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# Mathematical study of some systems of particles in a disordered medium

Raphaël Ducatez

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# THÈSE DE DOCTORAT

de l'Université de recherche Paris Sciences et Lettres  
PSL Research University

Préparée à Université Paris Dauphine

Analyse mathématique de divers systèmes de particules en milieu désordonné

**ED n°543**

L'ÉCOLE DOCTORALE DE DAUPHINE

**Spécialité** MATHÉMATIQUE

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le 18 septembre 2018

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A mes frères Franklin, Théophile et Aymeric.

Localisation...



...d'Anderson<sup>1</sup>

---

1.

(Fantastic Mr Fox, dans le film de Wes Anderson.)

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# Chapitre 1

## Introduction

### 1.1 Motivations physiques : de la localisation d'Anderson partout

On s'intéresse aux équations d'onde, omniprésentes dans toutes les branches de la physique. La problématique est la suivante : les matériaux ne sont jamais parfaitement homogènes et l'on retrouve partout des défauts, des impuretés et de l'inhomogénéité. Pour prendre en compte ces imperfections, qui sont impossibles en pratique à mesurer précisément, une tentative naturelle est de les modéliser avec un terme aléatoire. Ce fut le modèle que proposa le physicien Anderson dans les années 1950 [12] (prix Nobel en 1977), qui depuis fut étendu et extensivement développé pour des phénomènes physiques très divers tels que les ondes de spin, la lumière, les ultrasons, les gaz d'électrons ou les atomes froids. Dans un milieu inhomogène, l'onde ne se propage pas librement, elle est déviée, diffusée par les défauts qu'elle rencontre. On pourrait s'attendre à ce que l'onde diffuse dans tout le milieu mais Anderson prédit au contraire un phénomène très surprenant : lorsque l'inhomogénéité est importante, l'onde pouvait être complètement « gelée », coincée dans un petit espace comme si le désordre formait des miroirs tout autour. C'est ce phénomène que l'on appelle localisation d'Anderson. Dans les sections suivantes nous discuterons plus précisément certaines situations physiques de ce type.

#### 1.1.1 Les ondes de spin

Dans le papier historique d'Anderson [12], les ondes de spin furent le premier système considéré. Il s'intéressait plus particulièrement aux propagations dans les impuretés des semi conducteurs. Parce que les atomes sur le réseau interagissent les uns avec les autres, les spins de deux atomes côte à côte peuvent permuter. Ces permutations peuvent se produire en chaîne comme des dominos et on parle alors d'onde de spin [25]. L'équation qui décrit ce phénomène vient de la seconde quantification de la mécanique quantique et s'écrit sous la forme

$$i\partial_t\phi = \left( \sum_i v_i a_i^\dagger a_i + \sum_{i,j} \Delta_{i,j} a_i^\dagger a_j \right) \phi \quad (1.1)$$

où  $|\phi|^2$  décrit la probabilité de mesurer l'onde dans un certain état,  $a_i, a_i^+$  sont des opérateurs de permutation vers le bas et vers le haut,  $\Delta_{i,j}$  est l'énergie d'interaction entre le spin en  $i$  et en  $j$  et  $v_i$  est un terme énergie associé à chaque spin. Ce dernier dépend de la nature des impuretés et de la manière dont elles sont intégrées au réseau que l'on choisit de modéliser aléatoirement. On peut montrer que cette équation (1.1) est équivalente à l'équation (1.4) que nous étudierons plus loin, pour la configuration initiale  $\phi(t=0) = a_0^+|0\rangle$  si on se restreint à l'espace des fonctions d'onde avec un seul spin dirigé vers le haut.

### 1.1.2 Les électrons dans un semi conducteurs : le modèle des liaisons fortes

On étudie des électrons dans un métal ou un semi conducteur. L'hypothèse usuelle est de n'étudier qu'un seul électron à la fois, puis de supposer que les différents électrons diffusent indépendamment. Malheureusement, cette hypothèse n'est pas justifiée. En effet l'énergie d'interaction de Coulomb entre les électrons est comparable aux autres énergies du système et de plus elle ne diminue que très lentement avec la distance. Depuis les années 2000, se passer de cette hypothèse est au centre d'une forte activité de recherche [9, 42, 83], cependant le modèle à considérer (un gaz d'électrons où toutes les particules interagissent entre elles) est extrêmement compliqué et reste encore mal compris.

