscr{R}$ where $\mathscr{R}$ denotes the controllability matrix of the delay-free system, $\mathscr{R} = \left(B \vdots AB \vdots \quad \dots \quad \vdots A^{n-1}B\right)$. Proving that $\mathscr{R}_N$ corresponds to prove that $\mathrm{Ker}\{A^{-N}\} \cap \mathrm{Im}\{\mathscr{R}\} \equiv \{0\}$. The sufficiency of (i) and (ii) is straightforward as (ii) implies that A^{-N} is non singular and, thus, $\mathrm{Ker}\{A^{-N}\} \equiv \{0\}$. The necessity can be easily proven by contradiction by assuming that $\mathrm{Ker}\{A^{-N}\} \cap \mathrm{Im}\{\mathscr{R}\} - \{0\} \neq 0$ so that there exists $\bar{x} \in \mathrm{Im}\{\mathscr{R}\} - \{0\}$ such that $A^{-N}\bar{x} = 0$. Because of (i), one has that $\bar{x} = 0$, so contradicting the assumption. Thus, one has that any feedback that $u = Lx$ ensuring that $A + A^{-N}BL$ is Schur asymptotically stabilizes (29). Finally, to prove asymptotic stability of (27), one introduces the auxiliary state $v_{ik} = u_{k-N+1-i}$ (for $i = 1, \dots, N$) so that $v_{ik+1} = v_{i+1k}$ and consider the upper-triangular system

$$x_{k+1} = \hat{A}v_k + A\eta_k, \quad v_{k+1} = A_0 v_k + B_0 L\eta_k,$$
$$\eta_{k+1} = (A + A^{-N}BL)\eta_k$$

that is clearly asymptotically stable, so concluding the proof. ◁

Proposition 6.1 shows that the problem of stabilizing the retarded system (27) is reformulated as an eigenvalue placement problem over the reduced model (29) provided that A is invertible.

One can assign the spectrum of $A + A^{-N}BL$ as directly dependent on N so to guarantee stability for a fixed range of N. In this sense, one does not fully compensate the effect of the delay over the closed-loop system but might ensure that the closed-loop eigenvalues are stable for a fixed range of values of N so possibly improving robustness.

Remark 6.1. Asking for A to be invertible seems to be only a sufficient requirement as non invertibility of A corresponds to the presence of 0 eigenvalues corresponding to asymptotically stable modes. Thus, one might still define a suitable reduction over a lower dimensional state–space by leaving the stable part (associated to the 0 eigenvalues) unchanged.

Remark 6.2. Assumption (i) in Proposition 6.1 can be weakened to requiring only stabilizability of the couple (A, B) without affecting the result.

6.2. Input-affine-like dynamics

Consider the class of time-delay system (1) whose delay-free (F_0, G) representation is provided by [16]

$$x^+ = F_0(x), \quad x^+ := x^+(0) \tag{31a}$$

$$\frac{\partial x^+(u)}{\partial u} = G(x^+(u)) \tag{31b}$$

where the vector field G does not explicitly depend on u (i.e., $G(x, u) = G(x)$).

Remark 6.3. This class of systems has been shown to be equivalent (up to a coordinate change) to the difference map $x_{k+1} = F_0(x_k) + Bu_k$ for a suitable constant matrix B. However, it is of interest to exactly specify the proposed control solutions over this class which indeed well approximates larger classes of systems of the form (5).

By construction, the reduced model associated to (13) evolves according to the differential-difference representation

$$\eta^+ = F_0(\eta), \quad \eta^+ := \eta^+(0) \tag{32a}$$

$$\frac{\partial \eta^+(u)}{\partial u} = \bar{G}_N(\eta^+(u)) \tag{32b}$$

with $\bar{G}_N(\eta) = [\nabla_\eta F_0^{-N}(\eta)]_{\eta = F_0^N(\eta)} G(F_0^N(\eta))$.

Remark 6.4. The reduced dynamics (32) preserves the structure of the original system (31).

The following results specify Theorems 5.1 and 5.2 for this class of systems.

Corollary 6.1. *Let (31) verify Assumption 5.1 with quadratic Lyapunov function $V(x) = x^\top Px$ and $P > 0$; then, the feedback $u_k = L_r(N, \eta_k)$ solving the D-ILM problem is provided by*

$$L_r(N, \eta_k) = (\bar{G}_N^\top(\eta_k)\bar{G}_N(\eta_k))^{-1}\bar{G}_N^\top(\eta_k)G(\eta_k)\,\gamma(\eta_k). \tag{33}$$

Remark 6.5. The prediction-based feedback $u_k = F_0(\gamma(\eta_k))$ does not solve the D-ILM equality.

Corollary 6.2. *Let (31) verify Assumption 5.2 with $S(x) = \frac{1}{2}x^\top Qx$; then, the reduced model (32) is u-average passive with respect to the output $H(N, \eta) = \bar{G}_N^\top(\eta)Q$ and the stabilizing passivity-based feedback is provided by*

$$L_{av}(N, \eta_k) = -(I + \frac{1}{2}(\bar{G}_N^\top(\eta_k))^\top Q\bar{G}_N(\eta_k))^{-1}\bar{G}_N^\top(\eta_k)QF_0(\eta_k).$$

Remark 6.6. The dynamics (24) is of the form (32) so that the coordinate change $z = \mathrm{col}(e^{-\frac{1}{2}x_2}x_1, x_2)$ transforms the system into the form $z_{k+1} = F_0(z_k) + Bu_k$. Nevertheless, the non quadratic storage function $\tilde{S}(z) = \frac{1}{2}(z_2^2 + e^{3z_2}z_1^2)$ prevents from applying Corollary 6.2.

6.3. Sampled-data systems

The proposed strategy applies to nonlinear systems issued from sampling whenever the length of the delay is a multiple of the sampling period δ. For this purpose, consider the input-affine system

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t - \tau) \tag{34}$$

with $u(t) = u(k\delta) = u_k$ for $t \in [k\delta, (k+1)\delta[$ and affected by entire-delay (i.e., $\tau = N\delta$ for some $N \in \mathbb{N}$). The sampled-data equivalent-model is provided by

$$x_{k+1} = F^\delta(x_k, u_{k-N}) \tag{35}$$

with $F^{\delta}(x_k, u_k) = e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)} x\big|_{x_k}$ and $F_0^{\delta}(x) = e^{\delta \mathrm{L}_f} x$. A first analysis and design on this class of time-delay systems was developed in [13] through prediction-based feedback later improved via the notion of Immersion and Invariance.

Accordingly, an alternative approach to the aforementioned solution is provided by introducing the reduction map (13) in the form

$$\eta_k = e^{\delta(\mathrm{L}_f + u_{k-N} \mathrm{L}_g)} \circ \cdots \circ e^{\delta(\mathrm{L}_f + u_{k-1} \mathrm{L}_g)} \circ e^{-\tau \mathrm{L}_f} x\big|_{x_k} \tag{36}$$

through successive application of the Lie exponential operator. One gets that the reduced dynamics (14) is delay free and parametrized by δ and $\tau = N\delta$; i.e.

$$F_r^{\delta}(\tau, \eta_k, u_k) = e^{\tau \mathrm{L}_f} \circ e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)} \circ e^{-\tau \mathrm{L}_f} \eta\big|_{\eta_k} \tag{37}$$

or equivalently with $G^{\delta}(\cdot, u) = \int_0^{\delta} e^{-s\, ad_{f+gu}} g ds$ and $Ad_{F_0^{\tau}} Id = e^{-\tau ad_f} Id$

$$\eta^+ = F_0^{\delta}(\eta), \quad \eta^+ := \eta^+(0) \tag{38a}$$

$$\frac{\partial \eta^+(u)}{\partial u} = e^{-\tau ad_f} G^{\delta}(\eta^+(u), u). \tag{38b}$$

Once the reduction mapping is computed, the control can be designed on the dynamics (37) or equivalently (38) exploiting the exponential form representation which provides a useful way of computing approximate solutions in the form of power expansions in δ and τ (see [13] for further details).

7. Conclusions

This work extends the notion of reduction to discrete-time and nonlinear dynamics affected by a constant input delay and provides a way of designing the stabilizing feedback based on the properties of the delay-free system associated to the original dynamics. Future works are addressing the extension of this technique to the case of unknown time-delay and larger classes of time-delay systems (e.g., nonlinear systems affected by multi-channel delays and distributed delays).

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Sampled-Data Reduction of Nonlinear Input-Delayed Dynamics

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***Abstract*—A reduction approach on the discrete-time equivalent model of a nonlinear input delayed system is proposed to design a sampled-data stabilizing feedback. Global asymptotic stability of the feedback system is so achieved by solving the problem over the reduction state. Stabilization of the reduced dynamics is obtained through input-Lyapunov matching. Connections with prediction-based methods are established. A simulated example illustrates the performances.**

***Index Terms*—Sampled-data control, delay systems, algebraic/geometric methods.**

I. Introduction

When dealing with time delay systems, a huge number of challenges arise from both theoretical and practical problems (see, among others, [1]–[4] and references therein). In particular, two main classes of delays have been identified: discrete delays when the model depends on retarded variables at time $t - \tau$ ($\tau > 0$ denotes the delay length); distributed delays, when the model explicitly depends on the story of the retarded variables over the interval $[t - \tau, t[$.

This letter is concerned with systems affected by discrete delays over the input variables. Despite the wide literature, a lot of questions still remain unanswered, even for Linear Time Invariant (LTI) systems. This is mainly linked with the fact that the retarded system is intrinsically infinite dimensional. Different prediction and reduction-based design approaches have been proposed (e.g., [5]–[9]). In the first case, the stabilizing feedback is deduced by computing the delay-free feedback over the future trajectories of the system on the time window $[t, t+\tau[$. In the second case, the design of the reduction-based control is lead to a somehow *equivalent* reduced delay-free dynamics in a sense that depends on the control purpose.

More recently, an increasing focus has been devoted to sampled-data time-delay systems (e.g, [10] and [11]) when assuming that the control is piecewise constant and measures are available at discrete-time instants. This interest is mainly motivated by the fact that the retarded infinite dimensional continuous-time system admits a finite dimensional equivalent sampled-data model whenever there exists an explicit relation among the delay and the sampling period. In this context, several approaches have been proposed based on reduction (e.g., [12], only for the LTI case), prediction or, more recently, Immersion and Invariance methods (e.g., [13] and [14] for nonlinear systems). In the latter cases, the design is based on the assumption that the delay-length is an entire multiple of the sampling period (i.e., $\tau = N\delta$ for some $N \in \mathbb{N}$). This assumption has been recently relaxed in [15] by considering *non-entire* delays (namely, $\tau = N\delta + \sigma$ for some $N \in \mathbb{N}$ and $\sigma \in [0, \delta[$) and extending the prediction method to a non-entire time interval of length $N\delta + \sigma$. Moreover, to improve robustness, an Immersion and Invariance (I&I) approach, with the delay free dynamics corresponding to the I&I target dynamics, has been proposed in [16].

In spite of that, predictor-based strategies are hard to extend to much general classes of time-delay systems as, for example, LTI dynamics affected by multichannel delay [17], [18]. Following the work by Artstein [6], and for the first time at the best of authors' knowledge, a *sampled-data reduction*-based method is proposed in this letter for stabilizing nonlinear systems affected by input delay. Differently from the work in [15], the present strategy qualifies for extension to a larger class of time-delay systems, such as the multichannel case. Finally, the sampled-data design over the reduced model simplifies the task and allows the computation of approximate solutions that are actually implemented in practice.

Our contribution is two-fold: first, we define a *discrete-time reduction variable* exhibiting a delay-free dynamics which identifies the discrete-time *reduced model*; secondly, we prove that any discrete-time feedback stabilizing the reduced model guarantees stabilization at the sampling instants of the original system. The design of the control law is pursued via a suitably defined Input-Lyapunov Matching (ILM) problem [19], [20] when assuming smooth stabilizability of the delay-free system. It is also shown that a suitable choice of the reduction-based

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control enables one to recover the prediction-based feedback proposed in [15].

In Section II the problem is set and the instrumental definitions provided while the main result is stated in Section III. In Section IV, the design of the control law is developed and the case of LTI system is detailed as a case study in Section V. Simulations on the van der Pol oscillator are discussed in Section VI. Final comments in Section VII conclude this letter.

Notations and Definitions: All the functions and vector fields defining the dynamics are assumed smooth over the respective definition spaces. M_U (resp. M_U^I) denotes the space of measurable and locally bounded functions $u : \mathbb{R} \to U$ ($u : I \to U, I \subset \mathbb{R}$) with $U \subseteq \mathbb{R}$. $\mathscr{U}_\delta \subseteq M_U$ denotes the set of piecewise constant functions over time intervals of length $\delta \in]0, T^*[$, a finite time interval; i.e., $\mathscr{U}_\delta = \{u \in M_U \text{ s.t. } u(t) = u_k, \forall t \in [k\delta, (k+1)\delta[; k \geq 0\}$. Id and I denote the identity function and matrix respectively. Given a vector field f, L_f denotes the Lie derivative operator, $\mathrm{L}_f = \sum_{i=1}^n f_i(\cdot)\frac{\partial}{\partial x_i}$. Given two vector fields f and g, $ad_f g = [f, g]$ and iteratively $ad_f^i g = [f, ad_f^{i-1} g]$. The operator $e^{\mathrm{L}_f} Id$ denotes the associated Lie series operator, $e^{\mathrm{L}_f} := \mathrm{I} + \sum_{i\geq 1} \frac{\mathrm{L}_f^i}{i!}$. Given any smooth function $h : \mathbb{R}^n \to \mathbb{R}$ then $e^{\mathrm{L}_f} h(x) = h(e^{\mathrm{L}_f} Id\big|_x)$. The composition of functions is denoted by "$\circ$". A function $R(x, \delta) = O(\delta^p)$ is said of order $\delta^p; p \geq 1$ if whenever it is defined it can be written as $R(x, \delta) = \delta^{p-1}\tilde{R}(x, \delta)$ and there exist a function $\theta \in \mathscr{K}_\infty$ and $\delta^* > 0$ s. t. $\forall \delta \leq \delta^*$, $|\tilde{R}(x, \delta)| \leq \theta(\delta)$.

II. Problem Statement

Consider the nonlinear time-delay system

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t - \tau) \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $f(0) = 0$ and the *delay-free system*

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t). \tag{2}$$

The following standing assumptions are set: the delay-free system is forward complete so implying forward completeness of (1) [14]; denoting by δ the sampling period, measurements are available at sampling instants $t = k\delta$, $k \geq 0$ and $u \in \mathscr{U}_\delta$; the time delay τ is fixed, known and such that $\tau = N\delta + \sigma$ for some $N \in \mathbb{N}$ and $\sigma \in [0, \delta[$.

Denoting $x_k := x(k\delta)$ and $u_k := u(k\delta)$, one computes the sampled-data equivalent model of (1) as

$$\begin{aligned} x_{k+1} &= F^{\delta-\sigma}(\cdot, u_{k-N}) \circ F^{\sigma}(x_k, u_{k-N-1}) \\ &= F^{\delta-\sigma}(F^{\sigma}(x_k, u_{k-N-1}), u_{k-N}) \\ &:= F^{\delta}(\sigma, x_k, u_{k-N-1}, u_{k-N}) \end{aligned} \tag{3}$$

with $F^{\theta}(x, u) = e^{\theta(\mathrm{L}_f + u\mathrm{L}_g)} Id\big|_x$ and $u \in \mathscr{U}_\delta$.

