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Magda Khalile

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Problèmes spectraux avec conditions de Robin sur des domaines à
coins du plan

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Chapitre I

Introduction

I.1 Motivations

Soient un domaine $\Omega \subset \mathbb{R}^d$, $d \geq 2$, et $\gamma > 0$ un paramètre réel. Considérons le problème spectral

$$-\Delta u := -\sum_{j=1}^d \frac{\partial^2 u}{\partial x_j^2} = Eu \text{ sur } \Omega, \quad (\text{I.1.1})$$

$$\frac{\partial u}{\partial \nu} = \gamma u \text{ sur } \partial\Omega, \quad (\text{I.1.2})$$

où ν est la normale unitaire *sortante* de $\partial\Omega$, E est une valeur propre et u est une fonction propre associée. La condition de bord (I.1.2) est appelée condition de Robin. Le travail présenté ici a pour objet l'étude du comportement asymptotique de ces valeurs propres lorsque le paramètre γ est grand et que le domaine Ω admet des *coins*.

L'étude des valeurs propres du Laplacien est un thème classique en théorie spectrale. En effet, cet opérateur apparaît dans la modélisation mathématique de nombreux problèmes physiques comme l'équation de la chaleur, qui décrit l'évolution de la température d'un solide, l'équation de Schrödinger modélisant l'évolution d'une particule non relativiste ou encore l'équation des ondes, décrivant les vibrations d'une membrane. Dans ce dernier cas, il est aisé de mettre en évidence l'interprétation physique des valeurs propres du Laplacien : les petites variations d'une membrane $\Omega \subset \mathbb{R}^2$ sont décrites par l'équation aux dérivées partielles

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0.$$

La fonction $u(t, x)$ décrit le déplacement vertical de la membrane en vibration au temps $t \in \mathbb{R}_+$ et au point $x \in \Omega$. Si l'on cherche une solution particulière de cette équation à variables temporelles et spatiales séparées, $u(t, x) = g(t)f(x)$, on obtient alors le système

$$\begin{aligned} -g''(t) &= Eg(t), \\ -\Delta f(x) &= Ef(x). \end{aligned}$$

Une solution particulière de l'équation des ondes est donc de la forme

$$u(t, x) = \left(A \cos(\sqrt{E}t) + B \sin(\sqrt{E}t) \right) f(x), \quad A, B \in \mathbb{R},$$

et oscille à une fréquence \sqrt{E} . Dans son célèbre livre *The Theory of Sound* [Ray77], Lord Rayleigh conjectura que, lorsque l'aire est fixée, c'est la membrane circulaire qui a la plus

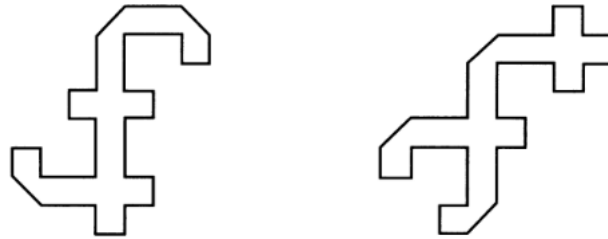


FIGURE I.1 – Domaines isospectraux pour les Laplaciens de Dirichlet et Neumann [GWW92].

basse fréquence. Mathématiquement, ceci veut dire que si nous considérons la première valeur propre $E_1(\Omega)$, appelée fréquence fondamentale, du problème suivant :

$$\begin{aligned} -\Delta u &= Eu \text{ sur } \Omega, \\ u &= 0 \text{ sur } \partial\Omega, \end{aligned}$$

nous devrions obtenir

$$E_1(B) \leq E_1(\Omega), \tag{I.1.3}$$

où l'on note $B \subset \mathbb{R}^2$ le disque de même aire que la membrane Ω . La conjecture de Rayleigh a été le point de départ de ce que nous appelons maintenant les inégalités isopérimétriques. L'inégalité (I.1.3) a été rigoureusement démontrée quelle que soit la dimension et est maintenant connue sous le nom d'inégalité de Faber-Krahn. Dans le système ci-dessus, la condition $u = 0$ sur $\partial\Omega$, appelée condition de bord de Dirichlet, décrit une membrane fixée au bord. Si l'on considère une membrane libre, c'est-à-dire non fixée à son bord, la condition devient $\frac{\partial u}{\partial \nu} = 0$, et est connue sous le nom de condition de bord de Neumann. L'étude des inégalités isopérimétriques a été étendue aux conditions de Neumann et de Robin pour des paramètres γ *négatifs*. Plus précisément, il a été prouvé que parmi les domaines réguliers à volume fixé et quelle que soit la dimension, la boule maximise la deuxième valeur propre, c'est-à-dire la première valeur propre non nulle, du Laplacien de Neumann [Sze54, Wei56], et qu'elle minimise la première valeur propre du Laplacien de Robin avec paramètre négatif [Bos88, Dan06]. Il est intéressant de noter qu'au contraire, la recherche d'optimiseurs à volume fixé de la première valeur propre du Laplacien de Robin à paramètre γ *positif* est un problème ouvert.

En 1966, Kac s'est intéressé au problème inverse : *Can one hear the shape of a drum?* [Kac71]. Autrement dit, si l'on connaît les valeurs propres du Laplacien d'un domaine Ω , peut-on en déduire sa forme? Malheureusement, la réponse est non en toute généralité, comme le montre le premier contre-exemple dans le plan donné par [GWW92], voir Figure I.1. Toutefois, des informations sur la géométrie du domaine peuvent être obtenues grâce à l'étude des valeurs propres du Laplacien. En 1911, Weyl démontra l'asymptotique suivante pour les valeurs propres de Dirichlet :

$$E_k(\Omega) \sim 4\pi^2 \left(\frac{k}{\omega_d |\Omega|} \right)^{\frac{2}{d}}, \text{ lorsque } k \rightarrow +\infty, \tag{I.1.4}$$

où l'on note ω_d le volume de la boule unité de \mathbb{R}^d . Cette asymptotique nous permet en particulier de conclure que deux domaines de volumes différents ne peuvent être isospectraux. Le volume est alors un invariant spectral du Laplacien de Dirichlet. Depuis

lors, des asymptotiques plus précises ont été obtenues pour les valeurs propres de Dirichlet, et la recherche d'invariants spectraux continue de susciter l'attention que ce soit pour le Laplacien avec condition de Dirichlet [LR15, LR16, Uca17], mais aussi pour d'autres conditions de bord comme celle de Neumann ou des conditions mixtes de Dirichlet-Neumann [HLR17, JLNP06, LPP06], et d'autres problèmes spectraux comme celui de Steklov [GPPS14, PS15]. L'étude de l'asymptotique des valeurs propres de Dirichlet a également donné naissance à la conjecture de Pólya. On dit qu'un domaine pave le plan si le plan peut être recouvert par ce même domaine, sans trou ni chevauchement, en autorisant les rotations, les translations et les réflexions. En 1961, Pólya établit l'inégalité suivante pour tout domaine $\Omega \subset \mathbb{R}^2$ pavant le plan :

$$E_k(\Omega) \geq \frac{4\pi k}{|\Omega|}, \text{ pour tout } k \in \mathbb{N}.$$

Il conjectura que cette inégalité reste vérifiée pour tout domaine borné de \mathbb{R}^d , ce qui est équivalent à dire que l'asymptotique de Weyl (I.1.4) minore les valeurs propres. A ce jour, ce problème reste ouvert.

Nous nous intéressons dans cette thèse à la condition de bord de Robin (I.1.2). Elle peut être considérée comme la condition de bord la plus générale, dans le sens où elle contient celle de Neumann lorsque $\gamma = 0$ et celle de Dirichlet lorsque $\gamma \rightarrow -\infty$. Le problème aux valeurs propres de Robin (I.1.1)-(I.1.2) apparaît dans de nombreuses applications comme l'estimation de la température critique des matériaux supraconducteurs [GS07] ou encore l'étude des équations de réaction-diffusion en temps longs [LOS98]. Comme le montre le récent article de synthèse [BFK17], la condition de Robin répulsive ($\gamma < 0$) a été grandement étudiée alors que la condition de Robin attractive ($\gamma > 0$) n'a attiré l'attention que plus récemment.

Dans un premier temps, définissons rigoureusement l'opérateur Laplacien de Robin. Notre intérêt se porte uniquement sur la condition de Robin *attractive*, et dans la suite le paramètre de Robin γ est strictement positif : $\gamma > 0$. Considérons un domaine $\Omega \subset \mathbb{R}^d$, $d \geq 2$, et la forme sesquilinéaire

$$q_\Omega^\gamma(u, u) = \int_\Omega |\nabla u|^2 dx - \gamma \int_{\partial\Omega} |u|^2 ds, \quad u \in H^1(\Omega),$$

où nous notons ds la mesure de Hausdorff $(d - 1)$ -dimensionnelle. Supposons que les conditions suivantes sont satisfaites :

- l'opérateur de trace $H^1(\Omega) \ni u \mapsto u|_{\partial\Omega} \in L^2(\partial\Omega)$ est bien défini,
- il existe une constante $K > 0$ telle que l'inégalité

$$\int_{\partial\Omega} |u|^2 ds \leq K \left(\epsilon \int_\Omega |\nabla u|^2 dx + \epsilon^{-1} \int_\Omega |u|^2 dx \right) \quad (\text{I.1.5})$$

est satisfaite quels que soient $u \in H^1(\Omega)$ et $\epsilon \in (0, 1)$.

Sous ces conditions, nous pouvons montrer que la forme sesquilinéaire q_Ω^γ est bornée inférieurement : il existe une constante $c > 0$ telle que, pour tout $u \in H^1(\Omega)$,

$$q_\Omega^\gamma(u, u) \geq -c \|u\|_{L^2(\Omega)}^2,$$

et que la norme induite par la forme q_Ω^γ et définie par le produit scalaire $\langle u, v \rangle_{q_\Omega^\gamma} := q_\Omega^\gamma(u, v) + (c + 1)\langle u, v \rangle$ est équivalente à la norme H^1 : il existe $c_1, c_2 > 0$ telles que

$$c_1 \|u\|_{H^1(\Omega)}^2 \leq q_\Omega^\gamma(u, u) + (c + 1) \|u\|_{L^2(\Omega)}^2 \leq c_2 \|u\|_{H^1(\Omega)}^2.$$

Lorsque ces deux propriétés sont vérifiées, la forme est dite *fermée*. D'après les résultats standards sur les formes sesquilinéaires, voir e.g. [RS80, Theorem VIII. 15], la forme q_Ω^γ définit un unique opérateur auto-adjoint dans $L^2(\Omega)$ noté Q_Ω^γ .

Soit $\Omega \subset \mathbb{R}^d$ un domaine borné et lipschitzien. Alors, l'opérateur de trace est bien défini [Gri85, Theorem 1.5.1.3], et l'inégalité (II.1.5) est satisfaite [Gri85, Theorem 1.5.1.10]. Il en résulte que la forme q_Ω^γ est fermée pour tout domaine borné lipschitzien. Comme nous le verrons plus tard, les secteurs infinis du plan vont jouer un rôle particulier dans notre étude, et nous pouvons déjà mentionner que la forme q_Ω^γ est fermée lorsque Ω est un secteur infini. Une preuve complète est donnée en Section IV.1. Enfin, ajoutons que la fermeture de q_Ω^γ a également été étudiée pour certains domaines non lipschitziens, et en particulier pour des domaines avec des pointes [Dan13, KP, NT13], pour lesquels l'opérateur de trace est bien défini si les pointes ne sont pas trop 'pointues'.

Dans toute la suite, Ω est un domaine lipschitzien borné. Grâce à une intégration par parties, nous pouvons montrer que l'opérateur Q_Ω^γ agit comme $u \mapsto -\Delta u$ sur le domaine

$$D(Q_\Omega^\gamma) := \left\{ u \in H^1(\Omega) : \Delta u \in L^2(\Omega), \frac{\partial u}{\partial \nu} = \gamma u \text{ sur } \partial\Omega \right\}.$$

L'injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ étant compacte [Gri85, Section 1.4.4], l'opérateur Q_Ω^γ est à résolvante compacte et donc, voir e.g. [RS78, Theorem XIII.64], son spectre est constitué d'une suite de valeurs propres

$$E_1(Q_\Omega^\gamma) \leq E_2(Q_\Omega^\gamma) \leq \dots \leq E_n(Q_\Omega^\gamma) \leq \dots \rightarrow +\infty,$$

où chaque valeur propre est répétée selon sa multiplicité finie, et les fonctions propres associées forment une base orthonormale de $L^2(\Omega)$. La caractérisation variationnelle des valeurs propres par les principes du max-min et du min-max, voir le Théorème III.1.3 ci-dessous, appliquée à Q_Ω^γ nous donne, pour tout $n \in \mathbb{N} := \{1, 2, \dots\}$,

$$E_n(Q_\Omega^\gamma) := \sup_{\psi_1, \dots, \psi_{n-1} \in L^2(\Omega)} \inf_{\substack{u \in H^1(\Omega), u \neq 0 \\ u \perp \psi_j, j=1, \dots, n-1}} \frac{q_\Omega^\gamma(u, u)}{\langle u, u \rangle} \equiv \inf_{\substack{G \subset H^1(\Omega) \\ \dim G = n}} \sup_{\substack{u \in G \\ u \neq 0}} \frac{q_\Omega^\gamma(u, u)}{\langle u, u \rangle}.$$

Quelques propriétés satisfaites par les valeurs propres sont immédiatement déduites de cette caractérisation. Chaque valeur propre, vue comme une fonction de $\gamma \in \mathbb{R}_+$, est décroissante. La décroissance est en réalité stricte en γ , comme le montre le papier [Roh14], dans lequel un résultat plus général est obtenu pour des paramètres de Robin qui sont des fonctions définies sur le bord, i.e. $\partial\Omega \ni s \mapsto \gamma(s)$. De plus, en dilatant le domaine par γ nous obtenons

$$E_n(Q_\Omega^\gamma) = \gamma^2 E_n(Q_{\gamma\Omega}^1), \text{ pour tout } n \in \mathbb{N},$$

où nous utilisons la notation $\gamma\Omega := \{\gamma x, x \in \Omega\}$. Il est important de remarquer qu'à cause de la condition de bord, les valeurs propres ne sont pas monotones par rapport au domaine. Nous pouvons également prouver le comportement asymptotique suivant lorsque γ devient grand.

Proposition I.1.1. *Soit $\Omega \subset \mathbb{R}^d$ un domaine borné lipschitzien. Pour tout $n \in \mathbb{N}$ fixé nous avons :*

$$E_n(Q_\Omega^\gamma) \rightarrow -\infty \text{ lorsque } \gamma \rightarrow +\infty.$$

Démonstration. Fixons $n \in \mathbb{N}$, et considérons n fonctions $\varphi_1, \dots, \varphi_n \in H^{\frac{1}{2}}(\partial\Omega)$. Comme l'opérateur de trace $H^1(\Omega) \ni u \mapsto u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)$ est surjectif [Gri85, Theorem 1.5.1.3], il existe $\psi_1, \dots, \psi_n \in H^1(\Omega)$ telles que pour tout $j = 1, \dots, n$ nous avons $(\psi_j)|_{\partial\Omega} = \varphi_j$. Alors,

par linéarité de la trace les fonctions ψ_1, \dots, ψ_n sont linéairement indépendantes. Soient $\psi \in \text{span}\{\psi_1, \dots, \psi_n\}$. Il existe $(c_1, \dots, c_n) \in \mathbb{C}^n$ tel que $\psi := \sum_{j=1}^n c_j \psi_j$. Notons

$$M^+ := \max_{j=1, \dots, n} \|\psi_j\|_{H^1(\Omega)}^2, \quad M^- := \min_{j=1, \dots, n} \|\psi_j\|_{L^2(\Omega)}^2, \quad m^- := \min_{j=1, \dots, n} \|\varphi_j\|_{L^2(\partial\Omega)}^2.$$

En utilisant l'inégalité de Cauchy-Schwarz nous obtenons

$$\|\nabla\psi\|_{L^2(\Omega)}^2 \leq (1 + 2n)M^+ \sum_{j=1}^n |c_j|^2,$$

et en utilisant le fait que les ψ_j sont linéairement indépendantes et que les φ_j le sont également, nous avons

$$\|\psi\|_{L^2(\partial\Omega)}^2 \geq m^- \sum_{j=1}^n |c_j|^2, \quad M^- \sum_{j=1}^n |c_j|^2 \leq \|\psi\|_{L^2(\Omega)}^2 \leq M^+ \sum_{j=1}^n |c_j|^2.$$

Le principe du min-max nous donne tout d'abord,

$$E_n(Q_\Omega^\gamma) \leq \sup_{\substack{(c_1, \dots, c_n) \in \mathbb{C}^n \\ (c_1, \dots, c_n) \neq (0, \dots, 0)}} \frac{q_\Omega^\gamma(\psi, \psi)}{\|\psi\|_{L^2(\Omega)}^2},$$

et en appliquant les estimées précédentes nous pouvons conclure :

$$E_n(Q_\Omega^\gamma) \leq \frac{(1 + 2n)M^+}{M^-} - \frac{m^-}{M^+} \gamma \rightarrow -\infty \text{ lorsque } \gamma \rightarrow +\infty.$$

□

Enfin, en utilisant l'inégalité (I.1.5) nous pouvons montrer que les valeurs propres ne décroissent pas plus vite que $-\gamma^2$.

Proposition I.1.2. *Soit $\Omega \subset \mathbb{R}^d$ un domaine borné lipschitzien. Alors, il existe une constante $C_\Omega > 0$ telle que, lorsque γ est grand,*

$$E_1(Q_\Omega^\gamma) \geq -C_\Omega \gamma^2.$$

L'opérateur de Robin unidimensionnel agissant dans $L^2(\mathbb{R}_+)$ sera important pour notre étude. L'opérateur $Q_{\mathbb{R}_+}^\gamma$ admet une unique valeur propre discrète $-\gamma^2$ associée à la fonction propre $e^{-\gamma t}$, voir Section III.5.1. Cette dernière est en particulier utilisée comme une fonction test afin de prouver l'inégalité suivante obtenue par [GS07] :

Proposition I.1.3. *Soit $\Omega \subset \mathbb{R}^d$ un domaine borné lipschitzien. Alors, pour tout $\gamma > 0$ nous avons :*

$$E_1(Q_\Omega^\gamma) \leq -\gamma^2.$$

Démonstration. Sans perte de généralité, nous pouvons supposer que $\Omega \subset \{x \in \mathbb{R}^d : x_1 > 0\}$. Soit $u(x) := e^{-\gamma x_1}$. Alors, $\int_\Omega |\nabla u|^2 dx = \gamma^2 \int_\Omega |u|^2 dx$. De plus, en appliquant le théorème de divergence au champ de vecteurs $F(x) := (e^{-2\gamma x_1}, 0, \dots, 0)$ nous obtenons

$$\int_{\partial\Omega} |u|^2 ds \geq \int_{\partial\Omega} F \nu ds = - \int_\Omega \text{div } F dx = 2\gamma \int_\Omega |u|^2 dx.$$

Le principe du min-max nous permet de conclure :

$$E_1(Q_\Omega^\gamma) \leq \frac{\int_\Omega |\nabla u|^2 - \gamma \int_{\partial\Omega} |u|^2 ds}{\int_\Omega |u|^2 dx} \leq -\gamma^2.$$

□

L'intérêt porté à l'étude de l'asymptotique des valeurs propres de Robin quand $\gamma \rightarrow +\infty$ s'est accru ces dernières années. Il semblerait que le premier article mentionnant ce problème ait été [LOS98]. Il se base sur l'observation suivante : si l'on note $B \subset \mathbb{R}^d$ la boule unité, alors l'asymptotique de la première valeur propre est donnée par

$$E_1(Q_B^\gamma) = -\gamma^2 - (d-1)\gamma + o(\gamma^2), \text{ lorsque } \gamma \rightarrow +\infty.$$

Ce résultat s'obtient en étudiant le comportement asymptotique de la fonction propre associée qui est une fonction de Bessel modifiée de paramètre $\frac{d-2}{2}$ [AS64]. Dans l'article [LOS98], il est prouvé que

$$E_1(Q_\Omega^\gamma) \sim -\gamma^2, \text{ lorsque } \gamma \rightarrow +\infty, \tag{I.1.6}$$

lorsque Ω est un domaine borné de classe C^2 dont le bord est difféomorphe à la sphère. Ces hypothèses peuvent être affaiblies, et en particulier [LZ04] ont prouvé que (I.1.6) reste vraie pour des domaines de classe C^1 . Plus tard, il a été démontré que le comportement asymptotique de chaque valeur propre est le même au premier ordre :

Théorème I.1.4 ([DK10]). *Soit $\Omega \subset \mathbb{R}^d$ un domaine borné de classe C^1 . Alors, pour tout $n \in \mathbb{N}$ fixé nous avons :*

$$E_n(Q_\Omega^\gamma) = -\gamma^2 + o(\gamma^2), \text{ lorsque } \gamma \rightarrow +\infty.$$

La preuve de [DK10] est purement variationnelle et s'appuie sur la construction de fonctions tests et du principe du min-max afin d'obtenir la majoration $E_n(Q_\Omega^\gamma) \leq -\gamma^2 + o(1)$ pour tout $n \in \mathbb{N}$ lorsque γ est grand. Cette estimée est suffisante pour conclure puisque nous avons également $E_1(Q_\Omega^\gamma) \sim -\gamma^2$, qui fournit la minoration attendue.

Le premier ordre dans l'asymptotique des valeurs propres ne dépend ni du volume du domaine, ni de la dimension de l'espace ambiant, ce qui peut être surprenant au premier abord. Toutefois, une simple observation peut expliquer en partie ce phénomène. Considérons un domaine $\Omega \subset \mathbb{R}^d$ borné et de classe C^2 . Pour $\delta > 0$, nous définissons $\Omega_\delta := \{x \in \Omega : d(x, \partial\Omega) < \delta\}$. Soient $Q_\delta^{\gamma, D/N}$ les opérateurs agissant dans $L^2(\Omega_\delta)$ comme le Laplacien avec la condition de Robin avec paramètre γ sur $\partial\Omega$ et la condition de Dirichlet/Neumann sur la partie restante du bord $\partial\Omega_\delta \setminus \partial\Omega$. Alors, grâce au principe du min-max nous obtenons

$$E_n(Q_\delta^{\gamma, N} \oplus Q^N) \leq E_n(Q_\Omega^\gamma) \leq E_n(Q_\delta^{\gamma, D} \oplus Q^D),$$

où $Q^{D/N}$ sont les Laplaciens de Neumann/Dirichlet agissant dans $L^2(\Omega \setminus \overline{\Omega_\delta})$, et ces opérateurs sont en particulier positifs. Cette technique est appelée *Dirichlet-Neumann bracketing*. Pour γ assez grand nous avons $E_n(Q_\delta^{\gamma, D/N}) < 0$, ce qui implique $E_n(Q_\delta^{\gamma, D/N} \oplus Q^{D/N}) = E_n(Q_\delta^{\gamma, D/N})$. Par conséquent, pour tout $n \in \mathbb{N}$ fixé et pour γ assez grand nous avons :

$$E_n(Q_\delta^{\gamma, N}) \leq E_n(Q_\Omega^\gamma) \leq E_n(Q_\delta^{\gamma, D}),$$

et ainsi l'étude est réduite à un voisinage du bord. En outre, il a également été prouvé pour des domaines du plan dans [HK17] que les fonctions propres associées sont localisées dans un voisinage du bord, et plus précisément près des points de courbure maximale.

Ces précédentes observations nous poussent alors à étudier les termes suivants de l'asymptotique des valeurs propres, dans lesquels les propriétés géométriques du bord, et plus particulièrement la courbure, apparaissent. Le premier article sur ce sujet a été [Pan13], dans lequel les deux premiers termes de l'asymptotique de $E_1(Q_\Omega^\gamma)$ sont obtenus

pour des domaines $\Omega \subset \mathbb{R}^2$ réguliers par morceaux. Puis il a été prouvé, dans [EMP14] pour $d = 2$ et dans [PP15] pour tout $d \geq 2$, que pour tout domaine borné $\Omega \subset \mathbb{R}^d$ de classe C^3 nous avons :

$$E_n(Q_\Omega^\gamma) = -\gamma^2 - (d-1)\gamma H_{\max} + \mathcal{O}(\gamma^{\frac{2}{3}}), \text{ lorsque } \gamma \rightarrow +\infty, \quad (\text{I.1.7})$$

où nous notons H_{\max} le maximum de la courbure moyenne de $\partial\Omega$. Le reste peut être remplacé par $\mathcal{O}(\gamma^{\frac{1}{2}})$ pour des domaines de classe C^4 . La preuve est basée sur un Dirichlet-Neumann bracketing afin de réduire l'étude à un voisinage tubulaire bien choisi du bord. L'asymptotique (I.1.7) a également été démontrée pour une certaine classe de domaines non bornés dans [EM14]. Lorsque le domaine est non borné, le spectre essentiel n'est pas vide, puisque l'opérateur Q_Ω^γ n'est plus à résolvante compacte. Ainsi, l'un des enjeux dans ce contexte est de déterminer l'existence ou non de spectre discret. Si le bord est compact et le domaine est non borné, alors $\text{spec}_{\text{ess}}(Q_\Omega^\gamma) = [0, +\infty)$, voir e.g. [KP13], et il existe un nombre fini de valeurs propres discrètes. De plus, pour tout $n \in \mathbb{N}$ fixé on a $E_n(Q_\Omega^\gamma) < 0$ lorsque γ est assez grand. En d'autres termes, chaque $E_n(Q_\Omega^\gamma)$ est une valeur propre discrète lorsque γ est grand. En revanche, ceci n'est plus vrai lorsque le bord n'est pas compact. Des exemples de domaines non bornés pour lesquels le spectre de Q_Ω^γ est purement essentiel sont exhibés dans [EM14], et il est important de noter que la concavité de ces domaines a une influence sur l'absence de valeurs propres. D'autre part, il est aussi prouvé que dans l'asymptotique $\gamma \rightarrow +\infty$ et pour des domaines assez réguliers, la convexité locale du bord peut créer une infinité de valeurs propres sous le seuil du spectre essentiel.

L'asymptotique (I.1.7) peut également être utilisée pour discuter de la validité des inégalités isopérimétriques dans le régime $\gamma \rightarrow +\infty$. Il a été conjecturé par [Bar77] que la boule maximise la première valeur propre $E_1(Q_\Omega^\gamma)$ à volume fixé. Cependant, il a été prouvé par [FK15] que cette conjecture est en réalité fautive pour de grandes valeurs du paramètre γ , et ceci peut facilement être retrouvé grâce à l'asymptotique (I.1.7). Considérons une boule $B \subset \mathbb{R}^d$ de rayon $r > 0$ et un anneau $S_R \subset \mathbb{R}^d$ de rayon extérieur $R > 0$ et de même volume que B . Alors nécessairement $R > r$ et donc $H_{\max}(B) = \frac{1}{r} > H_{\max}(S_R) = \frac{1}{R}$. Nous pouvons alors déduire de l'asymptotique (I.1.7) l'inégalité suivante :

$$E_1(Q_{S_R}^\gamma) > E_1(Q_B^\gamma), \text{ lorsque } \gamma \text{ est grand.}$$

Ce résultat est assez inattendu car il semblerait que ce soit le seul pour lequel la boule n'est pas un optimiseur de la première valeur propre du Laplacien à volume fixé. Il a également été démontré par [FNT16] que la conjecture est fautive pour des domaines de \mathbb{R}^3 difféomorphes à une boule, en prouvant que pour cette classe de domaines nous avons :

$$\inf\{H_{\max}(\Omega), |\Omega| = 1\} = 0.$$

Néanmoins, il existe des cas pour lesquels la conjecture est vérifiée. En particulier, elle est vraie pour des domaines bornés lipschitziens qui sont proches, en un certain sens bien défini, d'une boule comme le montre [FNT15], et également pour des domaines bornés du plan de classe C^2 lorsque γ est assez petit. Dans [PP15], il est aussi prouvé que la boule minimise le maximum de la courbure moyenne parmi les domaines bornés étoilés de classe C^2 à volume fixé, et la conjecture est donc vérifiée pour cette classe de domaines lorsque γ est grand.

L'influence du bord du domaine apparaît clairement dans le second terme de l'asymptotique (I.1.7), mais le numéro de la valeur propre n'est, par contre, pas visible dans les deux premiers termes. De ce fait les valeurs propres sont indistinguables. L'article [HK17] permet de résoudre ce problème en obtenant un développement asymptotique complet de chaque valeur propre lorsque Ω est un domaine du plan. Plus précisément, supposons que

$\Omega \subset \mathbb{R}^2$ est borné et de classe C^∞ , et que le bord $\partial\Omega$ est paramétré par la longueur de l'arc. Notons κ la courbure du bord et κ_{\max} son maximum. La courbure κ doit satisfaire deux conditions cruciales : κ atteint son maximum en un *unique* point $x_0 \in \partial\Omega$ et le maximum est non dégénéré : $\kappa''(x_0) < 0$.

Théorème I.1.5 ([HK17]). *Sous ces conditions, et pour tout $n \in \mathbb{N}$, il existe une suite $(\beta_{j,n})_{j \geq 0}$ de nombres réels telle que pour tout $M \in \mathbb{N}$ nous avons :*

$$E_n(Q_\Omega^\gamma) = -\gamma^2 - \gamma\kappa_{\max} + (2n-1)\sqrt{\frac{-\kappa''(x_0)}{2}}\sqrt{\gamma} + \sum_{j=0}^M \beta_{j,n}\gamma^{-\frac{j}{2}} + o(\gamma^{-\frac{M}{2}}), \text{ lorsque } \gamma \rightarrow +\infty.$$

La localisation des fonctions propres près du point de courbure maximale est un élément fondamental pour prouver cette asymptotique, et est en particulier utilisée pour construire des fonctions tests qui permettent d'obtenir une majoration des valeurs propres grâce au principe du min-max. Les termes $(2n-1)\sqrt{\frac{-\kappa''(x_0)}{2}}$ sont exactement les valeurs propres de l'oscillateur harmonique $-\frac{d^2}{ds^2} - \frac{\kappa''(x_0)}{2}s^2$ dans $L^2(\mathbb{R})$. Dans un certain sens, les premiers termes de l'asymptotique sont donnés par l'approximation harmonique de l'opérateur $-\gamma^2 - \partial_s^2 - \gamma\kappa(s)$ sur le bord, où s est la longueur d'arc, et où le potentiel $-\gamma\kappa(s)$ admet un *unique* point de minimum non dégénéré x_0 par hypothèse. Par analogie avec l'étude des opérateurs de Schrödinger semi-classiques [HS84], la courbure joue le rôle d'un puits de potentiel. Ainsi, un unique point de courbure maximale correspond à un unique puits. Il est alors naturel de se demander ce qu'il adviendrait pour l'asymptotique des valeurs propres si le bord présentait plusieurs puits. Dans [HKR17], le cas particulier d'un domaine régulier du plan admettant un axe de symétrie et exactement deux points de courbures maximales atteints loin de cette axe est étudié. C'est un exemple typique d'effet tunnel provoqué par la présence d'une symétrie du domaine, et le but de ce papier est d'établir une asymptotique pour la différence $E_2(Q_\Omega^\gamma) - E_1(Q_\Omega^\gamma)$ lorsque $\gamma \rightarrow +\infty$.

Dans l'article [PP16], l'influence du bord sur les valeurs propres de Q_Ω^γ est établie quelle que soit la dimension de la manière suivante :

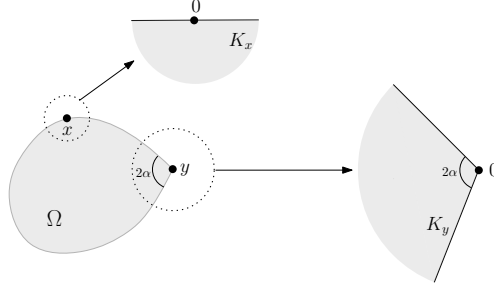
Theorem I.1.6 ([PP16]). *Soit $\Omega \subset \mathbb{R}^d$ un domaine borné de classe C^3 . Alors, pour tout $n \in \mathbb{N}$ fixé nous avons :*

$$E_n(Q_\Omega^\gamma) = -\gamma^2 + E_n(-\Delta_{\partial\Omega} - \gamma K) + \mathcal{O}(1), \text{ lorsque } \gamma \rightarrow +\infty,$$

où nous notons $-\Delta_{\partial\Omega}$ l'opérateur de Laplace-Beltrami dans $L^2(\partial\Omega, ds)$ et $s \mapsto K(s)$ la somme des courbures principales de $\partial\Omega$.

L'asymptotique de $E_n(Q_\Omega^\gamma)$ est donc déterminée par l'opérateur *effectif* $-\Delta_{\partial\Omega} - \gamma K$ sur $\partial\Omega$. Ici, aucune hypothèse n'est faite sur les points de courbures maximales, en comparaison avec le résultat précédant. De plus, des asymptotiques plus précises sont obtenues en étudiant l'opérateur effectif. On peut également ajouter que ce résultat reste valable pour certains domaines non bornés, pour lesquels le spectre discret n'est pas vide.

Comme le soulignent les résultats précédents, l'intérêt porté au Laplacien de Robin agissant sur des domaines lisses a été non négligeable ces dix dernières années, et nous pouvons également citer d'autres travaux en lien avec ce sujet [KP17, KL18, KL]. Cependant, il n'existe qu'un nombre très restreint de résultats portant sur des domaines non lisses, et en particulier des domaines avec des coins. Une simple observation peut nous convaincre que nous nous attendons, dans ce cas là, à un comportement asymptotique différent pour les valeurs propres. Considérons un cube $\mathcal{C}_d \subset \mathbb{R}^d$ d'arêtes de longueur 1. Par séparation de


 FIGURE I.2 – Secteurs tangents du polygone curviligne Ω .

variables, l'étude de $Q_{\mathcal{C}_d}^\gamma$ est en fait réduite à l'étude du Laplacien de Robin unidimensionnel agissant sur un intervalle de longueur 1, et nous obtenons par un simple calcul :

$$E_1(Q_{\mathcal{C}_d}^\gamma) = -d\gamma^2 + o(\gamma^2), \text{ lorsque } \gamma \rightarrow +\infty.$$

Dans les articles [LP08, BP16], l'étude du comportement asymptotique de $E_1(Q_\Omega^\gamma)$ est menée pour une classe de domaines lisses par morceaux admettant des coins à laquelle le cube appartient : les *domaines à coins*. Un cône $K \subset \mathbb{R}^d$ est un domaine lipschitzien invariant par les dilatations positives. Un domaine $\Omega \subset \mathbb{R}^d$ est un *domaine à coins* si son bord est lipschitzien, lisse par morceaux et si pour tout $y \in \partial\Omega$, il existe un cône $K_y \subset \mathbb{R}^d$ et un difféomorphisme régulier F_y tels que

$$F_y : \Omega \cap B(y, r) \rightarrow K_y \cap B(0, r), \quad F_y(\overline{\Omega \cap B(y, r)}) = \overline{K_y \cap B(0, r)}$$

pour $r > 0$ assez petit, $F_y(y) = 0$ et $\nabla F_y(y) = Id$. Le cône K_y est appelé *cône tangent* du bord $\partial\Omega$ en y . Si le point $y \in \partial\Omega$ est un point régulier, alors le cône K_y est simplement un demi-espace. En dimension deux, les cônes tangents sont le plan et les secteurs infinis paramétrés par leur angle de demi-ouverture $\alpha \in (0, \pi)$, et le demi-plan correspond à $\alpha = \frac{\pi}{2}$. Les domaines à coins du plan sont les *polygones curvilignes*. Ils admettent un nombre fini de coins d'ouverture dans $(0, \pi) \cup (\pi, 2\pi)$, ce qui comprend en particulier les domaines lisses, voir Figure I.2. En dimension trois, les domaines à coins admettent une large variété de cônes tangents comme l'espace entier, les demi-espaces, les cônes de révolution, les cônes pyramidaux, ... Prenons l'exemple du cube en dimension 3 : soit $y \in \partial\mathcal{C}_3$. Si y appartient à une face, le cône tangent K_y est alors un demi-espace. Si y appartient à une arête, le cône tangent est alors de la forme $K_y = \mathbb{R} \times U_{\frac{\pi}{4}}$ où $U_{\frac{\pi}{4}}$ est un secteur infini de demi-ouverture $\frac{\pi}{4}$. Enfin, si y est un sommet, le cône tangent est un cône pyramidal de section, c'est-à-dire d'intersection avec la sphère unité, un triangle sphérique équilatéral de longueur $\frac{\pi}{2}$. Nous renvoyons aux manuscrits [Dau88, BNDP16] pour une description précise des domaines à coins et de leurs propriétés. L'opérateur $Q_{K_y}^\gamma$ n'est pas à résolvante compacte, quel que soit le cône $K_y \subset \mathbb{R}^d$, et son spectre essentiel n'est donc pas vide. Nous définissons $\Lambda(K_y, \gamma) := \inf \text{spec}(Q_{K_y}^\gamma)$. L'invariance par dilatations de K_y nous donne $\Lambda(K_y, \gamma) = \gamma^2 \Lambda(K_y, 1)$.

Théorème I.1.7 ([LP08, BP16]). *Soit $\Omega \subset \mathbb{R}^d$ un domaine à coins. Alors,*

$$E_1(Q_\Omega^\gamma) = \left(\inf_{y \in \partial\Omega} \Lambda(K_y, 1) \right) \gamma^2 + o(\gamma^2), \text{ lorsque } \gamma \rightarrow +\infty.$$

Ce résultat nous montre que le premier ordre de l'asymptotique de $E_1(Q_\Omega^\gamma)$ est déterminé par les Laplaciens de Robin agissant sur les cônes tangents au bord, que nous appelons *opérateurs modèles*. Ainsi, une étude approfondie de ces opérateurs modèles nous permettra

d'obtenir des résultats plus précis pour les domaines à coins. En dimension deux, il est prouvé dans [LP08] que si l'on note α la demi-ouverture du secteur infini K_y , alors

$$\Lambda(K_y, 1) = -\frac{1}{\sin^2 \alpha} \text{ si } \alpha < \frac{\pi}{2}, \quad \Lambda(K_y, 1) = 1 \text{ sinon.} \quad (\text{I.1.8})$$

Lorsque $\alpha < \frac{\pi}{2}$, l'infimum du spectre est en réalité une valeur propre discrète associée à la fonction propre $e^{-\frac{x_1}{\sin \alpha}}$, où l'on note $(x_1, x_2) \in \mathbb{R}^2$ les coordonnées cartésiennes dans le plan et Ox_1 coïncide avec la bissectrice du secteur. En dimension trois, des estimations de $\Lambda(K_y, 1)$ sont prouvées dans [LP08] pour des cônes ayant une section régulière bornée et convexe. Nous pouvons également mentionner le cas des cônes circulaires, qui est bien connu : l'infimum du spectre est une valeur propre discrète $-\frac{1}{\sin^2 \alpha}$, où α est le rayon sphérique de la section, et la fonction propre associée est $e^{-\frac{z}{\sin \alpha}}$ où Oz coïncide avec l'axe de révolution du cône, ainsi que celui des cônes dont la section est un polygone sphérique admettant un cercle inscrit, puisque dans ce cas les estimations dans [LP08] deviennent des égalités, voir [LP08, Remark 5.3]. Revenons à l'exemple du cube de \mathbb{R}^3 . Si y appartient à une face, alors le cône tangent est simplement un demi-espace et grâce au fait que $E_1(Q_{\mathbb{R}_+}^1) = -1$, nous avons $\Lambda(K_y, 1) = -1$. Si y appartient à une arête, nous pouvons utiliser (I.1.8) afin d'obtenir $\Lambda(K_y, 1) = -2$. Enfin, si y est un sommet, le cône tangent en y est de section un triangle sphérique équilatéral et admet donc un cercle inscrit. On peut alors utiliser [LP08, Remark 5.3], ce qui nous donne $\Lambda(K_y, 1) = -3$. Grâce au Theorem II.1.7 nous retrouvons l'asymptotique :

$$E_1(Q_{\mathcal{C}_3}^\gamma) = -3\gamma^2 + o(\gamma^2), \text{ lorsque } \gamma \rightarrow +\infty.$$

En outre, l'étude de l'existence du spectre discret pour des cônes a été menée par [Pan16] quelle que soit la dimension, et il s'avère qu'il peut exister une infinité de valeurs propres discrètes. En dimension trois, le résultat se présente de la manière suivante, voir [Pan16, Corollary 9] :

Proposition I.1.8. *Soit $K \subset \mathbb{R}^3$ un cône donc la section est lisse et simplement connexe. Si $\mathbb{R}^3 \setminus K$ est convexe, alors le spectre discret de Q_K^γ est vide, sinon il existe une infinité de valeurs propres discrètes.*

Dans l'article [BPP18], la distribution asymptotique des valeurs propres discrètes pour des cônes en dimension trois a été étudiée. Il est en particulier prouvé que l'asymptotique de Weyl est déterminée par la courbure de la section du cône.

Grâce au Théorème I.1.7 et à la grande variété de singularités au bord existante, nous comprenons qu'une étude approfondie des valeurs propres de Robin doit être menée en fonction de la dimension de l'espace ambiant. Dans tout ce qui suit, nous nous concentrons sur les domaines du plan, i.e. $d = 2$. Revenons au résultat (I.1.8). Si nous considérons un polygone curviligne admettant des coins *convexes*, c'est-à-dire que leur demi-ouverture vérifie $\alpha \in (0, \frac{\pi}{2})$, alors le premier ordre de l'asymptotique de $E_1(Q_\Omega^\gamma)$ est donné par le coin *le plus convexe*. Au contraire, si l'on considère un polygone curviligne qui n'admet que des coins *non convexes*, le premier ordre est donné par $-\gamma^2$, ce qui correspond en particulier au premier ordre des domaines lisses (II.1.7). Ainsi, les coins non convexes n'influencent pas le premier ordre de l'asymptotique de $E_1(Q_\Omega^\gamma)$ lorsque $\gamma \rightarrow +\infty$. De manière informelle et en référence à la célèbre question de Kac, on peut aussi dire qu'il semble que l'on ne peut pas entendre les coins non convexes. Le résultat suivant va dans le sens de cette supposition :

Théorème I.1.9 ([Pan13]). *Soit $\Omega \subset \mathbb{R}^2$ un polygone curviligne de classe C^4 n'admettant que des coins non convexes. Alors,*

$$E_1(Q_\Omega^\gamma) = -\gamma^2 - \gamma\kappa_{\max} + \mathcal{O}(\gamma^{\frac{2}{3}}), \text{ lorsque } \gamma \rightarrow +\infty,$$

où nous notons κ_{\max} le maximum de la courbure de $\partial\Omega$.

La preuve est similaire à celle des domaines lisses. Elle se base sur un Dirichlet-Neumann bracketing afin de réduire l'étude à un voisinage du bord. D'un point de vue technique, il est aisé de comprendre pourquoi nous retrouvons la même asymptotique que dans le cas lisse. En effet, en présence de coins non convexes et lorsque le bord est lisse par morceaux, il est facile de voir que l'on peut construire un voisinage tubulaire du bord, alors qu'en présence de coins convexes ce n'est plus le cas. Dans le papier [Pan15], l'asymptotique de $E_n(Q_\Omega^\gamma)$ pour tout $n \in \mathbb{N}$ est étudiée lorsque $\Omega \subset \mathbb{R}^2$ est l'extérieur d'un polygone convexe. Il s'avère que le premier ordre est également donné par $-\gamma^2$, et le second terme est exactement la n ème valeur propre du Laplacien de Dirichlet agissant sur les arêtes du domaine. Au vu de ces résultats nous pouvons alors nous demander :

Quelle est l'influence des coins convexes sur les propriétés spectrales du Laplacien de Robin ?

L'objectif principal de cette thèse est d'étudier cette question en obtenant des asymptotiques plus précises pour les valeurs propres de Robin sur les polygones curvilignes.

Le calcul explicite des valeurs propres du Laplacien est une tâche difficile de manière générale. Toutefois, nous avons à notre disposition quelques exemples qui vont nous permettre d'avoir une intuition quant au comportement asymptotique des valeurs propres. Premièrement, dans son manuscrit [McC11], McCartin mène une étude complète du Laplacien agissant sur des triangles équilatéraux, et il obtient en particulier des asymptotiques pour les valeurs propres de Robin. Soit $\mathcal{T} \subset \mathbb{R}^2$ un triangle équilatéral. Il est prouvé dans [McC11, Section 7.4] que pour $n = 1, 2, 3$ nous avons :

$$E_n(Q_{\mathcal{T}}^\gamma) = -4\gamma^2 + o(\gamma^2), \text{ lorsque } \gamma \rightarrow +\infty,$$

et

$$E_4(Q_{\mathcal{T}}^\gamma) = -\gamma^2 + o(1), \text{ lorsque } \gamma \rightarrow +\infty.$$

Bien sûr, l'asymptotique de la première valeur propre nous était déjà connue grâce aux résultats de [LP08, BP16]. Il est intéressant de voir que les deux valeurs propres suivantes ont exactement le même premier terme que la première, alors que la quatrième présente le même premier terme que pour les domaines lisses. Deuxièmement, des résultats ont également été obtenus pour les triangles isocèles dans [HP15]. Soit $\mathcal{T}_\theta \subset \mathbb{R}^2$ un triangle isocèle dont les deux angles égaux sont $\theta \in (0, \frac{\pi}{3})$. Alors, il existe une constante $c > 0$ telle que pour $n = 1, 2$ nous avons :

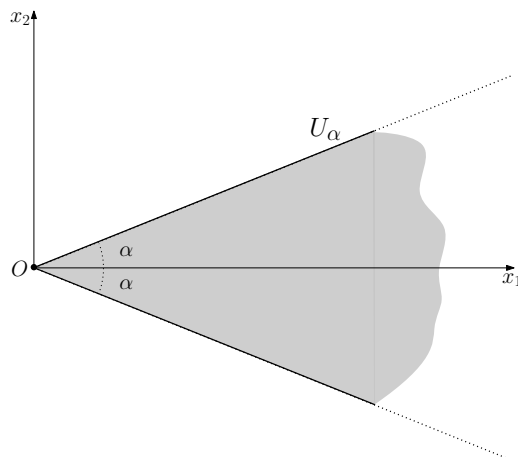
$$E_n(Q_{\mathcal{T}_\theta}^\gamma) = -\frac{\gamma^2}{\sin^2(\theta/2)} + \mathcal{O}(e^{-c\gamma}), \text{ lorsque } \gamma \rightarrow +\infty.$$

Des résultats plus précis sont obtenus par [HP15], et en particulier une asymptotique de la différence $E_2(Q_{\mathcal{T}_\theta}^\gamma) - E_1(Q_{\mathcal{T}_\theta}^\gamma)$ est prouvée. Troisièmement, nous pouvons également mentionner l'exemple du carré. Grâce à la séparation de variables, l'étude est simplement réduite au Laplacien de Robin unidimensionnel agissant sur un intervalle, et nous pouvons donc obtenir des asymptotiques précises pour chaque valeur propre, voir l'Exemple III.5.5 ci-dessous. Notons $\square_\ell \subset \mathbb{R}^2$ le carré de longueur ℓ . Pour $n = 1, 2, 3, 4$ il existe une constante $c > 0$ telle que

$$E_n(Q_{\square_\ell}^\gamma) = -2\gamma^2 + \mathcal{O}(e^{-c\gamma}), \text{ lorsque } \gamma \rightarrow +\infty,$$

et de plus, pour tout $j \in \mathbb{N}$,

$$E_{4+j}(Q_{\square_\ell}^\gamma) = -\gamma^2 + \mu_j + o(1), \text{ lorsque } \gamma \rightarrow +\infty,$$


 FIGURE I.3 – Le secteur infini U_α .

où μ_j est exactement le j ème élément de l'union disjointe de quatre copies de l'ensemble $\left\{ \left(\frac{\pi k}{\ell} \right)^2, k \in \mathbb{N} \right\}$, et en particulier $\mu_j > 0$. Ces exemples nous montrent que les valeurs propres des polygones présentent deux comportements différents : les premières semblent être influencées par la présence de coins convexes, d'après le résultat de [LP08, BP16]. Les suivantes au contraire, ont le même premier terme que les domaines lisses, mais le second ordre peut être très différent si l'on compare les cas du triangle équilatéral et du carré. Pour comprendre le comportement asymptotique des valeurs propres de Robin sur des polygones curvilignes, il semble alors nécessaire de séparer l'étude des valeurs propres influencées par les coins et des suivantes.

Dans la section suivante, nous présentons brièvement les résultats obtenus pour des polygones curvilignes. Une discussion détaillée de ces résultats pourra être trouvée dans la version anglaise en Section II.2. Grâce aux résultats de [LP08, BP16], nous comprenons qu'une étude complète des Laplaciens de Robin agissant sur des secteurs infinis, que nous appelons les opérateurs modèles, est nécessaire afin d'obtenir des informations plus précises sur les polygones curvilignes. Ainsi, nous présentons les résultats obtenus pour ces opérateurs en Section I.2.1. En Section I.2.2, nous décrivons les résultats obtenus pour les valeurs propres influencées par les coins. En Section I.2.3, nous présentons les résultats obtenus pour les valeurs propres suivantes. Enfin, la Section I.2.4 est dédiée à la distribution asymptotique des valeurs propres sur des polygones curvilignes.

I.2 Résultats principaux

I.2.1 Les Laplaciens de Robin sur des secteurs infinis

Soient $\alpha \in (0, \pi)$ et U_α le secteur infini d'ouverture 2α ,

$$U_\alpha = \{(x_1, x_2) \in \mathbb{R}^2 : |\arg(x_1 + ix_2)| < \alpha\},$$

voir Figure I.3. Nous nous intéressons aux propriétés spectrales du Laplacien de Robin associé, noté $T_\alpha^\gamma := Q_{U_\alpha}^\gamma$, qui est défini comme l'unique opérateur auto-adjoint dans $L^2(U_\alpha)$ associé à la forme sesquilinéaire

$$t_\alpha^\gamma(u, u) = \int_{U_\alpha} |\nabla u|^2 dx_1 dx_2 - \gamma \int_{\partial U_\alpha} |u|^2 ds, \quad u \in H^1(U_\alpha), \quad (\text{I.2.1})$$

où l'on note ds la mesure de Hausdorff unidimensionnelle, voir Théorème IV.1.1 pour une justification complète. L'étude de cet opérateur présente un intérêt pour plusieurs raisons. Premièrement, le domaine U_α peut être considéré comme le domaine le plus élémentaire présentant une singularité en dimension deux, et ne dépendant que d'un seul paramètre géométrique : la demi-ouverture α . Deuxièmement, comme mentionné ci-dessus, les propriétés spectrales de T_α^γ jouent un rôle important dans l'étude des domaines à coins lorsque $\gamma \rightarrow +\infty$. En effet, en dimension deux les seuls cônes tangents sont les secteurs infinis, et d'après les résultats [BP16, CGM11, LP08], une meilleure compréhension de cet opérateur nous permettra d'obtenir des résultats plus précis pour les polygones curvilignes. Finalement, nous pouvons aussi ajouter que le domaine U_α et son bord sont non compacts, ce qui pourrait amener à des comportements spectraux inhabituels.

Dans la suite, nous présentons les résultats obtenus sur T_α^γ , rassemblés dans *Eigenvalues of Robin Laplacians in infinite sectors* [KP18]. Les preuves sont données au Chapitre IV. Fixons $\gamma > 0$. Le seul résultat disponible dans la littérature se présentait comme suit, voir [LP08] :

$$\inf \text{spec } T_\alpha^\gamma = \begin{cases} -\gamma^2, & \alpha \geq \frac{\pi}{2}, \\ -\frac{\gamma^2}{\sin^2 \alpha}, & \alpha < \frac{\pi}{2}, \end{cases} \quad (\text{I.2.2})$$

et pour $\alpha < \frac{\pi}{2}$ l'infimum du spectre est une valeur propre discrète associée à la fonction propre $\exp(-\gamma x_1 / \sin \alpha)$. Le but du présent travail est de donner une analyse spectrale plus détaillée.

Premièrement, nous démontrons que le spectre essentiel de T_α^γ ne dépend pas de l'angle de demi-ouverture α ,

$$\text{spec}_{\text{ess}}(T_\alpha^\gamma) = [-\gamma^2, +\infty),$$

voir Théorème IV.1.1. Il découle de (I.2.2) que le spectre discret est non vide si et seulement si $\alpha < \frac{\pi}{2}$, i.e. si et seulement si le secteur est strictement inclus dans le demi-plan.

Dans le Théorème IV.2.1, nous montrons que le spectre discret est fini, ce qui est non trivial à cause de la non compacité du bord. De plus, ce résultat dépend de la dimension de l'espace ambiant, étant donné qu'en dimension supérieure les Laplaciens de Robin agissant sur des cônes peuvent avoir un spectre discret infini [BPP18, Pan16].

En Section IV.2.3, nous obtenons des résultats plus précis : nous montrons dans le Théorème IV.2.3 que chaque valeur propre est une fonction continue et strictement croissante en l'angle α , et qu'il n'existe qu'une unique valeur propre lorsque $\alpha \geq \frac{\pi}{6}$, voir Theorem IV.2.6. Il est important de noter que nous ne montrons pas que $\alpha \geq \frac{\pi}{6}$ est un saut de la fonction de comptage des valeurs propres de T_α^γ . Toutefois, certains arguments présentés au Chapitre VII nous laissent penser que c'est le cas.

En Section IV.3, nous étudions le comportement asymptotique des valeurs propres lorsque l'angle α est petit. Pour tout $\gamma > 0$ fixé, nous montrons que la n ème valeur propre $E_n(T_\alpha^\gamma)$ se comporte comme

$$E_n(T_\alpha^\gamma) = -\frac{\gamma^2}{(2n-1)^2 \alpha^2} + \mathcal{O}(1), \quad \alpha \rightarrow 0, \quad (\text{I.2.3})$$

voir Corollaire IV.3.3. Nous obtenons également dans le Théorème IV.3.15 un développement asymptotique complet pour chaque valeur propre en fonction des puissances de α^2 . L'asymptotique (I.2.3) nous permet également de conclure que le nombre de valeurs propres discrètes est minoré par κ/α , $\kappa > 0$, lorsque $\alpha \rightarrow 0$ et donc qu'il tend vers $+\infty$, voir Corollaire IV.3.2.

Enfin, nous prouvons que les fonctions propres associées sont localisées, dans un sens bien défini, près du sommet de U_α , voir Théorème IV.4.1.

I.2.2 Etats propres des Laplaciens de Robin influencés par la présence de coins

Cette section est dédiée à l'étude des valeurs propres de Robin influencées par les coins sur les polygones curvilignes. Les résultats sont rassemblés dans l'article *Spectral asymptotics for Robin Laplacians on polygonal domains* [Kha18].

Avant de présenter nos résultats, nous introduisons une définition des domaines étudiés adaptée à notre sujet, voir également [Gri85, Definition 1.4.5.1], [Dau88, Chapter 1] et [BNDP16, Section II. 3].

Définition I.2.1. Soit $\Omega \subset \mathbb{R}^2$ un domaine borné et simplement connexe. Le domaine Ω est un *polygone curviligne* si $\partial\Omega$ est lipschitzien et s'il existe $M \geq 1$ courbes non auto-intersectantes Γ_k , $k = 1, \dots, M$, telles que

$$\partial\Omega = \bigcup_{k=1}^M \overline{\Gamma_k},$$

et si l'on note l_k la longueur de Γ_k et ζ_k une paramétrisation de $\overline{\Gamma_k}$ par la longueur d'arc alors $\zeta_k \in C^4([0, l_k])$. De plus, si deux composantes Γ_k, Γ_j s'intersectent en un point $v := \Gamma_k(l_k) = \Gamma_j(0)$, alors deux situations sont autorisées : soit $\overline{\Gamma_k} \cup \overline{\Gamma_j}$ est C^4 près de v et ainsi v est un point régulier de $\partial\Omega$, ou l'angle d'ouverture en v , noté $2\alpha_v$, et mesuré à l'intérieur de Ω et formé par les demi-tangentes en v appartient à $(0, \pi) \cup (\pi, 2\pi)$. Dans ce dernier cas, v est un coin de Ω .

Les pointes (i.e. $\alpha = 0$) ne sont pas autorisées par notre définition puisque le bord est lipschitzien.

Introduisons l'ensemble des coins convexes de Ω :

$$\mathcal{V} := \left\{ v \in \partial\Omega, v \text{ est un coin de } \Omega \text{ et } \alpha_v \in (0, \frac{\pi}{2}) \right\}.$$

Alors, pour tout $v \in \mathcal{V}$, il existe $r_v > 0$ et un C^2 -difféomorphisme F_v tels que :

- (a) $F_v : \Omega \cap B(v, r_v) \rightarrow U_{\alpha_v} \cap B(0, r_v)$,
- (b) $F_v(\overline{\Omega \cap B(v, r_v)}) = \overline{U_{\alpha_v} \cap B(0, r_v)}$,
- (c) $F_v(v) = 0$ et $\nabla F_v(v) = I_2$,

où l'on note I_2 la matrice identité en dimension deux, $B(v, r_v)$ la boule de centre v et de rayon r_v dans \mathbb{R}^2 , et ∇F_v la matrice jacobienne de F_v . A rotation et translation près, le secteur U_{α_v} est le cône tangent à $\partial\Omega$ en v .

Soit $\Omega \subset \mathbb{R}^2$ un polygone curviligne. Définissons l'opérateur modèle :

$$T^\oplus := \bigoplus_{v \in \mathcal{V}} T_{\alpha_v}^1, \text{ and } \mathcal{N}^\oplus := \sum_{v \in \mathcal{V}} \mathcal{N}_v,$$

où l'on note $\mathcal{N}_v := \#\{n \in \mathbb{N} : E_n(T_{\alpha_v}^1) < -1\}$ le nombre de valeurs propres discrètes de $T_{\alpha_v}^1$. Nous avons $\mathcal{N}_v < +\infty$ par le Théorème IV.2.1.

Le résultat principal de cette section nous donne l'asymptotique des \mathcal{N}^\oplus premières valeurs propres lorsque $\gamma \rightarrow +\infty$.

Théorème I.2.2. Pour tout $n \in \{1, \dots, \mathcal{N}^\oplus\}$ nous avons :

$$E_n(Q_\Omega^\gamma) = E_n(T^\oplus)\gamma^2 + r(\gamma), \text{ lorsque } \gamma \rightarrow +\infty,$$

où $r(\gamma) = O(\gamma^{4/3})$, et nous avons $r(\gamma) = O(e^{-c\gamma})$ avec $c > 0$ si Ω est un polygone droit.

Pour un résultat plus précis voir Théorème V.2.1 pour les polygones droits et Théorème VI.3.1 pour le cas général. Dans le Théorème I.2.2, nous supposons que Ω est simplement connexe, voir la Définition I.2.1. Il apparaît clairement dans les preuves que cette hypothèse n'est faite que pour garder des notations simples. Le résultat reste valable pour une union finie de domaines connexes.

Nous montrons également que les fonctions propres associées sont localisées près des coins convexes de Ω , voir la Proposition V.3.2 pour les polygones droits et la Proposition VI.3.4 pour le cas général. De plus, dans le cas de polygones droits, elles sont également proches, en un sens bien défini, des fonctions propres tronquées de l'opérateur modèle, voir Théorème V.3.1.

I.2.3 Les valeurs propres suivantes

Nous souhaitons maintenant obtenir des informations sur le comportement asymptotique des valeurs propres suivantes, c'est-à-dire $E_{\mathcal{N}^{\oplus+j}}(Q_{\Omega}^{\gamma})$ lorsque $\gamma \rightarrow +\infty$. Le résultat ci-dessous nous montre en particulier que les coins n'influencent pas le premier ordre de l'asymptotique de ces valeurs propres.

Proposition I.2.3. *Soit $\Omega \subset \mathbb{R}^2$ un polygone curviligne. Pour tout $j \in \mathbb{N}$, nous avons*

$$E_{\mathcal{N}^{\oplus+j}}(Q_{\Omega}^{\gamma}) = -\gamma^2 + o(\gamma^2), \text{ lorsque } \gamma \rightarrow +\infty.$$

Par analogie avec le cas lisse, voir Théorème I.1.6, nous nous attendons au comportement asymptotique suivant pour les polygones curvilignes :

$$E_{\mathcal{N}^{\oplus+j}}(Q_{\Omega}^{\gamma}) = -\gamma^2 + E_j(\tilde{L}_{\gamma}) + R(\gamma), \text{ lorsque } \gamma \rightarrow +\infty,$$

où l'opérateur effectif \tilde{L}_{γ} agit dans $L^2(\partial\Omega)$ comme $u \mapsto -\partial^2 u - \gamma \kappa u$, où κ est la courbure de $\partial\Omega$, sur des fonctions satisfaisant des conditions de bord en chaque coin et $R(\gamma)$ est un petit reste. Nous présentons ici un premier résultat pour des polygones droits qui va dans le sens de cette conjecture. Le cas des polygones curvilignes est plus technique et sera étudié plus tard. Ces résultats sont issus d'une collaboration avec Konstantin Pankrashkin et Thomas Ourmières-Bonafos (Orsay).

Introduisons d'abord quelques notations. Nous allons définir une troncature particulière de U_{α} . Soient $0 < \alpha < \frac{\pi}{2}$ et $R > 0$. Considérons les points

$$A_R^{\pm} = R(\cos \alpha, \pm \sin \alpha) \in \partial U_{\alpha}, \quad B_R = R\left(\frac{1}{\cos \alpha}, 0\right) \in U_{\alpha},$$

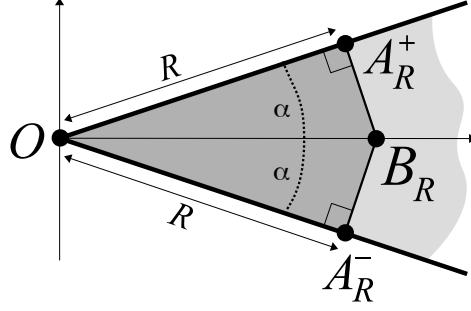
et notons $U_{\alpha,R}$ le quadrangle $OA_R^+B_R^+A_R^-$. Définis de cette manière, les côtés BA_R^{\pm} sont orthogonaux à ∂U_{α} et A_R^{\pm} , voir Figure I.4. Notons $T_{\alpha,R}^{\gamma}$ l'opérateur agissant dans $L^2(U_{\alpha,R})$ et défini comme l'unique opérateur auto-adjoint associé à

$$t_{\alpha,R}^{\gamma}(u, u) = \int_{U_{\alpha,R}} |\nabla u|^2 dx - \gamma \int_{\partial U_{\alpha,R} \cap \partial U_{\alpha}} |u|^2 ds, \quad D(t_{\alpha,R}^{\gamma}) := H^1(U_{\alpha,R}). \quad (\text{I.2.4})$$

L'opérateur $T_{\alpha,R}^{\gamma}$ agit comme le Laplacien sur $U_{\alpha,R}$ avec la condition de bord de Robin sur $\partial U_{\alpha,R} \cap \partial U_{\alpha}$ et la condition de Neumann sur le reste du bord. Cet opérateur joue un rôle important en Section V.4 et dans la définition suivante :

Définition I.2.4. Un angle $\alpha \in (0, \frac{\pi}{2})$ est dit *non résonant* si, pour $\gamma > 0$, il existe une constante $C > 0$ telle que

$$E_{\mathcal{N}_{\alpha+1}}(T_{\alpha,R}^{\gamma}) \geq -\gamma^2 + \frac{C}{R^2}, \text{ lorsque } R \text{ est grand.} \quad (\text{I.2.5})$$


 FIGURE I.4 – The truncated sector $U_{\alpha, R}$ is shaded.

Cette propriété ne dépend pas du choix de $\gamma > 0$ puisque nous avons l'égalité :

$$E_n(T_{\alpha, R}^\gamma) = \gamma^2 E_n(T_{\alpha, \gamma R}^1).$$

Le résultat suivant, basé sur la monotonie des valeurs propres en fonction de α , exhibe des angles non résonants explicites.

Proposition I.2.5. *Tout angle $\alpha \in [\frac{\pi}{4}, \frac{\pi}{2})$ est non résonant.*

La preuve est donnée en Section IV.5.

Soit $\Omega \subset \mathbb{R}^2$ un polygone droit admettant $V \in \mathbb{N}$ sommets. Notons les sommets $A_1, \dots, A_V \in \mathbb{R}^2$, et supposons qu'il sont énumérés de telle façon que le bord $\partial\Omega$ soit exactement l'union des V segments $[A_v, A_{v+1}]$, où l'on note :

$$A_{V+1} := A_1 \text{ and } A_0 := A_V.$$

Nous supposons qu'il n'y a pas de sommet artificiel. Introduisons également :

- l'angle de Ω au sommet A_v , i.e. l'angle entre les segments $[A_v, A_{v-1}]$ et $[A_v, A_{v+1}]$ mesuré à l'intérieur de Ω , est toujours noté $2\alpha_v$. Les hypothèses précédentes impliquent :

$$\alpha_v \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi) \text{ pour tout } v.$$

- $\mathcal{N}^\oplus := \mathcal{N}_{\alpha_1} + \dots + \mathcal{N}_{\alpha_V}$,
- La longueur $\ell_v = |A_{v+1} - A_v|$ du v ème côté de Ω .
- Le Laplacien de Dirichlet dans $L^2(0, \ell_v)$ noté D_v et agissant comme $f \mapsto -f''$ sur $(0, \ell_v)$,
- L'opérateur $D := D_1 \oplus \dots \oplus D_V$, qui jouera le rôle de l'opérateur effectif sur $\partial\Omega$.

Le résultat principal se présente comme suit :

Théorème I.2.6. *Supposons que $\alpha_v > \frac{\pi}{2}$ ou α_v est non résonant pour tout $v \in \{1, \dots, V\}$. Alors, pour tout $j \in \mathbb{N}$ fixé nous avons :*

$$E_{\mathcal{N}^\oplus + j}(Q_\Omega^\gamma) = -\gamma^2 + E_j(D) + \mathcal{O}\left(\frac{\log \gamma}{\sqrt{\gamma}}\right) \text{ lorsque } \gamma \rightarrow +\infty.$$

La preuve est donnée en Section V.4. Nous pouvons déduire de la Proposition II.2.7 le cas particulier suivant :

Corollaire I.2.7. *Si $\alpha_v \geq \frac{\pi}{4}$ pour tout v , alors pour tout $j \in \mathbb{N}$ fixé nous avons*

$$E_{\mathcal{N}^\oplus + j}(Q_\Omega^\gamma) = -\gamma^2 + E_j(D) + \mathcal{O}\left(\frac{\log \gamma}{\sqrt{\gamma}}\right) \text{ lorsque } \gamma \rightarrow +\infty.$$

I.2.4 Asymptotiques de Weyl pour les polygones curvilignes

Dans cette section, nous introduisons les résultats relatifs à la fonction de comptage des valeurs propres de l'opérateur Q_Ω^γ . Les preuves sont données en Section VI.4.

Soit A un opérateur auto-adjoint, borné inférieurement dans un espace de Hilbert \mathcal{H} . Pour un paramètre $\lambda \in \mathbb{R}$, nous notons $\mathcal{N}(A, \lambda)$ le nombre de valeurs propres, comptées avec multiplicités, de l'opérateur A dans $(-\infty, \lambda)$ si $\text{spec}_{\text{ess}}(A) \cap (-\infty, \lambda) = \emptyset$ et $\mathcal{N}(A, \lambda) = +\infty$ sinon.

L'étude de la distribution asymptotique des valeurs propres est un sujet classique en théorie spectrale, et le premier résultat sur le sujet est dû à Weyl en 1911. Notons Q_Ω^D le Laplacien dans $L^2(\Omega)$ avec la condition de bord de Dirichlet, i.e. $u = 0$ on $\partial\Omega$. Alors nous avons :

$$\mathcal{N}(Q_\Omega^D, \lambda) = \frac{\omega_d |\Omega|}{(2\pi)^d} \lambda^{\frac{d}{2}} (1 + o(1)), \text{ lorsque } \lambda \rightarrow +\infty, \quad (\text{I.2.6})$$

où ω_d est le volume de la boule unité de \mathbb{R}^d . Des asymptotiques plus précises ont été obtenues depuis, et nous renvoyons à l'article de synthèse [Ivr16] pour une discussion détaillée.

Nous nous intéressons ici à la distribution asymptotique des valeurs propres négatives du Laplacien de Robin lorsque $\gamma \rightarrow +\infty$. Pour des domaines lisses, des asymptotiques de Weyl ont été obtenues par [HKR17] : pour tout $E \in (-1, 0)$ nous avons

$$\mathcal{N}(Q_\Omega^\gamma, E\gamma^2) = \gamma \frac{|\partial\Omega| \sqrt{E+1}}{\pi} + R_1(\gamma), \quad R_1(\gamma) = \mathcal{O}(1), \text{ lorsque } \gamma \rightarrow +\infty, \quad (\text{I.2.7})$$

et pour tout $\lambda \in \mathbb{R}$ nous avons lorsque $\gamma \rightarrow +\infty$,

$$\mathcal{N}(Q_\Omega^\gamma, -\gamma^2 + \lambda\gamma) = \frac{\sqrt{\gamma}}{\pi} \int_{\partial\Omega} \sqrt{(\kappa(s) + \lambda)_+} ds + R_2(\gamma), \quad R_2(\gamma) = o(\sqrt{\gamma}), \quad (\text{I.2.8})$$

où κ est la courbure de $\partial\Omega$, voir (VI.2.1) pour une définition précise, et $(x)_+ := \max(x, 0)$. Des asymptotiques semblables ont également été démontrées en dimensions supérieures dans [KKR16]. Nous pouvons alors montrer que, lorsque Ω admet des coins, ces asymptotiques restent vraies.

Théorème I.2.8. *Les asymptotiques (I.2.7) et (I.2.8) sont vraies lorsque Ω est un polygone curviligne avec respectivement $R_1(\gamma) = \mathcal{O}(\gamma^\theta)$ pour tout $\theta \in (0, \frac{1}{2})$ et $R_2(\gamma) = \mathcal{O}(\gamma^{\frac{1}{4}})$.*

En comparaison avec les domaines lisses, ce résultat nous montre en particulier que les coins n'influencent pas l'asymptotique de Weyl, ce qui n'est pas surprenant puisque, d'après le Théorème II.2.3, ils ne créent qu'un nombre *fini* de valeurs propres.

Nous obtenons également l'asymptotique de Weyl suivante pour des seuils positifs.

Proposition I.2.9. *Pour tout $\mathcal{C} > 0$ nous avons lorsque γ est grand :*

$$\mathcal{N}(Q_\Omega^\gamma, \mathcal{C}\gamma^2) = \mathcal{C}\gamma^2 \frac{|\Omega|}{4\pi} + o(\gamma^2).$$

On remarque en particulier qu'il s'agit du même premier terme que celui de l'asymptotique de Weyl classique (I.2.6).

Chapter II

Introduction

II.1 Motivation

For a domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, and a real parameter $\gamma > 0$ we consider the spectral problem

$$-\Delta u := -\sum_{j=1}^d \frac{\partial^2 u}{\partial x_j^2} = Eu \text{ on } \Omega, \quad (\text{II.1.1})$$

$$\frac{\partial u}{\partial \nu} = \gamma u \text{ on } \partial\Omega, \quad (\text{II.1.2})$$

where ν is the *outward* unit normal of $\partial\Omega$, E is an eigenvalue and u is an associated eigenfunction. The boundary condition (II.1.2) is called the Robin boundary condition. The present work is motivated by the investigation of the asymptotic behavior of the eigenvalues for large γ when Ω is a planar domain with *corners*.

The study of the eigenvalues of the Laplace operator is a classical topic in spectral theory, as this operator appears in mathematical models of numerous physical phenomena. One can mention the heat equation, which models the evolution of the temperature in a solid, the Schrödinger equation, which describes the evolution of a quantum particle, or also the wave equation modeling the vibrations of a membrane. In the latter situation, the physical meaning of the eigenvalues of the Laplace operator is easy to understand. If we consider a planar membrane $\Omega \subset \mathbb{R}^2$, its small variations are described by the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0.$$

The function $u(t, x)$ denotes the height of the membrane at time $t \in \mathbb{R}_+$ and point $x \in \Omega$. An easy way to solve this equation is to separate the temporal and spatial variables: we look for a solution in the form $u(t, x) = g(t)f(x)$. This leads to the system

$$\begin{aligned} -g''(t) &= Eg(t), \\ -\Delta f(x) &= Ef(x). \end{aligned}$$

A particular solution of the wave equation is then given by

$$u(t, x) = \left(A \cos(\sqrt{E}t) + B \sin(\sqrt{E}t) \right) f(x), \quad A, B \in \mathbb{R},$$

which oscillates with a frequency \sqrt{E} . In his famous book *The Theory of Sound* [Ray77], Lord Rayleigh conjectured that, when the area is fixed, the circular membrane has the

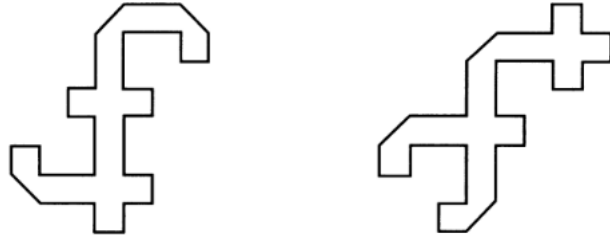


Figure II.1 – Isospectral domains for Dirichlet and Neumann Laplacians [GWW92].

lowest frequency. Mathematically, this means that considering the first eigenvalue $E_1(\Omega)$, called the fundamental frequency, of the following problem

$$\begin{aligned} -\Delta u &= Eu \text{ on } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

we should have

$$E_1(B) \leq E_1(\Omega), \tag{II.1.3}$$

where $B \subset \mathbb{R}^2$ is the disk of same area as the membrane Ω . The conjecture of Rayleigh was the starting point of what are now called the isoperimetric inequalities. The inequality (II.1.3) was rigorously proved in any dimension and is known as the Faber-Krahn inequality. In the above system, the boundary condition $u = 0$ on $\partial\Omega$, known as the Dirichlet boundary condition, models a membrane fixed at its boundary. Similarly, if one considers the vibrations of a free membrane, the boundary condition becomes $\frac{\partial u}{\partial \nu} = 0$. This boundary condition is known as the Neumann boundary condition. The isoperimetric inequalities were extended to the Neumann and the Robin case for a *negative* parameter γ . More precisely, it was proved in [Sze54, Wei56] that the ball maximizes the second eigenvalue, namely the first non-trivial eigenvalue, of the Neumann Laplacian among the class of smooth domains of same volume in any dimension. Similarly to the Dirichlet case, the ball is a minimizer for the first eigenvalue of the negative Robin Laplacian in any dimension [Bos88, Dan06]. However, it is interesting to note that the investigation of optimizers of the first eigenvalue of the Robin Laplacian with a *positive* parameter γ when the volume is fixed remains an open question.

In 1966, Kac was interested in the inverse problem: *Can one hear the shape of a drum?* [Kac71]. In other words, if one knows the eigenvalues of the Laplacian acting on a domain Ω , can we deduce its form? Unfortunately, without any additional assumptions this is not true as showed in particular by [GWW92] where a planar counterexample is exhibited, see Figure II.1. Nevertheless, some information on the geometry of the domain can be obtained by the study of the eigenvalues of the Laplacian. In 1911, Weyl proved the following asymptotics for the Dirichlet eigenvalues,

$$E_k(\Omega) \sim 4\pi^2 \left(\frac{k}{\omega_d |\Omega|} \right)^{\frac{2}{d}}, \text{ as } k \rightarrow +\infty, \tag{II.1.4}$$

where ω_d is the volume of the unit ball in \mathbb{R}^d . We deduce from this result that two domains with different volumes can never have the same spectrum, and the volume is then a spectral invariant for the Dirichlet Laplacian. Since then, more precise asymptotics were obtained for the Dirichlet eigenvalues and the search for spectral invariants still continues

to attract a lot of attention for the Dirichlet Laplacian [LR15, LR16, Uca17], but also for other boundary conditions as the Neumann or the mixed Dirichlet-Neumann ones [HLR17, JLNP06, LPP06], and other spectral problems as the so-called Steklov problem [GPPS14, PS15]. These considerations on the asymptotics on the Dirichlet eigenvalues also gave rise to the well-known Pólya conjecture. In 1961, Pólya proved in [P61] that for any plane-covering domain, that is to say any domain which covers the whole plane, without gaps and overlaps, allowing rotations, translations and reflections of itself, there holds

$$E_k(\Omega) \geq \frac{4\pi k}{|\Omega|}, \text{ for all } k \in \mathbb{N}.$$

He conjectured that the above inequality is valid for any bounded domain in \mathbb{R}^d , which is equivalent to say that the Weyl asymptotics (II.1.4) is a lower bound for the eigenvalues. The conjecture remains open to this day.

In this thesis, our interest is focused on the Robin boundary condition (II.1.2). In a certain sense, it can be viewed as the most general boundary condition, as it contains the Neumann one when $\gamma = 0$ and the Dirichlet one in the limit $\gamma \rightarrow -\infty$. The Robin eigenvalue problem (II.1.1)-(II.1.2) appears in several applications as the estimation of the critical temperature of superconductors [GS07], or the study of reaction-diffusion process in the long-time asymptotics [LOS98]. The repulsive case ($\gamma < 0$) has been studied in great details, as described in the recent review paper [BFK17], while the attractive case ($\gamma > 0$) is a relatively new topic.

Let us define more rigorously the Robin Laplacian. We are only interested in the *attractive* Robin boundary condition and, in what follows, the Robin parameter γ is strictly positive: $\gamma > 0$. Consider a domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, and the sesquilinear form

$$q_\Omega^\gamma(u, u) = \int_\Omega |\nabla u|^2 dx - \gamma \int_{\partial\Omega} |u|^2 ds, \quad u \in H^1(\Omega),$$

where ds stands for the $(d-1)$ -dimensional Hausdorff measure. Assume that the following conditions are satisfied:

- the trace map $H^1(\Omega) \ni u \mapsto u|_{\partial\Omega} \in L^2(\partial\Omega)$ is well-defined,
- there exists a constant $K > 0$ such that

$$\int_{\partial\Omega} |u|^2 ds \leq K \left(\epsilon \int_\Omega |\nabla u|^2 dx + \epsilon^{-1} \int_\Omega |u|^2 dx \right), \quad (\text{II.1.5})$$

holds for any $u \in H^1(\Omega)$ and $\epsilon \in (0, 1)$.

Under these assumptions, one can show that the sesquilinear form q_Ω^γ is semibounded from below: there exists a constant $c > 0$ such that, for any $u \in H^1(\Omega)$,

$$q_\Omega^\gamma(u, u) \geq -c \|u\|_{L^2(\Omega)}^2,$$

and that the norm induced by the form q_Ω^γ and defined by the scalar product $\langle u, v \rangle_{q_\Omega^\gamma} := q_\Omega^\gamma(u, v) + (c+1)\langle u, v \rangle$ is equivalent to the H^1 -norm: there exist $c_1, c_2 > 0$ satisfying

$$c_1 \|u\|_{H^1(\Omega)}^2 \leq q_\Omega^\gamma(u, u) + (c+1) \|u\|_{L^2(\Omega)}^2 \leq c_2 \|u\|_{H^1(\Omega)}^2.$$

When both of the above properties are satisfied, we say that the form is *closed*. By the theory of sesquilinear forms, see e.g. [RS80, Theorem VIII. 15], the form q_Ω^γ defines a unique self-adjoint operator in $L^2(\Omega)$ which is denoted by Q_Ω^γ .

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then, the trace map is well defined [Gri85, Theorem 1.5.1.3], and the inequality (II.1.5) is satisfied [Gri85, Theorem 1.5.1.10]. As a consequence, the form q_Ω^γ is closed for any bounded Lipschitz domain. Another prominent examples of domains for which the form q_Ω^γ is closed are the infinite sectors in two dimensions. They play a particular role in our study and we give a complete proof of the closedness of q_Ω^γ when $\Omega \subset \mathbb{R}^2$ is an infinite sector in Section IV.1. One can also mention that some investigations have been made for non-Lipschitz domains, and in particular for domains with cusps [Dan13, KP, NT13], for which the trace map is well defined if the cusps are not too sharp.

In the sequel, Ω is a bounded Lipschitz domain. An integration by parts shows that the operator Q_Ω^γ acts as the Laplacian $u \mapsto -\Delta u$ on the domain

$$D(Q_\Omega^\gamma) := \left\{ u \in H^1(\Omega) : \Delta u \in L^2(\Omega), \frac{\partial u}{\partial \nu} = \gamma u \text{ on } \partial\Omega \right\}.$$

Due to the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ [Gri85, Section 1.4.4], the operator Q_Ω^γ has a compact resolvent and hence, see e.g. [RS78, Theorem XIII.64], its spectrum consists of a sequence of eigenvalues

$$E_1(Q_\Omega^\gamma) \leq E_2(Q_\Omega^\gamma) \leq \dots \leq E_n(Q_\Omega^\gamma) \leq \dots \rightarrow +\infty,$$

where each eigenvalue is repeated according to its finite multiplicity, and the associated eigenfunctions form an orthonormal basis of $L^2(\Omega)$. The variational characterization of the eigenvalues by the max-min and the min-max principle, see Theorem III.1.3 below, holds for the operator Q_Ω^γ and we have, for any $n \in \mathbb{N} := \{1, 2, \dots\}$,

$$E_n(Q_\Omega^\gamma) := \sup_{\psi_1, \dots, \psi_{n-1} \in L^2(\Omega)} \inf_{\substack{u \in H^1(\Omega), u \neq 0 \\ u \perp \psi_j, j=1, \dots, n-1}} \frac{q_\Omega^\gamma(u, u)}{\langle u, u \rangle} \equiv \inf_{\substack{G \subset H^1(\Omega) \\ \dim G = n}} \sup_{\substack{u \in G \\ u \neq 0}} \frac{q_\Omega^\gamma(u, u)}{\langle u, u \rangle}.$$

We can deduce from this variational characterization some simple properties satisfied by the eigenvalues. Each eigenvalue viewed as a function of $\gamma \in \mathbb{R}_+$ is decreasing. In fact, the decrease is strict in γ as discussed e.g. in [Roh14] for the more general context of Robin parameters being functions on the boundary, i.e. $\partial\Omega \ni s \mapsto \gamma(s)$. Moreover, we have the following scaling with respect to the dilations of the domain:

$$E_n(Q_\Omega^\gamma) = \gamma^2 E_n(Q_{\gamma\Omega}^1), \text{ for any } n \in \mathbb{N},$$

where we use the notation $\gamma\Omega := \{\gamma x, x \in \Omega\}$. Due in particular to the boundary condition, notice that there is no domain monotonicity of the eigenvalues. We can also infer the asymptotic behavior as γ becomes large.

Proposition II.1.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. For each fixed $n \in \mathbb{N}$ there holds*

$$E_n(Q_\Omega^\gamma) \rightarrow -\infty \text{ as } \gamma \rightarrow +\infty.$$

Proof. Let $n \in \mathbb{N}$ be fixed. Consider n functions linearly independent $\varphi_1, \dots, \varphi_n \in H^{\frac{1}{2}}(\partial\Omega)$. As the trace map $H^1(\Omega) \ni u \mapsto u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)$ is surjective [Gri85, Theorem 1.5.1.3], there exist $\psi_1, \dots, \psi_n \in H^1(\Omega)$ such that for any $j = 1, \dots, n$ we have $(\psi_j)|_{\partial\Omega} = \varphi_j$. Thus, by linearity of the trace map the functions ψ_1, \dots, ψ_n are linearly independent. Let $\psi \in \text{span}\{\psi_1, \dots, \psi_n\}$. There exists $(c_1, \dots, c_n) \in \mathbb{C}^n$ such that $\psi := \sum_{j=1}^n c_j \psi_j$. Denote by

$$M^+ := \max_{j=1, \dots, n} \|\psi_j\|_{H^1(\Omega)}^2, \quad M^- := \min_{j=1, \dots, n} \|\psi_j\|_{L^2(\Omega)}^2, \quad m^- := \min_{j=1, \dots, n} \|\varphi_j\|_{L^2(\partial\Omega)}^2.$$

Using the Cauchy-Schwarz inequality we have

$$\|\nabla\psi\|_{L^2(\Omega)}^2 \leq (1+2n)M^+ \sum_{j=1}^n |c_j|^2,$$

and using the fact that the ψ_j are linearly independent and the φ_j are linearly independent as well, we also have

$$\|\psi\|_{L^2(\partial\Omega)}^2 \geq m^- \sum_{j=1}^n |c_j|^2, \quad M^- \sum_{j=1}^n |c_j|^2 \leq \|\psi\|_{L^2(\Omega)}^2 \leq M^+ \sum_{j=1}^n |c_j|^2.$$

The min-max characterization of the eigenvalues yields

$$E_n(Q_\Omega^\gamma) \leq \sup_{\substack{(c_1, \dots, c_n) \in \mathbb{C}^n \\ (c_1, \dots, c_n) \neq (0, \dots, 0)}} \frac{q_\Omega^\gamma(\psi, \psi)}{\|\psi\|_{L^2(\Omega)}^2},$$

and thanks to the previous estimates we obtain the upper bound

$$E_n(Q_\Omega^\gamma) \leq \frac{(1+2n)M^+}{M^-} - \frac{m^-}{M^+} \gamma \rightarrow -\infty \text{ as } \gamma \rightarrow +\infty.$$

□

Finally, a simple adaptation of (II.1.5) tells us that the eigenvalues do not decrease faster than $-\gamma^2$:

Proposition II.1.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then, there exists a constant $C_\Omega > 0$ such that for large γ ,*

$$E_1(Q_\Omega^\gamma) \geq -C_\Omega \gamma^2.$$

The one-dimensional Robin Laplacian acting on $L^2(\mathbb{R}_+)$ will be of importance for our purpose. The operator $Q_{\mathbb{R}_+}^\gamma$ admits a unique discrete eigenvalue $-\gamma^2$ associated with the eigenfunction $e^{-\gamma t}$, see Section III.5.1. This eigenfunction is in particular used as a test function to prove the following upper bound obtained in [GS07]:

Proposition II.1.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then, for any $\gamma > 0$ we have*

$$E_1(Q_\Omega^\gamma) \leq -\gamma^2.$$

Proof. Without loss of generality, we can suppose that $\Omega \subset \{x \in \mathbb{R}^d : x_1 > 0\}$. Let $u(x) := e^{-\gamma x_1}$. Then, $\int_\Omega |\nabla u|^2 dx = \gamma^2 \int_\Omega |u|^2 dx$. Moreover, applying the divergence theorem to the vector field $F(x) := (e^{-2\gamma x_1}, 0, \dots, 0)$ we get

$$\int_{\partial\Omega} |u|^2 ds \geq \int_{\partial\Omega} F \nu ds = - \int_\Omega \operatorname{div} F dx = 2\gamma \int_\Omega |u|^2 dx.$$

By the min-max principle we finally obtain

$$E_1(Q_\Omega^\gamma) \leq \frac{\int_\Omega |\nabla u|^2 - \gamma \int_{\partial\Omega} |u|^2 ds}{\int_\Omega |u|^2 dx} \leq -\gamma^2.$$

□

The study of the Robin eigenvalues for large γ has attracted a lot of attention and the first paper on this topic was probably [LOS98]. The starting point of this work relies on the simple observation that for the unit ball $B \subset \mathbb{R}^d$, the asymptotics of the first eigenvalue is easily shown to be

$$E_1(Q_B^\gamma) = -\gamma^2 - (d-1)\gamma + o(\gamma), \quad \gamma \rightarrow +\infty.$$

This can be proved studying the asymptotic behavior of the associated eigenfunction which turns out to be the modified Bessel function with parameter $\frac{d-2}{2}$ [AS64]. In [LOS98], the authors prove that

$$E_1(Q_\Omega^\gamma) \sim -\gamma^2, \quad \gamma \rightarrow +\infty, \quad (\text{II.1.6})$$

for C^2 -domains whose boundary is diffeomorphic to the sphere. These assumptions can be weakened and in particular [LZ04] proved that (II.1.6) still holds for C^1 -domains. Later on, this result has been extended to any Robin eigenvalue:

Theorem II.1.4 ([DK10]). *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^1 -domain. Then, for any fixed $n \in \mathbb{N}$ there holds*

$$E_n(Q_\Omega^\gamma) = -\gamma^2 + o(\gamma^2), \quad \text{as } \gamma \rightarrow +\infty.$$

The proof in [DK10] is purely variational and relies on the construction of suitable test functions and the min-max principle to obtain the upper bound $E_n(Q_\Omega^\gamma) \leq -\gamma^2 + o(1)$ for any $n \in \mathbb{N}$ and large γ . This is sufficient to obtain the desired asymptotics as $E_1(Q_\Omega^\gamma) \sim -\gamma^2$.

It might be surprising that the leading order in the asymptotics of the eigenvalues is not dependent on the volume of the domain, nor on the ambient dimension. However, a simple observation can explain this phenomenon. Let $\Omega \subset \mathbb{R}^d$ be a bounded C^2 -domain. For $\delta > 0$ define $\Omega_\delta := \{x \in \Omega : d(x, \partial\Omega) < \delta\}$. Consider the operators $Q_\delta^{\gamma, D/N}$ acting on $L^2(\Omega_\delta)$ as the Laplacian with the γ -Robin boundary condition on $\partial\Omega$ and the Dirichlet/Neumann boundary condition on the remaining part of the boundary $\partial\Omega_\delta \setminus \partial\Omega$. Thus, using the min-max characterization of the eigenvalues we arrive at

$$E_n(Q_\delta^{\gamma, N} \oplus Q^N) \leq E_n(Q_\Omega^\gamma) \leq E_n(Q_\delta^{\gamma, D} \oplus Q^D),$$

where $Q^{N/D}$ denote the Neumann/Dirichlet Laplacians acting on $L^2(\Omega \setminus \overline{\Omega_\delta})$ which are positive operators. This technique is called the Dirichlet-Neumann bracketing. For γ large enough we have $E_n(Q_\delta^{\gamma, N}) < 0$, which implies $E_n(Q_\delta^{\gamma, N} \oplus Q^N) = E_n(Q_\delta^{\gamma, N})$ and the same holds for the Dirichlet version. Thus, for any fixed $n \in \mathbb{N}$ we get for large γ

$$E_n(Q_\delta^{\gamma, N}) \leq E_n(Q_\Omega^\gamma) \leq E_n(Q_\delta^{\gamma, D}),$$

and the study is reduced to a neighborhood of the boundary. In addition, it is proved in [HK17] for planar domains that the associated eigenfunctions are localized in a small vicinity of the boundary and, more precisely, near points of maximal curvature of the boundary.

