HAL
open science

Control issues for some fluid-solid models

Jozsef Kolumban

To cite this version:

Jozsef Kolumban. Control issues for some fluid-solid models. Fluid mechanics [physics.class-ph]. Université Paris sciences et lettres, 2018. English. NNT : 2018PSLED012 . tel-01885627

HAL Id: tel-01885627

https://theses.hal.science/tel-01885627v1

Submitted on 2 Oct 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THÈSE DE DOCTORAT

de l'Université de recherche Paris Sciences et Lettres
PSL Research University

Préparée à l'Université Paris Dauphine

Control issues for some fluid-solid models

École doctorale n°543

ÉCOLE DOCTORALE DE DAUPHINE

Spécialité SCIENCES

COMPOSITION DU JURY :

Mme Céline GRANDMONT
Inria Paris, Présidente

M Enrique FERNÁNDEZ-CARA
Universidad de Sevilla, Rapporteur

M David GÉRARD-VARET
Université Paris Diderot, Rapporteur

M Olivier GLASS
Université Paris Dauphine, Directeur

M Franck SUEUR
Université de Bordeaux, Co-directeur

Mme Muriel BOULAKIA
Université Pierre et Marie Curie,
Membre du jury

M Pierre LISSY
Université Paris Dauphine,
Membre du jury

M Ping ZHANG
The Chinese Academy of Sciences,
Membre du jury

Soutenue par **József J. KOLUMBÁN**
le 28 Septembre 2018

Dirigée par **Olivier GLASS**
Franck SUEUR

DAUPHINE UNIVERSITÉ PARIS PSL RESEARCH UNIVERSITY PARIS

Acknowledgements

First and foremost I would like to sincerely thank Olivier Glass and Franck Sueur for taking me on as their PhD student and for guiding me during these last three years. I am grateful to them for giving me such interesting and challenging problems to work on, as well as for providing me with invaluable expertise, always being patiently available to help me out with any questions or uncertainties I had during my PhD, no matter how big or small.

I would especially like to thank Olivier for introducing me to the field of Fluid Mechanics when I was just a second year Master's student looking for a dissertation advisor. Although it is never an easy decision for an aspiring young mathematician to choose just one research area to work in for the next foreseeable n years, I was convinced by the manner in which Olivier had presented a couple of relevant problems the first time I walked into his office, a good three and a half years ago. Looking back, I have never regretted the decision to work in this beautiful field (where Mathematics meets Physics in a ratio that is just suitable to my tastes), nor the decision to work with Olivier, whose wisdom and depth of knowledge have definitely given me courage and inspired me to tackle deeper and deeper problems with more and more self-confidence.

I am equally grateful to Franck for the amount of time and effort invested in my PhD. They say that having two advisors is harder because it's twice the work, but in my case it never really felt like a burden, rather a gift or an opportunity, since they complemented each other really seamlessly, Olivier's regular reassuring discussions with Franck's occasional more "fast and furious" approach. I recall with fondness the intense working sessions we put in together when either he was in Paris or I was in Bordeaux, when I really learned that in this line of work you have to be ready to change, throw out or improvise new ideas in a heartbeat, even if that meant letting older ideas (that you might have worked on for weeks if not months) die. I do feel that our brainstorming episodes spurred me on and brought the most out of me creatively, and that the attitude I learned from him will also help me in my future work.

Overall I am proud to have been the doctoral student of two such outstanding Mathematicians, and if during these three years I have indeed successfully managed to take in just a small percentage of their knowledge and skills (and to make a convex combination out of it), then I can consider myself in an amazing position for making it on my own in the Academic World. I hope to continue keeping in touch and working with both of them after the completion of my PhD. Merci pour tout !

I also want to warmly thank Enrique Fernández-Cara and David Gérard-Varet for accepting to review this thesis, as well as for their appreciation of my work and their useful suggestions. I would equally like to thank Muriel Boulakia, Céline Grandmont, Pierre Lissy and Ping Zhang for accepting to be part of the jury.

I am extremely grateful to the Fondation Sciences Mathématiques de Paris, who have financed

both my Master's Scholarship and my PhD thesis, allowing me to not only study at such an elite establishment as Paris Dauphine University, but also to have had the opportunity to work at a prestigious institution such as the CEREMADE Research Laboratory, where many great mathematicians have conducted research in the past, and (I'm sure) will in the future. These past five years of my life have definitely been an experience which I will always recall with the utmost pride. I would like to once more thank the FSMP for this great opportunity, and especially all the help by the secretaries regarding settling in Paris the first time I arrived here five years ago, knowing next to no French whatsoever. I would also like to thank Filippo Santambrogio from Université d'Orsay for all his help and advice when I was just a Bachelor's student in Cluj-Napoca, as a consequence of which I was able to find and apply for the FSMP Scholarship, and by such to come to Paris in the first place.

I wish to take this opportunity to thank everyone from the CEREMADE for their friendship, kindness and openness.

In particular I would like to start by mentioning a few of my former professors from my Master's degree who have helped me and inspired me to stay on this career path, both through their advice and help regarding my Master's degree and my PhD application, as well as their interesting classes : Julien Salomon, Pierre Cardaliaguet, Jacques Féjoz, Eric Séré and Ivar Ekeland (I am especially grateful to Professor Ekeland for writing a letter of recommendation for my PhD application, as well as his kind words of encouragement). I would also like to mention that even though I have decided to go down a different career path, I have very much enjoyed the classes of Guillaume Carlier (on Optimal Control), as well as José Trashorras and Halim Doss (on Stochastic Differential Equations).

Moving forward chronologically, I have really enjoyed my teaching duties during my PhD, and I would like to thank Yannick Viossat, Katia Meziani, François Huveneers, Anne-Marie Boussion, Nejla Nouaili, Amic Frouvelle and Emeric Bouin for their collaboration and guidance. I also want to thank all my fellow PhD students/post-docs for the pleasant company, as well as for the various research seminars, summer schools and nights out held for us young researchers (in the last category I also thank Boris Haspot and Pierre Lissy for their company). Last but not least, I want to thank Vincent Rivoirard, the director of the CEREMADE, for his generosity and promptness regarding financing requests for various conferences, as well as the secretaries Isabelle Bellier, Marie Belle and César Faivre for their efficiency and hard work regarding administrative issues.

I also want to take the opportunity to thank everyone from the Institut de Mathématiques de Bordeaux (IMB) for being extremely kind and welcoming every time I visited Franck for research purposes. Special shout out to Marco, Edoardo and the other PhD students at Bordeaux for the camaraderie provided.

I am also thankful to my advisors for taking me to various conferences and summer schools to meet other people working in Fluid Mechanics and Control Theory, since ultimately Mathematics (like all proper Sciences) is about collaborating, nobody can take on the whole world by themselves. I am grateful to have met some of my advisors' collaborators and colleagues, and to have had the opportunity to listen to and learn from a lot of worldwide experts in the respective fields. Since not everything is about work, this was complemented by the odd football, dining or sight-seeing session, which I have thoroughly enjoyed.

In particular I wish to thank Marius Tucsnak from the IMB and Alexandre Munnier from Université de Lorraine for accepting to write letters of recommendation during my search for a post-doc position, as well as for their helpful and insightful discussions regarding my research.

Of course I would have never arrived where I am today without certain people from my home town Cluj-Napoca who have put me on and helped me progress on this path. I am especially grateful to Jutka Szilágyi, my high school Math teacher, for inspiring my early love of Mathematics, as well as Szilárd András, my mentor during my Bachelor's degree, for introducing me to Mathematical Research, for setting further examples in terms of work ethics, as well as for his friendship.

I am thankful for the support of my family, especially my Dad, also for his patience and unconditional love during this journey of coming into my own, not only as a Mathematician, but also as an adult human being.

Last but not least, I must thank all my friends for keeping my life balanced during these years of working on my PhD. Special shout out goes to Walid from Paris ; Paul (and his lovely family), Kevin, Stephanie and Scott from Manchester ; as well as Laci, Boti, Rigó, Bence, Zoli, Zsolt and Tudor from Cluj/Transylvania.

Thank you all !

Table des matières

1 Introduction Générale **9**

1.1 Théorie du contrôle 9

1.2 Problèmes de contrôle en mécanique des fluides 10

1.2.1 Le cas non visqueux 10

1.2.2 Le cas visqueux 12

1.3 Modèles d'interaction fluide-solide 14

1.3.1 La dynamique d'un corps rigide dans un fluide non visqueux, irrotationnel et incompressible 14

1.3.2 La dynamique d'un corps rigide dans un fluide visqueux incompressible 16

1.3.3 Quelques autres modèles fluide-solide, résultats connectés et autres références . . 17

1.4 Principaux résultats de la thèse 18

1.4.1 Contrôle du mouvement d'un corps rigide immergé dans un fluide bidimensionnel parfait irrotationnel 18

1.4.2 Contrôle du mouvement d'un corps rigide immergé dans un fluide bidimensionnel visqueux incompressible 19

1.5 Méthodologie 21

1.5.1 La construction dans le cas non visqueux : géodésiques et contrôle impulsif 21

1.5.2 La réduction du cas visqueux à l'état non visqueux par une méthode de viscosité évanescente 25

1.6 Directions futures et problèmes ouverts 29

2 General Introduction **33**

2.1 Control Theory 33

2.2 Control problems in fluid mechanics 34

2.2.1 The inviscid case 34

2.2.2 The viscous case 36

2.3 Fluid-solid interaction models 38

2.3.1 The dynamics of a rigid body in an irrotational inviscid incompressible fluid . . . 38

2.3.2 The dynamics of a rigid body in a viscous incompressible fluid 39

2.3.3 Some other fluid-solid models, related results and further references 40

2.4 Main results of the PhD Thesis 41

2.4.1 Control of the motion of a rigid body immersed in an irrotational perfect two-dimensional fluid 41

2.4.2 Control of the motion of a rigid body immersed in a viscous incompressible two-dimensional fluid . . . 43
2.5 Methodology . . . 44
2.5.1 The construction in the inviscid case : geodesics and impulsive control . . . 44
2.5.2 The reduction of the viscous case to the inviscid one via a vanishing viscosity method . . . 48
2.6 Future directions and open problems . . . 52

3 External boundary control of the motion of a rigid body immersed in a perfect two-dimensional fluid 55
3.1 Introduction and main result . . . 55
3.1.1 The model without control . . . 55
3.1.2 The control problem . . . 57
3.2 Reformulation of the solid's equation into an ODE . . . 60
3.2.1 A reminder of the uncontrolled case . . . 61
3.2.2 Extension to the controlled case . . . 64
3.3 Reduction to the case where the displacement, the velocities and the circulation are small 66
3.4 Reduction to an approximate controllability result . . . 67
3.5 Proof of the approximate controllability result . . . 68
3.5.1 First step . . . 68
3.5.2 Illustration of the method on a toy model . . . 69
3.5.3 Back to the complete model . . . 70
3.5.4 About Remark 3.1.1 . . . 72
3.6 Closeness of the controlled system to the geodesic . . . 72
3.6.1 Proof of Proposition 3.5.2 . . . 72
3.6.2 Proof of Proposition 3.6.1 . . . 73
3.6.3 Proof of Proposition 3.6.2 . . . 74
3.6.4 Proof of Proposition 3.6.3 . . . 76
3.7 Design of the control according to the solid position . . . 76
3.7.1 The case of a homogeneous disk . . . 76
3.7.2 The case when $\mathcal{S}_0$ is not a disk . . . 79

4 Control at a distance of the motion of a rigid body immersed in a two-dimensional viscous incompressible fluid 81
4.1 Introduction . . . 81
4.1.1 The mathematical model . . . 81
4.1.2 Definition of weak solutions . . . 82
4.1.3 Main result . . . 84
4.2 Preliminary reductions . . . 86
4.2.1 A reduction of Theorem 4.1.1 to the case of a fixed domain and small viscosity . 87
4.2.2 Proving Theorem 4.2.1 by the means of an asymptotic expansion for the solid trajectory . . . 89
4.2.3 Proving Theorem 4.2.2 - Constructing the control via the asymptotic expansion for the fluid velocity . . . 91

4.2.4 Regarding Remark 4.2.1 . . . 93
4.3 The inviscid term u^0 . . . 93
4.3.1 Reformulation of the solid's equation into an ODE . . . 95
4.3.2 Proof of the exact controllability result Theorem 4.3.1 . . . 98
4.4 The boundary layer profiles . . . 104
4.4.1 The physical boundary layer profile v . . . 104
4.4.2 The second boundary corrector w . . . 106
4.4.3 The inner domain corrector θ^ε . . . 106
4.5 The first order term u^1 . . . 107
4.5.1 Vorticity . . . 108
4.5.2 An ODE reformulation in the linearized case . . . 110
4.5.3 An impulsive control strategy to control the final velocity $p^1(T)$ - Proof of Theorem 4.5.1 . . . 113
4.6 Estimating the remainder . . . 115
4.6.1 The equation of the remainder and weak solutions . . . 115
4.6.2 Existence and continuity . . . 117
4.7 Conclusion . . . 123
Annexe 4.A Design of the controls . . . 125
4.A.1 Proof of Proposition 4.3.3 . . . 125
4.A.2 Proof of Proposition 4.5.3 . . . 129

Chapitre 1

Introduction Générale

1.1 Théorie du contrôle

Le but de la théorie du contrôle est d'étudier les systèmes dynamiques en fonction de certains paramètres de contrôle. De tels systèmes apparaissent dans les domaines de l'ingénierie, de la physique, de la biologie, de la chimie, etc. Plus précisément, on peut généralement définir un système de contrôle comme une équation d'évolution de la forme suivante :

$$\frac{d}{dt}x(t) = f(t, x(t), g(t)), \text{ pour } t \in [0, T], \tag{1.1}$$

où $T > 0$ est l'horizon temporel, $x : [0, T] \to \mathcal{X}$ est l'état du système, et $g : [0, T] \to \mathcal{G}$ représente le paramètre de contrôle (ou plutôt la fonction de contrôle) avec lequel nous aimerions agir sur l'évolution du système. Deux exemples mathématiques standard de telles équations d'évolution sont les systèmes de contrôle d'EDO (lorsque l'espace d'états $\mathcal{X}$ et l'espace de contrôle $\mathcal{G}$ sont de dimension finie) et les systèmes de contrôle d'EDP (quand l'état et le contrôle appartient à certains espaces fonctionnels de dimension infinie). De plus, en fonction des propriétés de la fonction f par rapport à x et g, on peut encore catégoriser les systèmes de contrôle linéaires et non linéaires.

La question générale de la théorie du contrôle est de savoir comment utiliser la fonction de contrôle g afin d'obtenir le comportement désiré du système (1.1). Voici une liste (non exhaustive) de ces problèmes.

1. Contrôlabilité exacte. Le système de contrôle peut-il être conduit d'un état initial donné à un état final donné à un moment donné ? Plus précisément, étant donné $T > 0$, $x_0, x_1 \in \mathcal{X}$, existe-t-il un contrôle $g : [0, T] \to \mathcal{G}$ tel que la solution $x : [0, T] \to \mathcal{X}$ du système (1.1) avec le contrôle g et les données initiales $x(0) = x_0$ vérifie $x(T) = x_1$?

2. Contrôlabilité approchée. Une question similaire à celle ci-dessus, mais permettant une erreur (arbitrairement petite) pour l'état final. Autrement dit, étant donné $T > 0$, $x_0, x_1 \in \mathcal{X}$ et $\varepsilon > 0$, existe-t-il un contrôle $g : [0, T] \to \mathcal{G}$ tel que la solution $x : [0, T] \to \mathcal{X}$ du système (1.1) avec le contrôle g et les données initiales $x(0) = x_0$ vérifie $\|x(T) - x_1\| \leq \varepsilon$?

3. Contrôlabilité à zéro. Peut-on mettre l'état du système au repos ? Mathématiquement cela peut être décrit de la même manière que le problème de contrôlabilité exacte ci-dessus, mais au lieu d'essayer d'atteindre un état final arbitraire x_1, nous voulons seulement avoir un état final nul, c'est-à-dire $x(T) = 0$.

4. Contrôlabilité aux trajectoires. Soit $\bar{x} : [0, T] \to \mathcal{X}$ une trajectoire donnée de (1.1) associée à un contrôle $\bar{g} : [0, T] \to \mathcal{G}$. Étant donné $x_0 \in \mathcal{X}$, existe-t-il $g : [0, T] \to \mathcal{G}$ tel que la solution associée

$x : [0,T] \to \mathcal{X}$ de (1.1) avec le contrôle g satisfait $x(T) = \bar{x}(T)$?

5. Stabilisation asymptotique. Pour simplifier supposons que nous sommes dans le cas autonome (c'est-à-dire $f = f(x,g)$ ne dépend pas de t), et que $(x_e, g_e) \in \mathcal{X} \times \mathcal{G}$ est un point d'équilibre du systeme, c'est à dire $f(x_e, g_e) = 0$. Imaginons que nous voulions contrôler le système à cet état d'équilibre, c'est-à-dire trouver un contrôle g et une solution associée x de (1.1) tel que $(x(T), g(T)) = (x_e, g_e)$. Le problème est qu'un tel contrôle serait sensible aux perturbations dans le modèle, telles que celles provenant du bruit, des imprécisions dans le modèle ou dans l'implémentation du contrôle. Par conséquent, si l'équilibre est instable, le système ne se comportera pas comme nous l'aurions voulu avec le contrôle g. Pour résoudre ce problème, nous recherchons un contrôle en boucle fermée, c'est-à-dire un contrôle de la forme $g = g[x]$, plus robuste aux perturbations.

La question de la stabilisabilité asymptotique devient la suivante : peut-on trouver un contrôle de feedback $g : \mathcal{X} \to \mathcal{G}$ tel que $g_e = g[x_e]$ et que la solution du système

$$\frac{d}{dt}x(t) = f(x(t), g[x(t)]), \text{ pour } t \in [0,T], \tag{1.2}$$

soit asymptotiquement stable à x_e ? Nous rappelons que la notion de stabilité asymptotique est la suivante :

(i) pour tout $\varepsilon > 0$ il existe $\delta > 0$ tel que, pour tout $x_0 \in \overline{B}(x_e, \delta)$ et toute solution de (1.2) avec $x(0) = x_0$, nous avons $x(t) \in \overline{B}(x_e, \varepsilon)$ pour tout $t \geq 0$;

(ii) pour tout $x_0 \in \mathcal{X}$ et toute solution maximale de (1.2) avec $x(0) = x_0$ nous avons que x est une solution globale et $\|x(t) - x_e\| \to 0$ quand $t \to +\infty$.

Notons que les questions ci-dessus ont été formulées dans un contexte global. Les versions locales peuvent être considérées en exigeant seulement que les données initiales et finales soient dans certaines boules fixes de petit rayon (au lieu d'être arbitraires) dans le cas de la question de contrôlabilité ; respectivement en demandant seulement la stabilité asymptotique locale à x_e dans le cas du problème de stabilisation en boucle fermée.

Nous différencions également les problèmes ci-dessus à ceux du contrôle optimal, où l'on cherche un contrôle qui satisfait également un critère d'optimalité (comme la minimisation d'un coût fonctionnel), plutôt que de simplement conduire le système à un état donné.

On peut trouver une description plus détaillée de ces problèmes, ainsi que l'état de l'art et les problèmes ouverts dans les livres de J.-M. Coron [21], J.-L. Lions [76], respectivement M. Tucsnak et G. Weiss [96], ainsi que dans le papier de D. L. Russell [89].

1.2 Problèmes de contrôle en mécanique des fluides

1.2.1 Le cas non visqueux

Présentons quelques exemples concrets de problèmes de contrôle EDP en mécanique des fluides. Nous commençons par un résultat classique concernant la contrôlabilité exacte de l'équation d'Euler en deux dimensions.

Soit $\Omega \subset \mathbb{R}^2$ un domaine lisse, borné, connexe (mais pas nécessairement simplement connexe), et considérons l'équation d'Euler incompressible dans Ω :

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi = 0 \quad \text{et} \quad \text{div } u = 0 \quad \text{pour} \quad t \in [0,T] \text{ et } x \in \Omega, \tag{1.3}$$

où $u : [0,T] \times \overline{\Omega} \to \mathbb{R}^2$ est le champ de vitesse du fluide et $\pi : [0,T] \times \overline{\Omega} \to \mathbb{R}$ le champ de pression. Ce système décrit l'évolution d'un fluide homogène, non visqueux soumis uniquement à la conservation de la masse et à l'incompressibilité. Puisque le système (1.3) est une EDP dans Ω, nous devons ajouter des conditions aux limites sur $\partial\Omega$ pour qu'il soit bien posé. Dans le cas sans contrôle, la condition d'imperméabilité $u \cdot n = 0$ sur $[0,T] \times \partial\Omega$ est souvent considérée comme une condition aux limites classique associée à (1.3), ce qui signifie que le fluide ne peut pas traverser la frontière $\partial\Omega$.

Cependant, puisque nous sommes intéressés par le contrôle du système (1.3), nous allons laisser du fluide entrer et sortir dans le domaine, ce qui sera notre contrôle. Plus précisément, une formulation mathématique de tels contrôles est due à Yudovich [98]. On considère $\Sigma \subset \partial\Omega$ comme une partie non vide et ouverte de la frontière, et on prescrit d'une part la vitesse normale sur Σ, c'est-à-dire

$$u(t,x) \cdot n(x) = g_n(t,x) \text{ sur } [0,T] \times \Sigma, \tag{1.4}$$

où $g_n \in C_0^\infty([0,T] \times \Sigma)$ avec $\int_\Sigma g_n = 0$, tandis que sur le reste de la frontière, nous avons la condition d'imperméabilité habituelle

$$u \cdot n = 0 \text{ sur } [0,T] \times (\partial\Omega \setminus \Sigma), \tag{1.5}$$

et d'autre part le tourbillon sur l'ensemble Σ^- des points de $[0,T] \times \Sigma$ où le champ de vitesse pointe à l'intérieur de Ω, c'est-à-dire

$$\operatorname{curl} u(t,x) = g_v(t,x) \text{ sur } \Sigma^-, \tag{1.6}$$

où $g_v \in C_0^\infty([0,T] \times \Sigma)$. Notons que $\Sigma^- = \{(t,x) \in [0,T] \times \Sigma : \ u \cdot n < 0\}$ est déduit immédiatement de g_n. Le contrôle associé au système (1.3) devient alors $g = (g_n, g_v)$.

Le fait que le système (1.3), (1.4), (1.5), (1.6) soit bien posé a été prouvé par Yudovich dans [98] pour des données initiales et contrôles suffisamment lisses, avec certaines hypothèses appropriées sur Σ.

Nous avons le résultat de contrôlabilité exacte global suivant dû à Coron [22].

Theorem 1.2.1. *Étant donné $T > 0$, $u_0, u_1 \in C^\infty(\overline{\Omega})$ telle que*

$$\operatorname{div} u_0 = \operatorname{div} u_1 = 0 \text{ dans } \Omega, \quad u_0 \cdot n = u_1 \cdot n = 0 \text{ sur } \partial\Omega,$$

le système (1.3), (1.4), (1.5), (1.6) *est contrôlable de manière exacte si et seulement si Σ rencontre chaque composant connexe de la frontière $\partial\Omega$.*

Par contrôlabilité exacte, nous entendons ici qu'il existe un contrôle au bord approprié g dans le sens de Yudovich tel que la solution lisse u de (1.3), (1.4), (1.5), (1.6) à partir de $u(0) = u_0$ satisfait $u(T) = u_1$. Mentionnons que dans le cas du contrôle d'EDP, il n'est pas rare de rechercher implicitement de tels contrôles. Autrement dit, nous recherchons une solution u de (1.3) satisfaisant seulement (1.5) et la propriété de contrôlabilité que nous souhaitons établir (par exemple une contrôlabilité exacte, c'est-à-dire $u(T) = u_1$ où u_1 est donné), qui est un système sous-déterminé, puis en déduisant implicitement la valeur du contrôle g à partir d'une telle solution u en utilisant (1.4), (1.6).

La preuve repose sur la "méthode de retour" de Coron pour les problèmes de contrôle non-linéaire, qui consiste à trouver une solution particulière $\bar{u}$ à l'équation satisfaisant $\bar{u}(0) = \bar{u}(T) = 0$ tel que le système linéarisé autour de $\bar{u}$ soit contrôlable, puis prouver l'existence d'une solution au problème de contrôle original qui est proche de $\bar{u}$.

Notons que la condition que Σ rencontre chaque composant connexe de la frontière $\partial\Omega$ est essentielle pour deux raisons principales. La première raison est que, grâce à la loi de Kelvin, nous savons que la

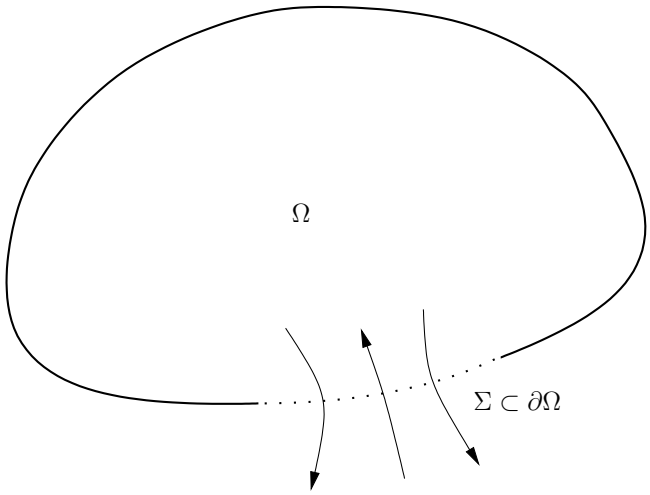

FIGURE 1.1: Le cadre du problème de contrôle

circulation de la vitesse du fluide est constante autour de toute courbe de Jordan suivant le flot. Par conséquent, s'il y avait une composante connexe de $\partial\Omega$ non touchée par Σ, la circulation autour de cette composante serait conservée quel que soit le choix du contrôle. Par conséquent, on ne pourrait pas contrôler la vitesse du fluide de u_0 à u_1 avec différentes circulations. La deuxième raison pour laquelle la vitesse du fluide n'est pas contrôlable si Σ ne touche pas tous les composants connexes de $\partial\Omega$ est que le tourbillon $\operatorname{curl} u$ est transporté le long du flot de u, de sorte que toute vorticité autour d'une composante non-contrôlée de $\partial\Omega$ serait également conservée.

Un résultat similaire à Theorem 1.2.1 a été établi par Glass dans [49] en ce qui concerne la contrôlabilité exacte de l'équation d'Euler en trois dimensions. Nous mentionnons aussi les résultats de [24, 25, 48] concernant la stabilisation asymptotique de l'équation d'Euler.

1.2.2 Le cas visqueux

Considérons maintenant le problème de contrôlabilité dans le cas où le fluide est visqueux, c'est-à-dire modélisé par les équations de Navier-Stokes incompressible. En raison de la viscosité ajoutée (que nous notons $\nu > 0$), notre modèle devient

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla\pi - \nu\Delta u = 0 \quad \text{et} \quad \operatorname{div} u = 0 \quad \text{pour} \quad t \in [0,T] \text{ et } x \in \Omega. \tag{1.7}$$

Pour l'EDP (1.7), qui est du second ordre en x, une condition aux limites naturelle est celle de Dirichlet ou condition de "non-glissement", c'est-à-dire $u = 0$ sur $[0,T] \times \partial\Omega$, qui a d'abord été considérée par Stokes dans [93]. Cette condition implique que les particules fluides adhèrent à la frontière et, par conséquent, engendrent des couches limites de grande amplitude (voir par exemple le travail de Prandtl [87]). Cependant, Serrin a souligné dans [92] que cette condition n'est pas toujours réaliste (par exemple lorsqu'il s'agit d'une pression modérée, comme dans le cas de l'aérodynamique à haute altitude).

D'autre part, Navier introduit dans [82] un autre type intéressant de conditions aux limites, les conditions dites de "glissement avec friction", lorsque le fluide peut glisser sur la limite, mais subit une certaine friction. Cela peut être décrit mathématiquement comme

$$u \cdot n = 0 \text{ et } (D(u)n)_{\tan} = -\mu(u)_{\tan} \text{ sur } [0,T] \times \partial\Omega, \tag{1.8}$$

où $\mu \geq 0$ est le coefficient de friction, et pour tout champ vectoriel f on définit le gradient symétrique, respectivement la partie tangentielle par

$$D(f) = \frac{1}{2}\left(\nabla f + (\nabla f)^T\right) \text{ et } (f)_{\tan} = f - (f \cdot n)n. \tag{1.9}$$

Malgré le fait que les conditions de "non-glissement" soient plus fréquentes dans la communauté mathématique, les conditions de Navier sont également bien justifiées et fréquemment considérées dans divers contextes physiques, comme pour étudier le comportement des écoulements près des murs rugueux (voir instance [4, 16, 73], ainsi que [75] pour des données expérimentales suggérant diverses possibilités pour que les phénomènes de glissement se produisent). De plus, il est plus facile d'établir des résultats de convergence de viscosité pour passer à l'équation d'Euler quand $\nu \to 0$ dans le cas de Navier par rapport au cas de Dirichlet, car les couches limites engendrées par la condition de glissement sont d'amplitude plus petite que celles engendrées par la condition d'adhérence (voir par exemple [70] et les références qui s'y trouvent), et l'équation des couches limites dans le cas de Dirichlet peut être mal posée (voir par exemple [31]). Notons également que la condition de Dirichlet peut être vue comme une limite des conditions de Navier lorque $\mu \to +\infty$, voir [74].

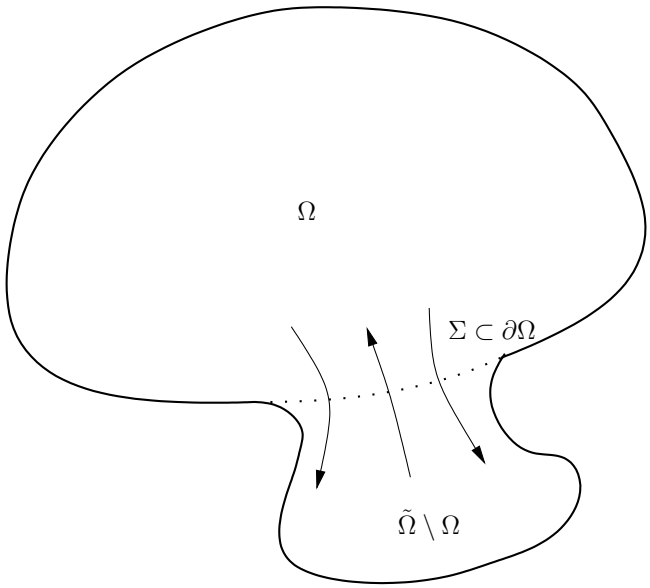

FIGURE 1.2: Le passage entre le contrôle aux frontières et le contrôle intérieur

Avant d'entrer dans les détails sur la contrôlabilité des équations de Navier-Stokes, citons une méthode courante dans le contrôle d'EDP pour changer la perspective de certains problèmes de contrôle. Comme nous l'avons mentionné précédemment pour l'équation d'Euler, on considère souvent que le contrôle prend la forme d'une condition aux limites non homogène sur une partie non vide et ouverte Σ de $\partial\Omega$. Clairement, nous pourrions définir des types de contrôles similaires pour le système (1.7) sous les conditions de Dirichlet, respectivement les conditions de Navier. Cependant, ce point de vue équivaut essentiellement à avoir un contrôle intérieur (c'est-à-dire un terme source de support compacte sur l'EDP jouant le rôle de contrôle), tout en maintenant des conditions aux limites homogènes. En effet, nous pouvons étendre le domaine Ω à un domaine borné $\tilde{\Omega}$ tel que $\Omega \subset \tilde{\Omega}$, $\Sigma \subset \tilde{\Omega}$ et $\partial\Omega \setminus \Sigma \subset \partial\tilde{\Omega}$. Maintenant, si nous considérons un terme source supporté dans $\tilde{\Omega} \setminus \Omega$, alors établir un résultat de contrôle sur $\tilde{\Omega}$ en utilisant ce contrôle intérieur nous donnera un résultat de contrôle aux frontières sur Ω, simplement en laissant circuler le fluide dans $\tilde{\Omega}$ (en particulier sur Σ) et en lisant la valeur appropriée de notre contrôle sur la frontière à partir de la vitesse du fluide passant par Σ.

Nous mentionnons quelques résultats de contrôlabilité concernant les équations de Navier-Stokes. Imanuvilov dans [71], respectivement Fernández-Cara, Guerrero, Imanuvilov et Puel dans [36, 37] ont obtenu des résultats de contrôlabilité locale aux trajectoires dans le cas de Dirichlet, alors qu'un résultat similaire a été établi par Guerrero dans [65] pour le cas des conditions de Navier. Ces résultats reposent sur la linéarisation de l'équation et l'utilisation des estimations de Carleman.

Cependant, au lieu de considérer le terme non linéaire comme une perturbation (comme dans les résultats mentionnés précédemment), une autre approche consiste à considérer le terme visqueux comme une perturbation du système non visqueux, en s'appuyant sur les résultats de contrôlabilité de l'équation d'Euler pour déduire la contrôlabilité du système de Navier-Stokes. Habituellement, de telles techniques donnent lieu à des résultats de contrôlabilité approchée globaux, qui peuvent ensuite être combinés avec les résultats locaux (mais exacts) mentionnés ci-dessus pour fournir une contrôlabilité exacte globale. Nous mentionnons les travaux de Coron [23] concernant la contrôlabilité approximative globale, ainsi que Coron et Fursikov [26], respectivement Chapouly [19], concernant les résultats de contrôlabilité nulle exacte globale dans le cas de conditions de Navier. Plus récemment, ces résultats ont été améliorés par Coron, Marbach et Sueur dans [27, 28] pour établir une contrôlabilité exacte globale à un petit temps, en utilisant une méthode de dissipation bien préparée pour gérer les couches limites. L'adaptation de ces méthodes au cas de Dirichlet pose toujours un problème ouvert difficile en raison des couches limites d'amplitude plus grandes qui sont créées. Cependant, Coron, Marbach, Sueur et Zhang ont récemment obtenu un résultat dans [29], où dans certains contextes géométriques particuliers un résultat similaire était possible avec des conditions de Dirichlet, mais avec un "terme de force fantôme" ajouté.

1.3 Modèles d'interaction fluide-solide

Dans cette Section, nous présentons quelques modèles fluide-solide, à la fois dans le cas non visqueux et visqueux.

1.3.1 La dynamique d'un corps rigide dans un fluide non visqueux, irrotationnel et incompressible

Commençons par un modèle bidimensionnel simple d'interaction fluide-solide. Soit $\Omega \subset \mathbb{R}^2$ un domaine régulier, ouvert, borné et simplement connexe (nous notons que la simple connexité est juste une hypothèse simplificatrice et non essentielle à l'analyse). Le domaine contient un système fluide-solide qui évolue par rapport au temps $t \in [0, T]$, plus précisément une partie ouverte $\mathcal{F}(t)$ contenant le fluide, et une partie fermée $\mathcal{S}(t) := \Omega \setminus \mathcal{F}(t)$ représentant le solide. Le fluide évolue selon l'équation d'Euler incompressible, tandis que le mouvement du corps est un mouvement rigide régi par la loi de Newton avec une force normale due à la pression du fluide agissant sur le corps.

Plus précisément, le mouvement du solide peut être complètement décrit par son centre de masse $h(t) \in \Omega$ et son angle de rotation $\vartheta(t) \in \mathbb{R}$ comme $\mathcal{S}(t) = h(t) + R(\vartheta(t))(\mathcal{S}_0 - h_0)$, où h_0 est le centre de masse au moment initial, et

$$R(\vartheta) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}.$$

En outre, la dynamique du modèle fluide-solide peut être écrite comme le système couplé EDO-EDP suivant.

Équations fluides :

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi = 0 \quad \text{et} \quad \text{div } u = 0 \text{ pour } t \in [0,T] \text{ et } x \in \mathcal{F}(t). \tag{1.10}$$

Équations solides :

$$mh''(t) = \int_{\partial \mathcal{S}(t)} \pi\, n\, d\sigma \quad \text{et} \quad \mathcal{J}\vartheta''(t) = \int_{\partial \mathcal{S}(t)} \pi\, (x - h(t))^{\perp} \cdot n\, d\sigma, \text{ pour } t \in [0,T]. \tag{1.11}$$

Conditions aux limites :

$$u \cdot n = 0 \text{ sur } \partial\Omega, \quad \text{et} \quad u \cdot n = u_S \cdot n \text{ sur } \partial\mathcal{S}(t), \text{ for } t \in [0,T], \tag{1.12}$$

où $u : [0,T] \times \overline{\Omega} \to \mathbb{R}^2$ désigne le champ de vitesse du fluide et $\pi : [0,T] \times \overline{\Omega} \to \mathbb{R}$ le champ de pression, en outre, $m > 0$ et $\mathcal{J} > 0$ désignent respectivement la masse et le moment d'inertie du solide, et la vitesse du solide est donnée par $u_S(t,x) = h'(t) + \vartheta'(t)(x - h(t))^{\perp}$, avec $(x_1, x_2)^{\perp} := (-x_2, x_1)$. Pour simplifier, on a supposé que le fluide est homogène de densité 1.

Le problème de Cauchy pour ce système avec des données initiales

$$u|_{t=0} = u_0 \text{ pour } x \in \mathcal{F}_0, \; h(0) = h_0, \; h'(0) = h'_0, \; \vartheta(0) = 0, \; \vartheta'(0) = \vartheta'_0, \tag{1.13}$$

est maintenant bien compris, voir par ex. [55, 61, 69, 84, 85], alors que le cas 3D a aussi été étudié dans [62, 88].

De plus, nous supposons que le fluide est irrotationnel au temps initial, c'est-à-dire $\operatorname{curl} u_0 = 0$, ce qui implique qu'il restera irrotationnel à tout moment, en conséquence du troisième théorème de Helmholtz, c'est-à-dire

$$\text{curl } u = 0 \text{ pour } x \in \mathcal{F}(t), \; \forall t \geq 0. \tag{1.14}$$

Enfin, en raison du théorème de Kelvin, la circulation autour du corps est conservée, c'est-à-dire

$$\int_{\partial \mathcal{S}(t)} u(t) \cdot \tau\, d\sigma = \int_{\partial \mathcal{S}_0} u_0 \cdot \tau\, d\sigma = \gamma \in \mathbb{R}, \; \forall t \geq 0, \tag{1.15}$$

où τ désigne le vecteur tangent dans le sens antihoraire.

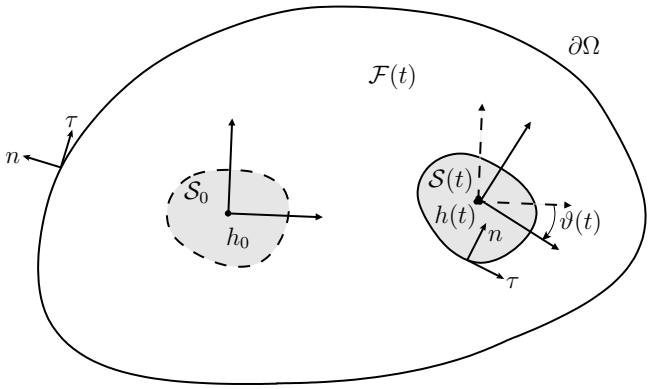

FIGURE 1.3: Les domaines Ω, $\mathcal{S}(t)$ et $\mathcal{F}(t) = \Omega \setminus \mathcal{S}(t)$

Il a été prouvé par Glass, Munnier et Sueur dans [57] que dans le cas irrotationnel, la dynamique du système (1.10)-(1.12) peut être reformulée comme une EDO pour les degrés de liberté du solide, à savoir $(h, \vartheta) \in \Omega \times \mathbb{R}$, que nous détaillerons dans la Section 1.5.1.

L'objectif principal de la première moitié de cette thèse est d'étudier la contrôlabilité du solide dans le système (1.10)-(1.12) par un contrôle sur la frontière $\partial\Omega$. Les résultats de cette analyse seront présentés dans la Section 1.4.1.

1.3.2 La dynamique d'un corps rigide dans un fluide visqueux incompressible

Présentons maintenant un problème d'interaction fluide-solide similaire à la Section 1.3.1, mais dans le cas où le fluide est supposé être visqueux incompressible (et pas nécessairement irrotationnel), avec les conditions aux limites de Navier. Nous maintenons les autres hypothèses faites au début de la Section 1.3.1. Le système peut alors être décrit comme suit.

Équations fluides :

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi - \nu \Delta u = 0 \quad \text{et} \quad \text{div } u = 0 \quad \text{pour} \quad t \in [0,T] \text{ et } x \in \mathcal{F}(t). \tag{1.16}$$

Équations solides :

$$\begin{aligned} mh''(t) &= -\int_{\partial \mathcal{S}(t)} (-\pi \mathrm{Id} + 2\nu D(u))\, n \; d\sigma, \\ \mathcal{J}\vartheta''(t) &= -\int_{\partial \mathcal{S}(t)} (x - h(t))^{\perp} \cdot (-\pi \mathrm{Id} + 2\nu D(u)) n \, d\sigma, \text{ pour } t \in [0,T]. \end{aligned} \tag{1.17}$$

Conditions aux limites :

$$\begin{aligned} u \cdot n &= u_S \cdot n \text{ et } (D(u)n)_{\tan} = -\mu(u - u_S)_{\tan} \text{ sur } \partial \mathcal{S}(t), \\ u \cdot n &= 0 \text{ et } (D(u)n)_{\tan} = -\mu(u)_{\tan} \text{ sur } \partial \Omega, \text{ pour } t \in [0,T], \end{aligned} \tag{1.18}$$

où $\nu > 0$ dénote le coefficient de viscosité, le gradient symétrique et la partie tangentielle d'un champ de vecteurs ont été définis dans (1.9), tandis que la vitesse du solide est donnée par $u_S(t,x) = h'(t) + \vartheta'(t)(x - h(t))^{\perp}$. Nous considérons le même type de données initiales $(u_0, h_0, h_0', \vartheta_0, \vartheta_0')$ que dans (1.13).

Notons qu'il n'y a pas de possibilité de découpler la partie ODE du système (1.16)-(1.18) comme cela est fait dans [57] pour le cas non visqueux irrotationnel. Cependant, en raison des conditions aux limites de Navier, on peut voir le système (1.10)-(1.12) comme la limite du système (1.16)-(1.18) quand $\nu \to 0$ (comme déjà mentionné dans la Section 1.2.2 pour le cas du fluide seul). En effet, dans [86] Planas et Sueur ont établi un tel résultat de convergence dans le cas 3D lorsque le domaine borné Ω contenant le système fluide-solide est remplacé par l'espace entier $\mathbb{R}^3$, mais ce résultat peut facilement être adapté au cas 2D.

Notons qu'un "paradoxe de non collision" peut se produire dans le cas de modèles fluide-solide avec des conditions aux limites de Dirichlet (voir [45, 67, 68]), c'est-à-dire qu'il n'y a pas de collision possible (en temps fini) entre le solide et la frontière externe $\partial\Omega$. Ceci est irréaliste car il contredit la poussée d'Archimède, cependant, il a été montré dans [46] que ce paradoxe peut être résolu en utilisant les conditions de Navier à la place de celles de Dirichlet.

On peut également trouver des résultats d'existence pour des solutions faibles à la variante 3D du système (1.16)-(1.18) dans les articles [44] et [86], pour le cas borné, respectivement non borné. Pour l'existence de solutions fortes dans un domaine 2D borné, nous citons [97]. Enfin, nous notons que l'existence de solutions (à la fois faibles et fortes, en 2D et 3D) dans le cas des conditions aux limites de Dirichlet a été étudiée dans de nombreux articles, citons par exemple [30] pour des solutions faibles, respectivement [20, 94] pour des solutions fortes.

La deuxième partie de cette thèse consiste à étudier la contrôlabilité du solide dans le système (1.16)-(1.18) par un contrôle intérieur. Les résultats de cette analyse seront présentés dans la Section 1.4.2.

1.3.3 Quelques autres modèles fluide-solide, résultats connectés et autres références

En dehors des résultats mentionnés pour les deux modèles dans les sections précédentes, présentons d'autres résultats connectés concernant les interactions fluide-solide.

Un autre type de résultat de contrôle fluide-solide est due à Glass et Rosier dans [58], concernant le contrôle du mouvement d'un bateau, où le fluide est gouverné par l'équation d'Euler bidimensionnelle, mais le contrôle est situé sur le bord du solide, contrairement au résultat de la Section 1.4.1 où l'on considère des contrôles de type Yudovich sur la frontière externe du domaine fluide, de même que (1.4)-(1.6).

En fait, le résultat de la Section 1.4.1 peut plutôt être vu comme une extension au cas d'un corps immergé des résultats [51, 52, 53] par Glass et Horsin, concernant la contrôlabilité lagrangienne des équations incompressibles d'Euler et de Stokes (c'est-à-dire contrôlant le mouvement d'un ensemble de particules fluides), où les auteurs considèrent également des contrôles de type Yudovich.

Comme dans le cas du fluide seul (comme mentionné dans la Section 1.2.2), dans le cas des conditions aux limites de Dirichlet pour les systèmes de contrôle "fluide visqueux+corps rigide", des résultats de contrôlabilité locale ont déjà été obtenus, dans 2D et 3D, voir par exemple les travaux de Boulakia et Guerrero, Boulakia et Osses, respectivement Imanuvilov et Takahashi [10, 11, 72]. Comme dans le cas du fluide seul, ces résultats reposent sur des estimations de Carleman sur l'équation linéarisée, et par conséquent sur le caractère parabolique de l'équation fluide. Un résultat similaire a été établi par Liu, Takahashi et Tucsnak dans [78] pour le cas de l'équation 1D visqueuse de Burgers avec une nouvelle stratégie introduite par les auteurs sans utiliser aucune estimation de Carleman, et comme indiqué dans cet article, ces méthodes peuvent être étendues aux autres systèmes paraboliques non linéaires. Cependant, notons que les résultats mentionnés ci-dessus concernent la contrôlabilité nulle locale pour la position du solide et les vitesses du solide et du fluide, alors que, alors que dans la Section 1.4.2 nous obtenons une contrôlabilité exacte globale pour la position et la vitesse du solide directement.

Mentionnons également quelques résultats par Badra et Takahashi concernant la stabilisation concernant les systèmes "fluide visqueux+corps rigide" dans un domaine borné avec des conditions aux limites de Dirichlet, voir [6] pour les cas 2D et 3D, respectivement [7] pour un modèle simplifié dans le cas 1D. Dans ces résultats, les auteurs stabilisent la position et la vitesse du solide ainsi que la vitesse du fluide en utilisant un contrôle de feedback sur la limite extérieure du domaine fluide, en supposant que les données initiales du système sont proches d'un état stationnaire, qui n'est pas nécessairement supposé d'être zéro.

Un type différent de problème concernant les interactions fluide-solide est celui d'un corps déformable dans un fluide, concernant la dynamique de la nage, voir par exemple les résultats dans [17, 40, 77, 90, 91] pour le cas visqueux, respectivement [18] par Chambrion et Munnier pour le cas non visqueux. Dans le cas de tels problèmes, le contrôle n'est plus à distance, il consiste plutôt en la déformation du corps lui-même. De plus, des modèles d'EDO simplifiés sont déduits et contrôlés par Alouges et al. dans [2, 3] pour le cas de robots micro-nageurs autopropulsés composés d'assemblages de billes reliés par des bras capables d'allonger et de rétrécir, contenus dans un fluide régi par l'équation de Stokes. Une autre étude intéressante des micro-nageurs peut être trouvée par Alouges et Giraldi dans [1], où le modèle considéré possède aussi un champ magnétique agissant sur les nageurs composés d'une tête et d'une queue déformable, et c'est le champ magnétique qui provoque la déformation de leur queue qui les propulse dans le fluide.

1.4 Principaux résultats de la thèse

Dans cette Section, nous présentons les principaux résultats de la thèse, concernant la contrôlabilité des modèles fluide-solide présentés dans les Sections 1.3.1 et 1.3.2.

1.4.1 Contrôle du mouvement d'un corps rigide immergé dans un fluide bidimensionnel parfait irrotationnel

Nous considérons le système (1.10)-(1.11) comme présenté dans la Section 1.3.1, mais avec des contrôles de type Yudovich (pareil que dans (1.4)-(1.6) dans le cas de fluide seul), c'est-à-dire les conditions aux limites suivantes :

$$u \cdot n = g \text{ on } \Sigma, \ u \cdot n = 0 \text{ sur } \partial\Omega \setminus \Sigma, \quad \text{et} \quad u \cdot n = u_S \cdot n \text{ sur } \partial\mathcal{S}(t), \text{ pour } t \in [0,T], \tag{1.19}$$

où $\Sigma \subset \partial\Omega$ est une partie non vide et ouverte de la frontière, et $g \in C_0^\infty([0,T] \times \Sigma)$ avec $\int_\Sigma g = 0$.

Puisque nous allons travailler avec des données initiales irrotationnelles, nous définissons la vorticité entrante sur Σ^- d'être zéro, c'est-à-dire

$$\operatorname{curl} u(t,x) = 0 \text{ on } \Sigma^-, \tag{1.20}$$

où $\Sigma^- = \{(t,x) \in [0,T] \times \Sigma : \ u \cdot n < 0\}$.

Cela garantira que la condition d'irrotationnalité (1.14) est toujours valide via le théorème de Helmholtz.

Nous nous intéressons à contrôler le solide d'une position donnée et d'une vitesse donnée à une autre position et vitesse prescrites via la fonction $g \in C_0^\infty([0,T] \times \Sigma)$. Notons que la raison pour laquelle nous n'essayons pas de contrôler la vitesse du fluide u en même temps est qu'il n'y a aucun espoir pour un tel résultat de contrôlabilité à tenir, puisque le théorème de Kelvin donne un invariant du système, indépendamment du contrôle, voir (1.15).

De plus, nous voulons nous assurer que la trajectoire solide reste loin de la frontière $\partial\Omega$, puisque le système (1.10)-(1.11) n'est plus valide s'il y a une collision entre le solide et la frontière extérieure. Par conséquent, nous définissons

$$\mathcal{Q} = \{q := (h,\vartheta) \in \Omega \times \mathbb{R} : \ d(h + R(\vartheta)(\mathcal{S}_0 - h_0), \partial\Omega) > 0\}. \tag{1.21}$$

Le premier résultat principal de la thèse est le théorème suivant, qui est prouvé dans le Chapitre 3.

Theorem 1.4.1. *Soit $T > 0$, $\mathcal{S}_0 \subset \Omega$ borné, fermé, simplement connexe et lisse, qui n'est pas un disque, et $u_0 \in C^\infty(\overline{\mathcal{F}_0}; \mathbb{R}^2)$, $\gamma \in \mathbb{R}$, $q_0 = (h_0, 0), q_1 = (h_1, \vartheta_1) \in \mathcal{Q}$, $h_0', h_1' \in \mathbb{R}^2, \vartheta_0', \vartheta_1' \in \mathbb{R}$, telle que $(h_0, 0)$ et (h_1, ϑ_1) appartiennent à la même composante connexe de $\mathcal{Q}$, et de plus*

$$\operatorname{div} u_0 = \operatorname{curl} u_0 = 0 \text{ dans } \mathcal{F}_0, \ u_0 \cdot n = 0 \text{ sur } \partial\Omega,$$
$$u_0 \cdot n = (h_0' + \vartheta_0'(x - h_0)^\perp) \cdot n \text{ sur } \partial\mathcal{S}_0, \ \int_{\partial\mathcal{S}_0} u_0 \cdot \tau \, d\sigma = \gamma.$$

Alors il existe $g \in C_0^\infty((0,T) \times \Sigma)$ et une solution

$$(h, \vartheta, u) \in C^\infty([0,T]; \mathcal{Q}) \times C^\infty([0,T]; C^\infty(\overline{\mathcal{F}(t)}; \mathbb{R}^2))$$

de (1.10), (1.11), (1.13), (1.14), (1.15), (1.19), (1.20) telle que

$$(h, h', \vartheta, \vartheta')(T) = (h_1, h_1', \vartheta_1, \vartheta_1').$$

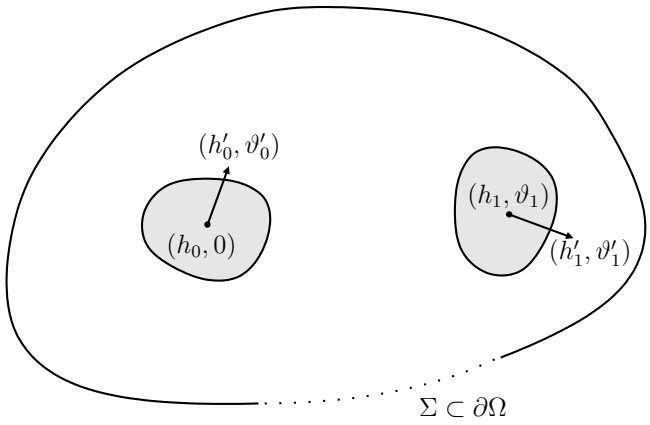

FIGURE 1.4: Les positions et les vitesses initiales et finales dans le problème de contrôle

Notons qu'il y a un léger abus de notation dans l'écriture $C^\infty([0,T]; C^\infty(\overline{\mathcal{F}(t)}; \mathbb{R}^2))$, puisque le domaine dans lequel le fluide évolue dépend également du temps.

De plus, nous observons que le cas où le solide est un disque est dégénéré : si le solide est un disque homogène par exemple, nous avons $(x - h(t))^\perp \cdot n = 0$ pour $t \in [0,T]$, $x \in \partial\mathcal{S}(t)$, donc la deuxième équation dans (1.11) implique simplement que ϑ est une fonction linéaire du temps. Cependant, un résultat similaire à Theorem 1.4.1 peut être établi pour h seul dans le cas d'un disque homogène, voir le Chapitre 3.

1.4.2 Contrôle du mouvement d'un corps rigide immergé dans un fluide bidimensionnel visqueux incompressible

La deuxième partie de la thèse consiste à étudier la contrôlabilité du solide dans le modèle (1.16)-(1.17). Cependant, pour des raisons techniques (voir Remarque 1.5.3 plus tard), au lieu de travailler dans un domaine borné $\Omega \subset \mathbb{R}^2$, nous considérerons plutôt le cas lorsque le modèle fluide-solide occupe l'espace entier $\mathbb{R}^2$, et au lieu d'utiliser des contrôles aux limites, nous utiliserons un contrôle intérieur (comme mentionné dans la Section 1.2.2). Par conséquent, nous définissons $\mathcal{F}(t) := \mathbb{R}^2 \setminus \mathcal{S}(t)$, pour tout $t \in [0,T]$.

Nous considérons le modèle suivant :

$$\begin{aligned}
&\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla\pi - \Delta u = \xi \quad \text{et} \quad \operatorname{div} u = 0 \quad \text{pour} \quad t \in [0,T] \text{ et } x \in \mathcal{F}(t),\\
&mh''(t) = -\int_{\partial\mathcal{S}(t)} (-\pi \mathrm{Id} + 2D(u))\, n \; d\sigma,\\
&\mathcal{J}\vartheta''(t) = -\int_{\partial\mathcal{S}(t)} (x - h(t))^\perp \cdot (-\pi \mathrm{Id} + 2D(u)) n \, d\sigma, \text{ pour } t \in [0,T],\\
&u \cdot n = u_S \cdot n \text{ et } (D(u)n)_{\tan} = -\mu(u - u_S)_{\tan} \text{ sur } \partial\mathcal{S}(t), \quad \lim_{|x| \to +\infty} |u| = 0, \text{ pour } t \in [0,T],
\end{aligned} \tag{1.22}$$

où la fonction $\xi : [0,T] \times \overline{\mathcal{F}(t)} \to \mathbb{R}^2$ agira comme le contrôle et est supportée dans un domaine compact, simplement connexe et lisse $\Omega_c \subset \mathbb{R}^2$ avec un intérieur non vide. Sans perte de généralité, nous pouvons considérer que les données initiales sont

$$u|_{t=0} = u_0 \text{ pour } x \in \mathcal{F}_0, \; h(0) = 0, \; h'(0) = h_0', \; \vartheta(0) = 0, \; \vartheta'(0) = \vartheta_0'. \tag{1.23}$$

Notons que dans ce cas le coefficient de viscosité est 1, alors que les autres notations sont les mêmes que dans la Section 1.3.2.

Pour que le modèle (1.22) soit valide, le solide doit rester loin du support du contrôle, c'est-à-dire

$$\text{supp } \xi(t,\cdot) \cap \mathcal{S}(t) = \emptyset, \ \forall t \in [0,T]. \tag{1.24}$$

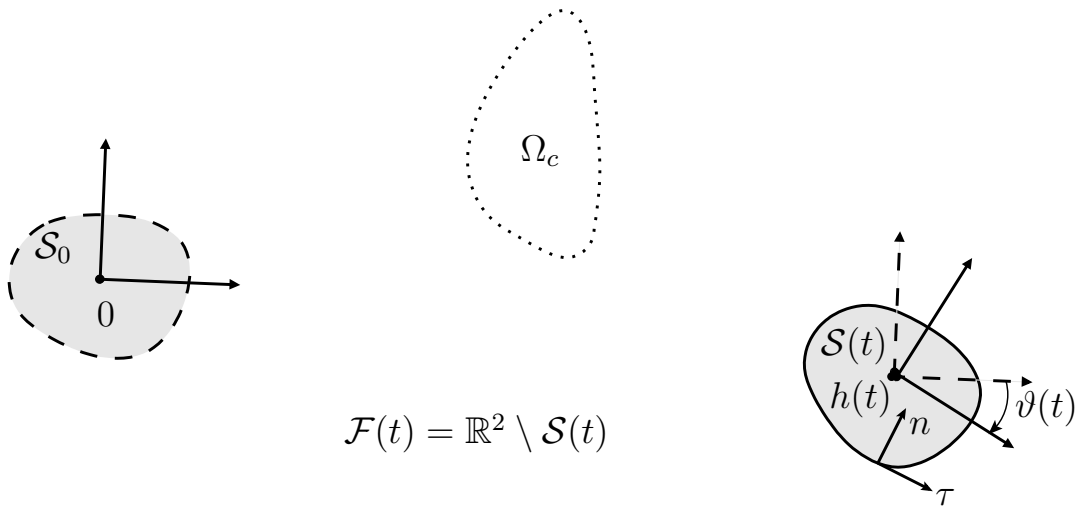

FIGURE 1.5: Le cadre du problème de contrôle

Nous allons travailler dans le cadre de solutions faibles pour le système (1.22), que nous ne définissons pas ici par souci de brièveté, mais nous notons que cela se fait d'une manière similaire à la définition des solutions faibles de type Leray pour le fluide seul dans le cas des équations de Navier-Stokes.

Notre but est encore une fois de contrôler le solide d'une position donnée et d'une vitesse donnée à une autre position et vitesse prescrites, en prescrivant un contrôle intérieur agissant sur le fluide. Afin de nous assurer que (1.24) est valide, nous notons que nous n'avons pas besoin de contrôler sur tout l'ensemble de Ω_c. Au lieu de cela, nous allons introduire un ensemble de positions admissibles pour le solide, de sorte que tant que la position finale du solide est dans cet ensemble, il existe un sous-ensemble ouvert fixe de Ω_c, qui ne touche pas le solide ni dans sa position initiale, ni dans sa position finale, et que nous utiliserons comme support des contrôles que nous construisons. Par conséquent, nous définissons

$$\mathcal{Q} = \left\{ q := (h,\vartheta) \in \mathbb{R}^3 : \ \text{int}(\Omega_c) \setminus \{(h + R(\vartheta)\mathcal{S}_0) \cup \mathcal{S}_0\} \neq \emptyset \right\}. \tag{1.25}$$

Comme $\mathcal{S}_0$ est simplement connexe, il est facile de vérifier que $\mathcal{Q}$ est connexe par arcs.

Nous énonçons le deuxième résultat principal de la thèse, qui est prouvé dans le Chapitre 4.

Theorem 1.4.2. *Soit $\mathcal{S}_0 \subset \mathbb{R}^2$ borné, fermé, simplement connexe et lisse, qui n'est pas un disque, et $u_0 \in H^4(\mathcal{F}_0;\mathbb{R}^2)$, $\text{curl}\, u_0 \in L^1(\mathcal{F}_0;\mathbb{R}^2)$, $q_0 = 0, q_f = (h_f,\vartheta_f) \in \mathcal{Q}$, $h_0', h_f' \in \mathbb{R}^2, \vartheta_0', \vartheta_f' \in \mathbb{R}$, telle que*

$$\text{div } u_0 = 0 \text{ in } \mathcal{F}_0, \quad \lim_{|x|\to+\infty} |u_0(x)| = 0,$$

$$u_0 \cdot n = (h_0' + \vartheta_0' x^\perp) \cdot n, \ (D(u_0)n)_{tan} = -\mu(u_0 - (h_0' + \vartheta_0' x^\perp))_{tan} \text{ sur } \partial\mathcal{S}_0.$$

Alors il existe $\tilde{T} > 0$ telle que pour chaque $T \in (0,\tilde{T}]$ il existe un contrôle $\xi \in L^2((0,T)\times\Omega_c)$, à support compact en temps, et une solution faible (u,h,ϑ) de (1.22) telle que (1.24) est valide, et de plus $(h,h',\vartheta,\vartheta')(T) = (h_f,h_f',\vartheta_f,\vartheta_f')$.

On remarque que la limite supérieure $\tilde{T}$ pour les horizons temporels implique que ce résultat est un résultat de contrôlabilité en petit temps, ce qui est courant dans le cas des systèmes de type Navier-Stokes, en raison des propriétés d'échelle du modèle, voir Chapitre 4 pour plus de détails ainsi que la possibilité de passer à temps arbitraire.

De plus, la condition que $\mathcal{S}_0$ n'est pas un disque est essentielle car une étape importante durant notre démonstration s'appuiera sur un résultat similaire à Théorème 1.4.1 (voir Section 1.5.2). Cependant, il est toujours possible de contrôler le centre de masse h seul dans le cas d'un disque homogène, en utilisant une stratégie similaire.

1.5 Méthodologie

Dans cette Section, nous donnons une présentation rapide des idées principales utilisées pour construire les contrôles pour prouver les Théorèmes 1.4.1 et 1.4.2 dans les Chapitres 3 et 4.

1.5.1 La construction dans le cas non visqueux : géodésiques et contrôle impulsif

Une structure géodésique sous-jacente

Il a été prouvé par Glass, Munnier et Sueur dans [57] que dans le cas irrotationnel sans contrôle, la dynamique du système (1.10)-(1.13) peut être reformulée comme une EDO pour les degrés de liberté du solide, à savoir $(h, \vartheta) \in \Omega \times \mathbb{R}$. Pour rendre cela plus précis, notons $q := (h, \vartheta)$, et puisque les domaines $\mathcal{F}(t)$ et $\mathcal{S}(t)$ ne dépendent du temps t qu'en raison de la dépendance sur $q(t)$, nous les notons aussi par $\mathcal{F}(q)$ et $\mathcal{S}(q)$.

Afin de préciser la reformulation EDO mentionnée ci-dessus, nous introduisons certains objets qui ne dépendent que de la géométrie et des constantes du système. En particulier, pour "se débarrasser" de la partie EDP du système nous résolvons certains problèmes de type elliptique sur un domaine en fonction de la position du solide.

— Les potentiels de Kirchhoff

$$\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3)(q, \cdot) \tag{1.26}$$

sont définis (à constante près) comme la solution des problèmes de Neumann

$$\begin{aligned} &\Delta\Phi_i(q,x) = 0 \text{ dans } \mathcal{F}(q), \quad \partial_n\Phi_i(q,x) = 0 \text{ sur } \partial\Omega, \text{ pour } i \in \{1,2,3\}, \\ &\partial_n\Phi_i(q,x) = \begin{cases} n_i \text{ sur } \partial\mathcal{S}(q), \text{ pour } i \in \{1,2\}, \\ (x-h)^\perp \cdot n \text{ sur } \partial\mathcal{S}(q), \text{ pour } i = 3, \end{cases} \end{aligned} \tag{1.27}$$

où tous les opérateurs différentiels sont par rapport à la variable x.

— La fonction de courant ψ pour le terme de circulation est définie de la manière suivante. Nous considérons d'abord la solution $\tilde{\psi}(q, \cdot)$ du problème de Dirichlet $\Delta\tilde{\psi}(q,x) = 0$ dans $\mathcal{F}(q)$, $\tilde{\psi}(q,x) = 0$ sur $\partial\Omega$, $\tilde{\psi}(q,x) = 1$ sur $\partial\mathcal{S}(q)$. Alors nous mettons

$$\psi(q,\cdot) = -\left(\int_{\partial\mathcal{S}(q)} \partial_n\tilde{\psi}(q,x)\, d\sigma\right)^{-1} \tilde{\psi}(q,\cdot), \tag{1.28}$$

de telle sorte que nous avons

$$\int_{\partial\mathcal{S}(q)} \partial_n\psi(q,x)\, d\sigma = -1,$$

en notant que le principe du maximum fort nous donne $\partial_n\tilde{\psi}(q,x) < 0$ sur $\partial\mathcal{S}(q)$.

— Nous définissons respectivement les matrices 3×3 de masse réelle et ajoutée par

$$\mathcal{M}_g = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mathcal{J} \end{pmatrix},$$

et, pour $q \in \mathcal{Q}$,

$$\mathcal{M}_a(q) = \left(\int_{\mathcal{F}(q)} \nabla\Phi_i(q,x) \cdot \nabla\Phi_j(q,x)\, dx \right)_{1\leqslant i,j\leqslant 3}.$$

Notons que $\mathcal{M}_a$ est une matrice de Gram symétrique, intuitivement qui code la quantité de fluide incompressible que le corps rigide doit accélérer autour de lui-même, d'où le terme "matrice de masse ajoutée".

— Nous définissons l'application bilinéaire symétrique $\Gamma(q)$ donnée par

$$\langle \Gamma(q), p, p\rangle = \left(\sum_{1\leq i,j\leq 3} \Gamma^k_{i,j}(q)\, p_i\, p_j \right)_{1\leq k\leq 3} \in \mathbb{R}^3,\ \forall p \in \mathbb{R}^3,$$

où, pour chaque $i, j, k \in \{1, 2, 3\}$, $\Gamma^k_{i,j}$ désigne les symboles Christoffel du premier type définis sur $\mathcal{Q}$ par

$$\Gamma^k_{i,j} = \frac{1}{2}\left(\frac{\partial(\mathcal{M}_a)_{k,j}}{\partial q_i} + \frac{\partial(\mathcal{M}_a)_{k,i}}{\partial q_j} - \frac{\partial(\mathcal{M}_a)_{i,j}}{\partial q_k} \right). \tag{1.29}$$

On peut vérifier que Γ est de classe C^∞ sur $\mathcal{Q}$.

— Nous introduisons les champs de vecteurs C^∞ sur $\mathcal{Q}$ avec des valeurs dans $\mathbb{R}^3$ suivants

$$E(q) = -\frac{1}{2}\int_{\partial\mathcal{S}(q)} |\partial_n\psi(q,\cdot)|^2 \partial_n\Phi(q,\cdot)\, d\sigma, \quad B(q) = \int_{\partial\mathcal{S}(q)} \partial_n\psi(q,\cdot)\,(\partial_n\Phi(q,\cdot)\times\partial_\tau\Phi(q,\cdot))\, d\sigma. \tag{1.30}$$

Le Théorème 2.2 de [57] indique ce qui suit.

Theorem 1.5.1 (Glass-Munnier-Sueur)**.** *Pour des solutions lisses, le système* (1.10)-(1.13) *peut être reformulé, jusqu'à collision, comme l'EDO de second ordre*

$$\Big(\mathcal{M}_g + \mathcal{M}_a(q)\Big)q'' + \langle \Gamma(q), q', q'\rangle = \gamma^2 E(q) + \gamma q' \times B(q), \tag{1.31}$$

avec $q(0) = q_0$, $q'(0) = q'_0$. Dans ce cas, la vitesse du fluide u peut être récupérée par

$$u(t,\cdot) = \nabla(q'(t)\cdot\Phi(q(t),\cdot)) + \gamma\nabla^\perp\psi(q(t),\cdot).$$

Notons que dans le cas où $\gamma = 0$, l'EDO (1.31) signifie que la particule q se déplace le long des géodésiques associées à la métrique riemannienne induite sur $\mathcal{Q}$ par le champ de matrice d'inertie totale $\mathcal{M}_g + \mathcal{M}_a(\cdot)$, c.f. [81]. De plus, si $\gamma \neq 0$, le membre de droite de (1.31) est une force qui nous rappelle la force de Lorentz dans l'électromagnétisme par sa structure (voir [57] pour plus de détails). Mentionnons également que l'ensemble du système "fluide incompressible + corps rigide" peut être réinterprété comme un flot géodésique sur une variété de dimension infinie, cf. [60]. Cependant, la reformulation établie dans (1.31) repose sur la variété de dimension finie $\mathcal{Q}$ et éclaire davantage la dynamique du corps rigide.

En outre, nous rappelons de Section 1.4.1 que dans le cas où $\mathcal{S}_0$ est un disque, l'équation (1.31) devient dégénérée. Si le disque n'est pas homogène alors le modèle devient encore plus compliqué, voir le Théorème 2.9 de [57].

De plus, notons que l'analyse présentée ci-dessus pourrait être répétée dans le cas avec vorticité, mais une reformulation purement EDO comme dans (1.31) ne serait pas possible dans ce cas, car il

faudrait encore résoudre une EDP d'évolution afin de déterminer le tourbillon du système (qui dépend aussi de q), ce qui aurait à son tour un effet sur l'équation solide pour q.

Dans [54], nous avons étendu le Théorème 1.5.1 au cas avec contrôle de la manière suivante. Nous fixons

$$\mathcal{C} := \left\{ g \in C_0^\infty(\Sigma;\mathbb{R}) \ \text{ telle que } \ \int_\Sigma g\, d\sigma = 0 \right\},$$

et nous définissons pour tout $q \in \mathcal{Q}$ et $g \in \mathcal{C}$ la solution unique $\overline{\alpha} := \mathcal{A}[q,g] \in C^\infty(\overline{\mathcal{F}(q)};\mathbb{R})$ au problème de Neumann suivant :

$$\Delta\overline{\alpha} = 0 \text{ dans } \mathcal{F}(q) \quad \text{ et } \quad \partial_n\, \overline{\alpha} = g\mathbb{1}_\Sigma \text{ sur } \partial\mathcal{F}(q), \tag{1.32}$$

de moyenne zéro.

Nous avons le résultat suivant.