Remark 1: When $N = 0$ and $\sigma = 0$, (3) recovers the sampled-data equivalent model to the delay-free (2) [21]; i.e.,

$$x_{k+1} = F^{\delta}(x_k, u_k) = e^{\delta(\mathrm{L}_f + u_k\mathrm{L}_g)} Id\big|_{x_k}. \tag{4}$$

Roughly speaking, from (3) one deduces that a discrete delay affecting (1) is transformed into a distributed delay on the equivalent discrete-time model (3).

The aim of this letter is to characterize a *discrete-time reduction variable* (or simply *reduction*), say y, which exhibits a discrete-time delay-free dynamics (the *discrete-time reduced model*) with the property that any of its stabilizing controller achieves stabilization of (3) in turn (i.e., sampled-data stabilization of the original system (1)).

Definition 1 (S-GAS): The equilibrium of a continuous-time dynamics $\dot{x} = f(x)$ is sampled-data globally asymptotically stable at the sampling instants $t = k\delta$ ($k \geq 0$), if the equilibrium of its discrete-time equivalent dynamics $x_{k+1} = e^{\delta\mathrm{L}_f} Id\big|_{x_k}$ is globally asymptotically stable (GAS).

III. Main Result

A. The Case $\tau = \sigma$ ($N = 0$)

When $N = 0$, the sampled model (3) reduces to

$$x_{k+1} = F^{\delta}(\sigma, x_k, u_{k-1}, u_k) = F^{\delta-\sigma}(F^{\sigma}(x_k, u_{k-1}), u_k). \tag{5}$$

Accordingly, one can define the mapping

$$y_k = F_0^{-\sigma}(F^{\sigma}(x_k, u_{k-1})) \tag{6}$$

with $F_0^{\theta}(x) = e^{\theta\mathrm{L}_f} Id\big|_x$ as a candidate reduction for (5). Computing (6) one-step ahead, one gets

$$y_{k+1} = F_0^{-\sigma}(F^{\sigma}(x_{k+1}, u_k)). \tag{7}$$

By rewriting (5) in terms of the reduction (6), one has

$$x_{k+1} = \bar{F}^{\delta}(\sigma, y_k, u_k) \tag{8}$$

with

$$\begin{aligned} \bar{F}^{\delta}(\sigma, y_k, u_k) &= F^{\delta-\sigma}(\cdot, u_k) \circ F_0^{\sigma}(y_k) \\ &= e^{\sigma\mathrm{L}_f} e^{(\delta-\sigma)(\mathrm{L}_f + u_k\mathrm{L}_g)} Id\big|_{y_k}. \end{aligned}$$

By substituting the above mappings into (7), one concludes that the dynamics of (6) is delay-free so that (6) is actually a reduction for (5). More in detail, the reduced model takes the form

$$y_{k+1} = F_r^{\delta}(\sigma, y_k, u_k) \tag{9}$$

with

$$\begin{aligned} F_r^{\delta}(\sigma, y, u) := {} & F_0^{-\sigma} \circ F^{\delta}(\cdot, u) \circ F_0^{\sigma}(y) \\ &= e^{\sigma\mathrm{L}_f} e^{\delta(\mathrm{L}_f + u\mathrm{L}_g)} e^{-\sigma\mathrm{L}_f} Id\big|_y. \end{aligned}$$

Proposition 1: Any feedback $u_k = \alpha(y_k)$ achieving GAS of the origin of (9) ensures GAS the origin of (5) and, thus, S-GAS of (1). Furthermore, suppose that $y_k = 0$, $k \geq \bar{k}$, then x_k goes to 0 in exactly $\bar{k} + 1$ steps.

Proof: Consider the original dynamics (5) equivalently rewritten in the form (8). First, we write the original dynamics (8) and the reduced model (9) as a strict-feedforward interconnection over $\mathbb{R}^n \times \mathbb{R}^n$ of the form

$$x_{k+1} = \bar{F}^{\delta}(\sigma, y_k, u_k) \tag{10a}$$

$$y_{k+1} = F_r^{\delta}(\sigma, y_k, u_k). \tag{10b}$$

Now, consider any feedback $u_k = \alpha(y_k)$ that makes the origin of the reduced model (10b) GAS and define the bicausal transformation $\zeta_k = x_k - \phi(y_k, y_{k-1})$ with

$$\phi(y_k, y_{k-1}) = F^{-\sigma}(\cdot, \alpha(y_{k-1})) \circ F_0^{\sigma}(y_k).$$

Under $u_k = \alpha(y_k)$, one has that $\phi(y_{k+1}, y_k) = \bar{F}^\delta(\sigma, y_k, \alpha(y_k))$ so implying that, in the (ζ, y) coordinates, the dynamics (10) in closed-loop rewrites as the composition of two decoupled dynamics

$$\begin{aligned}\zeta_{k+1} &= 0\\ y_{k+1} &= F_r^\delta(\sigma, y_k, \alpha(y_k))\end{aligned}$$

with GAS equilibrium at the origin. Consequently, GAS of the origin of the original system (5) (equivalently, (10a)) follows. By virtue of the feedforward structure, if $y_k = 0$ for any $k \geq \bar{k}$, then $x_k = 0$ for $k \geq \bar{k} + 1$. ■

B. The Case $\tau = N\delta + \sigma$, $(N > 0)$

The definition of the reduction is generalized to $N \geq 0$ as follows.

Proposition 2: Consider the continuous-time system (1) and let (3) be its sampled-data equivalent model. The map

$$\begin{aligned}y_k = F_0^{-\tau} \circ F^\delta(\cdot, u_{k-1}) \circ \cdots \\ \circ F^\delta(\cdot, u_{k-N}) \circ F^\sigma(x_k, u_{k-N-1})\end{aligned} \tag{11}$$

defines a *reduction* for (3) evolving according to the reduced dynamics

$$y_{k+1} = F_r^\delta(\tau, y_k, u_k) \tag{12}$$

with

$$\begin{aligned}F_r^\delta(\tau, y, u) &:= F_0^{-\tau} \circ F^\delta(\cdot, u) \circ F_0^\tau(y)\\ &= e^{\tau L_f} e^{\delta(L_f + u L_g)} e^{-\tau L_f} Id\big|_y.\end{aligned}$$

Proof: Computing (11) one-step ahead, we get

$$\begin{aligned}y_{k+1} = F_0^{-\tau} \circ F^\delta(\cdot, u_k) \circ \cdots \\ \circ F^\delta(\cdot, u_{k-N+1}) \circ F^\sigma(x_{k+1}, u_{k-N})\end{aligned} \tag{13}$$

while (5) rewrites as

$$x_{k+1} = \bar{F}^\delta(\sigma, y_k, u_{k-1}, \ldots, u_{k-N}) \tag{14}$$

with

$$\begin{aligned}\bar{F}^\delta(\sigma, y_k, u_{k-1}, \ldots, u_{k-N}) := F^{-\sigma}(\cdot, u_{k-N}) \circ F^{-\delta}(\cdot, u_{k-N+1})\\ \circ \cdots \circ F^{-\delta}(\cdot, u_{k-1}) \circ F_0^\tau(y_k).\end{aligned}$$

and, for $N = 1$, $\bar{F}^\delta(\sigma, y_k, u_{k-1}) := F^{-\sigma}(\cdot, u_{k-1}) \circ F_0^\tau(y_k)$. By substituting (14) into (13) one gets the result. ■

Remark 2: Again, when $\tau = 0$, $y \equiv x$ and the reduction dynamics (12) recovers the sampled-data delay-free one (4).

Remark 3: By exploiting the Lie exponential, (11) rewrites as

$$\begin{aligned}y_k = x_k + \sum_{s_1+\cdots+s_{N+2}>0} \frac{(-1)^{s_2}\sigma^{s_1+s_2}\delta^{s_3+\cdots+s_{N+2}}}{s_1!\ldots s_{N+2}!}\\ \times L_{f+u_{k-N-1}g}^{s_{N+2}} \ldots L_{f+u_{k-1}g}^{s_1} L_f^{s_2} Id\big|_{x_k}.\end{aligned}$$

Remark 4: By expanding (12), one gets

$$y_{k+1} = e^{\delta L_f}(y_k) + \delta u_k e^{\tau \mathrm{ad}_f} g(y_k) + O(u^2)$$

so explicitly recovering the Lie controllability directions $\mathrm{ad}_f^j g$ and their Lie brackets describing the sampled-data reduced dynamics (12) which is delay-free but generally nonlinear in the control u_k.

Proposition 1 extends to this case as follows.

Theorem 1: Consider the continuous-time system (1) with sampled-data equivalent model (3). Define the reduction y in the form (11) evolving according to (12). Then, any feedback $u_k = \alpha(y_k)$ achieving GAS of the origin of (12), ensures GAS (resp., S-GAS) of the origin of (3) (resp., (1)). Furthermore, suppose that $y_k = 0$ for $k \geq \bar{k}$, then x_k converges to 0 in exactly $\bar{k} + N + 1$ steps.

Proof: The proof proceeds along the lines of the one of Proposition 1 by considering (14) and exploiting the cascade structure

$$x_{k+1} = \bar{F}^\delta(\sigma, y_k, u_{k-1}, \ldots, u_{k-N}), \quad y_{k+1} = F_r^\delta(\tau, y_k, u_k).$$

■

Remark 5: The results in Proposition 2 and Theorem 1 hold in the case of entire delays (i.e., when $\sigma = 0$) so providing an alternative solution to the one presented in [14].

According to the previous result, stabilization of the reduced dynamics (12) ensures S-GAS in closed-loop of the original system (1). In the following, a possible choice of the feedback $u_k = \alpha(y_k)$ is proposed.

IV. On the Design of the Sampled-Data Feedback

The following assumption is introduced.

A. There exists a smooth continuous-time feedback $u(t) = \gamma(x(t))$ ensuring GAS of the equilibrium of the delay-free (2) with radially-unbounded strict Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that $L_g V(x) \neq 0$ for any $x \neq 0$.

As proved in [19], Assumption **A** implies the existence of a smooth sampled-data feedback stabilizing the origin of the delay-free system (4). Such a feedback is inferred via the notion of Input-Lyapunov Matching (ILM, [19], [20]).

Theorem 2 [19]: Let the delay-free dynamics (2) fulfil Assumption **A**. Then, there exists $\gamma^\delta : \mathbb{R}^n \to \mathbb{R}$ as the unique solution $u_k = \gamma^\delta(x_k)$, for any $x_k = x(k\delta)$, of the ILM equality

$$e^{\delta(L_f + u_k L_g)} V(x)\big|_{x_k} - V(x_k) = \int_{k\delta}^{(k+1)\delta} L_{f+\gamma(\cdot)g} V(x(s)) ds \tag{16}$$

with $x(s) = e^{s L_{f+\gamma(\cdot)g}} Id\big|_{x_k}$. Moreover, $\gamma^\delta(x)$ admits the power expansion

$$\gamma^\delta(x) = \gamma(x) + \sum_{i\geq 0} \frac{\delta^i}{(i+1)!}\gamma_i(x). \tag{17}$$

As a consequence, $u_k = \gamma^\delta(x_k)$ ensures GAS (resp. S-GAS) of the closed-loop delay-free dynamics (4) (resp., (2)).

In the following, we will show that Assumption **A** is sufficient to ensure the existence of a sampled-data reduction-based feedback yielding S-GAS of the equilibrium of the retarded system (1).

A. Reduction-Based Stabilization via ILM

The idea is to construct a sampled-data feedback over the dynamics (9) to ensure matching (at any sampling instant) of the Lyapunov function $V(x(t))$ along the closed-loop delay-free dynamics (2) when $u(t) = \gamma(x(t))$. For, we recall that when $\tau = 0$, $x \equiv y$ so that the following result can be stated.

Theorem 3: Consider the time-delay system (1) under Assumption **A** and let (3) be its sampled-data equivalent model. Introduce the reduction y as in (11) with reduced dynamics (12). Then, there exists a smooth mapping $K^\delta(\tau, \cdot) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ of the form

$$K^\delta(\tau, y) = \gamma(y) + \sum_{i+j>0} \frac{\delta^i \tau^j}{(i+1)!j!} K_{ij}(y) \tag{18}$$

that is the unique solution $u_k = K^\delta(\tau, y_k)$ of the ILM equality

$$V(F_r^\delta(\tau, y_k, u_k)) - V(y_k) = \int_{k\delta}^{(k+1)\delta} \mathrm{L}_{f+\gamma(\cdot)g} V(y(s)) \mathrm{d}s \tag{19}$$

for any $k \geq 0$ and $y(s) = e^{(s-k\delta)\mathrm{L}_{f+\gamma(\cdot)g}} Id\big|_{y_k}$. Moreover, the feedback $u_k = K^\delta(\tau, y_k)$ makes the closed-loop equilibrium of (3) (resp. (1)) GAS (resp. S-GAS).

Proof: The existence of a unique solution to (19) in the form (18) is deduced from direct application of the Implicit Function Theorem provided that Assumption **A** holds. Concerning closed-loop stability, because of matching, one gets that

$$\int_{k\delta}^{(k+1)\delta} \mathrm{L}_{f+\gamma(\cdot)g} V(y(s)) \mathrm{d}s = \int_{k\delta}^{(k+1)\delta} \mathrm{L}_{f+\gamma(\cdot)g} V(x(s)) \mathrm{d}s < 0$$

with $x(s) = e^{(s-k\delta)L_{f+\gamma(\cdot)g}} Id\big|_{x_k}$. Thus, by construction, one has that

$$V(F_r^\delta(\tau, y_k, K^\delta(\tau, y_k))) - V(y_k) < 0.$$

Thus, when $u_k = K^\delta(\tau, y_k)$, (12) has a GAS equilibrium of the origin. From Theorem 1, one concludes that such a feedback ensures GAS of (3) (resp., S-GAS of (1)) in closed-loop. ■

Remark 6: The final feedback $u_k = K^\delta(\tau, y)$ is smoothly parametrized by both δ and τ. When $\tau \to 0$, (19) coincides with (16) so implying that $K^\delta(0, x) = \gamma^\delta(x)$ in (17).

B. About Approximate Solutions

Theorem 3 proves that whenever one can compute a stabilizing smooth feedback for the continuous-time delay-free system (2), sampled-data stabilization in closed-loop of the time-delay dynamics (1) can be pursued by combining reduction-based and ILM arguments. Though, the final feedback comes in the form of a series expansions in powers of δ and τ. As a consequence, exact solutions cannot be computed in general and only approximation of (18) can be implemented in practice.