The next natural question is then to study the further terms in the asymptotics in which the geometric properties of the boundary, and more precisely the curvature, will appear. This question was first addressed in [Pan13] where a two-term asymptotics is obtained for $E_1(Q_\Omega^\gamma)$ when Ω is a piecewise smooth planar domain. Then, it was shown in [EMP14] for $d = 2$ and in [PP15] for any $d \geq 2$ that, if $\Omega \subset \mathbb{R}^d$ is a bounded C^3 -domain,

$$E_n(Q_\Omega^\gamma) = -\gamma^2 - (d-1)\gamma H_{\max} + \mathcal{O}(\gamma^{\frac{2}{3}}), \quad \text{as } \gamma \rightarrow +\infty, \quad (\text{II.1.7})$$

where H_{\max} denotes the maximum of the mean curvature of $\partial\Omega$. The remainder can be replaced by $\mathcal{O}(\gamma^{\frac{1}{2}})$ if Ω is C^4 . The proof is based on a Dirichlet-Neumann bracketing reducing the study to a well-chosen tubular neighborhood of the boundary. The asymptotics (II.1.7) was also proved in [EM14] for a particular class of unbounded domains. As in this context the Robin Laplacian does not have a compact resolvent, the essential spectrum is not empty anymore, and one of the main question is to investigate the existence of discrete spectrum. If the boundary is compact and the domain is unbounded, then $\text{spec}_{\text{ess}}(Q_{\Omega}^{\gamma}) = [0, +\infty)$, see e.g [KP13], and there is only a finite number of discrete eigenvalues. Moreover, if $n \in \mathbb{N}$ is fixed one can see that for γ sufficiently large $E_n(Q_{\Omega}^{\gamma}) < 0$, in other words each $E_n(Q_{\Omega}^{\gamma})$ is an eigenvalue for large γ . However, when the boundary is non-compact this observation does not hold anymore. In [EM14], examples of unbounded domains for which the spectrum of Q_{Ω}^{γ} is purely essential are given, and we can emphasize the importance of the concavity of the domain in these cases. On the contrary, it is shown that in the asymptotic regime $\gamma \rightarrow +\infty$ and for sufficiently smooth boundary, local convexity of the boundary produces infinitely many discrete eigenvalues.

The asymptotics (II.1.7) can also be used to discuss the reversed Faber-Krahn inequality for Robin Laplacians in the asymptotic regime. In this context, we look for a maximizer of the first eigenvalue $E_1(Q_{\Omega}^{\gamma})$. In [Bar77], it was first conjectured that, among the class of smooth bounded domains of same volume, the ball should maximizes the first eigenvalue $E_1(Q_{\Omega}^{\gamma})$. However, [FK15] disproved the conjecture for large γ and this can be easily seen using the asymptotics (II.1.7). Indeed, consider a ball $B \subset \mathbb{R}^d$ of radius $r > 0$ and a spherical shell $S_R \subset \mathbb{R}^d$ of outer radius $R > 0$ and of same volume. Necessarily one has $R > r$ and thus $H_{\max}(B) = \frac{1}{r} > H_{\max}(S_R) = \frac{1}{R}$. Due to (II.1.7), for large γ there holds,

$$E_1(Q_{S_R}^{\gamma}) > E_1(Q_B^{\gamma}).$$

This situation is quite unexpected as it seems that it is the only one for which the first eigenvalue of the Laplace operator is not optimized by a ball under a volume constraint. Additionally, it was shown in [FNT16] that the conjecture is false for three-dimensional domains diffeomorphic to a ball proving that among this class there holds

$$\inf\{H_{\max}(\Omega), |\Omega| = 1\} = 0.$$

Nevertheless, one can find some configurations in which the conjecture holds true. In particular, it appears to be true for bounded Lipschitz domains which are close in a certain sense to a ball as proved in [FNT15] as well as for bounded C^2 -planar domains for small γ , see [FK15]. It is also shown in [PP15] that among the class of bounded star-shaped C^2 -domains of same volume, the ball minimizes the maximum of the mean curvature, and the conjecture is then verified for this class of domains when γ is large.

The influence of the boundary is clearly established through the second term in the asymptotics (II.1.7), however the number of the eigenvalues does not appear in the two first terms and at this point we cannot distinguish them. The paper [HK17] focuses on this problem and gives an asymptotics of $E_{n+1}(Q_{\Omega}^{\gamma}) - E_n(Q_{\Omega}^{\gamma})$ where Ω is a planar domain by proving a complete asymptotic expansion of each eigenvalue. To be more specific, suppose that $\Omega \subset \mathbb{R}^2$ is a bounded C^{∞} -domain and that the boundary $\partial\Omega$ is parametrized by the arc-length. Denote by κ the curvature of $\partial\Omega$ and by κ_{\max} the maximum of the curvature. The most important assumptions are the following: κ attains its maximum κ_{\max} at a *unique* point $x_0 \in \partial\Omega$ and the maximum is non-degenerate: $\kappa''(x_0) < 0$.

Theorem II.1.5 ([HK17]). *Under these assumptions, and for any $n \in \mathbb{N}$, there exists a*

sequence $(\beta_{j,n})_{j \geq 0}$ of real numbers such that for any $M \in \mathbb{N}$ there holds

$$E_n(Q_\Omega^\gamma) = -\gamma^2 - \gamma\kappa_{\max} + (2n-1)\sqrt{\frac{-\kappa''(x_0)}{2}}\sqrt{\gamma} + \sum_{j=0}^M \beta_{j,n}\gamma^{-\frac{j}{2}} + o(\gamma^{-\frac{M}{2}}), \text{ as } \gamma \rightarrow +\infty.$$

The localization of the eigenfunctions near the point of maximal curvature is crucial to prove this asymptotics, and is used to construct approximate eigenfunctions, which associated with the min-max principle gives an upper bound for the eigenvalues. Remark that the terms $(2n-1)\sqrt{\frac{-\kappa''(x_0)}{2}}$ are the eigenvalues of the harmonic oscillator $-\frac{d^2}{ds^2} - \frac{\kappa''(x_0)}{2}s^2$ in $L^2(\mathbb{R})$. Actually, one can say that the first terms in the asymptotics are given by the harmonic approximation of the boundary operator $-\gamma^2 - \partial_s^2 - \gamma\kappa(s)$ where s is the arc-length, due to the assumption that the potential $-\gamma\kappa(s)$ admits a *unique* non-degenerate minimum at $x_0 \in \partial\Omega$. Following the denomination of [HS84] for semi-classical Schrödinger operators, the curvature acts as a potential well. Then, the situation of a unique point of maximal curvature corresponds to the existence of a unique well. A natural question is to consider the case of multiple wells. In [HKR17], the particular case of a smooth planar bounded domain admitting an axis of symmetry and exactly two points at which the curvature attains its maximum away from this axis is considered. This is a typical example of a tunneling effect induced by symmetries of the domain and the aim of this paper is to establish asymptotics for the spectral gap $E_2(Q_\Omega^\gamma) - E_1(Q_\Omega^\gamma)$ as $\gamma \rightarrow +\infty$.

In [PP16], the influence of the boundary on the eigenvalues of Q_Ω^γ is established in any dimension in the following form:

Theorem II.1.6 ([PP16]). *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^3 -domain. Then, for any fixed $n \in \mathbb{N}$ one has*

$$E_n(Q_\Omega^\gamma) = -\gamma^2 + E_n(-\Delta_{\partial\Omega} - \gamma K) + \mathcal{O}(1), \text{ as } \gamma \rightarrow +\infty,$$

where $-\Delta_{\partial\Omega}$ is the Laplace-Beltrami operator acting on $L^2(\partial\Omega, ds)$ and $s \mapsto K(s)$ denotes the sum of the principal curvatures of $\partial\Omega$.

The asymptotics of $E_n(Q_\Omega^\gamma)$ is then determined by the *effective* operator $-\Delta_{\partial\Omega} - \gamma K$ acting on $\partial\Omega$. Remark that here, no assumption on the points of maximal curvature is made in comparison with the previous result. Furthermore, more precise asymptotics are deduced studying the effective operator. We can also mention that this still holds for unbounded domains under additional assumptions on the boundary, providing in particular the existence of discrete spectrum.

As the above results indicate, the interest for Robin Laplacians acting on smooth domains was significant in the last decade and one can also mention the related papers [KP17, KL18, KL]. However, only a few results are available in the literature concerning non-smooth domains, and particularly domains with corners. A simple example can convince us that we expect a very different asymptotic behavior for the eigenvalues. Consider the cube $\mathcal{C}_d \subset \mathbb{R}^d$ with edges of length 1. By separation of variables, the study of $Q_{\mathcal{C}_d}^\gamma$ is reduced to the study of the one-dimensional Robin Laplacian acting on an interval of length 1 and an easy computation gives

$$E_1(Q_{\mathcal{C}_d}^\gamma) = -d\gamma^2 + o(\gamma^2), \text{ as } \gamma \rightarrow +\infty.$$

In [LP08, BP16], the asymptotic behavior of $E_1(Q_\Omega^\gamma)$ is investigated for a class of piecewise smooth domains admitting corners, to which the cube belongs to: the so-called *corner domains*. By a *cone*, we mean a Lipschitz domain $K \subset \mathbb{R}^d$ invariant by positive dilations.

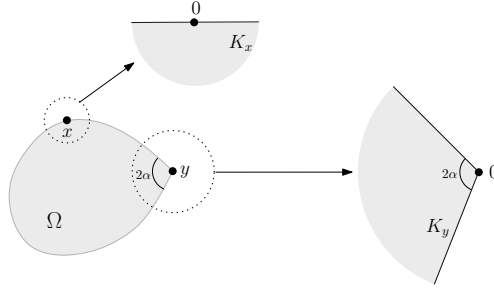


Figure II.2 – Tangent sectors of the curvilinear polygon Ω .

A domain $\Omega \subset \mathbb{R}^d$ is a *corner domain* if its boundary is Lipschitz, piecewise smooth and if for any $y \in \partial\Omega$, there exist a cone $K_y \subset \mathbb{R}^d$ and a smooth diffeomorphism F_y satisfying

$$F_y : \Omega \cap B(y, r) \rightarrow K_y \cap B(0, r), \quad F_y(\overline{\Omega \cap B(y, r)}) = \overline{K_y \cap B(0, r)}$$

for small $r > 0$, $F_y(y) = 0$ and $\nabla F_y(y) = Id$. We say that K_y is the *tangent cone* of $\partial\Omega$ at y . Notice that if $y \in \partial\Omega$ is a smooth point, the cone K_y is simply a half-space. In dimension two, the tangent cones are the full plane and the infinite sectors parametrized by their half-opening angle $\alpha \in (0, \pi)$, while the half-plane corresponds to $\alpha = \frac{\pi}{2}$. The corner domains in dimension two are the *curvilinear polygons*. They admit a finite number of corners of opening in $(0, \pi) \cup (\pi, 2\pi)$ and they include of course the smooth domains, see Figure II.2. In dimension three the corner domains admit a large variety of tangent cones as the full space, half-spaces, wedges, axisymmetric cones, ... For example, consider a point on the boundary of the three-dimensional cube: $y \in \partial\mathcal{C}_3$. The tangent cone K_y can be the half-space if y belongs to a face, a wedge if y belongs to an edge or a cone which section, that is to say the intersection with the unit sphere, is a spherical equilateral triangle of length $\frac{\pi}{2}$ if y is a vertex. We refer to [Dau88, BNDP16] for a more rigorous description of corner domains and their properties. As for any cone K_y the Robin Laplacian $Q_{K_y}^\gamma$ does not have a compact resolvent, the essential spectrum is not empty and we set $\Lambda(K_y, \gamma) := \inf \text{spec}(Q_{K_y}^\gamma)$. The invariance by dilations of K_y implies that $\Lambda(K_y, \gamma) = \gamma^2 \Lambda(K_y, 1)$.

Theorem II.1.7 ([LP08, BP16]). *Let $\Omega \subset \mathbb{R}^d$ be a corner domain. Then,*

$$E_1(Q_\Omega^\gamma) = \left(\inf_{y \in \partial\Omega} \Lambda(K_y, 1) \right) \gamma^2 + o(\gamma^2), \quad \text{as } \gamma \rightarrow +\infty.$$

In view of this asymptotics, one can say that the Robin Laplacians acting on cones play the role of *model operators* and understanding them better is needed to obtain more precise asymptotics for corner domains. In [LP08], it is proved that if $d = 2$ and α is the half-aperture of the infinite sector K_y then,

$$\Lambda(K_y, 1) = -\frac{1}{\sin^2 \alpha} \text{ if } \alpha < \frac{\pi}{2}, \quad \Lambda(K_y, 1) = 1 \text{ otherwise.} \quad (\text{II.1.8})$$

When $\alpha < \frac{\pi}{2}$, the infimum of the spectrum is in fact a discrete eigenvalue associated with the eigenfunction $e^{-\frac{x_1}{\sin \alpha}}$, where (x_1, x_2) denotes the cartesian coordinates in \mathbb{R}^2 and Ox_1 coincides with the bisector of the sector. In dimension three, some estimates on $\Lambda(K_y, 1)$ are given in [LP08] for cones admitting a smooth bounded convex section. One can also mention the case of circular cones which is well known: the bottom of the spectrum is the discrete eigenvalue $-\frac{1}{\sin^2 \alpha}$ with α being the spherical radius of the section and it is

associated with the eigenfunction $e^{-\frac{z}{\sin \alpha}}$ where Oz coincide with the axis of revolution of the cone, and the case of cones which section is a spherical polygon admitting an inscribed circle as the estimate in [LP08] becomes an equality, see [LP08, Remark 5.3]. Remark that, due to these results we have another way to recover the asymptotics for three-dimensional cubes. Let y be a point of the boundary of the three-dimensional cube \mathcal{C}_3 . If y belongs to a face, it is a smooth point and the associated tangent cone is the half-plane. Using the fact that $E_1(Q_{\mathbb{R}^+}^1) = -1$ we then get $\Lambda(K_y, 1) = -1$. If y belongs to an edge, we can write K_y as follows: $K_y = \mathbb{R} \times U_{\frac{\pi}{4}}$ where $U_{\frac{\pi}{4}}$ is an infinite sector of half-aperture $\frac{\pi}{4}$ and thus by (II.1.8) we have $\Lambda(K_y, 1) = -2$. Finally, if y is a vertex the associated cone has a section which is a spherical equilateral triangle and thus admits an inscribed circle. Using [LP08, Remark 5.3] one can prove that $\Lambda(K_y, 1) = -3$. We can now apply Theorem II.1.7 to get

$$E_1(Q_{\mathcal{C}_3}^\gamma) = -3\gamma^2 + o(\gamma^2), \text{ as } \gamma \rightarrow +\infty.$$

In addition, an investigation on the existence of the discrete spectrum of cones in any dimension was carried out in [Pan16], and it turns out that cones can admit infinitely many discrete eigenvalues. In dimension three this gives the following simple statement, see [Pan16, Corollary 9]:

Proposition II.1.8. *Let $K \subset \mathbb{R}^3$ be a cone with a simply connected smooth cross section. If $\mathbb{R}^3 \setminus K$ is convex, then the discrete spectrum of Q_K^γ is empty otherwise it is infinite.*

In [BPP18], the asymptotic distribution of the discrete eigenvalues of three-dimensional cones is studied for fixed γ . They prove in particular that the Weyl asymptotics for the eigenvalues is led by the curvature of the section of the cone.

Due to the asymptotics in Theorem II.1.7 and the large variety of boundary singularities of corner domains, a precise study of the asymptotic behavior of the eigenvalues should be performed with respect to the ambient dimension. From now on, we focus on planar domains, namely $d = 2$. Let us come back to the statement (II.1.8). If we consider curvilinear polygons with *convex* vertices, namely their half-aperture satisfies $\alpha \in (0, \frac{\pi}{2})$, then the first order in the asymptotics of $E_1(Q_\Omega^\gamma)$ is given by the *most convex* vertex. On the contrary, for curvilinear polygons which admit only *non-convex* corners the leading order is given by $-\gamma^2$, and is thus the same as the one for smooth domains, see (II.1.7). Hence, the non-convex vertices do not influence the leading order of $E_1(Q_\Omega^\gamma)$ in the asymptotic regime $\gamma \rightarrow +\infty$. We can also say, in reference to the famous question of Kac, that it seems that one cannot hear non-convex corners. A result in favor of this idea is the following:

Theorem II.1.9 ([Pan13]). *Let $\Omega \subset \mathbb{R}^2$ be a C^4 curvilinear polygon admitting only non-convex corners. Then,*

$$E_1(Q_\Omega^\gamma) = -\gamma^2 - \gamma\kappa_{\max} + \mathcal{O}(\gamma^{\frac{2}{3}}), \text{ as } \gamma \rightarrow +\infty,$$

where κ_{\max} is the maximum of the curvature of $\partial\Omega$.

The proof is similar to the one for smooth domains. It is based on a Dirichlet-Neumann bracketing, reducing the study to a small vicinity of the boundary. At a technical level, we can understand why we recover the same asymptotics as the one for smooth domains: indeed, when the curvilinear polygon admits only non-convex corners and a smooth boundary, we can easily construct a tubular neighborhood of the latter, whereas it is no longer the case when there are convex corners. Moreover in [Pan15], an asymptotics of any $E_n(Q_\Omega^\gamma)$ is obtained when $\Omega \subset \mathbb{R}^2$ is the exterior of a convex polygon. The first order is again given by $-\gamma^2$, while the second one is exactly the n -th eigenvalue of the Laplacian acting on the sides of the domain with Dirichlet boundary conditions at each vertex. A natural question that we can ask in view of these results is then:

What is the influence of the convex corners on the spectral properties of Robin Laplacians ?

The main goal of this thesis is to study this question by obtaining more precise asymptotics of the eigenvalues of curvilinear polygons.

Even if the computation of the eigenvalues of the Laplacian is a difficult task in general, we know some explicit examples which can help us to have an intuition on their asymptotic behavior. First, in his paper [McC11], McCartin makes a quite complete study of the Laplacian acting on equilateral triangles, and he obtains in particular some asymptotics for the eigenvalues of the Robin Laplacian. Let $\mathcal{T} \subset \mathbb{R}^2$ be an equilateral triangle. Then it is proved in [McC11, Section 7.4] that for any $n = 1, 2, 3$ there holds

$$E_n(Q_{\mathcal{T}}^\gamma) = -4\gamma^2 + o(\gamma^2), \text{ as } \gamma \rightarrow +\infty,$$

and

$$E_4(Q_{\mathcal{T}}^\gamma) = -\gamma^2 + o(1), \text{ as } \gamma \rightarrow +\infty.$$

Of course, the asymptotics of $E_1(Q_{\mathcal{T}}^\gamma)$ was already known by the result of [LP08, BP16], but we see here that the two next eigenvalues behave the same way while the fourth one has the same leading term as the one for smooth domain. One can also mention the result of [HP15], in which an isosceles triangle is studied. Denote by $\mathcal{T}_\theta \subset \mathbb{R}^2$ an isosceles triangle which two equal angles are $\theta \in (0, \frac{\pi}{3})$. Then, there exists a constant $c > 0$ such that for $n = 1, 2$ we have

$$E_n(Q_{\mathcal{T}_\theta}^\gamma) = -\frac{\gamma^2}{\sin^2(\theta/2)} + \mathcal{O}(e^{-c\gamma}), \text{ as } \gamma \rightarrow +\infty.$$

Actually more precise asymptotics are derived in [HP15], and in particular an asymptotics of the spectral gap $E_2(Q_{\mathcal{T}_\theta}^\gamma) - E_1(Q_{\mathcal{T}_\theta}^\gamma)$ is obtained. As a last example, we mention the square. Due to separation of variables, the study is reduced to the one-dimensional Robin Laplacian on an interval and we thus can obtain precise asymptotics of any eigenvalue, see Example III.5.5 below. Let $\square_\ell \subset \mathbb{R}^2$ be a square of length ℓ . For any $n = 1, 2, 3, 4$ there exists a constant $c > 0$ such that

$$E_n(Q_{\square_\ell}^\gamma) = -2\gamma^2 + \mathcal{O}(e^{-c\gamma}), \text{ as } \gamma \rightarrow +\infty,$$

and moreover for any $j \in \mathbb{N}$,

$$E_{4+j}(Q_{\square_\ell}^\gamma) = -\gamma^2 + \mu_j + o(1), \text{ as } \gamma \rightarrow +\infty,$$

where μ_j is exactly the j -th element of the union of the four disjoint copies of $\left\{ \left(\frac{\pi k}{\ell} \right)^2, k \in \mathbb{N} \right\}$, and in particular $\mu_j > 0$. These examples show that the eigenvalues of polygons exhibit two different behaviors: the first eigenvalues seem to be influenced by the presence of convex corners due to the result of [LP08, BP16]. The further ones however have the same leading term as the one for smooth domains and the next term can be very different, if we compare what happens for the equilateral triangle and the square. To understand the asymptotic behavior of the eigenvalues of curvilinear polygons, it seems that one has to carry out separately a study of the eigenvalues induced by the corners, and the further ones.

In the next section, we explain the new results obtained for curvilinear polygons. Due to [LP08, BP16], we understand that a complete study of what we called the model operators, that is to say the Robin Laplacians acting on infinite sectors, is needed to have a better

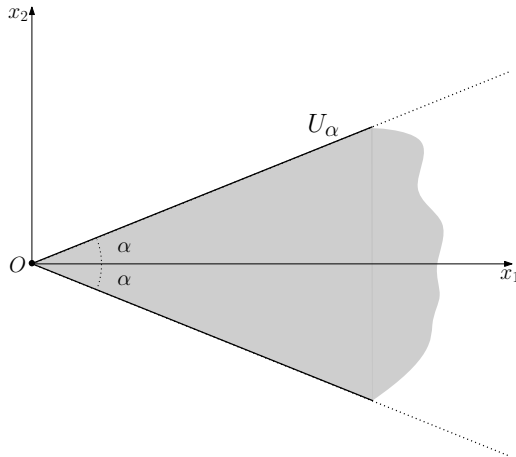


Figure II.3 – The infinite sector U_α .

understanding of Robin Laplacians acting on curvilinear polygons. Thus, in Section II.2.1 we list the results obtained for these operators. In Section II.2.2 we state the theorem concerning the asymptotics of the eigenvalues induced by corners. In Section II.2.3, we describe the results obtained for the further eigenvalues. Finally, we focus in Section II.2.4 on the asymptotic behavior of the associated eigenvalue counting function.

II.2 Main results

II.2.1 Robin Laplacians in infinite sectors

For $\alpha \in (0, \pi)$, let U_α denote the infinite sector of opening 2α ,

$$U_\alpha = \{(x_1, x_2) \in \mathbb{R}^2 : |\arg(x_1 + ix_2)| < \alpha\},$$

see Figure II.3. We are interested in the spectral properties of the associated Robin Laplacian, to be denoted by $T_\alpha^\gamma := Q_{U_\alpha}^\gamma$, which is defined as the unique self-adjoint operator in $L^2(U_\alpha)$ associated with the sesquilinear form

$$t_\alpha^\gamma(u, u) = \int_{U_\alpha} |\nabla u|^2 dx_1 dx_2 - \gamma \int_{\partial U_\alpha} |u|^2 ds, \quad u \in H^1(U_\alpha), \quad (\text{II.2.1})$$

where ds is the one-dimensional Hausdorff measure, see Theorem IV.1.1 for a complete justification.

The study of the above operator T_α^γ has several interesting aspects from the point of view of the existing results. First, it can be viewed as the simplest non-smooth domain in two dimensions and depending in an explicit way on the single geometric parameter α . Second, as already mentioned in the previous section, its spectral properties play an important role in the study of more general non-smooth domains in the strong coupling limit $\gamma \rightarrow +\infty$. Indeed, the sector U_α exhausts the whole family of possible tangent cones to the boundary in two dimensions and thus, according to the results of [BP16, CGM11, LP08], having a better understanding of T_α^γ will allow us to obtain more information about the Robin Laplacian acting on curvilinear polygons. Third, the domain U_α and its boundary are non-compact, which may potentially lead to quite unusual spectral properties.

In the sequel we state the main results obtained on T_α^γ and gathered in our paper *Eigenvalues of Robin Laplacians in infinite sectors* [KP18]. The proofs are given in

Chapter IV. Let $\gamma > 0$ be fixed. The only spectral result on T_α^γ available in the existing literature was as follows, see [LP08]:

$$\inf \operatorname{spec} T_\alpha^\gamma = \begin{cases} -\gamma^2, & \alpha \geq \frac{\pi}{2}, \\ -\frac{\gamma^2}{\sin^2 \alpha}, & \alpha < \frac{\pi}{2}, \end{cases} \quad (\text{II.2.2})$$

and for $\alpha < \frac{\pi}{2}$ the value indicated is an eigenvalue with an explicitly known eigenfunction $\exp(-\gamma x_1 / \sin \alpha)$. The aim of the present work is to provide a more detailed spectral analysis.

First, we prove that the essential spectrum of T_α^γ does not depend on the angle α ,

$$\operatorname{spec}_{\text{ess}}(T_\alpha^\gamma) = [-\gamma^2, +\infty),$$

see Theorem IV.1.1. The proof is purely variational, and we construct in particular a Weyl sequence to characterize the essential spectrum. It follows from (II.2.2) that the discrete spectrum is non-empty if and only if $\alpha < \frac{\pi}{2}$, i.e. if and only if the sector is strictly smaller than the half-plane.

In Theorem IV.2.1 we show that the discrete spectrum is always finite, which is a non-trivial result due to the non-compactness of the boundary. Moreover, this result is dimension-dependent in the sense that Robin Laplacians on cones may have an infinite discrete spectrum in higher dimensions, as shown in [BPP18, Pan16]. To prove Theorem IV.2.1, we use an idea proposed in [MT05] for a different operator involving a similar geometry. We first make an appropriate domain decomposition thanks to a partition of unity. Then, we perform a reduction of the dimension to reduce the study to a one-dimensional operator, using an orthogonal projection. We conclude with the help of a Bargmann-type estimate for the number of eigenvalues of one-dimensional Schrödinger operators, see e.g. [Sim76].

In Section IV.2.3 we obtain more detailed results: in Theorem IV.2.3 we show that each individual eigenvalue is a strictly increasing continuous function of the angle α , thanks to a suitable change of variables, and, moreover, that there is just one discrete eigenvalue for $\alpha \geq \frac{\pi}{6}$, see Theorem IV.2.6. To do that, we make a decomposition of the domain which permits us to compare the eigenvalues of $T_{\frac{\pi}{6}}^\gamma$ to the eigenvalues of Robin Laplacians acting on equilateral triangles. We conclude thanks to the results of [McC11] on equilateral triangles. Remark that we are not able to prove here that $\frac{\pi}{6}$ is a jump for the eigenvalue counting function of T_α^γ . However, some evidence let us think that it is the case, see Chapter VII.

In Section IV.3 we discuss the behavior of the discrete eigenvalues for small α . For any fixed $\gamma > 0$, we show that the n -th eigenvalue $E_n(T_\alpha^\gamma)$ behaves as

$$E_n(T_\alpha^\gamma) = -\frac{\gamma^2}{(2n-1)^2\alpha^2} + \mathcal{O}(1), \quad \alpha \rightarrow 0, \quad (\text{II.2.3})$$

see Corollary IV.3.3. The proof is based on the well-known Born-Oppenheimer strategy [CDS81, Ray17] and, similar to various problems involving small parameters [Bon05, DR12, DR14, HJ11, Naz14, Ray17], on a reduction to a one-dimensional effective operator, which in our case acts in $L^2(\mathbb{R}_+)$ as

$$f \mapsto \left(-\frac{d^2}{dr^2} - \frac{1}{4r^2} - \frac{1}{\alpha r} \right) f$$

with a suitable boundary condition at the origin. This first order asymptotics tells us in particular that each eigenvalue is simple as the angle is small. This allows us to apply the

standard perturbation theory to obtain a full asymptotic expansion up to any order with respect to the powers of α^2 , see Theorem IV.3.15. A second consequence of the first order asymptotics is that the number of discrete eigenvalues becomes arbitrarily large and is minorated by κ/α , $\kappa > 0$, as α approaches 0, see Corollary IV.3.2.

Finally, in Theorem IV.4.1 we show that the associated eigenfunctions are localized, in a suitable sense, near the vertex of U_α . This is done with the help of the standard Agmon-type approach using a suitable decomposition of the domain [Agm82].

II.2.2 Corner-induced eigenstates of Robin Laplacians on polygonal domains

This section is devoted to the corner-induced eigenvalues of Robin Laplacians acting on the planar corner domains: the curvilinear polygons. The results are gathered in the paper *Spectral asymptotics for Robin Laplacians on polygonal domains* [Kha18]. Only a few results exist for Robin Laplacians acting on non-smooth domains and many questions remain open. However, with this work we can answer to two questions asked in the recent review paper [BFK17, Open problem 4.19]:

Open problem II.2.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded, piecewise smooth domain having $L \geq 1$ corners with half-angles $\alpha_1 \leq \dots \leq \alpha_L < \frac{\pi}{2}$. Is it true that the first L eigenvalues have the asymptotic behavior*

$$E_n(Q_\Omega^\gamma) \sim -\frac{\gamma^2}{\sin^2 \alpha_n}, \quad \text{as } \gamma \rightarrow +\infty,$$

for $n = 1, \dots, L$? How does $E_n(Q_\Omega^\gamma)$ behave for fixed $n \geq L$? Investigate the corresponding situation in higher dimensions and for more general Ω .

It appears that the conjecture is not true as stated, and we can propose and prove a correct version. But before stating our results, let us first recall a rigorous definition of the curvilinear polygons, see e.g [Gri85, Definition 1.4.5.1], [Dau88, Chapter 1] or [BNDP16, Section II. 3], slightly adapted to our needs.

Definition II.2.2. Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded open set. We say that Ω is a *curvilinear polygon* if $\partial\Omega$ is Lipschitz and if there exist $M \geq 1$ non-intersecting connected arcs Γ_k , $k = 1, \dots, M$, such that

$$\partial\Omega = \bigcup_{k=1}^M \overline{\Gamma_k},$$

and if we denote by l_k the length of Γ_k and by ζ_k a parametrization of $\overline{\Gamma_k}$ by the arc length then $\zeta_k \in C^4([0, l_k])$. Moreover, if two components Γ_k, Γ_j intersect at some point $v := \Gamma_k(l_k) = \Gamma_j(0)$, then two cases are allowed: either $\overline{\Gamma_k} \cup \overline{\Gamma_j}$ is C^4 near v and then v is called a regular point of $\partial\Omega$, or the corner opening angle at v , denoted by $2\alpha_v$, measured inside Ω and formed by the one-sided tangents at v belongs to $(0, \pi) \cup (\pi, 2\pi)$. In the latter case, v is called a vertex of Ω .

Notice that cusps (zero angles) are not allowed by our definition as the boundary is Lipschitz.

We introduce the set of convex vertices of Ω by

$$\mathcal{V} := \left\{ v \in \partial\Omega, v \text{ is a vertex of } \Omega \text{ and } \alpha_v \in (0, \frac{\pi}{2}) \right\}.$$

It is then easy to see that for each $v \in \mathcal{V}$ there exist $r_v > 0$ and F_v a C^2 -diffeomorphism satisfying the following conditions:

- (a) $F_v : \Omega \cap B(v, r_v) \rightarrow U_{\alpha_v} \cap B(0, r_v)$,
- (b) $F_v(\overline{\Omega \cap B(v, r_v)}) = \overline{U_{\alpha_v} \cap B(0, r_v)}$,
- (c) $F_v(v) = 0$ and $\nabla F_v(v) = I_2$,

where I_2 stands for the identity matrix in two dimensions, $B(v, r_v)$ is the ball of center v and radius r_v in \mathbb{R}^2 , ∇F_v is the Jacobian matrix of F_v . Then, a suitably rotated copy of U_{α_v} becomes the tangent sector of Ω at v .

Let $\Omega \subset \mathbb{R}^2$ be a curvilinear polygon. We define the model operator

$$T^\oplus := \bigoplus_{v \in \mathcal{V}} T_{\alpha_v}^1, \text{ and } \mathcal{N}^\oplus := \sum_{v \in \mathcal{V}} \mathcal{N}_v,$$

where $\mathcal{N}_v := \#\{n \in \mathbb{N} : E_n(T_{\alpha_v}^1) < -1\}$ denotes the number of discrete eigenvalues of $T_{\alpha_v}^1$. Recall that $\mathcal{N}_v < +\infty$ by Theorem IV.2.1.

Our main result gives the asymptotic behavior of the \mathcal{N}^\oplus first eigenvalues of Q_Ω^γ as γ becomes large.

Theorem II.2.3. *For any $n \in \{1, \dots, \mathcal{N}^\oplus\}$ there holds*

$$E_n(Q_\Omega^\gamma) = E_n(T^\oplus)\gamma^2 + r(\gamma), \text{ as } \gamma \rightarrow +\infty,$$

where $r(\gamma) = O(\gamma^{4/3})$, and one can take $r(\gamma) = O(e^{-c\gamma})$ with $c > 0$ if Ω is a polygon with straight edges.

For a more detailed statement see Theorem V.2.1 for polygons with straight edges and Theorem VI.3.1 for the general case. In Theorem II.2.3, we make the assumption that Ω is simply connected, see Definition II.2.2. It will be clear from the proof that this assumption is made for the sake of simplicity. The result extends in fact to finite unions of connected domains.

By Theorem II.2.3, we see that the conjecture stated in Open problem II.2.1 becomes false if $E_2(T_v) < E_1(T_w)$ for some $v, w \in \mathcal{V}$, which happens for α_v small enough, see (II.2.3). However, it is possible to find a setting for which the conjecture holds true, which is due to the fact that T_α^γ admits a unique discrete eigenvalue for any $\alpha \geq \frac{\pi}{6}$.

Corollary II.2.4. *Let $\Omega \subset \mathbb{R}^2$ be a curvilinear polygon having $L \geq 1$ convex vertices with half angles $\frac{\pi}{6} \leq \alpha_1 \leq \dots \leq \alpha_L < \frac{\pi}{2}$. Then,*

$$E_n(Q_\Omega^\gamma) = -\frac{\gamma^2}{\sin^2 \alpha_n} + O\left(\gamma^{\frac{4}{3}}\right), \quad \gamma \rightarrow +\infty,$$

for all $n = 1, \dots, L$.

The proof of Theorem II.2.3 uses an idea of [BND06] in which a magnetic Laplacian acting on curvilinear polygons is considered. It mainly relies on the construction of quasi-modes, thanks to the eigenfunctions of the model operator. For $v \in \mathcal{V}$, let $\psi_n^{\gamma;v}$ be an eigenfunction of $T_{\alpha_v}^\gamma$ associated with a discrete eigenvalue $E_n(T_{\alpha_v}^\gamma)$, $n \leq \mathcal{N}_v$. Now denote by $\phi_n^{\gamma;v} := \psi_n^{\gamma;v} \circ F_v$ and for $\rho > 0$ small enough, introduce χ_v^ρ a smooth radial cut-off function centered in v which support is contained in $B(v, \rho)$. We can define the quasi-mode

$$\tilde{\phi}_n^{\gamma;v} := \chi_v^\rho \phi_n^{\gamma;v}.$$

It is easy to see that defined in this way, $\tilde{\phi}_n^{\gamma;v} \in D(Q_\Omega^\gamma)$. Using the decay property of $\psi_n^{\gamma;v}$, we can prove some useful estimates on the quasi-modes: they form a linearly independent

family for large γ and they satisfy $q_\Omega^\gamma(\tilde{\phi}_n^{\gamma,v}, \tilde{\phi}_n^{\gamma,v}) = E_n(T_{\alpha_v}^\gamma) \|\tilde{\phi}_n^{\gamma,v}\|_{L^2(\Omega)}^2 + \mathcal{O}(\gamma^\theta)$, where $\theta \in (0, 2)$. Applying the min-max principle, this gives easily an upper bound for the eigenvalues of Q_Ω^γ . The lower bound is obtained with a partition of unity of the domain. For polygon with straight edges, the partition of unity is independent of γ and the proof of the lower bound is quite straightforward. However, when the sides are curved, it is necessary to use a partition of unity depending on the parameter γ and for this reason the proof becomes more technical. Let us develop this idea. We will define a partition of unity $(\chi_\mathcal{V}^\gamma, \chi_0^\gamma)$ such that the support of $\chi_\mathcal{V}^\gamma$ is a union of balls centered in the convex vertices of Ω and the radii $r_\gamma \rightarrow 0$ as $\gamma \rightarrow +\infty$ and such that $\text{supp } \chi_0^\gamma \cap \mathcal{V} = \emptyset$. Putting the partition of unity in the sesquilinear form q_Ω^γ , this gives us a lower bound in terms of a direct sum of operators acting on truncated sectors of radii r_γ and on $\Omega_0^\gamma := \text{supp } \chi_0^\gamma \cap \Omega$ modulo a remainder. The new operators act as Laplacians with Robin boundary condition on the common part of the boundary with Ω and Dirichlet boundary condition on the remaining part. By monotony of the eigenvalues of Dirichlet Laplacians, the eigenvalues of operators acting on truncated sectors admit as lower bounds the eigenvalues of Robin Laplacians acting on the associated infinite sectors. To conclude, it remains to prove that the first eigenvalue of the operator acting on Ω_0^γ is greater than the ones of the infinite sectors. The simplest way to do this would be to use again the monotony of the Dirichlet eigenvalues and extend the domain Ω_0^γ in a smooth way to apply the result of [Pan13]: if Ω^{reg} is a curvilinear polygon admitting only non-convex vertices we have for large γ , $E_1(Q_{\Omega^{\text{reg}}}^\gamma) = -\gamma^2 - \kappa_{\max} \gamma + \mathcal{O}(\gamma^{\frac{2}{3}})$. However, in our case, Ω_0^γ depends on γ and an additional work is then needed.

In addition, we show in Proposition V.3.2 for polygons with straight edges and Proposition VI.3.4 for curvilinear polygons, that the \mathcal{N}^\oplus first associated eigenfunctions are localized, in a rigorously defined sense, near the convex vertices of Ω . The proofs use the same strategy as the one for the infinite sectors, namely the standard Agmon-type approach, based on a decomposition of the domain.

In Section V.3, we also prove that in the case of a polygon with straight edges, the \mathcal{N}^\oplus first eigenfunctions are exponentially close to linear combinations of the quasi-modes. To be more specific, let u_n be a normalized eigenfunction of Q_Ω^γ where $n \leq \mathcal{N}^\oplus$. We introduce the set of the *associated* quasi-modes,

$$\mathcal{F}_n^\gamma := \left\{ \tilde{\phi}_j^{\gamma,v} : E_j(T_{\alpha_v}^\gamma) = \gamma^2 E_n(T^\oplus) \right\}.$$

Denote by $P_{\mathcal{F}_n^\gamma}$ the orthogonal projection on \mathcal{F}_n^γ in $L^2(\Omega)$. Then, we can prove that for large γ ,

$$\|u_n - P_{\mathcal{F}_n^\gamma} u_n\| = \mathcal{O}(e^{-c\gamma}), \quad c > 0,$$

see Theorem V.3.1. The proof is based on the fact that in that case, the quasi-modes $\tilde{\phi}_n^{\gamma,v}$ belong to the domain of Q_Ω^γ because they satisfy the Robin boundary condition on $\partial\Omega$. The results in [HS84] on the distance between subspaces are relevant in this context and allow us to conclude.

II.2.3 The further eigenvalues

The next natural step of this work is to obtain the asymptotics of the further eigenvalues, namely $E_{\mathcal{N}^\oplus+j}(Q_\Omega^\gamma)$, as γ is large. As a first answer, we obtain the following result which particularly means that the corners do not contribute to the asymptotics at the first order.

Proposition II.2.5. *Let $\Omega \subset \mathbb{R}^2$ be a curvilinear polygon. For any $j \in \mathbb{N}$, there holds*

$$E_{\mathcal{N}^\oplus+j}(Q_\Omega^\gamma) = -\gamma^2 + o(\gamma^2), \quad \text{as } \gamma \rightarrow +\infty.$$

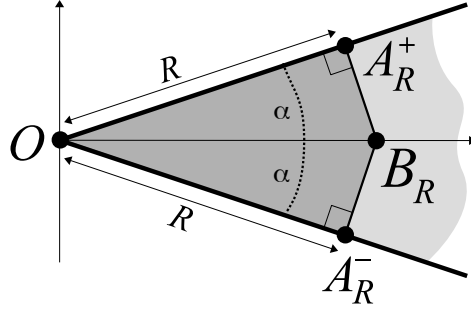


Figure II.4 – The truncated sector $U_{\alpha, R}$ is shaded.

Remark that this answers partially to the second question of the Open Problem II.2.1. The proof, see Section VI.5, is essentially a consequence of the preceding constructions.

By analogy with the smooth case, see Theorem II.1.6, one may expect the following asymptotics for curvilinear polygons,

$$E_{\mathcal{N}^{\oplus+j}}(Q_{\Omega}^{\gamma}) = -\gamma^2 + E_j(\tilde{L}_{\gamma}) + R(\gamma), \text{ as } \gamma \rightarrow +\infty,$$

where the effective operator \tilde{L}_{γ} acts in $L^2(\partial\Omega)$ as $u \mapsto -\partial^2 u - \gamma\kappa u$, where κ is the curvature of $\partial\Omega$, on functions satisfying some additional boundary conditions at the corners and $R(\gamma)$ is a small remainder. We obtain an asymptotics sustaining this conjecture for polygons with straight edges. The curvilinear case turns out to be much more technical and will be studied later. The results presented in this section come from a collaboration with Konstantin Pankrashkin and Thomas Ourmières-Bonafos (Orsay). We first need to introduce some intermediary notation.

Let us define a special truncation of U_{α} . Let $0 < \alpha < \frac{\pi}{2}$ and $R > 0$. Consider the points

$$A_R^{\pm} = R(\cos \alpha, \pm \sin \alpha) \in \partial U_{\alpha}, \quad B_R = R\left(\frac{1}{\cos \alpha}, 0\right) \in U_{\alpha},$$

and denote by $U_{\alpha, R}$ the quadrangle $OA_R^+B_R^+A_R^-$. Remark that the sides BA_R^{\pm} are orthogonal to ∂U_{α} at A_R^{\pm} , see Figure II.4. Denote by $T_{\alpha, R}^{\gamma}$ the operator acting on $L^2(U_{\alpha, R})$ defined as the unique self-adjoint operator associated with

$$t_{\alpha, R}^{\gamma}(u, u) = \int_{U_{\alpha, R}} |\nabla u|^2 dx - \gamma \int_{\partial U_{\alpha, R} \cap \partial U_{\alpha}} |u|^2 ds, \quad D(t_{\alpha, R}^{\gamma}) := H^1(U_{\alpha, R}). \quad (\text{II.2.4})$$

The operator $T_{\alpha, R}^{\gamma}$ acts as the Laplacian on $U_{\alpha, R}$ with the Robin boundary condition on $\partial U_{\alpha, R} \cap \partial U_{\alpha}$ and the Neumann boundary condition on the left part of the boundary. It plays a crucial role in Section V.4 and in the following definition:

Definition II.2.6. We say that an angle $\alpha \in (0, \frac{\pi}{2})$ is *non-resonant* if for some $\gamma > 0$ there exists a constant $C > 0$ such that

$$E_{\mathcal{N}_{\alpha+1}}(T_{\alpha, R}^{\gamma}) \geq -\gamma^2 + \frac{C}{R^2}, \text{ as } R \text{ is large.} \quad (\text{II.2.5})$$

Remark that this above property does not depend on a specific choice of $\gamma > 0$ due to the scale invariance

$$E_n(T_{\alpha, R}^{\gamma}) = \gamma^2 E_n(T_{\alpha, \gamma R}^1),$$

and we have the following simple result, based essentially on the eigenvalue monotony with respect to α :

Proposition II.2.7. *All angles $\alpha \in [\frac{\pi}{4}, \frac{\pi}{2})$ are non-resonant.*

The proof is given in Section IV.5.

Let $\Omega \subset \mathbb{R}^2$ be a polygon with straight edges admitting $V \in \mathbb{N}$ vertices. Denote its vertices by $A_1, \dots, A_V \in \mathbb{R}^2$, and assume that they are enumerated in such a way that the boundary $\partial\Omega$ is the union of the V line segments $[A_v, A_{v+1}]$, where we denote

$$A_{V+1} := A_1 \text{ and } A_0 := A_V,$$

and we assume that there are no artificial vertices. The following objects will be of importance:

- the angle of the polygon Ω at the vertex A_v , i.e. the angle between the line segments $[A_v, A_{v-1}]$ and $[A_v, A_{v+1}]$ measured *inside* Ω , is still denoted by $2\alpha_v$. Remark that by the above assumptions one has

$$\alpha_v \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi) \text{ for any } v.$$

- $\mathcal{N}^\oplus := \mathcal{N}_{\alpha_1} + \dots + \mathcal{N}_{\alpha_V}$,
- The length $\ell_v = |A_{v+1} - A_v|$ of the v th side of Ω .
- The Dirichlet Laplacian D_v acting as $f \mapsto -f''$ on $(0, \ell_v)$ viewed as a self-adjoint operator in $L^2(0, \ell_v)$,
- The operator $D := D_1 \oplus \dots \oplus D_V$, which will be interpreted as an effective operator on $\partial\Omega$.

Our main result is as follows.

Theorem II.2.8. *Assume that for each $v \in \{1, \dots, V\}$ one has either $\alpha_v > \frac{\pi}{2}$ or α_v is non-resonant, then for every fixed $j \in \mathbb{N}$ there holds*

$$E_{\mathcal{N}^\oplus+j}(Q_\Omega^\gamma) = -\gamma^2 + E_j(D) + \mathcal{O}\left(\frac{\log \gamma}{\sqrt{\gamma}}\right) \text{ as } \gamma \text{ tends to } +\infty.$$

The proof is given in Section V.4. In view of Proposition II.2.7 we have then the following particularly simple case:

Corollary II.2.9. *If $\alpha_v \geq \frac{\pi}{4}$ for all v , then for every fixed $j \in \mathbb{N}$ there holds*

$$E_{\mathcal{N}^\oplus+j}(Q_\Omega^\gamma) = -\gamma^2 + E_j(D) + \mathcal{O}\left(\frac{\log \gamma}{\sqrt{\gamma}}\right) \text{ as } \gamma \text{ tends to } +\infty.$$

It will be seen that the first step of the proof of Theorem II.2.8 is a Dirichlet-Neumann bracketing. Obtaining the upper bound for the eigenvalues is quite straightforward, due to the non-resonance of the half-angles α_v . As usual, obtaining a lower bound is a more difficult task. The non-resonance condition is used to show that the associated eigenfunctions near the corners are dominated, in a suitable sense, by their value in the interior of Ω , which is done in the spirit of the work [Pos05] where shrinking wave guides are studied. The study of general curvilinear polygons will be the main topic of an upcoming project on which we give some details in Chapter VII.

Let us make a last remark on the motivations of the presented work. The study of Robin Laplacians acting on curvilinear polygons was in part stimulated by the paper [BND06] in which a magnetic Neumann Laplacian is studied. Both of the magnetic Neumann

Laplacian and the Robin Laplacian appear in models in the theory of superconductivity [GS07] and it seems that these operators show some similarities in two dimensions. In particular, assuming that the magnetic field is constant but not zero, it is proved in [HM01, FH06] that the eigenfunctions of the magnetic Laplacian are localized near the point of maximal curvature of the domain when it is unique, and an analogue of (II.1.7) is derived. As mentioned in the previous section, the study of magnetic Neumann Laplacians acting on curvilinear polygons has been made in [BND06] and similar results are obtained, namely the model operators are the magnetic Neumann Laplacians on infinite sectors. However, one should remark that in that case, the number of discrete eigenvalues and the behavior of the infimum of the essential spectrum with respect to the angle of the model operators are not known. No result is available for the further eigenvalues of magnetic Laplacian on curvilinear polygons, and one can hope that the progress made for Robin Laplacians could give some new insights to study the magnetic ones.

II.2.4 Weyl-type asymptotics on curvilinear polygons

In this section, we introduce the results regarding the eigenvalue counting function of the operator Q_Ω^γ , see Section VI.4 for the proofs.

Let A be a self-adjoint operator semibounded from below acting on a Hilbert space \mathcal{H} . For $\lambda \in \mathbb{R}$, we denote by $\mathcal{N}(A, \lambda)$ the number of eigenvalues, counted with the multiplicities, of A in $(-\infty, \lambda)$ if $\text{spec}_{\text{ess}}(A) \cap (-\infty, \lambda) = \emptyset$, and $\mathcal{N}(A, \lambda) = +\infty$ otherwise.

The study of the asymptotic distribution of the eigenvalues of the Laplacian is a classical question in spectral theory. Indeed, the explicit computation of the eigenvalues is a difficult task in general and the eigenvalues are only known for very specific geometries as rectangles, circles or equilateral triangles. It is then natural to look for some information on the eigenvalue counting function. The Weyl law, obtained by Hermann Weyl in 1911, is the very first result obtained on this topic. Denote by Q_Ω^D the Laplacian acting on $L^2(\Omega)$ with the Dirichlet boundary condition on $\partial\Omega$, i.e. $u = 0$ on $\partial\Omega$. Then there holds

$$\mathcal{N}(Q_\Omega^D, \lambda) = \frac{\omega_d |\Omega|}{(2\pi)^d} \lambda^{\frac{d}{2}} (1 + o(1)), \quad \text{as } \lambda \rightarrow +\infty, \quad (\text{II.2.6})$$

where ω_d is the volume of the unit ball in \mathbb{R}^d . Some refinements of the asymptotics (II.2.6) were obtained, adding some assumptions on the domain Ω , and we refer to the review paper [Ivr16] for a detailed discussion.

In our case, we are interested in the distribution of the negative eigenvalues of the Robin Laplacian Q_Ω^γ in the asymptotic regime $\gamma \rightarrow +\infty$. The following Weyl-type asymptotics were proved in [HKR17] for smooth bounded $\Omega \subset \mathbb{R}^2$. For any $E \in (-1, 0)$ there holds

$$\mathcal{N}(Q_\Omega^\gamma, E\gamma^2) = \gamma \frac{|\partial\Omega| \sqrt{E+1}}{\pi} + R_1(\gamma), \quad R_1(\gamma) = \mathcal{O}(1), \quad \text{as } \gamma \rightarrow +\infty. \quad (\text{II.2.7})$$

For any $\lambda \in \mathbb{R}$ there holds as $\gamma \rightarrow +\infty$,

$$\mathcal{N}(Q_\Omega^\gamma, -\gamma^2 + \lambda\gamma) = \frac{\sqrt{\gamma}}{\pi} \int_{\partial\Omega} \sqrt{(\kappa(s) + \lambda)_+} ds + R_2(\gamma), \quad R_2(\gamma) = o(\sqrt{\gamma}), \quad (\text{II.2.8})$$

where κ is the signed curvature of $\partial\Omega$, see (VI.2.1) for a rigorous definition, and $(x)_+ := \max(x, 0)$. Higher dimensional analogues were considered in [KKR16]. It turns out that if Ω admits corners, we obtain the same first order asymptotics.

Theorem II.2.10. *The asymptotics (II.2.7) and (II.2.8) hold true when Ω is a curvilinear polygon with respectively $R_1(\gamma) = \mathcal{O}(\gamma^\theta)$ for any $\theta \in (0, \frac{1}{2})$ and $R_2(\gamma) = \mathcal{O}(\gamma^{\frac{1}{4}})$.*

This result particularly means that the vertices do not contribute to the Weyl law at the leading order. In fact, it is not surprising as we understood with the asymptotics of Theorem II.2.3 that the corners only create a *finite* number of eigenvalues. We can also prove the following asymptotics for a positive threshold.

Proposition II.2.11. *For any $\mathcal{C} > 0$ there holds for large γ*

$$\mathcal{N}(Q_\Omega^\gamma, \mathcal{C}\gamma^2) = \mathcal{C}\gamma^2 \frac{|\Omega|}{4\pi} + o(\gamma^2).$$

In order to compare these results with the classical Weyl law, let us make a simple rescaling. For any $n \in \mathbb{N}$, it is easy to see that $E_n(Q_\Omega^D) = \gamma^2 E_n(Q_{\gamma\Omega}^D)$ and $E_n(Q_\Omega^\gamma) = \gamma^2 E_n(Q_{\gamma\Omega}^1)$. Thus,

$$\mathcal{N}(Q_\Omega^D, \lambda\gamma^2) = \mathcal{N}(Q_{\gamma\Omega}^D, \lambda) \text{ and } \mathcal{N}(Q_\Omega^\gamma, \lambda\gamma^2) = \mathcal{N}(Q_{\gamma\Omega}^1, \lambda).$$

We can then interpret the previous results as asymptotic distributions of the eigenvalues under a fixed threshold as the domain Ω is dilated by γ . For a positive threshold λ , we find again the classical Weyl law, namely the leading order is γ^2 and the volume of Ω appears to be a spectral invariant. However, when the thresholds are negative, namely $E \in (-1, 0)$ or -1 , you can observe that, at the leading order, the characteristics of the boundary appear as the length and the curvature. Indeed, the interior part of the domain Ω does not create any negative eigenvalue. The proof of these Weyl asymptotics use a partition of unity and a Dirichlet bracketing in order to cut out the corners from the domain Ω . We are then led to study separately the corners and the rest of Ω . We show that the corners do not contribute to the asymptotics at the leading order using the same kind of arguments as in [KK13] where magnetic Laplacians are studied. Then, the first term in the asymptotics comes from the study of the rest of the domain, called abusively the *regular part*. To prove this, we adapt the sketch of the proof of [Pan13]. The idea, inspired by the proof of a classical Weyl law of a Schrödinger operator in [RS78], consists in a reduction to a well chosen neighborhood of the boundary. Notice that, here again we cannot directly apply the results obtained on the Weyl asymptotics for smooth domains to the *regular part* as it still depends on the parameter γ .

Chapter III

Strategy of the proofs

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III.1 Variational estimates for eigenvalues

Let A be a self-adjoint operator acting in a Hilbert space \mathcal{H} of infinite dimension. We assume that A is semibounded from below, $A \geq -c$, $c \in \mathbb{R}$. The domain and the form domain of A will be denoted by $D(A)$ and $D(a)$, respectively, where $a : D(a) \times D(a) \rightarrow \mathbb{C}$ denotes the associated sesquilinear form. We define the resolvent set $\rho(A)$ by

$$\rho(A) := \{z \in \mathbb{C}, A - z : D(A) \rightarrow \mathcal{H} \text{ is bijective and } (A - z)^{-1} \text{ is bounded}\}.$$

The spectrum of A is defined as the complement of $\rho(A)$ in \mathbb{C} :

$$\text{spec}(A) := \mathbb{C} \setminus \rho(A).$$

We introduce the following partition of the spectrum

$$\begin{aligned} \text{spec}_{\text{disc}}(A) &:= \{\lambda \in \text{spec}(A) : \lambda \text{ is an isolated eigenvalue of finite multiplicity}\}, \\ \text{spec}_{\text{ess}}(A) &:= \text{spec}(A) \setminus \text{spec}_{\text{disc}}(A), \end{aligned}$$

where $\text{spec}_{\text{disc}}(A)$ is called the discrete spectrum of A and $\text{spec}_{\text{ess}}(A)$ the essential spectrum of A . Denote

$$\Sigma := \begin{cases} \inf \text{spec}_{\text{ess}} A, & \text{if } \text{spec}_{\text{ess}} A \neq \emptyset, \\ +\infty, & \text{if } \text{spec}_{\text{ess}} A = \emptyset. \end{cases}$$

Generally, we can have an intuition on the value of Σ following the philosophy of the Persson's theorem, see e.g. [HS96, Theorem 14.11], saying that the essential spectrum of Laplacians 'comes from infinity'. To calculate it, the Weyl characterization by sequences, see e.g. [HS96, Theorem 7.2], is often used.

Theorem III.1.1 (Weyl's criterion). *The condition $\lambda \in \text{spec}_{\text{ess}}(A)$ is equivalent to the existence of a sequence $(u_n)_n \subset D(A)$ satisfying:*

1. $\|u_n\| \geq 1$,
2. u_n weakly converges to 0 as $n \rightarrow +\infty$,
3. $\|(A - \lambda)u_n\| \rightarrow 0$ as $n \rightarrow +\infty$.

This characterization is in particular used to prove the following result, see [BS87, Theorem 4, Section 1, Chapter 9].

Theorem III.1.2. *Let B be a self-adjoint operator acting on \mathcal{H} such that for some $z \in \rho(A) \cap \rho(B)$ the difference $K(z) := (A - z)^{-1} - (B - z)^{-1}$ is a compact operator. Then,*

$$\text{spec}_{\text{ess}}(A) = \text{spec}_{\text{ess}}(B).$$

Now, let us recall the min-max principle giving a variational characterization of eigenvalues. By $E_j(A)$ we denote the j th discrete eigenvalue of A when ordered in the non-decreasing order and counted with multiplicities. Recall that $D(a)$ is a Hilbert space when considered with the scalar product $\langle u, v \rangle_a := a(u, v) + (c + 1)\langle u, v \rangle$. The following result is a standard tool of the spectral theory, see e.g. [RS78, Section XIII.1].

Theorem III.1.3 (Max-min and min-max principle). *Let $n \in \mathbb{N}$ and D be a dense subset of the Hilbert space $D(a)$. Let $\Lambda_n(A)$ be the n th Rayleigh quotient of A , which is defined by*

$$\Lambda_n(A) := \sup_{\psi_1, \dots, \psi_{n-1} \in \mathcal{H}} \inf_{\substack{u \in D, u \neq 0 \\ u \perp \psi_j, j=1, \dots, n-1}} \frac{a(u, u)}{\langle u, u \rangle} \equiv \inf_{\substack{G \subset D \\ \dim G = n}} \sup_{\substack{u \in G \\ u \neq 0}} \frac{a(u, u)}{\langle u, u \rangle},$$

then one and only one of the following assertions is true:

1. $\Lambda_n(A) < \Sigma$ and $E_n(A) = \Lambda_n(A)$.
2. $\Lambda_n(A) = \Sigma$ and $\Lambda_m(A) = \Lambda_n(A)$ for all $m \geq n$.

The min-max principle is particularly useful to obtain some bounds on the eigenvalues and the infimum of the essential spectrum. It is also a powerful tool to compare the eigenvalues of different operators, and the following proposition is a basic tool for our study of Robin Laplacians.

Proposition III.1.4. *Let B be a semibounded from below and self-adjoint operator acting on \mathcal{H} satisfying $D(b) \subset D(a)$ and for any $u \in D(b)$, $a(u, u) \leq b(u, u)$. Suppose in addition that A and B have a compact resolvent. Thus for any $n \in \mathbb{N}$,*

$$E_n(A) \leq E_n(B).$$

Recall that for $\lambda \in \mathbb{R}$, $\mathcal{N}(A, \lambda)$ denotes the number of eigenvalues, counting the multiplicities, of A in $(-\infty, \lambda)$ if $\text{spec}_{\text{ess}}(A) \cap (-\infty, \lambda) = \emptyset$, and $\mathcal{N}(A, \lambda) = +\infty$ otherwise. The previous proposition can be used to compare the number of eigenvalues of A and B under the threshold λ .

Corollary III.1.5. *Under the assumptions of Proposition III.1.4 there holds*

$$\mathcal{N}(B, \lambda) \leq \mathcal{N}(A, \lambda).$$

In the following proposition, we compare the eigenvalues of two operators which form domains differ from a finite dimensional subspace of \mathcal{H} , see [BS87, Theorem 5, page 227].

Proposition III.1.6. *Let B be a semibounded from below and self-adjoint operator acting on \mathcal{H} and D be a subset of \mathcal{H} of dimension $d \in \mathbb{N}$. Suppose that A and B have a compact resolvent and that $D(a) = D(b) \oplus D$ and for any $u \in D(b)$ there holds $a(u, u) = b(u, u)$. Thus for any $n \in \mathbb{N}$,*

$$E_n(A) \leq E_n(B) \leq E_{n+d}(A).$$

We conclude this section with another simple application of the min-max principle, see e.g. Lemma 2.1 in [EP05] or Lemma 2.2 in [Pos05]. As the proof is elementary we recall it here.

Proposition III.1.7. *Let B and B' be non-negative self-adjoint operators, with compact resolvents, acting respectively in Hilbert spaces \mathcal{H} and \mathcal{H}' and generated by sesquilinear forms b and b' . Pick $n \in \mathbb{N}$ and assume there exists a linear map $J : D(b) \rightarrow D(b')$ and two constants $\epsilon_1, \epsilon_2 > 0$ such that $\epsilon_1 \leq (1 + E_n(B))^{-1}$ and that for any $u \in D(b)$ there holds*

$$\|u\|^2 - \|Ju\|^2 \leq \epsilon_1 (b(u, u) + \|u\|^2), \quad (\text{III.1.1})$$

$$b'(Ju, Ju) - b(u, u) \leq \epsilon_2 (b(u, u) + \|u\|^2), \quad (\text{III.1.2})$$

then

$$E_n(B') \leq E_n(B) + \frac{(E_n(B)\epsilon_1 + \epsilon_2)(1 + E_n(B))}{1 - (1 + E_n(B))\epsilon_1}.$$

Proof. Denote by F_n the space spanned by the eigenfunctions associated with $E_1(B), \dots, E_n(B)$ and let $\varphi \in F_n$, $\varphi \neq 0$. Then,

$$\frac{b(\varphi, \varphi)}{\|\varphi\|^2} \leq E_n(B). \quad (\text{III.1.3})$$

Moreover, using the inequality (III.1.2) one can write

$$\frac{b'(J\varphi, J\varphi)}{\|J\varphi\|^2} \leq \frac{b(\varphi, \varphi) + \epsilon_2 (b(\varphi, \varphi) + \|\varphi\|^2)}{\|J\varphi\|^2}.$$

As the sesquilinear b is positive we get using (III.1.1)

$$\frac{b'(J\varphi, J\varphi)}{\|J\varphi\|^2} \leq \frac{b(\varphi, \varphi) + \epsilon_2 (b(\varphi, \varphi) + \|\varphi\|^2)}{\|\varphi\|^2 - \epsilon_1 (b(\varphi, \varphi) + \|\varphi\|^2)}.$$

We can easily rewrite the right hand side as

$$\frac{b(\varphi, \varphi)}{\|\varphi\|^2} \frac{1}{1 - \epsilon_1 \left(\frac{b(\varphi, \varphi)}{\|\varphi\|^2} + 1 \right)} + \epsilon_2 \frac{\frac{b(\varphi, \varphi)}{\|\varphi\|^2} + 1}{1 - \epsilon_1 \left(\frac{b(\varphi, \varphi)}{\|\varphi\|^2} + 1 \right)}.$$

Thus, we have the upper bound, using also (III.1.3) and the inequality $\epsilon_1 \leq (1 + E_n(B))^{-1}$,

$$\begin{aligned} \frac{b'(J\varphi, J\varphi)}{\|J\varphi\|^2} - \frac{b(\varphi, \varphi)}{\|\varphi\|^2} &\leq E_n(B) \left(\frac{1}{1 - \epsilon_1 (E_n(B) + 1)} - 1 \right) + \epsilon_2 \frac{E_n(B) + 1}{1 - \epsilon_1 (E_n(B) + 1)} \\ &= \frac{(E_n(B) + 1)(E_n(B)\epsilon_1 + \epsilon_2)}{1 - \epsilon_1 (E_n(B) + 1)}. \end{aligned}$$

Notice that, by (III.1.1)

$$\|J\varphi\|^2 \geq \|\varphi\|^2 (1 - \epsilon_1(E_n(B) + 1)),$$

where the right hand side is positive by hypothesis. This implies that if $J\varphi = 0$ then $\varphi = 0$ and thus $\dim J(F_n) = n$. We now can apply the min-max principle:

$$\begin{aligned} E_n(B') &\leq \sup_{\varphi \in F_n} \frac{b'(J\varphi, J\varphi)}{\|\varphi\|^2} \\ &\leq \sup_{\varphi \in F_n} \frac{b(\varphi, \varphi)}{\|\varphi\|^2} + \frac{(E_n(B) + 1)(E_n(B)\epsilon_1 + \epsilon_2)}{1 - \epsilon_1(E_n(B) + 1)} \\ &\leq E_n(B) + \frac{(E_n(B) + 1)(E_n(B)\epsilon_1 + \epsilon_2)}{1 - \epsilon_1(E_n(B) + 1)}. \end{aligned}$$

□

Let us describe more precisely the techniques involved in this thesis.

III.2 Domain decomposition

For any bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, we denote by $Q_\Omega^{D/N}$ the Dirichlet, respectively Neumann, Laplacian acting on $L^2(\Omega)$. Recall that these operators are defined as the unique self-adjoint operators associated with the sesquilinear forms

$$q_\Omega^N(u, u) = \int_\Omega |\nabla u|^2 dx, \quad u \in H^1(\Omega),$$

and q_Ω^D has the same expression but acts on the smaller domain $H_0^1(\Omega)$.

We begin by explaining how we use Proposition III.1.4. It is easy to see that $H_0^1(\Omega) \subset H^1(\Omega)$, thus applying Proposition III.1.4 we directly have

$$E_n(Q_\Omega^N) \leq E_n(Q_\Omega^D).$$

Let $(\Omega_j)_{j \leq J}$ be a finite partition of Ω with each Ω_j being Lipschitz, namely $\bar{\Omega} = \cup_{j \leq J} \bar{\Omega}_j$ and $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. Define the sesquilinear forms associated with this partition:

$$q_j^N(u, u) = \int_{\Omega_j} |\nabla u|^2 dx, \quad u \in H^1(\Omega_j),$$

and q_j^D has the same expression but acts on the smaller domain $\tilde{H}_0^1(\Omega_j) := \{u \in H^1(\Omega_j) : u = 0 \text{ on } \partial\Omega_j \setminus \partial\Omega\}$. Recall that the form domain of Q_Ω^γ is $H^1(\Omega)$. We have the obvious embeddings

$$H^1(\Omega) \subset H^1(\cup_{j \leq J} \Omega_j) \text{ and } \tilde{H}_0^1(\cup_{j \leq J} \Omega_j) \subset H^1(\Omega),$$

and the spaces $H^1(\cup_{j \leq J} \Omega_j)$ and $\oplus_{j \leq J} H^1(\Omega_j)$, respectively $\tilde{H}_0^1(\cup_{j \leq J} \Omega_j)$ and $\oplus_{j \leq J} \tilde{H}_0^1(\Omega_j)$, are naturally isomorphic. Thus we obtain by applying Proposition III.1.4,

$$E_n(\oplus_{j \leq J} Q_j^N) \leq E_n(Q_\Omega^\gamma) \leq E_n(\oplus_{j \leq J} Q_j^D).$$

Notice that this also implies the following inequality for the eigenvalue counting functions:

$$\sum_{j \leq J} \mathcal{N}(Q_j^D, \lambda) \leq \mathcal{N}(Q_\Omega^\gamma, \lambda) \leq \sum_{j \leq J} \mathcal{N}(Q_j^N, \lambda).$$

This simple result, called the Dirichlet-Neumann bracketing, is however very useful in general. It is in particular a part of the argument in the proof of the classical Weyl law for Dirichlet and Neumann Laplacians, see e.g. [RS78, Theorem XIII.78], but also for some Schrödinger operators, see e.g. [RS78, Theorem XIII.79]. Notice that this technique is particularly adapted to our study. Indeed, we are mainly interested in the behavior of the *negative* eigenvalues of Q_Ω^γ as γ becomes large and adding some Dirichlet or Neumann barriers does not change the negative spectrum at the leading orders.

Sometimes, adding some Neumann barriers to obtain lower bounds on the eigenvalues of Q_Ω^γ could be a dead end if we cannot estimate the eigenvalues of the associated operator. Due to the following result, for which there is no generalization to Neumann Laplacian, it seems that it is easier to obtain, in certain cases, lower bounds for Dirichlet eigenvalues.

Lemma III.2.1. *Let $\Omega' \subset \mathbb{R}^d$ such that $\Omega \subset \Omega'$. Thus,*

$$E_n(Q_\Omega^D) \geq E_n(Q_{\Omega'}^D).$$

This is a direct consequence of the min-max principle. The following technical lemma, known as the IMS formula, is a useful tool to get lower bounds for the eigenvalues of Robin Laplacians adding some Dirichlet barriers, modulo a remainder.

Lemma III.2.2. *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain, bounded or with a suitable behavior at infinity ensuring that q_Ω^γ is semibounded from below, and $(\chi_j)_{j \in J}$ be an IMS partition of unity of Ω i.e. $\chi_j \in C^\infty(\Omega)$, $\chi_j \geq 0$ and $\sum_{j \in J} \chi_j^2 = 1$ on Ω . Then, for any $u \in H^1(\Omega)$,*

$$q_\Omega^\gamma(u, u) = \sum_{j \in J} \left(q_\Omega^\gamma(u\chi_j, u\chi_j) - \|u(\nabla\chi_j)\|_{L^2(\Omega)}^2 \right).$$

If we denote by $(\Omega_j)_j$ the partition associated with $(\chi_j)_j$ it is easy to see that $u\chi_j$ satisfies the Dirichlet boundary condition on $\partial\Omega_j$ for any $u \in H^1(\Omega)$.

Proof of Lemma III.2.2. Let $j \in J$, then

$$|\nabla(u\chi_j)|^2 = \chi_j^2 |\nabla u|^2 + |u|^2 |\nabla\chi_j|^2 + 2u\chi_j \nabla u \nabla\chi_j. \quad (\text{III.2.1})$$

As $\sum_{j \in J} \chi_j^2 = 1$ on Ω one can write

$$q_\Omega^\gamma(u, u) = \sum_{j \in J} \int_\Omega |\chi_j \nabla u|^2 dx - \gamma \int_{\partial\Omega} |\chi_j u|^2 ds,$$

and using (III.2.1) we obtain the equality

$$q_\Omega^\gamma(u, u) = \sum_{j \in J} \left(q_\Omega^\gamma(u\chi_j, u\chi_j) - \int_\Omega |u \nabla\chi_j|^2 dx \right) - \sum_{j \in J} \int_\Omega 2u\chi_j \nabla u \nabla\chi_j dx.$$

We can rewrite the last term as $2u\chi_j \nabla u \nabla\chi_j = u \nabla u \nabla(\chi_j^2)$ and thus

$$\sum_{j \in J} \int_\Omega 2u\chi_j \nabla u \nabla\chi_j dx = \int_\Omega u \nabla u \nabla \left(\sum_{j \in J} \chi_j^2 \right) = 0,$$

which finishes the proof. \square

III.3 Construction of quasi-modes

Let us first recall an application of the spectral theorem, which gives an estimate on the distance of any complex number to the spectrum of self-adjoint operators.

Theorem III.3.1. *Let $u \in D(A)$, $u \neq 0$. For any $\lambda \in \mathbb{C}$ one has the estimate*

$$\text{dist}(\lambda, \text{spec}(A)) \leq \frac{\|(A - \lambda)u\|}{\|u\|}.$$

This is particularly useful to approximate the spectrum of self-adjoint operators. Indeed, in a practical way we construct functions, also called approximated eigenfunctions or *quasi-modes*, satisfying $u \in D(A)$ and $\|(A - \lambda)u\| \leq \epsilon \|u\|$ for small $\epsilon > 0$ which implies

$$\text{dist}(\lambda, \text{spec}(A)) \leq \epsilon. \quad (\text{III.3.1})$$

In this thesis, the construction of quasi-modes is one of the main tools to obtain upper bounds of the eigenvalues of Robin Laplacians. In particular, we construct them using various truncations of the eigenfunctions of model operators associated with our problems. To obtain estimates like (III.3.1), we need to prove that the eigenfunctions admit some exponential decay and we use the standard method of Agmon estimates, see e.g. [Agm82]. The following result will help us to get the desired estimates. We state it for eigenfunctions of Laplacians with Robin boundary condition but it is easy to see that it can be adapted to Dirichlet, Neumann or mixed boundary conditions.

Lemma III.3.2. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, bounded or with a suitable behavior at infinity such that q_Ω^γ is semibounded from below. Let $u \in D(Q_\Omega^\gamma)$ be an eigenfunction of Q_Ω^γ associated with the eigenvalue E and $\varphi : \Omega \rightarrow \mathbb{R}$ be a Lipschitz function satisfying $\varphi \in L^\infty(\Omega)$, $\nabla \varphi \in L^\infty(\Omega)$. Then,*

$$q_\Omega^\gamma(ue^\phi, ue^\phi) = \int_\Omega (|\nabla \phi|^2 + E) |ue^\phi|^2 dx.$$

Proof. By a simple computation we have for almost every $x \in \Omega$,

$$|\nabla(ue^\phi)(x)|^2 = |\nabla u(x)|^2 e^{2\phi(x)} + |\nabla \phi(x)|^2 |u(x)e^{\phi(x)}|^2 + 2u(x)e^{2\phi(x)} \nabla u(x) \nabla \phi(x).$$

Remark that the last term can be rewritten as $2ue^{2\phi} \nabla u \nabla \phi = u \nabla u \nabla (e^{2\phi})$ and by integration by parts we get

$$\int_\Omega u \nabla u \nabla (e^{2\phi}) dx = - \int_\Omega (|\nabla u|^2 + u \Delta u) e^{2\phi} dx + \int_{\partial\Omega} u \partial_\nu u e^{2\phi} d\sigma.$$

We are now able to write,

$$q_\Omega^\gamma(ue^\phi, ue^\phi) = \int_\Omega (|\nabla \phi|^2 |u|^2 - u \Delta u) e^{2\phi} dx - \gamma \int_{\partial\Omega} |ue^\phi|^2 d\sigma + \int_{\partial\Omega} u \partial_\nu u e^{2\phi} d\sigma. \quad (\text{III.3.2})$$

The function u is an eigenfunction of Q_Ω^γ associated with E thus $-\Delta u = Eu$ and $\partial_\nu u = \gamma u$ on $\partial\Omega$. Replacing these equalities in (III.3.2) finishes the proof. \square

The spectral theorem gives some rough approximation of the spectrum of self-adjoint operators. To obtain precise asymptotics we need the following results.

Proposition III.3.3. *Let \mathcal{H} be a Hilbert space, A a self-adjoint operator in \mathcal{H} and $\lambda \in \mathbb{R}$. We suppose that there exists $\eta > 0$ and an orthonormal family $\psi_1, \dots, \psi_n \in D(A)$ satisfying*

$$\|(A - \lambda)\psi_j\| < \eta, \quad j = 1, \dots, n.$$

Then,

$$\dim \text{Ran } P_A(\lambda - \sqrt{n}\eta, \lambda + \sqrt{n}\eta) \geq n,$$

where $P_A(a, b)$ stands for the spectral projection of A on the interval $(a, b) \subset \mathbb{R}$.

Proof. For simplicity we denote $P := P_A(\lambda - \sqrt{n}\eta, \lambda + \sqrt{n}\eta)$. Let us make a proof by contradiction and suppose that $\dim \text{Ran } P \leq n - 1$. Let $\tilde{A} := A|_{\text{Ran}(1-P)}$. Then, $\text{spec}(\tilde{A}) \cap (\lambda - \sqrt{n}\eta, \lambda + \sqrt{n}\eta) = \emptyset$. Moreover, as we assumed $\dim \text{Ran } P \leq n - 1$, there exists $\psi \in \text{span}\{\psi_1, \dots, \psi_n\} \setminus \{0\}$ such that $\psi \in \text{Ran}(1 - P)$. Without loss of generality we assume that there exist $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ such that

$$\psi = \sum_{i=1}^n \alpha_i \psi_i \quad \text{with} \quad \sum_{i=1}^n |\alpha_i|^2 = 1.$$

Then,

$$\begin{aligned} \|(\tilde{A} - \lambda)\psi\|^2 &= \|(1 - P)(A - \lambda)\psi\|^2 \\ &\leq \|(A - \lambda)\psi\|^2 \\ &\leq \left(\sum_{i=1}^n |\alpha_i|^2 \right) \left(\sum_{i=1}^n \|(A - \lambda)\psi_i\|^2 \right) \\ &= \sum_{i=1}^n \|(A - \lambda)\psi_i\|^2. \end{aligned}$$

Hence $\|(\tilde{A} - \lambda)\psi\| \leq \sqrt{n}\eta$. Then, by Theorem III.3.1 we can conclude that \tilde{A} admits some spectrum in $(\lambda - \sqrt{n}\eta, \lambda + \sqrt{n}\eta)$, which gives a contradiction. \square

Corollary III.3.4. *If there exist $\varphi_1, \dots, \varphi_n \in D(A)$ linearly independent and satisfying*

$$\frac{\|(A - \lambda)\varphi_j\|}{\|\varphi_j\|} < \eta, \quad j = 1, \dots, n,$$

then

$$\dim \text{Ran } P_A\left(\lambda - n^{\frac{3}{2}}\eta\sqrt{\frac{\beta_{\max}}{\beta_{\min}}}, \lambda + n^{\frac{3}{2}}\eta\sqrt{\frac{\beta_{\max}}{\beta_{\min}}}\right) \geq n,$$

where β_{\min} (resp. β_{\max}) is the minimal (resp. maximal) eigenvalue of the Gramian matrix of the family $(\varphi_j)_{j=1}^n$. In particular, if $\text{spec}_{\text{ess}}(A) \cap (\lambda - n^{\frac{3}{2}}\eta\sqrt{\frac{\beta_{\max}}{\beta_{\min}}}, \lambda + n^{\frac{3}{2}}\eta\sqrt{\frac{\beta_{\max}}{\beta_{\min}}}) = \emptyset$ there exist at least n eigenvalues of A in $(\lambda - n^{\frac{3}{2}}\eta\sqrt{\frac{\beta_{\max}}{\beta_{\min}}}, \lambda + n^{\frac{3}{2}}\eta\sqrt{\frac{\beta_{\max}}{\beta_{\min}}})$.

Proof. The idea consists in using a specific orthonormalized family obtained from $(\varphi_j)_{j=1}^n$ and then use Proposition III.3.3. We denote by G the Gramian matrix of $(\varphi_j)_{j=1}^n$. It is known that G is strictly positive and then there exists a matrix R such that

$$G = R^2.$$

Let us define, for $j = 1, \dots, n$,

$$\psi_j = \sum_{l=1}^n (R^{-1})_{jl} \varphi_l.$$

Then $(\psi_j)_{j=1}^n$ is orthonormal :

$$\begin{aligned} \langle \psi_j, \psi_k \rangle &= \sum_{l=1}^n \sum_{m=1}^n (R^{-1})_{j,l} \overline{(R^{-1})_{k,m}} \langle \varphi_l, \varphi_m \rangle \\ &= \sum_{l=1}^n \sum_{m=1}^n (R^{-1})_{j,l} (R^{-1})_{m,k} G_{l,m} \\ &= (R^{-1} G R^{-1})_{j,k} \\ &= (I_n)_{j,k}. \end{aligned}$$

Moreover, for all $j = 1, \dots, n$,

$$\|(A - \lambda)\psi_j\|^2 \leq \eta^2 \sum_{k=1}^n |(R^{-1})_{j,k}|^2 \|\varphi_k\|^2 + 2\Re \sum_{l < k} (R^{-1})_{j,l} \overline{(R^{-1})_{j,k}} \langle (A - \lambda)\varphi_l, (A - \lambda)\varphi_k \rangle.$$

Applying Cauchy-Schwarz we get

$$\left| \sum_{l < k} (R^{-1})_{j,l} \overline{(R^{-1})_{j,k}} \langle (A - \lambda)\varphi_l, (A - \lambda)\varphi_k \rangle \right| \leq \eta^2 \sum_{l < k} |(R^{-1})_{j,l}| |(R^{-1})_{j,k}| \|\varphi_l\| \|\varphi_k\|,$$

which implies

$$\|(A - \lambda)\psi_j\|^2 \leq \eta^2 \left(\sum_{k=1}^n |(R^{-1})_{j,k}| \|\varphi_k\| \right)^2 \leq \eta^2 \left(\sum_{k=1}^n |(R^{-1})_{j,k}|^2 \right) \left(\sum_{k=1}^n \|\varphi_k\|^2 \right).$$

By definition $\|\varphi_k\|^2 = G_{k,k} \leq \beta_{\max}$ and $|(R^{-1})_{j,k}|^2 = (R^{-1})_{j,k} (R^{-1})_{k,j} = (G^{-1})_{j,j}$. Then, $|(R^{-1})_{j,k}|^2 \leq \frac{1}{\beta_{\min}}$ and we get

$$\|(A - \lambda)\psi_j\| \leq \eta n \sqrt{\frac{\beta_{\max}}{\beta_{\min}}},$$

which allows us to conclude using Proposition III.3.3. \square

When the construction of quasi-modes is not obvious, we have recourse to *weak quasi-modes*. They are approximated eigenfunctions which belong to the form domain $D(a)$, but not to the domain of the operator $D(A)$, and allow us to obtain some estimates on the eigenvalues using the min-max principle. The following proposition is an analogue of Theorem III.3.1 for weak quasi-modes.

Proposition III.3.5. *Let A be a self-adjoint operator in a Hilbert space \mathcal{H} such that $A \geq c > 0$, and let a be the associated sesquilinear form. Assume that there exist a non-zero $u \in D(a)$ and numbers $\lambda > 0$ and $\epsilon \in (0, 1)$ such that*

$$|a(u, v) - \lambda \langle u, v \rangle| \leq \epsilon \sqrt{a(u, u)} \sqrt{a(v, v)} \text{ for all } v \in D(a), \quad (\text{III.3.3})$$

then $\text{dist}(\lambda, \text{spec } A) \leq \frac{\epsilon}{1 - \epsilon} \lambda$.

Proof. Due to the spectral theorem for self-adjoint operators we have $D(a) = D(\sqrt{A})$, and for $f, g \in D(a)$ there holds $a(f, g) = \langle \sqrt{A}f, \sqrt{A}g \rangle$. Hence, we can rewrite the inequality (III.3.3) as

$$\left| \langle \sqrt{A}u, \sqrt{A}v \rangle - \lambda \langle A^{-1} \sqrt{A}u, \sqrt{A}v \rangle \right| \leq \epsilon \|\sqrt{A}u\| \cdot \|\sqrt{A}v\|,$$

and,

$$\left| \left\langle (1 - \lambda A^{-1})\sqrt{A}u, \frac{\sqrt{A}v}{\|\sqrt{A}v\|} \right\rangle \right| \leq \epsilon \|\sqrt{A}u\| \text{ for all } v \in D(a), \quad v \neq 0. \quad (\text{III.3.4})$$

As the vectors $\sqrt{A}v$ cover the whole of \mathcal{H} as v runs through $D(a)$, taking the supremum over v in (III.3.4) gives $\|(1 - \lambda A^{-1})\sqrt{A}u\| \leq \epsilon \|\sqrt{A}u\|$. As $\sqrt{A}u \neq 0$, by the spectral theorem for self-adjoint operators we have $\text{dist}(1, \text{spec}(\lambda A^{-1})) \leq \epsilon$, which means that there exists $\mu \in \text{spec } A \subset [c, +\infty)$ such that $|1 - \lambda\mu^{-1}| \leq \epsilon$. Hence, $|\mu - \lambda| \leq \epsilon\mu$. In particular, $\mu - \lambda \leq \epsilon\mu$ and $\mu \leq \lambda/(1 - \epsilon)$, which concludes the proof. \square

III.4 Distance between subspaces

Let E and F be closed subspaces of a Hilbert space \mathcal{H} and denote by P_E , respectively P_F , the orthogonal projector on E , respectively F , in \mathcal{H} . We define the non-symmetric distance between E and F as:

$$\text{dist}_{\mathcal{H}}(E, F) := \|P_E - P_E P_F\| \equiv \|P_E - P_F P_E\|,$$

where $\|P_E - P_E P_F\| = \inf_{x \in \mathcal{H}} \|P_E x - P_E P_F x\|_{\mathcal{H}}$. When there is no ambiguity we simply denote $\text{dist} := \text{dist}_{\mathcal{H}}$. We summarize some properties of the distance in the following proposition, see e.g. [HS84, Lemma 1.3 and Proposition 1.4].

Proposition III.4.1. *The distance between subspaces has the following properties:*

1. $\text{dist}(E, F) = 0$ if and only if $E \subset F$,
2. $\text{dist}(E, F) \leq \text{dist}(E, G) + \text{dist}(G, F)$ for any closed subspace G of \mathcal{H} ,
3. if $\text{dist}(E, F) < 1$, then the map $E \ni f \mapsto P_F f \in F$ is injective, and the map $F \ni f \mapsto P_E f \in E$ has a continuous right inverse,
4. if $\text{dist}(E, F) < 1$ and $\text{dist}(F, E) < 1$, then $\text{dist}(E, F) = \text{dist}(F, E)$, the map $F \ni f \mapsto P_E f \in E$ is bijective, and its inverse is continuous.

The following proposition can be used to estimate the distance between closed subspaces, see e.g. [HS84, Proposition 3.5].

Proposition III.4.2. *Let A be a self-adjoint operator in a Hilbert space \mathcal{H} . Let $I \subset \mathbb{R}$ be a compact interval, $\psi_1, \dots, \psi_n \in D(A)$ be linearly independent and $\mu_1, \dots, \mu_n \in I$. Suppose that there exists $\eta > 0$ such that*

$$A\psi_j = \mu_j\psi_j + r_j \quad \text{with} \quad \|r_j\| \leq \eta,$$

and $a > 0$ such that $\text{spec}(A) \cap ((I + B(0, 2a)) \setminus I) = \emptyset$. Then, if F is the subspace spanned by ψ_1, \dots, ψ_n and $E := \text{Ran } P_A(I)$,

$$\text{dist}(F, E) \leq \frac{\sqrt{n}\eta}{a\sqrt{\beta_{\min}}},$$

where β_{\min} stands for the minimal eigenvalue of the Gramian matrix of $(\psi_j)_j$.

III.5 One-dimensional Robin Laplacians

III.5.1 Robin Laplacian on the half-line

For $\gamma > 0$ denote by \mathcal{B}_γ the self-adjoint operator acting on $L^2(\mathbb{R}_+)$ by

$$\mathcal{B}_\gamma f := -f'', \quad D(\mathcal{B}_\gamma) := \left\{ f \in H^2(\mathbb{R}_+) : -f'(0) = \gamma f(0) \right\}.$$

Proposition III.5.1. *The operator \mathcal{B}_γ admits a unique discrete eigenvalue $E_1(\mathcal{B}_\gamma) = -\gamma^2$ with $f(t) = e^{-\gamma t}$ the associated eigenfunction and $\text{spec}_{\text{ess}}(\mathcal{B}_\gamma) = [0, +\infty)$.*

Proof. Denote by T^N the self-adjoint operator acting on $L^2(\mathbb{R}_+)$ as

$$T^N f := -f'', \quad D(T^N) := \left\{ f \in H^2(\mathbb{R}_+) : f'(0) = 0 \right\},$$

and by S the restriction of T^N to $D(S) := C_c^\infty(\mathbb{R}_+)$. For any $z \in \mathbb{C} \setminus \mathbb{R}$, introduce $K(z) := (\mathcal{B}_\gamma - z)^{-1} - (T^N - z)^{-1}$ and $L := \text{Ran}(S - z)$. It is easy to see that $K(z)$ is identically zero on L . Moreover, $L^\perp = \text{Ker}(S^* - \bar{z})$ where S^* acts as S and $D(S^*) = H^2(\mathbb{R}_+)$. Thus, $\dim(L^\perp) \leq 2$ and K is a compact operator. We can apply Theorem III.1.2 to get

$$\text{spec}_{\text{ess}}(\mathcal{B}_\gamma) = \text{spec}_{\text{ess}}(T^N) = [0, +\infty).$$

The second assertion is obvious, solving the eigenvalue problem $-f''(t) = \lambda f(t)$ with $-f'(0) = \gamma f(0)$ for $f \in L^2(\mathbb{R}_+)$. \square

Remark that the sesquilinear form associated with \mathcal{B}_γ is

$$b_\gamma(f, f) = \int_{\mathbb{R}_+} |f'(t)|^2 dt - \gamma |f(0)|^2, \quad f \in H^1(\mathbb{R}_+),$$

hence, for any $f \in H^1(\mathbb{R}_+)$,

$$\int_{\mathbb{R}_+} |f'(t)|^2 dt - \gamma |f(0)|^2 \geq -\gamma^2 \int_{\mathbb{R}_+} |f(t)|^2 dt. \quad (\text{III.5.1})$$

III.5.2 Robin Laplacians on an interval

We first study the asymptotics of the eigenvalues of Laplacians with mixed Robin-Neumann and Robin-Dirichlet boundary conditions acting on an interval.

Lemma III.5.2. *For $\gamma > 0$ and $\ell > 0$, denote by $\mathcal{N}_{\ell, \gamma}$ the operator acting on $L^2(0, \ell)$ as $f \mapsto -f''$ with $D(\mathcal{N}_{\ell, \gamma}) := \{f \in H^2(0, \ell) : -f'(0) = f'(\ell) - \gamma f(\ell) = 0\}$. Then, $E_1(\mathcal{N}_{\ell, \gamma})$ is the unique negative eigenvalue, and for any fixed $\ell > 0$ one has*

$$E_1(\mathcal{N}_{\ell, \gamma}) = -\gamma^2 - 4\gamma^2 e^{-2\gamma\ell} + 8\gamma^2(2\gamma\ell - 1)e^{-4\gamma\ell} + \mathcal{O}(\gamma^4 e^{-6\gamma\ell}), \quad \text{as } \gamma \rightarrow +\infty, \quad (\text{III.5.2})$$

and $t \mapsto \cosh(\sqrt{-E_1(\mathcal{N}_{\ell, \gamma})}t)$ is the associated eigenfunction. Moreover, the second eigenvalue satisfies

$$E_2(\mathcal{N}_{\ell, \gamma}) \geq \frac{\pi^2}{4\ell^2}, \quad (\text{III.5.3})$$

and for any $j \in \mathbb{N}$ one has for large γ ,

$$E_{1+j}(\mathcal{N}_{\ell, \gamma}) = \left(\frac{\pi(2j-1)}{2\ell} \right)^2 + o(1). \quad (\text{III.5.4})$$

Proof. We first prove (III.5.2). Let us denote by $E_1(\mathcal{N}_{\ell,\gamma}) = -k^2$, with $k > 0$. An eigenfunction associated with a negative eigenvalue is in the form $f(t) = Ae^{kt} + Be^{-kt}$ with $(A, B) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Using the boundary conditions we obtain the system

$$A = B, \quad k(Ae^{k\ell} - Be^{-k\ell}) = \gamma(Ae^{k\ell} + Be^{-k\ell}).$$

As a result, $f(t) = 2A \cosh(kt)$ and k must satisfy

$$(k - \gamma)e^{k\ell} = (k + \gamma)e^{-k\ell}. \quad (\text{III.5.5})$$

Remark that the previous equality can be rewritten in the form $k\ell \tanh(k\ell) = \gamma\ell$ and, as $\mathbb{R}_+ \ni t \mapsto t \tanh(t) \in \mathbb{R}_+$ is a bijection, there exists a unique solution $k > 0$, which means that there exists a unique negative eigenvalue. In order to obtain an asymptotics of k as $\gamma \rightarrow +\infty$, we first notice that (III.5.5) implies that $k > \gamma$. Moreover, (III.5.5) can be rewritten as $k - \gamma = 2\gamma e^{-2\gamma\ell} \frac{e^{-2\ell(k-\gamma)}}{1 - e^{-2\ell k}}$, which implies that

$$k = \gamma + \mathcal{O}(\gamma e^{-2\gamma\ell}). \quad (\text{III.5.6})$$

Using again (III.5.5), we can write

$$k = \gamma \left(1 + 2e^{-2k\ell} + \mathcal{O}(e^{-4\gamma\ell}) \right). \quad (\text{III.5.7})$$

Combining (III.5.6) and (III.5.7), we first obtain

$$k = \gamma + 2\gamma e^{-2\gamma\ell} + \mathcal{O}(\gamma^2 e^{-4\gamma\ell}). \quad (\text{III.5.8})$$

Taking into account another term in (III.5.5) we have

$$k = \gamma \left(1 + 2e^{-2k\ell} + 2e^{-4k\ell} + \mathcal{O}(e^{-6\gamma\ell}) \right). \quad (\text{III.5.9})$$

Combining (III.5.8) and (III.5.9) we finally obtain

$$k = \gamma + 2\gamma e^{-2\gamma\ell} + 2\gamma(1 - 4\gamma\ell)e^{-4\gamma\ell} + \mathcal{O}(\gamma^3 e^{-6\gamma\ell}).$$

As $E_1(\mathcal{N}_{\ell,\gamma}) = -k^2$, this finishes the proof.