Theorem 1.5.2 (Glass-Kolumbán-Sueur)**.** *Étant donné*

$$q \in C^\infty([0,T];\mathcal{Q}), \quad u \in C^\infty([0,T];C^\infty(\overline{\mathcal{F}(q(t))};\mathbb{R}^2)) \quad \text{ et } \quad g \in C_0^\infty([0,T];\mathcal{C}),$$

le couple (q,u) *est une solution de* (1.10)*,* (1.11)*,* (1.13)*,* (1.14)*,* (1.15)*,* (1.19)*,* (1.20) *si et seulement si l'EDO suivante est verifiée sur* $[0,T]$ *:*

$$\begin{aligned}\big(\mathcal{M}_g + \mathcal{M}_a(q)\big)q'' + \langle \Gamma(q), q', q'\rangle = \gamma^2 E(q) + \gamma q' \times B(q) \\ + F_1(q,q',\gamma)[\alpha] + F_2(q)[\partial_t\alpha],\end{aligned} \tag{1.33}$$

où F_1 *et* F_2 *sont réguliers, et où* $\alpha(t,x) := \mathcal{A}[q(t), g(t,\cdot)](x)$*. En outre, la vitesse du fluide* u *peut alors être récupérée par*

$$u(t,x) = \nabla(q'(t)\cdot\Phi(q(t),x)) + \gamma\nabla^\perp\psi(q(t),x) + \nabla\mathcal{A}[q(t),g(t,\cdot)](x). \tag{1.34}$$

Contrôle impulsif

La principale raison pour établir le Théorème 1.5.2 est la suivante. Supposons que nous ayons $\gamma = 0$ (si ce n'est pas le cas, on peut s'attendre au moins à être proche du cas sans circulation quand γ est assez petit, alors que pour le cas γ plus grand on pourrait exploiter les propriétés d'invariance d'échelle de l'équation (1.33)), et supposons que nous puissions trouver un contrôle approprié $g \in C_0^\infty([0,T];\mathcal{C})$ tel que le terme $F_1(q,q',0)[\alpha] + F_2(q)[\partial_t\alpha]$ dans (1.33) se comporte approximativement comme $v_0\delta_0(t) + v_1\delta_T(t)$, pour tout $v_0, v_1 \in \mathbb{R}^3$ données, où δ_0 et δ_T indiquent les distributions de Dirac au temps $t = 0$, respectivement $t = T$.

Alors, (1.33) sera proche (dans un sens approprié) du modèle-jouet formel suivant :

$$\big(\mathcal{M}_g + \mathcal{M}_a(\tilde{q})\big)\tilde{q}'' + \langle \Gamma(\tilde{q}), \tilde{q}', \tilde{q}'\rangle = v_0\delta_0 + v_1\delta_T, \tag{1.35}$$

et contrôler (1.33) (au moins de manière approchée) se réduira à contrôler (1.35) en utilisant les vecteurs $v_0, v_1 \in \mathbb{R}^3$ comme contrôle.

Expliquons rapidement comment la contrôlabilité de (1.35) peut être établie. Étant donné que $q_0, q_1 \in \mathcal{Q}$, il existe (au moins dans le cas où q_0 et q_1 sont suffisamment proches, le cas général peut alors être traité via un argument de recollement) une géodésique associée à la métrique riemannienne

induite sur $\mathcal{Q}$ par $\mathcal{M}_g + \mathcal{M}_a(\cdot)$, qui relie q_0 avec q_1. Plus précisément, il existe une fonction lisse unique $\bar{q}$ satisfaisant

$$\big(\mathcal{M}_g + \mathcal{M}_a(\bar{q})\big)\bar{q}'' + \langle \Gamma(\bar{q}), \bar{q}', \bar{q}' \rangle = 0 \text{ sur } [0,T], \text{ avec } \bar{q}(0) = q_0,\ \bar{q}(T) = q_1. \tag{1.36}$$

Ainsi, on peut arriver à la position finale désirée q_1, mais a priori la vitesse finale $\bar{q}'(T)$ diffère de q_1', et même la vitesse initiale $\bar{q}'(0)$ diffère de q_0'.

Alors, pour contrôler la solution $\tilde{q}$ de (1.35) de (q_0, q_0') à (q_1, q_1') on peut chercher à imposer $v_0 := \big(\mathcal{M}_g + \mathcal{M}_a(q_0)\big)(\bar{q}'(0) - q_0')$ et $v_1 := -\big(\mathcal{M}_g + \mathcal{M}_a(q_1)\big)(\bar{q}'(T) - q_1')$, qui transforme les vitesses initiales et finales $\tilde{q}'(0)$ et $\tilde{q}'(T)$ exactement aux vitesses désirées afin de réaliser la contrôlabilité.

Nous prouverons dans le Chapitre 3 que de tels contrôles $g \in C_0^\infty([0,T];\mathcal{C})$ peuvent en effet être construits. Plus précisément, nous pouvons construire g de la forme suivante :

$$g(t,x) = g_\varepsilon(t,x) := \beta_\varepsilon(t)\overline{g}(q_0,v_0)(x) + \beta_\varepsilon(T-t)\overline{g}(q_1,v_1)(x), \tag{1.37}$$

où $\beta_\varepsilon(t) := \frac{1}{\sqrt{\varepsilon}}\beta\left(\frac{t-\varepsilon}{\varepsilon}\right)$, pour ε in $(0,1)$, avec $\beta : \mathbb{R} \to \mathbb{R}$ une fonction lisse, non-négative supportée dans $[-1,1]$, telle que $\int_{-1}^{1} \beta(t)^2\, dt = 1$, et donc $(\beta_\varepsilon^2)_\varepsilon$ est une approximation de l'unité quand $\varepsilon \to 0^+$. De plus, la fonction $\overline{g}$ est construite via l'analyse complexe, satisfaisant certaines contraintes telle que la condition susmentionnée

$$F_1(q,q',0)[\alpha] + F_2(q)[\partial_t \alpha] \approx v_0\delta_0(t) + v_1\delta_T(t)$$

tient quand $\varepsilon \to 0^+$. En fait, en utilisant de tels contrôles g_ε, le terme dominant dans l'expression ci-dessus sera $F_1(q,q',0)[\alpha]$ et se comportera comme $\int_{\partial\mathcal{S}(q)} |\nabla\alpha|^2\, \partial_n\Phi(q,\cdot)\, d\sigma$, donc nous allons imposer une contrainte quadratique sur $\overline{g}$ (voir le Chapitre 3 pour plus de détails).

Une telle stratégie est appelée "contrôle impulsif" en raison de la grande amplitude du contrôle g_ε sur un support court dans le temps (notons que supp $g_\varepsilon \subset ([0,2\varepsilon] \cup [T-2\varepsilon, T]) \times \Sigma$), nous mentionnons [13] et les références qui y sont données pour de nombreux autres exemples sur le contrôle impulsif.

Conclusion

Comme déjà mentionné, la construction ci-dessus nous permet de déduire la contrôlabilité approchée de (1.33) dans le cas $\gamma = 0$, avec une erreur en terme de ε provenant des approximations de Dirac. On peut étendre ce résultat au cas de γ petit, l'erreur dépend alors de $\varepsilon > 0$ et de $|\gamma|$.

Cependant, puisque nous contrôlons la quantité de dimension finie $(q(T), q'(T)) \in \mathbb{R}^6$, on peut passer de la contrôlabilité approchée à la contrôlabilité exacte en utilisant un argument topologique de type Brouwer, comme le lemme suivant, qui a été utilisé dans [58, pages 32-33] à des fins similaires.

Lemma 1.5.1. *Soit $w_0 \in \mathbb{R}^n$, $\kappa > 0$, $f : \overline{B}(w_0,\kappa) \to \mathbb{R}^n$ une fonction continue telle que $|f(w) - w| \leq \frac{\kappa}{2}$ pour chaque x dans $\partial B(w_0,\kappa)$. Alors $B(w_0, \frac{\kappa}{2}) \subset f(\overline{B}(w_0,\kappa))$.*

Bien sûr, dans ce cas, nous devons également assurer une propriété de continuité appropriée de notre construction par rapport à la position et vitesse finale cible (q_1, q_1').

Par conséquent, nous pouvons conclure la contrôlabilité exacte de l'équation (1.33) dans le cas de γ assez petit. Le cas général peut être déduit par un argument d'invariance d'échelle similaire à celui utilisé par J.-M. Coron pour l'équation d'Euler [22], qui a également été utilisée dans [58] pour passer du cas potentiel au cas avec vorticité. On obtient que contrôler (1.33) avec une circulation arbitraire $\gamma \in \mathbb{R}$

sur l'intervalle de temps $[0, T]$ se réduit à contrôler (1.33) avec un petit γ sur un intervalle de temps plus court $[0, \lambda T]$, avec $\lambda \sim \frac{1}{|\gamma|}$. Cependant, il ne pose aucune difficulté pour obtenir l'intervalle de temps désiré, en utilisant le fait que cette équation (1.33) bénéficie de certaines propriétés d'invariance par translation et inversion temporelle, il suffit de recoller ensemble un certain nombre de solutions contrôlées appropriées définies chacune sur un intervalle de temps de longueur λT. Voir le Chapitre 3 pour plus de détails.

Remark 1.5.1. *Nous mentionnons ici que l'utilisation de contrôles de la forme* (1.37) *nous permet de nous assurer que notre solution aura un flux total arbitrairement petit à travers* Σ^-. *C'est-à-dire, pour tout* $T > 0$, *pour tout* $\nu > 0$, *il existe un contrôle* g *et une solution* (h, ϑ, u) *satisfaisant les propriétés de Théorème 1.4.1 et de telle sorte que de plus*

$$\left| \int_0^T \int_{\Sigma^-} u \cdot n \, d\sigma dt \right| < \nu.$$

En effet, en raison de la forme de β_ε *on obtient que le flux total à travers* Σ^-, *c'est-à-dire* $\int_0^T \int_{\Sigma^-} g_\varepsilon \, d\sigma dt$, *est d'ordre* $\sqrt{\varepsilon}$. *Ceci est également invariant par l'argument de changement d'échelle ou de recollement. La signification d'un tel résultat est que l'on peut dans un sens limiter la quantité de fluide échangé pendant la phase de contrôle. Nous mentionnons également qu'une telle condition de flux petit ne peut pas être garantie dans les résultats [22, 47, 49] en ce qui concerne la contrôlabilité de l'équation d'Euler.*

1.5.2 La réduction du cas visqueux à l'état non visqueux par une méthode de viscosité évanescente

Dans cette Section, nous présenterons une stratégie basée sur un développement asymptotique qui nous permet de réduire la preuve de Theorem 1.4.2 à contrôler certains systèmes de type Euler et Euler linéarisé, ce que nous ferons par une stratégie de contrôle impulsif similaire à celui dans la Section 1.5.1. Afin de réaliser ce développement asymptotique, nous allons d'abord remplacer le contrôle de l'équation d'évolution par un contrôle de la divergence de u in (1.22), puis transformer le domaine de l'EDP dans (1.22) en un domaine cylindrique, et introduire un petit paramètre de viscosité.

Relever le contrôle de l'équation d'évolution et le placer sur la divergence

Tout d'abord, observons qu'il est équivalent de contrôler l'EDP d'évolution pour u et la divergence de u. Nous rappelons la définition de l'opérateur Bogovskii (voir par exemple [9] ou [43]).

Definition 1.5.1. *Étant donné un domaine lisse, borné, simplement connexe* $\Omega \subset \mathbb{R}^2$, *il existe un opérateur* $B : C_0^\infty(\Omega) \to C_0^\infty(\Omega)^2$ *telle que, pour tout* $g \in C_0^\infty(\Omega)$ *avec* $\int g = 0$, *nous avons* $\operatorname{div} Bg = g$. *De plus,* $B \in \mathcal{L}(W_0^{s,p}(\Omega), W_0^{s+1,p}(\Omega)^2)$, *pour tout* $1 < p < +\infty$, $s \geq 0$. *Notons également que nous pouvons étendre* Bg *par* 0 *en dehors de* Ω.

A partir de là, soit B l'opérateur Bogovskii associé au domaine Ω_c. Maintenant, supposons que nous

ayons une solution $(U, \pi, h, \vartheta, g)$ au système

$$\begin{aligned}
&\frac{\partial U}{\partial t} + (U\cdot\nabla)U + \nabla\pi - \Delta U = 0 \quad \text{et} \quad \operatorname{div} u = g \quad \text{pour} \quad t\in[0,T] \text{ et } x\in\mathcal{F}(t),\\
&U\cdot n = u_S\cdot n, \quad (D(U)n)_{\tan} = -\mu(U-u_S)_{\tan} \quad \text{pour } x\in\partial\mathcal{S}(t), \lim_{|x|\to+\infty}|U| = 0,\\
&mh''(t) = -\int_{\partial\mathcal{S}(t)} (-\pi\mathrm{Id} + 2D(U))\, n\, d\sigma,\\
&\mathcal{J}\vartheta''(t) = -\int_{\partial\mathcal{S}(t)} (x-h(t))^{\perp}\cdot(-\pi\mathrm{Id}+2D(U))n)\, d\sigma, \text{ pour } t\in[0,T],
\end{aligned} \tag{1.38}$$

avec $\int g = 0$ et $\operatorname{supp} g \cap \mathcal{S}(t) = \emptyset$, pour tout $t \in [0,T]$. Si nous posons $u := U - Bg$ et $\xi := -\frac{\partial Bg}{\partial t} + Bg\cdot\nabla Bg - U\cdot\nabla Bg - Bg\cdot\nabla U + \Delta Bg$, nous obtenons que (u,π,h,ϑ,ξ) est une solution de (1.22).

La principale raison d'un tel changement de point de vue est que notre stratégie de construction d'une solution au système fluide-solide visqueux reposera sur une construction pour le système non visqueux. Dans le cas non visqueux, de même que dans (1.32), (1.34) de la Section 1.5.1, on peut décomposer linéairement l'effet d'un contrôle g sur la divergence en considérant une fonction $\overline{\alpha} \in C^\infty(\overline{\mathcal{F}_0};\mathbb{R})$ qui s'annule à l'infini et satisfait le problème elliptique suivant :

$$\Delta\overline{\alpha} = g\mathbb{1}_{R(\vartheta)^T(B_c-h)}(x) \text{ dans } \mathcal{F}_0, \quad \lim_{|x|\to+\infty}|\nabla\overline{\alpha}| = 0, \quad \text{et } \partial_n\,\overline{\alpha} = 0 \text{ sur } \partial\mathcal{S}_0.$$

De telles fonctions peuvent être facilement étudiées au moyen de l'analyse complexe.

Un changement de variables pour passer à un domaine fixe avec une petite viscosité

Comme mentionné précédemment, notre stratégie reposera sur le lien entre le cas visqueux et le cas non visqueux. Pour ce faire, nous aimerions introduire un petit paramètre de viscosité (noté $\varepsilon > 0$) que nous ferons tendre à zéro. En même temps, nous aimerions changer le domaine de l'EDP, $\mathcal{F}(t)$, qui dépend de la position solide, à un domaine qui est fixé. Heureusement, puisque le système fluide-solide occupe tout le plan $\mathbb{R}^2$, ceci peut être réalisé simplement par un mouvement rigide.

Étant donné $\varepsilon > 0$, on peut introduire un changement de variables qui consiste en un mouvement rigide dans la variable spatiale, correspondant au mouvement solide, et un changement d'échelle temporelle par rapport à ε, tel que (1.38) est équivalent au système suivant :

$$\begin{aligned}
&\frac{\partial u^\varepsilon}{\partial t} + (u^\varepsilon - u_S^\varepsilon)\cdot\nabla u^\varepsilon + r^\varepsilon(u^\varepsilon)^{\perp} + \nabla\pi^\varepsilon - \varepsilon\Delta u^\varepsilon = 0 \quad \text{et} \quad \operatorname{div} u^\varepsilon = g^\varepsilon \quad \text{pour } x\in\mathcal{F}_0,\\
&u^\varepsilon\cdot n = u_S^\varepsilon\cdot n, \quad (D(u^\varepsilon)n)_{\tan} = -\mu(u^\varepsilon - u_S^\varepsilon)_{\tan} \quad \text{pour } x\in\partial\mathcal{S}_0, \lim_{|x|\to+\infty}|u^\varepsilon| = 0,\\
&m(l^\varepsilon)' = -\int_{\partial\mathcal{S}_0} (-\pi^\varepsilon\mathrm{Id} + 2\varepsilon D(u^\varepsilon))\, n\, d\sigma - mr^\varepsilon(l^\varepsilon)^{\perp},\\
&\mathcal{J}\ (r^\varepsilon)' = -\int_{\partial\mathcal{S}_0} x^{\perp}\cdot(-\pi^\varepsilon\mathrm{Id} + 2\varepsilon D(u^\varepsilon))n\, d\sigma, \quad \text{pour } t\in[0,T/\varepsilon],
\end{aligned} \tag{1.39}$$

où $u_S^\varepsilon(t,x) = l^\varepsilon(t) + r^\varepsilon(t)x^{\perp}$, pour $t \in [0,T/\varepsilon]$, $u^\varepsilon(0,\cdot) = \varepsilon u_0(\cdot), (l^\varepsilon, r^\varepsilon)(0) = \varepsilon(h_0', \vartheta_0')$, et le terme de contrôle est maintenant $g^\varepsilon \in C_0^\infty((0,T/\varepsilon)\times\mathcal{F}_0)$ telle que $\operatorname{supp} g^\varepsilon(t,\cdot) \subset R(\vartheta^\varepsilon(t))^T(\Omega_c - h^\varepsilon(t))$ et $\int g^\varepsilon(t,\cdot)\, dx = 0$, pour tout $t \in [0,T/\varepsilon]$. De plus, nous pouvons associer la position solide $(h^\varepsilon, \vartheta^\varepsilon)$, qui ne joue plus un rôle (directement) dans la résolution du système (1.39), par

$$h^\varepsilon(t) = \int_0^t R(\vartheta^\varepsilon(s))l^\varepsilon(s)\, ds, \quad \vartheta^\varepsilon(t) = \int_0^t r^\varepsilon(s)\, ds, \tag{1.40}$$

et en particulier, nous avons la mise à l'échelle suivante de la trajectoire solide par rapport à ε,

$$(h^\varepsilon(t), \vartheta^\varepsilon(t)) = (h(\varepsilon t), \vartheta(\varepsilon t)), \quad (l^\varepsilon(t), r^\varepsilon(t)) = \varepsilon(R(\vartheta(\varepsilon t))^T h'(\varepsilon t), \vartheta'(\varepsilon t)), \tag{1.41}$$

pour $t \in [0, T/\varepsilon]$.

Notons que la résolution du système (1.22) sur un intervalle de temps $[0, T]$ se réduit alors à résoudre (1.39) sur $[0, T/\varepsilon]$. Cependant, étant donné que $T > 0$ est fixe, on peut chercher à résoudre (1.39) sur $[0, T]$ et déduire l'existence d'une solution à (1.22) sur $[0, \varepsilon T]$, ce qui prouvera Théorème 1.4.2 dans le temps εT, pour un certain $\varepsilon \in (0, 1)$. C'est exactement ce sur quoi notre stratégie s'appuiera.

Maintenant, comme $\varepsilon \to 0^+$, on s'attendrait à ce que (1.39) converge vers un système fluide-solide non visqueux dans tout le plan. Cependant, afin d'obtenir la contrôlabilité désirée pour le solide, nous devons étudier le comportement de u^ε comme un développement asymptotique jusqu'à des ordres supérieurs de ε.

Le développement asymptotique

Nous aimerions avoir le développement asymptotique suivant pour la trajectoire du solide :

$$\begin{aligned} h^\varepsilon = h^0 + \varepsilon h^1 + \varepsilon h_R^\varepsilon, \ \vartheta^\varepsilon = \vartheta^0 + \varepsilon \vartheta^1 + \varepsilon \vartheta_R^\varepsilon, \\ l^\varepsilon = l^0 + \varepsilon l^1 + \varepsilon l_R^\varepsilon, \ r^\varepsilon = r^0 + \varepsilon r^1 + \varepsilon r_R^\varepsilon, \end{aligned} \tag{1.42}$$

avec $(l^0, r^0), (l^1, r^1), (l_R^\varepsilon, r_R^\varepsilon) \in L^\infty(0, T)$. À chaque ordre de ε, nous aurons aussi une vitesse de fluide associée, à savoir u^0, u^1, u_R^ε, dont nous préciserons la construction plus tard. Cependant, nous pouvons déterminer une équation pour chacun de ces termes à l'ordre $\mathcal{O}(1)$, $\mathcal{O}(\varepsilon)$, $\mathcal{O}(\varepsilon^+)$ en branchant formellement le développement asymptotique dans l'équation de $(u^\varepsilon, l^\varepsilon, r^\varepsilon)$ et en séparant les termes en fonction des ordres de ε.

Nous obtenons ce qui suit.

— à l'ordre $\mathcal{O}(1)$, (u^0, l^0, r^0) satisfont un système "corps rigide + fluide non visqueux" où le fluide est gouverné par l'équation de Euler. Nous voudrions conduire, au moyen d'un contrôle approprié, $(h^0, \vartheta^0)(T)$ à (h_f, ϑ_f) et $(l^0, r^0)(T)$ à 0.

— à l'ordre $\mathcal{O}(\varepsilon)$, (u^1, l^1, r^1) satisfont un système fluide-solide qui est une linéarisation du système à $\mathcal{O}(1)$. Nous voudrions conduire, au moyen d'un contrôle approprié, $(l^1, r^1)(T)$ à $(R(\vartheta_f)^T h'_f, \vartheta'_f)$.

— à l'ordre $\mathcal{O}(\varepsilon^+)$, $(u_R^\varepsilon, l_R^\varepsilon, r_R^\varepsilon)$ satisfont une équation de reste. Nous voudrions prouver que $(l_R^\varepsilon, r_R^\varepsilon)(T) \to 0$ quand $\varepsilon \to 0^+$.

Si nous avions ce qui précède, en tenant compte de (1.41), il s'ensuit que nous avons (approximativement) conduit $(h^\varepsilon, \vartheta^\varepsilon)(T)$ à (h_f, ϑ_f) et $(l^\varepsilon, r^\varepsilon)(T)$ à $\varepsilon(R(\vartheta_f)^T h'_f, \vartheta'_f)$, pour $\varepsilon > 0$ assez petit .

On peut alors passer de la contrôlabilité approchée mentionnée ci-dessus à une contrôlabilité exacte au moyen d'un argument topologique de type Brouwer, comme dans Lemma 1.5.1 de la Section 1.5.1. Cela nécessite en outre une certaine propriété de continuité pour l'ensemble de notre construction par rapport aux données cibles pour la trajectoire solide. Enfin, en rappelant l'échelle dans (1.41), on peut conclure le résultat de contrôlabilité souhaité $(h, \vartheta, h', \vartheta')(T) = (h_f, \vartheta_f, h'_f, \vartheta'_f)$.

Pour obtenir un développement comme dans (1.42) pour la trajectoire solide, nous considérons les contrôles g^ε sous la forme de $g^\varepsilon = g^0 + \varepsilon g^1$, dans le style de [27], et nous cherchons le développement asymptotique suivant pour la vitesse et la pression du fluide :

$$u^\varepsilon = u^0 + \sqrt{\varepsilon}\{v\} + \varepsilon u^1 + \varepsilon u_R^\varepsilon, \quad \pi^\varepsilon = \pi^0 + \varepsilon\{Q\} + \varepsilon \pi^1 + \varepsilon \pi_R^\varepsilon, \tag{1.43}$$

où, pour $f = f(t,x,z)$, nous dénotons $\{f\}$ son évaluation à $z = \frac{\varphi(x)}{\sqrt{\varepsilon}}$, avec une fonction φ telle que $|\varphi(x)| = d(x, \mathcal{S}_0)$ dans un voisinage du solide (voir le Chapitre 4 pour plus de détails). Donc, la preuve du Théorème 1.4.2 se réduit à construire les termes du côté droit de (1.42), (1.43) d'une manière appropriée. Notons que nous allons utiliser une estimation d'énergie pour prouver la petitesse de $(l_R^\varepsilon, r_R^\varepsilon)$, qui est la raison principale pour étudier non seulement les termes du développement asymptotique de la trajectoire solide, mais aussi la termes dans le développement de u^ε, qui inclut un profil de couche limite v et la pression associée Q, apparaissant du fait que la solution du système fluide-solide non visqueux ne satisfait pas les conditions de Navier, comme mentionné dans la Section 1.2.2.

Par conséquent, notre stratégie sera la suivante :

— Nous construisons g^0 et une solution u^0 de l'équation d'Euler (avec le contrôle g^0), avec des données initiales nulles, donc zéro tourbillon et zéro circulation autour du solide, de telle sorte que nous avons un résultat de contrôlabilité exact pour $(h^0, \vartheta^0)(T)$ avec $(l^0, r^0)(T)$ nulle. Cependant, nous notons que contrairement à [27], notre stratégie ne reposera pas sur une méthode de retour pour u^0, nous préférons simplement utiliser u^0 pour contrôler la position solide $(h^0, \vartheta^0)(T)$, en utilisant une stratégie de contrôle impulsif similaire à celle présentée dans la Section 1.5.1 pour le cas d'un domaine borné.

— En raison des conditions aux limites de Navier, la couche limite v apparaîtra près du solide à l'ordre $\mathcal{O}(\sqrt{\varepsilon})$, avec sa pression Q à l'ordre $\mathcal{O}(\varepsilon)$. Notons qu'il n'y aura pas de contribution à l'ordre $\mathcal{O}(\sqrt{\varepsilon})$ dans (1.42), cependant v apparaîtra dans les équations solides pour (l^1, r^1). En outre, nous soulignons que puisque nous ne contrôlons pas la vitesse du fluide u^ε, il n'est pas nécessaire de contrôler v (contrairement à [27]), il suffira de prouver des estimations de régularité pour gérer l'effet des couches limites à l'ordre $\mathcal{O}(\varepsilon)$ dans les équations solides pour (l^1, r^1), et dans les équations du reste. Notez que v et Q ne dépendent que de (u^0, l^0, r^0) et leur existence est immédiate à partir de l'existence de $(u^0, \pi^0, l^0, r^0, g^0)$.

— Nous construisons g^1 et une solution u^1 d'une équation d'Euler linéarisée autour de u^0 (avec le contrôle g^1), avec des données initiales dérivées du système original (1.22), de telle sorte que nous aurons un résultat de contrôlabilité approchée pour $(l^1, r^1)(T)$ (nous nous contenterons ici d'une contrôlabilité approchée car cela simplifie notre construction). Un tel résultat peut être établi par une stratégie de contrôle impulsif plus simple que dans le cas du terme non visqueux u^0, puisque l'équation est linéaire, et nous voulons seulement contrôler la vitesse solide $(l^1, r^1)(T)$, pas la vitesse et la position en même temps.

— Nous construisons $(u_R^\varepsilon, l_R^\varepsilon, r_R^\varepsilon)$ comme la solution d'un système que nous déduisons des équations vérifiées par tous les autres termes dans (1.42), (1.43) (notons qu'à ce stade nous n'avons pas encore prouvé l'existence de $(u^\varepsilon, l^\varepsilon, r^\varepsilon)$, mais nous savons que cela devrait être une solution de (1.39) avec $g^\varepsilon = g^0 + \varepsilon g^1$). Nous prouvons au moyen d'une estimation d'énergie que $(u_R^\varepsilon, l_R^\varepsilon, r_R^\varepsilon)$ est petit dans $L^\infty((0,T); L^2(\mathcal{F}_0) \times \mathbb{R}^3)$, quand $\varepsilon > 0$ est petit.

À partir de la construction ci-dessus, nous pouvons définir $(u^\varepsilon, l^\varepsilon, r^\varepsilon)$ comme étant les côtés droits de (1.42), (1.43), puisque maintenant tous les termes respectifs sont construits et bien définis, et on peut conclure la preuve du Théorème 1.4.2. Voir le Chapitre 4 pour l'analyse détaillée.

Remark 1.5.2. *Notons que pendant toute la construction, nous devons également nous assurer que la condition de "non-collision"* (1.24) *est valide (il suffit d'assurer une condition similaire pour g^ε), et que $(h^\varepsilon, \vartheta^\varepsilon, l^\varepsilon, r^\varepsilon)(T)$ dépend continuellement de $(h_f, \vartheta_f, h'_f, \vartheta'_f)$, afin de pouvoir appliquer Lemme 1.5.1, comme mentionné précédemment.*

Remark 1.5.3. *Expliquons maintenant pourquoi nous avons choisi de travailler dans tout le plan* $\mathbb{R}^2$ *au lieu d'un domaine borné. La difficulté technique clé dans la gestion du cas d'un domaine borné avec une stratégie similaire serait l'étape de transformation du domaine mobile* $\mathcal{F}(t)$ *en un domaine fixe. Comme mentionné ci-dessus, dans le cas du plan, cela peut se faire par un simple mouvement rigide. Cependant, dans le cas borné, il faudrait aussi tenir compte du bord externe* $\partial\Omega$*, et construire un difféomorphisme qui est un mouvement rigide dans un voisinage du solide, mais qui laisse la frontière* $\partial\Omega$ *intacte. Ce difféomorphisme dépendrait clairement de la position du solide, et contribuerait à des termes non linéaires plus compliqués dans l'EDP (voir par exemple [12] ou [61] pour une telle construction). Le problème principal est alors d'étudier ce qu'il advient de ces termes quand on cherche un développement asymptotique de la forme de* (1.42) *pour la trajectoire solide, en plus de les séparer en termes d'ordres de* ε*. Pour le faire correctement, il faudrait établir un développement asymptotique rigoureux du difféomorphisme (et des termes associés dans l'EDP) par rapport à la position solide, ce qui est plutôt difficile.*

1.6 Directions futures et problèmes ouverts

Une généralisation naturelle des résultats de ce doctorat serait le passage du cas bidimensionnel au cas tridimensionnel, car il est souvent plus commun et réaliste de considérer des modèles tridimensionnels au lieu de bidimensionnels pour décrire des phénomènes réels. Les principales difficultés pour adapter les méthodes présentées dans les Chapitres 3 et 4 au cas tridimensionnel sont d'une part la théorie de Cauchy pour le système visqueux en 3D, et d'autre part l'utilisation d'une analyse complexe pour construire explicitement les parties des contrôles en fonction de la variable spatiale (par exemple dans (1.37)).

En effet, dans le cas borné par exemple (dans le même cadre que dans la Section 1.5.1, avec $\mathcal{F}(q) = \Omega \setminus \mathcal{S}(q)$), on veut construire une fonction $\overline{\alpha} \in C^\infty(\overline{\mathcal{F}(q)}; \mathbb{R})$ telle que

$$\Delta\overline{\alpha} = 0 \text{ dans } \mathcal{F}(q),\ \partial_n\overline{\alpha} = 0 \text{ sur } \partial\mathcal{F}(q) \setminus \Sigma, \text{ et } \int_{\partial\mathcal{S}(q)} |\nabla\overline{\alpha}|^2\, \partial_n\Phi(q,\cdot)\, d\sigma = v,$$

pour $v \in \mathbb{R}^3$ donné, et $q \in \mathcal{Q}$. Dans le cas bidimensionnel, ceci est fait en construisant d'abord une fonction définie uniquement dans un voisinage de $\partial\mathcal{S}(q)$ et en satisfaisant la contrainte intégrale ci-dessus avec v, en utilisant une série de Laurent. Ensuite, on peut déduire l'existence d'une fonction appropriée définie sur l'ensemble de $\overline{\mathcal{F}(q)}$ via le théorème de Runge.

Cependant, une direction méritant d'être étudiée serait de remplacer ces arguments par une construction de type Cauchy-Kovalevskaya et un théorème de type Runge en dimension supérieure (c.f. [38]), de sorte qu'un résultat similaire puisse être établi dans le cas tridimensionnel.

De plus, on pourrait aussi s'intéresser au contrôle de plusieurs solides. En effet, les arguments d'analyse complexe mentionnés ci-dessus sont assez locaux autour du solide (et ensuite rendus globaux via le théorème de Runge), donc on pourrait imaginer que des méthodes similaires pourraient être adaptées au cas des solides multiples tant qu'il n'y a pas de collision entre eux. Cependant, il serait également intéressant d'un point de vue de modélisation de rechercher s'il est encore possible d'utiliser l'une des trajectoires solides pour contrôler les autres (sans collision bien sûr). Notons que le modèle avec plusieurs solides lorsque le nombre de solides est assez grand pourrait être un bon moyen de modéliser les sprays par exemple.

Mentionnons également le problème ouvert suivant concernant la planification du mouvement pour le corps rigide, à la fois dans le cas visqueux et dans le cas non visqueux. Étant donné une courbe ξ dans $C^2([0,T];\Omega)$, existe-t-il un contrôle approprié et une solution au système fluide-solide qui satisfait $\xi = q$ sur $[0,T]$? Même la planification de mouvement approximative en C^2, c'est-à-dire la même question que ci-dessus mais avec $\|\xi - q\|_{C^2([0,T])} \leq \varepsilon$ (avec $\varepsilon > 0$ arbitraire) au lieu de $\xi = q$, est un problème ouvert. Ces problèmes ouverts sont intéressants du point de vue des applications, car il serait utile de pouvoir faire en sorte que la trajectoire solide reste proche d'une courbe donnée, d'effectuer certains mouvements prescrits, d'éviter certaines régions du domaine fluide. En revanche, les méthodes du Chapitre 3 s'appuient sur la trajectoire solide suivant une courbe géodésique qui est définie uniquement par la géométrie, et non par la construction du contrôle, qui est la principale difficulté à surmonté. Cependant, dans le Chapitre 4 nous avons développé une stratégie pour passer de la contrôlabilité approchée du solide à la contrôlabilité exacte en utilisant un argument de théorème de fonction inverse (par opposition à un argument topologique, comme mentionné dans Lemma 1.5.1). Nous espérons qu'une telle stratégie pourrait être adaptée pour résoudre ce problème en ce qui concerne la planification du mouvement du solide dans l'avenir.

Une autre direction intéressante serait d'étudier les modèles d'interaction fluide-solide où le fluide est compressible, car cela pourrait donner lieu à des applications où le fluide entourant le solide est en fait un gaz. La contrôlabilité approchée de la vitesse et de la densité du fluide pour les équations compressibles d'Euler a été établi dans [83], alors qu'il existe des résultats partiels pour le cas de Navier-Stokes (voir [32]). Cependant, il se pourrait que le contrôle de la trajectoire solide puisse être réalisé plus facilement. La principale différence entre les cas compressibles et incompressibles est que la reformulation de l'EDO pour la trajectoire solide de [54] ne s'appliquera plus, puisqu'elle repose sur certaines fonctions harmoniques. Néanmoins, il serait intéressant d'étudier si les équations solides dans le cas compressible peuvent encore être mises sous une forme qui permet une stratégie de contrôle impulsif. Un problème intermédiaire pourrait être d'étudier le cas des fluides incompressibles, mais à densité variable, souvent considérés pour la modélisation des océans, des rivières ou des fluides dans de grands domaines (voir par exemple [35] pour le contrôle du fluide seul).

Un autre suivi de la question de la contrôlabilité pourrait être celui de *stabilisation*, comme mentionné dans l'introduction, qui consiste à trouver une loi de rétroaction stationnaire $g = g[u,q]$, de telle sorte que si nous la mettons comme contrôle, le système fluide-solide en boucle fermée est globalement bien défini et asymptotiquement stable à un point d'équilibre (u_e, q_e, q_e'). Bien sûr, le système fluide-solide ne peut être stabilisé que sur un point où le solide et le fluide sont au repos. Cependant, il peut y avoir une certaine vorticité initiale dans le fluide, qui pourrait être éliminée par une stratégie similaire à celle utilisée par J.-M. Coron dans [24], tant que le tourbillon initial ne touche pas la frontière solide.

Nous mentionnons également que la contrôlabilité exacte globale d'un système "fluide visqueux + corps rigide" avec des conditions aux limites de Dirichlet "sans glissement" est complètement ouverte, et un problème très difficile dû au fait que les conditions aux limites de Dirichlet créent des couches limites avec une amplitude plus grande que dans le cas des conditions aux limites de Navier. Notons que même le problème de contrôle de la vitesse du fluide dans un tel contexte est ouvert et difficile, certaines avancées récentes ont été faites dans [29] dans le cas très particulier où le domaine est supposé être un rectangle, utilisant un force fantôme distribuée additionnelle. Comme nous l'avons déjà mentionné dans

la Section 1.3.3, nous différencions à nouveau ces types de résultats globaux avec les résultats locaux existants (comme dans [10, 11, 72]), qui sont basés sur une stratégie différente et pas sur l'étude des couches limites.

Cependant, le problème moins ambitieux de contrôler uniquement la vitesse du solide, et pas la position (qui peut être utilisée par exemple pour arrêter le solide) pourrait être abordé plus facilement dans le cas de Dirichlet, en utilisant une stratégie similaire à celle du Chapitre 4. En effet, si nous devions implémenter un développement asymptotique comme dans (1.42)-(1.43) pour le cas de Dirichlet, on aurait des couches limites apparaissant à $\mathcal{O}(1)$, à savoir le développement pour la vitesse du fluide serait de la forme

$$u^\varepsilon = u^0 + \{v\} + \varepsilon u^1 + \varepsilon u_R^\varepsilon.$$

Cependant, comme expliqué dans la stratégie présentée dans la Section 1.5.2, le contrôle de la position solide correspondrait à u^0. Puisque nous ne voulons plus faire cela, il s'ensuit que nous pouvons considérer $u^0 = 0$ pour réduire l'effet de v. Bien sûr, cela implique que l'équation pour u^1 se simplifie en $\partial_t u^1 + \nabla \pi^1 = 0$, mais peut-être on peut l'exploiter en utilisant des contrôles qui ne s'annulent pas à la fin de l'intervalle de temps afin de contrôler l'équation solide pour p^1.

De plus, dans l'esprit de [29], on pourrait considérer le mouvement d'un solide à l'intérieur d'un rectangle avec des conditions aux limites de Dirichlet, et permettre du fluide de sortir à travers deux côtés parallèles du rectangle. Il serait intéressant d'étudier la possibilité de faire sortir le solide du rectangle et de faire reposer le fluide à l'intérieur du rectangle.

Chapitre 2

General Introduction

2.1 Control Theory

The aim of control theory is to study dynamical systems depending on some control parameter. Such systems arise in the fields of engineering, physics, biology, chemistry, etc. More precisely, we may generally define a control system as an evolution equation of the form

$$\frac{d}{dt}x(t) = f(t, x(t), g(t)), \text{ for } t \in [0, T], \tag{2.1}$$

where $T > 0$ is the time horizon, $x : [0, T] \to \mathcal{X}$ is the state of the system, and $g : [0, T] \to \mathcal{G}$ represents the control parameter (or rather control function) with which we would like to act on the evolution of the system. Two standard mathematical examples of such evolution equations are ODE control systems (when both the state space $\mathcal{X}$ and the control space $\mathcal{G}$ are finite-dimensional), and PDE control systems (when the state and the control belong to some infinite-dimensional functional spaces). Furthermore, depending on the properties of the function f with respect to x and g, one can further categorize linear and nonlinear control systems.

The general question of control theory is how to use the control function g in order to achieve some desired behaviour of system (2.1). Let us present a (non-exhaustive) list of such problems.

1. Exact controllability. Can the control system be driven from any given initial state to any given final state in any given time ? More precisely, given $T > 0$, $x_0, x_1 \in \mathcal{X}$, does there exist a control $g : [0, T] \to \mathcal{G}$ such that the solution $x : [0, T] \to \mathcal{X}$ of system (2.1) with control g and initial data $x(0) = x_0$ satisfies $x(T) = x_1$?

2. Approximate controllability. A similar question as above, but allowing for some (arbitrarily small) error when reaching the final state. That is, given $T > 0$, $x_0, x_1 \in \mathcal{X}$ and $\varepsilon > 0$, does there exist a control $g : [0, T] \to \mathcal{G}$ such that the solution $x : [0, T] \to \mathcal{X}$ of system (2.1) with control g and initial data $x(0) = x_0$ satisfies $\|x(T) - x_1\| \leq \varepsilon$?

3. Null controllability. Can the state of the system be put to rest ? Mathematically this can be described in the same way as the problem of exact controllability above, but instead of trying to reach an arbitrary final state x_1, we only want to have a vanishing final state, i.e. $x(T) = 0$.

4. Controllability to trajectories. Let $\bar{x} : [0, T] \to \mathcal{X}$ be a given trajectory of (2.1) associated with a control $\bar{g} : [0, T] \to \mathcal{G}$. Given $x_0 \in \mathcal{X}$, does there exist $g : [0, T] \to \mathcal{G}$ such that the associated solution $x : [0, T] \to \mathcal{X}$ of (2.1) with control g satisfies $x(T) = \bar{x}(T)$?

5. Asymptotic stabilization. For simplicity let us assume that we are in the autonomous case (that is $f = f(x, g)$ does not depend on t), and that $(x_e, g_e) \in \mathcal{X} \times \mathcal{G}$ is an equilibrium of the system,

i.e. $f(x_e, g_e) = 0$. Imagine that we wanted to control the system to this equilibrium state, i.e. to find a control g and an associated solution x of (2.1) such that $(x(T), g(T)) = (x_e, g_e)$. The problem is that such a control would be sensitive to perturbations in the model, such as those arising from noise, imprecisions in the model or in the implementation of the control. Therefore, if the equilibrium is unstable, the system will not behave in the way we would have intended with the control g. To overcome this issue, we look for a closed-loop control, i.e. a control of the form $g = g[x]$, which is more robust to perturbations. The question of asymptotic stabilizability becomes the following : can one find a feedback control $g : \mathcal{X} \to \mathcal{G}$ such that $g_e = g[x_e]$ and the solution of the system

$$\frac{d}{dt}x(t) = f(x(t), g[x(t)]), \text{ for } t \in [0, T], \tag{2.2}$$

is asymptotically stable at x_e ? We recall that the notion of asymptotic stability consists of the following :

(i) for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $x_0 \in \overline{B}(x_e, \delta)$ and any solution of (2.2) with $x(0) = x_0$, we have $x(t) \in \overline{B}(x_e, \varepsilon)$ for any $t \geq 0$;

(ii) for any $x_0 \in \mathcal{X}$ and any maximal solution of (2.2) with $x(0) = x_0$ we have that x is a global solution and $\|x(t) - x_e\| \to 0$ as $t \to +\infty$.

Let us note that the questions above have been formulated in a global context. Local versions can be considered by only requiring that the initial and final data are in some fixed small radius balls (instead of being arbitrary) in the case of the controllability question ; respectively by only asking for local asymptotic stability at x_e in the case of the closed-loop stabilization problem.

We also contrast the above problems with that of optimal control, where one looks for a control that also satisfies some optimality criterion (such as minimizing a cost functional), rather than just driving the system to a given state.

One may find a more detailed description of such problems, as well as the state of the art and open problems in the books of J.-M. Coron [21], J.-L. Lions [76], respectively M. Tucsnak and G. Weiss [96], as well as in the paper of D. L. Russell [89].

2.2 Control problems in fluid mechanics

2.2.1 The inviscid case

Let us present some concrete examples of PDE control problems in fluid mechanics. We start with a classical result regarding the exact boundary controllability of the Euler equation in two dimensions.

Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded, connected (but not necessarily simply connected) domain, and consider the incompressible Euler equation in Ω, namely

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi = 0 \quad \text{and} \quad \text{div } u = 0 \quad \text{for} \quad t \in [0, T] \text{ and } x \in \Omega, \tag{2.3}$$

where $u : [0, T] \times \overline{\Omega} \to \mathbb{R}^2$ denotes the fluid velocity field and $\pi : [0, T] \times \overline{\Omega} \to \mathbb{R}$ the pressure field. This system describes the evolution of a homogeneous, inviscid fluid subject only to conservation of mass and incompressibility. Since system (2.3) is a PDE in Ω, we need to add boundary conditions on $\partial\Omega$ for it to be well-posed. In the case without control the impermeability condition $u \cdot n = 0$ on $[0, T] \times \partial\Omega$ is often considered as a classical boundary condition associated with (2.3), meaning that the fluid is not allowed to cross the boundary $\partial\Omega$.

However, since we are interested in controlling system (2.3), we are going to let some fluid go in and out of the domain, which will be used as the control. More precisely, a possibility of giving a

mathematical formulation of such controls is due to Yudovich [98]. One considers $\Sigma \subset \partial\Omega$ a nonempty, open part of the boundary, and one prescribes on the one hand the normal velocity on Σ, i.e.

$$u(t,x) \cdot n(x) = g_n(t,x) \text{ on } [0,T] \times \Sigma, \tag{2.4}$$

where $g_n \in C_0^\infty([0,T] \times \Sigma)$ with $\int_\Sigma g_n = 0$, while on the rest of the boundary we have the usual impermeability condition

$$u \cdot n = 0 \text{ on } [0,T] \times (\partial\Omega \setminus \Sigma), \tag{2.5}$$

and on the other hand the vorticity on the set Σ^- of points of $[0,T] \times \Sigma$ where the velocity field points inside Ω, i.e.

$$\operatorname{curl} u(t,x) = g_v(t,x) \text{ on } \Sigma^-, \tag{2.6}$$

where $g_v \in C_0^\infty([0,T] \times \Sigma)$. Note that $\Sigma^- = \{(t,x) \in [0,T] \times \Sigma : \ u \cdot n < 0\}$ is deduced immediately from g_n. The control associated with system (2.3) then becomes $g = (g_n, g_v)$.

The well-posedness of system (2.3), (2.4), (2.5), (2.6) is proven by Yudovich in [98] for sufficiently smooth initial data and control, as well as some appropriate assumptions on Σ.

We have the following global exact controllability result due to Coron [22].

Theorem 2.2.1. *Given $T > 0$, $u_0, u_1 \in C^\infty(\overline{\Omega})$ such that*

$$\operatorname{div} u_0 = \operatorname{div} u_1 = 0 \text{ in } \Omega, \quad u_0 \cdot n = u_1 \cdot n = 0 \text{ on } \partial\Omega,$$

then system (2.3), (2.4), (2.5), (2.6) *is exactly controllable if and only if Σ meets every connected component of the boundary $\partial\Omega$.*

By exact controllability here we mean that there exists an appropriate boundary control g in the sense of Yudovich such that the smooth solution u of (2.3), (2.4), (2.5), (2.6) starting from $u(0) = u_0$ satisfies $u(T) = u_1$. Let us mention that in the case of PDE control it is not uncommon to look for such controls implicitly. That is, we look for a solution u of (2.3) satisfying only (2.5) and the controllability property we wish to establish (for instance exact controllability, i.e. having $u(T) = u_1$ for some given u_1), which is an underdetermined system, and then deducing the value of the control g implicitly from such a solution u by using (2.4), (2.6).

The proof relies on Coron's seminal "return method" for non-linear control problems, which consists of finding a particular solution $\bar{u}$ to the equation satisfying $\bar{u}(0) = \bar{u}(T) = 0$ such that the linearized system around $\bar{u}$ is controllable, and then proving the existence of a solution to the original control problem which is close to $\bar{u}$.

Note that the condition that Σ meets every connected component of the boundary $\partial\Omega$ is essential for two main reasons. The first reason being that due to Kelvin's law we know that the circulation of the fluid velocity is constant around any Jordan curve in the flow. Consequently, if there was a connected component of $\partial\Omega$ not met by Σ, the circulation around this component would be conserved regardless of the choice of control. Hence, one could not control the fluid velocity from u_0 to u_1 with different circulations. The second reason for which the fluid velocity is not controllable if Σ doesn't meet every connected component of $\partial\Omega$ is that the vorticity $\operatorname{curl} u$ is conserved along the flow of u, so any vorticity around an uncontrolled component of $\partial\Omega$ would also be conserved.

A similar result to Theorem 2.2.1 has been established by Glass in [49] regarding the exact controllability of the Euler equation in three dimensions. We also mention the results from [24, 25, 48] regarding the asymptotic stabilization of the Euler equation.

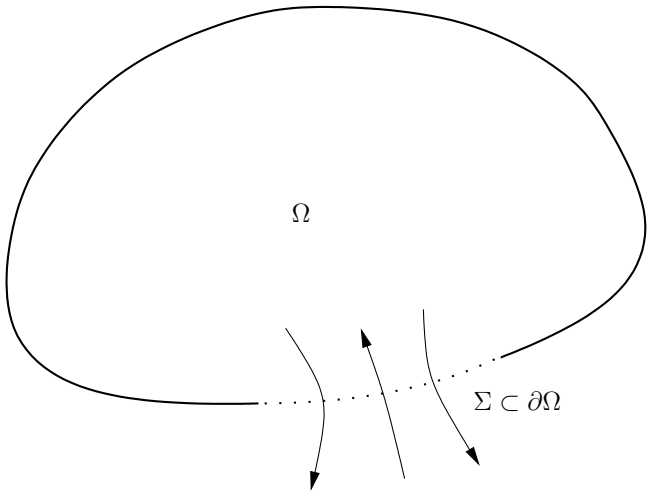

FIGURE 2.1: The setting of the boundary control problem

2.2.2 The viscous case

Next let us consider the controllability problem in the case when the fluid is viscous, i.e. modelled by the incompressible Navier-Stokes equations. Due to the added viscosity (which we denote by $\nu > 0$), our model becomes

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi - \nu \Delta u = 0 \quad \text{and} \quad \text{div } u = 0 \quad \text{for} \quad t \in [0, T] \text{ and } x \in \Omega, \tag{2.7}$$

gaining a dissipative second order term, which can actually be seen as a dissipation of momentum between the fluid particles. Since the PDE (2.7) is second order, appropriate boundary conditions have to be set (already in the case without control). The most classical possibility is that of Dirichlet or "no-slip" boundary condition, namely setting $u = 0$ on $[0, T] \times \partial\Omega$, which was first considered by Stokes in [93]. This condition implies that the fluid particles adhere to the boundary, and consequently generates large amplitude boundary layers (see for instance the work of Prandtl [87]). However, Serrin pointed out in [92] that this condition is not always realistic (for instance when moderate pressure is involved, such as in the case of high-altitude aerodynamics).

On the other hand, Navier introduced in [82] another interesting type of boundary conditions, the so-called "slip-with-friction" conditions, when the fluid is allowed to slip on the boundary, but subject to some friction. This can be described mathematically as

$$u \cdot n = 0 \text{ and } (D(u)n)_{\tan} = -\mu(u)_{\tan} \text{ on } [0, T] \times \partial\Omega, \tag{2.8}$$

where $\mu \geq 0$ is the friction coefficient, and for any vector field f we define the symmetric gradient, respectively the tangential part as

$$D(f) = \frac{1}{2}\left(\nabla f + (\nabla f)^T\right) \text{ and } (f)_{\tan} = f - (f \cdot n)n. \tag{2.9}$$

Despite the fact that the "no-slip" conditions are more common in the mathematical community, the Navier conditions are also well justified and frequently considered in various physical settings, such as for studying the behaviour of flows near rough walls (see for instance [4, 16, 73], as well as [75] for experimental data suggesting various possibilities for the slip phenomena to occur). Furthermore, it is easier to establish vanishing viscosity convergence results for passing to the Euler equation as $\nu \to 0$ in the case of the Navier conditions compared to the Dirichlet case, since the boundary layers generated

by the slip condition are of smaller amplitude than those generated by the adherence condition (see for instance [70] and the references therein), and the equation of the boundary layers in the Dirichlet case can be ill-posed (see for instance [31]). Also observe that the Dirichlet condition can be seen as a limit of the Navier conditions as $\mu \to +\infty$, see [74].

Before going into details on the controllability of the Navier-Stokes equations, let us mention a common method in PDE control for changing the perspective of certain control problems. As we have previously mentioned for the Euler equation, the control is often considered to take the form of a non-homogenous boundary condition on some nonempty, open part Σ of $\partial\Omega$. Clearly, we could define similar types of controls for system (2.7) under the Dirichlet, respectively Navier boundary conditions. However, this point of view is equivalent to having an interior control (i.e. a compactly supported source term on the PDE acting as the control), while maintaining homogenous boundary conditions. Indeed, we may extend the domain Ω to some bounded domain $\tilde{\Omega}$ such that $\Omega \subset \tilde{\Omega}$, $\Sigma \subset \tilde{\Omega}$ and $\partial\Omega \setminus \Sigma \subset \partial\tilde{\Omega}$. Now, if we consider a source term supported in $\tilde{\Omega} \setminus \Omega$, then establishing a control result on $\tilde{\Omega}$ using this interior control will give us a boundary control result in Ω, simply by letting the fluid circulate in $\tilde{\Omega}$ (in particular over Σ) and reading the appropriate value of our boundary control from the fluid velocity that passes over Σ.

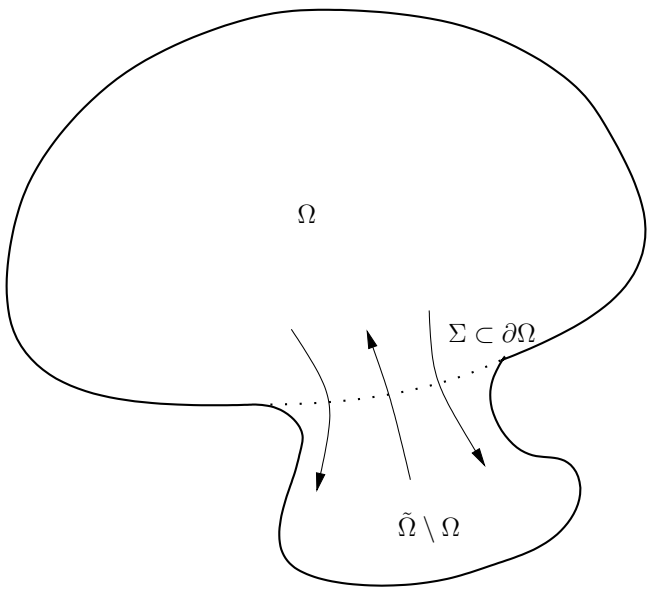

FIGURE 2.2: The passage between boundary control and interior control

We mention a couple of controllability results regarding the Navier-Stokes equations. Imanuvilov in [71], respectively Fernández-Cara, Guerrero, Imanuvilov and Puel in [36, 37] have obtained results of local controllability to trajectories in the case of Dirichlet condition, while a similar result has been established by Guerrero in [65] for the case of Navier conditions. These results rely on linearizing the equation and using so-called Carleman estimates.

However, instead of considering the nonlinear term as a perturbation (as done in the previously mentioned results), another approach is to consider the viscous term as a perturbation of the inviscid system, relying on controllability results for the Euler equation in order to deduce the controllability of the Navier-Stokes system. Usually such techniques give rise to global approximate controllability results, which can then be combined with the aforementionted local (but exact) results to provide global exact controllability. We mention the works of Coron [23] regarding global approximate controllability, as

well as Coron and Fursikov [26], respectively Chapouly [19], regarding global exact null controllability results in the case of the Navier conditions. More recently, these results were improved by Coron, Marbach and Sueur in [27, 28] to establish small-time global exact controllability, using a well-prepared dissipation method to handle the boundary layers. Adapting such methods to the Dirichlet case still poses a challenging open problem due to the larger amplitude boundary layers which are created. However, there has been a recent result by Coron, Marbach, Sueur and Zhang in [29], where in certain particular geometric settings a similar result was possible under Dirichlet boundary conditions, with an added "phantom force term".

2.3 Fluid-solid interaction models

In this section we present some common fluid-solid models, both in the inviscid and viscous case.

2.3.1 The dynamics of a rigid body in an irrotational inviscid incompressible fluid

Let us start with a simple two dimensional model of fluid-solid interaction. Let $\Omega \subset \mathbb{R}^2$ be a simply connected, bounded, open domain (we note that the simply connectedness is merely a simplifying assumption and not essential to the analysis). The domain contains a fluid-solid system which evolves with respect to time $t \in [0, T]$, namely an open part $\mathcal{F}(t)$ containing the fluid, and a closed part $\mathcal{S}(t) := \Omega \backslash \mathcal{F}(t)$ representing the solid. The fluid evolves according to the incompressible Euler equation, while the motion of the body is a rigid movement governed by Newton's law with a normal force due to the fluid's pressure acting on the body.

More precisely, the movement of the solid can be completely described by its center of mass $h(t) \in \Omega$ and angle of rotation $\vartheta(t) \in \mathbb{R}$, as $\mathcal{S}(t) = h(t) + R(\vartheta(t))(\mathcal{S}_0 - h_0)$, where h_0 is the center of mass at initial time, and

$$R(\vartheta) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}.$$

Furthermore, the dynamics of the fluid-solid model can be written as the following coupled ODE-PDE system.

Fluid equations :

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla\pi = 0 \quad \text{and} \quad \text{div } u = 0 \quad \text{for} \quad t \in [0, T] \text{ and } x \in \mathcal{F}(t). \tag{2.10}$$

Solid equations :

$$mh''(t) = \int_{\partial\mathcal{S}(t)} \pi\, n\, d\sigma \quad \text{and} \quad \mathcal{J}\vartheta''(t) = \int_{\partial\mathcal{S}(t)} \pi\, (x - h(t))^{\perp} \cdot n\, d\sigma, \text{ for } t \in [0, T]. \tag{2.11}$$

Boundary conditions :

$$u \cdot n = 0 \text{ on } \partial\Omega, \quad \text{and} \quad u \cdot n = u_S \cdot n \text{ on } \partial\mathcal{S}(t), \text{ for } t \in [0, T], \tag{2.12}$$

where once more $u : [0, T] \times \overline{\Omega} \to \mathbb{R}^2$ denotes the fluid velocity field and $\pi : [0, T] \times \overline{\Omega} \to \mathbb{R}$ the pressure field, furthermore, $m > 0$ and $\mathcal{J} > 0$ denote respectively the solid's mass and moment of inertia, and the solid velocity is given by $u_S(t, x) = h'(t) + \vartheta'(t)(x - h(t))^{\perp}$, with $(x_1, x_2)^{\perp} := (-x_2, x_1)$. For simplicity, the fluid is assumed to be homogeneous of density 1.

The Cauchy problem for this system with initial data

$$u|_{t=0} = u_0 \text{ for } x \in \mathcal{F}_0,\ h(0) = h_0,\ h'(0) = h'_0,\ \vartheta(0) = 0,\ \vartheta'(0) = \vartheta'_0, \tag{2.13}$$

is now well-understood, see e.g. [55, 61, 69, 84, 85], while the 3D case has also been studied in [62, 88].

Furthermore, we assume that the fluid is irrotational at initial time, i.e. $\operatorname{curl} u_0 = 0$, which implies that it will remain irrotational at all times, as a consequence of Helmholtz's third theorem, i.e.

$$\operatorname{curl} u = 0 \text{ for } x \in \mathcal{F}(t),\ \forall t \geq 0. \tag{2.14}$$

Lastly, we have that due to Kelvin's theorem the circulation around the body is conserved, i.e.

$$\int_{\partial\mathcal{S}(t)} u(t) \cdot \tau \, d\sigma = \int_{\partial\mathcal{S}_0} u_0 \cdot \tau \, d\sigma = \gamma \in \mathbb{R},\ \forall t \geq 0, \tag{2.15}$$

where τ denotes the unit counterclockwise tangent vector.

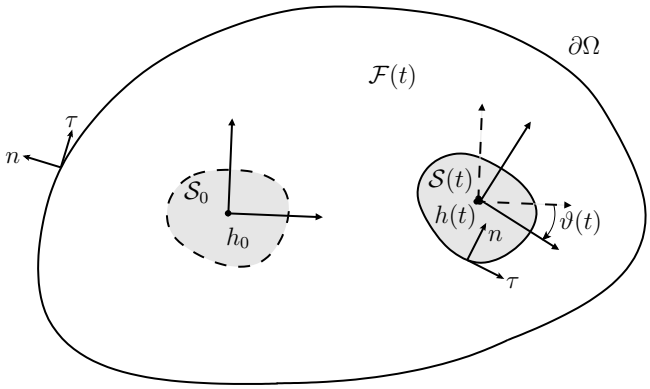

FIGURE 2.3: The domains Ω, $\mathcal{S}(t)$ and $\mathcal{F}(t) = \Omega \setminus \mathcal{S}(t)$

It has been proven by Glass, Munnier and Sueur in [57] that in the irrotational case, the dynamics of system (2.10)-(2.12) can be recast as an ODE for the degrees of freedom of the solid, namely $(h, \vartheta) \in \Omega \times \mathbb{R}$, which we will detail in Section 2.5.1.

The main goal of the first half of this thesis is to investigate the controllability of the solid in system (2.10)-(2.12) by the means of a boundary control on $\partial\Omega$. The results of that analysis will be presented in Section 2.4.1.

2.3.2 The dynamics of a rigid body in a viscous incompressible fluid

Let us now present a similar fluid-solid interaction problem as in Section 2.3.1, but in the case when the fluid is assumed to be viscous incompressible (and not necessarily irrotational), with Navier boundary conditions. We maintain the other assumptions from the beginning of Section 2.3.1. The system can then be described as follows.

Fluid equations :

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla\pi - \nu\Delta u = 0 \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{for} \quad t \in [0, T] \text{ and } x \in \mathcal{F}(t). \tag{2.16}$$

Solid equations :

$$\begin{aligned} mh''(t) &= -\int_{\partial\mathcal{S}(t)} (-\pi \mathrm{Id} + 2\nu D(u))\, n \, d\sigma, \\ \mathcal{J}\vartheta''(t) &= -\int_{\partial\mathcal{S}(t)} (x - h(t))^{\perp} \cdot (-\pi \mathrm{Id} + 2\nu D(u)) n \, d\sigma, \text{ for } t \in [0, T]. \end{aligned} \tag{2.17}$$

Boundary conditions :

$$\begin{aligned} &u \cdot n = u_S \cdot n \text{ and } (D(u)n)_{\tan} = -\mu(u - u_S)_{\tan} \text{ on } \partial\mathcal{S}(t), \\ &u \cdot n = 0 \text{ and } (D(u)n)_{\tan} = -\mu(u)_{\tan} \text{ on } \partial\Omega, \text{ for } t \in [0, T], \end{aligned} \tag{2.18}$$

where $\nu > 0$ denotes the viscosity coefficient, the symmetric gradient and the tangential part of a vector field were defined in (2.9), while the solid velocity is given by $u_S(t, x) = h'(t) + \vartheta'(t)(x - h(t))^{\perp}$. We consider the same type of initial data $(u_0, h_0, h_0', \vartheta_0, \vartheta_0')$ as in (2.13).

Note that there is no possibility of decoupling the ODE part from the system (2.16)-(2.18) as done in [57] for the irrotational inviscid case. However, due to the Navier boundary conditions, one could see system (2.10)-(2.12) as the limit of system (2.16)-(2.18) as $\nu \to 0$ (as mentioned already in Section 2.2.2 for the case of the fluid alone). Indeed, in [86] Planas and Sueur have established such a convergence result in the 3D case when the bounded domain Ω containing the fluid-solid system is replaced by the whole space $\mathbb{R}^3$, but the aforementioned result can easily be adapted to the 2D case as well.

We also mention that having a vanishing viscosity convergence is not the only reason for considering Navier conditions instead of the Dirichlet case. In fact, it has been proven in [45, 67, 68] that a so called "no-collision paradox" can occur in the case of fluid-solid models with Dirichlet boundary conditions, i.e. there is no collision possible (in finite time) between the solid and the outer boundary $\partial\Omega$. This is quite unrealistic since it contradicts Archimedes' principle, however, it has been shown in [46] that this paradox can be addressed by using the Navier conditions instead.

One can also find existence results for weak solutions to the 3D variant of system (2.16)-(2.18) in the papers [44] and [86], for the bounded, respectively unbounded case. For the existence of strong solutions in a bounded 2D domain, we cite [97]. Finally, we note that the existence of solutions (both weak and strong, in 2D and 3D) in the case of Dirichlet boundary conditions has been investigated in many papers, we cite for instance [30] for weak solutions, respectively [20, 94] for strong solutions.

The second part of this thesis consists in investigating the controllability of the solid in system (2.16)-(2.18) by the means of an interior control. The results of that analysis will be presented in Section 2.4.2.

2.3.3 Some other fluid-solid models, related results and further references

Apart from the results mentioned for the two models in the previous sections, let us present some other related results in fluid-solid interactions.

A different type of fluid-solid control result is due to Glass and Rosier in [58], regarding the control of the motion of a boat, where the fluid is governed by the two-dimensional Euler equation, but the control is located on the solid's boundary, contrary to the result in Section 2.4.1 where we consider Yudovich-type controls on the outer boundary of fluid domain, similarly to (2.4)-(2.6).

Actually, the result of Section 2.4.1 can rather be seen as an extension to the case of an immersed body of the results [51, 52, 53] by Glass and Horsin, concerning Lagrangian controllability of the incompressible Euler and Stokes equations (i.e. controlling the motion of a set of fluid particles), where the authors also consider external boundary controls of Yudovich type.

Similarly to the case of the fluid alone (as mentioned in Section 2.2.2), for the case of Dirichlet boundary conditions for "viscous fluid + rigid body" control systems, local controllability results have already been obtained in both 2D and 3D, see e.g. the works of Boulakia and Guerrero, Boulakia and Osses, respectively Imanuvilov and Takahashi [10, 11, 72]. As in the case of the fluid alone, these results

rely on Carleman estimates on the linearized equation, and consequently on the parabolic character of the fluid equation. A similar result has been established by Liu, Takahashi and Tucsnak in [78] for the case of the 1D viscous Burgers equation with a new strategy introduced by the authors without the use of any Carleman estimates, and as noted in the aforementioned article, those methods can be extended to other nonlinear parabolic systems. However, note that the results mentioned above concern local null-controllability for the solid position and the velocities of both the solid and the fluid, whereas in Section 2.4.2 we achieve global exact controllability for both the solid position and velocity directly.

Let us also mention some stabilization results by Badra and Takahashi regarding "viscous fluid + rigid body" systems in a bounded domain with Dirichlet boundary conditions, see [6] for the 2D and 3D cases, respectively [7] for a simplified model in the 1D case. In these results the authors stabilize the position and velocity of the solid and also the velocity of the fluid using a feedback control on the exterior boundary of the fluid domain, assuming that the initial data of the system is close to a stationary state, which is not necessarily assumed to be zero.

A different type of problem regarding fluid-solid interactions is that of a deformable body in a fluid, regarding the dynamics of swimming, see for instance the various results [17, 40, 77, 90, 91] for the viscous case, respectively [18] by Chambrion and Munnier for the inviscid case. In the case of such problems, the control is no longer at a distance, rather it consists of the deformation of the body itself. Furthermore, simplified ODE models are deduced and controlled in [2, 3] by Alouges et al. for the case of self-propelled microswimmer robots composed of assemblies of balls linked by arms which are capable of elongating and shrinking, contained in a fluid governed by the Stokes equation. Another interesting study of microswimmers can be found in [1] by Alouges and Giraldi, where the model considered also has a magnetic field acting on swimmers composed of a head and a deformable tail, causing the deformation of their tails which propels them forward in the fluid.

2.4 Main results of the PhD Thesis

In this section we present the main results of the PhD Thesis, regarding the controllability of the fluid-solid models presented in Sections 2.3.1 and 2.3.2.

2.4.1 Control of the motion of a rigid body immersed in an irrotational perfect two-dimensional fluid

We consider system (2.10)-(2.11) as presented in Section 2.3.1, but with Yudovich-type controls (similarly to (2.4)-(2.6) in the case of the fluid alone), i.e. the following boundary conditions :

$$u \cdot n = g \text{ on } \Sigma, \ u \cdot n = 0 \text{ on } \partial\Omega \setminus \Sigma, \quad \text{and} \quad u \cdot n = u_S \cdot n \text{ on } \partial\mathcal{S}(t), \ \text{for } t \in [0,T], \tag{2.19}$$

where $\Sigma \subset \partial\Omega$ is a nonempty, open part of the boundary, and $g \in C_0^\infty([0,T] \times \Sigma)$ with $\int_\Sigma g = 0$.

Since we will work with irrotational initial data, we set the entering vorticity on Σ^- to be zero, i.e.

$$\operatorname{curl} u(t,x) = 0 \text{ on } \Sigma^-, \tag{2.20}$$

where we recall that $\Sigma^- = \{(t,x) \in [0,T] \times \Sigma : \ u \cdot n < 0\}$. This will guarantee that the irrotationality condition (2.14) still holds via Helmholtz's theorem.

We are interested in controlling the solid from a given position and a given velocity to some other prescribed position and velocity via the function $g \in C_0^\infty([0,T] \times \Sigma)$. Note that the reason for which

we do not attempt to control the fluid velocity u at the same time is that there is no hope for such a controllability result to hold, since Kelvin's theorem gives an invariant of the system, regardless of the control, see (2.15).

Furthermore, we want to ensure that the solid trajectory stays away from the boundary $\partial\Omega$, since system (2.10)-(2.11) is no longer valid if there is a collision between the solid and the outer boundary. Therefore we introduce

$$\mathcal{Q} = \{q := (h, \vartheta) \in \Omega \times \mathbb{R} : \ d(h + R(\vartheta)(\mathcal{S}_0 - h_0), \partial\Omega) > 0\}. \tag{2.21}$$

The first main result of the PhD is the following Theorem, which is proved in Chapter 3.

Theorem 2.4.1. *Let $T > 0$. Consider $\mathcal{S}_0 \subset \Omega$ bounded, closed, simply connected with smooth boundary, which is not a disk, and $u_0 \in C^\infty(\overline{\mathcal{F}_0}; \mathbb{R}^2)$, $\gamma \in \mathbb{R}$, $q_0 = (h_0, 0), q_1 = (h_1, \vartheta_1) \in \mathcal{Q}$, $h_0', h_1' \in \mathbb{R}^2, \vartheta_0', \vartheta_1' \in \mathbb{R}$, such that $(h_0, 0)$ and (h_1, ϑ_1) belong to the same connected component of $\mathcal{Q}$ and*

$$\operatorname{div} u_0 = \operatorname{curl} u_0 = 0 \ \textit{in } \mathcal{F}_0, \ u_0 \cdot n = 0 \ \textit{on } \partial\Omega,$$
$$u_0 \cdot n = (h_0' + \vartheta_0'(x - h_0)^\perp) \cdot n \ \textit{on } \partial\mathcal{S}_0, \ \int_{\partial\mathcal{S}_0} u_0 \cdot \tau \, d\sigma = \gamma.$$

Then there exists a control $g \in C_0^\infty((0, T) \times \Sigma)$ and a solution

$$(h, \vartheta, u) \in C^\infty([0, T]; \mathcal{Q}) \times C^\infty([0, T]; C^\infty(\overline{\mathcal{F}(t)}; \mathbb{R}^2))$$

to (2.10), (2.11), (2.13), (2.14), (2.15), (2.19), (2.20) *which satisfies*

$$(h, h', \vartheta, \vartheta')(T) = (h_1, h_1', \vartheta_1, \vartheta_1').$$

Note that there is a slight abuse of notation in writing $C^\infty([0, T]; C^\infty(\overline{\mathcal{F}(t)}; \mathbb{R}^2))$, since the domain in which the fluid evolves is also time-dependent.