Definition 2: An *approximate solution of order p* $K_N^{\delta[p]}(\tau, \cdot)$ to (19) is defined as the truncation of the series (18) at any finite $p := i + j$ in $\delta^i \tau^j$; i.e.,

$$K^{\delta[p]}(\tau, y) = \sum_{i=0, j=0}^{j+i=p} \frac{\delta^i \tau^j}{(i+1)!j!} K_{ij}(y).$$

Each term K_{ij} can be computed via an iterative procedure by developing both sides of (19) and equating the terms with the same power $\delta^i \tau^j$. Accordingly, at each step, one has to solve a linear equation in the unknown K_{ij} as a function of the previous terms. For the first terms one gets

$$K_{01} = \frac{\gamma(y)}{\mathrm{L}_g V} \mathrm{L}_{ad_f g} V, \quad K_{20} = \ddot{\gamma}(y) + \frac{\dot{\gamma}(y)}{2\mathrm{L}_g V} \mathrm{L}_{ad_f g} V \tag{20a}$$

$$K_{02} = \frac{2K_{01}}{\mathrm{L}_g V} \mathrm{L}_{ad_f g} V - \frac{\gamma(x)}{\mathrm{L}_g V} (\mathrm{L}_g \mathrm{L}_f^2 - 2\mathrm{L}_f \mathrm{L}_g \mathrm{L}_f + \mathrm{L}_f^2 \mathrm{L}_g) V$$

$$K_{10} = \dot{\gamma}(y) = \mathrm{L}_{f+\gamma g} \gamma(y) \tag{20b}$$

$$\begin{aligned} K_{11} = &- \frac{K_{01}}{\mathrm{L}_g V} (\mathrm{L}_g \mathrm{L}_f + \mathrm{L}_f \mathrm{L}_g) V - \frac{2K_{00} K_{01}}{\mathrm{L}_g V} \mathrm{L}_g^2 V \\ &+ \frac{K_{10}}{\mathrm{L}_g V} \mathrm{L}_{ad_f g} V - \frac{K_{00}}{\mathrm{L}_g V} (\mathrm{L}_g \mathrm{L}_f^2 - \mathrm{L}_f^2 \mathrm{L}_g) V \\ &- \frac{K_{00}^2}{\mathrm{L}_g V} (\mathrm{L}_g^2 \mathrm{L}_f - \mathrm{L}_f \mathrm{L}_g^2) V \end{aligned} \tag{20c}$$

with $\ddot{\gamma}(y) = \mathrm{L}_{f+\gamma g}^2 \gamma(y)$.

Although global results are in general lost under approximate solutions, those controls still yield interesting properties in closed-loop, such as practical-GAS or Input-to-State Stability [14], [20], [22].

C. Reduction and Prediction-Based Stabilization

In the sequel a comparison with respect to the predictor-based approach proposed in [15] is developed. As a matter of fact, by suitably defining $u = \alpha(y)$ in Theorem 1, the predictor feedback is recovered. For this purpose, we note that the reduction variable y_k in (11) rewrites as $y_k = F_0^{-\tau}(x(k\delta + \tau))$ where

$$x(k\delta + \tau) = F^\delta(\cdot, u_{k-1}) \circ \cdots \circ F^\delta(\cdot, u_{k-N}) \circ F^\sigma(x_k, u_{k-N-1})$$

defines the prediction of the state at $t = k\delta + \tau$ from x_k.

Based on the above relation, it turns out that reduction can be interpreted as prediction of the state at $t = k\delta + \tau$ that is projected backward via the free evolution $F_0^{-\tau}(\cdot)$; namely, $y_k = F_0^{-\tau}(x(k\delta + \tau))$.

The following statement settles the result in [15] in terms of reduction.

Theorem 4: Consider the time-delay system (1) under Assumption **A** and let (5) be its equivalent sampled-data model. Let the reduction state y in (11) evolve according to (12). Then, the feedback $u_k = \gamma^\delta(F_0^\tau(y_k))$, where $\gamma^\delta : \mathbb{R}^n \to \mathbb{R}$ is computed as the unique solution to (16), ensures GAS of (1) at the time instants $t = k\delta + \sigma$ with $k \geq 0$.

Proof: In order to prove the result, one has to prove that the feedback $u_k = \gamma^\delta(F_0^\tau(y_k))$ coincides with the predictor-based feedback proposed in [15]. For, introduce the coordinates change $z_k = F_0^\tau(y_k)$ so that $u_k = \gamma^\delta(z_k)$ while the dynamics (12) takes the form $z_{k+1} = F^\delta(z_k, u_k)$ with $F^\delta(z_k, u_k) = e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)} Id\big|_{z_k}$. Thus, the predictor feedback is recovered. Since γ^δ is the solution of an ILM problem, $u = \gamma^\delta(z)$ stabilizes the predictor dynamics. Thus, such a feedback ensures GAS of (1) in closed loop at the time instants $t = k\delta + \sigma$, $k \geq 0$. ■

By virtue of the above result, we note that, whenever the system (1) is driftless, the reduction and predictor-based solutions coincide.

Contrarily to prediction, the reduction-based feedback only requires the knowledge of the state at the sampling instants. Indeed, the former control is based on the knowledge of the state at the inter sampling instant $t = k\delta + \sigma$ that is not available from measures. Thus, the feedback in [15] needs a further prediction over the inter sampling interval.

Moreover, the prediction-based controller [15] ensures sampled-data stabilization at the inter sampling instants $t = k\delta + \sigma$ $(k \geq 0)$ while the proposed reduction feedback ensures stabilization at the sampling instants $t = k\delta$ and, thus, S-GAS. By virtue of this, the prediction-based control should be more sensible to the variation of σ and, thus, on τ.

V. LTI Systems As a Case Study

Consider the case in which (1) is a LTI system

$$\dot{x}(t) = Ax(t) + Bu(t - \tau) \tag{21}$$

under the standing assumptions presented in Section II plus

$\mathbf{A_L}$. the couple (A, B) is controllable.

The sampled-data equivalent model of (21) is provided by

$$x_{k+1} = A^{\delta}x_k + A^{\delta-\sigma}B^{\sigma}u_{k-N-1} + B^{\delta-\sigma}u_{k-N}. \tag{22}$$

reducing to, for $\tau = 0$,

$$x_{k+1} = A^{\delta}x_k + B^{\delta}u_k \tag{23}$$

with $A^s = e^{sA}$, $B^s = \int_0^s e^{A\mu}\mathrm{d}\mu B$ and $A^{\delta-\sigma}B^{\sigma} + B^{\delta-\sigma} = B^{\delta}$.

Remark 7: Assumption $\mathbf{A_L}$ is necessary and sufficient to guarantee that the delay-free sampled-data couple (A^{δ}, B^{δ}) is controllable almost everywhere [23]. This can be relaxed by only requiring stabilizability of the couple (A, B) without affecting our result.

Accordingly, Theorem 2 specifies as follows.

Corollary 1: Consider the LTI system (21) under Assumption $\mathbf{A_L}$. Then,

$$y_k = x_k + A^{-\sigma}B^{\sigma}u_{k-N-1} + \sum_{j=k-N}^{k-1} A^{(k-N-j-1)\delta-\sigma}B^{\delta}u_j \tag{24}$$

is a reduction for (22) evolving according to the dynamics

$$y_{k+1} = A^{\delta}y_k + A^{-\tau}B^{\delta}u_k. \tag{25}$$

From Theorem 1, it turns out that, whenever (25) is controllable, one can compute a control $u_k = F^{\delta}y_k$ so that $A^{\delta}+A^{-\tau}B^{\delta}F^{\delta}$ is Schur and, as a consequence, (21) is S-GAS in closed-loop. As a consequence, the problem of stabilizing the retarded system is brought back to assigning the eigenvalues of the reduced model.

In the following, it is shown that controllability of the delay-free continuous-time system ensures controllability (almost everywhere) of (25).

Proposition 3: Consider the LTI system (21) under Assumption $\mathbf{A_L}$ and introduce the reduction (24) with dynamics (25). Then, (25) is controllable almost everywhere and any feedback $u_k = F^{\delta}y_k$ such that $A^{\delta} + A^{-\tau}B^{\delta}F^{\delta}$ is Schur ensures that (22) (resp., (21)) is GAS (resp., S-GAS).

Proof: One has to show that (25) is controllable. By computing the controllability matrix $\mathscr{R}(A^{\delta}, A^{-\tau}B^{\delta})$, one can easily verify that $\mathscr{R}(A^{\delta}, A^{-\tau}B^{\delta}) = A^{-\tau}\mathscr{R}(A^{\delta}, B^{\delta})$ where $\mathscr{R}(A^{\delta}, B^{\delta})$ denotes the nonsingular controllability matrix of the delay-free system (23). Thus, one can compute a control $u_k = F^{\delta}y_k$ so that $A^{\delta} + A^{-\tau}B^{\delta}F^{\delta}$ is Schur. In order to guarantee asymptotic stability of (22), introduce the auxiliary states $v = \mathrm{col}(v^1, \dots, v^{N+1})$ with $v_k^i = u_{k-N+i-2}$ for $i = 1, \dots, N+1$ and consider the extended (x, v, y)-dynamics under $u_k = F^{\delta}y_k$

$$\begin{pmatrix} x_{k+1} \\ v_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & A_{12} & A^{\delta} \\ \mathbf{0} & \hat{A} & \hat{B}F^{\delta} \\ \mathbf{0} & \mathbf{0} & A^{\delta} + A^{-\tau}B^{\delta}F^{\delta} \end{pmatrix} \begin{pmatrix} x_k \\ v_k \\ y_k \end{pmatrix}$$

with

$$A_{12} = \begin{pmatrix} 0 & -A^{-\sigma}B^{\sigma} & -A^{-(\delta+\sigma)}B^{\delta} & \dots & -A^{-(N-1)\delta-\sigma}B^{\delta} \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} \mathbf{0} & \mathrm{I}_{\mathrm{N}\times\mathrm{N}} \\ 0 & \mathbf{0} \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}.$$

It is clear that the overall dynamical matrix is Schur so proving the result. ∎

VI. The van der Pol Oscillator

Consider the case of the van der Pol oscillator whose dynamics is provided by

$$\dot{x}_1 = x_2 - x_2^2u(t - \tau), \quad \dot{x}_2 = u(t - \tau) \tag{26}$$

with $x = \mathrm{col}(x_1, x_2)$, $\tau = \delta + \sigma$ and sampled-data equivalent model described in [14] and [15]. Accordingly, the sampled-data reduction state $y = \mathrm{col}(y_1, y_2)$ gets the form

$$\begin{aligned} y_1 = x_1 &- \frac{\sigma^3}{3}u_{k-2}^3 - \sigma(x_2 + \delta u_{k-1} + \sigma u_{k-2}) + \delta(x_2 + \sigma u_{k-2} \\ &- u_{k-1}(x_2 + \sigma u_{k-2})^2) - \sigma(u_{k-2}x_2^2 - x_2) - \frac{\delta^3}{3}u_{k-1}^3 \\ &- \frac{\sigma^2}{2}u_{k-2}(2u_{k-2}x_2 - 1) \\ &- \frac{\delta^2}{2}u_{k-1}(2u_{k-1}(x_2 + \sigma u_{k-2}) - 1) \\ y_2 = x_2 &+ \sigma u_{k-2} + (\delta - \sigma)u_{k-1} \end{aligned}$$

so evolving according to

$$\begin{aligned} y_{1k+1} &= y_1 + \delta(y_1 - y_1^2u - (\sigma + \delta)u) \\ &\quad + \frac{\delta^2}{2}(1 - 2y_2u)u - \frac{\delta^3}{3}u^3 \\ y_{2k+1} &= y_2 + \delta u. \end{aligned}$$

For feedback design, it was shown in [15] that (26) verifies Assumption $\mathbf{A}$ with $\gamma(x) = -3x_1 - \frac{x_1^3}{3} - x_2$ and Lyapunov function $V(x) = x_1^2 + \frac{x_1^4}{3} + x_1x_2 + \frac{1}{2}x_2^2$. Accordingly, the result in Theorem 3 applies and one can compute the resulting feedback $u_k = K^{\delta}(\delta + \sigma, y_k)$.

Partial simulations are reported in Figure 1 providing an interesting comparison of the closed-loop performances yielded by the approximate reduction-based (RB) and prediction-based (PB, [15]) feedback laws. In particular, the approximate control law $u_k = K^{\delta[2]}(\sigma, y)$ of Theorem 3 has been applied. Although further simulations show that both strategies behave similarly for small δ, prediction-based control yields degrading performances (1) as δ and σ increase. Moreover, further simulations underline that the evolutions of the Lyapunov function under reduction-based feedback are

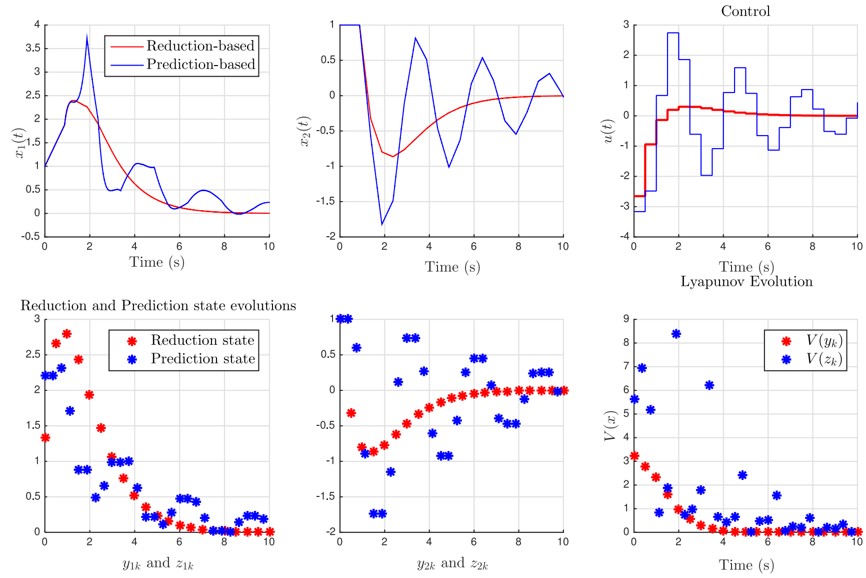

Fig. 1. $\delta = 0.5$ s, $N = 1$ and $\sigma = 0, 4$ s.

decreasing, at the sampling instants, even for higher values of the sampling period.

VII. Conclusion

This letter introduces a sampled-data reduction approach for stabilizing nonlinear dynamics affected by non-entire input delay as a generalization of the prediction-based methodologies presented in [14] and [15]. Further investigations will address robustness with respect to variations of the delay length and extensions to more general classes of time-delayed systems. Finally work is in progress toward nonlinear time-delay discrete-time dynamics.

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Reduction of discrete-time two-channel delayed systems

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Abstract— **In this paper, the reduction method is extended to time-delay systems affected by two mismatched input delays. To this end, the intrinsic feedback structure of the retarded dynamics is exploited to deduce a reduced dynamics which is free of delays. Moreover, among other possibilities, an Immersion and Invariance feedback over the reduced dynamics is designed for achieving stabilization of the original dynamics. A chained sampled-data dynamics is used to show the effectiveness of the proposed control strategy through simulations.**

Index Terms— **Delay systems, Sampled-data control, Stability of nonlinear systems**

I. INTRODUCTION

Nonlinear discrete-time dynamics with input delays exhibit a strict feedback form over an extended state space [1], [2], [3], [4], [5], [6], [7]. Taking advantage of this peculiar cascade structure, several stabilizing controllers have been proposed. In [8], Immersion and Invariance (I&I) based controllers have been designed for discrete-time nonlinear dynamics with input delays while improving prediction-based methodologies. Those invariance-based strategies are generally easier to deduce than predictor-based ones. Indeed, the necessity of computing a prediction of the state trajectories over the delayed window is replaced by requiring convergence to some suitably shaped set over which the closed-loop dynamics recovers the ideal delay-free one.