Pour le modèle à une particule, on écrit souvent le modèle des liaisons fortes qui donne une version discrétisée de l'équation de Schrödinger. On commence avec le Hamiltonien général

$$H = -\partial_{xx} - \partial_{yy} - \partial_{zz} + V_0$$

où  $V_0$  est le potentiel effectif créé par les noyaux des atomes dans le cristal et par les électrons de « cœur » qui gravitent autour des noyaux. Autour de chaque atome, on diagonalise cet Hamiltonien en négligeant l'influence des atomes qui l'entourent. On obtient ainsi un ensemble de vecteurs propres que les chimistes et physiciens dénomment orbitales. L'idée du modèle des liaisons fortes est d'une part, de ne pas considérer les orbitales de plus basse énergie qui sont complètement saturées par les électrons de cœur et ne participent pas à la conduction, d'autre part de se restreindre à l'espace engendré par les orbitales qui, parmi celles encore autorisées, minimisent l'énergie pour chaque atome. Cela est justifié dans la mesure où, pour les phénomènes considérés, les niveaux de plus haute énergie ne sont jamais excités. On peut alors, en modifiant légèrement ces orbitales minimales, obtenir une base orthonormée sur laquelle le Hamiltonien restreint s'écrit maintenant sur  $\mathbb{Z}^d$  (ou le réseau discret correspondant) sous la forme

$$H = -\Delta + V$$

avec  $\Delta_{i,j}$  le terme de saut entre le site  $i$  et le site  $j$  parce que les orbitales débordent un peu sur les atomes voisins et  $V_i$  qui est l'énergie associée à l'orbitale minimale de l'atome  $i$ . En ajoutant à ce modèle de l'aléa, on obtient de nouveau le modèle d'Anderson. Ce modèle fut particulièrement étudié par Mott [114] qui a obtenu une formule de la conductivité  $\sigma$  en fonction de la température

$$\sigma = \sigma_0 e^{-\left(\frac{T_0}{T}\right)^{\frac{1}{4}}}$$

qui apparaît dans le régime où l'on observe le phénomène de localisation d'Anderson [89, 70].

### 1.1.3 Localisation de la lumière

Curieusement l'étude du phénomène de localisation d'Anderson pour la lumière est relativement récente. La propagation de la lumière est gouvernée par les équations de Maxwell

$$\partial_{tt}u + \operatorname{div}(c^2\nu(x)\nabla u) = j$$

où  $u$  est le potentiel électromagnétique,  $c$  la vitesse de la lumière,  $j$  est un terme de source et  $\nu(x)$  un facteur qui dépend des propriétés diélectrique et diamagnétique du milieu et qui n'est pas constant si le milieu est inhomogène. Comparé au modèle précédent, il se trouve que la lumière interagit très peu avec elle-même et le modèle linéaire est beaucoup plus justifié. Le point négatif du point de vue expérimental est que la lumière ne se conserve pas et dans la plupart des milieux, l'absorption domine. L'absence de diffusion est alors due à la partie imaginaire du coefficient de diffusion et non au phénomène de localisation d'Anderson.

Une variante intéressante du modèle est le cas où l'aléa se trouve dans la définition du domaine de propagation. Plutôt que  $\mathbb{R}^d$ , on définit le domaine  $\Omega = \mathbb{R}^d - \cup_i A_i$  où les  $A_i$  sont des formes choisies de manière aléatoire et on souhaiterait résoudre les équations de Maxwell avec les conditions au bord de Dirichlet

$$\begin{cases} \partial_{tt}u + \operatorname{div}(c^2\nabla u) = j & \text{pour } x \in \mathring{\Omega} \\ u = 0 & \text{pour } x \in \partial\Omega \end{cases}$$

On a pu observer la localisation d'Anderson en envoyant un laser dans de la poudre de GaAs [133], chaque grain se comportant comme un miroir presque parfait. Pour cette expérience l'ensemble des  $A_i$  correspond à l'espace occupé par les grains de GaAs et  $j$  est le laser entrant.

*Remarque 1.1.* Dans le cas sans désordre et avec un environnement périodique il existe également un équivalent pour la lumière avec les cristaux photoniques. Ces derniers présentent des bandes de longueur d'onde interdites décrites par la théorie de Bloch-Floquet. Dans ces plages de longueurs d'onde, la lumière ne peut pas se propager dans le cristal.