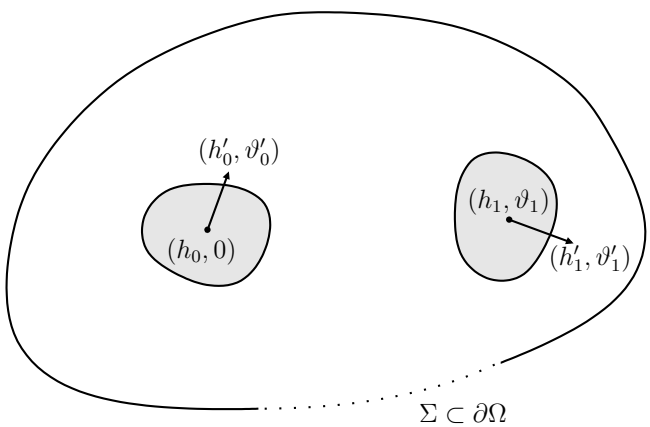

FIGURE 2.4: The initial and final positions and velocities in the control problem

Note that the case when the solid is a disk is a degenerate one, if the solid is a homogeneous disk for example, we have that $(x - h(t))^\perp \cdot n = 0$ for $t \in [0, T]$, $x \in \partial\mathcal{S}(t)$, so the second equation in (2.11) just implies that ϑ is a linear function of time. However, a similar result to Theorem 2.4.1 can be established for h alone in the case of a homogeneous disk, see Chapter 3.

2.4.2 Control of the motion of a rigid body immersed in a viscous incompressible two-dimensional fluid

The second part of the PhD consists in investigating the controllability of the solid in model (2.16)-(2.17). However, for technical reasons (see Remark 2.5.3 later), instead of working in a bounded domain $\Omega \subset \mathbb{R}^2$, we will instead consider the case when the fluid-solid model occupies the whole space $\mathbb{R}^2$, and instead of using boundary controls, we will use an interior control (as mentioned in Section 2.2.2). Therefore, we set $\mathcal{F}(t) := \mathbb{R}^2 \setminus \mathcal{S}(t)$, for any $t \in [0,T]$.

We consider the following model :

$$\begin{aligned}
&\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla\pi - \Delta u = \xi \quad \text{and} \quad \operatorname{div} u = 0 \quad \text{for} \quad t\in[0,T] \text{ and } x \in \mathcal{F}(t),\\
&mh''(t) = -\int_{\partial\mathcal{S}(t)} (-\pi \mathrm{Id} + 2D(u))\, n \; d\sigma,\\
&\mathcal{J}\vartheta''(t) = -\int_{\partial\mathcal{S}(t)} (x-h(t))^{\perp}\cdot(-\pi\mathrm{Id}+2D(u))n\, d\sigma, \text{ for } t\in[0,T],\\
&u\cdot n = u_S\cdot n \text{ and } (D(u)n)_{\tan} = -\mu(u-u_S)_{\tan} \text{ on } \partial\mathcal{S}(t), \quad \lim_{|x|\to+\infty}|u| = 0, \text{ for } t\in[0,T],
\end{aligned} \tag{2.22}$$

where the function $\xi : [0,T]\times\overline{\mathcal{F}(t)} \to \mathbb{R}^2$ is supported in a smooth, compact, simply connected set $\Omega_c \subset \mathbb{R}^2$ with non-empty interior, and will act as the control. Without loss of generality, we may consider the initial data to be

$$u|_{t=0} = u_0 \text{ for } x\in\mathcal{F}_0,\ h(0)=0,\ h'(0)=h'_0,\ \vartheta(0)=0,\ \vartheta'(0)=\vartheta'_0. \tag{2.23}$$

Note that in this case the viscosity coefficient is 1, while the other notations are the same as in Section 2.3.2.

For the model (2.22) to be valid, the solid needs to stay away from the support of the control, i.e. the following has to hold :

$$\operatorname{supp}\xi(t,\cdot)\cap\mathcal{S}(t) = \emptyset,\ \forall t\in[0,T]. \tag{2.24}$$

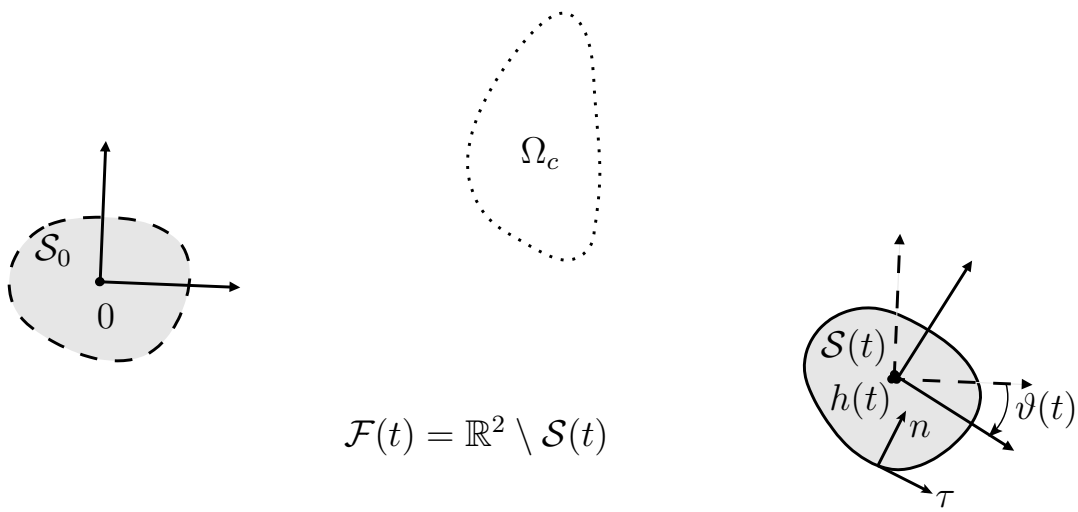

FIGURE 2.5: The setting of the control problem

We will be working in the setting of weak solutions for system (2.22), which for the sake of brevity we do not define here, instead we note that it is done in a similar manner as defining the Leray-type weak solutions for the fluid alone in the case of the Navier-Stokes equations.

Our goal is once again to control the solid from a given position and a given velocity to some other prescribed position and velocity, by the means of prescribing an interior control ξ acting on the fluid. In order to ensure that (2.24) holds, we note that we do not need to control on the whole of Ω_c. Instead, we will introduce a set of admissible positions for the solid, such that as long as the final position of the solid is in this set, there exists a fixed open subset of Ω_c, which does not touch the solid neither in its initial nor in its final position, and which we will use as the support of the controls we construct. Therefore, we introduce

$$\mathcal{Q} = \left\{ q := (h, \vartheta) \in \mathbb{R}^3 : \ \mathrm{int}(\Omega_c) \setminus \{(h + R(\vartheta)\mathcal{S}_0) \cup \mathcal{S}_0\} \neq \emptyset \right\}. \tag{2.25}$$

Since $\mathcal{S}_0$ is simply connected, it can be easily checked that $\mathcal{Q}$ is path-connected.

We state the second main result of the PhD, which is proved in Chapter 4.

Theorem 2.4.2. *Consider $\mathcal{S}_0 \subset \mathbb{R}^2$ bounded, closed, simply connected with smooth boundary, which is not a disk, and $u_0 \in H^4(\mathcal{F}_0; \mathbb{R}^2)$, $\mathrm{curl}\, u_0 \in L^1(\mathcal{F}_0; \mathbb{R}^2)$, $q_0 = 0, q_f = (h_f, \vartheta_f) \in \mathcal{Q}$, $h_0', h_f' \in \mathbb{R}^2, \vartheta_0', \vartheta_f' \in \mathbb{R}$, such that*

$$\mathrm{div}\ u_0 = 0 \ \textit{in}\ \mathcal{F}_0, \ \lim_{|x| \to +\infty} |u_0(x)| = 0,$$

$$u_0 \cdot n = (h_0' + \vartheta_0' x^\perp) \cdot n, \ (D(u_0)n)_{tan} = -\mu(u_0 - (h_0' + \vartheta_0' x^\perp))_{tan} \ \textit{on}\ \partial\mathcal{S}_0.$$

Then there exists $\tilde{T} > 0$, such that for any $T \in (0, \tilde{T}]$, there exists a control $\xi \in L^2((0,T) \times \Omega_c)$, compactly supported in time, and a weak solution (u, h, ϑ) of system (2.22) such that (2.24) holds and we have $(h, h', \vartheta, \vartheta')(T) = (h_f, h_f', \vartheta_f, \vartheta_f')$.

We note that the upper limit $\tilde{T}$ for the time horizons implies that this result is a small time controllability result, which is common in the case of Navier-Stokes type systems, due to the scaling properties of the model, see Chapter 4 for further details as well as a possibility of passing to arbitrary time.

Furthermore, the condition that $\mathcal{S}_0$ is not a disk is essential because a significant step during our proof will rely on a similar result to Theorem 2.4.1 (see Section 2.5.2). However, it is still possible to control the center of mass h alone in the case of a homogeneous disk, using a similar strategy.

2.5 Methodology

In this section we give a quick presentation of the main ideas used to construct the controls to prove Theorems 2.4.1 and 2.4.2 in Chapters 3 and 4.

2.5.1 The construction in the inviscid case : geodesics and impulsive control

An underlying geodesic structure

It has been proven by Glass, Munnier and Sueur in [57] that in the irrotational case with no control, the dynamics of system (2.10)-(2.13) can be recast as an ODE for the degrees of freedom of the solid, namely $(h, \vartheta) \in \Omega \times \mathbb{R}$. To make this more precise, let us denote $q := (h, \vartheta)$, and since the domains $\mathcal{F}(t)$ and $\mathcal{S}(t)$ only depend on time t due to the dependence on $q(t)$, we also denote them by $\mathcal{F}(q)$ and $\mathcal{S}(q)$. Furthermore, recall the definition of the admissible set of positions for q, denoted by $\mathcal{Q}$ given in (2.21).

In order to precise the ODE reformulation mentioned above, we introduce certain objects which only depend on the geometry and the constants in the system. In particular, to "get rid of" the PDE part of the system we solve some elliptic-type problems on a domain depending on the solid position.

— The Kirchhoff potentials

$$\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3)(q, \cdot) \tag{2.26}$$

are defined (up to a constant) as the solution of the Neumann problems

$$\begin{aligned} &\Delta\Phi_i(q,x) = 0 \text{ in } \mathcal{F}(q), \quad \partial_n\Phi_i(q,x) = 0 \text{ on } \partial\Omega, \text{ for } i \in \{1,2,3\}, \\ &\partial_n\Phi_i(q,x) = \begin{cases} n_i \text{ on } \partial\mathcal{S}(q), \text{ for } i \in \{1,2\}, \\ (x-h)^{\perp} \cdot n \text{ on } \partial\mathcal{S}(q), \text{ for } i = 3, \end{cases} \end{aligned} \tag{2.27}$$

where all differential operators are with respect to the variable x.

— The stream function ψ for the circulation term is defined in the following way. First we consider the solution $\tilde{\psi}(q,\cdot)$ of the Dirichlet problem $\Delta\tilde{\psi}(q,x) = 0$ in $\mathcal{F}(q)$, $\tilde{\psi}(q,x) = 0$ on $\partial\Omega$, $\tilde{\psi}(q,x) = 1$ on $\partial\mathcal{S}(q)$. Then we set

$$\psi(q,\cdot) = -\left(\int_{\partial\mathcal{S}(q)} \partial_n\tilde{\psi}(q,x)\, d\sigma\right)^{-1} \tilde{\psi}(q,\cdot), \tag{2.28}$$

such that we have

$$\int_{\partial\mathcal{S}(q)} \partial_n\psi(q,x)\, d\sigma = -1,$$

noting that the strong maximum principle gives us $\partial_n\tilde{\psi}(q,x) < 0$ on $\partial\mathcal{S}(q)$.

— We respectively define the genuine and added mass 3×3 matrices by

$$\mathcal{M}_g = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mathcal{J} \end{pmatrix},$$

and, for $q \in \mathcal{Q}$,

$$\mathcal{M}_a(q) = \left(\int_{\mathcal{F}(q)} \nabla\Phi_i(q,x) \cdot \nabla\Phi_j(q,x)\, dx\right)_{1\leqslant i,j\leqslant 3}.$$

Note that $\mathcal{M}_a$ is a symmetric Gram matrix, which intuitively encodes the amount of incompressible fluid that the rigid body has to accelerate around itself, hence the term "added mass matrix".

— We define the symmetric bilinear map $\Gamma(q)$ given by

$$\langle \Gamma(q), p, p\rangle = \left(\sum_{1\leq i,j\leq 3} \Gamma^k_{i,j}(q)\, p_i\, p_j\right)_{1\leq k\leq 3} \in \mathbb{R}^3, \ \forall p \in \mathbb{R}^3,$$

where, for each $i,j,k \in \{1,2,3\}$, $\Gamma^k_{i,j}$ denotes the Christoffel symbols of the first kind defined on $\mathcal{Q}$ by

$$\Gamma^k_{i,j} = \frac{1}{2}\left(\frac{\partial(\mathcal{M}_a)_{k,j}}{\partial q_i} + \frac{\partial(\mathcal{M}_a)_{k,i}}{\partial q_j} - \frac{\partial(\mathcal{M}_a)_{i,j}}{\partial q_k}\right). \tag{2.29}$$

It can be checked that Γ is of class C^∞ on $\mathcal{Q}$.

— We introduce the following C^∞ vector fields on $\mathcal{Q}$ with values in $\mathbb{R}^3$ by

$$E(q) = -\frac{1}{2}\int_{\partial\mathcal{S}(q)} |\partial_n\psi(q,\cdot)|^2\partial_n\Phi(q,\cdot)\,d\sigma, \quad B(q) = \int_{\partial\mathcal{S}(q)} \partial_n\psi(q,\cdot)\,(\partial_n\Phi(q,\cdot)\times\partial_\tau\Phi(q,\cdot))\,d\sigma. \tag{2.30}$$

Theorem 2.2 from [57] states the following.

Theorem 2.5.1 (Glass-Munnier-Sueur)**.** *For smooth solutions, system* (2.10)-(2.13) *can be recast, up to collision, as the second order ODE*

$$\Big(\mathcal{M}_g + \mathcal{M}_a(q)\Big)q'' + \langle\Gamma(q), q', q'\rangle = \gamma^2 E(q) + \gamma q'\times B(q), \tag{2.31}$$

with $q(0) = q_0$, $q'(0) = q_0'$. *In this case the fluid velocity* u *can be recovered as*

$$u(t,\cdot) = \nabla(q'(t)\cdot\Phi(q(t),\cdot)) + \gamma\nabla^\perp\psi(q(t),\cdot).$$

Note that in the case where $\gamma = 0$, the ODE (2.31) means that the particle q is moving along the geodesics associated with the Riemannian metric induced on $\mathcal{Q}$ by the total inertia matrix field $\mathcal{M}_g + \mathcal{M}_a(\cdot)$, c.f. [81]. Furthermore, if $\gamma \neq 0$, the right-hand side of (2.31) is a force which reminds us of the Lorentz force in electromagnetism by its structure (see [57] for further details). Let us also mention that the whole "inviscid fluid + rigid body" system can be reinterpreted as a geodesic flow on an infinite dimensional manifold, cf. [60]. However the reformulation established in (2.31) relies on the finite dimensional manifold $\mathcal{Q}$ and sheds more light on the dynamics of the rigid body.

Furthermore, we recall from Section 2.4.1 that in the case when $\mathcal{S}_0$ is a disk, equation (2.31) becomes degenerate. If the disk is not homogeneous then the model becomes even more complicated, see Theorem 2.9 from [57].

Also, note that the analysis presented above could be repeated in the case with vorticity, however a purely ODE reformulation as in (2.31) would not be possible in that case, since one would have to still solve an evolution PDE in order to determine the vorticity of the system (which also depends on q), which would in turn have an effect on the solid equation for q.

In [54] we have extended Theorem 2.5.1 to the case with control in the following manner. We set

$$\mathcal{C} := \left\{ g \in C_0^\infty(\Sigma;\mathbb{R}) \ \text{ such that } \ \int_\Sigma g\,d\sigma = 0 \right\},$$

and define for any $q \in \mathcal{Q}$ and $g \in \mathcal{C}$ the unique solution $\overline{\alpha} := \mathcal{A}[q,g] \in C^\infty(\overline{\mathcal{F}(q)};\mathbb{R})$ to the following Neumann problem :

$$\Delta\overline{\alpha} = 0 \text{ in } \mathcal{F}(q) \quad \text{ and } \quad \partial_n\,\overline{\alpha} = g\mathbb{1}_\Sigma \text{ on } \partial\mathcal{F}(q), \tag{2.32}$$

with zero mean.

We have the following result.

Theorem 2.5.2 (Glass-Kolumbán-Sueur)**.** *Given*

$$q \in C^\infty([0,T];\mathcal{Q}), \quad u \in C^\infty([0,T];C^\infty(\overline{\mathcal{F}(q(t))};\mathbb{R}^2)) \quad \textit{and} \quad g \in C_0^\infty([0,T];\mathcal{C}),$$

we have that (q,u) *is a solution to* (2.10), (2.11), (2.13), (2.14), (2.15), (2.19), (2.20) *if and only if the following ODE holds on* $[0,T]$ *:*

$$\begin{aligned}\big(\mathcal{M}_g + \mathcal{M}_a(q)\big)q'' + \langle\Gamma(q), q', q'\rangle = \gamma^2 E(q) + \gamma q'\times B(q)\\ + F_1(q,q',\gamma)[\alpha] + F_2(q)[\partial_t\alpha],\end{aligned} \tag{2.33}$$

where F_1 and F_2 are regular, respectively $\alpha(t,x) := \mathcal{A}[q(t), g(t,\cdot)](x)$. Furthermore, the fluid velocity u can then be recovered as

$$u(t,x) = \nabla(q'(t)\cdot\Phi(q(t),x)) + \gamma\nabla^{\perp}\psi(q(t),x) + \nabla\mathcal{A}[q(t),g(t,\cdot)](x). \tag{2.34}$$

Impulsive control

The main reason for establishing Theorem 2.5.2 is the following. Suppose we have $\gamma = 0$ (if this is not the case, one can at least expect to be close in some sense to the case without circulation when γ is small enough, while for γ large one can exploit the time-rescale properties of equation (2.33)), and suppose that we can find some appropriate control $g \in C_0^\infty([0,T];\mathcal{C})$ such that the term $F_1(q,q',0)[\alpha]+F_2(q)[\partial_t\alpha]$ in (2.33) behaves approximately like $v_0\delta_0(t)+v_1\delta_T(t)$, for any given $v_0, v_1 \in \mathbb{R}^3$, where δ_0 and δ_T denote the Dirac distributions at time $t=0$, respectively $t=T$.

Then, (2.33) is going to be close (in an appropriate sense) to the following formal toy model :

$$\big(\mathcal{M}_g + \mathcal{M}_a(\tilde{q})\big)\tilde{q}'' + \langle\Gamma(\tilde{q}),\tilde{q}',\tilde{q}'\rangle = v_0\delta_0 + v_1\delta_T, \tag{2.35}$$

and controlling (2.33) (at least approximately) reduces to controlling (2.35) by using the vectors $v_0, v_1 \in \mathbb{R}^3$ as our control.

Let us quickly explain how such a controllability can be established. Given $q_0, q_1 \in \mathcal{Q}$, there exists (at least in the case when q_0 and q_1 are sufficiently close, the general case can then be treated via a gluing argument) a geodesic associated with the Riemannian metric induced on $\mathcal{Q}$ by $\mathcal{M}_g + \mathcal{M}_a(\cdot)$, which connects q_0 with q_1. More precisely, there exists a unique smooth function $\bar{q}$ satisfying

$$\big(\mathcal{M}_g + \mathcal{M}_a(\bar{q})\big)\bar{q}'' + \langle\Gamma(\bar{q}),\bar{q}',\bar{q}'\rangle = 0 \text{ on } [0,T], \text{ with } \bar{q}(0) = q_0,\ \bar{q}(T) = q_1. \tag{2.36}$$

So, one can arrive at the desired final position q_1, but a priori the final velocity $\bar{q}'(T)$ differs from q_1', furthermore even the initial velocity $\bar{q}'(0)$ differs from q_0'.

Then, controlling the solution $\tilde{q}$ of (2.35) from (q_0, q_0') to (q_1, q_1') just amounts to setting $v_0 := \big(\mathcal{M}_g + \mathcal{M}_a(q_0)\big)(\bar{q}'(0) - q_0')$ and $v_1 := -\big(\mathcal{M}_g + \mathcal{M}_a(q_1)\big)(\bar{q}'(T) - q_1')$, which transforms the initial and final velocities $\tilde{q}'(0)$ and $\tilde{q}'(T)$ exactly to the desired velocities in order to achieve controllability.

We will prove in Chapter 3 that such controls $g \in C_0^\infty([0,T];\mathcal{C})$ can indeed be constructed. More precisely, we can construct g of the following form :

$$g(t,x) = g_\varepsilon(t,x) := \beta_\varepsilon(t)\overline{g}(q_0,v_0)(x) + \beta_\varepsilon(T-t)\overline{g}(q_1,v_1)(x), \tag{2.37}$$

where $\beta_\varepsilon(t) := \frac{1}{\sqrt{\varepsilon}}\beta\left(\frac{t-\varepsilon}{\varepsilon}\right)$, for ε in $(0,1)$, with $\beta : \mathbb{R} \to \mathbb{R}$ being a smooth, non-negative function supported in $[-1,1]$, such that $\int_{-1}^{1}\beta(t)^2\,dt = 1$, so that $(\beta_\varepsilon^2)_\varepsilon$ is an approximation of the unity when $\varepsilon \to 0^+$. Furthermore, the function $\overline{g}$ is constructed via complex analysis, satisfying certain constraints such that the aforementioned condition

$$F_1(q,q',0)[\alpha] + F_2(q)[\partial_t\alpha] \approx v_0\delta_0(t) + v_1\delta_T(t)$$

holds when $\varepsilon \to 0^+$. In fact, using such controls g_ε, the dominant term in the expression above will be $F_1(q,q',0)[\alpha]$ and will behave like $\int_{\partial\mathcal{S}(q)} |\nabla\alpha|^2\,\partial_n\Phi(q,\cdot)\,d\sigma$, so we will impose an appropriate quadratic constraint on $\overline{g}$ (see Chapter 3 for further details).

Such a strategy is called "impulsive control" due to the large amplitude of the control g_ε over a short support in time (note that supp $g_\varepsilon \subset ([0,2\varepsilon] \cup [T-2\varepsilon, T]) \times \Sigma$), we mention [13] and the references therein for many more examples on impulsive control.

Conclusion

As already mentioned, the construction above allows us to deduce the approximate controllability of (2.33) in the case $\gamma = 0$, due to there being an error in terms of ε coming from the Dirac approximations. One can extend this result to the case of small γ, the error in the approximate controllability will then depend on $\varepsilon > 0$ and $|\gamma|$.

However, since we are controlling the finite dimensional quantity $(q(T), q'(T)) \in \mathbb{R}^6$, one can pass from approximate controllability to exact controllability by using a Brouwer-type topological result such as the following lemma, which was used in [58, pages 32-33] for similar purposes.

Lemma 2.5.1. *Let $w_0 \in \mathbb{R}^n$, $\kappa > 0$, $f : \overline{B}(w_0, \kappa) \to \mathbb{R}^n$ a continuous map such that we have $|f(w) - w| \leq \frac{\kappa}{2}$ for any x in $\partial B(w_0, \kappa)$. Then $B(w_0, \frac{\kappa}{2}) \subset f(\overline{B}(w_0, \kappa))$.*

Of course in this case we also need to ensure an appropriate continuity property of our construction with respect to the target final position and velocity (q_1, q_1').

Therefore, we may conclude the exact controllability of equation (2.33) in the case of γ small enough. The general case can be deduced via a time-rescaling argument similar to the one used by J.-M. Coron for the Euler equation [22], which has been also used in [58] in order to pass from the potential case to the case with vorticity. One obtains that controlling (2.33) with arbitrary circulation $\gamma \in \mathbb{R}$ over the time interval $[0, T]$ reduces to controlling (2.33) with small γ over a shorter time interval $[0, \lambda T]$, with $\lambda \sim \frac{1}{|\gamma|}$. However, it poses no difficulty to get the desired time interval, by using that equation (2.33) enjoys some invariance properties by translation and time-reversal, it is sufficient to glue together a number of appropriate controlled solutions each defined on a time interval of length λT. See Chapter 3 for further details.

Remark 2.5.1. *We mention here that using controls of the form* (2.37) *allows us to further ensure that our solution will have an arbitrarily small total flux through Σ^-. That is, for any $T > 0$, for any $\nu > 0$, there exists a control g and a solution (h, ϑ, u) satisfying the properties of Theorem 2.4.1 and such that moreover*

$$\left| \int_0^T \int_{\Sigma^-} u \cdot n \, d\sigma dt \right| < \nu.$$

Indeed, due to the expression of β_ε one obtains that the total flux through Σ^-, that is $\int_0^T \int_{\Sigma^-} g_\varepsilon \, d\sigma dt$, is of order $\sqrt{\varepsilon}$. This is also left invariant by any time-rescaling or gluing arguments. The significance of such a result is that one can in a sense limit the amount of exchanged fluid during the control phase. We also mention that such a small flux condition cannot be guaranteed in the results [22, 47, 49] regarding the controllability of the Euler equations.

2.5.2 The reduction of the viscous case to the inviscid one via a vanishing viscosity method

In this section we will present a strategy based on an asymptotic expansion which allows us to reduce the proof of Theorem 2.4.2 to controlling certain Euler- and linearized Euler-type systems, which we will do by a similar impulsive control strategy as in Section 2.5.1. In order to achieve such an asymptotic expansion, we will first replace the control on the evolution equation with a control on the divergence of u in (2.22), then transform the domain of the PDE in (2.22) into a cylindrical domain, and introduce a small viscosity parameter.

Lifting the control from the evolution equation and placing it on the divergence

First, let us observe that it is equivalent to control the evolution PDE for u with controlling instead the divergence of u. We recall the definition of the Bogovskii operator (see for instance [9] or [43]).

Definition 2.5.1. *Given a smooth, bounded, simply connected domain $\Omega \subset \mathbb{R}^2$, there exists an operator $B : C_0^\infty(\Omega) \to C_0^\infty(\Omega)^2$ such that, for any $g \in C_0^\infty(\Omega)$ with $\int g = 0$, we have* div $Bg = g$. *Furthermore, $B \in \mathcal{L}(W_0^{s,p}(\Omega), W_0^{s+1,p}(\Omega)^2)$, for any $1 < p < +\infty$, $s \geq 0$. Also observe that we may extend Bg by 0 outside of Ω.*

From here on let B denote the Bogovskii operator associated with the domain Ω_c. Now, suppose we have a solution $(U, \pi, h, \vartheta, g)$ to the system

$$\begin{aligned}
&\frac{\partial U}{\partial t} + (U \cdot \nabla)U + \nabla\pi - \Delta U = 0 \ \text{ and } \ \operatorname{div} u = g \ \text{ for } \ t \in [0,T] \text{ and } x \in \mathcal{F}(t),\\
&U \cdot n = u_S \cdot n, \quad (D(U)n)_{\tan} = -\mu(U - u_S)_{\tan} \ \text{ for } x \in \partial\mathcal{S}(t), \ \lim_{|x|\to+\infty} |U| = 0,\\
&mh''(t) = -\int_{\partial\mathcal{S}(t)} (-\pi \mathrm{Id} + 2D(U))\, n \, d\sigma,\\
&\mathcal{J}\vartheta''(t) = -\int_{\partial\mathcal{S}(t)} (x - h(t))^\perp \cdot (-\pi \mathrm{Id} + 2D(U))n) \, d\sigma, \ \text{ for } t \in [0,T],
\end{aligned} \tag{2.38}$$

with $\int g = 0$ and supp $g \cap \mathcal{S}(t) = \emptyset$, for all $t \in [0,T]$. Setting $u := U - Bg$ and $\xi := -\frac{\partial Bg}{\partial t} + Bg \cdot \nabla Bg - U \cdot \nabla Bg - Bg \cdot \nabla U + \Delta Bg$, we have that $(u, \pi, h, \vartheta, \xi)$ is a solution to (2.22).

The main reason for such a shift in the point of view is that our strategy for constructing a solution to the viscous fluid-solid system will rely on a construction for the inviscid system. In the inviscid case however, similarly to (2.32), (2.34) from Section 2.5.1, one may linearly decompose the effect of a control g on the divergence by considering a function $\overline{\alpha} \in C^\infty(\overline{\mathcal{F}_0}; \mathbb{R})$ which vanishes at infinity and satisfies the following elliptic problem :

$$\Delta\overline{\alpha} = g\mathbb{1}_{R(\vartheta)^T(B_c - h)}(x) \text{ in } \mathcal{F}_0, \ \lim_{|x|\to+\infty} |\nabla\overline{\alpha}| = 0, \ \text{ and } \partial_n \overline{\alpha} = 0 \text{ on } \partial\mathcal{S}_0.$$

Such functions can be easily studied by the means of complex analysis.

A change of variables for passing to a fixed domain with small viscosity

As mentioned previously, our strategy will rely on linking the viscous case to the inviscid one. For this purpose, we would like to introduce a small viscosity parameter (denoted by $\varepsilon > 0$), which we will make go to zero. At the same time, we would like to change the domain of the PDE, $\mathcal{F}(t)$, which depends on the solid position, to a domain which is fixed. Luckily, since the fluid-solid system occupies the whole plane $\mathbb{R}^2$, this can be achieved simply by a rigid movement.

Given $\varepsilon > 0$, one may introduce a change of variables which consists of a rigid movement in the space variable, corresponding to the solid movement, and a time-rescale with respect to ε, such that

(2.38) is equivalent to the following system :

$$\begin{aligned}
&\frac{\partial u^\varepsilon}{\partial t} + (u^\varepsilon - u_S^\varepsilon)\cdot\nabla u^\varepsilon + r^\varepsilon (u^\varepsilon)^\perp + \nabla\pi^\varepsilon - \varepsilon\Delta u^\varepsilon = 0 \quad \text{and} \quad \operatorname{div} u^\varepsilon = g^\varepsilon \quad \text{for } x\in\mathcal{F}_0,\\
&u^\varepsilon\cdot n = u_S^\varepsilon\cdot n, \quad (D(u^\varepsilon)n)_{\tan} = -\mu(u^\varepsilon - u_S^\varepsilon)_{\tan} \quad \text{for } x\in\partial\mathcal{S}_0, \quad \lim_{|x|\to+\infty}|u^\varepsilon| = 0,\\
&m(l^\varepsilon)' = -\int_{\partial\mathcal{S}_0}(-\pi^\varepsilon \mathrm{Id} + 2\varepsilon D(u^\varepsilon))\, n\, d\sigma - m r^\varepsilon (l^\varepsilon)^\perp,\\
&\mathcal{J}\,(r^\varepsilon)' = -\int_{\partial\mathcal{S}_0} x^\perp\cdot(-\pi^\varepsilon\mathrm{Id} + 2\varepsilon D(u^\varepsilon))n\, d\sigma, \quad \text{for } t\in[0,T/\varepsilon],
\end{aligned} \tag{2.39}$$

where $u_S^\varepsilon(t,x) = l^\varepsilon(t) + r^\varepsilon(t)x^\perp$, for $t\in[0,T/\varepsilon]$, $u^\varepsilon(0,\cdot) = \varepsilon u_0(\cdot)$, $(l^\varepsilon, r^\varepsilon)(0) = \varepsilon(h_0', \vartheta_0')$, and the control term is now $g^\varepsilon \in C_0^\infty((0,T/\varepsilon)\times\mathcal{F}_0)$ such that supp $g^\varepsilon(t,\cdot)\subset R(\vartheta^\varepsilon(t))^T(\Omega_c - h^\varepsilon(t))$ and $\int g^\varepsilon(t,\cdot)\,dx = 0$, for any $t\in[0,T/\varepsilon]$. Furthermore, we may associate the solid position $(h^\varepsilon,\vartheta^\varepsilon)$, which no longer plays a role directly in solving system (2.39), through

$$h^\varepsilon(t) = \int_0^t R(\vartheta^\varepsilon(s))l^\varepsilon(s)\,ds, \quad \vartheta^\varepsilon(t) = \int_0^t r^\varepsilon(s)\,ds, \tag{2.40}$$

and in particular we have the following scaling for the solid trajectory with respect to ε,

$$(h^\varepsilon(t),\vartheta^\varepsilon(t)) = (h(\varepsilon t),\vartheta(\varepsilon t)), \quad (l^\varepsilon(t), r^\varepsilon(t)) = \varepsilon(R(\vartheta(\varepsilon t))^T h'(\varepsilon t), \vartheta'(\varepsilon t)), \tag{2.41}$$

for $t\in[0,T/\varepsilon]$.

Notice that solving system (2.22) on a time interval $[0,T]$ then reduces to solving (2.39) on $[0,T/\varepsilon]$. However, given a fixed $T>0$, one may instead look to solve (2.39) on $[0,T]$ and deduce the existence of a solution to (2.22) on $[0,\varepsilon T]$, which would prove Theorem 2.4.2 in time εT, for some $\varepsilon\in(0,1)$. This is exactly what our strategy will rely on.

Now, as $\varepsilon\to 0^+$, one would expect (2.39) to converge to an inviscid fluid-solid system in the whole plane. However, in order to achieve our desired controllability for the solid, we need to further investigate the behaviour of u^ε as an asymptotic expansion up to higher orders of ε.

Asymptotic expansion

We would like to have the following asymptotic expansion for the solid trajectory :

$$\begin{aligned}
h^\varepsilon = h^0 + \varepsilon h^1 + \varepsilon h_R^\varepsilon,\ \vartheta^\varepsilon = \vartheta^0 + \varepsilon\vartheta^1 + \varepsilon\vartheta_R^\varepsilon,\\
l^\varepsilon = l^0 + \varepsilon l^1 + \varepsilon l_R^\varepsilon,\ r^\varepsilon = r^0 + \varepsilon r^1 + \varepsilon r_R^\varepsilon,
\end{aligned} \tag{2.42}$$

with $(l^0,r^0),(l^1,r^1),(l_R^\varepsilon,r_R^\varepsilon)\in L^\infty(0,T)$. At each order of ε we will also have an associated fluid velocity, namely u^0, u^1, u_R^ε, the construction of which we specify later. However, we can determine an equation for each of these terms at order $\mathcal{O}(1)$, $\mathcal{O}(\varepsilon)$, $\mathcal{O}(\varepsilon^+)$ by formally plugging the asymptotic expansion into the equation of $(u^\varepsilon,l^\varepsilon,r^\varepsilon)$ and separating the terms in function of orders of ε.

We then obtain the following.

— at order $\mathcal{O}(1)$, (u^0,l^0,r^0) satisfy a "rigid body + inviscid fluid" system where the fluid is governed by the Euler equation. We would like to drive, by the means of an appropriate control, $(h^0,\vartheta^0)(T)$ to (h_f,ϑ_f) and $(l^0,r^0)(T)$ to 0.

— at order $\mathcal{O}(\varepsilon)$, (u^1,l^1,r^1) satisfy a fluid-solid system which is a linearization of the system at $\mathcal{O}(1)$. We would like to drive, by the means of an appropriate control, $(l^1,r^1)(T)$ to $(R(\vartheta_f)^T h_f', \vartheta_f')$.

— at order $\mathcal{O}(\varepsilon^+)$, $(u_R^\varepsilon, l_R^\varepsilon, r_R^\varepsilon)$ satisfy a remainder equation. We would like to prove that $(l_R^\varepsilon, r_R^\varepsilon)(T) \to 0$ as $\varepsilon \to 0^+$.

If we had the above, taking into account (2.41), it would follow that we have (approximately) driven $(h^\varepsilon, \vartheta^\varepsilon)(T)$ to (h_f, ϑ_f) and $(l^\varepsilon, r^\varepsilon)(T)$ to $\varepsilon(R(\vartheta_f)^T h'_f, \vartheta'_f)$, for $\varepsilon > 0$ small enough.

We may then pass from the above mentioned approximate controllability to exact controllability by the means of a topological argument of Brouwer-type, as in Lemma 2.5.1 from Section 2.5.1. This further requires some continuity property for our whole construction with respect to the target data for the solid trajectory. Finally, recalling the scaling (2.41), we may conclude the desired controllability result $(h, \vartheta, h', \vartheta')(T) = (h_f, \vartheta_f, h'_f, \vartheta'_f)$.

In order to achieve an expansion as in (2.42) for the solid trajectory, we consider controls g^ε in the form of $g^\varepsilon = g^0 + \varepsilon g^1$, in the style of [27], and we look for the following asymptotic expansion for the fluid velocity and pressure :

$$u^\varepsilon = u^0 + \sqrt{\varepsilon}\{v\} + \varepsilon u^1 + \varepsilon u_R^\varepsilon, \quad \pi^\varepsilon = \pi^0 + \varepsilon\{Q\} + \varepsilon\pi^1 + \varepsilon\pi_R^\varepsilon, \tag{2.43}$$

where, for $f = f(t, x, z)$, we denote $\{f\}$ its evaluation at $z = \frac{\varphi(x)}{\sqrt{\varepsilon}}$, with some function φ satisfying $|\varphi(x)| = d(x, \mathcal{S}_0)$ in some neighbourhood of the solid (see Chapter 4 for further details). Hence, proving Theorem 2.4.2 reduces to constructing the terms in the right-hand side of (2.42), (2.43) in an appropriate way. Note that we will use an energy estimate to prove the smallness of $(l_R^\varepsilon, r_R^\varepsilon)$, which is the main reason for investigating not only the terms in the asymptotic expansion for the solid trajectory, but also the terms in the expansion for u^ε, which includes a boundary layer profile v and the associated pressure Q, appearing due to the fact that the solution of the inviscid fluid-solid system does not satisfy the Navier boundary condition, as mentioned in Section 2.2.2.

Therefore, our strategy will be the following :

— We construct g^0 and a smooth solution u^0 to the Euler equation (with control g^0), with zero initial data, hence zero vorticity and zero circulation around the solid, such that we have an exact controllability result for $(h^0, \vartheta^0)(T)$ with $(l^0, r^0)(T) = 0$. However, we note that contrary to [27], our strategy will not rely on a return method for u^0, we will rather just use u^0 to control the solid position $(h^0, \vartheta^0)(T)$, by using an impulsive control strategy similar to the one presented in Section 2.5.1 for the bounded case.

— Due to the Navier slip-with-friction boundary conditions, the fluid velocity boundary layer v will appear near the solid at order $\mathcal{O}(\sqrt{\varepsilon})$, together with its pressure Q at order $\mathcal{O}(\varepsilon)$. Note that there will be no contribution at order $\mathcal{O}(\sqrt{\varepsilon})$ in the solid equations due to the boundary layers, however v will appear in the solid equations for (l^1, r^1). Furthermore, we stress that since we do not control the fluid velocity u^ε, there is no need to control v (contrary to [27]), it will suffice to prove some regularity estimates to handle the effect of the boundary layers at $\mathcal{O}(\varepsilon)$ in the solid equations for (l^1, r^1), and in the equations of the remainder. Note that v and Q only depend on (u^0, l^0, r^0) and their existence is immediate from the existence of $(u^0, \pi^0, l^0, r^0, g^0)$.

— We construct g^1 and a smooth solution u^1 to a linearized Euler equation around u^0 (with control g^1), with initial data derived from the original system (2.22), such that we have an approximate controllability result for $(l^1, r^1)(T)$ (we settle for an approximate controllability here because it simplifies our construction). Such a result can be established by a simpler impulsive control strategy than in the case of the inviscid term u^0, since the equation is linear, and we only want to control the solid velocity $(l^1, r^1)(T)$, not the velocity and the position at the same time.

— We construct $(u_R^\varepsilon, l_R^\varepsilon, r_R^\varepsilon)$ as the solution of a system which we deduce from the equations verified by all the other terms in (2.42), (2.43) (note that at this point we have not yet proven the existence of $(u^\varepsilon, l^\varepsilon, r^\varepsilon)$, but we know that it should be a solution of (2.39) with $g^\varepsilon = g^0 + \varepsilon g^1$). We prove by the means of an energy estimate that $(u_R^\varepsilon, l_R^\varepsilon, r_R^\varepsilon)$ is small in $L^\infty((0,T); L^2(\mathcal{F}_0)\times\mathbb{R}^3)$, when $\varepsilon > 0$ is small.

From the above construction we may define $(u^\varepsilon, l^\varepsilon, r^\varepsilon)$ as the right-hand sides of (2.42), (2.43), since now all the respective terms are constructed and well-defined, and conclude the proof of Theorem 2.4.2. See Chapter 4 for the detailed analysis.

Remark 2.5.2. *Note that during the whole construction we also need to ensure that the "no-collision" condition* (2.24) *holds (it suffices to ensure a similar condition for g^ε), and that $(h^\varepsilon, \vartheta^\varepsilon, l^\varepsilon, r^\varepsilon)(T)$ depends continuously on $(h_f, \vartheta_f, h'_f, \vartheta'_f)$, in order to be able to apply Lemma 2.5.1, as mentioned previously.*

Remark 2.5.3. *Let us now explain why we chose to work in the whole plane $\mathbb{R}^2$ instead of a bounded domain. The key technical difficulty in handling the case of a bounded domain with a similar strategy would be the step of transforming the moving domain $\mathcal{F}(t)$ into a fixed domain. As mentioned above, in the case of the plane this can be done through a simple rigid movement. However, in the bounded case one would also need to account for the outer boundary $\partial\Omega$, and construct a diffeomorphism which is a rigid movement in a neighbourhood of the solid, but leaves the boundary $\partial\Omega$ intact. This diffeomorphism would clearly depend on the solid position, as well as contribute more complicated nonlinear terms in the PDE (see for instance [12] or [61] for such a construction). The main problem then is investigating what happens to these terms when we look for an asymptotic expansion of the form of* (2.42) *for the solid trajectory, moreover separating them in terms of orders of ε. To properly do this, one would need to establish a rigorous asymptotic expansion of the diffeomorphism (and the associated terms in the PDE) with respect to the solid position, which is rather difficult.*

2.6 Future directions and open problems

A natural generalisation of the results of this PhD would be the passage from the two-dimensional case to the three-dimensional one, since often it is is more common and realistic to consider three-dimensional models instead of two-dimensional ones for describing real-world phenomena. The main difficulties in adapting the methods presented in Chapters 3 and 4 to the three-dimensional case are on hand the Cauchy theory for the viscous system in 3D, and on the other hand the use of complex analysis to explicitly construct the parts of the controls depending on the space variable (for instance in (2.37)).

Indeed, in the bounded case for instance (in the same setting as in Section 2.5.1, with $\mathcal{F}(q) = \Omega \setminus \mathcal{S}(q)$), one wishes to construct a function $\overline{\alpha} \in C^\infty(\overline{\mathcal{F}(q)}; \mathbb{R})$ satisfying

$$\Delta\overline{\alpha} = 0 \text{ in } \mathcal{F}(q),\ \partial_n\overline{\alpha} = 0 \text{ on } \partial\mathcal{F}(q) \setminus \Sigma, \text{ and } \int_{\partial\mathcal{S}(q)} |\nabla\overline{\alpha}|^2\, \partial_n\Phi(q,\cdot)\, d\sigma = v,$$

for any given $v \in \mathbb{R}^3$, and fixed $q \in \mathcal{Q}$. In the two-dimensional case this is done by first constructing a function only defined in a neighbourhood of $\partial\mathcal{S}(q)$ and satisfying the integral constraint above with v, by using a Laurent series. Then, one can deduce the existence of an appropriate function defined on the whole of $\overline{\mathcal{F}(q)}$ via Runge's theorem.

However, one direction worth investigating would be replacing these arguments by a Cauchy-Kovalevskaya type construction and a higher dimensional generalization of Runge's theorem (c.f. [38]), so that a similar result can be established in the three-dimensional case.

Furthermore, one could also be interested in controlling several solids. Indeed, the complex analysis arguments mentioned above are quite local around the solid (and then made global via Runge's theorem), so one would imagine that similar methods could be adapted to the case of multiple solids, as long as there is no collision between them. However, it would be also interesting from a modelling point of view to investigate whether it is still possible to use one of the solid trajectories to control the others (without collision of course). Note that the model with multiple solids when the number of solids is quite large could be a good way to model sprays for instance.

Let us also mention the following open problem regarding the motion planning for the rigid body, both in the viscous and inviscid case. Given a curve ξ in $C^2([0,T];\Omega)$, does there exist an appropriate control and a solution to the fluid-solid system which satisfies $\xi = q$ on $[0,T]$? Even the approximate motion planning in C^2, i.e. the same question as above but with $\|\xi - q\|_{C^2([0,T])} \leq \varepsilon$ (with $\varepsilon > 0$ arbitrary) instead of $\xi = q$, is an open problem. These open problems are interesting from the point of view of applications, as it would be useful to be able to make the solid trajectory stay close to a given curve, to perform certain prescribed movement, perhaps avoid certain regions of the fluid domain. By contrast, the methods in Chapter 3 rely on the solid trajectory following a geodesic curve which is defined solely by the geometry of the setting, and not by the construction of the control, which is the main difficulty that would need to be overcome. However, in Chapter 4 we have developed a strategy for passing from approximate controllability of the solid to exact controllability by using an inverse function theorem argument (as opposed to a topological one, as mentioned in Lemma 2.5.1). We hope that such a strategy could be adapted to tackle this problem regarding the motion planning of the solid in the future.

Another interesting direction would be studying fluid-solid interaction models where the fluid is compressible, as this could give rise to applications where the fluid surrounding the solid is in fact a gas. The approximate controllability of the fluid velocity and density for the compressible Euler equations has been estabilshed in [83], while there are some partial results for the Navier-Stokes case (see [32]). However, it could be that controlling the solid trajectory alone might be achieved more easily. The main difference between the compressible and incompressible cases is that the ODE reformulation for the solid trajectory from [54] will no longer apply, since it relies on certain harmonic functions. Nonetheless, it would be worth investigating if the solid equations in the compressible case can still be put into a form where perhaps some sort of impulsive control strategy could be implemented. An intermediary problem could be to investigate the case of incompressible, but variable density fluids, which are often considered for modelling oceans, rivers or fluids in large domains (see for instance [35] for the control of the fluid alone).

A further follow-up to the question of controllability could be that of *stabilization*, as mentioned in the Introduction, that is finding a stationary feedback law $g = g[u,q]$, such that setting this as control, the closed-loop fluid-solid system is globally well-defined and asymptotically stable at an equilibrium point (u_e, q_e, q_e'). Of course, the fluid-solid system can only be stabilized around a point where both the solid and the fluid are at rest. However, there can be some initial vorticity present in the fluid, which

could be eliminated by a strategy similar to the one used by J.-M. Coron in [24], as long as the initial vorticity does not touch the solid boundary.

We also mention that the global exact controllability of a "viscous fluid + rigid body" system with Dirichlet "no slip" boundary conditions is completely open, and a very challenging problem due to the fact that the Dirichlet boundary conditions create boundary layers with a larger amplitude than in the case of Navier slip-with-friction boundary conditions. Note that even the problem of controlling only the fluid velocity in such a context is open and similarly challenging, some recent advances have been made in [29] in the very particular case when the domain is assumed to be a rectangle, using an added distributed phantom force. As already mentioned in Section 2.3.3, we once again contrast these global type of results with the existing local result (as in [10, 11, 72]), which are based on a different strategy and therefore do not require the study of boundary layers.

However, perhaps the less ambitious problem of only controlling the solid velocity, and not the position (which can be used for instance to stop the solid) could be tackled more easily even in the Dirichlet case, using a strategy similar to that of Chapter 4. Indeed, if we were to implement an asymptotic expansion as in (2.42)-(2.43) for the Dirichlet case, one would have boundary layers appearing at $\mathcal{O}(1)$, namely the expansion for the fluid velocity would be of the form

$$u^\varepsilon = u^0 + \{v\} + \varepsilon u^1 + \varepsilon u_R^\varepsilon.$$

However, as explained in the strategy presented in Section 2.5.2, controlling the solid position would correspond to controlling u^0. Since we no longer want to do this, it follows that we may consider $u^0 = 0$ to reduce the effect of v. Of course this implies that the equation for u^1 simplifies into $\partial_t u^1 + \nabla \pi^1 = 0$, but perhaps one may exploit this by using controls which do not vanish at the end of the time interval in order to control the solid equation for p^1.

Furthermore, in the spirit of [29], one could consider the movement of a solid inside a rectangle with Dirichlet boundary conditions, and allow fluid in an out through two parallel sides of the rectangle. It would be worth investigating the possibility of making the solid leave the rectangle and putting the fluid inside the rectangle to rest.

Chapitre 3

External boundary control of the motion of a rigid body immersed in a perfect two-dimensional fluid

3.1 Introduction and main result

3.1.1 The model without control

A simple model of fluid-solid evolution is that of a single rigid body surrounded by a perfect incompressible fluid. Let us describe this system. We consider a two-dimensional bounded, open, smooth and simply connected[1] domain $\Omega \subset \mathbb{R}^2$. The domain Ω is composed of two disjoint parts : the open part $\mathcal{F}(t)$ filled with fluid and the closed part $\mathcal{S}(t)$ representing the solid. These parts depend on time t. Furthermore, we assume that $\mathcal{S}(t)$ is also smooth and simply connected. On the fluid part $\mathcal{F}(t)$, the velocity field $u : [0,T] \times \overline{\mathcal{F}(t)} \to \mathbb{R}^2$ and the pressure field $\pi : [0,T] \times \overline{\mathcal{F}(t)} \to \mathbb{R}$ satisfy the incompressible Euler equation :

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi = 0 \quad \text{and} \quad \text{div } u = 0 \quad \text{for} \quad t \in [0,T] \text{ and } x \in \mathcal{F}(t). \tag{3.1}$$

We consider impermeability boundary conditions, namely, on the solid boundary, the normal velocity coincides with the solid normal velocity

$$u \cdot n = u_S \cdot n \text{ on } \partial\mathcal{S}(t), \tag{3.2}$$

where u_S denotes the solid velocity described below, while on the outer part of the boundary we have

$$u \cdot n = 0 \text{ on } \partial\Omega, \tag{3.3}$$

where n is the unit outward normal vector on $\partial\mathcal{F}(t)$. The solid $\mathcal{S}(t)$ is obtained by a rigid movement from $\mathcal{S}(0)$, and one can describe its position by the center of mass, $h(t)$, and the angle variable with respect to the initial position, $\vartheta(t)$. Consequently, we have

$$\mathcal{S}(t) = h(t) + R(\vartheta(t))(\mathcal{S}_0 - h_0), \tag{3.4}$$

1. The condition of simple connectedness is actually not essential and one could generalize the present result to the case where Ω is merely open and connected at the price of long but straightforward modifications.

where h_0 is the center of mass at initial time, and

$$R(\vartheta) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}.$$

Moreover the solid velocity is hence given by

$$u_S(t,x) = h'(t) + \vartheta'(t)(x - h(t))^{\perp}, \tag{3.5}$$

where for $x = (x_1, x_2)$ we denote $x^{\perp} = (-x_2, x_1)$.

The solid evolves according to Newton's law, and is influenced by the fluid's pressure on the boundary :

$$mh''(t) = \int_{\partial\mathcal{S}(t)} \pi\, n\, d\sigma \quad \text{and} \quad \mathcal{J}\vartheta''(t) = \int_{\partial\mathcal{S}(t)} \pi\,(x - h(t))^{\perp} \cdot n\, d\sigma. \tag{3.6}$$

Here the constants $m > 0$ and $\mathcal{J} > 0$ denote respectively the mass and the moment of inertia of the body, where the fluid is supposed to be homogeneous of density 1, without loss of generality. Furthermore, the circulation around the body is constant in time, that is

$$\int_{\partial\mathcal{S}(t)} u(t) \cdot \tau\, d\sigma = \int_{\partial\mathcal{S}_0} u_0 \cdot \tau\, d\sigma = \gamma \in \mathbb{R},\ \forall t \geq 0, \tag{3.7}$$

due to Kelvin's theorem, where τ denotes the unit counterclockwise tangent vector.

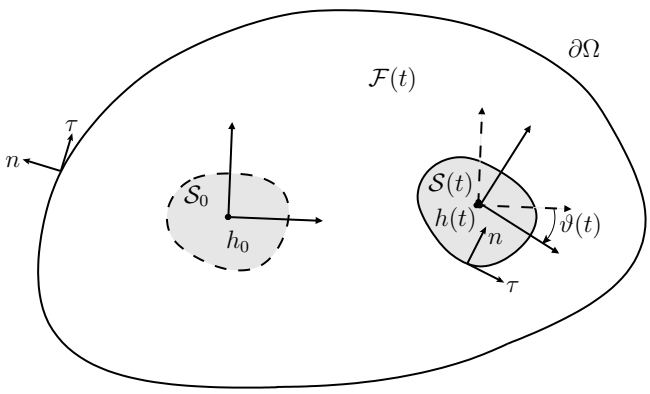

FIGURE 3.1: The domains Ω, $\mathcal{S}(t)$ and $\mathcal{F}(t) = \Omega \setminus \mathcal{S}(t)$

The Cauchy problem for this system with initial data

$$\begin{aligned} u|_{t=0} &= u_0 \text{ for } x \in \mathcal{F}(0), \\ h(0) = h_0,\ h'(0) = h_0',\ \vartheta(0) &= 0,\ \vartheta'(0) = \vartheta_0', \end{aligned} \tag{3.8}$$

is now well-understood, see e.g. [55, 61, 69, 84, 85]. Furthermore, the 3D case has also been studied in [62, 88]. Note in passing that it is our convention used throughout the Chapter that $\vartheta(0) = 0$.

In this Chapter, we will furthermore assume that the fluid is irrotational at the initial time, that is curl $u_0 = 0$ in $\mathcal{F}(0)$, which implies that it stays irrotational at all times, due to Helmholtz's third theorem, i.e.

$$\text{curl } u = 0 \text{ for } x \in \mathcal{F}(t),\ \forall t \geq 0. \tag{3.9}$$

3.1.2 The control problem

We are now in position to state our main result.

Our goal is to investigate the possibility of controlling the solid by means of a boundary control acting on the fluid. Consider Σ a nonempty, open part of the outer boundary $\partial\Omega$. Suppose that one can choose some non-homogeneous boundary conditions on Σ. One natural possibility is due to Yudovich (see [98]), which consists in prescribing on the one hand the normal velocity on Σ, i.e. choosing some function $g \in C_0^\infty([0,T]\times\Sigma)$ with $\int_\Sigma g = 0$ and imposing that

$$u(t,x)\cdot n(x) = g(t,x) \text{ on } [0,T]\times\Sigma, \tag{3.10}$$

while on the rest of the boundary we have the usual impermeability condition

$$u\cdot n = 0 \text{ on } [0,T]\times(\partial\Omega\setminus\Sigma), \tag{3.11}$$

and on the other hand the vorticity on the set Σ^- of points of $[0,T]\times\Sigma$ where the velocity field points inside Ω. Note that Σ^- is deduced immediately from g.

Since we are interested in the vorticity-free case, we will actually consider here a null control in vorticity, that is

$$\operatorname{curl} u(t,x) = 0 \text{ on } \Sigma^- = \{(t,x)\in[0,T]\times\Sigma \text{ such that } u(t,x)\cdot n(x) < 0\}. \tag{3.12}$$

Condition (3.12) enforces the validity of (3.9) as in the uncontrolled setting despite the fact that some fluid is entering the domain.

The general question of this Chapter is how to control the solid's movement by using the above boundary control (that is, the function g). In particular we raise the question of driving the solid from a given position and a given velocity to some other prescribed position and velocity. Remark that we cannot expect to control the fluid velocity in the situation described above : for instance, Kelvin's theorem gives an invariant of the dynamics, regardless of the control.

Throughout this Chapter we will only consider solid trajectories which stay away from the boundary. Therefore we introduce

$$\mathcal{Q} = \{q := (h,\vartheta)\in\Omega\times\mathbb{R} : \ d(h + R(\vartheta)(\mathcal{S}_0 - h_0), \partial\Omega) > 0\}.$$

The main result of this Chapter is the following statement.

Theorem 3.1.1. *Let $T>0$. Consider $\mathcal{S}_0\subset\Omega$ bounded, closed, simply connected with smooth boundary, which is not a disk, and $u_0\in C^\infty(\overline{\mathcal{F}(0)};\mathbb{R}^2)$, $\gamma\in\mathbb{R}$, $q_0=(h_0,0), q_1=(h_1,\vartheta_1)\in\mathcal{Q}$, $h_0', h_1'\in\mathbb{R}^2, \vartheta_0',\vartheta_1'\in\mathbb{R}$, such that $(h_0,0)$ and (h_1,ϑ_1) belong to the same connected component of $\mathcal{Q}$ and*

$$\operatorname{div} u_0 = \operatorname{curl} u_0 = 0 \text{ in } \mathcal{F}(0), \ u_0\cdot n = 0 \text{ on } \partial\Omega,$$
$$u_0\cdot n = (h_0' + \vartheta_0'(x-h_0)^\perp)\cdot n \text{ on } \partial\mathcal{S}_0, \ \int_{\partial\mathcal{S}_0} u_0\cdot\tau\, d\sigma = \gamma.$$

Then there exists a control $g\in C_0^\infty((0,T)\times\Sigma)$ and a solution

$$(h,\vartheta,u)\in C^\infty([0,T];\mathcal{Q})\times C^\infty([0,T];C^\infty(\overline{\mathcal{F}(t)};\mathbb{R}^2))$$

to (3.1), (3.2), (3.6), (3.7), (3.8), (3.9), (3.10), (3.11), *which satisfies $(h,h',\vartheta,\vartheta')(T) = (h_1,h_1',\vartheta_1,\vartheta_1')$.*

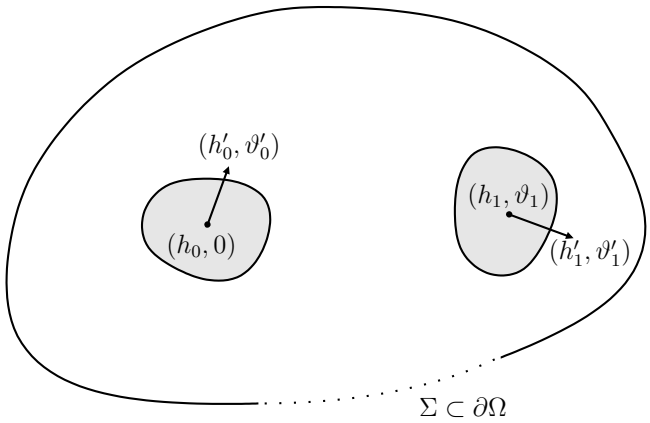

FIGURE 3.2: The initial and final positions and velocities in the control problem

Note that there is a slight abuse of notation in writing $C^\infty([0,T]; C^\infty(\overline{\mathcal{F}(t)}; \mathbb{R}^2))$, since the domain in which the fluid evolves is also time-dependent.

Remark 3.1.1. *In Theorem 3.1.1 the control g can be chosen with an arbitrary small total flux through Σ^-, that is for any $T > 0$, for any $\nu > 0$, there exists a control g and a solution (h, ϑ, u) satisfying the properties of Theorem 3.1.1 and such that moreover*

$$\left| \int_0^T \int_{\Sigma^-} u \cdot n \, d\sigma dt \right| < \nu.$$

See Section 3.5.4 for more explanations. Let us mention that such a small flux condition cannot be guaranteed in the results [22, 47, 49] regarding the controllability of the Euler equations.

When $\mathcal{S}_0$ is a disk, the second equation in (3.6) becomes degenerate, so it needs to be treated separately. For instance, in the case of a homogeneous disk, i.e. when the center of mass coincides with the center of the disk and we have $(x - h(t))^\perp \cdot n = 0$, for any $x \in \partial\mathcal{S}(t),\ t \geq 0$, hence we cannot control ϑ. However, we have a similar result for controlling the center of mass h.

Theorem 3.1.2. *Let $T > 0$. Given a homogeneous disk $\mathcal{S}_0 \subset \Omega$, $u_0 \in C^\infty(\overline{\mathcal{F}(0)}; \mathbb{R}^2)$, $\gamma \in \mathbb{R}$, $h_0, h_1 \in \Omega$, $h_0', h_1' \in \mathbb{R}^2$, such that $(h_0, 0)$ and $(h_1, 0)$ are in the same connected component of $\mathcal{Q}$, and* $\operatorname{div} u_0 = \operatorname{curl} u_0 = 0$ *in $\mathcal{F}(0)$, $u_0 \cdot n = 0$ on $\partial\Omega$, $u_0 \cdot n = h_0' \cdot n$ on $\partial\mathcal{S}_0$, $\int_{\partial\mathcal{S}_0} u_0 \cdot \tau \, d\sigma = \gamma$, there exists $g \in C_0^\infty((0,T) \times \Sigma)$ and a solution (h, u) in $C^\infty([0,T]; \Omega) \times C^\infty([0,T]; C^\infty(\overline{\mathcal{F}(t)}; \mathbb{R}^2))$ of (3.1), (3.2), (3.6), (3.7), (3.9), (3.10), (3.11), (3.12) with initial data (h_0, h_0', u_0), which satisfies $(h, h')(T) = (h_1, h_1')$.*

The proof is similar to that of Theorem 3.1.1, with the added consideration that $(x - h(t))^\perp \cdot n = 0$, for any $x \in \partial\mathcal{S}(t),\ t \geq 0$. We therefore omit the proof. In the case where the disk is non-homogeneous the analysis is technically more intricate already in the uncontrolled setting, see [57], and we will let aside this case in this Chapter.

References. Let us mention a few results of boundary controllability of a fluid alone, that is without any moving body. The problem is then finding a boundary control which steers the fluid velocity from u_0 to some prescribed state u_1. For the incompressible Euler equations small-time global exact boundary controllability has been obtained in [22, 49] in the 2D, respectively 3D case. This result has been recently extended to the case of the incompressible Navier-Stokes equation with Navier slip-with-friction boundary conditions in [27], see also [28] for a gentle exposition. Note that the proof there

relies on the previous results for the Euler equations by means of a rapid and strong control which drives the system in a high Reynolds regime. This strategy was initiated in [23], where an interior controllability result was already established. For "viscous fluid + rigid body" control systems (with Dirichlet boundary conditions), local controllability results have already been obtained in both 2D and 3D, see e.g. [10, 11, 72]. These results rely on Carleman estimates on the linearized equation, and consequently on the parabolic character of the fluid equation.

A different type of fluid-solid control result can be found in [58], where the fluid is governed by the two-dimensional Euler equation. However in this Chapter the control is located on the solid's boundary which makes the situation quite different.

Actually, the results of Theorem 3.1.1 and Theorem 3.1.2 can rather be seen as some extensions to the case of an immersed body of the results [51, 52, 53] concerning Lagrangian controllability of the incompressible Euler and Stokes equations, where the control takes the same form as here.

Generalizations and open problems. First, as we mentioned before, using the techniques of this Chapter, the result could be straightforwardly generalized for non simply connected domains. One could also manage in the same way the control of several solids (the reader may in particular see that the argument using Runge's theorem in Section 3.7 is local around the solid). We would also like to underline that the absence of vorticity is not central here. This may surprise the reader acquainted with the Euler equation, but actually following the arguments of Coron [22, 23], one knows how to control the full model when one can control the irrotational one. This is by the way the technique that we use to take care of the circulation γ (see in particular Section 3.3). But the presence of vorticity makes a lot of complications from the point of view of the initial boundary problem, in particular for what concerns the uniqueness issue, see Yudovich [98]. To avoid these unnecessary technical complications, we restrain ourselves to the irrotational problem. But the full problem could undoubtedly be treated in the same way.

There remain also many open problems. Considering the recent progresses on the controllability in the viscous case, a natural question is whether or not the results in this Chapter could be adapted to the case where a rigid body is moving in a fluid driven by the incompressible Navier-Stokes equation, with Navier slip-with-friction boundary conditions. We hope that the analysis performed here in the case of "inviscid fluid + rigid body" control systems could be used in order to get small-time global controllability results of "viscous fluid + rigid body" control systems.

Let us mention the following open problem regarding the motion planning of a rigid body immersed in an inviscid incompressible irrotational flow.

Open problem 3.1.1. *Let $T > 0$, $(h_0, 0)$ in $\mathcal{Q}$, ξ in $C^2([0,T];\mathcal{Q})$, with $\xi(0) = (h_0, 0)$. Let us decompose $\xi'(0)$ into $\xi'(0) = (h_0', \vartheta_0')$. Consider $\mathcal{S}_0 \subset \Omega$ bounded, closed, simply connected with smooth boundary, which is not a disk, $\gamma \in \mathbb{R}$, and $u_0 \in C^\infty(\overline{\mathcal{F}(0)};\mathbb{R}^2)$ such that* $\operatorname{div} u_0 = \operatorname{curl} u_0 = 0$ *in $\mathcal{F}(0)$, $u_0 \cdot n = 0$ on $\partial\Omega$, $u_0 \cdot n = (h_0' + \vartheta_0'(x - h_0)^\perp) \cdot n$ on $\partial\mathcal{S}_0$ and $\int_{\partial\mathcal{S}_0} u_0 \cdot \tau \, d\sigma = \gamma$. Do there exist $g \in C_0([0,T] \times \Sigma)$ and a solution $(h, \vartheta, u) \in C^2([0,T];\mathcal{Q}) \times C^\infty([0,T]; C^1(\overline{\mathcal{F}(t)};\mathbb{R}^2))$ to (3.1), (3.2), (3.6), (3.7), (3.8), (3.9), (3.10), (3.11), which satisfies $\xi = (h, \vartheta)$?*

Even the approximate motion planning in C^2, i.e. the same statement as above but with $\|\xi - (h,\vartheta)\|_{C^2([0,T])} \leq \varepsilon$ (with $\varepsilon > 0$ arbitrary) instead of $\xi = (h, \vartheta)$, is an open problem.

Plan of the Chapter. The Chapter is organized as follows. In Section 3.2 we first recall from [57] a reformulation of the Newton equations (3.6) as an ODE in the uncontrolled case and then extend it to the case with control. In particular in the case with zero circulation and no control this ODE is the geodesic equation associated with a metric which encodes the added mass phenomenon.

In Section 3.3 we prove that Theorem 3.1.1 can be deduced from a simpler result, namely Theorem 3.3.1, where the solid displacement, the initial and final solid velocities and the circulation are assumed to be small.

In Section 3.4 we prove that another reduction is possible, as we prove that an approximate controllability result (rather than an exact one), namely Theorem 3.4.1, allows to deduce Theorem 3.3.1.

Section 3.5 is devoted to the proof of Theorem 3.4.1 and is the core of the Chapter.

In Section 3.6 we prove a Proposition that is important for Theorem 3.4.1, namely that we can approximate the whole system by a simpler one in a certain regime.

Section 3.7 explains how one can construct the control by means of complex analysis : it can be considered as the cornerstone of our control strategy.

3.2 Reformulation of the solid's equation into an ODE

In this section we establish a reformulation of the Newton equations (3.6) as an ODE for the three degrees of freedom of the rigid body with coefficients obtained by solving some elliptic-type problems on a domain depending on the solid position. Indeed the fluid velocity can be recovered from the solid position and velocity by an elliptic-type problem, so that the fluid state may be seen as solving an auxiliary steady problem, where time only appears as a parameter, instead of the evolution equation (3.1). The Newton equations can therefore be rephrased as a second-order differential equation on the solid position whose coefficients are determined by the auxiliary fluid problem.

Such a reformulation in the case without boundary control was already achieved in [57] and we will start by recalling this case in Section 3.2.1, cf. Proposition 3.2.1 below. A crucial fact in the analysis is that in the ODE reformulation the pre-factor of the body's accelerations is the sum of the inertia of the solid and of the so-called "added inertia" which is a symmetric positive-semidefinite matrix depending only on the body's shape and position, and which encodes the amount of incompressible fluid that the rigid body has also to accelerate around itself. Remarkably enough in the case without control and where the circulation is 0 it turns out that the solid equations can be recast as a geodesic equation associated with the metric given by the total inertia.

Then we will extend this analysis to the case where there is a control on a part of the external boundary in Section 3.2.2, cf. Theorem 3.2.1. In particular we will establish that the remote influence of the external boundary control translates into two additional force terms in the second-order ODE for the solid position ; indeed we will distinguish one force term associated with the control velocity and another one associated with its time derivative.

To simplify notations, we denote the positions and velocities $q = (h, \vartheta)$, $q' = (h', \vartheta')$, and

$$\mathcal{S}(q) = h + R(\vartheta)(\mathcal{S}_0 - h_0) \text{ and } \mathcal{F}(q) = \Omega \setminus \mathcal{S}(q),$$

since the dependence in time of the domain occupied by the solid comes only from the position q. Furthermore, we denote $q(t) = (h(t), \vartheta(t))$.

3.2.1 A reminder of the uncontrolled case

We first recall that in the case without any control the fluid velocity satisfies (3.2), (3.3), (3.7) and (3.9). Therefore at each time t the fluid velocity u satisfies the following div/curl system :

$$\begin{cases} \text{div } u = \text{curl}\, u = 0 \ \text{ in } \mathcal{F}(q), \\ u \cdot n = 0 \text{ on } \partial\Omega \ \text{ and } \ u \cdot n = \left(h' + \vartheta'(x-h)^{\perp}\right) \cdot n \text{ on } \partial\mathcal{S}(q), \\ \displaystyle\int_{\partial\mathcal{S}(q)} u \cdot \tau \, d\sigma = \gamma, \end{cases} \tag{3.13}$$

where the dependence in time is only due to the one of q and q'. Given the solid position q and the right hand sides, the system (3.13) uniquely determines the fluid velocity u in the space of C^∞ vector fields on the closure of $\mathcal{F}(q)$. Moreover thanks to the linearity of the system with respect to its right hand sides, its unique solution u can be uniquely decomposed with respect to the following functions which depend only on the solid position $q = (h, \vartheta)$ in $\mathcal{Q}$ and encode the contributions of elementary right hand sides.

— The Kirchhoff potentials

$$\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3)(q, \cdot) \tag{3.14}$$

are defined as the solution of the Neumann problems

$$\begin{aligned} &\Delta\Phi_i(q,x) = 0 \text{ in } \mathcal{F}(q), \quad \partial_n\Phi_i(q,x) = 0 \text{ on } \partial\Omega, \text{ for } i \in \{1,2,3\}, \\ &\partial_n\Phi_i(q,x) = \begin{cases} n_i \text{ on } \partial\mathcal{S}(q), \text{ for } i \in \{1,2\}, \\ (x-h)^{\perp} \cdot n \text{ on } \partial\mathcal{S}(q), \text{ for } i = 3, \end{cases} \end{aligned} \tag{3.15}$$

where all differential operators are with respect to the variable x.

— The stream function ψ for the circulation term is defined in the following way. First we consider the solution $\tilde{\psi}(q,\cdot)$ of the Dirichlet problem $\Delta\tilde{\psi}(q,x) = 0$ in $\mathcal{F}(q)$, $\tilde{\psi}(q,x) = 0$ on $\partial\Omega$, $\tilde{\psi}(q,x) = 1$ on $\partial\mathcal{S}(q)$. Then we set

$$\psi(q,\cdot) = -\left(\int_{\partial\mathcal{S}(q)} \partial_n\tilde{\psi}(q,x)\, d\sigma\right)^{-1} \tilde{\psi}(q,\cdot), \tag{3.16}$$

such that we have

$$\int_{\partial\mathcal{S}(q)} \partial_n\psi(q,x)\, d\sigma = -1,$$

noting that the strong maximum principle gives us $\partial_n\tilde{\psi}(q,x) < 0$ on $\partial\mathcal{S}(q)$.