Those methodologies apply to sampled-data dynamics as well when affected by a constant input delay. In this scenario, the continuous-time dynamics is controlled through piecewise constant input signals while measures of the state are available only at the sampling instants. Accordingly, the stabilizing sampled-data feedback can be designed over an extended discrete-time equivalent dynamics which exhibits a strict-feedback form, too. In this scenario, predictor and I&I-based control laws are discussed and compared in [9], [10]. Truncated expansions in powers of the sampling period δ are also proposed to approximate the exact solutions which are difficult to compute in practice.

A more recent approach concerns reduction based methods aimed at reducing the input delayed dynamics to a delay free one that is equivalent (from the point of view of stability) to the original one [11]. Because reduction implicitly relies on prediction, stabilization of the reduced dynamics implies stabilization of the delayed one. However, an interesting feature of reduction stands in the simplification of the design because the reduced dynamics is by construction delay-free. Moreover, contrarily to prediction dynamics, the reduced model is not a delay-free copy of the system dynamics but differs in the controlled vector fields that come to be explicitly parametrized by the delay-length so leaving space for a further redesign.

Up to now, the discussion has been referring to single-input dynamics though extensions to the case of multiple inputs is straightforward whenever the input channels are uniformly delayed (i.e., affected by the same delay) as developed in [12]. In continuous time, predictor-based techniques for multi-input linear time-invariant systems affected by distinct input delays have been proposed in [13], [14], [15] with extensions to nonlinear dynamics in [16].

The aim of this paper is to address this problem in the nonlinear context when considering dynamics affected by two distinct input delays. The contribution relies upon the possibility of extending the reduction method [12] to this class of dynamics by taking advantage of the feedback structure underlying the evolutions of the retarded system. First, a state augmentation is used to make the delayed dynamics uniform in the action of the delays (i.e., the extended system is affected by the same delay); then, a modified reduction variable is exhibited so to transform the input delayed dynamics into a delay free one over the extended state-space. Finally, among the other possibilities, an I&I design procedure is worked out for stabilizing the reduced dynamics. The cascade structure allows to conclude that stabilization of the reduced dynamics implies stabilization of the input-delayed one.

The paper is organized as follows. In Section II, recalls on the discrete-time reduction method are provided when the inputs are affected by the same delay. The case of two-channel time delays systems is studied in Section III by exhibiting a reduced dynamics which is free of delays. An I&I-based design procedure over the reduced model is then presented in Section IV. In Section V, the case of a chained dynamics is considered as a case of study while Section VI concludes the paper.

Notations: $\mathbf{0}_{i\times j}$ denotes the $i\times j$-dimensional matrix whose entries are zero, I_N stands for the N dimensional identity matrix while $\mathbf{1}_j$ the column vector whose entries are all ones. Maps and vector fields are assumed smooth. Given $i,j\in\mathbb{N}$ such that $j<i$, $u_{[k-i,k-j[}$denotes the history of the discrete variable u over the window $[k-i,k-j[$ (i.e., $u_{[k-i,k-j[}=\{u(k-i),\dots,u(k-j-1)\}$). The symbol "$\circ$"

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denotes the composition of functions. Given a vector field f, L_f denotes the Lie derivative operator, $\mathrm{L}_f = \sum_{i=1}^n f_i(\cdot)\nabla_{x_i}$ with $\nabla_{x_i} := \frac{\partial}{\partial x_i}$ while $\nabla = (\nabla_{x_1},\dots,\nabla_{x_n})$. $e^{\mathrm{L}_f} Id$ denotes the associated Lie series operator, $e^{\mathrm{L}_f} := \mathrm{I} + \sum_{i\geq 1} \frac{\mathrm{L}_f^i}{i!}$

II. Recalls on discrete-time reduction

Consider the nonlinear discrete-time system

$$x(k+1) = F(x(k), u(k-N)) \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, possessing an equilibrium at the origin and affected by a discrete delay $N \geq 0$ uniformly affecting each input channel. Invertibility of the function $F_0(\cdot) = F(\cdot, 0)$ with respect to the state vector x is assumed.

It was proven in [12] that the problem of finding a stabilizer for (1) can be settled toward a new dynamics who is equivalent to the original one as far as stability properties are concerned. For, we introduce the so-called *reduction* variable

$$\eta(k) := F_0^{-N}(\cdot) \circ F^N(x(k), u_{[k-N,k[}) \tag{2}$$

where $u_{[k-N,k[}$ denotes the history of the control signal and

$$F_0^N(x) = \underbrace{F_0(\cdot)\circ\cdots\circ F_0(x)}_{\text{N times}},\ F_0^{-N}(x) = \underbrace{F_0^{-1}(\cdot)\circ\cdots\circ F_0^{-1}(x)}_{\text{N times}}$$

represent the usual N-times composition of the drift term and the corresponding inverse. Composing N steps ahead the full dynamics (1), one computes for $N \geq 1$

$$\begin{aligned} F^N(x(k), u_{[k-N,k[}) :=& F^{N-1}(\cdot, u_{[k-N+1,k[}) \circ F(x(k), u(k-N)) \\ :=& F(\cdot, u(k-1)) \circ \cdots \circ F(x(k), u(k-N)) \end{aligned}$$

with $F^1(x(k), u_{[k-1,k[}) := F(x(k), u(k-1))$. It is a matter of computations to verify that (2) evolves according to the *reduced dynamics*

$$\eta(k+1) = F_r(\eta(k), u(k)) \tag{3}$$

with $F_r(\eta, u) := F_0^{-N}(\cdot) \circ F(\cdot, u) \circ F_0^N(\eta)$.

The reduced dynamics (3) is delay free with the same drift as (1) but modified controlled vector field. More precisely, when assuming $p = 1$ for the sake of simplicity, (3) rewrites as the N-depending dynamics

$$\begin{aligned} \eta(k+1) =& F_0(\eta(k)) \\ &+ \int_0^{u(k)} \nabla_u \big(F_0^{-N}(\cdot) \circ F(\cdot, u) \circ F_0^N(\eta(k))\big)\mathrm{d}u. \end{aligned} \tag{4}$$

Hence, the problem stands in finding a feedback $u(k) = \alpha(\eta(k))$ stabilizing the equilibrium of (3) so getting in turn stabilization of (1) in closed loop as, by construction, $x(k+N) = F_0^N(\eta(k))$. In fact, one gets in closed loop the cascade structure

$$\begin{aligned} x(k+1) =& F_0^N(\eta_1(k)) \\ \eta_1(k+1) =& \eta_2(k), \quad \dots, \quad \eta_{N-1}(k+1) = \eta(k) \\ \eta(k+1) =& F_r(\eta(k), \alpha(\eta(k))) \end{aligned}$$

with $\eta_1(k) = F(x(k), \alpha(F_0^{-N}(x(k))))$.

Several strategies aimed at computing the reduction-based feedback have been discussed in [12] by exploiting on the properties of time-delay system (1) in free evolution.

III. Two-channel time-delay systems

In the sequel we address the problem of stabilizing the time-delay system

$$x(k+1) = F(x(k), u_1(k-N_1), u_2(k-N_2)) \tag{5}$$

whose input channels $u_i \in \mathbb{R}^{p_i}$, $i = 1, 2$ with $p = p_1 + p_2$, are affected by different time delays verifying, after possible index sorting, $N_2 - N_1 = N > 0$; $F_0(\cdot) := F(\cdot, 0, 0)$ is assumed to be invertible over $\mathbb{R}^n$.

The design we propose is based on three steps: first, we introduce a dynamics extension over the control u_1 so to compensate the mismatch among the two input delays; then, we extend the reduction method as recalled in Section II over the extended dynamics; finally, we design one possible reduction-based feedback by carrying out an I&I design.

Remark 3.1: The presented results apply to sampled-data systems affected by entire delays; namely, continuous-time dynamics of the form

$$\dot{x}(t) = f(x(t)) + g_1(x(t))u_1(t-\tau_1) + g_2(x(t))u_2(t-\tau_2)$$

with $u_i(t) = u_i(k\delta) = u_i(k)$ for $t \in [k\delta, (k+1)\delta[$ and $\tau_i = N_i\delta$ for some $N_i \in \mathbb{N}$ $(i = 1, 2)$. Then, for $x(k) = x(k\delta)$, the discrete-time equivalent model gets the δ-dependent form

$$\begin{aligned} x(k+1) =& F^\delta(x(k), u_1(k-N_1), u_2(k-N_2)) \\ =& e^{\delta(\mathrm{L}_f + \mathrm{L}_{g_1 u_1(k-N_1)} + \mathrm{L}_{g_2 u_2(k-N_2)})} x\big|_{x(k)}. \end{aligned}$$

A. The dynamical extension

Let us introduce the new state $\xi := (\xi_1, \dots, \xi_N)^\top \in \mathbb{R}^{p_2 N}$ (with $N = N_2 - N_1$ being the mismatch between the two delays) evolving as the linear dynamics

$$\xi(k+1) = A\xi(k) + Bu_2(k)$$

with $\xi_i(k) = u_2(k-N+i-1)$ for $i = 1, \dots, N$ and

$$A = \begin{pmatrix} \mathbf{0}_{p_2(N-1)\times p_2} & \mathrm{I}_{p_2(N-1)} \\ \mathbf{0}_{p_2\times p_2} & \mathbf{0}^\top_{p_2(N-1)\times p_2} \end{pmatrix}, \quad B = \begin{pmatrix} \mathbf{0}_{p_2(N-1)\times p_2} \\ \mathrm{I}_{p_2} \end{pmatrix}.$$

Accordingly, the extended delayed system exhibits a cascade structure affected by both state and input delays of the same length N_1; i.e., when setting $\xi_1(k-N_1) = u_2(k-N_2)$

$$x(k+1) = F(x(k), u_1(k-N_1), \xi_1(k-N_1)) \tag{6a}$$

$$\xi(k+1) = A\xi(k) + Bu_2(k) \tag{6b}$$

with $\xi_N(k+1) = \xi_1(k+N) = u_2(k)$ for $k \geq 0$.

B. The reduced dynamics

Because of the cascade structure of (6), we introduce the extended reduction variable $\eta_e^\top = (\eta^\top,\ \xi^\top)$ composed of two components: the usual one defined for the x-dynamics (6a) over the N_1-steps delay; a mere copy of the state extension ξ. Accordingly, one gets

$$\begin{aligned} \eta(k) :=& F_0^{-N_1}(\cdot) \circ F^{N_1}(x(k), u_{1[k-N_1,k[}, \xi_{1[k-N_1,k[}) \\ =& F_0^{-N_1}(\cdot) \circ F(\cdot, u_1(k-1), \xi_1(k-1)) \circ \dots \\ & \circ F(x(k), u_1(k-N_1), \xi_1(k-N_1)). \end{aligned}$$

By construction, the η_e-dynamics is delay free with respect to the control variables $u = (u_1, u_2)$. One computes the *extended reduced dynamics* as

$$\eta(k+1) = F_r(\eta(k), u_1(k), \xi_1(k)) \tag{8a}$$
$$\xi(k+1) = A\xi(k) + Bu_2(k) \tag{8b}$$

with $F_r(\eta, u_1, \xi_1) = F_0^{-N_1}(\cdot) \circ F(\cdot, u_1, \xi_1) \circ F_0^{N_1}(\eta)$ and a copy of (6b) which is free of delays itself. Moreover, (8) exhibits a cascade structure with connection variable ξ_1 and unchanged drift term $F_r(\eta, 0, 0) = F_0(\eta)$. The following result can be thus given while the proof is omitted as it follows the lines of [12] by exploiting the cascade structure of (8) when suitably interconnected to the original dynamics (5) in closed loop.

Theorem 3.1: Consider the two-channel input delayed dynamics (5) with invertible drift term $F_0(\cdot)$. Any feedback $u = (\alpha_1(\eta, \xi), \alpha_2(\eta, \xi))$ achieving Global Asymptotic Stability (GAS) of the equilibrium of the reduced model (8) ensures GAS of the equilibrium of (5).

The cascade structure is the core of the stabilizing design over the reduced dynamics we shall present in Section IV among other possibilities.

Remark 3.2: As an alternative reduction design, one might introduce an artificial delay over the less retarded input channel u_1 so to directly compensate the delay mismatch and then apply the standard methodology in [12]. However, this approach induces a dynamical feedback over u_1.

C. An alternative differential/difference representation

Assuming for the sake of simplicity $p_1 = p_2 = 1$ and following [17], one can equivalently describe the dynamics (8) via the so-called (F_0, G)-representation. Denoting by $\eta_e^+(u_1, u_2)$ any curve in $\mathbb{R}^{n+N}$ parametrized by (u_1, u_2), an equivalent representation of (8) is provided through two coupled difference-differential equations over $\mathbb{R}^{n+N}$ as

$$\eta^+ = F_0(\eta), \quad \eta^+ := \eta^+(0,0) \tag{9a}$$
$$\xi^+ = A\xi, \quad \xi^+ := \xi^+(0,0) \tag{9b}$$
$$\frac{\partial \eta^+(u_1,u_2)}{\partial u_1} = G_1(\eta^+(u_1,u_2), \xi_1^+(u_1,u_2), u_1, u_2) \tag{9c}$$
$$\frac{\partial \xi^+(u_1,u_2)}{\partial u_1} = 0 \tag{9d}$$
$$\frac{\partial \eta^+(u_1,u_2)}{\partial u_2} = 0, \quad \frac{\partial \xi^+(u_1,u_2)}{\partial u_2} = B \tag{9e}$$

with $G_1(\eta, \xi_1, u_1, u_2)$ being a vector field over $\mathbb{R}^{n+N}$, parameterized by (u_1, u_2) and verifying [1]

$$G_1(F_r(\eta, u_1, \xi_1), \xi_2, u_1, u_2) := \nabla_{u_1} F_r(\eta, \xi_1, u_1). \tag{10}$$

Thus, for any $(k, \eta_e(k), u_1(k), u_2(k))$, one recovers (8) by integrating (9c)-(9d) over the interval $[0, u_1(k)[$ and (9e) over the interval $[0, u_2(k)[$ and initial condition (9a)-(9b) with $\eta_e = \eta_e(k)$; i.e. $\eta_e(k+1) = \eta_e^+(u_1(k), u_2(k))$ with

$$\eta_e(k+1) = \eta_e^+(0,0) + \int_0^{u_1(k)} G_{e1}(\eta_e^+(u_1,0), u_1, 0)\mathrm{d}u_1$$
$$+ \int_0^{u_2(k)} G_{e2}(\eta_e^+(u_1(k), u_2), u_1(k), u_2)\mathrm{d}u_2 \tag{11}$$

with $G_{e1} = (G_1, 0)$, $G_{e2} = (0, B)$.

Remark 3.3: The integral form (11) rewrites as (see [17])

$$\eta_e(k+1) = \eta_e^+(0,0)$$
$$+ \int_0^{u_1(k)} G_{e1}(\eta_e^+(u_1, u_2(k)), u_1, u_2(k))\mathrm{d}u_1$$
$$+ \int_0^{u_2(k)} G_{e2}(\eta_e^+(0, u_2), 0, u_2)\mathrm{d}u_2$$

because by definition the vector fields $G_{e1}(\eta_e, u_1, u_2)$ and $G_{e2}(\eta_e, u_1, u_2)$ verify the so called compatibility conditions

$$\nabla_{u_1} G_{e2}(\cdot, u_1, u_2) - \nabla_{u_2} G_{e1}(\cdot, u_1, u_2)$$
$$= \left[G_{e1}(\cdot, u_1, u_2), G_{e2}(\cdot, u_1, u_2)\right]$$

with $\left[G_{e1}, G_{e2}\right] = (\nabla_{\eta_e} G_{e2})G_{e1} - (\nabla_{\eta_e} G_{e1})G_{e2}$.