Let us focus on (III.5.3). Denote by T_ℓ^{DN} the operator acting as $f \mapsto -f''$ with the domain $D(T_\ell^{DN}) := \{f \in H^2(0, \ell) : f'(0) = f(\ell) = 0\}$. The form domain of $\mathcal{N}_{\ell,\gamma}$ is $H^1(0, \ell)$ whereas the form domain of T_ℓ^{DN} is $\tilde{H}_0^1 := \{f \in H^1(0, \ell) : f(\ell) = 0\}$. We have $H^1(0, \ell) = \tilde{H}_0^1 \oplus \mathbb{C}$ and the forms of $\mathcal{N}_{\ell,\gamma}$ and T_ℓ^{DN} coincide on $\tilde{H}_0^1(0, \ell)$. Thus applying Proposition III.1.6 we obtain

$$E_2(\mathcal{N}_{\ell,\gamma}) \geq E_1(T_\ell^{DN}) = \frac{\pi^2}{4\ell^2}.$$

We now prove (III.5.4). For $j \in \mathbb{N}$, if we denote by $E_{1+j}(\mathcal{N}_{\ell,\gamma}) = k^2$ with $k > 0$. Then, k^2 is a positive eigenvalue of $\mathcal{N}_{\ell,\gamma}$ if and only if k satisfies $k\ell \tan(k\ell) = -\gamma\ell$. Denote by $h(t) := t \tan(t)$ defined on $[0, \frac{\pi}{2}) \cup_{j \in \mathbb{N}} I_j$, with $I_j := \{x \in \mathbb{R} : \frac{\pi}{2} + (j-1)\pi < t < \frac{\pi}{2} + j\pi\}$. For any $t \in [0, \frac{\pi}{2})$ we have $h(t) \geq 0$ and for any $j \in \mathbb{N}$, h is strictly increasing, $h(t) \rightarrow -\infty$ as $t \rightarrow \frac{\pi}{2} + (j-1)\pi$ and $h(t) \rightarrow +\infty$ as $t \rightarrow \frac{\pi}{2} + j\pi$. Thus, for any $j \in \mathbb{N}$ there exists a unique solution k to $k\ell \tan(k\ell) = -\gamma\ell$ satisfying $k\ell \in I_j$ and $k \rightarrow \pi \frac{2j-1}{2\ell}$ as $\gamma \rightarrow +\infty$. As a consequence we finally have for any $j \in \mathbb{N}$,

$$E_{1+j}(\mathcal{N}_{\ell,\gamma}) = \left(\frac{\pi(2j-1)}{2\ell} \right)^2 + o(1), \text{ as } \gamma \rightarrow +\infty.$$

□

Lemma III.5.3. For $\gamma > 0$ and $\ell > 0$, denote by $\mathcal{D}_{\ell,\gamma}$ the operator acting on $L^2(0, \ell)$ as $f \mapsto -f''$ with $D(\mathcal{D}_{\ell,\gamma}) := \{f \in H^2(0, \ell) : f(0) = f'(\ell) - \gamma f(\ell) = 0\}$. Then, $E_1(\mathcal{D}_{\ell,\gamma}) < 0$ if and only if $\gamma\ell > 1$, and in that case it is the unique negative eigenvalue. For any fixed $\ell > 0$ one has

$$E_1(\mathcal{D}_{\ell,\gamma}) = -\gamma^2 + 4\gamma^2 e^{-2\gamma\ell} + 8\gamma^2(2\gamma\ell - 1)e^{-4\gamma\ell} + \mathcal{O}(\gamma^4 e^{-6\gamma\ell}), \text{ as } \gamma \rightarrow +\infty, \quad (\text{III.5.10})$$

and $t \mapsto \sinh(\sqrt{-E_1(\mathcal{D}_{\ell,\gamma})}t)$ is the associated eigenfunction. Moreover, the further eigenvalues satisfy for large γ ,

$$E_{1+j}(\mathcal{D}_{\ell,\gamma}) = \left(\frac{\pi j}{\ell}\right)^2 + o(1). \quad (\text{III.5.11})$$

Proof. We first prove (III.5.10). Let us denote by $E_1(\mathcal{D}_{\ell,\gamma}) = -k^2$, with $k > 0$. An eigenfunction associated with a negative eigenvalue is in the form $f(t) = Ae^{kt} + Be^{-kt}$ with $(A, B) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Using the boundary conditions we obtain the system

$$A = -B, \quad k(Ae^{k\ell} - Be^{-k\ell}) = \gamma(Ae^{k\ell} + Be^{-k\ell}).$$

As a result, $f(t) = 2A \sinh(kt)$ and k must satisfy

$$(\gamma - k)e^{k\ell} = (\gamma + k)e^{-k\ell}. \quad (\text{III.5.12})$$

The previous equality can be rewritten in the form $k\ell \cotanh(k\ell) = \gamma\ell$ and, as $\mathbb{R}_+ \ni t \mapsto t \cotanh(t) \in (1, +\infty)$ is a bijection, there exists a solution if and only if $\gamma\ell > 1$, and in that case the solution is unique which means that there exists a unique negative eigenvalue. In the following we always assume that $\gamma\ell > 1$. In order to obtain an asymptotics of k as $\gamma \rightarrow +\infty$, we first notice that (III.5.12) implies that $\gamma > k$. Moreover, we can rewrite (III.5.12) as follows

$$\ln(\gamma - k) - \ln(\gamma + k) + 2k\ell = 0.$$

Let us define $h(t) := \ln(\gamma - t) - \ln(\gamma + t) + 2t\ell$ for $t \in [0, \gamma)$. It is easy to see that $h(0) = 0$ and $h(t) \rightarrow +\infty$ as $t \rightarrow \gamma$. The equation $h'(t) = 0$ admits a unique solution $t_0 := \gamma\sqrt{1 - (\gamma\ell)^{-1}}$. This implies that the unique solution k of (III.5.12) satisfies

$$\gamma\sqrt{1 - (\gamma\ell)^{-1}} < k < \gamma. \quad (\text{III.5.13})$$

We can use again (III.5.12) to write $\gamma - k = 2k \frac{e^{-2k\ell}}{1 - e^{-2k\ell}}$ and thus we obtain by (III.5.13),

$$k = \gamma + \mathcal{O}(\gamma e^{-2\gamma\ell}).$$

Following the same steps as in the proof of Lemma III.5.2 allow us to get the result.

Let us prove (III.5.11). We denote by $E_{1+j}(\mathcal{D}_{\ell,\gamma}) = k^2$ for $j \in \mathbb{N}$ with $k > 0$. Then, k^2 is a positive eigenvalue of $\mathcal{D}_{\ell,\gamma}$ if and only if k satisfies $k\ell \cotan(k\ell) = \gamma\ell$. Denote by $g(t) := t \cotan(t)$ defined on $(0, \pi) \cup_{j \in \mathbb{N}} I_j$ with $I_j := (j\pi, (j+1)\pi)$. The function g is decreasing on $(0, \pi)$ and $g(t) \rightarrow 1$ as $t \rightarrow 0$. Recall that here, we suppose that $\gamma\ell > 1$. Thus, the equation $k\ell \cotan(k\ell) = \gamma\ell$ does not admit any solution in $(0, \pi)$. For any $j \in \mathbb{N}$, g is strictly decreasing on I_j , $g(t) \rightarrow +\infty$ as $t \rightarrow j\pi$ and $t \rightarrow (j+1)\pi$ and $g(t) \rightarrow -\infty$ as $t \rightarrow (j+1)\pi$. As a result, for any $j \in \mathbb{N}$ there exists a unique solution k of $k\ell \cotan(k\ell) = \gamma\ell$ satisfying $k\ell \in I_j$ and $k \rightarrow \frac{\pi j}{\ell}$ as $\gamma \rightarrow +\infty$, and thus we have

$$E_{1+j}(\mathcal{D}_{\ell,\gamma}) = \left(\frac{\pi j}{\ell}\right)^2 + o(1), \text{ as } \gamma \rightarrow +\infty.$$

□

We can now pass to the description of the classical Robin Laplacian on an interval. For $\ell > 0$ and $\gamma > 0$, consider the operator $\mathcal{R}_{\ell,\gamma} := Q_{(-\ell,\ell)}^\gamma$ acting on $L^2(-\ell, \ell)$ as $f \mapsto -f''$ with

$$D(\mathcal{R}_{\ell,\gamma}) := \{f \in H^2(0, \ell) : -f'(-\ell) - \gamma f(-\ell) = f'(\ell) - \gamma f(\ell) = 0\}.$$

The operator $\mathcal{R}_{\ell,\gamma}$ has a compact resolvent and its spectrum is purely discrete and consists of the simple eigenvalues

$$E_j(\ell, \gamma) := E_j(\mathcal{R}_{\ell,\gamma}) \quad j \in \mathbb{N},$$

numbered in the increasing order. Remark that due to the scaling we have

$$E_j(\ell, \gamma) = \frac{1}{\ell^2} E_j(1, \gamma\ell), \quad j \in \mathbb{N}. \quad (\text{III.5.14})$$

An easy application of the min-max principle shows that the maps $\mathbb{R} \ni \gamma \mapsto E_j(1, \gamma)$ are continuous, in particular $E_j(1, 0)$ coincides with the j th eigenvalue of the Neumann Laplacian on $(-1, 1)$, hence

$$E_1(1, 0) = 0, \quad (\text{III.5.15})$$

$$E_2(1, 0) = \frac{\pi^2}{4}. \quad (\text{III.5.16})$$

The following proposition gives the asymptotics of each eigenvalue as γ is large.

Proposition III.5.4. *The first eigenvalue of $\mathcal{R}_{\ell,\gamma}$ is strictly negative, the associated eigenfunctions is*

$$\Phi_\ell(\gamma, t) := \cosh(\sqrt{-E_1(\ell, \gamma)}t),$$

and the second eigenvalue is strictly negative if and only if $\gamma\ell > 1$. Moreover, for any $j \in \mathbb{N}$, there holds for large γ ,

$$E_1(\ell, \gamma) = -\gamma^2 - 4\gamma^2 e^{-2\gamma\ell} + 8\gamma^2(2\gamma\ell - 1)e^{-4\gamma\ell} + \mathcal{O}(\gamma^4 e^{-6\gamma\ell}), \quad (\text{III.5.17})$$

$$E_2(\ell, \gamma) = -\gamma^2 + 4\gamma^2 e^{-2\gamma\ell} + 8\gamma^2(2\gamma\ell - 1)e^{-4\gamma\ell} + \mathcal{O}(\gamma^4 e^{-6\gamma\ell}), \quad (\text{III.5.18})$$

and,

$$E_{2+j}(\ell, \gamma) = \left(\frac{\pi j}{2\ell}\right)^2 + o(1).$$

Proof. By symmetry, it is easy to see that $\mathcal{R}_{\ell,\gamma}$ is unitarily equivalent to $\mathcal{N}_{\ell,\gamma} \oplus \mathcal{D}_{\ell,\gamma}$. As $\mathcal{N}_{\ell,\gamma}$ is a rank-one perturbation of $\mathcal{D}_{\ell,\gamma}$, we can again apply Proposition III.1.6 to have the the following inequality for any $j \in \mathbb{N}$,

$$E_j(\mathcal{N}_{\ell,\gamma}) \leq E_j(\mathcal{D}_{\ell,\gamma}) \leq E_{j+1}(\mathcal{N}_{\ell,\gamma}).$$

Therefore, $E_1(\ell, \gamma) = E_1(\mathcal{N}_{\ell,\gamma})$ and $E_2(\ell, \gamma) = E_1(\mathcal{D}_{\ell,\gamma})$, and the associated asymptotics follow from Lemma III.5.2 and III.5.3.

Let us focus on the positive eigenvalues. Noticing that

$$\left\{ \left(\frac{\pi(2j-1)}{2\ell}\right)^2, j \in \mathbb{N} \right\} \cup \left\{ \left(\frac{\pi j}{\ell}\right)^2, j \in \mathbb{N} \right\} = \left\{ \left(\frac{\pi j}{2\ell}\right)^2, j \in \mathbb{N} \right\},$$

and using (III.5.4) and (III.5.11) gives us the asymptotics of the further eigenvalues $E_{2+j}(\ell, \gamma)$. \square

Example III.5.5. Thanks to these asymptotics, we are able to compute the asymptotics for any eigenvalue of the Robin Laplacian on a square. For $\ell > 0$, denote by \mathcal{R}_ℓ the square of length 2ℓ . By separation of variables, the Robin Laplacian on \mathcal{R}_ℓ denoted by $Q_{\mathcal{R}_\ell}^\gamma$ is unitarily equivalent to $\mathcal{R}_{\ell,\gamma} \otimes 1 + 1 \otimes \mathcal{R}_{\ell,\gamma}$ and using Proposition III.5.4 we have the following asymptotics for large γ ,

$$\begin{aligned} E_1(Q_{\mathcal{R}_\ell}^\gamma) &= -2\gamma^2 - 8\gamma^2 e^{-2\gamma\ell} + 16\gamma^2(2\gamma\ell - 1)e^{-4\gamma\ell} + \mathcal{O}(\gamma^4 e^{-6\gamma\ell}), \\ E_2(Q_{\mathcal{R}_\ell}^\gamma) &= E_3(Q_{\mathcal{R}_\ell}^\gamma) = -2\gamma^2 + 16\gamma^2(2\gamma\ell - 1)e^{-4\gamma\ell} + \mathcal{O}(\gamma^4 e^{-6\gamma\ell}), \\ E_4(Q_{\mathcal{R}_\ell}^\gamma) &= -2\gamma^2 + 8\gamma^2 e^{-2\gamma\ell} + 16\gamma^2(2\gamma\ell - 1)e^{-4\gamma\ell} + \mathcal{O}(\gamma^4 e^{-6\gamma\ell}), \end{aligned}$$

and for any $j \in \mathbb{N}$,

$$E_{4+j}(Q_{\mathcal{R}_\ell}^\gamma) = -\gamma^2 + \mu_j + o(1),$$

where μ_j is exactly the j -th element of the union of the four disjoint copies of $\left\{ \left(\frac{\pi k}{2\ell} \right)^2, k \in \mathbb{N} \right\}$.

Let us introduce some quantities to be used later in the text. For $\gamma \in \mathbb{R}_+$, denote

$$m(\gamma) := \sqrt{-E_1(1, \gamma)},$$

then we have

$$E_1(\ell, \gamma) = -\frac{m(\gamma\ell)^2}{\ell^2}, \quad \Phi_\ell(\gamma, t) = \cosh\left(\frac{m(\gamma\ell)t}{\ell}\right),$$

and a simple direct computation gives

$$\int_{-\ell}^{\ell} |\Phi_\ell(\gamma, t)|^2 dt = \ell \left(\frac{\sinh(2m(\gamma\ell))}{2m(\gamma\ell)} + 1 \right).$$

The first eigenfunction $\tilde{\Phi}_\ell(\gamma, \cdot)$ of $\mathcal{R}_{\ell,\gamma}$, chosen positive and normalized, is then given by

$$\tilde{\Phi}_\ell(\gamma, t) = C_\ell(\gamma) \Phi_\ell(\gamma, t) \quad \text{with} \quad C_\ell(\gamma) := \frac{1}{\sqrt{\ell}} \left(\frac{\sinh(2m(\gamma\ell))}{2m(\gamma\ell)} + 1 \right)^{-\frac{1}{2}}.$$

It follows from the proof of Proposition III.5.4 that $\gamma \ni \mathbb{R}_+ \mapsto E_1(1, \gamma)$ is real-analytic, as it is defined as the unique solution of $k \tanh(k) = \gamma$. Then, for any fixed ℓ and t the functions $\gamma \mapsto C_\ell(\gamma)$ and $\gamma \mapsto \Phi_\ell(\gamma, t)$ are smooth, and, by direct computation,

$$\begin{aligned} \frac{\partial}{\partial \gamma} C_\ell(\gamma) &= -\sqrt{\ell} m'(\gamma\ell) \left(\frac{\sinh(2m(\gamma\ell))}{2m(\gamma\ell)} + 1 \right)^{-\frac{3}{2}} \left(\frac{\cosh(2m(\gamma\ell))}{2m(\gamma\ell)} - \frac{\sinh(2m(\gamma\ell))}{(2m(\gamma\ell))^2} \right), \\ \frac{\partial}{\partial \gamma} \Phi_\ell(\gamma, t) &= m'(\gamma\ell) t \sinh\left(m(\gamma\ell) \frac{t}{\ell}\right). \end{aligned}$$

Moreover, as $\gamma \mapsto E_1(\ell, \gamma)$ is simple, we can compute its first derivative in the standard way, namely, consider the implicit equation satisfied by $\Phi_\ell(\gamma, \cdot)$ and $E_1(\ell, \gamma)$:

$$-\frac{\partial^2}{\partial t^2} \Phi_\ell(\gamma, t) = E_1(\ell, \gamma) \Phi_\ell(\gamma, t), \quad t \in (-\ell, \ell) \quad (\text{III.5.19})$$

$$\pm \frac{\partial}{\partial t} \Phi_\ell(\gamma, \pm\ell) = \gamma \Phi_\ell(\gamma, \pm\ell). \quad (\text{III.5.20})$$

We take the derivative of (III.5.19) with respect to γ and, after multiplying by $\Phi_\ell(\gamma, \cdot)$, we integrate it over $(-\ell, \ell)$. After two integrations by part we obtain the following equality

$$\begin{aligned} & \int_{-\ell}^{\ell} \partial_\gamma \Phi_\ell(\gamma, t) \left(-\frac{\partial^2}{\partial t^2} \Phi_\ell(\gamma, t) \right) dt \\ & - \left[(\partial_t \partial_\gamma \Phi_\ell(\gamma, t)) (\Phi_\ell(\gamma, t)) \right]_{-\ell}^{\ell} + \left[(\partial_\gamma \Phi_\ell(\gamma, t)) (\partial_t \Phi_\ell(\gamma, t)) \right]_{-\ell}^{\ell} \\ & = \partial_\gamma E_1(\ell, \gamma) \int_{-\ell}^{\ell} |\Phi_\ell(\gamma, t)|^2 dt + E_1(\ell, \gamma) \int_{-\ell}^{\ell} \partial_\gamma \Phi_\ell(\gamma, t) \Phi_\ell(\gamma, t) dt. \end{aligned} \quad (\text{III.5.21})$$

We now take the derivative of (III.5.20) with respect to γ and get

$$\begin{aligned} \partial_\gamma \partial_t \Phi_\ell(\gamma, \ell) &= \Phi_\ell(\gamma, \ell) + \gamma \partial_\gamma \Phi_\ell(\gamma, \ell), \\ -\partial_\gamma \partial_t \Phi_\ell(\gamma, -\ell) &= \Phi_\ell(\gamma, -\ell) + \gamma \partial_\gamma \Phi_\ell(\gamma, -\ell). \end{aligned}$$

After replacing these expressions in (III.5.21) we finally have

$$\frac{\partial}{\partial \gamma} E_1(\ell, \gamma) = -\frac{|\Phi_\ell(\gamma, -\ell)|^2 + |\Phi_\ell(\gamma, \ell)|^2}{\|\Phi_\ell(\gamma, \cdot)\|_{L^2(-\ell, \ell)}^2} = -\frac{2 \cosh^2(m(\gamma \ell))}{\ell \left(\frac{\sinh(2m(\gamma \ell))}{2m(\gamma \ell)} + 1 \right)}. \quad (\text{III.5.22})$$

In particular,

$$\frac{\partial}{\partial \gamma} E_1(1, \gamma) \Big|_{\gamma=0} = -1. \quad (\text{III.5.23})$$

Furthermore, due to the preceding consideration, the following representation is valid:

Proposition III.5.6. *There exists $\phi \in C^\infty(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ such that $E_1(1, \gamma) = -\gamma + \gamma^2 \phi(\gamma)$ for all $\gamma \in \mathbb{R}_+$.*

Proof. Define $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ by $\phi(\gamma) = \gamma^{-2}(E_1(1, \gamma) + \gamma)$. Due to (III.5.17), the function ϕ is bounded at infinity. For γ near 0 one has, due to (III.5.15) and (III.5.23) and due to the analyticity, $E_1(1, \gamma) = -\gamma + O(\gamma^2)$, which shows that ϕ is bounded near 0. As ϕ is continuous, the result follows. \square

We conclude this section by giving a simple result on a Robin Laplacian acting on an interval with distinct Robin parameters at each end-points.

Proposition III.5.7. *For $\gamma, \beta, \ell > 0$ denote by $\mathcal{R}_{\ell, \gamma, \beta}$ the operator acting on $L^2(0, \ell)$ as $f \mapsto -f''$ with $D(\mathcal{R}_{\ell, \gamma, \beta}) := \{f \in H^2(0, \ell), -f'(0) - \gamma f(0) = f'(\ell) - \beta f(\ell) = 0\}$. If $\gamma > \beta$ and $\beta \ell < 1$, then $\mathcal{R}_{\ell, \gamma, \beta}$ has a unique negative eigenvalue. Moreover, for any $\ell, \beta > 0$ there holds*

$$E_1(\mathcal{R}_{\ell, \gamma, \beta}) = -\gamma^2 + \mathcal{O}(\gamma^2 e^{-\gamma \ell}), \quad \gamma \rightarrow +\infty. \quad (\text{III.5.24})$$

Proof. Suppose that $\gamma > \beta$ and $\beta \ell < 1$. We first prove that $\mathcal{R}_{\ell, \gamma, \beta}$ admits a unique negative eigenvalue. The form domain of $\mathcal{R}_{\ell, \gamma, \beta}$ is $H^1(0, \ell)$. Consider $\tilde{H}_0^1 := \{f \in H^1(0, \ell) : f(0) = 0\}$. Then, \tilde{H}_0^1 is the form domain of $\mathcal{D}_{\ell, \beta}$. We have $H^1(0, \ell) = \tilde{H}_0^1 \oplus \mathbb{C}$ and the forms associated with $\mathcal{R}_{\ell, \gamma, \beta}$ and $\mathcal{D}_{\ell, \beta}$ coincide on \tilde{H}_0^1 . We can apply Proposition III.1.6 to get, for any $n \in \mathbb{N}$,

$$E_n(\mathcal{R}_{\ell, \gamma, \beta}) \leq E_n(\mathcal{D}_{\ell, \beta}) \leq E_{n+1}(\mathcal{R}_{\ell, \gamma, \beta}). \quad (\text{III.5.25})$$

By Lemma III.5.3, we know that $E_1(\mathcal{D}_{\ell,\beta}) \geq 0$ as $\beta\ell < 1$, which implies due to (III.5.25) that

$$E_2(\mathcal{R}_{\ell,\gamma,\beta}) \geq 0.$$

Let us prove that $E_1(\mathcal{R}_{\ell,\gamma,\beta}) < 0$. Let us denote by $E_1(\mathcal{R}_{\ell,\gamma,\beta}) = -k^2$, with $k > 0$. An eigenfunction associated with a negative eigenvalue is in the form $f(t) = Ae^{kt} + Be^{-kt}$ where $(A, B) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Using the boundary conditions we obtain the system:

$$k(-A + B) = \gamma(A + B), \quad k(Ae^{k\ell} - Be^{-k\ell}) = \beta(Ae^{k\ell} + Be^{-k\ell}).$$

It admits a non-zero solution if and only if its determinant vanishes, namely if k satisfies

$$(k + \gamma)(k + \beta)e^{-k\ell} = (k - \gamma)(k - \beta)e^{k\ell}. \quad (\text{III.5.26})$$

For $t \in (\gamma, +\infty)$, consider the functions

$$h(t) := \frac{t - \beta}{t + \beta}e^{2t\ell}, \quad g(t) := \frac{t + \gamma}{t - \gamma}.$$

One may rewrite (III.5.26) as follows: $g(k) = h(k)$. The function g is strictly decreasing on $(\gamma, +\infty)$, $g(t) \rightarrow +\infty$ as $t \rightarrow \gamma$ and $g(t) \rightarrow 1$ as $t \rightarrow +\infty$. Recall that $\gamma > \beta$. Then, the function h is strictly increasing on $(\gamma, +\infty)$, $h(\gamma) = \frac{\gamma - \beta}{\gamma + \beta}e^{2\gamma\ell} > 0$ and $h(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. The properties of the functions h and g imply that (III.5.26) admits a unique solution $k > \gamma$ and thus $\mathcal{R}_{\ell,\gamma,\beta}$ admits a unique negative eigenvalue satisfying

$$E_1(\mathcal{R}_{\ell,\gamma,\beta}) < -\gamma^2. \quad (\text{III.5.27})$$

Let us prove (III.5.24). Notice that as $\gamma > \beta$ we have immediately

$$E_1(\mathcal{R}_{\ell,\gamma,\beta}) \geq E_1(\mathcal{R}_{\ell,\gamma}).$$

Combining (III.5.17) and (III.5.27) finishes the proof. \square

III.6 Summary of the important notations

In this thesis, the positive constants which *do not depend* on a large (e.g. γ) or a small (e.g. ϵ) parameter are denoted by $C > 0$.

In Table III.1 we recall the notation associated with a self-adjoint and semibounded from below operator A acting on a Hilbert space \mathcal{H} . Recall that $\mathbb{N} := \{1, 2, 3, \dots\}$.

Operator	Sesquilinear form	Eigenvalues	Eigenvalue counting function	Spectral projection
$(A, D(A))$	$(a, D(a))$	$E_n(A)$	$\mathcal{N}(A, \lambda) := \#\{n \in \mathbb{N} : E_n(A) < \lambda\}$	$P_A(a, b)$

Table III.1 – General notation for a self-adjoint and semibounded from below operator A .

Definition III.6.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Suppose that the boundary is composed by two components Γ_1, Γ_2 such that $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$. For a parameter $\gamma > 0$, we say that the operator A defined on $L^2(\Omega)$ acts as the Laplacian with the γ -Robin boundary condition on Γ_1 and the Neumann, respectively the Dirichlet, boundary condition on Γ_2 if it is defined as the unique self-adjoint operator associated with

$$a(u, u) = \int_{\Omega} |\nabla u|^2 dx - \gamma \int_{\Gamma_1} |u|^2 ds,$$

with $D(a) := H^1(\Omega)$ if A satisfies the Neumann boundary condition on Γ_2 and $D(a) := \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_2\}$ if A satisfies the Dirichlet boundary condition on Γ_2 .

In Table III.2, we make a summary of the important operators.

Operator	Form domain	Quadratic form
\mathcal{B}_γ	$H^1(\mathbb{R}_+)$	$f \mapsto \int_{\mathbb{R}_+} f'(t) ^2 dt - \gamma f(0) ^2$
$\mathcal{N}_{\ell,\gamma}$	$H^1(0, \ell)$	$f \mapsto \int_0^\ell f'(t) ^2 dt - \gamma f(\ell) ^2$
$\mathcal{D}_{\ell,\gamma}$	$\{f \in H^1(0, \ell) : f(0) = 0\}$	$f \mapsto \int_0^\ell f'(t) ^2 dt - \gamma f(\ell) ^2$
$\mathcal{R}_{\ell,\gamma}$	$H^1(-\ell, \ell)$	$f \mapsto \int_{-\ell}^\ell f'(t) ^2 dt - \gamma f(-\ell) ^2 - \gamma f(\ell) ^2$
$\mathcal{R}_{\ell,\gamma,\beta}$	$H^1(0, \ell)$	$f \mapsto \int_0^\ell f'(t) ^2 dt - \gamma f(0) ^2 - \beta f(\ell) ^2$
Q_Ω^γ	$H^1(\Omega)$	$u \mapsto \int_\Omega \nabla u ^2 dx - \gamma \int_{\partial\Omega} u ^2 ds$
T_α^γ	$H^1(U_\alpha)$	$u \mapsto \int_{U_\alpha} \nabla u ^2 dx - \gamma \int_{\partial U_\alpha} u ^2 ds$
$T_{\alpha,R}^\gamma$	$H^1(U_{\alpha,R})$	$u \mapsto \int_{U_{\alpha,R}} \nabla u ^2 dx - \gamma \int_{\partial U_{\alpha,R} \cap \partial U_\alpha} u ^2 ds$

Table III.2 – Important operators.

Chapter IV

Robin Laplacians in infinite sectors

Abstract

Let $\alpha \in (0, \pi)$ and $\gamma > 0$. This chapter is devoted to the study of the Laplacian with the Robin boundary condition $\partial_\nu u = \gamma u$, where ν is the *outward* unit normal, acting on the infinite sector U_α and denoted by T_α^γ . In Section IV.1 we give a precise definition of the operator and show that the essential spectrum equals $[-\gamma^2, +\infty)$ and that the discrete spectrum exists iff $\alpha < \frac{\pi}{2}$. In Section IV.2, we prove that the discrete spectrum is finite and that each individual eigenvalue is a continuous strictly increasing function of the angle α . In particular, there is just one eigenvalue for $\alpha \geq \frac{\pi}{6}$. In Section IV.3, we give a first order asymptotics and a complete asymptotic expansion for the eigenvalues as the angle α becomes small. In Section IV.4, we prove that the associated eigenfunctions are localized near the origin. Finally, in Section IV.5 we prove some useful results on a Laplacian with a mixed Robin-Neumann boundary condition acting on a truncated sector, which will be used in the subsequent chapter.

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IV.1 First properties

IV.1.1 Essential spectrum and existence of the discrete spectrum

The following theorem is a starting point for our considerations. The points (b) and (c) were already discussed in [LP08], and a study of Robin Laplacians acting on cones in any dimension is made in [BP16]. However, we prefer to give a complete proof in order to keep the presentation self-contained.

Theorem IV.1.1. *For any $\alpha \in (0, \pi)$ and any $\gamma > 0$ the sesquilinear form t_α^γ given by (II.2.1) is closed and semibounded from below, hence, the associated operator T_α^γ is self-adjoint in $L^2(U_\alpha)$. Furthermore,*

(a) $\text{spec}_{\text{ess}} T_\alpha^\gamma = [-\gamma^2, +\infty)$ for any $\alpha \in (0, \pi)$.

(b) If $\alpha \in (0, \frac{\pi}{2})$, then

$$E_1(T_\alpha^\gamma) = -\frac{\gamma^2}{\sin^2 \alpha},$$

and $u(x_1, x_2) = \exp(-\gamma x_1 / \sin \alpha)$ is an associated eigenfunction.

(c) for $\alpha \in [\frac{\pi}{2}, \pi)$, the discrete spectrum of T_α^γ is empty.

Therefore, the discrete spectrum of T_α^γ is non-empty if and only if $\alpha < \frac{\pi}{2}$.

IV.1.2 Proof of Theorem IV.1.1

IV.1.2.1 Closedness and semiboundedness.

We need the following density result, which is quite standard (see Appendix A.1 for the proof):

Lemma IV.1.2. *The set*

$$\mathcal{F} = \left\{ u \in C^\infty(\overline{U_\alpha}) : \text{there exist } R_1, R_2 > 0 \text{ such that } u = 0 \text{ for } |x| < R_1 \text{ and for } |x| > R_2 \right\}$$

is dense in $H^1(U_\alpha)$.

Let $\gamma > 0$, $\alpha \in (0, \pi)$ and $u \in \mathcal{F}$. Using the one-dimensional inequality (III.5.1) we have

$$\begin{aligned} t_\alpha^\gamma(u, u) &= \int_{x_2 \in \mathbb{R}} \int_{x_1 > \frac{|x_2|}{\tan \alpha}} |\nabla u(x_1, x_2)|^2 dx - \gamma \int_{\mathbb{R}} |u\left(\frac{|x_2|}{\tan \alpha}, x_2\right)|^2 \frac{dx_2}{\sin \alpha} \\ &\geq \int_{\mathbb{R}} \left(\int_{x_1 > \frac{|x_2|}{\tan \alpha}} \left| \frac{\partial u}{\partial x_1}(x_1, x_2) \right|^2 dx_1 - \frac{\gamma}{\sin \alpha} |u\left(\frac{|x_2|}{\tan \alpha}, x_2\right)|^2 \right) dx_2, \quad (\text{IV.1.1}) \\ &\geq -\frac{\gamma^2}{\sin^2 \alpha} \|u\|_{L^2(U_\alpha)}^2. \end{aligned}$$

Writing $\gamma = 1/\epsilon$ we conclude that for all $u \in \mathcal{F}$ and for all $\epsilon > 0$ we have

$$\int_{\partial U_\alpha} |u|^2 ds \leq \epsilon \int_{U_\alpha} |\nabla u|^2 dx + \frac{1}{\epsilon \sin^2 \alpha} \int_{U_\alpha} |u|^2 dx,$$

which means that the trace can be extended to a bounded linear map from $H^1(U_\alpha)$ to $L^2(\partial U_\alpha)$. Furthermore, the boundary term in t_α^γ is then infinitesimally small with respect to the gradient term, hence, t_α^γ is closed on $H^1(U_\alpha)$ due to the KLMN theorem, see [RS75, Theorem X.17].

IV.1.2.2 Bottom of the spectrum for $\alpha < \frac{\pi}{2}$.

Let us show that $\inf \text{spec } T_\alpha^\gamma = -\gamma^2(\sin^2 \alpha)^{-1}$ for $\alpha < \frac{\pi}{2}$. By (IV.1.1) we have the inequality $T_\alpha^\gamma \geq -\gamma^2(\sin^2 \alpha)^{-1}$. On the other hand, by the explicit computation, $t_\alpha^\gamma(u_0, u_0) = -\gamma^2(\sin^2 \alpha)^{-1} \|u_0\|_{L^2(U_\alpha)}^2$ for $u_0(x_1, x_2) = e^{-\gamma x_1 / \sin \alpha}$. It follows that the lower bound is optimal and that the bottom of the spectrum is an eigenvalue, and u_0 is an associated eigenfunction, which proves the point (b) of the theorem.

IV.1.2.3 Lower bound for $\alpha \geq \frac{\pi}{2}$.

Let us show that $\inf \text{spec } T_\alpha^\gamma \geq -\gamma^2$ for $\alpha \in [\frac{\pi}{2}, \pi)$. Decompose U_α into the following three pieces:

$$D_1 = \left\{ x \in \mathbb{R}^2 : |\arg(x_1 + ix_2)| < \alpha - \frac{\pi}{2} \right\},$$

$$D_2 = (U_\alpha \setminus \overline{D_1}) \cap (\mathbb{R} \times \mathbb{R}_+), \quad D_3 = (U_\alpha \setminus \overline{D_1}) \cap (\mathbb{R} \times \mathbb{R}_-).$$

Then $\overline{U_\alpha} = \overline{D_1 \cup D_2 \cup D_3}$ and we define, for $j = 1, 2, 3$,

$$q_j(u, u) = \int_{D_j} |\nabla u|^2 dx - \gamma \int_{\partial U_\alpha \cap \partial D_j} |u|^2 ds, \quad u \in H^1(D_j).$$

By the min-max principle and the inclusion $H^1(U_\alpha) \subset H^1(D_1 \cup D_2 \cup D_3)$ we have

$$\inf \text{spec } T_\alpha^\gamma \geq \inf \text{spec } Q, \tag{IV.1.2}$$

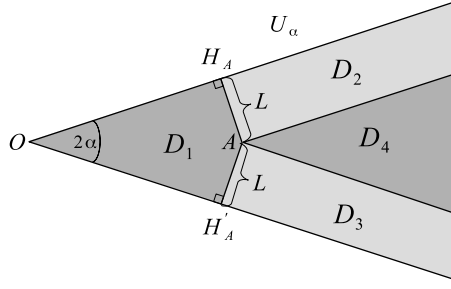
where Q is the self-adjoint operator acting on $L^2(U_\alpha)$ associated with the sesquilinear form defined for $u \in H^1(D_1 \cup D_2 \cup D_3)$ by $q(u, u) = q_1(u, u) + q_2(u, u) + q_3(u, u)$. We have then $Q = Q_1 \oplus Q_2 \oplus Q_3$, where Q_j are the self-adjoint operators associated with q_j and acting in $L^2(D_j)$. Notice that Q_1 is positive as $\partial U_\alpha \cap \partial D_1 = \emptyset$ and Q_2 and Q_3 are unitarily equivalent and have the same spectrum. Furthermore, Q_2 is unitarily equivalent to $T^N \otimes 1 + 1 \otimes \mathcal{B}_\gamma$, where T^N is the Neumann Laplacian in $L^2(\mathbb{R}_+)$ and \mathcal{B}_γ is the Robin Laplacian in $L^2(\mathbb{R}_+)$, defined in subsection III.5.1. This immediately gives $\inf \text{spec } Q_2 = -\gamma^2$. Therefore, $\inf \text{spec } Q = \min\{\inf \text{spec } Q_1, \inf \text{spec } Q_2, \inf \text{spec } Q_3\} = -\gamma^2$, and (IV.1.2) gives the result.

IV.1.2.4 Lower bound for the essential spectrum as $\alpha < \frac{\pi}{2}$.

Let us show the lower bound

$$\inf \text{spec}_{\text{ess}} T_\alpha^\gamma \geq -\gamma^2. \tag{IV.1.3}$$

Let $A = (a, 0)$ with $a > 0$. We denote by C_A the sector obtained after a translation of U_α along the vector OA . Let H_A be the orthogonal projection of A on the half-line $\mathbb{R}_+(1, \tan \alpha)$ and $L := |AH_A| \equiv a \sin \alpha$, and, in particular, $L \rightarrow +\infty$ for $a \rightarrow +\infty$. In the same way, we


 Figure IV.1 – Partition of U_α , $\alpha < \frac{\pi}{2}$.

define H'_A the orthogonal projection of A on $\mathbb{R}_+(1, -\tan \alpha)$. Consider the following four domains:

$$\begin{aligned} D_1 &= OH_AAH'_A, & D_2 &= \left(U_\alpha \setminus (\overline{C_A \cup D_1}) \right) \cap (\mathbb{R} \times \mathbb{R}_+), \\ D_3 &= \left(U_\alpha \setminus (\overline{C_A \cup D_1}) \right) \cap (\mathbb{R} \times \mathbb{R}_-), & D_4 &= C_A, \end{aligned}$$

see Fig IV.1. Clearly, $\overline{U_\alpha} = \overline{D_1 \cup D_2 \cup D_3 \cup D_4}$. Define for $j \in \{1, 2, 3, 4\}$ the sesquilinear forms

$$q_j(u, u) = \int_{D_j} |\nabla u|^2 dx - \gamma \int_{\partial U_\alpha \cap \partial D_j} |u|^2 ds, \quad D(q_j) = H^1(D_j).$$

By the min-max principle, the inclusion $H^1(U_\alpha) \subset H^1(D_1 \cup D_2 \cup D_3 \cup D_4)$ implies the inequality

$$\inf \text{spec}_{\text{ess}} T_\alpha^\gamma \geq \inf \text{spec}_{\text{ess}} Q, \quad (\text{IV.1.4})$$

where Q is the self-adjoint operator acting in $L^2(U_\alpha)$ and associated with the sesquilinear form defined for $u \in H^1(D_1 \cup D_2 \cup D_3 \cup D_4)$ by $q(u, u) = q_1(u, u) + q_2(u, u) + q_3(u, u) + q_4(u, u)$. (Remark that we use the convention $\inf \emptyset = +\infty$.) We have then $Q = Q_1 \oplus Q_2 \oplus Q_3 \oplus Q_4$, where each Q_j is the self-adjoint operator acting in $L^2(D_j)$ and associated with q_j , and (IV.1.4) implies

$$\inf \text{spec}_{\text{ess}} T_\alpha^\gamma \geq \min_{j \in \{1, 2, 3, 4\}} \inf \text{spec}_{\text{ess}} Q_j. \quad (\text{IV.1.5})$$

The operator Q_1 has a compact resolvent, then its essential spectrum is empty and $\inf \text{spec}_{\text{ess}} Q_1 = +\infty$, and the operator Q_4 is non-negative, hence, $\inf \text{spec}_{\text{ess}} Q_4 \geq 0$.

The operators Q_2 et Q_3 are unitarily equivalent, so we have to study Q_2 only. By applying an anticlockwise rotation of angle $\theta = \frac{\pi}{2} - \alpha$ we see that Q_2 is unitarily equivalent to the self-adjoint operator \tilde{Q}_2 acting in $H^1((0, L) \times \mathbb{R}_+)$ and associated with the sesquilinear form

$$\tilde{q}_2(u, u) = \int_0^{+\infty} \int_0^L |\nabla u|^2 dx - \gamma \int_0^{+\infty} |u(0, x_2)|^2 dx_2, \quad u \in H^1((0, L) \times \mathbb{R}_+).$$

Clearly, $\tilde{Q}_2 = \mathcal{N}_{L, \gamma} \otimes 1 + 1 \otimes T^N$, where $\mathcal{N}_{L, \gamma}$ is the one-dimensional Robin-Neumann Laplacian acting in $L^2(0, L)$ defined in Lemma III.5.2 and T^N is the Neumann Laplacian in $L^2(\mathbb{R}_+)$. As $\text{spec } T^N = [0, +\infty)$, we have $\text{spec } Q_2 = \text{spec}_{\text{ess}} Q_2 = [E_1(\mathcal{N}_{L, \gamma}), +\infty)$, and by the asymptotics (III.5.2) we have $E_1(\mathcal{N}_{L, \gamma}) = -\gamma^2 + o(1)$ as $L \rightarrow +\infty$. Hence, for any $\epsilon > 0$ there exists $L_\epsilon > 0$ such as

$$\inf \text{spec}_{\text{ess}} Q_2 \geq -\gamma^2 - \epsilon \text{ for } L \geq L_\epsilon.$$

As (IV.1.5) is valid for any $L > 0$, we have $\inf \text{spec}_{\text{ess}} T_\alpha^\gamma \geq -\gamma^2 - \epsilon$ for any $\epsilon > 0$, which implies (IV.1.3).

IV.1.2.5 Description of the essential spectrum

Due to the results of subsections IV.1.2.3 and IV.1.2.4 we have $\text{spec}_{\text{ess}} T_\alpha^\gamma \subset [-\gamma^2, +\infty)$ for any α . In order to conclude the proof of the point (a), it is sufficient to show the inclusion

$$[-\gamma^2, +\infty) \subset \text{spec } T_\alpha^\gamma. \quad (\text{IV.1.6})$$

It is more convenient to work with the rotated sector

$$\tilde{U}_\alpha = \{x \in \mathbb{R}^2 : 0 < \arg(x_1 + ix_2) < 2\alpha\}$$

and the associated Robin Laplacian \tilde{T}_α^γ in $L^2(\tilde{U}_\alpha)$ corresponding to the sesquilinear form

$$\tilde{t}_\alpha^\gamma(u, u) = \int_{\tilde{U}_\alpha} |\nabla u|^2 dx - \gamma \int_{\partial \tilde{U}_\alpha} |u|^2 ds, \quad u \in H^1(\tilde{U}_\alpha).$$

Clearly, \tilde{T}_α^γ is unitarily equivalent to T_α^γ . The proof of (IV.1.6) consists in finding, for each $k \in \mathbb{R}$, a family of functions $f_N \in D(\tilde{T}_\alpha^\gamma)$ with

$$\frac{\|\tilde{T}_\alpha^\gamma f_N - (k^2 - \gamma^2)f_N\|_{L^2(\tilde{U}_\alpha)}^2}{\|f_N\|_{L^2(\tilde{U}_\alpha)}^2} \rightarrow 0 \text{ for } N \rightarrow +\infty, \quad (\text{IV.1.7})$$

then the result follows by the spectral theorem for self-adjoint operators, see Theorem III.3.1. Let $\phi \in C^\infty(\mathbb{R})$ satisfying $\phi(t) = 0$ for $t \leq 0$ and $\phi(t) = 1$ for $t \geq 1$. For $N \in \mathbb{N}$, define a function $f_N : \tilde{U}_\alpha \rightarrow \mathbb{C}$ by

$$f_N(x_1, x_2) = e^{ikx_1 - \gamma x_2} \phi(2N - x_1) \phi(x_1 - N) \phi(aN - x_2),$$

where $a = \sin 2\alpha$ for $\alpha < \frac{\pi}{4}$ and $a > 0$ is arbitrary otherwise. The function f_N is smooth, compactly supported and satisfies the Robin boundary condition $\partial f_N / \partial \nu = \gamma f_N$ at $\partial \tilde{U}_\alpha$, hence, it belongs to $D(\tilde{T}_\alpha^\gamma)$, and $\tilde{T}_\alpha^\gamma f_N = -\Delta f_N$. Easy estimates show that for large N we have

$$\|f_N\|_{L^2(\tilde{U}_\alpha)}^2 \geq cN \text{ with } c > 0, \quad \|-\Delta f_N - (k^2 - \gamma^2)f_N\|_{L^2(\tilde{U}_\alpha)}^2 = O(1),$$

which gives (IV.1.7).

Finally, the combination of (IV.1.6) with the result of subsection IV.1.2.3 gives the point (c) of the theorem.

IV.2 Qualitative spectral properties

We are interested in the discrete spectrum of T_α^γ . Our principal aim is to obtain a more detailed information on the number of discrete eigenvalues and on their behavior with respect to the angle α . In view of Theorem IV.1.1, we can restrict ourselves to the case $\alpha < \frac{\pi}{2}$, *which will be assumed in the rest of this chapter*. Moreover, due to the invariance by dilations of U_α the operator T_α^γ is unitarily equivalent to $\gamma^2 T_\alpha^1$. It is then sufficient in what follows to study

$$T_\alpha := T_\alpha^1.$$

Our first objective is to show that the discrete spectrum of T_α is finite. Later, in section IV.2.3, we prove that the eigenvalues of T_α are monotone continuous functions of α .

IV.2.1 Reduction by parity

We start with a decomposition of T_α due to the symmetry of the sector. Consider the upper half U_α^+ of U_α , $U_\alpha^+ = U_\alpha \cap (\mathbb{R} \times \mathbb{R}_+)$, and the unitary map

$$\begin{aligned} \mathcal{U} : L^2(U_\alpha) &\ni u \mapsto (g, h) \in L^2(U_\alpha^+) \oplus L^2(U_\alpha^+), \\ g(x_1, x_2) &:= \frac{u(x_1, x_2) + u(x_1, -x_2)}{\sqrt{2}}, \quad h(x_1, x_2) := \frac{u(x_1, x_2) - u(x_1, -x_2)}{\sqrt{2}}. \end{aligned}$$

By a direct computation, for $u \in D(t_\alpha)$ one has $t_\alpha(u, u) = t_\alpha^N(g, g) + t_\alpha^D(h, h)$ with

$$t_\alpha^N(g, g) = \int_{U_\alpha^+} |\nabla g(x_1, x_2)|^2 dx - \int_{\mathbb{R}^+} \left| g\left(\frac{x_2}{\tan \alpha}, x_2\right) \right|^2 \frac{dx_2}{\sin \alpha}, \quad g \in H^1(U_\alpha^+), \quad (\text{IV.2.1})$$

and t_α^D is given by the same expression but acts on the smaller domain

$$D(t_\alpha^D) := \{h \in H^1(U_\alpha^+) : h(\cdot, 0) = 0\},$$

and $D(t_\alpha^N) = P_1 \mathcal{U} D(t_\alpha)$ and $D(t_\alpha^D) = P_2 \mathcal{U} D(t_\alpha)$, where $P_j : L^2(U_\alpha^+) \oplus L^2(U_\alpha^+) \rightarrow L^2(U_\alpha^+)$ is the projection onto the j th component. Hence, if T_α^N and T_α^D are the self-adjoint operators acting in $L^2(U_\alpha^+)$ and associated with t_α^N and t_α^D , respectively, then, by construction, $T_\alpha = \mathcal{U}^*(T_\alpha^N \oplus T_\alpha^D)\mathcal{U}$, and it is sufficient to study separately the spectra of T_α^D and T_α^N .

Let us show first that

$$\inf \text{spec } T_\alpha^D \geq -1. \quad (\text{IV.2.2})$$

To see this, consider the half-plane

$$P_\alpha = \{(x_1, x_2) \in \mathbb{R}^2, \quad x_1 \geq \frac{x_2}{\tan \alpha}\} \quad (\text{IV.2.3})$$

and remark that if one takes $u \in D(t_\alpha^D)$ and denotes by \tilde{u} its extension by zero to P_α , then

$$t_\alpha^D(u, u) = q_{P_\alpha}(\tilde{u}, \tilde{u}),$$

where

$$q_{P_\alpha}(u, u) = \int_{\mathbb{R}} \int_{\frac{x_2}{\tan \alpha}}^{+\infty} |\nabla u(x_1, x_2)|^2 dx - \int_{\mathbb{R}} \left| u\left(\frac{x_2}{\tan \alpha}, x_2\right) \right|^2 \frac{dx_2}{\sin \alpha}, \quad D(q_{P_\alpha}) := H^1(P_\alpha).$$

If Q_{P_α} is the self-adjoint operator associated with q_{P_α} and acting in $L^2(P_\alpha)$, then $\inf \text{spec } T_\alpha^D \geq \inf \text{spec } Q_{P_\alpha}$. On the other side, by applying a rotation one sees that Q_{P_α} is unitarily equivalent to $Q = \mathcal{B}_1 \otimes 1 + 1 \otimes (-\Delta_{\mathbb{R}})$ acting in $L^2(\mathbb{R} \times \mathbb{R}_+) \simeq L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)$, where \mathcal{B}_1 is the Robin Laplacian acting on $L^2(\mathbb{R}_+)$ defined in subsection III.5.1 and $-\Delta_{\mathbb{R}}$ is the free Laplacian in $L^2(\mathbb{R})$. In particular, $\text{spec } Q = \text{spec } \mathcal{B}_1 + \text{spec}(-\Delta_{\mathbb{R}}) = [-1, +\infty)$, which proves (IV.2.2). Therefore, we have

$$\text{spec } T_\alpha \cap (-\infty, -1) = \text{spec } T_\alpha^N \cap (-\infty, -1).$$

Furthermore, in view of Theorem IV.1.1 we have

$$\text{spec}_{\text{ess}} T_\alpha^N \subset \text{spec}_{\text{ess}} T_\alpha = [-1, +\infty),$$

in particular,

$$\text{spec}_{\text{disc}} T_\alpha = \text{spec}_{\text{disc}} T_\alpha^N \cap (-\infty, -1), \quad (\text{IV.2.4})$$

and the eigenvalue multiplicities are preserved. It also follows that all eigenfunctions of T_α associated with the discrete eigenvalues are even with respects to x_2 .

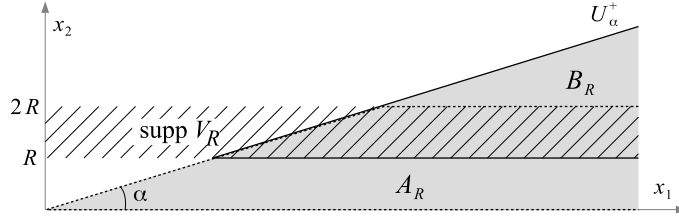


Figure IV.2 – Covering of U_α^+ by the domains A_R (surrounded by the dash line) and B_R (surrounded by the solid line). The support of V_R is hatched.

IV.2.2 Finiteness of the discrete spectrum

Theorem IV.2.1. *The discrete spectrum of T_α is finite for any $\alpha \in (0, \frac{\pi}{2})$.*

Proof. In view of (IV.2.4), it is sufficient to show that the operator T_α^N has only a finite number of eigenvalues in $(-\infty, -1)$. Recall that for A a self-adjoint operator associated with a semibounded from below sesquilinear form a and $\lambda \in \mathbb{R}$, we denote by $\mathcal{N}(A, \lambda)$ or $\mathcal{N}(a, \lambda)$ the number of the eigenvalues (counting the multiplicities) of A in $(-\infty, \lambda)$ for $\text{spec}_{\text{ess}} A \cap (-\infty, \lambda) = \emptyset$, and set $\mathcal{N}(A, \lambda) = +\infty$ otherwise. The proof scheme is inspired by [MT05, Theorem 2.1]. The idea is to perform a dimensional reduction in order to compare the operator with a one-dimensional one and to conclude using a Bargmann-type estimate.

We first introduce a decomposition of U_α^+ . Let χ_0 and χ_1 be smooth real-valued functions defined on \mathbb{R}_+ such that

$$\chi_0(t) = 1 \text{ for } 0 < t < 1, \quad \chi_0(t) = 0 \text{ for } t > 2, \quad \chi_0^2 + \chi_1^2 = 1.$$

For $R > 1$, to be determined later, consider the functions $\chi_{0,R}$ and $\chi_{1,R}$ defined on U_α^+ by

$$\chi_{j,R}(x_1, x_2) := \chi_j\left(\frac{x_2}{R}\right), \quad j = 0, 1,$$

and the following subdomains of U_α^+ ,

$$A_R = \{(x_1, x_2) \in U_\alpha^+ : 0 < x_2 < 2R\}, \quad B_R = \{(x_1, x_2) \in U_\alpha^+ : x_2 > R\},$$

see Fig IV.2. We get easily applying Lemma III.2.2 to the smooth partition $(\chi_{0,R}, \chi_{1,R})$,

$$t_\alpha^N(u, u) = t_\alpha^N(\chi_{0,R}u, \chi_{0,R}u) + t_\alpha^N(\chi_{1,R}u, \chi_{1,R}u) - \|u\nabla\chi_{0,R}\|_{L^2(U_\alpha)}^2 - \|u\nabla\chi_{1,R}\|_{L^2(U_\alpha)}^2,$$

for all $u \in D(t_\alpha^N)$, so that

$$\begin{aligned} t_\alpha^N(u, u) &\geq \int_{A_R} (|\nabla(u\chi_{0,R})|^2 - V_R|u\chi_{0,R}|^2) dx - \int_{\partial U_\alpha \cap \partial A_R} |u\chi_{0,R}|^2 ds \\ &\quad + \int_{B_R} (|\nabla(u\chi_{1,R})|^2 - V_R|u\chi_{1,R}|^2) dx - \int_{\partial U_\alpha \cap \partial B_R} |u\chi_{1,R}|^2 ds, \end{aligned} \quad (\text{IV.2.5})$$

where

$$V_R(x_1, x_2) = \sum_{j=0,1} |\nabla\chi_{j,R}(x_1, x_2)|^2 \equiv \frac{1}{R^2} \sum_{j=0,1} \left| \chi'_j\left(\frac{x_2}{R}\right) \right|^2, \quad \|V_R\|_\infty \leq \frac{C}{R^2}, \quad C := \|\chi'_0\|_\infty^2 + \|\chi'_1\|_\infty^2.$$

We define the following sesquilinear forms:

$$\begin{aligned} q_{A_R}(u, u) &= \int_{A_R} (|\nabla u|^2 - V_R |u|^2) dx - \int_{\partial U_\alpha \cap \partial A_R} |u|^2 ds, \\ D(q_{A_R}) &:= \{u \in H^1(A_R) : u(\cdot, 2R) = 0\}, \\ q_{B_R}(u, u) &= \int_{B_R} (|\nabla u|^2 - V_R |u|^2) dx - \int_{\partial U_\alpha \cap \partial B_R} |u|^2 ds, \\ D(q_{B_R}) &:= \{u \in H^1(B_R) : u(\cdot, R) = 0\}. \end{aligned}$$

Recall that the n -th Rayleigh quotient of a self-adjoint and semibounded from below operator A is denoted by $\Lambda_n(A)$ and is defined in Theorem III.1.3. Due to

$$u\chi_{0,R} \in D(q_{A_R}), \quad u\chi_{1,R} \in D(q_{B_R}), \quad \|u\chi_{0,R}\|_{L^2(A_R)}^2 + \|u\chi_{1,R}\|_{L^2(B_R)}^2 = \|u\|_{L^2(U_\alpha^+)}^2,$$

the min-max principle, see Theorem III.1.3, and (IV.2.5) give for any $n \in \mathbb{N}$:

$$\begin{aligned} \Lambda_n(T_\alpha^N) &\geq \min_{\substack{G \subset D(t_\alpha^N) \\ \dim(G)=n}} \max_{\substack{u \in G \\ u \neq 0}} \frac{q_{A_R}(u\chi_{0,R}, u\chi_{0,R}) + q_{B_R}(u\chi_{1,R}, u\chi_{1,R})}{\|u\chi_{0,R}\|_{L^2(A_R)}^2 + \|u\chi_{1,R}\|_{L^2(B_R)}^2} \\ &\geq \min_{\substack{G \subset D(q_{A_R} \oplus q_{B_R}) \\ \dim(G)=n}} \max_{\substack{(u_0, u_1) \in G \\ (u_0, u_1) \neq (0,0)}} \frac{q_{A_R}(u_0, u_0) + q_{B_R}(u_1, u_1)}{\|u_0\|_{L^2(A_R)}^2 + \|u_1\|_{L^2(B_R)}^2} \\ &= \Lambda_n(Q_{A_R} \oplus Q_{B_R}), \end{aligned}$$

where Q_{A_R} and Q_{B_R} are the self-adjoint operators acting respectively in $L^2(A_R)$ and $L^2(B_R)$ and produced by the forms q_{A_R} and q_{B_R} . Then,

$$\mathcal{N}(T_\alpha^N, -1) \leq \mathcal{N}(q_{A_R}, -1) + \mathcal{N}(q_{B_R}, -1). \quad (\text{IV.2.6})$$

Let us first estimate $\mathcal{N}(q_{A_R}, -1)$. We consider the following two domains:

$$A_R^0 = \left\{ (x_1, x_2) \in A_R : 0 < x_1 < \frac{2R}{\tan \alpha} \right\}, \quad A_R^1 = \left\{ (x_1, x_2) \in A_R : x_1 > \frac{2R}{\tan \alpha} \right\},$$

and the sesquilinear forms

$$\begin{aligned} q_{A_R^0}(u, u) &= \int_{A_R^0} (|\nabla u|^2 - |u|^2 V_R) dx - \int_{\partial A_R^0 \cap \partial U_\alpha} |u|^2 ds, \quad D(q_{A_R^0}) := H^1(A_R^0), \\ q_{A_R^1}(u) &= \int_{A_R^1} (|\nabla u|^2 - |u|^2 V_R) dx, \quad D(q_{A_R^1}) := \{u \in H^1(A_R^1) : u(\cdot, 2R) = 0\}. \end{aligned}$$

By the min-max principle we have

$$\mathcal{N}(q_{A_R}, -1) \leq \mathcal{N}(q_{A_R^0}, -1) + \mathcal{N}(q_{A_R^1}, -1). \quad (\text{IV.2.7})$$

On one hand, the operator $Q_{A_R^0}$ associated with $q_{A_R^0}$ has a compact resolvent, which implies

$$\mathcal{N}(Q_{A_R^0}, -1) < +\infty. \quad (\text{IV.2.8})$$

On the other hand, the operators $Q_{A_R^1}$ associated with $q_{A_R^1}$ can be represented as

$$Q_{A_R^1} = \tilde{T}^N \otimes 1 + 1 \otimes T^{V_R},$$

where \tilde{T}^N is the Neumann Laplacian in $L^2(2R/\tan\alpha, +\infty)$ and T^{V_R} is the self-adjoint operator in $L^2(0, 2R)$ associated with the sesquilinear form

$$t^{V_R}(u, u) = \int_0^{2R} (|u'(t)|^2 - V_R(t)|u(t)|^2) dt, \quad u \in D(t^{V_R}) := \left\{ u \in H^1(0, 2R), u(2R) = 0 \right\}.$$

One has $\inf \text{spec } \tilde{T}^N = 0$ and $\inf \text{spec } T^{V_R} \geq -\|V_R\|_\infty$. Recall that $\|V_R\|_\infty \leq \frac{C}{R^2}$. Setting $R_1 = \sqrt{C}$ and taking $R > R_1$ implies $\inf \text{spec } T^{V_R} \geq -1$ and then

$$\mathcal{N}(Q_{A_R^1}, -1) = 0 \text{ for } R > R_1, \quad (\text{IV.2.9})$$

and we conclude by (IV.2.7), (IV.2.8) and (IV.2.9) that $\mathcal{N}(q_{A_R}, -1) < +\infty$, for all $R > R_1$.

Now let us estimate $\mathcal{N}(q_{B_R}, -1)$ for $R > R_1$. Let us introduce the sesquilinear form

$$q_{P_\alpha}^R(u, u) = \int_{P_\alpha} (|\nabla u|^2 - V_R|u|^2) dx - \int_{\mathbb{R}} \left| u\left(\frac{x_2}{\tan\alpha}, x_2\right) \right|^2 \frac{dx_2}{\sin\alpha}, \quad D(q_{P_\alpha}^R) := H^1(P_\alpha),$$

where P_α is the half-plane given by (IV.2.3), then $\mathcal{N}(q_{B_R}, -1) \leq \mathcal{N}(q_{P_\alpha}^R, -1)$ by the min-max principle. If we make an anti-clockwise rotation of angle $\frac{\pi}{2} - \alpha$ of P_α , then we obtain the half-plane $\mathbb{R}_+ \times \mathbb{R}$, and the operator $Q_{P_\alpha}^R$ associated with $q_{P_\alpha}^R$ is then unitarily equivalent to the operator Q^R associated with the sesquilinear form

$$q^R(u, u) = \int_{\mathbb{R}_+ \times \mathbb{R}} (|\nabla u|^2 - \tilde{V}_R|u|^2) dx - \int_{\mathbb{R}} |u(0, x_2)|^2 dx_2, \quad u \in D(q^R) := H^1(\mathbb{R}_+ \times \mathbb{R}),$$

where

$$\begin{aligned} \tilde{V}_R(x_1, x_2) &= V_R(x_1 \sin\alpha + x_2 \cos\alpha, x_2 \sin\alpha - x_1 \cos\alpha) \equiv v_R(x_2 \sin\alpha - x_1 \cos\alpha), \\ v_R(t) &= \frac{1}{R^2} \sum_{j=1,2} \left| \chi_j' \left(\frac{t}{R} \right) \right|^2 \end{aligned}$$

One has, for $u \in D(q^R)$,

$$q^R(u, u) = q(u, u) - \int_{\mathbb{R}_+ \times \mathbb{R}} \tilde{V}_R|u|^2 dx,$$

where the operator associated with q is $Q = \mathcal{B}_1 \otimes 1 + 1 \otimes (-\Delta_{\mathbb{R}})$ with \mathcal{B}_1 defined in subsection III.5.1 and $-\Delta_{\mathbb{R}}$ is the free Laplacian in $L^2(\mathbb{R})$. Let us consider the orthogonal projections Π and P in $L^2(\mathbb{R}_+ \times \mathbb{R})$,

$$\Pi u(x_1, x_2) = \sqrt{2} e^{-x_1} \psi(x_2), \quad \psi(x_2) = \sqrt{2} \int_{\mathbb{R}_+} u(t, x_2) e^{-t} dt,$$

$$Pu = u - \Pi u.$$

Remark that $\Pi = \pi \otimes 1$, where π is the spectral projector of \mathcal{B}_1 on $\{-1\}$. For $u \in D(q^R)$ there holds $\Pi u, Pu \in D(q^R)$, and

$$q^R(u, u) = q^R(\Pi u, \Pi u) + q^R(Pu, Pu) - 2\Re \int_{\mathbb{R}_+ \times \mathbb{R}} \tilde{V}_R \overline{\Pi u} Pu dx,$$

as $q(\Pi u, Pu) = 0$ by the spectral theorem. Writing

$$W_R(x_2) = 2 \int_{\mathbb{R}_+} e^{-2x_1} \tilde{V}_R(x_1, x_2) dx_1$$

we have

$$q^R(\Pi u, \Pi u) = \int_{\mathbb{R}} \left(|\psi'(x_2)|^2 - W_R(x_2) |\psi(x_2)|^2 \right) dx_2 - \|\psi\|_{L^2(\mathbb{R})}^2.$$

By the spectral theorem applied to \mathcal{B}_1 , for a.e. $x_2 \in \mathbb{R}$ one has,

$$\int_{\mathbb{R}_+} \left| \frac{\partial P u}{\partial x_1}(x_1, x_2) \right|^2 dx_1 - |P u(0, x_2)|^2 \geq 0,$$

and, finally,

$$q^R(P u, P u) \geq \int_{\mathbb{R}} \int_{\mathbb{R}_+} \left(\left| \frac{\partial P u}{\partial x_2} \right|^2 - \tilde{V}_R |P u|^2 \right) dx.$$

For any $\epsilon \in \mathbb{R}_+$ one can estimate

$$2 \left| \int_{\mathbb{R}_+ \times \mathbb{R}} \overline{\Pi u} P u \tilde{V}_R dx \right| \leq \epsilon \|P u\|_{L^2(\mathbb{R}_+ \times \mathbb{R})}^2 + \frac{1}{\epsilon} \|\Pi u \tilde{V}_R\|_{L^2(\mathbb{R}_+ \times \mathbb{R})}^2.$$

Then, using the equality $\|\psi\|_{L^2(\mathbb{R})} = \|\Pi u\|_{L^2(\mathbb{R}_+ \times \mathbb{R})}$, we get

$$q^R(u, u) \geq \int_{\mathbb{R}} \left(|\psi'(x_2)|^2 - Z_R(x_2) |\psi(x_2)|^2 \right) dx_2 - \|\Pi u\|_{L^2(\mathbb{R}_+ \times \mathbb{R})}^2 - \left(\epsilon + \frac{C}{R^2} \right) \|P u\|_{L^2(\mathbb{R}_+ \times \mathbb{R})}^2,$$

where

$$Z_R(x_2) = W_R(x_2) + \frac{1}{\epsilon} \int_{\mathbb{R}_+} 2e^{-2x_1} \tilde{V}_R^2(x_1, x_2) dx_1.$$

We can choose $R_2 > R_1$ and $\epsilon > 0$ such that $\epsilon + C/R_2^2 \leq 1$, then for $R > R_2$ one arrives at

$$q^R(u, u) \geq \int_{\mathbb{R}} \left(|\psi'|^2 - Z_R |\psi|^2 \right) dx_2 - \|u\|_{L^2(\mathbb{R}_+ \times \mathbb{R})}^2. \quad (\text{IV.2.10})$$

We introduce the sesquilinear form

$$a^R(\psi, \psi) = \int_{\mathbb{R}} \left(|\psi'(x_2)|^2 - Z_R(x_2) |\psi(x_2)|^2 \right) dx_2, \quad u \in H^1(\mathbb{R}),$$

then, by (IV.2.10) and the min-max principle we have

$$\mathcal{N}(q^R, -1) \leq \mathcal{N}(a^R, 0), \quad R > R_2. \quad (\text{IV.2.11})$$

In order to show that the number of negative eigenvalues of a^R is finite, we want to use a Bargmann-type estimate, see e.g. [Sim76, Eq.(8)]:

$$\mathcal{N}(a^R, 0) \leq 2 + \int_{\mathbb{R}} |x_2| Z_R(x_2) dx_2.$$

We can write, using the fact that $\text{supp } \chi'_0 \cup \text{supp } \chi'_1 \subset [1, 2]$,

$$\int_{\mathbb{R}} |x_2| Z_R(x_2) dx_2 = \int_{\mathbb{R}_+} 2e^{-2x_1} \left(\int_{\frac{R+x_1 \cos \alpha}{\sin \alpha}}^{\frac{2R+x_1 \cos \alpha}{\sin \alpha}} |x_2| \left(\tilde{V}_R(x_1, x_2) + \frac{1}{\epsilon} \tilde{V}_R^2(x_1, x_2) \right) dx_2 \right) dx_1.$$

Using the boundedness of \tilde{V}_R we finally get the following upper bound:

$$\int_{\mathbb{R}} |x_2| Z_R(x_2) dx_2 \leq \frac{C}{R \sin^2 \alpha} \left(1 + \frac{C}{R^2 \epsilon} \right) \int_{\mathbb{R}_+} e^{-2x_1} (R + 2x_1 \cos \alpha) dx_1 < +\infty.$$

Hence, $\mathcal{N}(a^R, 0) < +\infty$ and (IV.2.11) implies that $\mathcal{N}(q^R, -1) < +\infty$ for $R > R_2$. By (IV.2.6) we conclude that $\mathcal{N}(T_\alpha^N, -1) < +\infty$. \square

IV.2.3 Continuity and monotonicity with respect to the angle

Let us discuss first the monotonicity of the Rayleigh quotients of T_α^N , defined in Theorem III.1.3 and denoted by $\Lambda_n(T_\alpha^N)$, with respect to α .

Proposition IV.2.2. *For any $n \in \mathbb{N}$ the function $(0, \frac{\pi}{2}) \ni \alpha \mapsto \Lambda_n(T_\alpha^N)$ is non-decreasing and continuous.*

Proof. We denote by \tilde{t}_α^N the sesquilinear form obtained after the anti-clockwise rotation of angle $\frac{\pi}{2} - \alpha$ of U_α^+ . Then \tilde{t}_α^N is unitarily equivalent to t_α^N and we have

$$\tilde{t}_\alpha^N(g, g) = \int_{\tilde{U}_\alpha^+} |\nabla g|^2 dx - \int_{\mathbb{R}_+} |g|^2(0, x_2) dx_2,$$

for all $g \in H^1(\tilde{U}_\alpha^+)$, where $\tilde{U}_\alpha^+ = \{(x_1, x_2) \in (\mathbb{R}_+)^2, x_1 \leq x_2 \tan \alpha\}$. After the scaling $t = x_2 \tan \alpha$ and writing $\tilde{g}(x_1, t) = g(x_1, \frac{t}{\tan \alpha})$ we have

$$\tilde{t}_\alpha^N(g, g) = \int_{\tilde{U}_{\frac{\pi}{4}}^+} (|\partial_{x_1} \tilde{g}(x_1, t)|^2 + \tan^2 \alpha |\partial_t \tilde{g}(x_1, t)|^2) \frac{dx_1 dt}{\tan \alpha} - \int_{\mathbb{R}_+} |\tilde{g}(0, t)|^2 \frac{dt}{\tan \alpha},$$

and $\|g\|_{L^2(\tilde{U}_\alpha^+)}^2 = \frac{1}{\tan \alpha} \|\tilde{g}\|_{L^2(\tilde{U}_{\frac{\pi}{4}}^+)}^2$. Then, we can define

$$q_\alpha(v, v) = \int_{\tilde{U}_{\frac{\pi}{4}}^+} (|\partial_{x_1} v(x_1, t)|^2 + \tan^2 \alpha |\partial_t v(x_1, t)|^2) dx_1 dt - \int_{\mathbb{R}_+} |v(0, t)|^2 dt,$$

for $v \in D(q_\alpha) = H^1(\tilde{U}_{\frac{\pi}{4}}^+)$ and Q_α the associated operator in $L^2(\tilde{U}_{\frac{\pi}{4}}^+)$. By construction, we have $\Lambda_n(T_\alpha) = \Lambda_n(Q_\alpha)$. The dependence of Q_α on α only appears through the coefficient $(\tan \alpha)^2$, which gives the result due to the min-max principle. \square

Let us now state the result for the eigenvalues of T_α .

Theorem IV.2.3. *Assume that for some $n \in \mathbb{N}$ and $\alpha_n \in (0, \frac{\pi}{2})$ the operator T_{α_n} has at least n discrete eigenvalues, then T_α has at least n discrete eigenvalues for all $\alpha < \alpha_n$, and the function $(0, \alpha_n) \ni \alpha \mapsto E_n(T_\alpha)$ is continuous and strictly increasing.*

Proof. Let $n \in \mathbb{N}$ be fixed and suppose that T_{α_n} has at least n discrete eigenvalues. Then, due to (IV.2.4) one has $E_n(T_{\alpha_n}) = E_n(T_{\alpha_n}^N)$. By Proposition IV.2.2, we have for any $\alpha \leq \alpha_n$: $\Lambda_n(T_\alpha^N) \leq E_n(T_{\alpha_n}^N) < -1$. Hence, $\Lambda_n(T_\alpha^N)$ is the n -th discrete eigenvalue of T_α^N by the min-max principle. This implies due to (IV.2.4) that T_α admits at least n discrete eigenvalues for any $\alpha \in (0, \alpha_n)$. The weak monotonicity and continuity of $\alpha \mapsto E_n(T_\alpha)$ also follow from Proposition IV.2.2.

Let us show the strict monotonicity of the eigenvalues. Let α_1, α_2 such that $\alpha_1 < \alpha_2$ and $E_n(T_{\alpha_2}) < -1$. We continue using the notation of the proof of Proposition IV.2.2, then $E_n(Q_{\alpha_2}) = E_n(T_{\alpha_2}) < -1$, and we need to show the strict inequality $E_n(Q_{\alpha_1}) < E_n(Q_{\alpha_2})$. For all $\varphi \in H^1(\tilde{U}_{\frac{\pi}{4}}^+)$ one has, with $\kappa = \tan^2 \alpha_2 - \tan^2 \alpha_1 > 0$,

$$\frac{q_{\alpha_1}(\varphi, \varphi)}{\|\varphi\|_{L^2(\tilde{U}_{\frac{\pi}{4}}^+)}^2} = \frac{q_{\alpha_2}(\varphi, \varphi)}{\|\varphi\|_{L^2(\tilde{U}_{\frac{\pi}{4}}^+)}^2} - \kappa \frac{\int_{\tilde{U}_{\frac{\pi}{4}}^+} |\partial_t \varphi|^2 dx_1 dt}{\|\varphi\|_{L^2(\tilde{U}_{\frac{\pi}{4}}^+)}^2}. \quad (\text{IV.2.12})$$

Let $\varphi_1, \dots, \varphi_n$ be an orthonormal basis in $\mathcal{K}_n := \sum_{k=1}^n \text{Ker}(Q_{\alpha_2} - E_k(Q_{\alpha_2}))$. On one hand, for all $\varphi \in \mathcal{K}_n \setminus \{0\}$ we have

$$\frac{q_{\alpha_2}(\varphi, \varphi)}{\|\varphi\|_{L^2(\tilde{U}_{\frac{\pi}{4}}^+)}^2} \leq E_n(Q_{\alpha_2}).$$

On the other hand, using the min-max principle and (IV.2.12) we have

$$E_n(Q_{\alpha_1}) \leq \sup_{\substack{\varphi \in \mathcal{K}_n \\ \varphi \neq 0}} \frac{q_{\alpha_1}(\varphi, \varphi)}{\|\varphi\|_{L^2(\tilde{U}_{\frac{\pi}{4}}^+)}^2} \leq E_n(Q_{\alpha_2}) - \kappa \inf_{\substack{\varphi \in \mathcal{K}_n \setminus \{0\} \\ \varphi \neq 0}} \frac{\int_{\tilde{U}_{\frac{\pi}{4}}^+} |\partial_t \varphi|^2 dx_1 dt}{\|\varphi\|_{L^2(\tilde{U}_{\frac{\pi}{4}}^+)}^2}.$$

Assume that $E_n(Q_{\alpha_1}) = E_n(Q_{\alpha_2})$, then the second term on the right-hand side is zero. As the unit ball of \mathcal{K}_n is compact, there must exist $\varphi \in \mathcal{K}_n$ with $\|\varphi\| = 1$ such that

$$\int_{\tilde{U}_{\frac{\pi}{4}}^+} |\partial_t \varphi|^2 dx_1 dt = 0,$$

i.e. $\partial_t \varphi(x_1, t) = 0$. Then φ depends on the x_1 variable only, but as $\varphi \in L^2(\tilde{U}_{\frac{\pi}{4}}^+)$ we necessarily have $\varphi = 0$, which contradicts the normalization $\|\varphi\| = 1$. \square

Another important corollary is as follows:

Corollary IV.2.4. *Assume that for some α_1 the operator T_{α_1} has a unique discrete eigenvalue, then T_α has a unique discrete eigenvalue for all $\alpha \in (\alpha_1, \frac{\pi}{2})$.*

A natural candidate for α_1 is $\frac{\pi}{4}$. In fact, the respective operator $T_{\frac{\pi}{4}}$ admits a separation of variables and is unitarily equivalent to $\mathcal{B}_1 \otimes 1 + 1 \otimes \mathcal{B}_1$ where \mathcal{B}_1 , given in subsection III.5.1, admits a unique discrete eigenvalue -1 and its essential spectrum is $[0, +\infty)$. Hence, $T_{\frac{\pi}{4}}$ has a unique discrete eigenvalue (-2) , which shows

Corollary IV.2.5. *For $\alpha \in [\frac{\pi}{4}, \frac{\pi}{2})$, the operator T_α admits a unique discrete eigenvalue.*

In fact, we can obtain a better estimate:

Theorem IV.2.6. *The operator T_α has a unique discrete eigenvalue for $\alpha \in [\frac{\pi}{6}, \frac{\pi}{2})$.*

Proof. In view of Corollary IV.2.4 it is sufficient to consider $\alpha = \frac{\pi}{6}$. We continue using the domain \tilde{U}_α^+ from the proof of Proposition IV.2.2. Let Q_α^L be the Laplacian in $L^2(\tilde{U}_{\frac{\pi}{6}}^+)$ with the Robin boundary condition $\partial u / \partial \nu = u$ at $x_1 = 0$, the Neumann boundary condition at the line $x_1 = x_2 \tan \alpha$ and with the Neumann boundary condition at the both sides of the lines $x_2 = L$ and $x_1 = L/\sqrt{3}$, then by the min-max principle for any $L > 0$ and $k \in \mathbb{N}$ one has the inequality

$$\Lambda_k(T_{\frac{\pi}{6}}^N) = \Lambda_k(Q_{\frac{\pi}{6}}) \geq \Lambda_k(Q_{\frac{\pi}{6}}^L). \quad (\text{IV.2.13})$$

Let us argue by contradiction. Assume that $\Lambda_2(T_{\frac{\pi}{6}}) < -1$, then it follows from (IV.2.13) that

$$\limsup_{L \rightarrow +\infty} \Lambda_2(Q_{\frac{\pi}{6}}^L) < -1. \quad (\text{IV.2.14})$$

Remark that $Q_{\frac{\pi}{6}}^L = A_{1,L} \oplus A_{2,L} \oplus A_{3,L}$, where $A_{1,L}$ is the Laplacian in the triangle

$$\Omega_L = \left\{ (x_1, x_2) : 0 < x_1 < \frac{x_2}{\sqrt{3}}, 0 < x_2 < L \right\}$$

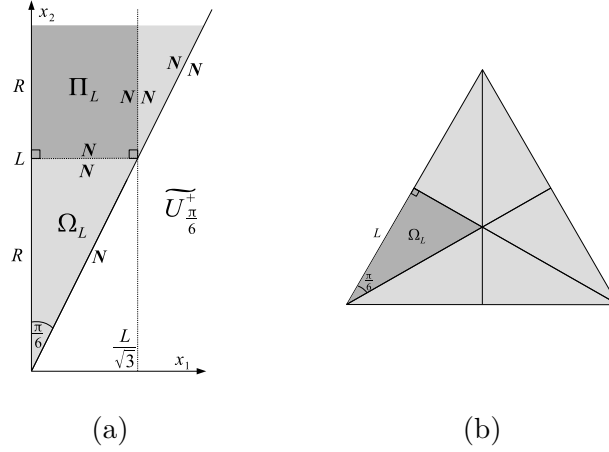


Figure IV.3 – (a) Decomposition of $\widetilde{U}_{\frac{\pi}{6}}^+$. The symbols R and N indicate respectively the Robin and Neumann boundary conditions. (b) The triangle $2L\Theta$.

with the Robin boundary condition at $x_1 = 0$ and with the Neumann boundary condition on the other two sides, the operator $A_{2,L}$ in the Laplacian in the half-strip $\Pi_L = (0, L/\sqrt{3}) \times (L, \infty)$ with the Robin boundary condition at $x_1 = 0$ and with the Neumann boundary condition at the remaining part of the boundary, and $A_{3,L}$ is the Neumann Laplacian in $\widetilde{U}_{\frac{\pi}{6}}^+ \setminus \overline{\Omega_L \cap \Pi_L}$, see Fig.IV.3(a).

The spectrum of $A_{2,L}$ can be easily estimated using the separation of variables, and then $A_{2,L} \geq -1 + o(1)$ for $L \rightarrow +\infty$. Furthermore, $A_{3,L} \geq 0$, and Eq. (IV.2.14) implies

$$\limsup_{L \rightarrow +\infty} \Lambda_2(A_{1,L}) < -1. \quad (\text{IV.2.15})$$

Remark that each eigenfunction of $A_{1,L}$ can be extended, using the reflections with respect to the Neumann sides, to an eigenfunction of the Laplacian K_{2L} with the Robin boundary condition $\partial u / \partial \nu = u$ on the equilateral triangle $2L\Theta$ composed from six copies on Ω_L , see Figure IV.3(b), where Θ is an equilateral triangle of side length 1.

Therefore, $\Lambda_2(A_{1,L}) = \Lambda_2(K_{2L,s})$, where $K_{2L,s}$ is the restriction of K_{2L} to the functions which are invariant under the reflections with respect to the medians and with respect to the rotations by $\frac{2\pi}{3}$. The eigenvalues of K_{2L} in the limit $L \rightarrow +\infty$ were analyzed in [McC11, Section 7]. In particular, the first eigenfunction of K_{2L} has the above-mentioned symmetries, hence, $\Lambda_1(K_{2L}) = \Lambda_1(K_{2L,s})$. On the other hand, the second eigenvalue of K_{2L} is double-degenerate, and no associated eigenfunction has the required symmetries: there is just one eigenfunction, noted $T_s^{0,1}$ in [McC11, Section 7], which is even with respect to one of the medians, but it does not possess the other symmetries. Therefore, $\Lambda_2(K_{2L,s}) \geq \Lambda_4(K_{2L})$ for large L . On the other hand, it is shown in [McC11, Subsection 7.4] that $\liminf_{L \rightarrow +\infty} \Lambda_j(K_{2L}) \geq -1$ for $j \geq 4$, which contradicts (IV.2.15). This contradiction shows that the inequality $\Lambda_2(T_{\frac{\pi}{6}}) \equiv \Lambda_2(Q_{\frac{\pi}{6}}) < -1$ is not possible, and $T_{\frac{\pi}{6}}$ has a unique discrete eigenvalue. \square

IV.3 Asymptotics of eigenvalues for small angle

The present section is devoted to the study of the discrete spectrum of T_α as α tends to 0.

IV.3.1 First order asymptotics

We are going to show the following result giving an estimate for the Rayleigh quotients of T_α :

Theorem IV.3.1. *There exists $\alpha_0 \in (0, \frac{\pi}{2})$ and $\mathcal{C} > 0$ such that for all $\alpha \in (0, \alpha_0)$ and for all $n \in \mathbb{N}$ there holds*

$$\left| \Lambda_n(T_\alpha) + \frac{1}{(2n-1)^2\alpha^2} \right| \leq \mathcal{C}.$$

Before passing to the proof let us discuss the most important consequences. Recall that $\mathcal{N}(T_\alpha, -1)$ is the number of discrete eigenvalues of T_α , and it is finite in virtue of Theorem IV.2.1.

Corollary IV.3.2. *There exists $\kappa > 0$ such that $\mathcal{N}(T_\alpha, -1) \geq \kappa/\alpha$ as α is small. In particular, $\mathcal{N}(T_\alpha, -1)$ tends to $+\infty$ as α tends to 0.*

Proof. For all $\alpha \in (0, \alpha_0)$ and any $n \in \mathbb{N}$ one has

$$\Lambda_n(T_\alpha) \leq \mathcal{C} - \frac{1}{(2n-1)^2\alpha^2}.$$

By the min-max principle, the number $\Lambda_n(T_\alpha)$ is the n th eigenvalue iff it is strictly less than (-1) . Notice that the right-hand side is smaller than (-1) for all $n < n_\alpha$,

$$n_\alpha := \frac{1}{2} \left(\frac{1}{\alpha\sqrt{1+\mathcal{C}}} + 1 \right),$$

and then $\mathcal{N}(T_\alpha, -1)$ is not smaller than the integer part of $n_\alpha - 1$. □

Another obvious corollary is

Corollary IV.3.3. *For any $n \in \mathbb{N}$ there holds*

$$E_n(T_\alpha) = -\frac{1}{(2n-1)^2\alpha^2} + \mathcal{O}(1) \text{ as } \alpha \text{ tends to 0.}$$

In Theorem IV.3.15 below we show that $E_n(T_\alpha)$ admits a full asymptotic expansion in powers of α^2 .

IV.3.1.1 Model one-dimensional operator

The main idea of the proof of Theorem IV.3.1 is to compare the operator T_α with some one-dimensional operator. Namely, for $a > 0$ consider the following operator acting in $L^2(\mathbb{R}_+)$:

$$(H_a v)(r) = \left(-\frac{d^2}{dr^2} - \frac{1}{4r^2} - \frac{1}{ar} \right) v(r), \quad v \in C_c^\infty(\mathbb{R}_+)$$

and h_a be the associated sesquilinear form,

$$h_a(v, v) = \langle v, H_a v \rangle_{L^2(\mathbb{R}_+)} = \int_0^\infty \left(|v'|^2 - \frac{|v|^2}{4r^2} - \frac{|v|^2}{ar} \right) dr, \quad v \in C_c^\infty(\mathbb{R}_+).$$

Denote by H_a^∞ the Friedrichs extension of H_a . It is known (see Appendix A.2) that its essential spectrum equals $[0, +\infty)$ and that the discrete spectrum consists of the simple negative eigenvalues

$$\mathcal{E}_n(a) := E_n(H_a^\infty) = -\frac{1}{(2n-1)^2 a^2}, \quad n \in \mathbb{N},$$

while the respective eigenfunctions ψ_n are

$$\psi_n(r) = \sqrt{r} e^{-\frac{r}{(2n-1)a}} L_{n-1}\left(\frac{2r}{(2n-1)a}\right)$$

with L_m being the Laguerre polynomials.

IV.3.1.2 Polar coordinates

Denote by $V_\alpha := (0, +\infty) \times (-\alpha, \alpha)$ and define a unitary operator $\mathcal{U}: L^2(U_\alpha, dx_1 dx_2) \rightarrow L^2(V_\alpha, dr d\theta)$,

$$\mathcal{U}u(r, \theta) = r^{\frac{1}{2}} u(r \cos \theta, r \sin \theta).$$

Recall that according to Lemma IV.1.2

$$\mathcal{F} := \left\{ u \in C^\infty(\overline{U_\alpha}) : \text{there exist } R_1, R_2 > 0 \text{ such that } u = 0 \text{ for } |x| < R_1 \text{ and for } |x| > R_2 \right\}$$

is dense in $H^1(U_\alpha)$. We have

$$\mathcal{U}(\mathcal{F}) = \mathcal{G} := \{v \in C^\infty(\overline{V_\alpha}) : \exists R_1, R_2 > 0 \text{ such that } u(r, \theta) = 0 \text{ for } r < R_1 \text{ and for } r > R_2\}.$$

We define the new sesquilinear form $q_\alpha(v, v) := t_\alpha(\mathcal{U}^*v, \mathcal{U}^*v)$,

$$q_\alpha(v, v) = \int_{V_\alpha} \left(|v_r|^2 - \frac{1}{4} \frac{|v|^2}{r^2} + \frac{|v_\theta|^2}{r^2} \right) dr d\theta - \int_{\mathbb{R}_+} \left(\frac{|v(r, \alpha)|^2}{r} + \frac{|v(r, -\alpha)|^2}{r} \right) dr, \quad v \in \mathcal{G},$$

defined initially on \mathcal{G} . As the set \mathcal{F} is dense in the form domain of T_α , the operator Q_α corresponding to the closure of q_α writes as $Q_\alpha = \mathcal{U}T_\alpha\mathcal{U}^*$, and $\Lambda_n(T_\alpha) = \Lambda_n(Q_\alpha)$ for all $n \in \mathbb{N}$.

Recall that the numbers $E_j(\ell, \gamma)$, defined in subsection III.5.2, are the eigenvalues of the one-dimensional Robin Laplacian $\mathcal{R}_{\ell, \gamma}$ in $L^2(-\alpha, \alpha)$. The next proposition is a direct application of the min-max principle.

Proposition IV.3.4. *For all $v \in \mathcal{G}$ we have*

$$q_\alpha(v, v) \geq \int_{V_\alpha} \left(|v_r|^2 - \frac{1}{4} \frac{|v|^2}{r^2} + \frac{E_1(1, r\alpha)}{(r\alpha)^2} |v|^2 \right) dr d\theta.$$

Proof. Let $v \in \mathcal{G}$, then

$$q_\alpha(v, v) = \int_{V_\alpha} |v_r|^2 - \frac{1}{4} \frac{|v|^2}{r^2} dr d\theta + \int_{\mathbb{R}_+} \frac{1}{r^2} \left(\int_{-\alpha}^{\alpha} |v_\theta|^2 d\theta - r|v(r, \alpha)|^2 - r|v(r, -\alpha)|^2 \right) dr.$$

We recognize in the bracket the sesquilinear form associated with the Robin Laplacian $\mathcal{R}_{\alpha, r}$, defined in subsection III.5.2, acting on $L^2(-\alpha, \alpha)$ with the Robin boundary condition $-v'(-\alpha) - rv(-\alpha) = v'(\alpha) - rv(\alpha) = 0$. As $v \in \mathcal{G}$, we have $\theta \mapsto v(\cdot, \theta) \in C^\infty(-\alpha, \alpha) \subset H^1(-\alpha, \alpha)$. Then we can apply the min-max principle to obtain

$$q_\alpha(v, v) \geq \int_{V_\alpha} \left(|v_r|^2 - \frac{1}{4} \frac{|v|^2}{r^2} \right) dr d\theta + \int_{\mathbb{R}_+} \left(\frac{E_1(\alpha, r)}{r^2} \int_{-\alpha}^{\alpha} |v|^2 d\theta \right) dr.$$

The conclusion is due to the equality (III.5.14). \square

IV.3.1.3 Upper bound of Theorem IV.3.1.

The operator $\mathcal{R}_{\alpha,r}$ defined in subsection III.5.2 will play a special role. Some of its spectral properties are gathered in Proposition III.5.4. Recall that $E_1(\alpha, r) < 0$, and the associated normalized eigenfunction is

$$\tilde{\Phi}_\alpha(r, \theta) = C_\alpha(r)\Phi_\alpha(r, \theta) = C_\alpha(r) \cosh\left(\frac{m(r\alpha)\theta}{\alpha}\right) \text{ for } \theta \in (-\alpha, \alpha),$$

where $m(r\alpha) := \sqrt{-E_1(1, r\alpha)}$. In this section, we will omit the lower index and denote

$$\Phi(r, \theta) := \Phi_\alpha(r, \theta), \quad C(r) := C_\alpha(r) \text{ and } \tilde{\Phi}(r, \theta) := C(r)\Phi(r, \theta).$$

Define two orthogonal projections Π and P in $L^2(V_\alpha)$ by

$$\begin{aligned} \Pi v(r, \theta) &:= f(r)\tilde{\Phi}(r, \theta), \quad f(r) := \int_{-\alpha}^{\alpha} v(r, \theta)\tilde{\Phi}(r, \theta)d\theta, \\ P v(r, \theta) &:= v(r, \theta) - \Pi v(r, \theta). \end{aligned} \tag{IV.3.1}$$

During the proof, the functions v and f will always be related by (IV.3.1).

Proposition IV.3.5. *For all $v \in \mathcal{G}$ we have $\Pi v \in \mathcal{G}$, and*

$$q_\alpha(\Pi v, \Pi v) = \int_{\mathbb{R}_+} |f'(r)|^2 + \left(K_\alpha(r) - \frac{1}{4r^2} + \frac{E_1(1, \alpha r)}{(\alpha r)^2} \right) |f(r)|^2 dr,$$

where $K_\alpha(r) := \int_{-\alpha}^{\alpha} |\partial_r \tilde{\Phi}(r, \theta)|^2 d\theta$.