*Remarque 1.2.* Les ondes sonores obéissent formellement aux mêmes équations que pour la lumière [131],[80].

### 1.1.4 Localisation d'Anderson dans un gaz d'atomes froids

Depuis les progrès expérimentaux fulgurants des gaz d'atomes à ultra basse température, ils sont devenus un outil presque universel pour poser et réétudier certaines questions fondamentales de mécanique quantique. L'exemple le plus spectaculaire fut bien sûr la production d'un condensat de Bose Einstein dans les années 90. Le gaz d'atomes froids est facilement manipulable avec des lasers, il peut par exemple être très allongé de telle sorte à reproduire un système à une dimension et, pour le désordre, on ajoute des petites taches de lumière. Le condensat de Bose-Einstein est gouverné par l'équation de Gross-Pitaevskii

$$i\partial_t\phi = -\Delta\phi + V\phi + a|\phi|^2\phi$$

le terme en  $a|\phi|^2$  est dû aux interactions entre les atomes constituant le condensat qui est ici modélisé avec une approximation de champ moyen. Cependant pour un gaz suffisamment dilué, on peut en première approximation négliger ce terme pour retrouver le modèle d'Anderson [21].

## 1.2 Présentation de la thèse

Cette thèse est divisée en 4 projets qui ont chacun abouti à une publication ou à une prépublication. Les trois premiers projets traitent de différentes versions du modèle d'Anderson. Le premier rediscute le modèle à une dimension en proposant une preuve simplifiée pour la localisation. Le deuxième propose une solution approchée au problème à plusieurs particules en interaction, en utilisant une approximation de Hartree-Fock. Le troisième étudie l'influence d'une perturbation périodique en temps sur le modèle d'Anderson, on cherche alors à comprendre la conductivité AC du matériau. Le dernier projet s'attaque au modèle du Jellium, un système de particules en interaction en une dimension. L'originalité de ce dernier travail fut de traiter le modèle dans un cadre plus général avec une densité de charge inhomogène. Il s'intègre ainsi dans la thèse en permettant de travailler avec un milieu généré aléatoirement.

### 1.2.1 Modèle d'Anderson à une dimension : un processus avant / arrière

Nous avons revisité le modèle d'Anderson à une dimension et son lien avec les produits de matrices aléatoires. Notre objectif est de réécrire à la limite la norme des vecteurs propres comme l'exponentielle d'un processus Brownien avec une dérive, la dérive correspondant au résultat classique de localisation avec décroissance exponentielle. Ce résultat est connu pour le produit de matrices aléatoires indépendantes [103] mais à notre connaissance n'avait pas été généralisé pour le modèle d'Anderson. Rifkind et Virag ont récemment trouvé la loi exacte de la norme des vecteurs propres pour le régime critique [120]. Dans [52], nous avons obtenu une formule qui permet une interprétation nouvelle pour certains résultats antérieurs et nous avons pu l'appliquer pour donner la répartition de la température du modèle d'Anderson couplé à deux bains thermiques.

### 1.2.2 Modèle d'Anderson dans l'approximation de Hartree-Fock

Ce projet est la suite du modèle construit et étudié par Cancès, Lahbabi et Lewin [32, 100, 33]. Considérons un matériau isolant c'est à dire dont le spectre présente un « trou » et remplissons d'électrons les états propres jusqu'au niveau de Fermi situé au milieu de ce gap. Ajoutons à ce modèle du désordre et des interactions entre les électrons. Dans l'approximation de Hartree-Fock, il existe un unique état qui minimise l'énergie totale du système. La répartition des électrons à ce minimum fait apparaître un potentiel effectif qui écrante en partie le désordre. La conclusion de notre travail est que si le matériau est fortement isolant (le trou spectral est important) et les interactions sont faibles, alors ces dernières sont incapables de lisser suffisamment le désordre et on conserve l'existence de la localisation d'Anderson.

### 1.2.3 Modèle d'Anderson avec une perturbation périodique en temps

Au système décrit par le modèle d'Anderson, on ajoute une perturbation périodique en temps. La question centrale est alors d'étudier la diffusion en temps long d'une particule. Sans le terme perturbatif, il est bien connu que la localisation d'Anderson annihile tout caractère diffusif. La conductivité pour un courant continu est alors nulle. En ajoutant un terme oscillant en temps, on s'attend à observer une résonance entre différents niveaux d'énergie et à ce que la particule puisse sauter de site en site sur ces états résonnants. Cependant, on a été capable de montrer la localisation d'Anderson sur l'espace augmenté : position + fréquence en temps. Une conséquence est l'absence de diffusion dans ce modèle.