IV. Stabilization of the extended reduced dynamics-an I&I approach

Hereinafter, we discuss the design of a stabilizing controller for the reduced dynamics (8) by assuming the existence of a stabilizing feedback when there is no mismatch in the delays acting over the input channels of (5) (i.e., when $N = N_2 - N_1 = 0$).

Assumption 4.1 (Uniform delay): When $N = N_2 - N_1 = 0$, there exists a feedback $u_1 = \gamma_1(\eta)$, $u_2 = \gamma_2(\eta)$ which makes the origin a GAS equilibrium for the "ideal" reduced-dynamics

$$\eta(k+1) = F_r(\eta(k), u_1(k), u_2(k)) \tag{12}$$

computed over the uniformly delayed system (1).

Remark 4.1: Assumption 4.1 can be inferred from the stabilizability of the delay-free dynamics associated to (5). For further details the reader is referred to [12].

In the following, we denote $\gamma(\cdot) = (\gamma_1(\cdot), \gamma_2(\cdot))$. Under Assumption 4.1, the existence of a stabilizing feedback over the multi-delayed dynamics (5) can be proved by defining an I&I feedback over the extended reduced model (8). For this purpose, the I&I design over the dynamics (8) proceeds along the steps sketched below.

Target dynamics - One deduces the target dynamics over $\mathbb{R}^n$ from Assumption 4.1 so getting

$$\zeta(k+1) = F_r(\zeta(k), \gamma_1(\zeta(k)), \gamma_2(\zeta(k))) \tag{13}$$

which possesses a GAS equilibrium at the origin.

Immersion mapping - The immersion mapping $\pi : \mathbb{R}^n \to \mathbb{R}^{n+p_2N}$ is defined as

$$\pi(\zeta) = \begin{pmatrix} \zeta(k) \\ \pi_2(\zeta(k)) \end{pmatrix} = \begin{pmatrix} \zeta(k) \\ \gamma_2(\zeta(k)) \\ \gamma_2(\zeta(k+1)) \\ \dots \\ \gamma_2(\zeta(k+N-1)) \end{pmatrix}$$

[1] Because $F_0(\cdot)$ admits an inverse, then $F_r(\cdot, u_1, u_2)$ is smooth enough and admits a inverse $F_r^{-1}(\cdot, u_1, u_2)$ for $(u_1, u_2) \in \mathbb{R}^2$ sufficiently small.

where, for $i = 1,\dots,N$

$$\zeta(k+i) = \underbrace{F_r(\cdot,\gamma(\cdot))\circ\cdots\circ F_r(\zeta(k),\gamma(\zeta(k)))}_{i\text{ times}}.$$

The on-the set feedback is thus given by $c(\zeta(k)) = (c_1(\zeta(k)), c_2(\zeta(k))) = (\gamma_1(\zeta(k)), \gamma_2(\zeta(k+N)))$ so that the following invariance condition is verified

$$\begin{pmatrix} F_r(\zeta(k), c_1(\zeta(k)), \gamma_2(\zeta(k)) \\ A\pi_2(\zeta(k)) + Bc_2(\zeta(k)) \end{pmatrix} = \pi(F_r(\zeta(k),\gamma(\zeta(k)))).$$

Invariant set - The invariant set is described as the null set of the mapping $\phi(\eta,\xi): \mathbb{R}^{n+p_2N} \to \mathbb{R}^{p_2N}$ with $\phi(\eta,\xi) = \mathrm{col}\{\phi_1(\eta,\xi),\dots,\phi_N(\eta,\xi)\}$ and for any $i = 1,\dots,N$

$$\phi_i(\eta(k),\xi(k)) = \xi_i(k) - \gamma_2(\eta(k+i))$$

with for $i = 1,\dots,N$ and $u_1 = \gamma_1(\eta)$

$$\eta(k+i) = F_r(\cdot,\gamma_1(\cdot),\xi_i(k))\circ\cdots\circ F_r(\eta(k),\gamma_1(\eta(k)),\xi_1(k))$$

i.e., one sets

$$\mathscr{M} = \{(\eta,\xi)\in\mathbb{R}^n\times\mathbb{R}^{p_2N} \text{ s.t. } \phi_i(\eta,\xi) = 0 \text{ for } i = 1,\dots,N\}.$$

Accordingly, the off-the-set component is defined over $\mathbb{R}^{p_2N}$ as $z = \mathrm{col}(z_1,\dots,z_N)$ with $z_i = \phi_i(\eta,\xi)$ for $i = 1,\dots,N$.

The following result can now be enhanced by showing that Assumption 4.1 is sufficient to infer I&I stabilizability of the extended reduced dynamics (8). The proof is omitted as it follows the lines of [8].

Proposition 4.1: Under Assumption 4.1, any feedback $\psi(\eta_e,z): \mathbb{R}^{n+p_2N}\times\mathbb{R}^{p_2N}\to\mathbb{R}^p$ making the trajectories of the closed-loop system

$$\begin{aligned} z(k+1) =& Az(k) + B\psi_2(\eta_e(k),z(k)) \\ \eta(k+1) =& F_r(\eta(k),\psi_1(\eta_e(k),z(k)),\xi_1(k)) \\ \xi(k+1) =& A\xi(k) + B\psi_2(\eta_e(k),z(k)) \end{aligned}$$

bounded for all $k\ge 0$ with $\lim_{k\to\infty} z(k) = 0$ and $\psi(\pi(\zeta),0) = c(\zeta)$ ensures that the reduced dynamics (8) is I&I stabilizable. Accordingly, the equilibrium of

$$\eta(k+1) = F_r(\eta(k),\psi_1(\eta_e(k),\phi(\eta_e(k))),\xi_1(k)) \tag{15a}$$
$$\xi(k+1) = A\xi(k) + B\psi_2(\eta_e(k),\phi(\eta_e(k))) \tag{15b}$$

is GAS in closed loop.
The I&I stabilizing feedback is given in the theorem below.

Theorem 4.1: Let the system (5) verify Assumption 4.1. Then, the reduced model (8) is I&I stabilizable with target dynamics (13) under the I&I feedback $u = \psi(\eta_e,z)$

$$u_1(k) = \psi_1(\eta_e(k),z(k)) = \gamma_1(\eta(k)) \tag{16a}$$
$$u_2(k) = \psi_2(\eta_e(k),z(k)) = \ell z(k) + \gamma_2(\eta(k+N)) \tag{16b}$$

with ℓ making $A+B\ell$ Schur and verifying $\psi(\pi(\zeta),0) = c(\zeta)$.

Proof: The proof follows the lines of the main result in [9]. It is a matter of computations to verify that by construction of the immersion mapping, invariance of the closed-loop dynamics is ensured by the choice $c_1(\zeta(k)) = \gamma_1(\zeta(k))$, $c_2(\zeta(k)) = \gamma_2(\zeta(k+N))$. Thus, the associated set is thus feedback invariant and the overall design aims at making it attractive while ensuring boundedness of the extended dynamics

$$\begin{aligned} z(k+1) =& Az(k) + B(u_2(k) - \gamma_2(\eta(k+N))) \\ \eta(k+1) =& F_r(\eta(k),u_1(\eta(k)),\gamma_2(\eta(k))) + z_1(k)) \\ =& F_r(\eta(k),u_1(k),\gamma_2(\eta(k))) + \mathscr{F}(\eta(k),u_1(k),z_1(k))) \\ \xi(k+1) =& A\xi(k) + Bu_2(k) \end{aligned}$$

with

$$\mathscr{F}(\eta,u_1,z_1) := \sum_{i=1}^{p_2}\int_0^{z_1^i} \nabla_{v_i} F_r(\eta,u_1,\mathbf{c}^i\gamma_2(\eta) + \mathbf{v}^i)\mathrm{d}v_i$$

with $z_1 = \mathrm{col}(z_1^1,\dots,z_1^{p_2})$, $\mathbf{c}^i = (\mathbf{1}^\top_{p_2-i+1}\ \ \mathbf{0}_{1\times i-1})$ and $\mathbf{v}^i = (z_1^1,\dots,z_1^{i-1},v_i,\mathbf{0}_{1\times i-1})$. As a result, I&I stability is ensured by any feedback of the form (16) making $A+B\ell$ Schur. ■

Remark 4.2: Contrarily to classical prediction methods, the feedback (16) requires the computation of the trajectories of the reduced dynamics (8) over N steps ahead by also minimizing the prediction horizon.

V. A CHAINED DYNAMICS AS AN EXAMPLE

As an example consider the chained dynamics [18]

$$\begin{aligned} &\dot{x}_1(t) = x_2(t),\ \dot{x}_2(t) = u_1(t-\tau_1),\ \dot{x}_3(t) = x_5(t) \\ &\dot{x}_4(t) = x_6(t),\ \dot{x}_5(t) = u_2(t-\tau_2) \\ &\dot{x}_6(t) = -x_3(t)(1+u_1(t-\tau_1)) \end{aligned}$$

and let the control be piecewise constant over time intervals of length δ (the sampling period) with respective delays $\tau_i = N_i\delta, N_i\in\mathbb{N}$ for $i = 1,2$.

Remark 5.1: The above system might represent the dynamics provided after suitable coordinates change and feedback as described in [18].

Setting $x = \mathrm{col}(x_1,\dots,x_6)$ and $x(k) := x(k\delta)$, one exactly computes the sampled-data equivalent model as

$$\begin{aligned} x(k+1) =& A^\delta x(k) + B_0^\delta(u_1(k-N_1),u_2(k-N_2)) \\ &+ B_1^\delta(u_1(k-N_1))x(k) \end{aligned} \tag{18}$$

with

$$A^\delta = \begin{pmatrix} 1 & \delta & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \delta & 0 \\ 0 & 0 & -\frac{\delta^2}{2} & 1 & -\frac{\delta^3}{6} & \delta \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\delta & 0 & -\frac{\delta^2}{2} & 1 \end{pmatrix}$$

$$B_0^\delta(u_1,u_2) = \begin{pmatrix} \frac{\delta^2 u_1}{2} & \delta u_1 & \frac{\delta^2 u_2}{2} & \frac{-\delta^4(1+u_1)u_2}{24} & \delta u_2 & \frac{-\delta^3(1+u_1)u_2}{6} \end{pmatrix}^\top$$

$$B_1^\delta(u_1) = \begin{pmatrix} & & \mathbf{0}_{3\times 6} & & & \\ 0 & 0 & -\frac{\delta^2}{2}u_1 & 0 & -\frac{\delta^3}{6}u_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta u_1 & 0 & -\frac{\delta^2}{2}u_1 & 0 \end{pmatrix}.$$

A. A reduced feedback for the uniformly delayed case

Assuming $N_1 = N_2 = 1$, we want to solve a steering problem for the chained dynamics. Basically, we aim at defining a feedback law so that the full state $x(k)$ reaches a desired value $x_d = (x_d^1,\ 0,\ 0,\ x_d^2,\ 0,\ 0)^\top$ in exactly one step over δ (deadbeat). For this purpose, we first rewrite the error dynamics as $\varepsilon(k) = x(k) - x_d$ and compute the corresponding dynamics

$$\begin{aligned}\varepsilon(k+1) =& A^\delta[\varepsilon(k)+x_d] + B_0^\delta(u_1(k-1),u_2(k-1)) \\ &+ B_1^\delta(u_1(k-1))[\varepsilon(k)+x_d] - x_d\end{aligned} \tag{19}$$

possessing an equilibrium at the origin to be stabilized. Accordingly, we apply our procedure to stabilize (19). By noticing that $B_1^\delta(u_1)x_d = 0$, one defines the reduction as

$$\begin{aligned}\eta(k) :=& \varepsilon(k) + A^{-\delta}B_0^\delta(u_1(k-1),u_2(k-1)) \\ &+ A^{-\delta}B_1^\delta(u_1(k-1))\varepsilon(k)\end{aligned}$$

and the corresponding reduced dynamics as

$$\begin{aligned}\eta(k+1) =& A^\delta\eta(k) + A^{-\delta}B_0^\delta(u_1(k),u_2(k)) \\ &+ A^{-\delta}B_1^\delta(u_1(k))A^\delta\eta(k).\end{aligned} \tag{20}$$

As far as control design is concerned, we first build the feedback stabilizing (20) in the uniformly delayed case (i.e., when $N_1 = N_2$) so to guarantee the requirements in Assumption 4.1. For this purpose, we set up a multi-rate strategy of orders $m_1 = 2$ and $m_2 = 4$ over, respectively, u_1 and u_2 by setting

$$\begin{aligned}u_1(t) &= u_1^j(k),\ t\in[(k+\frac{j-1}{2})\delta,(k+\frac{j}{2})\delta[,\ \ j=1,2 \\ u_2(t) &= u_2^j(k),\ t\in[(k+\frac{j-1}{4})\delta,(k+\frac{j}{4})\delta[,\ \ j=1,\dots,4.\end{aligned}$$

At any sampling instant $t = k\delta$ by denoting $\bar\delta = \frac{\delta}{4}$ and by dropping the k-argument in the right hand side, the multi-rate reduced model gets the form

$$\begin{aligned}\eta(k+1) =& (A^{\bar\delta}+A^{-\bar\delta}B_1^{\bar\delta}(u_1^2)A^{\bar\delta})^2(A^{\bar\delta}+A^{-\bar\delta}B_1^{\bar\delta}(u_1^1)A^{\bar\delta})^2\eta(k) \\ &+ (A^{\bar\delta}+A^{-\bar\delta}B_1^{\bar\delta}(u_1^2)A^{\bar\delta})^2(I+A^{-\bar\delta}B_1^{\bar\delta}(u_1^1))B_0^{\bar\delta}(u_1^1,u_2^1) \\ &+ (A^{\bar\delta}+A^{-\bar\delta}B_1^{\bar\delta}(u_1^2)A^{\bar\delta})^2B_0^{\bar\delta}(u_1^1,u_2^2) \\ &+ (I+A^{-\bar\delta}B_1^{\bar\delta}(u_1^1))B_0^{\bar\delta}(u_1^2,u_2^3) + A^{-\bar\delta}B_0^{\bar\delta}(u_1^2,u_2^4)\end{aligned}$$

with six control inputs. Accordingly, one computes the feedback $u_1^i(k) = \gamma_1^i(\eta(k))$ and $u_2^j(k) = \gamma_2^j(\eta(k))$ ($i = 1,2$ and $j = 1,\dots,4$) as the unique solution to $\eta(k+1) \equiv 0$ also ensuring global exponential stability of (19) when $N_1 = N_2 = 1$ and, thus, Assumption 4.1.