Proof. Let $v \in \mathcal{G}$, then $f \in C_c^\infty(\mathbb{R}_+)$. As $\tilde{\Phi}$ is smooth, one has $\Pi v = f\tilde{\Phi} \in \mathcal{G}$. Moreover,

$$\begin{aligned} \partial_r \Pi v(r, \theta) &= f'(r)\tilde{\Phi}(r, \theta) + f(r)\partial_r \tilde{\Phi}(r, \theta), \\ \partial_\theta \Pi v(r, \theta) &= f(r)\partial_\theta \tilde{\Phi}(r, \theta). \end{aligned} \tag{IV.3.2}$$

The evaluation of q_α on Πv gives

$$\begin{aligned} q_\alpha(\Pi v, \Pi v) &= \int_{V_\alpha} \left(|\partial_r \Pi v(r, \theta)|^2 + \frac{|\partial_\theta \Pi v(r, \theta)|^2}{r^2} - \frac{|\Pi v(r, \theta)|^2}{4r^2} \right) d\theta dr \\ &\quad - \int_{\mathbb{R}_+} \left(\frac{|\Pi v(r, \alpha)|^2}{r} + \frac{|\Pi v(r, -\alpha)|^2}{r} \right) dr. \end{aligned}$$

By (IV.3.2),

$$\int_{V_\alpha} |\partial_r \Pi v(r, \theta)|^2 d\theta dr = \int_{V_\alpha} \left(|f'(r)\tilde{\Phi}(r, \theta)|^2 + |f(r)\partial_r \tilde{\Phi}(r, \theta)|^2 + 2f'(r)f(r)\tilde{\Phi}(r, \theta)\partial_r \tilde{\Phi}(r, \theta) \right) d\theta dr.$$

For all $r \in \mathbb{R}_+$, $\int_{-\alpha}^{\alpha} |\tilde{\Phi}(r, \theta)|^2 d\theta = 1$, then

$$\partial_r \left(\int_{-\alpha}^{\alpha} |\tilde{\Phi}(r, \theta)|^2 d\theta \right) = 2 \int_{-\alpha}^{\alpha} \tilde{\Phi}(r, \theta)\partial_r \tilde{\Phi}(r, \theta) d\theta = 0.$$

Applying Fubini's theorem we obtain

$$\begin{aligned} q_\alpha(\Pi v, \Pi v) &= \int_{\mathbb{R}_+} \left(|f'(r)|^2 + K_\alpha(r)|f(r)|^2 - \frac{|f(r)|^2}{4r^2} \right) dr \\ &\quad + \int_{\mathbb{R}_+} \frac{|f(r)|^2}{r^2} \left\{ \int_{-\alpha}^{\alpha} |\partial_\theta \tilde{\Phi}(r, \theta)|^2 d\theta - r \left(|\tilde{\Phi}(r, \alpha)|^2 + |\tilde{\Phi}(r, -\alpha)|^2 \right) \right\} dr, \end{aligned}$$

and the expression in the curly brackets equals $E_1(\alpha, r)$ due to the choice of $\tilde{\Phi}$. Then,

$$q_\alpha(\Pi v, \Pi v) = \int_{\mathbb{R}_+} \left(|f'(r)|^2 + \left(K_\alpha(r) - \frac{1}{4r^2} + \frac{E_1(\alpha, r)}{r^2} \right) |f(r)|^2 \right) dr.$$

To finish the proof we use (III.5.14). \square

In order to obtain an upper bound for the form q_α we need to study the quantity K_α .

Lemma IV.3.6. *There exists $A > 0$ such that for all $r > 0$ and $\alpha \in (0, \frac{\pi}{2})$ we have $K_\alpha(r) \leq A\alpha^2$.*

Proof. Recall that $C(r)$ is defined by the equality

$$\|\Phi(r, \cdot)\|_{L^2(-\alpha, \alpha)}^2 = \frac{1}{(C(r))^2}, \text{ and } m(r\alpha) := \sqrt{-E_1(1, r\alpha)},$$

see Section III.5.2. We first notice that K_α is continuous on \mathbb{R}_+ . In addition,

$$\begin{aligned} K_\alpha(r) &= \int_{-\alpha}^{\alpha} |C'(r)\Phi(r, \theta)|^2 d\theta + \int_{-\alpha}^{\alpha} |C(r)\partial_r \Phi(r, \theta)|^2 d\theta + 2 \int_{-\alpha}^{\alpha} C'(r)C(r)\Phi(r, \theta)\partial_r \Phi(r, \theta) d\theta \\ &\leq 2 \int_{-\alpha}^{\alpha} |C'(r)\Phi(r, \theta)|^2 d\theta + 2 \int_{-\alpha}^{\alpha} |C(r)\partial_r \Phi(r, \theta)|^2 d\theta \\ &:= 2T_1 + 2T_2. \end{aligned}$$

Then,

$$T_1 = \left| \frac{C'(r)}{C(r)} \right|^2 = \alpha^2 |m'(r\alpha)|^2 \left(\frac{\sinh(2m(r\alpha))}{2m(r\alpha)} + 1 \right)^{-2} \left(\frac{\cosh(2m(r\alpha))}{2m(r\alpha)} - \frac{\sinh(2m(r\alpha))}{(2m(r\alpha))^2} \right).$$

We also give an upper bound for the second term T_2 :

$$T_2 \leq |C(r)|^2 |m'(r\alpha)|^2 \alpha^2 \int_{-\alpha}^{\alpha} \sinh^2 \left(m(r\alpha) \frac{\theta}{\alpha} \right) d\theta = |C(r)|^2 |m'(r\alpha)|^2 \alpha^3 \left(\frac{\sinh(2m(r\alpha))}{2m(r\alpha)} - 1 \right).$$

Finally, we arrive at $K_\alpha(r) \leq 2\alpha^2 F(r\alpha)$ with

$$\begin{aligned} F(r\alpha) &:= |m'(r\alpha)|^2 \left(\frac{\sinh(2m(r\alpha))}{2m(r\alpha)} + 1 \right)^{-2} \left(\frac{\cosh(2m(r\alpha))}{2m(r\alpha)} - \frac{\sinh(2m(r\alpha))}{(2m(r\alpha))^2} \right)^2 \\ &\quad + |m'(r\alpha)|^2 \left(\frac{\sinh(2m(r\alpha))}{2m(r\alpha)} + 1 \right)^{-1} \left(\frac{\sinh(2m(r\alpha))}{2m(r\alpha)} - 1 \right). \end{aligned}$$

In order to conclude it is sufficient to show that F is bounded on \mathbb{R}_+ . As the function F is continuous, we only have to prove that it admits finite limits in 0 and $+\infty$. Let $x = r\alpha$. The function $x \mapsto m(x)$ is $C^1(\mathbb{R}_+)$ with $m(0) = 0$, $m(x) \sim x$ for $x \rightarrow +\infty$, and

$$m'(x) = -\frac{\partial_x E_1(1, x)}{2\sqrt{-E_1(1, x)}}.$$

Furthermore, we have by (III.5.22)

$$\partial_x E_1(1, x) = -2 \frac{\cosh^2 m(x)}{\frac{\sinh(2m(x))}{2m(x)} + 1}.$$

Then we can write $F(x) = G(x) + H(x)$, where

$$G(x) = \frac{\cosh^4(m(x))}{m^2(x)} \left(\frac{\sinh(2m(x))}{2m(x)} + 1 \right)^{-4} \left(\frac{\cosh(2m(x))}{2m(x)} - \frac{\sinh(2m(x))}{(2m(x))^2} \right)^2,$$

$$H(x) = \frac{\cosh^4(m(x))}{m^2(x)} \frac{\frac{\sinh(2m(x))}{2m(x)} - 1}{\left(\frac{\sinh(2m(x))}{2m(x)} + 1 \right)^3}.$$

After a direction computation using the behavior of m as $x \rightarrow 0$ and $x \rightarrow +\infty$ we get

$$\begin{aligned} G(x) &\rightarrow \frac{4}{9} \text{ as } x \rightarrow 0, & H(x) &\rightarrow \frac{1}{12} \text{ as } x \rightarrow 0, \\ G(x) &\rightarrow 4 \text{ as } x \rightarrow +\infty, & H(x) &\rightarrow \frac{1}{4} \text{ as } x \rightarrow +\infty. \end{aligned}$$

Then F admits finite limits too, which concludes the proof. \square

Corollary IV.3.7. *For all $v \in \mathcal{G}$ and for all $\alpha \in (0, \frac{\pi}{2})$,*

$$q_\alpha(\Pi v, \Pi v) \leq \int_{\mathbb{R}_+} \left(|f'(r)|^2 + \left(-\frac{1}{4r^2} + \frac{E_1(1, r\alpha)}{(r\alpha)^2} \right) |f(r)|^2 \right) dr + \alpha^2 A \|\Pi v\|_{L^2(V_\alpha)}^2.$$

The next proposition gives an upper bound for the Rayleigh quotients of T_α in terms of $\mathcal{E}_n(\alpha)$, the discrete eigenvalues of H_α^∞ defined in Section IV.3.1.1.

Proposition IV.3.8. *There exist $M > 0$ and $\alpha_0 > 0$ such that $\Lambda_n(T_\alpha) \leq \mathcal{E}_n(\alpha) + M$ for all $n \in \mathbb{N}$ and all $\alpha \in (0, \alpha_0)$.*

Proof. By Proposition III.5.6, there exists $\phi \in C^\infty(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ such that $E_1(1, r\alpha) = -r\alpha + (r\alpha)^2\phi(r\alpha)$ for all $r, \alpha \in \mathbb{R}_+$. By Corollary IV.3.7, for all $v \in \mathcal{G}$ there holds

$$q_\alpha(\Pi v, \Pi v) \leq \int_{\mathbb{R}_+} \left(|f'(r)|^2 - \frac{1}{4r^2} |f(r)|^2 - \frac{1}{\alpha r} |f(r)|^2 \right) dr + \|\phi\|_\infty \|f\|_{L^2(\mathbb{R}_+)}^2 + \alpha^2 A \|\Pi v\|_{L^2(V_\alpha)}^2.$$

Let $M = \|\phi\|_\infty + \alpha^2 A$, then,

$$q_\alpha(\Pi v, \Pi v) \leq \int_{\mathbb{R}_+} \left(|f'(r)|^2 - \frac{1}{4r^2} |f(r)|^2 - \frac{1}{\alpha r} |f(r)|^2 \right) dr + M \|\Pi v\|_{L^2(V_\alpha)}^2. \quad (\text{IV.3.3})$$

Let $n \in \mathbb{N}$ and $G \subset C_c^\infty(\mathbb{R}_+)$ such that $\dim G = n$. Denote $\tilde{G} = \{v = g\tilde{\Phi} : g \in G\}$, then $\tilde{G} \subset \mathcal{G}$ and $\dim \tilde{G} = n$. By (IV.3.3) we obtain

$$\sup_{\substack{v \in \tilde{G} \\ v \neq 0}} \frac{q_\alpha(v, v)}{\|v\|_{L^2(V_\alpha)}^2} \leq \sup_{\substack{g \in G \\ g \neq 0}} \frac{h_\alpha(g, g)}{\|g\|_{L^2(\mathbb{R}_+)}^2} + M,$$

and by the min-max principle,

$$\Lambda_n(T_\alpha) \leq \inf_{\substack{G \subset C_c^\infty(\mathbb{R}_+) \\ \dim G = n}} \sup_{\substack{v \in \tilde{G} \\ v \neq 0}} \frac{q_\alpha(v, v)}{\|v\|_{L^2(V_\alpha)}^2}.$$

Recall that $C_c^\infty(\mathbb{R}_+)$ is dense in the form domain of H_α^∞ as H_α^∞ is the Friedrichs extension of H_α (see subsection IV.3.1.1), hence,

$$\inf_{\substack{G \subset C_c^\infty(\mathbb{R}_+) \\ \dim G = n}} \sup_{\substack{g \in G \\ g \neq 0}} \frac{h_\alpha(g, g)}{\|g\|_{L^2(\mathbb{R}_+)}^2} = \mathcal{E}_n(\alpha),$$

which concludes the proof. \square

IV.3.1.4 Lower bound of Theorem IV.3.1.

Here we still use the orthogonal projections Π and P defined in (IV.3.1). We recall that for $v \in \mathcal{G}$ we have $\Pi v = f\tilde{\Phi} \in \mathcal{G}$, $Pv = v - \Pi v \in \mathcal{G}$, and $\|\Pi v\|_{L^2(V_\alpha)} = \|f\|_{L^2(\mathbb{R}_+)}$.

Proposition IV.3.9. *For all $v \in \mathcal{G}$ and for all $\alpha \in (0, \frac{\pi}{2})$,*

$$\begin{aligned} q_\alpha(v, v) &\geq \int_{\mathbb{R}_+} \left(\left(1 - \alpha^2 \frac{r}{r+1}\right) |f'(r)|^2 + \left(\frac{E_1(1, \alpha r)}{(\alpha r)^2} - \frac{1}{4r^2}\right) |f(r)|^2 \right) dr \\ &\quad + \int_{V_\alpha} \left((1 - \alpha) |\partial_r Pv(r, \theta)|^2 + \left(\frac{E_2(1, \alpha r)}{(\alpha r)^2} - \frac{1}{4r^2} - A \frac{r+1}{r}\right) |Pv(r, \theta)|^2 \right) d\theta dr \\ &\quad - \alpha A \|\Pi v\|_{L^2(V_\alpha)}^2, \end{aligned}$$

with the constant A from Lemma IV.3.6.

Proof. Let $v \in \mathcal{G}$, then

$$q_\alpha(v, v) = q_\alpha(\Pi v, \Pi v) + q_\alpha(Pv, Pv) + 2\Re q_\alpha(\Pi v, Pv). \quad (\text{IV.3.4})$$

The first term is known thanks to proposition IV.3.5. Then we have

$$\begin{aligned} q_\alpha(Pv, Pv) &= \int_{V_\alpha} |\partial_r Pv(r, \theta)|^2 - \frac{|Pv(r, \theta)|^2}{4r^2} dr d\theta \\ &\quad + \int_{\mathbb{R}_+} \frac{1}{r^2} \left(\int_{-\alpha}^{\alpha} |\partial_\theta Pv(r, \theta)|^2 d\theta - r |Pv(r, \alpha)|^2 - r |Pv(r, -\alpha)|^2 \right) dr. \end{aligned}$$

Applying the spectral theorem to the operator $\mathcal{R}_{\alpha, r}$ we obtain

$$\int_{-\alpha}^{\alpha} |\partial_\theta Pv(r, \theta)|^2 d\theta - r |Pv(r, \alpha)|^2 - r |Pv(r, -\alpha)|^2 \geq E_2(\alpha, r) \|Pv\|_{L^2(-\alpha, \alpha)}^2, \quad r > 0.$$

Using the equality $E_2(\alpha, r) = \alpha^{-2} E_2(1, r\alpha)$ we finally get

$$q_\alpha(Pv, Pv) \geq \int_{V_\alpha} \left(|\partial_r Pv(r, \theta)|^2 + \left(\frac{E_2(1, r\alpha)}{(r\alpha)^2} - \frac{1}{4r^2}\right) |Pv(r, \theta)|^2 d\theta \right) dr. \quad (\text{IV.3.5})$$

To estimate the last term in (IV.3.4) we write

$$\begin{aligned} q_\alpha(\Pi v, Pv) &= \int_{V_\alpha} \left(\overline{(\partial_r \Pi v)} (\partial_r Pv) - \frac{\overline{\Pi v} Pv}{4r^2} \right) d\theta dr \\ &\quad + \int_{\mathbb{R}_+} \left(\frac{1}{r^2} \int_{-\alpha}^{\alpha} \overline{(\partial_\theta \Pi v)} (\partial_\theta Pv) d\theta - r (\overline{\Pi v}(r, \alpha) Pv(r, \alpha) + \overline{\Pi v}(r, -\alpha) Pv(r, -\alpha)) \right) dr. \end{aligned} \quad (\text{IV.3.6})$$

The functions Πv and Pv are orthogonal with respect to the scalar product of $L^2(-\alpha, \alpha)$,

$$\int_{-\alpha}^{\alpha} \overline{\Pi v} Pv d\theta = 0,$$

and with respect to the form associated with $\mathcal{R}_{\alpha, r}$, i.e.

$$\int_{-\alpha}^{\alpha} \overline{(\partial_\theta \Pi v)} (\partial_\theta Pv) d\theta - r (\overline{\Pi v}(r, \alpha) Pv(r, \alpha) + \overline{\Pi v}(r, -\alpha) Pv(r, -\alpha)) = 0.$$

Then (IV.3.6) becomes

$$\begin{aligned} q_\alpha(\Pi v, Pv) &= \int_{V_\alpha} \overline{(\partial_r \Pi v)} (\partial_r Pv) d\theta dr \\ &= \int_{V_\alpha} \left(\overline{f'(r)} \tilde{\Phi}(r, \theta) + \overline{f(r)} \partial_r \tilde{\Phi}(r, \theta) \right) \partial_r Pv(r, \theta) d\theta dr. \end{aligned} \quad (\text{IV.3.7})$$

In addition, $\int_{-\alpha}^{\alpha} Pv(r, \theta) \tilde{\Phi}(r, \theta) d\theta = 0$, and the derivative in r gives

$$\int_{-\alpha}^{\alpha} (\partial_r Pv(r, \theta)) \tilde{\Phi}(r, \theta) d\theta = - \int_{-\alpha}^{\alpha} Pv(r, \theta) \partial_r \tilde{\Phi}(r, \theta) d\theta.$$

The substitution into (IV.3.7) gives:

$$q_\alpha(\Pi v, Pv) = \int_{V_\alpha} \overline{f(r)} \left(\partial_r \tilde{\Phi}(r, \theta) \right) (\partial_r Pv(r, \theta)) dr d\theta - \int_{V_\alpha} \overline{f'(r)} \left(\partial_r \tilde{\Phi}(r, \theta) \right) Pv(r, \theta) dr d\theta. \quad (\text{IV.3.8})$$

To get a lower bound, we notice that

$$\left| \int_{V_\alpha} \overline{f(r)} \partial_r \tilde{\Phi}(r, \theta) \partial_r Pv(r, \theta) dr d\theta \right| \leq \frac{1}{2\alpha} \int_{V_\alpha} |f(r)|^2 |\partial_r \tilde{\Phi}(r, \theta)|^2 dr d\theta + \frac{\alpha}{2} \int_{V_\alpha} |\partial_r Pv(r, \theta)|^2 dr d\theta,$$

and using lemma IV.3.6 we obtain

$$\left| \int_{V_\alpha} \overline{f(r)} \partial_r \tilde{\Phi}(r, \theta) \partial_r Pv(r, \theta) dr d\theta \right| \leq \frac{\alpha A}{2} \int_{\mathbb{R}_+} |f(r)|^2 dr + \frac{\alpha}{2} \int_{V_\alpha} |\partial_r Pv(r, \theta)|^2 dr d\theta. \quad (\text{IV.3.9})$$

Furthermore for any $\epsilon(r) > 0$ we have

$$\begin{aligned} \left| \int_{V_\alpha} \overline{f'(r)} Pv(r, \theta) \partial_r \tilde{\Phi}(r, \theta) dr d\theta \right| \\ \leq \frac{1}{2} \int_{V_\alpha} \frac{1}{\epsilon(r)} |Pv(r, \theta)|^2 dr d\theta + \frac{1}{2} \int_{V_\alpha} \epsilon(r) |f'(r)|^2 |\partial_r \tilde{\Phi}(r, \theta)|^2 dr d\theta, \end{aligned}$$

and using again Lemma IV.3.6 and $\epsilon(r) := \frac{r}{A(r+1)}$ we get

$$\begin{aligned} \left| \int_{V_\alpha} \overline{f'(r)} Pv(r, \theta) \partial_r \tilde{\Phi}(r, \theta) dr d\theta \right| \\ \leq \frac{1}{2} \int_{V_\alpha} \frac{A(r+1)}{r} |Pv(r, \theta)|^2 dr d\theta + \frac{\alpha^2}{2} \int_{\mathbb{R}_+} \frac{r}{r+1} |f'(r)|^2 dr. \end{aligned} \quad (\text{IV.3.10})$$

The substitution of (IV.3.9) and (IV.3.10) into (IV.3.8) gives us the lower bound for the cross-term. Combining it with (IV.3.5) finally gives us the following lower bound for q_α :

$$\begin{aligned} q_\alpha(v, v) &\geq \int_{\mathbb{R}_+} \left((1 - \alpha^2 \frac{r}{r+1}) |f'(r)|^2 + \left(K_\alpha(r) + \frac{E_1(1, \alpha r)}{(\alpha r)^2} - \frac{1}{4r^2} \right) |f(r)|^2 \right) dr \\ &\quad + \int_{V_\alpha} \left((1 - \alpha) |\partial_r Pv(r, \theta)|^2 + \left(\frac{E_2(1, \alpha r)}{(\alpha r)^2} - \frac{1}{4r^2} - A \frac{r+1}{r} \right) |Pv(r, \theta)|^2 \right) d\theta dr \\ &\quad - \alpha A \|f\|_{L^2(0, +\infty)}^2. \end{aligned}$$

We notice that K_α is positive, which concludes the proof. \square

The next proposition gives us a lower bound of q_α in terms of the one-dimensional operator H_a^∞ defined in subsection IV.3.1.1.

Proposition IV.3.10. *There exists $K \in \mathbb{R}_+$ such that for all $\alpha \in (0, 1)$ and for all $v \in \mathcal{G}$,*

$$q_\alpha(v, v) \geq (1 - \alpha^2) \int_{\mathbb{R}_+} \left[|f'(r)|^2 - \left(\frac{1}{4r^2} + \frac{1}{r\alpha(1 - \alpha^2)} \right) |f(r)|^2 \right] dr - K \|v\|_{L^2(V_\alpha)}^2.$$

In order to prove Proposition IV.3.10, we will need some preliminary constructions. Let us introduce a new sesquilinear form,

$$a_1(u, u) = \int_{V_\alpha} (1 - \alpha) \left(|\partial_r u(r, \theta)|^2 + W(r, \alpha) |u(r, \theta)|^2 \right) dr d\theta, \quad u \in \mathcal{G},$$

where

$$W(r, \alpha) = \frac{E_2(1, r\alpha)}{(r\alpha)^2} - \frac{1}{4r^2} - A \frac{r+1}{r}.$$

Lemma IV.3.11. *There exists $C_1 \in \mathbb{R}_+$ such that, for all $\alpha \in (0, 1)$ and for all $u \in D(a_1)$,*

$$a_1(u, u) \geq -C_1 \|u\|_{L^2(V_\alpha)}^2.$$

Proof. It is sufficient to check that, for all $\alpha \in (0, 1)$, the potential $r \mapsto W(r, \alpha)$ is uniformly semibounded from below on \mathbb{R}_+ . Notice that

$$\inf_{r \in \mathbb{R}_+} W(r, \alpha) = \inf_{r \in \mathbb{R}_+} W\left(\frac{r}{\alpha}, \alpha\right) = \inf_{x \in \mathbb{R}_+} \frac{E_2(1, x)}{x^2} - \frac{\alpha^2}{4x^2} - A \frac{x+\alpha}{x} \geq \inf_{x \in \mathbb{R}_+} h(x) - A, \quad (\text{IV.3.11})$$

with

$$h(x) := \frac{E_2(1, x)}{x^2} - \frac{1}{4x^2} - \frac{A}{x}.$$

Clearly, h is continuous on \mathbb{R}_+ . In addition, by (III.5.18) we have $h(x) \rightarrow -1$ as $x \rightarrow +\infty$. Furthermore, $E_2(1, 0) = \frac{\pi^2}{4}$ by (III.5.16). Hence, $h(x) \sim \frac{\pi^2-1}{4x^2} \rightarrow +\infty$ as $x \rightarrow 0$, and we can conclude that h admits a finite lower bound on \mathbb{R}_+ . Then, by (IV.3.11), there exists $C_1 \in \mathbb{R}_+$ such that $\inf_{r \in \mathbb{R}_+} W(r, \alpha) \geq -C_1$ for all $\alpha \in (0, 1)$. \square

Introduce another sesquilinear form,

$$a_2(u, u) = \int_{\mathbb{R}_+} \left[\left(1 - \alpha^2 \frac{r}{r+1} \right) |u'(r)|^2 + \left(\frac{E_1(1, r\alpha)}{(r\alpha)^2} - \frac{1}{4r^2} \right) |u(r)|^2 \right] dr, \quad u \in C_c^\infty(\mathbb{R}_+).$$

Lemma IV.3.12. *There exists $C_2 \in \mathbb{R}_+$ such that for all $\alpha \in (0, 1)$ and for all $u \in D(a_2)$,*

$$a_2(u, u) \geq (1 - \alpha^2) \int_{\mathbb{R}_+} \left[|u'(r)|^2 + \left(-\frac{1}{4r^2} - \frac{1}{r\alpha(1 - \alpha^2)} \right) |u(r)|^2 \right] dr - C_2 \|u\|_{L^2(\mathbb{R}_+)}^2.$$

Proof. Using the function ϕ from Proposition III.5.6 we can write, for all $u \in D(a_2)$,

$$a_2(u, u) \geq \int_{\mathbb{R}_+} \left[\left(1 - \alpha^2 \frac{r}{r+1} \right) |u'(r)|^2 + \left(-\frac{1}{4r^2} - \frac{1}{r\alpha} \right) |u(r)|^2 \right] dr - \|\phi\|_\infty \|u\|_{L^2(\mathbb{R}_+)}^2.$$

We want to study the right hand side of this inequality. To do that we separate the integral in two parts: a first integral on $(0, 1)$ and a second one on $(1, +\infty)$.

On one hand, we notice that

$$\int_0^1 \frac{r}{r+1} |u'(r)|^2 dr \leq \int_0^1 |u'(r)|^2 r dr.$$

And, by an improved Hardy type inequality from [BM97, Lemma A.1], as $u \in C_c^\infty(\mathbb{R}_+)$,

$$\int_0^1 \left(|u'(r)|^2 - \frac{|u(r)|^2}{4r^2} \right) dr \geq \int_0^1 \left(|u'(r)|^2 + \frac{|u(r)|^2}{4r^2} \right) r dr. \quad (\text{IV.3.12})$$

Then we have

$$\int_0^1 |u'(r)|^2 r dr \leq \int_0^1 \left[|u'(r)|^2 - \left(\frac{1}{4r^2} + \frac{1}{4r} \right) |u(r)|^2 \right] dr,$$

and we can write

$$\begin{aligned} & \int_0^1 \left[\left(1 - \alpha^2 \frac{r}{r+1} \right) |u'(r)|^2 + \left(-\frac{1}{4r^2} - \frac{1}{r\alpha} \right) |u(r)|^2 \right] dr \\ & \geq (1 - \alpha^2) \int_0^1 |u'(r)|^2 dr + \alpha^2 \int_0^1 \left(\frac{1}{4r^2} + \frac{1}{4r} \right) |u(r)|^2 dr + \int_0^1 \left(-\frac{1}{4r^2} - \frac{1}{r\alpha} \right) |u(r)|^2 dr. \end{aligned}$$

Finally,

$$\begin{aligned} & \int_0^1 \left[\left(1 - \alpha^2 \frac{r}{r+1} \right) |u'(r)|^2 + \left(-\frac{1}{4r^2} - \frac{1}{r\alpha} \right) |u(r)|^2 \right] dr \\ & \geq (1 - \alpha^2) \int_0^1 \left[|u'(r)|^2 - \left(\frac{1}{4r^2} + \frac{1}{\alpha(1 - \alpha^2)r} \right) |u(r)|^2 \right] dr. \quad (\text{IV.3.13}) \end{aligned}$$

On the other side, the integrals on $(1, +\infty)$ can be estimated by

$$\begin{aligned} & \int_1^{+\infty} \left[\left(1 - \alpha^2 \frac{r}{r+1} \right) |u'(r)|^2 - \left(\frac{1}{4r^2} + \frac{1}{r\alpha} \right) |u(r)|^2 \right] dr \\ & \geq (1 - \alpha^2) \int_1^{+\infty} \left[|u'(r)|^2 - \left(\frac{1}{4r^2(1 - \alpha^2)} + \frac{1}{r\alpha(1 - \alpha^2)} \right) |u(r)|^2 \right] dr. \end{aligned}$$

Using

$$\frac{1}{4r^2(1 - \alpha^2)} = \frac{1}{4r^2} + \frac{\alpha^2}{4r^2(1 - \alpha^2)}$$

we obtain

$$\begin{aligned} & \int_1^{+\infty} \left[\left(1 - \alpha^2 \frac{r}{r+1} \right) |u'(r)|^2 - \left(\frac{1}{4r^2} + \frac{1}{r\alpha} \right) |u(r)|^2 \right] dr \\ & \geq (1 - \alpha^2) \int_1^{+\infty} \left[|u'(r)|^2 - \left(\frac{1}{4r^2} + \frac{1}{r\alpha(1 - \alpha^2)} \right) |u(r)|^2 \right] dr - \frac{\alpha^2}{4} \|u\|_{L^2(\mathbb{R}_+)}^2. \quad (\text{IV.3.14}) \end{aligned}$$

Putting (IV.3.13) and (IV.3.14) together we obtain the result with $C_2 = \|\phi\|_\infty + \frac{\alpha^2}{4}$. \square

Proof of Proposition IV.3.10. Let $v \in \mathcal{G}$, then $f \in D(a_2)$ and $Pv \in D(a_1)$, and, by Proposition IV.3.9,

$$q_\alpha(v, v) \geq a_2(f, f) + a_1(Pv, Pv) - \alpha A \|\Pi v\|_{L^2(V_\alpha)}^2.$$

We can conclude thanks to Lemmas IV.3.11 and IV.3.12 and by writing $K = \max(A + C_2, C_1)$. \square

The following proposition gives a lower bound of the Rayleigh quotients of T_α in terms of $\mathcal{E}_n(\alpha)$, the eigenvalues of H_α^∞ , defined in Section IV.3.1.1.

Proposition IV.3.13. *There exist $\alpha_0 > 0$ and $M_0 > 0$ such that, for all $\alpha \in (0, \alpha_0)$ and for all $n \in \mathbb{N}$,*

$$\Lambda_n(T_\alpha) \geq \mathcal{E}_n(\alpha) - M_0.$$

The proof is again based on the use of the min-max principle, and we need a preliminary assertion.

Lemma IV.3.14. *We define, for all $(g, \varphi) \in C_c^\infty(\mathbb{R}_+) \times \mathcal{G}$, the sesquilinear form*

$$h^{\text{diag}}((g, \varphi), (g, \varphi)) = \int_0^{+\infty} \left[|g'(r)|^2 - \left(\frac{1}{4r^2} + \frac{1}{r\alpha(1-\alpha^2)} \right) |g(r)|^2 \right] dr,$$

and let H^{diag} be the self-adjoint operator associated with its closure in $L^2(\mathbb{R}_+) \times L^2(V_\alpha)$. Then $\Lambda_n(H^{\text{diag}}) \geq \mathcal{E}_n(\alpha(1-\alpha^2))$ for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. By the min-max principle,

$$\Lambda_n(H^{\text{diag}}) = \inf_{\substack{G \subset C_c^\infty(\mathbb{R}_+) \times \mathcal{G} \\ \dim(G)=n}} \sup_{\substack{(g, \varphi) \in G \\ (g, \varphi) \neq (0, 0)}} \frac{h^{\text{diag}}((g, \varphi), (g, \varphi))}{\|g\|_{L^2(\mathbb{R}_+)}^2 + \|\varphi\|_{L^2(\mathbb{R}^2)}^2}.$$

Moreover, thanks again to the min-max principle, we also have:

$$\begin{aligned} \mathcal{E}_n(\alpha(1-\alpha^2)) &= \inf_{\substack{G \subset C_c^\infty(\mathbb{R}_+) \\ \dim(G)=n}} \sup_{\substack{g \in G \\ g \neq 0}} \frac{\langle g, H_{\alpha(1-\alpha^2)}^\infty g \rangle}{\|g\|_{L^2(\mathbb{R}_+)}^2} \\ &= \inf_{\substack{G \subset C_c^\infty(\mathbb{R}_+) \times \{0\} \\ \dim(G)=n}} \sup_{\substack{(g, 0) \in G \\ g \neq 0}} \frac{h^{\text{diag}}((g, 0), (g, 0))}{\|g\|_{L^2(\mathbb{R}_+)}^2} \geq \Lambda_n(H^{\text{diag}}), \end{aligned}$$

where $H_{\alpha(1-\alpha^2)}^\infty$ is the Friedrichs extension of $H_{\alpha(1-\alpha^2)}$ defined in Section IV.3.1.1. As $\mathcal{E}_n(\alpha(1-\alpha^2)) < 0$ for all $n \in \mathbb{N}$, one has $\Lambda_n(H^{\text{diag}}) < 0$, and

$$\begin{aligned} \Lambda_n(H^{\text{diag}}) &= \inf_{\substack{G \subset C_c^\infty(\mathbb{R}_+) \times \mathcal{G} \\ \dim(G)=n \\ h_{|G}^{\text{diag}} < 0}} \sup_{\substack{(g, \varphi) \in G \\ (g, \varphi) \neq (0, 0)}} \frac{h^{\text{diag}}((g, \varphi), (g, \varphi))}{\|g\|_{L^2(\mathbb{R}_+)}^2 + \|\varphi\|_{L^2(\mathbb{R}^2)}^2} \\ &\geq \inf_{\substack{G \subset C_c^\infty(\mathbb{R}_+) \times \mathcal{G} \\ \dim(G)=n \\ h_{|G}^{\text{diag}} < 0}} \sup_{\substack{(g, \varphi) \in G \\ (g, \varphi) \neq (0, 0)}} \frac{h^{\text{diag}}((g, 0), (g, 0))}{\|g\|_{L^2(\mathbb{R}_+)}^2}. \end{aligned}$$

On the other hand,

$$\inf_{\substack{G \subset C_c^\infty(\mathbb{R}_+) \times \mathcal{G} \\ \dim(G)=n \\ h_{|G}^{\text{diag}} < 0}} \sup_{\substack{(g, \varphi) \in G \\ (g, \varphi) \neq (0, 0)}} \frac{h^{\text{diag}}((g, 0), (g, 0))}{\|g\|_{L^2(\mathbb{R}_+)}^2} = \inf_{\substack{G \subset C_c^\infty(\mathbb{R}_+) \\ \dim(G)=n \\ h_{|G}^{\text{diag}} < 0}} \sup_{\substack{g \in G \\ g \neq 0}} \frac{\langle g, H_{\alpha(1-\alpha^2)}^\infty g \rangle}{\|g\|_{L^2(\mathbb{R}_+)}^2} = \mathcal{E}_n(\alpha(1-\alpha^2)),$$

namely $\Lambda_n(H^{\text{diag}}) = \mathcal{E}_n(\alpha(1-\alpha^2))$. \square

We are now able to give the proof of the lower bound of the eigenvalues.

Proof of Proposition IV.3.13. Let $n \in \mathbb{N}$. By Proposition IV.3.10 and the min-max principle we have

$$\Lambda_n(T_\alpha) \geq (1 - \alpha^2) \inf_{\substack{G \subset \mathcal{G} \\ \dim(G)=n}} \sup_{\substack{v \in G \\ v \neq 0}} \frac{h^{\text{diag}}((f, Pv), (f, Pv))}{\|f\|_{L^2(\mathbb{R}_+)}^2 + \|Pv\|_{L^2(U_\alpha)}^2} - K.$$

Consider the map $\mathcal{J} : \mathcal{G} \ni v \mapsto (f, Pv) \in C_c^\infty(\mathbb{R}_+) \times \mathcal{G}$, where f and Pv are defined in (IV.3.1). Then,

$$\begin{aligned} \Lambda_n(T_\alpha) &\geq (1 - \alpha^2) \inf_{\substack{G' \subset \mathcal{J}(\mathcal{G}) \\ \dim(G')=n}} \sup_{\substack{(g, \varphi) \in G' \\ (g, \varphi) \neq (0,0)}} \frac{h^{\text{diag}}((g, \varphi), (g, \varphi))}{\|g\|_{L^2(\mathbb{R}_+)}^2 + \|\varphi\|_{L^2(V_\alpha)}^2} - K \\ &\geq (1 - \alpha^2) \inf_{\substack{G' \subset C_c^\infty(\mathbb{R}_+) \times \mathcal{G} \\ \dim(G')=n}} \sup_{\substack{(g, \varphi) \in G' \\ (g, \varphi) \neq (0,0)}} \frac{h^{\text{diag}}((g, \varphi), (g, \varphi))}{\|g\|_{L^2(\mathbb{R}_+)}^2 + \|\varphi\|_{L^2(V_\alpha)}^2} - K \\ &\geq (1 - \alpha^2) \mathcal{E}_n(\alpha(1 - \alpha^2)) - K, \end{aligned}$$

thanks to Lemma IV.3.14. As

$$(1 - \alpha^2) \mathcal{E}_n(\alpha(1 - \alpha^2)) = -\frac{1}{\alpha^2(2n - 1)^2} - \frac{1}{(2n - 1)^2(1 - \alpha^2)},$$

we can estimate

$$(1 - \alpha^2) \mathcal{E}_n(\alpha(1 - \alpha^2)) \geq \mathcal{E}_n(\alpha) - \frac{1}{1 - \alpha^2} - K. \quad \square$$

Now we are able to finish the proof of Theorem IV.3.1. Thanks to proposition IV.3.8 and IV.3.13, there exist $\alpha_0 \in (0, 1)$, $M \in \mathbb{R}$ and $m \in \mathbb{R}$ such that, for all $\alpha \in (0, \alpha_0)$ and for all $n \in \mathbb{N}$ one has $\mathcal{E}_n(\alpha) + m \leq \Lambda_n(T_\alpha) \leq \mathcal{E}_n(\alpha) + M$. Taking $\mathcal{C} = \max(M, |m|)$ we arrive at the result.

IV.3.2 Complete asymptotic expansion for eigenvalues

Theorem IV.3.1 and Corollary IV.3.3 give a first order asymptotics for the eigenvalues. In particular, it follows that each discrete eigenvalue is simple as the angle is small. This can be used to apply the standard perturbation theory to obtain a full asymptotic expansion.

Theorem IV.3.15. *For any $n \in \mathbb{N}$ there exist $\lambda_{j,n} \in \mathbb{R}$, $j \in \mathbb{N} \cup \{0\}$, such that for any $N \in \mathbb{N}$ one has the asymptotics*

$$E_n(T_\alpha) = \frac{1}{\alpha^2} \sum_{j=0}^N \lambda_{j,n} \alpha^{2j} + O(\alpha^{2N}) \text{ as } \alpha \rightarrow 0,$$

$$\text{and } \lambda_{0,n} = -\frac{1}{(2n - 1)^2}.$$

Proof. Let us consider the operator Q_α acting on $L^2(V_\alpha)$ and defined in Section IV.3.1.2:

$$Q_\alpha v = -\partial_r^2 v - \frac{1}{4r^2} - \frac{1}{r^2} \partial_\theta^2 v, \quad \pm \frac{1}{r} \partial_\theta v(r, \pm\alpha) = v(r, \pm\alpha).$$

Using the scaling, $\theta = \alpha\eta$, $r = \alpha t$, one shows that Q_α is unitarily equivalent to $\alpha^{-2}L_\alpha$, where L_α acts in $L^2(V_1)$ by

$$L_\alpha v := -\partial_t^2 v - \frac{v}{4t^2} - \frac{1}{\alpha^2 t^2} \partial_\eta^2 v, \quad \pm \partial_\eta v(t, \pm 1) = \alpha^2 t v(t, \pm 1),$$

and is associated with the sesquilinear form

$$\ell_\alpha(u, v) = \int_{V_1} \left(\overline{u_t} v_t - \frac{\overline{u} v}{4t^2} + \frac{\overline{u_\eta} v_\eta}{\alpha^2 t^2} \right) dt d\eta - \int_{\mathbb{R}_+} \frac{\overline{u(t, 1)} v(t, 1) + \overline{u(t, -1)} v(t, -1)}{t} dt.$$

For the eigenvalues one has $E_n(T_\alpha) = \alpha^{-2} E_n(L_\alpha)$, $n \in \mathbb{N}$, and we prefer to work with L_α in what follows. Remark that, in view of Theorem IV.1.1, for any $\gamma > 0$ one has

$$\begin{aligned} \int_{\mathbb{R}_+} \frac{|v(t, 1)|^2 + |v(t, -1)|^2}{t} dt \\ \leq \frac{1}{\gamma} \int_{V_1} \left(|v_t|^2 - \frac{|v|^2}{4t^2} + \frac{|v_\eta|^2}{\alpha^2 t^2} \right) dt d\eta + \frac{\gamma \alpha^2}{\sin^2 \alpha} \int_{V_1} |v|^2 dt d\eta, \quad v \in D(\ell_\alpha). \end{aligned}$$

In particular, there exists $b_0 > 0$ and $b > 0$ such that for small α there holds

$$\int_{\mathbb{R}_+} \frac{|v(t, 1)|^2 + |v(t, -1)|^2}{t} dt \leq b_0 \left(\ell_\alpha(v, v) + b \|v\|_{L^2(V_1)}^2 \right), \quad v \in D(\ell_\alpha). \quad (\text{IV.3.15})$$

Introduce the following differential expressions

$$\mathcal{L}_{-1} := -\frac{1}{t^2} \partial_\eta^2, \quad \mathcal{L}_0 := -\partial_t^2 - \frac{1}{4t^2},$$

then $L_\alpha u = \mathcal{L}_0 u + \alpha^{-2} \mathcal{L}_{-1} u$. We look for a formal approximate eigenpair $(E_\alpha, \varphi_\alpha)$ of L_α of the form

$$E_\alpha \underset{\alpha \rightarrow 0}{\sim} \sum_{j \geq 0} \lambda_j \alpha^{2j}, \quad \varphi_\alpha \underset{\alpha \rightarrow 0}{\sim} \sum_{j \geq 0} u_j \alpha^{2j},$$

satisfying in the sense of formal series the following eigenvalue problem

$$L_\alpha \varphi_\alpha \underset{\alpha \rightarrow 0}{\sim} E_\alpha \varphi_\alpha, \quad \pm \partial_\eta \varphi_\alpha(t, \pm 1) \underset{\alpha \rightarrow 0}{\sim} \alpha^2 t \varphi_\alpha(t, \pm 1).$$

By collecting the terms according to the powers of α , one arrives at an infinite system of partial differential equations.

To determine (λ_0, u_0) we collect the terms containing α^{-2} , then we have to solve

$$\mathcal{L}_{-1} u_0 = 0, \quad \pm \partial_\eta u_0(t, \pm 1) = 0.$$

As a consequence, u_0 only depends on t . Collecting the terms corresponding to α^0 we get

$$\mathcal{L}_0 u_0 + \mathcal{L}_{-1} u_1 = \lambda_0 u_0, \quad (\text{IV.3.16})$$

$$\pm \partial_\eta u_1(t, \pm 1) = t u_0(t, \pm 1). \quad (\text{IV.3.17})$$

Let us denote by \mathcal{L}_{-1}^N the operator acting as \mathcal{L}_{-1} on $L^2(V_1)$ whose domain is

$$D(\mathcal{L}_{-1}^N) = \{u \in L^2(V_1), \quad \mathcal{L}_{-1}^N u \in L^2(V_1), \quad \pm \partial_\eta u(t, \pm 1) = 0\}.$$

We already know that $u_0 \in \text{Ker}(\mathcal{L}_{-1}^N)$ and we notice that the orthogonal projections on $\text{Ker}(\mathcal{L}_{-1}^N)$ and the differential expression \mathcal{L}_0 commute. Thus we can integrate (IV.3.16) on $(-1, 1)$ and obtain

$$-2\mathcal{L}_0 u_0 - \frac{1}{t^2} \int_{-1}^1 \partial_\eta^2 u_1 d\eta = 2\lambda_0 u_0. \quad (\text{IV.3.18})$$

Using the boundary condition (IV.3.17), the equality (IV.3.18) becomes

$$-\partial_t^2 u_0 - \frac{1}{4t^2} u_0 - \frac{1}{t} u_0 = \lambda_0 u_0.$$

Here, we recognize a one-dimensional differential operator H_1^∞ defined in section IV.3.1.1. Hence, we are lead to choose

$$\lambda_0 = -\frac{1}{(2n-1)^2},$$

where $n \in \mathbb{N}$ is fixed for the rest of the proof, and u_0 is the associated normalized eigenfunction, see Appendix A.2. We then can get an expression for u_1 rewriting (IV.3.16) as

$$\mathcal{L}_{-1} u_1 = -\frac{1}{t} u_0.$$

Integrating it two times in η and using the boundary condition (IV.3.17) we obtain

$$u_1(t, \eta) = \frac{u_0(t)}{2} t \eta^2 + C_1(t),$$

where C_1 has to be determined in the next step. Notice that the function $t \mapsto t u_0(t)$ belongs to $L^2(\mathbb{R}_+)$ as u_0 decays exponentially, see Appendix A.2.

We now can give the proof of the existence of the further terms. Let $k \in \mathbb{N}$ and suppose that $(\lambda_1, \dots, \lambda_{k-1})$ and (u_1, \dots, u_{k-1}) are known and satisfy

$$u_l = \sum_{i=1}^l f_l^i(u_0(t), t) \eta^{2i} + C_l(t) \in L^2(V_1), \quad l = 1, \dots, k-1, \quad C_l \in \text{Ker} \left(\mathcal{L}_0 - \frac{1}{t} - \lambda_0 \right)^\perp,$$

and the functions $t \mapsto C_l(t)$ and $t \mapsto f_l^i(u_0(t), t)$ decay exponentially, for all $i \leq l$ and $l \leq k-1$. We want to determine (λ_k, u_k) . We first use the equation obtained by collecting the terms in α^{2k-2} , i.e.

$$\mathcal{L}_{-1} u_k + \mathcal{L}_0 u_{k-1} = \sum_{i+j=k-1} \lambda_j u_i, \quad (\text{IV.3.19})$$

$$\pm \partial_\eta u_k(t, \pm 1) = t u_{k-1}(t, \pm 1). \quad (\text{IV.3.20})$$

Using (IV.3.19) and the hypotheses we have

$$\begin{aligned} \mathcal{L}_{-1} u_k &= \sum_{\substack{i+j=k-1 \\ i \neq 0}} \lambda_j \left(\sum_{m=1}^i f_i^m(u_0(t), t) \eta^{2m} + C_i(t) \right) + \lambda_{k-1} u_0(t) \\ &\quad - \sum_{m=1}^{k-1} (\mathcal{L}_0 f_{k-1}^m(u_0(t), t)) \eta^{2m} - \mathcal{L}_0 C_{k-1}(t). \end{aligned}$$

We integrate it two times in η and we use the boundary condition (IV.3.20) to cancel the term corresponding to η , then there exists $t \mapsto \tilde{C}_k(t)$ such that

$$\begin{aligned} u_k(t) &= -t^2 \left(\sum_{\substack{i+j=k-1 \\ i \neq 0}} \lambda_j \left(\sum_{m=1}^i f_i^m(u_0(t), t) \right) \frac{\eta^{2m+2}}{(2m+1)(2m+2)} \right. \\ &\quad \left. - \sum_{m=1}^{k-1} (\mathcal{L}_0 f_{k-1}^m(u_0(t), t)) \frac{\eta^{2m+2}}{(2m+1)(2m+2)} \right. \\ &\quad \left. + \left(\sum_{\substack{i+j=k-1 \\ i \neq 0}} \lambda_j C_i(t) - \mathcal{L}_0 C_{k-1}(t) + \lambda_{k-1} u_0(t) \right) \frac{\eta^2}{2} + \tilde{C}_k(t) \right). \quad (\text{IV.3.21}) \end{aligned}$$

We set

$$f_k^m(x, t) := -\frac{t^2}{2m(2m-1)} \left(\sum_{i=m-1}^{k-1} \lambda_{k-1-i} f_i^{m-1}(x, t) - \mathcal{L}_0 f_{k-1}^{m-1}(x, t) \right), \quad m = 2, \dots, k,$$

$$f_k^1(x, t) := -\frac{t^2}{2} \left(\sum_{\substack{i+j=k-1 \\ i \neq 0}} \lambda_j C_i(t) - \mathcal{L}_0 C_{k-1}(t) + \lambda_{k-1} x \right),$$

$$C_k(t) := -t^2 \tilde{C}_k(t).$$

Notice that each $t \mapsto f_k^m(u_0(t), t)$, $m = 1, \dots, k$, is then in $L^2(\mathbb{R}_+)$ and decay exponentially due to the hypothesis on $(f_i^m(u_0(\cdot), \cdot))_{m=1}^i$, $i = 1, \dots, k-1$, and $(C_i(\cdot))_{i=1}^{k-1}$. Then (IV.3.21) can be written in the form

$$u_k(t, \eta) = \sum_{m=1}^k f_k^m(u_0(t), t) \eta^{2m} + C_k(t).$$

We have now to determine λ_k and C_k . Let us consider the equation obtained after collecting the terms in α^{2k} :

$$\mathcal{L}_{-1} u_{k+1} + \mathcal{L}_0 u_k = \sum_{i+j=k} \lambda_j u_i, \quad (\text{IV.3.22})$$

$$\pm \partial_\eta u_{k+1}(t, \pm 1) = t u_k(t, \pm 1). \quad (\text{IV.3.23})$$

The integration of (IV.3.22) on $(-1, 1)$ with respect to η and the boundary conditions (IV.3.23) give,

$$\begin{aligned} \left(\mathcal{L}_0 - \frac{1}{t} - \lambda_0 \right) C_k(t) &= \frac{1}{t} \sum_{m=1}^k f_k^m(u_0(t), t) - \sum_{m=1}^k \frac{1}{2m+1} \mathcal{L}_0 f_k^m(u_0(t), t) \\ &+ \sum_{\substack{i+j=k \\ i \neq 0, j \neq 0}} \lambda_j \left(\sum_{l=1}^i \frac{1}{2l+1} f_i^l(u_0(t), t) + C_i(t) \right) + \lambda_k u_0(t). \end{aligned} \quad (\text{IV.3.24})$$

This equation admits a solution $C_k \in \text{Ker}(H_1^\infty - \lambda_0)^\perp$ iff the right hand side belongs to $\text{Ker}(H_1^\infty - \lambda_0)^\perp$. Thus, λ_k is uniquely determined by

$$\begin{aligned} \lambda_k &= \left\langle -\frac{1}{t} \sum_{m=1}^k f_k^m(u_0(\cdot), \cdot) + \sum_{m=1}^k \frac{1}{2m+1} \mathcal{L}_0 f_k^m(u_0(\cdot), \cdot) \right. \\ &\quad \left. - \sum_{\substack{i+j=k \\ i \neq 0, j \neq 0}} \lambda_j \left(\sum_{l=1}^i \frac{1}{2l+1} f_i^l(u_0(\cdot), \cdot) + C_i(\cdot) \right), u_0 \right\rangle_{L^2(\mathbb{R}_+)}. \end{aligned}$$

As C_k satisfies the inhomogeneous equation (IV.3.24), a standard application of the variation of constants shows that it is exponentially decaying, which concludes the construction of the formal asymptotics.

Now we are going to show that the above formal expression for E_α provides an asymptotics for the eigenvalues of L_α . Now let us fix $N \in \mathbb{N}$ and consider the finite sums

$$E_N = \sum_{j=0}^N \lambda_j \alpha^{2j}, \quad \varphi_N = \sum_{j=0}^N u_j \alpha^{2j}.$$

By the preceding constructions one has

$$\begin{aligned} (\mathcal{L}_0 + \alpha^{-2}\mathcal{L}_{-1})\varphi_N &= E_N\varphi_N + \alpha^{2N}\psi_N, \quad \psi_N := \mathcal{L}_0 u_N - \sum_{k=N}^{2N} \left(\sum_{i+j=k} \lambda_i u_j \right) \alpha^{2(k-N)}, \\ \pm \partial_\eta \varphi_N(t, \pm 1) &= \alpha^2 t \varphi_N(t, \pm 1) - \alpha^{2N+2} t u_N(t, \pm 1). \end{aligned}$$

Remark that φ_N does not belong to the domain of L_α as it does not satisfy the boundary condition, but belongs to the form domain of L_α , and for any $v \in D(\ell_\alpha)$ one has, using the integration by parts,

$$\begin{aligned} \ell_\alpha(\varphi_N, v) - E_N \int_{V_1} \overline{\varphi_N} v \, dt \, d\eta \\ = \alpha^{2N} \left(\int_{V_1} \overline{\psi_N} v \, dt \, d\eta - \int_{\mathbb{R}_+} \frac{\overline{u_N(t, 1)} v(t, 1) + \overline{u_N(t, -1)} v(t, -1)}{t} \, dt \right). \end{aligned} \quad (\text{IV.3.25})$$

Recall that $\inf \text{spec } L_\alpha = \alpha^2 \inf \text{spec } Q_\alpha = -\alpha^2 (\sin^2 \alpha)^{-1}$. Furthermore, without loss of generality we may assume that the constant b in (IV.3.15) is such that $L'_\alpha := L_\alpha + b \geq 1$ for small α and consider the associated shifted sesquilinear form

$$\ell'_\alpha(u, v) = \ell_\alpha(u, v) + b \langle u, v \rangle_{L^2(V_1)}, \quad D(\ell'_\alpha) = D(\ell_\alpha).$$

Using the preceding estimates and the Cauchy-Schwarz inequality one deduces from (IV.3.15) and (IV.3.25) that for any $v \in D(\ell'_\alpha)$ there holds, with some $a_1 > 0$,

$$\begin{aligned} |\ell'_\alpha(\varphi_N, v) - (E_N + b) \langle \varphi_N, v \rangle_{L^2(V_1)}| \\ \leq a_1 \alpha^{2N} \left(\ell'_\alpha(\psi_N, \psi_N)^{\frac{1}{2}} \ell'_\alpha(v, v)^{\frac{1}{2}} + \ell'_\alpha(u_N, u_N)^{\frac{1}{2}} \ell'_\alpha(v, v)^{\frac{1}{2}} \right) \end{aligned} \quad (\text{IV.3.26})$$

as α is sufficiently small. Remark that for small α the values $\ell'_\alpha(\psi_N, \psi_N)$ and $\ell'_\alpha(u_N, u_N)$ can be estimated as $O(\alpha^{-2})$. On the other hand, $\ell'_\alpha(\varphi_N, \varphi_N) \geq \|\varphi_N\|^2 = 1 + O(1)$ for small α , and it follows from (IV.3.26) that, with some $a_2 > 0$,

$$|\ell'_\alpha(\varphi_N, v) - (E_N + b) \langle \varphi_N, v \rangle_{L^2(V_1)}| \leq a_2 \alpha^{2N-1} \ell'_\alpha(\varphi_N, \varphi_N)^{\frac{1}{2}} \ell'_\alpha(v, v)^{\frac{1}{2}}, \quad v \in D(\ell'_\alpha).$$

A simple application of the spectral theorem, see Proposition III.3.5 in Section III.3, shows that

$$\text{dist}(\text{spec } L_\alpha, E_N) \leq \frac{a_2 \alpha^{2N-1}}{1 - a_2 \alpha^{2N-1}} (E_N + b) = O(\alpha^{2N-1}) \text{ as } \alpha \text{ tends to 0.}$$

By Theorem IV.3.1, the only point of the spectrum of L_α which can satisfy the above estimate is the n th eigenvalue $E_n(L_\alpha)$. As N is arbitrary, the result follows. \square

IV.4 Decay of eigenfunctions

Let $\alpha \in (0, \frac{\pi}{2})$ be fixed. The following proposition (Agmon-type estimate) shows that the eigenfunctions of T_α corresponding to the discrete eigenvalues are localized near the vertex of the sector.

Theorem IV.4.1. *Let $\alpha \in (0, \frac{\pi}{2})$. Let E be a discrete eigenvalue of T_α and \mathcal{V} be an associated eigenfunction, then for any $\epsilon \in (0, 1)$ one has*

$$\int_{U_\alpha} (|\nabla \mathcal{V}|^2 + |\mathcal{V}|^2) e^{2(1-\epsilon)\sqrt{-1-E}|x|} dx < +\infty. \quad (\text{IV.4.1})$$

We remark that the term $(-1 - E)$ appearing in the exponential is exactly the distance between the eigenvalue and the bottom of the essential spectrum of T_α .

Proof. Let $\epsilon \in (0, 1)$ and $L > 0$. Define $\phi_L(x) = \sqrt{-1 - E} \min(|x|, L)$. We are going to show first that there exists $K_\epsilon > 0$ such that

$$\|\mathcal{V}e^{(1-\epsilon)\phi_L}\|_{H^1(U_\alpha)}^2 \leq K_\epsilon. \quad (\text{IV.4.2})$$

The proof follows essentially the same steps as in [HP15, Proposition 2.8] where a similar result for the lowest eigenvalue of corner domains is shown.

Let χ_0 and χ_1 be smooth functions of \mathbb{R}_+ satisfying

$$\chi_0(t) = 1 \text{ for } 0 < t < 1, \quad \chi_0(t) = 0 \text{ for } t > 2, \quad \chi_0^2(t) + \chi_1^2(t) = 1.$$

For $R > 0$, consider the functions $\chi_{j,R}(x) := \chi_j(|x|/R)$ defined on U_α , $j = 0, 1$. We get easily, applying Lemma III.2.2, for $u \in H^1(U_\alpha)$ and for $\gamma > 0$,

$$t_\alpha^\gamma(u, u) = t_\alpha^\gamma(u\chi_{0,R}, u\chi_{0,R}) + t_\alpha^\gamma(u\chi_{1,R}, u\chi_{1,R}) - \sum_{j=0,1} \|u\nabla\chi_{j,R}\|_{L^2(U_\alpha)}^2.$$

In particular, by definition of $\chi_{j,R}$, there exists $C > 0$ such that for any $R > 0$,

$$t_\alpha^\gamma(u, u) \geq t_\alpha^\gamma(u\chi_{0,R}, u\chi_{0,R}) + t_\alpha^\gamma(u\chi_{1,R}, u\chi_{1,R}) - \frac{C}{R^2} \|u\|_{L^2(U_\alpha)}^2, \quad u \in H^1(U_\alpha). \quad (\text{IV.4.3})$$

For $\delta \in (0, 1)$ we have $t_\alpha(u, u) = \delta \|\nabla u\|_{L^2(U_\alpha)}^2 + (1 - \delta)t_\alpha^{\frac{1}{1-\delta}}(u, u)$, and (IV.4.3) leads to

$$t_\alpha(u, u) \geq \delta \|\nabla u\|_{L^2(U_\alpha)}^2 + (1 - \delta) \left(t_\alpha^{\frac{1}{1-\delta}}(u\chi_{0,R}, u\chi_{0,R}) + t_\alpha^{\frac{1}{1-\delta}}(u\chi_{1,R}, u\chi_{1,R}) - \frac{C}{R^2} \|u\|_{L^2(U_\alpha)}^2 \right). \quad (\text{IV.4.4})$$

We are going to provide a lower bound for the first two terms in the bracket. As $u\chi_{0,R} \in H^1(U_\alpha)$, we have immediatly by the min-max principle

$$t_\alpha^{\frac{1}{1-\delta}}(u\chi_{0,R}, u\chi_{0,R}) \geq E_1(T_\alpha^{\frac{1}{1-\delta}}) \|u\chi_{0,R}\|_{L^2(U_\alpha)}^2 = -\frac{1}{(1-\delta)^2 \sin^2 \alpha} \|u\chi_{0,R}\|_{L^2(U_\alpha)}^2. \quad (\text{IV.4.5})$$

To estimate $t_\alpha^{\frac{1}{1-\delta}}(u\chi_{1,R}, u\chi_{1,R})$ we introduce the domain $U_\alpha^R = \{x \in U_\alpha, |x| \geq R\}$. Let C_R be the sector obtained by translation of vector $(R, 0)$ of U_α . Define

$$D^+ = (U_\alpha^R \setminus \overline{C_R}) \cap (\mathbb{R} \times \mathbb{R}_+), \quad D^- = (U_\alpha^R \setminus \overline{C_R}) \cap (\mathbb{R} \times \mathbb{R}_-),$$

see Figure IV.4, then $\overline{U_\alpha^R} = \overline{D^+ \cup D^- \cup C_R}$. Consider the sesquilinear forms

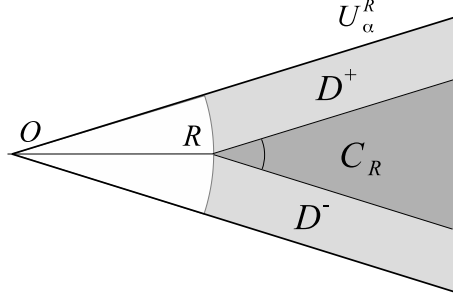
$$q_R^\pm(u, u) = \int_{D^\pm} |\nabla u|^2 dx - \frac{1}{1-\delta} \int_{\partial U_\alpha \cap \partial D^\pm} |u|^2 ds, \quad D(q_R^\pm) = \{u \in H^1(D^\pm), u = 0 \text{ if } |x| = R\},$$

$$q_{C_R}(u, u) = \int_{C_R} |\nabla u|^2 dx, \quad D(q_{C_R}) = H^1(C_R),$$

and denote by Q_R^\pm and Q_{C_R} the associated self-adjoint operators in $L^2(D^\pm)$ and $L^2(C_R)$ respectively. As Q_R^+ and Q_R^- are unitarily equivalent and q_{C_R} is non-negative, we have by the min-max principle,

$$t_\alpha^{\frac{1}{1-\delta}}(u\chi_{1,R}, u\chi_{1,R}) = q_R^+(u\chi_{1,R}, u\chi_{1,R}) + q_R^-(u\chi_{1,R}, u\chi_{1,R}) + q_{C_R}(u\chi_{1,R}, u\chi_{1,R})$$

$$\geq \Lambda_1(Q_R^+) \left(\|u\chi_{1,R}\|_{L^2(D^+)}^2 + \|u\chi_{1,R}\|_{L^2(D^-)}^2 \right).$$


 Figure IV.4 – Partition of U_α^R .

Define

$$\mathcal{R}_\alpha = \left\{ x \in \mathbb{R}^2 : \frac{x_2}{\tan \alpha} \leq x_1 \leq \frac{x_2}{\tan \alpha} + R \right\}.$$

Remark that, if one takes $u \in D(q_R^+)$ and denotes by \tilde{u} its extension by zero to \mathcal{R}_α then, $q_R^+(u, u) = q_{\mathcal{R}_\alpha}(\tilde{u}, \tilde{u})$, where

$$q_{\mathcal{R}_\alpha}(u, u) = \int_{\mathcal{R}_\alpha} |\nabla u|^2 dx - \frac{1}{1-\delta} \int_{\mathbb{R}} \left| u\left(\frac{x_2}{\tan \alpha}, x_2\right) \right|^2 \frac{dx_2}{\sin \alpha}, \quad u \in H^1(\mathcal{R}_\alpha).$$

Then $\inf \text{spec } Q_R^+ \geq \inf \text{spec } Q_{\mathcal{R}_\alpha}$, where $Q_{\mathcal{R}_\alpha}$ is the self-adjoint operator associated with $q_{\mathcal{R}_\alpha}$ acting in $L^2(\mathcal{R}_\alpha)$. The domain \mathcal{R}_α is simply a straight strip of width $R \sin \alpha$. Thus, by separation of variables one can see that $Q_{\mathcal{R}_\alpha}$ is unitarily equivalent to $\mathcal{N}_{R \sin \alpha, (1-\delta)^{-1}} \otimes 1 + 1 \otimes (-\Delta_{\mathbb{R}})$, where $\mathcal{N}_{R \sin \alpha, (1-\delta)^{-1}}$ is the one-dimensional Robin-Neumann Laplacian, defined in Section III.5.2, acting in $L^2(0, R \sin \alpha)$ and $-\Delta_{\mathbb{R}}$ is the free Laplacian in $L^2(\mathbb{R})$. Then, $\inf \text{spec } Q_{\mathcal{R}_\alpha} = \inf \text{spec } \mathcal{N}_{R \sin \alpha, (1-\delta)^{-1}}$. By Lemma III.5.2 we have

$$E_1(\mathcal{N}_{R \sin \alpha, (1-\delta)^{-1}}) = -\frac{1}{(1-\delta)^2} - \frac{4}{(1-\delta)^2} e^{-2\frac{R \sin \alpha}{1-\delta}} + O(R e^{-4\frac{R \sin \alpha}{1-\delta}}), \quad R \rightarrow \infty,$$

and there exist $R_0 > 0$ and $C_0 > 0$ such that for all $R \geq R_0$ we have

$$E_1(\mathcal{N}_{R \sin \alpha, (1-\delta)^{-1}}) \geq -\frac{1}{(1-\delta)^2} - C_0 e^{-2\frac{R \sin \alpha}{1-\delta}}.$$

Finally, for all $u \in H^1(U_\alpha)$ we have

$$t_\alpha^{\frac{1}{1-\delta}}(u\chi_{1,R}, u\chi_{1,R}) \geq \left(-\frac{1}{(1-\delta)^2} - C_0 e^{-2\frac{R \sin \alpha}{1-\delta}} \right) \|u\chi_{1,R}\|_{L^2(U_\alpha)}^2, \quad R \geq R_0. \quad (\text{IV.4.6})$$

Combining (IV.4.4) with (IV.4.5) and (IV.4.6) we get, for all $u \in H^1(U_\alpha)$ and $R \geq R_0$,

$$\begin{aligned} t_\alpha(u, u) &\geq \delta \|\nabla u\|_{L^2(U_\alpha)}^2 - \frac{1}{(1-\delta) \sin^2 \alpha} \|u\chi_{0,R}\|_{L^2(U_\alpha)}^2 \\ &\quad - \left(\frac{1}{1-\delta} + (1-\delta)C_0 e^{-2\frac{R \sin \alpha}{1-\delta}} \right) \|u\chi_{1,R}\|_{L^2(U_\alpha)}^2 - \frac{C(1-\delta)}{R^2} \|u\|_{L^2(U_\alpha)}^2. \end{aligned} \quad (\text{IV.4.7})$$

Denote by $\psi_{L,\epsilon} := (1-\epsilon)\phi_L$. The functions $\psi_{L,\epsilon}$ and $\nabla \psi_{L,\epsilon}$ belong to $L^\infty(\mathbb{R}^2)$ and

$$\|\nabla \psi_{L,\epsilon}\|_\infty^2 \leq (1-\epsilon)^2 (-1-E). \quad (\text{IV.4.8})$$

Thus we can apply Lemma III.3.2 to obtain

$$t_\alpha(\mathcal{V}e^{\psi_{L,\epsilon}}, \mathcal{V}e^{\psi_{L,\epsilon}}) = \int_{U_\alpha} |\mathcal{V}e^{\psi_{L,\epsilon}}|^2 (E + |\nabla \psi_{L,\epsilon}|^2) dx.$$

As $\mathcal{V}e^{\psi_{L,\epsilon}} \in H^1(U_\alpha)$, we obtain by (IV.4.7), for all $R \geq R_0$,

$$\begin{aligned} & \int_{U_\alpha} |\mathcal{V}e^{\psi_{L,\epsilon}}|^2 \left(E + |\nabla \psi_{L,\epsilon}|^2 \right) dx \\ & \geq \delta \|\nabla(\mathcal{V}e^{\psi_{L,\epsilon}})\|_{L^2(U_\alpha)}^2 - \frac{1}{(1-\delta)\sin^2 \alpha} \|\mathcal{V}e^{\psi_{L,\epsilon}} \chi_{0,R}\|_{L^2(U_\alpha)}^2 \\ & \quad - \left(\frac{1}{1-\delta} + (1-\delta)C_0 e^{-\frac{2R\sin \alpha}{1-\delta}} \right) \|\mathcal{V}e^{\psi_{L,\epsilon}} \chi_{1,R}\|_{L^2(U_\alpha)}^2 - \frac{C(1-\delta)}{R^2} \|\mathcal{V}e^{\psi_{L,\epsilon}}\|_{L^2(U_\alpha)}^2, \end{aligned}$$

which can be transformed in virtue of (IV.4.8) into

$$A_0 \|\mathcal{V}e^{\psi_{L,\epsilon}} \chi_{0,R}\|_{L^2(U_\alpha)}^2 \geq \delta \|\nabla(\mathcal{V}e^{\psi_{L,\epsilon}})\|_{L^2(U_\alpha)}^2 + A_1 \|\mathcal{V}e^{\psi_{L,\epsilon}} \chi_{1,R}\|_{L^2(U_\alpha)}^2, \quad R \geq R_0, \quad (\text{IV.4.9})$$

with

$$\begin{aligned} A_0 &= (\epsilon^2 - 2\epsilon)(-1 - E) + \frac{\cos^2 \alpha + \delta \sin^2 \alpha}{(1-\delta)\sin^2 \alpha} + \frac{C(1-\delta)}{R^2}, \\ A_1 &= (2\epsilon - \epsilon^2)(-1 - E) - \frac{\delta}{1-\delta} - (1-\delta)C_0 e^{-\frac{2R\sin \alpha}{1-\delta}} - \frac{C(1-\delta)}{R^2}. \end{aligned}$$

As $\epsilon \in (0, 1)$, we have $0 < 2\epsilon - \epsilon^2 < 1$. In addition, $0 < -1 - E \leq -1 - E_1(T_\alpha) = \cot^2 \alpha$. Therefore, one can find $R_\epsilon > R_0$ and $\delta_\epsilon \in (0, 1)$ such that $A_0 > 0$ and $A_1 > 0$ for all $R \geq R_\epsilon$. Furthermore, there exists $m_\epsilon > 0$ such that $A_1 \geq m_\epsilon$ for $R \geq R_\epsilon$. For the same δ_ϵ we can show that there exists $M_\epsilon > 0$ such that $A_0 \leq M_\epsilon$ for all $R \geq R_\epsilon$. The inequality (IV.4.9) implies then

$$M_\epsilon \|\mathcal{V}e^{\psi_{L,\epsilon}} \chi_{0,R}\|_{L^2(U_\alpha)}^2 \geq \delta_\epsilon \|\nabla(\mathcal{V}e^{\psi_{L,\epsilon}})\|_{L^2(U_\alpha)}^2 + m_\epsilon \|\mathcal{V}e^{\psi_{L,\epsilon}} \chi_{1,R}\|_{L^2(U_\alpha)}^2, \quad R \geq R_\epsilon,$$

and, finally,

$$C_\epsilon \|\mathcal{V}e^{\psi_{L,\epsilon}} \chi_{0,R}\|_{L^2(U_\alpha)}^2 \geq \|\mathcal{V}e^{\psi_{L,\epsilon}}\|_{H^1(U_\alpha)}^2, \quad R \geq R_\epsilon,$$

with $C_\epsilon = \frac{M_\epsilon}{\delta_\epsilon + m_\epsilon} + 1$. Notice that,

$$\|\mathcal{V}e^{\psi_{L,\epsilon}} \chi_{0,R}\|_{L^2(U_\alpha)}^2 \leq e^{4(1-\epsilon)\sqrt{-1-E}R} \|\mathcal{V} \chi_{0,R}\|_{L^2(U_\alpha)}^2 \leq e^{4(1-\epsilon)\sqrt{-1-E}R} \|\mathcal{V}\|_{L^2(U_\alpha)}^2,$$

which gives (IV.4.2).

Now let us pass from (IV.4.2) to (IV.4.1). We have

$$\|\mathcal{V}e^{\psi_{L,\epsilon}}\|_{H^1(U_\alpha)}^2 = \int_{U_\alpha} \left| \nabla \mathcal{V} + \mathcal{V} \nabla \psi_{L,\epsilon} \right|^2 e^{2\psi_{L,\epsilon}} dx + \int_{U_\epsilon} |\mathcal{V}|^2 e^{2\psi_{L,\epsilon}} dx.$$

Using the inequality

$$2 \left| \int_{U_\alpha} e^{2\psi_{L,\epsilon}} \mathcal{V} (\nabla \mathcal{V} \cdot \nabla \psi_{L,\epsilon}) dx \right| \leq \sigma \|\nabla \mathcal{V} e^{\psi_{L,\epsilon}}\|_{L^2(U_\alpha)}^2 + \frac{1}{\sigma} \|\nabla \psi_{L,\epsilon} \mathcal{V} e^{\psi_{L,\epsilon}}\|_{L^2(U_\alpha)}^2, \quad \sigma > 0,$$

with $\sigma = \frac{1}{2}$ we get

$$\|\mathcal{V}e^{\psi_{L,\epsilon}}\|_{H^1(U_\alpha)}^2 \geq \int_{U_\alpha} \left(1 - |\nabla \psi_{L,\epsilon}|^2 \right) |\mathcal{V}|^2 e^{2\psi_{L,\epsilon}} dx + \frac{1}{2} \int_{U_\alpha} |\nabla \mathcal{V}|^2 e^{2\psi_{L,\epsilon}} dx,$$

and then

$$\|\mathcal{V}e^{\psi_{L,\epsilon}}\|_{H^1(U_\alpha)}^2 + \|\nabla \psi_{L,\epsilon} \mathcal{V} e^{\psi_{L,\epsilon}}\|_{L^2(U_\alpha)}^2 \geq \frac{1}{2} \int_{U_\alpha} \left(|\nabla \mathcal{V}|^2 + |\mathcal{V}|^2 \right) e^{2\psi_{L,\epsilon}} dx.$$

By combining (IV.4.2) with (IV.4.8) we arrive at

$$\int_{U_\alpha} \left(|\nabla \mathcal{V}|^2 + |\mathcal{V}|^2 \right) e^{2\psi_{L,\epsilon}} dx \leq 2K_\epsilon \left(1 + (1-\epsilon)^2(-1-E) \right).$$

As the right-hand side does not depend on L , one can pass to the limit as $L \rightarrow +\infty$ using the monotone convergence, which gives the result. \square

IV.5 Robin-Neumann Laplacians on truncated convex sectors

This last section of the present chapter is dedicated to the operator $T_{\alpha,R}^\gamma$, defined in Section II.2.3. The results presented here will be used in Section V.4 for the study of Robin Laplacians acting on polygons with straight edges.

Recall that $T_{\alpha,R}^\gamma$, defined by (II.2.4), acts as the Laplacian on the quadrangle $U_{\alpha,R}$ with the γ -Robin boundary condition on $\partial U_{\alpha,R} \cap \partial U_\alpha$ and the Neumann boundary condition on the remaining part of the boundary, according to Definition III.6.1. It is easy to see that, by change of variables, the operator $T_{\alpha,R}^\gamma$ is unitarily equivalent to $\gamma^2 T_{\alpha,\gamma R}^1$. Thus, in what follows we restrict the study to the operator

$$T_{\alpha,R} := T_{\alpha,R}^1.$$

We also use the simpler notation

$$\mathcal{N}_\alpha := \mathcal{N}(T_\alpha, -1).$$

Recall that $\mathcal{N}_\alpha < +\infty$ by Theorem IV.2.1. The following proposition gives some explicit *non-resonant* angles, see Definition II.2.6.

Proposition IV.5.1. *All angles $\alpha \in [\frac{\pi}{4}, \frac{\pi}{2})$ are non-resonant.*

Proof. The proof is divided in two steps. First we prove that $\frac{\pi}{4}$ is non-resonant. Then, we use an argument of monotonicity to conclude.

Let $\alpha = \frac{\pi}{4}$. As $\mathcal{N}_{\frac{\pi}{4}} = 1$, see Corollary IV.2.5, we have to prove that there exists $C > 0$ such that for large R ,

$$E_2(T_{\frac{\pi}{4},R}) \geq -1 + \frac{C}{R^2}. \quad (\text{IV.5.1})$$

By separation of variables, it is easy to see that for any $n \in \mathbb{N}$,

$$E_n(T_{\frac{\pi}{4},R}) = E_n(\mathcal{N}_{R,1} \otimes 1 + 1 \otimes \mathcal{N}_{R,1}),$$

where $\mathcal{N}_{R,1}$ is defined in Lemma III.5.2. In particular, $E_2(T_{\frac{\pi}{4},R}) = E_1(\mathcal{N}_{R,1}) + E_2(\mathcal{N}_{R,1})$ and due to the asymptotics (III.5.2) and the inequality $E_2(\mathcal{N}_{R,1}) \geq \frac{\pi^2}{4R^2}$, see (III.5.3), we obtain (IV.5.1).

Now, let $\alpha \geq \frac{\pi}{4}$. Then $\mathcal{N}_\alpha = 1$, see Corollary IV.2.5. We start with a decomposition of $T_{\alpha,R}$ with respect to the axial symmetry of $U_{\alpha,R}$. Consider $U_{\alpha,R}^+ := U_{\alpha,R} \cap (\mathbb{R}_+ \times \mathbb{R}_+)$ and the unitary map

$$\begin{aligned} \mathcal{U} : L^2(U_{\alpha,R}) &\ni u \mapsto (g, h) \in L^2(U_{\alpha,R}^+) \oplus L^2(U_{\alpha,R}^+), \\ g(x_1, x_2) &:= \frac{u(x_1, x_2) + u(x_1, -x_2)}{\sqrt{2}}, \quad h(x_1, x_2) := \frac{u(x_1, x_2) - u(x_1, -x_2)}{\sqrt{2}}. \end{aligned}$$

By a direct computation, for $u \in D(t_{\alpha,R})$ one has $t_{\alpha,R}(u, u) = t_{\alpha,R}^N(g, g) + t_{\alpha,R}^D(h, h)$, with

$$t_{\alpha,R}^N(g, g) = \int_{U_{\alpha,R}^+} |\nabla g|^2 dx - \int_{\partial U_{\alpha,R}^+ \cap \partial U_\alpha} |g|^2 ds, \quad g \in H^1(U_{\alpha,R}^+),$$

and $t_{\alpha,R}^D$ is given by the same expression but acts on the smaller domain

$$D(t_{\alpha,R}^D) := \left\{ h \in H^1(U_{\alpha,R}^+) : h(\cdot, 0) = 0 \right\}.$$

Then by construction, the operator $T_{\alpha,R}$ is unitarily equivalent to the direct sum of $T_{\alpha,R}^D$ and $T_{\alpha,R}^N$,

$$T_{\alpha,R} = \mathcal{U}^* T_{\alpha,R}^D \oplus T_{\alpha,R}^N \mathcal{U}.$$

Let us first study the infimum of the spectrum of $T_{\alpha,R}^D$. Remark that $U_{\alpha,R}^+$ is included in a rectangle of length $R \tan \alpha$ and width R . Extending $h \in D(t_{\alpha,R}^D)$ by zero and using separation of variables we easily get

$$E_1(T_{\alpha,R}^D) \geq E_1(\mathcal{D}_{R \tan \alpha, 1} \otimes 1 + 1 \otimes T_R^{DN}),$$

where T_R^{DN} is the Dirichlet-Neumann Laplacian acting on $L^2(0, R)$ as $f \mapsto -f''$ with $f(0) = f'(R) = 0$ and $\mathcal{D}_{R \tan \alpha, 1}$ is defined in Lemma III.5.3. Using the asymptotics (III.5.10) we have

$$E_1(T_{\alpha,R}^D) \geq E_1(\mathcal{D}_{R \tan \alpha, 1}) + E_1(T_R^{DN}) \geq -1 + \frac{C}{R^2}.$$

Thus,

$$\text{spec}(T_{\alpha,R}) \cap (-\infty, -1 + \frac{C}{R^2}) = \text{spec}(T_{\alpha,R}^N) \cap (-\infty, -1 + \frac{C}{R^2}).$$

Let us now focus on $T_{\alpha,R}^N$. After an anti-clockwise rotation of angle $\frac{\pi}{2} - \alpha$ of $U_{\alpha,R}^+$, one can see that $T_{\alpha,R}^N$ is unitarily equivalent to $\tilde{T}_{\alpha,R}^N$ defined as the unique self-adjoint operator associated with

$$\tilde{t}_{\alpha,R}^N(u, u) = \int_{\tilde{U}_{\alpha,R}^+} |\nabla u|^2 dx - \int_0^R |u(0, x_2)|^2 dx_2, \quad u \in H^1(\tilde{U}_{\alpha,R}^+),$$

where $\tilde{U}_{\alpha,R}^+ := \{x_1 \geq 0, x_2 \in (0, R) : x_1 \leq x_2 \tan \alpha\}$. Using the scaling $t = x_2 \tan \alpha$, and writing $v := u(x_1, \frac{t}{\tan \alpha})$ we have

$$\tilde{t}_{\alpha,R}^N(u, u) = \frac{1}{\tan \alpha} \tilde{t}_{\frac{\pi}{4}, R \tan \alpha}^N(v, v) + (\tan^2 \alpha - 1) \int_{\tilde{U}_{\frac{\pi}{4}}^+} |\partial_t v|^2 dx_1 \frac{dt}{\tan \alpha},$$

and $\|u\|_{L^2(\tilde{U}_{\alpha,R}^+)}^2 = \frac{1}{\tan \alpha} \|v\|_{L^2(\tilde{U}_{\frac{\pi}{4}, R \tan \alpha}^+)}^2$. As $\alpha \geq \frac{\pi}{4}$, we have $\tan^2 \alpha - 1 \geq 0$ and by the min-max principle we get for any $n \in \mathbb{N}$,

$$E_n(\tilde{T}_{\alpha,R}^N) \geq E_n(\tilde{T}_{\frac{\pi}{4}, R \tan \alpha}^N). \quad (\text{IV.5.2})$$

In particular, as we already proved that $\frac{\pi}{4}$ is non-resonant, there exists $C > 0$ such that $E_2(\tilde{T}_{\frac{\pi}{4}, R \tan \alpha}^N) \geq -1 + \frac{C}{R^2}$ for large R , which implies due to (IV.5.2) that α is non-resonant. \square

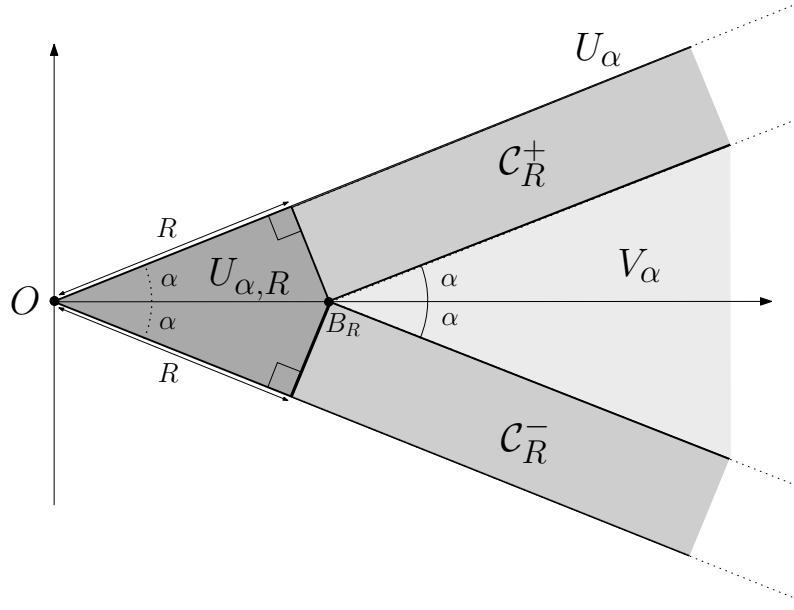
The following result gives the asymptotics of the \mathcal{N}_α first eigenvalues of $T_{\alpha,R}$ as R is large.

Proposition IV.5.2. *Let $0 < \alpha < \frac{\pi}{2}$. For any $n \leq \mathcal{N}_\alpha$ there holds*

$$E_n(T_{\alpha,R}) = E_n(T_\alpha) + o(1), \quad \text{as } R \rightarrow +\infty.$$

Proof. Let $n \in \{1, \dots, \mathcal{N}_\alpha\}$ be fixed. Let us prove the upper bound, namely:

$$E_n(T_{\alpha,R}) \leq E_n(T_\alpha) \quad \text{for large } R. \quad (\text{IV.5.3})$$


 Figure IV.5 – Partition of U_α .

The idea is to introduce an appropriate decomposition of the infinite sector U_α and the associated Neumann barriers to obtain the desired estimate. Let V_α be the infinite sector obtained by translation of vector $R(\cos^{-1}\alpha, 0)$ of U_α , and consider for $j \in \{+, -\}$,

$$\mathcal{C}_R^j := \left(U_\alpha \setminus \overline{U_{\alpha,R}} \cup V_\alpha \right) \cap (\mathbb{R}_+ \times \mathbb{R}_j).$$

We thus obtain a partition of U_α , see Figure IV.5,

$$U_\alpha = \overline{U_{\alpha,R}} \cup V_\alpha \cup \mathcal{C}_R^+ \cup \mathcal{C}_R^-.$$

Define the sesquilinear forms

$$q^\pm(u, u) = \int_{\mathcal{C}_R^\pm} |\nabla u|^2 dx - \gamma \int_{\partial \mathcal{C}_R^\pm \cap \partial U_\alpha} |u|^2 ds, \quad D(q^\pm) := H^1(\mathcal{C}_R^\pm),$$

and

$$q_{V_\alpha}(u, u) = \int_{V_\alpha} |\nabla u|^2 dx, \quad D(q_{V_\alpha}) := H^1(V_\alpha).$$

By the min-max principle we have immediately

$$E_n(T_\alpha) \geq E_n(T_{\alpha,R} \oplus Q^+ \oplus Q^- \oplus Q_{V_\alpha}).$$

Remark that the operator Q_{V_α} is positive. But, for $n \leq \mathcal{N}_\alpha$ we have $E_n(T_\alpha) < -1$, thus $E_n(T_{\alpha,R} \oplus Q^+ \oplus Q^- \oplus Q_{V_\alpha}) = E_n(T_{\alpha,R} \oplus Q^+ \oplus Q^-)$ and

$$E_n(T_\alpha) \geq E_n(T_{\alpha,R} \oplus Q^+ \oplus Q^-). \quad (\text{IV.5.4})$$

By symmetry it is easy to see that Q^+ and Q^- are unitarily equivalent. It is thus sufficient to focus on Q^+ . By separation of variables one can write

$$\inf \text{spec}(Q^+) = \inf \text{spec}(T^N \otimes 1 + 1 \otimes \mathcal{N}_{R \tan \alpha, 1}),$$

where T^N is the Neumann Laplacian acting on $L^2(\mathbb{R}_+)$ as $f \mapsto -f''$ with $f'(0) = 0$ and $\mathcal{N}_{R \tan \alpha, 1}$ is defined in Lemma III.5.2. Due to the asymptotics (III.5.2), there exist two

constants $C, c > 0$ such that $E_1(\mathcal{N}_{R \tan \alpha, 1}) \geq -1 - Ce^{-cR}$ as R is large. As a consequence, $\inf \text{spec}(Q^+) \geq -1 - Ce^{-cR}$. Due to (IV.5.4) and the fact that $E_n(T_\alpha) < -1$, this necessarily implies (IV.5.3).

Let us pass to the proof of the lower bound, namely: there exists a constant $C > 0$ such that

$$E_n(T_{\alpha, R}) \geq E_n(T_\alpha) - \frac{C}{R^2}, \text{ for large } R. \quad (\text{IV.5.5})$$

Here we use a partition of unity of $U_{\alpha, R}$ to add some Dirichlet barriers in order to obtain the estimate. To be more specific, let $\varphi \in C^\infty(\mathbb{R}_+)$ satisfying $\varphi(t) = 1$ if $t \leq \frac{1}{4}$ and $\varphi(t) = 0$ if $t \geq \frac{1}{2}$ and $0 \leq \varphi \leq 1$. We introduce $\tilde{\chi}_0(x) := \varphi\left(\frac{|x|}{R}\right)$, and $\tilde{\chi}_1(x) := 1 - \tilde{\chi}_0(x)$ for $x \in U_{\alpha, R}$. Let (χ_0, χ_1) be a partition of unity of $U_{\alpha, R}$ defined as

$$\chi_0(x) := \frac{\tilde{\chi}_0(x)}{\sqrt{(\tilde{\chi}_0(x))^2 + (\tilde{\chi}_1(x))^2}}, \quad \chi_1(x) := \frac{\tilde{\chi}_1(x)}{\sqrt{(\tilde{\chi}_0(x))^2 + (\tilde{\chi}_1(x))^2}}.$$

Defined like this $\chi_0, \chi_1 \in C^\infty(U_{\alpha, R})$ and $(\chi_0)^2 + (\chi_1)^2 = 1$ on $U_{\alpha, R}$. We also define the following subdomains of $U_{\alpha, R}$:

$$\mathcal{A}_\alpha := U_\alpha \cap B\left(0, \frac{R}{2}\right), \quad \mathcal{B}_R := \overline{U_{\alpha, R} \setminus B\left(0, \frac{R}{4}\right)}.$$

We now can apply Lemma III.2.2 to obtain, for any $u \in H^1(U_{\alpha, R})$,

$$t_{\alpha, R}(u, u) = t_{\alpha, R}(u\chi_0, u\chi_0) + t_{\alpha, R}(u\chi_1, u\chi_1) - \int_{U_{\alpha, R}} \left(|\nabla\chi_0|^2 + |\nabla\chi_1|^2\right) |u|^2 dx.$$

By definition of (χ_0, χ_1) there exists a constant $C > 0$ satisfying

$$|\nabla\chi_0|^2 + |\nabla\chi_1|^2 \leq \frac{C}{R^2},$$

and we have the obvious inequality

$$t_{\alpha, R}(u, u) \geq t_{\alpha, R}(u\chi_0, u\chi_0) + t_{\alpha, R}(u\chi_1, u\chi_1) - \frac{C}{R^2} \|u\|_{L^2(U_{\alpha, R})}^2. \quad (\text{IV.5.6})$$

Let us define the sesquilinear forms

$$q_{\mathcal{A}_\alpha}(u, u) = \int_{\mathcal{A}_\alpha} \left(|\nabla u|^2 - \frac{C}{R^2} |u|^2\right) dx - \int_{\partial\mathcal{A}_\alpha \cap \partial U_\alpha} |u|^2 ds,$$

with $D(q_{\mathcal{A}_\alpha}) := \{u \in H^1(\mathcal{A}_\alpha) : u = 0 \text{ on } \partial\mathcal{A}_\alpha \setminus \partial U_\alpha\}$, and

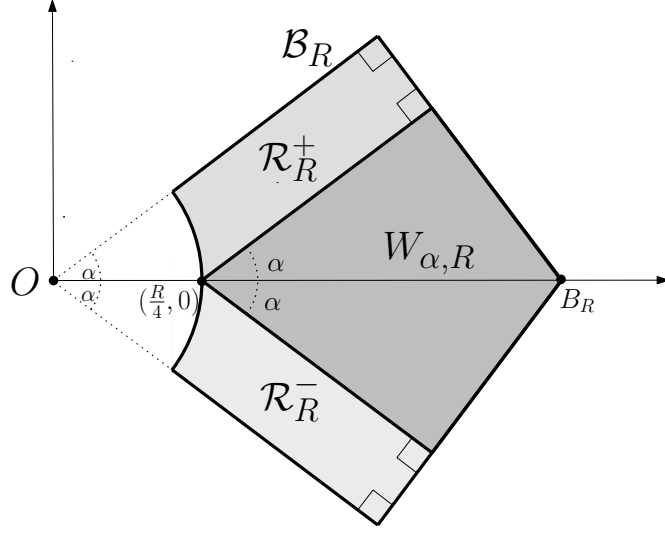
$$q_{\mathcal{B}_R}(u, u) = \int_{\mathcal{B}_R} \left(|\nabla u|^2 - \frac{C}{R^2} |u|^2\right) dx - \int_{\partial\mathcal{B}_R \cap \partial U_\alpha} |u|^2 ds,$$

with $D(q_{\mathcal{B}_R}) := \{u \in H^1(\mathcal{B}_R) : u = 0 \text{ on } \partial\mathcal{B}_R \cap \partial B(0, \frac{R}{4})\}$. Due to

$$u\chi_0 \in D(q_{\mathcal{A}_\alpha}), \quad u\chi_1 \in D(q_{\mathcal{B}_R}), \quad \|u\chi_0\|_{L^2(\mathcal{A}_\alpha)}^2 + \|u\chi_1\|_{L^2(\mathcal{B}_R)}^2,$$

inequality (IV.5.6) and the min-max principle we get

$$E_n(T_{\alpha, R}) \geq E_n(Q_{\mathcal{A}_\alpha} \oplus Q_{\mathcal{B}_R}). \quad (\text{IV.5.7})$$


 Figure IV.6 – Partition of \mathcal{B}_R .