### 1.2.4 Modèle du Jellium à une dimension dans un milieu inhomogène

En utilisant la théorie spectrale appliquée à un opérateur de transfert, on peut montrer de nombreuses propriétés sur le Jellium en milieu homogène à une dimension. Dans notre travail, nous avons utilisé une autre approche, à savoir la distance de Hilbert et le théorème de Birkhoff-Hopf pour généraliser ces résultats au modèle inhomogène. En particulier nous avons pu retrouver la régularité de l'énergie libre, la décroissance des corrélations ainsi qu'un théorème de limite centrale. Nous avons également pu aborder la version quantique du modèle.

Dans la section 1.5, nous décrivons plus en détail les différents résultats mais avant, nous définissons plus en détail la localisation d'Anderson.

## 1.3 Le modèle d'Anderson : formulation mathématique et résultats antérieurs

*“En théorie, la pratique fonctionne. Mais en pratique, la théorie ne fonctionne pas.”* (proverbe de physiciens)

### 1.3.1 Version continue du modèle

Comme vu précédemment, pour modéliser la lumière, le son, un électron sans faire l'approximation des liaisons fortes ou pour le gaz d'atomes froids on cherche à résoudre l'équation :

$$\begin{cases} \phi(t=0) = \phi_0 \\ i\partial_t \phi(t, x) = H\phi(t, x), \end{cases} \quad (1.2)$$

avec  $H$  le Hamiltonien donné par

$$H = -\Delta + V_0 + V_\omega,$$

où  $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \dots + \partial_{x_n}^2$  est le Laplacien usuel en dimension  $d$  et  $V$  le potentiel extérieur que l'on décompose en  $V_0$ , un potentiel déterministe périodique qui est dû à l'organisation régulière du cristal, et  $V_\omega$  que l'on doit ajouter pour prendre en compte le désordre. On suppose qu'il existe un sous groupe discret



$G$  de  $\mathbb{R}^d$  (qui représente le réseau du cristal) tel que pour tout  $K \in G$  et tout  $x \in \mathbb{R}^d$   $V_0(x + K) = V_0(x)$ . On construit ensuite  $V_\omega$  de la manière suivante autour de chaque site du réseau. L'amplitude est tirée aléatoirement de manière indépendante et identiquement distribuée (IID)

$$V_\omega(x) = \sum_{y \in G} v_y(\omega) c_0(x - y) \quad (1.3)$$

où  $c_0$  est une fonction définie de  $\mathbb{R}^d$  dans  $\mathbb{R}$  telle que  $c_0(x) = 0$  si  $\|x\| \geq r$  où  $r$  est à peu près la distance entre deux sommets du réseau et  $v_y$  sont des variables aléatoires réelles IID.

### 1.3.2 Version discrétisée du modèle

Il est souvent plus facile de travailler avec une version discrétisée du modèle et qui s'applique bien pour les ondes de spin et pour les électrons dans le modèle des liaisons fortes. On travaille alors sur  $\mathbb{Z}^d$  (ou sur le sous groupe  $G$ ) à la place de  $\mathbb{R}^d$ . Formellement on a toujours

$$H = -\Delta + V_0 + V_\omega, \quad (1.4)$$

mais le Laplacien discret est ici défini par :

$$\forall x, y \in \mathbb{Z}^d : -\Delta(x, y) = \begin{cases} 1 & \text{si } |x - y| = 1 \\ 0 & \text{sinon.} \end{cases}$$

Agissant sur un vecteur  $\phi$ , il donne

$$(-\Delta\phi)(x) = \sum_{|x-y|=1} \phi(y).$$

Pour le potentiel on a

$$V_\omega = \sum_{x \in \mathbb{Z}^d} v_x(\omega) \delta_x$$

qui agit sur un vecteur  $\phi$  comme

$$(V\phi)(x) = V(x)\phi(x).$$

Comme pour le modèle continu, on supposera que les  $v_x$  sont des variables réelles IID.