Remark 3.2.1. *The Kirchhoff potentials $\mathbf{\Phi}$ and the stream function ψ are C^∞ as functions of q on $\mathcal{Q}$. We will use several times some properties of regularity with respect to the domain of solutions to linear elliptic problems, included for another potential $\mathcal{A}[q,g]$ associated with the control, see Definition 3.2.1 below. We will mention along the proof the properties which will be used and we refer to [18, 66, 79] for more on this material which is now standard in fluid-structure interaction.*

The following statement is an immediate consequence of the definitions above.

Lemma 3.2.1. *For any $q = (h, \vartheta)$ in $\mathcal{Q}$, for any $p = (\ell, \omega)$ in $\mathbb{R}^2 \times \mathbb{R}$ and for any γ, the unique solution u in $C^\infty(\overline{\mathcal{F}(q)})$ to the following system :*

$$\begin{cases} \operatorname{div} u = \operatorname{curl} u = 0 \quad \text{in } \mathcal{F}(q), \\ u \cdot n = 0 \text{ on } \partial\Omega \text{ and } u \cdot n = \left(\ell + \omega(x-h)^\perp\right) \cdot n \text{ on } \partial\mathcal{S}(q), \\ \displaystyle\int_{\partial\mathcal{S}(q)} u \cdot \tau \, d\sigma = \gamma, \end{cases} \tag{3.17}$$

is given by the following formula, for x in $\overline{\mathcal{F}(q)}$,

$$u(x) = \nabla(p \cdot \Phi(q, x)) + \gamma \nabla^\perp \psi(q, x). \tag{3.18}$$

Above $p \cdot \Phi(q, x)$ denotes the inner product $p \cdot \Phi(q, x) = \sum_{i=1}^3 p_i \Phi_i(q, x)$.

Let us now address the solid dynamics. The solid motion is driven by the Newton equations (3.6) where the influence of the fluid on the solid appears through the fluid pressure. The pressure can in turn be related to the fluid velocity thanks to the Euler equations (3.1). The contributions to the solid dynamics of the two terms in the right hand side of the fluid velocity decomposition formula (3.18) are very different. On the one hand the potential part, i.e. the first term in the right hand side of (3.18), contributes as an added inertia matrix, together with a connection term which ensures a geodesic structure (see [81]), whereas on the other hand the contribution of the term due to the circulation, i.e. the second term in the right hand side of (3.18), turns out to be a force which reminds us of the Lorentz force in electromagnetism by its structure (see [57]). We therefore introduce the following notations.

— We respectively define the genuine and added mass 3×3 matrices by

$$\mathcal{M}_g = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mathcal{J} \end{pmatrix},$$

and, for $q \in \mathcal{Q}$,

$$\mathcal{M}_a(q) = \left(\int_{\mathcal{F}(q)} \nabla\Phi_i(q, x) \cdot \nabla\Phi_j(q, x) \, dx \right)_{1 \leqslant i, j \leqslant 3}.$$

Note that $\mathcal{M}_a$ is a symmetric Gram matrix and is C^∞ on $\mathcal{Q}$.

— We define the symmetric bilinear map $\Gamma(q)$ given by

$$\langle \Gamma(q), p, p \rangle = \left(\sum_{1 \leq i, j \leq 3} \Gamma_{i,j}^k(q) \, p_i \, p_j \right)_{1 \leq k \leq 3} \in \mathbb{R}^3, \ \forall p \in \mathbb{R}^3,$$

where, for each $i, j, k \in \{1, 2, 3\}$, $\Gamma_{i,j}^k$ denotes the Christoffel symbols of the first kind defined on $\mathcal{Q}$ by

$$\Gamma_{i,j}^k = \frac{1}{2} \left(\frac{\partial(\mathcal{M}_a)_{k,j}}{\partial q_i} + \frac{\partial(\mathcal{M}_a)_{k,i}}{\partial q_j} - \frac{\partial(\mathcal{M}_a)_{i,j}}{\partial q_k} \right). \tag{3.19}$$

It can be checked that Γ is of class C^∞ on $\mathcal{Q}$.

— We introduce the following C^∞ vector fields on $\mathcal{Q}$ with values in $\mathbb{R}^3$ by

$$E(q) = -\frac{1}{2} \int_{\partial\mathcal{S}(q)} |\partial_n \psi(q, \cdot)|^2 \partial_n \Phi(q, \cdot) \, d\sigma, \ B(q) = \int_{\partial\mathcal{S}(q)} \partial_n \psi(q, \cdot) \left(\partial_n \Phi(q, \cdot) \times \partial_\tau \Phi(q, \cdot) \right) d\sigma. \tag{3.20}$$

We recall that the notation Φ was given in (3.14).

The reformulation of the model as an ODE is given in the following result, which was first established in [81] in the case $\gamma = 0$ and in [57] in the case $\gamma \in \mathbb{R}$.

Theorem 3.2.1. *Given $q = (h, \vartheta) \in C^\infty([0,T]; \mathcal{Q})$, $u \in C^\infty([0,T]; C^\infty(\overline{\mathcal{F}(q(t))}; \mathbb{R}^2))$ we have that (q, u) is a solution to (3.1), (3.2), (3.3), (3.6), (3.7) and (3.9) if and only if q satisfies the following ODE on $[0,T]$*

$$\Big(\mathcal{M}_g + \mathcal{M}_a(q)\Big)q'' + \langle \Gamma(q), q', q' \rangle = \gamma^2 E(q) + \gamma q' \times B(q), \tag{3.21}$$

and u is the unique solution to the system (3.13). Moreover the total kinetic energy $\frac{1}{2}\Big(\mathcal{M}_g + \mathcal{M}_a(q)\Big)q' \cdot q'$ is conserved in time for smooth solutions of (3.21), at least as long as there is no collision.

Note that in the case where $\gamma = 0$, the ODE (3.21) means that the particle q is moving along the geodesics associated with the Riemannian metric induced on $\mathcal{Q}$ by the matrix field $\mathcal{M}_g + \mathcal{M}_a(q)$. Note that, since $\mathcal{Q}$ is a manifold with boundary and the metric $\mathcal{M}_g + \mathcal{M}_a(q)$ may become singular at the boundary of $\mathcal{Q}$, the Hopf-Rinow theorem does not apply and geodesics may not be global. However we will make use only of local geodesics.

Remark 3.2.2. *Let us also mention that the whole "inviscid fluid + rigid body" system can be reinterpreted as a geodesic flow on an infinite dimensional manifold, cf. [60]. However the reformulation established by Theorem 3.2.1 relies on the finite dimensional manifold $\mathcal{Q}$ and sheds more light on the dynamics of the rigid body.*

Below we provide a sketch of the proof of Theorem 3.2.1 ; this will be useful in Section 3.2.2 when extending the analysis to the controlled case.

Proof. Let us focus on the direct part of the proof for sake of clarity but all the subsequent arguments can be arranged in order to ensure the converse part of the statement as well. Using Green's first identity and the properties of the Kirchhoff functions, the Newton equations (3.6) can be rewritten as

$$\mathcal{M}_g\, q'' = \int_{\mathcal{F}(q)} \nabla\pi \cdot \nabla\Phi(q,x)\, dx. \tag{3.22}$$

Moreover when u is irrotational, Equation (3.1) can be rephrased as

$$\nabla\pi = -\partial_t u - \frac{1}{2}\nabla_x |u|^2, \qquad \text{for } x \text{ in } \mathcal{F}(q(t)), \tag{3.23}$$

and Lemma 3.2.1 shows that for any t in $[0,T]$,

$$u(t,\cdot) = \nabla(q'(t) \cdot \Phi(q(t),\cdot)) + \gamma\nabla^\perp\psi(q(t),\cdot). \tag{3.24}$$

Substituting (3.24) into (3.23) and then the resulting decomposition of $\nabla\pi$ into (3.22) we get

$$\begin{aligned} \mathcal{M}_g\, q'' = &-\int_{\mathcal{F}(q)} \left(\partial_t \nabla(q' \cdot \Phi(q,x)) + \frac{\nabla|\nabla(q' \cdot \Phi(q,x))|^2}{2}\right) \cdot \nabla\Phi(q,x)\, dx \\ &-\gamma \int_{\mathcal{F}(q)} \Big(\partial_t \nabla^\perp\psi(q,x) + \nabla\Big(\nabla(q' \cdot \Phi(q,x)) \cdot \nabla^\perp\psi(q,x)\Big)\Big) \cdot \nabla\Phi(q,x)\, dx \\ &-\gamma^2 \int_{\mathcal{F}(q)} \frac{\nabla|\nabla\psi(q,x)|^2}{2} \cdot \nabla\Phi(q,x)\, dx. \end{aligned} \tag{3.25}$$

According to Lemmas 32, 33 and 34 in [57], the terms in the three lines of the right-hand side above are respectively equal to $-\mathcal{M}_a(q)q'' - \langle \Gamma(q), q', q' \rangle$, $\gamma q' \times B(q)$ and $\gamma^2 E(q)$, so that we easily deduce the ODE (3.21) from (3.25).

The conservation of the kinetic energy $\frac{1}{2}\Big(\mathcal{M}_g + \mathcal{M}_a(q)\Big)q' \cdot q'$ is then simply obtained by multiplying the ODE (3.21) by q' and observing that

$$\Big(\Big(\mathcal{M}_g + \mathcal{M}_a(q)\Big)q'' + \langle \Gamma(q), q', q' \rangle\Big) \cdot q' = \Big(\frac{1}{2}\Big(\mathcal{M}_g + \mathcal{M}_a(q)\Big)q' \cdot q'\Big)'. \tag{3.26}$$

□

3.2.2 Extension to the controlled case

We now tackle the case where a control is imposed on the part Σ of the external boundary $\partial\Omega$. At any time this control has to be compatible with the incompressibility of the fluid meaning that the flux through Σ has to be zero. We therefore introduce the set

$$\mathcal{C} := \left\{ g \in C_0^\infty(\Sigma; \mathbb{R}) \ \text{ such that } \ \int_\Sigma g \, d\sigma = 0 \right\}.$$

The decomposition of the fluid velocity u then involves a new potential term involving the following function.

Definition 3.2.1. *With any $q \in \mathcal{Q}$ and $g \in \mathcal{C}$ we associate the unique solution $\overline{\alpha} := \mathcal{A}[q, g] \in C^\infty(\overline{\mathcal{F}(q)}; \mathbb{R})$ to the following Neumann problem :*

$$\Delta \overline{\alpha} = 0 \ \textit{in } \mathcal{F}(q) \quad \textit{and} \quad \partial_n \overline{\alpha} = g \mathbb{1}_\Sigma \ \textit{on } \partial\mathcal{F}(q), \tag{3.27}$$

with zero mean on $\mathcal{F}(q)$.

Let us mention that the zero mean condition above allows to determine a unique solution to the Neumann problem but plays no role in the sequel.

Now Lemma 3.2.1 can be modified as follows.

Lemma 3.2.2. *For any $q = (h, \vartheta)$ in $\mathcal{Q}$, for any $p = (\ell, \omega)$ in $\mathbb{R}^2 \times \mathbb{R}$, for any $\overline{g}$ in $\mathcal{C}$, the unique solution u in $C^\infty(\overline{\mathcal{F}(q)})$ to*

$$\begin{aligned}
&\operatorname{div} u = \operatorname{curl} u = 0 \quad \textit{in } \mathcal{F}(q), \\
&u \cdot n = \mathbb{1}_\Sigma \overline{g} \ \textit{on } \partial\Omega \ \textit{and } u \cdot n = \Big(\ell + \omega(x-h)^\perp\Big) \cdot n \ \textit{on } \partial\mathcal{S}(q), \\
&\int_{\partial\mathcal{S}(q)} u \cdot \tau \, d\sigma = \gamma,
\end{aligned}$$

is given by

$$u(x) = \nabla(p \cdot \Phi(q, x)) + \gamma \nabla^\perp \psi(q, x) + \nabla \mathcal{A}[q, \overline{g}](x), \ \textit{for } x \in \overline{\mathcal{F}(q)}. \tag{3.28}$$

Let us avoid a possible confusion by mentioning that the ∇ operator above has to be considered with respect to the space variable x. The function $\mathcal{A}[q, \overline{g}]$ and its time derivative will respectively be involved into the arguments of the following force terms.

Definition 3.2.2. *We define, for any q in $\mathcal{Q}$, p in $\mathbb{R}^3$, α in $C^\infty(\overline{\mathcal{F}(q)};\mathbb{R})$ and γ in $\mathbb{R}$, $F_1(q,p,\gamma)[\alpha]$ and $F_2(q)[\alpha]$ in $\mathbb{R}^3$ by*

$$F_1(q,p,\gamma)[\alpha] := -\frac{1}{2}\int_{\partial\mathcal{S}(q)} |\nabla\alpha|^2\,\partial_n\Phi(q,\cdot)\,d\sigma \tag{3.29}$$
$$- \int_{\partial\mathcal{S}(q)} \nabla\alpha\cdot\Big(\nabla(p\cdot\Phi(q,\cdot)) + \gamma\nabla^\perp\psi(q,\cdot)\Big)\,\partial_n\Phi(q,\cdot)\,d\sigma,$$
$$F_2(q)[\alpha] := -\int_{\partial\mathcal{S}(q)} \alpha\,\partial_n\Phi(q,\cdot)\,d\sigma. \tag{3.30}$$

Observe that Formulas (3.29) and (3.30) only require α and $\nabla\alpha$ to be defined on $\partial\mathcal{S}(q)$. Moreover when these formulas are applied to $\alpha = \mathcal{A}[q,g]$ for some g in $\mathcal{C}$, then only the trace of α and the tangential derivative $\partial_\tau\alpha$ on $\partial\mathcal{S}(q)$ are involved, since the normal derivative of α vanishes on $\partial\mathcal{S}(q)$ by definition, cf. (3.27).

We define our notion of controlled solution of the "fluid+solid" system as follows.

Definition 3.2.3. *We say that (q,g) in $C^\infty([0,T];\mathcal{Q})\times C_0^\infty([0,T];\mathcal{C})$ is a controlled solution if the following ODE holds true on $[0,T]$:*

$$\big(\mathcal{M}_g + \mathcal{M}_a(q)\big)q'' + \langle\Gamma(q),q',q'\rangle = \gamma^2 E(q) + \gamma q'\times B(q) + F_1(q,q',\gamma)[\alpha] + F_2(q)[\partial_t\alpha], \tag{3.31}$$

where $\alpha(t,\cdot) := \mathcal{A}[q(t),g(t,\cdot)]$.

We have the following result for reformulating the model as an ODE.

Proposition 3.2.1. *Given*

$$q\in C^\infty([0,T];\mathcal{Q}),\quad u\in C^\infty([0,T];C^\infty(\overline{\mathcal{F}(q(t))};\mathbb{R}^2))\quad \text{and}\quad g\in C_0^\infty([0,T];\mathcal{C}),$$

we have that (q,u) is a solution to (3.1), (3.2), (3.6), (3.7), (3.8), (3.9), (3.10), (3.11), (3.12) if and only if (q,g) is a controlled solution and u is the unique solution to the unique div/curl type problem :

$$\begin{aligned}
&\operatorname{div} u = \operatorname{curl} u = 0 \quad \text{in } \mathcal{F}(q),\\
&u\cdot n = \mathbb{1}_\Sigma\, g \text{ on } \partial\Omega \text{ and } u\cdot n = \Big(h' + \vartheta'(x-h)^\perp\Big)\cdot n \text{ on } \partial\mathcal{S}(q),\\
&\int_{\partial\mathcal{S}(q)} u\cdot\tau\,d\sigma = \gamma,
\end{aligned}$$

with $q=(h,\vartheta)$.

Proposition 3.2.1 therefore extends Theorem 3.2.1 to the case with an external boundary control (in particular one recovers Theorem 3.2.1 in the case where g is identically vanishing).

Proof. We proceed as in the proof of Theorem 3.2.1 recalled above, with some modifications due to the extra term involved in the decomposition of the fluid velocity, compare (3.18) and (3.28). In particular some extra terms appear in the right hand side of (3.25) after substituting the right hand side of (3.28) for u in (3.23). Using some integration by parts and the properties of the Kirchhoff functions we obtain integrals on $\partial\mathcal{S}(q)$ whose sum precisely gives $F_1(q,q',\gamma)[\alpha(t,\cdot)] + F_2(q)[\partial_t\alpha(t,\cdot)]$. This allows to conclude. □

3.3 Reduction to the case where the displacement, the velocities and the circulation are small

For $\delta > 0$, we introduce the set

$$Q_\delta = \{q \in \Omega \times \mathbb{R} : \ d(\mathcal{S}(q), \partial\Omega) > \delta\}. \tag{3.32}$$

The goal of this section is to prove that Theorem 3.1.1 can be deduced from the following result. The balls have to be understood for the Euclidean norm (rather than for the metric $\mathcal{M}_g + \mathcal{M}_a(q)$).

Theorem 3.3.1. *Given $\delta > 0$, $\mathcal{S}_0 \subset \Omega$ bounded, closed, simply connected with smooth boundary, which is not a disk, q_0 in $\mathcal{Q}_\delta$ and $T > 0$, there exists $r > 0$ such that for any q_1 in $B(q_0, r)$, for any $\gamma \in \mathbb{R}$ with $|\gamma| \leq r$ and for any $q_0', q_1' \in B(0, r)$, there is a controlled solution (q, g) in $C^\infty([0, T]; \mathcal{Q}_\delta) \times C_0^\infty([0, T] \times \Sigma)$ such that $(q, q')(0) = (q_0, q_0')$ and $(q, q')(T) = (q_1, q_1')$.*

Remark in particular that for $r > 0$ small enough, $B(q_0, r)$ is included in the connected component of $\mathcal{Q}_\delta$ containing q_0.

Proof of Theorem 3.1.1 from Theorem 3.3.1. We proceed in two steps : first we use a time-rescaling argument in order to deduce from Theorem 3.3.1 a more general result covering the case where the initial and final velocities q_0' and q_1' and the circulation γ are large. This argument is reminiscent of a time-rescaling argument used by J.-M. Coron for the Euler equation [22], which has been also used in [58] in order to pass from the potential case to the case with vorticity. Then we use a compactness argument in order to deal with the case where q_0 and q_1 are remote (but of course in the same connected component of $\mathcal{Q}_\delta$).

The time-rescaling argument relies on the following observation : it follows from (3.31) that (q, g) is a controlled solution on $[0, T]$ with circulation γ if and only if (q^λ, g^λ) is a controlled solution on $[0, \lambda T]$ with circulation $\frac{\gamma}{\lambda}$, where (q^λ, g^λ) is defined by

$$q^\lambda(t) := q\left(\frac{t}{\lambda}\right) \text{ and } g^\lambda(t, x) := \frac{1}{\lambda} g\left(\frac{t}{\lambda}, x\right). \tag{3.33}$$

Of course the initial and final conditions

$$(q, q')(0) = (q_0, q_0') \text{ and } (q, q')(T) = (q_1, q_1')$$

translate respectively into

$$(q^\lambda, (q^\lambda)')(0) = \left(q_0, \frac{q_0'}{\lambda}\right) \text{ and } (q^\lambda, (q^\lambda)')(\lambda T) = \left(q_1, \frac{q_1'}{\lambda}\right). \tag{3.34}$$

Now consider q_0 in $\mathcal{Q}_\delta$ and q_1 in $\overline{B}(q_0, r)$ in the same connected component of $\mathcal{Q}_\delta$ as q_0, with $r > 0$ as in Theorem 3.3.1, and q_0', q_1' and γ without size constraint. For λ small enough, $(q_0, \lambda q_0')$, $(q_1, \lambda q_1')$ and $\lambda\gamma$ satisfy the assumptions of Theorem 3.3.1. Hence there exists a controlled solution (q, g) on $[0, T]$, achieving $(q, q')(0) = (q_0, \lambda q_0')$ and $(q, q')(T) = (q_1, \lambda q_1')$. On the other hand, the corresponding trajectory q^λ constructed above will satisfy the conclusions of Theorem 3.1.1 on $[0, \lambda T]$, in particular that $(q^\lambda, (q^\lambda)')(0) = (q_0, q_0')$ and $(q^\lambda, (q^\lambda)')(\lambda T) = (q_1, q_1')$. Moreover we can assume that it is the case without loss of generality that λ is small, and in particular that $\lambda \leq 1$. Thus the result is obtained but in a shorter time interval.

To get to the desired time interval, using that Equation (3.31) enjoys some invariance properties by translation and time-reversal (up to the change of the sign of γ) it is sufficient to glue together an odd number, say $2N+1$ with N in $\mathbb{N}^*$, of appropriate controlled solutions each defined on a time interval of length λT with $\lambda = \frac{1}{2N+1}$, going back and forth between (q_0, q_0') and (q_1, q_1') until time $T = (2N+1)\lambda T$. Moreover one can see that the gluings are not only C^2 but even C^∞.

We have therefore already proven that Theorem 3.1.1 is true in the case where q_1 is close to q_0, or more precisely for any q_0 in $\mathcal{Q}_\delta$ and q_1 in $\overline{B}(q_0, r_{q_0})$.

For the general case where q_0 and q_1 are in the same connected component of $\mathcal{Q}_\delta$ for some $\delta > 0$, without the closeness condition, we use again a gluing process. Consider indeed a smooth curve from q_0 to q_1. For each point q on this curve, there is a $r_q > 0$ such that for any $\tilde{q}$ in $B(q, r_q)$, any q_0', q_1' and any γ, one can connect (q, q_0') to $(\tilde{q}, q_1')$ by a solution of the system, for any time $T > 0$. Extract a finite subcover of the curve by the balls $B(q, r_q)$. Therefore we find $N \geq 2$ and $(q_{\frac{i}{N}})_{i=1,\dots,N-1}$ in the same connected component of $\mathcal{Q}_\delta$ as q_0 such that for any $i = 1, \dots, N$, $q_{\frac{i}{N}}$ is in $\overline{B}(q_{\frac{i-1}{N}}, r_{q_{\frac{i-1}{N}}})$ (note that this includes q_0 and q_1). Therefore, using again the local result obtained above, there exist some controlled solutions from $(q_{\frac{i-1}{N}}, 0)$ to $(q_{\frac{i}{N}}, 0)$ (for $i = 1$ and $i = N$ we use $(q_0, \frac{q_0'}{N})$ and $(q_1, \frac{q_1'}{N})$ rather than $(q_0, 0)$ and $(q_1, 0)$), each on a time interval of length T associated with circulation $\frac{\gamma}{N}$. One deduces by time-rescaling some controlled solutions associated with circulation γ on a time interval of length $\frac{T}{N}$. Gluing them together leads to the desired controlled solution. □

3.4 Reduction to an approximate controllability result

The goal of this section is to prove that Theorem 3.3.1 can be deduced from the following approximate controllability result thanks to a topological argument already used in [58], see Lemma 3.4.1 below. Let us mention that a similar argument has also been used for control purposes but in other contexts, see e.g. [5, 15, 63, 64].

Theorem 3.4.1. *Given $\delta > 0$, $\mathcal{S}_0 \subset \Omega$ bounded, closed, simply connected with smooth boundary, which is not a disk, q_0 in $\mathcal{Q}_\delta$ and $T > 0$, there is $\tilde{r} > 0$ such that $B(q_0, \tilde{r})$ is included in the same connected component of $\mathcal{Q}_\delta$ as q_0, and furthermore, for any $\eta > 0$, there exists $r' = r'(\eta) > 0$ such that for any $\gamma \in \mathbb{R}$ with $|\gamma| \leq r'$ and for any q_0' in $\overline{B}(0, \tilde{r})$, there is a mapping*

$$\mathcal{T} : \overline{B}\big((q_0, q_0'), \tilde{r}\big) \to C^\infty([0, T]; \mathcal{Q}_\delta)$$

which to (q_1, q_1') associates q where (q, g) is a controlled solution associated with the initial data (q_0, q_0'), such that the mapping

$$(q_1, q_1') \in \overline{B}\big((q_0, q_0'), \tilde{r}\big) \mapsto \big(\mathcal{T}(q_1, q_1'), \mathcal{T}(q_1, q_1')'\big)(T) \in \mathcal{Q}_\delta \times \mathbb{R}^3$$

is continuous and such that for any (q_1, q_1') in $\overline{B}\big((q_0, q_0'), \tilde{r}\big)$,

$$\|\big(\mathcal{T}(q_1, q_1'), \mathcal{T}(q_1, q_1')'\big)(T) - (q_1, q_1')\| \leqslant \eta.$$

The proof of Theorem 3.4.1 will be given in Section 3.5. Here we prove that Theorem 3.3.1 follows from Theorem 3.4.1.

Proof of Theorem 3.3.1 from Theorem 3.4.1. Let $\delta > 0$, $\mathcal{S}_0 \subset \Omega$ bounded, closed, simply connected with smooth boundary, which is not a disk, q_0 in $\mathcal{Q}_\delta$ and $T > 0$. Let $\tilde{r} > 0$ as in Theorem 3.4.1 and

$\eta = \frac{\tilde{r}}{2}$. We deduce that for any $\gamma \in \mathbb{R}$ with $|\gamma| \leq r' = r'(\frac{\tilde{r}}{2})$ and q_0' in $\overline{B}(0, \tilde{r})$, there is a mapping $\mathcal{T} : \overline{B}\big((q_0, q_0'), \tilde{r}\big) \to C^\infty([0,T]; \mathcal{Q}_\delta)$ which maps (q_1, q_1') to q where (q, g) is a controlled solution associated with the initial data (q_0, q_0'), such that for any (q_1, q_1') in $\overline{B}\big((q_0, q_0'), \tilde{r}\big)$, $\|\big(\mathcal{T}(q_1, q_1'), \mathcal{T}(q_1, q_1')'\big)(T) - (q_1, q_1')\| \leqslant \frac{\tilde{r}}{2}$. We define a mapping f from $\overline{B}\big((q_0, q_0'), \tilde{r}\big)$ to $\mathbb{R}^6$ which maps (q_1, q_1') to $f(q_1, q_1') := \big(\mathcal{T}(q_1, q_1'), \mathcal{T}(q_1, q_1')'\big)(T)$. Then we apply the following lemma borrowed from [58, pages 32-33], to $w_0 = (q_0, q_0')$ and $\kappa = \tilde{r}$.

Lemma 3.4.1. *Let $w_0 \in \mathbb{R}^n$, $\kappa > 0$, $f : \overline{B}(w_0, \kappa) \to \mathbb{R}^n$ a continuous map such that we have $|f(w) - w| \leq \frac{\kappa}{2}$ for any x in $\partial B(w_0, \kappa)$. Then $B(w_0, \frac{\kappa}{2}) \subset f(\overline{B}(w_0, \kappa))$.*

This allows to conclude the proof of Theorem 3.3.1 by setting $r = \min\Big\{\frac{\tilde{r}}{2\sqrt{5}}, r'(\frac{\tilde{r}}{2})\Big\}$, since the conditions $q_1 \in B(q_0, r)$, $|\gamma| \leq r$ and $q_0', q_1' \in B(0, r)$ imply $|\gamma| \leq r'(\frac{\tilde{r}}{2})$ and $(q_1, q_1') \in B((q_0, q_0'), \frac{\tilde{r}}{2})$. $\square$

3.5 Proof of the approximate controllability result

In this section we prove Theorem 3.4.1 by exploiting the geodesic feature of the uncontrolled system with zero circulation, cf. the observation below Theorem 3.2.1. To do so, we will use some well-chosen impulsive controls which allow to modify the velocity q' in a short time interval and put the state of the system on a prescribed geodesic (and use that $|\gamma|$ is small). We mention here [13] and the references therein for many more examples on the impulsive control strategy.

3.5.1 First step

We consider $\mathcal{S}_0 \subset \Omega$ as before and consider $\delta > 0$ so that $q_0 \in \mathcal{Q}_\delta$. We let $r_1 > 0$ be small enough so that $B(q_0, r_1) \subset \mathcal{Q}_\delta$. We also let $T > 0$.

The first step consists in considering the geodesics associated to the uncontrolled, potential case ($\gamma = 0$). The following classical result regarding the existence of geodesics can be found for instance in [80, Section 7.5], see also [39] for the continuity feature.

Lemma 3.5.1. *There exists r_2 in $(0, \frac{1}{2}r_1)$ such that for any q_1 in $\overline{B}(q_0, r_2)$ there exists a unique C^∞ solution $\bar{q}(t)$ lying in $B(q_0, \frac{1}{2}r_1)$ to*

$$\big(\mathcal{M}_g + \mathcal{M}_a(\bar{q})\big)\bar{q}'' + \langle \Gamma(\bar{q}), \bar{q}', \bar{q}' \rangle = 0 \text{ on } [0,T], \text{ with } \bar{q}(0) = q_0, \ \bar{q}(T) = q_1. \tag{3.35}$$

Furthermore the map $q_1 \in \overline{B}(q_0, r_2) \mapsto (c_0, c_1) \in \mathbb{R}^6$ given by $c_0 = \bar{q}'(0)$, $c_1 = \bar{q}'(T)$ is continuous.

Let us fix r_2 as in the lemma before. Let q_0' in $\overline{B}(0, r_2)$ and (q_1, q_1') in $\overline{B}\big((q_0, q_0'), r_2\big)$.

Our goal is to make the system follow approximately such a geodesic $\bar{q}$ which we consider fixed during this section. For the geodesic equation in (3.35), q_0 and q_1 determine the initial and final velocities (which of course differ in general from q_0' and q_1'). But we will see that is possible to use the penultimate term of (3.31) in order to modify the initial and final velocities of the system. Precisely, the control will be used so that the right hand side of (3.31) behaves like two Dirac masses at time close to 0 and T, driving the velocity q' from the initial and final velocities to the ones of the geodesic in two short time intervals close to 0 and T.

3.5.2 Illustration of the method on a toy model

Let us illustrate this strategy on a toy model. We will later on adapt the analysis to the complete model, cf. Proposition 3.6.1.

Let $\beta : \mathbb{R} \to \mathbb{R}$ be a smooth, non-negative function supported in $[-1, 1]$, such that $\int_{-1}^{1} \beta(t)^2\, dt = 1$ and, for ε in $(0, 1)$, $\beta_\varepsilon(t) := \frac{1}{\sqrt{\varepsilon}}\beta\left(\frac{t-\varepsilon}{\varepsilon}\right)$, so that [2] $(\beta_\varepsilon^2)_\varepsilon$ is an approximation of the unity when $\varepsilon \to 0^+$.

For a function f defined on $[0, T]$, we will denote

$$\|f\|_{T,\varepsilon} := \|f\|_{C^0([0,T])} + \|f\|_{C^1([2\varepsilon,T-2\varepsilon])}. \tag{3.36}$$

Lemma 3.5.2. *Let q_0, r_2, q_1, q_0' and q_1' as above. Let*

$$v_0 := \big(\mathcal{M}_g + \mathcal{M}_a(q_0)\big)(c_0(q_1) - q_0') \text{ and } v_1 := -\big(\mathcal{M}_g + \mathcal{M}_a(q_1)\big)(c_1(q_1) - q_1'). \tag{3.37}$$

Let, for ε in $(0, 1)$, q_ε the maximal solution to the following Cauchy problem :

$$\big(\mathcal{M}_g + \mathcal{M}_a(q_\varepsilon)\big)q_\varepsilon'' + \langle \Gamma(q_\varepsilon), q_\varepsilon', q_\varepsilon'\rangle = \beta_\varepsilon^2(\cdot)\, v_0 + \beta_\varepsilon^2(T - \cdot)v_1, \tag{3.38}$$

with $q_\varepsilon(0) = q_0$ and $q_\varepsilon'(0) = q_0'$. Then for ε small enough, $q_\varepsilon(t)$ lies in $B(q_0, r_1)$ for t in $[0, T]$ and, as $\varepsilon \to 0^+$, $\|q_\varepsilon - \bar{q}\|_{T,\varepsilon} \to 0$ and $(q_\varepsilon, q_\varepsilon')(T) \to (q_1, q_1')$.

Proof. For ε in $(0, 1)$, let us denote $T_\varepsilon = \sup\{\hat{T} > 0$ such that $q_\varepsilon(t) \in B(q_0, r_1)$ for $t \in (0, \hat{T})\}$. Let us first prove that there exists $\tilde{T} > 0$ such that for any ε in $(0, 1)$, $T_\varepsilon \geq \tilde{T}$. Using the identity (3.26), we obtain indeed, for any ε in $(0, 1)$, for any $t \in (0, T_\varepsilon)$,

$$\Big(\mathcal{M}_g + \mathcal{M}_a(q_\varepsilon(t))\Big)q_\varepsilon'(t) \cdot q_\varepsilon'(t) = \Big(\mathcal{M}_g + \mathcal{M}_a(q_0)\Big)q_0' \cdot q_0' + 2\int_0^t \big(\beta_\varepsilon^2(\cdot)\, v_0 + \beta_\varepsilon^2(T - \cdot)v_1\big) \cdot q_\varepsilon',$$

Moreover, relying on Remark 3.2.1, we see that there exists $c > 0$ (which depends on δ) such that for any q in $\mathcal{Q}_\delta$, for any p in $\mathbb{R}^3$,

$$c|p|^2 \leq \Big(\mathcal{M}_g + \mathcal{M}_a(q)\Big)p \cdot p \leq c^{-1}|p|^2. \tag{3.39}$$

Therefore using Gronwall's lemma we obtain that there exists $C > 0$ such that for any ε in $(0, 1)$, for any $t \in (0, T_\varepsilon)$, $\sup_{t\in(0,T_\varepsilon)} \|q_\varepsilon'(t)\| \leq C$. Therefore by the mean value theorem for $\tilde{T} := r_1/2C$, one has for any ε in $(0, 1)$, $T_\varepsilon \geq \tilde{T}$.

We now prove in the same time that for $\varepsilon > 0$ small enough, $T_\varepsilon \geq T$, and the convergence results stated in Lemma 3.5.2. In order to exploit the supports of the functions $\beta_\varepsilon(\cdot)$ and $\beta_\varepsilon(T - \cdot)$ in the right hand side of the equation (3.38) we compare the dynamics of q_ε and $\bar{q}$ during the three time intervals $[0, 2\varepsilon]$, $[2\varepsilon, T - 2\varepsilon]$ and $[T - 2\varepsilon, T]$.

For $\varepsilon_1 := \tilde{T}/2$ and ε in $(0, \varepsilon_1)$, one already has that $T_\varepsilon \geq 2\varepsilon$ and we can therefore simply compare the dynamics of q_ε and $\bar{q}$ on the first interval $[0, 2\varepsilon]$. Indeed using again the mean value theorem we obtain that $\sup_{t\in[0,2\varepsilon]} |q_\varepsilon - q_0|$ converges to 0 as ε goes to 0. Moreover integrating the equation (3.38) on $[0, 2\varepsilon]$ and taking into account the choice of v_0 in (3.37), we obtain

$$\begin{aligned}\Big(\mathcal{M}_g + \mathcal{M}_a(q_\varepsilon(2\varepsilon))\Big)q_\varepsilon'(2\varepsilon) =& \Big(\mathcal{M}_g + \mathcal{M}_a(q_0)\Big)c_0(q_1) \\ &- \int_0^{2\varepsilon} \Big(D\mathcal{M}_a(q_\varepsilon) \cdot q_\varepsilon'\Big) \cdot q_\varepsilon'\, dt - \int_0^{2\varepsilon} \langle \Gamma(q_\varepsilon), q_\varepsilon', q_\varepsilon'\rangle\, dt.\end{aligned} \tag{3.40}$$

2. In the next lemma we are going to make use only of the square function β_ε^2 but we will also have to deal with the function β_ε itself in the sequel, see below Proposition 3.5.1.

Now, there exists $C > 0$ such that for any q in $\mathcal{Q}_\delta$, for any p in $\mathbb{R}^3$,

$$|\big(D\mathcal{M}_a(q)\cdot p\big)\cdot p| + |\langle \Gamma(q), p, p\rangle| \leq C|p|^2. \tag{3.41}$$

Combining this and the bound on q'_ε we see that the two terms of the last line of (3.40) above converge to 0 as ε goes to 0. Since $q \mapsto \mathcal{M}_a(q)$ is continuous on $\mathcal{Q}_\delta$ and $q_\varepsilon(2\varepsilon)$ converges to q_0 as $\varepsilon \to 0$, the matrix $\mathcal{M}_a(q_\varepsilon)$ converges to $\mathcal{M}_a(q_0)$ as $\varepsilon \to 0$. Therefore, using that the matrix $\mathcal{M}_g + \mathcal{M}_a(q_0)$ is invertible we deduce that $q'_\varepsilon(2\varepsilon)$ converges to $c_0(q_1)$ as ε goes to 0.

During the time interval $[2\varepsilon, T-2\varepsilon]$, the right hand side of the equation (3.38) vanishes and the equation therefore reduces to the geodesic equation in (3.35). Since this equation is invariant by translation in time, one may use the following elementary result on the continuous dependence on the data, with a time shift of 2ε.

Lemma 3.5.3. *There exists $\eta > 0$ such that for any $(\tilde{q}_0, \tilde{q}'_0)$ in $B((q_0, c_0(q_1)), \eta)$ there exists a unique C^∞ solution $\tilde{q}(t)$ lying in $B(q_0, r_1)$ to $\big(\mathcal{M}_g + \mathcal{M}_a(\tilde{q})\big)\tilde{q}'' + \langle \Gamma(\tilde{q}), \tilde{q}', \tilde{q}'\rangle = 0$ on $[0,T]$, with $\tilde{q}(0) = \tilde{q}_0$, $\tilde{q}'(0) = \tilde{q}'_0$. Furthermore $\|\tilde{q} - \overline{q}\|_{C^1([0,T])} \to 0$ as $(\tilde{q}_0, \tilde{q}'_0) \to (q_0, c_0(q_1))$.*

Since $q_\varepsilon(2\varepsilon)$ and $q'_\varepsilon(2\varepsilon)$ respectively converge to q_0 and $c_0(q_1)$, according to Lemma 3.5.3 there exists ε_2 in $(0, \varepsilon_1)$ such that for ε in $(0, \varepsilon_2)$, there exists a unique C^∞ solution $\tilde{q}_\varepsilon(t)$ lying in $B(q_0, r_1)$ to $\big(\mathcal{M}_g + \mathcal{M}_a(\tilde{q}_\varepsilon)\big)\tilde{q}''_\varepsilon + \langle \Gamma(\tilde{q}_\varepsilon), \tilde{q}'_\varepsilon, \tilde{q}'_\varepsilon\rangle = 0$ on $[0,T]$, with $\tilde{q}_\varepsilon(0) = q_\varepsilon(2\varepsilon)$, $\tilde{q}'_\varepsilon(0) = q'_\varepsilon(2\varepsilon)$ and $\|\tilde{q}_\varepsilon - \overline{q}\|_{C^1([0,T])} \to 0$ as $\varepsilon \to 0$.

Since the function defined by $\hat{q}_\varepsilon(t) = q_\varepsilon(t + 2\varepsilon)$ also satisfies $\big(\mathcal{M}_g + \mathcal{M}_a(\hat{q}_\varepsilon)\big)\hat{q}''_\varepsilon + \langle \Gamma(\hat{q}_\varepsilon), \hat{q}'_\varepsilon, \hat{q}'_\varepsilon\rangle = 0$ on $[0, T-4\varepsilon]$, with $\hat{q}_\varepsilon(0) = q_\varepsilon(2\varepsilon)$, $\hat{q}'_\varepsilon(0) = q'_\varepsilon(2\varepsilon)$, by the uniqueness part in the Cauchy-Lipschitz theorem one has that $T_\varepsilon \geq T - 2\varepsilon$ and $\hat{q}_\varepsilon$ and $\tilde{q}_\varepsilon$ coincide on $[0, T-4\varepsilon]$, so that, shifting back in time, $\|q_\varepsilon - \overline{q}(\cdot - 2\varepsilon)\|_{C^1([2\varepsilon, T-2\varepsilon])} \to 0$ as $\varepsilon \to 0$. Since $\overline{q}$ is smooth, this entails that $\|q_\varepsilon - \overline{q}\|_{C^1([2\varepsilon, T-2\varepsilon])} \to 0$ as $\varepsilon \to 0$.

Finally one deals with the time interval $[T-2\varepsilon, T]$ in the same way as the first step. In particular, reducing ε one more time if necessary one obtains, by an energy estimate, a Gronwall estimate and the mean value theorem, that $T_\varepsilon \geq T$. Moreover the choice of the vector v_1 in (3.37) allows to reorient the velocity q'_ε from $c_1(q_1)$ to q'_1 whereas the position is not much changed (due to the uniform bound of q'_ε and the mean value theorem) so that the value of q_ε at time T converges to q_1 as ε goes to 0. □

3.5.3 Back to the complete model

Now in order to mimic the right hand side of (3.38) we are going to use one part of the force term F_1 introduced in Definition 3.2.2. Let us therefore introduce some notations for the different contributions of the force term F_1. We define, for any q in $\mathcal{Q}$, p in $\mathbb{R}^3$, α in $C^\infty(\overline{\mathcal{F}(q)}; \mathbb{R})$,

$$F_{1,a}(q)[\alpha] := -\frac{1}{2}\int_{\partial\mathcal{S}(q)} |\nabla\alpha|^2\, \partial_n\Phi(q,\cdot)\, d\sigma, \tag{3.42}$$

$$F_{1,b}(q,p)[\alpha] := -\int_{\partial\mathcal{S}(q)} \nabla\alpha\cdot\nabla(p\cdot\Phi(q,\cdot))\, \partial_n\Phi(q,\cdot)\, d\sigma, \tag{3.43}$$

$$F_{1,c}(q)[\alpha] := -\int_{\partial\mathcal{S}(q)} \nabla\alpha\cdot\nabla^\perp\psi(q,\cdot)\, \partial_n\Phi(q,\cdot)\, d\sigma, \tag{3.44}$$

so that for any γ in $\mathbb{R}$,

$$F_1(q,p,\gamma)[\alpha] = F_{1,a}(q)[\alpha] + F_{1,b}(q,p)[\alpha] + \gamma F_{1,c}(q)[\alpha].$$

The part which will allow us to approximate the right hand side of (3.38) is $F_{1,a}$. More precisely we are going to see (cf. Proposition 3.5.2) that there exists a control α (chosen below as $\alpha = \mathcal{A}[q, g_\varepsilon]$ with g_ε given by (3.48)) such that in the appropriate regime the dynamics of (3.31) behaves like the equation with only $F_{1,a}$ on the right hand side. Moreover the following lemma, where the time parameter does not appear, proves that the operator $F_{1,a}(q)[\cdot]$ can actually attain any value v in $\mathbb{R}^3$. Recall that $\delta > 0$ has been fixed at the beginning of Section 3.5.1.

Proposition 3.5.1. *There exists a continuous mapping $\overline{g} : \mathcal{Q}_\delta \times \mathbb{R}^3 \to \mathcal{C}$ such that for any (q, v) in $\mathcal{Q}_\delta \times \mathbb{R}^3$ the function $\overline{\alpha} := \mathcal{A}[q, \overline{g}(q, v)]$ in $C^\infty(\overline{\mathcal{F}(q)}; \mathbb{R})$ satisfies :*

$$\Delta\overline{\alpha} = 0 \text{ in } \mathcal{F}(q), \text{ and } \partial_n\overline{\alpha} = 0 \text{ on } \partial\mathcal{F}(q) \setminus \Sigma, \tag{3.45}$$

$$\int_{\partial\mathcal{S}(q)} |\nabla\overline{\alpha}|^2 \, \partial_n\Phi(q, \cdot) \, d\sigma = v, \tag{3.46}$$

$$\int_{\partial\mathcal{S}(q)} \overline{\alpha} \, \partial_n\Phi(q, \cdot) \, d\sigma = 0. \tag{3.47}$$

We recall that the operator $\mathcal{A}$ was introduced in Definition 3.2.1. The result above will be proved in Section 3.7. Note that when $\mathcal{S}(q)$ is a homogeneous disk, an adapted version of Proposition 3.5.1 still holds, see Proposition 3.7.1 in Section 3.7. The condition (3.47) will be useful to cancel out the last term of (3.31).

We define

$$g_\varepsilon(t, x) := \beta_\varepsilon(t)\overline{g}(q_0, -2v_0)(x) + \beta_\varepsilon(T - t)\overline{g}(q_1, -2v_1)(x), \tag{3.48}$$

where v_0 and v_1 defined in (3.37), for (q_1, q_1') in $\overline{B}\big((q_0, q_0'), r_2\big)$, and $\overline{g}$ is given by Proposition 3.5.1. The goal is to prove that for ε and $|\gamma|$ small enough, this control drives the system (3.31) with $\alpha = \mathcal{A}[q, g_\varepsilon]$ from (q_0, q_0') to (q_1, q_1'), approximately.

1. We first observe that

$$F_{1,a}(q)[\mathcal{A}[q, g_\varepsilon]] = \beta_\varepsilon^2(t)F_{1,a}(q)\Big[\mathcal{A}[q, \overline{g}(q_0, -2v_0)]\Big] + \beta_\varepsilon^2(T - t)F_{1,a}(q)\Big[\mathcal{A}[q, \overline{g}(q_1, -2v_1)]\Big], \tag{3.49}$$

and is therefore a good candidate to approximate the right hand side of (3.38) if q is near q_0 for t near 0 and if q is near q_1 for t near T. One then may indeed expect that

$$F_{1,a}(q)\Big[\mathcal{A}[q, \overline{g}(q_0, -2v_0)]\Big] \text{ and } F_{1,a}(q)\Big[\mathcal{A}[q, \overline{g}(q_1, -2v_1)]\Big] \text{ are close to}$$
$$F_{1,a}(q_0)\Big[\mathcal{A}[q_0, \overline{g}(q_0, -2v_0)]\Big] \text{ and } F_{1,a}(q_1)\Big[\mathcal{A}[q_1, \overline{g}(q_1, -2v_1)]\Big], \text{ respectively,}$$

on the respective supports of $\beta_\varepsilon(\cdot)$ and $\beta_\varepsilon(T - \cdot)$. Moreover, according to Proposition 3.5.1 these last two terms are equal to v_0 and v_1 (see (3.42) and (3.46)).

2. Next we will rigorously prove in Proposition 3.6.1 below that the conclusion of Lemma 3.5.2 for the toy system also holds when one substitutes the term $F_{1,a}(q)[\mathcal{A}[q, g_\varepsilon]]$ in (3.49). This corresponds also to (3.31) with $\gamma = 0$ and the term $F_{1,b}$ and F_2 put to zero.

3. Finally it will appear that in an appropriate regime, in particular for small ε and $|\gamma|$, the second last term of (3.31) is dominant with respect to the other terms of the right hand side (here the condition (3.47) above will be essential in order to deal with the last term of (3.31)).

Let us state a proposition summarizing the claims above. According to the Cauchy-Lipschitz theorem there exists a controlled solution $q_{\varepsilon,\gamma}$ associated with the control g_ε introduced in (3.48), starting with the initial condition $q_{\varepsilon,\gamma}(0) = q_0$ and $q'_{\varepsilon,\gamma}(0) = q'_0$, with circulation γ, and lying in $B(q_0, r_1)$ up to some positive time $T_{\varepsilon,\gamma}$. More explicitly $q_{\varepsilon,\gamma}$ satisfies on $[0, T_{\varepsilon,\gamma}]$,

$$\begin{aligned}\big(\mathcal{M}_g + \mathcal{M}_a(q_{\varepsilon,\gamma})\big)q''_{\varepsilon,\gamma} + \langle \Gamma(q_{\varepsilon,\gamma}), q'_{\varepsilon,\gamma}, q'_{\varepsilon,\gamma}\rangle &= \gamma^2 E(q_{\varepsilon,\gamma}) + \gamma q'_{\varepsilon,\gamma} \times B(q_{\varepsilon,\gamma}) \\ &\quad + F_1(q_{\varepsilon,\gamma}, q'_{\varepsilon,\gamma}, \gamma)\big[\mathcal{A}[q_{\varepsilon,\gamma}, g_\varepsilon]\big] + F_2(q_{\varepsilon,\gamma})\big[\partial_t \mathcal{A}[q_{\varepsilon,\gamma}, g_\varepsilon]\big]. \quad (3.50)\end{aligned}$$

Observe that due to the choice of the control g_ε in (3.48) the function $q_{\varepsilon,\gamma}$ also depends on (q_1, q'_1) through v_0 and v_1, see their definition in (3.37).

We have the following approximation result.

Proposition 3.5.2. *For ε and $|\gamma|$ small enough, $T_{\varepsilon,\gamma} \geqslant T$ and, as ε and $|\gamma|$ converge to 0^+, $\|q_{\varepsilon,\gamma} - \bar{q}\|_{T,\varepsilon} \to 0$ and $(q_{\varepsilon,\gamma}, q'_{\varepsilon,\gamma})(T) \to (q_1, q'_1)$, uniformly for (q_1, q'_1) in $\overline{B}\big((q_0, q'_0), r_2\big)$.*

This result will be proved in Section 3.6. Once Proposition 3.5.2 is proved, Theorem 3.4.1 follows rapidly. Indeed, let us set $\tilde{r} = r_2$, according to Proposition 3.5.2, for $\eta > 0$, there exists $\varepsilon = \varepsilon(\eta) > 0$ and $r' = r'(\eta)$ in $(0, \tilde{r})$ such that for any $\gamma \in \mathbb{R}$ with $|\gamma| \leq r'$ and for any q'_0 in $\overline{B}(0, \tilde{r})$ the mapping $\mathcal{T}$ defined on $\overline{B}\big((q_0, q'_0), \tilde{r}\big)$ by setting $\mathcal{T}(q_1, q'_1) = q_{\varepsilon,\gamma}$, has the desired properties. In particular the continuity of $\mathcal{T}$ follows from the regularity of c_0 in Lemma 3.5.1 and of the solution of ODEs on their initial data. This ends the proof of Theorem 3.4.1.

3.5.4 About Remark 3.1.1

Now that we presented the scheme of proof of Theorem 3.1.1 let us explain how to obtain the improvement mentioned in Remark 3.1.1. It is actually a direct consequence of the explicit formula for $g_\varepsilon(t, x)$ given in (3.48) and of a change of variable in time. Due to the expression of β_ε given at the beginning of Section 3.5.2 one obtains that the total flux through Σ^-, that is $\int_0^T \int_{\Sigma^-} g_\varepsilon \, d\sigma dt$, is of order $\sqrt{\varepsilon}$. Hence one can reduce ε again in order to satisfy the requirement of Remark 3.1.1.

On the other hand observe that the time-rescaling argument used in the proof of Theorem 3.1.1 from Theorem 3.3.1, cf. (3.33), leaves the total flux through Σ^- invariant, while the number N of steps involved in the end of the same proof does not depend on ε.

3.6 Closeness of the controlled system to the geodesic

In this section, we prove Proposition 3.5.2.

3.6.1 Proof of Proposition 3.5.2

The proof of Proposition 3.5.2 is split in several parts. To compare $q_{\varepsilon,\gamma}$ and $\overline{q}$, we are going to consider an "intermediate trajectory" $\tilde{q}_\varepsilon$ which imitates the trajectory q_ε of the toy model of Lemma 3.5.2, by using the part $F_{1,a}$ of the force term. More precisely we define $\tilde{q}_\varepsilon$ by

$$\Big(\mathcal{M}_g + \mathcal{M}_a(\tilde{q}_\varepsilon)\Big)\tilde{q}''_\varepsilon + \langle \Gamma(\tilde{q}_\varepsilon), \tilde{q}'_\varepsilon, \tilde{q}'_\varepsilon\rangle = F_{1,a}(\tilde{q}_\varepsilon)\big[\mathcal{A}[\tilde{q}_\varepsilon, g_\varepsilon]\big], \quad \text{with } \tilde{q}_\varepsilon(0) = q_0,\ \tilde{q}'_\varepsilon(0) = q'_0, \quad (3.51)$$

where g_ε was defined in (3.48) and where the operator $\mathcal{A}$ was introduced in Definition 3.2.1. Note that due to the definition of g_ε, the function $\tilde{q}_\varepsilon$ also depends on q_1, q_1'. The statement below is an equivalent of Lemma 3.5.2 for $\tilde{q}_\varepsilon$, comparing $\tilde{q}_\varepsilon$ to the "target geodesic" $\overline{q}$.

Proposition 3.6.1. *There exists $\varepsilon_1 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_1]$, for any (q_1, q_1') in $\overline{B}\big((q_0, q_0'), r_2\big)$, the solution $\tilde{q}_\varepsilon$ given by* (3.51) *lies in the ball $B(q_0, r_1)$ at least up to T. Moreover $\|\tilde{q}_\varepsilon - \bar{q}\|_{T,\varepsilon}$ converges to* 0 *and $(\tilde{q}_\varepsilon, \tilde{q}_\varepsilon')(T)$ converges to (q_1, q_1') when ε converges to 0^+, uniformly for (q_1, q_1') in $\overline{B}\big((q_0, q_0'), r_2\big)$ for both convergences.*

We recall that the norm $\|\cdot\|_{T,\varepsilon}$ was defined in (3.36). The proof of Proposition 3.6.1 can be found in Subsection 3.6.2.

The following result allows us to deduce the closeness of the trajectories $q_{\varepsilon,0}$, given by (3.50) with $\gamma = 0$, and $\tilde{q}_\varepsilon$ given by (3.51). Let us recall that by the definition of $T_{\varepsilon,\gamma}$ that comes along (3.50), $q_{\varepsilon,0}$ lies in $B(q_0, r_1)$ up to the time $T_{\varepsilon,0}$, which depends on q_1, q_1'.

Proposition 3.6.2. *There exists ε_2 in $(0, \varepsilon_1]$ such that for any $\varepsilon \in (0, \varepsilon_2]$, one has $T_{\varepsilon,0} \geq T$. Moreover $\|\tilde{q}_\varepsilon - q_{\varepsilon,0}\|_{C^1([0,T])} \to 0$ when $\varepsilon \to 0^+$, uniformly for (q_1, q_1') in $\overline{B}\big((q_0, q_0'), r_2\big)$.*

The proof of Proposition 3.6.2 can be found in Subsection 3.6.3.

Finally, we have the following estimation of the deviation due to the circulation γ, which will be proved in Subsection 3.6.4.

Proposition 3.6.3. *There exists ε_3 in $(0, \varepsilon_2]$ such that for all $\varepsilon \in (0, \varepsilon_3]$, there exists $\gamma_0 > 0$ such that for any $\gamma \in [-\gamma_0, \gamma_0]$, we have $T_{\varepsilon,\gamma} \geq T$ and $\|q_{\varepsilon,\gamma} - q_{\varepsilon,0}\|_{C^1[0,T]}$ converges to* 0 *when $\gamma \to 0$, uniformly for (q_1, q_1') in $\overline{B}\big((q_0, q_0'), r_2\big)$.*

Propositions 3.6.1, 3.6.2 and 3.6.3 give us directly the result of Proposition 3.5.2.

3.6.2 Proof of Proposition 3.6.1

We proceed as in the proof of Lemma 3.5.2 with a few extra complications related to the fact that the right hand side of the equation (3.51) is more involved than the one of the equation (3.38) and to the fact that we need to obtain uniform convergences with respect to (q_1, q_1') in $\overline{B}\big((q_0, q_0'), r_2\big)$.

As in the proof of Lemma 3.5.2 we introduce, for ε in $(0, 1)$, the time

$$T_\varepsilon = \sup\{\hat{T} > 0 \text{ such that } \tilde{q}_\varepsilon(t) \in B(q_0, r_1) \text{ for } t \in (0, \hat{T})\}$$

and we first prove that there exists $\tilde{T} > 0$ such that for any ε in $(0, 1)$, $T_\varepsilon \geq \tilde{T}$ thanks to an energy estimate. In order to deal with the term coming from (3.49) in the right hand side of the energy estimate, recalling Remark 3.2.1 and the definition of $F_{1,a}$ in (3.42), we observe that for any $R > 0$, there exists $C > 0$ such that for any $q, \tilde{q}$ in $\mathcal{Q}_\delta$, for any v in $B(0, R)$,

$$|\, F_{1,a}(q)\big[\mathcal{A}[q, \overline{g}(\tilde{q}, v)]\big] \,| \leq C. \tag{3.52}$$

This allows to deduce from the expressions of v_0 and v_1 in (3.37) that there exists $\tilde{T} > 0$ and $C > 0$ such that for any (q_1, q_1') in $\overline{B}\big((q_0, q_0'), r_2\big)$, for any ε in $(0, 1)$, $T_\varepsilon \geq \tilde{T}$ and $\|\tilde{q}_\varepsilon'\|_{C([0,T_\varepsilon])} \leq C$. We deduce that for $\varepsilon_1 := \tilde{T}/2$ and ε in $(0, \varepsilon_1)$, $T_\varepsilon \geq 2\varepsilon$ and that $\sup_{t\in[0,2\varepsilon]} |\tilde{q}_\varepsilon - q_0|$ converges to 0 as ε goes to 0 uniformly in (q_1, q_1') in $\overline{B}\big((q_0, q_0'), r_2\big)$.

Now let us prove that $\tilde{q}'_\varepsilon(2\varepsilon)$ converges to $c_0(q_1)$ as ε goes to 0 uniformly in (q_1, q'_1) in $\overline{B}\big((q_0, q'_0), r_2\big)$. We integrate the equation (3.51) on $[0, 2\varepsilon]$. Thus

$$\Big(\mathcal{M}_g + \mathcal{M}_a(\tilde{q}_\varepsilon(2\varepsilon))\Big)\tilde{q}'_\varepsilon(2\varepsilon) = \Big(\mathcal{M}_g + \mathcal{M}_a(q_0)\Big)q'_0 \\ - \int_0^{2\varepsilon} \Big(D\mathcal{M}_a(\tilde{q}_\varepsilon)\cdot \tilde{q}'_\varepsilon\Big)\cdot \tilde{q}'_\varepsilon\, dt - \int_0^{2\varepsilon} \langle \Gamma(\tilde{q}_\varepsilon), \tilde{q}'_\varepsilon, \tilde{q}'_\varepsilon\rangle\, dt + \int_0^{2\varepsilon} F_{1,a}(\tilde{q}_\varepsilon)\big[\mathcal{A}[\tilde{q}_\varepsilon, g_\varepsilon]\big]\, dt. \quad (3.53)$$

Then we pass to the limit as ε goes to 0^+ in the last equality. Here we use two extra arguments with respect to the corresponding argument in the proof of Lemma 3.5.2. On the one hand we see that the convergences of $\mathcal{M}_a(\tilde{q}_\varepsilon(2\varepsilon))$ to $\mathcal{M}_a(q_0)$ and of the two first terms of the last line to 0, already obtained in the proof of Lemma 3.5.2, hold uniformly with respect to (q_1, q'_1) in $\overline{B}\big((q_0, q'_0), r_2\big)$, as a consequence of the uniform estimates of $\tilde{q}_\varepsilon - q_0$ and $\tilde{q}'_\varepsilon$ obtained above. On the other hand the term $F_{1,a}$ enjoys the following regularity property with respect to q : we have that $q \mapsto F_{1,a}(q)\Big[\mathcal{A}[q, \overline{g}(q_0, v)]\Big]$ is Lipschitz with respect to q in $\mathcal{Q}_\delta$ uniformly for v in bounded sets of $\mathbb{R}^3$. Therefore using that $\sup_{t\in[0,2\varepsilon]} |\tilde{q}_\varepsilon - q_0|$ converges to 0 as ε goes to 0 uniformly in (q_1, q'_1) in $\overline{B}\big((q_0, q'_0), r_2\big)$, the expressions of v_0 and v_1 in (3.37) and that $F_{1,a}(q_0)\Big[\mathcal{A}[q_0, \overline{g}(q_0, -2v_0)\Big] = v_0$, according to Proposition 3.5.1 we deduce that

$$\sup_{t\in[0,2\varepsilon]} \Big|F_{1,a}(\tilde{q}_\varepsilon)\Big[\mathcal{A}[\tilde{q}_\varepsilon, \overline{g}(q_0, -2v_0)]\Big] - v_0\Big|$$

converges to 0 as ε goes to 0 uniformly in (q_1, q'_1) in $\overline{B}\big((q_0, q'_0), r_2\big)$. Since for t in $[0, 2\varepsilon]$, the equation (3.49) applied to $q = \tilde{q}_\varepsilon$ is simplified into

$$F_{1,a}(\tilde{q}_\varepsilon)[\mathcal{A}[\tilde{q}_\varepsilon, g_\varepsilon]] = \beta_\varepsilon^2(t) F_{1,a}(\tilde{q}_\varepsilon)\Big[\mathcal{A}[\tilde{q}_\varepsilon, \overline{g}(q_0, -2v_0)]\Big],$$

and that $\int_0^{2\varepsilon} \beta_\varepsilon^2(t)\, dt = 1$, we get that the last term in (3.53) converges to v_0 when ε goes to 0. Moreover, due to the choice of v_0 the first and last term of the right hand side of (3.53) can be combined at the limit to get $\Big(\mathcal{M}_g + \mathcal{M}_a(q_0)\Big)c_0(q_1)$.

Therefore, inverting the matrix in the right hand side of (3.53) and passing to the limit, we see that $\tilde{q}'_\varepsilon(2\varepsilon)$ converges to $c_0(q_1)$ as ε goes to 0 uniformly in (q_1, q'_1) in $\overline{B}\big((q_0, q'_0), r_2\big)$.

When t is in $[2\varepsilon, T - 2\varepsilon]$, the equation (3.51) reduces to a geodesic equation so that the same arguments as in the proof of Lemma 3.5.2 apply.

Finally for the last step, for t in $[T - 2\varepsilon, T]$, we proceed in the same way as in the first step. This ends the proof of Proposition 3.6.1.

3.6.3 Proof of Proposition 3.6.2

We begin with the following lemma, which provides a uniform boundedness for the trajectories $q_{\varepsilon,0}$ satisfying (3.50) with $\gamma = 0$, that is

$$\big(\mathcal{M}_g + \mathcal{M}_a(q_{\varepsilon,0})\big)q''_{\varepsilon,0} + \langle \Gamma(q_{\varepsilon,0}), q'_{\varepsilon,0}, q'_{\varepsilon,0}\rangle = F_{1,a}(q_{\varepsilon,0})\big[\mathcal{A}[q_{\varepsilon,0}, g_\varepsilon]\big] \\ + F_{1,b}(q_{\varepsilon,0}, q'_{\varepsilon,0})\big[\mathcal{A}[q_{\varepsilon,0}, g_\varepsilon]\big] + F_2(q_{\varepsilon,0})\big[\partial_t \mathcal{A}[q_{\varepsilon,0}, g_\varepsilon]\big]. \quad (3.54)$$

We recall that g_ε is given by (3.48) with v_0 and v_1 given by (3.37). The terms $F_{1,a}$ and $F_{1,b}$ were defined in (3.42)-(3.43), F_2 in (3.29). Also we recall that by definition of $T_{\varepsilon,0}$ (see the definition of $T_{\varepsilon,\gamma}$ in the end of Subsection 3.5.3), during the time interval $[0, T_{\varepsilon,0}]$, $q_{\varepsilon,0}$ remains in $B(q_0, r_1)$.

Lemma 3.6.1. *There exists $\varepsilon_a > 0$ such that*

$$\sup_{\substack{(q_1,q_1')\in\overline{B}((q_0,q_0'),r_2),\\ \varepsilon\in(0,\varepsilon_a]}} \|q'_{\varepsilon,0}\|_{C([0,T_{\varepsilon,0}])} < +\infty.$$

Proof. First we see that the mappings

$$q \mapsto F_{1,a}(q)[\mathcal{A}[q,\overline{g}(q_0,v)]] \text{ and } q \mapsto F_{1,b}(q,\cdot)[\mathcal{A}[q,\overline{g}(q_0,v)]]$$

are bounded for q in $\mathcal{Q}_\delta$, uniformly for v in bounded sets of $\mathbb{R}^3$. Let us now focus on the F_2 term. For t in $[0,2\varepsilon]$, $g_\varepsilon(t) = \beta_\varepsilon(t)\overline{g}(q_0,-2v_0)$ so that, by the chain rule, for t in $[0,\min(2\varepsilon,T_{\varepsilon,0})]$,

$$\partial_t\mathcal{A}[q_{\varepsilon,0},g_\varepsilon] = \beta_\varepsilon D_q\mathcal{A}[q_{\varepsilon,0},\overline{g}(q_0,-2v_0)]\cdot q'_{\varepsilon,0} + \beta'_\varepsilon\mathcal{A}[q_{\varepsilon,0},\overline{g}(q_0,-2v_0)].$$

For what concerns F_2 we have, using the property (3.47),

$$\begin{aligned} F_2(q_{\varepsilon,0})\big[\partial_t\mathcal{A}[q_{\varepsilon,0},g_\varepsilon]\big] &= \beta_\varepsilon\int_{\partial\mathcal{S}(q_{\varepsilon,0})}\Big(D_q\mathcal{A}[q_{\varepsilon,0},\overline{g}(q_0,-2v_0)]\cdot q'_{\varepsilon,0}\Big)\,\partial_n\Phi(q_{\varepsilon,0},\cdot)\,d\sigma \\ &\quad+\beta'_\varepsilon\Big(\int_{\partial\mathcal{S}(q_{\varepsilon,0})}\mathcal{A}[q_{\varepsilon,0},\overline{g}(q_0,-2v_0)]\,\partial_n\Phi(q_{\varepsilon,0},\cdot)\,d\sigma - \int_{\partial\mathcal{S}(q_0)}\mathcal{A}[q_0,\overline{g}(q_0,-2v_0)]\,\partial_n\Phi(q_0,\cdot)\,d\sigma\Big). \end{aligned}$$

Using that the mapping $q\mapsto\int_{\partial\mathcal{S}(q)}\nabla_q\mathcal{A}[q,\overline{g}(q_0,v)]\otimes\partial_n\Phi(q,\cdot)\,d\sigma$ is bounded for q over $\mathcal{Q}_\delta$ and that the mapping $q\mapsto\int_{\partial\mathcal{S}(q)}\mathcal{A}[q,\overline{g}(q_0,v)]\,\partial_n\Phi(q,\cdot)\,d\sigma$ is Lipschitz with respect to q in $\mathcal{Q}_\delta$, both uniformly for v in bounded sets of $\mathbb{R}^3$, we see that this involves (recalling the expression of β_ε given at the beginning of Section 3.5.2)

$$\big|F_2(q_{\varepsilon,0})\big[\partial_t\mathcal{A}[q_{\varepsilon,0},g_\varepsilon]\big]\big| \lesssim C\left(\frac{1}{\varepsilon^{1/2}}|q'_{\varepsilon,0}| + \frac{1}{\varepsilon^{3/2}}|q_{\varepsilon,0}-q_0|\right), \tag{3.55}$$

uniformly for (q_1,q_1') in $\overline{B}\big((q_0,q_0'),r_2\big)$. Then, multiplying (3.54) by $q'_{\varepsilon,0}$ and using once more the identity (3.26), we obtain, for any ε in $(0,1)$, for t in $[0,\min(2\varepsilon,T_{\varepsilon,0})]$,

$$\begin{aligned} \Big(\mathcal{M}_g+\mathcal{M}_a(q_{\varepsilon,0}(t))\Big)q'_{\varepsilon,0}(t)\cdot q'_{\varepsilon,0}(t) &= \Big(\mathcal{M}_g+\mathcal{M}_a(q_0)\Big)q_0'\cdot q_0' \\ &\quad+2\int_0^t\Big(F_{1,a}(q_{\varepsilon,0})\big[\mathcal{A}[q_{\varepsilon,0},g_\varepsilon]\big]+F_{1,b}(q_{\varepsilon,0},q'_{\varepsilon,0})\big[\mathcal{A}[q_{\varepsilon,0},g_\varepsilon]\big]+F_2(q_{\varepsilon,0})\big[\partial_t\mathcal{A}[q_{\varepsilon,0},g_\varepsilon]\big]\Big)\cdot q'_{\varepsilon,0}, \end{aligned} \tag{3.56}$$

Then, using (3.39), the boundedness of the mappings $q\mapsto F_{1,a}(q)[\mathcal{A}[q,\overline{g}(q_0,v)]]$, $F_{1,b}(q,\cdot)[\mathcal{A}[q,\overline{g}(q_0,v)]]$ already mentioned above, the definition of β_ε and the bound (3.55), we get

$$|q'_{\varepsilon,0}(t)|^2 \le C\left(1+\frac{1}{\varepsilon^{1/2}}\int_0^t|q'_{\varepsilon,0}(s)|^2\,ds+\frac{1}{\varepsilon^{3/2}}\int_0^t|q'_{\varepsilon,0}(s)||q_{\varepsilon,0}(s)-q_0|\,ds\right).$$

Then using the mean value theorem and that $t\le 2\varepsilon$, we have that

$$|q'_{\varepsilon,0}(t)|^2 \le C\left(1+\varepsilon^{1/2}\sup_{[0,\min(2\varepsilon,T_{\varepsilon,0})]}|q'_{\varepsilon,0}|^2\right),$$

so that for ε small enough, and for t in $[0,\min(2\varepsilon,T_{\varepsilon,0})]$, $|q'_{\varepsilon,0}(t)|\le C$, uniformly for (q_1,q_1') in $\overline{B}\big((q_0,q_0'),r_2\big)$. As a consequence of the usual blow-up criterion for ODEs, we have that $T_{\varepsilon,0}\ge 2\varepsilon$.

During the next phase, i.e. for t in $[2\varepsilon,T-2\varepsilon]$, the control is inactive so that the equation (3.54) is a geodesic equation. Then by a simple energy estimate we get again that $|q'_{\varepsilon,0}(t)|\le C$ on $[0,\min(T-2\varepsilon,T_{\varepsilon,0})]$.

Finally if $T_{\varepsilon,0}\ge T-2\varepsilon$, then we deal with the last phase as in the first phase. This concludes the proof of Lemma 3.6.1. □

We then conclude the proof of Proposition 3.6.2 by a classical comparison argument using Gronwall's lemma and the Lipschitz regularity with respect to q of the various mappings involved ($\mathcal{M}_a$, Γ, $F_{1,a}$, $F_{1,b}$ and F_2). This allows to prove that there exists ε_2 in $(0, \varepsilon_1]$ such that for any $\varepsilon \in (0, \varepsilon_2]$, $T_{\varepsilon,0} \geq T$ and $\|\tilde{q}_\varepsilon - q_{\varepsilon,0}\|_{C^1([0,T])} \to 0$ when $\varepsilon \to 0^+$, uniformly for (q_1, q_1') in $\overline{B}\big((q_0, q_0'), r_2\big)$. This ends the proof of Proposition 3.6.2.

3.6.4 Proof of Proposition 3.6.3

First we may extend Lemma 3.6.1 to the solutions $q_{\varepsilon,\gamma}$ to (3.50) in the following manner.