B. The multichannel case

Assuming now $N_1 = 1$ and $N_2 = 2$, one computes the extended reduced model of the error dynamics under multi-rate sampling as

$$\begin{aligned}\eta(k+1) =& (A^{\bar\delta}+A^{-\bar\delta}B_1^{\bar\delta}(u_1^2)A^{\bar\delta})^2(A^{\bar\delta}+A^{-\bar\delta}B_1^{\bar\delta}(u_1^1)A^{\bar\delta})^2\eta \\ &+ (A^{\bar\delta}+A^{-\bar\delta}B_1^{\bar\delta}(u_1^2)A^{\bar\delta})^2(I+A^{-\bar\delta}B_1^{\bar\delta}(u_1^1))B_0^{\bar\delta}(u_1^1,\xi^1) \\ &+ (A^{\bar\delta}+A^{-\bar\delta}B_1^{\bar\delta}(u_1^2)A^{\bar\delta})^2B_0^{\bar\delta}(u_1^1,\xi^2) \\ &+ (I+A^{-\bar\delta}B_1^{\bar\delta}(u_1^1))B_0^{\bar\delta}(u_1^2,\xi^3) + A^{-\bar\delta}B_0^{\bar\delta}(u_1^2,\xi^4) \\ \xi^j(k+1) =& u_2^j(k), \quad i = 1,2,3,4.\end{aligned}$$

By applying the I&I procedure in Section IV, one computes the off-the-set component as $z^j(k) = \xi^j(k) - \gamma_2^j(\eta(k))$. Accordingly, for $i = 1,2$ and $j = 1,\dots,4$, the final multi-rate feedback gets the form

$$u_1^i(k) = \gamma_1^i(\eta(k)),\ u_2^j(k) = \gamma_2^j(\eta(k+1)) + \ell_j z^j(k), \quad |\ell_j| < 1.$$

C. Simulations

Simulations of the proposed deadbeat maneuver are reported when applying the I&I reduced feedback and setting $\ell_j = 0$ for $j = 1,2,3,4$ and desired final configuration $x_d^\top = (10,0,0,10,0,0)$ when starting from the origin. The red solid lines represent the evolution of the target and the controls when a uniform delay affects all of the input channels, while the blue solid lines represent the actual behavior in the multichannel (MC) case with $N_1 = 1$ and $N_2 = 2$.

The proposed strategy ensures convergence of the dynamical system toward the desired final position in the desired number of steps while ensuring $\eta(k) \equiv 0$ in exactly one step (simulations of this last scenario are omitted for the sake of space). Furthermore, we note that the proposed feedback still ensures stability for larger values of the sampling period while still guaranteeing small control effort.

VI. Conclusions

In this paper, we show how to extend the reduction approach to handle time-delay systems affected by two distinct input-delays. Moreover, we exhibit one among the possible controllers by combining reduction and Immersion and Invariance arguments for achieving stabilization in closed loop. The proposed methodology applies to sampled-data delayed dynamics under entire delays. Future works are toward different directions: sampled-data systems under non-entire delays [11]; a comparison with the continuous-time prediction framework with special emphasis on the cascade like representations provided by transport PDEs [14], [16] with respect to the discrete-time one [9]; the specialization of this methodology to different scenarios where time delays are unavoidable as in networked systems [19].

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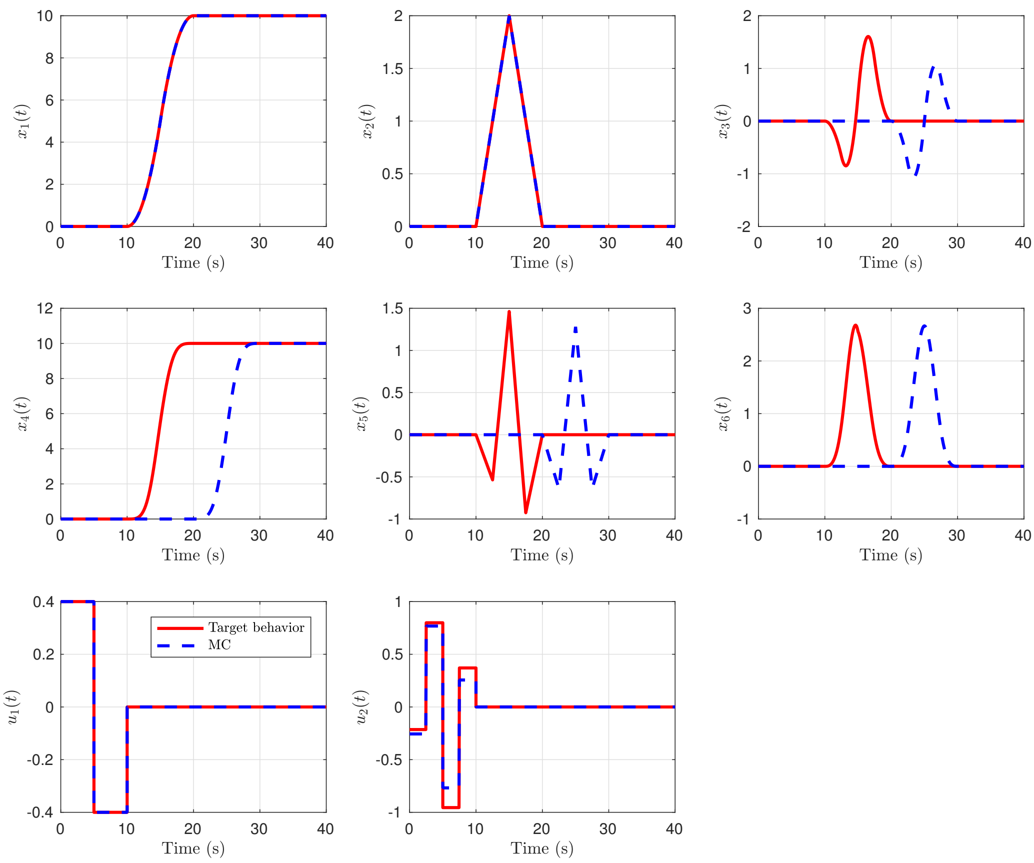

Fig. 1. $\delta = 10$ seconds and $x_d = (10, 0, 0, 10, 0, 0)$

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On partially minimum phase systems and nonlinear sampled-data control

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Abstract— **The concept of partially minimum phase systems is introduced and used with reference to the class of nonlinear systems exhibiting a linear output. It turns out that input-output feedback linearization with stability of the internal dynamics can be pursued via the use of a dummy output with respect to which the system is minimum-phase. The design strategy is extended to multirate sampled-data control and a working example illustrates the performances.**

Index Terms— **Feedback linearization; Nonlinear output feedback; Sampled-data control**

I. INTRODUCTION

A huge number of control strategies is about assigning a target dynamic to a given system. Basically, the concerned design techniques require the inversion of some intrinsic dynamics of the plant that might filter the required behavior ([1], [2], [3], [4], [5], [6]). In the linear case, this corresponds to designing a feedback that assigns part of the eigenvalues coincident with the zeros of the system so making the corresponding dynamics unobservable. In the nonlinear case, similar considerations can be made via the inversion under feedback of the so-called zero-dynamics [7]. It results that the so-defined control will ensure stability in closed loop if and only if the zero-dynamics are asymptotically stable.

Though, the linear case suggests that stability in closed loop can be still pursued under state feedback via partial dynamic cancellation. As a matter of fact, one might design a feedback so to cancel only the stable zeros while leaving the remaining ones unchanged so performing a filtering action that should not compromise the required closed-loop behavior. Based on this idea, we consider non minimum phase nonlinear single-input single-output (SISO) systems that are controllable in first approximation and settle the problem in the context of Input-Output linearization. In that case, because the zero-dynamics are unstable, classical techniques cannot be implemented to solve the problem with stability. Based on the notion of *partially minimum phase* systems, the design we propose proceeds in two steps: considering the linear tangent model (LTM) of the original system, we first define a dummy output based on a suitable factorization of the numerator of its transfer function so that the corresponding linearized system is minimum-phase; then, we perform classical input-output linearization of the locally minimum-phase nonlinear system with the aforementioned dummy output. Finally, we show that when applying the resulting feedback to the original system, input-output linearization still holds with respect to the actual output while guaranteeing stability of the internal dynamics.

The proposed methodology is then applied to the sampled-data context; namely, measures of the output (say the state) are available only at some time instants and the control is piecewise constant over the sampling period. In this context, the problem under study is even more crucial because of the further zero-dynamics intrinsically induced by sampling that are generally unstable [8]. As a consequence, the minimum-phase property of a given nonlinear continuous-time system is not preserved by its sampled-data equivalent ([9], [10], [11], [12]). To overcome those issues, several solutions were proposed based on different sampling procedures ([10], [13], [14], [15], [16], [17]). Among these, the first one was based on multirate sampling in which the control signal is sampled-faster (say r times) than the measured variables. Accordingly, this sampling procedure introduces further degrees of freedom and prevents from the appearance of the unstable sampling zero dynamics while preserving the continuous-time relative degree ([9], [18]). As an alternative, in [13], [16] the authors exploited sampling via generalized hold function (GHF) in order to arbitrarily assign the zero-dynamics of the corresponding sampled-data equivalent system. Though, the relative degree is still not preserved in this case and the GHF method can be seen as a particular case of multirate sampling.

The paper is organized as follows: The problem is settled in Section II and motivated in Section III; the main result is in Section IV and extended to the sampled-data context in Section V. A simulated example is in Section VI. Section VII concludes the paper.

Notation and definitions: All the functions and vector fields defining the dynamics are assumed smooth and complete over the respective definition spaces. M_U (resp. M_U^I) denotes the space of measurable and locally bounded functions $u : \mathbb{R} \to U$ $(u : I \to U,\ I \subset \mathbb{R})$ with $U \subseteq \mathbb{R}$. $\mathscr{U}_\delta \subseteq M_U$ denotes the set of piecewise constant functions over time intervals of fixed length $\delta \in]0,T^*[$; i.e. $\mathscr{U}_\delta = \{u \in M_U \text{ s.t. } u(t) = u_k, \forall t \in [k\delta,(k+1)\delta[; k \geq 0\}$. Given a vector field f, L_f denotes the Lie derivative operator, $\mathrm{L}_f = \sum_{i=1}^n f_i(\cdot)\frac{\partial}{\partial x_i}$. $e^{\mathrm{L}_f}x$ denotes the associated Lie series operator, $e^{\mathrm{L}_f}x := x + \sum_{i\geq 1}\frac{\mathrm{L}_f^i x}{i!}$. A function $R(x,\delta) = O(\delta^p)$ is

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said to be of order δ^p ($p \geq 1$) if whenever it is defined it can be written as $R(x,\delta) = \delta^{p-1}\tilde{R}(x,\delta)$ and there exist function $\theta \in \mathscr{K}_\infty$ and $\delta^* > 0$ s. t. $\forall \delta \leq \delta^*$, $|\tilde{R}(x,\delta)| \leq \theta(\delta)$.

II. Problem settlement

We consider nonlinear feedback linearizable input-affine dynamics with linear output map of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \quad x \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R} \\ y &= Cx \end{aligned} \tag{1}$$

verifying the following assumptions: (1) has relative degree $r \leq n$ and is partially minimum phase[1]; the Linear Tangent Model (LTM) at the origin

$$\begin{aligned} A &= \frac{\partial f}{\partial x}\Big|_0 = \begin{pmatrix} \mathbf{0} & I_{r-1} \\ & -\mathbf{a} \end{pmatrix}, \; B = g(0) = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \\ C &= \begin{pmatrix} b_0 & \dots & b_m & \mathbf{0} \end{pmatrix} \end{aligned} \tag{2}$$

is controllable. $\mathbf{a} = (a_0 \; \dots \; a_{n-1})$ is a row vector containing the coefficients of the associated characteristic polynomial. As a consequence, (2) rewrites

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{3}$$

and has relative degree $\hat{r}$ coinciding, at least locally, with r.

Remark 2.1: If (A,B,C) is not in the canonical controllable form (2), one preliminarily applies to (1) the linear transformation

$$\xi = Tx, \quad T = \begin{pmatrix} \gamma^\top & (\gamma A)^\top & \dots (\gamma A^{n-1})^\top \end{pmatrix}^\top$$

with $\gamma = \begin{pmatrix} \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} B & AB & \dots A^{n-1}B \end{pmatrix}^{-1}$ so transforming the system into the required form.

In this setting, one looks for a continuous-time feedback that ensures input-output linearization of (1) while guaranteeing stability of the internal dynamics. This will be achieved via partial dynamics cancellation. Then, the strategy will be extended to the sampled-data context through multirate sampled-data feedback.

III. Partial zero-dynamics cancellation

Let us start discussing how partial cancellation of the zero dynamics can be used to assign the dynamics under feedback. For, let (3) be the LTM at the origin of (1). Since (A,B) is controllable, the transfer function of the system is provided by

$$W(s) = C(sI - A)^{-1}B = \frac{N(s)}{D(s)}$$

with $N(s) = b_0 + b_1 s + \dots + b_m s^m$ and $D(s) = a_0 + a_1 s + \dots + a_{n-1}s^{n-1} + s^n$ and relative degree $\hat{r} = n - m$.

Given any factorization of the numerator $N(s) = N_1(s)N_2(s)$ and fixed $D(s)$, the dummy output $y_i = C_i x$ with $C_i = (b_0^i \dots b_{m_i}^i \; \mathbf{0})$ corresponds to the transfer function having

$$N_i(s) := b_0^i + b_1^i s + \dots + b_{m_i}^i s^{m_i}$$

[1]Consider a nonminimum phase nonlinear system (1) with LTM model at the origin (3) whose zeros are the roots of a not Hurwitz polynomial N(s); we say that it is *partially minimum phase* there exists a factorization of $N(s) = N_1(s)N_2(s)$ so that $N_2(s)$ is Hurwitz.

($i = 1,2$) as numerator and relative degree $r_i = n - m_i$ ($i = 1,2$). Accordingly, the outputs y, y_1 and y_2 are related by

$$y(t) = N_1(\mathrm{d})y_2(t), \quad y(t) = N_2(\mathrm{d})y_1(t)$$

so getting for $j \neq i$ and $\mathrm{d} = \frac{\mathrm{d}}{\mathrm{d}t}$

$$y(t) = b_0^j y_i + b_1^j \frac{\mathrm{d}}{\mathrm{d}t} y_i + \dots + b_{m_j}^j \frac{\mathrm{d}^{m_j}}{\mathrm{d}t^{m_j}} y_i.$$

Remark 3.1: The feedback

$$u_i = F_i x + v, \quad F_i = -\frac{C_i A^{r_i}}{C_i A^{r_i - 1} B}, \quad i = 1,2$$

transforms (3) into a system with closed-loop transfer function given by

$$\begin{aligned} W^{F_i}(s) &= C(sI - A - BF_i)^{-1}B \\ &= \frac{N_j(s)}{s_i^r} = \frac{b_0 + b_1^j s + .. + b_{m_j}^j s^{m_j}}{s^{r_i}}, \quad j \neq i. \end{aligned}$$

Remark 3.2: It is a matter of computations to verify that the feedback $u = F_i x$ coincides with the one deduced from the Ackermann formula assigning the poles of the system to the roots of $p_i^*(s) = s^{r_i} N_i(s)$. As a consequence, it rewrites $u_i = F_i x$ with $F_i = -\gamma p_i^*(A)$ and $\gamma = \begin{pmatrix} \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}^{-1}$.