We can now study separately the operators $Q_{\mathcal{A}_\alpha}$ and $Q_{\mathcal{B}_R}$. Remark that, extending $u \in D(q_{\mathcal{A}_\alpha})$ by zero, it is easy to see that

$$E_n(Q_{\mathcal{A}_\alpha}) \geq E_n(T_\alpha) - \frac{C}{R^2}. \quad (\text{IV.5.8})$$

To estimate the infimum of the spectrum of $Q_{\mathcal{B}_R}$ we introduce a partition of \mathcal{B}_R . Let W_α be the infinite sector obtained by translation of vector $(\frac{R}{4}, 0)$ of U_α and $W_{\alpha, R} := W_\alpha \cap U_{\alpha, R}$. We introduce the new domains for $j \in \{+, -\}$,

$$\mathcal{R}_R^j := \left(\mathcal{B}_R \setminus \overline{W_{\alpha, R}} \right) \cap (\mathbb{R}_+ \times \mathbb{R}_j).$$

We thus obtain a partition of \mathcal{B}_R , see Figure IV.6,

$$\mathcal{B}_R = \overline{\mathcal{R}_R^+ \cup \mathcal{R}_R^- \cup W_{\alpha, R}}.$$

We define the associated sesquilinear forms

$$q_R^\pm(u, u) = \int_{\mathcal{R}_R^\pm} |\nabla u|^2 dx - \int_{\partial \mathcal{R}_R^\pm \cap \partial U_\alpha} |u|^2 ds,$$

with $D(q_R^\pm) := \left\{ u \in H^1(\mathcal{R}_R^\pm) : u = 0 \text{ on } \partial \mathcal{R}_R^\pm \cap \partial B(0, \frac{R}{4}) \right\}$, and

$$q_{W_{\alpha, R}}(u, u) = \int_{W_{\alpha, R}} |\nabla u|^2, \quad D(q_{W_{\alpha, R}}) := H^1(W_{\alpha, R}).$$

We immediately get by the min-max principle

$$E_n(Q_{\mathcal{B}_R}) \geq E_n(Q_R^+ \oplus Q_R^- \oplus Q_{W_{\alpha, R}}).$$

Notice that $Q_{W_{\alpha, R}}$ is a non-negative operator. By symmetry the operators Q_R^+ and Q_R^- are unitarily equivalent and as a result,

$$E_1(Q_{\mathcal{B}_R}) \geq \min(E_1(Q_R^+), E_1(Q_{W_{\alpha, R}})). \quad (\text{IV.5.9})$$

The domain \mathcal{R}_R^+ is included in a rectangle of length $R(1 - \frac{\cos \alpha}{4})$ and width $\frac{R}{4} \sin \alpha$. Thus, extending $u \in D(q_R^+)$ by zero and using separation of variables, we obtain the inequality

$$E_1(Q_R^+) \geq E_1(\mathcal{N}_{\frac{R}{4} \sin \alpha, 1} \otimes 1 + 1 \otimes T_R^{DN}),$$

where T_R^{DN} is the Dirichlet-Neumann Laplacian acting on $L^2(0, R(1 - \frac{\cos \alpha}{4}))$ as $f \mapsto -f''$ with $f(0) = 0$ and $f'(R(1 - \frac{\cos \alpha}{4})) = 0$ and $\mathcal{N}_{\frac{R}{4} \sin \alpha, 1}$ is defined in Lemma III.5.2. Due to the asymptotics (III.5.2), there exist two constants $C, c > 0$ such that $E_1(\mathcal{N}_{\frac{R}{4} \sin \alpha, 1}) \geq -1 - Ce^{-cR}$ and thus

$$E_1(Q_R^+) \geq -1 + \frac{C}{R^2} \text{ for large } R,$$

where $C > 0$. This implies, due to (IV.5.9), that $E_1(Q_{\mathcal{B}_R}) \geq -1 + \frac{C}{R^2}$ for large R . We now can conclude the proof: recall that we already proved in the first part of the demonstration that for $n \leq \mathcal{N}_\alpha$ we have $E_n(T_{\alpha, R}) < -1$. Due to (IV.5.7), this implies, as we just proved that $E_1(Q_{\mathcal{B}_R}) \geq -1 + \frac{C}{R^2}$,

$$E_n(T_{\alpha, R}) \geq E_n(Q_{\mathcal{A}_\alpha} \oplus Q_{\mathcal{B}_R}) = E_n(Q_{\mathcal{A}_\alpha}).$$

This finishes the proof of (IV.5.5) thanks to the inequality (IV.5.8).

Gathering (IV.5.3) and (IV.5.5) we have for large R

$$E_n(T_\alpha) - \frac{C}{R^2} \leq E_n(T_{\alpha, R}) \leq E_n(T_\alpha),$$

which finishes the proof of the proposition passing to the limit $R \rightarrow +\infty$. \square

Let us introduce a new operator, which is in a sense the Dirichlet counterpart of $T_{\alpha, R}^\gamma$. Consider the sesquilinear form

$$t_{\alpha, R}^{D, \gamma}(u, u) = \int_{U_{\alpha, R}} |\nabla u|^2 - \gamma \int_{\partial U_{\alpha, R} \cap \partial U_\alpha} |u|^2 ds, \quad (\text{IV.5.10})$$

with $D(t_{\alpha, R}^{D, \gamma}) := \{u \in H^1(U_{\alpha, R}) : u = 0 \text{ on } \partial U_{\alpha, R} \setminus \partial U_\alpha\}$. Defined in this way, the form $q_{\alpha, R}^{D, \gamma}$ is closed and hence defines a unique self-adjoint operator in $L^2(U_{\alpha, R})$ denoted by $T_{\alpha, R}^{D, \gamma}$. Remark that this operator acts as the Laplacian with the γ -Robin boundary condition on $\partial U_{\alpha, R} \cap \partial U_\alpha$ and the Dirichlet boundary condition on the remaining part of the boundary of $U_{\alpha, R}$, according to Definition III.6.1. We have an analogue of Proposition IV.5.2 for $T_{\alpha, R}^{D, \gamma}$.

Proposition IV.5.3. *Let $\gamma > 0$ be fixed and $0 < \alpha < \frac{\pi}{2}$. For any $n \leq \mathcal{N}_\alpha$ there holds*

$$E_n(T_{\alpha, R}^{D, \gamma}) = E_n(T_\alpha^\gamma) + o(1), \text{ as } R \rightarrow +\infty.$$

Proof. Let $n \in \{1, \dots, \mathcal{N}_\alpha\}$ be fixed. Let us prove the lower bound, namely:

$$E_n(T_{\alpha, R}^{D, \gamma}) \geq E_n(T_\alpha^\gamma). \quad (\text{IV.5.11})$$

Let $u \in D(t_{\alpha, R}^{D, \gamma})$. If we denote by \tilde{u} the extension by zero of u to U_α , we have $t_{\alpha, R}^{D, \gamma}(u, u) = t_\alpha^\gamma(\tilde{u}, \tilde{u})$. The min-max principle gives immediately (IV.5.11).

Let us now prove the upper bound, i.e: there exists a constant $C > 0$ such that

$$E_n(T_{\alpha, R}^{D, \gamma}) \leq E_n(T_\alpha^\gamma) + \frac{C}{R^2} \text{ for large } R. \quad (\text{IV.5.12})$$

To do that, we introduce a partition of unity of the infinite sector U_α to add some Dirichlet barriers. The decomposition of U_α associated with the partition will be the same as the one in the proof of Theorem IV.4.1, see Figure IV.4, up to a constant. Thus, some results involved in this proof will be used here. Let $\varphi \in C^\infty(\mathbb{R}_+)$ satisfying $\varphi(t) = 1$ if $t \leq \frac{1}{4}$ and $\varphi(t) = 0$ if $t \geq \frac{1}{2}$ and $0 \leq \varphi \leq 1$. We introduce $\tilde{\chi}_0(x) := \varphi\left(\frac{|x|}{R}\right)$, and $\tilde{\chi}_1(x) := 1 - \tilde{\chi}_0(x)$ for $x \in U_\alpha$. Let (χ_0, χ_1) be a partition of unity of U_α defined as

$$\chi_0(x) := \frac{\tilde{\chi}_0(x)}{\sqrt{(\tilde{\chi}_0(x))^2 + (\tilde{\chi}_1(x))^2}}, \quad \chi_1(x) := \frac{\tilde{\chi}_1(x)}{\sqrt{(\tilde{\chi}_0(x))^2 + (\tilde{\chi}_1(x))^2}}.$$

Defined like this $\chi_0, \chi_1 \in C^\infty(U_\alpha)$ and $(\chi_0)^2 + (\chi_1)^2 = 1$ on U_α . We can thus apply Lemma III.2.2 to the form t_α^γ to obtain for any $u \in H^1(U_\alpha)$,

$$t_\alpha^\gamma(u, u) = t_\alpha^\gamma(u\chi_0, u\chi_0) + t_\alpha^\gamma(u\chi_1, u\chi_1) - \int_{U_\alpha} (|\nabla\chi_0|^2 + |\nabla\chi_1|^2) |u|^2 dx.$$

By definition of (χ_0, χ_1) there exists a constant $C > 0$ such that $|\nabla\chi_0|^2 + |\nabla\chi_1|^2 \leq \frac{C}{R^2}$ and thus we obtain the inequality

$$t_\alpha^\gamma(u, u) \geq t_\alpha^\gamma(u\chi_0, u\chi_0) + t_\alpha^\gamma(u\chi_1, u\chi_1) - \frac{C}{R^2} \|u\|_{L^2(U_\alpha)}^2. \quad (\text{IV.5.13})$$

Let us define the subdomains

$$(U_\alpha)_{0, \frac{R}{2}} := U_\alpha \cap B(0, \frac{R}{2}), \quad U_\alpha^{\frac{R}{4}} := \left\{ x \in U_\alpha : |x| \geq \frac{R}{4} \right\},$$

and the associated sesquilinear forms

$$q_{0, \frac{R}{2}}(u, u) = \int_{(U_\alpha)_{0, \frac{R}{2}}} \left(|\nabla u|^2 - \frac{C}{R^2} |u|^2 \right) dx - \gamma \int_{\partial(U_\alpha)_{0, \frac{R}{2}} \cap \partial U_\alpha} |u|^2 ds,$$

with $D(q_{0, \frac{R}{2}}) := \left\{ u \in H^1\left((U_\alpha)_{0, \frac{R}{2}}\right) : u = 0 \text{ on } \partial(U_\alpha)_{0, \frac{R}{2}} \cap \partial B(0, \frac{R}{2}) \right\}$, and

$$q_{U_\alpha^{\frac{R}{4}}}(u, u) = \int_{U_\alpha^{\frac{R}{4}}} \left(|\nabla u|^2 - \frac{C^2}{R} |u|^2 \right) dx - \gamma \int_{\partial U_\alpha^{\frac{R}{4}} \cap \partial U_\alpha} |u|^2 ds,$$

with $D(q_{U_\alpha^{\frac{R}{4}}}) := \left\{ u \in H^1(U_\alpha^{\frac{R}{4}}) : u = 0 \text{ on } \partial U_\alpha^{\frac{R}{4}} \cap \partial B(0, \frac{R}{4}) \right\}$. Thanks to the fact that for any $u \in H^1(U_\alpha)$ we have

$$u\chi_0 \in D(q_{0, \frac{R}{2}}), \quad u\chi_1 \in D(q_{U_\alpha^{\frac{R}{4}}}), \quad \|u\chi_0\|_{L^2(U_\alpha)}^2 + \|u\chi_1\|_{L^2(U_\alpha)}^2 = \|u\|_{L^2(U_\alpha)}^2,$$

and using the min-max principle and inequality (IV.5.13), we have for any $n \leq \mathcal{N}_\alpha$,

$$E_n(T_\alpha^\gamma) \geq E_n(Q_{0, \frac{R}{2}} \oplus Q_{U_\alpha^{\frac{R}{4}}}). \quad (\text{IV.5.14})$$

Notice that the operator $Q_{U_\alpha^{\frac{R}{4}}}$ was already studied in the proof of Theorem IV.4.1. In particular, using the notation of this latter we have

$$\inf \text{spec}(Q_{U_\alpha^{\frac{R}{4}}}) = \Lambda_1(Q_{\frac{R}{4}}^+) - \frac{C}{R^2} = -\gamma^2 - \frac{C}{R^2}, \text{ for large } R.$$

Recall that for any $n \leq \mathcal{N}_\alpha$ there holds $E_n(T_\alpha^\gamma) < -\gamma^2$. We necessarily have, due to (IV.5.14), $E_n(T_\alpha^\gamma) \geq E_n(Q_{0, \frac{R}{2}})$. Let $u \in D(q_{0, \frac{R}{2}})$ and denote by \tilde{u} its extension by zero to

$U_{\alpha,R}$. Thus, one can write $q_{0,\frac{R}{2}}(u, u) = t_{\alpha,R}^{D,\gamma}(\tilde{u}, \tilde{u}) - \frac{C}{R^2} \|\tilde{u}\|_{L^2(U_{\alpha,R})}^2$ and by the min-max principle we get $E_n(Q_{0,\frac{R}{2}}) \geq E_n(T_{\alpha,R}^{D,\gamma}) - \frac{C}{R^2}$ for large R , which finishes the proof of the upper bound (IV.5.12).

Finally, gathering (IV.5.11) and (IV.5.12) gives

$$E_n(T_\alpha^\gamma) \leq E_n(T_{\alpha,R}^{D,\gamma}) \leq E_n(T_\alpha^\gamma) + \frac{C}{R^2},$$

which ends the proof passing to the limit $R \rightarrow +\infty$. \square

In the following proposition, we prove that the eigenfunctions of $T_{\alpha,R}$ associated with the \mathcal{N}_α first eigenvalues are localized near the origin. The strategy of the proof is the same as the one of Theorem IV.4.1.

Proposition IV.5.4. *Let φ be an eigenfunction of $T_{\alpha,R}$ associated with $E := E_n(T_{\alpha,R})$, $n \leq \mathcal{N}_\alpha$. Then, there exist $\eta > 0$ and $C > 0$ such that*

$$\int_{U_{\alpha,R}} (|\nabla\varphi|^2 + |\varphi|^2) e^{\eta|x|} dx \leq C \|\varphi\|_{L^2(U_{\alpha,R})}^2, \text{ as } R \text{ is large.}$$

Proof. Let $\eta > 0$ to be chosen later and define $\phi(x) := \eta|x|$. We are going to prove first that

$$\|\varphi e^\phi\|_{H^1(U_{\alpha,R})}^2 \leq C \|\varphi\|_{L^2(U_{\alpha,R})}^2, \text{ as } R \text{ is large.} \quad (\text{IV.5.15})$$

As $\phi, \nabla\phi \in L^\infty(U_{\alpha,R})$ and by adapting Lemma III.3.2 we have the useful equality

$$t_{\alpha,R}(\varphi e^\phi, \varphi e^\phi) = \int_{U_{\alpha,R}} (|\nabla\phi|^2 + E) |\varphi e^\phi|^2 dx. \quad (\text{IV.5.16})$$

For $\beta \in (0, 1)$ to be fixed later, we also have

$$t_{\alpha,R}(\varphi e^\phi, \varphi e^\phi) = \beta \|\nabla(\varphi e^\phi)\|_{L^2(U_{\alpha,R})}^2 + (1 - \beta) t_{\alpha,R}^{\frac{1}{1-\beta}}(\varphi e^\phi, \varphi e^\phi). \quad (\text{IV.5.17})$$

Let us focus on the second term of the right hand side of this equality. We want to obtain a lower bound involving $\|\varphi e^\phi\|_{L^2(U_{\alpha,R})}^2$. To do this, we introduce a partition of $U_{\alpha,R}$ and the associated Neumann barriers. Let $0 < L < R$ and define

$$D_L := U_{\alpha,R} \setminus \overline{U_{\alpha,L}},$$

and the sesquilinear form

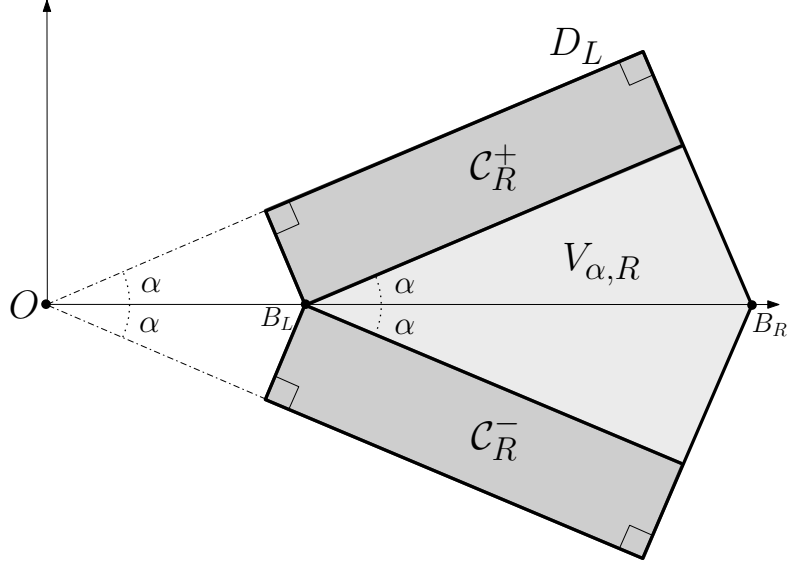
$$q_{D_L}^{\frac{1}{1-\beta}}(u, u) = \int_{D_L} |\nabla u|^2 dx - \frac{1}{1-\beta} \int_{\partial D_L \cap \partial U_\alpha} |u|^2 ds, \quad D(q_{D_L}) := H^1(D_L).$$

Thus, one can write for any $u \in H^1(U_{\alpha,R})$,

$$t_{\alpha,R}^{\frac{1}{1-\beta}}(u, u) = t_{\alpha,L}^{\frac{1}{1-\beta}}(u, u) + q_{D_L}^{\frac{1}{1-\beta}}(u, u). \quad (\text{IV.5.18})$$

We have the obvious inequality

$$t_{\alpha,L}^{\frac{1}{1-\beta}}(u, u) \geq \frac{1}{(1-\beta)^2} E_1(T_{\alpha, \frac{L}{1-\beta}}) \|u\|_{L^2(U_{\alpha,L})}^2.$$


 Figure IV.7 – Partition of D_L .

By Proposition IV.5.2, for any $\epsilon > 0$, there exists $L_0 > 0$ such that $E_1(T_{\alpha, \frac{L_0}{1-\beta}}) \geq E_1(T_\alpha) - \epsilon$, and then

$$t_{\alpha, L_0}^{\frac{1}{1-\beta}}(u, u) \geq \frac{1}{(1-\beta)^2} \left(-\frac{1}{\sin^2(\alpha)} - \epsilon \right) \|u\|_{L^2(U_{\alpha, L_0})}^2, \quad (\text{IV.5.19})$$

as $E_1(T_\alpha) = -\frac{1}{\sin^2 \alpha}$. We now focus on the lower bound of $E_1(Q_{D_L}^{\frac{1}{1-\beta}})$ and to this purpose we introduce a partition of the domain D_L . Let V_α be the infinite sector obtained by translation of vector $(\frac{L}{\cos \alpha}, 0)$ of U_α and $V_{\alpha, R} := V_\alpha \cap D_L$. Consider the domains, for $j \in \{+, -\}$,

$$\mathcal{C}_R^j := (D_L \setminus \overline{V_{\alpha, R}}) \times (\mathbb{R}_+ \times \mathbb{R}_j).$$

We obtain a partition of D_L , see Figure IV.7,

$$D_L = \overline{\mathcal{C}_R^+ \cup \mathcal{C}_R^- \cup V_{\alpha, R}}.$$

Define the associated sesquilinear forms

$$q_R^\pm(u, u) = \int_{\mathcal{C}_R^\pm} |\nabla u|^2 dx - \frac{1}{1-\beta} \int_{\partial \mathcal{C}_R^\pm \cap \partial U_\alpha} |u|^2 ds, \quad D(q_R^\pm) := H^1(\mathcal{C}_R^\pm),$$

and

$$q_{V_{\alpha, R}}(u, u) = \int_{V_{\alpha, R}} |\nabla u|^2 dx, \quad D(q_{V_{\alpha, R}}) := H^1(V_{\alpha, R}).$$

By the min-max principle we immediately get

$$E_1(Q_{D_L}^{\frac{1}{1-\beta}}) \geq E_1(Q_R^+ \oplus Q_R^- \oplus Q_{V_{\alpha, R}}).$$

Remark that the operator $Q_{V_{\alpha, R}}$ is non-negative and that, by symmetry, Q_R^+ and Q_R^- are unitarily equivalent. Moreover, by separation of variables one can write

$$E_1(Q_R^+) = E_1(\mathcal{N}_{L \tan \alpha, \frac{1}{1-\beta}} \otimes 1 + 1 \otimes T_R^N),$$

where the operator T_R^N is the Neumann Laplacian acting on $L^2(R-L)$ as $f \mapsto -f''$ with $-f'(0) = f'(R-L) = 0$ and $\mathcal{N}_{L \tan \alpha, \frac{1}{1-\beta}}$ is defined in Lemma III.5.2. By the asymptotics (III.5.2), for any $\epsilon > 0$ there exists $L_1 > L_0$ such that $E_1(\mathcal{N}_{L_1 \tan \alpha, \frac{1}{1-\beta}}) \geq \frac{1}{(1-\beta)^2}(-1-\epsilon)$ and thus

$$E_1(Q_{D_{L_1}}^{\frac{1}{1-\beta}}) \geq E_1(Q_R^+) \geq \frac{1}{(1-\beta)^2}(-1-\epsilon).$$

Gathering this inequality and (IV.5.19) we obtain by (IV.5.18) for R large enough such that $R > L_1$,

$$t_{\alpha, R}^{\frac{1}{1-\beta}}(u, u) \geq \frac{1}{(1-\beta)^2} \left(-\frac{1}{\sin^2 \alpha} - \epsilon \right) \|u\|_{L^2(U_{\alpha, L_1})}^2 + \frac{1}{(1-\beta)^2}(-1-\epsilon) \|u\|_{L^2(D_{L_1})}^2.$$

To finish the proof of the estimate (IV.5.15), we apply the previous inequality to φe^ϕ and we use (IV.5.16) and (IV.5.17) to get

$$\begin{aligned} \int_{U_{\alpha, R}} \left(|\nabla \phi|^2 + E \right) |\varphi e^\phi|^2 dx &\geq \beta \|\nabla(\varphi e^\phi)\|_{L^2(U_{\alpha, R})}^2 \\ &+ \frac{1}{1-\beta} \left(\left(-\frac{1}{\sin^2 \alpha} - \epsilon \right) \|\varphi e^\phi\|_{L^2(U_{\alpha, L_1})}^2 + (-1-\epsilon) \|\varphi e^\phi\|_{L^2(D_{L_1})}^2 \right). \end{aligned}$$

As $|\nabla \phi|^2 \leq \eta^2$, we can rewrite it as follows

$$A_1 \|\varphi e^\phi\|_{L^2(U_{\alpha, L_1})}^2 \geq \beta \|\nabla(\varphi e^\phi)\|_{L^2(U_{\alpha, R})}^2 + A_2 \|\varphi e^\phi\|_{L^2(D_{L_1})}^2,$$

with

$$\begin{aligned} A_1 &:= \eta^2 + E + \frac{1}{1-\beta} \left(\frac{1}{\sin^2 \alpha} + \epsilon \right), \\ A_2 &:= -\eta^2 - E + \frac{1}{1-\beta} (-1-\epsilon). \end{aligned}$$

Recall that $E \geq E_1(T_{\alpha, R}) \geq E_1(T_\alpha) - \epsilon$ for large R . Thus it is easy to see that $A_1 > 0$. On the other hand, $E < -1$ and rewriting A_2 as follows

$$A_2 = -\eta^2 - E - 1 - \frac{\beta + \epsilon}{1-\beta},$$

one can see that for small β , there exists $\eta < \sqrt{-E-1}$ such that $A_2 > 0$. With these choices of η and β , there exists $C > 0$ satisfying

$$C \|\varphi e^\phi\|_{L^2(U_{\alpha, L_1})}^2 \geq \|\nabla(\varphi e^\phi)\|_{L^2(U_{\alpha, R})}^2 + \|\varphi e^\phi\|_{L^2(D_{L_1})}^2.$$

For $x \in U_{\alpha, L_1}$, we have $|\phi(x)| \leq \eta \frac{L_1}{\cos \alpha}$, and thus

$$2C e^{\eta \frac{L_1}{\cos \alpha}} \|\varphi\|_{L^2(U_{\alpha, R})}^2 \geq \|\varphi e^\phi\|_{H^1(U_{\alpha, R})}^2, \text{ for large } R,$$

which finishes the proof of (IV.5.15). In order to finish the proof of the proposition, we write

$$\|\nabla(\varphi e^\phi)\|_{L^2(U_{\alpha, R})}^2 = \int_{U_{\alpha, R}} \left(|\nabla \varphi|^2 e^{2\phi} + |\nabla \phi|^2 |\varphi e^\phi|^2 + 2\varphi e^{2\phi} \nabla \varphi \nabla \phi \right) dx.$$

Using the estimate

$$2 \left| \int_{U_{\alpha, R}} \varphi e^{2\phi} \nabla \varphi \nabla \phi dx \right| \leq 2 \int_{U_{\alpha, R}} |\nabla \phi|^2 |\varphi e^\phi|^2 dx + \frac{1}{2} \int_{U_{\alpha, R}} |\nabla \varphi|^2 e^{2\phi} dx,$$

we obtain

$$\|\nabla(\varphi e^\phi)\|_{L^2(U_{\alpha,R})}^2 \geq \frac{1}{2} \int_{U_{\alpha,R}} |\nabla\varphi|^2 e^{2\phi} dx - \int_{U_{\alpha,R}} |\nabla\phi|^2 |\varphi e^\phi|^2 dx.$$

Thus,

$$\int_{U_{\alpha,R}} (|\nabla\varphi|^2 + |\varphi|^2) e^{2\phi} dx \leq 2\|\varphi e^\phi\|_{H^1(U_{\alpha,R})}^2 + 2 \int_{U_{\alpha,R}} |\nabla\phi|^2 |\varphi e^\phi|^2 dx,$$

and we can finish the proof thanks to $|\nabla\phi|^2 \leq \eta^2$ and applying (IV.5.15). □

Appendix A

Technical proofs of Chapter IV

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A.1 Proof of Lemma IV.1.2

By standard arguments, $C_c^\infty(\mathbb{R}^2)$ is dense in $H^1(U_\alpha)$. In order to prove Lemma IV.1.2 it is then sufficient to show that any function $v \in C_c^\infty(\mathbb{R}^2)$ can be approximated by the functions from \mathcal{F} in the norm of $H^1(U_\alpha)$.

Let $v \in C_c^\infty(\mathbb{R}^2)$. Pick a smooth function $\psi \in C^\infty(\mathbb{R}_+)$ with

$$\psi(s) = 1 \quad \text{if } s \leq \frac{1}{2}, \quad \psi(s) = 0 \quad \text{if } s \geq 1, \quad 0 \leq \psi \leq 1,$$

and set, for small $\epsilon > 0$,

$$\chi_\epsilon(x) = \psi\left(\left|\frac{\ln|x|}{\ln \epsilon}\right|\right).$$

Finally, set $v_\epsilon(x) = \chi_\epsilon(x)v(x)$.

Notice that χ_ϵ is radial and then there exists $\varphi_\epsilon \in C_c^\infty(\mathbb{R}_+)$ such that $\chi_\epsilon(x) = \varphi_\epsilon(|x|)$. In addition, φ_ϵ satisfies

$$\varphi_\epsilon(r) = 1 \quad \text{if } r \in \left(\sqrt{\epsilon}, \frac{1}{\sqrt{\epsilon}}\right), \quad \varphi_\epsilon(r) = 0 \quad \text{if } r \leq \epsilon \quad \text{or} \quad r \geq \frac{1}{\epsilon},$$

and $v_\epsilon \in \mathcal{F}$. Let us show that $\|v - v_\epsilon\|_{H^1(U_\alpha)} \rightarrow 0$ as $\epsilon \rightarrow 0$.

Denote $u(r, \theta) := v(r \cos \theta, r \sin \theta)$ and $u_\epsilon(r, \theta) := v_\epsilon(r \cos \theta, r \sin \theta)$, then

$$\begin{aligned} \|v - v_\epsilon\|_{L^2(U_\alpha)}^2 &= \int_{-\alpha}^{\alpha} \int_0^{+\infty} |u(r, \theta) - u_\epsilon(r, \theta)|^2 r dr d\theta \\ &= \int_{-\alpha}^{\alpha} \int_0^{\sqrt{\epsilon}} |u|^2 |1 - \varphi_\epsilon|^2 r dr d\theta + \int_{-\alpha}^{\alpha} \int_{\frac{1}{\sqrt{\epsilon}}}^{+\infty} |u|^2 |1 - \varphi_\epsilon|^2 r dr d\theta, \end{aligned}$$

and the right-hand side tends to 0 as ϵ is small due to the dominated convergence. Furthermore,

$$\begin{aligned} \|\nabla v - \nabla v_\epsilon\|_{L^2(U_\alpha)}^2 &\leq 2 \int_{-\alpha}^{\alpha} \int_0^{+\infty} |\nabla v(r \cos \theta, r \sin \theta)|^2 |1 - \varphi_\epsilon(r)|^2 r dr d\theta \\ &\quad + 2 \int_{-\alpha}^{\alpha} \int_0^{+\infty} |v(r \cos \theta, r \sin \theta)|^2 |\nabla \chi_\epsilon(r \cos \theta, r \sin \theta)|^2 r dr d\theta, \end{aligned}$$

and the first term tends to 0 by the dominated convergence. On the other hand,

$$|\nabla \chi_\epsilon(r \cos \theta, r \sin \theta)|^2 = |\varphi'_\epsilon(r)|^2 = \frac{1}{r^2 |\ln \epsilon|^2} \left| \psi' \left(\left| \frac{\ln r}{\ln \epsilon} \right| \right) \right|^2,$$

and

$$\begin{aligned} \int_{-\alpha}^{\alpha} \int_0^{+\infty} |v(r \cos \theta, r \sin \theta)|^2 |\nabla \chi_\epsilon(r \cos \theta, r \sin \theta)|^2 r dr d\theta &= \\ \int_{-\alpha}^{\alpha} \int_\epsilon^{\sqrt{\epsilon}} |u(r, \theta)|^2 |\varphi'_\epsilon(r)|^2 r dr d\theta &+ \int_{-\alpha}^{\alpha} \int_{\frac{1}{\sqrt{\epsilon}}}^{\frac{1}{\epsilon}} |u(r, \theta)|^2 |\varphi'_\epsilon(r)|^2 r dr d\theta. \end{aligned}$$

The functions u and ψ' are bounded, and we can get the following upper bound:

$$\int_{-\alpha}^{\alpha} \int_0^{+\infty} |v(r \cos \theta, r \sin \theta)|^2 |\nabla \chi_\epsilon(r \cos \theta, r \sin \theta)|^2 r dr d\theta \leq \alpha \frac{\|u\|_\infty^2 \|\chi'\|_\infty^2}{|\ln \epsilon|} \xrightarrow{\epsilon \rightarrow 0} 0,$$

which concludes the proof.

A.2 Study of the operator H_a^∞

We consider the operator acting on $L^2(0, +\infty)$ defined by:

$$H_a = -\frac{d^2}{dr^2} - \frac{1}{4r^2} - \frac{1}{ar}, \quad D(H_a) = C_c^\infty(\mathbb{R}_+).$$

and the associated sesquilinear form h_a ,

$$h_a(u, u) = \int_{\mathbb{R}_+} \left(|u'(r)|^2 dr - \frac{|u(r)|^2}{4r^2} - \frac{|u(r)|^2}{ar} \right) dr, \quad u \in C_c^\infty(\mathbb{R}_+).$$

Some parts of the analysis of the operator T_α^γ are based on the spectral properties of the Friedrichs extension H_a^∞ of H_a . Various parts of the description and of the spectral analysis of H_a^∞ are spread through the literature, and in the present section we give a compact presentation of the necessary results.

A.2.1 The adjoint of H_a

Let H_a^* be the adjoint of H_a . Recall the deficiency subspaces of H_a are defined by

$$\mathcal{K}_\pm = \text{Ker}(H_a^* \mp i) = \text{Ran}(H_a \pm i)^\perp,$$

and their dimensions $n_\pm = \dim(\mathcal{K}_\pm)$ are called the deficiency indices of H_a . The following proposition gives us the existence of self-adjoint extensions of H_a .

Proposition A.2.1. *The operator H_a^* is given by the same differential expression as H_a and acts on the domain*

$$D(H_a^*) = \{u \in L^2(\mathbb{R}_+) : u \in H^2(\epsilon, +\infty) \text{ for any } \epsilon > 0 \text{ and } H_a^*u \in L^2(\mathbb{R}_+)\}. \quad (\text{A.2.1})$$

The deficiency indices of H_a are equal to 1, and H_a admits self-adjoint extensions.

Proof. Denote by \mathcal{D} the set on the right-hand side of (A.2.1). Let $v \in D(H^*)$, then for all $u \in D(H_a)$ one has

$$\langle H_a u, v \rangle_{L^2(\mathbb{R}_+)} = \int_{\mathbb{R}_+} \overline{\left(-u''(r) - \frac{1}{4r^2}u(r) - \frac{1}{ar}u(r)\right)} v(r) dr = \langle u, H_a^* v \rangle_{L^2(\mathbb{R}_+)},$$

and

$$H_a^* v(r) = -v''(r) - \frac{1}{4r^2}v(r) - \frac{1}{ar}v(r) \quad \text{in } \mathcal{D}'(\mathbb{R}_+).$$

In particular, $D(H_a^*) \subset \{v \in L^2(\mathbb{R}_+), H_a^*v \in L^2(\mathbb{R}_+)\}$. Let $u \in L^2(\mathbb{R}_+)$ satisfy $H_a^*u \in L^2(\mathbb{R}_+)$, then, for all $\epsilon > 0$, we have $u/r^2 \in L^2(\epsilon, +\infty)$ and $u/r \in L^2(\epsilon, +\infty)$. Due to

$$-u''(r) = H_a^*u(r) + \frac{1}{4r^2}u(r) + \frac{1}{a}u(r) \in L^2(\epsilon, +\infty),$$

we also have $u \in H^2(\epsilon, +\infty)$ for all $\epsilon > 0$. This shows the inclusion $D(H_a^*) \subset \mathcal{D}$.

Let us prove the reverse inclusion. Let $v \in \mathcal{D}$. After an integration by parts we have, for all $\epsilon > 0$ and for all $u \in D(H_a)$,

$$\begin{aligned} \langle H_a u, v \rangle_{L^2(\mathbb{R}_+)} &= \int_0^\epsilon \overline{H_a u(r)} v(r) dr + \int_\epsilon^{+\infty} \overline{u(r)} \left(-v''(r) - \frac{1}{4r^2}v(r) - \frac{1}{ar}v(r)\right) dr \\ &\quad + \overline{u'(\epsilon)}v(\epsilon) - \overline{u(\epsilon)}v'(\epsilon). \end{aligned}$$

Notice that the boundary terms are well defined as u, v, u', v' are continuous. Moreover, as $u \in C_c^\infty(\mathbb{R}_+)$ we have

$$\lim_{\epsilon \rightarrow 0} \left(\overline{u'(\epsilon)}v(\epsilon) - \overline{u(\epsilon)}v'(\epsilon)\right) = 0.$$

Hence, in the limit $\epsilon \rightarrow 0$ we obtain

$$\langle H_a u, v \rangle_{L^2(\mathbb{R}_+)} = \int_{\mathbb{R}_+} \overline{u(r)} \left(-v''(r) - \frac{1}{4r^2}v(r) - \frac{1}{ar}v(r)\right) dr,$$

for all $u \in D(H_a)$. Then $\mathcal{D} \subset D(H_a^*)$ and, finally, $D(H_a^*) = \mathcal{D}$.

Recall that a symmetric operator admits self-adjoint extensions if and only if his deficiency indices are equal. In our case, H_a^* commutes with the complex conjugation, which gives immediately $n_+ = n_-$, as \mathcal{K}_+ and \mathcal{K}_- are mutually complex conjugate.

It remains to determine the deficiency indices. The functions $u \in \text{Ker}(H_a^* + i)$ are the solutions to the differential equation

$$-u'' - \frac{1}{4r^2}u - \frac{1}{ar}u + iu = 0.$$

Represent $u(r) = w(2e^{i\frac{\pi}{4}}r)$, then the function w is a solution to the Whittaker's equation,

$$w''(y) + \left(\frac{1}{4y^2} + \frac{1}{2ae^{i\frac{\pi}{4}}y} - \frac{1}{4}\right)w(y) = 0.$$

whose linearly independent solutions are expressed in terms on the confluent hypergeometric functions, and the space of L^2 solutions is one-dimensional, see [AS64, Eq. 13.1.31]. \square

A.2.2 Self-adjoint extensions of H_a

To describe the self-adjoint extensions of the operator H_a we use the boundary triple approach, see e.g. [BGP08, GG91]. The integration by parts gives

$$\begin{aligned} \langle H_a^* \phi, \psi \rangle_{L^2(\mathbb{R}_+)} - \langle \phi, H_a^* \psi \rangle_{L^2(\mathbb{R}_+)} &= \delta(\phi, \psi), \quad \psi, \phi \in D(H_a^*), \\ \text{with } \delta(\phi, \psi) &= \lim_{r \rightarrow 0^+} \left(\overline{\phi'(r)} \psi(r) - \overline{\phi(r)} \psi'(r) \right). \end{aligned}$$

The self-adjoint extensions of H_a are restrictions of H_a^* to maximal subspaces D satisfying $D(H_a) \subset D \subset D(H_a^*)$ with the property that $\delta(\phi, \psi) = 0$ for all $\phi, \psi \in D$. Notice that $\delta(\phi, \psi)$ is finite but $\lim_{x \rightarrow 0} \phi(x)$ or $\lim_{x \rightarrow 0} \phi'(x)$ could be infinite.

As shown in e.g. [GTV12, page 301], based on the asymptotic behavior of the confluent hypergeometric functions, there exists linear maps $a_1, a_2 : D(H_a^*) \rightarrow \mathbb{C}$ such that for any $\psi \in D(H_a^*)$ one has, as $x \rightarrow 0$,

$$\begin{aligned} \psi(r) &= a_1(\psi) \sqrt{r} + a_2(\psi) \sqrt{r} \ln r + O(r^{\frac{3}{2}} \ln r), \\ \psi'(r) &= \frac{a_1(r)}{2\sqrt{r}} + \frac{a_2(\psi)}{\sqrt{r}} \left(\frac{\ln r}{2} + 1 \right) + O(\sqrt{r} \ln r), \end{aligned}$$

or, equivalently,

$$a_2(\psi) = \lim_{r \rightarrow 0^+} \frac{\psi(r)}{\sqrt{r} \ln r}, \quad a_1(\psi) = \lim_{r \rightarrow 0^+} \frac{\psi(r) - a_2(\psi) \sqrt{r} \ln r}{\sqrt{r}},$$

and a direct computation gives the equality $\delta(\phi, \psi) = \overline{a_2(\phi)} a_1(\psi) - \overline{a_1(\phi)} a_2(\psi)$. Furthermore, for any $(b_1, b_2) \in \mathbb{C}^2$ there is $\psi \in D(H_a^*)$ such that $a_1(\psi) = b_1$ and $a_2(\psi) = b_2$. In the language of [BGP08], the triple (\mathbb{C}, a_1, a_2) is a boundary triple for H_a , and any self-adjoint extension of H_a is a restriction of H_a^* to the functions ψ satisfying the boundary condition $a_1(\psi) \cos \vartheta = a_2(\psi) \sin \vartheta$, where ϑ is a real-valued parameter.

By [BG85, Theorem 3.1], the Friedrichs extension H_a^∞ corresponds to the boundary condition $a_2(\psi) = 0$, i.e. to $\vartheta = \frac{\pi}{2}$. The spectral properties for this case are completely analyzed in [GTV12, Subsection 8.3.3]: the essential spectrum is $[0, +\infty)$, and the discrete spectrum consists of the simple eigenvalues $\mathcal{E}_n(a)$ with the associated eigenfunctions ψ_n given by

$$\mathcal{E}_n(a) = -\frac{1}{a^2} \frac{1}{(2n-1)^2}, \quad \psi_n(r) = \sqrt{r} \exp\left(-\frac{r}{(2n-1)a}\right) L_{n-1}\left(\frac{2r}{(2n-1)a}\right), \quad n \in \mathbb{N},$$

where L_m are the Laguerre polynomials [AS64, Section 22.1].

Chapter V

Robin Laplacians on polygons

Abstract

Let $\Omega \subset \mathbb{R}^2$ be a simply connected polygon with straight edges. In this chapter we study the Robin Laplacian Q_Ω^γ acting on $L^2(\Omega)$ as $Q_\Omega^\gamma u = -\Delta u$ with the Robin boundary condition $\partial_\nu u = \gamma u$ on $\partial\Omega$, where $\gamma > 0$ and ν is the *outward* unit normal. This operator has a compact resolvent and thus its spectrum is purely discrete: we denote by $E_n(Q_\Omega^\gamma)$ its eigenvalues enumerated in the non-decreasing order and counting the multiplicities. We are interested in the behavior of the eigenvalues of Q_Ω^γ as γ becomes large. In Section V.1 we introduce some useful notation and the test functions, called the quasi-modes, used in the proof of this chapter. In Section V.2, we prove that there exists $N_\Omega \in \mathbb{N}$ such that the N_Ω first eigenvalues of Q_Ω^γ are exponentially close to those of model operators associated with the vertices: the Robin Laplacians acting on the tangent sectors associated with $\partial\Omega$. In Section V.3, we prove that the associated eigenfunctions are concentrated near the convex vertices of Ω and that they are exponentially close, in a certain sense, to the quasi-modes. Finally in Section V.4, we obtain a second order asymptotics for the further eigenvalues $E_{N_\Omega+j}(Q_\Omega^\gamma)$ when the polygon Ω admits only *non-resonant* or concave corners.

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V.1 Preliminaries

V.1.1 Notation

In this section we recall and introduce some important notation which will be used in this chapter.

Let Ω be a simply connected polygon with straight edges. As there is no ambiguity, we denote

$$Q^\gamma := Q_\Omega^\gamma$$

in the rest of this chapter. For a vertex $v \in \partial\Omega$, denote by α_v the half aperture of Ω at v measured *inside* Ω . Recall that the set of *convex* vertices of Ω denoted by \mathcal{V} was introduced in Section II.2.2,

$$\mathcal{V} := \{v \in \partial\Omega \text{ is a vertex and } \alpha_v \in (0, \frac{\pi}{2})\}.$$

For each $v \in \mathcal{V}$, there exists \tilde{U}_{α_v} an infinite sector of half aperture α_v and of vertex v such that, for $r > 0$ small enough,

$$\Omega \cap B(v, r) = \tilde{U}_{\alpha_v} \cap B(v, r),$$

and there exists F_v a rotation composed by a translation satisfying

$$U_{\alpha_v} = \{F_v(x), x \in \tilde{U}_{\alpha_v}\}.$$

Recall that, as U_{α_v} is invariant by dilations, the operator $T_{\alpha_v}^\gamma$ is unitarily equivalent to $\gamma^2 T_{\alpha_v}^1$. In the following, we keep the same notation as in Chapter IV and denote by

$$T_{\alpha_v} := T_{\alpha_v}^1.$$

The essential spectrum of T_{α_v} is equal to $[-1, +\infty)$, see Theorem IV.1.1, and its discrete spectrum is finite as shown in Theorem IV.2.1. The number of discrete eigenvalues of T_{α_v} is denoted by

$$\mathcal{N}_v := \mathcal{N}(T_{\alpha_v}, -1) < +\infty.$$

For $v \in \mathcal{V}$, we denote by $(\psi_n^v)_{n \leq \mathcal{N}_v}$ the orthonormalized eigenfunctions of T_{α_v} associated with the \mathcal{N}_v discrete eigenvalues. Then, $(\psi_n^{\gamma, v})_{n \leq \mathcal{N}_v}$ defined as

$$\psi_n^{\gamma, v}(x) := \gamma \psi_n^v(\gamma x), \tag{V.1.1}$$

are the eigenfunctions of $T_{\alpha_v}^\gamma$ associated with the \mathcal{N}_v discrete eigenvalues satisfying

$$\|\psi_n^{\gamma, v}\|_{L^2(U_{\alpha_v})} = 1, \quad \|\nabla \psi_n^{\gamma, v}\|_{L^2(U_{\alpha_v})} = \gamma \|\nabla \psi_n^v\|_{L^2(U_{\alpha_v})}, \quad \|\psi_n^{\gamma, v}\|_{L^2(\partial U_{\alpha_v})} = \sqrt{\gamma} \|\psi_n^v\|_{L^2(\partial U_{\alpha_v})}.$$

The model operator T^\oplus is defined as the direct sum of Robin Laplacians acting on tangent sectors associated with the convex vertices of $\partial\Omega$,

$$T^\oplus := \bigoplus_{v \in \mathcal{V}} T_{\alpha_v},$$

acting on $\bigoplus_{v \in \mathcal{V}} L^2(U_{\alpha_v})$. Then $\text{spec}(T^\oplus) = \bigcup_{v \in \mathcal{V}} \text{spec}(T_{\alpha_v})$. We denote by

$$\mathcal{N}^\oplus := \sum_{v \in \mathcal{V}} \mathcal{N}_v, \text{ and } \Lambda^\oplus := \{\lambda_l, 1 \leq l \leq K^\oplus\},$$

the eigenvalues of T^\oplus ordered in the increasing way and counted *without* multiplicity, namely : $\lambda_1 < \lambda_2 < \dots < \lambda_{K^\oplus}$. For $1 \leq l \leq K^\oplus$ we introduce

$$\mathcal{S}_l := \{(n, v) : v \in \mathcal{V}, 1 \leq n \leq \mathcal{N}_v : E_n(T_{\alpha_v}) = \lambda_l\}, \text{ and } m_l := \#\mathcal{S}_l.$$

Defined in this way, m_l is then the multiplicity of λ_l as an eigenvalue of T^\oplus and $\sum_{l=1}^{K^\oplus} m_l = \mathcal{N}^\oplus$. Finally we denote by $E^{\max} := E_{\mathcal{N}^\oplus}(T^\oplus)$.

V.1.2 Construction of quasi-modes

Let $v \in \mathcal{V}$. For $n \in \{1, \dots, \mathcal{N}_v\}$ we set

$$\phi_n^{\gamma,v} := \psi_n^{\gamma,v} \circ F_v.$$

Let us introduce

$$\rho_v := \frac{\text{dist}(v, \mathcal{V} \setminus \{v\})}{2}, \text{ and } \rho := \frac{\min_{v \in \mathcal{V}} \rho_v}{2}.$$

Let $\varphi \in C^\infty(\mathbb{R}_+)$ be a smooth cut-off function satisfying $0 \leq \varphi \leq 1$, $\varphi(t) = 1$ if $0 \leq t \leq 1$, and $\varphi(t) = 0$ if $t \geq 2$. We introduce the smooth radial cut-off function χ_v defined as follows:

$$\chi_v(x) = \varphi\left(\frac{|x-v|}{\rho}\right), \quad x \in \Omega. \quad (\text{V.1.2})$$

Notice that, for $v \neq v'$, $\text{supp } \chi_v \cap \text{supp } \chi_{v'} = \emptyset$. Finally we set, for $1 \leq n \leq \mathcal{N}_v$,

$$\tilde{\phi}_n^{\gamma,v} := \phi_n^{\gamma,v} \chi_v \text{ on } \Omega.$$

Proposition V.1.1. *For any $\epsilon \in (0, 1)$, there exists $C > 0$ such that for $v \in \mathcal{V}$, $n \in \{1, \dots, \mathcal{N}_v\}$ and $\gamma > 0$ we have $\tilde{\phi}_n^{\gamma,v} \in D(Q^\gamma)$, and*

$$1 - Ce^{-2\gamma(1-\epsilon)\sqrt{-1-E^{\max}}\rho} \leq \|\tilde{\phi}_n^{\gamma,v}\|_{L^2(\Omega)}^2 \leq 1, \quad (\text{V.1.3})$$

$$\frac{\|Q^\gamma \tilde{\phi}_n^{\gamma,v} - \gamma^2 E_n(T_{\alpha_v}) \tilde{\phi}_n^{\gamma,v}\|_{L^2(\Omega)}^2}{\|\tilde{\phi}_n^{\gamma,v}\|_{L^2(\Omega)}^2} \leq Ce^{-2\gamma(1-\epsilon)\sqrt{-1-E^{\max}}\rho}. \quad (\text{V.1.4})$$

Proof. The main tool is the property of localization of the eigenfunctions $\psi_n^{\gamma,v}$ near the vertex v , proved in Section IV.4.

We start by proving (V.1.3). We immediately see that $\|\tilde{\phi}_n^{\gamma,v}\|_{L^2(\Omega)}^2 \leq \|\psi_n^{\gamma,v}\|_{L^2(U_{\alpha_v})}^2 = 1$. On the other hand, we have

$$\|\tilde{\phi}_n^{\gamma,v}\|_{L^2(\Omega)}^2 \geq \int_{\Omega \cap B(v, \rho)} |\phi_n^{\gamma,v}|^2 dx = \int_{U_{\alpha_v}} |\psi_n^{\gamma,v}|^2 dx - \int_{U_{\alpha_v} \setminus \overline{B(0, \rho)}} |\psi_n^{\gamma,v}|^2 dx.$$

We now can apply Theorem IV.4.1 to $\psi_n^{\gamma,v}$ to get:

$$\int_{U_{\alpha_v} \setminus \overline{B(0, \rho)}} |\psi_n^{\gamma,v}|^2 dx \leq Ce^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho},$$

which gives us the lower bound for $\|\tilde{\phi}_n^{\gamma,v}\|_{L^2(\Omega)}^2$ and concludes the proof of (V.1.3).

To prove that $\tilde{\phi}_n^{\gamma,v} \in D(Q^\gamma)$ we have to show that $-\Delta \tilde{\phi}_n^{\gamma,v} \in L^2(\Omega)$, which is easily checked as χ_v is smooth, and that $\partial_\nu \tilde{\phi}_n^{\gamma,v} = \gamma \tilde{\phi}_n^{\gamma,v}$ on $\partial\Omega$. As χ_v is radial, $\partial_\nu \chi_v = 0$ on $\partial(\Omega \cap B(v, 2\rho)) \setminus \partial B(v, \rho)$ and then $\tilde{\phi}_n^{\gamma,v}$ satisfies the Robin boundary condition. Thus we can write

$$Q^\gamma \tilde{\phi}_n^{\gamma,v} = -\Delta(\phi_n^{\gamma,v})\chi_v - \Delta(\chi_v)\phi_n^{\gamma,v} - 2\nabla\phi_n^{\gamma,v}\nabla\chi_v,$$

and,

$$-\Delta\phi_n^{\gamma,v} = \gamma^2 E_n(T_{\alpha_v})\phi_n^{\gamma,v}.$$

Using the fact that $\text{supp } \Delta(\chi_v) \subset \text{supp } \nabla\chi_v \subset B(v, 2\rho) \setminus \overline{B(v, \rho)}$ and Theorem IV.4.1, we obtain

$$\|\Delta(\chi_v)\phi_n^{\gamma,v}\|_{L^2(\Omega)}^2 \leq \|\Delta(\chi_v)\|_\infty^2 Ce^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho},$$

and,

$$\|\nabla\phi_n^{\gamma,v}\nabla\chi_v\|_{L^2(\Omega)}^2 \leq \|\nabla\chi_v\|_\infty^2 Ce^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho}.$$

Gathering the two previous inequalities gives us

$$\|Q_\gamma\tilde{\phi}_n^{\gamma,\alpha_s} - \gamma^2 E_n(T_{\alpha_s})\tilde{\phi}_n^{\gamma,\alpha_s}\|_{L^2(\Omega)}^2 \leq Ce^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho}. \quad (\text{V.1.5})$$

Putting (V.1.3) and (V.1.5) together finishes the proof. \square

Corollary V.1.2. *For any $\epsilon \in (0, 1)$ there exists a constant $C > 0$ such that*

$$\text{dist}(\gamma^2 E_n(T_{\alpha_v}), \text{spec}(Q^\gamma)) \leq Ce^{-(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho}.$$

Proof. This is a consequence of the Theorem III.3.1 due to (V.1.4). \square

V.1.3 Properties of quasi-modes

In order to prove the main results of this chapter we need some properties satisfied by the quasi-modes gathered in the following lemma.

Lemma V.1.3. *Let $\epsilon \in (0, 1)$. There exists a constant $C > 0$ such that, for all $v \in \mathcal{V}$, for all $n \in \{1, \dots, \mathcal{N}_v\}$ and for all $i \neq j$, $(i, j) \in \{1, \dots, \mathcal{N}_v\}^2$ we have for γ large enough,*

$$|q^\gamma(\tilde{\phi}_n^{\gamma,v}, \tilde{\phi}_n^{\gamma,v}) - \gamma^2 E_n(T_{\alpha_v})| \leq \gamma Ce^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho}, \quad (\text{V.1.6})$$

$$|\langle \tilde{\phi}_i^{\gamma,v}, \tilde{\phi}_j^{\gamma,v} \rangle_{L^2(\Omega)}| \leq Ce^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho}, \quad (\text{V.1.7})$$

$$|q^\gamma(\tilde{\phi}_i^{\gamma,v}, \tilde{\phi}_j^{\gamma,v})| \leq \gamma Ce^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho}. \quad (\text{V.1.8})$$

Proof. We start by proving (V.1.6). Let us first expand $q^\gamma(\tilde{\phi}_n^{\gamma,v}, \tilde{\phi}_n^{\gamma,v})$:

$$\begin{aligned} q^\gamma(\tilde{\phi}_n^{\gamma,v}, \tilde{\phi}_n^{\gamma,v}) &= \int_\Omega |\chi_v|^2 |\nabla\phi_n^{\gamma,v}|^2 dx - \gamma \int_{\partial\Omega} |\chi_v|^2 |\phi_n^{\gamma,v}|^2 ds \\ &\quad + \int_\Omega |\nabla\chi_v|^2 |\phi_n^{\gamma,v}|^2 dx + 2\Re \int_\Omega \chi_v \nabla\phi_n^{\gamma,v} \overline{\nabla\chi_v \phi_n^{\gamma,v}} dx. \end{aligned}$$

As $\text{supp } \nabla\chi_v \subset B(v, 2\rho) \setminus \overline{B(v, \rho)}$, we can use Theorem IV.4.1 to bound the cross-term:

$$\begin{aligned} & \left| \int_\Omega |\nabla\chi_v|^2 |\phi_n^{\gamma,v}|^2 dx + 2\Re \int_\Omega \chi_v \nabla\phi_n^{\gamma,v} \overline{\nabla\chi_v \phi_n^{\gamma,v}} dx \right| \\ & \leq \left(2\|\nabla\chi_v\|_\infty^2 \int_{U_{\alpha_v} \setminus \overline{B(v, \rho)}} |\psi_n^{\gamma,v}|^2 dx + \|\chi_v\|_\infty^2 \int_{U_{\alpha_v} \setminus \overline{B(v, \rho)}} |\nabla\psi_n^{\gamma,v}|^2 dx \right) \\ & \leq Ce^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho}. \end{aligned}$$

We now focus on the main term:

$$\int_\Omega |\chi_v|^2 |\nabla\phi_n^{\gamma,v}|^2 dx \leq \int_{U_{\alpha_v}} |\nabla\psi_n^{\gamma,v}|^2 dx, \text{ and } \int_{\partial\Omega} |\chi_v|^2 |\phi_n^{\gamma,v}|^2 ds \leq \int_{\partial U_{\alpha_v}} |\psi_n^{\gamma,v}|^2 ds.$$

On the other hand,

$$\begin{aligned} \int_\Omega |\chi_v|^2 |\nabla\phi_n^{\gamma,v}|^2 dx &\geq \int_{U_{\alpha_v} \cap B(0, \rho)} |\nabla\psi_n^{\gamma,v}|^2 dx \\ &\geq \int_{U_{\alpha_v}} |\nabla\psi_n^{\gamma,v}|^2 dx - Ce^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho}, \end{aligned}$$

and,

$$\begin{aligned} \int_{\partial\Omega} |\chi_v|^2 |\phi_n^{\gamma,v}|^2 ds &\geq \int_{\partial(U_{\alpha_v} \cap B(0,\rho)) \setminus \partial B(0,\rho)} |\psi_n^{\gamma,v}|^2 ds \\ &= \int_{\partial U_{\alpha_v}} |\psi_n^{\gamma,v}|^2 ds - \int_{\partial(U_{\alpha_v} \setminus \overline{B(0,\rho)}) \setminus \partial B(0,\rho)} |\psi_n^{\gamma,v}|^2 ds. \end{aligned}$$

Notice that, as $U_{\alpha_v} \setminus \overline{B(0,\rho)}$ is a Lipschitz domain there exists $K > 0$ such that

$$\|\psi_n^{\gamma,v}\|_{L^2(\partial(U_{\alpha_v} \setminus \overline{B(0,\rho)}))}^2 \leq K \|\psi_n^{\gamma,v}\|_{H^1(U_{\alpha_v} \setminus \overline{B(0,\rho)})}^2. \quad (\text{V.1.9})$$

Then, using (V.1.9) and Theorem IV.4.1 we obtain

$$\int_{\partial\Omega} |\chi_v|^2 |\phi_n^{\gamma,v}|^2 ds \geq \int_{\partial U_{\alpha_v}} |\psi_n^{\gamma,v}|^2 ds - C e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho}.$$

As $t_{\alpha_v}^\gamma(\psi_n^{\gamma,v}, \psi_n^{\gamma,v}) = \gamma^2 E_n(T_{\alpha_v})$,

$$\left| \int_{\Omega} |\chi_v|^2 |\nabla \phi_n^{\gamma,v}|^2 dx - \gamma \int_{\partial\Omega} |\chi_v|^2 |\phi_n^{\gamma,v}|^2 ds - \gamma^2 E_n(T_{\alpha_v}) \right| \leq \gamma C e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho},$$

which concludes the proof combining this estimate with the one on the cross-term.

Let us now prove (V.1.7). As $i \neq j$, $\int_{U_{\alpha_v}} \psi_i^{\gamma,v} \overline{\psi_j^{\gamma,v}} dx = 0$. Then,

$$\langle \tilde{\phi}_i^{\gamma,v}, \tilde{\phi}_j^{\gamma,v} \rangle_{L^2(\Omega)} = \int_{U_{\alpha_v} \setminus \overline{B(0,2\rho)}} (|\chi_v \circ F_v^{-1}|^2 - 1) \psi_i^{\gamma,v} \overline{\psi_j^{\gamma,v}} dx.$$

We can conclude, as $||\chi_v|^2 - 1| \leq 1$, using Cauchy-Schwarz inequality and Theorem IV.4.1.

To finish, let now focus on (V.1.8). Let $i \neq j$, we have

$$q^\gamma(\tilde{\phi}_i^{\gamma,v}, \tilde{\phi}_j^{\gamma,v}) = \int_{\Omega} |\chi_v|^2 \nabla \phi_i^{\gamma,v} \overline{\nabla \phi_j^{\gamma,v}} dx - \gamma \int_{\partial\Omega} |\chi_v|^2 \phi_i^{\gamma,v} \overline{\phi_j^{\gamma,v}} ds + I(\gamma),$$

where

$$I(\gamma) = \int_{\Omega} \left\{ |\nabla \chi_v|^2 \phi_i^{\gamma,v} \overline{\phi_j^{\gamma,v}} + \phi_i^{\gamma,v} \nabla \chi_v \overline{\nabla \phi_j^{\gamma,v}} \chi_v + \chi_v \nabla \phi_i^{\gamma,v} \overline{\nabla \chi_v \phi_j^{\gamma,v}} \right\} dx.$$

Using the fact that $\text{supp } \nabla \chi_v \subset B(v, 2\rho) \setminus \overline{B(v, \rho)}$, Cauchy-Schwarz and Theorem IV.4.1 we get

$$|I(\gamma)| \leq C e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho}. \quad (\text{V.1.10})$$

By the spectral theorem we have $t_{\alpha_v}^\gamma(\psi_i^{\gamma,v}, \psi_j^{\gamma,v}) = 0$, which implies

$$\begin{aligned} &\int_{\Omega} |\chi_v|^2 \nabla \phi_i^{\gamma,v} \overline{\nabla \phi_j^{\gamma,v}} dx - \gamma \int_{\partial\Omega} |\chi_v|^2 \phi_i^{\gamma,v} \overline{\phi_j^{\gamma,v}} ds \\ &= \int_{U_{\alpha_v} \setminus \overline{B(0,2\rho)}} (|\chi_v \circ F_v^{-1}|^2 - 1) \nabla \psi_i^{\gamma,v} \overline{\nabla \psi_j^{\gamma,v}} dx - \gamma \int_{\partial(U_{\alpha_v} \setminus \overline{B(0,2\rho)}) \setminus \partial B(0,2\rho)} (|\chi_v \circ F_v^{-1}|^2 - 1) \psi_i^{\gamma,v} \overline{\psi_j^{\gamma,v}} ds. \end{aligned}$$

We can use the same arguments as before and (V.1.9) to obtain

$$\left| \int_{\Omega} |\chi_v|^2 \nabla \phi_i^{\gamma,v} \overline{\nabla \phi_j^{\gamma,v}} dx - \gamma \int_{\partial\Omega} |\chi_v|^2 \phi_i^{\gamma,v} \overline{\phi_j^{\gamma,v}} ds \right| \leq \gamma C e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho}. \quad (\text{V.1.11})$$

Putting (V.1.10) and (V.1.11) together finishes the proof of (V.1.8). \square

Lemma V.1.4. *For γ large enough the family $(\tilde{\phi}_n^{\gamma,v})_{(n,v) \in \cup_{l=1}^{K^\oplus} \mathcal{S}_l}$ is linearly independent.*

Proof. Let us denote by G the Gramian matrix associated with $(\tilde{\phi}_n^{\gamma,v})_{(n,v) \in \cup_{l=1}^{K^\oplus} \mathcal{S}_l}$ which entries are $G_{i,j} = \langle \tilde{\phi}_{n_i}^{\gamma,v_i}, \tilde{\phi}_{n_j}^{\gamma,v_j} \rangle$, where $(n_i, v_i), (n_j, v_j) \in \cup_{l=1}^{K^\oplus} \mathcal{S}_l$. First, the diagonal is simply composed of $1 + o(1)$ as $\gamma \rightarrow +\infty$, according to (V.1.3). Secondly, if $(n_i, v_i) \neq (n_j, v_j)$ then $v_i = v_j$ and $n_i \neq n_j$ or $v_i \neq v_j$. In the first case, we already know by (V.1.7) that $G_{i,j} = o(1)$ as $\gamma \rightarrow +\infty$. In the second case, $\text{supp } \chi_{v_i} \cap \text{supp } \chi_{v_j} = \emptyset$, then $G_{i,j} = 0$. Necessarily, $\det(G) = 1 + o(1)$ as $\gamma \rightarrow +\infty$. In particular, $\det(G) \neq 0$ for γ large enough, which gives us the result. \square

V.2 Asymptotic behavior of the corner-induced eigenvalues on polygons

This section is devoted to the proof of the asymptotics of the \mathcal{N}^\oplus first eigenvalues of Q^γ , see Theorem II.2.3, for Ω being a polygon with straight edges.

Theorem V.2.1. *Let $\Omega \subset \mathbb{R}^2$ be a polygon with straight edges. For any $\epsilon \in (0, 1)$ there exists $C > 0$ such that for all $n \in \{1, \dots, \mathcal{N}^\oplus\}$ and for γ large enough,*

$$|E_n(Q^\gamma) - \gamma^2 E_n(T^\oplus)| \leq C e^{-(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}.$$

Proof. The proof of Theorem V.2.1 requires two steps. First, we prove the upper bound and a lower bound for the eigenvalues of Q^γ , using respectively the properties of the quasi-modes and a partition of unity. The remainder obtained in the first lower bound appears to be a constant, and in order to improve it to the exponential one we use a spectral approximation result.

Recall that Λ^\oplus is the set of the eigenvalues of the operator T^\oplus enumerated in the increasing order and counted *without* multiplicity, $K^\oplus := \#\Lambda^\oplus$ and we denote by m_l the multiplicity of $\lambda_l \in \Lambda^\oplus$ as an eigenvalue of T^\oplus , see Section V.1.1.

Proposition V.2.2. *For any $\epsilon \in (0, 1)$ there exist $C > 0$ and $c > 0$ such that, for all $0 \leq l \leq K^\oplus$ and for γ large enough,*

$$E_{m_1+\dots+m_l}(Q^\gamma) \leq \gamma^2 \lambda_l + C \gamma^2 e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}, \quad (\text{V.2.1})$$

$$E_{m_0+\dots+m_{l+1}}(Q^\gamma) \geq \gamma^2 \lambda_{l+1} - c, \quad (\text{V.2.2})$$

with the convention $m_0 = 0$.

Proof. We begin by proving (V.2.1). Let $l \in \{1, \dots, K^\oplus\}$ be fixed. In the sequel we denote by $d := \sum_{j=1}^l m_j$. By the min-max principle:

$$E_d(Q^\gamma) = \inf_{\substack{F \subset D(Q^\gamma) \\ \dim(F)=d}} \sup_{\substack{\psi \in F \\ \psi \neq 0}} \frac{q^\gamma(\psi, \psi)}{\|\psi\|^2}.$$

We introduce

$$\mathcal{F}^\gamma := \text{span}\{\tilde{\phi}_n^{\gamma,v}, (n, v) \in \cup_{j=1}^l \mathcal{S}_j\}.$$

For simplicity we denote by $(\tilde{\phi}_1, \dots, \tilde{\phi}_d)$ the elements of $\{\tilde{\phi}_n^{\gamma, v}, (n, v) \in \cup_{j=1}^l \mathcal{S}_j\}$. By Lemma V.1.4, $\dim(\mathcal{F}^\gamma) = d$ for γ large enough and

$$E_d(Q^\gamma) \leq \sup_{\substack{\psi \in \mathcal{F}^\gamma \\ \psi \neq 0}} \frac{q^\gamma(\psi, \psi)}{\|\psi\|^2} = \sup_{\substack{(c_1, \dots, c_d) \in \mathbb{C}^d \\ (c_1, \dots, c_d) \neq (0, \dots, 0)}} \frac{q^\gamma(\sum_{j=1}^d c_j \tilde{\phi}_j, \sum_{j=1}^d c_j \tilde{\phi}_j)}{\|\sum_{j=1}^d c_j \tilde{\phi}_j\|^2}. \quad (\text{V.2.3})$$

Expanding the numerator we get

$$q^\gamma\left(\sum_{j=1}^d c_j \tilde{\phi}_j, \sum_{j=1}^d c_j \tilde{\phi}_j\right) = \sum_{j=1}^d |c_j|^2 q^\gamma(\tilde{\phi}_j, \tilde{\phi}_j) + 2\Re \sum_{j < k} c_j \bar{c}_k q^\gamma(\tilde{\phi}_j, \tilde{\phi}_k).$$

We can use (V.1.6) and (V.1.8) to obtain

$$\begin{aligned} q^\gamma\left(\sum_{j=1}^d c_j \tilde{\phi}_j, \sum_{j=1}^d c_j \tilde{\phi}_j\right) &\leq \sum_{j=1}^d |c_j|^2 (\gamma^2 \lambda_l + \gamma C e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}) \\ &\quad + 2 \sum_{j < k} |c_j c_k| \gamma C e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}. \end{aligned}$$

As $\sum_{j=1}^d \sum_{k=j+1}^d |c_j c_k| \leq d \sum_{j=1}^d |c_j|^2$, we can write

$$q^\gamma\left(\sum_{j=1}^d c_j \tilde{\phi}_j, \sum_{j=1}^d c_j \tilde{\phi}_j\right) \leq \left(\gamma^2 \lambda_l + \gamma C e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}\right) \sum_{j=1}^d |c_j|^2. \quad (\text{V.2.4})$$

The denominator expands as

$$\left\|\sum_{j=1}^d c_j \tilde{\phi}_j\right\|^2 = \sum_{j=1}^d |c_j|^2 \|\tilde{\phi}_j\|^2 + 2\Re \sum_{j < k} c_j \bar{c}_k \langle \tilde{\phi}_j, \tilde{\phi}_k \rangle.$$

Then, using (V.1.3) and (V.1.7) we have

$$\left| \left\|\sum_{j=1}^d c_j \tilde{\phi}_j\right\|^2 - \sum_{j=1}^d |c_j|^2 \right| \leq C e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}} \left(\sum_{j=1}^l |c_j|^2 \right). \quad (\text{V.2.5})$$

Combining (V.2.4) and (V.2.5) we first get :

$$\begin{aligned} \frac{q^\gamma(\sum_{j=1}^d c_j \tilde{\phi}_j, \sum_{j=1}^d c_j \tilde{\phi}_j)}{\left\|\sum_{j=1}^d c_j \tilde{\phi}_j\right\|^2} &\leq \left(\frac{\gamma^2 \lambda_l}{\left\|\sum_{j=1}^d c_j \tilde{\phi}_j\right\|^2} + \frac{\gamma C e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}}{\left\|\sum_{j=1}^d c_j \tilde{\phi}_j\right\|^2} \right) \sum_{j=1}^d |c_j|^2 \\ &\leq \frac{\gamma^2 \lambda_l}{1 + C e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}} + \frac{\gamma C e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}}{1 - C e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}}. \end{aligned}$$

Recall that there exists $(n, v) \in \mathcal{S}_l$ such that $\gamma^2 \lambda_l = \gamma^2 E_n(T_{\alpha_v})$. Then, $-\gamma^2 \lambda_l \leq -\gamma^2 \min_{v \in \mathcal{V}} E_1(T_{\alpha_v})$ and one has, for γ large enough

$$\frac{q^\gamma(\sum_{j=1}^d c_j \tilde{\phi}_j, \sum_{j=1}^d c_j \tilde{\phi}_j)}{\left\|\sum_{j=1}^d c_j \tilde{\phi}_j\right\|^2} \leq \gamma^2 \lambda_l + C \gamma^2 e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}.$$

This concludes the proof of (V.2.1) thanks to (V.2.3).

We now focus on the lower bound. Here $l \in \{0, \dots, K^\oplus\}$ and $d := \sum_{j=0}^l m_j$. Using the $(\chi_v)_{v \in \mathcal{V}}$ defined in (V.1.2), we introduce $\chi_0 := 1 - \sum_{v \in \mathcal{V}} \chi_v$ and for $v \in \mathcal{V} \cup \{0\}$,

$$\tilde{\chi}_v(x) := \frac{\chi_v(x)}{\sqrt{\sum_{v \in \mathcal{V} \cup \{0\}} \chi_v^2(x)}}, \quad x \in \Omega.$$

Defined in this way, $\tilde{\chi}_v \in C^\infty(\Omega)$ and $\sum_{v \in \mathcal{V} \cup \{0\}} \tilde{\chi}_v^2 = 1$ on Ω . We can apply Lemma III.2.2 to have

$$q^\gamma(\psi, \psi) = \sum_{v \in \mathcal{V} \cup \{0\}} q^\gamma(\psi \tilde{\chi}_v, \psi \tilde{\chi}_v) - \sum_{v \in \mathcal{V} \cup \{0\}} \|\psi \nabla \tilde{\chi}_v\|_{L^2(\Omega)}^2, \quad \psi \in D(q^\gamma). \quad (\text{V.2.6})$$

Let us introduce some notation. Let

$$V(x) := \sum_{v \in \mathcal{V} \cup \{0\}} |\nabla \tilde{\chi}_v(x)|^2, \quad \Omega_0 := \Omega \setminus \bigcup_{v \in \mathcal{V}} \overline{B(v, \rho)},$$

and for $v \in \mathcal{V}$ we denote by

$$\Omega_{v, 2\rho} := \Omega \cap B(v, 2\rho), \quad \Gamma_{v, 2\rho} := \partial\Omega_{v, 2\rho} \setminus \partial B(v, 2\rho).$$

By definition of V , there exists $c > 0$ such that for all $x \in \Omega$,

$$\|V\|_\infty \leq c. \quad (\text{V.2.7})$$

We also introduce

$$q_v^{\gamma, V}(\psi, \psi) = \int_{\Omega_{v, 2\rho}} (|\nabla \psi|^2 - V(x)|\psi|^2) dx - \gamma \int_{\Gamma_{v, 2\rho}} |\psi|^2 ds,$$

where $D(q_v^{\gamma, V}) := \{\psi \in H^1(\Omega_{v, 2\rho}), \psi(x) = 0 \text{ for } x \in \partial\Omega_{v, 2\rho} \setminus \Gamma_{v, 2\rho}\}$ and

$$q_0^{\gamma, V}(\psi, \psi) = \int_{\Omega_0} (|\nabla \psi|^2 - V(x)|\psi|^2) dx - \gamma \int_{\partial\Omega_0 \cap \partial\Omega} |\psi|^2 ds,$$

where $D(q_0^{\gamma, V}) := \{\psi \in H^1(\Omega_0), \psi(x) = 0 \text{ for } x \in \partial\Omega_0 \setminus \partial\Omega\}$. Notice that, if $\psi \in \bigoplus_{v \in \mathcal{V} \cup \{0\}} D(q_v^{\gamma, V})$, then $\psi \in D(q^\gamma)$ and $\|\psi\|_{L^2(\Omega)}^2 = \sum_{v \in \mathcal{V} \cup \{0\}} \|\psi \tilde{\chi}_v\|_{L^2(\Omega)}^2$. By the min-max principle and (V.2.6) we can write for all $n \in \mathbb{N}$,

$$E_n(Q^\gamma) \geq E_n\left(\bigoplus_{v \in \mathcal{V} \cup \{0\}} Q_v^{\gamma, V}\right).$$

By definition, $\Omega_0 \cap \mathcal{V} = \emptyset$. Moreover, Ω_0 does not depend on γ and we can extend it in a smooth way to obtain a domain with a Lipschitz, piecewise smooth boundary without any convex corners, which we call Ω^{reg} . We define

$$q^{\gamma, \text{reg}}(\psi, \psi) = \int_{\Omega^{\text{reg}}} |\nabla \psi|^2 dx - \gamma \int_{\partial\Omega^{\text{reg}}} |\psi|^2 ds, \quad \psi \in H^1(\Omega^{\text{reg}}).$$

By [Pan13, Theorem 1], we know that there exists $C > 0$ such that, for γ large enough,

$$E_1(Q^{\gamma, \text{reg}}) \geq -\gamma^2 - C\gamma.$$

In addition, by the min-max principle and (V.2.7) we also have, for all $n \in \mathbb{N}$, $E_n(Q_0^{\gamma,V}) \geq E_n(Q^{\gamma,\text{reg}}) - c$. Then, for γ large enough,

$$E_1(Q_0^{\gamma,V}) \geq -\gamma^2 - C\gamma - c. \quad (\text{V.2.8})$$

On the other hand, extending $\psi \in D(q_v^{\gamma,V})$ by zero and using the min-max principle and (V.2.7), one can write for all $n \leq \mathcal{N}_v$,

$$E_n(Q_v^{\gamma,V}) \geq \gamma^2 E_n(T_{\alpha_v}) - c.$$

In particular,

$$E_{d+1}(\bigoplus_{v \in \mathcal{V}} Q_v^{\gamma,V}) \geq \gamma^2 E_{d+1}(T^\oplus) - c = \gamma^2 \lambda_{l+1} - c.$$

Combining it with (V.2.8), we finally get $E_{d+1}(Q^\gamma) \geq \gamma^2 \lambda_{l+1} - c$, since $\lambda_{l+1} < -1$. This concludes the proof of (V.2.2). \square

This proposition tells us that the eigenvalues of Q^γ are gathered in clusters. For each $n \in \{1, \dots, \mathcal{N}^\oplus\}$, there exists $l \in \{0, \dots, K^\oplus - 1\}$ such that $m_0 + m_1 + \dots + m_l + 1 \leq n \leq m_0 + \dots + m_{l+1}$. Then, $\lambda_{l+1} = E_n(T^\oplus)$ and

$$E_n(Q^\gamma) \in I_{l+1} := (\gamma^2 \lambda_{l+1} - c, \gamma^2 \lambda_{l+1} + C\gamma^2 e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}),$$

for γ large enough. Notice that, as $\lambda_l < \lambda_{l+1}$ we have $I_l \cap I_{l+1} = \emptyset$ for large γ : the I_l are disjoint sets.

In order to conclude, we can now use the spectral approximation result stated in Section III.3. Let us recall (V.1.4). For all $1 \leq l \leq K^\oplus$, and for all $(n, v) \in \mathcal{S}_l$ there holds,

$$\|(Q^\gamma - \gamma^2 \lambda_l) \tilde{\phi}_n^{\gamma,v}\| \leq C e^{-(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}} \|\tilde{\phi}_n^{\gamma,v}\|.$$

Let $\eta := C e^{-(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}$. Recall that the family $(\tilde{\phi}_n^{\gamma,v})_{(n,v) \in \mathcal{S}_l}$ is linearly independent for large γ by Lemma V.1.4. Denote by $(\beta_i)_{i=1}^{m_l}$ the eigenvalues of the Gramian matrix of $(\tilde{\phi}_n^{\gamma,v})_{(n,v) \in \mathcal{S}_l}$ with $\beta_1 = \beta_{\min}$ and $\beta_{m_l} = \beta_{\max}$. Recall that $P_A(a, b)$ denotes the spectral projection of the operator A on the interval $(a, b) \subset \mathbb{R}$. Then we can apply Corollary III.3.4 to obtain for large γ ,

$$\dim \text{Ran } P_{Q^\gamma}(\gamma^2 \lambda_l - m_l^{3/2} \eta \sqrt{\frac{\beta_{\max}}{\beta_{\min}}}, \gamma^2 \lambda_l + m_l^{3/2} \eta \sqrt{\frac{\beta_{\max}}{\beta_{\min}}}) \geq m_l.$$

Notice that, by (V.1.3) and (V.1.7), we have for all $1 \leq i \leq m_l$,

$$|\beta_i - 1| \leq C e^{-2(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}, \quad (\text{V.2.9})$$

and it follows that $\sqrt{\frac{\beta_{\max}}{\beta_{\min}}} = 1 + o(1)$, as $\gamma \rightarrow +\infty$. Moreover, as $\text{spec}_{\text{ess}}(Q^\gamma) = \emptyset$ the operator Q^γ admits at least m_l eigenvalues in $\tilde{I}_l := (\gamma^2 \lambda_l - m_l^{3/2} \eta \sqrt{\frac{\beta_{\max}}{\beta_{\min}}}, \gamma^2 \lambda_l + m_l^{3/2} \eta \sqrt{\frac{\beta_{\max}}{\beta_{\min}}})$. But, as $\tilde{I}_l \cap I_{l+1} = \emptyset$, $\tilde{I}_l \cap \tilde{I}_{l-1} = \emptyset$ and $\tilde{I}_l \cap I_l \neq \emptyset$, we can conclude by the previous corollary that Q^γ admits exactly m_l eigenvalues in \tilde{I}_l and these eigenvalues correspond to the l th cluster mentioned above, which concludes the proof. \square

V.3 Approximation of eigenspaces

We are now going to prove that the corresponding \mathcal{N}^\oplus first eigenfunctions of Q^γ are, in a sense, exponentially close to linear combinations of the above quasi-modes $(\tilde{\phi}_j^{\gamma,v})$.

Recall that the distance between two closed subspaces E and F of a Hilbert space \mathcal{H} is defined in Section III.4 and is denoted by $\text{dist}(E, F)$, and that $P_A(a, b)$ denotes the spectral projection of a self-adjoint operator A on the interval $(a, b) \subset \mathbb{R}$. We introduce

$$\mathcal{F}_l^\gamma := \text{span}\{\tilde{\phi}_j^{\gamma,v}, (j, v) \in \mathcal{S}_l\},$$

and

$$\mathcal{E}_l^\gamma := \text{Ran } P_{Q^\gamma}(\gamma^2 \lambda_l - C e^{-(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho}, \gamma^2 \lambda_l + C e^{-(1-\epsilon)\gamma\sqrt{-1-E^{\max}}\rho}).$$

We can now state the theorem on the eigenspaces.

Theorem V.3.1. *For any $l \in \{1, \dots, K^\oplus\}$, there exists a constant $c > 0$ such that,*

$$\text{dist}(\mathcal{E}_l^\gamma, \mathcal{F}_l^\gamma) = \mathcal{O}(e^{-c\gamma}), \text{ as } \gamma \text{ is large.}$$

The proof is based on the fact that the \mathcal{N}^\oplus first eigenfunctions of Q^γ are exponentially decreasing outside neighborhoods of the vertices. This is due to an Agmon-type estimate satisfied by the eigenfunctions. The proof of this estimate follows the same steps as the one of Theorem IV.4.1 and Proposition IV.5.4 and it is proved for more general domains, the curvilinear polygons, see Proposition VI.3.4 below. Denote

$$d_v(x) := \text{dist}(x, v), \quad d(x) := \min_{v \in \mathcal{V}} d_v(x).$$

Proposition V.3.2. *Let φ be an eigenfunction of Q^γ associated with $E := E_n(Q^\gamma)$, where $n \leq \mathcal{N}^\oplus$. Then, for any $\theta \in (0, 1)$ there exist $\eta > 0$, $C > 0$ and $c > 0$ satisfying, as γ is large,*

$$\int_{\Omega} (|\nabla \varphi|^2 + |\varphi|^2) e^{\eta\gamma d(x)} dx \leq C e^{c\gamma^\theta} \|\varphi\|_{L^2(\Omega)}^2.$$

Proof of Theorem V.3.1. Recall that $\chi_v(x) := \varphi\left(\frac{|x-v|}{\rho}\right)$. We introduce

$$\chi_{\mathcal{V}}(x) := \sum_{v \in \mathcal{V}} \chi_v(x).$$

Consider the subspace

$$\tilde{\mathcal{E}}_l^\gamma := \chi_{\mathcal{V}} \mathcal{E}_l^\gamma.$$

Using the assertion (2) of Proposition III.4.1, one can write

$$\text{dist}(\mathcal{E}_l^\gamma, \mathcal{F}_l^\gamma) \leq \text{dist}(\mathcal{E}_l^\gamma, \tilde{\mathcal{E}}_l^\gamma) + \text{dist}(\tilde{\mathcal{E}}_l^\gamma, \mathcal{F}_l^\gamma). \quad (\text{V.3.1})$$

Let $u \in \mathcal{E}_l^\gamma$ be normalized and $\chi_0 := 1 - \chi_{\mathcal{V}}$. We can write $u = \chi_0 u + \chi_{\mathcal{V}} u$, and

$$\|u - P_{\tilde{\mathcal{E}}_l^\gamma} u\|_{L^2(\Omega)}^2 \leq 2\|\chi_{\mathcal{V}} u - P_{\tilde{\mathcal{E}}_l^\gamma} u\|_{L^2(\Omega)}^2 + 2\|\chi_0 u\|_{L^2(\Omega)}^2.$$

Remark that for any $x \in \text{supp } \chi_0$, the distance of x to the corners of Ω satisfies $d(x) \geq \rho$. By Proposition V.3.2, there exists $\tilde{c} > 0$ and $\eta > 0$ such that

$$\|\chi_0 u\|_{L^2(\Omega)}^2 \leq C e^{\gamma(\tilde{c}\gamma^{\theta-1} - \eta\rho)},$$

for $\theta \in (0, 1)$. In particular, for γ large enough there exists $c > 0$ such that

$$\|\chi_0 u\|_{L^2(\Omega)}^2 = \mathcal{O}(e^{-c\gamma}).$$

As $P_{\widetilde{\mathcal{E}}_l^\gamma}(\chi_\nu u) = \chi_\nu u$, we obtain

$$\|\chi_\nu u - P_{\widetilde{\mathcal{E}}_l^\gamma} u\|_{L^2(\Omega)}^2 = \|P_{\widetilde{\mathcal{E}}_l^\gamma}(\chi_\nu u - u)\|_{L^2(\Omega)}^2 \leq \|\chi_\nu u - u\|_{L^2(\Omega)}^2 = \mathcal{O}(e^{-c\gamma}),$$

which finally implies

$$\text{dist}(\mathcal{E}_l^\gamma, \widetilde{\mathcal{E}}_l^\gamma) = \mathcal{O}(e^{-c\gamma}).$$

In view of (V.3.1), it remains to prove

$$\text{dist}(\widetilde{\mathcal{E}}_l^\gamma, \mathcal{F}_l^\gamma) = \mathcal{O}(e^{-c\gamma}). \quad (\text{V.3.2})$$

Let $\phi \in \mathcal{E}_l^\gamma$, then

$$\chi_\nu \phi = \sum_{v \in \mathcal{V}} \chi_v \phi,$$

and for each $v \in \mathcal{V}$, $(\chi_v \phi) \circ F_v^{-1} \in L^2(U_{\alpha_v})$, where F_v , defined in Section V.1.1, is a diffeomorphism mapping $\Omega \cap B(v, \rho)$ onto $U_{\alpha_v} \cap B(0, \rho)$. Introduce

$$E_l^\gamma := \left\{ \sum_{v \in \mathcal{V}} (\chi_v \phi) \circ F_v^{-1}, \phi \in \mathcal{E}_l^\gamma \right\} \text{ and } F_l^\gamma := \{ \widetilde{\phi}_n^{\gamma, v} \circ F_v^{-1}, (n, v) \in \mathcal{S}_l \}.$$

Thus, it is easy to see that, by change of variables we have

$$\text{dist}_{L^2(\Omega)}(\widetilde{\mathcal{E}}_l^\gamma, \mathcal{F}_l^\gamma) = \text{dist}_{\oplus_{v \in \mathcal{V}} L^2(U_{\alpha_v})}(E_l^\gamma, F_l^\gamma).$$

Denote by G_l^γ the subspace of $\oplus_{v \in \mathcal{V}} L^2(U_{\alpha_v})$ spanned by the eigenfunctions of $\gamma^2 T^\oplus := \oplus_{v \in \mathcal{V}} T_{\alpha_v}^\gamma$ associated with $\gamma^2 \lambda_l$, i.e.:

$$G_l^\gamma := \text{span}\{\psi_n^{\gamma, v} : (n, v) \in \mathcal{S}_l\}.$$

Using again the assertion (2) of Proposition III.4.1, we can write

$$\text{dist}(E_l^\gamma, F_l^\gamma) \leq \text{dist}(E_l^\gamma, G_l^\gamma) + \text{dist}(G_l^\gamma, F_l^\gamma).$$

We can prove that $\text{dist}(G_l^\gamma, F_l^\gamma)$ is exponentially small. Indeed, let $u \in G_l^\gamma$ be normalized and $\varphi_0 := 1 - \sum_{v \in \mathcal{V}} \chi_v \circ F_v^{-1}$. One can write $u = \varphi_0 u + (\sum_{v \in \mathcal{V}} \chi_v \circ F_v^{-1})u$ and

$$\|u - P_{F_l^\gamma} u\|_{\oplus_{v \in \mathcal{V}} L^2(U_{\alpha_v})}^2 \leq 2 \left\| \left(\sum_{v \in \mathcal{V}} \chi_v \circ F_v^{-1} \right) u - P_{F_l^\gamma} u \right\|_{\oplus_{v \in \mathcal{V}} L^2(U_{\alpha_v})}^2 + 2 \|\varphi_0 u\|_{\oplus_{v \in \mathcal{V}} L^2(U_{\alpha_v})}^2.$$

For any $x \in \text{supp } \varphi_0 \cap U_{\alpha_v}$, the distance of x to the origin satisfies $d(x, 0) \geq \rho$, and then using Theorem IV.4.1 we have

$$\|\varphi_0 u\|_{\oplus_{v \in \mathcal{V}} L^2(U_{\alpha_v})}^2 = \mathcal{O}(e^{-c\gamma}), \quad c > 0.$$

As $P_{F_l^\gamma}((\sum_{v \in \mathcal{V}} \chi_v \circ F_v^{-1})u) = (\sum_{v \in \mathcal{V}} \chi_v \circ F_v^{-1})u$, we also have

$$\left\| \left(\sum_{v \in \mathcal{V}} \chi_v \circ F_v^{-1} \right) u - P_{F_l^\gamma} u \right\|_{\oplus_{v \in \mathcal{V}} L^2(U_{\alpha_v})}^2 \leq \left\| \left(\sum_{v \in \mathcal{V}} \chi_v \circ F_v^{-1} \right) u - u \right\|_{\oplus_{v \in \mathcal{V}} L^2(U_{\alpha_v})}^2 = \mathcal{O}(e^{-c\gamma}), \quad c > 0,$$

which finally implies

$$\text{dist}(G_l^\gamma, F_l^\gamma) = \mathcal{O}(e^{-c\gamma}), \text{ for large } \gamma.$$

Let us now focus on $\text{dist}(E_l^\gamma, G_l^\gamma)$. Let $\phi_j \in \mathcal{E}_l^\gamma$ be normalized eigenfunction of Q^γ associated with $E_{n_j}(Q^\gamma)$, where n_j is such that $E_{n_j}(T^\oplus) = \lambda_l$. Denote by

$$\psi_j := \sum_{v \in \mathcal{V}} (\chi_v \phi_j) \circ F_v^{-1} \in E_l^\gamma.$$

As the cut-off functions χ_v are radial we have $\partial_\nu \psi_j = \gamma \psi_j$ on each ∂U_{α_v} and thus $\psi_j \in D(\gamma^2 T^\oplus)$. Moreover, by change of variables

$$\|\gamma^2 T^\oplus \psi_j - E_{n_j}(Q^\gamma) \psi_j\|_{\oplus_{v \in \mathcal{V}} L^2(U_{\alpha_v})}^2 = \|-\Delta(\chi_v \phi_j) - E_{n_j}(Q^\gamma) \chi_v \phi_j\|_{L^2(\Omega)}^2. \quad (\text{V.3.3})$$

By direct computation we have

$$-\Delta(\chi_v \phi_j) = (-\Delta \phi_j) \chi_v + \phi_j (-\Delta \chi_v) - \nabla \phi_j \nabla \chi_v.$$

Notice that $-\Delta \phi_j = E_{n_j}(Q^\gamma)$ on $\text{supp } \chi_v$. In order to estimate on the last terms, remark that $\text{supp } \Delta \chi_v \subset \text{supp } \nabla \chi_v \subset \bigcup_{v \in \mathcal{V}} B(v, 2\rho) \setminus \overline{B(v, \rho)}$ and there exists $C > 0$ such that

$$|\nabla \chi_v|^2 \leq C, \text{ and } |\Delta \chi_v|^2 \leq C.$$

We thus can apply Proposition V.3.2 to obtain

$$\begin{aligned} \|\phi_j \Delta \chi_v\|_{L^2(\Omega)}^2 &= \mathcal{O}(e^{-c\gamma}), \\ \|\nabla \phi_j \nabla \chi_v\|_{L^2(\Omega)}^2 &= \mathcal{O}(e^{-c\gamma}). \end{aligned}$$

Gathering these estimates we obtain

$$\|-\Delta(\chi_v \phi_j) - E_{n_j}(Q^\gamma) \chi_v \phi_j\|_{L^2(\Omega)}^2 = \mathcal{O}(e^{-c\gamma}),$$

and then by (V.3.3)

$$\|\gamma^2 T^\oplus \psi_j - E_{n_j}(Q^\gamma) \psi_j\|_{\oplus_{v \in \mathcal{V}} L^2(U_{\alpha_v})}^2 = \mathcal{O}(e^{-c\gamma}). \quad (\text{V.3.4})$$

Recall that m_l is the multiplicity of λ_l viewed as an eigenvalue of T^\oplus . Let us prove that $(\phi_1 \chi_v, \dots, \phi_{m_l} \chi_v)$ is linearly independent. It is easy to see that

$$\|\phi_j \chi_v\|_{L^2(\Omega)}^2 \leq \|\phi_j\|_{L^2(\Omega)} = 1.$$

Recall that $\rho > 0$ is chosen such that $\text{supp } \chi_v \cap \text{supp } \chi_{v'} = \emptyset$ if $v \neq v'$. Thus, $\|\phi_j \chi_v\|_{L^2(\Omega)}^2 = \sum_{v \in \mathcal{V}} \|\phi_j \chi_v\|_{L^2(\Omega)}^2$. Moreover,

$$\sum_{v \in \mathcal{V}} \int_{\Omega} |\phi_j \chi_v|^2 dx \geq \sum_{v \in \mathcal{V}} \int_{\Omega \cap B(v, \rho)} |\phi_j|^2 dx = \int_{\Omega} |\phi_j|^2 dx - \int_{\Omega \setminus \bigcup_{v \in \mathcal{V}} B(v, \rho)} |\phi_j|^2 dx.$$

using again Proposition V.3.2 we have

$$\|\phi_j \chi_v\|_{L^2(\Omega)}^2 \geq 1 - C e^{-c\gamma}.$$

Let $i \in \{1, \dots, m_l\}$, $i \neq j$. One can write, as ϕ_i and ϕ_j are orthonormal in $L^2(\Omega)$,

$$\langle \phi_j \chi_v, \phi_i \chi_v \rangle_{L^2(\Omega)} = \sum_{v \in \mathcal{V}} \int_{\Omega} (|\chi_v|^2 - 1) \phi_j \overline{\phi_i} dx.$$

As $\|\chi_v\|^2 - 1 \leq 1$ and using Cauchy-Schwarz inequality we get

$$\left| \langle \phi_j \chi_v, \phi_i \chi_v \rangle_{L^2(\Omega)} \right| \leq \|\phi_i\|_{L^2(\Omega \setminus \bigcup_{v \in \mathcal{V}} B(v, \rho))} \|\phi_j\|_{L^2(\Omega \setminus \bigcup_{v \in \mathcal{V}} B(v, \rho))},$$

which gives, using again Proposition V.3.2,

$$\langle \phi_j \chi_V, \phi_i \chi_V \rangle_{L^2(\Omega)} = \mathcal{O}(e^{-c\gamma}).$$

If we denote by G the Gramian matrix of the family $(\phi_1 \chi_V, \dots, \phi_{m_l} \chi_V)$, which entries are $G_{j,i} := \langle \phi_j \chi_V, \phi_i \chi_V \rangle_{L^2(\Omega)}$, we just proved that $G_{j,i} = \delta_{j,i} + \mathcal{O}(e^{-c\gamma})$, where $\delta_{j,i}$ is the kronecker symbol. Thus, we necessarily have $\det(G) = 1 + \mathcal{O}(e^{-c\gamma})$ and as a consequence the family is linearly independent for large γ . Notice that this also proves that

$$\beta_i = 1 + \mathcal{O}(e^{-c\gamma}),$$

where $(\beta_i)_1^{m_l}$ are the eigenvalues of the Gramian matrix G . By change of variables, using the diffeomorphism F_v , this implies that the family $(\psi_1, \dots, \psi_{m_l})$ is linearly independent. In order to apply Proposition III.4.1, we need to choose a suitable spectral interval I . Recall that, by Theorem V.2.1, we know that

$$|E_{n_j}(Q^\gamma) - \gamma^2 \lambda_l| = \mathcal{O}(e^{-c\gamma}).$$

Thus, we introduce $I := (\gamma^2 \lambda_l - Ce^{-c\gamma}, \gamma^2 \lambda_l + Ce^{-c\gamma})$ and $a := \min\left(\frac{\lambda_l - \lambda_{l-1}}{4}, \frac{\lambda_{l+1} - \lambda_l}{4}\right)$. Defined in this way, $\text{spec}(\gamma^2 T^\oplus) \cap (I + B(0, 2a)) \setminus I = \emptyset$ and by Proposition III.4.1 we obtain

$$\text{dist}(E_l^\gamma, G_l^\gamma) \leq \frac{\sqrt{m_l} C e^{-c\gamma}}{a \sqrt{\beta_1}},$$

which concludes the proof of (V.3.2) as $a = \mathcal{O}(1)$ and $\beta_1 = 1 + o(1)$ and the proof of the theorem. \square

Recall that $\text{dist}(\mathcal{E}_l^\gamma, \mathcal{F}_l^\gamma) = \|P_{\mathcal{E}_l^\gamma} - P_{\mathcal{F}_l^\gamma} P_{\mathcal{E}_l^\gamma}\|$. Thus, if $\phi \in \mathcal{E}_l^\gamma$ is normalized one can write

$$\|\phi - P_{\mathcal{F}_l^\gamma} \phi\|_{L^2(\Omega)} = \|(P_{\mathcal{E}_l^\gamma} - P_{\mathcal{F}_l^\gamma} P_{\mathcal{E}_l^\gamma}) \phi\|_{L^2(\Omega)} = \mathcal{O}(e^{-c\gamma}),$$

and we say that ϕ is exponentially close to linear combinations of quasi-modes. Using again Proposition III.4.2, we can also prove that the quasi-modes are exponentially close to the eigenspaces of Q^γ .