Lemma 3.6.2. *There exists ε_b in $(0, \varepsilon_2)$ such that $\|q'_{\varepsilon,\gamma}\|_{C([0,T_{\varepsilon,\gamma}])}$ is bounded uniformly in $\varepsilon \in (0, \varepsilon_b]$, for any $\gamma \in [-1, 1]$, and for $(q_1, q_1') \in \overline{B}((q_0, q_0'), r_2)$.*

It is indeed a matter of adding the "electric field" E in (3.56), and noting that E is bounded on Q_δ ; the "magnetic field" B gives no contribution to the energy.

We now finish the proof of Proposition 3.6.3. Using a comparison argument we obtain that there exists ε_3 in $(0, \varepsilon_b]$ such that for all $\varepsilon \in (0, \varepsilon_3]$, there exists $\gamma_0 > 0$ such that for any $\gamma \in [-\gamma_0, \gamma_0]$, we have $T_{\varepsilon,\gamma} \geq T$ and $\|q_{\varepsilon,\gamma} - q_{\varepsilon,0}\|_{C^1[0,T]}$ converges to 0 when $\gamma \to 0$, uniformly for (q_1, q_1') in $\overline{B}\big((q_0, q_0'), r_2\big)$. This concludes the proof of Proposition 3.6.3.

3.7 Design of the control according to the solid position

This section is devoted to the proof of Proposition 3.5.1.

3.7.1 The case of a homogeneous disk

Before proving Proposition 3.5.1 we establish the following similar result concerning the simpler case where the solid is a homogeneous disk. In that case, the statement merely considers q of the form $q = (h, 0)$. Thus in order to simplify the writing, we introduce

$$\mathcal{Q}_\delta^h := \{h \in \mathbb{R}^2 \ \text{ such that } \ (h, 0) \in \mathcal{Q}_\delta\}.$$

Also in all this section when we will write q, it will be understood that q is associated with h by $q = (h, 0)$.

Proposition 3.7.1. *Let $\delta > 0$. Then there exists a continuous mapping $\overline{g} : \mathcal{Q}_\delta^h \times \mathbb{R}^2 \to \mathcal{C}$ such that the function $\overline{\alpha} := \mathcal{A}[q, \overline{g}(q, v)]$ in $C^\infty(\overline{\mathcal{F}(q)}; \mathbb{R})$ satisfies :*

$$\Delta\overline{\alpha}(q, x) = 0 \ \text{ in } \mathcal{F}(q), \ \text{ and } \partial_n\overline{\alpha}(q, x) = 0 \ \text{ on } \partial\mathcal{F}(q) \setminus \Sigma, \tag{3.57}$$

$$\int_{\partial\mathcal{S}(q)} |\nabla\overline{\alpha}(q, x)|^2 \, n \, d\sigma = v, \tag{3.58}$$

$$\int_{\partial\mathcal{S}(q)} \overline{\alpha}(q, x) \, n \, d\sigma = 0. \tag{3.59}$$

In order to prove Proposition 3.7.1, the mapping $\overline{g}$ will be constructed using a combination of some elementary functions which we introduce in several lemmas.

To begin with, we will make use of the elementary geometrical property that $\{n(q_0, x) : \ x \in \partial\mathcal{S}(q_0)\}$ is the unit circle $\mathbb{S}^1$ and of the following lemma.

Lemma 3.7.1. *There exist three vectors* $e_1, e_2, e_3 \in \{n(q_0, x) : \ x \in \partial\mathcal{S}(q_0)\}$ *and positive* C^∞ *maps* $(\mu_i)_{1\leqslant i\leqslant 3} : \mathbb{R}^2 \to \mathbb{R}_+$ *such that for any* $v \in \mathbb{R}^2$,

$$\sum_{i=1}^{3} \mu_i(v)e_i = v. \tag{3.60}$$

Proof. One may consider for instance $e_1 := (1, 0)$, $e_2 := (0, 1)$, $e_3 := (-1, -1)$, and

$$\mu_1(v) = v_1 + \sqrt{1 + |v_1|^2 + |v_2|^2},\ \mu_2(v) = v_2 + \sqrt{1 + |v_1|^2 + |v_2|^2} \text{ and } \mu_3(v) = \sqrt{1 + |v_1|^2 + |v_2|^2}.$$

□

In the next lemma, we introduce some functions that are defined in a neighbourhood of $\partial\mathcal{S}(q_0)$ (for some $q_0 = (h_0, 0)$ fixed), satisfying some counterparts of the properties (3.57) and (3.58).

Lemma 3.7.2. *There exist families of functions* $(\tilde{\alpha}_\varepsilon^{i,j})_{\varepsilon\in(0,1)}$, $i, j \in \{1, 2, 3\}$, *such that for any* $i, j \in \{1, 2, 3\}$, *for any* $\varepsilon \in (0, 1)$, $\tilde{\alpha}_\varepsilon^{i,j}$ *is defined and harmonic in a closed neighbourhood* $\mathcal{V}_\varepsilon^{i,j}$ *of* $\partial\mathcal{S}(q_0)$, *satisfies* $\partial_n\tilde{\alpha}_\varepsilon^{i,j} = 0$ *on* $\partial\mathcal{S}(q_0)$, *and moreover one has for any* i, j, k, l *in* $\{1, 2, 3\}$,

$$\int_{\partial\mathcal{S}(q_0)} \nabla\tilde{\alpha}_\varepsilon^{i,j} \cdot \nabla\tilde{\alpha}_\varepsilon^{k,l}\, n\, d\sigma \to \delta_{(i,j),(k,l)}\, e_i \qquad \text{as } \varepsilon \to 0^+.$$

Proof. Without loss of generality, we may suppose that $\mathcal{S}(q_0)$ is the unit disk. Consider the parameterisation $\{c(s) = (\cos(s), \sin(s)),\ s \in [0, 2\pi]\}$ of $\partial\mathcal{S}(q_0)$ and the corresponding s_i such that $n(q_0, c(s_i)) = e_i,\ i \in \{1, 2, 3\}$.

We consider families of smooth functions $\beta_\varepsilon^{i,j} : [0, 2\pi] \to \mathbb{R}$, $i, j \in \{1, 2, 3\}$, $\varepsilon \in (0, 1)$, such that $\text{supp}\ \beta_\varepsilon^{i,j} \cap \text{supp}\ \beta_\varepsilon^{k,l} = \emptyset$ whenever $(i, j) \neq (k, l)$, $\text{diam}\left(\text{supp}\ \beta_\varepsilon^{i,j}\right) \to 0$ as $\varepsilon \to 0^+$,

$$\int_0^{2\pi} \beta_\varepsilon^{i,j}(s)\, d\sigma = 0 \text{ and } \left|\int_0^{2\pi} |\beta_\varepsilon^{i,j}(s)|^2\, n(q_0, c(s))ds - e_i\right| \to 0 \text{ as } \varepsilon \to 0^+.$$

Then we define $\tilde{\alpha}_\varepsilon^{i,j}$ in polar coordinates as the truncated Laurent series :

$$\tilde{\alpha}_\varepsilon^{i,j}(r, \theta) := \frac{1}{2} \sum_{0<k\leq K} \frac{1}{k}\left(r^k + \frac{1}{r^k}\right)(-\hat{b}_{k,\varepsilon}^{i,j} \cos(k\theta) + \hat{a}_{k,\varepsilon}^{i,j} \sin(k\theta)),$$

where $\hat{a}_{k,\varepsilon}^{i,j}$ and $\hat{b}_{k,\varepsilon}^{i,j}$ denote the k-th Fourier coefficients of the function $\beta_\varepsilon^{i,j}$. It is elementary to check that the function $\tilde{\alpha}_\varepsilon^{i,j}$ satisfies the required properties for an appropriate choice of K. □

Now, for any $h \in \mathcal{Q}_\delta^h$, we may define $\mathcal{V}_\varepsilon^{i,j}(q) := \mathcal{V}_\varepsilon^{i,j} - h_0 + h$, which is a neighborhood of $\partial\mathcal{S}(q)$, and $\tilde{\alpha}_\varepsilon^{i,j}(q, x) := \tilde{\alpha}_\varepsilon^{i,j}(x + h_0 - h)$, for each $x \in \mathcal{V}_\varepsilon^{i,j}(q)$. We have for i, j, k, l in $\{1, 2, 3\}$,

$$\int_{\partial\mathcal{S}(q)} \nabla\tilde{\alpha}_\varepsilon^{i,j}(q, x) \cdot \nabla\tilde{\alpha}_\varepsilon^{k,l}(q, x)\, n(q, x)\, d\sigma = \int_{\partial\mathcal{S}(q_0)} \nabla\tilde{\alpha}_\varepsilon^{i,j}(x) \cdot \nabla\tilde{\alpha}_\varepsilon^{k,l}(x)\, n(q_0, x)\, d\sigma.$$

Proceeding as in [47] (see also [50, p. 147-149]) and relying in particular Runge's theorem, we have the following result which asserts the existence of harmonic approximate extensions on the whole fluid domain.

Lemma 3.7.3. *There exists a family of functions* $(\alpha_\eta^{i,j})_{\eta\in(0,1)}$, $i, j \in \{1, 2, 3\}$, *harmonic in* $\mathcal{F}(q)$, *satisfying* $\partial_n\alpha_\eta^{i,j}(q, x) = 0$ *on* $\partial\mathcal{F}(q) \setminus \Sigma$, *with for any* k *in* $\mathbb{N}$,

$$\|\alpha_\eta^{i,j}(q, \cdot) - \tilde{\alpha}_\varepsilon^{i,j}(q, \cdot)\|_{C^k(\mathcal{V}_\varepsilon^{i,j}(q)\cap\overline{\mathcal{F}(q)})} \to 0 \text{ when } \eta \to 0^+. \tag{3.61}$$

We now check that the above construction can be made continuous in q.

Lemma 3.7.4. *For any $\nu > 0$, there exist continuous mappings $h \in \mathcal{Q}_\delta^h \mapsto \overline{\alpha}^{i,j}(q,\cdot) \in C^\infty(\overline{\mathcal{F}(q)})$ where $q = (h,0)$, $i,j \in \{1,2,3\}$, such that for any $h \in \mathcal{Q}_\delta^h$, $\Delta_x \overline{\alpha}^{i,j}(q,x) = 0$ in $\mathcal{F}(q)$, $\partial_n \overline{\alpha}^{i,j}(q,x) = 0$ on $\partial\mathcal{F}(q) \setminus \Sigma$ and*

$$\left| \int_{\partial\mathcal{S}(q)} \nabla\overline{\alpha}^{i,j}(q,\cdot) \cdot \nabla\overline{\alpha}^{k,l}(q,\cdot)\, n\, d\sigma - \delta_{(i,j),(k,l)}\, e_i \right| \leq \nu. \tag{3.62}$$

Proof. Let us assume that the functions $\alpha_\eta^{i,j}$ were previously defined not only for $h \in Q_\delta^h$ but for $h \in \overline{Q_\delta^h}$; this is possible by using a smaller δ. Hence we may for each $h \in \overline{Q_\delta^h}$ find functions $\alpha_\eta^{i,j}$ (for some $\eta > 0$) satisfying the properties above, and in particular such that (3.62) is valid.

Next we observe that for any $h \in \overline{\mathcal{Q}_\delta^h}$, setting $q = (h,0)$, the unique solution $\hat{\alpha}_\eta^{i,j}(\tilde{q},q,\cdot)$ (up to an additive constant) to the Neumann problem $\Delta_x \hat{\alpha}_\eta^{i,j}(\tilde{q},q,x) = 0$ in $\mathcal{F}(\tilde{q})$, $\partial_n \hat{\alpha}_\eta^{i,j}(\tilde{q},q,x) = 0$ on $\partial\mathcal{F}(\tilde{q}) \setminus \Sigma$, $\partial_n \hat{\alpha}_\eta^{i,j}(\tilde{q},q,x) = \partial_n \alpha_\eta^{i,j}(q,x)$ on Σ, is continuous with respect to $\tilde{q} \in \mathcal{Q}_\delta$. It follows that when a family of functions $\alpha_\eta^{i,j}$ satisfies (3.62) at some point $h \in \overline{Q_\delta^h}$, it satisfies (3.62) (with perhaps 2ν in the right hand side) in some neighborhood of h. Since $\overline{Q_\delta^h}$ is compact and can be covered with such neighborhoods, one can extract a finite subcover and use a partition of unity (according to the variable q) adapted to this subcover to conclude : one gets an estimate like (3.62) with $C\nu$ on the right hand side (for some constant C). It is then just a matter of considering ν/C rather than ν at the beginning. □

Finally our basic bricks to prove Proposition 3.7.1 are given in the following lemma, where we can add the constraint (3.59).

Lemma 3.7.5. *For any $\nu > 0$, there exist continuous mappings $q = (h,0) \in \mathcal{Q}_\delta \mapsto \overline{\alpha}^i(q,\cdot) \in C^\infty(\overline{\mathcal{F}(q)})$, $i \in \{1,2,3\}$, such that for any $q = (h,0) \in \mathcal{Q}_\delta$, $\Delta_x \overline{\alpha}^i(q,x) = 0$ in $\mathcal{F}(q)$, $\partial_n \overline{\alpha}^i(q,x) = 0$ on $\partial\mathcal{F}(q) \setminus \Sigma$ and*

$$\left| \int_{\partial\mathcal{S}(q)} \nabla\overline{\alpha}^i(q,\cdot) \cdot \nabla\overline{\alpha}^j(q,\cdot)\, n\, d\sigma - \delta_{i,j}\, e_i \right| \leq \nu, \tag{3.63}$$

$$\int_{\partial\mathcal{S}(q)} \overline{\alpha}^i(q,\cdot)\, n\, d\sigma = 0. \tag{3.64}$$

Proof. Consider the functions $\overline{\alpha}^{i,j}$ given by Lemma 3.7.4. For any $q = (h,0) \in \mathcal{Q}_\delta$, for any $i \in \{1,2,3\}$, the three vectors $\int_{\partial\mathcal{S}(q)} \overline{\alpha}^{i,j}(q,\cdot)\, n\, d\sigma$, where $j \in \{1,2,3\}$, are linearly dependent in $\mathbb{R}^2$; therefore there exists $\lambda^{i,j}(q) \in \mathbb{R}$ such that

$$\sum_{j=1}^{3} \lambda^{i,j}(q) \int_{\partial\mathcal{S}(q)} \overline{\alpha}^{i,j}(q,\cdot)\, n\, d\sigma = 0 \text{ and } \sum_{j=1}^{3} |\lambda^{i,j}(q)|^2 = 1, \tag{3.65}$$

Then one defines $\overline{\alpha}^i(q,\cdot) := \sum_{j=1}^3 \lambda^{i,j}(q)\overline{\alpha}^{i,j}(q,\cdot)$, and one checks that it satisfies (3.63) with some $C\nu$ in the right hand side. Again changing ν in ν/C allows to conclude. □

We are now in position to prove Proposition 3.7.1.

Proof of Proposition 3.7.1. Let $\delta > 0$. Let $\nu > 0$. We define the mapping $\mathcal{S}$ which to $(h,v) \in \mathcal{Q}_\delta^h \times \mathbb{R}^2$ associates the function

$$\tilde{\alpha}(q,\cdot) := \sum_{i=1}^{3} \sqrt{\mu^i(v)}\, \overline{\alpha}^i(q,\cdot),$$

in $C^\infty(\overline{\mathcal{F}(q)})$, where the functions μ^i were introduced in Lemma 3.7.1 and the functions $\overline{\alpha}^i$ were introduced in Lemma 3.7.5. Next we define $\mathcal{T} : \mathcal{Q}_\delta^h \times \mathbb{R}^2 \to \mathcal{Q}_\delta^h \times \mathbb{R}^2$ by

$$(h, v) \mapsto (\mathcal{T}_1, \mathcal{T}_2)(h, v) := \left(h, \int_{\partial\mathcal{S}(q)} |\nabla\tilde{\alpha}(q, \cdot)|^2 \, n \, d\sigma \right), \quad \text{where} \quad \tilde{\alpha} = \mathcal{S}(h, v).$$

Using (3.60) and (3.63), one checks that $\mathcal{T}$ is smooth and that

$$\frac{\partial \mathcal{T}_2}{\partial v} = \mathrm{Id} + \mathcal{O}(\nu).$$

Hence taking ν sufficiently small, we see that $\frac{\partial \mathcal{T}_2}{\partial v}$ is invertible, hence $\frac{\partial \mathcal{T}}{\partial (h,v)}$ is invertible. Consequently one can use the inverse function theorem on $\mathcal{T}$: for each $h_0 \in \overline{Q_\delta^h}$ it realizes a local diffeomorphism at $(h_0, 0)$, and hence on $\overline{Q_\delta^h} \times B(0, r)$ for $r > 0$ small enough. This gives the result of Proposition 3.7.1 for v small : given $(h, v) \in \overline{Q_\delta^h} \times B(0, r)$, we let $(h, \tilde{v}) := \mathcal{T}^{-1}(h, v)$. Then the functions $\overline{\alpha} := \sum_{i=1}^3 \sqrt{\mu^i(\tilde{v})}\, \overline{\alpha}^i(q, \cdot)$ and $\overline{g} := \mathbb{1}_\Sigma \, \partial_n \overline{\alpha}$ satisfy the requirements. The general case follows by linearity of (3.57) and (3.59) and by homogeneity of (3.58). This ends the proof of Proposition 3.7.1. □

3.7.2 The case when $\mathcal{S}_0$ is not a disk

We now get back to the proof of Proposition 3.5.1. We will denote by $\mathrm{coni}(A)$ the conical hull of A, namely

$$\mathrm{coni}(A) := \left\{ \sum_{i=1}^k \lambda_i a_i, \ k \in \mathbb{N}^*, \ \lambda_i \geq 0, \ a_i \in A \right\},$$

The first step is the following elementary geometric lemma.

Lemma 3.7.6. *Let $\mathcal{S}_0 \subset \Omega$ bounded, closed, simply connected with smooth boundary, which is not a disk. Then coni$\{(n(x), (x - h_0)^\perp \cdot n(x)), \ x \in \partial\mathcal{S}_0\} = \mathbb{R}^3$.*

Proof. Suppose the contrary. Then there exists a plane separating (in the large sense) the origin in $\mathbb{R}^3$ from the set $\mathrm{coni}(\{(n(x), (x - h_0)^\perp \cdot n(x)), \ x \in \partial\mathcal{S}_0\})$. We claim that a normal vector to this plane can be put in the form $(a, b, 1)$, with $a, b \in \mathbb{R}$. Indeed, otherwise it would need to be of the form $(a, b, 0)$, and the separation inequality would give $(a, b) \cdot n(x) \geq 0, \ \forall x \in \partial\mathcal{S}_0$. However, since $\partial\mathcal{S}_0$ is a smooth, closed curve, the set $\{n(x) : \ x \in \partial\mathcal{S}_0\}$ is the unit circle of $\mathbb{R}^2$, therefore we have a contradiction.

Now we deduce that we have the following separation property :

$$(a, b) \cdot n(x) + (x - h_0)^\perp \cdot n(x) \geq 0, \quad \forall x \in \partial\mathcal{S}_0.$$

Denoting $w = (a, b) - h_0^\perp$, this translates into $(w + x^\perp) \cdot n(x) \geq 0$. But using Green's formula, we get

$$0 \leq \int_{\partial\mathcal{S}_0} (w + x^\perp) \cdot n(x) \, d\sigma = \int_{\mathcal{S}_0} \mathrm{div}(w + x^\perp) \, dx = 0,$$

and consequently, we deduce that $(w + x^\perp) \cdot n(x) = 0$ for all x in $\partial\mathcal{S}_0$. This is equivalent to $(x - w^\perp) \cdot \tau(x) = 0$ for all x in $\partial\mathcal{S}_0$. Parameterizing the translated curve $\partial\mathcal{S}_0 - w^\perp$ by $\{c(s), \ s \in [0, 1]\}$, it follows that $c(s) \cdot \dot{c}(s) = 0$, for all s in $[0, 1]$, and therefore $|c(s)|^2$ is constant. This means that $\partial\mathcal{S}_0 - w^\perp$ is a circle, so $\mathcal{S}_0$ is a disk, which is a contradiction. □

Fix $q_0 \in Q_\delta$. Recalling the definitions of the Kirchhoff potentials in (3.14) and (3.15), we infer from the previous lemma that

$$\mathrm{coni}\{\partial_n \Phi(q_0, x), \ x \in \partial\mathcal{S}_0\} = \mathbb{R}^3.$$

In place of Lemma 3.7.1, we have the following lemma which is a straightforward consequence of Lemma 3.7.6 and of a repeated application of Carathéodory's theorem on the convex hull.

Lemma 3.7.7. *There are some* $(x_i)_{i\in\{1,\dots,16\}}$ *in* $\partial\mathcal{S}_0$ *and positive continuous mappings* $\mu_i : \mathbb{R}^3 \to \mathbb{R},\ 1 \leqslant i \leqslant 16,\ v \mapsto \mu_i(v)$ *such that* $\sum_{i=1}^{16} \mu_i(v)\partial_n\Phi(q_0, x_i) = v$.

We are now in position to establish Proposition 3.5.1. We deduce from Lemma 3.7.7 that for any $q := (h, \vartheta) \in \overline{Q_\delta}$, for any v in $\mathbb{R}^3$,

$$\sum_{i=1}^{16} \mu_i(\mathcal{R}(\vartheta)v)\, \partial_n\Phi(q, x_i(q)) = \mathcal{R}(\vartheta)v,$$

where $x_i(q) := R(\vartheta)(x_i - h_0) + h$ and $\mathcal{R}(\vartheta)$ denotes the 3×3 rotation matrix defined by

$$\mathcal{R}(\vartheta) := \begin{pmatrix} R(\vartheta) & 0 \\ 0 & 1 \end{pmatrix}.$$

Due to the Riemann mapping theorem, there exists a biholomorphic mapping $\Psi : \overline{\mathbb{C}} \setminus B(0,1) \to \overline{\mathbb{C}} \setminus \mathcal{S}(q)$ with $\partial\mathcal{S}(q) = \Psi(\partial B(0,1))$, where $\overline{\mathbb{C}}$ denotes the Riemann sphere. We consider the parametrisations $\{c(s) = (\cos(s), \sin(s)),\ s \in [0, 2\pi]\}$ of $\partial B(0,1)$, respectively $\{\Psi(c(s)),\ s \in [0, 2\pi]\}$ of $\partial\mathcal{S}(q)$, and the corresponding s_i such that $x_i(q) = \Psi(c(s_i))$, for $i \in \{1, \dots, 16\}$.

Then, for any smooth function $\alpha : \partial\mathcal{S}(q) \to \mathbb{R}$, due to the Cauchy-Riemann relations, we have the following :

$$\partial_n\alpha(\Psi(x)) = \frac{1}{\sqrt{|\det(D\Psi(x))|}}\partial_{n_B}(\alpha \circ \Psi)(x),$$

$$\int_{\partial\mathcal{S}(q)} |\nabla\alpha(x)|^2\, \partial_n\Phi(q, x)\, d\sigma = \int_{\partial B(0,1)} |\nabla(\alpha(\Psi(x)))|^2\, \partial_n\Phi(q, \Psi(x))\, \frac{1}{\sqrt{|\det(D\Psi(x))|}}\, d\sigma,$$

for any $x \in \partial B(0,1)$, where n and n_B respectively denote the normal vectors on $\partial\mathcal{S}(q)$ and $\partial B(0,1)$. Note that, since Ψ is invertible, we have $|\det(D\Psi(x))| > 0$, for any $x \in \partial B(0,1)$.

For each $\varepsilon > 0$, $i \in \{1, \dots, 16\}, j \in \{1, 2, 3, 4\}$ (here the index j belongs to $\{1, 2, 3, 4\}$ rather than $\{1, 2, 3\}$ in order to adapt the linear dependence argument of Lemma 3.7.5 to the case of the three linear constraints (3.47)), we consider families of smooth functions $\beta_\varepsilon^{i,j} : [0, 2\pi] \to \mathbb{R}$ satisfying $\operatorname{supp} \beta_\varepsilon^{i,j} \cap \operatorname{supp} \beta_\varepsilon^{k,l} = \emptyset$ for $(i,j) \neq (k,l)$, $\operatorname{diam}\left(\operatorname{supp} \beta_\varepsilon^{i,j}\right) \to 0$ as $\varepsilon \to 0^+$,

$$\int_0^{2\pi} \beta_\varepsilon^{i,j}(s)\, ds = 0,$$

and

$$\left| \int_0^{2\pi} |\beta_\varepsilon^{i,j}(s)|^2\, \partial_n\Phi(q, \Psi(c(s)))\frac{1}{\sqrt{|\det(D\Psi(c(s)))|}}\, ds - \tilde{e}_i \right| \to 0 \text{ as } \varepsilon \to 0^+,$$

where

$$\tilde{e}_i := \frac{1}{\sqrt{|\det(D\Psi(c(s_i)))|}}\partial_n\Phi(q, x_i(q)).$$

Then one may proceed essentially as in the proof of Proposition 3.7.1. The details are therefore left to the reader.

Chapitre 4

Control at a distance of the motion of a rigid body immersed in a two-dimensional viscous incompressible fluid

4.1 Introduction

In this section we present the fluid-solid model we consider and we state our main results.

4.1.1 The mathematical model

We split the two dimensional plane into two disjoint parts : the closed part $\mathcal{S}(t)$ representing the solid and the open part $\mathcal{F}(t) = \mathbb{R}^2 \setminus \mathcal{S}(t)$ filled with fluid. These parts depend on time $t \in [0, T]$, where $T > 0$. Furthermore, we assume that $\mathcal{S}(t)$ is smooth and simply connected. On the fluid part $\mathcal{F}(t)$, the velocity field $u : [0, T] \times \overline{\mathcal{F}(t)} \to \mathbb{R}^2$ and the pressure field $\pi : [0, T] \times \overline{\mathcal{F}(t)} \to \mathbb{R}$ satisfy the incompressible Navier-Stokes equations with an added source term $\xi : [0, T] \times \overline{\mathcal{F}(t)} \to \mathbb{R}^2$, that is

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla\pi - \Delta u = \xi \quad \text{and} \quad \text{div } u = 0 \quad \text{for} \quad t \in [0, T] \text{ and } x \in \mathcal{F}(t). \tag{4.1}$$

Furthermore, we assume that the support of ξ is in a smooth, compact, simply connected set $\Omega_c \subset \mathbb{R}^2$ with non-empty interior.

We consider an impermeability boundary condition and a Navier slip-with-friction condition on the solid boundary, namely,

$$u \cdot n = u_S \cdot n \ , \quad (D(u)n)_{\tan} = -\mu(u - u_S)_{\tan} \text{ on } \partial\mathcal{S}(t), \tag{4.2}$$

where u_S denotes the solid velocity described below, n is the unit outward normal vector on $\partial\mathcal{S}(t)$, $\mu \geq 0$ is the coefficient of friction, and for any vector field f, we have

$$D(f) = \frac{1}{2}\left(\nabla f + (\nabla f)^T\right) \text{ and } (f)_{\tan} = f - (f \cdot n)n.$$

Furthermore, we consider a zero limit condition at infinity, namely,

$$|u| \to 0 \text{ as } |x| \to +\infty. \tag{4.3}$$

The solid $\mathcal{S}(t)$ is obtained by a rigid movement from $\mathcal{S}(0) = \mathcal{S}_0$, and one can describe its position by the center of mass, $h(t)$, and the angle variable with respect to the initial position, $\vartheta(t)$. Consequently, we have

$$\mathcal{S}(t) = h(t) + R(\vartheta(t))\mathcal{S}_0, \tag{4.4}$$

where the center of mass at initial time is assumed to be $h_0 = 0$ without loss of generality, and

$$R(\vartheta) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}.$$

Moreover the solid velocity is hence given by

$$u_S(t,x) = h'(t) + \vartheta'(t)(x - h(t))^{\perp}, \tag{4.5}$$

where for $x = (x_1, x_2)$ we denote $x^{\perp} = (-x_2, x_1)$.

The solid evolves according to Newton's law, and is influenced by the Cauchy stress tensor on the boundary :

$$\begin{aligned} mh''(t) &= -\int_{\partial\mathcal{S}(t)} (-\pi \mathrm{Id} + 2D(u))\, n \; d\sigma, \\ \mathcal{J}\vartheta''(t) &= -\int_{\partial\mathcal{S}(t)} (x - h(t))^{\perp} \cdot (-\pi \mathrm{Id} + 2D(u)) n \, d\sigma. \end{aligned} \tag{4.6}$$

Here the constants $m > 0$ and $\mathcal{J} > 0$ denote respectively the mass and the moment of inertia of the body, where the fluid is supposed to be homogeneous of density 1, without loss of generality.

We consider the following initial conditions :

$$\begin{aligned} u|_{t=0} &= u_0 \text{ for } x \in \mathcal{F}_0, \\ h(0) = 0,\ h'(0) = h'_0,\ \vartheta(0) &= 0,\ \vartheta'(0) = \vartheta'_0. \end{aligned} \tag{4.7}$$

For the initial data we will asume $u_0 \in H^4(\mathcal{F}_0)$ and $\operatorname{curl} u_0 \in L^1(\mathcal{F}_0)$, satisfying the compatibility conditions div $u_0 = 0$ in $\mathcal{F}_0$, $u_0 \cdot n = (h'_0 + \vartheta'_0 x^{\perp}) \cdot n$ and $(D(u_0)n)_{\tan} = -\mu(u_0 - (h'_0 + \vartheta'_0 x^{\perp}))_{\tan}$ on $\partial\mathcal{S}_0$. The integrability of the initial vorticity is assumed in order to guarantee that the circulation at infinity is well-defined.

Throughout this Chapter we will only consider solid trajectories which stay away from the control zone, therefore our construction will satisfy the following condition :

$$\operatorname{supp} \xi(t,\cdot) \cap \mathcal{S}(t) = \emptyset,\ \forall t \in [0,T]. \tag{4.8}$$

4.1.2 Definition of weak solutions

We will now present a notion of Leray-type weak solution to the fluid-solid system.

Let $\xi \in L^2((0,T) \times \Omega_c)$ be fixed. In order to define a notion of weak solution to (4.1), (4.2), (4.3), (4.6), (4.7), we introduce for each $t \in [0,T]$ the following spaces :

$$\begin{aligned} \mathcal{H}(t) &:= \left\{ \phi \in L^2(\mathbb{R}^2;\mathbb{R}^2) :\ \operatorname{div} \phi = 0 \text{ in } \mathbb{R}^2,\ D(\phi) = 0 \text{ in } \mathcal{S}(t) \right\}, \\ \mathcal{V}(t) &:= \left\{ \phi \in \mathcal{H}(t) :\ \nabla\phi \in L^2(\mathcal{F}(t)) \right\}. \end{aligned} \tag{4.9}$$

Note that the above spaces depend on the solution itself through the domains $\mathcal{F}(t)$ and $\mathcal{S}(t)$. Furthermore, according to Lemma 11 in [95], for any $\phi \in \mathcal{H}(t)$, there exists $(l_\phi, r_\phi) \in \mathbb{R}^3$ (which may depend

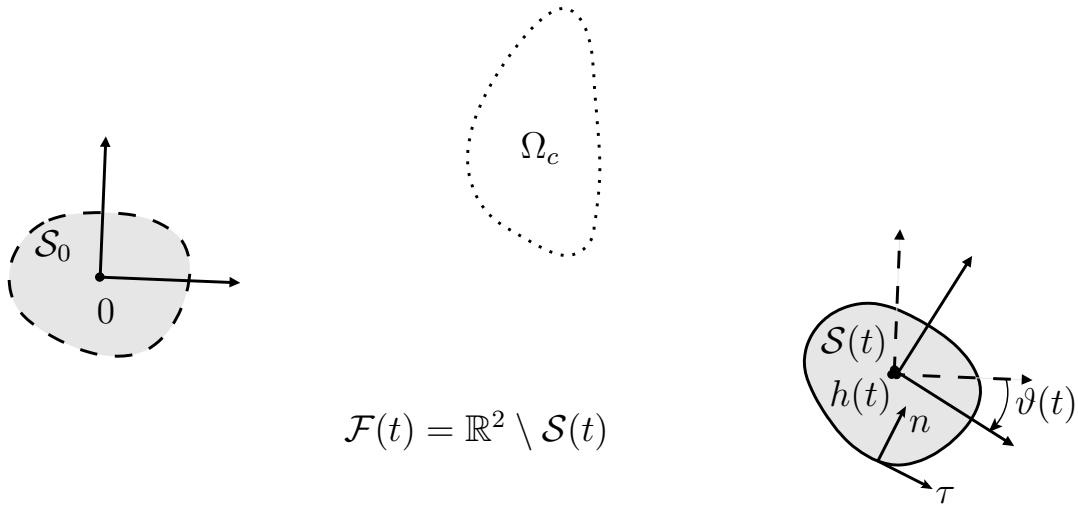

FIGURE 4.1: The setting of the control problem

on t) such that $\phi(x) = \phi_S(x) := l_\phi + r_\phi(x - h(t))^\perp$ in $\mathcal{S}(t)$. Therefore, we extend the initial data u_0 by $h_0' + \vartheta_0' x^\perp$ in $\mathcal{S}_0$.

We further define the following scalar product, which endows $\mathcal{H}(t)$ with a Hilbert space structure,

$$(u, v)_{\mathcal{H}(t)} := \int_{\mathcal{F}(t)} u \cdot v \, dx + m l_u \cdot l_v + \mathcal{J} r_u r_v.$$

We give the following definition of a weak solution "à la Leray".

Definition 4.1.1. *We say that*

$$u \in C([0,T]; \mathcal{H}(t)) \cap L^2((0,T); \mathcal{V}(t))$$

is a weak solution to the system (4.1), (4.2), (4.3), (4.6), (4.7) *if* (4.8) *holds, and we have that for all* $\phi \in C^\infty([0,T]; \mathcal{H}(t))$ *such that* $\phi|_{\overline{\mathcal{F}(t)}} \in C^\infty([0,T]; C_0^\infty(\overline{\mathcal{F}(t)}; \mathbb{R}^2))$, *the following holds on* $[0,T]$,

$$\begin{aligned}(u(t,\cdot), \phi(t,\cdot))_{\mathcal{H}(t)} - (u_0, \phi(0,\cdot))_{\mathcal{H}(0)} &= \int_0^t (u(s,\cdot), \partial_t \phi(s,\cdot))_{\mathcal{H}(s)} \, ds + \int_0^t \int_{\Omega_c} \xi \cdot \phi \, dx \, ds \\ &+ \int_0^t \int_{\mathcal{F}(s)} u \cdot \nabla \phi \cdot u \, dx \, dt - 2 \int_0^t \int_{\mathcal{F}(s)} D(u) : D(\phi) \, dx \, ds \\ &-2\mu \int_0^t \int_{\partial \mathcal{S}(s)} (u(s,\cdot) - u_S(s,\cdot)) \cdot (\phi(s,\cdot) - \phi_S(s,\cdot)) \, d\sigma \, ds,\end{aligned}$$

where the spaces $\mathcal{H}(t)$ *are associated with* $\mathcal{S}(t)$ *as in* (4.9), *and we have that* $\mathcal{S}$ *is transported by the flow of* u_S.

It can be easily checked, by performing some integration by parts, that a strong solution to (4.1), (4.2), (4.3), (4.6), (4.7) is also a weak solution in the above sense (see for instance [86] for further details).

Note that there is a slight abuse of notation in writing $C([0,T]; \mathcal{H}(t))$ for instance, since the domain in which the fluid evolves is also time-dependent. Furthermore, the test functions also depend on the solution u as noted above in the definition of the spaces $\mathcal{H}(t)$ and $\mathcal{V}(t)$.

4.1.3 Main result

Our goal is to investigate the possibility of controlling the solid by the means of prescribing an interior control ξ acting on the fluid. In particular we raise the question of driving the solid from a given position and a given velocity to some other prescribed position and velocity.

In order to ensure that (4.8) holds, we note that we do not need to control on the whole of Ω_c. Instead, we will introduce a set of admissible positions for the solid, such that as long as the final position of the solid is in this set, there exists a fixed open subset of Ω_c, which does not touch the solid neither in its initial nor in its final position, and which we will use as the support of the controls we construct. More precisely, we set

$$\mathcal{Q} = \left\{ q := (h, \vartheta) \in \mathbb{R}^3 : \ \mathrm{int}(\Omega_c) \setminus \{(h + R(\vartheta)\mathcal{S}_0) \cup \mathcal{S}_0\} \neq \emptyset \right\}. \tag{4.10}$$

Since $\mathcal{S}_0$ is simply connected, it can be easily checked that $\mathcal{Q}$ is path-connected.

We are now in position to state our main result regarding the small-time global exact controllability of the solid position and velocity.

Theorem 4.1.1. *Consider $\mathcal{S}_0 \subset \mathbb{R}^2$ bounded, closed, simply connected with smooth boundary, which is not a disk, and $u_0 \in H^4(\mathcal{F}_0; \mathbb{R}^2)$, $\mathrm{curl}\, u_0 \in L^1(\mathcal{F}_0; \mathbb{R}^2)$, $q_0 = 0, q_f = (h_f, \vartheta_f) \in \mathcal{Q}$, $h_0', h_f' \in \mathbb{R}^2, \vartheta_0', \vartheta_f' \in \mathbb{R}$, such that*

$$\mathrm{div}\ u_0 = 0 \ \textit{in}\ \mathcal{F}_0, \quad \lim_{|x| \to +\infty} |u_0(x)| = 0,$$

$$u_0 \cdot n = (h_0' + \vartheta_0' x^\perp) \cdot n, \ (D(u_0)n)_{tan} = -\mu(u_0 - (h_0' + \vartheta_0' x^\perp))_{tan} \ \textit{on}\ \partial\mathcal{S}_0.$$

Then there exists $\tilde{T} > 0$ such that, for any $T \in (0, \tilde{T}]$, there exists a control $\xi \in L^2((0,T) \times \Omega_c)$, compactly supported in time, and a weak solution $u \in C([0,T]; \mathcal{H}(t)) \cap L^2((0,T); \mathcal{V}(t))$ in the sense of Definition 4.1.1 to the system (4.1), (4.2), (4.3), (4.6), (4.7) *with* (4.8), *such that we have $(h, h', \vartheta, \vartheta')(T) = (h_f, h_f', \vartheta_f, \vartheta_f')$.*

Note that the condition that $\mathcal{S}_0$ is not a disk is essential because a significant step during our proof will rely heavily on a strategy similar to the one presented in Chapter 3. As mentioned in the aforementioned Chapter, it is possible to treat the case of a homogeneous disk with a similar strategy, controlling only the center of mass h.

Furthermore, we will present in Section 4.7 a possible strategy for passing to arbitrary time controllability, given a sufficiently strong autoregularization property of the system (that would also allow for less regular initial data), which up to our best knowledge is currently still an open problem in the literature for this type of systems.

We also note that the reason for working in the whole plane instead of a bounded domain will be given in Remark 4.2.3 in Section 4.2 below, once the main ideas behind our proof have been presented.

References. Our result can be contrasted with the result from [27] (see also [28] for a gentle exposition) regarding the controllability of the fluid velocity alone in the case of the incompressible Navier-Stokes equation with Navier slip-with-friction boundary conditions. However, in our case we are only interested in controlling the solid position and velocity, and unlike in [27] we achieve this without any control on the solid boundary. Note that the proof relies on the previous results for the Euler equations by means of a rapid and strong control which drives the system in a high Reynolds regime, a strategy which

originates from [23], where an interior controllability result was already established. For this purpose in our case we generalize the control result regarding 2D "perfect fluid + rigid body" systems from Chapter 3 to treat the viscous case with a similar strategy, but without using the return method, in contrast to [23, 27]. Note also that the main result in Chapter 3 was presented in the absence of vorticity, however, it was mentioned that the effect of the vorticity could be handled by some appropriate control strategy. In this Chapter there will indeed be some vorticity created at the solid boundary, which we will handle by the means of an asymptotic boundary layer expansion.

For "viscous fluid + rigid body" control systems (with Dirichlet boundary conditions), local null-controllability results have already been obtained in both 2D and 3D, see e.g. the works of Boulakia and Guerrero, Boulakia and Osses, respectively Imanuvilov and Takahashi [10, 11, 72]. These results rely on Carleman estimates on the linearized equation, and consequently on the parabolic character of the fluid equation. A similar result has been established in [78] for the case of the 1D viscous Burgers equation with a new strategy introduced by the authors without the use of any Carleman estimates, and as noted in the respective article, those methods can be extended to other nonlinear parabolic systems. However, note that the results mentioned above concern local null-controllability for the solid position and the velocities of both the solid and the fluid, whereas in this Chapter we achieve global exact controllability for both the solid position and velocity directly.

Let us also mention some stabilization results regarding "viscous fluid + rigid body" systems in a bounded domain with Dirichlet boundary conditions, see [6] for the 2D and 3D cases, respectively [7] for a simplified model in the 1D case. In the respective articles the authors stabilize the position and velocity of the solid and the velocity of the fluid using a feedback control on the exterior boundary of the fluid domain, assuming that the initial data of the system is close to a stationary state (which is not necessarily assumed to be zero).

A different type of problem regarding fluid-solid interactions is that of a deformable body in a fluid, regarding the dynamics of swimming, see for instance [17, 40, 77, 90, 91] for the viscous case, respectively [18] for the inviscid case. In the case of such problems, the control is no longer at a distance, rather it consists of the deformation of the body itself.

Generalizations and open problems. A natural generalization of the problems above would be the passage from the two-dimensional case to the three-dimensional one. The main difficulty in adapting our methods to the 3D case (apart from the Cauchy theory of the 3D system) is the use of complex analysis to explicitly construct the spacial part of the control (see the Appendices of the Chapter). However, one could replace these arguments by a Cauchy-Kovalevskaya type construction and a higher dimensional generalization of Runge's theorem (c.f. [38]), so that a similar result could be established in the three-dimensional case.

Furthermore, one could also be interested in controlling several solids. Indeed, one can see in the Appencides of the Chapter that the construction of the spacial part of the vector field associated with control is quite local around the solid, so the arguments in our proof should be adaptable to the case of multiple solids, also guaranteeing that there is no collision between the solids.

Another interesting open problem is that of the motion planning of a rigid body immersed in a viscous incompressible fluid. Namely, suppose that we have a fixed curve $\Gamma \in C^2([0,T];\mathbb{R}^3)$, and the conditions of Theorem 4.1.1 are satisfied with $(h_0, \vartheta_0, h_0', \vartheta_0') = (\Gamma(0), \Gamma'(0))$, $(h_f, \vartheta_f, h_f', \vartheta_f') = (\Gamma(T), \Gamma'(T))$. Does there exist a control and a solution to the fluid-solid system as described in Theorem

4.1.1, satisfying in addition $\Gamma = (h, \vartheta)$ on $[0, T]$? Even the approximate motion planning in C^2, i.e. the same statement as above but with $\|\Gamma - (h, \vartheta)\|_{C^2([0,T])} \leq \varepsilon$ (with $\varepsilon > 0$ arbitrary) instead of $\Gamma = (h, \vartheta)$, is an open problem. Furthermore, as mentioned in Chapter 3, the motion planning for a rigid body in an inviscid fluid is also open. However, there might be some hope to adapt certain techniques presented in this chapter (specifically in Section 4.3 to pass from approximate controllability to exact controllability), in order to tackle this problem in the future.

Finally, we mention that the global exact controllability of a "viscous fluid + rigid body" system with Dirichlet "no slip" boundary conditions is completely open, and a very challenging problem due to the fact that the Dirichlet boundary conditions create boundary layers with a larger amplitude than in the case of Navier slip-with-friction boundary conditions. Note that even the problem of controlling only the fluid velocity in such a context is open and similarly challenging, some recent advances have been made in [29] in the very particular case when the domain is assumed to be a rectangle, using an added distributed phantom force.

Plan of the Chapter. The Chapter is organized as follows.

In Section 4.2 we give some preliminary results, such as reducing Theorem 4.1.1 to the case of a fixed domain and small viscosity, which we further reduce to constructing an appropriate asymptotic expansion with respect to the viscosity.

In Section 4.3 we construct the terms of order $\mathcal{O}(1)$ in the asymptotic expansion by a generalization of the geodesic method used in Chapter 3.

In Section 4.4 we construct the boundary layer profiles associated with u^0, which appear in particular at order $\mathcal{O}(\sqrt{\varepsilon})$ and $\mathcal{O}(\varepsilon)$.

In Section 4.5 we construct the linearized terms of $\mathcal{O}(\varepsilon)$ in the asymptotic expansion by a different impulsive control strategy.

In Section 4.6 we construct and estimate the remainder in the asymptotic expansion and prove that it converges to zero in an appropriate space.

We conclude the Chapter in Section 4.7 with some visual representations of the controls constructed during our strategy, as well as some remarks regarding the passage to arbitrary time controllability.

In Appendix 4.A we explicitly construct the controls by the means of complex analysis.

4.2 Preliminary reductions

In this section we will prove that Theorem 4.1.1 can be reduced to the case where the fluid domain is fixed and the fluid viscosity is small. Furthermore, we will show that in this case one can introduce a vanishing viscosity asymptotic expansion for the solid trajectory in order to prove our main result.

4.2.1 A reduction of Theorem 4.1.1 to the case of a fixed domain and small viscosity

The goal of this section is to prove that Theorem 4.1.1 can be deduced from a controllability result for the following system :

$$\begin{aligned}
&\frac{\partial u^\varepsilon}{\partial t} + (u^\varepsilon - u_S^\varepsilon)\cdot\nabla u^\varepsilon + r^\varepsilon (u^\varepsilon)^\perp + \nabla\pi^\varepsilon - \varepsilon\Delta u^\varepsilon = 0 \ \text{ and } \ \operatorname{div} u^\varepsilon = g^\varepsilon \ \text{ for } x\in\mathcal{F}_0,\\
&\quad u^\varepsilon\cdot n = u_S^\varepsilon\cdot n, \ \ (D(u^\varepsilon)n)_{\tan} = -\mu(u^\varepsilon - u_S^\varepsilon)_{\tan} \ \text{ for } x\in\partial\mathcal{S}_0, \ \lim_{|x|\to+\infty}|u^\varepsilon| = 0,\\
&\quad m(l^\varepsilon)' = -\int_{\partial\mathcal{S}_0}(-\pi^\varepsilon \mathrm{Id} + 2\varepsilon D(u^\varepsilon))\, n\, d\sigma - m r^\varepsilon (l^\varepsilon)^\perp,\\
&\quad \mathcal{J}\,(r^\varepsilon)' = -\int_{\partial\mathcal{S}_0} x^\perp\cdot(-\pi^\varepsilon\mathrm{Id} + 2\varepsilon D(u^\varepsilon))n\, d\sigma,
\end{aligned} \tag{4.11}$$

where $u_S^\varepsilon(t,x) = l^\varepsilon(t) + r^\varepsilon(t)x^\perp$, for $t\in[0,T]$, with $\varepsilon>0$, $u^\varepsilon(0,\cdot) = \varepsilon u_0(\cdot), (l^\varepsilon, r^\varepsilon)(0) = \varepsilon(h_0', \vartheta_0')$, and the control term is now $g^\varepsilon \in C_0^\infty((0,T)\times\mathcal{F}_0)$ such that supp $g^\varepsilon(t,\cdot)\subset R(\vartheta^\varepsilon(t))^T(\Omega_c - h^\varepsilon(t))$ and $\int g^\varepsilon(t,\cdot)\,dx = 0$, for any $t\in[0,T]$.

Note that in (4.11) the viscosity coefficient ε appears in front of the term $D(u^\varepsilon)$ in the solid equations (as part of the Cauchy stress tensor), but not in the Navier slip-with-friction boundary condition on $\partial\mathcal{S}_0$. This will be essential in the asymptotic expansion presented in Section 4.2.3.

Furthermore, we may associate the solid position $(h^\varepsilon, \vartheta^\varepsilon)$, which no longer plays a role directly in solving system (4.11), through

$$h^\varepsilon(t) = \int_0^t R(\vartheta^\varepsilon(s))l^\varepsilon(s)\,ds, \quad \vartheta^\varepsilon(t) = \int_0^t r^\varepsilon(s)\,ds, \tag{4.12}$$

for $t\in[0,T]$.

We have the following adaptation of (4.8) for g^ε,

$$\text{supp } g^\varepsilon(t,\cdot)\cap\mathcal{S}_0 = \emptyset, \ \forall t\in[0,T]. \tag{4.13}$$

Let us first give a definition of so-called "very weak solutions" to viscous fluid-solid models as (4.11) with non-zero divergence (similarly to the same notion for the fluid alone, as done for instance in [33]). To do so, we further introduce the following spaces :

$$\mathcal{H} := \left\{\phi\in L^2(\mathbb{R}^2;\mathbb{R}^2) : \ D(\phi) = 0 \text{ and } \operatorname{div}\phi = 0 \text{ in } \mathcal{S}_0\right\}, \ \mathcal{V} := \ \mathcal{H}\cap H^1(\mathcal{F}_0).$$

Once more, note that for any $\phi\in\mathcal{H}$, there exists $(l_\phi, r_\phi)\in\mathbb{R}^3$ such that $\phi(x) = \phi_S(x) := l_\phi + r_\phi x^\perp$ in $\mathcal{S}_0$. Therefore, we may once again extend the initial data u_0 by $h_0' + \vartheta_0' x^\perp$ in $\mathcal{S}_0$. We consider the following scalar product, which endows $\mathcal{H}$ with a Hilbert space structure,

$$(u,v)_{\mathcal{H}} := \int_{\mathcal{F}_0} u\cdot v\,dx + m l_u\cdot l_v + \mathcal{J} r_u r_v.$$

Note that $(\cdot,\cdot)_{\mathcal{H}}$ coincides with the scalar product for the Hilbert space $\mathcal{H}(0)$ defined in (4.9), however, $\mathcal{H}(0)$ is a strict subspace of $\mathcal{H}$.

Definition 4.2.1. *We say that*

$$u^\varepsilon \in C([0,T];\mathcal{H})\cap L^2((0,T);\mathcal{V})$$

is a very weak solution to the system (4.11) *if* (4.13) *holds, if we have*

$$\operatorname{div} u^\varepsilon = g^\varepsilon, \text{ for a.e. } t \in [0,T],$$

and if, for all $\phi \in C^\infty([0,T];\mathcal{H}(0))$ *such that* $\phi|_{\overline{\mathcal{F}_0}} \in C^\infty([0,T];C_0^\infty(\overline{\mathcal{F}_0};\mathbb{R}^2))$*, the following holds on* $[0,T]$*,*

$$\begin{aligned}(u^\varepsilon(t,\cdot),\phi(t,\cdot))_{\mathcal{H}} - \varepsilon(u_0,\phi(0,\cdot))_{\mathcal{H}} &= \int_0^t (u^\varepsilon(s,\cdot),\partial_t\phi(s,\cdot))_{\mathcal{H}}\,ds + \int_0^t\int_{\mathcal{F}_0} g^\varepsilon\, u^\varepsilon\cdot\phi\,dx\,ds\\ &+\int_0^t\int_{\mathcal{F}_0}\left[(u^\varepsilon-u_S^\varepsilon)\cdot\nabla\phi\cdot u^\varepsilon - r^\varepsilon(u^\varepsilon)^\perp\cdot\phi\right]dx\,ds - \int_0^t mr^\varepsilon(l^\varepsilon)^\perp\cdot l_\phi\,ds\\ &-2\varepsilon\int_0^t\int_{\mathcal{F}_0} D(u^\varepsilon):D(\phi)\,dx\,ds - 2\varepsilon\mu\int_0^t\int_{\partial\mathcal{S}_0}(u^\varepsilon-u_S^\varepsilon)\cdot(\phi-\phi_S)\,d\sigma\,ds.\end{aligned}$$

Note that in fact the above definition can be extended to less regular divergence terms g^ε (as done in [33] for example). But for our purposes the case of smooth g^ε will suffice, since our construction for such solutions will rely on a linear decomposition into a smooth term which has divergence g^ε, and the remaining term defined as the weak solution of a Navier-Stokes type system with zero divergence (in the Leray sense, since the divergence is considered to be zero).

We claim that Theorem 4.1.1 follows from the following result.

Theorem 4.2.1. *Consider* $T>0$*,* $\mathcal{S}_0\subset\mathbb{R}^2$ *bounded, closed, simply connected with smooth boundary, which is not a disk, and* $u_0\in H^4(\mathcal{F}_0;\mathbb{R}^2)$*,* $\operatorname{curl} u_0\in L^1(\mathcal{F}_0;\mathbb{R}^2)$*,* $q_0=0, q_f=(h_f,\vartheta_f)\in\mathcal{Q}$*,* $h_0',h_f'\in\mathbb{R}^2,\vartheta_0',\vartheta_f'\in\mathbb{R}$*, such that*

$$\operatorname{div} u_0 = 0 \text{ in } \mathcal{F}_0,\quad \lim_{|x|\to+\infty}|u_0(x)|=0,$$
$$u_0\cdot n = (h_0'+\vartheta_0'x^\perp)\cdot n,\ (D(u_0)n)_{tan} = -\mu(u_0-(h_0'+\vartheta_0'x^\perp))_{tan} \text{ on } \partial\mathcal{S}_0.$$

Then there exists $\bar\varepsilon\in(0,1)$ *such that, for any* $\varepsilon\in(0,\bar\varepsilon]$*, there exists a control* $g^\varepsilon\in C_0^\infty((0,T)\times\mathcal{F}_0)$ *with* $\int g^\varepsilon=0$*, supp* $g^\varepsilon(t,\cdot)\subset R(\vartheta^\varepsilon(t))^T(\Omega_c-h^\varepsilon(t))$*, and a very weak solution* $u^\varepsilon\in C([0,T];\mathcal{H})\cap L^2((0,T);\mathcal{V})$ *in the sense of Definition 4.2.1 to* (4.11) *such that we have*

$$(h^\varepsilon,l^\varepsilon,\vartheta^\varepsilon,r^\varepsilon)(T) = (h_f,\varepsilon R(\vartheta_f)^T h_f',\vartheta_f,\varepsilon\vartheta_f'), \tag{4.14}$$

with $(h^\varepsilon,\vartheta^\varepsilon)$ *given by* (4.12)*.*

Proof of Theorem 4.1.1 from Theorem 4.2.1. We will show that given $T>0$, g^ε as above, there exists some $\xi\in L^2((0,\varepsilon T)\times\Omega_c)$ and some appropriate transformations such that we may deduce the existence of a solution to system (4.1), (4.2), (4.3), (4.6), (4.7) on $[0,\varepsilon T]$ from the existence of a solution to system (4.11), (4.12) on $[0,T]$. We will introduce a change of variables for passing from small viscosity to viscosity 1 and for passing from a fixed domain to a moving domain, and we will switch from a control on the divergence to a control on the evolution equation via the Bogovskii operator (see for instance [9] or [43]), defined as follows.

Definition 4.2.2. *Given a smooth, bounded domain* $\Omega\subset\mathbb{R}^2$*, there exists an operator* $B: C_0^\infty(\Omega)\to C_0^\infty(\Omega)^2$ *such that, for any* $g\in C_0^\infty(\Omega)$ *with* $\int g=0$*, we have* $\operatorname{div} Bg = g$*. Furthermore,* $B\in\mathcal{L}(W_0^{s,p}(\Omega),W_0^{s+1,p}(\Omega)^2)$*, for any* $1<p<+\infty$*,* $s\geq 0$*. Also observe that we may extend* Bg *by* 0 *outside of* Ω*.*

From now on we consider B to be the Bogovskii operator associated with Ω_c. Given $g^\varepsilon \in C_0^\infty((0,T)\times \mathcal{F}_0)$ with supp $g^\varepsilon(t,\cdot) \subset R(\vartheta^\varepsilon(t))^T(\Omega_c - h^\varepsilon(t))$ and $u^\varepsilon \in C([0,T];\mathcal{H})\cap L^2((0,T);\mathcal{V})$ as in Theorem 4.2.1, we define

$$\begin{aligned} g(t,x) &:= \varepsilon^{-1} g^\varepsilon\left(\varepsilon^{-1}t, R\left(\vartheta^\varepsilon\left(\varepsilon^{-1}t\right)\right)^T\left(x - h^\varepsilon\left(\varepsilon^{-1}t\right)\right)\right), \\ U(t,x) &:= \varepsilon^{-1} R\left(\vartheta^\varepsilon\left(\varepsilon^{-1}t\right)\right) u^\varepsilon\left(\varepsilon^{-1}t, R\left(\vartheta^\varepsilon\left(\varepsilon^{-1}t\right)\right)^T\left(x - h^\varepsilon\left(\varepsilon^{-1}t\right)\right)\right), \\ h'(t) &:= \varepsilon^{-1} R\left(\vartheta^\varepsilon\left(\varepsilon^{-1}t\right)\right) l^\varepsilon\left(\varepsilon^{-1}t\right), \quad \vartheta'(t) := \varepsilon^{-1} r^\varepsilon\left(\varepsilon^{-1}t\right), \end{aligned} \tag{4.15}$$

for $t \in [0,\varepsilon T]$ and $x \in \overline{\mathcal{F}(t)}$.

We set

$$\xi := -\frac{\partial Bg}{\partial t} + Bg\cdot\nabla Bg - U\cdot\nabla Bg - Bg\cdot\nabla U + \Delta Bg \in L^2((0,\varepsilon T)\times\Omega_c) \tag{4.16}$$

and $u := U - Bg$. It can be easily checked that (u,h,ϑ,ξ) defined in this way allow us to deduce a weak solution in the sense of Definition 4.1.1.

Finally, we may conclude the proof by noting that we have $(h,\vartheta)(\varepsilon T) = (h^\varepsilon,\vartheta^\varepsilon)(T) = (h_f,\vartheta_f)$ and $(h',\vartheta')(\varepsilon T) = \varepsilon^{-1}(R(\vartheta^\varepsilon(T))l^\varepsilon(T), r^\varepsilon(T)) = (h'_f,\vartheta'_f)$, by using (4.14). Therefore, the conclusions of Theorem 4.1.1 are satisfied with εT instead of T. □

Remark 4.2.1. *In Theorem 4.2.1 the control g^ε can be chosen with an arbitrary small total flux through its support, that is for any $\delta_c > 0$, there exists a control g^ε and a very weak solution u^ε satisfying the properties of Theorem 4.2.1 and such that moreover*

$$\left|\int_0^T \int_{R(\vartheta^\varepsilon(t))^T(\Omega_c - h^\varepsilon(t))} (g^\varepsilon(t,x))_-\ dx\,dt\right| < \delta_c.$$

See Section 4.2.4 for more explanations. Let us mention that such a small flux condition cannot be guaranteed in the results [23, 27] regarding the controllability of the Navier-Stokes equations.

4.2.2 Proving Theorem 4.2.1 by the means of an asymptotic expansion for the solid trajectory

We introduce the following asymptotic expansion for the solid trajectory :

$$\begin{aligned} h^\varepsilon = h^0 + \varepsilon h^1 + \varepsilon h^\varepsilon_R,\ \vartheta^\varepsilon = \vartheta^0 + \varepsilon\vartheta^1 + \varepsilon\vartheta^\varepsilon_R, \\ l^\varepsilon = l^0 + \varepsilon l^1 + \varepsilon l^\varepsilon_R,\ r^\varepsilon = r^0 + \varepsilon r^1 + \varepsilon r^\varepsilon_R, \end{aligned} \tag{4.17}$$

with $(l^0,r^0),(l^1,r^1),(l^\varepsilon_R,r^\varepsilon_R) \in L^\infty(0,T)$.

Now suppose that we could exactly drive $(h^0,\vartheta^0)(T)$ to (h_f,ϑ_f), while we had $(h^1,\vartheta^1)(T)$ and $(h^\varepsilon_R,\vartheta^\varepsilon_R)(T)$ bounded in $\varepsilon > 0$. It would follow that we have approximately driven $(h^\varepsilon,\vartheta^\varepsilon)(T)$ to (h_f,ϑ_f), for $\varepsilon > 0$ small enough.

Furthermore, suppose that at the same time we managed to exactly drive $(l^0,r^0)(T)$ to 0 and (approximately) drive $(l^1,r^1)(T)$ to $(R(\vartheta_f)^T h'_f,\vartheta'_f)$, while $(l^\varepsilon_R,r^\varepsilon_R)(T) \to 0$ as $\varepsilon \to 0^+$. It would follow that we have approximately driven $(l^\varepsilon,r^\varepsilon)(T)$ to $\varepsilon(R(\vartheta_f)^T h'_f,\vartheta'_f)$, for $\varepsilon > 0$ small enough.

However, we want to prove the exact controllability in (4.14). To do so, we may pass from the above mentioned approximate controllability to exact controllability by the means of a topological argument of Brouwer-type, as done in Chapter 3. This further requires some continuity property for our whole

construction with respect to the target data for the solid trajectory. Therefore, we will realize the above construction not only for $(h_f, \vartheta_f, R(\vartheta_f)^T h'_f, \vartheta'_f)$ as given in Theorem 4.2.1, but for any $(h_1, \vartheta_1, l_1, r_1)$ in a small enough ball centered at $(h_f, \vartheta_f, R(\vartheta_f)^T h'_f, \vartheta'_f)$, such that the construction depends continuously on $(h_1, \vartheta_1, l_1, r_1)$.

More precisely, we claim that Theorem 4.2.1 follows from the following controllability result.

Theorem 4.2.2. *Suppose that the conditions of Theorem 4.2.1 are verified. Let $\kappa > 0$ such that*

$$int(\Omega_c) \setminus \bigcup_{(h,\vartheta)\in\overline{B}((h_f,\vartheta_f),\kappa)} \{(h + R(\vartheta)\mathcal{S}_0) \cup \mathcal{S}_0\} \neq \emptyset.$$

For any $\nu > 0$ there exists $\varepsilon_0 = \varepsilon_0(\nu) > 0$, which only depends on $\nu > 0$, such that for any $\varepsilon \in (0, \varepsilon_0]$, for any $(h_1, \vartheta_1, l_1, r_1) \in \overline{B}((h_f, \vartheta_f, R(\vartheta_f)^T h'_f, \vartheta'_f), \kappa)$, there exists a control $g^\varepsilon \in C_0^\infty((0,T) \times \mathcal{F}_0)$ with $\int g^\varepsilon = 0$, supp $g^\varepsilon(t,\cdot) \subset R(\vartheta^\varepsilon(t))^T(\Omega_c - h^\varepsilon(t))$, and a very weak solution $u^\varepsilon \in C([0,T];\mathcal{H}) \cap L^2((0,T);\mathcal{V})$ to (4.11), (4.12), such that (4.17) holds, and we have

$$\begin{aligned} (h^0, \vartheta^0, l^0, r^0)(T) = (h_1, \vartheta_1, 0, 0), \quad |(l^1, r^1)(T) - (l_1, r_1)| \leq \nu, \\ |(h^1, \vartheta^1)(T)| \leq C, \quad |(l^\varepsilon_R, r^\varepsilon_R)(T)| \leq C\varepsilon^{1/8}, \end{aligned} \tag{4.18}$$

where $C > 0$ can depend on $\nu > 0$, but is independent of $(h_1, \vartheta_1, l_1, r_1) \in \overline{B}((h_f, \vartheta_f, R(\vartheta_f)^T h'_f, \vartheta'_f), \kappa)$ and $\varepsilon > 0$. Furthermore, the map

$$(h_1, \vartheta_1, l_1, r_1) \in \overline{B}((h_f, \vartheta_f, R(\vartheta_f)^T h'_f, \vartheta'_f), \kappa) \mapsto (h^\varepsilon, \vartheta^\varepsilon, l^\varepsilon, r^\varepsilon)(T) \tag{4.19}$$

is continuous.

Remark 4.2.2. *We note that we settle for approximate controllability for (l^1, r^1) because it simplifies elements of the proof of Theorem 4.2.2 and it will be sufficient to prove Theorem 4.2.1. Furthermore, the introduction of $\varepsilon_0(\nu) > 0$ serves the purpose of making the trajectory $(h^\varepsilon, \vartheta^\varepsilon)$ stay sufficiently close to (h^0, ϑ^0) such that in order to guarantee condition (4.13), it suffices to guarantee a similar condition for (h^0, ϑ^0), which we will detail during the proof of Theorem 4.2.2.*

Proof of Theorem 4.2.1 from Theorem 4.2.2. We will conclude the proof of Theorem 4.2.1 by a topological argument based on the result below borrowed from [58, pages 32-33].

Lemma 4.2.1. *Let $w_0 \in \mathbb{R}^n$, $\kappa > 0$, $f : \overline{B}(w_0, \kappa) \to \mathbb{R}^n$ a continuous map such that we have $|f(w) - w| \leq \frac{\kappa}{2}$ for any x in $\partial B(w_0, \kappa)$. Then $B(w_0, \frac{\kappa}{2}) \subset f(\overline{B}(w_0, \kappa))$.*

We set $w_0 = (h_f, \vartheta_f, R(\vartheta_f)^T h'_f, \vartheta'_f)$ as in Theorem 4.2.1, and $\kappa > 0$ as in Theorem 4.2.2.

Let $T > 0$, $\nu > 0$ and $(h_1, \vartheta_1, l_1, r_1) \in \overline{B}(w_0, \kappa)$. We apply Theorem 4.2.2 to deduce that, for $\varepsilon > 0$ small enough, we have

$$\begin{aligned} |(h^\varepsilon, \vartheta^\varepsilon)(T) - (h_1, \vartheta_1)| &= |(h^\varepsilon, \vartheta^\varepsilon)(T) - (h^0, \vartheta^0)(T)| = \varepsilon|(h^1, \vartheta^1)(T) + (h^\varepsilon_R, \vartheta^\varepsilon_R)(T)| \leq C\varepsilon, \\ |\varepsilon^{-1}(l^\varepsilon, r^\varepsilon)(T) - (l_1, r_1)| &\leq \varepsilon^{-1}|(l^\varepsilon, r^\varepsilon)(T) - (l^0, r^0)(T) - \varepsilon(l^1, r^1)(T)| + \nu \\ &= |(l^\varepsilon_R, r^\varepsilon_R)(T)| + \nu \leq C\varepsilon^{\frac{1}{8}} + \nu, \end{aligned} \tag{4.20}$$

where, $C = C(\nu) > 0$ is independent of ε and $(h_1, \vartheta_1, l_1, r_1)$.

Therefore, we fix $\nu = \frac{\kappa}{2\sqrt{5}}$ and $\bar{\varepsilon} \in (0, \varepsilon_0(\nu)]$ such that $C\bar{\varepsilon}^{\frac{1}{8}} \leq \frac{\kappa}{2\sqrt{5}}$, and for any $\varepsilon \in (0, \bar{\varepsilon}]$ set

$$f(h_1, \vartheta_1, l_1, r_1) = (h^\varepsilon, \vartheta^\varepsilon, \varepsilon^{-1} l^\varepsilon, \varepsilon^{-1} r^\varepsilon)(T).$$

It follows from (4.20) that $|f(h_1,\vartheta_1,l_1,r_1)-(h_1,\vartheta_1,l_1,r_1)| \leq \frac{\kappa}{2}$, uniformly for $(h_1,\vartheta_1,l_1,r_1) \in \overline{B}(w_0,\kappa)$, and f is continuous due to Theorem 4.2.2. We may conclude the exact controllability result of Theorem 4.2.1 by setting applying Lemma 4.2.1 to deduce that $w_0 = (h_f, \vartheta_f, R(\vartheta_f)^T h'_f, \vartheta'_f)$ is in Range(f).

□

4.2.3 Proving Theorem 4.2.2 - Constructing the control via the asymptotic expansion for the fluid velocity

In order to achieve an expansion as in (4.17) for the solid trajectory, we consider controls g^ε in the form of $g^\varepsilon = g^0 + \varepsilon g^1$, in the style of [27], and we look for the following asymptotic expansion for the fluid velocity and pressure :

$$\begin{aligned} u^\varepsilon &= u^0 + \sqrt{\varepsilon}\{v\} + \varepsilon u^1 + \varepsilon \nabla\theta^\varepsilon + \varepsilon\{w\} + \varepsilon u^\varepsilon_R, \\ \pi^\varepsilon &= \pi^0 + \varepsilon\{Q\} + \varepsilon\pi^1 + \varepsilon\rho^\varepsilon + \varepsilon\pi^\varepsilon_R, \end{aligned} \tag{4.21}$$

where, for $f = f(t,x,z)$, we denote $\{f\}$ its evaluation at $z = \frac{\varphi(x)}{\sqrt{\varepsilon}}$, with some function φ to be specifyd in Section 4.4.1. Therefore, proving Theorem 4.2.2 reduces to constructing the terms in the right-hand side of (4.17), (4.21) in an appropriate way. Note that we will use an energy estimate to prove the smallness of $(l^\varepsilon_R, r^\varepsilon_R)$ as stated in Theorem 4.2.2, which is the main reason for investigating not only the terms in the asymptotic expansion for the solid trajectory, but also the terms in the expansion for u^ε, which include certain boundary layer profiles v, w and $\nabla\theta^\varepsilon$.

Furthermore, let us emphasise that our whole construction will be done in order to have that the fluid and solid velocities $(u^\varepsilon_R, l^\varepsilon_R, r^\varepsilon_R)$ associated with the remainder satisfy a Navier-Stokes-type fluid-solid system with small initial data, some added small source term and small viscosity ("small" with respect to $\varepsilon > 0$), such that an appropriate energy estimate can be achieved.

Therefore, our strategy will be the following :

— We construct g^0 and a smooth solution u^0 to the Euler equation (with control g^0), with zero initial data, hence zero vorticity and zero circulation around the solid, such that we have an exact controllability result for $(h^0,\vartheta^0)(T)$ with $(l^0,r^0)(T) = 0$. However, we note that contrary to [27], our strategy will not rely on a return method for u^0, we will rather just use u^0 to control the solid position $(h^0,\vartheta^0)(T)$. See Theorem 4.3.1 in Section 4.3.

— Due to the Navier slip-with-friction boundary conditions, the fluid velocity boundary layer v will appear near the solid at order $\mathcal{O}(\sqrt{\varepsilon})$, together with its pressure Q at order $\mathcal{O}(\varepsilon)$. Furthermore, at order $\mathcal{O}(\varepsilon)$ we introduce a boundary corrector w and an inner domain corrector $\nabla\theta^\varepsilon$, together with its pressure ρ^ε, as done in [27]. Note that there will be no contribution at order $\mathcal{O}(\sqrt{\varepsilon})$ in the solid equations due to the boundary layers, however, at order $\mathcal{O}(\varepsilon)$ the solid position and velocity $(h^1,\vartheta^1,l^1,r^1)$, and therefore the fluid velocity u^1, will depend on ε in a subtle manner due to ρ^ε. However, we mention that, for simplicity of notation, we will not write this dependence explicitly in the notations u^1, h^1, etc. Furthermore, we stress that since we do not control the fluid velocity u^ε, there is no need to control v (contrary to [27]), it will suffice to prove some regularity estimates to handle the effect of the boundary layers at $\mathcal{O}(\varepsilon)$ in the solid equations for (l^1,r^1), and in the equations of the remainder. Note that v, w and $\nabla\theta^\varepsilon$ only depend on (u^0,l^0,r^0) and their existence is immediate from the existence of (u^0,π^0,l^0,r^0,g^0). See Section 4.4 for some regularity estimates for these boundary layer profiles, which we will use in the energy estimate in Section 4.6, but also to estimate the above-mentioned impact of ρ^ε on (l^1,r^1).