Dans la suite on travaillera exclusivement sur le modèle discret, cependant la plupart des résultats et théorèmes que l'on énoncera sont également valides pour le modèle continu.

### 1.3.3 Localisation, différentes définitions

Pour résoudre l'équation de Schrödinger il « suffit » de diagonaliser le Hamiltonien  $H$ . Il faut remarquer que pour le modèle continu, dans un domaine borné avec les conditions de Dirichlet le Hamiltonien est bien auto adjoint et donc diagonalisable dans une base orthonormée.

Tout d'abord, il se trouve que l'on peut explicitement diagonaliser  $H$  avec la théorie Bloch-Floquet [60, 119] dans le cas sans désordre

### 1.3. FORMULATION MATHÉMATIQUE ET RÉSULTATS ANTÉRIEURS 15

**Théorème 1.3.** Avec  $V_\omega = 0$ , on peut trouver une base de vecteurs propres généralisés  $\phi_\lambda$ ,  $\lambda$  la valeur propre, de  $H$  défini par (1.4) qui sont quasi-périodiques. C'est-à-dire qu'il existe  $\alpha_\lambda \in \mathbb{R}^d$  tel que pour tout  $K \in G$  et  $x \in \mathbb{R}^d$

$$\phi_\lambda(x + K) = \phi_\lambda(x)e^{i\alpha_\lambda \cdot K}.$$

Ces fonctions sont également réparties sur tout l'espace et on dit qu'elles sont complètement délocalisées. Une conséquence est que les ondes se propagent librement qu'il y ait un potentiel périodique ou non. On parle de transport balistique.

**Définition 1.4.** (Transport balistique) Si  $\phi$  est une solution de (1.2) alors il existe  $C > 0$  tel que pour tout temps  $t$

$$\int_{\mathbb{R}^d} \|x\|^2 |\phi(t, x)|^2 dx \geq Ct^2$$

et

$$\lim_{t \rightarrow \infty} \int_{B(0, r)} |\phi(t, x)|^2 dx = 0$$

pour tout rayon  $r > 0$ .

Ces intégrales sont des sommes pour le modèle discret sur  $\mathbb{Z}^d$ .

Le théorème 1.3 est une conséquence de l'invariance par translation suivant le sous groupe  $G$  du Hamiltonien. Cette invariance est brisée lorsqu'il y a de l'aléa. Au contraire on est capable de prouver la localisation :

**Définition 1.5.** (Localisation spectrale) : On dit que l'on a localisation spectrale (sur  $I \subset \mathbb{R}$ ) si le spectre de  $H$  (restreint à  $I$ ) est purement ponctuel et que les vecteurs propres décroissent exponentiellement depuis leur centre de localisation. C'est à dire, pour tout  $\phi_\lambda$  vecteur propre de  $H$ , il existe  $x_\lambda \in \mathbb{Z}^d$  (le centre) et  $\gamma_\lambda > 0$ ,  $c_\lambda > 0$  tel que

$$|\phi_\lambda(x)| \leq c_\lambda e^{-\gamma_\lambda \|x - x_\lambda\|}$$

pour tout  $x \in \mathbb{Z}^d$ .

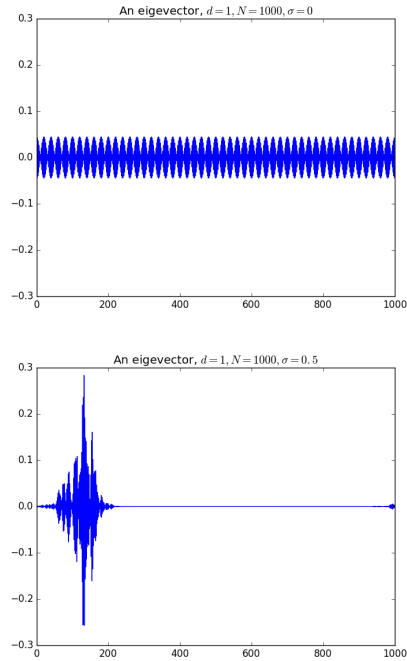
Dans le modèle d'Anderson, il est souvent possible de montrer la disparition de la diffusion lorsque l'on a la localisation spectrale (d'autres hypothèses sont cependant nécessaires), on parle de localisation dynamique.