Theorem V.3.3. *For any $l \in \{1, \dots, K^\oplus\}$, there exists $c > 0$ such that*

$$\text{dist}(\mathcal{F}_l^\gamma, \mathcal{E}_l^\gamma) = \mathcal{O}(e^{-c\gamma}), \text{ as } \gamma \text{ is large.}$$

Proof. By (V.1.5), there exists $r_{n,v} \in L^2(\Omega)$ such that

$$Q^\gamma \tilde{\phi}_n^{\gamma,v} = \gamma^2 \lambda_l \tilde{\phi}_n^{\gamma,v} + r_{n,v},$$

and $\|r_{n,v}\| \leq C e^{-(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}$, for any $(n, v) \in \mathcal{S}_l$. Let $a := \min\left(\frac{\lambda_{l+1} - \lambda_l}{4}, \frac{\lambda_l - \lambda_{l-1}}{4}\right)$.

Denote by

$$\tilde{I}_l := (\gamma^2 \lambda_l - C e^{-(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}, \gamma^2 \lambda_l + C e^{-(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}).$$

By Theorem V.2.1, we have $\text{spec}(Q^\gamma) \cap ((\tilde{I}_l + B(0, 2a)) \setminus \tilde{I}_l) = \emptyset$. Then by Proposition III.4.2,

$$\text{dist}(\mathcal{F}_l^\gamma, \mathcal{E}_l^\gamma) \leq \frac{\sqrt{m_l} C e^{-(1-\epsilon)\gamma\sqrt{-1-E^{\max}\rho}}}{a \sqrt{\beta_{\min}}}.$$

As by (V.2.9) we have $\beta_{\min} = 1 + o(1)$, this concludes the proof. \square

Notice that by the assertion (4) of Proposition III.4.1, we have the following corollary.

Corollary V.3.4. *For any $l \in \{1, \dots, K^\oplus\}$, there holds for large γ ,*

$$\text{dist}(\mathcal{E}_l^\gamma, \mathcal{F}_l^\gamma) = \text{dist}(\mathcal{F}_l^\gamma, \mathcal{E}_l^\gamma) = \mathcal{O}(e^{-c\gamma}).$$

V.4 Effective operator for the further Robin eigenvalues on polygons

This section is devoted to the proof the asymptotics of the further eigenvalues $E_{\mathcal{N}^{\oplus+j}}(Q^\gamma)$ stated in Theorem II.2.8.

Some important notation were introduced in Section II.2.3. In particular, the definition of the truncated sector $U_{\alpha,R}$ is given, see Figure II.4, and the one of the operator $T_{\alpha,R}^\gamma$, see (II.2.4). Some properties of this operator are given in Section IV.5 and will be used here. Finally, the definition of the *non-resonant* angles is introduced in Definition II.2.6 and will be of importance.

In this section, Ω is a polygon with straight edges admitting only *non-resonant* angles and non-convex corners, see the assumptions in Theorem II.2.8. The operator Q_Ω^γ is the γ -Robin Laplacian in $L^2(\Omega)$ and as there is no ambiguity, we still use the notation

$$Q^\gamma := Q_\Omega^\gamma.$$

We keep the same notation introduced in Section II.2.3. The proof of Theorem II.2.8 decomposes in two steps: we first prove the upper bound in Section V.4.2, which is a direct consequence of the non-resonance condition by a Dirichlet bracketing, and we prove the lower bound in Section V.4.3. Let us first introduce a decomposition of the polygon and the associated notation.

V.4.1 Decomposition of the polygon

Let us pick some small $\delta > 0$ and denote

$$\Omega_\delta := \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}, \quad \Gamma_\delta := \partial\Omega_\delta \setminus \partial\Omega.$$

In this section we denote by A_1, \dots, A_V the vertices of Ω , by ℓ_v the length of the side $A_v A_{v+1}$ and by α_v the angle of Ω at the vertex A_v , measured *inside* Ω . We also use the convention

$$\begin{aligned} v+1 &= 1 \text{ if } v = V, \\ v-1 &= V \text{ if } v = 1. \end{aligned}$$

Our aim is to construct a special neighborhood for each vertex A_v included into Ω_δ . The construction will be different for convex and concave vertices.

- Let A_v be a convex vertex of Ω , that is to say $\alpha_v \in (0, \frac{\pi}{2})$. Denote by $A_{v,\delta}^\pm$ the point lying on $[A_v, A_{v\pm 1}]$ at the distance $\delta \cot \alpha_v$ from A_v . Let $\Pi_{v,\delta}^\pm$ be the half-line originating from $A_{v,\delta}^\pm$, orthogonal to $[A_v, A_{v\pm 1}]$ and pointing to the interior of Ω . Denote by $B_{v,\delta}$ the intersection point of $\Pi_{v,\delta}^-$ and $\Pi_{v,\delta}^+$, then $B_{v,\delta}$ is at the distance δ from $[A_v, A_{v\pm 1}]$. We set

$$Z_{v,\delta} := \text{the quadrangle } A_v A_{v,\delta}^+ B_{v,\delta} A_{v,\delta}^-.$$

Remark that $Z_{v,\delta}$ is just a rotated copy of the truncated sector $U_{\alpha_v, \delta \cot \alpha_v}$, introduced in Section II.2.3, and that $Z_{v,\delta} \subset \Omega$ if δ is sufficiently small. The constructions are illustrated in Figure V.1(a). We denote

$$\lambda_v = \cot \alpha_v \text{ if } A_v \text{ is convex.}$$

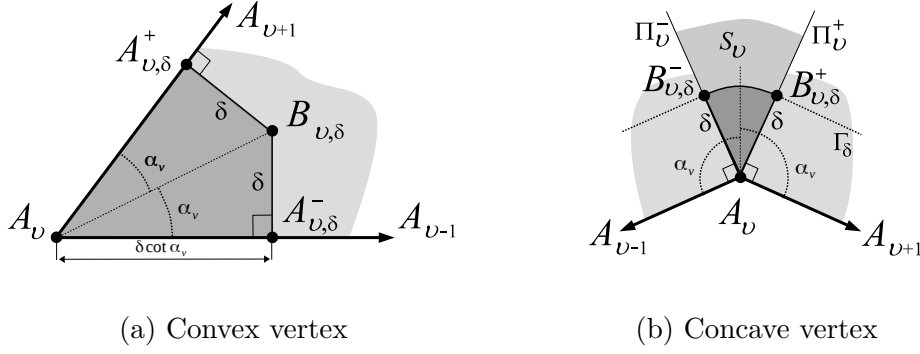


Figure V.1 – The neighborhood $Z_{v,\delta}$ is shaded.

- Let A_v be a concave vertex of Ω . Denote by Π_v^\pm the half-line originating from A_v , orthogonal to $[A_v, A_{v\pm 1}]$ and pointing to the interior of Ω , and let S_v be the convex sector bounded by Π_v^- and Π_v^+ and $B_{v,\delta}^\pm$ be the intersection points of Π_v^\pm with Γ_δ . We denote

$$Z_{v,\delta} := S_v \cap \Omega_\delta,$$

which is just a sector of radius δ and opening angle $2\alpha_v - \pi$, see Figure V.1(b). We denote

$$\lambda_v = 0 \text{ if } A_v \text{ is concave.}$$

Now we decompose Ω_δ as follows, see Figure V.2. Remark that the set $\Omega_\delta \setminus \overline{\bigcup_{v=1}^V Z_{v,\delta}}$ represents the union of V disjoint rectangles $W_{v,\delta}$ glued to the sides $[A_v, A_{v+1}]$. One easily sees that each rectangle $W_{v,\delta}$ is isometric to $I_{v,\delta} \times (0, \delta)$ with

$$I_{v,\delta} = (\lambda_v \delta, \ell_v - \lambda_{v+1} \delta).$$

In what follows, on each $W_{v,\delta}$ we will use the coordinates $(s_v, t_v) \in I_{v,\delta} \times (0, \delta)$ defined in the natural way, e.g. t_v is the distance between (s_v, t_v) and the side $[A_v, A_{v+1}]$, and s_v is the coordinate along the side, such that the points with $s_v = \lambda_v \delta$ are attached to $Z_{v,\delta}$.

We will also need the common boundaries between the rectangles and the vertex neighborhoods,

$$\Sigma_{v,\delta}^- = \overline{Z_{v,\delta}} \cap \overline{W_{v-1,\delta}}, \quad \Sigma_{v,\delta}^+ = \overline{Z_{v,\delta}} \cap \overline{W_{v,\delta}}.$$

Using the coordinates (s_v, t_v) , the set $\Sigma_{v,\delta}^-$ corresponds to $s_{v-1} = \ell_{v-1} - \lambda_v \delta$, while $\Sigma_{v,\delta}^+$ corresponds to $s_v = \lambda_v \delta$.

V.4.2 Dirichlet-Neumann bracketing and the upper bound

For subsequent constructions δ should depend on γ . *In the rest of this section we assume that,*

$$\delta = c_0 \frac{\log \gamma}{\gamma}, \quad c_0 > 0. \tag{V.4.1}$$

Later on we will choose a suitable value of c_0 which will be sufficiently large. A discussion on the choice of δ is made in Remark V.4.12.

Denote by $Q_{D/N,\delta}^\gamma$ the self-adjoint operators acting in $L^2(\Omega_\delta)$ as $u \mapsto -\Delta u$ on Ω_δ with the γ -Robin boundary condition at $\partial\Omega$, i.e.: $\partial_\nu u = \gamma u$ on $\partial\Omega$, and with the Dirichlet boundary condition $u = 0$ (denoted by the letter D), or the Neumann boundary condition

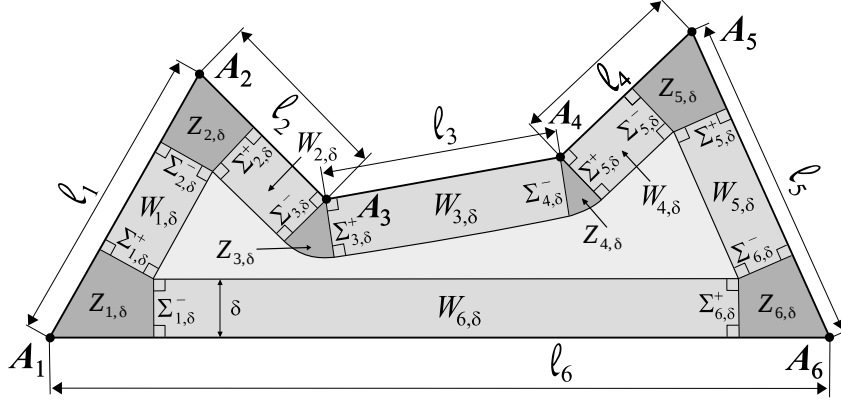


Figure V.2 – An example of the decomposition of Ω_δ for a polygon with six vertices.

$\partial_\nu u = 0$ (denoted by the letter N) at the remaining part of the boundary $\partial\Omega_\delta \setminus \partial\Omega$. Notice that they are well defined in the sense of Definition III.6.1. By the min-max principle one has for all negative eigenvalues

$$E_n(Q_{N,\delta}^\gamma) \leq E_n(Q^\gamma) \leq E_n(Q_{D,\delta}^\gamma).$$

We introduce the unions

$$Z_\delta = \bigcup_{v=1}^V Z_{v,\delta}, \quad W_\delta = \bigcup_{v=1}^V W_{v,\delta}, \quad \Sigma_\delta = \bigcup_{v=1}^V \bigcup_{j=\pm} \Sigma_{v,\delta}^j.$$

Denote $B_{D/N,Z_{v,\delta}}^\gamma$ the Laplacian in $L^2(Z_{v,\delta})$ with the γ -Robin boundary condition at $\partial Z_{v,\delta} \cap \partial\Omega$ and the Dirichlet/Neumann boundary condition at the remaining part of the boundary. Note that, if v is a concave vertex, $B_{D/N,Z_{v,\delta}}^\gamma$ is simply the Dirichlet/Neumann Laplacian on $Z_{v,\delta}$ as $\partial Z_{v,\delta} \cap \partial\Omega = \{A_v\}$, see Figure V.2. If the vertex v is convex, then $Z_{v,\delta}$ is a rotated copy of the truncated sector $U_{\alpha_v,\delta \cot \alpha_v}$. It follows that $B_{N,Z_{v,\delta}}^\gamma$ is unitary equivalent to $T_{\alpha_v,\delta \cot \alpha_v}^\gamma$, defined by (II.2.4), while $B_{D,Z_{v,\delta}}^\gamma$ is unitary equivalent to $T_{\alpha_v,\delta \cot \alpha_v}^{D,\gamma}$, defined by (IV.5.10). We introduce the following operators acting in $\bigoplus_{v=1}^V L^2(Z_{v,\delta})$:

$$B_{D/N,Z_\delta}^\gamma := \bigoplus_{v=1}^V B_{D/N,Z_{v,\delta}}^\gamma.$$

Furthermore, denote by $B_{D/N,W_\delta}^\gamma$ the Laplacian in $L^2(W_\delta)$ with the γ -Robin boundary condition at $\partial W_\delta \cap \partial\Omega$ and the Dirichlet/Neumann boundary condition at the remaining part of the boundary, which in turn represents the direct sum of respective operators in $L^2(W_{v,\delta})$, and the part on each $L^2(W_{v,\delta})$ admits a separation of variables using the one-dimensional operators $\mathcal{N}_{\delta,\gamma}$ and $\mathcal{D}_{\delta,\gamma}$ defined in Section III.5.2 and the Dirichlet/Neumann Laplacian on $I_{v,\delta}$ denoted respectively by $D_{v,\delta}$ and $N_{v,\delta}$.

By the min-max principle one has

$$E_n(B_{N,Z_\delta}^\gamma \oplus B_{N,W_\delta}^\gamma) \leq E_n(Q_{N,\delta}^\gamma) \leq E_n(Q^\gamma) \leq E_n(Q_{D,\delta}^\gamma) \leq E_n(B_{D,Z_\delta}^\gamma \oplus B_{D,W_\delta}^\gamma). \quad (\text{V.4.2})$$

Remark that due to the assumption that all convex vertices are non-resonant, see Definition II.2.6, one has, with a suitable $c > 0$ and for large γ ,

$$E_{\mathcal{N}^{\oplus+1}}(B_{D,Z_\delta}^\gamma) \geq E_{\mathcal{N}^{\oplus+1}}(B_{N,Z_\delta}^\gamma) \geq -\gamma^2 + \frac{c}{\delta^2}. \quad (\text{V.4.3})$$

Recall that D_v denotes the Dirichlet Laplacian in $L^2(0, \ell_v)$ and $D := D_1 \oplus \cdots \oplus D_V$. The non-resonance condition gives a simple proof of the upper bound of the eigenvalues $E_{\mathcal{N}^{\oplus+j}}(Q^\gamma)$:

Proposition V.4.1. *For any fixed $j \in \mathbb{N}$ one has*

$$E_{\mathcal{N}^{\oplus+j}}(Q^\gamma) \leq -\gamma^2 + E_j(D) + \mathcal{O}(\delta).$$

Proof. Using first (V.4.2) one has

$$E_{\mathcal{N}^{\oplus+j}}(Q^\gamma) \leq E_{\mathcal{N}^{\oplus+j}}(B_{D, Z_\delta}^\gamma \oplus B_{D, W_\delta}^\gamma).$$

Furthermore, the separation of variables gives

$$E_j(B_{D, W_\delta}^\gamma) = E_j\left(\bigoplus_{v=1}^V (D_{v, \delta} \otimes 1 + 1 \otimes \mathcal{D}_{\delta, \gamma}),\right)$$

with $D_{v, \delta}$ the Dirichlet Laplacian on $I_{v, \delta}$ and $\mathcal{D}_{\delta, \gamma}$ being defined in Lemma III.5.3. In particular, for any fixed j one has

$$E_j(D_{v, \delta}) = E_j(D_v) + \mathcal{O}(\delta).$$

Using the asymptotics (III.5.10) one has

$$E_j(B_{D, W_\delta}^\gamma) = -\gamma^2 + E_j(D_1 \oplus \cdots \oplus D_V) + \mathcal{O}(\delta). \quad (\text{V.4.4})$$

By Proposition IV.5.3 and the scale invariance $E_n(T_{\alpha, R}^{D, \gamma}) = \gamma^2 E_n(T_{\alpha, \gamma R}^{D, 1})$ we know that for any $n \leq \mathcal{N}^{\oplus}$ and $\epsilon > 0$ there holds

$$E_n(B_{D, Z_\delta}^\gamma) < E_n\left(\bigoplus_{v=1}^V T_{\alpha_v}^\gamma\right) + \epsilon\gamma^2, \text{ for large } \gamma.$$

In particular, due to (V.4.4) one has for any $n \leq \mathcal{N}^{\oplus}$ and for large γ ,

$$E_n(B_{D, Z_\delta}^\gamma) < E_1(B_{D, W_\delta}^\gamma).$$

On the other hand, by (V.4.3) we also have for any fixed j

$$E_j(B_{D, W_\delta}^\gamma) < E_{\mathcal{N}^{\oplus+1}}(B_{D, Z_\delta}^\gamma).$$

Thus, one can conclude

$$E_{\mathcal{N}^{\oplus+j}}(B_{D, Z_\delta}^\gamma \oplus B_{D, W_\delta}^\gamma) = E_j(B_{D, W_\delta}^\gamma),$$

which finishes the proof thanks to (V.4.4). □

V.4.3 The lower bound

We still assume the above choice of δ made in (V.4.1), and the validity of the inequality (V.4.3), and we are going to obtain a lower bound for $E_{\mathcal{N}^{\oplus+j}}(Q_{N, \delta}^\gamma)$. All estimates below are for large γ . In view of (V.4.2), to get a lower bound we need to study the operators B_{N, Z_δ}^γ and B_{N, W_δ}^γ . As there is no ambiguity, we denote in the rest of this chapter

$$Q := Q_{N, \delta}^\gamma, \quad B^W := B_{N, W_\delta}^\gamma, \quad B^Z := B_{N, Z_\delta}^\gamma.$$

Similar to the preceding constructions we obtain

$$E_j(B^W) = E_j\left(\bigoplus_{v=1}^V (N_{v,\delta} \otimes 1 + 1 \otimes \mathcal{N}_{\delta,\gamma})\right)$$

with $N_{v,\delta}$ being the Neumann Laplacian on $I_{v,\delta}$ and $\mathcal{N}_{\delta,\gamma}$ being defined in Lemma III.5.2. This implies, for any fixed j ,

$$E_j(B^W) = E_1(\mathcal{N}_{\delta,\gamma}) + E_j(N_{1,\delta} \oplus \dots \oplus N_{V,\delta}),$$

in particular, by the asymptotics (III.5.2), for any $m > 0$ there exists $c_0 > 0$ large enough such that

$$E_1(B^W) = E_1(\mathcal{N}_{\delta,\gamma}) = -\gamma^2 + \mathcal{O}(\gamma^{-m}) < E_{\mathcal{N}^{\oplus+1}}(B^Z).$$

Moreover, by Proposition IV.5.2 and the scale invariance $E_n(T_{\alpha,R}^\gamma) = \gamma^2 E_n(T_{\alpha,\gamma R}^1)$, we also have for any $\epsilon > 0$ and $n \leq \mathcal{N}^{\oplus}$, $E_n(B^Z) < E_n(\bigoplus_{v=1}^V T_{\alpha_v}^\gamma) + \epsilon\gamma^2$ for large γ , which implies

$$E_n(B^Z) < E_1(B^W), \quad n \leq \mathcal{N}^{\oplus}.$$

Hence, due to (V.4.2),

$$E_{\mathcal{N}^{\oplus+1}}(Q) \geq E_{\mathcal{N}^{\oplus+1}}(B^Z \oplus B^W) = E_1(B^W) = E_1(\mathcal{N}_{\delta,\gamma}). \quad (\text{V.4.5})$$

V.4.3.1 Localization estimates for the eigenfunctions

Denote

$$d_v(x) := \text{dist}(x, A_v), \quad d(x) := \min d_v(x). \quad (\text{V.4.6})$$

Denote by:

- Λ the subspace of $L^2(\Omega_\delta)$ spanned by the first \mathcal{N}^{\oplus} eigenfunctions of Q ,
- Λ_Z the subspace of $L^2(\Omega_\delta)$ spanned by the first \mathcal{N}^{\oplus} eigenfunctions of $B^Z \oplus B^W$.

Remark that the \mathcal{N}^{\oplus} first eigenvalues of $B^Z \oplus B^W$ are given by the \mathcal{N}^{\oplus} first eigenvalues of B^Z and each function of Λ_Z vanishes in W_δ . The following proposition is a consequence of Proposition IV.5.4.

Proposition V.4.2. *There exist $\eta > 0$ and $C > 0$ such that there holds, as γ is large,*

$$\int_{\Omega_\delta} (|\nabla u|^2 + \gamma^2 |u|^2) e^{\eta\gamma d(x)} dx \leq \gamma^2 C \|u\|_{L^2(\Omega_\delta)}^2, \quad u \in \Lambda_Z.$$

Proof. Notice that, as the \mathcal{N}^{\oplus} first eigenvalues of $B^Z \oplus B^W$ are given by the \mathcal{N}^{\oplus} first eigenvalues of B^Z , it is sufficient to give the proof for u being an eigenfunction of B^Z . But, as B^Z is simply a direct sum of Robin-Neumann Laplacians on the truncated sectors $Z_{v,\delta}$, it is sufficient to give the proof for u being an eigenfunction of $T_{\alpha,\delta \cot \alpha}^\gamma$, $\alpha \in (0, \frac{\pi}{2})$. Recall that in Section IV.5, we introduced the notation

$$T_{\alpha,R} := T_{\alpha,R}^1.$$

For $\alpha \in (0, \frac{\pi}{2})$, let v be an eigenfunction of $T_{\alpha,c_0 \log \gamma \cot \alpha}$, associated with $E_n(T_{\alpha,c_0 \log \gamma \cot \alpha})$, $n \leq \mathcal{N}^{\oplus}$. By Proposition IV.5.4, there exists $\eta > 0$ and $C > 0$ such that for large γ there holds

$$\int_{U_{\alpha,c_0 \log \gamma \cot \alpha}} (|\nabla v|^2 + |v|^2) e^{\eta|x|} \leq C \|u\|_{L^2(U_{\alpha,c_0 \log \gamma \cot \alpha})}^2.$$

Now define $u(x) := v(\gamma x)$. It is easy to check that u is then an eigenfunction of $T_{\alpha, \delta \cot \alpha}^\gamma$ associated with $E_n(T_{\alpha, \delta \cot \alpha}^\gamma) = \gamma^2 E_n(T_{\alpha, c_0 \log \gamma \cot \alpha})$. By change of variables,

$$\int_{U_{\alpha, c_0 \log \gamma \cot \alpha}} (|\nabla v|^2 + |v|^2) e^{\eta|x|} dx = \int_{U_{\alpha, \delta \cot \alpha}} (|\nabla u|^2 + \gamma^2 |u|^2) e^{\gamma \eta|x|} dx,$$

and $\|v\|_{L^2(U_{\alpha, c_0 \log \gamma \cot \alpha})}^2 = \gamma^2 \|u\|_{L^2(U_{\alpha, \delta \cot \alpha})}^2$, which finishes the proof. \square

The following proposition tells us that the first \mathcal{N}^\oplus eigenfunctions of $B^Z \oplus B^W$ are close, in the sense of the distance dist defined in Section III.4, to the first \mathcal{N}^\oplus eigenfunctions of Q .

Proposition V.4.3. *For any $m > 0$, there exists $c_0 > 0$ such that*

$$\text{dist}(\Lambda_Z, \Lambda) = \mathcal{O}(\gamma^{-m}).$$

Proof. Let $a \in (0, \frac{1}{2})$ be fixed with $2a < \cot \alpha_v$ for all convex A_v . Let χ be a cutoff function satisfying

$$\chi : \mathbb{R} \rightarrow [0, 1], \quad \chi(t) = 1 \text{ for } t < a, \quad \chi(t) = 0 \text{ for } t > 2a,$$

and set

$$\chi_\delta(x) := \sum_{v=1}^V \chi\left(\frac{d_v(x)}{\delta}\right).$$

Consider the intermediate subspace

$$\Lambda'_Z = \chi_\delta \Lambda_Z \subset L^2(\Omega_\delta).$$

Using the properties of the distance between subspaces, assertion (2) in Proposition III.4.1, one can write

$$\text{dist}(\Lambda_Z, \Lambda) \leq \text{dist}(\Lambda_Z, \Lambda'_Z) + \text{dist}(\Lambda'_Z, \Lambda). \quad (\text{V.4.7})$$

Let $u \in \Lambda_Z$ be normalized and $\chi_0 := 1 - \chi_\delta$. We can write $u = \chi_0 u + \chi_\delta u$ and

$$\|u - P_{\Lambda'_Z} u\|_{L^2(\Omega_\delta)}^2 \leq 2\|\chi_\delta u - P_{\Lambda'_Z} u\|_{L^2(\Omega_\delta)}^2 + 2\|\chi_0 u\|_{L^2(\Omega_\delta)}^2.$$

Notice that, for any $x \in \text{supp } \chi_0 \cap \Omega_\delta$ we have

$$d(x) \geq \delta a,$$

where $d(x)$ denotes the distance of x to the corners of Ω introduced in (V.4.6). Hence, by Proposition V.4.2 we obtain the upper bound

$$\|\chi_0 u\|_{L^2(\Omega_\delta)}^2 \leq C\gamma^2 e^{-\eta a \delta} \|u\|_{L^2(\Omega_\delta)}^2.$$

Recall that $\delta := c_0 \frac{\log \gamma}{\gamma}$. For any $m > 0$, there exists $c_0 > 0$ such that

$$\|\chi_0 u\|_{L^2(\Omega_\delta)} \leq C\gamma^{-m}. \quad (\text{V.4.8})$$

As $P_{\Lambda'_Z}(\chi_\delta u) = \chi_\delta u$ we obtain, using (V.4.8), $\|\chi_\delta u - P_{\Lambda'_Z} u\|_{L^2(\Omega_\delta)} \leq \|\chi_\delta u - u\|_{L^2(\Omega_\delta)} \leq C\gamma^{-m}$, which implies

$$\text{dist}(\Lambda_Z, \Lambda'_Z) = \mathcal{O}(\gamma^{-m}).$$

In view of (V.4.7), we now need to estimate $\text{dist}(\Lambda'_Z, \Lambda)$ in order to conclude. We want to apply Proposition III.4.2 and we have to prove first some estimates on Λ'_Z .

Let $j \in \{1, \dots, \mathcal{N}^\oplus\}$ and $u_j \in \Lambda_Z$ be a normalized eigenfunction of $B^Z \oplus B^W$ associated with $E_j(B^Z \oplus B^W) = E_j(B^Z)$. Then, $u_j \chi_\delta \in \Lambda'_Z$ and with this choice of χ_δ it is easy to see that $u_j \chi_\delta \in D(Q)$: indeed as χ_δ is radial, $u_j \chi_\delta$ still satisfies the γ -Robin boundary condition on $\partial\Omega_\delta \cap \partial\Omega$. Moreover,

$$Q(u_j \chi_\delta) = (-\Delta u_j) \chi_\delta + (-\Delta \chi_\delta) u_j - 2\nabla u_j \nabla \chi_\delta.$$

Notice that $-\Delta u_j = E_j(B^Z) u_j$ on $\text{supp } \chi_\delta$. In order to estimate the last terms, remark that $\text{supp } \Delta \chi_\delta \subset \text{supp } \nabla \chi_\delta \subset \bigcup_{v=1}^V (B(A_v, 2\delta a) \setminus \overline{B(A_v, \delta a)})$ and

$$|\nabla \chi_\delta|^2 = \mathcal{O}(\gamma^2), \text{ and } |\Delta \chi_\delta|^2 = \mathcal{O}(\gamma^4).$$

Thus, by Proposition V.4.2 there exists $c_0 > 0$ such that

$$\|u_j \Delta \chi_\delta\|_{L^2(\Omega_\delta)}^2 \leq C\gamma^{-m}. \quad (\text{V.4.9})$$

Using again Proposition V.4.2, we also have

$$\|\nabla u_j \nabla \chi_\delta\|_{L^2(\Omega_\delta)}^2 \leq C\gamma^{-m}. \quad (\text{V.4.10})$$

Gathering (V.4.9) and (V.4.10) we obtain

$$\|Q(u_j \chi_\delta) - E_j(B^Z) u_j \chi_\delta\|_{L^2(\Omega_\delta)} \leq C\gamma^{-m}. \quad (\text{V.4.11})$$

Let us prove that $(u_1 \chi_\delta, \dots, u_{\mathcal{N}^\oplus} \chi_\delta)$ is linearly independent. It is easy to see that

$$\|u_j \chi_\delta\|_{L^2(\Omega_\delta)}^2 \leq 1.$$

On the other hand,

$$\int_{\Omega_\delta} |u_j \chi_\delta|^2 dx \geq \int_{\Omega_\delta \cap (\bigcup_{v=1}^V B(A_v, a\delta))} |u_j|^2 dx = \int_{\Omega_\delta} |u_j|^2 dx - \int_{\Omega_\delta \setminus (\bigcup_{v=1}^V B(A_v, a\delta))} |u_j|^2 dx.$$

Using Proposition V.4.2, we have the upper bound

$$\int_{\Omega_\delta \setminus (\bigcup_{v=1}^V B(A_v, a\delta))} |u_j|^2 dx \leq C\gamma^{-m},$$

and then

$$\|u_j \chi_\delta\|_{L^2(\Omega_\delta)}^2 = 1 + \mathcal{O}(\gamma^{-m}).$$

Let $i \in \{1, \dots, \mathcal{N}^\oplus\}$ and $i \neq j$. One can write, using the fact that u_j and u_i are orthogonal in $L^2(\Omega_\delta)$,

$$\langle \chi_\delta u_j, \chi_\delta u_i \rangle_{L^2(\Omega_\delta)} = \int_{\Omega_\delta} (|\chi_\delta|^2 - 1) u_j \bar{u}_i dx.$$

As $||\chi_\delta|^2 - 1| \leq 1$ and by the Cauchy-Schwarz inequality we have

$$|\langle \chi_\delta u_j, \chi_\delta u_i \rangle_{L^2(\Omega_\delta)}| \leq \|u_j\|_{L^2(\Omega_\delta \setminus (\bigcup_{v=1}^V B(A_v, a\delta))} \|u_i\|_{L^2(\Omega_\delta \setminus (\bigcup_{v=1}^V B(A_v, a\delta))},$$

and we can conclude using again Proposition V.4.2 that

$$|\langle \chi_\delta u_j, \chi_\delta u_i \rangle_{L^2(\Omega_\delta)}| = \mathcal{O}(\gamma^{-m}).$$

If we denote by G the Gramian matrix of $(u_j \chi_\delta)_j$ which entries are $G_{j,i} := \langle u_j \chi_\delta, u_i \chi_\delta \rangle_{L^2(\Omega_\delta)}$ then we just proved that $G_{j,i} = \delta_{j,i} + \mathcal{O}(\gamma^{-m})$, where $\delta_{j,i}$ is the Kronecker symbol.

Necessarily, $\det(G) = 1 + \mathcal{O}(\gamma^{-m}) \neq 0$ and $(u_j \chi_\delta)_j$ is linearly independent for γ large enough. Notice that this also proves the following estimate

$$\beta_i = 1 + \mathcal{O}(\gamma^{-m}), \quad (\text{V.4.12})$$

where $(\beta_i)_{i=1}^{\mathcal{N}^\oplus}$ are the eigenvalues, counted in the non-decreasing order, of the matrix G . Let us now introduce the spectral interval I as in Proposition III.4.2. By Proposition IV.5.2, and the scale invariance $E_n(T_{\alpha,R}^\gamma) = \gamma^2 E_n(T_{\alpha,\gamma R})$, we know that for any $n \in \{1, \dots, \mathcal{N}^\oplus\}$ and $\epsilon > 0$,

$$|E_n(B^Z) - \gamma^2 E_n(\bigoplus_{v=1}^V T_{\alpha_v})| \leq \gamma^2 \epsilon. \quad (\text{V.4.13})$$

Moreover, in Theorem V.2.1, it is proved that, for any $n \leq \mathcal{N}^\oplus$

$$|E_n(Q^\gamma) - \gamma^2 E_n(\bigoplus_{v=1}^V T_{\alpha_v})| \leq \epsilon.$$

By (V.4.2), we know that $E_n(B^Z \oplus B^W) \leq E_n(Q) \leq E_n(Q^\gamma)$, thus gathering the above estimate and (V.4.13) we have

$$|E_n(Q) - \gamma^2 E_n(\bigoplus_{v=1}^V T_{\alpha_v})| \leq \gamma^2 \epsilon.$$

We introduce $I := (\gamma^2 E_1(\bigoplus_{v=1}^V T_{\alpha_v}) - \gamma^2 \epsilon, \gamma^2 E_{\mathcal{N}^\oplus}(\bigoplus_{v=1}^V T_{\alpha_v}) + \gamma^2 \epsilon)$. Notice that, due to (V.4.5),

$$E_{\mathcal{N}^\oplus+1}(Q) \geq -\gamma^2 + \mathcal{O}(\gamma^{-m}).$$

As a consequence, for any constant $a > 0$ we get

$$\text{spec}(Q) \cap (I + B(0, 2a)) \setminus I = \emptyset.$$

We are now able to apply Proposition III.4.2 to obtain

$$\text{dist}(\Lambda'_Z, \Lambda) \leq C\gamma^{-m},$$

which finishes the proof due to (V.4.7). \square

V.4.3.2 Construction of the identification map

To obtain the sought lower bound we will use the technique introduced in Proposition III.1.7. We thus have to define the appropriate operators B and B' and the identification map J .

In what follows we simply put

$$E = E_1(\mathcal{N}_{\delta,\gamma}).$$

Remark that in view of the properties of the 1D operator $\mathcal{N}_{\delta,\gamma}$, see Lemma III.5.2, the inequality (V.4.3) can be rewritten as

$$E_{\mathcal{N}^\oplus+1}(B^Z) \geq E + \frac{c}{\delta^2}, \quad c > 0. \quad (\text{V.4.14})$$

We are going to apply Proposition III.1.7 for:

- B the restriction of $Q - E$ to the orthogonal complement of Λ .
- $B' := D_{1,\delta} \oplus \dots \oplus D_{V,\delta}$.

In other words, the Hilbert spaces \mathcal{H} and \mathcal{H}' will be

$$\mathcal{H} := \text{the orthogonal complement of } \Lambda \text{ in } L^2(\Omega_\delta), \quad \mathcal{H}' = \bigoplus_{v=1}^V L^2(I_{v,\delta}),$$

and the forms b and b' are given by

$$b(u, u) = \int_{\Omega_\delta} |\nabla u|^2 dx - \gamma \int_{\partial\Omega_\delta \cap \partial\Omega} |u|^2 ds - E \int_{\Omega_\delta} |u|^2 dx, \quad D(b) := H^1(\Omega_\delta) \cap \mathcal{H},$$

and

$$b'(f, f) = \sum_{v=1}^V \int_{I_{v,\delta}} |f'_v|^2 ds, \quad D(b') = \bigoplus_{v=1}^V H_0^1(I_{v,\delta}).$$

Remark that due to (V.4.5) we have $b \geq 0$, and the inequality $b' \geq 0$ is obvious. Moreover, $E_n(B') = E_n(D) + \mathcal{O}(\delta)$ for any fixed n . Due to (V.4.2) and Proposition V.4.1 one has $(1 + E_n(B))^{-1} \geq \frac{1}{E_n(D)+1}(1 + \mathcal{O}(\delta))$, namely there exists a constant $C > 0$ such that

$$(1 + E_n(B))^{-1} \geq C. \quad (\text{V.4.15})$$

In order to construct the identification operator J we proceed as follows. Let $\Phi \in L^2(0, \delta)$ be the positive normalized eigenfunction associated with the first eigenvalue of $\mathcal{N}_{\delta,\gamma}$. Define

$$P : L^2(W_\delta) \rightarrow \mathcal{H}', \quad (Pu)(s_v) = \int_0^\delta \Phi(t_v) u(s_v, t_v) dt_v, \quad s_v \in I_{v,\delta}.$$

Remark that $L^2(W_\delta) \ni u \mapsto (Pu)\Phi \in L^2(W_\delta)$ is simply the orthogonal projection on the subspace $\bigoplus_{v=1}^V L^2(I_{v,\delta}) \otimes \text{span}\{\Phi\} \subset L^2(W_\delta)$. We have the following simple result:

Lemma V.4.4. *For any $u \in \mathcal{H}$,*

$$\sum_{v=1}^V |Pu(\lambda_v \delta)|^2 + |Pu(\ell_v - \lambda_{v+1} \delta)|^2 \leq \int_{\Sigma_\delta} |u|^2 ds.$$

Proof. Let $u \in \mathcal{H}$ and $v \in \{1, \dots, V\}$. By Cauchy-Schwarz, and as Φ is normalized we have,

$$|Pu(\lambda_v \delta)|^2 + |Pu(\ell_v - \lambda_{v+1} \delta)|^2 \leq \int_{\Sigma_{v,\delta}^+} |u|^2 ds + \int_{\Sigma_{v+1,\delta}^-} |u|^2 ds, \quad (\text{V.4.16})$$

with the convention $\Sigma_{V+1,\delta}^- = \Sigma_{1,\delta}^-$. Summing (V.4.16) on v gives the result. \square

Let $\rho \in C^\infty(\mathbb{R})$ with $\rho(0) = 1$ and $\rho(s) = 0$ for $s \geq a$, with $a < \ell_v/2$ for all v , then set

$$\rho_v^-(s) = \rho(s - \lambda_v \delta), \quad \rho_v^+(s) = \rho(\ell_v - \lambda_{v+1} \delta - s).$$

Finally as an identification map J we take

$$(Ju)(s_v) = (Pu)(s_v) - (Pu)(\lambda_v \delta) \rho_v^-(s_v) - (Pu)(\ell_v - \lambda_{v+1} \delta) \rho_v^+(s_v), \quad s_v \in I_{v,\delta}. \quad (\text{V.4.17})$$

Clearly, by construction one has $Ju \in D(b')$ for any $u \in D(b)$.

V.4.3.3 Estimates for the identification map

In order to proceed with estimates, let us first decompose the above expression of the form b as follows:

$$b(u, u) = (b^Z - E)(u, u) + (b^W - E)(u, u), \quad u \in D(b),$$

with $b^{Z/W}$ the quadratic forms associated with the operators $B^{Z/W}$ where we use the notation $(b^{Z/W} - E)(u, u) := b^{Z/W}(u, u) - E\|u\|_{L^2(Z_\delta/W_\delta)}^2$ and denote the restriction of u to Z_δ/W_δ by the same symbol.

We now want to control the trace on Σ_δ of functions in $D(b)$ using the non-resonance condition. We represent, for any $u \in D(b)$,

$$u = v_\delta^0 + v_\delta, \quad v_\delta^0 \in \Lambda_Z, \quad v_\delta \in (\Lambda_Z)^\perp. \quad (\text{V.4.18})$$

Recall that v_δ^0 vanishes on W_δ , i.e.

$$\|v_\delta^0\|_{L^2(Z_\delta)} = \|v_\delta^0\|_{L^2(\Omega_\delta)}.$$

The following result is a consequence of the closeness between Λ_Z and Λ .

Lemma V.4.5. *For any $m > 0$, there exists $c_0 > 0$ and $C > 0$ such that for any $u \in \mathcal{H}$,*

$$\|v_\delta^0\|_{L^2(\Omega_\delta)} \leq C\gamma^{-m}\|u\|_{L^2(\Omega_\delta)}.$$

Proof. Let $u \in \mathcal{H}$. Then, by definition $u = (1 - P_\Lambda)u$ and $v_\delta^0 = P_{\Lambda_Z}u$. One can write

$$\|v_\delta^0\|_{L^2(\Omega_\delta)}^2 = \|P_{\Lambda_Z}(1 - P_\Lambda)u\|_{L^2(\Omega_\delta)}^2.$$

By Proposition V.4.3, for any $m > 0$ there exists $c_0 > 0$ such that $d(\Lambda_Z, \Lambda) := \|P_{\Lambda_Z} - P_\Lambda P_\Lambda\| = \mathcal{O}(\gamma^{-m})$. Using this estimate we finally obtain

$$\|v_\delta^0\|_{L^2(\Omega_\delta)} \leq C\gamma^{-m}\|u\|_{L^2(\Omega_\delta)}.$$

□

We are now able to get an estimate on the trace of u on Σ_δ , which is a crucial point for our purpose.

Proposition V.4.6. *There exists $C > 0$ such that for any $u \in D(b)$,*

$$\|u\|_{L^2(\Sigma_\delta)}^2 \leq C\delta^2\gamma \left((b^Z - E)(u, u) + \|u\|_{L^2(\Omega_\delta)}^2 \right).$$

Proof. The proof needs several steps that we decompose in intermediary lemmas.

Lemma V.4.7. *There exists a constant $A > 0$ such that for any $u \in D(b)$ there holds*

$$\|u\|_{L^2(\Sigma_\delta)}^2 \leq \frac{1}{\gamma} \left((b^Z - E)(u, u) + A\gamma^2\|u\|_{L^2(Z_\delta)}^2 \right).$$

Proof. Consider the Laplacian with the γ -Robin boundary condition in $L^2(Z_\delta)$ denoted by $Q_{Z_\delta}^\gamma$. We have the trivial inequality for any $u \in H^1(Z_\delta)$,

$$q_{Z_\delta}^\gamma(u, u) \geq E_1(Q_{Z_\delta}^\gamma)\|u\|_{L^2(Z_\delta)}^2. \quad (\text{V.4.19})$$

Now, by change of variables one has $E_1(Q_{Z_\delta}^\gamma) = \frac{1}{\delta^2}E_1(Q_{Z_1}^{\delta\gamma})$. Using the asymptotics of Theorem V.2.1, there exists a constant $A > 0$ such that $E_1(Q_{Z_1}^{\delta\gamma}) \geq -A(\delta\gamma)^2$ which gives

$$E_1(Q_{Z_\delta}^\gamma) \geq -A\gamma^2.$$

Rewriting (V.4.19), one has for any $u \in H^1(Z_\delta)$,

$$\int_{Z_\delta} |\nabla u|^2 dx - \gamma \int_{\Sigma_\delta} |u|^2 ds - \gamma \int_{\partial Z_\delta \cap \partial \Omega} |u|^2 ds \geq -A\gamma^2 \|u\|_{L^2(Z_\delta)}^2.$$

As $b^Z(u, u) = \int_{Z_\delta} |\nabla u|^2 dx - \gamma \int_{\partial Z_\delta \cap \partial \Omega} |u|^2 ds$, this can be transformed into

$$b^Z(u, u) - \gamma \|u\|_{L^2(\Sigma_\delta)}^2 \geq -A\gamma^2 \|u\|_{L^2(Z_\delta)}^2,$$

and as $E < 0$ this ends the proof. \square

Lemma V.4.8. *For any $m > 0$, there exists $c_0 > 0$ such that*

$$(b^Z - E)(v_\delta^0, v_\delta^0) = \mathcal{O}(\gamma^{-m}) \|u\|_{L^2(\Omega_\delta)}^2.$$

Proof. By the min-max principle, as $v_\delta^0 \in \Lambda_Z$, one has

$$E_1(B^Z) \|v_\delta^0\|_{L^2(Z_\delta)}^2 \leq b^Z(v_\delta^0, v_\delta^0) \leq E_{\mathcal{N}^\oplus}(B^Z) \|v_\delta^0\|_{L^2(Z_\delta)}^2.$$

Due to Proposition IV.5.2, we also have $E_1(B^Z) = \mathcal{O}(\gamma^2)$ and $E_{\mathcal{N}^\oplus}(B^Z) = \mathcal{O}(\gamma^2)$. Moreover, by the asymptotics (III.5.2), for any $m > 0$, there exists $c_0 > 0$ such that $E = -\gamma^2 + \mathcal{O}(\gamma^{-m}) > E_{\mathcal{N}^\oplus}(B^Z) \geq E_1(B^Z)$ and thus

$$(b^Z - E)(v_\delta^0, v_\delta^0) = \mathcal{O}(\gamma^2) \|v_\delta^0\|_{L^2(Z_\delta)}^2,$$

which concludes the proof thanks to Lemma V.4.5. \square

We are now able to give the proof of Proposition V.4.6. Let $u \in D(b)$. Then, for $u = v_\delta^0 + v_\delta$ one has

$$\|u\|_{L^2(\Sigma_\delta)}^2 \leq 2\|v_\delta^0\|_{L^2(\Sigma_\delta)}^2 + 2\|v_\delta\|_{L^2(\Sigma_\delta)}^2.$$

Applying Lemma V.4.7 to v_δ^0 and v_δ this gives

$$\begin{aligned} \|u\|_{L^2(\Sigma_\delta)}^2 &\leq \frac{2}{\gamma} \left((b^Z - E)(v_\delta^0, v_\delta^0) + A\gamma^2 \|v_\delta^0\|_{L^2(Z_\delta)}^2 \right) \\ &\quad + \frac{2}{\gamma} \left((b^Z - E)(v_\delta, v_\delta) + A\gamma^2 \|v_\delta\|_{L^2(Z_\delta)}^2 \right). \end{aligned} \quad (\text{V.4.20})$$

The first term on the right hand side is easily controlled: using Lemma V.4.8 together with Lemma V.4.5, we know that for any $m > 0$ there exists $c_0 > 0$ such that

$$(b^Z - E)(v_\delta^0, v_\delta^0) + A\gamma^2 \|v_\delta^0\|_{L^2(Z_\delta)}^2 = \mathcal{O}(\gamma^{-m}) \|u\|_{L^2(\Omega_\delta)}^2. \quad (\text{V.4.21})$$

Let us now focus on the second term. Writing $(b^Z - E)(v_\delta, v_\delta) = (b^Z - E)(u, u) - (b^Z - E)(v_\delta^0, v_\delta^0)$, we have by Lemma V.4.8,

$$(b^Z - E)(v_\delta, v_\delta) = (b^Z - E)(u, u) + \mathcal{O}(\gamma^{-m}) \|u\|_{L^2(\Omega_\delta)}^2. \quad (\text{V.4.22})$$

Moreover, as $v_\delta \in (\Lambda_Z)^{\perp n}$, the spectral theorem implies

$$b^Z(v_\delta, v_\delta) \geq E_{\mathcal{N}^\oplus+1}(B^Z) \|v_\delta\|_{L^2(Z_\delta)}^2,$$

which gives, due to the non-resonance condition (V.4.14),

$$\|v_\delta\|_{L^2(Z_\delta)}^2 \leq C\delta^2 (b^Z - E)(v_\delta, v_\delta), \quad C > 0. \quad (\text{V.4.23})$$

We can apply (V.4.22) to the above inequality to have

$$\|v_\delta\|_{L^2(Z_\delta)}^2 \leq C\delta^2 (b^Z - E)(u, u) + \mathcal{O}(\gamma^{-m})\|u\|_{L^2(\Omega_\delta)}^2. \quad (\text{V.4.24})$$

Gathering (V.4.22) and (V.4.23) gives the upper bound

$$(b^Z - E)(v_\delta, v_\delta) + A\gamma^2\|v_\delta\|_{L^2(Z_\delta)}^2 \leq C\log^2\gamma (b^Z - E)(u, u) + \mathcal{O}(\gamma^{-m})\|u\|_{L^2(\Omega_\delta)}^2.$$

This estimate and (V.4.21) finish the proof in view of (V.4.20). \square

In order to apply Proposition III.1.7 to our situation, we need to compare $\|u\|_{\mathcal{H}}^2$ and $\|Ju\|_{\mathcal{H}'}$.

Lemma V.4.9. *There exists $C > 0$ such that for any $u \in D(b)$ one has*

$$\|u\|_{\mathcal{H}}^2 - \|Ju\|_{\mathcal{H}'}^2 \leq C\frac{\log\gamma}{\sqrt{\gamma}} (b(u, u) + \|u\|_{\mathcal{H}}^2).$$

Proof. We first give a lower bound for $\|Ju\|_{\mathcal{H}'}$. Let $\eta > 0$ to be chosen later. Using the inequality

$$(a_1 - a_2)^2 \geq (1 - \eta)a_1^2 - \frac{1}{\eta}a_2^2, \quad a_1, a_2 \geq 0,$$

we get

$$\|Ju\|_{\mathcal{H}'}^2 \geq (1 - \eta) \sum_{v=1}^V \int_{I_{v,\delta}} |Pu(s_v)|^2 ds_v - \frac{2R}{\eta} \sum_{v=1}^V (|Pu(\lambda\delta)|^2 + |Pu(l_v - \lambda_{v+1}\delta)|^2),$$

with

$$R := \max_{v=1, \dots, V} \left(\max \left(\int_{I_{v,\delta}} |\rho^-(s_v)|^2 ds_v, \int_{I_{v,\delta}} |\rho^+(s_v)|^2 ds_v \right) \right).$$

Applying Lemma V.4.4 to the last summand we obtain the lower bound

$$\|Ju\|_{\mathcal{H}'}^2 \geq (1 - \eta) \sum_{v=1}^V \int_{I_{v,\delta}} |Pu(s_v)|^2 ds_v - \frac{2R}{\eta} \int_{\Sigma_\delta} |u|^2 ds.$$

Using Cauchy-Schwarz inequality and the fact that Φ is normalized we have

$$\sum_{v=1}^V \int_{I_{v,\delta}} |Pu(s_v)|^2 ds_v = \sum_{v=1}^V \int_{I_{v,\delta}} \left| \int_0^\delta \Phi(t_v)u(s_v, t_v) dt_v \right|^2 ds_v \leq \|u\|_{L^2(W_\delta)}^2,$$

and we can gather the previous inequalities to obtain

$$\begin{aligned} \|u\|_{\mathcal{H}}^2 - \|Ju\|_{\mathcal{H}'}^2 &\leq \|u\|_{L^2(Z_\delta)}^2 + \|u\|_{L^2(W_\delta)}^2 - \sum_{v=1}^V \int_{I_{v,\delta}} |Pu(s_v)|^2 ds_v \\ &\quad + \eta\|u\|_{L^2(W_\delta)}^2 + \frac{2R}{\eta} \int_{\Sigma_\delta} |u|^2 ds. \end{aligned} \quad (\text{V.4.25})$$

By the spectral theorem, as $E_2(\mathcal{A}_{\delta,\gamma}) \geq 0$ we have

$$\int_{I_{v,\delta}} \left(\int_0^\delta |\partial_{t_v} u|^2 dt_v - \gamma|u(s_v, \delta)|^2 - E\|Pu(s_v)\Phi\|_{L^2(0,\delta)}^2 \right) ds_v \geq 0,$$

and then

$$b^W(u, u) - E \sum_{v=1}^V \int_{I_{v,\delta}} \|Pu(s_v)\Phi\|_{L^2(0,\delta)}^2 ds_v \geq 0.$$

As $\int_{I_{v,\delta}} \|Pu(s_v)\Phi\|_{L^2(0,\delta)}^2 ds_v = \int_{I_{v,\delta}} |Pu(s_v)|^2 ds_v$ and $E < 0$ we can rewrite this inequality as follows:

$$\|u\|_{L^2(W_\delta)}^2 - \sum_{v=1}^V \int_{I_{v,\delta}} |Pu(s_v)|^2 ds_v \leq -\frac{1}{E} \left(b^W(u, u) - E\|u\|_{L^2(W_\delta)}^2 \right).$$

Using the fact that $E = \mathcal{O}(\gamma^2)$, this gives

$$\|u\|_{L^2(W_\delta)}^2 - \sum_{v=1}^V \int_{I_{v,\delta}} |Pu(s_v)|^2 ds_v \leq \frac{C}{\gamma^2} (b^W - E)(u, u), \quad C > 0. \quad (\text{V.4.26})$$

Moreover writing $\|u\|_{L^2(Z_\delta)}^2 = \|v_\delta\|_{L^2(Z_\delta)}^2 + \|v_\delta^0\|_{L^2(Z_\delta)}^2$, we can apply Lemma V.4.5 and (V.4.24) to obtain

$$\|u\|_{L^2(Z_\delta)}^2 \leq C\delta^2 (b^Z - E)(u, u) + \mathcal{O}(\gamma^{-m})\|u\|_{L^2(\Omega_\delta)}^2.$$

Using this upper bound and (V.4.26), the inequality (V.4.25) becomes

$$\begin{aligned} \|u\|_{\mathcal{H}}^2 - \|Ju\|_{\mathcal{H}'}^2 &\leq C\delta^2 \left((b^Z - E)(u, u) + (b^W - E)(u, u) \right) + \eta\|u\|_{L^2(W_\delta)}^2 \\ &\quad + \frac{2R}{\eta} \int_{\Sigma_\delta} |u|^2 ds \|u\|_{L^2(Z_\delta)}^2 + \mathcal{O}(\gamma^{-m})\|u\|_{L^2(\Omega_\delta)}^2. \end{aligned}$$

Applying Proposition V.4.6 to the trace term, we can rewrite it as

$$\begin{aligned} \|u\|_{\mathcal{H}}^2 - \|Ju\|_{\mathcal{H}'}^2 &\leq C\left(\delta^2 + \frac{\log^2 \gamma}{\gamma\eta}\right)(b^Z - E)(u, u) + C\delta^2(b^W - E)(u, u) \\ &\quad + C\left(\eta + \frac{\log^2 \gamma}{\gamma\eta}\right)\|u\|_{L^2(\Omega)}^2 + \mathcal{O}(\gamma^{-m})\|u\|_{L^2(\Omega_\delta)}^2. \end{aligned}$$

We conclude the proof by taking $\eta = \frac{\log \gamma}{\sqrt{\gamma}}$. □

Now we need to compare $b'(Ju, Ju)$ and $b(u, u)$.

Lemma V.4.10. *There exists $C > 0$ such that for any $u \in D(b)$ there holds*

$$b'(Ju, Ju) - b(u, u) \leq C \frac{\log \gamma}{\sqrt{\gamma}} \left(b(u, u) + \|u\|_{\mathcal{H}}^2 \right).$$

Proof. Let us first estimate $b'(Ju, Ju)$. By Cauchy-Schwarz inequality and the fact that Φ is normalized we have

$$\begin{aligned} \sum_{v=1}^V \int_{I_{v,\delta}} |(Pu)'(s_v)|^2 ds_v &= \sum_{v=1}^V \int_{I_{v,\delta}} \left| \int_0^\delta \partial_{s_v} u(s_v, t_v) \Phi(t_v) dt_v \right|^2 ds_v \\ &\leq \int_{W_\delta} |\partial_{s_v} u|^2 ds_v dt_v. \end{aligned} \quad (\text{V.4.27})$$

As $\int_0^\delta |f'|^2 dt_v - \gamma |f(\delta)|^2 - E \|f\|_{L^2(0,\delta)}^2 \geq 0$ for any $f \in H^1(0,\delta)$, one can write for any $u \in \dot{D}(b)$,

$$\int_{W_\delta} |\partial_{t_v} u|^2 dx - \gamma \int_{\partial W_\delta \cap \partial \Omega} |u|^2 ds - E \|u\|_{L^2(W_\delta)}^2 \geq 0,$$

and we can add this positive term to the right hand side of (V.4.27) to obtain

$$\sum_{v=1}^V \int_{I_{v,\delta}} |(Pu)'(s_v)|^2 ds_v \leq b^W(u, u) - E \|u\|_{L^2(W_\delta)}^2. \quad (\text{V.4.28})$$

Moreover, using Lemma V.4.4 we get

$$\begin{aligned} \sum_{v=1}^V \int_{I_{v,\delta}} |Pu(\lambda_v \delta)(\rho^-)'(s_v)|^2 ds_v \\ + \int_{I_{v,\delta}} |Pu(l_v - \lambda_{v+1} \delta)(\rho^+)'(s_v)|^2 ds_v \leq R' \int_{\Sigma_\delta} |u|^2 ds, \end{aligned} \quad (\text{V.4.29})$$

where

$$R' := \max_{v=1,\dots,V} \left(\max \left(\int_{I_{v,\delta}} |(\rho^-)'(s_v)|^2 ds_v, \int_{I_{v,\delta}} |(\rho^+)'(s_v)|^2 ds_v \right) \right).$$

Let $\beta \in (0, 1)$ to be chosen later. Using the inequality

$$(a_1 + a_2)^2 \leq (1 + \beta) a_1^2 + \frac{2}{\beta} a_2^2, \quad a_1, a_2 \geq 0,$$

and gathering the inequalities (V.4.28) and (V.4.29) we obtain the upper bound

$$b'(Ju, Ju) \leq (1 + \beta) \left(b^W(u, u) - E \|u\|_{L^2(W_\delta)}^2 \right) + \frac{4R'}{\beta} \int_{\Sigma_\delta} |u|^2 ds. \quad (\text{V.4.30})$$

On the other hand writing $u = v_\delta^0 + v_\delta$ as in (V.4.18) one has, due to the non-resonance condition (V.4.14),

$$(b^Z - E)(v_\delta, v_\delta) \geq \left(E_{\mathcal{N}^{\oplus+1}}(B^Z) - E \right) \|v_\delta\|_{L^2(Z_\delta)}^2 \geq 0,$$

and by Lemma V.4.8 we also have

$$(b^Z - E)(v_\delta^0, v_\delta^0) \geq -C\gamma^{-m} \|u\|_{L^2(\Omega_\delta)}^2, \quad C > 0.$$

We thus obtain, gathering the above inequalities,

$$(b^Z - E)(u, u) \geq -C\gamma^{-m} \|u\|_{L^2(\Omega_\delta)}^2. \quad (\text{V.4.31})$$

Using (V.4.30) and (V.4.31) we arrive at

$$b'(Ju, Ju) - b(u, u) \leq \beta \left(b^W - E \right) (u, u) + \frac{4R'}{\beta} \int_{\Sigma_\delta} |u|^2 ds + \mathcal{O}(\gamma^{-m}) \|u\|_{L^2(\Omega_\delta)}^2.$$

Applying Lemma V.4.6 to the trace term gives

$$\begin{aligned} b'(Ju, Ju) - b(u, u) &\leq \beta \left(b^W - E \right) (u, u) \\ &\quad + C \frac{\log^2 \gamma}{\beta \gamma} \left((b^Z - E)(u, u) + \|u\|_{L^2(\Omega_\delta)}^2 \right) + \mathcal{O}(\gamma^{-m}) \|u\|_{L^2(\Omega_\delta)}^2. \end{aligned}$$

This concludes the proof choosing $\beta = \frac{\log \gamma}{\sqrt{\gamma}}$.

□

Now let $\epsilon_1 = \epsilon_2 = C \frac{\log \gamma}{\gamma}$. Due to the inequality (V.4.15), we have

$$(1 + E_n(B))^{-1} \geq \epsilon_1,$$

and thanks to Lemma V.4.9 and Lemma V.4.10 we can apply Proposition III.1.7 to obtain

$$E_n(B') \leq E_n(B) + \mathcal{O}\left(\frac{\log \gamma}{\sqrt{\gamma}}\right).$$

Recall that $E_n(B) \leq E_{\mathcal{N}^{\oplus+n}}(Q) + \gamma^2 + \mathcal{O}(\gamma^{-m})$ and $E_n(B') = E_n(D) + \mathcal{O}(\delta)$. This proves the following proposition which finishes the proof of Theorem II.2.8.

Proposition V.4.11. *For any fixed $j \in \mathbb{N}$ one has*

$$E_{\mathcal{N}^{\oplus+j}}(Q^\gamma) \geq -\gamma^2 + E_j(D) + \mathcal{O}\left(\frac{\log \gamma}{\sqrt{\gamma}}\right).$$

Remark V.4.12. Let us make a small comment on $\delta := c_0 \frac{\log \gamma}{\gamma}$. The choice of δ is lead by two conditions which have to be satisfied:

$$\delta \rightarrow 0 \text{ and } \delta\gamma \rightarrow +\infty \text{ as } \gamma \rightarrow +\infty.$$

Multiple choices of δ are then possible and our choice may be not optimal. However, one can see easily that the obvious choice $\delta = \mathcal{O}(\gamma^{-m})$ with $m \in (0, 1)$ does not improve our result. On one hand, the Dirichlet bracketing will give a remainder of order $\mathcal{O}(\gamma^{-m})$ in the upper bound. On the other hand, the remainder in the lower bound is given by Proposition V.4.6. Indeed, in Proposition V.4.6 we obtain an upper bound for the trace of functions $u \in D(b)$ involving $\delta^2\gamma$, and the remainder in the lower bound is in fact given by $\epsilon_1 = \epsilon_2 = \sqrt{\delta^2\gamma} = \mathcal{O}(\gamma^{-m+\frac{1}{2}})$.

Chapter VI

Robin Laplacians on curvilinear polygons

Abstract

Let $\Omega \subset \mathbb{R}^2$ be a curvilinear polygon. In this chapter we study the Robin Laplacian Q_Ω^γ acting on $L^2(\Omega)$ as $Q_\Omega^\gamma u = -\Delta u$ with the Robin boundary condition $\partial_\nu u = \gamma u$ on $\partial\Omega$, where $\gamma > 0$ and ν is the *outward* unit normal. This operator has a compact resolvent and thus its spectrum is purely discrete: we denote by $E_n(Q_\Omega^\gamma)$ its eigenvalues enumerated in the non-decreasing order and counting the multiplicities. We are interested in the behavior of the eigenvalues of Q_Ω^γ as γ becomes large. In Section VI.1 and Section VI.2 we introduce the test functions used in the proofs of this chapter and some technical results. In Section VI.3, we prove that there exists $N_\Omega \in \mathbb{N}$ such that the asymptotics of the N_Ω first eigenvalues of Q_Ω^γ are determined at the leading order by those of model operators associated with the vertices: the Robin Laplacians acting on the tangent sectors associated with $\partial\Omega$. We also prove that the associated eigenfunctions are localized near the convex vertices of Ω . In Section VI.4, we prove a Weyl asymptotics for the eigenvalue counting function of Q_Ω^γ for a threshold depending on γ . When the threshold is negative the leading term is the same as the one for smooth domains, while we obtain the same leading term as the one of the classical Weyl law for Dirichlet or Neumann Laplacians when the threshold is positive. Finally, in Section VI.5 we give the first order in the asymptotics of the further eigenvalues $E_{N_\Omega+j}(Q_\Omega^\gamma)$.

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VI.1 Preliminaries

VI.1.1 Construction and properties of weak quasi-modes

In this chapter Ω denotes a curvilinear polygon in the sense of Definition II.2.2. As there is no ambiguity we still denote

$$Q^\gamma := Q_\Omega^\gamma.$$

We keep the same notation as in Chapter V, namely the model operator is defined by

$$T^\oplus := \bigoplus_{v \in \mathcal{V}} T_{\alpha_v}, \text{ and } \mathcal{N}^\oplus := \sum_{v \in \mathcal{V}} \mathcal{N}_v.$$

We denote by $\Lambda^\oplus := \{\lambda_l, 1 \leq l \leq K^\oplus\}$ the eigenvalues of T^\oplus enumerated in the increasing order and counted *without* multiplicity, namely : $\lambda_1 < \lambda_2 < \dots < \lambda_{K^\oplus}$. For $1 \leq l \leq K^\oplus$ we introduce

$$\mathcal{S}_l := \{(n, v) : v \in \mathcal{V}, 1 \leq n \leq \mathcal{N}_v : E_n(T_{\alpha_v}) = \lambda_l\},$$

and $m_l := \#\mathcal{S}_l$. Defined in this way, m_l is then the multiplicity of λ_l as an eigenvalue of T^\oplus and $\sum_{l=1}^{K^\oplus} m_l = \mathcal{N}^\oplus$. Finally we denote by $E^{\max} := E_{\mathcal{N}^\oplus}(T^\oplus)$.

We can now introduce the test functions which will play the role of the quasi-modes we used in the proofs for polygons with straight edges. Recall that $\psi_n^{\gamma, v}$ are the eigenfunctions of the γ -Robin Laplacian acting on the infinite sector U_{α_v} introduced in Section V.1.1. For $v \in \mathcal{V}$ and $1 \leq n \leq \mathcal{N}_v$ we set

$$\phi_n^{\gamma, v} := \psi_n^{\gamma, v} \circ F_v.$$

where F_v is the C^2 -diffeomorphism mapping $\overline{B(v, r_v) \cap \Omega}$ onto $\overline{B(0, r_v) \cap U_{\alpha_v}}$, as described in Section II.2.2. Then, $\phi_n^{\gamma, v} \in H^1(\Omega \cap B(v, r_v))$. Let $\varphi \in C^\infty(\mathbb{R}_+)$ be a smooth cut-off function satisfying $0 \leq \varphi \leq 1$, $\varphi(t) = 1$ if $0 \leq t \leq 1$, and $\varphi = 0$ if $t \geq 2$. We introduce the smooth radial cut-off function χ_v^γ defined as follows:

$$\chi_v^\gamma(x) = \varphi(|x - v|/\gamma^\beta), \quad x \in \Omega, \tag{VI.1.1}$$

where $\beta \in (1/2, 1)$ will be chosen later. Notice that for γ large enough $\text{supp } \chi_v^\gamma \subset B(v, r_v)$ and $\text{supp } \chi_v^\gamma \cap \text{supp } \chi_{v'}^\gamma = \emptyset$ for $v \neq v'$. In the following, γ will be supposed large enough such that these conditions are satisfied. We define

$$\tilde{\phi}_n^{\gamma, v} := \phi_n^{\gamma, v} \chi_v^\gamma \quad \text{on } \Omega. \tag{VI.1.2}$$

Defined as above, $\tilde{\phi}_n^{\gamma, v} \in D(q^\gamma) := H^1(\Omega)$ but $\tilde{\phi}_n^{\gamma, v}$ does not belong to the domain of the operator Q^γ as it does not satisfy the Robin boundary condition: we call it a *weak quasi-mode*.

In order to list some properties of the weak quasi-modes we will need some additional results and notations.

Notation.

- (a) If $D \subset \mathbb{R}^2$, we set $D_{y,r} := D \cap B(y, r)$.
- (b) Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be C^1 . We denote by Jg the determinant of the Jacobian matrix of g , namely

$$Jg := \det(\nabla g).$$

Lemma VI.1.1. *Let $v \in \mathcal{V}$, $\psi \in H^1(U_{\alpha_v})$ and $\phi := \psi \circ F_v$. Then, $\phi \in H^1(\Omega_{v,2\gamma^{-\beta}})$ for γ large enough and there exists a constant $C > 0$ such that*

$$\left| \int_{\Omega_{v,2\gamma^{-\beta}}} |\phi(x)|^2 dx - \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} |\psi(u)|^2 du \right| \leq C\gamma^{-\beta} \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} |\psi(u)|^2 du, \quad (\text{VI.1.3})$$

$$\left| \int_{\Omega_{v,2\gamma^{-\beta}}} |\nabla\phi(x)|^2 dx - \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} |\nabla\psi(u)|^2 du \right| \leq C\gamma^{-\beta} \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} |\nabla\psi(u)|^2 du, \quad (\text{VI.1.4})$$

$$\left| \int_{\Gamma_{v,2\gamma^{-\beta}}} |\phi(s)|^2 ds - \int_{\Sigma_{v,2\gamma^{-\beta}}} |\psi(s)|^2 ds \right| \leq C\gamma^{-\beta} \int_{\Sigma_{v,2\gamma^{-\beta}}} |\psi(s)|^2 ds, \quad (\text{VI.1.5})$$

where $\Gamma_{v,r} := \partial\Omega_{v,r} \setminus \partial B(v,r)$ and $\Sigma_{v,r} := \partial(U_{\alpha_v})_{0,r} \setminus \partial B(0,r)$.

Proof. We first want to estimate the L^2 -norm of ϕ . By change of variables,

$$\int_{\Omega_{v,2\gamma^{-\beta}}} |\phi(x)|^2 dx = \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} |\psi(u)|^2 |JF_v^{-1}(u)| du.$$

As F_v^{-1} is C^2 , then $u \mapsto JF_v^{-1}(u)$ is also C^2 and by Taylor-Lagrange for all $u \in B(0,2\gamma^{-\beta})$ we have

$$|JF_v^{-1}(u) - 1| \leq C\gamma^{-\beta}. \quad (\text{VI.1.6})$$

Writing

$$\left| \int_{\Omega_{v,2\gamma^{-\beta}}} |\phi(x)|^2 dx - \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} |\psi(u)|^2 du \right| \leq \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} |\psi(u)|^2 \left| |JF_v^{-1}(u)| - 1 \right| du,$$

finishes the proof of (VI.1.3).

We now estimate the L^2 -norm of $\nabla\phi$. By definition of ϕ we have $\nabla\phi(x) = \nabla\psi(F_v(x))\nabla F_v(x)$. Then,

$$\int_{\Omega_{v,2\gamma^{-\beta}}} |\nabla\phi(x)|^2 dx = \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} |\nabla\psi(u)\nabla F_v(F_v^{-1}(u))|^2 |JF_v^{-1}(u)| du.$$

Using again Taylor-Lagrange, we know that for all $u \in B(0,2\gamma^{-\beta})$,

$$|\nabla F_v(F_v^{-1}(u)) - I_2| \leq C\gamma^{-\beta}. \quad (\text{VI.1.7})$$

We denote $I := \left| \int_{\Omega_{v,2\gamma^{-\beta}}} |\nabla\phi(x)|^2 dx - \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} |\nabla\psi(u)|^2 du \right|$. Then,

$$\begin{aligned} I \leq & \left| \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} \left(|\nabla\psi(u)\nabla F_v(F_v^{-1}(u))|^2 - |\nabla\psi(u)|^2 \right) |JF_v^{-1}(u)| du \right| \\ & + \left| \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} |\nabla\psi(u)|^2 (|JF_v^{-1}(u)| - 1) du \right|. \end{aligned}$$

First we have using (VI.1.7),

$$\left| |\nabla\psi(u)\nabla F_v(F_v^{-1}(u))|^2 - |\nabla\psi(u)|^2 \right| \leq C\gamma^{-\beta} |\nabla\psi(u)|^2,$$

and using (VI.1.6) we also get

$$\left| |\nabla\psi(u)\nabla F_v(F_v^{-1}(u))|^2 - |\nabla\psi(u)|^2 \right| |JF^{-1}(u)| \leq C\gamma^{-\beta} |\nabla\psi(u)|^2,$$

which gives us the upper bound for the first term of I . We can use again (VI.1.6) for the second term and we get (VI.1.4).

We are now interested in the integral along the boundary. Recall that by assumption $\partial\Omega = \bigcup_{k=1}^M \overline{\Gamma_k}$. Without loss of generality, we suppose that two components Γ_j, Γ_k intersect iff $k = j + 1$ or $k = j - 1$. Then, there exists $j \in \{1, \dots, M\}$ such that $v = \overline{\Gamma_j} \cap \overline{\Gamma_{j+1}}$ and $\Gamma_{v,2\gamma^{-\beta}}$ is composed by two connected C^4 components

$$\Gamma_{v,2\gamma^{-\beta}}^j := \Gamma_j \cap B(v, 2\gamma^{-\beta}), \text{ and } \Gamma_{v,2\gamma^{-\beta}}^{j+1} := \Gamma_{j+1} \cap B(v, 2\gamma^{-\beta}),$$

such that $\overline{\Gamma_{v,2\gamma^{-\beta}}^j} \cup \overline{\Gamma_{v,2\gamma^{-\beta}}^{j+1}} = \overline{\Gamma_{v,2\gamma^{-\beta}}}$. We can introduce

$$\overline{\Sigma_{v,2\gamma^{-\beta}}^j} := F_v(\overline{\Gamma_{v,2\gamma^{-\beta}}^j}), \text{ and } \overline{\Sigma_{v,2\gamma^{-\beta}}^{j+1}} := F_v(\overline{\Gamma_{v,2\gamma^{-\beta}}^{j+1}}),$$

such that $\overline{\Sigma_{v,2\gamma^{-\beta}}} = \overline{\Sigma_{v,2\gamma^{-\beta}}^j} \cup \overline{\Sigma_{v,2\gamma^{-\beta}}^{j+1}}$. Notice that

$$\Sigma_{v,2\gamma^{-\beta}}^l = \{(t, \pm \tan \alpha_v t), t \in (0, 2\gamma^{-\beta} \cos(\alpha_v))\}, \quad l = j, j + 1.$$

Thus,

$$\int_{\Gamma_{v,2\gamma^{-\beta}}} |\phi(s)|^2 ds = \sum_{l=j, j+1} \int_{\Gamma_{v,2\gamma^{-\beta}}^l} |\phi(s)|^2 ds = \sum_{l=j, j+1} \int_0^{2\gamma^{-\beta} \cos(\alpha_v)} |\psi(t, \pm \tan \alpha_v t)|^2 \rho^l(t) dt,$$

where

$$\rho^l(t) := \sqrt{\det \left(\left(\frac{d}{dt} (F_v^{-1}(t, \pm \tan \alpha_v t)) \right)^T \left(\frac{d}{dt} (F_v^{-1}(t, \pm \tan \alpha_v t)) \right) \right)} = \left| \nabla F_v^{-1}(t, \pm \tan \alpha_v t) T^l \right|,$$

with $T^l = \begin{pmatrix} 1 \\ \pm \tan \alpha_v \end{pmatrix}$. Finally we can write

$$\begin{aligned} \int_{\Gamma_{v,2\gamma^{-\beta}}} |\phi(s)|^2 ds &= \sum_{l=j, j+1} \int_{\Sigma_{v,2\gamma^{-\beta}}^l} |\psi(s)|^2 |\nabla F_v^{-1}(s) T^l| \cos(\alpha_v) ds \\ &= \int_{\Sigma_{v,2\gamma^{-\beta}}} |\psi(s)|^2 |\nabla F_v^{-1}(s) T(s)| \cos(\alpha_v) ds, \end{aligned}$$

where $T(s) = T^l$ when $s \in \Sigma_{v,2\gamma^{-\beta}}^l$, $l = j, j + 1$. As $t \mapsto \nabla F_v^{-1}(t, \pm \tan \alpha_v t)$ is C^1 on $(0, 2\gamma^{-\beta} \cos(\alpha_v))$, by Taylor-Lagrange, for all $t \in (0, 2\gamma^{-\beta} \cos(\alpha_v))$,

$$|\nabla F_v^{-1}(t, \pm \tan \alpha_v t) - I_2| \leq C\gamma^{-\beta}. \quad (\text{VI.1.8})$$

Then, $|\rho^l(t) - \frac{1}{\cos \alpha_v}| \leq \frac{C\gamma^{-\beta}}{\cos \alpha_v}$, as $|T^l| = \cos^{-1} \alpha_v$. Finally,

$$\left| \int_{\Gamma_{v,2\gamma^{-\beta}}} |\phi(s)|^2 ds - \int_{\Sigma_{v,2\gamma^{-\beta}}} |\psi(s)|^2 ds \right| \leq \int_{\Sigma_{v,2\gamma^{-\beta}}} |\psi(s)|^2 \left| |\nabla F_v^{-1}(s) T(s)| \cos(\alpha_v) - 1 \right| ds,$$

which concludes the proof of (VI.1.5). \square

Remark VI.1.2. Let us briefly explain the choice of the cut-off functions χ_v^γ with $\beta < 1$. Following the steps of the proof of Theorem V.2.1, we need to prove some estimates on the weak quasi-modes to apply the min-max principle. To prove these estimates, we use the exponential decay of the eigenfunctions $\psi_n^{\gamma,v}$, stated in Theorem IV.4.1. Then, it is necessary that the size of the supports of the cut-off functions decreases with respect to γ , in order to control the error term due to the change of variable F_v at each convex vertex v . With our choice, $\text{supp } \chi_v^\gamma \subset B(v, 2\gamma^{-\beta})$ and we are lead to estimate, after change of variables,

$$\int_{U_{\alpha_v} \cap B(0, 2\gamma^{-\beta})} |\psi_n^{\gamma,v}|^2 du.$$

Using the same tricks as before, this rewrites as

$$\int_{U_{\alpha_v} \cap B(0, 2\gamma^{-\beta})} |\psi_n^{\gamma,v}|^2 du = \int_{U_{\alpha_v}} |\psi_n^{\gamma,v}|^2 du - \int_{U_{\alpha_v} \setminus \overline{B(0, 2\gamma^{-\beta})}} |\psi_n^{\gamma,v}|^2 du,$$

and the second term of the right hand side has to be small as γ is large. To prove this, let us apply Theorem IV.4.1:

$$\int_{U_{\alpha_v} \setminus \overline{B(0, 2\gamma^{-\beta})}} |\psi_n^{\gamma,v}|^2 du \leq C e^{-2(1-\epsilon)\gamma\sqrt{-1-E_n(T_{\alpha_v})}2\gamma^{-\beta}},$$

and we immediately see that $\beta < 1$ is necessary to have a suitable exponential decay. The choice $\beta > 1/2$ becomes clear later.

We can now summarize some properties of the weak quasi-modes. The ideas are the same as the ones for polygons with straight edges and the following results are based on the decay property of the eigenfunctions of the Robin Laplacian acting on infinite sectors, see Theorem IV.4.1.

Proposition VI.1.3. *Let $\beta \in (\frac{1}{2}, 1)$. For all $i, j \in \{1, \dots, \mathcal{N}_v\}$ and for γ large enough we have*

$$|\langle \tilde{\phi}_i^{\gamma,v}, \tilde{\phi}_j^{\gamma,v} \rangle_{L^2(\Omega)} - \delta_{i,j}| \leq C\gamma^{-\beta}, \quad (\text{VI.1.9})$$

$$|q^\gamma(\tilde{\phi}_i^{\gamma,v}, \tilde{\phi}_j^{\gamma,v}) - \delta_{i,j}\gamma^2 E_i(T_{\alpha_v})| \leq C\gamma^{2-\beta}, \quad (\text{VI.1.10})$$

where $\delta_{i,j}$ denotes the Kronecker symbol.

Proof. Let us first estimate the L^2 -norm of $\tilde{\phi}_i^{\gamma,v}$. We have immediately

$$\|\tilde{\phi}_i^{\gamma,v}\|_{L^2(\Omega)}^2 = \int_{\Omega_{v, 2\gamma^{-\beta}}} |\chi_v^\gamma|^2 |\phi_i^{\gamma,v}|^2 dx \leq \int_{(U_{\alpha_v})_{0, 2\gamma^{-\beta}}} |\psi_i^{\gamma,v}|^2 |JF_v^{-1}| du.$$

We conclude thanks to (VI.1.6). For the lower bound we remark that

$$\|\tilde{\phi}_i^{\gamma,v}\|_{L^2(\Omega)}^2 \geq \int_{\Omega_{v, \gamma^{-\beta}}} |\phi_i^{\gamma,v}|^2 dx = \int_{(U_{\alpha_v})_{0, \gamma^{-\beta}}} |\psi_i^{\gamma,v}|^2 |JF_v^{-1}| du.$$

We can use again (VI.1.6) to obtain

$$\|\tilde{\phi}_i^{\gamma,v}\|_{L^2(\Omega)}^2 \geq \int_{(U_{\alpha_v})_{0, \gamma^{-\beta}}} |\psi_i^{\gamma,v}|^2 du - C\gamma^{-\beta}.$$

Writing

$$\int_{(U_{\alpha_v})_{0, \gamma^{-\beta}}} |\psi_i^{\gamma,v}|^2 du = \int_{U_{\alpha_v}} |\psi_i^{\gamma,v}|^2 du - \int_{U_{\alpha_v} \setminus \overline{B(0, \gamma^{-\beta})}} |\psi_i^{\gamma,v}|^2 du,$$

and using the Agmon-type estimate for $\psi_i^{\gamma,v}$ proved in Theorem IV.4.1 to get a lower bound for the second term permits us to conclude the proof of (VI.1.9) when $i = j$.

Now, if $i \neq j$,

$$\left| \langle \tilde{\phi}_i^{\gamma,v}, \tilde{\phi}_j^{\gamma,v} \rangle - \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} \psi_i^{\gamma,v} \overline{\psi_j^{\gamma,v}} du \right| = \left| \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} \psi_i^{\gamma,v} \overline{\psi_j^{\gamma,v}} \left((\chi_v^\gamma \circ F_v^{-1}(u))^2 |JF_v^{-1}(u)| - 1 \right) du \right|.$$

By Taylor-Lagrange, for all $u \in B(0, 2\gamma^{-\beta})$,

$$\left| (\chi_v^\gamma \circ F_v^{-1}(u))^2 |JF_v^{-1}(u)| - 1 \right| \leq C\gamma^{-\beta}. \quad (\text{VI.1.11})$$

Then we get, using the Cauchy-Schwarz inequality,

$$\left| \langle \tilde{\phi}_i^{\gamma,v}, \tilde{\phi}_j^{\gamma,v} \rangle - \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} \psi_i^{\gamma,v} \overline{\psi_j^{\gamma,v}} du \right| \leq C\gamma^{-\beta}.$$

We have now to estimate $\int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} \psi_i^{\gamma,v} \overline{\psi_j^{\gamma,v}} du$ more precisely. As $(\psi_n^{\gamma,v})_{n \leq \mathcal{N}_v}$ is orthonormal we can write

$$\int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} \psi_i^{\gamma,v} \overline{\psi_j^{\gamma,v}} du = - \int_{U_{\alpha_v} \setminus B(0,2\gamma^{-\beta})} \psi_i^{\gamma,v} \overline{\psi_j^{\gamma,v}} du.$$

We then obtain, using again Cauchy-Schwarz and Theorem IV.4.1,

$$\left| \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} \psi_i^{\gamma,v} \overline{\psi_j^{\gamma,v}} du \right| \leq \int_{U_{\alpha_v} \setminus B(0,\gamma^{-\beta})} |\psi_i^{\gamma,v} \overline{\psi_j^{\gamma,v}}| du \leq C e^{-2(1-\epsilon)\gamma^{1-\beta} \sqrt{-1-E^{\max}}},$$

which ends the proof of (VI.1.9).

Let us focus of (VI.1.10). We first expand $q^\gamma(\tilde{\phi}_i^{\gamma,v}, \tilde{\phi}_i^{\gamma,v})$:

$$\begin{aligned} q^\gamma(\tilde{\phi}_i^{\gamma,v}, \tilde{\phi}_i^{\gamma,v}) &= \int_{\Omega} |\chi_v^\gamma|^2 |\nabla \phi_i^{\gamma,v}|^2 dx - \gamma \int_{\partial\Omega} |\chi_v^\gamma|^2 |\phi_i^{\gamma,v}|^2 ds \\ &\quad + \int_{\Omega} |\nabla \chi_v^\gamma|^2 |\phi_i^{\gamma,v}|^2 dx + 2\Re \int_{\Omega} \chi_v^\gamma \nabla \phi_i^{\gamma,v} \cdot \overline{\nabla \chi_v^\gamma \phi_i^{\gamma,v}} dx. \end{aligned}$$

Notice that $\text{supp } \nabla \chi_v^\gamma \subset A_v := B(v, 2\gamma^{-\beta}) \setminus \overline{B(v, \gamma^{-\beta})}$ and by definition of χ_v^γ we also have $|\nabla \chi_v^\gamma|^2 \leq \gamma^{2\beta} \|\varphi'\|_\infty^2$. Then using (VI.1.6) we can write

$$\left| \int_{\Omega} |\nabla \chi_v^\gamma|^2 |\phi_i^{\gamma,v}|^2 dx \right| \leq \gamma^{2\beta} (1 + C\gamma^{-\beta}) \|\varphi'\|_\infty^2 \int_{U_{\alpha_v} \setminus B(0,\gamma^{-\beta})} |\psi_i^{\gamma,v}|^2 du.$$

We use Theorem IV.4.1 to obtain

$$\left| \int_{\Omega} |\nabla \chi_v^\gamma|^2 |\phi_i^{\gamma,v}|^2 dx \right| \leq \gamma^{2\beta} (1 + C\gamma^{-\beta}) \|\varphi'\|_\infty^2 C e^{-2(1-\epsilon)\gamma^{1-\beta} \sqrt{-1-E^{\max}}}. \quad (\text{VI.1.12})$$

We can obtain the same kind of upper bound for the cross-term using Cauchy-Schwarz,

$$\left| \int_{\Omega} \chi_v^\gamma \nabla \phi_i^{\gamma,v} \cdot \overline{\nabla \chi_v^\gamma \phi_i^{\gamma,v}} dx \right| \leq \left(\int_{\Omega \cap A_v} |\chi_v^\gamma|^2 |\nabla \phi_i^{\gamma,v}|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega \cap A_v} |\nabla \chi_v^\gamma|^2 |\phi_i^{\gamma,v}|^2 dx \right)^{\frac{1}{2}},$$

the estimate (VI.1.6) and Theorem IV.4.1,

$$\left| \int_{\Omega} \chi_v^\gamma \nabla \phi_i^{\gamma,v} \cdot \overline{\nabla \chi_v^\gamma \phi_i^{\gamma,v}} dx \right| \leq \gamma^\beta \|\varphi'\|_\infty (1 + C\gamma^{-\beta}) C e^{-2(1-\epsilon)\gamma^{1-\beta} \sqrt{-1-E^{\max}}}. \quad (\text{VI.1.13})$$

Combining (VI.1.12) and (VI.1.13) we get

$$\left| \int_{\Omega} |\nabla \chi_v^\gamma|^2 |\phi_i^{\gamma,v}|^2 dx + 2\Re \int_{\Omega} \chi_v^\gamma \nabla \phi_i^{\gamma,v} \cdot \overline{\nabla \chi_v^\gamma \phi_i^{\gamma,v}} dx \right| \leq \gamma^{2\beta} C e^{-2(1-\epsilon)\gamma^{1-\beta}\sqrt{-1-E^{\max}}}. \quad (\text{VI.1.14})$$

Let us now focus on the main term. First, using (VI.1.4) we obtain

$$\int_{\Omega} |\chi_v^\gamma|^2 |\nabla \phi_i^{\gamma,v}|^2 dx \leq \int_{U_{\alpha_v}} |\nabla \psi_i^{\gamma,v}|^2 du + C\gamma^{2-\beta} \|\nabla \psi_i^{1,v}\|_{L^2(U_{\alpha_v})}^2,$$

and

$$\begin{aligned} \int_{\Omega} |\chi_v^\gamma|^2 |\nabla \phi_i^{\gamma,v}|^2 dx &\geq \int_{U_{\alpha_v}} |\nabla \psi_i^{\gamma,v}|^2 du - C\gamma^{2-\beta} \|\nabla \psi_i^{1,v}\|_{L^2(U_{\alpha_v})}^2 \\ &\quad - (1 - C\gamma^{-\beta}) C e^{-2(1-\epsilon)\gamma^{1-\beta}\sqrt{-1-E^{\max}}}. \end{aligned}$$

For the boundary term, we use (VI.1.5) to obtain

$$\int_{\partial\Omega} |\chi_v^\gamma|^2 |\phi_i^{\gamma,v}|^2 ds \leq \int_{\partial U_{\alpha_v}} |\psi_i^{\gamma,v}|^2 ds + C\gamma^{1-\beta} \int_{\partial U_{\alpha_v}} |\psi_i^{1,v}|^2 ds,$$

and,

$$\int_{\partial\Omega} |\chi_v^\gamma|^2 |\phi_i^{\gamma,v}|^2 ds \geq \int_{\partial U_{\alpha_v}} |\psi_i^{\gamma,v}|^2 ds - C\gamma^{1-\beta} \int_{\partial U_{\alpha_v}} |\psi_i^{1,v}|^2 ds - (1 - C\gamma^{-\beta}) \int_{\partial U_{\alpha_v} \setminus \Sigma_{v,\gamma^{-\beta}}} |\psi_i^{\gamma,v}|^2 ds.$$

As $U_{\alpha_v} \setminus \overline{B(v, \gamma^{-\beta})}$ is a Lipschitz domain, there exists a constant K such that

$$\|\psi_i^{\gamma,v}\|_{L^2(\partial U_{\alpha_v} \setminus \Sigma_{v,\gamma^{-\beta}})}^2 \leq \gamma^\beta K \|\psi_i^{\gamma,v}\|_{H^1(U_{\alpha_v} \setminus \overline{B(v, \gamma^{-\beta})})}^2,$$

for all $v \in \mathcal{V}$ and $i \leq \mathcal{N}_v$. Then we can use Theorem IV.4.1 to get

$$\int_{\partial\Omega} |\chi_v^\gamma|^2 |\phi_i^{\gamma,v}|^2 ds \geq \int_{\partial U_{\alpha_v}} |\psi_i^{\gamma,v}|^2 ds - C\gamma^{1-\beta} \int_{\partial U_{\alpha_v}} |\psi_i^{1,v}|^2 ds - C\gamma^\beta e^{-2(1-\epsilon)\gamma^{1-\beta}\sqrt{-1-E^{\max}}}.$$

This concludes the proof of (VI.1.10) when $i = j$ as $t_{\alpha_v}^\gamma(\psi_i^{\gamma,v}, \psi_i^{\gamma,v}) = \gamma^2 E_i(T_{\alpha_v})$.