The feedback $u = F_i x + v$ places r_i eigenvalues of the system coincident with the zeros of $N_i(s)$ and the remaining ones to 0 so that stabilization in closed loop can be achieved via a further feedback v if and only if $N_i(s)$ is Hurwitz. The previous argument is the core idea of assigning the dynamics of the system via feedback through cancellation of the stable zeros only. Accordingly, if $N(s)$ is not Hurwitz (i.e. $N_j(s)$ has positive real part zeros) the closed-loop system will still have non stable zeros that will play an important role in filtering actions but that will not affect closed-loop stability. Concluding, given any controllable linear system one can pursue stabilization in closed loop via partial zeros cancelation: starting from a suitable factorization of the polynomial defining the zeros, this is achieved via the definition a dummy output with respect to which the system is minimum phase.

IV. Continuous-time feedback linearization of partially minimum phase systems

In what follows, we show how the idea developed in the linear context can be settled in the one of feedback linearization of nonlinear dynamics of the form (1) that are not minimum phase in first approximation.

Lemma 4.1: Consider the nonlinear system (1) and suppose that its LTM at the origin is controllable in the form (2) and non minimum phase with relative degree r. Denote by $N(s) = b_0 + b_1 s + \dots b_{n-r} s^{n-r}$ the not Hurwitz polynomial identifying the zeros of the LTM of (1) at the origin. Consider the maximal factorization of $N(s) = N_1(s)N_2(s)$

$$N_i(s) = b_0^i + b_1^i s + \dots b_{n-r_i}^i s^{n-r_i}, \quad i = 1,2 \tag{4}$$

such that $N_2(s)$ is a Hurwitz polynomial of degree $n - r_2$. Then, the system

$$\dot{x} = f(x) + g(x)u, \quad y_2 = C_2 x. \tag{5}$$

$C_2 = \begin{pmatrix} b_0^2 & b_1^2 & \dots & b_{n-r_2}^2 & \mathbf{0} \end{pmatrix}$ has relative degree r_2 and is locally minimum-phase.

Proof: By computing the linear approximation at the origin of (5), one gets that the matrices (A,B,C_2) are in the form (2) so that the entries of C_2 are the coefficients of $N_2(s)$ that is the numerator of the corresponding transfer function. By construction, $N_2(s)$ is a Hurwitz polynomial of degree $n-r_2$. It follows that, in a nearby of the origin, the relative degree of (5) is r_2. Furthermore, since the linear approximation of the zero-dynamics of (5) coincides with the zero-dynamics of its LTM model at the origin, one gets that (5) is minimum-phase. ∎

Lemma 4.2: Consider the nonlinear system (5) and introduce the normal-form associated to $h_2(x) = C_2 x$

$$\begin{pmatrix} \zeta \\ \eta \end{pmatrix} = \phi(x) = \begin{pmatrix} h_2(x) & \dots & L_f^{r_2-1} h_2(x) & \phi_2^\top(x) \end{pmatrix}^\top \quad (6)$$

with $\phi_2(x)$ such that $L_g \phi_2(x) = 0$ so that

$$\dot{\zeta} = \hat{A}\zeta + \hat{B}(b(\zeta,\eta) + a(\zeta,\eta)u) \quad (7a)$$

$$\dot{\eta} = q(\zeta,\eta) \quad (7b)$$

$$y_2 = \begin{pmatrix} 1 & \mathbf{0} \end{pmatrix} \zeta. \quad (7c)$$

Then, the feedback

$$u = \frac{1}{a(\zeta,\eta)}(v - a(\zeta,\eta)) \quad (8)$$

solves the Input-Output Linearization problem with stable zero-dynamics.

Proof: The proof is straightforward from construction of y_2 in Lemma 4.1. ∎

Remark 4.1: We recall that, in the original coordinates, the feedback (8) rewrites as

$$u = \gamma(x,v) := \frac{v - \mathrm{L}_f^{r_2} h_2(x)}{\mathrm{L}_g \mathrm{L}_f^{r_2-1} h_2(x)}. \quad (9)$$

Remark 4.2: By invoking the arguments in Section III, the original output $y = Cx$ rewrites as $y = N_1(\mathrm{d}) y_2$.

Theorem 4.1: Consider the nonlinear system (1) and suppose that its LTM at the origin is controllable in the form (2) and non minimum phase with relative degree r. Define the dummy output $y_i = h_i(x) = C_i x$ $(i = 1,2)$ as in Lemma 4.1 and the state transformation (6) that puts the system into the form

$$\dot{\zeta} = \hat{A}\zeta + \hat{B}(b(\zeta,\eta) + a(\zeta,\eta)u) \quad (10a)$$

$$\dot{\eta} = q(\zeta,\eta) \quad (10b)$$

$$y = N_1(\mathrm{d}) y_2. \quad (10c)$$

Then, the feedback (8) solves the input-output linearization problem with stability of the internal dynamics.

Proof: From Lemmas 4.1 and 4.2, by expliciting $y = N_1(\mathrm{d}) y_2$ and exploiting (6) one gets

$$y = b_0^1 y_2 + b_1^1 \dot{y}_2 + \dots + b_{r_2-r}^1 y_2^{(r_2-r)} = \begin{pmatrix} C_1 & \mathbf{0} \end{pmatrix} \zeta$$

so that in closed loop (1) rewrites as

$$\dot{\zeta} = \hat{A}\zeta + \hat{B}v \quad (11a)$$

$$\dot{\eta} = q(\zeta,\eta) \quad (11b)$$

$$y = \begin{pmatrix} C_1 & \mathbf{0} \end{pmatrix} \zeta \quad (11c)$$

that exhibits a linear input-output behavior. Moreover, by construction, $y_2 \equiv 0$ implies $y \equiv 0$ so that the restriction of the trajectories of (11) onto the manifold identified by $y \equiv 0$ is described by the dynamics $\dot{\eta} = q(0,\eta)$ that has a locally asymptotically stable equilibrium by construction. Accordingly, when setting $v = F\zeta$ so that $\sigma(\hat{A} + \hat{B}F) \subset \mathbb{C}^-$, the closed-loop system has an asymptotically stable equilibrium at the origin. ∎

The previous result shows that even if a nonlinear system is non-minimum phase, a suitable partition of the output can be performed on its LTM at the origin so that feedback linearization of the input-output behavior can be pursued while preserving stability of the internal dynamics.

Remark 4.3: It is a matter of computations to verify that the LTM model of the closed-loop system (11) has transfer function $W(s) = \frac{N_1(s)}{s^{r_2}}$. Accordingly, one can interpret the nonlinear feedback (8) as the counterpart of the linear feedback presented in Section III; roughly speaking, when applying (8) to the original plant (1), one is inverting only the stable component of the zero-dynamics associated to y. As a consequence, as $y \to 0$, the trajectories of the closed-loop system are constrained onto the stable manifold associated to the dummy output $y_2 = C_2 x$ where they evolve according to $\dot{\eta} = q(0,\eta)$.

V. Feedback linearization of partially minimum phase systems under sampling

We now address the problem of preserving input-output linearization of (1) with stability under sampling by suitably exploiting the result in Theorem 4.1. As recalled in the introduction, the problem cannot be solved via standard (also known as single-rate) sampling procedures. In fact, considering $u(t) \in \mathscr{U}_\delta$ and $y(t) = y(k\delta)$ for $t \in [k\delta, (k+1)\delta[$ (δ the sampling period), the dynamics of (1) at the sampling instants is described by the single-rate sampled-data equivalent model

$$x_{k+1} = F^\delta(x_k, u_k), \quad y_k = h(x_k) \quad (12)$$

with $x_k := x(k\delta)$, $y_k := y(k\delta)$, $u_k := u(k\delta)$, $h(x) = Cx$ and $F^\delta(x_k,u_k) = e^{\delta(\mathrm{L}_f + u_k \mathrm{L}_g)} x\big|_{x_k}$. It is a matter of computations to verify that

$$y_{k+1} = h(x_k) + \sum_{i=1}^{r} \frac{\delta^i}{i!} \mathrm{L}_f^i h(x)\big|_{x_k} + \frac{\delta^r}{r!} u_k \mathrm{L}_g \mathrm{L}_f^r h(x)\big|_{x_k} + O(\delta^{r+1})$$

so that

$$\frac{\partial y_{k+1}}{\partial u_k} = \frac{\delta^r}{r!} \mathrm{L}_g \mathrm{L}_f^r h(x)\big|_{x_k} + O(\delta^{r+1}) \neq 0.$$

Thus, the relative degree of the sampled-data equivalent model of (1) is always falling to $r_d = 1$, despite the continuous-time one. As a consequence, whenever $r > 1$, the sampling process induces a further zero-dynamics of dimension $r-1$ (i.e., the so-called *sampling zero dynamics*, [9]) that is in general unstable for $r>1$. As a consequence, feedback linearization via single-rate sampling cannot be achieved while guaranteeing internal stability.

Multirate sampling enables us to preserve the relative degree and to avoid the appearance of the unstable sampling zero dynamics. Accordingly, one sets $u(t)=u_k^i$ for $t\in[(k+i-1)\delta,(k+i)\delta[$ for $i=1,\ldots,r$ and $y(t)=y_k$ for $t\in[k\delta,(k+1)\delta[$ so that the multirate equivalent model of order r_2 of (1) gets the form

$$x_{k+1}=F_m^{\bar\delta}(x_k,u_k^1,\ldots,u_k^{r_2}) \tag{13}$$

where $\bar\delta=\frac{\delta}{r_2}$ and

$$\begin{aligned}F_m^{\bar\delta}(x_k,u_k^1,\ldots,u_k^{r_2})=&e^{\bar\delta(\mathrm{L}_f+u_k^1\mathrm{L}_g)}\ldots e^{\bar\delta(\mathrm{L}_f+u_k^{r_2}\mathrm{L}_g)}x\big|_{x_k}=\\&F_m^{\bar\delta}(\cdot,u_k^{r_2})\circ\cdots\circ F^{\bar\delta}(x_k,u_k^1).\end{aligned}$$

In the sequel, we show how multirate feedback can be suitably employed with the arguments in Theorem 4.1 to achieve input-output linearization of (1) at the sampling instant $t=k\delta$ $(k\geq 0)$ with stability regardless the minimum-phase property. Accordingly, we first design a multirate feedback $\mathbf{u}_k=\gamma(\bar\delta,x_k,\mathbf{v}_k)$ ($\mathbf{u}=\mathrm{col}(u^1,\ldots,u^{r_2})$ and $\mathbf{v}=\mathrm{col}(v^1,\ldots,v^{r_2})$) so to ensure input/output linearization of the $\mathbf{v}$-y_2 behavior of (5), at the sampling instants. This is achieved by considering the sampled-data dynamics (13) with augmented dummy output $Y_{2k}=H_2(x_k)$ composed of $y_2=C_2x$ and its first r_2-1 derivatives; namely, we consider

$$x_{k+1}=F_m^{\bar\delta}(x_k,u_k^1,\ldots,u_k^{r_2}),\quad Y_{2k}=H_2(x_k) \tag{14}$$

with $\bar\delta=\frac{\delta}{r_2}$ and output vector

$$H_2(x)=\begin{pmatrix}h_2(x) & \mathrm{L}_fh_2(x) & \ldots & \mathrm{L}_f^{r_2-1}h_2(x)\end{pmatrix}^\top$$

that has by construction a vector relative degree $r^\delta=(1,\ldots,1)$.

In this Section we refer to ([19], [18]) where these concepts are introduced and similar manipulations detailed with analog motivations.

At first, we compute the feedback $\mathbf{u}_k=\gamma(\bar\delta,x_k,\mathbf{v}_k)$ so that to reproduce, at the sampling instants $t=k\delta$, the trajectories of the dummy output of (5) and of its first r_2-1 derivatives in closed loop under the continuous-time linearizing feedback (9). The existence of the sampled-data control is stated in the following result.

Lemma 5.1: Consider the nonlinear system (5) under the hypotheses of Lemma 4.2 with multirate equivalent model of order r_2 provided by (14). Then, there exists a unique solution

$$\mathbf{u}^{\bar\delta}=\gamma(\bar\delta,x,\mathbf{v})=(\gamma^1(\bar\delta,x,\mathbf{v})\ \ldots\ \gamma^{r_2}(\bar\delta,x,\mathbf{v}))^\top \tag{15}$$

to the input-output Matching (I-OM) equality

$$\begin{aligned}&H_2(F_m^{\bar\delta}(x_k,\gamma^1(\bar\delta,x_k,\mathbf{v_k}),\ldots,\gamma^{r_2}(\bar\delta,x_k,\mathbf{v_k}))=\\&e^{r_2\bar\delta(\mathrm{L}_f+\gamma(\cdot,v)\mathrm{L}_g)}H_2(x)\big|_{x_k}\end{aligned} \tag{16}$$

for any $x_k=x(k\delta)$ and $v(t)=v(k\delta):=v_k$, $\mathbf{v_k}=(v_k,\ldots,v_k)$. Such a solution is in the form of a series expansion in powers of $\bar\delta$ around the continuous-time $\gamma(x,v)$ in (9); i.e., for $i=1,\ldots,r_2$

$$\gamma^i(\bar\delta,x,\mathbf{v})=\gamma(x,\mathbf{v})+\sum_{j\geq 1}\frac{\bar\delta}{(j+1)!}\gamma_j^i(x,\mathbf{v}). \tag{17}$$

As a consequence, the feedback $\mathbf{u}_k^\delta=\gamma(\bar\delta,x_k,\mathbf{v}_k)$ ensures Input-Output linearization of (14) with stability of the internal dynamics.