— We construct g^1 and a smooth solution u^1 to a linearized Euler equation around u^0 (with control g^1) such that we have an have an approximate controllability result for $(l^1, r^1)(T)$ (we settle for an approximate controllability here because it simplifies our construction). It would be natural to assume that the initial data of the original system, that is (4.7), would be the initial data in the equation of the linearized term (u^1, l^1, r^1). However, we only have $u_0 \in H^4$, and for simplifying reasons, we would like to work with an initial data which has smooth and compactly supported curl, so that the vorticity associated with the linearized equation for u^1 stays smooth and compactly supported at all times.
This can be achieved with the following modification. We construct a family of divergence-free $u^* \in C^\infty(\overline{\mathcal{F}_0}) \cap L^2(\mathcal{F}_0)$ which is bounded in C^2 with respect to $\varepsilon > 0$ such that we have $\operatorname{curl} u^* \in C_0^\infty(\overline{\mathcal{F}_0})$, $u^* \cdot n = u_0 \cdot n$ on $\partial\mathcal{S}_0$, $|u^*(x)| \to 0$ as $|x| \to +\infty$, and

$$\|u_0 - u^*\|_2 \leq \varepsilon^{1/8}. \tag{4.22}$$

Indeed such a u^* can be straightforwardly constructed using a Helmholtz decomposition and an appropriate mollification of $\operatorname{curl} u_0$.
We then set the initial data for (u^1, l^1, r^1) to be $(u^*, h_0', \vartheta_0')$, and implicitly leave the remaining $u_0 - u^*$ in the initial data for u_R^ε, namely we consider $(u_R^\varepsilon, l_R^\varepsilon, r_R^\varepsilon)(0) = (u_0 - u^*, 0, 0)$. Note that u^* implicitly depends once more on $\varepsilon > 0$, and this gives rise to further dependence of u^1 with respect to $\varepsilon > 0$, which we once again omit from the notations for the sake of simplicity. However, since we have that u^* is bounded in C^2 uniformly with respect to $\varepsilon > 0$, we expect this dependence to be slight enough such that we can have some uniform estimates for u^1 with respect to $\varepsilon > 0$. See Theorem 4.5.1 in Section 4.5.

— We construct $(u_R^\varepsilon, l_R^\varepsilon, r_R^\varepsilon)$ as a weak solution (in the sense of Leray) of a system which we deduce from the equations verified by all the other terms in (4.17), (4.21) (note that at this point we have not yet proven the existence of $(u^\varepsilon, l^\varepsilon, r^\varepsilon)$, but we know that it should be a very weak solution in the sense of Definition 4.2.1 with $g^\varepsilon = g^0 + \varepsilon g^1$). We prove by the means of an energy estimate that $(u_R^\varepsilon, l_R^\varepsilon, r_R^\varepsilon)$ is small in $L^\infty((0,T); L^2(\mathcal{F}_0) \times \mathbb{R}^3)$, when $\varepsilon > 0$ is small. In particular, we have $|(l_R^\varepsilon, r_R^\varepsilon)(T)| \leq C\varepsilon^{1/8}$, and we may conclude the estimates (4.18) in Theorem 4.2.2. See Proposition 4.6.1 in Section 4.6.

From the above construction we may define $(u^\varepsilon, l^\varepsilon, r^\varepsilon)$ as the right-hand sides of (4.17), (4.21), since now all the respective terms are constructed and well-defined. Furthermore, in order to ensure the continuity of the map (4.19), we make sure that the terms on the right-hand side of (4.17) at time T are constructed continuously with respect to $(h_1, \vartheta_1, l_1, r_1)$ in the steps above. In particular it is sufficient to guarantee that $(l^0, r^0), (l^1, r^1), (l_R^\varepsilon, r_R^\varepsilon) \in L^\infty(0,T)$ depend continuously on $(h_1, \vartheta_1, l_1, r_1)$, which gives the continuity of the map (4.19) by using (4.12).

However, as mentioned in Remark 4.2.2, $(u^\varepsilon, l^\varepsilon, r^\varepsilon)$ defined in such a way will only qualify as a very weak solution in the sense of Definition 4.2.1 if (4.13) is verified as well, so we proceed in the following manner.

We fix an open ball $B_c \subset \operatorname{int}(\Omega_c) \setminus \bigcup_{(h,\vartheta)\in\overline{B}((h_f,\vartheta_f),\kappa)} \{(h + R(\vartheta)\mathcal{S}_0) \cup \mathcal{S}_0\}$ such that $d(B_c, \Omega_c) > 0$. During our construction we make sure that $g^0(t,\cdot)$ and $g^1(t,\cdot)$ are supported in $R(\vartheta^0(t))^T(B_c - h^0(t))$ and that we have

$$B_c \cap (R(\vartheta^0(t))\mathcal{S}_0 + h^0(t)) = \emptyset, \ \forall t \in [0,T]. \tag{4.23}$$

Since (h^1, ϑ^1) and $(h_R^\varepsilon, \vartheta_R^\varepsilon)$ are bounded in $L^\infty(0,T)$ by $C = C(\nu) > 0$, there exists $\varepsilon_0 = \varepsilon_0(\nu) > 0$ such that (4.23) implies (4.13), for any $\varepsilon \in (0, \varepsilon_0]$, with $g^\varepsilon(t,\cdot) = g^0(t,\cdot) + \varepsilon g^1(t,\cdot)$ supported in $R(\vartheta^\varepsilon(t))^T(\Omega_c - h^\varepsilon(t))$, for any $t \in [0,T]$.

This concludes the proof of Theorem 4.2.2.

Remark 4.2.3. *Let us now explain why we chose to work in the whole plane $\mathbb{R}^2$ instead of a bounded domain. The key technical difficulty in handling the case of a bounded domain with a similar strategy would be the step of transforming the moving domain $\mathcal{F}(t)$ into a fixed domain. As mentioned above, in the case of the plane this can be done through a simple rigid movement. However, in the bounded case one would also need to account for the outer boundary $\partial\Omega$, and construct a diffeomorphism which is a rigid movement in a neighbourhood of the solid, but leaves the boundary $\partial\Omega$ intact. This diffeomorphism would clearly depend on the solid position, as well as contribute more complicated nonlinear terms in the PDE (see for instance [12] or [61] for such a construction). The main problem then is to investigate what happens to these terms when we look for an asymptotic expansion of the form of* (4.17) *for the solid trajectory, moreover separating them in terms of orders of ε. To properly do this, one would need to establish a rigorous asymptotic expansion of the diffeomorphism (and the associated terms in the PDE) with respect to the solid position, which is rather difficult.*

4.2.4 Regarding Remark 4.2.1

In order to prove that the small flux condition mentioned in Remark 4.2.1 can indeed be achieved, we ensure during our construction that g^0 satisfies in addition the small flux condition

$$\left| \int_0^T \int_{R(\vartheta^0(t))^T(B_c - h^0(t))} \big(g^0(t,x)\big)_- \; dx\, dt \right| < \frac{\delta_c}{2}. \tag{4.24}$$

This can be achieved by similar arguments as in Chapter 3 which we will detail in Remark 4.3.1 at the end of Section 4.3.2.

Consequently, we have

$$\begin{aligned} \left| \int_0^T \int_{R(\vartheta^\varepsilon(t))^T(\Omega_c - h^\varepsilon(t))} (g^\varepsilon(t,x))_- \; dx\, dt \right| &\le \left| \int_0^T \int_{R(\vartheta^0(t))^T(B_c - h^0(t))} \big(g^0(t,x)\big)_- \; dx\, dt \right| \\ &\quad + \varepsilon \left| \int_0^T \int_{R(\vartheta^0(t))^T(B_c - h^0(t))} \big(g^1(t,x)\big)_- \; dx\, dt \right| < \frac{\delta_c}{2} + \varepsilon C, \end{aligned}$$

where $C \in (0, +\infty)$ is independent of $\varepsilon > 0$, therefore we may in fact further reduce the $\bar{\varepsilon} > 0$ in the proof of Theorem 4.2.1 from Theorem 4.2.1 from Section 4.2.2, so that it satisfies $C\bar{\varepsilon} < \frac{\delta_c}{2}$. This allows us to prove the small flux condition from Remark 4.2.1.

4.3 The inviscid term u^0

In this section we construct a controlled solution to the inviscid terms appearing in the asymptotic expansion (4.17), (4.21).

At order $\mathcal{O}(1)$, we look for $(u^0, \pi^0, l^0, r^0, g^0)$ satisfying the following system.

$$\begin{aligned} &\frac{\partial u^0}{\partial t} + (u^0 - u_S^0)\cdot\nabla u^0 + r^0(u^0)^\perp + \nabla\pi^0 = 0 \ \text{ and } \ \operatorname{div} u^0 = g^0 \ \text{ for } x \in \mathcal{F}_0, \\ &u^0\cdot n = u_S^0\cdot n \ \text{ for } x\in\partial\mathcal{S}_0, \ \lim_{|x|\to+\infty}|u^0| = 0, \\ &m(l^0)' = \int_{\partial\mathcal{S}_0}\pi^0\, n\, d\sigma - m r^0 (l^0)^\perp \ \text{ and } \ \mathcal{J}\,(r^0)' = \int_{\partial\mathcal{S}_0}\pi^0\, x^\perp\cdot n\, d\sigma, \end{aligned} \tag{4.25}$$

where $u_S^0(t,x) = l^0(t) + r^0(t)x^\perp$, for $t\in[0,T]$, with $u^0(0,\cdot) = 0, (l^0, r^0)(0) = 0$. The position of the solid is associated through the system

$$h^0(t) = \int_0^t R(\vartheta^0(s))l^0(s)\, ds, \quad \vartheta^0(t) = \int_0^t r^0(s)\, ds, \tag{4.26}$$

for $t\in[0,T]$.

We introduce the following notation.

Definition 4.3.1. *The space C^k_∞, $k\geq 0$ is defined as follows : for any bounded set $A\subset\mathbb{R}^2$, we define*

$$C^k_\infty(\mathbb{R}^2\setminus A) := \left\{ f\in C^k(\mathbb{R}^2\setminus A) \text{ such that } \lim_{|x|\to+\infty}\left|\nabla^i f(x)\right| = 0,\ \forall i\in\{0,\dots,k\}\right\}.$$

Note that, for any $f\in C^k_\infty(\mathbb{R}^2\setminus A)$, we have $\|f\|_{C^k_\infty} = \sup_{x\in\mathbb{R}^2\setminus A}\max_{0\leq i\leq k}|\nabla^i f(x)| < +\infty$.

In the sequel we will use some regularity results with respect to the position q for certain integral terms, which only hold on a bounded set of admissible positions. Therefore, we consider q_f and $\kappa>0$ as in Theorem 4.2.2, and we pick some open ball $B\subset\mathbb{R}^3$ such that $\overline{B}(q_f,\kappa)\subset\mathcal{Q}\cap B$, and for $\delta>0$ we introduce the set

$$Q_\delta = \{q = (h,\vartheta)\in B:\ d(h + R(\vartheta)\mathcal{S}_0, B_c) > \delta\}. \tag{4.27}$$

Consequently, as long as q^0 stays in Q_δ, condition (4.23) will hold. For $\delta>0$ small enough, $\mathcal{Q}_\delta$ is clearly path-connected.

Furthermore, we will look for solutions (l^0, r^0, u^0) satisfying the following additional condition, which will not be needed for the construction of u^0, however it will be helpful in the construction of u^1, which as already mentioned, is a linearized solution around u^0. We will therefore look for solutions satisfying

$$\operatorname{span}\left\{(n(x), x^\perp\cdot n(x)),\ x\in\partial\mathcal{S}_0\cap\operatorname{supp}\left\{u^0(T/2,\cdot) - u_S^0(T/2,\cdot)\right\}\right\} = \mathbb{R}^3. \tag{4.28}$$

We have the following exact controllability result for $(h^0, \vartheta^0, l^0, r^0)$.

Theorem 4.3.1. *Let $T>0$, $\delta>0$ small enough such that $\mathcal{Q}_\delta$ is path-connected, $\mathcal{S}_0\subset\mathbb{R}^2$ bounded, closed, simply connected with smooth boundary, which is not a disk, and $q_0, q_1\in\mathcal{Q}_\delta$ with $q_0 = 0$ and $q_1 = (h_1,\vartheta_1)$. There exists a control $g^0\in C^\infty_0((0,T)\times\mathcal{F}_0)$ and a solution $(h^0,\vartheta^0,u^0)\in C^\infty([0,T];\mathcal{Q}_\delta)\times C^\infty([0,T]\times\overline{\mathcal{F}_0};\mathbb{R}^2)$ to (4.25), (4.26) with zero initial conditions for $(h^0,\vartheta^0,l^0,r^0,u^0)$, such that (4.28) holds, $u^0\in C([0,T];L^2(\mathcal{F}_0))$ and*

$$(h^0,\vartheta^0,l^0,r^0)(T) = (h_1,\vartheta_1,0,0),\ \textit{supp}\ g^0(t,\cdot)\subset R(\vartheta^0(t))^T(B_c - h^0(t)),\ \forall t\in[0,T].$$

Furthermore, one may define a continuous map $(h_1,\vartheta_1)\mapsto(l^0,r^0,u^0)\in C^3([0,T];\mathbb{R}^3\times C^5_\infty(\overline{\mathcal{F}_0}))$.

The proof will be given in Section 4.3.2. Note that the specific regularity $C^3(C^5_\infty)$ will serve to establish appropriate higher order energy estimates for the boundary layer profiles constructed in Section 4.4.

4.3.1 Reformulation of the solid's equation into an ODE

In this section we establish a reformulation of the solid equations from (4.25) as an ODE for the three degrees of freedom of the rigid body with coefficients obtained by solving some elliptic-type problems.

To simplify notations, we denote the positions and velocities $q^0 = (h^0, \vartheta^0)$, $p^0 = (l^0, r^0)$. Observe that a smooth solution u^0 of (4.25) satisfies the following div/curl type system :

$$\begin{cases} \text{div } u^0 = g^0, \text{ curl}\, u^0 = 0 \ \text{ in } \mathcal{F}_0, \\ u^0 \cdot n = \left(l^0 + r^0 x^\perp\right) \cdot n \text{ on } \partial\mathcal{S}_0, \lim\limits_{|x|\to+\infty} |u^0| = 0, \\ \displaystyle\int_{\partial\mathcal{S}_0} u^0 \cdot \tau \, d\sigma = 0, \end{cases} \tag{4.29}$$

for $t \in [0, T]$, the last equation above coming from Kelvin's theorem regarding the conservation of the circulation around the body.

We observe that the unique smooth solution of the above system can be uniquely decomposed in a linear manner. We introduce the Kirchhoff potentials, which in our case are simplified due to the fact that we have moved our evolution PDE onto a cylindrical domain. Let $\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3) \in C^\infty(\overline{\mathcal{F}_0}) \cap C^6_\infty(\overline{\mathcal{F}_0})$ be the solution (up to a constant) of the elliptic problems

$$\begin{aligned} &\Delta\Phi_i(x) = 0 \text{ in } \mathcal{F}_0, \ \lim_{|x|\to+\infty} |\nabla\Phi_i(x)| = 0, \quad \text{for } i \in \{1,2,3\}, \\ &\partial_n \Phi_i(x) = \begin{cases} n_i \text{ on } \partial\mathcal{S}_0, \text{ for } i \in \{1,2\}, \\ x^\perp \cdot n \text{ on } \partial\mathcal{S}_0, \text{ for } i = 3. \end{cases} \end{aligned} \tag{4.30}$$

Note that $\nabla\Phi(x) = \mathcal{O}(1/|x|^2)$ as $|x| \to +\infty$, implying that $\nabla\Phi$ is in fact square-integrable (see for instance Section 2.3 in [55]).

Furthermore, we will also be looking for potential flows to handle the term in the decomposition of u^0 due to the control. In order to satisfy the condition supp $g^0(t,\cdot) \subset R(\vartheta^0(t))^T(B_c - h^0(t))$, for all $t \in [0, T]$ in Theorem 4.3.1, we introduce, for any $q = (h, \vartheta) \in \mathcal{Q}_\delta$,the set

$$\mathcal{C}(q) := \left\{ g \in C^\infty_0(R(\vartheta)^T(B_c - h); \mathbb{R}) \ \text{ such that } \ \int g = 0 \right\},$$

and we only consider potential flows of the following type.

Definition 4.3.2. *With any $q = (h, \vartheta) \in \mathcal{Q}_\delta$ and $g \in \mathcal{C}(q)$ we associate the unique solution $\overline{\alpha} := \mathcal{A}[q, g] \in C^\infty(\overline{\mathcal{F}_0}; \mathbb{R})$ which vanishes at infinity to the following elliptic problem :*

$$\Delta\overline{\alpha} = g \mathbb{1}_{R(\vartheta)^T(B_c - h)}(x) \ \textit{in } \mathcal{F}_0, \ \lim_{|x|\to+\infty} |\nabla\overline{\alpha}| = 0, \quad \textit{and } \partial_n \overline{\alpha} = 0 \ \textit{on } \partial\mathcal{S}_0. \tag{4.31}$$

Note that since $\overline{\alpha}$ is harmonic outside of a compact set and $\lim\limits_{|x|\to+\infty} |\nabla\overline{\alpha}| = 0$, in particular we also have $\overline{\alpha} \in C^6_\infty$, by using a Laurent series development to investigate its behaviour at infinity. Furthermore, noting that $\int_{\partial\mathcal{S}_0} \nabla\overline{\alpha} \cdot \tau \, d\sigma = \int_{\partial\mathcal{S}_0} \nabla\overline{\alpha} \cdot n \, d\sigma = 0$, we may in fact conclude as in Lemma A8 from [56] that $\nabla\overline{\alpha}(x) = \mathcal{O}(1/|x|^2)$ as $|x| \to +\infty$, implying that $\nabla\overline{\alpha}$ is in fact square-integrable. Finally, we observe that the map $q \mapsto \mathcal{A}[q, g]$ is smooth.

The following statement is an immediate consequence of the definitions above.

Lemma 4.3.1. *For any $q = (h, \vartheta)$ in $\mathcal{Q}_\delta$, for any $p = (l, r)$ in $\mathbb{R}^2 \times \mathbb{R}$, and $g \in \mathcal{C}(q)$, the unique solution u in $C^\infty(\overline{\mathcal{F}_0})$ to the following system :*

$$\begin{cases} \operatorname{div} u = g \mathbb{1}_{R(\vartheta)^T(B_c - h)}, \quad \operatorname{curl} u = 0 \quad \text{in } \mathcal{F}_0, \\ u \cdot n = \left(l + r x^\perp\right) \cdot n \text{ on } \partial\mathcal{S}_0, \quad \lim\limits_{|x| \to +\infty} |u| = 0, \\ \displaystyle\int_{\partial\mathcal{S}_0} u \cdot \tau \, d\sigma = 0, \end{cases} \tag{4.32}$$

is given by the following formula, for x in $\overline{\mathcal{F}_0}$,

$$u(x) = \nabla(p \cdot \Phi(x)) + \nabla\mathcal{A}[q, g](x). \tag{4.33}$$

Above $p \cdot \Phi(x)$ denotes the inner product $p \cdot \Phi(x) = \sum_{i=1}^3 p_i \Phi_i(x)$.

Let us now address the solid dynamics. We aim for a reformulation as in Chapter 3, however due to the fact that we are now in a domain which does not depend on q^0, there will be terms that become simplified. We introduce the following notations.

Definition 4.3.3. *We respectively define the genuine and added mass 3×3 matrices by*

$$\mathcal{M}_g = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \mathcal{J} \end{pmatrix},$$

and,

$$\mathcal{M}_a = \left(\int_{\mathcal{F}_0} \nabla\Phi_i(x) \cdot \nabla\Phi_j(x) \, dx \right)_{1 \leqslant i,j \leqslant 3},$$

and we denote their sum by $\mathcal{M}$.

We define the symmetric bilinear map Γ for any $p = (l, r), \tilde{p} = (\tilde{l}, \tilde{r}) \in \mathbb{R}^3$ by

$$\langle \Gamma, p, \tilde{p} \rangle = \frac{1}{2} \int_{\partial\mathcal{S}_0} (\nabla(p \cdot \Phi(x))) \cdot (\nabla(\tilde{p} \cdot \Phi(x))) \partial_n \Phi(x) \, d\sigma + \langle \tilde{\Gamma}^{sym}, p, \tilde{p} \rangle \in \mathbb{R}^3,$$

where $\tilde{\Gamma}^{sym}$ denotes the symmetric part of the bilinear map $\tilde{\Gamma}$ defined by

$$\langle \tilde{\Gamma}, p, \tilde{p} \rangle = -\int_{\partial\mathcal{S}_0} (l + r x^\perp) \cdot \nabla(\tilde{p} \cdot \Phi(x)) \partial_n \Phi(x) \, d\sigma + (m \tilde{r} l^\perp, 0) \in \mathbb{R}^3.$$

Note that $\tilde{\Gamma}$ is no longer symmetric, however, we have $2\langle \tilde{\Gamma}^{sym}, p, \tilde{p} \rangle = \langle \tilde{\Gamma}, p, \tilde{p} \rangle + \langle \tilde{\Gamma}, \tilde{p}, p \rangle$, for all $p, \tilde{p} \in \mathbb{R}^3$.

Let us first give the reformulation of the model as an ODE when there is no control.

Proposition 4.3.1. *Given $p^0 = (l^0, r^0) \in C^\infty([0, T]; \mathbb{R}^3)$, $u^0 \in C^\infty([0, T] \times \overline{\mathcal{F}_0}; \mathbb{R}^2)$, we have that (u^0, p^0) is a solution to (4.25) with $g^0 = 0$ and zero initial conditions if and only if p^0 satisfies the following ODE on $[0, T]$*

$$\mathcal{M}(p^0)' + \langle \Gamma, p^0, p^0 \rangle = 0 \tag{4.34}$$

and u^0 is the unique smooth solution to the system (4.29) with $g^0 = 0$.

Observe that the position of the solid plays no role in this case.

Proof. The proof is straightforward based on the above definitions, and due to the fact that (4.25) is on a fixed domain, contrary to the case of Chapter 3 or [81].

We recall Lamb's form : for any differentiable functions v_1, v_2 defined on a subset of $\mathbb{R}^2$ with values in $\mathbb{R}^2$ we have

$$\nabla(v_1 \cdot v_2) = v_1 \cdot \nabla v_2 + v_2 \cdot \nabla v_1 - \operatorname{curl} v_1 (v_2)^\perp - \operatorname{curl} v_2 (v_1)^\perp, \tag{4.35}$$

and we use it to obtain that the gradient of the pressure π^0 in (4.25) with $g^0 = 0$ can be expressed as $\nabla \pi^0 = -\partial_t u^0 - \frac{1}{2}\nabla(|u^0|^2) + \nabla(u^0 \cdot u_S^0)$. Note that the solid equations can be rewritten as

$$\begin{aligned}\mathcal{M}(p^0)' + (mr^0(l^0)^\perp, 0) &= \int_{\partial\mathcal{S}_0} \pi\, \partial_n \Phi\, d\sigma = \int_{\mathcal{F}_0} \nabla\pi \cdot \nabla\Phi\, dx \\ &= \int_{\mathcal{F}_0} \left(-\partial_t u^0 - \frac{1}{2}\nabla(|u^0|^2) + \nabla(u^0 \cdot u_S^0)\right) \cdot \nabla\Phi\, dx,\end{aligned}$$

since even though u_S^0 grows like x at infinity, the integration by parts above is justified since $\nabla\Phi$ (and implicitly u^0) behaves like $1/|x|^2$ as $|x| \to +\infty$. We may conclude by using Lemma 4.3.1, integrating once more by parts and rearranging the appropriate terms to get (4.34). □

Now we may move to the case with control. We introduce the following force terms.

Definition 4.3.4. *We define, for any $p = (l, r)$ in $\mathbb{R}^3$, α in $C^\infty(\overline{\mathcal{F}_0}; \mathbb{R})$, $F_1(p)[\alpha]$ and $F_2[\alpha]$ in $\mathbb{R}^3$ by $F_1(p)[\alpha] = F_{1,a}[\alpha] + F_{1,b}(p)[\alpha]$, where*

$$\begin{aligned}F_{1,a}[\alpha] &= -\frac{1}{2}\int_{\partial\mathcal{S}_0} |\nabla\alpha(x)|^2\, \partial_n\Phi(x)\, d\sigma, \\ F_{1,b}(p)[\alpha] &= -\int_{\partial\mathcal{S}_0} \nabla\alpha(x) \cdot \nabla(p \cdot \Phi(x))\, \partial_n\Phi(x) d\sigma + \int_{\partial\mathcal{S}_0} (l + rx^\perp) \cdot \nabla\alpha(x) \partial_n\Phi(x)\, d\sigma, \\ F_2[\alpha] &= -\int_{\partial\mathcal{S}_0} \alpha(x)\, \partial_n\Phi(x)\, d\sigma.\end{aligned} \tag{4.36}$$

Observe that Formulas (4.36) only require α and $\nabla\alpha$ to be defined on $\partial\mathcal{S}_0$. Moreover when these formulas are applied to $\alpha = \mathcal{A}[q, g]$ for some g in $\mathcal{C}$, then only the trace of α and the tangential derivative $\partial_\tau\alpha$ on $\partial\mathcal{S}_0$ are involved, since the normal derivative of α vanishes on $\partial\mathcal{S}_0$ by definition, cf. (4.31).

We define our notion of controlled solution of the "fluid+solid" system as follows.

Definition 4.3.5. *We say that (q^0, p^0, g^0) in $C^\infty([0,T]; \mathcal{Q}_\delta \times \mathbb{R}^3) \times C_0^\infty((0,T); \mathcal{C}(q^0(t)))$ is a controlled solution associated with* (4.25), (4.26), *if the following ODE holds true on* $[0, T]$ *:*

$$\begin{aligned}\mathcal{M}(p^0)' + \langle \Gamma, p^0, p^0 \rangle &= F_1(p)[\alpha^0] + F_2[\partial_t \alpha^0], \\ (q^0)' &= \mathcal{R}(q^0) p^0,\end{aligned} \tag{4.37}$$

where $\alpha^0(t, \cdot) := \mathcal{A}[q^0(t), g^0(t, \cdot)]$ and

$$\mathcal{R}(q^0) = \mathcal{R}(\vartheta^0) := \begin{pmatrix} R(\vartheta^0) & 0 \\ 0 & 1 \end{pmatrix}.$$

We have the following result for reformulating the model as an ODE.

Proposition 4.3.2. *Given $(q^0, p^0) \in C^\infty([0,T]; \mathcal{Q}_\delta \times \mathbb{R}^3)$, $u^0 \in C^\infty([0,T] \times \overline{\mathcal{F}_0}; \mathbb{R}^2)$ and $g^0 \in C_0^\infty((0,T); \mathcal{C}(q^0(t)))$, we have that (u^0, q^0, p^0, g^0) is a solution to* (4.25), (4.26), *if and only if (q^0, p^0, g^0) is a controlled solution and*

$$u^0(t, x) = \nabla(p^0(t) \cdot \Phi(x)) + \nabla\mathcal{A}[q^0(t), g^0(t, x)](x),$$

for any $(t, x) \in [0, T] \times \overline{\mathcal{F}_0}$, with Φ and $\mathcal{A}$ given in (4.30), *respectively* (4.31).

The proof is a straightforward adaptation of the proof of Proposition 4.3.1, noting that the regularity of the functions involved allows us to perform all the integration by parts, and is therefore omitted.

4.3.2 Proof of the exact controllability result Theorem 4.3.1

In this section we observe that it is possible to prove Theorem 4.3.1 by exploiting the geodesic feature of the uncontrolled system and an impulsive control strategy, as done in Chapter 3. We will skip certain straightforward parts of the proof and refer the reader to the aforementioned chapter for more details. We will instead focus on presenting the strategy and highlighting the main differences between our case and the bounded case in Chapter 3.

We observe that Theorem 4.3.1 can be deduced from a simpler approximative controllability result, namely Theorem 4.3.2, where the solid displacement is assumed to be small. The proof of passing from small solid displacement to an arbitrary one is a straightforward adaptation of Section 3.4 in Chapter 3, therefore will be omitted.

As mentioned in the proof of Theorem 4.2.1 in this chapter, respectively in Section 3.4 of Chapter 3, it is possible to pass from approximate controllability of the final position and velocity to exact controllability using a topological argument (see Lemma 4.2.1). In this section however we will proof that this argument could be replaced by a local inversion argument, both in our case and in the case of Chapter 3. For simplicity, we will present the method only in the case of (u^0, q^0, p^0), where the fluid is irrotational and the circulation around the body is zero, however, it can be extended to the general case through some straightforward but technical modifications.

We have the approximate controllability result below, which can be seen as a generalization of Theorem 3.4.1 from Chapter 3.

Theorem 4.3.2. *Consider $\delta > 0$, $\mathcal{S}_0 \subset \mathbb{R}^2$ bounded, closed, simply connected with smooth boundary, which is not a disk, q_0 in $\mathcal{Q}_\delta$ and $T > 0$. Then there exists $\tilde{r} > 0$ such that $B(q_0, \tilde{r}) \subset \mathcal{Q}_\delta$, and for any $\nu > 0$, there exists a mapping*

$$\mathcal{T} : \overline{B}\big((q_0, 0), \tilde{r}\big) \to C^\infty([0, T]; \mathcal{Q}_\delta \times \mathbb{R}^3)$$

which with (q_1, q_1') associates (q^0, p^0) where (q^0, p^0, g^0) is a controlled solution associated with (4.25), (4.26)*, and the initial data $(q_0, 0)$, such that we have the following :*

- (4.28) *holds and the map $q_1 \mapsto (l^0, r^0, u^0) \in C^3([0, T]; \mathbb{R}^3) \times C^3([0, T]; C^5_\infty(\overline{\mathcal{F}_0}))$ is continuous, where u^0 is the associated fluid velocity given by Proposition 4.3.2 ;*
- *the mapping*

$$(q_1, q_1') \in \overline{B}\big((q_0, 0), \tilde{r}\big) \mapsto \mathcal{T}(q_1, q_1')(T) \in \mathcal{Q}_\delta \times \mathbb{R}^3$$

  is C^1, and for any w in $\overline{B}\big((q_0, 0), \tilde{r}\big)$, we have

$$\frac{\partial}{\partial w}\mathcal{T}(w)(T) = Id + \mathcal{O}(\nu). \tag{4.38}$$

Using this result, we may prove Theorem 4.3.1 in the case of small solid displacement. Indeed, we set $\mathbb{T}(w) = \mathcal{T}(w)(T)$, for any w in $\overline{B}\big((q_0, 0), \tilde{r}\big)$. Taking ν sufficiently small, we see that $\frac{\partial \mathbb{T}}{\partial w}(w)$ is invertible. Consequently one can use the inverse function theorem on $\mathbb{T}$: there exists $r > 0$ small enough such that if $q_1 \in \overline{B}(q_0, r)$, then $\mathbb{T}$ is invertible at $w = (q_1, 0)$. Therefore, $\mathcal{T}(\mathbb{T}^{-1}(q_1, 0))$ is a trajectory associated with a controlled solution which at time $T > 0$ takes exactly the value $(q_1, 0)$. This concludes the proof of Theorem 4.3.1.

Proof of Theorem 4.3.2 without (4.28)

For simplicity, we will organise the proof of Theorem 4.3.2 into two steps. In this subsection we first give our construction without trying to satisfy (4.28). In Section 4.3.2, we show how our construction can be straightforwardly modified to ensure that (4.28) holds, without interfering with the conclusions presented in this subsection.

The choice of the form of the control - constructing the operator $\mathcal{T}$. Let us first present our impulsive control strategy based on the intuition that we want to exploit the underlying geodesic structure of (4.37). See [13] and the references therein for examples of impulsive control strategies.

Let $w = (q_1, q_1') \in \overline{B}\big((q_0, 0), \tilde{r}\big)$. We consider controls of the form

$$g_\eta^0(t,x) := \beta_\eta(t) g_0(x) + \beta_\eta(T-t) g_1(x), \tag{4.39}$$

where $\beta_\eta \geq 0$, $(\beta_\eta^2)_\eta$ is compactly supported in $(0, 2\eta)$ and is an approximation of the unity when $\eta \to 0^+$. Furthermore, we denote by (q^0, p^0) the solution of (4.37) with control g_η^0, dropping the dependence with respect to $\eta > 0$ from the solid trajectory for simplicity of notation.

It can be proven that, for a right choice of g_0, g_1, the trajectory (q^0, p^0) will be close to the solution $(\tilde{q}, \tilde{p})$ of the following toy system :

$$\mathcal{M}\tilde{p}' + \langle \Gamma, \tilde{p}, \tilde{p} \rangle = \beta_\eta^2(\cdot)\, v_0 + \beta_\eta^2(T - \cdot) v_1, \;\; \tilde{q}' = \mathcal{R}(\tilde{q})\tilde{p}, \tag{4.40}$$

with zero initial conditions and some $v_0, v_1 \in \mathbb{R}^3$ given.

On the other hand, we claim that there exists a local solution in $\mathcal{Q}_\delta$ to the ODE system

$$\mathcal{M}(\bar{p})' + \langle \Gamma, \bar{p}, \bar{p} \rangle = 0, \;\; (\bar{q})' = \mathcal{R}(\bar{q})\bar{p}, \;\; \text{on } [0,T], \;\; \text{with } \bar{q}(0) = q_0, \;\; \bar{q}(T) = q_1, \tag{4.41}$$

where the map $q_1 \in \overline{B}\big(q_0, \tilde{r}\big) \mapsto (c_0, c_1) \in \mathbb{R}^6$ given by $c_0(q_1) = \bar{p}(0)$, $c_1(q_1) = \bar{p}(T)$, is C^1. Note that, contrary to Chapter 3, $\mathcal{M}$ and Γ do not depend on $\bar{q}$ here, so proving the existence of $(\bar{q}, \bar{p})$ at first glance does not seem to follow as in Chapter 3, at least not directly.

However, a simple way to prove the existence of a solution to (4.41) is to first introduce a change of coordinates which corresponds to changing back from a fixed domain to a domain which moves according to the trajectory $(\bar{q}, \bar{p})$. Namely, setting

$$\bar{\tau}(t,x) = R(\bar{\vartheta}(t))^T (x - \bar{h}(t)), \;\; \bar{u}(t,x) = R(\bar{\vartheta}(t)) \nabla \left(\bar{p}(t) \cdot \Phi(\bar{\tau}(t,x)) \right), \tag{4.42}$$

allows us to switch back to the setting where the solid position (and implicitly the fluid domain) evolves and the control zone stays fixed in time. More precisely, $(\bar{u}, \bar{q})$ will satisfy an "inviscid fluid + rigid body" system as in Chapter 3, but in the whole plane, without any control, with zero initial data for $\bar{u}$, and the endpoints of $\bar{q}$ fixed as q_0 and q_1. For a geodesic reformulation of the equation of $\bar{q}$ in this setting, see Section 4.1 in [57], and note that the existence of $\bar{q}$ then follows in the same way as the proof of Lemma 3.35 from Chapter 3.

Setting

$$v_i = (-1)^i \mathcal{M}(c_i(q_1) - q_i'), \;\; i = 0, 1, \tag{4.43}$$

we obtain, similarly as in Section 3.5.2 of Chapter 3, that the solution $(\tilde{q}, \tilde{p})$ of (4.40) satisfies

$$\lim_{\eta \to 0^+} \|(\tilde{q}, \tilde{p}) - (\bar{q}, \bar{p})\|_{C([2\eta, T-2\eta])} = 0 \tag{4.44}$$

and $(\tilde{q},\tilde{p})(T) \to (q_1, q_1')$ as $\eta \to 0$.

The construction of the controls g_0, g_1 relies on the observation that when we approximate the solution (q^0, p^0) with $(\tilde{q},\tilde{p})$, the term from (4.37) which will behave like $\beta_\eta^2 v_i$ is $F_{1,a}$, and we use a complex analysis argument to prove that it can attain any direction $v \in \mathbb{R}^3$. Since we are in the unbounded case, the construction of potential flows needs to be adapted in order to guarantee that the flow velocity is in $C^5_\infty(\overline{\mathcal{F}_0})$ for instance.

We consider $\tilde{B}_c \subset B_c$ closed such that $d(\tilde{B}_c, \partial B_c) > 0$ and set

$$\tilde{\mathcal{C}}(q) := \left\{ g \in C^1_0(R(\vartheta)^T(\tilde{B}_c - h); \mathbb{R}) \ \text{ such that } \int g = 0 \right\}, \tag{4.45}$$

for any $q \in \mathcal{Q}_\delta$. Note that the purpose of introducing $\tilde{\mathcal{C}}(q)$ is twofold : on one hand it allows us to construct our controls more robustly with respect to $q \in \mathcal{Q}_\delta$, since if $g \in \tilde{\mathcal{C}}(q)$ and q is close to $\hat{q}$, it follows that $g \in \mathcal{C}(\hat{q})$, in particular this will be useful to prove that the support of the control we construct will stay sufficiently far away from $\mathcal{S}_0$; and on the other hand we will construct a C^1 map with respect to g below, which is easier to accomplish on the Banach space $\tilde{\mathcal{C}}(q)$. Furthermore, we will extend $g \in \tilde{\mathcal{C}}(q)$ by 0 in $B_c \setminus \tilde{B}_c$.

We have the following generalization of Proposition 3.5.1 from Chapter 3, which will be proved in Appendix 4.A.1. Recall that $\mathcal{A}$ was defined in Definition 4.3.2.

Proposition 4.3.3. *There exists a C^1 mapping $\overline{g}^0 : \mathcal{Q}_\delta \times \mathbb{R}^3 \to \tilde{\mathcal{C}}(\mathcal{Q}_\delta)$ such that for any $q \in \mathcal{Q}_\delta$ we have Range$(\overline{g}^0(q,\cdot)) \subset \mathcal{C}(q) \cap \tilde{\mathcal{C}}(q)$, and for any (q,v) in $\mathcal{Q}_\delta \times \mathbb{R}^3$ the function $\overline{\alpha}^0 := \mathcal{A}[q, \overline{g}^0(q,v)]$ in $C^\infty(\overline{\mathcal{F}_0}; \mathbb{R}) \cap C^6_\infty(\overline{\mathcal{F}_0}; \mathbb{R})$ satisfies :*

$$\Delta \overline{\alpha}^0 = 0 \text{ in } \mathcal{F}_0 \setminus R(\vartheta)^T(\tilde{B}_c - h), \lim_{|x| \to +\infty} |\nabla \overline{\alpha}^0| = 0 \text{ and } \partial_n \overline{\alpha}^0 = 0 \text{ on } \partial\mathcal{S}_0, \tag{4.46}$$

$$\int_{\partial\mathcal{S}_0} \left|\nabla \overline{\alpha}^0\right|^2 \partial_n \Phi \, d\sigma = v, \tag{4.47}$$

$$\int_{\partial\mathcal{S}_0} \overline{\alpha}^0 \, \partial_n \Phi \, d\sigma = 0, \tag{4.48}$$

$$\text{span}\left\{ (n(x), x^\perp \cdot n(x)), \ x \in \text{supp } \nabla \overline{\alpha}^0(q,\cdot) \cap \partial\mathcal{S}_0 \right\} = \mathbb{R}^3. \tag{4.49}$$

Using Proposition 4.3.3, we set

$$g_0 = \bar{g}^0(q_0, -2v_0), \quad g_1 = \bar{g}^0(q_1, -2v_1)$$

in (4.39), with v_i as in (4.43).

It follows that for $\eta > 0$ small enough, since the solid displacement during the control phase $[0, 2\eta] \cup [T - 2\eta, T]$ is small enough, the interior control g_η^0 given by (4.39) is truly supported inside the control zone $R(\vartheta^0(\cdot))^T(B_c - h^0(\cdot))$.

With the choice of control presented above, we set $\mathcal{T}(w) := (q^0, p^0)$ and $\tilde{\mathcal{T}} := (\tilde{q}, \tilde{p})$ the solutions of (4.37), respectively (4.40). We will prove that given any $\nu > 0$, for $\eta > 0$ small enough, the following hold

(i) The toy model satisfies a similar condition to (4.38) which we want to prove for $\mathcal{T}$, namely

$$\frac{\partial}{\partial w} \tilde{\mathcal{T}}(w)(T) = \text{Id} + \mathcal{O}(\nu). \tag{4.50}$$

(ii) The differentials with respect to $w \in \overline{B}\big((q_0,0),\tilde{r}\big)$ of the toy model and the real trajectory are close in $C([0,T];\mathbb{R}^{6\times 6})$, uniformly with respect to $w \in \overline{B}\big((q_0,0),\tilde{r}\big)$, namely we have

$$\left\| \frac{\partial}{\partial w}\mathcal{T}(w)(\cdot) - \frac{\partial}{\partial w}\tilde{\mathcal{T}}(w)(\cdot) \right\|_{C([0,T])} \leq \nu, \ \forall w \in \overline{B}\big((q_0,0),\tilde{r}\big). \tag{4.51}$$

The result of Theorem 4.3.2 without (4.28) clearly follows from (i) and (ii), by fixing $\eta = \eta_0 > 0$ small enough, and by observing that once η is fixed, we may easily deduce the continuity result with respect to q_1 for (p^0, u^0) mentioned in Theorem 4.3.2. Indeed, it follows from (4.30), (4.31), (4.37) and Proposition 4.3.2 that the only issue which needs further investigation in order to establish such a continuity result is what happens when we differentiate $\mathcal{A}[q^0(t), g^0_\eta(t,x)](x)$ with respect to t. Using (4.31), the fact that g^0_η defined in (4.39) does not depend on q^0, we obtain that

$$\partial_t A[q^0(t), g^0_\eta(t,x)](x) = \partial_q A[q^0(t), g^0_\eta(t,x)](x)\cdot (q^0)'(t) + A[q^0(t), \partial_t g^0_\eta(t,x)](x) = A[q^0(t), \partial_t g^0_\eta(t,x)](x).$$

From here it is straightforward to conclude that the map

$$q_1 \mapsto (l^0, r^0, u^0) \in C^3([0,T];\mathbb{R}^3) \times C^3([0,T]; C^5_\infty(\overline{\mathcal{F}_0}))$$

is continuous.

Differentiating the toy model with respect to $w \in \overline{B}\big((q_0,0),\tilde{r}\big)$. We will prove the following Lemma, which exactly implies (4.50).

Lemma 4.3.2. *Let $w = (q_1, q_1') \in \overline{B}\big((q_0,0),\tilde{r}\big)$, and consider the solutions $(\tilde{q},\tilde{p})$ of (4.40) with $v_i = (-1)^i\mathcal{M}(c_i(q_1) - q_i')$, $i = 0,1$, respectively $(\bar{q},\bar{p})$ of (4.41). Then $(\tilde{q},\tilde{p})$ and $(\bar{q},\bar{p})$ are differentiable with respect to w, and we have that*

$$\lim_{\eta\to 0^+} \|(\partial_w\tilde{q}, \partial_w\tilde{p}) - (\partial_w\bar{q}, \partial_w\bar{p})\|_{C([2\eta, T-2\eta])} = 0 \tag{4.52}$$

and $(\partial_w\tilde{q}, \partial_w\tilde{p})(T) \to Id$ as $\eta \to 0^+$.

Proof. The existence of the differentials of $(\tilde{q},\tilde{p})$ and $(\bar{q},\bar{p})$ with respect to $w = (q_1, q_1') \in \overline{B}\big((q_0,0),\tilde{r}\big)$ is immediate. Furthermore, we may formally differentiate (4.40) and (4.41) to find that the aforementioned differentials satisfy the following ODE systems :

$$\begin{aligned} \mathcal{M}(\partial_w\tilde{p})' + 2\left(\langle \Gamma, \tilde{p}, \partial_{w_i}\tilde{p}\rangle\right)_{1\leq i\leq 6} &= \beta_\eta^2\partial_w v_0 + \tilde{\beta}_\eta^2\partial_w v_1, \\ (\partial_w\tilde{q})' = \left((D\mathcal{R}(\tilde{q})\cdot\partial_{w_i}\tilde{q})\,\tilde{p}\right)_{1\leq i\leq 6} &+ \mathcal{R}(\tilde{q})\partial_w\tilde{p}, \end{aligned} \tag{4.53}$$

on $[0,T]$, with $\tilde{\beta}_\eta = \beta_\eta(T-\cdot)$ and $(\partial_w\tilde{q}, \partial_w\tilde{p})(0) = 0$; respectively,

$$\mathcal{M}(\partial_w\bar{p})' + 2\left(\langle \Gamma, \bar{p}, \partial_{w_i}\bar{p}\rangle\right)_{1\leq i\leq 6} = 0, \ (\partial_w\bar{q})' = \left((D\mathcal{R}(\bar{q})\cdot\partial_{w_i}\bar{q})\,\bar{p}\right)_{1\leq i\leq 6} + \mathcal{R}(\bar{q})\partial_w\bar{p}, \tag{4.54}$$

on $[0,T]$, with $\partial_{q_1}\bar{q}(0) = 0$, $\partial_{q_1}\bar{q}(T) = \mathrm{Id}$, $\partial_{q_1'}\bar{q}(0) = 0$, $\partial_{q_1'}\bar{q}(T) = 0$. In particular note that due to $\partial_{q_1'}\bar{q}(0) = 0$, $\partial_{q_1'}\bar{q}(T) = 0$, we in fact have that $\partial_{q_1'}\bar{q}$ and $\partial_{q_1'}\bar{p}$ vanish on $[0,T]$. It further follows from (4.41) that $\partial_w\bar{p}(0) = \partial_w c_0(q_1) = (\partial_{q_1}c_0(q_1), 0)$ and $\partial_w\bar{p}(T) = \partial_w c_1(q_1) = (\partial_{q_1}c_1(q_1), 0)$.

On the other hand, we may further develop $\partial_w v_i = (\partial_{q_1}v_i, \partial_{q_1'}v_i)$ that appear in (4.53), recalling the forms of v_i in the statement of the Lemma. We have

$$\begin{aligned} \partial_{q_1}v_i &= \partial_{q_1}\left((-1)^i\mathcal{M}(c_i(q_1) - q_i'\right) = (-1)^i\mathcal{M}\partial_{q_1}c_i(q_1), \\ \partial_{q_1'}v_i &= \partial_{q_1'}\left((-1)^i\mathcal{M}(c_i(q_1) - q_i'\right) = i\mathcal{M}. \end{aligned} \tag{4.55}$$

Before we proceed to comparing the trajectories $(\partial_w \tilde{q}, \partial_w \tilde{p})$ and $(\partial_w \bar{q}, \partial_w \bar{p})$, let us explain how one can prove that they are uniformly bounded in $L^\infty(0,T)$ with respect to $\eta \in (0,1)$ and $w \in \overline{B}\big((q_0,0),\tilde{r}\big)$. Through similar methods as in Section 3.5 of Chapter 3, it can be checked that $\|\tilde{p}\|_\infty$ is uniformly bounded with respect to (η, w), respectively $\|\bar{p}\|_\infty$ is uniformly bounded with respect to w. Using Gronwall estimates for (4.53) and (4.54), we get that $\|\partial_w \tilde{p}\|_\infty$ is uniformly bounded with respect to (η, w), respectively $\|\partial_w \bar{p}\|_\infty$ is uniformly bounded with respect to w. We refer the reader to Chapter 3 for further details.

Now we are ready to compare the trajectories $(\partial_w \tilde{q}, \partial_w \tilde{p})$ and $(\partial_w \bar{q}, \partial_w \bar{p})$, which we will do on three time intervals, namely $[0, 2\eta]$, $[2\eta, T-2\eta]$ and $[T-2\eta, T]$, in order to exploit the supports of the functions β_η and $\tilde{\beta}_\eta$.

Integrating (4.53) on $[0, 2\eta]$ and taking into account (4.55), we get that

$$\mathcal{M}\partial_w \tilde{p}(2\eta) = \mathcal{M}\partial_{q_1} c_0(q_1) - 2\int_0^{2\eta} \left(\langle \Gamma, \tilde{p}, \partial_{w_i}\tilde{p}\rangle\right)_{1\le i\le 6}\, dt.$$

Using that $\|\tilde{p}\|_\infty$ and $\|\partial_w \tilde{p}\|_\infty$ are uniformly bounded with respect to (η, w), it follows that $\partial_w \tilde{p}(2\eta)$ converges to $\partial_{q_1} c_0(q_1)$ as $\eta \to 0^+$, uniformly for $w \in \overline{B}\big((q_0,0),\tilde{r}\big)$, while the position $\partial_w \tilde{q}(2\eta)$ converges to $\partial_w \tilde{q}(0) = 0$ as $\eta \to 0^+$.

During the time interval $[2\eta, T-2\eta]$, the right-hand side of the first line in (4.53) vanishes, so the equation satisfied by $(\partial_w \tilde{q}, \partial_w \tilde{p})$ is close to (4.54), provided that $\tilde{p}$ is close to $\bar{p}$ (in the L^∞ norm). We may take the difference of (4.53) and (4.54), use a Gronwall argument, (4.44) and the boundedness of $\|\tilde{p}\|_\infty$, $\|\bar{p}\|_\infty$, $\|\partial_w \tilde{p}\|_\infty$, respectively $\|\partial_w \bar{p}\|_\infty$, to conclude that (4.52) holds. Implicitly, we have that

$$(\partial_w \tilde{q}, \partial_w \tilde{p})(T-2\eta) \to \begin{pmatrix} \mathrm{Id} & \partial_{q_1} c_1(q_1) \\ 0 & 0 \end{pmatrix}, \text{ as } \eta \to 0^+. \tag{4.56}$$

Finally, during the interval $[T-2\eta, T]$ we proceed as we did during $[0, 2\eta]$. More precisely, integrating (4.53), using a Gronwall estimate and the form of $\partial_w v_1$ from (4.55) allows us to reorient the final velocity $\partial_w \tilde{p}(T)$ from $(\partial_{q_1} c_1(q_1), 0)$ to $(0, \mathrm{Id})$, while the position $\partial_w \tilde{q}(T)$ converges to $(\mathrm{Id}, 0)$ as $\eta \to 0^+$, which concludes the proof. □

Differentiating and estimating the real model. As we have also noted at the beginning of Section 4.3.2, a lot of the arguments we will use, in particular those using Gronwall type estimates to prove the boundedness or closeness of certain trajectories, work in the same way as in Section 3.6 of Chapter 3, so for the sake of brevity we will often skip such details, only sketch the main elements of the proof and refer the reader to Chapter 3.

For simplicity of notation, we rewrite (4.37) into the following form, in which we only consider the dependences with respect to q^0, p^0, w and β_η. We have

$$\begin{aligned} \mathcal{M}(p^0)' + \langle \Gamma, p^0, p^0\rangle &= \beta_\eta^2 \hat{F}_{1,0}(q^0, w) + \tilde{\beta}_\eta^2 \hat{F}_{1,1}(q^0, w) + (\beta_\eta + \tilde{\beta}_\eta)\hat{F}_2(q^0, p^0, w) + (\beta_\eta' + \tilde{\beta}_\eta')\hat{F}_3(q^0), \\ (q^0)' &= \mathcal{R}(q^0)p^0, \end{aligned} \tag{4.57}$$

where $\tilde{\beta}_\eta = \beta_\eta(T - \cdot)$, the maps $\hat{F}_{1,0}, \hat{F}_{1,1}, \hat{F}_2, \hat{F}_3$ can be deduced from (4.37) and are of class C^∞ in (q^0, p^0) and C^1 in w. Furthermore, note that we have

$$\hat{F}_{1,i}(q, w) = F_{1,a}\left[\mathcal{A}\left[q, \overline{g}^0\left(q_i, -2v_i\right)\right]\right], \quad i = 0, 1, \tag{4.58}$$

where $\mathcal{A}$ was given in Definition 4.31, $F_{1,a}$ in Definition 4.36, and v_i in (4.43).

Using similar estimates as in Section 3.6 of Chapter 3, it can be shown that $\|p^0\|_\infty$ is uniformly bounded with respect to $0 < \eta << 1$ and $w \in \overline{B}\big((q_0, 0), \tilde{r}\big)$.

The differentiability with respect to w of the solution (q^0, p^0) to the ODE system (4.57) can be inferred from the regularity of the maps involved. We deduce the following ODE system for $(\partial_w q^0, \partial_w p^0)$.

$$\begin{aligned}
\mathcal{M}(\partial_w p^0)' + 2\left(\langle \Gamma, p^0, \partial_{w_i} p^0\rangle\right)_{1\le i\le 6} &= \beta_\eta^2 \left(\partial_q \hat{F}_{1,0}(q^0, w)\partial_w q^0 + \partial_w \hat{F}_{1,0}(q^0, w)\right) \\
&+ \tilde{\beta}_\eta^2 \left(\partial_q \hat{F}_{1,1}(q^0, w)\partial_w q^0 + \partial_w \hat{F}_{1,1}(q^0, w)\right) \\
&+ (\beta_\eta + \tilde{\beta}_\eta)\nabla \hat{F}_2(q^0, p^0, w)\cdot(\partial_w q^0, \partial_w p^0, \mathrm{Id}) \\
&+ (\beta_\eta' + \tilde{\beta}_\eta')\partial_q \hat{F}_3(q^0)\partial_w q^0, \\
(\partial_w q^0)' &= \left(\left(D\mathcal{R}(q^0)\cdot \partial_{w_i} q^0\right) p^0\right)_{1\le i\le 6} + \mathcal{R}(q^0)\partial_w p^0,
\end{aligned} \tag{4.59}$$

with $(\partial_w q^0, \partial_w p^0)(0) = 0$. As before, it can be shown that $\|\partial_w p^0\|_\infty$ is uniformly bounded with respect to $0 < \eta << 1$ and $w \in \overline{B}\big((q_0, 0), \tilde{r}\big)$, by a Gronwall estimate and the uniform boundedness of $\|p^0\|_\infty$.

Let us show that $(\partial_w q^0, \partial_w p^0)$ is close to $(\partial_w \tilde{q}, \partial_w \tilde{p})$ in $C([0, T])$ by the means of a Gronwall argument. In order to see this, we claim that the only terms in (4.59) that need further investigation are $\partial_w \hat{F}_{1,i}(q^0, w)$, $i = 0, 1$, since all the other terms are sufficiently regular for a Gronwall-type estimate, uniformly with respect to $\eta > 0$. Indeed, $(p^0, \partial_w p^0, w)$ is uniformly bounded in (w, η), the other terms on the right-hand side of (4.59) are regular with respect to (q^0, p^0, w), we have $\beta_\eta \in L^2$ uniformly with respect to $\eta > 0$, furthermore, we may estimate

$$|\partial_w q^0| \le \int_{[0,2\eta]\cup[T-2\eta,T]} |\partial_w p^0| \le C\eta$$

on the support of $\beta_\eta' + \tilde{\beta}_\eta'$, so that we can obtain $\left|\int (\beta_\eta' + \tilde{\beta}_\eta')\partial_q \hat{F}_3(q^0)\partial_w q^0\right| \le C\sqrt{\eta}$.

For the terms $\partial_w \hat{F}_{1,i}(q^0, w)$, let us first observe that, since $\|p^0\|_\infty$ is uniformly bounded with respect to $0 < \eta << 1$ and $w \in \overline{B}\big((q_0, 0), \tilde{r}\big)$, there exists $C > 0$ independent of (w, η) such that $|q^0(t) - q_0| \le C\eta$ for $t \in [0, 2\eta]$ and $|q^0(t) - q_1| \le C\eta$ for $t \in [T - 2\eta, T]$. Since $\partial_w \hat{F}_{1,i}$ are Lipschitz in q, we get that

$$\left|\partial_w \hat{F}_{1,i}(q^0(t), w) - \partial_w \hat{F}_{1,i}(q_i, w)\right| \le C\eta \text{ for } t \in [i(T - 2\eta), i(T - 2\eta) + 2\eta],\ i = 0, 1. \tag{4.60}$$

Therefore, it suffices to estimate $\partial_w \hat{F}_{1,i}(q_i, w)$ on $[0, 2\eta] \cup [T - 2\eta, T]$ instead of $\partial_w \hat{F}_{1,i}(q^0(t), w)$. We use (4.58) and recall that $w = (q_1, q_1')$, to infer from Proposition 4.3.3 and Definition 4.36 that

$$\partial_v F_{1,a}\left[\mathcal{A}\left[q_i, \overline{g}^0\left(q_i, v\right)\right]\right] = -\frac{1}{2}\mathrm{Id}, \quad \partial_{q_1} F_{1,a}\left[\mathcal{A}\left[q_i, \overline{g}^0\left(q_i, v\right)\right]\right] = 0. \tag{4.61}$$

On the other hand, using (4.58), we get that

$$\partial_w \hat{F}_{1,i}(q_i, w) = \partial_w v_i. \tag{4.62}$$

We may conclude by using (4.53), (4.59), the respective uniform bounds for the solutions of the two systems, furthermore, (4.60), (4.62), and a Gronwall argument, that $\|(\partial_w q^0, \partial_w p^0) - (\partial_w \tilde{q}, \partial_w \tilde{p})\|_{C([0,T])} \to 0$ as $\eta \to 0^+$, uniformly with respect to $w \in \overline{B}\big((q_0, 0), \tilde{r}\big)$, which exactly gives us (4.51).

Proof of Theorem 4.3.2 with (4.28)

Finally, in order to ensure that (4.28) holds, we will add another impulsive control around time $T/2$ which we will show does not affect (4.37) in any substantial manner.

More precisely, we change the control given by (4.39) to the following

$$g_\eta^0(t,x) := \beta_\eta(t)\overline{g}^0(q_0,-2v_0)(x) + \beta_\eta(t-T/2+\eta)\overline{g}^0\left(q^0(T/2-\eta),0\right)(x) + \beta_\eta(T-t)\overline{g}^0(q_1,-2v_1)(x), \tag{4.63}$$

where we note that due to Proposition 4.3.3, the function $\overline{g}^0\left(q^0(T/2-\eta),0\right)$ is non-trivial and uniformly bounded with respect to $\eta > 0$, due to the boundedness of $q^0(T/2-\eta)$. By the same considerations as in Section 4.3.2, the associated solution (q^0,p^0) will be converge to the solution $(\tilde{q},\tilde{p})$ of (4.40) in $C([0,T])$ as $\eta \to 0^+$, since we deduced $\overline{g}^0\left(q^0(T/2-\eta),0\right)$ by applying Proposition 4.3.3 with $v=0$, so its effect on the solid equation is neglectable as $\eta \to 0^+$. Furthermore, we have

$$\partial_w F_{1,a}\left[\mathcal{A}\left[q^0(T/2-\eta),\overline{g}^0\left(q^0(T/2-\eta),0\right)\right]\right] = 0,$$

so we may repeat the same arguments as in Section 4.3.2 for $(\partial_w q^0, \partial_w p^0)$ to arrive at the same conclusion.

To ensure that (4.28) holds, we note that from Proposition 4.3.3 and by continuity it follows that

$$\text{span}\left\{(n(x), x^\perp\cdot n(x)),\ x\in \text{supp}\ \nabla\mathcal{A}\left[q^0(T/2),\overline{g}^0\left(q^0(T/2-\eta),0\right)\right]\cap\partial\mathcal{S}_0\right\} = \mathbb{R}^3,$$

for $\eta > 0$ small enough. Due to (4.63), we have

$$u^0(T/2,\cdot) = \nabla\left(p^0(T/2)\cdot\Phi\right) + \beta_\eta(\eta)\nabla\mathcal{A}\left[q^0(T/2),\overline{g}^0\left(q^0(T/2-\eta),0\right)\right].$$

Since $\beta_\eta(\eta) = \mathcal{O}(1/\sqrt{\eta})$ and $p^0 \in L^\infty$ is uniformly bounded with respect to $\eta > 0$, it follows that for $\eta > 0$ small enough (4.28) holds.

We conclude the proof of Theorem 4.3.2 by further reducing (if necessary) and fixing $\eta = \eta_0 > 0$ such that (4.28) also holds.

Remark 4.3.1. *The fact that one can guarantee the small flux condition* (4.24) *is a direct consequence of the explicit formula for* $g_\eta^0(t,x)$ *given in* (4.39) *and of a change of variables in time. Due to the properties of* β_η *given at the beginning of Section 4.3.2 one obtains that the flux is of order* $\sqrt{\eta}$*. Hence one can reduce* η *again in order to satisfy* (4.24)*. It can be easily seen that this argument is also invariant for passing from small solid displacement to an arbitrary one.*

4.4 The boundary layer profiles

In this section we prove the existence of the boundary layer profiles appearing near the solid, and provide some regularity estimates which we will use in the sequel to handle the boundary layers' effect on the linearized and remainder terms in the asymptotic development (4.17), (4.21).

4.4.1 The physical boundary layer profile v

At order $\mathcal{O}(\sqrt{\varepsilon})$, we look for $v = v(t,x,z)$ which satisfies the following equations

$$\begin{aligned}
&\partial_t v + v\cdot\nabla u^0 + (u^0-u_S^0)\cdot\nabla_x v + r^0 v^\perp - \partial_z Q n - \frac{(u^0-u_S^0)\cdot n}{\varphi} z\partial_z v = \partial_{zz}^2 v \text{ in } [0,T]\times\overline{\mathcal{F}_0}\times\mathbb{R}_+,\\
&\partial_z v(\cdot,\cdot,0)_{\tan} = 2\chi(\cdot)\left(D(u^0)n + \mu(u^0-u_S^0)\right)_{\tan},\ \text{in } [0,T]\times\overline{\mathcal{F}_0},\\
&v(0,\cdot,\cdot) = 0 \text{ in } \overline{\mathcal{F}_0}\times\mathbb{R}_+.
\end{aligned} \tag{4.64}$$

Here $\varphi : \mathbb{R}^2 \to \mathbb{R}$ denotes a smooth function which is introduced in the spirit of [27] and [70] such that $\varphi = 0$ on $\partial\mathcal{S}_0$, $\varphi > 0$ in $\mathcal{F}_0$, $\varphi < 0$ in the interior of $\mathcal{S}_0$, and $|\varphi(x)| = \mathrm{dist}(x, \partial\mathcal{S}_0)$ in a small neighbourhood $\mathcal{V}$ of $\partial\mathcal{S}_0$, such that the normal n can be computed as $n = -\nabla\varphi$ close to the boundary and extended smoothly in $\mathcal{F}_0$. Furthermore, we consider $\chi \in C_0^\infty(\overline{\mathcal{F}_0}; [0,1])$ such that $\mathrm{supp}\, \chi \subset \mathcal{V}$ and $\chi = 1$ in a neighbourhood of $\partial\mathcal{S}_0$, constructed as in Section 3.4 of [27] such that the x-support of v does not include points where $\nabla\varphi$ vanishes.

Note that the introduction of such a boundary layer profile v in (4.21) is due to the fact that the PDE satisfied by u^0 is of first order, hence it can only satisfy a single scalar boundary condition on $\partial\mathcal{S}_0$, namely $u^0 \cdot n = (l^0 + r^0 x^\perp) \cdot n$, and therefore the full Navier slip-with-friction condition does not hold for u^0. This issue is corrected by v at order $\mathcal{O}(\sqrt{\varepsilon})$ satisfying system (4.64). In particular, due to the definition of χ in the paragraph above, v is compactly supported in a neighbourhood of $\partial\mathcal{S}_0$.

We further introduce the spaces in which we will look for a solution v to (4.64), namely the weighted anisotropic Sobolev spaces $H^{k,m,p}$, $k, m, p \in \mathbb{N}$, defined by their norms

$$\|v\|_{k,m,p}^2 = \sum_{|\alpha|\le m,\ j\le p} \int_0^{+\infty} \int_{\mathcal{F}_0} (1+z^2)^k |\partial_x^\alpha \partial_z^j v|^2 \, dx\, dz.$$

We have the following existence and regularity result for v, which can be seen as a generalization of Proposition 5 from [70].

Proposition 4.4.1. *Let $(h^0, \vartheta^0, u^0) \in C^\infty([0,T]; \mathcal{Q}_\delta) \times C^\infty([0,T] \times \overline{\mathcal{F}_0}; \mathbb{R}^2)$ as given by Theorem 4.3.1. There exist a unique solution $v \in L^2((0,T); H^{k,2,1}) \cap L^\infty((0,T); H^{k,2,0})$ to (4.64) such that $\partial_z v \in L^\infty([0,T] \times \mathcal{F}_0 \times \mathbb{R}_+)$ and v satisfies the orthogonality property $v \cdot n = 0$, for any $z \ge 0$. Furthermore, we have that v is bounded in $L^2((0,T); H^{k,5,3})$, for any $k \in \mathbb{N}$, and one may define a continuous map $(h_1, \vartheta_1) \mapsto v \in L^2((0,T); H^{k,5,3})$.*

Scheme of proof. We follow the same strategy as in the proof of Proposition 5 from [70].

We first show that the pressure Q in (4.64) can be separated from the first equation, so we may simply consider v as the solution of the linear PDE

$$\begin{aligned}
&\partial_t v + [v \cdot \nabla u^0 + (u^0 - u_S^0) \cdot \nabla_x v + r^0 v^\perp]_{\tan} - \frac{(u^0 - u_S^0)\cdot n}{\varphi} z \partial_z v = \partial_{zz}^2 v \text{ in } [0,T] \times \overline{\mathcal{F}_0} \times \mathbb{R}_+,\\
&\partial_z v(\cdot,\cdot,0)_{\tan} = 2\chi(\cdot)\,(D(u^0)n + \mu(u^0 - u_S^0))_{\tan}, \text{ in } [0,T] \times \overline{\mathcal{F}_0},\\
&v(0,\cdot,\cdot) = 0 \text{ in } \overline{\mathcal{F}_0} \times \mathbb{R}_+.
\end{aligned} \tag{4.65}$$

Therefore it suffices to prove the conclusions of Proposition 4.4.1 for v satisfying (4.65). Indeed, once the existence of such a v is determined, we may associate Q as the unique function that vanishes as $z \to +\infty$ and satisfies

$$\partial_z Q = [v \cdot \nabla u^0 + (u^0 - u_S^0) \cdot \nabla_x v + r^0 v^\perp] \cdot n, \text{ in } [0,T] \times \overline{\mathcal{F}_0} \times \mathbb{R}_+. \tag{4.66}$$

Since v vanishes for x outside of $\mathcal{V}$, so does Q. Furthermore, it is easy to check that we have

$$\|Q(t,\cdot,\cdot)\|_{1,1,1} \lesssim \|v(t,\cdot,\cdot)\|_{3,2,0}, \text{ for any } t \in [0,T], \tag{4.67}$$

where " $\lesssim$ " denotes an estimation with a constant which does not depend on the parameter (h_1, ϑ_1).

To deduce the existence of v, we note that the fact that $\mathcal{F}_0$ is unbounded does not interfere with the methods used in [70] to prove the existence of such boundary layers, since as mentioned before, v

is compactly supported in $\mathcal{V}$, due to the definition of χ. Therefore, the same methods as used in the proof of Proposition 5 from [70] can be applied to deduce that there exists a unique solution v of (4.65) which satisfies $v \in L^2((0,T); H^{k,2,1}) \cap L^\infty((0,T); H^{k,2,0})$, $\partial_z v \in L^\infty([0,T] \times \mathcal{F}_0 \times \mathbb{R}_+)$ and $v \cdot n = 0$, for any $z \geq 0$.

To obtain the higher regularity estimates in Proposition 4.4.1, we bootstrap the methods from the proof of Proposition 5 from [70] in the following manner.

First, we note that having higher regularity with respect to x follows simply from the regularity of u^0 and repeating the same methods used to obtain $v \in L^2((0,T); H^{k,2,1})$. This allows us to prove $v \in L^2((0,T); H^{k,5,1})$.

Next, we differentiate (4.65) with respect to t to obtain similar estimates on $\partial_t v$, by using the higher regularity of (u^0, p^0). This in turn allows us to estimate $\partial^2_{zz} v$, using (4.65). We iterate the process once more to achieve the desired regularity $v \in L^2((0,T); H^{k,5,3})$.

The continuity part of the result is also straightforward due to Theorem 4.3.1. The details are left to the reader.

□

4.4.2 The second boundary corrector w

As in [27], v generates a non-vanishing slow divergence and tangential boundary flux, i.e. $\{\text{div } v\}$ in $\mathcal{F}_0$, and $(D(v(\cdot,\cdot,0))n + \mu v)_{\tan}$ on $\mathcal{S}_0$ do not vanish a priori. This is undesirable if we take into account (4.17), (4.21), since we would like to have that the boundary layers do not contribute to the divergence of u^ε and that the Navier condition holds for u^ε.

To address this we introduce w given by

$$w(t,x,z) := -2e^{-z}(D(v(t,x,0))n(x) + \mu v(t,x,0))_{\tan} - n(x) \int_z^{+\infty} \text{div } v(t,x,z')\, dz'. \tag{4.68}$$

It is easy to check that we have

$$(D(v(\cdot,\cdot,0))n + \mu v)_{\tan} - \frac{1}{2}(\partial_z w)_{\tan} = 0 \text{ on } \mathcal{S}_0, \text{ and } \{\text{div } v\} - n \cdot \{\partial_z w\} = 0 \text{ in } \mathcal{F}_0. \tag{4.69}$$

Furthermore, w also vanishes for x outside of $\mathcal{V}$, and for any $k, m, p \in \mathbb{N}$ we have

$$\|w(t,\cdot,\cdot)\|_{k,m,p} \lesssim \|v(t,\cdot,\cdot)\|_{k+2,m+1,p+1}, \text{ for any } t \in [0,T], \tag{4.70}$$

where once again we recall that we use " $\lesssim$ " to denote an estimation with a constant which does not depend on the parameter (h_1, ϑ_1).

4.4.3 The inner domain corrector θ^ε

Finally, we note that w generates a non-vanishing boundary flux $w \cdot n$ on $\partial\mathcal{S}_0$ and a slow divergence. To address this, for fixed time $t \in [0,T]$, we define θ^ε as the solution to

$$\Delta\theta^\varepsilon = -\{\text{div } w\} \text{ in } \mathcal{F}_0,\ \partial_n \theta^\varepsilon = -w(t,\cdot,0) \cdot n \text{ on } \partial\mathcal{S}_0,\ \lim_{|x| \to +\infty} |\nabla\theta^\varepsilon| = 0, \tag{4.71}$$

where we recall that for $f = f(t,x,z)$, we denote $\{f\}$ its evaluation at $z = \frac{\varphi(x)}{\sqrt{\varepsilon}}$.