Let $i \neq j$. We can write

$$q^\gamma(\tilde{\phi}_i^{\gamma,v}, \tilde{\phi}_j^{\gamma,v}) = \int_{\Omega} |\chi_v^\gamma|^2 \nabla \phi_i^{\gamma,v} \overline{\nabla \phi_j^{\gamma,v}} dx - \gamma \int_{\partial\Omega} |\chi_v^\gamma|^2 \phi_i^{\gamma,v} \overline{\phi_j^{\gamma,v}} ds + I(\gamma),$$

where

$$I(\gamma) := \int_{\Omega} \left(\chi_v^\gamma \nabla \phi_i^{\gamma,v} \overline{\nabla \chi_v^\gamma \phi_j^{\gamma,v}} dx + \int_{\Omega} \phi_i^{\gamma,v} \nabla \chi_v^\gamma \overline{\nabla \phi_j^{\gamma,v} \chi_v^\gamma} + \int_{\Omega} |\nabla \chi_v^\gamma|^2 \phi_i^{\gamma,v} \overline{\phi_j^{\gamma,v}} \right) dx$$

We first estimate $I(\gamma)$ using the same tools as before. We have

$$\left| \int_{\Omega} |\nabla \chi_v^\gamma|^2 \phi_i^{\gamma,v} \overline{\phi_j^{\gamma,v}} dx \right| \leq \gamma^{2\beta} \|\varphi'\|_\infty^2 (1 + C\gamma^{-\beta}) C e^{-2(1-\epsilon)\gamma^{1-\beta}\sqrt{-1-E^{\max}}}.$$

Notice that the other terms in the brackets are symmetric with respect to i and j , it is then sufficient to estimate one of them

$$\left| \int_{\Omega} \chi_v^\gamma \nabla \phi_i^{\gamma,v} \overline{\nabla \chi_v^\gamma \phi_j^{\gamma,v}} dx \right| \leq \gamma^\beta \|\varphi'\|_\infty \left((1 + C\gamma^{-\beta})(1 + C\gamma^{-\beta}) \right)^{\frac{1}{2}} C e^{-2(1-\epsilon)\gamma^{1-\beta}\sqrt{-1-E^{\max}}}.$$

Then,

$$|I(\gamma)| \leq \gamma^{2\beta} C e^{-2(1-\epsilon)\gamma^{1-\beta}\sqrt{-1-E^{\max}}}. \quad (\text{VI.1.15})$$

Let us now focus on the main term. By Taylor-Lagrange, for all $u \in B(0, 2\gamma^{-\beta})$ and for all $t \in (0, 2\gamma^{-\beta})$ we have

$$|(\chi_v^\gamma \circ F_v^{-1}(u))^2 JF_v^{-1}(u) \nabla F_v(F_v^{-1}(u)) - I_2| \leq C\gamma^{-\beta}, \quad (\text{VI.1.16})$$

$$|(\chi_v^\gamma \circ F_v^{-1}(t, \pm \tan \alpha_v t))^2 \nabla F_v^{-1}(t, \pm \tan \alpha_v t) - I_2| \leq C\gamma^{-\beta}. \quad (\text{VI.1.17})$$

Then by (VI.1.16), (VI.1.17) and Theorem IV.4.1 we get

$$\left| \int_{\Omega} |\chi_v^\gamma|^2 \nabla \phi_i^{\gamma,v} \overline{\nabla \phi_j^{\gamma,v}} dx - \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} \nabla \psi_i^{\gamma,v} \overline{\nabla \psi_j^{\gamma,v}} dx \right| \leq C\gamma^{2-\beta} + C e^{-2(1-\epsilon)\gamma^{1-\beta}\sqrt{-1-E^{\max}}},$$

and

$$\left| \int_{\partial\Omega} |\chi_v^\gamma|^2 \phi_i^{\gamma,v} \overline{\phi_j^{\gamma,v}} ds - \int_{\Sigma_{v,2\gamma^{-\beta}}} \psi_i^{\gamma,v} \overline{\psi_j^{\gamma,v}} ds \right| \leq C\gamma^{2-\beta} + C e^{-2(1-\epsilon)\gamma^{1-\beta}\sqrt{-1-E^{\max}}}.$$

As $i \neq j$, by the spectral theorem $t_{\alpha_v}^\gamma(\psi_i^{\gamma,v}, \psi_j^{\gamma,v}) = 0$. Then,

$$\left| \int_{\Omega} |\chi_v^\gamma|^2 \nabla \phi_i^{\gamma,v} \overline{\nabla \phi_j^{\gamma,v}} dx - \gamma \int_{\partial\Omega} |\chi_v^\gamma|^2 \phi_i^{\gamma,v} \overline{\phi_j^{\gamma,v}} ds \right| \leq C\gamma^{2-\beta} + C e^{-2(1-\epsilon)\gamma^{1-\beta}\sqrt{-1-E^{\max}}},$$

which concludes the proof using (VI.1.15). \square

Lemma VI.1.4. *For γ large enough the family $(\tilde{\phi}_n^{\gamma,v})_{(n,v) \in \cup_{l=1}^K \mathcal{S}_l}$ is linearly independent.*

Proof. Let us denote by G the Gramian matrix associated with $(\tilde{\phi}_n^{\gamma,v})_{(n,v) \in \cup_{l=1}^K \mathcal{S}_l}$ which entries are $G_{i,j} = \langle \tilde{\phi}_{n_i}^{\gamma,v_i}, \tilde{\phi}_{n_j}^{\gamma,v_j} \rangle$, where $(n_i, v_i), (n_j, v_j) \in \cup_{l=1}^K \mathcal{S}_l$. On one hand, the diagonal is simply composed by $G_{i,i} = 1 + O(\gamma^{-\beta})$, according to (VI.1.9). On the other hand, if $(n_i, v_i) \neq (n_j, v_j)$ then two cases are allowed: $v_i = v_j$ and $n_i \neq n_j$ or $v_i \neq v_j$. In the first case, we already know by (VI.1.9) that $G_{i,j} = O(\gamma^{-\beta})$. In the second case, $\text{supp } \chi_{v_i}^\gamma \cap \text{supp } \chi_{v_j}^\gamma = \emptyset$ for γ large enough which implies $G_{i,j} = 0$. Then we can conclude that $\det(G) = 1 + O(\gamma^{-\beta})$ and in particular $\det(G) \neq 0$ for γ large enough. \square

VI.1.2 Cutting out the vertices

This section is a prelude to the study of the asymptotics of the eigenvalues of Q^γ (Section VI.3) and the Weyl asymptotics (Section VI.4). We show how to separate the convex vertices from the rest of Ω , which we call *regular part*, using a partition of unity and a Dirichlet bracketing.

Let us introduce the smooth function χ_0^γ defined by $\chi_0^\gamma := 1 - \sum_{v \in \mathcal{V}} \chi_v^\gamma$, where the χ_v^γ are defined in (VI.1.1). For all $v \in \mathcal{V} \cup \{0\}$, let

$$\tilde{\chi}_v^\gamma(x) := \frac{\chi_v^\gamma(x)}{\sqrt{\sum_{v \in \mathcal{V} \cup \{0\}} (\chi_v^\gamma(x))^2}}, \quad x \in \Omega.$$

Defined in this way, $\tilde{\chi}_v^\gamma \in C^\infty(\Omega)$ and $\sum_{v \in \mathcal{V} \cup \{0\}} (\tilde{\chi}_v^\gamma)^2 = 1$ on Ω . Thus we can apply Lemma III.2.2 to obtain, for any $\phi \in H^1(\Omega)$,

$$q^\gamma(\phi, \phi) = \sum_{v \in \mathcal{V} \cup \{0\}} q^\gamma(\phi \tilde{\chi}_v^\gamma, \phi \tilde{\chi}_v^\gamma) - \int_{\Omega} V(x) |\phi|^2 dx, \quad (\text{VI.1.18})$$

with $V(x) := \sum_{v \in \mathcal{V} \cup \{0\}} |\nabla \tilde{\chi}_v^\gamma(x)|^2$. Notice that, by definition of $\tilde{\chi}_v^\gamma$, there exists $c > 0$ such that

$$\|V\|_\infty \leq c\gamma^{2\beta}. \quad (\text{VI.1.19})$$

Let us define the *regular part*, which is γ -dependent,

$$\Omega_0 := \Omega \setminus \bigcup_{v \in \mathcal{V}} \overline{B(v, \gamma^{-\beta})}. \quad (\text{VI.1.20})$$

For all $v \in \mathcal{V}$ we introduce the sesquilinear forms

$$q_{v, 2\gamma^{-\beta}}^{\gamma, V}(\phi, \phi) = \int_{\Omega_{v, 2\gamma^{-\beta}}} (|\nabla \phi|^2 - V(x) |\phi|^2) dx - \gamma \int_{\Gamma_{v, 2\gamma^{-\beta}}} |\phi|^2 ds,$$

with $D(q_{v, 2\gamma^{-\beta}}^{\gamma, V}) := \{\phi \in H^1(\Omega_{v, 2\gamma^{-\beta}}), \phi(x) = 0 \text{ for } x \in \partial\Omega_{v, 2\gamma^{-\beta}} \setminus \Gamma_{v, 2\gamma^{-\beta}}\}$, and

$$q_0^{\gamma, V}(\phi, \phi) = \int_{\Omega_0} (|\nabla \phi|^2 - V(x) |\phi|^2) dx - \gamma \int_{\Gamma_0} |\phi|^2 ds, \quad (\text{VI.1.21})$$

where $\Gamma_0 := \partial\Omega_0 \cap \partial\Omega$ and $D(q_0^{\gamma, V}) := \{\phi \in H^1(\Omega_0), \phi(x) = 0 \text{ for } x \in \partial\Omega_0 \setminus \Gamma_0\}$.

Lemma VI.1.5. *For all $n \in \mathbb{N}$ and for all $\gamma > 0$,*

$$E_n(Q^\gamma) \geq E_n \left(\left(\bigoplus_{v \in \mathcal{V}} Q_{v, 2\gamma^{-\beta}}^{\gamma, V} \right) \oplus Q_0^{\gamma, V} \right).$$

Proof. Notice that $\phi \tilde{\chi}_v^\gamma \in D(q_{v, 2\gamma^{-\beta}}^{\gamma, V})$ for all $v \in \mathcal{V}$ and $\phi \tilde{\chi}_0^\gamma \in D(q_0^{\gamma, V})$. Thanks to (VI.1.18) one can write

$$q^\gamma(\phi, \phi) \geq \sum_{v \in \mathcal{V}} q_{v, 2\gamma^{-\beta}}^{\gamma, V}(\phi \tilde{\chi}_v^\gamma, \phi \tilde{\chi}_v^\gamma) + q_0^{\gamma, V}(\phi \tilde{\chi}_0^\gamma, \phi \tilde{\chi}_0^\gamma), \quad \phi \in H^1(\Omega). \quad (\text{VI.1.22})$$

Moreover, $\|\phi\|_{L^2(\Omega)}^2 = \sum_{v \in \mathcal{V}} \|\phi \tilde{\chi}_v^\gamma\|_{L^2(\Omega_{v, 2\gamma^{-\beta}})}^2 + \|\phi \tilde{\chi}_0^\gamma\|_{L^2(\Omega_0)}^2$. Then, by (VI.1.22) and the min-max principle we get, for all $n \in \mathbb{N}$,

$$\begin{aligned} E_n(Q^\gamma) &\geq \min_{\substack{G \subset D(q^\gamma) \\ \dim(G)=n}} \max_{\substack{\phi \in G \\ \phi \neq 0}} \frac{\sum_{v \in \mathcal{V}} q_{v, 2\gamma^{-\beta}}^{\gamma, V}(\phi \tilde{\chi}_v^\gamma, \phi \tilde{\chi}_v^\gamma) + q_0^{\gamma, V}(\phi \tilde{\chi}_0^\gamma, \phi \tilde{\chi}_0^\gamma)}{\sum_{v \in \mathcal{V}} \|\phi \tilde{\chi}_v^\gamma\|_{L^2(\Omega_{v, 2\gamma^{-\beta}})}^2 + \|\phi \tilde{\chi}_0^\gamma\|_{L^2(\Omega_0)}^2} \\ &\geq \min_{\substack{G \subset D \left(\left(\bigoplus_{v \in \mathcal{V}} q_{v, 2\gamma^{-\beta}}^{\gamma, V} \right) \oplus q_0^{\gamma, V} \right) \\ \dim(G)=n}} \max_{\substack{((\phi_v)_{v \in \mathcal{V}}, \phi_0) \in G \\ ((\phi_v)_{v \in \mathcal{V}}, \phi_0) \neq (0, \dots, 0)}} \frac{\sum_{v \in \mathcal{V}} q_{v, 2\gamma^{-\beta}}^{\gamma, V}(\phi_v, \phi_v) + q_0^{\gamma, V}(\phi_0, \phi_0)}{\sum_{v \in \mathcal{V}} \|\phi_v\|_{L^2(\Omega_{v, 2\gamma^{-\beta}})}^2 + \|\phi_0\|_{L^2(\Omega_0)}^2} \\ &= E_n \left(\left(\bigoplus_{v \in \mathcal{V}} Q_{v, 2\gamma^{-\beta}}^{\gamma, V} \right) \oplus Q_0^{\gamma, V} \right), \end{aligned}$$

which concludes the proof. \square

Let us now introduce the sesquilinear forms

$$q_{v,\gamma^{-\beta}}^\gamma(\phi, \phi) = \int_{\Omega_{v,\gamma^{-\beta}}} |\nabla\phi|^2 dx - \gamma \int_{\Gamma_{v,\gamma^{-\beta}}} |\phi|^2 ds,$$

with $D(q_{v,\gamma^{-\beta}}^\gamma) := \{\phi \in H^1(\Omega_{v,\gamma^{-\beta}}), \phi(x) = 0 \text{ for } x \in \partial\Omega_{v,\gamma^{-\beta}} \setminus \Gamma_{v,\gamma^{-\beta}}\}$ and

$$q_0^\gamma(\phi, \phi) = \int_{\Omega_0} |\nabla\phi|^2 dx - \gamma \int_{\Gamma_0} |\phi|^2 ds, \quad (\text{VI.1.23})$$

where $D(q_0^\gamma) := D(q_0^{\gamma,V})$.

Lemma VI.1.6. *For all $n \in \mathbb{N}$ and for all $\gamma > 0$,*

$$E_n(Q^\gamma) \leq E_n \left(\bigoplus_{v \in \mathcal{V}} Q_{v,\gamma^{-\beta}}^\gamma \bigoplus Q_0^\gamma \right).$$

Proof. It is a consequence of the min-max principle, noticing that $(\bigoplus_{v \in \mathcal{V}} q_{v,\gamma^{-\beta}}^\gamma) \bigoplus q_0^\gamma$ is a restriction of q^γ . \square

Recall that for each component of the boundary Γ_k , we denote by l_k its length and by κ_k its signed curvature, see (VI.2.1) for a rigorous definition. The following lemma gives us some estimates concerning the operators acting on the *regular part* of Ω , i.e. the operators $Q_0^{\gamma,V}$ and Q_0^γ . The proof is given in the next section.

Lemma VI.1.7. *For γ large enough and for all $\beta \in (1/2, 1)$ one has*

$$E_1(Q_0^{\gamma,V}) \geq -\gamma^2 + \mathcal{O}(\gamma^{2\beta}). \quad (\text{VI.1.24})$$

Moreover, for any $E \in (-1, 0)$, $\lambda \in \mathbb{R}$ and for any $\beta \in (1/2, 1)$, one has as $\gamma > 0$ is large enough,

$$\mathcal{N}(Q_0^{\gamma,V}, E\gamma^2) \leq \gamma \frac{|\partial\Omega| \sqrt{E+1}}{\pi} + \mathcal{O}(\gamma^{1-\beta}), \quad (\text{VI.1.25})$$

$$\mathcal{N}(Q_0^\gamma, E\gamma^2) \geq \gamma \frac{|\partial\Omega| \sqrt{E+1}}{\pi} + \mathcal{O}(\gamma^{1-\beta}), \quad (\text{VI.1.26})$$

and,

$$\mathcal{N}(Q_0^{\gamma,V}, -\gamma^2 + \lambda\gamma) \leq \frac{\sqrt{\gamma}}{\pi} \sum_{k=1}^M \int_0^{l_k} \sqrt{(\kappa_k(s) + \lambda)_+} ds + \mathcal{O}(\gamma^{\beta-1/2}), \quad (\text{VI.1.27})$$

$$\mathcal{N}(Q_0^\gamma, -\gamma^2 + \lambda\gamma) \geq \frac{\sqrt{\gamma}}{\pi} \sum_{k=1}^M \int_0^{l_k} \sqrt{(\kappa_k(s) + \lambda)_+} ds + \mathcal{O}(1). \quad (\text{VI.1.28})$$

VI.2 Proof of Lemma VI.1.7

This section is devoted to the proof of Lemma VI.1.7. We first introduce some notation. The boundary of Ω is composed by M connected arcs: $\partial\Omega = \bigcup_{k=1}^M \overline{\Gamma_k}$. We consider ζ_k the parametrization by the arc length of $\overline{\Gamma_k}$, namely : $\zeta_k \in C^4([0, l_k], \mathbb{R}^2)$ is injective, $\zeta_k(s) = (\zeta_{k,1}(s), \zeta_{k,2}(s)) \in \overline{\Gamma_k}$ and $|\zeta_k'(s)| = 1$ for all $s \in [0, l_k]$. Let $T_k(s) := (\zeta_{k,1}'(s), \zeta_{k,2}'(s))$ be the unit tangent vector of Γ_k at the point $\zeta_k(s)$. We choose the anti-clockwise orientation of the boundary so that

$$\nu_k(s) := (\zeta_{k,2}'(s), -\zeta_{k,1}'(s)),$$

is the unit vector, normal to the boundary and pointing *outward* at the point $\zeta_k(s)$. The *signed curvature* of Γ_k is defined by the identity

$$T'_k(s) = -\kappa_k(s)\nu_k(s),$$

which gives

$$\kappa_k(s) = \zeta'_{k,1}(s)\zeta''_{k,2}(s) - \zeta'_{k,2}(s)\zeta''_{k,1}(s). \quad (\text{VI.2.1})$$

Defined in this way, κ_k is non-negative on convex domains. We also denote

$$\kappa_{\max} := \max_{k=1,\dots,M} \left(\max_{s \in [0, l_k]} \kappa_k(s) \right). \quad (\text{VI.2.2})$$

Let us introduce the map

$$\varphi_k : (0, l_k) \times \mathbb{R} \rightarrow \mathbb{R}^2, \quad \varphi_k(s, t) = \zeta_k(s) - t\nu_k(s).$$

There exists $a_k > 0$ such that, for all $a < a_k$, φ_k is a diffeomorphism between $\square_a^k := (0, l_k) \times (0, a)$ and $\Omega_a^k := \varphi_k(\square_a^k)$. We define

$$\tilde{\Omega}_a^k := \Omega_a^k \cap \Omega_0. \quad (\text{VI.2.3})$$

Recall that Ω_0 , defined in (VI.1.20), is γ -dependent and that the parameter $\beta \in (1/2, 1)$ was introduced in (VI.1.1). For $\epsilon \in (0, 1 - \beta)$, let $a_\gamma := \gamma^{-1+\epsilon}$. Then, for γ large enough and for $k \neq k'$ we have

$$\tilde{\Omega}_{a_\gamma}^k \cap \tilde{\Omega}_{a_\gamma}^{k'} = \emptyset. \quad (\text{VI.2.4})$$

In the following, $\gamma > 0$ is large enough such that (VI.2.4) is satisfied and $a_\gamma < \min_{k=1,\dots,M} a_k$.

VI.2.1 Proof of (VI.1.24), (VI.1.25) and (VI.1.27)

Notice that in order to prove (VI.1.24) we cannot use the same trick used in the proof of Proposition V.2.2, namely extending the domain in a smooth way to apply [Pan13, Theorem 1], as Ω_0 depends on γ . Instead, we study the problem manually by following the construction of [Pan13]. The difficulty remains in the fact that we have to control the potential V appearing in the expression of the sesquilinear form $q_0^{\gamma, V}$.

Let $k \in \{1, \dots, M\}$ be fixed. We define

$$q_k^{\gamma, N, V}(\phi, \phi) = \int_{\Omega_{a_\gamma}^k} (|\nabla \phi|^2 - V(x)|\phi|^2) dx - \gamma \int_{\Gamma_k} |\phi|^2 ds,$$

where $D(q_k^{\gamma, N, V}) := \{\phi \in H^1(\Omega_{a_\gamma}^k), \phi(x) = 0 \text{ for } x \in \Gamma_l^k \cup \Gamma_r^k\}$, $\Gamma_l^k := \{\varphi(0, t), t \in (0, a_\gamma)\}$ and $\Gamma_r^k := \{\varphi_k(l_k, t), t \in (0, a_\gamma)\}$.

Proposition VI.2.1. *For γ large enough and for any $\beta \in (1/2, 1)$, one has*

$$E_1(Q_k^{\gamma, N, V}) \geq -\gamma^2 + \mathcal{O}(\gamma^{2\beta}). \quad (\text{VI.2.5})$$

Moreover, for any $E \in (-1, 0)$,

$$\mathcal{N}(Q_k^{\gamma, N, V}, E\gamma^2) \leq \gamma \frac{l_k \sqrt{E+1}}{\pi} + \mathcal{O}(\gamma^{1-\beta}), \quad (\text{VI.2.6})$$

and for any $\lambda \in \mathbb{R}$,

$$\mathcal{N}(Q_k^{\gamma, N, V}, -\gamma^2 + \lambda\gamma) \leq \frac{\sqrt{\gamma}}{\pi} \int_0^{l_k} \sqrt{(\kappa_k(s) + \lambda)_+} ds + \mathcal{O}(\gamma^{\beta-1/2}). \quad (\text{VI.2.7})$$

Proof. Step 1. As the study is the same for all $k \in \{1, \dots, M\}$ we omit the indices k in the proof. The idea is to perform a change of variables thanks to the diffeomorphism φ in order to work in \square_{a_γ} which will allow us to use the separation of variables. But, in order to avoid the weight in the integrals due to the Jacobian of the change of variables, we first introduce a unitary transform, following the idea of [Pan13] and [EMP14]. Define $U_{a_\gamma} : L^2(\Omega_{a_\gamma}) \rightarrow L^2(\square_{a_\gamma})$ by

$$U_{a_\gamma}(\phi)(s, t) := \sqrt{1 - t\kappa(s)} (\phi \circ \varphi)(s, t).$$

It is easy to check that $U_{a_\gamma}(D(q^{\gamma, N, V})) = \{\phi \in H^1(\square_{a_\gamma}), \phi(0, \cdot) = \phi(l, \cdot) = 0\}$. Let us define

$$\begin{aligned} p^{\gamma, N, \tilde{V}}(\phi, \phi) &= \int_{\square_{a_\gamma}} \left(\frac{1}{(1 - t\kappa(s))^2} |\partial_s \phi|^2 + |\partial_t \phi|^2 - \tilde{V}(s, t) |\phi|^2 - P(s, t) |\phi|^2 \right) ds dt \\ &\quad + \frac{1}{2} \int_0^l \frac{\kappa(s)}{1 - a_\gamma \kappa(s)} |\phi(s, a_\gamma)|^2 ds - \int_0^l \left(\frac{\kappa(s)}{2} + \gamma \right) |\phi(s, 0)|^2 ds, \end{aligned}$$

where $D(p^{\gamma, N, \tilde{V}}) := U_{a_\gamma}(D(q^{\gamma, N, V}))$, $\tilde{V}(s, t) = V \circ \varphi(s, t)$ and

$$P(s, t) = \frac{\kappa^2(s)}{4(1 - t\kappa(s))^2} + \frac{t\kappa''(s)}{2(1 - t\kappa(s))^3} + \frac{5t^2(\kappa'(s))^2}{4(1 - t\kappa(s))^4}. \quad (\text{VI.2.8})$$

We want to prove that $Q^{\gamma, N, V}$ and $P^{\gamma, N, \tilde{V}}$ are unitarily equivalent. To be more specific, we will prove the following equality

$$q^{\gamma, N, V}(\phi, \phi) = p^{\gamma, N, \tilde{V}}(U_{a_\gamma} \phi, U_{a_\gamma} \phi), \quad \phi \in D(q^{\gamma, N, V}). \quad (\text{VI.2.9})$$

By change of variables we can write for any $\phi \in D(q^{\gamma, N, V})$

$$\begin{aligned} q^{\gamma, N, V}(\phi, \phi) &= \int_{\square_{a_\gamma}} \left(|\partial_{x_1} \phi \circ \varphi|^2 + |\partial_{x_2} \phi \circ \varphi|^2 - V \circ \varphi |\phi \circ \varphi|^2 \right) (1 - t\kappa(s)) ds dt \\ &\quad - \gamma \int_0^l |\phi \circ \varphi(s, 0)|^2 ds. \end{aligned}$$

Moreover, by a direct computation we have

$$\begin{aligned} |\partial_{x_1} \phi \circ \varphi|^2 + |\partial_{x_2} \phi \circ \varphi|^2 &= \frac{1}{(1 - t\kappa(s))^3} |\partial_s U_{a_\gamma} \phi|^2 + \frac{1}{(1 - t\kappa(s))} |\partial_t U_{a_\gamma} \phi|^2 \\ &\quad + \left(\frac{(t\kappa'(s))^2}{4(1 - t\kappa(s))^5} + \frac{\kappa^2(s)}{4(1 - t\kappa(s))^3} \right) |U_{a_\gamma} \phi|^2 \\ &\quad + \frac{t\kappa'(s)}{(1 - t\kappa(s))^4} \partial_s U_{a_\gamma} \phi U_{a_\gamma} \phi + \frac{\kappa(s)}{(1 - t\kappa(s))^2} \partial_t U_{a_\gamma} \phi U_{a_\gamma} \phi. \end{aligned}$$

Remark that we can rewrite $\partial_s (U_{a_\gamma} \phi) (U_{a_\gamma} \phi) = \frac{1}{2} \partial_s ((U_{a_\gamma} \phi)^2)$. Thus, by integration by parts we obtain

$$\int_0^l (\partial_s U_{a_\gamma} \phi) (U_{a_\gamma} \phi) \frac{t\kappa'(s)}{(1 - t\kappa(s))^3} ds = -\frac{1}{2} \int_0^l |U_{a_\gamma} \phi|^2 \left(\frac{t\kappa''(s)}{(1 - t\kappa(s))^3} + 3 \frac{t^2(\kappa'(s))^2}{(1 - t\kappa(s))^4} \right) ds,$$

as $U_{a_\gamma} \phi(0, t) = U_{a_\gamma} \phi(l, t) = 0$. Using the same trick we also have

$$\begin{aligned} \int_0^{a_\gamma} (\partial_t U_{a_\gamma} \phi) (U_{a_\gamma} \phi) \frac{\kappa(s)}{(1 - t\kappa(s))} dt &= \frac{1}{2} |U_{a_\gamma} \phi(s, a_\gamma)|^2 \frac{\kappa(s)}{1 - a_\gamma \kappa(s)} - \frac{1}{2} |U_{a_\gamma} \phi(s, 0)|^2 \kappa(s) \\ &\quad - \frac{1}{2} \int_0^{a_\gamma} |U_{a_\gamma} \phi|^2 \frac{\kappa^2(s)}{(1 - t\kappa(s))^2} dt. \end{aligned}$$

This finishes the proof of (VI.2.9) gathering all the previous equalities. As U_{a_γ} is a unitary map, we immediately get by the min-max principle, for all $n \in \mathbb{N}$,

$$E_n(Q^{\gamma,N,V}) = E_n(P^{\gamma,N,\tilde{V}}). \quad (\text{VI.2.10})$$

Step 2. Let us make a remark on the potential \tilde{V} . Recall that $V(x) = \sum_{v \in \mathcal{V} \cup \{0\}} |\nabla \tilde{\chi}_v^\gamma(x)|^2$. Then, $\text{supp } V \subset \bigcup_{v \in \mathcal{V}} B(v, 2\gamma^{-\beta}) \setminus \overline{B(v, \gamma^{-\beta})}$. This implies that there exists a constant $b > 0$ such that

$$\text{supp } \tilde{V} \subset \left((0, b\gamma^{-\beta}) \times (0, a_\gamma) \right) \bigcup \left((l - b\gamma^{-\beta}, l) \times (0, a_\gamma) \right).$$

We denote $s_\gamma := b\gamma^{-\beta}$ and introduce

$$\xi(s) := \begin{cases} 1, & \text{if } s \in (0, s_\gamma) \text{ or } s \in (l - s_\gamma, l), \\ 0, & \text{otherwise.} \end{cases}$$

By (VI.1.19) we can write, for all $(s, t) \in \square_{a_\gamma}$,

$$\tilde{V}(s, t) \leq c\gamma^{2\beta} \xi(s).$$

We now give some estimates which will simplify the study. As $\kappa \in C^2([0, l], \mathbb{R})$, there exist $\mathcal{K} > 0$ and $C > 0$ such that we have for all $(s, t) \in \square_{a_\gamma}$:

$$\left| \frac{\kappa(s)}{1 - t\kappa(s)} \right| \leq 2\mathcal{K}, \quad 1 - a_\gamma C < \frac{1}{(1 - t\kappa(s))^2} < 1 + a_\gamma C, \quad |P(s, t)| \leq C. \quad (\text{VI.2.11})$$

Thus for all $\phi \in D(p^{\gamma,N,\tilde{V}})$ we have $p^{\gamma,N,\tilde{V}}(\phi, \phi) \geq h^{\gamma,N,\xi_\gamma}(\phi, \phi)$, where

$$\begin{aligned} h^{\gamma,N,\xi_\gamma}(\phi, \phi) &= \int_{\square_{a_\gamma}} \left((1 - a_\gamma C) |\partial_s \phi|^2 + |\partial_t \phi|^2 - \xi_\gamma(s) |\phi|^2 - C |\phi|^2 \right) ds dt \\ &\quad - \mathcal{K} \int_0^l |\phi(s, a_\gamma)|^2 ds - \int_0^l \left(\frac{\kappa(s)}{2} + \gamma \right) |\phi(s, 0)|^2 ds, \end{aligned}$$

where $D(h^{\gamma,N,\xi_\gamma}) := H^1(\square_{a_\gamma})$ and

$$\xi_\gamma(s) := c\gamma^{2\beta} \xi(s).$$

We now can conclude by the min-max principle that, for all $n \in \mathbb{N}$,

$$E_n(P^{\gamma,N,\tilde{V}}) \geq E_n(H^{\gamma,N,\xi_\gamma}). \quad (\text{VI.2.12})$$

In order to control the potential ξ_γ , we have to introduce some new sesquilinear forms. We define

$$\begin{aligned} h^{\gamma,N,1}(\phi, \phi) &= \int_0^{s_\gamma} \int_0^{a_\gamma} \left((1 - a_\gamma C) |\partial_s \phi|^2 + |\partial_t \phi|^2 - c\gamma^{2\beta} |\phi|^2 - C |\phi|^2 \right) ds dt \\ &\quad - \mathcal{K} \int_0^{s_\gamma} |\phi(s, a_\gamma)|^2 ds - \int_0^{s_\gamma} \left(\frac{\kappa(s)}{2} + \gamma \right) |\phi(s, 0)|^2 ds, \quad \phi \in H^1((0, s_\gamma) \times (0, a_\gamma)), \end{aligned}$$

$$\begin{aligned} h^{\gamma,N,2}(\phi, \phi) &= \int_{s_\gamma}^{l-s_\gamma} \int_0^{a_\gamma} \left((1 - a_\gamma C) |\partial_s \phi|^2 + |\partial_t \phi|^2 - C |\phi|^2 \right) ds dt \\ &\quad - \mathcal{K} \int_{s_\gamma}^{l-s_\gamma} |\phi(s, a_\gamma)|^2 ds - \int_{s_\gamma}^{l-s_\gamma} \left(\frac{\kappa(s)}{2} + \gamma \right) |\phi(s, 0)|^2 ds, \quad \phi \in H^1((s_\gamma, l - s_\gamma) \times (0, a_\gamma)), \end{aligned}$$

and

$$\begin{aligned} h^{\gamma,N,3}(\phi, \phi) &= \int_{l-s_\gamma}^l \int_0^{a_\gamma} \left((1 - a_\gamma C) |\partial_s \phi|^2 + |\partial_t \phi|^2 - c_\gamma^{2\beta} |\phi|^2 - C |\phi|^2 \right) ds dt \\ &\quad - \mathcal{K} \int_{l-s_\gamma}^l |\phi(s, a_\gamma)|^2 ds - \int_{l-s_\gamma}^l \left(\frac{\kappa(s)}{2} + \gamma \right) |\phi(s, 0)|^2 ds, \quad \phi \in H^1((l-s_\gamma, l) \times (0, a_\gamma)). \end{aligned}$$

Using the min-max principle we obtain the following inequality for all $n \in \mathbb{N}$,

$$E_n(H^{\gamma,N,\xi_\gamma}) \geq E_n \left(\bigoplus_{i=1}^3 H^{\gamma,N,i} \right). \quad (\text{VI.2.13})$$

Step 3. Let us introduce, for simplicity, the more general sesquilinear form

$$\begin{aligned} h^{\gamma,N}(\phi, \phi) &= \int_0^L \int_0^{a_\gamma} \left((1 - a_\gamma C) |\partial_s \phi|^2 + |\partial_t \phi|^2 - c_\gamma |\phi|^2 \right) ds dt \\ &\quad - \mathcal{K} \int_0^L |\phi(s, a_\gamma)|^2 ds - \int_0^L \left(\frac{g(s)}{2} + \gamma \right) |\phi(s, 0)|^2 ds, \quad \phi \in H^1((0, L) \times (0, a_\gamma)), \end{aligned}$$

where $L > 0$, $c_\gamma > 0$ depends on γ and will play the role of the potentials $c_\gamma^{2\beta} + C$ or C and $g \in C^2([0, L], \mathbb{R})$. We first prove some results on $H^{\gamma,N}$, namely estimates on the first eigenvalue and the eigenvalue counting function, and then apply them to the $H^{\gamma,N,i}$. For any $K \in \mathbb{N}$, we denote

$$\delta := \frac{L}{K}, \quad I_j := (\delta(j-1), \delta j), \quad j \in \{1, \dots, K\},$$

and

$$g_j^+ := \sup_{s \in I_j} g(s).$$

We begin by defining the sesquilinear forms associated with the partition of $(0, L)$. For $j \in \{1, \dots, K\}$, let us consider

$$\begin{aligned} t_j^{\gamma,N}(\phi, \phi) &= \int_{I_j} \int_0^{a_\gamma} \left((1 - a_\gamma C) |\partial_s \phi|^2 + |\partial_t \phi|^2 - c_\gamma |\phi|^2 \right) ds dt \\ &\quad - \mathcal{K} \int_{I_j} |\phi(s, a_\gamma)|^2 ds - \int_{I_j} \left(\frac{g_j^+}{2} + \gamma \right) |\phi(s, 0)|^2 ds, \quad \phi \in H^1(I_j \times (0, a_\gamma)). \end{aligned}$$

Clearly we have for any $\phi \in D(h^{\gamma,N})$,

$$h^{\gamma,N}(\phi, \phi) \geq \sum_{j=1}^K t_j^{\gamma,N}(\phi, \phi),$$

and then, by the min-max principle we get for all $n \in \mathbb{N}$,

$$E_n(H^{\gamma,N}) \geq E_n \left(\bigoplus_{j=1}^K T_j^{\gamma,N} \right). \quad (\text{VI.2.14})$$

Let us fix $j \in \{1, \dots, K\}$. By separation of variables, it is easy to see that $E_n(T_j^{\gamma,N}) = E_n(\mathcal{L}^N \otimes 1 + 1 \otimes \mathcal{R}_j)$, where the operator \mathcal{L}^N acts on $L^2(0, \delta)$ as

$$\mathcal{L}^N f = -(1 - a_\gamma C) f'' - c_\gamma f, \quad D(\mathcal{L}^N) := \{f \in H^2(0, \delta) : -f'(0) = f'(\delta) = 0\}.$$

The operator $\mathcal{R}_j := \mathcal{R}_{a_\gamma, \frac{g_j^+}{2} + \gamma, \mathcal{K}}$ is defined in Proposition III.5.7 and acts on $L^2(0, a_\gamma)$ as $f \mapsto -f''$ with

$$D(\mathcal{R}_j) := \{f \in H^2(0, a_\gamma) : -f'(0) - (\frac{g_j^+}{2} + \gamma)f(0) = f'(a_\gamma) - \mathcal{K}f(a_\gamma) = 0\}.$$

Recall that $a_\gamma := \gamma^{-1+\epsilon}$, where $\epsilon \in (0, 1 - \beta)$. Thus, there exists $\gamma_1 > 0$ such that for all $\gamma > \gamma_1$ we have $(\frac{g_j^+}{2} + \gamma) > \mathcal{K}$ and $a_\gamma \mathcal{K} < 1$. Applying Proposition III.5.7, we know that for any $\gamma > \gamma_1$, $E_1(\mathcal{R}_j)$ is the unique negative eigenvalue of \mathcal{R}_j . Moreover, there exists a constant $C > 0$ such that, for γ large enough,

$$E_1(\mathcal{R}_j) > -(\frac{g_j^+}{2} + \gamma)^2 - C(\frac{g_j^+}{2} + \gamma)^2 e^{-(\frac{g_j^+}{2} + \gamma)a_\gamma}. \quad (\text{VI.2.15})$$

As $\inf \text{spec}(\mathcal{L}^N) = -c_\gamma$, we get

$$E_1(T_j^{\gamma, N}) = E_1(\mathcal{R}_j) - c_\gamma. \quad (\text{VI.2.16})$$

Using (VI.2.16) and (VI.2.15), there exists $\gamma_2 > \gamma_1$ such that for all $\gamma > \gamma_2$,

$$E_1(T_j^{\gamma, N}) \geq -(\frac{g_j^+}{2} + \gamma)^2 - c_\gamma - C.$$

For all $j \in \{1, \dots, K\}$, $g_j^+ \leq g_{\max} := \max_{s \in [0, L]} g(s)$ and by (VI.2.14) we can conclude that for all $\gamma > \gamma_2$,

$$E_1(H^{\gamma, N}) \geq -\gamma^2 - \gamma g_{\max} - c_\gamma - C.$$

Notice that it is easy to apply the previous result to the operators $H^{\gamma, N, i}$ by making a translation and considering, for $i = 2$, $g(s) := \kappa(s + s_\gamma)$ and for $i = 3$, $g(s) := \kappa(s + (l - s_\gamma))$. Thus, for $\gamma > \gamma_2$ we have

$$E_1(H^{\gamma, N, i}) \geq -\gamma^2 - \gamma \kappa_{\max} - c\gamma^{2\beta} - C, \quad i = 1, 3,$$

and

$$E_1(H^{\gamma, N, 2}) \geq -\gamma^2 - \gamma \kappa_{\max} - C.$$

There exists $\gamma_3 > \gamma_2$ such that for all $\gamma > \gamma_3$ we have, thanks to (VI.2.13),

$$E_1(H^{\gamma, N, \xi_\gamma}) \geq -\gamma^2 - C\gamma^{2\beta},$$

as $\beta \in (1/2, 1)$. Finally, this concludes the proof of (VI.2.5) thanks to (VI.2.10) and (VI.2.12).

Step 4. Let us now focus on the eigenvalue counting function. Let $E \in (-1, 0)$ be fixed. Thanks to the fact that $E_1(\mathcal{R}_j)$ is the unique negative eigenvalue of the operator \mathcal{R}_j as $\gamma > \gamma_1$ and using estimate (VI.2.15) one can write

$$\mathcal{N}(T_j^{\gamma, N}, E\gamma^2) \leq \frac{\delta}{\pi\sqrt{1 - a_\gamma C}} \sqrt{(E + 1)\gamma^2 + g_j^+ \gamma + c_\gamma + C} + 1.$$

Thus, summing on $k \in \{1, \dots, K\}$ and using (VI.2.14) we obtain for γ large enough,

$$\mathcal{N}(H^{\gamma, N}, E\gamma^2) \leq \gamma \frac{L\sqrt{E + 1}}{\pi\sqrt{1 - a_\gamma C}} + LCc_\gamma \gamma^{-1} + K.$$

Recall that $a_\gamma := \gamma^{-1+\epsilon}$ with $\epsilon \in (0, 1 - \beta)$. We have

$$(1 - a_\gamma C)^{-1/2} = 1 + \frac{1}{2}C\gamma^{-1+\epsilon} + O(\gamma^{-2+2\epsilon}) \text{ as } \gamma \rightarrow +\infty.$$

Then,

$$\mathcal{N}(H^{\gamma,N}, E\gamma^2) \leq \gamma \frac{L\sqrt{E+1}}{\pi} + LC\gamma^\epsilon + LCc_\gamma\gamma^{-1} + K.$$

We can now apply this previous result to the operators $H^{\gamma,N,i}$ with $c_\gamma = c\gamma^{2\beta} + C$ and $L = b\gamma^{-\beta}$ for $i = 1, 3$ and $c_\gamma = C$ and $L = l - 2b\gamma^{-\beta}$ for $i = 2$. We finally obtain, choosing $K \in [\gamma^\epsilon, 2\gamma^\epsilon] \cap \mathbb{N}$,

$$\mathcal{N}(H^{\gamma,N,\xi_\gamma}, E\gamma^2) \leq \gamma \frac{l\sqrt{E+1}}{\pi} + O(\gamma^\epsilon), \quad \gamma \rightarrow +\infty,$$

with $\epsilon < 1 - \beta$. This finishes the proof of (VI.2.6) thanks to (VI.2.10) and (VI.2.12).

Let us prove (VI.2.7). Let $\lambda \in \mathbb{R}$ be fixed. There exists $\gamma_4 > \gamma_1$ such that for all $\gamma > \gamma_4$ we have $-\gamma^2 + \lambda\gamma < 0$. We can write, using again (VI.2.15),

$$\mathcal{N}(T_j^{\gamma,N}, -\gamma^2 + \lambda\gamma) \leq \frac{\delta}{\pi\sqrt{1-a_\gamma C}} \sqrt{\gamma(g_j^+ + \lambda)_+} + C\delta c_\gamma\gamma^{-1/2} + 1.$$

We can sum the inequalities on $j \in \{1, \dots, K\}$ and apply it to the operators $H^{\gamma,N,i}$. We obtain, for $i = 1, 3$,

$$\mathcal{N}(H^{\gamma,N,i}, -\gamma^2 + \lambda\gamma) \leq C\gamma^{1/2-\beta}K + C\gamma^{\beta-1/2} + K,$$

and

$$\mathcal{N}(H^{\gamma,N,2}, -\gamma^2 + \lambda\gamma) \leq \frac{\sqrt{\gamma}}{\pi\sqrt{1-a_\gamma C}} \frac{l - 2b\gamma^{-\beta}}{K} \sum_{j=1}^K \sqrt{(g_j^+ + \lambda)_+} + C\gamma^{-1/2} + K,$$

where $g(s) = \kappa(s + s_\gamma)$. Notice that $(0, L) \ni s \mapsto \sqrt{(g + \lambda)_+}$ is Lipschitz, thus we can use the convergence of Riemann sum to have

$$\int_0^L \sqrt{(g + \lambda)_+} ds = \frac{L}{K} \sum_{j=1}^K \sqrt{(g_j^+ + \lambda)_+} + O\left(\frac{1}{K}\right), \quad K \rightarrow +\infty.$$

Recall that $\beta \in (1/2, 1)$. Let us choose $K \in [\gamma^{\beta-1/2}, 2\gamma^{\beta-1/2}] \cap \mathbb{N}$. Then for γ large enough,

$$\mathcal{N}(H^{\gamma,N,2}, -\gamma^2 + \lambda\gamma) \leq \frac{\sqrt{\gamma}}{\pi} \int_{s_\gamma}^{l-s_\gamma} \sqrt{(\kappa(s) + \lambda)_+} ds + C\gamma^{\beta-1/2}.$$

In addition we have

$$\int_0^l \sqrt{(\kappa(s) + \lambda)_+} ds = \int_{s_\gamma}^{l-s_\gamma} \sqrt{(\kappa(s) + \lambda)_+} ds + O(\gamma^{-\beta}), \quad \gamma \rightarrow +\infty,$$

as $\sqrt{(\kappa(s) + \lambda)_+} \leq \sqrt{(\kappa_{\max} + \lambda)_+}$ for all $s \in (0, l)$. Finally,

$$\mathcal{N}(H^{\gamma,N,\xi_\gamma}, -\gamma^2 + \lambda\gamma) \leq \frac{\sqrt{\gamma}}{\pi} \int_0^l \sqrt{(\kappa(s) + \lambda)_+} ds + O(\gamma^{\beta-1/2}), \quad \gamma \rightarrow +\infty.$$

This concludes the proof of (VI.2.7) thanks to (VI.2.10) and (VI.2.12). \square

Proof of (VI.1.24), (VI.1.25) and (VI.1.27). Recall that the regular part Ω_0 is defined in (VI.1.20) and $\tilde{\Omega}_{a_\gamma}^k$ is defined in (VI.2.3).

We introduce $\tilde{\Omega}_0 := \Omega_0 \setminus \bigcup_{k=1}^M \overline{\tilde{\Omega}_{a_\gamma}^k}$ and the closed sesquilinear forms

$$\tilde{q}_k^{\gamma, N, V}(\phi, \phi) = \int_{\tilde{\Omega}_{a_\gamma}^k} (|\nabla \phi|^2 - V(x)|\phi|^2) dx - \gamma \int_{\Gamma_k \cap \tilde{\Omega}_{a_\gamma}^k} |\phi|^2 ds,$$

with $D(\tilde{q}_k^{\gamma, N, V}) := \{\phi \in H^1(\tilde{\Omega}_{a_\gamma}^k), \phi(x) = 0 \text{ for } x \in (\partial \tilde{\Omega}_{a_\gamma}^k \cap \partial \Omega_0) \setminus \Gamma_0\}$ and

$$\tilde{q}_0^{N, V}(\phi, \phi) = \int_{\tilde{\Omega}_0} (|\nabla \phi|^2 - V(x)|\phi|^2) dx,$$

with $D(\tilde{q}_0^{N, V}) := \{\phi \in H^1(\tilde{\Omega}_0), \phi(x) = 0 \text{ for } x \in \partial \tilde{\Omega}_0 \cap \partial \Omega_0\}$. Noticing that $D(q_0^{\gamma, V}) \subset \bigoplus_{k=1}^M D(\tilde{q}_k^{\gamma, N, V}) \oplus D(\tilde{q}_0^{N, V})$ and thanks to (VI.2.4), we can use the min-max principle and immediately obtain, for all $n \in \mathbb{N}$,

$$E_n(Q_0^{\gamma, V}) \geq E_n \left(\left(\bigoplus_{k=1}^M \tilde{Q}_k^{\gamma, N, V} \right) \oplus \tilde{Q}_0^{N, V} \right). \quad (\text{VI.2.17})$$

Notice that, by (VI.1.19), $\mathcal{N}(\tilde{Q}_0^{N, V}, E\gamma^2 + \lambda\gamma) \leq \mathcal{N}(\tilde{Q}_0^N, E\gamma^2 + \lambda\gamma + c\gamma^{2\beta})$, where \tilde{Q}_0^N is the unique self-adjoint operator associated with the sesquilinear form

$$\tilde{q}_0^N(\phi, \phi) = \int_{\tilde{\Omega}_0} |\nabla \phi|^2 dx, \quad \phi \in D(\tilde{q}_0^N) := D(\tilde{q}_0^{N, V}).$$

The operator \tilde{Q}_0^N is positive. As $\beta < 1$, there exists $\gamma_0 > 0$ such that, for all $\gamma > \gamma_0$ we have $\tilde{E}\gamma^2 + \lambda\gamma + c\gamma^{2\beta} < 0$, with $\tilde{E} \in [-1, 0)$ and $\lambda \in \mathbb{R}$. Then, for all $\gamma > \gamma_0$, $\mathcal{N}(\tilde{Q}_0^N, \tilde{E}\gamma^2 + \lambda\gamma + c\gamma^{2\beta}) = 0$ and by (VI.2.17),

$$\mathcal{N}(Q_0^{\gamma, V}, \tilde{E}\gamma^2 + \lambda\gamma) \leq \sum_{k=1}^M \mathcal{N}(\tilde{Q}_k^{\gamma, N, V}, \tilde{E}\gamma^2 + \lambda\gamma). \quad (\text{VI.2.18})$$

As $\tilde{\Omega}_{a_\gamma}^k \subset \Omega_{a_\gamma}^k$, extending $\phi \in D(\tilde{q}_k^{\gamma, N, V})$ by zero we obtain, by the min-max principle and for all $n \in \mathbb{N}$,

$$E_n(\tilde{Q}_k^{\gamma, N, V}) \geq E_n(Q_k^{\gamma, N, V}). \quad (\text{VI.2.19})$$

We are now able to conclude. On one hand, noticing that $E_1(\tilde{Q}_0^{N, V}) \geq -c\gamma^{2\beta}$ and by (VI.2.5) we have for γ large enough,

$$E_1 \left(\bigoplus_{k=1}^M \tilde{Q}_k^{\gamma, N, V} \oplus \tilde{Q}_0^{N, V} \right) \geq -\gamma^2 - C\gamma^{2\beta},$$

which finishes the proof of (VI.1.24) thanks to (VI.2.17). On the other hand, combining the estimates (VI.2.6) and (VI.2.7) of Proposition VI.2.1 and (VI.2.18) and (VI.2.19) finishes the proof of (VI.1.25) and (VI.1.27). \square

VI.2.2 Proof of (VI.1.26) and (VI.1.28)

We still follow the ideas of the proof of [Pan13], but this proof is easier than the previous one as there is no potential V in the sesquilinear form q_0^γ to control.

Let us introduce the sesquilinear form

$$q_k^{\gamma,D}(\phi, \phi) = \int_{\tilde{\Omega}_{a_\gamma}^k} |\nabla \phi|^2 dx - \gamma \int_{\Gamma_k \cap \tilde{\Omega}_{a_\gamma}^k} |\phi|^2 ds, \quad (\text{VI.2.20})$$

where $D(q_k^{\gamma,D}) := \{\phi \in H^1(\tilde{\Omega}_{a_\gamma}^k), \phi(x) = 0 \text{ for } x \in \partial\tilde{\Omega}_{a_\gamma}^k \setminus \Gamma_k\}$, and the domain $\tilde{\Omega}_{a_\gamma}^k$ is defined in (VI.2.3). Notice that even if the strategy of the proofs will be same as in Section VI.2.1, we have to work with $\tilde{\Omega}_{a_\gamma}^k$ instead of $\Omega_{a_\gamma}^k$, as the trick we used previously does not apply here.

Remark VI.2.2. Let k be such that Γ_k links two convex corners. Then, by definition of $\tilde{\Omega}_{a_\gamma}^k$, there exists $b > 0$ such that $\tilde{\Omega}_{a_\gamma}^k = \varphi_k(\tilde{\square}_{a_\gamma}^k)$, with

$$\tilde{\square}_{a_\gamma}^k := (b\gamma^{-\beta}, l_k - b\gamma^{-\beta}) \times (0, a_\gamma).$$

In the following we denote by $s_\gamma := b\gamma^{-\beta}$. Notice that it is sufficient to study the case where Γ_k links two convex corners. Indeed, in the two other cases (namely Γ_k links one convex corner and one non-convex corner or two non-convex corners) we have $\tilde{\square}_{a_\gamma}^k := (s_\gamma, l) \times (0, a_\gamma)$ or $\tilde{\square}_{a_\gamma}^k := (0, l) \times (0, a_\gamma)$ and the study is then the same.

Proposition VI.2.3. *Let $k \in \{1, \dots, M\}$. For any $E \in (-1, 0)$ and $\lambda \in \mathbb{R}$, one has for $\gamma > 0$ large enough,*

$$\mathcal{N}(Q_k^{\gamma,D}, E\gamma^2) \geq \gamma \frac{l_k \sqrt{E+1}}{\pi} + \mathcal{O}(\gamma^{1-\beta}), \quad (\text{VI.2.21})$$

and

$$\mathcal{N}(Q_k^{\gamma,D}, -\gamma^2 + \lambda\gamma) \geq \frac{\sqrt{\gamma}}{\pi} \int_{s_\gamma}^{l_k - s_\gamma} \sqrt{(\kappa_k(s) + \lambda)_+} ds + \mathcal{O}(1). \quad (\text{VI.2.22})$$

Proof. Step 1. In the following we omit the indices k . Let $E \in (-1, 0)$ and $\lambda \in \mathbb{R}$. We want to perform a change of variables in order to work in $\tilde{\square}_{a_\gamma}$. Following the same steps as in the previous section, we first introduce a unitary transform. Define $\tilde{U}_{a_\gamma} : L^2(\tilde{\Omega}_{a_\gamma}) \rightarrow L^2(\tilde{\square}_{a_\gamma})$,

$$\tilde{U}_{a_\gamma} \phi(s, t) := \sqrt{1 - \kappa(s)} (\phi \circ \varphi)(s, t).$$

Then, $\tilde{U}_{a_\gamma}(D(q^{\gamma,D})) = \{\phi \in H^1(\tilde{\square}_{a_\gamma}), \phi(s_\gamma, t) = \phi(l - s_\gamma, t) = \phi(s, a_\gamma) = 0\}$. It is easy to prove that, after a using integration by parts, $q^{\gamma,D}(\phi, \phi) = p^{\gamma,D}(\tilde{U}_{a_\gamma} \phi, \tilde{U}_{a_\gamma} \phi)$ with $D(q^{\gamma,D}) := \tilde{U}_{a_\gamma}(D(q^{\gamma,D}))$ and $p^{\gamma,D}$ is given by the following expression:

$$\begin{aligned} p^{\gamma,D}(\phi, \phi) &= \int_{\tilde{\square}_{a_\gamma}} \left(\frac{1}{(1 - t\kappa(s))^2} |\partial_s \phi|^2 + |\partial_t \phi|^2 - P(s, t) |\phi|^2 \right) ds dt \\ &\quad - \int_{s_\gamma}^{l - s_\gamma} \left(\frac{\kappa(s)}{2} + \gamma \right) |\phi(s, 0)|^2 ds, \end{aligned}$$

where the potential P is given by the expression in (VI.2.8), namely

$$P(s, t) = \frac{\kappa^2(s)}{4(1 - t\kappa(s))^2} + \frac{t\kappa''(s)}{2(1 - t\kappa(s))^3} + \frac{5t^2(\kappa'(s))^2}{4(1 - t\kappa(s))^4}.$$

As \tilde{U}_{a_γ} is a unitary map we have, for all $n \in \mathbb{N}$,

$$E_n(Q^{\gamma,D}) = E_n(P^{\gamma,D}). \quad (\text{VI.2.23})$$

Step 2. We use the estimates (VI.2.11) mentioned in the proof of Proposition VI.2.1 to simplify the study. Recall that, as $\kappa \in C^2([0, l], \mathbb{R}^2)$ there exists $C > 0$ such that for all $(s, t) \in \tilde{\square}_{a_\gamma}$

$$1 - a_\gamma C < \frac{1}{(1 - t\kappa(s))^2} < 1 + a_\gamma C, \quad |P(s, t)| \leq C. \quad (\text{VI.2.24})$$

Thus, for all $\phi \in D(p^{\gamma, D})$ we have $p^{\gamma, D}(\phi, \phi) \leq h^{\gamma, D}(\phi, \phi)$, where

$$\begin{aligned} h^{\gamma, D}(\phi, \phi) &= \int_{\tilde{\square}_{a_\gamma}} ((1 + a_\gamma C)|\partial_s \phi|^2 + |\partial_t \phi|^2 + C|\phi|^2) ds dt \\ &\quad - \int_{s_\gamma}^{l-s_\gamma} \left(\frac{\kappa(s)}{2} + \gamma \right) |\phi(s, 0)|^2 ds, \quad \phi \in D(h^{\gamma, D}) := D(p^{\gamma, D}). \end{aligned}$$

We obtain, by the min-max principle,

$$E_n(P^{\gamma, D}) \leq E_n(H^{\gamma, D}). \quad (\text{VI.2.25})$$

Let us introduce $L_\gamma := l - 2s_\gamma$ and $\tilde{\kappa}(s) := \kappa(s + s_\gamma)$, for $s \in (0, L_\gamma)$. Then, $H^{\gamma, D}$ is unitarily equivalent to the operator $\tilde{H}^{\gamma, D}$ acting on $L^2((0, L_\gamma) \times (0, a_\gamma))$ and defined as the unique self-adjoint operator associated with the sesquilinear form

$$\begin{aligned} \tilde{h}^{\gamma, D}(\phi, \phi) &= \int_0^{L_\gamma} \int_0^{a_\gamma} ((1 + a_\gamma C)|\partial_s \phi|^2 + |\partial_t \phi|^2 + C|\phi|^2) ds dt \\ &\quad - \int_0^{L_\gamma} \left(\frac{\tilde{\kappa}(s)}{2} + \gamma \right) |\phi(s, 0)|^2 ds, \end{aligned}$$

where $D(\tilde{h}^{\gamma, D}) := \{\phi \in H^1((0, L_\gamma) \times (0, a_\gamma)), \phi(0, t) = \phi(L_\gamma, t) = \phi(s, a_\gamma) = 0\}$. By the min-max principle we obtain the equality

$$E_n(H^{\gamma, D}) = E_n(\tilde{H}^{\gamma, D}). \quad (\text{VI.2.26})$$

Step 3. Let us now introduce a partition of $(0, L_\gamma)$. For any $K \in \mathbb{N}$, we denote

$$\delta := \frac{L_\gamma}{K}, \quad I_j := (\delta(j-1), \delta j), \quad j \in \{1, \dots, K\},$$

and,

$$\tilde{\kappa}_j^- := \inf_{s \in I_j} \tilde{\kappa}(s).$$

We define the new sesquilinear forms adapted to this partition,

$$\begin{aligned} t_j^{\gamma, D}(\phi, \phi) &= \int_{I_j} \int_0^{a_\gamma} ((1 + a_\gamma C)|\partial_s \phi|^2 + |\partial_t \phi|^2 + C|\phi|^2) ds dt \\ &\quad - \int_{I_j} \left(\frac{\tilde{\kappa}_j^-}{2} + \gamma \right) |\phi(s, 0)|^2 ds, \end{aligned}$$

where $D(t_j^{\gamma, D}) := \{\phi \in H^1(I_j \times (0, a_\gamma)), \phi((j-1)\delta, t) = \phi(\delta j, t) = \phi(s, a_\gamma) = 0\}$. Clearly we have for any $\phi \in \bigoplus_{j=1}^K D(t_j^{\gamma, D})$,

$$\tilde{h}^{\gamma, D}(\phi, \phi) \leq \sum_{j=1}^K t_j^{\gamma, D}(\phi, \phi),$$

and by the min-max principle we get

$$E_n(\tilde{H}^{\gamma,D}) \leq E_n(\oplus_{j=1}^K T_j^{\gamma,D}). \quad (\text{VI.2.27})$$

Let us fix $j \in \{1, \dots, K\}$. It is easy to see that, by separation of variables, $E_n(T_j^{\gamma,D}) = E_n(\mathcal{L}^D \otimes 1 + 1 \otimes \mathcal{D}_j)$, where the operator \mathcal{L}^D acts on $L^2(0, \delta)$ as

$$\mathcal{L}^D f = -(1 + a_\gamma C) f'' + C f, \quad D(\mathcal{L}^D) := H^2(0, \delta) \cap H_0^1(0, \delta).$$

The operator $\mathcal{D}_j := \mathcal{D}_{a_\gamma, \frac{\tilde{\kappa}_j^-}{2} + \gamma}$ defined in Lemma III.5.3 acts on $L^2(0, a_\gamma)$ as $f \mapsto -f''$ with

$$D(\mathcal{D}_j) = \{f \in H^2(0, a_\gamma) : -f'(0) - (\frac{\tilde{\kappa}_j^-}{2} + \gamma)f(0) = f(a_\gamma) = 0\}.$$

There exists $\gamma_1 > 0$ such that, for all $\gamma > \gamma_1$ we have $(\frac{\tilde{\kappa}_j^-}{2} + \gamma)a_\gamma > 1$. Then, using Lemma III.5.3 we know that $E_1(\mathcal{D}_j)$ is the unique negative eigenvalue of \mathcal{D}_j and we have the following estimate, for all $\gamma > \gamma_1$,

$$E_1(\mathcal{D}_j) \leq -(\frac{\tilde{\kappa}_j^-}{2} + \gamma)^2 + 4(\frac{\tilde{\kappa}_j^-}{2} + \gamma)^2 e^{-2(\frac{\tilde{\kappa}_j^-}{2} + \gamma)a_\gamma}. \quad (\text{VI.2.28})$$

Let $E \in (-1, 0)$ be fixed. For all $\gamma > \gamma_1$ one can write, using estimate (VI.2.28),

$$\mathcal{N}(T_j^{\gamma,D}, E\gamma^2) \geq \gamma \frac{\delta \sqrt{E+1}}{\pi \sqrt{1+a_\gamma C}} - C.$$

We immediately have summing on $j \in \{1, \dots, K\}$,

$$\mathcal{N}(H^{\gamma,D}, E\gamma^2) \geq \gamma \frac{L_\gamma \sqrt{E+1}}{\pi \sqrt{1+a_\gamma C}} - C.$$

Recall that $a_\gamma := \gamma^{-1+\epsilon}$, and then

$$(1 + a_\gamma C)^{-1/2} = 1 - \frac{1}{2} C \gamma^{-1+\epsilon} + O(\gamma^{-2+2\epsilon}), \text{ as } \gamma \rightarrow +\infty.$$

Moreover, $L_\gamma := l - 2b\gamma^{-\beta}$. Thus we have

$$\mathcal{N}(H^{\gamma,D}, E\gamma^2) \geq \gamma \frac{l \sqrt{E+1}}{\pi} - C\gamma^\epsilon,$$

with $\epsilon \in (0, 1 - \beta)$. This concludes the proof of (VI.2.21) thanks to (VI.2.23) and (VI.2.25).

Let $\lambda \in \mathbb{R}$. We have

$$\mathcal{N}(H^{\gamma,D}, -\gamma^2 + \lambda\gamma) \geq \sqrt{\gamma} \frac{1}{\pi \sqrt{1-a_\gamma C}} \frac{L_\gamma}{K} \sum_{j=1}^K \sqrt{(\tilde{\kappa}_j^- + \lambda)_+} - C.$$

Again, we can use the convergence of Riemman sum of the Lipschitz function $s \mapsto \sqrt{(\tilde{\kappa}(s) + \lambda)_+}$ to write

$$\int_0^{L_\gamma} \sqrt{(\tilde{\kappa}(s) + \lambda)_+} ds = \frac{L_\gamma}{K} \sum_{j=1}^K \sqrt{(\tilde{\kappa}_j^- + \lambda)_+} + O(\frac{1}{K}), \quad K \rightarrow +\infty.$$

This concludes the proof of (VI.2.22) taking $K \in [\gamma, 2\gamma] \cap \mathbb{N}$. \square

Proof of (VI.1.26) and (VI.1.28). Let us introduce the sesquilinear form

$$\tilde{q}_0^D(\phi, \phi) = \int_{\tilde{\Omega}_0} |\nabla \phi|^2 dx, \quad \phi \in H_0^1(\tilde{\Omega}_0),$$

with $\tilde{\Omega}_0 := \Omega_0 \setminus \bigcup_{k=1}^M \overline{\tilde{\Omega}_{a_\gamma}^k}$. For any $\phi \in \bigoplus_{k=1}^M D(q_k^{\gamma, D}) \oplus D(\tilde{q}_0^D)$ one has $\phi \in D(q_0^\gamma)$ and

$$q_0^\gamma(\phi, \phi) \leq \tilde{q}_0^D(\phi, \phi) + \sum_{k=1}^M q_k^{\gamma, D}(\phi, \phi).$$

Then, we obtain by the min-max principle for all $n \in \mathbb{N}$,

$$E_n(Q_0^\gamma) \leq E_n(\bigoplus_{k=1}^M Q_k^{\gamma, D} \oplus \tilde{Q}_0^D). \quad (\text{VI.2.29})$$

As $s_\gamma = b\gamma^{-\beta}$ we have,

$$\int_{s_\gamma}^{l_k - s_\gamma} \sqrt{(\kappa_k(s) + \lambda)_+} ds = \int_0^{l_k} \sqrt{(\kappa_k(s) + \lambda)_+} ds + O(\gamma^{-\beta}), \quad \gamma \rightarrow +\infty.$$

Combining it with Proposition VI.2.3 and (VI.2.29) finishes the proof. \square

VI.3 Asymptotic behavior of the corner-induced eigenvalues on curvilinear polygons

This section is devoted to the proof of the asymptotics of the \mathcal{N}^\oplus first eigenvalues of Q^γ , see Theorem II.2.3, for Ω being a curvilinear polygon (Definition II.2.2).

Theorem VI.3.1. *Let $\Omega \subset \mathbb{R}^2$ be a curvilinear polygon. There exists $C > 0$ such that, for all $n \in \{1, \dots, \mathcal{N}^\oplus\}$ and for γ large enough we have,*

$$|E_n(Q^\gamma) - \gamma^2 E_n(T^\oplus)| \leq C\gamma^{\frac{4}{3}}.$$

The proof follows the same steps as the one for polygons with straight edges: the weak quasi-modes $\tilde{\phi}_n^{\gamma, v}$ defined in (VI.1.2) are used to obtain the upper bound and we use a partition of unity to get the lower bound.

Recall that Λ^\oplus is the set of the eigenvalues of the operator T^\oplus enumerated in the increasing order and counted *without* multiplicity, $K^\oplus := \#\Lambda^\oplus$ and we denote by m_l the multiplicity of $\lambda_l \in \Lambda^\oplus$ as an eigenvalue of T^\oplus , see Section VI.1.1. We need the following intermediary result.

Proposition VI.3.2. *For all $l \in \{0, \dots, K^\oplus\}$ and for γ large enough we have,*

$$E_{m_1 + \dots + m_l}(Q^\gamma) \leq \gamma^2 \lambda_l + C\gamma^{4/3}, \quad (\text{VI.3.1})$$

$$E_{m_0 + \dots + m_{l+1}}(Q^\gamma) \geq \gamma^2 \lambda_{l+1} - C\gamma^{4/3}, \quad (\text{VI.3.2})$$

with the convention $m_0 = 0$.

Theorem VI.3.1 is easily deduced from Proposition VI.3.2 as follows:

Proof of Theorem VI.3.1. For each $n \in \{1, \dots, \mathcal{N}^\oplus\}$, there exists $l \in \{0, \dots, K^\oplus - 1\}$ such that $m_0 + \dots + m_l + 1 \leq n \leq m_0 + \dots + m_{l+1}$ and $\lambda_{l+1} = E_n(T^\oplus)$. We get the result by Proposition VI.3.2 and the fact that the eigenvalues are ordered in the increasing way. \square

Proof of Proposition VI.3.2. We begin with the proof of (VI.3.1). We introduce $d := \sum_{j=1}^l m_j$ and

$$\mathcal{F}^\gamma := \text{span}\{\tilde{\phi}_n^{\gamma,v}, (n, v) \in \bigcup_{j=1}^l \mathcal{S}_j\}.$$

For simplicity we denote by $(\tilde{\phi}_1, \dots, \tilde{\phi}_d)$ the elements of $\{\tilde{\phi}_n^{\gamma,v}, (n, v) \in \bigcup_{j=1}^l \mathcal{S}_j\}$. By Lemma VI.1.4, $\dim(\mathcal{F}^\gamma) = d$ for γ large enough. Then, by the min-max principle, for γ large enough we have

$$E_d(Q^\gamma) \leq \sup_{\substack{\psi \in \mathcal{F}^\gamma \\ \psi \neq 0}} \frac{q^\gamma(\psi, \psi)}{\|\psi\|^2} = \sup_{\substack{(c_1, \dots, c_d) \in \mathbb{C}^d \\ (c_1, \dots, c_d) \neq (0, \dots, 0)}} \frac{q^\gamma(\sum_{j=1}^d c_j \tilde{\phi}_j, \sum_{j=1}^d c_j \tilde{\phi}_j)}{\|\sum_{j=1}^d c_j \tilde{\phi}_j\|^2}. \quad (\text{VI.3.3})$$

Let us first expand the numerator:

$$q^\gamma\left(\sum_{j=1}^d c_j \tilde{\phi}_j, \sum_{j=1}^d c_j \tilde{\phi}_j\right) = \sum_{j=1}^d |c_j|^2 q^\gamma(\tilde{\phi}_j, \tilde{\phi}_j) + 2\Re \sum_{j < k} c_j \bar{c}_k q^\gamma(\tilde{\phi}_j, \tilde{\phi}_k).$$

We use (VI.1.10) to obtain

$$q^\gamma\left(\sum_{j=1}^d c_j \tilde{\phi}_j, \sum_{j=1}^d c_j \tilde{\phi}_j\right) \leq \left(\gamma^2 \lambda_l + C\gamma^{2-\beta}\right) \sum_{j=1}^d |c_j|^2. \quad (\text{VI.3.4})$$

The denominator expands as

$$\left\|\sum_{j=1}^d c_j \tilde{\phi}_j\right\|^2 = \sum_{j=1}^d |c_j|^2 \|\tilde{\phi}_j\|^2 + 2\Re \sum_{j < k} c_j \bar{c}_k \langle \tilde{\phi}_j, \tilde{\phi}_k \rangle,$$

and by (VI.1.9) we get

$$\left| \left\|\sum_{j=1}^d c_j \tilde{\phi}_j\right\|^2 - \sum_{j=1}^d |c_j|^2 \right| \leq C\gamma^{-\beta} \left(\sum_{j=1}^d |c_j|^2 \right). \quad (\text{VI.3.5})$$

Combining (VI.3.4) and (VI.3.5), we get for γ large enough,

$$\frac{q^\gamma(\sum_{j=1}^d c_j \tilde{\phi}_j, \sum_{j=1}^d c_j \tilde{\phi}_j)}{\|\sum_{j=1}^d c_j \tilde{\phi}_j\|^2} \leq \gamma^2 \lambda_l + C\gamma^{2-\beta},$$

which concludes the proof on the upper bound thanks to (VI.3.3) and taking $\beta = \frac{2}{3}$.

Let us now focus on the lower bound (VI.3.2). In the following $d := \sum_{j=0}^l m_l$. Thanks to Lemma VI.1.5 we can write

$$E_{d+1}(Q^\gamma) \geq E_{d+1} \left(\bigoplus_{v \in \mathcal{V}} Q_{v, 2\gamma^{-\beta}}^{\gamma, V} \oplus Q_0^{\gamma, V} \right). \quad (\text{VI.3.6})$$

Moreover, by (VI.1.19), we have the lower bound, for all $n \in \mathbb{N}$,

$$E_n(Q_{v, 2\gamma^{-\beta}}^{\gamma, V}) \geq E_n(Q_{v, 2\gamma^{-\beta}}^\gamma) - c\gamma^{2\beta}, \quad (\text{VI.3.7})$$

where $Q_{v, 2\gamma^{-\beta}}^\gamma$ acts on $L^2(\Omega_{v, 2\gamma^{-\beta}})$ and is defined as the unique self-adjoint operator associated with

$$q_{v, 2\gamma^{-\beta}}^\gamma(\phi, \phi) = \int_{\Omega_{v, 2\gamma^{-\beta}}} |\nabla \phi|^2 dx - \gamma \int_{\Gamma_{v, 2\gamma^{-\beta}}} |\phi|^2 ds, \quad \phi \in D(q_{v, 2\gamma^{-\beta}}^\gamma) := D(q_{v, 2\gamma^{-\beta}}^{\gamma, V}).$$

Let us fix $v \in \mathcal{V}$. In order to study $Q_{v,2\gamma^{-\beta}}^\gamma$ we perform a change of variables. For $\phi \in D(q_{v,2\gamma^{-\beta}}^\gamma)$, we introduce $\psi(u) := \phi \circ F_v^{-1}(u)$ for all $u \in U_{\alpha_v} \cap B(0, 2\gamma^{-\beta})$. By Taylor-Lagrange, for all $u \in B(0, 2\gamma^{-\beta})$ we have

$$|(\nabla F_v^{-1}(u))^{-1} - I_2| \leq C\gamma^{-\beta}. \quad (\text{VI.3.8})$$

Thanks to the estimates (VI.1.4), (VI.1.5) and (VI.3.8) we get, for all $\phi \in D(q_{v,2\gamma^{-\beta}}^\gamma)$,

$$q_{v,2\gamma^{-\beta}}^\gamma(\phi, \phi) \geq (1 - C\gamma^{-\beta}) \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} |\nabla \psi|^2 du - \gamma(1 + C\gamma^{-\beta}) \int_{\Sigma_{v,2\gamma^{-\beta}}} |\psi| ds. \quad (\text{VI.3.9})$$

We now introduce the sesquilinear form

$$t_{v,2\gamma^{-\beta}}^{f_\beta(\gamma)}(\psi, \psi) = \int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} |\nabla \psi|^2 du - f_\beta(\gamma) \int_{\Sigma_{v,2\gamma^{-\beta}}} |\psi|^2 ds,$$

with $D(t_{v,2\gamma^{-\beta}}^{f_\beta(\gamma)}) := \{\psi \in H^1((U_{\alpha_v})_{0,2\gamma^{-\beta}}), \psi(u) = 0 \text{ for } u \in \partial(U_{\alpha_v})_{0,2\gamma^{-\beta}} \setminus \Sigma_{v,2\gamma^{-\beta}}\}$, and

$$f_\beta(\gamma) := \gamma + C\gamma^{1-\beta}.$$

Lemma VI.3.3. *For any $n \in \mathbb{N}$ and for γ large enough,*

$$E_n(Q_{v,2\gamma^{-\beta}}^\gamma) \geq (1 - C\gamma^{-\beta}) E_n(T_{v,2\gamma^{-\beta}}^{f_\beta(\gamma)}).$$

Proof. First, we have to notice that if $\phi \in D(q_{v,2\gamma^{-\beta}}^\gamma)$, then $\psi \in D(t_{v,2\gamma^{-\beta}}^{f_\beta(\gamma)})$. We can use (VI.3.9) and the min-max principle to obtain,

$$E_n(Q_{v,2\gamma^{-\beta}}^\gamma) \geq (1 - C\gamma^{-\beta}) \inf_{\substack{\mathcal{G} \subset D(q_{v,2\gamma^{-\beta}}^\gamma) \\ \dim(\mathcal{G})=n}} \sup_{\substack{\phi \in \mathcal{G} \\ \phi \neq 0}} \frac{t_{v,2\gamma^{-\beta}}^{f_\beta(\gamma)}(\psi, \psi)}{\|\phi\|_{L^2(\Omega_{v,2\gamma^{-\beta}})}^2}.$$

By (VI.1.3) we have

$$\frac{\int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} |\nabla \psi|^2 dx}{\|\phi\|_{L^2(\Omega_{v,2\gamma^{-\beta}})}^2} \geq \frac{\int_{(U_{\alpha_v})_{0,2\gamma^{-\beta}}} |\nabla \psi|^2 dx}{(1 + C\gamma^{-\beta}) \|\psi\|_{L^2((U_{\alpha_v})_{0,2\gamma^{-\beta}})}^2},$$

and

$$-f_\beta(\gamma) \frac{\int_{\Sigma_{v,2\gamma^{-\beta}}} |\psi|^2 ds}{\|\phi\|_{L^2(\Omega_{v,2\gamma^{-\beta}})}^2} \geq -f_\beta(\gamma) \frac{\int_{\Sigma_{v,2\gamma^{-\beta}}} |\psi|^2 ds}{(1 - C\gamma^{-\beta}) \|\psi\|_{L^2((U_{\alpha_v})_{0,2\gamma^{-\beta}})}^2}.$$

Thus, we first obtain

$$E_n(Q_{v,2\gamma^{-\beta}}^\gamma) \geq (1 - C\gamma^{-\beta}) \inf_{\substack{\mathcal{G} \subset D(q_{v,2\gamma^{-\beta}}^\gamma) \\ \dim(\mathcal{G})=n}} \sup_{\substack{\phi \in \mathcal{G} \\ \phi \neq 0}} \frac{t_{v,2\gamma^{-\beta}}^{f_\beta(\gamma)}(\psi, \psi)}{\|\psi\|_{L((U_{\alpha_v})_{0,2\gamma^{-\beta}})}^2}. \quad (\text{VI.3.10})$$

If we denote $\mathcal{J} := \{\psi = \phi \circ F_v^{-1}, \phi \in \mathcal{G}\}$ and if (ϕ_1, \dots, ϕ_n) is an orthonormal basis of \mathcal{G} , then using again (VI.1.6) we obtain

$$|\langle \phi_i, \phi_j \rangle_{L^2(\Omega_{v,2\gamma^{-\beta}})} - \langle \psi_i, \psi_j \rangle_{L^2((U_{\alpha_v})_{0,2\gamma^{-\beta}})}| \leq C\gamma^{-\beta} |\langle \phi_i, \phi_j \rangle_{L^2(\Omega_{v,2\gamma^{-\beta}})}|,$$

where $\psi_k = \phi_k \circ F_v^{-1}$. Then, $(\psi_k)_{k=1}^n$ is linearly independent if γ is large enough which implies that $\dim(\mathcal{J}) = n$ for large γ . Thus, we can use the min-max principle to get the inequality

$$E_n(T_{v,2\gamma^{-\beta}}^{f_\beta(\gamma)}) \leq \inf_{\substack{\mathcal{G} \subset D(t_{v,2\gamma^{-\beta}}^\gamma) \\ \dim(\mathcal{G})=n}} \sup_{\substack{\phi \in \mathcal{G} \\ \phi \neq 0}} \frac{t_{v,2\gamma^{-\beta}}^{f_\beta(\gamma)}(\psi, \psi)}{\|\psi\|_{L^2((U_{\alpha_v})_{0,2\gamma^{-\beta}})}^2},$$

which concludes the proof combining it with (VI.3.10). \square

Extending $\psi \in D(t_{v,2\gamma^{-\beta}}^{f_\beta(\gamma)})$ by 0, we immediately have for all $n \leq \mathcal{N}^\oplus$ and for γ large enough, thanks to Lemma VI.3.3 and the min-max principle,

$$E_n(Q_{v,2\gamma^{-\beta}}^\gamma) \geq (1 - C\gamma^{-\beta})E_n(T_{\alpha_v}^{f_\beta(\gamma)}). \quad (\text{VI.3.11})$$

In particular,

$$\begin{aligned} E_{d+1}\left(\bigoplus_{v \in \mathcal{V}} Q_{v,2\gamma^{-\beta}}^\gamma\right) &\geq (1 - C\gamma^{-\beta})(f_\beta(\gamma))^2 E_{d+1}(T^\oplus) \\ &= (1 - C\gamma^{-\beta})(f_\beta(\gamma))^2 \lambda_{l+1}. \end{aligned}$$

Notice that $(1 - C\gamma^{-\beta})(f_\beta(\gamma))^2 = \gamma^2 + O(\gamma^{2-\beta})$, as $\gamma \rightarrow +\infty$. Then, for γ large enough,

$$E_{d+1}\left(\bigoplus_{v \in \mathcal{V}} Q_{v,2\gamma^{-\beta}}^\gamma\right) \geq \gamma^2 \lambda_{l+1} - C\gamma^{2-\beta}. \quad (\text{VI.3.12})$$

To finish the proof, in view of (VI.3.6), we need a lower bound of the first eigenvalue of $Q_0^{\gamma,V}$. By the inequality (VI.1.24) of Lemma VI.1.7 we know that $E_1(Q_0^{\gamma,V}) \geq -\gamma^2 - C\gamma^{2\beta}$ for γ large enough and $\beta \in (1/2, 1)$. As $\lambda_{l+1} < -1$, we finally obtain

$$E_{d+1}(Q^\gamma) \geq \gamma^2 \lambda_{l+1} - C\gamma^{2-\beta} - c\gamma^{2\beta}.$$

Taking $\beta = 2/3$ gives us the result. \square

We can also prove that the associated eigenfunctions are concentrated, in a suitable sense, near the convex vertices of Ω . This is due to the standard Agmon-type approach and the proof is similar to the ones of Theorem IV.4.1 and Proposition IV.5.4. Let us introduce the distance to the convex vertices of Ω ,

$$d_v(x) := \text{dist}(x, v), \quad d(x) := \min_{v \in \mathcal{V}} d_v(x).$$

Proposition VI.3.4. *Let φ be an eigenfunction of Q^γ associated with $E := E_n(Q^\gamma)$, where $n \leq \mathcal{N}^\oplus$. Then, for any $\theta \in (0, 1)$ there exist $\eta > 0$, $C > 0$ and $c > 0$ such that there holds*

$$\int_{\Omega} (|\nabla \varphi|^2 + |\varphi|^2) e^{\eta \gamma d(x)} dx \leq C e^{c\gamma^\theta} \|\varphi\|_{L^2(\Omega)}^2, \text{ for large } \gamma.$$

Proof. Let $\eta > 0$ to be fixed later and denote by $\phi(x) := \eta \gamma d(x)$. We are going to prove first

$$\|\varphi e^\phi\|_{H^1(\Omega)}^2 \leq C e^{\eta \gamma^\theta} \|\varphi\|_{L^2(\Omega)}^2 \text{ as } \gamma \text{ is large.} \quad (\text{VI.3.13})$$

For $\xi \in (0, 1)$ to be fixed later we have for any $u \in H^1(\Omega)$,

$$q^\gamma(u, u) = \xi \|\nabla u\|_{L^2(\Omega)}^2 + (1 - \xi) q^{\frac{\gamma}{1-\xi}}(u, u). \quad (\text{VI.3.14})$$

To obtain a lower bound for the last term of this equality we use the previous results. Let $\beta \in (0, 1)$. Recall that $\tilde{\chi}_v^\gamma$ and $\tilde{\chi}_0^\gamma$ are defined in (VI.1.1) and in Section VI.1.2. By the inequality (VI.1.22) we have

$$q^{\frac{\gamma}{1-\xi}}(u, u) \geq \sum_{v \in \mathcal{V}} q_{v, 2\gamma^{-\beta}}^{\frac{\gamma}{1-\xi}, V}(u\tilde{\chi}_v^\gamma, u\tilde{\chi}_v^\gamma) + q_0^{\frac{\gamma}{1-\xi}, V}(u\tilde{\chi}_0^\gamma, u\tilde{\chi}_0^\gamma).$$

In particular,

$$q_{v, 2\gamma^{-\beta}}^{\frac{\gamma}{1-\xi}, V}(u\tilde{\chi}_v^\gamma, u\tilde{\chi}_v^\gamma) \geq E_1(Q_{v, 2\gamma^{-\beta}}^{\frac{\gamma}{1-\xi}, V}) \|u\tilde{\chi}_v^\gamma\|_{L^2(\Omega)}^2,$$

and by (VI.3.7) and (VI.3.11) we have

$$E_1(Q_{v, 2\gamma^{-\beta}}^{\frac{\gamma}{1-\xi}, V}) \geq \frac{(1 - C\gamma^{-\beta})}{(1 - \xi)^2} E_1(T_{\alpha_v}^{f_\beta(\gamma)}) - c\gamma^{2\beta}, \quad c > 0,$$

where $f_\beta(\gamma) := \gamma + C\gamma^{1-\beta}$. On the other hand, by (VI.1.24) we also have

$$q_0^{\frac{\gamma}{1-\xi}, V}(u\tilde{\chi}_0^\gamma, u\tilde{\chi}_0^\gamma) \geq \left(-\frac{\gamma^2}{(1 - \xi)^2} - C\gamma^{2\beta}\right) \|u\tilde{\chi}_0^\gamma\|_{L^2(\Omega)}^2, \quad C > 0.$$

Gathering these estimates we can write

$$\begin{aligned} q^{\frac{\gamma}{1-\xi}}(u, u) \geq & \left(\frac{1 - C\gamma^{-\beta}}{(1 - \xi)^2} f_\beta^2(\gamma) E_1(T^\oplus) - c\gamma^{2\beta} \right) \sum_{v \in \mathcal{V}} \|u\tilde{\chi}_v^\gamma\|_{L^2(\Omega)}^2 \\ & + \left(-\frac{\gamma^2}{(1 - \xi)^2} - C\gamma^{2\beta} \right) \|u\tilde{\chi}_0^\gamma\|_{L^2(\Omega)}^2. \end{aligned} \quad (\text{VI.3.15})$$

As $\phi, \nabla\phi \in L^\infty(\Omega)$, we can apply Lemma III.3.2 to obtain, as φ is an eigenfunction of Q^γ associated with E ,

$$q^\gamma(\varphi e^\phi, \varphi e^\phi) = \int_\Omega (|\nabla\phi|^2 + E) |\varphi e^\phi|^2 dx.$$

We can now apply the estimates (VI.3.14) and (VI.3.15) to φe^ϕ and use the previous equality to get

$$\begin{aligned} \int_\Omega (|\nabla\phi|^2 + E) |\varphi e^\phi|^2 dx \geq & \xi \|\nabla(\varphi e^\phi)\|_{L^2(\Omega)}^2 \\ & + \left(\frac{1 - C\gamma^{-\beta}}{1 - \xi} f_\beta^2(\gamma) E_1(T^\oplus) - c\gamma^{2\beta} \right) \sum_{v \in \mathcal{V}} \|\varphi e^\phi \tilde{\chi}_v^\gamma\|_{L^2(\Omega)}^2 \\ & + \left(-\frac{\gamma^2}{1 - \xi} - C\gamma^{2\beta} \right) \|\varphi e^\phi \tilde{\chi}_0^\gamma\|_{L^2(\Omega)}^2. \end{aligned}$$

As $|\nabla\phi|^2 \leq \gamma^2 \eta^2$, we can rewrite it as follows

$$\gamma^2 A_1 \sum_{v \in \mathcal{V}} \|\varphi e^\phi \tilde{\chi}_v^\gamma\|_{L^2(\Omega)}^2 \geq \xi \|\nabla(\varphi e^\phi)\|_{L^2(\Omega)}^2 + \gamma^2 A_2 \|\varphi e^\phi \tilde{\chi}_0^\gamma\|_{L^2(\Omega)}^2,$$

where

$$\begin{aligned} A_1 & := \eta^2 + E\gamma^{-2} - \gamma^{-2} \left(\frac{1 - C\gamma^{-\beta}}{1 - \xi} f_\beta^2(\gamma) E_1(T^\oplus) - c\gamma^{2\beta} \right), \\ A_2 & := -\eta^2 - E\gamma^{-2} + \gamma^{-2} \left(-\frac{\gamma^2}{1 - \xi} - C\gamma^{2\beta} \right). \end{aligned}$$

Recall that, by Theorem VI.3.1 we have $E \leq \gamma^2 E_n(T^\oplus) + C\gamma^{\frac{4}{3}}$ and thus

$$E\gamma^{-2} < -1.$$

It is easy to see that there exists $\eta < \sqrt{-E\gamma^{-2} - 1}$ such that $A_2 > 0$ for $\xi \in (0, 1)$ small enough. Recall that $f_\beta(\gamma) = \gamma + C\gamma^{1-\beta}$. By Theorem VI.3.1 we have the inequality $E \geq \gamma^2 E_1(T^\oplus) - C\gamma^{\frac{4}{3}}$. Thus we obtain a first lower bound for A_1 ,

$$A_1 \geq \eta^2 + E_1(T^\oplus) - C\gamma^{-\frac{2}{3}} - \frac{1}{1-\xi} E_1(T^\oplus) - C\gamma^{-\beta} - c\gamma^{2\beta-2},$$

and we can conclude that $A_1 > 0$ for γ large enough and ξ sufficiently small. With these choices of η and ξ , there exists $C > 0$ satisfying

$$C\gamma^2 \sum_{v \in \mathcal{V}} \|\varphi e^\phi \tilde{\chi}_v^\gamma\|_{L^2(\Omega)}^2 \geq \|\nabla(\varphi e^\phi)\|_{L^2(\Omega)}^2 + \|\varphi e^\phi \tilde{\chi}_0^\gamma\|_{L^2(\Omega)}^2.$$

Notice that for any $x \in \text{supp} \sum_{v \in \mathcal{V}} \tilde{\chi}_v^\gamma$ we have $\phi(x) \leq 2\eta\gamma^{1-\beta}$. Using the fact that $\sum_{v \in \mathcal{V}} \|\varphi e^\phi \tilde{\chi}_v^\gamma\|_{L^2(\Omega)}^2 + \|\varphi e^\phi \tilde{\chi}_0^\gamma\|_{L^2(\Omega)}^2 = \|\varphi e^\phi\|_{L^2(\Omega)}^2$, we finally obtain

$$\|\varphi e^\phi\|_{H^1(\Omega)}^2 \leq C\gamma^2 e^{2\eta\gamma^{1-\beta}} \|\varphi\|_{L^2(\Omega)}^2,$$

for any $\beta \in (0, 1)$, which proves (VI.3.13). We finish the proof following the same steps as in Theorem IV.4.1 and Proposition IV.5.4. \square

VI.4 Eigenvalue counting functions of curvilinear polygons and truncated sectors

In this section, we give the proofs concerning the eigenvalue counting function of Q^γ in Theorem VI.4.4 for the negative thresholds and Proposition VI.4.5 for the positive one. We need some estimates on the truncated sectors $\Omega_{v,2\gamma^{-\beta}} := \Omega \cap B(v, 2\gamma^{-\beta})$ gathered in the following section.