Proof: First, we rewrite (16) as a formal series equality in the unknown $\mathbf{u}^{\bar\delta}$; i.e.,

$$\begin{pmatrix}\bar\delta^{r_2}S_1^{\bar\delta}(x,\mathbf{u}^{\bar\delta}) & \ldots & \bar\delta S_1^{\bar\delta}(x,\mathbf{u}^{\bar\delta})\end{pmatrix}^\top \tag{18}$$

with, for $i=1,\ldots,r_2$,

$$\begin{aligned}\bar\delta^iS_i^\delta(x,\mathbf{u}^{\bar\delta})=&e^{\bar\delta(\mathrm{L}_f+u^1\mathrm{L}_g)}\ldots e^{\bar\delta(\mathrm{L}_f+u^1\mathrm{L}_g)}\mathrm{L}_f^{i-1}h_2(x)\\&-e^{r_2\bar\delta(\mathrm{L}_f+\gamma(\cdot,v)\mathrm{L}_g)}\mathrm{L}_f^{i-1}h_2(x).\end{aligned}$$

Thus one looks for $\mathbf{u}=\gamma(\delta,x,v)$ satisfying

$$S^{\bar\delta}(x,\mathbf{u}^{\bar\delta})=\begin{pmatrix}S_1^{\bar\delta}(x,\mathbf{u}^{\bar\delta}) & \ldots & S_1^{\bar\delta}(x,\mathbf{u}^{\bar\delta})\end{pmatrix}^\top=\mathbf{0} \tag{19}$$

where each term rewrites as $S_i^\delta(x,\mathbf{u}^{\bar\delta})=\sum_{s\geq 0}\delta^sS_{ij}(x,\mathbf{u}^{\bar\delta})$ with

$$S_{i0}(x,\mathbf{u}^{\bar\delta})=\Big(\Delta_j\mathbf{u}^{\bar\delta}-r_2^{r_2-i+1}\gamma(x,v)\Big)\mathrm{L}_g\mathrm{L}_f^{r_2-1}h_2(x) \tag{20}$$

and $\frac{\Delta_j}{j!}=(\frac{j^{r_2-j+1}-(j-1)^{r_2-j+1}}{j!}\ \frac{(j-1)^{r_2-j+1}-(j-2)^{r_2-j+1}}{j!}\ \ldots\ \frac{1}{j!})$. It results that $\mathbf{u}^\delta=\gamma(\delta,x,v)=(\gamma(x,v),\ldots,\gamma(x,v))^\top$ solves (19) as $\delta\to 0$. More precisely, as $\delta\to 0$, one gets the equation

$$S^{\bar\delta\to 0}(x,\mathbf{u}^{\bar\delta})=\Big(\Delta\mathbf{u}^{\bar\delta}-D\gamma(x,v)\Big)\mathrm{L}_g\mathrm{L}_f^{r_2-1}h_2(x)$$

with $\Delta=(\Delta_1^\top,\ldots\Delta_{r_2}^\top)^\top$ and $D=\mathrm{diag}(r_2^{r_2},\ldots,r_2)$. Furthemore, the Jacobian of $S^{\bar\delta}$ with respect to $\mathbf{u}^{\bar\delta}$ is

$$\nabla_{\mathbf{u}^{\bar\delta}}S^{\bar\delta}(x,(\gamma(x,v),\ldots,\gamma(x,v))^\top)\big|_{\bar\delta\to 0}=\Delta\ \mathrm{L}_g\mathrm{L}_f^{r_2-1}h_2(x)$$

is full rank by definition of the continuous-time relative degree r_2 and because Δ is invertible (see [10] for details) so concluding, from the Implicit Function Theorem, the existence of $\delta\in]0,T^*[$ so that (16) admits a unique solution of the form (17) around the continuous-time solution $\gamma(x,v)$. Stability of the zero-dynamics is ensured by multirate sampling as proven in [10]. ■

The feedback control is in the form of a series expansion in powers of $\bar\delta$. Thus, iterative procedures can be carried out by substituting (17) into (16) and equating the terms with the same powers of $\bar\delta$ (see [19] where the explicit expression for the first terms are given). Unfortunately, only approximate solutions $\gamma^{[p]}(\bar\delta,x,v)$ can be implemented in practice through truncations of the series (17)) at finite order p in $\bar\delta$; namely, setting $\gamma^{[p]}(\bar\delta,x,v)=(\gamma^{1[p]}(\bar\delta,x,v),\ldots,\gamma^{r_2[p]}(\bar\delta,x,v)$, one gets for $i=1,\ldots,r_2$

$$\gamma^{i[p]}(\bar\delta,x,\mathbf{v})=\gamma(x,\mathbf{v})+\sum_{j=1}^{p}\frac{\bar\delta}{(j+1)!}\gamma_j^i(x,\mathbf{v}). \tag{21}$$

When $p = 0$, one recovers the sample-and-hold (or emulated) solution $\gamma^{i[0]}(\bar{\delta}, x_k, \mathbf{v_k}) = \gamma(x(k\delta), v(k\delta))$. Preservation of performances under approximate solutions has been discussed in [20] by showing that, although global asymptotic stability is lost, input-to-state stability (ISS) and practical global asymptotic stability can be deduced in closed loop even throughout the inter sampling instants.

Similarly to the continuous-time case, the next result shows that applying the feedback (15) to (1) ensures input-output linearization of the input-output behavior at any sampling instant $t = k\delta$ $(k \geq 0)$ while preserving stability of the internal dynamics.

Theorem 5.1: Consider the nonlinear system (1) under the hypotheses of Theorem 4.1 with multirate equivalent model of order r_2 provided by

$$x_{k+1} = F_m^{\bar{\delta}}(x_k, u_k^1, \ldots, u_k^{r_2}), \quad y_k = \begin{pmatrix} C_1 & \mathbf{0} \end{pmatrix} H_2(x_k) \tag{22}$$

and let the feedback (15) be the unique solution to the I-OM equality (16). Then the feedback $\mathbf{u}_k^{\delta} = \gamma(\bar{\delta}, x_k, \mathbf{v}_k)$ ensures Input-Output linearization of (22) with stability of the internal dynamics.

Proof: We first note that y_k rewrites as a linear combination of Y_2. As a consequence, because the $\mathbf{v}$-Y_{2k} behavior is linear under (15), the $\mathbf{v}_k$-y_k is linear by construction. Moreover, we observe that $Y_2 \equiv 0$ implies $y_k \equiv 0$ by definition. Thus, by construction of (15), as $y_k \to 0$, the closed-loop trajectories of (22) are forced onto the zero-manifold defined by $Y_2 \equiv 0$ over which they are asymptotically stable. ∎

Remark 5.1: Denote by z_i^c the zeros of the non Hurwitz polynomial $N_1(s)$ in Lemma 4.1. When considering the LTM model of (22) in closed loop under (15), one gets that, as $\bar{\delta} \to 0$, the closed-loop linearized system has exactly $r_2 - r$ zeros asymptotically approaching to the origin as $e^{\bar{\delta} z_i^c}$ (namely, as $\bar{\delta} \to 0$, $z_i^{\delta} \to e^{\bar{\delta} z_i^c}$, $i = 1, \ldots, r_2$). Accordingly, by applying this result in the linear case, one gets that the feedback (15) is the one that assigns $n - r_2$ poles coincident with the stable zeros, without affecting the unstable ones.

Remark 5.2: Along the lines of the continuous-time case, when controlling (22) via the multirate feedback (15) one is constraining the trajectories of the closed-loop system onto the stable part of the zero-manifold identified by the non-minimum phase output.

Remark 5.3: A purely digital single-rate feedback might be computed over (12) by settling Lemma 4.1 to this context. Assuming, for simplicity, that (1) is locally minimum-phase, one might define a partition of the original output $y_k = Cx_k$ based on the numerator $N^{\delta}(z)$ of transfer function of its LTM at the origin. Accordingly, one might deduce $y_2^{\delta} = C_2^{\delta} x_k$ with respect to which the original dynamics has no sampling zero dynamics and the $y = N(q) y_2^{\delta}$ where q denotes the shift operator and $N(q)$ is the polynomial defining the sampling zeros of the LTM. Though, an exact partition of the original output is hard to be found and only approximate solutions can be found based on the concept of limiting sampling zeros ([8], [16])

VI. The TORA example

An academic working example is proposed on the basis of the TORA system described in [21] (Section 4.4.1, model (4.4.2)). In this context, we consider the fictitious output

$$y = \left(\frac{2}{\varepsilon}(\varepsilon^2 - 1) \;\; 0 \;\; 1 - \varepsilon^2 \;\; 1 - \varepsilon^2\right) x$$

with respect to which the system is non-minimum phase and has relative degree $r = 1$. By applying first the coordinates transformation in Remark 2.1 and following the lines of Section IV, we define the partition $N_1(s) = s - 1$ and $N_2(s) = s^2 + 2s + 1$ so that, in the original coordinates, we define the dummy

$$y_2 = \left(0 \;\; -\frac{2}{\varepsilon}(\varepsilon^2 - 1) \;\; 1 - \varepsilon^2 \;\; 0\right) x$$

with respect to which the system is minimum-phase in first approximation and has relative degree $r_2 = 2$. Accordingly, by applying Theorem 4.1, the feedback (8) with

$$\begin{aligned} \mathrm{L}_g \mathrm{L}_f h_2(x) =& \frac{\varepsilon^2 - 1}{\varepsilon^2 \cos^2(x_3) - 1} \\ \mathrm{L}_f^2 h_2(x) =& \frac{2x_2(\varepsilon^2 - 1)}{\varepsilon} - 2x_4 \cos(x_3)(\varepsilon^2 - 1) + \\ &+ \frac{\varepsilon \cos(x3)(\varepsilon^2 - 1)(x_1 - \varepsilon \sin(x3)(x_4^2 + 1))}{\varepsilon^2 \cos(x_3)^2 - 1} \end{aligned}$$

and $v = -k_1 h_2(x) - k_2 \mathrm{L}_f h_2(x)$ achieves local asymptotic stabilization in closed loop for $k_1, k_2 > 0$.

To solve the problem under sampling, the multirate feedback $\gamma^{[1]}(\delta, x, \mathbf{v})$ in (21) is computed with first corrective terms

$$\gamma_1^1(x, v) = \frac{1}{3}\dot{\gamma}(x, \mathbf{v}), \quad \gamma_1^2(x, v) = \frac{5}{3}\dot{\gamma}(x, v)$$

and $\dot{\gamma}(x, v) = (\mathrm{L}_f + \gamma(x, v)\mathrm{L}_g)\gamma(x, v)$.

Figures 1 and 2 depict simulations of the aforementioned situations under the continuous-time feedback (8) and the approximate sampled-data one (21) with $p = 1$ and for different values of the sampling period. The sample and hold solution is reported as well in a comparative sense. In particular, setting by $\eta = (\eta_1, \; \eta_2, \; \eta_3)^{\top}$, we denote the internal dynamics corresponding to the simulated situations. It is clear from Figure 1 that the continuous-time feedback computed via partial dynamic inversion yields feedback linearization while ensuring asymptotic stability in cosed-loop. Concerning sampled-data control, we note that, as δ increases, the emulated based solution fails in stabilizing (and linearizing the input-output behavior) in closed loop while the presented multirate strategy yields more than acceptable performances even in that case.

VII. Conclusions

The notion of partially minimum-phase systems is used to get feedback input-output linearization while preserving stability. The proposed approach is introduced in continuous time and extended to the sampled-data context through multirate to overcome the well-known pathologies induced by the sampling zero dynamics. The extension to systems

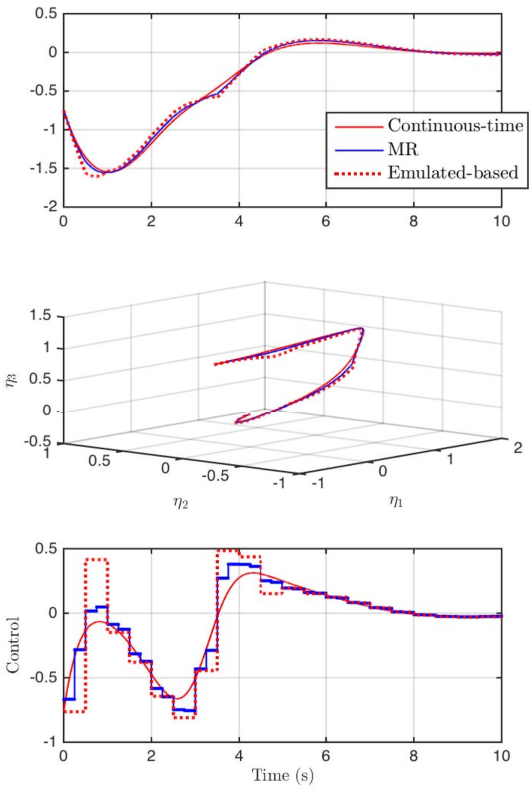

Fig. 1. $\delta = 0.5$ s.

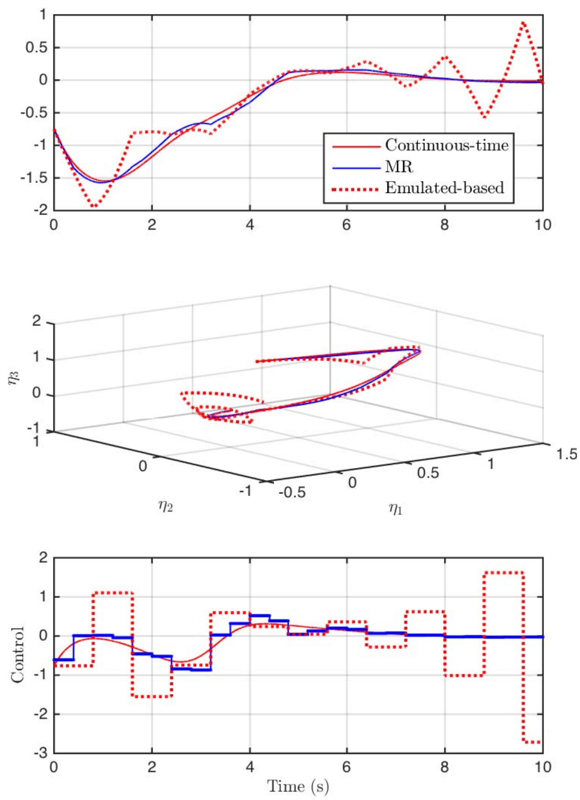

Fig. 2. $\delta = 0.7$ s.

exhibiting a nonlinear output mapping is the objective of further investigations.

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Title: Stabilization of cascaded nonlinear systems under sampling and delays

Abstract: Over the last decades, the methodologies of dynamical systems and control theory have been playing an increasingly relevant role in a lot of situations of practical interest. Though, a lot of theoretical problem still remain unsolved. Among all, the ones concerning stability and stabilization are of paramount importance. In order to stabilize a physical (or not) system, it is necessary to acquire and interpret heterogeneous information on its behavior in order to correctly intervene on it. In general, those information are not available through a continuous flow but are provided in a synchronous or asynchronous way. This issue has to be unavoidably taken into account for the design of the control action. In a very natural way, all those heterogeneities define an hybrid system characterized by both continuous and discrete dynamics. This thesis is contextualized in this framework and aimed at proposing new methodologies for the stabilization of sampled-data nonlinear systems with focus toward the stabilization of cascade dynamics. In doing so, we shall propose a small number of tools for constructing sampled-data feedback laws stabilizing the origin of sampled-data nonlinear systems admitting cascade interconnection representations. To this end, we shall investigate on the effect of sampling on the properties of the continuous-time system while enhancing design procedures requiring no extra assumptions over the sampled-data equivalent model. Finally, we shall show the way sampling positively affects nonlinear retarded dynamics affected by a fixed and known time-delay over the input signal by enforcing on the implicit cascade representation the sampling process induces onto the retarded system.

Keywords: Nonlinear systems, Sampled-data systems, Time-delay systems.

Titre: Stabilisation des systèmes échantillonnés en cascade et avec retards

Résumé: Les méthodologies de l'automatique ont joué au cours des dernières décennies un rôle essentiel au sein de nombreux secteurs technologiques avancés. Cependant, de nombreuse questions restent ouvertes. Parmi celles-ci, celles concernant la stabilité et la stabilisation de systèmes non linéaires sont d'intérêt primordial. Afin de stabilizer un système (physique ou non), il est nécessaire de capter et interpreter en temps réel les informations hétérogènes caractérisant son fonctionnement afin intervenir efficacement. Actuellement ces informations ne sont pas captées en temps continu, mais de façon synchrone ou asynchrone et ceci est valable aussi pour les actuateurs. De façon très naturelle, on définit donc un système hybride, caractérisé par des dynamiques à la fois discrètes et continues. Dans ce contexte, cette thèse est orientée au développement de nouvelles méthodologies pour la stabilisation de systèmes échantillonnés non linéaires en se focalisant sur la stabilisation de formes en cascades qui se retrouvent dans de nombreuse situations concretes. Pour cela, on étudiera l'effet de l'échantillonnage sur les propriétés de la dynamique continue et l'on proposera des méthodologies pour la conception de lois de commande qui ne requièrent pas d'assumptions supplémentaires au cas continu. Enfin, on étudiera l'effet de l'échantillonnage sur des systèmes présentant de retards sur les entrées. On développera des lois de commande stabilisantes exploitant la structure en cascade induite par l'échantillonnage.

Mots clés: Systèmes non linéaires, Systèmes à données échantillonnées, Systèmes à retards.