**Définition 1.6.** (Localisation dynamique) On dit qu'il y a localisation dynamique si pour  $\phi$  solution de (1.2), il existe  $p > 0$  et  $C > 0$  tels que

$$\forall t : \int_{\mathbb{R}^d} \|x\|^p |\phi(t, x)|^2 dx \leq C$$

Les résultats mathématiques plus précis sur l'existence de la localisation d'Anderson sont présentés plus loin à la Section 1.5.

Figure 1.1 – Un vecteur propre quelconque pour le cas sans et avec désordre



### 1.3.4 Quelques illustrations numériques

Voici quelques simulations numériques faites pour le modèle en une dimension sur le réseau  $\mathbb{Z}$  avec un domaine  $[0, N]$ , sans potentiel périodique et avec comme loi la loi uniforme sur  $[-\sigma/2, \sigma/2]$ .

- Dans la Figure 1.1, on a tracé un vecteur propre dans les deux cas : avec et sans désordre.
- Dans les Figures 1.2 et 1.3, on a résolu l'équation de Schrödinger et tracé la solution pour différents temps dans les deux cas : avec désordre et sans désordre.

### 1.3. FORMULATION MATHÉMATIQUE ET RÉSULTATS ANTÉRIEURS 17

Figure 1.2 – L'évolution temporelle de  $\phi$  solution de (1.2) sans désordre.

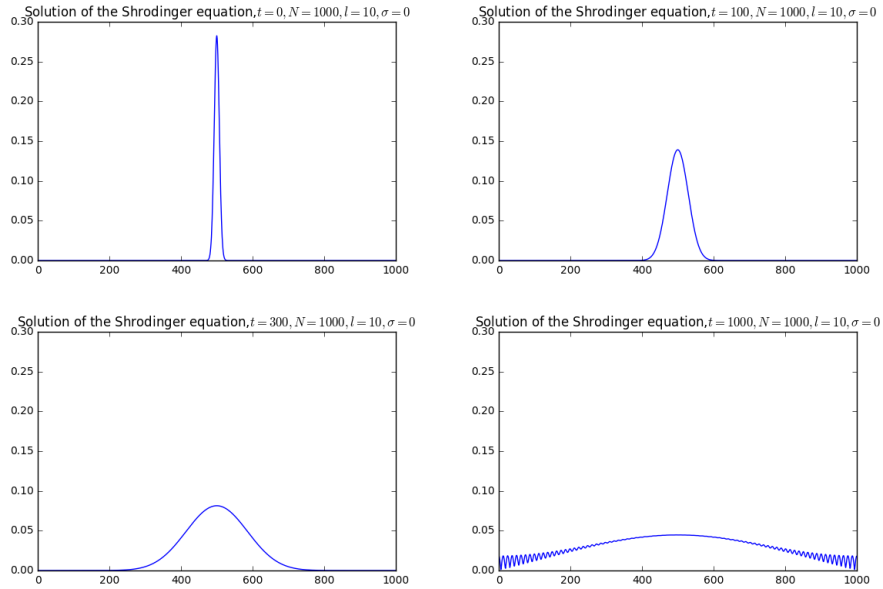
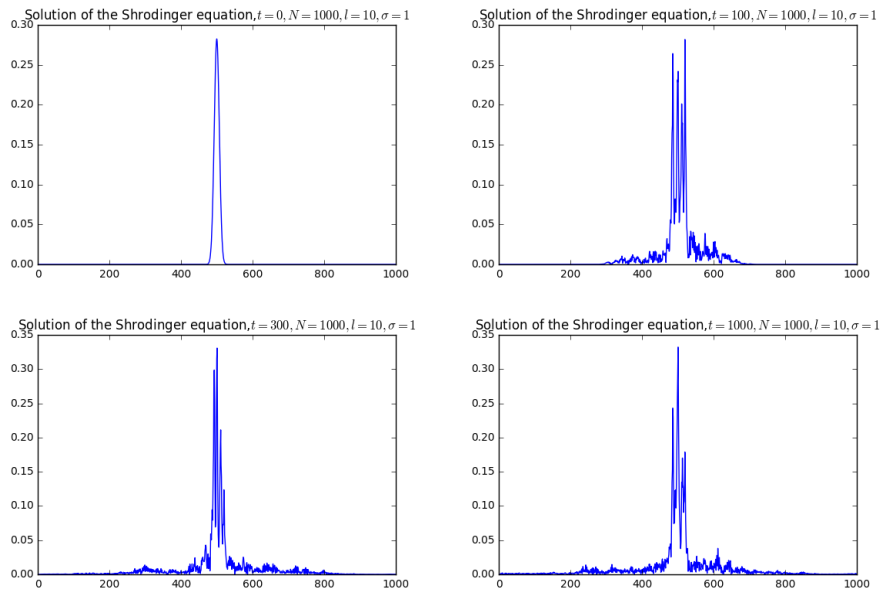


FIGURE 1.3 – L'évolution temporelle de  $\phi$  solution de (1.2) avec désordre.