VI.4.1 Estimates on the truncated sectors

Let us recall some notation. The operator $Q_{v,2\gamma^{-\beta}}^\gamma$ is defined as the unique self-adjoint operator in $L^2(\Omega_{v,2\gamma^{-\beta}})$ associated with the sesquilinear form

$$q_{v,2\gamma^{-\beta}}^\gamma(u, u) = \int_{\Omega_{v,2\gamma^{-\beta}}} |\nabla u|^2 dx - \gamma \int_{\Gamma_{v,2\gamma^{-\beta}}} |u|^2 ds,$$

where $\Gamma_{v,2\gamma^{-\beta}} := \partial\Omega \cap \partial\Omega_{v,2\gamma^{-\beta}}$ and

$$D(q_{v,2\gamma^{-\beta}}^\gamma) := \left\{ u \in H^1(\Omega_{v,2\gamma^{-\beta}}) : u(x) = 0 \text{ on } \partial\Omega_{v,2\gamma^{-\beta}} \setminus \Gamma_{v,2\gamma^{-\beta}} \right\}.$$

The following lemma is a direct consequence of Lemma VI.3.3.

Lemma VI.4.1. *For any $v \in \mathcal{V}$, $\beta \in (1/2, 1)$, $C \in \mathbb{R}$ and for large γ there holds*

$$\mathcal{N}(Q_{v,2\gamma^{-\beta}}^\gamma, C) \leq \mathcal{N}(T_{v,2\gamma^{-\beta}}^{f_\beta(\gamma)}, \frac{C}{1 - C\gamma^{-\beta}}),$$

where $f_\beta(\gamma) := \gamma + C\gamma^{1-\beta}$.

The following result gives an estimate on the eigenvalue counting function of truncated sectors for negative thresholds.

Proposition VI.4.2. *For any $v \in \mathcal{V}$, $C_1 \in (0, 1)$, $\beta \in (1/2, 1)$ and for large γ we have,*

$$\mathcal{N}(Q_{v,2\gamma^{-\beta}}^\gamma, -C_1\gamma^2) = \mathcal{O}(\gamma^{1-\beta}).$$

Proof. By Lemma VI.4.1 one can write

$$\mathcal{N}(Q_{v,2\gamma^{-\beta}}^\gamma, -C_1\gamma^2) \leq \mathcal{N}(T_{v,2\gamma^{-\beta}}^{f_\beta(\gamma)}, -C_1\gamma^2), \quad (\text{VI.4.1})$$

and we are now lead to study the eigenvalue counting function of $T_{v,2\gamma^{-\beta}}^{f_\beta(\gamma)}$.

We introduce

$$U_{\alpha_v}^+ := U_{\alpha_v} \cap (\mathbb{R}_+ \times \mathbb{R}_+) \quad \text{and} \quad \Sigma_{v,2\gamma^{-\beta}}^+ := \Sigma_{v,2\gamma^{-\beta}} \cap \partial U_{\alpha_v}^+.$$

Due to the symmetry of the domain $(U_{\alpha_v})_{0,2\gamma^{-\beta}}$ with respect to the x_1 -axis, it is easy to see that

$$\mathcal{N}(T_{v,2\gamma^{-\beta}}^{f_\beta(\gamma)}, -C_1\gamma^2) \leq 2\mathcal{N}(T_{v,2\gamma^{-\beta}}^{f_\beta(\gamma),+}, -C_1\gamma^2), \quad (\text{VI.4.2})$$

where $T_{v,2\gamma^{-\beta}}^{f_\beta(\gamma),+}$ is the unique self-adjoint operator associated with

$$t_{v,2\gamma^{-\beta}}^{f_\beta(\gamma),+}(\psi, \psi) = \int_{(U_{\alpha_v}^+)_{0,2\gamma^{-\beta}}} |\nabla \psi|^2 du - f_\beta(\gamma) \int_{\Sigma_{v,2\gamma^{-\beta}}^+} |\psi|^2 ds,$$

with

$$D(t_{v,2\gamma^{-\beta}}^{f_\beta(\gamma),+}) := \left\{ \psi \in H^1 \left((U_{\alpha_v}^+)_{0,2\gamma^{-\beta}} \right) : \psi(u) = 0 \text{ for } u \in \partial(U_{\alpha_v}^+)_{0,2\gamma^{-\beta}} \cap \partial B(0, 2\gamma^{-\beta}) \right\}.$$

Let us now introduce a partition of $(U_{\alpha_v}^+)_{0,2\gamma^{-\beta}}$. Consider the points $A_\gamma := (\gamma^{-1}, 0)$, and H_{A_γ} defined as the orthogonal projection of A_γ on $\Sigma_{v,2\gamma^{-\beta}}^+$. Let V be the infinite sector obtained by translation of vector $(\gamma^{-1}, 0)$ of U_{α_v} . We denote $V^+ := V \cap (\mathbb{R}_+ \times \mathbb{R}_+)$. Notice that H_{A_γ} is well defined as \mathcal{V} is the set of *convex* vertices of Ω : then $\alpha_v \in (0, \pi/2)$. We introduce the two new domains:

- D_γ^1 the right-angled triangle defined by its vertices $(0, 0)$, A_γ and H_{A_γ} ;
- $D^2 := (U_{\alpha_v}^+)_{0,2\gamma^{-\beta}} \setminus (\overline{D_\gamma^1} \cup V^+)$.

Hence we have, see Figure VI.1,

$$\overline{(U_{\alpha_v}^+)_{0,2\gamma^{-\beta}}} = \overline{D_\gamma^1 \cup D^2 \cup (V^+)_{0,2\gamma^{-\beta}}},$$

where $(V^+)_{0,2\gamma^{-\beta}} := V^+ \cap B(0, 2\gamma^{-\beta})$.

We will obtain an upper bound of $\mathcal{N}(T_{v,2\gamma^{-\beta}}^{f_\beta(\gamma),+}, -C_1\gamma^2)$ by adding some Neumann barriers at the boundaries of these new domains. To do this, we introduce the sesquilinear forms associated with this covering. Recall that $f_\beta(\gamma) := \gamma + C\gamma^{1-\beta}$. Hence, there exists $\mu > 0$ such that

$$f_\beta(\gamma) \leq \mu\gamma \text{ for any } \gamma > 0.$$

Consider the sesquilinear forms

$$h(\psi, \psi) = \int_{(V^+)_{0,2\gamma^{-\beta}}} |\nabla \psi|^2 du,$$

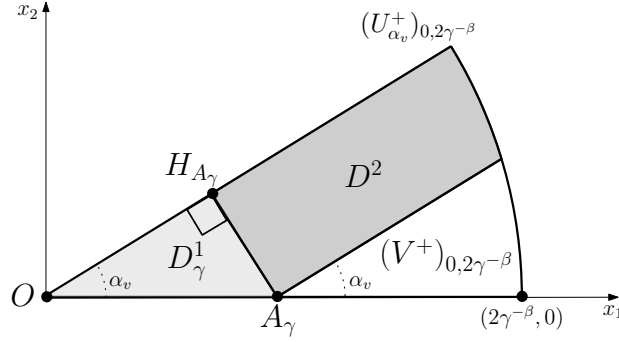


Figure VI.1 – Partition of $(U_{\alpha_v}^+)_{0,2\gamma^{-\beta}}$ by the domains D_γ^1 , D^2 and $(V^+)_{0,2\gamma^{-\beta}}$.

with

$$D(h) := \left\{ \psi \in H^1 \left((V^+)_{0,2\gamma^{-\beta}} \right) : \psi(u) = 0 \text{ for } u \in \partial(V^+)_{0,2\gamma^{-\beta}} \cap \partial B(0, 2\gamma^{-\beta}) \right\},$$

and

$$p^{\mu\gamma}(\psi, \psi) = \int_{D^2} |\nabla \psi|^2 du - \mu\gamma \int_{\partial D^2 \cap \partial U_{\alpha_v}^+} |\psi|^2 ds,$$

with

$$D(p^{\mu\gamma}) := \left\{ \psi \in H^1(D^2) : \psi(u) = 0 \text{ for } u \in \partial D^2 \cap \partial B(0, 2\gamma^{-\beta}) \right\}.$$

Furthermore, we introduce the following notation. For a Lipschitz bounded domain $D \subset \mathbb{R}^2$ we define

$$r_D^{\mu\gamma}(\psi, \psi) = \int_D |\nabla \psi|^2 du - \mu\gamma \int_{\partial D \cap \partial U_{\alpha_v}} |\psi|^2 ds, \quad \psi \in D(r_D^{\mu\gamma}) := H^1(D).$$

It is easy to see that $T_{v,2\gamma^{-\beta}}^{f_\beta(\gamma),+} \geq R_{D_\gamma^1}^{\mu\gamma} \oplus H \oplus P^{\mu\gamma}$. Moreover, as the operator H is positive we have

$$\mathcal{N}(H, -C_1\gamma^2) = 0.$$

Thus we obtain,

$$\mathcal{N}(T_{v,2\gamma^{-\beta}}^{f_\beta(\gamma),+}, -C_1\gamma^2) \leq \mathcal{N}(R_{D_\gamma^1}^{\mu\gamma}, -C_1\gamma^2) + \mathcal{N}(P^{\mu\gamma}, -C_1\gamma^2). \quad (\text{VI.4.3})$$

We perform the change of variables $u = \gamma^{-1}v$ in the sesquilinear form $r_{D_\gamma^1}^{\mu\gamma}$. Then, for all $n \in \mathbb{N}$ we have $E_n(R_{D_\gamma^1}^{\mu\gamma}) = \gamma^2 E_n(R_{D^1}^\mu)$, where D^1 is a right-angled triangle defined by its vertices $(0,0)$, $A := (1,0)$ and H_A the orthogonal projection of A on $\partial U_{\alpha_v} \cap (\mathbb{R}_+ \times \mathbb{R}_+)$. In particular,

$$\mathcal{N}(R_{D_\gamma^1}^{\mu\gamma}, -C_1\gamma^2) = \mathcal{N}(R_{D^1}^\mu, -C_1) = \mathcal{O}(1).$$

Let us now focus on the operator $P^{\mu\gamma}$. Notice that D^2 is included in a rectangle of length $2\gamma^{-\beta} - \gamma^{-1} \cos \alpha_v$ and width $\gamma^{-1} \sin \alpha_v$. Extending $\phi \in D(p^{\mu\gamma})$ by 0 and using the min-max principle, we obtain

$$\mathcal{N}(P^{\mu\gamma}, -C_1\gamma^2) \leq \mathcal{N}(\mathcal{L}^{ND} \otimes 1 + 1 \otimes \mathcal{N}_{\gamma^{-1} \sin \alpha_v, \mu\gamma}, -C_1\gamma^2),$$

where \mathcal{L}^{ND} is the operator acting on $L^2(0, 2\gamma^{-\beta} - \gamma^{-1} \cos \alpha_v)$ as $f \mapsto -f''$ on

$$D(\mathcal{L}^{ND}) := \left\{ f \in H^2(0, 2\gamma^{-\beta} - \gamma^{-1} \cos \alpha_v) : -f'(0) = f(2\gamma^{-\beta} - \gamma^{-1} \cos \alpha_v) = 0 \right\}.$$

The operator $\mathcal{N}_{\gamma^{-1} \sin \alpha_v, \mu \gamma}$ defined in Lemma III.5.2 acts on $L^2(0, \gamma^{-1} \sin \alpha_v)$ as $f \mapsto -f''$ on

$$D(\mathcal{N}_{\gamma^{-1} \sin \alpha_v, \mu \gamma}) := \left\{ H^2(0, \gamma^{-1} \sin \alpha_v) : -f'(0) - \mu \gamma f(0) = f'(\gamma^{-1} \sin \alpha_v) = 0 \right\}.$$

We know by Lemma III.5.2 that $E_1(\mathcal{N}_{\gamma^{-1} \sin \alpha_v, \mu \gamma})$ is the unique strictly negative eigenvalue of $\mathcal{N}_{\gamma^{-1} \sin \alpha_v, \mu \gamma}$ and $E_1(\mathcal{N}_{\gamma^{-1} \sin \alpha_v, \mu \gamma}) = \gamma^2 E_1(\mathcal{N}_{\sin \alpha_v, \mu})$. Thus, we have

$$\mathcal{N}(P^{\mu \gamma}, -C_1 \gamma^2) \leq \# \left\{ n \in \mathbb{N}, E_n(\mathcal{L}^{ND}) + \gamma^2 E_1(\mathcal{N}_{\sin \alpha_v, \mu}) < -C_1 \gamma^2 \right\},$$

which implies

$$\mathcal{N}(P^{\mu \gamma}, -C_1 \gamma^2) \leq \gamma^{1-\beta} \frac{\sqrt{|C_1 + E_1(\mathcal{N}_{\sin \alpha_v, \mu})|}}{\pi} + C.$$

Combining it with (VI.4.1), (VI.4.2) and (VI.4.3) finishes the proof. \square

We now give an estimate on the eigenvalue counting function of truncated sectors for positive thresholds.

Proposition VI.4.3. *For any $v \in \mathcal{V}$, $C_2 > 0$, $\beta \in (1/2, 1)$ and large γ there holds,*

$$\mathcal{N}(Q_{v, 2\gamma^{-\beta}}^\gamma, C_2 \gamma^2) = \mathcal{O}(\gamma^{2-2\beta}).$$

Proof. The proof is based on the decomposition used in the proof of [KK13, Lemma 2.5].

Using Lemma VI.4.1, there exists $C_3 > 0$ such that, for large γ we have

$$\mathcal{N}(Q_{v, 2\gamma^{-\beta}}^\gamma, C_2 \gamma^2) \leq \mathcal{N}(T_{v, 2\gamma^{-\beta}}^{f_\beta(\gamma)}, C_3 \gamma^2).$$

Let $\varphi \in C^\infty(\mathbb{R}_+)$ be a smooth cut-off function satisfying $0 \leq \varphi \leq 1$, $\varphi(t) = 1$ if $0 \leq t \leq 1$ and $\varphi(t) = 0$ if $t \geq 2$. Let us define $\tilde{\chi}_0^\gamma(x) := \varphi(|x|\gamma)$ and $\tilde{\chi}_1^\gamma(x) := 1 - \tilde{\chi}_0^\gamma(x)$. We introduce the smooth partition $(\chi_0^\gamma, \chi_1^\gamma)$ as follows:

$$\chi_0^\gamma(x) := \frac{\tilde{\chi}_0^\gamma(x)}{\sqrt{(\tilde{\chi}_0^\gamma)^2 + (\tilde{\chi}_1^\gamma)^2}}, \quad \chi_1^\gamma(x) := \frac{\tilde{\chi}_1^\gamma(x)}{\sqrt{(\tilde{\chi}_0^\gamma)^2 + (\tilde{\chi}_1^\gamma)^2}}.$$

Defined in this way, it satisfies $(\chi_0^\gamma)^2 + (\chi_1^\gamma)^2 = 1$ on $(U_{\alpha_v})_{0, 2\gamma^{-\beta}}$ and there exists $C > 0$ such that

$$|\nabla \chi_0^\gamma|^2 + |\nabla \chi_1^\gamma|^2 \leq C \gamma^2. \quad (\text{VI.4.4})$$

Using Lemma III.2.2, one can write for any $u \in D(t_{v, 2\gamma^{-\beta}}^{f_\beta(\gamma)})$,

$$\begin{aligned} t_{v, 2\gamma^{-\beta}}^{f_\beta(\gamma)}(u, u) &= t_{v, 2\gamma^{-\beta}}^{f_\beta(\gamma)}(u \chi_0^\gamma, u \chi_0^\gamma) + t_{v, 2\gamma^{-\beta}}^{f_\beta(\gamma)}(u \chi_1^\gamma, u \chi_1^\gamma) \\ &\quad - \int_{(U_{\alpha_v})_{0, 2\gamma^{-\beta}}} (|\nabla \chi_0^\gamma|^2 + |\nabla \chi_1^\gamma|^2) |u|^2 dx. \end{aligned} \quad (\text{VI.4.5})$$

Let us introduce the subdomains

$$(U_{\alpha_v})_{0, 2\gamma^{-1}} := U_{\alpha_v} \cap B(0, 2\gamma^{-1}), \quad A_\beta := (U_{\alpha_v})_{0, 2\gamma^{-\beta}} \setminus \overline{(U_{\alpha_v})_{0, \gamma^{-1}}},$$

and the sesquilinear form

$$q_{A_\beta}(u, u) = \int_{A_\beta} |\nabla u|^2 - f_\beta(\gamma) \int_{\partial A_\beta \cap \partial U_{\alpha_v}} |u|^2 ds,$$

with $D(q_{A_\beta}) := \left\{ u \in H^1(A_\beta) : u(x) = 0 \text{ on } (\partial A_\beta \cap \partial B(0, \gamma^{-1})) \cup (\partial A_\beta \cap \partial B(0, 2\gamma^{-\beta})) \right\}$.
 Let $u \in D(t_{v, 2\gamma^{-\beta}}^{f_\beta(\gamma)})$. Then, $u\chi_0^\gamma \in D(t_{v, 2\gamma^{-1}}^{f_\beta(\gamma)})$, $u\chi_1^\gamma \in D(q_{A_\beta})$, and

$$\|u\chi_0^\gamma\|_{L^2((U_{\alpha_v})_{0, 2\gamma^{-\beta}})}^2 + \|u\chi_1^\gamma\|_{L^2((U_{\alpha_v})_{0, 2\gamma^{-\beta}})}^2 = \|u\|_{L^2((U_{\alpha_v})_{0, 2\gamma^{-\beta}})}^2.$$

Applying the min-max principle, and taking into account (VI.4.5) and (VI.4.4), there exists $C_4 > 0$ such that

$$\mathcal{N}(T_{v, 2\gamma^{-\beta}}^{f_\beta(\gamma)}, C_3\gamma^2) \leq \mathcal{N}(T_{v, 2\gamma^{-1}}^{f_\beta(\gamma)}, C_4\gamma^2) + \mathcal{N}(Q_{A_\beta}, C_4\gamma^2).$$

Remark that we simply have $(U_{\alpha_v})_{0, 2\gamma^{-1}} = \gamma^{-1}(U_{\alpha_v})_{0, 2}$. By change of variables, one can write for any $n \in \mathbb{N}$,

$$E_n(T_{v, 2\gamma^{-1}}^{f_\beta(\gamma)}) = \gamma^2 E_n(T_{v, 2}^{f_\beta(\gamma)\gamma^{-1}}),$$

and $f_\beta(\gamma)\gamma^{-1} = 1 + C\gamma^{-\beta}$. In particular, for γ large enough one has

$$E_n(T_{v, 2}^{f_\beta(\gamma)\gamma^{-1}}) \geq E_n(T_{v, 2}^2),$$

and thus

$$\mathcal{N}(T_{v, 2\gamma^{-1}}^{f_\beta(\gamma)}, C_4\gamma^2) \leq \mathcal{N}(T_{v, 2}^2, C_4) = \mathcal{O}(1).$$

Let us focus on the operator Q_{A_β} . We want to introduce a partition of A_β . Let us introduce the polar coordinates $(r, \theta) \in \mathbb{R}_+ \times [0, 2\pi)$ which are more convenient to describe the truncated shell A_β . In polar coordinates it is described by

$$A_\beta := \left\{ x = (r \cos \theta, r \sin \theta) : r \in (\gamma^{-1}, 2\gamma^{-\beta}), \theta \in (-\alpha_v, \alpha_v) \right\}.$$

We define the domain

$$S_\beta := \left\{ x = (r \cos \theta, r \sin \theta) : r \in (\gamma^{-1}, 2\gamma^{-\beta}), \theta \in (0, 2\alpha_v) \right\},$$

which is a rotated copy of A_β . We denote by

$$\Sigma := \left\{ x \in \overline{A_\beta} : |x| = \gamma^{-1} \right\} \cup \left\{ x \in \overline{A_\beta} : |x| = 2\gamma^{-\beta} \right\}, \quad \Gamma := \partial S_\beta \setminus \Sigma.$$

such that $\partial S_\beta = \overline{\Gamma \cup \Sigma}$ and we introduce the sesquilinear form

$$q_{S_\beta}(u, u) = \int_{S_\beta} |\nabla u|^2 dx - f_\beta(\gamma) \int_\Gamma |u|^2 ds,$$

with $D(q_{S_\beta}) := \{u \in H^1(S_\beta) : u = 0 \text{ on } \Sigma\}$. One can see that Q_{A_β} and Q_{S_β} are unitary equivalent and thus

$$\mathcal{N}(Q_{A_\beta}, C_4\gamma^2) = \mathcal{N}(Q_{S_\beta}, C_4\gamma^2).$$

We will introduce a partition of unity depending only on the angle of aperture $\theta \in (0, 2\alpha_v)$. More precisely, we want to define a partition satisfying, if we write $x = (r \cos \theta, r \sin \theta)$,

$$\begin{aligned} \chi_1(x) &= 1, \text{ if } \theta \in (0, \alpha_v), \\ \chi_1(x) &= 0, \text{ if } \theta \in (\alpha_v + \epsilon, 2\alpha_v), \\ \chi_1^2 + \chi_2^2 &= 1 \text{ on } S_\beta, \end{aligned} \tag{VI.4.6}$$

for a suitable $\epsilon > 0$. Let us now construct this partition of unity using the cartesian coordinates. Let

$$0 < \epsilon < \frac{\pi}{2} - \alpha_v. \quad (\text{VI.4.7})$$

Pick a cut-off function $\varphi \in C^\infty(\mathbb{R}_+)$ satisfying $0 \leq \varphi \leq 1$, $\varphi(t) = 1$ if $t \leq \alpha_v$ and $\varphi(t) = 0$ if $t > \alpha_v + \epsilon$. We introduce

$$\tilde{\chi}_1(x) := \varphi \left(\arccos \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}} \right) \right), \quad \tilde{\chi}_2 := 1 - \tilde{\chi}_1,$$

and the smooth partition (χ_1, χ_2) by

$$\chi_1(x) := \frac{\tilde{\chi}_1(x)}{\sqrt{(\tilde{\chi}_1)^2 + (\tilde{\chi}_2)^2}}, \quad \chi_2(x) := \frac{\tilde{\chi}_2(x)}{\sqrt{(\tilde{\chi}_1)^2 + (\tilde{\chi}_2)^2}}.$$

Defined in this way it satisfies the three properties of (VI.4.6), and there exists $C > 0$ such that

$$|\nabla \chi_1|^2 + |\nabla \chi_2|^2 \leq C\gamma^2 \text{ on } S_\beta. \quad (\text{VI.4.8})$$

Consider the subdomains, described in polar coordinates for simplicity,

$$\begin{aligned} S_\beta^1 &:= \left\{ x = (r \cos \theta, r \sin \theta) : r \in (\gamma^{-\beta}, 2\gamma^{-\beta}), \theta \in (0, \alpha_v + \epsilon) \right\}, \\ S_\beta^2 &:= \left\{ x = (r \cos \theta, r \sin \theta) : r \in (\gamma^{-1}, 2\gamma^{-\beta}), \theta \in (\alpha_v, 2\alpha_v) \right\}, \end{aligned}$$

and the associated sesquilinear forms, for $j = 1, 2$,

$$q_\beta^j(u, u) = \int_{S_\beta^j} |\nabla u|^2 dx - f_\beta(\gamma) \int_{\partial S_\beta^j \cap \Gamma} |u|^2 ds,$$

with $D(q_\beta^j) := \left\{ u \in H^1(S_\beta^j) : u = 0 \text{ on } \partial S_\beta^j \setminus \Gamma \right\}$. Applying Lemma III.2.2 to q_{S_β} and (χ_1, χ_2) we first obtain, for any $u \in H^1(S_\beta)$,

$$q_{S_\beta}(u, u) = q_{S_\beta}(u\chi_1, u\chi_1) + q_{S_\beta}(u\chi_2, u\chi_2) - \int_{S_\beta} \left(|\nabla \chi_1|^2 + |\nabla \chi_2|^2 \right) |u|^2 dx.$$

Now using (VI.4.8), this gives

$$q_{S_\beta}(u, u) \geq q_{S_\beta}(u\chi_1, u\chi_1) + q_{S_\beta}(u\chi_2, u\chi_2) - C\gamma^2 \|u\|_{L^2(S_\beta)}^2. \quad (\text{VI.4.9})$$

Notice that, for any $u \in H^1(S_\beta)$ we have $u\chi_1 \in D(q_\beta^1)$, $u\chi_2 \in D(q_\beta^2)$,

$$\|u\chi_1\|_{L^2(S_\beta)}^2 + \|u\chi_2\|_{L^2(S_\beta)}^2 = \|u\|_{L^2(S_\beta)}^2,$$

and thus, applying the min-max principle and using (VI.4.9) we obtain

$$\mathcal{N}(Q_{S_\beta}, C_4\gamma^2) \leq \mathcal{N}(Q_\beta^1, C_5\gamma^2) + \mathcal{N}(Q_\beta^2, C_5\gamma^2), \quad C_5 > 0.$$

Let us study the operators Q_β^1 and Q_β^2 . The domain S_β^1 is a truncated shell of total aperture $\alpha_v + \epsilon < \frac{\pi}{2}$ due to the choice of ϵ in (VI.4.7), and the domain S_β^2 is a truncated shell of total aperture $\alpha_v < \frac{\pi}{2}$ by definition of α_v . Hence, both of these domains are included in a square of length $2\gamma^{-\beta}$. Moreover, the respective operators act as the Laplacian with the

$f_\beta(\gamma)$ -Robin boundary condition on $S_\beta^j \cap \Gamma$ and the Dirichlet boundary condition on the remaining part of the boundary. This implies that, extending $u \in D(Q_\beta^j)$ by zero, applying the min-max principle and using the separation of variables one has, for $j = 1, 2$,

$$\mathcal{N}(Q_\beta^j, C_5\gamma^2) \leq \mathcal{N}(\mathcal{D}_{2\gamma^{-\beta}, f_\beta(\gamma)} \otimes 1 + 1 \otimes \mathcal{L}^D, C_5\gamma^2),$$

where \mathcal{L}^D is the Dirichlet Laplacian in $L^2(0, 2\gamma^{-\beta})$ acting as $f \mapsto -f''$ with $f(0) = f(2\gamma^{-\beta}) = 0$ and the operator $\mathcal{D}_{2\gamma^{-\beta}, f_\beta(\gamma)}$ was introduced in Lemma III.5.3. First, we can write

$$\begin{aligned} \mathcal{N}(\mathcal{D}_{2\gamma^{-\beta}, f_\beta(\gamma)} \otimes 1 + 1 \otimes \mathcal{L}^D, C_5\gamma^2) &= \mathcal{N}(\mathcal{D}_{2\gamma^{-\beta}, f_\beta(\gamma)} \otimes 1 + 1 \otimes \mathcal{L}^D, 0) \\ &\quad + \#\left\{(n, m) \in \mathbb{N}^2 : 0 \leq E_n(\mathcal{D}_{2\gamma^{-\beta}, f_\beta(\gamma)}) + E_m(\mathcal{L}^D) < C_5\gamma^2\right\}. \end{aligned}$$

Let us first count the negative eigenvalues. Recall that $f_\beta(\gamma) := \gamma + C\gamma^{1-\beta}$. Thus, for γ large enough $2\gamma^{-\beta}f_\beta(\gamma) > 1$ and by Lemma III.5.3, we then know that $E_1(\mathcal{D}_{2\gamma^{-\beta}, f_\beta(\gamma)})$ is the unique negative eigenvalue of $\mathcal{D}_{2\gamma^{-\beta}, f_\beta(\gamma)}$. Moreover, by the asymptotics (III.5.10), we have

$$E_1(\mathcal{D}_{2\gamma^{-\beta}, f_\beta(\gamma)}) \geq -\gamma^2 - C\gamma^{1-\beta}, \quad C > 0.$$

As $E_m(\mathcal{L}^D) = \left(\frac{\pi m}{2\gamma^{-\beta}}\right)^2$ we obtain

$$\mathcal{N}(\mathcal{D}_{2\gamma^{-\beta}, f_\beta(\gamma)} \otimes 1 + 1 \otimes \mathcal{L}^D, 0) = \mathcal{O}(\gamma^{1-\beta}).$$

In order to count the positive eigenvalues, we can use (III.5.11) to write, for any $j \in \mathbb{N}$,

$$E_{1+j}(\mathcal{D}_{2\gamma^{-\beta}, f_\beta(\gamma)}) = \left(\frac{\pi j}{2\gamma^{-\beta}}\right)^2 + o(1), \quad \text{as } \gamma \rightarrow +\infty.$$

This implies

$$\begin{aligned} \#\left\{(n, m) \in \mathbb{N}^2 : 0 \leq E_n(\mathcal{D}_{2\gamma^{-\beta}, f_\beta(\gamma)}) + E_m(\mathcal{L}^D) < C_5\gamma^2\right\} \\ \leq \#\left\{(n, m) \in \mathbb{N}^2 : 0 \leq \left(\frac{\pi n}{2\gamma^{-\beta}}\right)^2 + \left(\frac{\pi m}{2\gamma^{-\beta}}\right)^2 < C\gamma^2\right\}, \end{aligned}$$

and the right hand side admits as an upper bound the area of the circle of radius $R := \frac{2\sqrt{C}\gamma^{1-\beta}}{\pi}$. Finally we obtain

$$\#\left\{(n, m) \in \mathbb{N}^2 : 0 \leq E_n(\mathcal{D}_{2\gamma^{-\beta}, f_\beta(\gamma)}) + E_m(\mathcal{L}^D) < C_5\gamma^2\right\} = \mathcal{O}(\gamma^{2-2\beta}),$$

which concludes the proof. \square

VI.4.2 Weyl-type asymptotics for Robin Laplacians on curvilinear polygons

This section is devoted to the proof of the Weyl-type asymptotics for the eigenvalue counting function of $Q_\Omega^\gamma := Q_\Omega^\gamma$. We first present the asymptotics for two negative thresholds similar to the ones involved in [HKR17].

Theorem VI.4.4. *For any $E \in (-1, 0)$ and $\beta \in (1/2, 1)$ there holds*

$$\mathcal{N}(Q^\gamma, E\gamma^2) = \gamma \frac{|\partial\Omega|\sqrt{E+1}}{\pi} + O(\gamma^{1-\beta}), \quad \text{as } \gamma \rightarrow +\infty. \quad (\text{VI.4.10})$$

For any $\lambda \in \mathbb{R}$ there holds

$$\mathcal{N}(Q^\gamma, -\gamma^2 + \lambda\gamma) = \frac{\sqrt{\gamma}}{\pi} \sum_{k=1}^M \int_0^{l_k} \sqrt{(\kappa_k(s) + \lambda)_+} ds + O(\gamma^{1/4}), \quad \text{as } \gamma \rightarrow +\infty. \quad (\text{VI.4.11})$$

Proof. Let $E \in (-1, 0)$ be fixed. Using Lemma VI.1.5, we have the upper bound

$$\mathcal{N}(Q^\gamma, E\gamma^2) \leq \sum_{v \in \mathcal{V}} \mathcal{N}(Q_{v, 2\gamma^{-\beta}}^{\gamma, V}, E\gamma^2) + \mathcal{N}(Q_0^{\gamma, V}, E\gamma^2). \quad (\text{VI.4.12})$$

By (VI.3.7) we also have

$$\mathcal{N}(Q_{v, 2\gamma^{-\beta}}^{\gamma, V}, E\gamma^2) \leq \mathcal{N}(Q_{v, 2\gamma^{-\beta}}^\gamma, E\gamma^2 + c\gamma^{2\beta}), \quad c > 0,$$

and as $\beta < 1$ this gives, using the asymptotics in Proposition VI.4.2,

$$\mathcal{N}(Q_{v, 2\gamma^{-\beta}}^{\gamma, V}, E\gamma^2) \leq C\gamma^{1-\beta}, \quad C > 0. \quad (\text{VI.4.13})$$

Taking into account (VI.4.13) and the upper bound (VI.1.25), the inequality (VI.4.12) becomes

$$\mathcal{N}(Q^\gamma, E\gamma^2) \leq \gamma \frac{|\partial\Omega|\sqrt{E+1}}{\pi} + C\gamma^{1-\beta}, \quad C > 0. \quad (\text{VI.4.14})$$

By Lemma VI.1.6, we have the lower bound

$$\mathcal{N}(Q^\gamma, E\gamma^2) \geq \mathcal{N}(Q_0^\gamma, E\gamma^2),$$

and using (VI.1.26) this gives

$$\mathcal{N}(Q^\gamma, E\gamma^2) \geq \gamma \frac{|\partial\Omega|\sqrt{E+1}}{\pi} - C\gamma^{1-\beta}, \quad C > 0. \quad (\text{VI.4.15})$$

Gathering the upper bound (VI.4.14) and the lower bound (VI.4.15) finishes the proof of (VI.4.10).

The proof of (VI.4.11) follows the same steps. Let $\lambda \in \mathbb{R}$. First, we have the following upper bound thanks to Lemma VI.1.5 and (VI.3.7):

$$\mathcal{N}(Q^\gamma, -\gamma^2 + \lambda\gamma) \leq \sum_{v \in \mathcal{V}} \mathcal{N}(Q_{v, 2\gamma^{-\beta}}^\gamma, -\gamma^2 + \lambda\gamma + c\gamma^{2\beta}) + \mathcal{N}(Q_0^{\gamma, V}, -\gamma^2 + \lambda\gamma), \quad c > 0.$$

There exists $C \in (0, 1)$ such that $-\gamma^2 + \lambda\gamma + c\gamma^{2\beta} < -C\gamma^2$, and we can apply Proposition VI.4.2 and the upper bound VI.1.27 to get

$$\mathcal{N}(Q^\gamma, -\gamma^2 + \lambda\gamma) \leq \frac{\sqrt{\gamma}}{\pi} \sum_{k=1}^M \int_0^{l_k} \sqrt{(\kappa_k(s) + \lambda)_+} ds + C\gamma^{\beta-\frac{1}{2}} + C_0\gamma^{1-\beta}, \quad C_0, C > 0.$$

We can take $\beta = \frac{3}{4}$ to have

$$\mathcal{N}(Q^\gamma, -\gamma^2 + \lambda\gamma) \leq \frac{\sqrt{\gamma}}{\pi} \sum_{k=1}^M \int_0^{l_k} \sqrt{(\kappa_k(s) + \lambda)_+} ds + C\gamma^{\frac{1}{4}}, \quad C > 0. \quad (\text{VI.4.16})$$

To obtain the lower bound, we apply Lemma VI.1.6 and (VI.1.28) and get

$$\mathcal{N}(Q^\gamma, -\gamma^2 + \lambda\gamma) \geq \frac{\sqrt{\gamma}}{\pi} \sum_{k=1}^M \int_0^{l_k} \sqrt{(\kappa_k(s) + \lambda)_+} ds - C, \quad C > 0. \quad (\text{VI.4.17})$$

The upper bound (VI.4.16) and the lower bound (VI.4.17) conclude the proof of (VI.4.11). \square

Let us now prove the Weyl-type asymptotics for a positive threshold.

Proposition VI.4.5. *For any $C > 0$ there holds*

$$\mathcal{N}(Q^\gamma, C\gamma^2) = C\gamma^2 \frac{|\Omega|}{4\pi} + o(\gamma^2), \quad \text{as } \gamma \rightarrow \infty.$$

Proof. The proof is composed by two steps: first we give an upper bound for $\mathcal{N}(Q^\gamma, C\gamma^2)$ and then a lower bound. We keep the same notation as in Section VI.1.2 and Section VI.2.

Step 1. Let us prove the upper bound. Recall that by Lemma VI.1.5, we first have

$$\mathcal{N}(Q^\gamma, C\gamma^2) \leq \sum_{v \in \mathcal{V}} \mathcal{N}(Q_{v, 2\gamma^{-\beta}}^{\gamma, V}, C\gamma^2) + \mathcal{N}(Q_0^{\gamma, V}, C\gamma^2). \quad (\text{VI.4.18})$$

Let us focus on $Q_0^{\gamma, V}$. First, remark that due to (VI.1.19), we have

$$\mathcal{N}(Q_0^{\gamma, V}, C\gamma^2) \leq \mathcal{N}(Q_0^\gamma, C\gamma^2 + c\gamma^{2\beta}), \quad c > 0,$$

where Q_0^γ is defined in (VI.1.23). Let $\varphi \in C^\infty(\mathbb{R}_+)$ be a smooth cut-off function satisfying $0 \leq \varphi \leq 1$, $\varphi(t) = 1$ if $0 \leq t \leq \frac{1}{2}$ and $\varphi(t) = 0$ if $t \geq 1$. We introduce the smooth cut-off functions $\tilde{\psi}_k$, $k = 1, \dots, M$ as follows:

$$\tilde{\psi}_k(x) := \varphi\left(\text{dist}(x, \Gamma_k)\gamma^{1-\epsilon}\right), \quad x \in \Omega_0,$$

where $\epsilon \in (0, 1 - \beta)$. Recall that each Γ_k is C^4 and thus, for γ large enough the cut-off functions $\tilde{\psi}_k$ are smooth. Let $\tilde{\psi}_0 := 1 - \sum_{k=1}^M \tilde{\psi}_k$. For any $k = 0, 1, \dots, M$ we define

$$\psi_k := \frac{\tilde{\psi}_k}{\sqrt{\sum_{k=0}^M \tilde{\psi}_k^2}}.$$

Defined in this way the ψ_k are smooth, satisfy $\sum_{k=0}^M \psi_k^2 = 1$ on Ω_0 and there exists a constant $c_0 > 0$ such that

$$\sum_{k=0}^M |\nabla \psi_k|^2 \leq c_0 \gamma^{2-2\epsilon}. \quad (\text{VI.4.19})$$

Moreover, in accordance with notation (VI.2.3) we have for any $k = 1, \dots, M$,

$$\overline{\text{supp } \psi_k \cap \Omega_0} \subset \overline{\tilde{\Omega}_{a_\gamma}^k}. \quad (\text{VI.4.20})$$

Define $\Omega_0^* := \Omega_0 \setminus \left(\bigcup_{k=1}^M \text{supp } \psi_k\right)$ and the sesquilinear form

$$q_{\Omega_0^*}^D(u, u) = \int_{\Omega_0^*} |\nabla u|^2 dx, \quad u \in H_0^1(\Omega_0^*).$$

We can apply Lemma III.2.2 to q_0^γ and the smooth partition $(\psi_k)_{k=0}^M$ together with the estimate (VI.4.19) to get, for any $u \in D(q_0^\gamma)$,

$$q_0^\gamma(u, u) \geq \sum_{k=0}^M q_0^\gamma(u\psi_k, u\psi_k) - c_0\gamma^{2-2\epsilon}\|u\|_{L^2(\Omega_0)}^2. \quad (\text{VI.4.21})$$

Recall that the forms $q_k^{\gamma, D}$ were defined in (VI.2.20). For any $u \in D(q_0^\gamma)$, we have thanks to the inclusion (VI.4.20), $u\psi_k \in D(q_k^{\gamma, D})$ for $k = 1, \dots, M$ and $u\psi_0 \in D(q_{\Omega_0^*}^D)$, and

$$\sum_{k=0}^M \|u\psi_k\|_{L^2(\Omega_0)}^2 = \|u\|_{L^2(\Omega_0)}^2.$$

We can apply the min-max principle and use (VI.4.21) to obtain

$$\begin{aligned} \mathcal{N}(Q_0^\gamma, \mathcal{C}\gamma^2 + c\gamma^{2\beta}) &\leq \sum_{k=1}^M \mathcal{N}(Q_k^{\gamma, D}, \mathcal{C}\gamma^2 + c_0\gamma^{2-2\epsilon} + c\gamma^{2\beta}) \\ &\quad + \mathcal{N}(Q_{\Omega_0^*}^D, \mathcal{C}\gamma^2 + c_0\gamma^{2-2\epsilon} + c\gamma^{2\beta}). \end{aligned}$$

By the choice of $\epsilon \in (0, 1 - \beta)$, and $\beta \in (1/2, 1)$, there exists $C > 0$ such that for large γ there holds

$$\gamma^2 + c_0\gamma^{2-2\epsilon} + c\gamma^{2\beta} \leq C\gamma^2.$$

Thus, the previous inequality becomes

$$\mathcal{N}(Q_0^\gamma, \mathcal{C}\gamma^2 + c\gamma^{2\beta}) \leq \sum_{k=1}^M \mathcal{N}(Q_k^{\gamma, D}, \mathcal{C}\gamma^2) + \mathcal{N}(Q_{\Omega_0^*}^D, \mathcal{C}\gamma^2 + c_0\gamma^{2-2\epsilon} + c\gamma^{2\beta}). \quad (\text{VI.4.22})$$

The operator $Q_{\Omega_0^*}^D$ acts simply as the Dirichlet Laplacian in $L^2(\Omega_0^*)$ and $\Omega_0^* \subset \Omega$. Thus, using the monotonicity of the Dirichlet eigenvalues we have

$$\mathcal{N}(Q_{\Omega_0^*}^D, \mathcal{C}\gamma^2 + c_0\gamma^{2-2\epsilon} + c\gamma^{2\beta}) \leq \mathcal{N}(Q_\Omega^D, \mathcal{C}\gamma^2 + c_0\gamma^{2-2\epsilon} + c\gamma^{2\beta}).$$

We can now apply the classical Weyl law, see e.g. [RS78, Theorem XIII. 78], to Q_Ω^D and obtain

$$\mathcal{N}(Q_\Omega^D, \mathcal{C}\gamma^2 + c_0\gamma^{2-2\epsilon} + c\gamma^{2\beta}) = \mathcal{C}\gamma^2 \frac{|\Omega|}{4\pi} + o(\gamma^2), \text{ as } \gamma \rightarrow +\infty. \quad (\text{VI.4.23})$$

In view of (VI.4.22), we now have to obtain an upper bound for $\mathcal{N}(Q_k^{\gamma, D}, \mathcal{C}\gamma^2)$. First, we can apply (VI.2.23) to write

$$\mathcal{N}(Q_k^{\gamma, D}, \mathcal{C}\gamma^2) = \mathcal{N}(P_k^{\gamma, D}, \mathcal{C}\gamma^2).$$

Define the sesquilinear form

$$\tilde{h}_k^{\gamma, D}(\phi, \phi) = \int_{\square_{a_\gamma}^k} (|\partial_s \phi|^2 + |\partial_t \phi|^2 - C|\phi|^2) ds dt - \int_0^{l_k} \left(\frac{\kappa_{\max}}{2} + \gamma \right) |\phi(s, 0)|^2 ds,$$

with $D(\tilde{h}_k^{\gamma, D}) := \left\{ \phi \in H^1(\square_{a_\gamma}^k) : \phi(0, t) = \phi(l_k, t) = \phi(s, a_\gamma) = 0 \right\}$. Recall that

$$\kappa_{\max} := \max_{k=1, \dots, M} \left(\max_{s \in [0, l_k]} \kappa_k(s) \right).$$

The functions in the domain of $P_k^{\gamma,D}$ satisfy a Robin boundary condition on $\{(s, 0), s \in (0, l_k)\}$ and the Dirichlet boundary condition on the remaining part of the boundary of $\tilde{\square}_{a_\gamma}^k$. Due to the inclusion $\tilde{\square}_{a_\gamma}^k \subset \square_{a_\gamma}^k$ and the monotonicity of the Dirichlet eigenvalues, and using the estimates (VI.2.24), one can write

$$\mathcal{N}(P_k^{\gamma,D}, C\gamma^2) \leq \mathcal{N}(\tilde{H}_k^{\gamma,D}, C\gamma^2).$$

It is easy to see that by separation of variables we have, for any $n \in \mathbb{N}$,

$$E_n(\tilde{H}_k^{\gamma,D}) = E_n(\mathcal{L}^D \otimes 1 + 1 \otimes \mathcal{D}).$$

The operator \mathcal{L}^D acts on $L^2(0, l_k)$ as

$$\mathcal{L}^D f = -(1 - a_\gamma C)f'' - Cf, \quad D(\mathcal{L}^D) := H^2(0, l_k) \cap H_0^1(0, l_k),$$

and the operator $\mathcal{D} := \mathcal{D}_{a_\gamma, \frac{\kappa_{\max}}{2} + \gamma}$ is defined in Lemma III.5.3. Thus,

$$\begin{aligned} \mathcal{N}(\tilde{H}_k^{\gamma,D}, C\gamma^2) &= \#\{(n, m) \in \mathbb{N}^2 : E_n(\mathcal{L}^D) + E_m(\mathcal{D}) < 0\} \\ &\quad + \#\{(n, m) \in \mathbb{N}^2 : 0 \leq E_n(\mathcal{L}^D) + E_m(\mathcal{D}) < C\gamma^2\}. \end{aligned}$$

For any $n \in \mathbb{N}$, $E_n(\mathcal{L}^D) = \left(\frac{\pi n}{l_k}\right)^2$. There exists $\gamma_0 > 0$ such that for any $\gamma \geq \gamma_0$ we have $(\frac{\kappa_{\max}}{2} + \gamma)a_\gamma > 1$. Thus, by Lemma III.5.3 we know that for $\gamma \geq \gamma_0$, $E_1(\mathcal{D})$ is the unique negative eigenvalue of \mathcal{D} . Moreover, due to (III.5.10) we have for large γ ,

$$E_1(\mathcal{D}) \geq -\gamma^2 - Ce^{-c\gamma^\epsilon}, \quad C, c > 0.$$

Finally, this gives

$$\#\{(n, m) \in \mathbb{N}^2 : E_n(\mathcal{L}^D) + E_m(\mathcal{D}) < 0\} = \mathcal{O}(\gamma).$$

In order to count the positive eigenvalues, we can use (III.5.11) to write for any $j \in \mathbb{N}$,

$$E_{1+j}(\mathcal{D}) = \left(\frac{\pi j}{a_\gamma}\right)^2 + o(1), \quad \text{as } \gamma \rightarrow +\infty.$$

This gives,

$$\begin{aligned} \#\{(n, m) \in \mathbb{N}^2 : 0 \leq E_n(\mathcal{L}^D) + E_m(\mathcal{D}) < C\gamma^2\} \\ \leq \#\{(n, j) \in \mathbb{N}^2 : \left(\frac{\pi n}{l_k}\right)^2 + \left(\frac{\pi j}{a_\gamma}\right)^2 < C\gamma^2\}, \end{aligned}$$

and the right hand side admits as an upper bound the area of the ellipse of radii $A := \frac{\sqrt{C}l_k\gamma}{\pi}$

and $B := \frac{\sqrt{C}a_\gamma\gamma}{\pi}$, where $a_\gamma := \gamma^{-1+\epsilon}$, with $\epsilon \in (0, 1 - \beta)$. The area is given by πAB and then

$$\#\{(n, m) \in \mathbb{N}^2 : 0 \leq E_n(\mathcal{L}^D) + E_m(\mathcal{D}) < C\gamma^2\} = \mathcal{O}(\gamma^{1+\epsilon}).$$

Gathering the previous estimates we finally have

$$\mathcal{N}(\tilde{H}_k^{\gamma,D}, C\gamma^2) = \mathcal{O}(\gamma^{1+\epsilon}),$$

and using also (VI.4.23) we obtain the upper bound

$$\mathcal{N}(Q_0^{\gamma,V}, \mathcal{C}\gamma^2) \leq \mathcal{C}\gamma^2 \frac{|\Omega|}{4\pi} + o(\gamma^2). \quad (\text{VI.4.24})$$

Due to (VI.4.18), we now need an estimate on $Q_{v,2\gamma^{-\beta}}^{\gamma,V}$. First, by (VI.3.7) we have

$$\mathcal{N}(Q_{v,2\gamma^{-\beta}}^{\gamma,V}, \mathcal{C}\gamma^2) \leq \mathcal{N}(Q_{v,2\gamma^{-\beta}}^{\gamma}, \mathcal{C}\gamma^2 + c\gamma^{2\beta}), \quad c > 0,$$

and we can apply Proposition VI.4.3 to get the upper bound

$$\mathcal{N}(Q_{v,2\gamma^{-\beta}}^{\gamma,V}, \mathcal{C}\gamma^2) \leq C\gamma^{2-2\beta}, \quad C > 0. \quad (\text{VI.4.25})$$

Gathering (VI.4.24) and (VI.4.25), we finally obtain the upper bound

$$\mathcal{N}(Q^{\gamma}, \mathcal{C}\gamma^2) \leq \mathcal{C}\gamma^2 \frac{|\Omega|}{4\pi} + o(\gamma^2). \quad (\text{VI.4.26})$$

Step 2. Let us now focus on the lower bound. By the min-max principle, we have for any $n \in \mathbb{N}$,

$$E_n(Q^{\gamma}) \leq E_n(Q_{\Omega}^D),$$

where Q_{Ω}^D is the Laplacian in $L^2(\Omega)$ with the Dirichlet boundary condition on $\partial\Omega$. Thus, applying the classical Weyl law to Q_{Ω}^D , see e.g. [RS78, Theorem XIII. 78], we immediately have

$$\mathcal{N}(Q^{\gamma}, \mathcal{C}\gamma^2) \geq \mathcal{C}\gamma^2 \frac{|\Omega|}{4\pi} + o(\gamma^2),$$

and this finishes the proof of the lower bound and of the proposition due to (VI.4.26). \square

VI.5 The further eigenvalues

In Theorem VI.3.1, we proved that the asymptotics of the \mathcal{N}^{\oplus} first eigenvalues of the operator Q^{γ} is determined by Robin Laplacians acting on the tangent sectors. The next natural step would be to understand what happens for the further eigenvalues. More precisely, we would like to obtain an asymptotics for $E_{\mathcal{N}^{\oplus+j}}(Q^{\gamma})$ as γ becomes large. For now, we can give a first answer stating that the corners do not contribute at the leading order to the asymptotics.

Proposition VI.5.1. *For each $j \geq 1$, there exists $C > 0$ such that, for γ large enough,*

$$-\gamma^2 - C\gamma^{4/3} \leq E_{\mathcal{N}^{\oplus+j}}(Q^{\gamma}) \leq -\gamma^2 - \kappa_{\min}\gamma + C,$$

where $\kappa_{\min} := \min_{k=1,\dots,M} \left(\min_{s \in [0, l_k]} \kappa_k(s) \right)$. Consequently,

$$E_{\mathcal{N}^{\oplus+j}}(Q^{\gamma}) = -\gamma^2 + o(\gamma^2), \quad \text{as } \gamma \rightarrow +\infty.$$

Proof. We obtain the lower bound by Proposition VI.3.2: using (VI.3.2) with $l = K^{\oplus}$ we immediately have, for γ large enough,

$$E_{\mathcal{N}^{\oplus+1}}(Q^{\gamma}) \geq -\gamma^2 - C\gamma^{4/3}.$$

Let us now focus on the upper bound. We use the notations of Section VI.2.2. Let $k \in \{1, \dots, M\}$. Recall that $D(q_k^{\gamma,D}) := \{\phi \in H^1(\tilde{\Omega}_{a_{\gamma}}^k), \phi = 0 \text{ on } \partial\tilde{\Omega}_{a_{\gamma}}^k \setminus \Gamma_k\}$. As $D(q_k^{\gamma,D}) \subset H^1(\Omega)$ we get by the min-max principle and for all $n \in \mathbb{N}$,

$$E_n(Q^{\gamma}) \leq E_n(Q_k^{\gamma,D}).$$

Following the steps of the proof of Proposition VI.2.3, we know that $E_n(Q_k^{\gamma,D}) \leq E_n(\tilde{H}_k^{\gamma,D})$. We now introduce

$$\begin{aligned} \tilde{h}_k^{\gamma,D,-}(\phi, \phi) &= \int_0^{L_\gamma} \int_0^{a_\gamma} \left((1 + a_\gamma C) |\partial_s \phi|^2 + |\partial_t \phi|^2 + C |\phi|^2 \right) ds dt \\ &\quad - \left(\frac{\kappa_{k,\min}}{2} + \gamma \right) \int_0^{L_\gamma} |\phi(s, 0)|^2 ds, \quad \phi \in D(\tilde{h}_k^{\gamma,D,-}), \end{aligned}$$

where $\kappa_{k,\min} := \min_{s \in [0, l_k]} \kappa_k(s)$. Then, by the min-max principle,

$$E_n(Q^\gamma) \leq E_n(\tilde{H}_k^{\gamma,D,-}).$$

By separation of variables it is easy to see that $\tilde{H}_k^{\gamma,D,-}$ is unitarily equivalent to the operator $\mathcal{L}_{L_\gamma}^D \otimes 1 + 1 \otimes \mathcal{D}^-$ where the one-dimensional operator $\mathcal{L}_{L_\gamma}^D$ acts on $L^2(0, L_\gamma)$ as $f \mapsto -(1 + a_\gamma C) f'' + C f$ with

$$D(\mathcal{L}_{L_\gamma}^D) := H^2(0, L_\gamma) \cap H_0^1(0, L_\gamma).$$

The operator $\mathcal{D}^- := \mathcal{D}_{a_\gamma, \frac{\kappa_{k,\min}}{2} + \gamma}$ defined in Lemma III.5.3 acts on $L^2(0, a_\gamma)$ as $f \mapsto -f''$ with

$$D(\mathcal{D}^-) := \{f \in H^2(0, a_\gamma), -f'(0) - \left(\frac{\kappa_{k,\min}}{2} + \gamma\right) f(0) = f(a_\gamma) = 0\}.$$

There exists $\gamma_0 > 0$ such that for all $\gamma > \gamma_0$ we have $\left(\frac{\kappa_{k,\min}}{2} + \gamma\right) a_\gamma > 1$. Then for all $\gamma > \gamma_0$, we know by Lemma III.5.3 that $E_1(\mathcal{D}^-)$ is the unique negative eigenvalue of \mathcal{D}^- and we have the following estimate, thanks to the asymptotics (III.5.10),

$$E_1(\mathcal{D}^-) \leq -\left(\frac{\kappa_{k,\min}}{2} + \gamma\right)^2 + C.$$

As $\text{spec}(\mathcal{L}_{L_\gamma}^D) \subset \mathbb{R}_+$, we then have for γ large enough,

$$E_n(Q^\gamma) \leq E_n(\mathcal{L}_{L_\gamma}^D) + E_1(\mathcal{D}^-).$$

Using the previous estimate on $E_1(\mathcal{D}^-)$ we get for any $k \in \{1, \dots, M\}$,

$$E_n(Q^\gamma) \leq -\gamma^2 - \kappa_{k,\min} \gamma + C.$$

We can take the minimum over k to conclude the proof. □

Chapter VII

Prospects

To conclude this manuscript, we discuss in this section some open problems arising from this thesis, and which may lead to future research projects.

VII.1 Embedded eigenvalues on infinite sectors

In Chapter IV, we do not discuss the (non-)existence of eigenvalues embedded into the continuous spectrum. Some partial information can be easily obtained, for example, the Rellich-Type result [BBDFHT16] implies the absence of positive embedded eigenvalues for $\alpha \geq \frac{\pi}{2}$. Moreover, by a simple adaptation of the virial theorem [RS78, Theorem XIII.59], we are able to prove that this still holds for any $\alpha \in (0, \pi)$.

Proposition VII.1.1. *For any $\alpha \in (0, \pi)$ and $\gamma > 0$, the operator T_α^γ has no positive embedded eigenvalue.*

Proof. Suppose that there exist $k \in \mathbb{R}$ and $u \in D(T_\alpha^\gamma)$ satisfying

$$\begin{aligned} -\Delta u &= k^2 u \text{ on } U_\alpha, \\ \partial_\nu u &= \gamma u \text{ on } \partial U_\alpha. \end{aligned}$$

Denote by $u_a(x) := u(ax)$ for $a \in \mathbb{R}_+$, $a \neq 1$. Then $u_a \in D(T_\alpha^{a\gamma})$ and satisfies $-\Delta u_a = a^2 k^2 u_a$ on U_α and $\partial_\nu u_a = a\gamma u_a$ on ∂U_α . Let us consider

$$t_\alpha^\gamma(u, u_a) = \int_{U_\alpha} \nabla u \nabla u_a dx - \gamma \int_{\partial U_\alpha} u u_a ds.$$

By integration by parts we get on one hand

$$t_\alpha^\gamma(u, u_a) = k^2 \int_{U_\alpha} u u_a dx,$$

and on the other hand

$$t_\alpha^\gamma(u, u_a) = a^2 k^2 \int_{U_\alpha} u u_a dx - (1-a)\gamma \int_{\partial U_\alpha} u u_a ds.$$

Gathering both of the previous equalities one can write

$$(a+1)k^2 \int_{U_\alpha} u u_a dx = -\gamma \int_{\partial U_\alpha} u u_a ds.$$

Passing to the limit $a \rightarrow 1$ gives $2k^2 \|u\|_{L^2(U_\alpha)}^2 = -\gamma \|u\|_{L^2(\partial U_\alpha)}^2$ which concludes the proof. \square

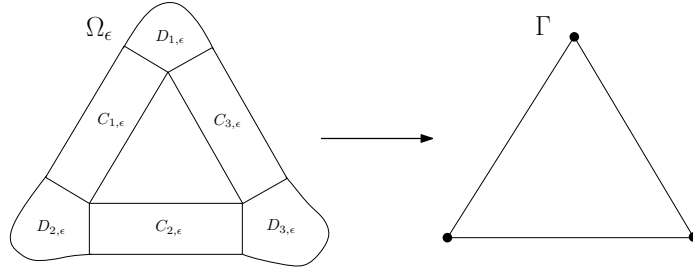


Figure VII.1 – An example of Ω_ϵ with $J = 3$ and the associated graph Γ .

This method cannot be adapted to negative embedded eigenvalues and for now this question remains open. One can remark that, as the spectra of the Robin Laplacian and the Dirichlet-to-Neumann operator are linked, this question also remains open for this latter operator, as mentioned by [Ivr, Remark A.18].

VII.2 Effective operator for Robin eigenvalues on curvilinear polygons

As already mentioned in Section II.2.3, we expect that, for any curvilinear polygon $\Omega \subset \mathbb{R}^2$, the further eigenvalues of Q_Ω^γ behave as

$$E_{\mathcal{N}^{\oplus+j}}(Q_\Omega^\gamma) = -\gamma^2 + E_j(\tilde{L}_\gamma) + R(\gamma), \text{ as } \gamma \rightarrow +\infty,$$

where the operator \tilde{L}_γ acts in $L^2(\partial\Omega)$ as $u \mapsto -\partial_s^2 u - \gamma \kappa u$, where κ is the signed curvature of $\partial\Omega$, on functions satisfying some additional boundary conditions at the corners and $R(\gamma)$ is a small remainder. This intuition comes from the fact that the Robin Laplacian seems to make some links with the Dirichlet Laplacian acting on domains collapsing on metric graphs. Let us develop this idea.

For $\epsilon > 0$, consider the domain $\Omega_\epsilon \subset \mathbb{R}^2$ defined as follows: Ω_ϵ is composed by $J \in \mathbb{N}$ cylinders $C_{j,\epsilon} := I_j \times (0, \epsilon)$, with $I_j := (0, \ell_j)$, 'glued' to J bounded Lipschitz domains $D_{j,\epsilon} \subset \mathbb{R}^2$ where each $D_{j,\epsilon}$ is isometric to ϵD_j , with $D_j \subset \mathbb{R}^2$ being an ϵ -independent bounded Lipschitz domain. The underlying metric graph Γ is defined by its J edges formed by the cylinders and its J vertices corresponding to the domains D_j , see Figure VII.1. Denote by $Q_\epsilon^D := Q_{\Omega_\epsilon}^D$ the Dirichlet Laplacian acting on Ω_ϵ and by $E_n(\epsilon)$ its eigenvalues, enumerated in the increasing order and counting the multiplicities. We are interested in the asymptotic behavior of $E_n(\epsilon)$ as $\epsilon \rightarrow 0$. In a sense, the metric graph Γ can be viewed as an asymptotic support of the problem and it seems natural to expect that the properties of $E_n(\epsilon)$ as ϵ is small are determined by the properties of Γ . Such a relation was studied in details in more general contexts and in any dimension in [Gri08, MV07, Pos05], and it was proved that the asymptotics of the eigenvalues is determined by an effective operator acting on Γ . In the situation described above, the results are as follows. First, by rescaling one can see that

$$\Omega_\epsilon = \epsilon X^N, \quad N := \frac{1}{\epsilon},$$

where X^N is formed by the cylinders $(0, N\ell_j) \times (0, 1)$ attached to the domains D_j . Remark that this rescaling implies in particular: $\epsilon^2 E_n(\epsilon) = E_n(Q_{X^N}^D)$ for any $n \in \mathbb{N}$. The limit object $\lim_{N \rightarrow +\infty} X^N$ will be of importance:

$$X^\infty := \cup_{j=1, \dots, J} X_j^\infty,$$

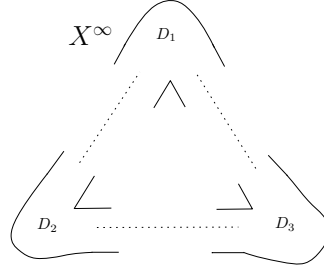


Figure VII.2 – The domain $X^\infty := \cup_{j=1,2,3} X_j^\infty$.

where X_j^∞ is formed by the domain D_j attached to the infinite cylinders $[0, +\infty) \times (0, 1)$, see Figure VII.2. Following the denomination of [Pan17b], the domain X_j^∞ is called a *star waveguide*. The spectrum of $Q_{X^\infty}^D$ is no longer discrete. In particular, its essential spectrum is equal to $[\pi^2, +\infty)$ and its discrete spectrum is finite. Denote by $E_j(Q_{X^\infty}^D)$ its discrete eigenvalues and by

$$\mathcal{N}^\infty := \#\text{spec}_{\text{disc}}(Q_{X^\infty}^D).$$

Then, it was proved in [Gri08, MV07] that there exists $N \geq \mathcal{N}^\infty$ such that for small ϵ there holds

1. for $n \leq N$,

$$E_n(\epsilon) = \frac{\tau_n}{\epsilon^2} + \mathcal{O}(e^{-\frac{\epsilon}{c}}), \quad \tau_n \in (0, \pi], \quad c > 0,$$

2. and for any $j \in \mathbb{N}$,

$$E_{N+j}(\epsilon) = \frac{\pi^2}{\epsilon^2} + \mu_j + \mathcal{O}(\epsilon),$$

where μ_j is the j -th eigenvalue of the self-adjoint operator L acting on $\bigoplus_{j=1}^J L^2(0, \ell_j)$ as $u \mapsto -u''$ with additional boundary conditions at the vertices of Γ which are determined by the scattering matrices of $Q_{X_j^\infty}^D$ at the threshold π^2 .

The asymptotics of the N first eigenvalues of Q_ϵ^D arise from the discrete spectrum of $Q_{X^\infty}^D$ and the bottom of the essential spectrum π , while the further ones arise from the interior of the essential spectrum, and the determination of the boundary conditions, also called transmission conditions, of the effective operator L is a difficult task in general.

In our context, the eigenfunctions of the Robin Laplacian are easily shown to be concentrated in a small vicinity of the boundary of the domain. Hence, the boundary can be considered as a kind of asymptotic support, and it is natural to expect that the existence of an effective operator acting on the boundary will determine the asymptotics of the eigenvalues in the strong coupling limit $\gamma \rightarrow +\infty$. This was in particular rigorously proved in any dimension for smooth domains by [PP16]. For the case of planar domains with corners, the only result available is the one presented in Section II.2.3, which holds for a particular class of curvilinear polygons, and the proof cannot be adapted to general ones. Our idea is to adapt the machinery proposed in [Gri08] to the Robin case, in order to prove an asymptotics for the Robin eigenvalues on any curvilinear polygon. By analogy, we believe that the tangent sectors play the role of the star waveguides X_j^∞ . Indeed, similarly to the asymptotics of the N first $E_n(\epsilon)$, the asymptotics of the \mathcal{N}^\oplus eigenvalues of Q_Ω^γ are led by the discrete spectrum of the Robin Laplacians acting on the tangent sectors, see Theorem II.2.3. Following this analogy, it seems that we will have to carry out a precise study of scattering-type problems on infinite sectors to understand the behavior of the further Robin eigenvalues.

Meanwhile, we also intend to prove that our definition of *non-resonant* angles, see Definition II.2.6, makes sense in this context, and more precisely that it is linked to an intrinsic characteristic of Robin Laplacians on infinite sectors. Actually, the definition of non-resonant angles comes from a naive adaptation of a sufficient condition for the absence of threshold resonance in star waveguides. Let us introduce some definition and notation to explain this link.

For any domain $\Omega \subset \mathbb{R}^2$, we define the subset $H_{loc}^2(\Omega)$ as follows:

$$H_{loc}^2(\Omega) := \{u : u \in H^2(K \cap \Omega) \text{ for any compact } K \subset \mathbb{R}^2\}.$$

Let $X \subset \mathbb{R}^2$ be a star waveguide. Then, u is a *threshold resonance* of Q_X^D if $u \in L^\infty(X) \cap H_{loc}^2(X)$ is a non-trivial solution of

$$\begin{aligned} -\Delta u &= \pi^2 u \text{ on } X, \\ u &= 0 \text{ on } \partial X. \end{aligned}$$

The non-existence of threshold resonance is linked to the behavior of the scattering matrix of Q_X^D at the threshold π^2 and thus to the asymptotic behavior of $E_n(\epsilon)$ as proved in [Gri08, MV07]: suppose that for any $j = 1, \dots, J$, the operator $Q_{X_j}^D$ does not admit any threshold resonance. Then, there holds for small ϵ ,

1. for $n \leq \mathcal{N}^\infty$,

$$E_n(\epsilon) = \frac{E_n(Q_{X^\infty}^D)}{\epsilon^2} + \mathcal{O}(e^{-\frac{c}{\epsilon}}), \quad c > 0,$$

2. and for any $j \in \mathbb{N}$,

$$E_{\mathcal{N}^\infty + j}(\epsilon) = \frac{\pi^2}{\epsilon^2} + \mu_j + \mathcal{O}(\epsilon),$$

where μ_j is the j -th eigenvalue of the Dirichlet Laplacian acting on $\bigoplus_{j=1}^J L^2(0, \ell_j)$.

This result means that when there is no threshold resonance, the effective operator is decoupled. Hence, the question of whether or not there exist some resonances becomes of importance and some investigations have been made in [BN, Pan17a]. Let us come back to the Robin case.

Definition VII.2.1. Let $\alpha \in (0, \frac{\pi}{2})$. We say that T_α admits a threshold resonance if there exists a non-trivial $u \in L^\infty(U_\alpha) \cap H_{loc}^2(U_\alpha)$ satisfying

$$\begin{aligned} -\Delta u &= -u \text{ on } U_\alpha, \\ \partial_\nu u &= u \text{ on } \partial U_\alpha. \end{aligned}$$

By analogy with the conditions given in [BN, Pan17a] for star waveguides, we intend to prove:

Conjecture VII.2.2. The angle $\alpha \in (0, \frac{\pi}{2})$ is non-resonant if and only if T_α does not admit any threshold resonance.

For now, this problem remains open. Nevertheless, we can prove a first result, which is an adaptation of [Pan17a, Proposition 11], and which allows us to rephrase Conjecture VII.2.2.

Proposition VII.2.3. *If T_α admits a threshold resonance for some $\alpha \in (0, \frac{\pi}{2})$, then the eigenvalue counting function $\theta \mapsto \mathcal{N}(T_\theta, -1)$ admits a jump at α .*

The proof is given below. Due to this proposition, the sufficiency condition in Conjecture VII.2.2 becomes

if $\alpha \in (0, \frac{\pi}{2})$ is *resonant*, then α is a jump of $\theta \mapsto \mathcal{N}(T_\theta, -1)$.

Recall that in his paper [McC11], McCartin made a study of the asymptotic behavior of the Robin eigenvalues on equilateral triangles. For an equilateral triangle $\mathcal{T} \subset \mathbb{R}^2$, we have $\mathcal{N}^\oplus = 3$ and McCartin proved in [McC11, Section 7.4] that

$$E_4(Q_\mathcal{T}^\gamma) = -\gamma^2 + o(1), \text{ as } \gamma \rightarrow +\infty.$$

Due to the asymptotics of Theorem II.2.8, we see that $\frac{\pi}{6}$ is necessarily a *resonant* angle. Then, combining Conjecture VII.2.2 and Proposition VII.2.3 we arrive at

Conjecture VII.2.4. The angle $\frac{\pi}{6}$ is a jump of $\theta \mapsto \mathcal{N}(T_\theta, -1)$.

Proof of Proposition VII.2.3. Let us recall some notation of Chapter IV. We denote by $\mathcal{N}_\alpha := \mathcal{N}(T_\alpha, -1)$. Let $U_\alpha^+ := U_\alpha \cap (\mathbb{R}_+ \times \mathbb{R}_+)$. The self-adjoint operator T_α^N defined in $L^2(U_\alpha^+)$, and introduced in (IV.2.1), acts as the Laplacian on U_α^+ with the Robin boundary condition on $\partial U_\alpha^+ \cap U_\alpha$ and the Neumann boundary condition on the remaining part of the boundary. In Section IV.2.1, we proved that

$$\mathcal{N}_\alpha = \mathcal{N}(T_\alpha^N, -1), \tag{VII.2.1}$$

see (IV.2.4).

We want to prove that $\mathcal{N}_\beta \geq \mathcal{N}_\alpha + 1$ for any $\beta \in (0, \alpha)$. In view of (VII.2.1), it is sufficient to prove

$$\mathcal{N}(T_\beta^N, -1) \geq \mathcal{N}(T_\alpha^N, -1) + 1. \tag{VII.2.2}$$

The proof follows the same steps as in [Pan17a, Proposition 11]. The idea is to construct a subspace $W \subset H^1(U_\beta^+)$ of dimension $\mathcal{N}_\alpha + 1$ satisfying

$$\sup_{\substack{v \in W \\ v \neq 0}} \frac{t_\beta^N(v, v)}{\|v\|_{L^2(U_\beta^+)}^2} < -1, \tag{VII.2.3}$$

which proves (VII.2.2) by the min-max principle.

Step 1. To construct such a subspace W , we first need to construct a family of $\mathcal{N}_\alpha + 1$ linearly independent functions. By assumptions, there exists a non-zero $u \in L^\infty(U_\alpha) \cap H_{loc}^2(U_\alpha)$ satisfying

$$\begin{aligned} -\Delta u &= -u \text{ on } U_\alpha, \\ \partial_\nu u &= u \text{ on } \partial U_\alpha. \end{aligned}$$

We introduce the even part of u_0 ,

$$v_0(x) := \frac{u(x) + u(-x)}{2}, \quad x \in U_\alpha^+.$$

Then, $v_0 \in L^\infty(U_\alpha^+) \cap H_{loc}^2(U_\alpha^+)$ and satisfies

$$\begin{aligned} -\Delta v_0 &= -v_0 \text{ on } U_\alpha^+, \\ \partial_\nu v_0 &= v_0 \text{ on } \partial U_\alpha^+ \cap \partial U_\alpha, \\ \partial_\nu v_0 &= 0 \text{ on } \partial U_\alpha^+ \setminus \partial U_\alpha. \end{aligned} \tag{VII.2.4}$$

For simplicity we denote by $\mathcal{N}_\alpha^N := \mathcal{N}(T_\alpha^N, -1)$ and $E_j := E_j(T_\alpha^N)$, $j = 1, \dots, \mathcal{N}_\alpha^N$ the discrete eigenvalues of T_α^N . Let $(v_1, \dots, v_{\mathcal{N}_\alpha^N})$ be the associated orthonormalized eigenfunctions of T_α^N . Then, we have the following result:

Lemma VII.2.5. *The functions $v_0, v_1, \dots, v_{\mathcal{N}_\alpha^N}$ are linearly independent on any open subset of U_α^+ .*

Proof. Let us make the proof by contradiction. Suppose that there exists a non-empty domain $\Omega \subset U_\alpha^+$ and $\xi := (\xi_0, \xi_1, \dots, \xi_{\mathcal{N}_\alpha^N}) \in \mathbb{R}^{\mathcal{N}_\alpha^N+1} \setminus \{0\}$ such that

$$\sum_{j=0}^{\mathcal{N}_\alpha^N} \xi_j v_j = 0 \text{ on } \Omega. \quad (\text{VII.2.5})$$

Denote by $E_0 = -1$. Let $E \in \Lambda := \{E_0, E_1, \dots, E_{\mathcal{N}_\alpha^N}\}$ be fixed and apply the differential expression $(-\Delta - \lambda)$ successively for any $\lambda \in \Lambda \setminus \{E\}$ to (VII.2.5). This gives

$$\left(\prod_{\lambda \in \Lambda \setminus \{E\}} (E - \lambda) \right) \varphi_E = 0 \text{ on } \Omega, \quad \varphi_E := \sum_{j: E_j = E} \xi_j v_j.$$

Due to this equality, the function φ_E vanishes on Ω . Moreover, it satisfies $-\Delta \varphi_E = E \varphi_E$ on U_α^+ and by the unique continuation principle this implies

$$\varphi_E = 0 \text{ on } U_\alpha^+.$$

In particular, if $E = E_0$ we have $\xi_0 v_0 = 0$ on U_α^+ which implies $\xi_0 = 0$ as v_0 is not identically zero. If $E = E_j$ with $j \neq 0$, this implies $\xi_k = 0$ for all k such that $E_k = E_j$ as the eigenfunctions $(v_1, \dots, v_{\mathcal{N}_\alpha^N})$ are orthogonal. At the end, we obtain $\xi_j = 0$ for any $j = 0, \dots, \mathcal{N}_\alpha^N$, which is a contradiction with $\xi \neq 0$. \square

Let $\varphi \in C^\infty(\mathbb{R}_+)$ be a smooth cut-off function satisfying $0 \leq \varphi \leq 1$, $\varphi(t) = 1$ if $t \leq 1$ and $\varphi(t) = 0$ if $t > 2$. For $\epsilon > 0$, define

$$\chi_\epsilon(x) := \varphi(|x|^\epsilon). \quad (\text{VII.2.6})$$

Then χ_ϵ satisfies:

- $\chi_\epsilon(x) = 1$ if $|x| \leq 1$, and $\chi_\epsilon(x) = 0$ if $|x| > 2^{\frac{1}{\epsilon}}$,
- $\text{supp } \nabla \chi_\epsilon \subset A_{1, 2^{\frac{1}{\epsilon}}} := B(0, 2^{\frac{1}{\epsilon}}) \setminus \overline{B(0, 1)}$,
- $|\nabla \chi_\epsilon(x)|^2 = \frac{\epsilon^2}{|x|^{2-2\epsilon}} |\varphi'(|x|^\epsilon)|^2$,
- $\Delta \chi_\epsilon(x) = \frac{\epsilon^2}{|x|^{2-\epsilon}} \varphi'(|x|^\epsilon) + \frac{\epsilon^2}{|x|^{2-2\epsilon}} \varphi''(|x|^\epsilon)$, and $|\Delta \chi_\epsilon|^2 \leq C \frac{\epsilon^4}{|x|^{4-4\epsilon}}$, $C > 0$.

As $v_0 \in L^\infty(U_\alpha^+) \cap H_{loc}^2(U_\alpha^+)$ we have $v_0 \chi_\epsilon \in L^2(U_\alpha^+)$ and

Lemma VII.2.6. *The functions $v_0 \chi_\epsilon, v_1, \dots, v_{\mathcal{N}_\alpha^N}$ are linearly independent in $L^2(U_\alpha^+)$ for any $\epsilon > 0$.*

Proof. One has $v_0 \chi_\epsilon = v_0$ on $U_\alpha^+ \cap B(0, 1)$ for any $\epsilon > 0$, and the result follows from Lemma VII.2.5. \square

Step 2. We introduce the sector

$$V_\alpha^+ := \{(x_1, x_2) \in (\mathbb{R}_+)^2 : x_1 \leq x_2 \tan \alpha\},$$

the following components of its boundary

$$\Sigma_R := \{x = (0, x_2), x_2 \in \mathbb{R}_+\}, \quad \Sigma_N := \{x = (x_2 \tan \alpha, x_2), x_2 \in \mathbb{R}_+\},$$

and the sesquilinear form

$$q_\alpha^N(u, u) = \int_{V_\alpha^+} |\nabla u|^2 dx - \int_{\Sigma_R} |u|^2 ds, \quad u \in H^1(V_\alpha^+).$$

Remark that there exists a rotation \mathcal{R}_α such that $U_\alpha^+ = \mathcal{R}_\alpha(V_\alpha^+)$, and the operators T_α^N and Q_α^N are thus unitary equivalent. Let $\beta \in (0, \alpha)$ and consider the unitary map

$$\begin{aligned} \mathcal{U} : L^2(V_\alpha^+) &\ni v \mapsto u \in L^2(V_\beta^+), \\ u &:= \sqrt{\frac{\tan \beta}{\tan \alpha}} v(x_1, x_2 \frac{\tan \beta}{\tan \alpha}). \end{aligned}$$

For any $v \in L^2(V_\alpha^+)$ one has

$$q_\beta^N(\mathcal{U}v, \mathcal{U}v) = q_\alpha^N(v, v) - \kappa \int_{V_\alpha^+} |\partial_{x_2} u|^2 dx, \quad (\text{VII.2.7})$$

with $\kappa := 1 - \left(\frac{\tan \beta}{\tan \alpha}\right)^2 > 0$. Define

$$u_0 := \chi_\epsilon(v_0 \circ \mathcal{R}_\alpha), \quad u_j := v_j \circ \mathcal{R}_\alpha, \quad j = 1, \dots, \mathcal{N}_\alpha^N,$$

and $w_j := \mathcal{U}u_j$, $j = 0, 1, \dots, \mathcal{N}_\alpha^N$. As \mathcal{U} is an isomorphism, the functions $w_0, w_1, \dots, w_{\mathcal{N}_\alpha^N}$ are linearly independent by Lemma VII.2.6. Denote by $W := \text{span}\{w_0, w_1, \dots, w_{\mathcal{N}_\alpha^N}\}$. The operators T_β^N and Q_β^N are unitary equivalent, hence to show (VII.2.3) we have to prove

$$\sup_{\substack{v \in W \\ v \neq 0}} \frac{q_\beta^N(v, v)}{\|v\|_{L^2(V_\beta^+)}^2} < -1,$$

and due to (VII.2.7) it is in fact sufficient to prove the following estimate:

$$\begin{aligned} \sup_{\substack{\xi \in \mathbb{R}^{\mathcal{N}_\alpha^N+1} \\ |\xi|=1}} \left(q_\alpha^N \left(\sum_{j=0}^{\mathcal{N}_\alpha^N} \xi_j u_j, \sum_{j=0}^{\mathcal{N}_\alpha^N} \xi_j u_j \right) + \left\| \sum_{j=0}^{\mathcal{N}_\alpha^N} \xi_j u_j \right\|_{L^2(V_\alpha^+)}^2 - \kappa \left\| \sum_{j=0}^{\mathcal{N}_\alpha^N} \xi_j \partial_{x_2} u_j \right\|_{L^2(V_\alpha^+)}^2 \right) \\ \equiv \sup_{\substack{\xi \in \mathbb{R}^{\mathcal{N}_\alpha^N+1} \\ |\xi|=1}} \langle \xi, (A - \kappa B) \xi \rangle < 0, \end{aligned} \quad (\text{VII.2.8})$$

where $A = (a_{i,j})$, $B = (b_{i,j})$,

$$a_{i,j} := \int_{V_\alpha^+} \nabla u_i \nabla u_j dx - \int_{\Sigma_R} u_i u_j ds + \int_{V_\alpha^+} u_i u_j dx \quad b_{i,j} := \int_{V_\alpha^+} \partial_{x_2} u_i \partial_{x_2} u_j dx, \quad i, j = 0, \dots, \mathcal{N}_\alpha^N.$$

Step 3. Let us prove (VII.2.8). For any $i, j = 1, \dots, \mathcal{N}_\alpha^N$ we have immediately

$$a_{i,j} = \delta_{i,j}(E_j + 1), \quad b_{i,j} = \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker symbol. In particular $a_{i,i} < 0$ for any $i \in \{1, \dots, \mathcal{N}_\alpha^N\}$. The coefficients involving u_0 need more attention. To estimate $a_{0,0}$ we decompose the first term

$$\int_{V_\alpha^+} |\nabla u_0|^2 dx = \int_{U_\alpha^+} |\nabla(v_0 \chi_\epsilon)|^2 dx = \int_{U_\alpha^+} (|\chi_\epsilon \nabla v_0|^2 + |v_0 \nabla \chi_\epsilon|^2 + 2v_0 \chi_\epsilon \nabla v_0 \nabla \chi_\epsilon) dx.$$

An integration by parts gives

$$\begin{aligned} 2 \int_{U_\alpha^+} v_0 \chi_\epsilon \nabla v_0 \nabla \chi_\epsilon dx &= \int_{U_\alpha^+} (v_0 \nabla v_0) \nabla (\chi_\epsilon^2) dx \\ &= - \int_{U_\alpha^+} (|\nabla v_0|^2 + \Delta v_0) |\chi_\epsilon|^2 dx + \int_{\partial U_\alpha^+ \cap \partial U_\alpha} |v_0 \chi_\epsilon|^2 dx, \end{aligned}$$

where we use $\partial_\nu v_0 = 0$ on $\partial U_\alpha^+ \setminus \partial U_\alpha$, and $\partial_\nu v_0 = v_0$ on $\partial U_\alpha^+ \cap U_\alpha$. As $-\Delta v_0 = -v_0$ we can conclude that

$$a_{0,0} = \int_{U_\alpha^+} |v_0 \nabla \chi_\epsilon|^2 dx.$$

By definition of χ_ϵ , see (VII.2.6), we have

$$\int_{U_\alpha^+} |\nabla \chi_\epsilon|^2 dx = \int_1^{2^{\frac{1}{\epsilon}}} \int_0^\alpha \frac{\epsilon^2}{r^{2-2\epsilon}} |\varphi'(r^\epsilon)|^2 r d\theta dr = \mathcal{O}(\epsilon),$$

and thus $a_{0,0} = \mathcal{O}(\epsilon)$ as $\epsilon \rightarrow 0$. To estimate $a_{0,j}$ with $j \neq 0$ we first make an integration by parts to get

$$\int_{V_\alpha^+} \nabla u_0 \nabla v_j dx = \int_{U_\alpha^+} \nabla(v_0 \chi_\epsilon) \nabla v_j dx = - \int_{U_\alpha^+} \Delta(v_0 \chi_\epsilon) v_j dx + \int_{\partial U_\alpha^+ \cap \partial U_\alpha} \chi_\epsilon v_0 v_j ds,$$

where we use the fact that $\partial_\nu \chi_\epsilon = 0$ on ∂U_α^+ as χ_ϵ is radial, $\partial_\nu v_0 = 0$ on $\partial U_\alpha^+ \setminus \partial U_\alpha$, and $\partial_\nu v_0 = v_0$ on $\partial U_\alpha^+ \cap U_\alpha$. Moreover,

$$- \int_{U_\alpha^+} \Delta(v_0 \chi_\epsilon) v_j dx = - \int_{U_\alpha^+} v_0 \chi_\epsilon v_j dx - \int_{U_\alpha^+} (v_0 \Delta \chi_\epsilon v_j + 2 \nabla v_0 \nabla \chi_\epsilon v_j) dx,$$

which gives

$$a_{0,j} = - \int_{U_\alpha^+} (v_0 \Delta \chi_\epsilon + 2 \nabla v_0 \nabla \chi_\epsilon) v_j dx.$$

Using again an integration by parts, we arrive at

$$a_{0,j} = \int_{U_\alpha^+} v_0 (v_j \Delta \chi_\epsilon + \nabla \chi_\epsilon \nabla v_j) dx.$$

Using the Cauchy-Schwarz inequality we can estimate

$$\left| \int_{U_\alpha^+} v_0 v_j \Delta \chi_\epsilon dx \right| \leq \|v_j\|_{L^2(U_\alpha^+)} \sqrt{\int_{U_\alpha^+} |v_0 \Delta \chi_\epsilon|^2 dx}.$$

By definition of χ_ϵ we have

$$\int_{U_\alpha^+} |\Delta \chi_\epsilon|^2 dx \leq C \epsilon^4 \int_1^{2^{\frac{1}{\epsilon}}} \int_0^\alpha \frac{1}{r^{4-4\epsilon}} r d\theta dr = \mathcal{O}(\epsilon^4).$$

In the same way,

$$\int_{U_\alpha^+} v_0 \nabla \chi_\epsilon \nabla v_j dx = \mathcal{O}(\epsilon),$$

and finally this gives $a_{0,j} = \mathcal{O}(\epsilon)$ as $\epsilon \rightarrow 0$. As the matrix A is symmetric, we just proved that $A = \text{diag}(0, E_1 + 1, \dots, E_{\mathcal{N}_\alpha^N + 1} + 1) + \mathcal{O}(\epsilon)$ which implies that there exists a constant $c > 0$ such that

$$\sup_{\substack{\xi \in \mathbb{R}^{\mathcal{N}_\alpha^N + 1} \\ |\xi| = 1}} \langle \xi, A\xi \rangle \leq c\epsilon, \text{ as } \epsilon \rightarrow 0. \quad (\text{VII.2.9})$$

To conclude the proof, let us show that there exists $b > 0$ such that

$$\inf_{\substack{\xi \in \mathbb{R}^{\mathcal{N}_\alpha^N + 1} \\ |\xi| = 1}} \langle \xi, B\xi \rangle \geq b. \quad (\text{VII.2.10})$$

First, notice that by definition of χ_ϵ , we have

$$\langle \xi, B\xi \rangle \geq \int_{V_\alpha^+ \cap B(0,1)} |\xi_0 \partial_{x_2}(v_0 \circ \mathcal{R}_\alpha) + \sum_{j=1}^{\mathcal{N}_\alpha^N} \xi_j \partial_{x_2} u_j|^2 dx := I(\xi),$$

and to prove (VII.2.10) it is then sufficient to prove that $\inf_{x \in \mathbb{R}^{\mathcal{N}_\alpha^N + 1}, |\xi| = 1} I(\xi) > 0$. Let us make the proof by contradiction and assume that

$$\inf_{\xi \in \mathbb{R}^{\mathcal{N}_\alpha^N + 1}, |\xi| = 1} I(\xi) = 0.$$

By compactness of the unit ball of $\mathbb{R}^{\mathcal{N}_\alpha^N + 1}$, there exists $\xi := (\xi_0, \dots, \xi_{\mathcal{N}_\alpha^N})$ satisfying $|\xi| = 1$ and $I(\xi) = 0$, which implies

$$\partial_{x_2} \psi(x_1, x_2) = 0 \text{ on } V_\alpha^+ \cap B(0, 1),$$

where $\psi(x_1, x_2) := \xi_0(v_0 \circ \mathcal{R}_\alpha) + \sum_{j=1}^{\mathcal{N}_\alpha^N} \xi_j u_j$. Thus, ψ depends only on the x_1 variable on $V_\alpha^+ \cap B(0, 1)$: there exists φ satisfying

$$\psi(x_1, x_2) = \varphi(x_1) \text{ on } V_\alpha^+ \cap B(0, 1).$$

Notice that by elliptic regularity, see e.g. [Dan09, Corollary 4.2], we have $u_j \in C(\overline{V_\alpha^+ \cap B(0, 1)})$, and due to the fact that $v_0 \in L^\infty(U_\alpha) \cap H_{loc}^2(U_\alpha)$ we also have $v_0 \circ \mathcal{R}_\alpha \in C(\overline{V_\alpha^+ \cap B(0, 1)})$. Thus $\varphi \in C([0, \sin \alpha])$. Moreover, as $v_0 \circ \mathcal{R}_\alpha$ and the eigenfunctions u_j satisfy the Neumann boundary condition on Σ_N we have

$$\varphi'(x_1) = 0, \quad x_1 \in [0, \sin \alpha],$$

which implies that φ is a constant. Moreover, $v_0 \circ \mathcal{R}_\alpha$ and the eigenfunctions u_j also satisfy the Robin boundary condition on Σ_R , which gives

$$-\varphi'(0) = \varphi(0),$$

and then $\varphi \equiv 0$ on $[0, \sin \alpha]$. Hence we have $\psi \equiv 0$ on $V_\alpha^+ \cap B(0, 1)$. By Lemma VII.2.5, we know that $v_0 \circ \mathcal{R}_\alpha, u_1, \dots, u_{\mathcal{N}_\alpha^N}$ are linearly independent on $V_\alpha^+ \cap B(0, 1)$ which implies $\xi = 0$. This is a contradiction with $|\xi| = 1$ and (VII.2.10) is proved. Finally, (VII.2.9) and (VII.2.10) show that (VII.2.8) holds for small ϵ and this concludes the proof of the proposition. \square

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Titre : Problèmes spectraux avec conditions de Robin sur des domaines à coins du plan

Mots clés : Laplacien, condition de bord de Robin, analyse asymptotique, géométrie spectrale, coins

Résumé : Dans cette thèse, nous étudions les propriétés spectrales du Laplacien avec la condition de bord de Robin attractive, $\partial_\nu u = \gamma u$, où ν est la normale unitaire sortante au domaine et $\gamma > 0$, défini sur des domaines du plan à coins. Notre but est de comprendre l'influence des coins convexes sur l'asymptotique des valeurs propres de cet opérateur lorsque $\gamma \rightarrow +\infty$. Nous montrons en particulier que l'asymptotique des premières valeurs propres de Robin sur des polygones curvilignes est déterminée par des opérateurs modèles : les Laplaciens agissant sur les secteurs tangents au domaine. Pour une certaine classe de polygones droits, nous montrons l'existence d'un opérateur effectif sur le bord du domaine qui détermine l'asymptotique des valeurs propres suivantes. Enfin, des asymptotiques de Weyl pour différents seuils dépendant de γ sont obtenues.

Title: Spectral problems with Robin boundary conditions on planar domains with corners

Keys words: Laplacian, Robin boundary condition, asymptotic analysis, spectral geometry, corners

Abstract: In this thesis, we are interested in the spectral properties of the Laplacian with the attractive Robin boundary condition, $\partial_\nu u = \gamma u$, where ν is the unit outward normal of the boundary and $\gamma > 0$, on planar domains with corners. The aim is to understand the influence of the convex corners on the spectral properties of this operator when $\gamma \rightarrow +\infty$. In particular, we show that the asymptotics of the first Robin eigenvalues on curvilinear polygons is determined by model operators: the Robin Laplacians acting on infinite sectors. For a particular class of polygons with straight edges, we prove the existence of an effective operator acting on the boundary of the domain and determining the asymptotics of the further eigenvalues. Finally, some Weyl-type asymptotics for different thresholds depending on γ are obtained.