Proceeding as in [27], we see that $\nabla\theta^\varepsilon$ satisfies

$$\begin{aligned}
&\partial_t\nabla\theta^\varepsilon + (u^0 - u_S^0)\cdot\nabla(\nabla\theta^\varepsilon) + \nabla\theta^\varepsilon\cdot\nabla u^0 + r^0\nabla^\perp\theta^\varepsilon + \nabla\rho^\varepsilon = 0 \text{ for } x\in\mathcal{F}_0,\\
&\text{div } \theta^\varepsilon = -\{\text{div } w\} \text{ for } x\in\mathcal{F}_0,\\
&\nabla\theta^\varepsilon\cdot n = -w(t,\cdot,0)\cdot n \text{ for } x\in\partial\mathcal{S}_0,\ \lim_{|x|\to+\infty}|\nabla\theta^\varepsilon| = 0.
\end{aligned} \tag{4.72}$$

with pressure term $\rho^\varepsilon := -\partial_t\theta^\varepsilon - (u^0 - u_S^0)\cdot\nabla\theta^\varepsilon$. Furthermore, we have

$$\|\theta^\varepsilon(t,\cdot)\|_{H^{m+2}} \lesssim \varepsilon^{\frac{1-2m}{4}}\|w(t,\cdot)\|_{0,m+2,m} + \|v(t,\cdot)\|_{1,m+1,0}, \tag{4.73}$$

for any $t\in[0,T],\ m=0,1,2$.

Gathering (4.69), (4.71), and the fact that $v\cdot n = 0$ for any (t,x,z), we may conclude that we have

$$\begin{aligned}
\text{div }\left(\sqrt{\varepsilon}\{v\} + \varepsilon\{w\} + \varepsilon\nabla\theta^\varepsilon\right) &= 0 \text{ in } \mathcal{F}_0,\\
(D(v(\cdot,\cdot,0))n + \mu v)_{\tan} - \frac{1}{2}(\partial_z w)_{\tan} &= 0 \text{ on } \mathcal{S}_0,\\
(\sqrt{\varepsilon}\{v\} + \varepsilon\{w\} + \varepsilon\nabla\theta^\varepsilon)\cdot n &= 0 \text{ on } \mathcal{S}_0.
\end{aligned}$$

Taking into account (4.25), (4.64), this allows us to look for a linearized term u^1 (with divergence g^1) and a divergence-free remainder u_R^ε satisfying the boundary conditions

$$\begin{aligned}
u^1\cdot n &= u_S^1\cdot n,\\
u_R^\varepsilon\cdot n &= u_{R,S}^\varepsilon\cdot n,\\
(D(u_R^\varepsilon)n)_{\tan} &= -\mu(u_R^\varepsilon - u_{R,S}^\varepsilon)_{\tan} - \left(D\left(u^1 + \nabla\theta^\varepsilon + w(\cdot,\cdot,0)\right)n\right) + \mu\left(u^1 - u_S^1 + \nabla\theta^\varepsilon + w(\cdot,\cdot,0)\right)_{\tan},
\end{aligned}$$

on $\mathcal{S}_0$, so that u^ε defined via (4.17), (4.21) satisfies the Navier boundary conditions in (4.11) and has divergence $g^0 + \varepsilon g^1$.

4.5 The first order term u^1

In this section we construct a controlled solution to the linearized terms appearing in the asymptotic expansion (4.17), (4.21).

At order $\mathcal{O}(\varepsilon)$, we look for (u^1,π^1,l^1,r^1,g^1) satisfying the following system.

$$\begin{aligned}
&\frac{\partial u^1}{\partial t} + (u^0 - u_S^0)\cdot\nabla u^1 + (u^1 - u_S^1)\cdot\nabla u^0 + r^0(u^1)^\perp + r^1(u^0)^\perp + \nabla\pi^1 = \nabla g^0,\\
&\text{div } u^1 = g^1 \ \text{ for } x\in\mathcal{F}_0,\ u^1\cdot n = u_S^1\cdot n \ \text{ for } x\in\partial\mathcal{S}_0,\ \lim_{|x|\to+\infty}|u^1| = 0,\\
&m(l^1)' = \int_{\partial\mathcal{S}_0}(\pi^1 + Q(\cdot,\cdot,0) + \rho^\varepsilon)\,n\,d\sigma - \int_{\partial\mathcal{S}_0}(2D(u^0)n - \partial_z v(\cdot,\cdot,0))\,d\sigma\\
&\qquad - mr^0(l^1)^\perp - mr^1(l^0)^\perp,\\
&\mathcal{J}\ (r^1)' = \int_{\partial\mathcal{S}_0}(\pi^1 + Q(\cdot,\cdot,0) + \rho^\varepsilon)\,x^\perp\cdot n\,d\sigma - \int_{\partial\mathcal{S}_0} x^\perp\cdot(2D(u^0)n - \partial_z v(\cdot,\cdot,0))\,d\sigma,
\end{aligned} \tag{4.74}$$

where $u_S^1(t,x) = l^1(t) + r^1(t)x^\perp$, for $t\in[0,T]$, with

$$u^1(0,\cdot) = u^*, \quad (l^1,r^1)(0) = (h_0',\vartheta_0'). \tag{4.75}$$

Note once again the subtle dependence on $\varepsilon > 0$ due to ρ^ε and u^*, which we have ommited from the notation, and recall that u^* was constructed as per (4.22) and is uniformly bounded in C^2 with respect to $\varepsilon > 0$. Furthermore, $\nabla g^0 = \Delta u^0$, since u^0 is irrotational.

The position of the solid $q^1 = (h^1, \vartheta^1)$ can be associated analogously with (4.12), however, we remark that the solid position q^1 does not play an important role in our control strategy, as long as it is bounded independently of ε, due to the scaling in (4.17). Once again, observe that using Lamb's form from (4.35), the gradient of the pressure π^1 in (4.74) can be expressed as

$$\nabla \pi^1 = -\partial_t u^1 - \nabla(u^0 \cdot u^1) + \nabla(u^1 \cdot u_S^0) + \nabla(u^0 \cdot u_S^1) + \nabla g^0 - \omega^1 (u^0 - u_S^0)^\perp, \tag{4.76}$$

where $\omega^1 = \operatorname{curl} u^1$.

We have the following approximate controllability result regarding the solid velocity (l^1, r^1).

Theorem 4.5.1. *Let $T > 0$, $\mathcal{S}_0 \subset \mathbb{R}^2$ bounded, closed, simply connected with smooth boundary, which is not a disk, and $u^* \in C^\infty(\overline{\mathcal{F}_0}) \cap L^2(\mathcal{F}_0)$, $\gamma \in \mathbb{R}$ constructed as per (4.22), $p_0 = (h_0', \vartheta_0'), p_1 = (l_1, r_1) \in \mathbb{R}^3$, such that*

$$\operatorname{div} u^* = 0 \text{ in } \mathcal{F}_0, \ \operatorname{curl} u^* \in C_0^\infty(\overline{\mathcal{F}_0}), \lim_{|x| \to +\infty} |u^*(x)| = 0,$$
$$u^* \cdot n = (h_0' + \vartheta_0' x^\perp) \cdot n \text{ on } \partial \mathcal{S}_0, \ \int_{\partial \mathcal{S}_0} u^* \cdot \tau \, d\sigma = \gamma.$$

Given $\nu > 0$, $\delta > 0$, $\varepsilon > 0$ and a solution $(u^0, h^0, \vartheta^0, g^0)$ as described in Theorem 4.3.1, there exists a control $g^1 \in C_0^\infty((0,T) \times \mathcal{F}_0)$, and a solution $(l^1, r^1, u^1) \in C^\infty([0,T]; \mathbb{R}^3 \times C^\infty(\overline{\mathcal{F}_0}; \mathbb{R}^2))$ to (4.74), (4.75), which satisfies $u^1 \in C([0,T]; L^2(\mathcal{F}_0))$ and

$$|(l^1, r^1)(T) - (l_1, r_1)| \le \nu, \text{ supp } g^1(t, \cdot) \subset R(\vartheta^0(t))^T (B_c - h^0(t)), \text{ for all } t \in [0,T].$$

Furthermore, (l^1, r^1, u^1) is uniformly bounded in $L^\infty(\mathbb{R}^3) \times L^\infty(C_\infty^2(\overline{\mathcal{F}_0}))$ with respect to $\varepsilon \in (0,1]$; and one may define a map $(h_1, \vartheta_1, l_1, r_1) \mapsto (l^1, r^1, u^1) \in L^\infty(\mathbb{R}^3) \times L^\infty(C_\infty^2(\overline{\mathcal{F}_0}))$ which is continuous, uniformly with respect to $\varepsilon \in (0,1]$.

The rest of the section is dedicated to proving Theorem 4.5.1. The proof will be based on a similar strategy as that of Theorem 4.3.1 before, with a slight simplification due to the fact that we do not want to control the solid position, but only the velocity. Hence one impulsive control will be enough.

4.5.1 Vorticity

First we consider ω^1 as the regular solution of the vorticity equation

$$\begin{aligned} &\frac{\partial \omega^1}{\partial t} + (u^0 - u_S^0) \cdot \nabla \omega^1 + \omega^1 g^0 = 0 \ \text{ for } x \in \mathcal{F}_0, \\ &\omega^1(0, \cdot) = \operatorname{curl} u^*(\cdot). \end{aligned} \tag{4.77}$$

Note that for ω^1 we have once more omitted the dependence with respect to $\varepsilon > 0$ from the notation.

We recall the definition of the hydrodynamic Biot-Savart operator on $\mathcal{F}_0$, which can be seen as the "inverse" of the curl operator, namely for $\omega \in C_0^\infty(\overline{\mathcal{F}_0})$, we consider $K_H[\omega] \in C^\infty(\overline{\mathcal{F}_0})$ given as the unique solution of the system

$$\begin{cases} \operatorname{div} K_H[\omega] = 0, \ \operatorname{curl} K_H[\omega] = \omega \ \text{ in } \mathcal{F}_0, \\ K_H[\omega] \cdot n = 0 \text{ on } \partial \mathcal{S}_0, \lim\limits_{|x| \to +\infty} |K_H[\omega]| = 0, \\ \displaystyle\int_{\partial \mathcal{S}_0} K_H[\omega] \cdot \tau \, d\sigma = 0. \end{cases} \tag{4.78}$$

We have the following existence and regularity result regarding the vorticity, in particular to investigate the dependence with respect to $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$ and $\varepsilon \in (0, 1]$.

Proposition 4.5.1. *Let $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$, (q^0, p^0, u^0) as in Theorem 4.3.1. There exists a unique smooth solution $\omega^1 \in C^\infty([0, T]; C_0^\infty(\overline{\mathcal{F}_0}))$ of (4.77) such that we have the following.*

(i) $K_H[\omega^1]$ is bounded in $L^\infty(C^2_\infty(\overline{\mathcal{F}_0}))$, ω^1 is bounded in $C([0, T] \times \overline{\mathcal{F}_0})$, both uniformly with respect to $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$ and $\varepsilon \in (0, 1]$, furthermore, the maps $(h_1, \vartheta_1) \mapsto K_H[\omega^1] \in L^\infty(C^2_\infty(\overline{\mathcal{F}_0}))$, $(h_1, \vartheta_1) \mapsto \omega^1 \in C([0, T] \times \overline{\mathcal{F}_0})$ are continuous, uniformly with respect to $\varepsilon \in (0, 1]$.

(ii) There exists $R > 0$, which does not depend on $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$ or $\varepsilon \in (0, 1]$, such that supp $\omega^1(t, \cdot) \subset$ supp $\omega^1(0, \cdot) + \overline{B}(0, R)$, for all $t \in [0, T]$.

(iii) $K_H[\partial_t \omega^1]$ is uniformly bounded in $L^2(L^3(\mathcal{F}_0))$ with respect to $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$ and $\varepsilon \in (0, 1]$, and the map $(h_1, \vartheta_1) \mapsto K_H[\partial_t \omega^1] \in L^2(L^3(\mathcal{F}_0))$ is continuous, uniformly with respect to $\varepsilon \in (0, 1]$.

Proof. Note that the existence and regularity of ω^1 is not in question here, it is classical that if the initial vorticity is smooth and compactly supported, then there exists a unique smooth solution to (4.77) which is also compactly supported at any time. However, tracking the effect of the control g^0 and initial data u^* in (4.77) to get uniform estimates with respect to (h_1, ϑ_1) and $\varepsilon \in (0, 1]$ is not trivial. Recall that u^* is uniformly bounded in C^2 with respect to $\varepsilon \in (0, 1]$.

(i) It is classical that, for any $\alpha \in (0, 1)$ and for any $\omega \in C_0^{1,\alpha}$, we have

$$\|K_H[\omega]\|_{2,\alpha} \lesssim \|\omega\|_{1,\alpha}. \tag{4.79}$$

On the other hand, from (4.77) it follows (c.f. [8]) that

$$\frac{d}{dt_+}\|\omega^1(t, \cdot)\|_{0,\alpha} \leq \|g^0(t, \cdot)\, \omega^1(t, \cdot)\|_{0,\alpha} + \alpha\|\nabla(u^0 - u^0_S)(t, \cdot)\|_\infty \|\omega^1(t, \cdot)\|_{0,\alpha},$$

where $\frac{d}{dt_+}$ denote the right derivative. A Gronwall estimate gives us

$$\|\omega^1(t, \cdot)\|_{0,\alpha} \leq C\|\omega^1(0, \cdot)\|_{0,\alpha}\, e^{\|u^0\|_{L^1(C^2_\infty)} + \|p^0\|_\infty} \leq C, \tag{4.80}$$

due to the regularity given by Theorem 4.3.1. Iterating this argument once again by differentiating (4.77) with respect to x gives us

$$\begin{aligned}\frac{d}{dt_+}\|\nabla\omega^1(t, \cdot)\|_{0,\alpha} \leq{}& \|\nabla g^0(t, \cdot)\, \omega^1(t, \cdot)\|_{0,\alpha} + \|g^0(t, \cdot)\nabla\omega^1(t, \cdot)\|_{0,\alpha} + \|\nabla\omega^1(t, \cdot) \cdot \nabla u^0(t, \cdot)\|_{0,\alpha} \\ &+ 2|r^0(t)|\|\nabla\omega^1(t, \cdot)\|_{0,\alpha} + \alpha\|\nabla(u^0 - u^0_S)(t, \cdot)\|_\infty \|\nabla\omega^1(t, \cdot)\|_{0,\alpha},\end{aligned}$$

from where we deduce once more by a Gronwall estimate and (4.80) that

$$\|\omega^1(t, \cdot)\|_{1,\alpha} \leq C\|\omega^1(0, \cdot)\|_{1,\alpha}\, e^{\|u^0\|_{L^1(C^3_\infty)} + \|p^0\|_\infty} \leq C, \tag{4.81}$$

thanks to Theorem 4.3.1. Therefore, (i) follows.

(ii) We define $\phi_t(\cdot) \in C^1(\overline{\mathcal{F}_0})$, $t \in [0, T]$, as the flow of $u^0 - u^0_S$,

$$\frac{d}{dt}\phi_t(x) = (u^0 - u^0_S)(t, \phi_t(x)), \quad \phi_0(x) = x. \tag{4.82}$$

It is classical that, for any $t \in [0, T]$, $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1-diffeomorphism, and (4.77) can be solved by the method of characteristics, namely we have

$$\omega^1(t, x) = \omega^1(0, \phi_t^{-1}(x))\, e^{-\int_0^t g^0(s, \phi_s \circ \phi_t^{-1}(x))\, ds} \tag{4.83}$$

Therefore, the support of ω^1 is transported by ϕ_t, and for any $x \in \text{supp}\ \omega(0,\cdot)$ we have

$$|\phi_t(x) - x| \leq \|u^0\|_{L^1(C^0_\infty)} + T\|l^0\|_\infty + T\|r^0\|_\infty \max_{x \in \text{supp}\ \omega(0,\cdot)} |x| + \int_0^t |r^0(s)||\phi_s(x) - x|\, ds,$$

once again a Gronwall estimate gives us

$$|\phi_t(x) - x| \leq R,$$

with $R > 0$ uniform in $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$ and $\varepsilon \in (0,1]$, due to the regularity given by Theorem 4.3.1. This concludes the proof of (ii).

(iii) From (ii), (4.77) and (4.81) we deduce that

$$\|\partial_t \omega^1(t,\cdot)\|_\infty \leq C(|p^0(t)| + \|u^0(t,\cdot)\|_\infty + \|g^0(t,\cdot)\|_\infty),$$

for some $C > 0$ which is uniform in $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$ and $\varepsilon \in (0,1]$. Therefore, using the fact that ω^1 is supported in supp $\omega^1(0,\cdot) + \overline{B}(0,R)$, we have

$$\|\partial_t \omega^1\|_{L^2(L^{6/5})} \leq C. \tag{4.84}$$

Finally, we apply Lemma 1 from [59] with $p = \frac{6}{5}$ and $\gamma = 0$ to get that

$$\|K_H[\partial_t \omega^1]\|_{L^2(L^3)} \leq C\|\partial_t \omega^1\|_{L^2(L^{6/5})} \leq C,$$

with $C > 0$ uniform in $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$ and $\varepsilon \in (0,1]$. The continuity part of the result follows once again from Theorem 4.3.1.

□

4.5.2 An ODE reformulation in the linearized case

In this section we will give a different ODE reformulation of the solid equations in (4.74) compared to Section 4.3.1, due to the fact that the equations are now linear.

We introduce the stream function $\psi \in C^\infty(\overline{\mathcal{F}_0})$ for the circulation term in the following way. First we consider the solution $\tilde{\psi}$ of the Dirichlet problem $\Delta\tilde{\psi} = 0$ in $\mathcal{F}_0$, $\tilde{\psi} = 1$ on $\partial\mathcal{S}_0$, $\lim_{|x|\to+\infty} |\nabla\psi(x)| = 0$. Then we set

$$\psi = -\left(\int_{\partial\mathcal{S}_0} \partial_n \tilde{\psi}\, d\sigma\right)^{-1} \tilde{\psi}, \tag{4.85}$$

such that we have

$$\int_{\partial\mathcal{S}_0} \partial_n \psi\, d\sigma = -1,$$

noting that the strong maximum principle gives us $\partial_n\tilde{\psi} < 0$ on $\partial\mathcal{S}_0$. Note that we have $\nabla^\perp\psi \in C^2_\infty(\overline{\mathcal{F}_0})$.

We remind the reader of the definition of the Kirchhoff potentials given in (4.30), as well as $\mathcal{M}$ from Definition 4.3.3. We further introduce the following notations for some new force terms appearing in the solid equations in (4.74), which do not depend on the control.

Definition 4.5.1. *Given a solution $p^0 = (l^0, r^0) \in C^\infty([0,T];\mathbb{R}^3)$, $u^0 \in C^\infty([0,T] \times \overline{\mathcal{F}_0};\mathbb{R}^2)$ to (4.25) with zero initial conditions, we define for any $p = (l,r) \in \mathbb{R}^3$, $\omega, \tilde{\omega} \in C^\infty_0(\overline{\mathcal{F}_0})$, $\varepsilon \in (0,1]$ and $t \in [0,T]$,*

the functions

$$
\begin{aligned}
A(t) :=& \mathcal{M}^{-1} \int_{\partial \mathcal{S}_0} \begin{pmatrix} Q(t,\cdot,0)\, n - 2D(u^0(t,\cdot))n + \partial_z v(t,\cdot,0) \\ x^\perp \cdot \big(Q(t,\cdot,0)\, n - 2D(u^0(t,\cdot))n + \partial_z v(t,\cdot,0)\big) \end{pmatrix} d\sigma, \\
B_\varepsilon(t) :=& \mathcal{M}^{-1} \int_{\partial \mathcal{S}_0} \rho^\varepsilon(t,\cdot)\, \partial_n \Phi \, d\sigma, \\
C[\omega,\tilde{\omega}](t) :=& -\mathcal{M}^{-1} \int_{\mathcal{F}_0} \Big(K_H[\tilde{\omega}] + \omega \left(u^0(t,\cdot) - u_S^0(t,\cdot)\right)^\perp \Big) \nabla \Phi \, dx \\
& - \mathcal{M}^{-1} \int_{\partial \mathcal{S}_0} (u^0(t,\cdot) - u_S^0(t,\cdot)) \cdot \Big(\gamma \nabla^\perp \psi(x) + K_H[\omega](x) \Big) \partial_n \Phi \, d\sigma, \\
\langle L(t), p \rangle :=& \mathcal{M}^{-1} \int_{\partial \mathcal{S}_0} \Big((l + r x^\perp) \cdot u^0(t,\cdot) - (u^0(t,\cdot) - u_S^0(t,\cdot)) \cdot \nabla \left(p \cdot \Phi\right) \Big) \, \partial_n \Phi \, d\sigma \\
& - \mathcal{M}^{-1} \Big(m r^0(t) l^\perp + m r (l^0(t))^\perp, 0 \Big).
\end{aligned}
$$

The following Lemma gives us the regularity of the terms introduced above with respect to $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$.

Lemma 4.5.1. *Let $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$, (q^0, p^0, u^0) as in Theorem 4.3.1 and ω^1 be the solution of (4.77). We have $A, B_\varepsilon, C[\omega^1, \partial_t \omega^1]$ and L are uniformly bounded in $L^2(0,T)$ with respect to $\varepsilon \in (0,1]$ and $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$. Furthermore, the map $(h_1, \vartheta_1) \in \mathcal{Q}_\delta \mapsto (A, B_\varepsilon, C[\omega^1, \partial_t \omega^1], L) \in \left(L^2(0,T)\right)^4$ is uniformly continuous with respect to $\varepsilon \in (0,1]$.*

Proof. We have

$$
|A(t)| \lesssim \|Q(t)\|_{1,1,1} + \|\nabla u^0(t)\|_\infty + \|\partial_z v(t)\|_{1,1,1} \lesssim \|v(t)\|_{3,2,1} + \|\nabla u^0(t)\|_\infty + \|v(t)\|_{1,1,2}.
$$

due to (4.67). The boundedness result for A follows from Theorem 4.3.1 and Proposition 4.4.1, so does the continuity in (h_1, ϑ_1).

Note that we have $\rho^\varepsilon = -\partial_t \theta^\varepsilon - (u^0 - u_S^0) \cdot \nabla \theta^\varepsilon$, so the result for B_ε holds as soon as $\partial_t \theta^\varepsilon$ and $\nabla \theta^\varepsilon$ are uniformly bounded in $L^2(H^1)$ with respect to $\varepsilon > 0$ and $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$.

Since θ^ε depends linearly on w, which in turn depends linearly on v, similarly to (4.70) and (4.73), we may estimate

$$
\|\partial_t \theta^\varepsilon(t)\|_{H^1} \lesssim \|\partial_t v(t)\|_{2,3,1} + \|\partial_t v(t)\|_{1,0,1}.
$$

We estimate $\partial_t v$ by using equation (4.65), to get that

$$
\|\partial_t v(t)\|_{k,m,p} \lesssim \Big(1 + |p^0(t)| + \|u^0(t)\|_{C_\infty^{m+1}} \Big) \|v(t)\|_{k+1,m+1,p+2},
$$

therefore, using Theorem 4.3.1 and Proposition 4.4.1, we get that $\partial_t \theta^\varepsilon$ is bounded in $L^2(H^1)$, uniformly with respect to $\varepsilon > 0$ and $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$.

Finally, using (4.70) and (4.73), we estimate

$$
\|\nabla \theta^\varepsilon(t)\|_{H^1} \lesssim \|v(t)\|_{2,3,1},
$$

and conclude the boundedness and continuity in (h_1, ϑ_1) of B_ε by once again using Proposition 4.4.1 and Theorem 4.3.1.

The results for $C[\omega^1, \partial_t \omega^1]$ and L follow from Theorem 4.3.1 and Proposition 4.5.1. In particular, for the term $\int_{\mathcal{F}_0} K_H[\partial_t \omega^1] \nabla \Phi \, dx$ recall that $\nabla \Phi(x) = \mathcal{O}(1/|x|^2)$ as $|x| \to +\infty$, therefore $\nabla \Phi \in L^{3/2}(\mathcal{F}_0)$. $\square$

In order to treat the effect of the control, we recall F_2 from Definition 4.3.4, and complete it with the following.

Definition 4.5.2. *Given a solution $p^0 = (l^0, r^0) \in C^\infty([0,T];\mathbb{R}^3)$, $u^0 \in C^\infty([0,T]\times\overline{\mathcal{F}_0};\mathbb{R}^2)$ to (4.25) with zero initial conditions, we define for any $\alpha \in C^\infty(\overline{\mathcal{F}_0};\mathbb{R})$ the map*

$$\tilde{F}_1[\alpha](t) = -\int_{\partial\mathcal{S}_0} \nabla\alpha\cdot(u^0(t,\cdot) - u_S^0(t,\cdot))\,\partial_n\Phi\,d\sigma.$$

We define our notion of controlled solution of the linearized fluid-solid system as follows.

Definition 4.5.3. *Given a controlled solution (q^0, p^0, g^0) associated with* (4.25), (4.26), *we say that (p^1, g^1) in $C^\infty([0,T];\mathbb{R}^3)\times C_0^\infty((0,T);\mathcal{C}(q^0(t)))$ is a controlled solution associated with* (4.74), (4.75) *if the following ODE holds true on $[0,T]$:*

$$(p^1)' = S_\varepsilon + \langle L, p^1\rangle + \mathcal{M}^{-1}\tilde{F}_1[\alpha^1] + \mathcal{M}^{-1}F_2[\partial_t\alpha^1], \tag{4.86}$$

with $p^1(0) = (h_0', \vartheta_0')$, where $\alpha^1(t,\cdot) := \mathcal{A}[q^0(t), g^1(t,\cdot)]$ with $\mathcal{A}$ given in (4.31), *and $S_\varepsilon := A + B_\varepsilon + C[\omega^1, \partial_t\omega^1]$.*

We have the following reformulation result for the linearized fluid-solid system.

Proposition 4.5.2. *Given a controlled solution (q^0, p^0, g^0) associated with* (4.25), (4.26),

$$p^1 \in C^\infty([0,T];\mathbb{R}^3),\ u^1 \in C^\infty([0,T]\times\overline{\mathcal{F}_0};\mathbb{R}^2) \quad \text{and} \quad g^1 \in C_0^\infty((0,T);\mathcal{C}(q^0(t))),$$

we have that (u^1, p^1, g^1) is a solution to (4.74), (4.75) *if and only if (p^1, g^1) is a controlled solution associated with* (4.74), (4.75) *and u^1 is the unique smooth solution to the div/curl type problem*

$$\begin{cases} \operatorname{div} u^1 = g^1\mathbb{1}_{R(\vartheta^0)^T(B_c-h^0)},\ \operatorname{curl} u^1 = \omega^1 \quad \text{in } \mathcal{F}_0, \\ u^1\cdot n = \left(l^1 + r^1x^\perp\right)\cdot n \text{ on } \partial\mathcal{S}_0, \lim\limits_{|x|\to+\infty}|u^1| = 0, \\ \displaystyle\int_{\partial\mathcal{S}_0} u^1\cdot\tau\,d\sigma = \int_{\partial\mathcal{S}_0} u^*\cdot\tau\,d\sigma =: \gamma, \end{cases} \tag{4.87}$$

for $t \in [0,T]$.

Remark 4.5.1. *The last equation in* (4.87) *can be easily checked as a generalization of Kelvin's theorem of conservation of the circulation around the body for the linearized Euler equation* (4.74), (4.75).

Remark 4.5.2. *Integrating* (4.77) *on $\mathcal{F}_0$, using the fact that $g^0 = \operatorname{div}(u^0 - u_S^0)$ and the divergence theorem, it follows that $\int\omega^1(t,\cdot)$ is conserved for $t \in [0,T]$. Since we have $u_0 \in L^2(\mathcal{F}_0)$ and that the circulation around the solid is conserved in* (4.87), *this implies that we have $u^1(t,\cdot) \in L^2(\mathcal{F}_0)$ for $t \in [0,T]$.*

Scheme of the proof of Proposition 4.5.2. We have the following result for solving div/curl systems of type (4.87), which is a simple consequence of (4.30), (4.31), (4.78) and (4.85).

Lemma 4.5.2. *For any $q = (h,\vartheta)$ in $\mathcal{Q}_\delta$, $\gamma \in \mathbb{R}$, $p = (l,r)$ in $\mathbb{R}^2\times\mathbb{R}$, $g \in \mathcal{C}(q)$ and $\omega \in C_0^\infty(\overline{\mathcal{F}_0})$, the unique solution u in $C^\infty(\overline{\mathcal{F}_0})$ to the following system :*

$$\begin{cases} \operatorname{div} u = g\mathbb{1}_{R(\vartheta)^T(B_c-h)},\ \operatorname{curl} u = \omega \quad \text{in } \mathcal{F}_0, \\ u\cdot n = \left(l + rx^\perp\right)\cdot n \text{ on } \partial\mathcal{S}_0, \lim\limits_{|x|\to+\infty}|u| = 0, \\ \displaystyle\int_{\partial\mathcal{S}_0} u\cdot\tau\,d\sigma = \gamma \end{cases} \tag{4.88}$$

is given by the following formula, for x in $\overline{\mathcal{F}_0}$,

$$u(x) = \nabla(p \cdot \Phi(x)) + \nabla\mathcal{A}[q, g](x) + \gamma\nabla^{\perp}\psi(x) + K_H[\omega](x). \tag{4.89}$$

From here on it is straightforward to conclude the proof of Proposition 4.5.2 in the same manner as for the proof of Proposition 4.3.1, by using (4.76) to express the pressure appearing in the solid equations in (4.74), expressing u^1 appearing in (4.76) with the help of Lemma 4.5.2, and using integration by parts (by noting the regularity at infinity of the functions involved) to obtain the terms given in Definitions 4.5.1 and 4.5.3. The details are left to the reader.

□

4.5.3 An impulsive control strategy to control the final velocity $p^1(T)$ - Proof of Theorem 4.5.1

Similarly to the case of (q^0, p^0), we will control the ODE (4.86) by the means of impulsive control, however, due to the fact that we only want to control $p^1(T)$ and that equation (4.86) is linear, the geodesic argument can be omitted and it is sufficient to use one impulsive control at for instance time $T/2$ (hence condition (4.28)). The form of the control will also change from (4.39) due to the linearity of (4.86) with respect to the control.

Let $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$ and (q^0, p^0, u^0) be given as in Theorem 4.3.1. We consider controls of the form

$$g^1_\eta(t, x) := \beta^2_\eta(t - T/2 + \eta)\overline{g}^1[u^0(T/2, \cdot) - u^0_S(T/2, \cdot)]\left(q^0(T/2 - \eta), v\right)(x), \tag{4.90}$$

where $v \in \mathbb{R}^3$, $(\beta^2_\eta)_\eta$ is supported in $[0, 2\eta]$ and is an approximation of the unity when $\eta \to 0^+$, and $\overline{g}^1$ is deduced from the following result.

Proposition 4.5.3. *Let $\mathcal{K}$ be a compact subset of $C^1(\partial\mathcal{S}_0; \mathbb{R}^2)$ such that for any $V \in \mathcal{K}$ we have $V \cdot n = 0$ on $\partial\mathcal{S}_0$ and*

$$span\,\{\partial_n\Phi(x),\ x \in \partial\mathcal{S}_0 \cap supp\ V\} = \mathbb{R}^3. \tag{4.91}$$

For any $V \in \mathcal{K}$, there exists a continuous mapping $\overline{g}^1[V] : \mathcal{Q}_\delta \times \mathbb{R}^3 \to \tilde{\mathcal{C}}(\mathcal{Q}_\delta)$ such that for any $q \in \mathcal{Q}_\delta$ we have Range$(\overline{g}^1[V](q, \cdot)) \subset \mathcal{C}(q) \cap \tilde{\mathcal{C}}(q)$, and for any (q, v) in $\mathcal{Q}_\delta \times \mathbb{R}^3$, the function $\overline{\alpha}^1 := \mathcal{A}[q, \overline{g}^1[V](q, v)]$ in $C^\infty(\overline{\mathcal{F}_0}; \mathbb{R}) \cap C^3_\infty(\overline{\mathcal{F}_0}; \mathbb{R})$ satisfies :

$$\Delta\overline{\alpha}^1 = 0 \text{ in } \mathcal{F}_0 \setminus R(\vartheta)^T(\tilde{B}_c - h), \lim_{|x| \to +\infty} |\nabla\overline{\alpha}^1| = 0 \text{ and } \partial_n\,\overline{\alpha}^1 = 0 \text{ on } \partial\mathcal{S}_0, \tag{4.92}$$

$$\int_{\partial\mathcal{S}_0} \nabla\overline{\alpha}^1 \cdot V\,\partial_n\Phi\,d\sigma = v. \tag{4.93}$$

Furthermore, the map $V \in \mathcal{K} \mapsto \overline{g}^1[V] \in C(\mathcal{Q}_\delta \times \mathbb{R}^3; \mathcal{C}(\mathcal{Q}_\delta))$ is also continuous.

Proposition 4.5.3 will be proved in Appendix 4.A.2. Note that to deduce $\overline{g}^1$ in (4.90) we apply Proposition 4.5.3 with $\mathcal{K} := \left\{u^0(T/2, \cdot) - u^0_S(T/2, \cdot),\ (h_1, \vartheta_1) \in \mathcal{Q}_\delta\right\}$. Therefore, we have that the map $(h_1, \vartheta_1) \in \mathcal{Q}_\delta \mapsto \overline{g}^1[u^0(T/2, \cdot) - u^0_S(T/2, \cdot)]\left(q^0(T/2 - \eta), \cdot\right) \in C(\mathbb{R}^3; \mathcal{C}(q^0(T/2 - \eta)))$ is also continuous, thanks to Theorem 4.3.1.

We recall that $\tilde{B}_c \subset B_c$ and $\tilde{\mathcal{C}}$ were introduced in (4.45) to reduce the size of the activ control zone B_c enough such that for $\eta > 0$ small enough, as long as the solid displacement during the control phase $[T/2 - \eta, T/2 + \eta]$ is small, we have that $g^1(t, \cdot)$ given by (4.90) is supported in $R(\vartheta^0(t))^T(B_c - h^0(t))$, for any $t \in [0, T]$.

We set

$$
\begin{aligned}
v := &- \mathcal{M}\left(e^{-\int_{\frac{T}{2}+\eta}^{T} L(t)\,dt} p_1 - \int_{\frac{T}{2}+\eta}^{T} e^{-\int_{\frac{T}{2}+\eta}^{t} L(s)\,ds} S_\varepsilon(t)\,dt \right) \\
&+ \mathcal{M} e^{\int_0^{\frac{T}{2}-\eta} L(t)\,dt} \left(p_0 + \int_0^{\frac{T}{2}-\eta} e^{-\int_0^t L(s)\,ds} S_\varepsilon(t)\,dt \right)
\end{aligned}
\tag{4.94}
$$

in (4.90), which is uniformly bounded with respect to $\eta, \varepsilon \in (0,1]$ and $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$ due to Lemma 4.5.1. We define p^1 to be the smooth solution of (4.86) with control (4.90) and initial data (h_0', ϑ_0'), dropping once more the dependence on $\eta > 0$ from the notation for the sake of simplicity.

We will prove that $p^1(T) \to p_1$ as $\eta \to 0^+$, uniformly with respect to $\varepsilon \in (0,1]$ and $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$. First let us observe that on the time interval $[0, T/2 - \eta]$ the control vanishes, so using Duhamel's principle on (4.86), we may express

$$
p^1(T/2 - \eta) = e^{\int_0^{\frac{T}{2}-\eta} L(t)\,dt} \left(p_0 + \int_0^{\frac{T}{2}-\eta} e^{-\int_0^t L(s)\,ds} S_\varepsilon(t)\,dt \right). \tag{4.95}
$$

On the other hand, integrating (4.86) on $[T/2 - \eta, T/2 + \eta]$ and using that g_η^1 vanishes at the endpoints of this interval in order to eliminate the term $\mathcal{M}^{-1} F_2[\partial_t \overline{\alpha}^1]$, we get

$$
\begin{aligned}
p^1(T/2 + \eta) = p^1(T/2 - \eta) &+ \int_{\frac{T}{2}-\eta}^{\frac{T}{2}+\eta} S_\varepsilon(t)\,dt \\
&+ \int_{\frac{T}{2}-\eta}^{\frac{T}{2}+\eta} \beta_\eta^2(t - T/2 + \eta) \mathcal{M}^{-1} \tilde{F}_1[\mathcal{A}(q^0(t), \overline{g}^1(q^0(T/2 - \eta), v)](t)\,dt.
\end{aligned}
$$

Using again Lemma 4.5.1 we get that the first integral term in the right-hand side above converges to zero as $\eta \to 0^+$, uniformly with respect to $\varepsilon \in (0,1]$ and $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$. On the other hand, from Theorem 4.3.1, Proposition 4.5.3 and the uniform boundedness of v it follows that the second integral term converges to $-\mathcal{M}^{-1} v$ as $\eta \to 0^+$, uniformly with respect to $\varepsilon \in (0,1]$ and $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$. We deduce that

$$
p^1(T/2 + \eta) - p^1(T/2 - \eta) \to -\mathcal{M}^{-1} v \text{ as } \eta \to 0^+, \tag{4.96}
$$

uniformly with respect to $\varepsilon \in (0,1]$ and $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$.

Finally, applying Duhamel's principle on $[T/2 + \eta, T]$ gives us

$$
p^1(T) = e^{\int_{\frac{T}{2}+\eta}^{T} L(t)\,dt} \left(p^1(T/2 + \eta) + \int_{\frac{T}{2}+\eta}^{T} e^{-\int_{\frac{T}{2}+\eta}^{t} L(s)\,ds} S_\varepsilon(t)\,dt \right). \tag{4.97}
$$

We may therefore deduce from (4.94), (4.95), (4.96) and (4.97) that

$$
p^1(T) \to p_1 \text{ as } \eta \to 0^+,
$$

uniformly with respect to $\varepsilon \in (0,1]$ and $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$.

We conclude the proof of Theorem 4.5.1 by fixing $\eta = \eta_1 > 0$ in function of $\nu > 0$, associating an appropriate fluid velocity u^1 with (p^1, g^1_η) as in Proposition 4.5.2, and observing that the regularity with respect to $\varepsilon \in (0,1]$ and $(h_1, \vartheta_1) \in \mathcal{Q}_\delta$ stated in Theorem 4.5.1 is guaranteed by our construction (by using Theorem 4.3.1, Proposition 4.5.1, Lemma 4.5.1 and Proposition 4.5.3). In particular we note that since $\eta > 0$ is now fixed, there is no issue with establishing the regularity with respect to ε and (h_1, ϑ_1) of the term $\mathcal{M}^{-1}F_2[\partial_t \overline{\alpha}^1]$ via Proposition 4.5.3, which we did not need to investigate previously in our strategy.

4.6 Estimating the remainder

In this section we establish the existence (in a weak sense) of the remainder terms in (4.17), (4.21), such that they satisfy an appropriate energy estimate and continuity property with respect to $(h_1, \vartheta_1, l_1, r_1)$.

4.6.1 The equation of the remainder and weak solutions

As noted before, at this point we have not yet proven the existence of the terms $(u^\varepsilon, \pi^\varepsilon, l^\varepsilon, r^\varepsilon)$, however our goal is to define a weak solution $(u^\varepsilon_R, l^\varepsilon_R, r^\varepsilon_R)$ for the remainder in an appropriate way such that defining $(u^\varepsilon, l^\varepsilon, r^\varepsilon)$ by the right-hand sides of the asymptotic development (4.17), (4.21) gives us a very weak solution to (4.11) in the sense of Definition 4.2.1.

Therefore, formally we can look for the equations of the remainder $(u^\varepsilon_R, \pi^\varepsilon_R, l^\varepsilon_R, r^\varepsilon_R)$ by replacing $(u^\varepsilon, \pi^\varepsilon, l^\varepsilon, r^\varepsilon)$ in (4.11) with the terms given in the expansion (4.17), (4.21), and by simplifying the equations using the systems (4.25), (4.64), (4.68), (4.72) and (4.74) satisfied by the respective terms in the expansion, which we have constructed in the previous sections. We obtain the following.

$$\begin{aligned}
&\frac{\partial u^\varepsilon_R}{\partial t} + \varepsilon(u^\varepsilon_R - u^\varepsilon_{R,S}) \cdot \nabla u^\varepsilon_R + \varepsilon r^\varepsilon_R (u^\varepsilon_R)^\perp + \nabla \pi^\varepsilon_R - \varepsilon \Delta u^\varepsilon_R = \{f^\varepsilon\} - A^\varepsilon(u^\varepsilon_R, p^\varepsilon_R),\\
&\qquad \operatorname{div} u^\varepsilon_R = 0 \ \text{ for } x \in \mathcal{F}_0, \ \lim_{|x|\to+\infty} |u^\varepsilon_R| = 0,\\
&(D(u^\varepsilon_R)n)_{\tan} = -\mu(u^\varepsilon_R - u^\varepsilon_{R,S})_{\tan} + (N^\varepsilon)_{\tan}, \ u^\varepsilon_R \cdot n = u^\varepsilon_{R,S} \cdot n \ \text{ for } x \in \partial\mathcal{S}_0,\\
&\qquad m(l^\varepsilon_R)' = \int_{\partial\mathcal{S}_0} \pi^\varepsilon_R \, n \, d\sigma - 2\sqrt{\varepsilon} \int_{\partial\mathcal{S}_0} \Sigma^\varepsilon \, d\sigma - 2\varepsilon \int_{\partial\mathcal{S}_0} D\left(u^\varepsilon_R\right) n \, d\sigma\\
&\qquad\qquad - m\varepsilon r^\varepsilon_R (l^\varepsilon_R)^\perp - F^\varepsilon_C(p^\varepsilon_R),\\
&\qquad \mathcal{J}\ (r^\varepsilon_R)' = \int_{\partial\mathcal{S}_0} \pi^\varepsilon_R \, x^\perp \cdot n \, d\sigma - 2\sqrt{\varepsilon} \int_{\partial\mathcal{S}_0} x^\perp \cdot \Sigma^\varepsilon \, d\sigma - 2\varepsilon \int_{\partial\mathcal{S}_0} x^\perp \cdot D\left(u^\varepsilon_R\right) n \, d\sigma,
\end{aligned} \tag{4.98}$$

where $u^\varepsilon_{R,S}(t,x) = l^\varepsilon_R(t) + r^\varepsilon_R(t)x^\perp$, for $t \in [0,T]$, with $u^\varepsilon_R(0,\cdot) = u_0 - u^*, (l^\varepsilon_R, r^\varepsilon_R)(0) = 0$, and recall that as per (4.22), we have

$$\|u^\varepsilon_R(0,\cdot)\|_2 \le \varepsilon^{1/8}.$$

Furthermore, we have

$$N^\varepsilon = -\left(D\left(u^1 + \nabla\theta^\varepsilon + w(\cdot,\cdot,0)\right) n\right) + \mu\left(u^1 - u^1_S + \nabla\theta^\varepsilon + w(\cdot,\cdot,0)\right), \tag{4.99}$$

$$\Sigma^\varepsilon = D\left(v(\cdot,\cdot,0)\right) n - \frac{1}{2}\partial_z w(\cdot,\cdot,0) + \sqrt{\varepsilon} D\left(u^1 + \nabla\theta^\varepsilon + w(\cdot,\cdot,0)\right) n, \tag{4.100}$$

$$\begin{aligned} A^\varepsilon(u_R^\varepsilon, p_R^\varepsilon) &= (u_R^\varepsilon - u_{R,S}^\varepsilon)\cdot\nabla(u^0 + \sqrt{\varepsilon}\{v\} + \varepsilon u^1 + \varepsilon\{w\} + \varepsilon\nabla\theta^\varepsilon) \\ &+ r_R^\varepsilon(u^0 + \sqrt{\varepsilon}\{v\} + \varepsilon u^1 + \varepsilon\{w\} + \varepsilon\nabla\theta^\varepsilon)^\perp + (r^0 + \varepsilon r^1)(u_R^\varepsilon)^\perp \\ &+ (u^0 + \sqrt{\varepsilon}\{v\} + \varepsilon u^1 + \varepsilon\{w\} + \varepsilon\nabla\theta^\varepsilon - u_S^0 - \varepsilon u_S^1)\cdot\nabla u_R^\varepsilon, \end{aligned} \tag{4.101}$$

$$F_C^\varepsilon(p_R^\varepsilon) = m((r^0 + \varepsilon r^1)(l_R^\varepsilon)^\perp + \varepsilon r^1(l^1)^\perp + r_R^\varepsilon(l^0 + \varepsilon l^1)^\perp), \tag{4.102}$$

and

$$f^\varepsilon = f_\Delta^\varepsilon + f_\nabla^\varepsilon + \tilde{f}^\varepsilon, \tag{4.103}$$

where

$$\begin{aligned} f_\Delta^\varepsilon &= \left(\Delta\varphi\partial_z v - 2(n\cdot\nabla)\partial_z v + \partial_{zz}^2 w\right) + \sqrt{\varepsilon}\left(\Delta v + \Delta\varphi\partial_z w - 2(n\cdot\nabla)\partial_z w\right) \\ &\quad + \varepsilon\left(\Delta w + \Delta u^1 + \Delta\nabla\theta^\varepsilon\right), \\ f_\nabla^\varepsilon &= -\left(v + \sqrt{\varepsilon}\left(u^1 - u_S^1 + w + \nabla\theta^\varepsilon\right)\right)\cdot\nabla\left(v + \sqrt{\varepsilon}\left(u^1 + w + \nabla\theta^\varepsilon\right)\right) \\ &\quad - (u^0 - u_S^0)\cdot\nabla w - w\cdot\nabla u^0 + \left(u^1 - u_S^1 + w + \nabla\theta^\varepsilon\right)\cdot n\partial_z(v + \sqrt{\varepsilon}w) \\ &\quad - \sqrt{\varepsilon}r^1 v^\perp - \varepsilon r^1(u^1 + w + \nabla\theta^\varepsilon)^\perp - r^0 w^\perp, \\ \tilde{f}^\varepsilon &= -\nabla Q - \partial_t w. \end{aligned} \tag{4.104}$$

Note that the solid position $q_R^\varepsilon = (h_R^\varepsilon, \vartheta_R^\varepsilon)$ can be associated analogously with (4.12), but in order to get the desired convergence in (4.17), we do not need to work with it explicitly.

Recall the definition of the spaces $\mathcal{H}(0)$, $\mathcal{V}(0)$ from (4.9). We define a notion of weak solution to (4.98) in the following manner.

Definition 4.6.1. *We say that*

$$u_R^\varepsilon \in C([0,T];\mathcal{H}(0)) \cap L^2((0,T);\mathcal{V}(0))$$

is a weak solution to the system (4.98) *if we have that, for all* $\phi \in C^\infty([0,T];\mathcal{H}(0))$ *such that* $\phi|_{\overline{\mathcal{F}_0}} \in C^\infty([0,T];C_0^\infty(\overline{\mathcal{F}_0};\mathbb{R}^2))$*, the following holds on* $[0,T]$*,*

$$\begin{aligned} (u_R^\varepsilon(t,\cdot),\phi(t,\cdot))_{\mathcal{H}(0)} - (u_R^\varepsilon(0,\cdot),\phi(0,\cdot))_{\mathcal{H}(0)} &= \int_0^t (u_R^\varepsilon(s,\cdot),\partial_t\phi(s,\cdot))_{\mathcal{H}(0)}\,ds \\ &+ \varepsilon\int_0^t\int_{\mathcal{F}_0}\left[(u_R^\varepsilon - u_{R,S}^\varepsilon)\cdot\nabla\phi\cdot u_R^\varepsilon - r_R^\varepsilon(u_R^\varepsilon)^\perp\cdot\phi\right]dx\,ds \\ -2\varepsilon\int_0^t\int_{\mathcal{F}_0} D(u_R^\varepsilon):D(\phi)\,dx\,ds - 2\varepsilon\mu\int_0^t\int_{\partial\mathcal{S}_0}&(u_R^\varepsilon - u_{R,S}^\varepsilon)\cdot(\phi - \phi_S)\,d\sigma\,ds \\ -\int_0^t\left(m\varepsilon r_R^\varepsilon(l_R^\varepsilon)^\perp\cdot l_\phi + F_C^\varepsilon(p_R^\varepsilon)\cdot l_\phi\right)ds &+ \int_0^t\int_{\mathcal{F}_0}(\{f^\varepsilon\} - A^\varepsilon(u_R^\varepsilon,p_R^\varepsilon))\cdot\phi\,dx\,ds \\ -2\sqrt{\varepsilon}\int_0^t\int_{\partial\mathcal{S}_0}\begin{pmatrix}\Sigma^\varepsilon \\ x^\perp\cdot\Sigma^\varepsilon\end{pmatrix}d\sigma\cdot(l_\phi,r_\phi)\,ds &+ 2\varepsilon\int_0^t\int_{\partial\mathcal{S}_0}(N^\varepsilon\cdot\tau)\left((\phi-\phi_S)\cdot\tau\right)d\sigma\,ds. \end{aligned}$$

In order to see that the above definition of a weak solution is justified, we suppose that $(u_R^\varepsilon, \pi_R^\varepsilon, l_R^\varepsilon, r_R^\varepsilon)$ is a sufficiently regular strong solution to (4.98) and that ϕ is a test function as in Definition 4.6.1. In this case one may multiply the PDE in (4.98) by ϕ and perform the usual operations for deriving the variational formulation for Leray-type weak solutions for viscous fluid-solid systems. The terms on the right-hand side of the PDE in (4.98) can be treated as a source term and as such pose no problem.

However, let us explain how to treat the terms containing N^ε and Σ^ε. One may integrate by parts and proceed as in Lemma 1 from [86] to obtain

$$\begin{aligned}\int_{\mathcal{F}_0} \Delta u_R^\varepsilon \cdot \phi = &-2\int_{\mathcal{F}_0} D(u_R^\varepsilon) : D(\phi) + 2\int_{\partial\mathcal{S}_0} \begin{pmatrix} D(u_R^\varepsilon)n \\ x^\perp \cdot (D(u_R^\varepsilon)n) \end{pmatrix} d\sigma \cdot (l_\phi, r_\phi) \\ &+ 2\int_{\partial\mathcal{S}_0} ((D(u_R^\varepsilon)n)\cdot\tau)\,((\phi-\phi_S)\cdot\tau)\, d\sigma.\end{aligned} \tag{4.105}$$

Combined with $(D(u_R^\varepsilon)n)\cdot\tau = -\mu(u_R^\varepsilon - u_{R,S}^\varepsilon)\cdot\tau + N^\varepsilon\cdot\tau$, the solid equations from (4.98), and the usual formula for the pressure term

$$\int_{\mathcal{F}_0} \nabla\pi_R^\varepsilon \cdot \phi = \int_{\partial\mathcal{S}_0} \pi_R^\varepsilon \begin{pmatrix} n \\ x^\perp \cdot n \end{pmatrix} d\sigma \cdot (l_\phi, r_\phi), \tag{4.106}$$

one can easily check that the terms with N^ε and Σ^ε in Definition 4.6.1 are justified.

Finally, note once more that if $u_R^\varepsilon \in C([0,T];\mathcal{H}(0)) \cap L^2((0,T);\mathcal{V}(0))$ is a weak solution in the sense of Definition 4.6.1, then $u^\varepsilon \in C([0,T];\mathcal{H}) \cap L^2((0,T);\mathcal{V})$ constructed via (4.17), (4.21) is a very weak solution in the sense of Defintion 4.2.1.

4.6.2 Existence and continuity

We have the following result for the existence of the remainder.

Proposition 4.6.1. *There exists $\bar{\varepsilon}_0 > 0$, independent of $(h_1, \vartheta_1, l_1, r_1)$, such that for any $\varepsilon \in (0, \bar{\varepsilon}_0]$, there exists a unique weak solution $u_R^\varepsilon \in C([0,T];\mathcal{H}(0)) \cap L^2((0,T);\mathcal{V}(0))$ to (4.98) in the sense of Definition 4.6.1 with initial data given by $u_R^\varepsilon(0,\cdot) = u_0 - u^*$, $(l_R^\varepsilon, r_R^\varepsilon)(0) = 0$. Furthermore, we have the following :*

(i) This unique solution satisfies the following energy inequality at time T :

$$|p_R^\varepsilon(T)|^2 + \|u_R^\varepsilon(T,\cdot)\|_2^2 + \varepsilon\|D(u_R^\varepsilon)\|_{L^2(L^2)}^2 + \varepsilon\mu\int_0^T\int_{\partial\mathcal{S}_0} \left|u_R^\varepsilon - u_{R,S}^\varepsilon\right|^2 d\sigma\, ds \le C\varepsilon^{1/4}, \tag{4.107}$$

where $C = C(\nu) > 0$ is independent of $\varepsilon > 0$, and depends continuously on $(h_1, \vartheta_1, l_1, r_1)$.

(ii) The map $(h_1, \vartheta_1, l_1, r_1) \mapsto p_R^\varepsilon \in L^\infty(0,T)$ is continuous.

Note that to prove Theorem 4.2.2 we only need the estimate $|(l_R^\varepsilon, r_R^\varepsilon)(T)| \le C\varepsilon^{1/8}$. However, the PDE satisfied by u_R^ε in (4.98) is no longer of Euler- or linearized Euler-type, but truly of Navier-Stokes-type. That is why we estimate the fluid and solid velocities of the remainder together via an energy estimate, instead of examining only the solid equations and using some well-constructed decomposition of the fluid velocity as we have done in the previous sections, which would not work in this case.

We split the proof of Proposition 4.6.1 into three parts : first we give an a priori estimate for the added source terms in Definition 4.6.1 ; then we prove the existence result with the energy estimate from (i) ; and finally we explain how to obtain the continuity result from (ii).

Estimation of the source terms

The proof of the existence relies on the classical Faedo-Galerkin method for Navier-Stokes-type problems, therefore it requires an energy estimate on the whole time interval $[0,T]$ for weak solutions in the sense of Definition 4.6.1.

In preparation for such an energy estimate, we will bound the terms containing N^ε, Σ^ε, A^ε and f^ε in Definition 4.6.1 using the following Lemma (recall the definition of the space $\mathcal{H}(0)$ from (4.9)).

Lemma 4.6.1. *There exist $C > 0$ and $b \in C([0,T];\mathbb{R}_+)$ which depend continuously on $(h_1, \vartheta_1, l_1, r_1)$, such that*

(i) $\displaystyle\int_0^T \int_{\partial\mathcal{S}_0} \left(|\Sigma^\varepsilon|^2 + |N^\varepsilon|^2\right) d\sigma\, dt \le C(1+\varepsilon^{-1/4})$,

(ii) $\displaystyle\left|\int_{\mathcal{F}_0} A^\varepsilon(\phi, p_\phi)\cdot\phi\right| \le b(t)\left(|p_\phi(t)|^2 + \|\phi(t,\cdot)\|_2^2\right),\ \forall t\in[0,T]$,

(iii) $\displaystyle\int_0^t \left|\int_{\mathcal{F}_0} \{f^\varepsilon\}\cdot\phi\right| ds \le C\varepsilon^{1/4}\left(1 + \max_{s\in[0,t]}\left(|p_\phi(s)|^2 + \|\phi(s,\cdot)\|_2^2\right)\right),\ \forall t\in[0,T]$,

uniformly with respect to $\varepsilon\in(0,1]$, for all $\phi\in C([0,T];\mathcal{H}(0))$ such that $\phi(t,\cdot)$ is compactly supported in $\mathbb{R}^2$ for every $t\in[0,T]$.

Proof. (i) We have the following, by using (4.70) and (4.73),

$$\begin{aligned}
\|D(v(t,\cdot,0))\|_{H^1} &\lesssim \|v(t)\|_{1,2,1},\\
\|\partial_z w(t,\cdot,0)\|_{H^1} &\lesssim \|w(t)\|_{1,1,2} \lesssim \|v(t)\|_{3,2,3},\\
\|D(w(t,\cdot,0))\|_{H^1} &\lesssim \|w(t)\|_{1,2,1} \lesssim \|v(t)\|_{3,3,2},\\
\|D(\nabla\theta^\varepsilon(t))\|_{H^1} &\lesssim \|\theta^\varepsilon(t)\|_{H^3} \lesssim \varepsilon^{-1/4}\|v(t)\|_{2,4,2} + \|v(t)\|_{1,2,0},\\
\int_0^T\int_{\partial\mathcal{S}_0} |D(u^1)|^2\, d\sigma\, dt &\lesssim \|u^1\|^2_{L^2(C^1_\infty)}.
\end{aligned}$$

Similar estimates hold for the lower order terms in (4.99), noting that we have $p^1 \in L^\infty$. The result follows from Theorem 4.3.1 and Proposition 4.4.1.

(ii) Since ϕ is compactly supported in the space variable, we have

$$\int_{\mathcal{F}_0} (u^0 + \sqrt{\varepsilon}\{v\} + \varepsilon u^1 + \varepsilon\{w\} + \varepsilon\nabla\theta^\varepsilon - u^0_S - \varepsilon u^1_S)\cdot\nabla\phi\cdot\phi\, dx = -\frac{1}{2}\int_{\mathcal{F}_0}(g^0 + \varepsilon g^1)|\phi|^2\, dx,$$

which can be estimated by $C\left(\|u^0(t)\|_{C^1_\infty} + \|u^1(t)\|_{C^1_\infty}\right)\|\phi(t,\cdot)\|_2^2$.

Next we set

$$V^\varepsilon := u^0 + \varepsilon u^1 + \varepsilon\nabla\theta^\varepsilon.$$

We may estimate

$$\left|\int_{\mathcal{F}_0} \phi\cdot\nabla V^\varepsilon\cdot\phi\right| \le C(\|\nabla u^0(t)\|_\infty + \|\nabla u^1(t)\|_\infty + \varepsilon\|\theta^\varepsilon(t)\|_{H^4})\|\phi(t)\|_2^2,$$

where $C > 0$ does not depend on ϕ.

Recall that (4.70) and (4.73) give us

$$\varepsilon\|\theta^\varepsilon(t)\|_{H^4} \lesssim \varepsilon^{1/4}\|v(t)\|_{2,5,3} + \varepsilon\|v(t)\|_{1,3,0}.$$

We use Lamb's form from (4.35) to get

$$-\nabla\left(\phi_S\cdot V^\varepsilon\right) = -\phi_S\cdot\nabla V^\varepsilon + r_\phi(V^\varepsilon)^\perp + \varepsilon\omega^1(\phi_S)^\perp. \tag{4.108}$$

On the other hand,

$$\int_{\mathcal{F}_0} \nabla\left(\phi_S\cdot V^\varepsilon\right)\cdot\phi = \int_{\partial\mathcal{S}_0} \phi_S\cdot V^\varepsilon\, \partial_n\Phi\, d\sigma\cdot p_\phi. \tag{4.109}$$

Therefore, we may conclude by using once again that ω^1 is smooth and compactly supported, that

$$\begin{aligned}
&\left|\int_{\mathcal{F}_0}\left((\phi-\phi_S)\cdot\nabla(u^0+\varepsilon u^1) + r_\phi(u^0+\varepsilon u^1)^\perp\right)\cdot\phi\right|\\
&\qquad\le C\left(\|u^0(t)\|_{C^1_\infty} + \|u^1(t)\|_{C^1_\infty} + \|v(t)\|_{2,5,3}\right)\left(|p_\phi(t)|^2 + \|\phi(t)\|_2^2\right).
\end{aligned} \tag{4.110}$$

Finally, all the remaining terms in (4.101) contain some derivatives of either $\{v\}$ or $\{w\}$, which are compactly supported. Therefore, we may estimate the terms containing ϕ_S more straightforwardly, since the procedure above was done only because we do not have $\phi_S \in L^2(\mathcal{F}_0)$ and we wanted estimates independent of the support of ϕ. Taking the $L^\infty(\mathcal{F}_0)$ norm of all the derivatives of $\{v\}$ and $\{w\}$ in the remaining terms in (4.101), then using once again (4.70) and (4.73) with the right Sobolev embeddings, we get

$$\left|\int_{\mathcal{F}_0} A^\varepsilon(\phi, p_\phi)\cdot\phi\right| \leq b(t)\left(|p_\phi(t)|^2 + \|\phi(t,\cdot)\|_2^2\right), \tag{4.111}$$

with

$$b(t) = C\left(\|u^0(t)\|_{C^1_\infty} + \|u^1(t)\|_{C^1_\infty} + \|v(t)\|_{2,5,3}\right),$$

where $C > 0$ does not depend on any other parameter. We conclude the proof of (ii) by using Theorem 4.3.1, Proposition 4.4.1 and Theorem 4.5.1.

(iii) As in the previous point, let us first estimate the contribution of the terms in f^ε_∇ from (4.104) which are not compactly supported in $\mathcal{F}_0$. We set

$$W^\varepsilon := u^1 + \nabla\theta^\varepsilon.$$

Using Lamb's form from (4.35), we observe that

$$\left(W^\varepsilon - u^1_S\right)\cdot\nabla W^\varepsilon + r^1\left(W^\varepsilon\right)^\perp = \frac{1}{2}\nabla\left(|W^\varepsilon|^2\right) - \nabla\left(W^\varepsilon\cdot u^1_S\right) + \omega^1(W^\varepsilon - u^1_S)^\perp,$$

therefore we may estimate

$$\left|\int_{\mathcal{F}_0}\nabla\left(|W^\varepsilon|^2\right)\cdot\phi\right| = \left|\int_{\partial\mathcal{S}_0}|W^\varepsilon|^2\,\partial_n\Phi\,d\sigma\cdot p_\phi\right| \leq C\left(\|u^1\|^2_{C^0_\infty(\mathcal{F}_0)} + \|\theta^\varepsilon(t)\|^2_{H^3}\right)|p_\phi|.$$

Hence, using (4.70) and (4.73), we get

$$\begin{aligned}\varepsilon\int_0^t\left|\int_{\mathcal{F}_0}\nabla\left(|W^\varepsilon|^2\right)\cdot\phi\right|ds &\leq C\left(\varepsilon\|u^1\|^2_{L^2(C^0_\infty)} + \varepsilon^{3/4}\|v\|^2_{L^2(H^{2,4,2})}\right)\max_{s\in[0,t]}|p_\phi(s)| \\ &\leq C\varepsilon^{3/4}\left(\|u^1\|^4_{L^2(C^0_\infty)} + \|v\|^4_{L^2(H^{2,4,2})} + \max_{s\in[0,t]}|p_\phi(s)|^2\right).\end{aligned}$$

Proceeding similarly for the term containing $\nabla\left(W^\varepsilon\cdot u^1_S\right)$, and using that ω^1 is smooth and compactly supported, we get that

$$\begin{aligned}&\varepsilon\int_0^t\left|\int_{\mathcal{F}_0}\left(\left(W^\varepsilon - u^1_S\right)\cdot\nabla W^\varepsilon + r^1\left(W^\varepsilon\right)^\perp\right)\cdot\phi\right|ds \\ &\qquad\leq C\varepsilon^{3/4}\left(\|u^1\|^4_{L^2(C^1_\infty)} + \|v\|^4_{L^2(H^{2,4,2})} + \|p^1\|^4_\infty + \max_{s\in[0,t]}\left(|p_\phi(s)|^2 + \|\phi(s,\cdot)\|_2^2\right)\right).\end{aligned}$$

For the rest of the terms in f^ε_∇ from (4.104), respectively for f^ε_Δ and $\tilde{f}^\varepsilon$, the same estimates can be applied as in Section 4.3 of [27], noting that the terms appearing at $\mathcal{O}(1)$ benefit from a fast variable scaling gain of $\varepsilon^{1/4}$ in $L^2(\mathcal{F}_0)$. One can then conclude the proof of Lemma 4.6.1 by using Theorem 4.3.1, Proposition 4.4.1 and Theorem 4.5.1. The details are left to the reader. □

A Faedo-Galerkin method for proving the existence of weak solutions

Now we are in position to prove the existence result from Proposition 4.6.1. Since the proof uses classical methods, we will only focus on the parts that are different from the existing literature, for the other details we refer the reader to [86] and the references therein.

The proof consists of a straightforward generalization of the methods presented in [86], where the authors give an extension of Leray's theorem to prove the existence of solutions to a similar fluid-solid system in the three-dimensional case with no source term, based on the method of Faedo-Galerkin approximations. We will show that being in the two-dimensional case and having some extra terms in (4.98) due to N^ε, Σ^ε, A^ε, F_C^ε and f^ε will pose no difficulty in adapting the same proof to our case.

To simplify notations we define, for any $u, v, w \in \mathcal{H}_0$ such that $v, w \in C_0^\infty(\mathbb{R}^2)$, the quantities

$$\begin{aligned}
\mathbb{a}(u,v) &= -2\int_{\mathcal{F}_0} D(u) : D(v)\,dx - 2\mu \int_{\partial\mathcal{S}_0} (u-u_S)\cdot(v-v_S)\,d\sigma,\\
\mathbb{b}(u,v,w) &= \int_{\mathcal{F}_0} \Big[(u-u_S)\cdot\nabla w\cdot v - r_u v^\perp\cdot w\Big]\,dx - m r_u (l_v)^\perp\cdot l_w,\\
\mathbb{c}^\varepsilon(u,v) &= -2\sqrt{\varepsilon}\int_{\partial\mathcal{S}_0}\begin{pmatrix}\Sigma^\varepsilon\\ x^\perp\cdot\Sigma^\varepsilon\end{pmatrix} d\sigma\cdot p_v + 2\varepsilon\int_{\partial\mathcal{S}_0}(N^\varepsilon\cdot\tau)\left((v-v_S)\cdot\tau\right)\,d\sigma\\
&\quad + \int_{\mathcal{F}_0}(\{f^\varepsilon\} - A^\varepsilon(u,p_u))\cdot v\,dx - F_C^\varepsilon(p_u)\cdot l_v.
\end{aligned} \tag{4.112}$$

Therefore, the equation satisfied by the weak solution u_R^ε from Definition 4.6.1 can be reformulated as

$$(u_R^\varepsilon,\phi)_{\mathcal{H}(0)} - (u_R^\varepsilon(0,\cdot),\phi(0,\cdot))_{\mathcal{H}(0)} = \int_0^t \left[(u_R^\varepsilon,\partial_t\phi)_{\mathcal{H}(0)} + \varepsilon\mathbb{a}(u_R^\varepsilon,\phi) + \varepsilon\mathbb{b}(u_R^\varepsilon,u_R^\varepsilon,\phi) + \mathbb{c}^\varepsilon(u_R^\varepsilon,\phi)\right]\,ds.$$

Following the same methodology as in the proof of Theorem 1 from [86], we first consider a truncated system in which the term $u_S(t,\cdot)$ in $\mathbb{b}$ above is modified in order that it becomes bounded in $L^\infty(\mathbb{R}^2)$. More precisely, let $M_0 \in \mathbb{R}$ such that $\mathcal{S}_0 \subset B\left(0, \frac{M_0}{2}\right)$, and for $M > M_0$ we consider $\chi_M : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\chi_M(x) = x$ in $B(0,M)$ and $\chi_M(x) = \frac{M}{\|x\|}x$ in $\mathbb{R}^2 \setminus B(0,M)$. We then truncate the term $\mathbb{b}$ from (4.112) by

$$\mathbb{b}_M(u,v,w) = \int_{\mathcal{F}_0}\Big[(u-(l_u + r_u\chi_M^\perp))\cdot\nabla w\cdot v - r_u v^\perp\cdot w\Big]\,dx - m r_u (l_v)^\perp\cdot l_w.$$

We claim that for any $M > M_0$ there exists a solution $u_M \in C([0,T];\mathcal{H}(0)) \cap L^2((0,T);\mathcal{V}(0))$ (dropping the dependence on ε from the notation for simplicity) to the truncated system, i.e. satisfying for any $\phi \in C^\infty([0,T];\mathcal{H}(0))$ such that $\phi|_{\overline{\mathcal{F}_0}} \in C^\infty([0,T];C_0^\infty(\overline{\mathcal{F}_0};\mathbb{R}^2))$, the equation below on $[0,T]$,

$$(u_M,\phi)_{\mathcal{H}(0)} - (u_M(0,\cdot),\phi(0,\cdot))_{\mathcal{H}(0)} = \int_0^t \left[(u_M,\partial_t\phi)_{\mathcal{H}(0)} + \varepsilon\mathbb{a}(u_M,\phi) + \varepsilon\mathbb{b}_M(u_M,u_M,\phi) + \mathbb{c}^\varepsilon(u_M,\phi)\right]\,ds. \tag{4.113}$$

Indeed, we consider a Hilbert basis $(w_j)_{j\geq 1}$ of $\mathcal{V}(0)$ such that

$$w_j \in \{\phi \in C_0^\infty(\mathbb{R}^2) :\ \text{div}\ \phi = 0 \text{ in } \mathcal{F}_0,\ D(\phi) = 0 \text{ in } \mathcal{S}_0\},\ j \geq 0.$$

Using a Faedo-Galerkin method, we may construct a sequence of approximate solutions $(u_N)_{N\geq 1} \subset C([0,T];\mathcal{H}(0)) \cap L^2((0,T);\mathcal{V}(0))$ (dropping the dependence on M from the notation for simplicity), $u_N = \sum_{i=1}^N g_{iN}(t)w_i$ satisfying, for any $j \in \{1,\dots,N\}$,

$$(\partial_t u_N, w_j)_{\mathcal{H}(0)} = \varepsilon\mathbb{a}(u_N,w_j) + \varepsilon\mathbb{b}_M(u_N,u_N,w_j) + \mathbb{c}^\varepsilon(u_N,w_j),$$

with initial data u_{N0}, which is defined as the projection of $u_R^\varepsilon(0)$ onto the space spanned by $w_1, \dots, w_N$. It follows that $u_N(t,\cdot)|_{\overline{\mathcal{F}_0}} \in C_0^\infty(\overline{\mathcal{F}_0})$, for all $t \in [0,T]$, due to regularity of $w_1, \dots, w_N$.

It can be checked that u_N will satisfy the following energy equality.

$$\frac{1}{2}\|u_N(t,\cdot)\|_2^2 + \frac{1}{2}m|l_N(t)|^2 + \frac{1}{2}\mathcal{J}|r_N(t)|^2 + 2\varepsilon\int_0^t\int_{\mathcal{F}_0}|D(u_N)|^2\,dx\,ds$$
$$+ 2\varepsilon\mu\int_0^t\int_{\partial\mathcal{S}_0}|u_N - u_{N,S}|^2\,d\sigma\,ds = \frac{1}{2}\|u_{N0}\|_2^2 + \int_0^t \mathfrak{c}^\varepsilon(u_N,u_N)\,ds. \quad (4.114)$$

Furthermore, one has $\|u_{N0}\|_2 \le \|u_R^\varepsilon(0,\cdot)\|_2 = \|u_0 - u^*\|_2$. Note that as long as we can bound the left-hand side of (4.114) by a constant that does not depend on the approximation or the truncation, one can straightforwardly adapt the rest of the proof of Theorem 1 from [86] to establish the existence part of the result.

Let us prove that this is indeed the case. In the rest of the section, $C > 0$ will denote a generic constant, which we usually deduce by using Theorem 4.3.1 and Theorem 4.5.1 to estimate the respective norms of u^0, p^0, u^1, p^1, and which can depend on $(h_1, \vartheta_1, l_1, r_1)$, but in a continuous manner. Furthermore, C is independent of $\varepsilon \in (0,1]$.

Since $p^0, p^1 \in L^\infty$, we may use (4.102) to estimate

$$|F_C^\varepsilon(p_N)\cdot l_N| \le C(\varepsilon^2 + |p_N(t)|^2). \quad (4.115)$$

On the other hand,

$$\left|\int_{\partial\mathcal{S}_0}\begin{pmatrix}\Sigma^\varepsilon\\ x^\perp\cdot\Sigma^\varepsilon\end{pmatrix}d\sigma\cdot p_N\right| \le C\int_{\partial\mathcal{S}_0}|\Sigma^\varepsilon(t)|^2\,d\sigma + |p_N(t)|^2. \quad (4.116)$$

Finally, we have

$$\left|\int_{\partial\mathcal{S}_0} N^\varepsilon\cdot\tau\left((u_N - u_{N,S})\cdot\tau\right)\,d\sigma\right| \le C\int_{\partial\mathcal{S}_0}|N^\varepsilon(t)|^2\,d\sigma + \frac{1}{2C_K}\|\nabla u_N(t,\cdot)\|_2^2 + C|p_N(t)|^2, \quad (4.117)$$

where $C_K > 0$ is the constant on the right hand side of the Korn inequality :

$$\|h\|_{H^1}^2 \le C_K\left(\|h\|_2^2 + \|D(h)\|_2^2\right), \text{ for any } h \in H^1(\mathcal{F}_0).$$

We sum up (4.114), (4.115), (4.116) and (4.117) and use Lemma 4.6.1 to obtain

$$\left(1 - C\varepsilon^{1/4}\right)\max_{s\in[0,t]}\frac{1}{2}\left(\|u_N(s,\cdot)\|_2^2 + m|l_N(s)|^2 + \mathcal{J}|r_N(s)|^2\right) + \varepsilon\|D(u_N)\|_{L^2(L^2)}^2$$
$$+2\varepsilon\mu\int_0^t\int_{\partial\mathcal{S}_0}|u_N - u_{N,S}|^2\,d\sigma\,ds \le C\|u_0 - u^*\|_2^2 + C\varepsilon^{1/4} \quad (4.118)$$
$$+C\int_0^t b(s)\left(\|u_N(s,\cdot)\|_2^2 + |p_N(s)|^2\right)\,ds,$$

for some $C > 0$ and $b \in C([0,T];\mathbb{R}_+)$ which depend continuously on $(h_1, \vartheta_1, l_1, r_1)$, and do not depend on $M > M_0$ used for the truncation or $N \in \mathbb{N}$ used in the Faedo-Galerkin method. Therefore, there exists $\bar{\varepsilon}_0 > 0$, uniform for $(h_1, \vartheta_1, l_1, r_1) \in \overline{B}((h_f, \vartheta_f, R(\vartheta_f)^T h_f', \vartheta_f'), \kappa)$, such that for any $\varepsilon \in (0, \bar{\varepsilon}_0]$ we have $\frac{1}{2} \le (1 - C\varepsilon^{1/4})$.

Using the fact that $\|u_0 - u^*\|_2 \le \varepsilon^{1/8}$ as per (4.22), a Gronwall argument for (4.118), and further reducing the left-hand side to get rid of any unnecessary constants, we get that

$$|p_N(t)|^2 + \|u_N(t,\cdot)\|_2^2 + \varepsilon\|D(u_N)\|_{L^2(L^2)}^2 + \varepsilon\mu\int_0^t\int_{\partial\mathcal{S}_0}|u_N - u_{N,S}|^2\,d\sigma\,ds \le C\varepsilon^{1/4}, \quad (4.119)$$

for any $t \in [0, T]$.

Since the bound in the right-hand side above is uniform for $N \geq 1$ and $M > M_0$, we may on one hand conclude the proof of the existence of a weak solution u_R^ε to (4.98) in the sense of Definition 4.6.1 by the same methods as used in the proof of Theorem 1 from [86]. More precisely, we may first pass to the limit as $N \to +\infty$ to deduce the existence of a solution u_M to (4.113), for any $M > M_0$. Then we may extract a convergent subsequence $(u_{M_k})_{k\geq 0}$ with limit u_R^ε and conclude that we may pass to the limit in (4.113) as $k \to +\infty$, in particular using once more the energy inequality (4.118) and Lemma 4.6.1 when needed in order to bound the time derivative of u_{M_k} in a similar fashion as in the aforementioned result (we refer the reader to [86] for further details). Therefore, we obtain that u_R^ε satisfies (4.107) and also (4.119). This concludes the proof of the existence and the energy estimate, the uniqueness of the solution follows from classical theory since we are in the 2D case.

Proving the continuity result

The continuity result (ii) from Proposition 4.6.1 follows in a straightforward (but lengthy) manner by first observing that the same arguments used to prove Lemma 4.6.1 can be adapted to prove the following result regarding the continuity of the terms in (4.98) which depend on $(h_1, \vartheta_1, l_1, r_1)$.

Lemma 4.6.2. *The elements*

- $\Sigma^\varepsilon, N^\varepsilon \in L^2((0,T); L^2(\partial\mathcal{S}_0))$,
- $F_C^\varepsilon \in \mathcal{L}(\mathbb{R}^3; \mathbb{R}^2)$,
- $A^\varepsilon(\cdot) \in \mathcal{L}(L^2((0,T); \mathcal{H}(0)_c); L^2((0,T) \times \mathcal{F}_0))$,
- $\displaystyle\int_0^t \langle \{f^\varepsilon\}, \cdot\rangle_{L^2(\mathcal{F}_0)}\, ds \in \mathcal{L}(C([0,T]; \mathcal{H}(0)); \mathbb{R})$, *for any* $t \in [0,T]$,

depend continuously on $(h_1, \vartheta_1, l_1, r_1) \in \overline{B}((h_f, \vartheta_f, R(\vartheta_f)^T h_f', \vartheta_f'), \kappa)$, *where* $\mathcal{H}(0)_c$ *denotes all functions in* $\mathcal{H}(0)$ *which are compactly supported in* $\mathbb{R}^2$, *respectively* $\mathcal{L}(\cdot;\cdot)$ *is used to denote the space of linear continuous operators between two given spaces.*

From here, the standard method for concluding the continuity is the following. One may consider two weak solutions associated with two different values of the parameters $(h_1, \vartheta_1, l_1, r_1)$ and compare them using an energy estimate of their difference, proving that the right-hand side of the energy estimate will go to zero as the difference of the parameters goes to zero (as done for instance in standard methods to prove the uniqueness of weak solutions, see eg. [12, 61] for the - more complicated - bounded case). However, one needs to take care when handling the trilinear convective terms, so one may consider the energy estimate first using the truncated approximate solutions corresponding to the two weak solutions, as explained in the construction above for the proof of the existence result. This allows one to deduce that p_N (as well as $u_N \in L^\infty(L^2) \cap L^2(H^1)$) depends continuously on $(h_1, \vartheta_1, l_1, r_1)$, for any $N \geq 1$. However, since the right-hand side of (4.119) does not depend on $(h_1, \vartheta_1, l_1, r_1)$, nor on N, it follows that the convergence of (u_N, p_N) to $(u_R^\varepsilon, p_R^\varepsilon)$ mentioned in the previous section is uniform with respect to $(h_1, \vartheta_1, l_1, r_1)$, from where the desired continuity result follows for p_R^ε. The details are left to the reader.

This concludes the proof of Proposition 4.6.1.

4.7 Conclusion

It follows from Sections 4.3 to 4.6 that we have completed the construction of the asymptotic expansion from Section 4.2.3, which gives us the result of Theorem 4.1.1 due to the reductions presented in Sections 4.2.1 and 4.2.2. We present below an overview of the amplitude with respect to time of the controls g^0 and g^1 constructed in Theorem 4.3.1, respectively Theorem 4.5.1.

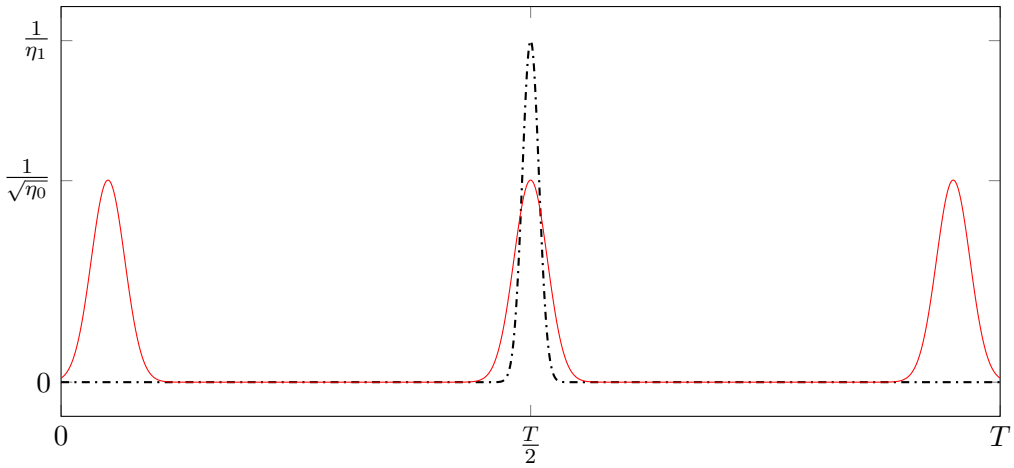

FIGURE 4.2: A comparison of the amplitudes of the controls g^0 (red continuous line) and g^1 (black dotted line) over the time interval $[0, T]$

We recall that the forms of $g^0 = g^0_{\eta_0}$ and $g^1 = g^1_{\eta_1}$ were given in (4.63), respectively (4.90), where $\beta_\eta \geq 0$, $\|\beta_\eta\|_\infty = \mathcal{O}(1/\sqrt{\eta})$, $(\beta^2_\eta)_\eta$ is supported in $[0, 2\eta]$ and is an approximation of the unity when $\eta \to 0^+$. The parameter $\eta_0 > 0$ was fixed at the end of the proof of Theorem 4.3.1 in Section 4.3.2, while the parameter $\eta_1 > 0$ was fixed (in function of the parameter $\nu > 0$, roughly such that $\eta_1 \leq \mathcal{O}(\nu)$) at the end of the proof of Theorem 4.5.1 in Section 4.5.3. However, for the purpose of illustration in Figure 4.2 and without loss of generality, we may assume that $0 < \eta_1 < \eta_0$.

A possibility of passing to arbitrary time via autoregularization

Let us present a possibility for passing to arbitrary time in Theorem 4.1.1, i.e. deducing a control result in any given time $T > 0$. Since we know how to control in small-time, one possibility would be to let the system evolve without control for a long time period and control during a short time interval at the end of $[0, T]$. The main technical issue with this strategy is the following. For our small-time controllability result Theorem 4.1.1 to hold, we assume H^4 regularity on the initial data u_0. Intuitively, one would think that due to the smoothing properties of the Navier-Stokes equations, even with initial data that is only L^2, after a short time the solution would become H^4 and stay that way for all times until T. However, up to our knowledge no such results exist in the literature for fluid-solid interaction problems such as the one considered in this chapter. We have tried to establish such a result ourselves, but we have ran into some technical difficulties due to the fact that we are in a moving exterior domain with Navier conditions on the solid boundary, which we will precise below.

So, for the time being let us formulate this autoregularization property as the following Open Problem.

Open problem 4.7.1. *Given $u_0 \in \mathcal{H}(0)$, $T > 0$, the associated weak solution $\bar{u} \in C([0, T]; \mathcal{H}(t)) \cap L^2((0, T); \mathcal{V}(t))$ in the sense of Definition 4.1.1 satisfies $(\bar{u}, \bar{h}', \bar{\vartheta}') \in C((0, T]; H^4(\mathcal{F}(t)) \times \mathbb{R}^3)$.*

If Open Problem 4.7.1 is proven, then the following result holds.

Theorem 4.7.1. *Given $T > 0$, $\mathcal{S}_0 \subset \mathbb{R}^2$ bounded, closed, simply connected with smooth boundary, which is not a disk, and $u_0 \in \mathcal{H}(0)$, $\operatorname{curl} u_0 \in L^1(\mathcal{F}_0;\mathbb{R}^2)$, $q_0 = 0, q_f = (h_f, \vartheta_f) \in \mathcal{Q}$, $h_0', h_f' \in \mathbb{R}^2, \vartheta_0', \vartheta_f' \in \mathbb{R}$, such that*

$$\operatorname{div} u_0 = 0 \text{ in } \mathcal{F}_0, \quad \lim_{|x|\to+\infty} |u_0(x)| = 0,$$

$$u_0 \cdot n = (h_0' + \vartheta_0' x^\perp) \cdot n, \ (D(u_0)n)_{tan} = -\mu(u_0 - (h_0' + \vartheta_0' x^\perp))_{tan} \text{ on } \partial\mathcal{S}_0.$$

Then there exists a control $\xi \in L^2((0,T) \times \Omega_c)$, compactly supported in time, and a weak solution $u \in C([0,T];\mathcal{H}(t)) \cap L^2((0,T);\mathcal{V}(t))$ in the sense of Definition 4.1.1 to the system (4.1), (4.2), (4.3), (4.6), (4.7) *with* (4.8), *such that we have $(h, h', \vartheta, \vartheta')(T) = (h_f, h_f', \vartheta_f, \vartheta_f')$.*

Proof. First we observe that $\tilde{T} > 0$ in Theorem 4.1.1 can be made uniform for $(h_0', \vartheta_0', u_0)$ in compact sets of $\mathbb{R}^3 \times H^4(\mathcal{F}_0;\mathbb{R}^2)$. Indeed, the initial data only comes into effect on the level of u^1 in our construction. However, it is easy to see that (l^1, r^1, u^1) depend continuously on $(h_0', \vartheta_0', u_0)$, since they satisfy a linear equation (and therefore depend continuously on the initial data) and the control g^1 constructed in Section 4.5 depends continuously on (h_0', ϑ_0'). This regularity is carried over to the remainder terms in Section 4.6, since they can be estimated by various norms of (l^1, r^1, u^1), which are all uniformly bounded for $(h_0', \vartheta_0', u_0)$ in compact sets of $\mathbb{R}^3 \times H^4(\mathcal{F}_0;\mathbb{R}^2)$.

Using Open Problem 4.7.1 we deduce the existence of a weak solution $\bar{u} \in C([0,T];\mathcal{H}(t)) \cap L^2((0,T);\mathcal{V}(t))$. For $\lambda > 0$ we then consider the following compact set

$$S := \{(l, r, v) \in \mathbb{R}^3 \times H^4(\mathcal{F}_0;\mathbb{R}^2), \ |(l, r) - (\bar{h}'(T), \bar{\vartheta}'(T))| + \|v - \bar{u}(T,\cdot)\|_{H^4} \le \lambda\}.$$

It follows that we may apply Theorem 4.1.1 with initial data in this set (without loss of generality we may assume that the initial position of the solid is once more the origin, since else one may simply apply a translation and a rotation to the system of coordinates in $\mathbb{R}^2$) to deduce $\tilde{T} > 0$, which is now uniform for all initial data in S.

All that needs to be proven is that there exists $T_\lambda \in (T - \tilde{T}, T)$ such that we have

$$(\bar{h}'(T_\lambda), \bar{\vartheta}'(T_\lambda), \bar{u}(T_\lambda, \cdot)) \in S,$$

since then we can let the system evolve without any control on $[0, T_\lambda]$ to ensure the regularization property of the equation, then use Theorem 4.1.1 to deduce the existence of a control on $[T_\lambda, T]$ which drives the solid to the desired position and velocity. However, the existence of such a T_λ follows from the regularity in Open Problem 4.7.1, therefore the proof of Theorem 4.7.1 follows.

□

Let us give a few remarks regarding the difficulties in proving Open Problem 4.7.1.

A direct possibility for the proof would be to consider $(\bar{u}, \bar{h}', \bar{\vartheta}')$ as the solution of a Stokes system and use for instance the regularity of the Stokes operator with Navier conditions in an exterior domain proven in [34], in particular the analyticity of the associated semigroup, to conclude the desired regularity. Of course, in order to do this, one transforms the PDE onto a fixed domain (with transformations similar to (4.15)), and then put all the unwanted terms into an inhomogeneous force term acting on the Stokes equation, which one would then estimate. The problem with this approach is that the change of variables creates a term of the type $u_S \cdot \nabla u$ (as seen for instance in the PDE (4.11)), which is not

even L^2 when $u \in H^1$. Note that this is not an issue in the Galerkin method presented in Section 4.6 or in [86] for instance, since the trilinear term $\int_{\mathcal{F}_0}(u - u_S) \cdot \nabla u \cdot u \, dx$ can be defined as 0 as a limit of approximations.

Another, more "manual" approach would be to bootstrap the regularity by classical Navier-Stokes methods. One obtains that after arbitrarily small time, the solution becomes H^1, and then using some appropriate test functions, the H^2 regularity can also be obtained after some manipulation of the weak solutions. However, for the H^3 regularity to hold, one usually differentiates the equation with respect to $t \in [0, T]$, proves an energy estimate for $\partial_t \bar{u}$ and then concludes by the means of a stationary Stokes problem (which in theory could be also done on a moving domain, without a change of variables). And it is at this point where this strategy breaks down in our setting. If one differentiates the equation on $\mathcal{F}(t)$, without switching to a fixed domain, then higher order unwanted terms appear in the Navier boundary conditions, which pose a problem when trying to estimate them in the energy inequality (we contrast this with the Dirichlet case, for instance in [10], which is more robust to differentiation with respect to time, and where such a strategy worked without issue). On the other hand if one switches to the fixed domain $\mathcal{F}_0$ to avoid this issue, then the term $\partial_t u_S \cdot \nabla u$ will be created, which will pose the same problems as mentioned above.

We plan to investigate this problem in a future article, and perhaps by a more lengthy and technical approach of considering a different change of variables, which behaves like a rigid movement near the solid, but behaves like the identity operator outside of a large enough ball (as done for instance in [42] for the Dirichlet case, where the authors only went as far as H^2 regularity) some sort of breakthrough could be made. This echoes the issues presented in Remark 4.2.3 for the bounded case, since such a change of variables would still create extra terms on the PDE which need to be estimated, but since this step is separate from the asymptotic expansion, perhaps those terms can be handled with sufficiently strong estimates, similarly to [42].

4.A Design of the controls

4.A.1 Proof of Proposition 4.3.3

We will use the notation of Section 3.7 of Chapter 3 for the rest of this proof, in contrast to the previous notations of this Chapter. Therefore, for the sake of simplicity, we will be working in the case when the solid is a homogeneous disk, however, using the same methods as in Section 3.7 of Chapter 3, our construction can be adapted to the general case as well. We recall from (4.45) that we had $\tilde{B}_c \subset B_c$ and, for $q \in \mathcal{Q}_\delta$, we defined

$$\mathcal{C}(q) = \left\{ g \in C_0^\infty(R(\vartheta)^T(B_c - h); \mathbb{R}) : \int g = 0 \right\}, \ \tilde{\mathcal{C}}(q) \ = \left\{ g \in C_0^1(R(\vartheta)^T(\tilde{B}_c - h); \mathbb{R}) : \int g = 0 \right\}.$$

Let us give the corresponding result in the case of a homogeneous disk.

Proposition 4.A.1. *There exists a C^1 mapping $\overline{g}^0 : \left(\mathcal{Q}_\delta \cap \{q = (h, 0) \in \mathbb{R}^3\}\right) \times \mathbb{R}^2 \to \tilde{\mathcal{C}}(\mathcal{Q}_\delta)$ such that for any $q = (h, 0) \in \mathcal{Q}_\delta$ we have Range$(\overline{g}^0(q, \cdot)) \subset \mathcal{C}(q) \cap \tilde{\mathcal{C}}(q)$, and for any $((h, 0), v)$ in $\mathcal{Q}_\delta \times \mathbb{R}^2$ the*

function $\overline{\alpha}^0 := \mathcal{A}[q, \overline{g}^0(q,v)]$ *in* $C^\infty(\overline{\mathcal{F}_0};\mathbb{R}) \cap C^6_\infty(\overline{\mathcal{F}_0};\mathbb{R})$ *satisfies :*

$$\Delta\overline{\alpha}^0 = 0 \text{ in } \mathcal{F}_0 \setminus (\tilde{B}_c - h), \lim_{|x|\to+\infty} |\nabla\overline{\alpha}^0| = 0 \text{ and } \partial_n\,\overline{\alpha}^0 = 0 \text{ on } \partial\mathcal{S}_0, \tag{4.120}$$

$$\int_{\partial\mathcal{S}_0} \left|\nabla\overline{\alpha}^0\right|^2 n\, d\sigma = v, \tag{4.121}$$

$$\int_{\partial\mathcal{S}_0} \overline{\alpha}^0\, n\, d\sigma = 0, \tag{4.122}$$

$$\text{span}\left\{n(x),\ x \in \text{supp } \nabla\overline{\alpha}^0(q,\cdot) \cap \partial\mathcal{S}_0\right\} = \mathbb{R}^2. \tag{4.123}$$

We follow the same construction as in Section 3.7 of Chapter 3. We have the following Lemma, which appears as Lemma 3.7.1 in Chapter 3, and also applies in our case without any modification, we refer the reader to the aforementioned Chapter for the proof.

Lemma 4.A.1. *There exist three vectors* $e_1, e_2, e_3 \in \{n(x) : x \in \partial\mathcal{S}_0\}$ *and positive* C^∞ *maps* $(\mu_i)_{1\leqslant i\leqslant 3} : \mathbb{R}^2 \to [1,+\infty)$ *such that for any* $v \in \mathbb{R}^2$,

$$\sum_{i=1}^{3} \mu_i(v) e_i = v. \tag{4.124}$$

Note that there exist $x_i \in \partial\mathcal{S}_0$ such that $n(x_i) = e_i$, $i = 1,2,3$. We will now modify Lemma 3.7.2 from Chapter 3 in order to guarantee (4.123). We have the following result.

Lemma 4.A.2. *There exist families of functions* $(\tilde{\alpha}^{i,j}_\varepsilon)_{\varepsilon\in(0,1)}$, $i,j \in \{1,2,3\}$, *such that for any* $i,j \in \{1,2,3\}$, *for any* $\varepsilon \in (0,1)$, $\tilde{\alpha}^{i,j}_\varepsilon$ *is defined and harmonic in a closed neighbourhood* $\mathcal{V}^{i,j}_\varepsilon$ *of* $\partial\mathcal{S}_0$, *satisfies* $\partial_n\tilde{\alpha}^{i,j}_\varepsilon = 0$ *on* $\partial\mathcal{S}_0$, *and moreover one has the following :*

(i) for any i,j,k,l *in* $\{1,2,3\}$,

$$\int_{\partial\mathcal{S}_0} \nabla\tilde{\alpha}^{i,j}_\varepsilon \cdot \nabla\tilde{\alpha}^{k,l}_\varepsilon\, n\, d\sigma \to \delta_{(i,j),(k,l)}\, e_i \qquad \text{as } \varepsilon \to 0^+;$$

(ii) for any i,j,k,l *in* $\{1,2,3\}$, $(i,j) \neq (k,l)$,

$$\|\nabla\tilde{\alpha}^{i,j}_\varepsilon \cdot \nabla\tilde{\alpha}^{k,l}_\varepsilon\|_{C(\partial\mathcal{S}_0)} \to 0 \qquad \text{as } \varepsilon \to 0^+;$$

(iii) for any i,j *in* $\{1,2,3\}$ *there exist* $x^{i,j}_\varepsilon \in \partial\mathcal{S}_0$ *such that* $|\nabla\tilde{\alpha}^{i,j}_\varepsilon(x^{i,j}_\varepsilon)| > 1$ *and* $x^{i,j}_\varepsilon \to x_i$ *as* $\varepsilon \to 0^+$.

Proof. The proof is essentially the same as that of Lemma 3.7.2 from Chapter 3, noting that the manner in which the functions $\beta^{i,j}_\varepsilon$ were constructed in the aforementioned proof allows us to deduce points (ii) and (iii). The details are left to the reader. □

Next we adapt Lemma 3.7.3 from Chapter 3 to our setting. We have the following result.

Lemma 4.A.3. *Let* $q = (h,0) \in \mathcal{Q}_\delta$. *There exists a family of functions* $(\alpha^{i,j}_\eta(q,\cdot))_{\eta\in(0,1)}$, $i,j \in \{1,2,3\}$, *harmonic in* $\mathcal{F}_0 \setminus (\tilde{B}_c - h)$, *satisfying* $\lim_{|x|\to+\infty} |\nabla\alpha^{i,j}_\eta(q,\cdot)| = 0$, $\partial_n\alpha^{i,j}_\eta(q,\cdot) = 0$ *on* $\partial\mathcal{S}_0$ *and* $\alpha^{i,j}_\eta(q,\cdot) \in C^6_\infty(\overline{\mathcal{F}_0}) \cap C^\infty(\overline{\mathcal{F}_0})$, *such that, for any* k *in* $\mathbb{N}$,

$$\|\alpha^{i,j}_\eta(q,\cdot) - \tilde{\alpha}^{i,j}_\varepsilon(\cdot)\|_{C^k(\mathcal{V}^{i,j}_\varepsilon\cap\overline{\mathcal{F}_0})} \to 0 \text{ when } \eta \to 0^+. \tag{4.125}$$

Proof. We construct $\alpha^{i,j}_\eta$ from $\tilde{\alpha}^{i,j}_\varepsilon$ using an approximation by rational functions, as mentioned in Chapter 3 and as used in [50], pages 147-149. We use the following generalization of Runge's theorem on the Riemann sphere (see for instance [41], page 238).

Theorem 4.A.1. *Let K be a compact subset of the Riemann sphere $\overline{\mathbb{C}}$, containing at least two points, and let P be a subset of $\overline{\mathbb{C}}$. A necessary and sufficient condition in order that for each holomorphic function f on K and each $\eta > 0$, there be a rational function r, whose poles lie in P, such that $\|f - r\|_{C^k(K)} < \eta$, for any $k \geq 0$, is that P meet each connected component of $\overline{\mathbb{C}} \setminus K$.*

We use the above result with $K = \left(\overline{\mathbb{C}} \setminus B(0,R)\right) \cup \mathcal{V}_\varepsilon^{i,j}$, for some $R > 0$ large enough such that $\left((\tilde{B}_c - h) \cup \mathcal{V}_\varepsilon^{i,j}\right) \subset B(0,R)$, set $f = 0$ on $\overline{\mathbb{C}} \setminus B(0,R)$ and $f = \partial_{x_1}\tilde{\alpha}_\varepsilon^{i,j} - i\partial_{x_2}\tilde{\alpha}_\varepsilon^{i,j}$ on $\mathcal{V}_\varepsilon^{i,j}$, and choose P such that it is made up of a point from int $(\tilde{B}_c - h)$ and a point from (int $\mathcal{S}_0) \setminus \mathcal{V}_\varepsilon^{i,j}$. We conclude as in [50] that there exists some function θ (ignoring the dependence on the parameters for this notation) which is harmonic on $\mathcal{F}_0 \setminus (\tilde{B}_c - h)$, $\lim\limits_{|x|\to+\infty} \nabla\theta(x) = C_\theta \in \mathbb{R}^2$, $|C_\theta| = \mathcal{O}(\eta)$, and $\partial_n\theta = \mathcal{O}(\eta)$ on $\partial\mathcal{S}_0$ in any C^k norm, such that $\theta \in C^k_\infty(\overline{\mathcal{F}_0})$ and

$$\|\theta - \tilde{\alpha}_\varepsilon^{i,j}\|_{C^k(\mathcal{V}_\varepsilon^{i,j}\cap\overline{\mathcal{F}_0})} \to 0 \text{ when } \eta \to 0^+, \text{ for any } k \geq 0.$$

Proceeding as in [50], we introduce a corrector function, to ensure the vanishing boundary conditions of our potential flow, by setting

$$\Delta\Psi = 0 \text{ on } \mathcal{F}_0, \quad \lim_{|x|\to+\infty} \nabla\Psi(x) = C_\theta, \text{ and } \partial_n\Psi = \partial_n\theta \text{ on } \partial\mathcal{S}_0.$$

Note that the function $\theta - \Psi$ satisfies all the conclusions of Lemma 4.A.3, except the smoothness on $\mathcal{F}_0$, since it has a singularity in $\tilde{B}_c - h$ as per Runge's theorem above. To address this issue, we use Whitney's extension theorem (see for instance [14], Chapter 2) to deduce that for any function $F \in C^\infty\left(\overline{\mathcal{F}_0 \setminus (\tilde{B}_c - h)}\right)$ there exists a smooth extension $E(F) \in C^\infty(\overline{\mathcal{F}_0})$ satisfying $E(F) = F$ on $\overline{\mathcal{F}_0 \setminus (\tilde{B}_c - h)}$. We conclude the proof by setting $\alpha_\eta^{i,j}(q,\cdot) = E(\theta - \Psi)$.

□

We have the following result, which corresponds to the adaptation of Lemma 3.7.4 from Chapter 3 to our setting.

Lemma 4.A.4. *For any $\nu > 0$, there exist C^1 mappings $q = (h,0) \in \mathcal{Q}_\delta \mapsto \overline{\alpha}^{i,j}(q,\cdot) \in C^2_\infty(\overline{\mathcal{F}_0})$, $i,j \in \{1,2,3\}$, such that for any $q = (h,0) \in \mathcal{Q}_\delta$, $\Delta_x\overline{\alpha}^{i,j}(q,\cdot) = 0$ in $\mathcal{F}_0 \setminus (\tilde{B}_c - h)$, $\lim\limits_{|x|\to+\infty} |\nabla\, \overline{\alpha}^{i,j}(q,\cdot)| = 0$, $\partial_n\overline{\alpha}^{i,j}(q,\cdot) = 0$ on $\partial\mathcal{S}_0$, $\overline{\alpha}^{i,j}(q,\cdot) \in C^6_\infty(\overline{\mathcal{F}_0}) \cap C^\infty(\overline{\mathcal{F}_0})$, and the following hold :*

(i) for any i,j,k,l in $\{1,2,3\}$,

$$\left|\int_{\partial\mathcal{S}_0} \nabla\overline{\alpha}^{i,j}(q,\cdot)\cdot\nabla\overline{\alpha}^{k,l}(q,\cdot)\, n\, d\sigma - \delta_{(i,j),(k,l)}\, e_i\right| \leq \nu;$$

(ii) for any i,j,k,l in $\{1,2,3\}$, $(i,j) \neq (k,l)$,

$$\|\nabla\overline{\alpha}^{i,j}(q,\cdot)\cdot\nabla\overline{\alpha}^{k,l}(q,\cdot)\|_{C(\partial\mathcal{S}_0)} \leq \nu;$$

(iii) for any i,j in $\{1,2,3\}$ there exist $x^{i,j} \in \partial\mathcal{S}_0$ such that $|\nabla\overline{\alpha}^{i,j}(x^{i,j})| > 1$ and $|x^{i,j} - x_i| \leq \nu$.

Proof. The proof of Lemma 3.7.4 from Chapter 3 can be easily adapted to our setting, simply by considering an elliptic problem of the type (4.31) instead of the Neumann problem mentioned in its proof. It can be easily checked that the solutions obtained this way are still in $C^6_\infty(\overline{\mathcal{F}_0})$ and that the map $q = (h,0) \in \mathcal{Q}_\delta \mapsto \overline{\alpha}^{i,j}(q,\cdot) \in C^2_\infty(\overline{\mathcal{F}_0})$ is C^1. Finally, points (ii) and (iii) follow from Lemma 4.A.2 and Lemma 4.A.3. □

Next we have the following result, which corresponds to Lemma 3.7.5 from Chapter 3.

Lemma 4.A.5. *For any $\nu > 0$, there exist C^1 mappings $q = (h, 0) \in \mathcal{Q}_\delta \mapsto \overline{\alpha}^i(q, \cdot) \in C^2_\infty(\overline{\mathcal{F}_0})$, $i \in \{1, 2, 3\}$, such that for any $q = (h, 0) \in \mathcal{Q}_\delta$, $\Delta_x \overline{\alpha}^i(q, \cdot) = 0$ in $\mathcal{F}_0 \setminus (\tilde{B}_c - h)$, $\lim_{|x|\to+\infty} |\nabla\, \overline{\alpha}^i(q, \cdot)| = 0$, $\partial_n \overline{\alpha}^i(q, \cdot) = 0$ on $\partial\mathcal{S}_0$, $\overline{\alpha}^i(q, \cdot) \in C^6_\infty(\overline{\mathcal{F}_0}) \cap C^\infty(\overline{\mathcal{F}_0})$, and the following hold :*

(i) for any i, j in $\{1, 2, 3\}$,

$$\left| \int_{\partial\mathcal{S}_0} \nabla\overline{\alpha}^i(q, \cdot) \cdot \nabla\overline{\alpha}^j(q, \cdot)\, n\, d\sigma - \delta_{i,j}\, e_i \right| \le \nu, \quad \int_{\partial\mathcal{S}_0} \overline{\alpha}^i(q, \cdot)\, n\, d\sigma = 0;$$

(ii) for any i, j in $\{1, 2, 3\}$, $i \neq j$,

$$\|\nabla\overline{\alpha}^i(q, \cdot) \cdot \nabla\overline{\alpha}^j(q, \cdot)\|_{C(\partial\mathcal{S}_0)} \le \nu;$$

(iii) for any i in $\{1, 2, 3\}$ there exist $\overline{x}^i \in \partial\mathcal{S}_0$ such that $|\nabla\overline{\alpha}^i(q, \overline{x}^i)|^2 > 1 - \nu$ and $|\overline{x}^i - x_i| \le \nu$.

Proof. Consider the functions $\overline{\alpha}^{i,j}$ given by Lemma 4.A.4. For any $q = (h, 0) \in \mathcal{Q}_\delta$, for any $i \in \{1, 2, 3\}$, the three vectors $\int_{\partial\mathcal{S}(q)} \overline{\alpha}^{i,j}(q, \cdot)\, n\, d\sigma$, where $j \in \{1, 2, 3\}$, are linearly dependent in $\mathbb{R}^2$; therefore there exist $\lambda^{i,j}(q) \in \mathbb{R}$ such that

$$\sum_{j=1}^{3} \lambda^{i,j}(q) \int_{\partial\mathcal{S}(q)} \overline{\alpha}^{i,j}(q, \cdot)\, n\, d\sigma = 0 \text{ and } \sum_{j=1}^{3} |\lambda^{i,j}(q)|^2 = 3, \tag{4.126}$$

Then one defines $\overline{\alpha}^i(q, \cdot) := \sum_{j=1}^3 \lambda^{i,j}(q)\overline{\alpha}^{i,j}(q, \cdot)$, and one checks that it satisfies (i) and (ii) with some $C\nu$ in the right hand side.

On the other hand, for each $i \in \{1, 2, 3\}$ let $k \in \arg\max_j |\lambda^{i,j}(q)|^2$, so that we have $|\lambda^{i,k}(q)|^2 \ge 1$. Using (ii) and (iii) from Lemma 4.A.4 it follows that

$$|\nabla\overline{\alpha}^i(q, x^{i,k})|^2 \ge |\lambda^{i,k}(q)|^2 |\nabla\overline{\alpha}^{i,k}(q, x^{i,k})|^2 - \tilde{C}\nu > 1 - \tilde{C}\nu,$$

for some $\tilde{C} > 0$.

Changing ν in $\min\{\nu/C, \nu/\tilde{C}, \nu\}$ allows to conclude. □

Before concluding the proof of Proposition 4.A.1, let us show how condition (4.123) can be satisfied. Let $v \in \mathbb{R}^2$, using Lemma 4.A.1 we set

$$\tilde{\alpha}(q, \cdot) := \sum_{i=1}^{3} \sqrt{\mu^i(v)}\, \overline{\alpha}^i(q, \cdot).$$

It follows from (ii) and (iii) in Lemma 4.A.5, and the fact that $\mu^i(v) \ge 1$, $i \in \{1, 2, 3\}$, that

$$|\nabla\tilde{\alpha}(q, \overline{x}^i)|^2 \ge |\nabla\overline{\alpha}^i(q, \overline{x}^i)|^2 - C\nu > 1 - (C + 1)\nu,$$

and $|\overline{x}^i - x_i| \le \nu$, for any $i \in \{1, 2, 3\}$. Since $\text{span}\{n(x_i),\ i = 1, 2, 3\} = \mathbb{R}^2$, it follows by continuity that, for $\nu > 0$ small enough, we have $\text{span}\{n(\overline{x}^i),\ i = 1, 2, 3\} = \mathbb{R}^2$, and $\overline{x}^i \in \text{supp}\, \nabla\tilde{\alpha}(q, \cdot)$, for any $i \in \{1, 2, 3\}$. Therefore, (4.123) holds for $\tilde{\alpha}$.

To conclude the proof of Proposition 4.A.1, we proceed as at the end of the proof of Proposition 3.7.1 from Chapter 3, namely that when $\nu > 0$ is small enough, one may apply a local inversion argument. The details are left to the reader.

Remark 4.A.1 (A remark on the case when the solid is not a disk)**.** *Lemma 3.7.7 from Chapter 3 can be improved in the following way : there exist vectors* $e_i \in \{\partial_n \Phi(x) : \ x \in \partial\mathcal{S}_0\}$, $1 \leqslant i \leqslant 16$, *a constant* $M > 0$, *and positive* C^∞ *maps* $(\mu_i)_{1\leqslant i\leqslant 16} : \mathbb{R}^2 \to [M, +\infty)$ *such that for any* $v \in \mathbb{R}^3$,

$$\sum_{i=1}^{16} \mu_i(v) e_i = v.$$

This allows us to prove (4.49) *similarly as we have done above in the case of a disk.*

4.A.2 Proof of Proposition 4.5.3

Again, for simplicity we will present the proof in the case when the solid is assumed to be a homogeneous disk, then one can deduce the general case in a similar manner as in Section 3.7 from Chapter 3, which will be explained at the end of the section. Furthermore, the notations of this section will be self-contained.

We prove the following adaptation of Proposition 4.5.3 to the case of a a homogeneous disk.

Proposition 4.A.2. *Let* $\mathcal{K}$ *be a compact subset of* $C^1(\partial\mathcal{S}_0; \mathbb{R}^2)$ *such that for any* $V \in \mathcal{K}$ *we have* $V \cdot n = 0$ *on* $\partial\mathcal{S}_0$ *and*

$$span\,\{n(x),\ x \in \partial\mathcal{S}_0 \cap supp\ V\} = \mathbb{R}^2. \tag{4.127}$$

For any $V \in \mathcal{K}$, *there exists a continuous mapping* $\overline{g}^1[V] : \big(\mathcal{Q}_\delta \cap \{q = (h, 0) \in \mathbb{R}^3\}\big) \times \mathbb{R}^2 \to \tilde{\mathcal{C}}(\mathcal{Q}_\delta)$ *such that for any* $q = (h, 0) \in \mathcal{Q}_\delta$ *we have Range*$(\overline{g}^1[V](q, \cdot)) \subset \mathcal{C}(q) \cap \tilde{\mathcal{C}}(q)$, *and for any* (q, v) *in* $\mathcal{Q}_\delta \times \mathbb{R}^2$, $q = (h, 0)$, *the function* $\overline{\alpha}^1 := \mathcal{A}[q, \overline{g}^1[V](q, v)]$ *in* $C^\infty(\overline{\mathcal{F}_0}; \mathbb{R}) \cap C^3_\infty(\overline{\mathcal{F}_0}; \mathbb{R})$ *satisfies :*

$$\Delta\overline{\alpha}^1 = 0 \ \textit{in}\ \mathcal{F}_0 \setminus (\tilde{B}_c - h), \lim_{|x|\to+\infty} |\nabla\overline{\alpha}^1| = 0 \ \textit{and}\ \partial_n\,\overline{\alpha}^1 = 0 \ \textit{on}\ \partial\mathcal{S}_0, \tag{4.128}$$

$$\int_{\partial\mathcal{S}_0} \nabla\overline{\alpha}^1 \cdot V\, n\, d\sigma = v. \tag{4.129}$$

Furthermore, the map $V \in \mathcal{K} \mapsto \overline{g}^1[V] \in C(\mathcal{Q}_\delta \times \mathbb{R}^2; \mathcal{C})$ *is also continuous.*

We may suppose without loss of generality that $\mathcal{S}_0$ is the unit disk, parametrized by $c(s) = (\cos(s), \sin(s))$. We have the following geometrical property.

Lemma 4.A.6. *Given* $V \in \mathcal{K}$, *there exist* $x_i \in \partial\mathcal{S}_0 \cap supp\ V$, $s_i \in [0, 2\pi]$, *such that* $n(x_i) = x_i = (\cos(s_i), \sin(s_i))$, $i = 1, 2$, *and that span*$\{b_1, b_2\} = \mathbb{R}^2$, *where*

$$b_i = \tau(x_i) \cdot V(x_i)\, n(x_i) - \frac{1}{\pi} \int_{\partial\mathcal{S}_0} \tau(x) \cdot V(x)\, n(x)\, d\sigma. \tag{4.130}$$

Proof. For each $i \in \{1, 2\}$, using (4.127) and $V \cdot n = 0$ on $\partial\mathcal{S}_0$, there exist $\bar{x}_i = (\cos(\bar{s}_i), \sin(\bar{s}_i))$ such that we have $\tau(\bar{x}_i) \cdot V(\bar{x}_i) \neq 0$ and

$$\text{span}\{\tau(\bar{x}_i) \cdot V(\bar{x}_i)\, n(\bar{x}_i),\ i = 1, 2\} = \mathbb{R}^2. \tag{4.131}$$

Clearly, $\frac{1}{\pi} \int_{\partial\mathcal{S}_0} \tau(x) \cdot V(x)\, n(x)\, d\sigma$ does not depend on the choice of $\bar{x}_i$, so either $\bar{b}_i = \tau(\bar{x}_i) \cdot V(\bar{x}_i)\, n(\bar{x}_i) - \frac{1}{\pi} \int_{\partial\mathcal{S}_0} \tau(x) \cdot V(x)\, n(x)\, d\sigma,\ i = 1, 2$, are collinear or they span $\mathbb{R}^2$.

If they are collinear, we may modify the one out of the two which has the smaller norm (in order to also handle the case in which one of them is zero) in the following way in order to break the

collinearity. Without loss of generality we may suppose that $|\bar{b}_1| \leq |\bar{b}_2|$. We observe that (4.131) is robust to perturbations, due to (4.127) and the continuity of V, i.e. there exists $\eta > 0$ such that for any $s \in (\bar{s}_1 - \eta, \bar{s}_1 + \eta)$ we have

$$\mathrm{span}\{\tau(c(s)) \cdot V(c(s))\, n(c(s)), \tau(\bar{x}_2) \cdot V(\bar{x}_2)\, n(\bar{x}_2)\} = \mathbb{R}^2.$$

It follows that there exists some $x_1 = (\cos(s_1), \sin(s_1))$ with $s_1 \in (\bar{s}_1 - \eta, \bar{s}_1 + \eta)$, such that setting $x_2 = \bar{x}_2$, the vectors b_i given by (4.130) are no longer collinear. □

We will base our construction on the existence of such vectors b_i given by Lemma 4.A.6, in order to satisfy (4.129). We have the following result.

Lemma 4.A.7. *There exist families of functions $(\tilde{\alpha}^i_\varepsilon)_{\varepsilon\in(0,1)}$, $i \in \{1,2\}$, such that for any $i \in \{1,2\}$, for any $\varepsilon \in (0,1)$, $\tilde{\alpha}^i_\varepsilon$ is defined and harmonic in a closed neighbourhood $\mathcal{V}^i_\varepsilon$ of $\partial\mathcal{S}_0$, satisfies $\partial_n \tilde{\alpha}^i_\varepsilon = 0$ on $\partial\mathcal{S}_0$, and moreover one has the following :*

$$\int_{\partial\mathcal{S}_0} \nabla\tilde{\alpha}^i_\varepsilon \cdot V\, n\, d\sigma \to b_i \qquad \text{as } \varepsilon \to 0^+.$$

Proof. For each $i \in \{1,2\}$ we consider families of smooth functions $\delta^i_\varepsilon \in C^\infty_0((0,2\pi);\mathbb{R})$, $\varepsilon \in (0,1)$, such that $\int \delta^i_\varepsilon = 1$, $\mathrm{diam}\left(\mathrm{supp}\ \delta^{i,j}_\varepsilon\right) = C\varepsilon$, and

$$\left|\int_0^{2\pi} \delta^i_\varepsilon(s)\, f(s) ds - f(s_i)\, ds\right| \leq C\|f'\|_\infty \varepsilon,$$

for any $f \in C^1([0,2\pi])$. Then we define $\tilde{\alpha}^{i,j}_\varepsilon$ in polar coordinates as the truncated Laurent series :

$$\tilde{\alpha}^i_\varepsilon(r,\theta) := \frac{1}{2} \sum_{0<k\leq K} \frac{1}{k}\left(r^k + \frac{1}{r^k}\right)(-\hat{b}^i_{k,\varepsilon}\cos(k\theta) + \hat{a}^i_{k,\varepsilon}\sin(k\theta)), \tag{4.132}$$

where $\hat{a}^i_{k,\varepsilon}$ and $\hat{b}^i_{k,\varepsilon}$ denote the k-th Fourier coefficients of the function δ^i_ε. It is elementary to check that the function $\tilde{\alpha}^i_\varepsilon$ satisfies the required properties for an appropriate choice of K. In particular, noting that $\nabla_x \tilde{\alpha}^i_\varepsilon = \partial_\tau \tilde{\alpha}^i_\varepsilon\, \tau$ on $\partial\mathcal{S}_0$, for each $\theta \in [0,2\pi]$ we have

$$\partial_\tau \tilde{\alpha}^i_\varepsilon(\cos(\theta), \sin(\theta)) = \partial_\theta \tilde{\alpha}^i_\varepsilon(1,\theta) = \delta^i_\varepsilon(\theta) - \frac{1}{\pi}\int_0^{2\pi} \delta^i_\varepsilon(\theta)\, d\theta + \mathcal{O}(\varepsilon),$$

and due to the properties of δ^i_ε, this implies

$$\left|\int_{\partial\mathcal{S}_0} \nabla\tilde{\alpha}^i_\varepsilon \cdot V\, n\, d\sigma - b_i\right| \leq C\left(\|V\|_{C^1(\partial\mathcal{S}_0)} + 1\right)\varepsilon.$$

□

We combine the methods used to prove Lemma 4.A.3 and Lemma 4.A.4 from Section 4.A.1 to get the following result.

Lemma 4.A.8. *There exist continuous mappings $(q,V) \in \left(\mathcal{Q}_\delta \cap \{q = (h,0) \in \mathbb{R}^3\}\right) \times \mathcal{K} \mapsto \overline{\alpha}_i[V](q,\cdot) \in C^\infty(\overline{\mathcal{F}_0})$, such that for any $(q,V) \in \mathcal{Q}_\delta \times \mathcal{K}$, $q = (h,0)$, we have $\Delta_x \overline{\alpha}_i[V](q,\cdot) = 0$ in $\mathcal{F}_0 \setminus (\tilde{B}_c - h)$, $\lim_{|x|\to+\infty} |\nabla\overline{\alpha}_i[V](q,\cdot)| = 0$, $\partial_n \overline{\alpha}_i[V](q,\cdot) = 0$ on $\partial\mathcal{S}_0$, $\overline{\alpha}_i[V](q,\cdot) \in C^3_\infty(\overline{\mathcal{F}_0})$, $i \in \{1,2\}$, and*

$$\mathit{span}\left\{\int_{\partial\mathcal{S}_0} \nabla\overline{\alpha}_i[V](q,\cdot) \cdot V\, n\, d\sigma,\ i = 1,2\right\} = \mathbb{R}^2. \tag{4.133}$$

Proof. The proof follows the same principles as the proofs of Lemma 4.A.3 and Lemma 4.A.4 from Section 4.A.1. The only difference is that instead of using a compact covering and partition of unity argument for $q \in \mathcal{Q}_\delta$, we use one for the pair $(q, V) \in \mathcal{Q}_\delta \times \mathcal{K}$ by observing that if $\overline{\alpha}_i[V]$ satisfy (4.133) for some fixed $V \in \mathcal{K}$, then there exists $\delta_V > 0$ such that

$$\operatorname{span}\left\{\int_{\partial\mathcal{S}_0} \nabla\overline{\alpha}_i[V](q,\cdot)\cdot\tilde{V}\, n\, d\sigma,\ i=1,2\right\} = \mathbb{R}^2.$$

holds for any $\tilde{V} \in \mathcal{K}$ with $\|V - \tilde{V}\|_{C^1} \leq \delta_V$.

This allows us to make our construction continuous with respect to V.

□

We conclude the proof of Proposition 4.A.2 by using Lemma 4.A.8 above to deduce $\overline{\alpha}_i(q,\cdot)$, $i = 1, 2$. For each $v \in \mathbb{R}^2$ we may obtain $\lambda_i[V](q, v) \in \mathbb{R}$, $i = 1, 2$, such that

$$\overline{\alpha}^1[V](q,\cdot) = \lambda_1[V](q,v)\overline{\alpha}_1[V](q,\cdot) + \lambda_2[V](q,v)\overline{\alpha}_2[V](q,\cdot),$$

which satisfies the required properties of Proposition 4.A.2, in particular the regularity with respect to q, v and V follows from the construction above and the fact that λ_i, $i = 1, 2$, are the solutions of a linear system whose coefficients are regular with respect to the aforementioned parameters.

Remark 4.A.2 (The case when $\mathcal{S}_0$ is not a disk)**.** *We may follow a similar construction as in Section 3.7 of Chapter 3 to reduce the general case to the case of a disk, by using a conformal mapping* $\Psi : \overline{\mathbb{C}} \setminus B(0,1) \to \overline{\mathbb{C}} \setminus \mathcal{S}_0$. *The key observation is that the condition* $\tau(x)\cdot V(x) \neq 0$ *for* $x \in \partial\mathcal{S}_0 \cap \mathit{supp}\ V$, *will be conserved by* Ψ *since it is a conformal mapping. Therefore we can deduce*

$$\mathit{span}\{\tau(x)\cdot V(x)\partial_n\,\Phi(x)\} = \mathbb{R}^3,$$

and proceed as in Lemma 4.A.6 to prove that subtracting the vector $\dfrac{1}{\pi}\displaystyle\int_{\partial\mathcal{S}_0} \tau(x)\cdot V(x)\,\partial_n\Phi(x)\,d\sigma$ *does not change the above span. The details are left to the reader.*

Bibliographie

[1] F. Alouges, A. DeSimone, L. Giraldi, M. Zoppello, Can magnetic multilayers propel artificial micro-swimmers mimicking sperm cells ?, Soft Robotics, Sep 2015, ahead of print.

[2] F. Alouges, A. DeSimone, L. Heltai, A. Lefebvre-Lepot, B. Merlet, Optimally swimming stokesian robots, Discrete and Continuous Dynamical Systems - B, 2013, 18 (5) : 1189–1215.

[3] F. Alouges, L. Giraldi, Enhanced controllability of low Reynolds number swimmers in the presence of a wall, Acta Applicandae Mathematicae 128 (1), 2013, 153-179.

[4] Y. Amirat, D. Bresch, J. Lemoine, J. Simon, Effect of rugosity on a flow governed by stationary Navier-Stokes equations, Q. Appl. Math. 59(4), 769-785, 2001.

[5] G. Aronsson, Global Controllability and Bang-Bang Steering of Certain Nonlinear Systems, SIAM Journal on Control, 11 (1973), no. 4, 607–619.

[6] Mehdi Badra, Takéo Takahashi, Feedback Stabilization of a Fluid-Rigid body Interaction System, Advances in Differential Equations, Khayyam Publishing, 2014, 19 (11/12), 1137-1184.

[7] Mehdi Badra, Takéo Takahashi, Feedback stabilization of a simplified 1d fluid- particle system, Annales de l'Institut Henri Poincaré (C) Non Linear Analysis, Elsevier, 2014, 31 (2), 369-389.

[8] C. Bardos, U. Frisch, Finite time regularity for bounded unbounded ideal incompressible fluids using Hölder estimates, Proc. of "Journées Mathématiques sur la Turbulence", R. TEMAM ed., Springer (1975).

[9] M. E. Bogovskiĭ, Solution of the first boundary value problem for an equation of continuity of an incompressible medium. Dokl. Akad. Nauk SSSR. 248 (1979), (5), 1037-1040.

[10] M. Boulakia, S. Guerrero, Local null controllability of a fluid-solid interaction problem in dimension 3, J. European Math Society, 15 (2013), no. 3, 825–856.

[11] M. Boulakia, A. Osses, Local null controllability of a two-dimensional fluid-structure interaction problem, ESAIM Control Optim. Calc. Var., 14 (2008), no. 1, 1–42.

[12] M. Bravin, On the weak uniqueness of "viscous incompressible fluid + rigid body" system with Navier slip-with-friction conditions in a 2D bounded domain, preprint 2018, arXiv :1803.08765.

[13] A. Bressan, Impulsive Control Systems, Nonsmooth Analysis and Geometric Methods in Deterministic Optimal Control, Volume 78 of the series The IMA Volumes in Mathematics and its Applications, 1–22.

[14] A. Brudnyi, Y. Brudnyi, Methods of Geometric Analysis in Extension and Trace Problems : Volume 1, Birkhäuser Basel, 2012.

[15] P. Brunovský, C. Lobry, Controlabilité Bang Bang, controlabilité différentiable, et perturbation des systèmes non linéaires. Ann. Mat. Pura Appl. (4) 105 (1975), 93–119.

[16] D. Bucur, E. Feireisl, Š Necasová, J. Wolf, On the asymptotic limit of the Navier-Stokes system on domains with rough boundaries, J. Differ. Equ. 244(11), 2890-2908, 2008.

[17] J. Carling, T. Williams, G. Bowtell, Self-propelled anguilliform swimming : simultaneous solution of the two-dimentional Navier-Stokes equations and Newton's laws of motion. J. of Experimental Biology, 201 :3143-3166, 1998.

[18] T. Chambrion, A. Munnier, Generic controllability of 3d swimmers in a perfect fluid. SIAM Journal on Control and Optimization, 50(5) (2012), 2814–2835.

[19] M. Chapouly, On the global null controllability of a Navier-Stokes system with Navier slip boundary conditions, J. Differential Equations 247 (2009), no. 7, 2094-2123.

[20] C. Conca, J. A. San Martín, M. Tucsnak, Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid, Comm. Partial Diff. Eq. 25 (5,6) 2000, 1019-1042.

[21] J.-M. Coron, Control and nonlinearity, Mathematical Surveys and Monographs volume 136, American Mathematical Society, Providence, RI, 2007.

[22] J.-M. Coron, Exact boundary controllability of the Euler equations of incompressible perfect fluids in dimension two, C. R. Acad. Sci. Paris, Serie I, 317 (1993), 271–276.

[23] J.-M. Coron, On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions, ESAIM Contrôle Optim. Calc. Var. 1 (1995/96), 35–75.

[24] J.-M. Coron, On the null asymptotic stabilization of 2-D incompressible Euler equation in a simply connected domain, SIAM J. Control Optim. 37 (1999), no. 6, 1874-1896.

[25] J.-M. Coron, Sur la stabilization des fluides parfaits incompressibles bidimensionnels, Séminaire : Équations aux Dérivées Partielles, 1998-1999, Exp. No. VII, École Polytechnique, Palaiseau, 1999.

[26] J.-M. Coron, A. V. Fursikov, Global exact controllability of the 2D Navier-Stokes equations on a manifold without boundary, Russian J. Math. Phys. 4 (1996), no. 4, 429-448.

[27] J.-M. Coron, F. Marbach, F. Sueur, Small time global exact null controllability of the Navier-Stokes equation with Navier slip-with-friction boundary conditions, to appear in Journal of EMS, 2017. http ://arxiv.org/abs/1612.08087.

[28] J.-M. Coron, F. Marbach, F. Sueur, On the controllability of the Navier-Stokes equation in spite of boundary layers. Proceeding of the RIMS conference "Mathematical Analysis of Viscous Incompressible Fluid". Preprint 2017. http ://arxiv.org/abs/1703.07265.

[29] J.-M. Coron, F. Marbach, F. Sueur, P. Zhang, Controllability of the Navier-Stokes equation in a rectangle with a little help of a distributed phantom force. Preprint 2018. https ://arxiv.org/abs/1801.01860.

[30] B. Desjardins, M. J. Esteban, Existence of weak solutions for the motion of rigid bodies in a viscous fluid, Arch. Ration. Mech. Anal. 146 (1999), 59–71.

[31] E. Dormy, D. Gérard-Varet, On the ill-posedness of the Prandtl equation, J. Amer. Math. Soc., 23(2) :591–609, 2010.

[32] S. Ervedoza, O. Glass, S. Guerrero, J.-P. Puel, Local exact controllability for the 1-D compressible Navier Stokes equation. Arch. Ration. Mech. Anal., October 2012, Volume 206, Issue 1, pp 189-238.

[33] R. Farwig, G. Galdi, G, H. Sohr, A new class of weak solutions of the Navier-Stokes equations with nonhomogeneous data, J. Math. Fluid Mech., 8(3) :423–444, 2006.

[34] R. Farwig, V. Rosteck, Resolvent estimates of the Stokes system with Navier boundary conditions in general unbounded domains, Adv. Differential Equations Volume 21, Number 5/6 (2016), 401–428.

[35] E. Fernández-Cara, Motivation, analysis and control of the variable density Navier-Stokes equations. En : Discrete and Continuous Dynamical Systems, Series S. 2012. Vol. 5. Núm. 6. Pag. 1021–1090.

[36] E. Fernández-Cara, S. Guerrero, O. Imanuvilov, J.-P. Puel, Local exact controllability of the Navier-Stokes system, J. Math. Pures Appl. (9), 83(12) :1501-1542, 2004.

[37] E. Fernández-Cara, S. Guerrero, O. Imanuvilov, J.-P. Puel, Some Controllability Results for the N-Dimensional Navier–Stokes and Boussinesq systems with N-1 scalar controls, SIAM Journal on Control and Optimization 45(1) :146-173, 2006.

[38] J. E. Fornaess, F. Forstneric, E. Fornaess Wold, Holomorphic approximation : the legacy of Weierstrass, Runge, Oka-Weil, and Mergelyan, preprint 2018, arXiv :1802.03924.

[39] R. Gaines, Continuous dependence for two-point boundary value problems, Pacific J. Math. 28 (1969), no. 2, 327–336.

[40] G. P. Galdi, On the steady self-propelled motion of a body in a viscous incompressible fluid. Arch. Ration. Mech. Anal., 148(1) :53-88, 1999.

[41] P. M. Gauthier, Uniform approximation, Complex Potential Theory (Montreal, PQ, 1993), Kluwer, Dordrecht, 1994, pp. 235–271.

[42] M. Geissert, K. Götze, M. Hieber, Lp-theory for strong solutions to fluid-rigid body interaction in Newtonian and generalized Newtonian fluids, Transactions of the American Mathematical Society, 2013, 365(3), 1393-1439. DOI : 10.1090/S0002-9947-2012-05652-2

[43] M. Geissert, H. Heck, M. Hieber, On the Equation div u = g and Bogovskii's Operator in Sobolev Spaces of Negative Order, Koelink E., van Neerven J., de Pagter B., Sweers G., Luger A., Woracek H. (eds) Partial Differential Equations and Functional Analysis. Operator Theory : Advances and Applications, vol 168. Birkhäuser Basel (2006).

[44] D. Gérard-Varet, M. Hillairet, Existence of Weak Solutions Up to Collision for Viscous Fluid-Solid Systems with Slip, Commun. Pur. Appl. Math. (2014), 67 : 2022-2076.

[45] D. Gérard-Varet, M. Hillairet, Regularity issues in the problem of fluid structure interaction, Arch. Rat. Mech. Anal. 19, 375-407 (2010).

[46] D. Gérard-Varet, M. Hillairet, C. Wang, The influence of boundary conditions on the contact problem in a 3d Navier-Stokes flow, 103, 1–38 (2015). Journal Math. Pures Appl.

[47] O. Glass, An addendum to a J. M. Coron theorem concerning the controllability of the Euler system for 2D incompressible inviscid fluids. J. Math. Pures Appl. (9) 80 (2001), no. 8, 845–877.

[48] O. Glass, Asymptotic stabilizability by stationary feedback of the two-dimensional Euler equation : the multiconnected case, SIAM J. Control Optim. 44 (2005), no. 3, 1105-1147.

[49] O. Glass, Exact boundary controllability of the 3D Euler equation, ESAIM : Control, Optimization and Calculus of variations, 2000.

[50] O. Glass, Some questions of control in fluid mechanics. Control of partial differential equations, 131–206, Lecture Notes in Math. 2048, Fond. CIME/CIME Found. Subser., Springer, Heidelberg, 2012.

[51] O. Glass, T. Horsin, Prescribing the motion of a set of particles in a 3D perfect fluid, SIAM J. Control Optim. 50 (2012), No. 5, 2726–2742.

[52] O. Glass, T. Horsin, Approximate Lagrangian controllability for the 2-D Euler equation. Application to the control of the shape of vortex patches. J. Math. Pures Appl. (9) 93 (2010), no. 1, 61–90.

[53] O. Glass, T. Horsin, Lagrangian controllability at low Reynolds number. ESAIM Control Optim. Calc. Var. 22 (2016), no. 4, 1040–1053.

[54] O. Glass, J. J. Kolumbán, F. Sueur, External boundary control of the motion of a rigid body immersed in a perfect two-dimensional fluid, preprint 2017, arXiv :1707.05093.

[55] O. Glass, C. Lacave, F. Sueur, On the motion of a small body immersed in a two dimensional incompressible perfect fluid, Bull. Soc. Math. France 142 (2014), no. 2, 1–48.

[56] O. Glass, C. Lacave, F. Sueur, On the motion of a small light body immersed in a two dimensional incompressible perfect fluid with vorticity, Communications in Math. Physics. Volume 341, Issue 3, 1015-1065, 2016

[57] O. Glass, A. Munnier, F. Sueur, Dynamics of a point vortex as limits of a shrinking solid in an irrotational fluid, preprint 2014, to appear in Inventiones Mathematicae, `arXiv:1402.5387`.

[58] O. Glass, L. Rosier, On the control of the motion of a boat, Math. Mod. Meth. Appl. Sci. 23 (2013), no. 4, 617–670.

[59] O. Glass, F. Sueur, Low regularity solutions for the two-dimensional "rigid body + incompressible Euler" system, Differential and Integral Equations, Volume 27, Issue 7/8, 625-642, 2014.

[60] O. Glass, F. Sueur, The movement of a solid in an incompressible perfect fluid as a geodesic flow, Proc. Amer. Math. Soc. 140 (2012), 2155–2168.

[61] O. Glass, F. Sueur, Uniqueness results for weak solutions of two-dimensional fluid-solid systems, Archive for Rational Mechanics and Analysis. Volume 218 (2015), Issue 2, 907–944.

[62] O. Glass, F. Sueur, T. Takahashi, Smoothness of the motion of a rigid body immersed in an incompressible perfect fluid, Ann. Sci. de l'Ecole Normale Supérieure Volume 45, fascicule 1 (2012), 1–51.

[63] K. A. Grasse, Nonlinear perturbations of control-semilinear control systems, SIAM J. Control Optim. 20 (1982), No. 3, 311–327.

[64] K. A. Grasse, Perturbations of nonlinear controllable systems, SIAM J. Control Optim. 19 (1981), No. 2, 203–220.

[65] S. Guerrero, Local exact controllability to the trajectories of the Navier-Stokes system with nonlinear Navier-slip boundary conditions, ESAIM Control Optim. Calc. Var., 12(3) :484-544 (electronic), 2006.

[66] A. Henrot, M. Pierre, Variation et optimisation de formes, Une analyse géométrique, Springer 2005 (in French).

[67] M.Hillairet, Lack of collision between solid bodies in a 2D incompressible viscous flow, Comm. Partial Differential Equations, 32 (2007), 1345–1371.

[68] M. Hillairet, T. Takahashi, Collisions in 3D fluid structure interaction problems, SIAM J. Math. Anal., 40 (2009), 2451–2477.

[69] J.-G. Houot, J. San Martin, M. Tucsnak, Existence and uniqueness of solutions for the equations modelling the motion of rigid bodies in a perfect fluid. J. Funct. Anal., 259(11) :2856–2885, 2010.

[70] D. Iftimie, F. Sueur, Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions. Arch. Ration. Mech. Anal., 199(1), 2011 : 145–175.

[71] O. Imanuvilov, Remarks on exact controllability for the Navier-Stokes equations, ESAIM Control Optim. Calc. Var., 6 :39-72 (electronic), 2001.

[72] O. Imanuvilov, T. Takahashi, Exact controllability of a fluid-rigid body system. J. Math. Pures Appl. 87, Issue 4 (2007), 408-437.

[73] W. Jager, A. Mikelic, On the roughness-induced effective boundary conditions for an incompressible viscous flow, J. Differ. Equ. 170(1), 96-122, 2001.

[74] J. Kelliher, Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane, SIAM J. Math. Anal., 38(1) :210-232 (electronic), 2006.

[75] E. Lauga, M. P. Brenner and H. A. Stone, Microfluidics : The no-slip boundary condition Handbook of Experimental Fluid Dynamics, Springer, 2006.

[76] J.-L. Lions, Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués, Recherches en Mathématiques Appliquées, Masson, Paris, 1988.

[77] H. Liu and K. Kawachi, A numerical study of undulatory swimming. J. comput. phys., 155(2) :223-247, 1999.

[78] Y. Liu, T. Takahashi, M. Tucsnak, Single input controllability of a simplified fluid-structure interaction model, ESAIM : Control, Optimisation and Calculus of Variations, EDP Sciences, 2013, 19 (01), pp.20-42.

[79] J. Lohéac, A. Munnier. Controllability of 3D low Reynolds number swimmers. ESAIM : Control, Optimisation and Calculus of Variations, 20 (2014) no. 1, 236-268.

[80] J. E. Marsden, T. Ratiu, Introduction to mechanics and symmetry : a basic exposition of classical mechanical systems (Vol. 17). Springer Science & Business Media. 2013.

[81] A. Munnier, Locomotion of Deformable Bodies in an Ideal Fluid : Newtonian versus Lagrangian Formalisms. J. Nonlinear Sci (2009) 19 : 665-715.

[82] C.L.M.H. Navier, Mémoire sur les lois du mouvement des fluides, Mem. Acad. R. Sci. Inst. 6, 389-416, 1823.

[83] H. Nersisyan, Controllability of the 3D compressible Euler system, Communications in Partial Differential Equations, Vol. 36, Issue 9 (2011), 1544-1564.

[84] J. Ortega, L. Rosier, T. Takahashi, On the motion of a rigid body immersed in a bidimensional incompressible perfect fluid. Ann. Inst. H. Poincaré Anal. Non Linéaire, 24 (2007), no. 1, 139–165.

[85] J. H. Ortega, L. Rosier, T. Takahashi. Classical solutions for the equations modelling the motion of a ball in a bidimensional incompressible perfect fluid. M2AN Math. Model. Numer. Anal., 39 (2005), no. 1, 79–108.

[86] G. Planas, F. Sueur, On the “viscous incompressible fluid+rigid body" system with Navier conditions, Annales de l'I.H.P. Analyse non linéaire, Volume 31 (2014) no. 1, p. 55–80.

[87] L. Prandtl, Über Flüssigkeitsbewegung bei sehr kleiner Reibung. Verhaldlg III Int. Math. Kong, pages 484-491, 1904.

[88] C. Rosier, L. Rosier. Smooth solutions for the motion of a ball in an incompressible perfect fluid. Journal of Functional Analysis, 256 (2009), no. 5, 1618–1641.

[89] D. L. Russell, Controllability and stabilizability theory for linear partial differential equations : recent progress and open questions, SIAM Rev., 20(4) :639–739, 1978.

[90] J. San Martin, J. F. Scheid, T. Takahashi, M. Tucsnak, An initial and boundary problem modeling fish-like swimming, Arch. Ration. Mech. Anal., June 2008, Volume 188, Issue 3, pp 429-455.

[91] J. San Martin, T. Takahashi, M. Tucsnak, A control theoretic approach to the swimming of microscopic organisms. Quart. Appl. Math., 65(3) :405-424, 2007.

[92] J. Serrin, Mathematical principles of classical fluid mechanics. Handbuch der Physik (herausgegeben von S. Flügge), Bd. 8/1, Strömungsmechanik I (Mitherausgeber C. Truesdell), 125-263, Springer-Verlag, 1959.

[93] G. Stokes, On the Theories of Internal Friction of Fluids in Motion and of the Equilibrium and Motion of Elastic Solids, Trans. Camb. Phil. Soc. 8, 287-319, 1845.

[94] T. Takahashi, Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain, Adv. Diff. Eq. 8 (12) 2003, 1499-1532.

[95] R. Temam, Problèmes mathématiques en plasticité, Méthodes Mathématiques de l'Informatique, 12. Gauthier-Villars, 1983.

[96] M. Tucsnak, G. Weiss, Observation and control for operator semigroups, Birkhäuser Advanced Texts : Basel Textbooks, Birkhäuser Verlag, Basel, 2009.

[97] C. Wang, Strong solutions for the fluid-solid systems in a 2-D domain, Asymptotic Analysis, 89 (2014) 263-306.

[98] V. I. Yudovich, The flow of a perfect, incompressible liquid through a given region. Dokl. Akad. Nauk SSSR 146 (1962), 561–564 (Russian). English translation in Soviet Physics Dokl. 7 (1962), 789–791.

Résumé

Dans cette thèse nous étudions certains problèmes de contrôle des modèles d'interactions fluide-solide. Dans la première partie nous démontrons la contrôlabilité exacte de la position et de la vitesse d'un corps rigide immergé dans un fluide parfait bidimensionnel, irrotationnel et incompressible. Le système occupe un domaine borné avec une condition d'imperméabilité sur une partie de la frontière externe. Sur l'autre partie de la frontière, nous utilisons des contrôles de type Yudovich (permettant du fluide d'entrer et de sortir du domaine). Dans la seconde partie nous considérons le mouvement d'un corps rigide immergé dans un fluide visqueux incompressible bidimensionnel avec des conditions de Navier à la frontière du solide. Le système fluide-solide occupe tout le plan. Nous prouvons la contrôlabilité exacte de la position et de la vitesse du solide lorsque le contrôle est une force supportée dans un sous-ensemble compact (d'intérieur non-vide) du domaine fluide, loin du corps.

Abstract

In this PhD thesis we study certain control problems related to fluid-solid interaction models. In the first part of the thesis we prove the exact controllability of the position and velocity of a rigid body immersed in a two-dimensional irrotational perfect incompressible fluid. The system is assumed to be confined in a bounded domain with an impermeable condition on a part of the external boundary. On the remaining part of the boundary we use Yudovich-type controls (allowing some fluid going in and out the domain). In the second part of the thesis we consider the motion of a rigid body immersed in a two-dimensional viscous incompressible fluid with Navier slip-with-friction conditions at the solid boundary. The fluid-solid system occupies the whole plane. We prove the exact controllability of the position and velocity of the solid when the control takes the form of a distributed force supported in a compact subset (with nonvoid interior) of the fluid domain, away from the body.

Mots Clés

Interaction fluide-solide, mécanique des fluides, théorie du contrôle, système couplé EDO-EDP.

Keywords

Fluid-solid interaction, fluid mechanics, control theory, coupled ODE-PDE system.