## 1.4 Statistique des valeurs propres

### 1.4.1 La densité d'états intégrée

Le spectre obtenu en diagonalisant  $H$  est en principe aléatoire. D'habitude, on étudie d'abord le système sur un domaine borné  $\Lambda_L = [0, L]^d$  puis on fait tendre  $L$  vers l'infini. On cherche alors à caractériser la loi limite. Pour cela on définit la mesure spectrale de  $H_\Lambda$  sur  $\mathbb{R}$  comme la somme de fonctions de Dirac sur chacune des valeurs propres de  $H_\Lambda$  :

$$\mu_{H,\Lambda} = \frac{1}{|\Lambda|} \sum_{\lambda \in \sigma(H_\Lambda)} \delta_\lambda.$$

Nous avons les propriétés suivantes :

$$\frac{1}{|\Lambda|} \text{Tr}(1_{[x,y]}(H_\Lambda)) = \mu_{H,\Lambda}([x,y]),$$

$$\frac{1}{|\Lambda|} \text{Tr}(e^{-sH_\Lambda}) = \int_{\mathbb{R}} e^{-sy} \mu_{H,\Lambda}(dy).$$

Le premier résultat est que, comme pour les matrices de Wigner, il existe une mesure spectrale limite vers laquelle on converge avec probabilité 1 [90, 37, 10].

**Théorème 1.7.** (*Densité d'états intégrée*) *Il existe  $\mu_H$  une mesure sur  $\mathbb{R}$  et un ensemble de probabilité 1 tel que pour toute réalisation aléatoire de cet ensemble et pour toute fonction  $f$  continue bornée à support compact dans  $\mathbb{R}$  on a*

$$\int_{\mathbb{R}} f(y) \mu_{H,\Lambda}(dy) \rightarrow \int_{\mathbb{R}} f(y) \mu_H(dy)$$

lorsque  $\Lambda \rightarrow \mathbb{Z}^d$ .

Il est rare que l'on puisse calculer explicitement cette mesure limite  $\mu_H$ , cependant on connaît son support [90, 37] dans les cas suivants pour le modèle discret avec  $\nu$  la loi du potentiel aléatoire dans (1.3) :

**Proposition 1.8.** *Soit  $\mu$  la densité d'états intégrée définie dans Théorème 1.7. On a alors :*

- Si  $\text{Supp}(\nu) = [a, b]$  alors  $\text{Supp}(\mu_H) = \sigma(-\Delta + V_0) + [a, b]$ .
- Si  $V_0 = 0$  alors  $\text{Supp}(\mu_H) = \sigma(-\Delta) + \text{Supp}(\nu)$ .

On rappelle que sur  $\mathbb{Z}^d$   $\sigma(-\Delta) = [-2d, 2d]$

Les extrema du spectre de  $\mu_H$  sont cependant difficile à atteindre. Pour qu'il existe un vecteur propre de  $H_\Lambda$  tel que la valeur propre  $\lambda$  soit très proche du minimum (resp maximum) de  $\text{Supp}(\mu_H)$ , il doit exister un espace suffisamment grand sur lequel tous les  $v_x(\omega)$  sont très petits (resp très grands). Lorsque le domaine s'étend à l'infini il est garanti de trouver un tel espace. Cependant sur un domaine fini cette réalisation obéit à un principe de grande déviation. Cela se traduit pour la densité d'états intégrée par une décroissance très rapide vers zéro aux extrémité du spectre [108].

**Proposition 1.9.** (*Queue de Lifshitz [108]*) *Soit  $E_0$  le minimum du spectre de  $H$ , alors*

$$\lim_{E' \rightarrow E_0^+} \left| \frac{\log(\log(\mu(1_{[E_0, E']})))}{\log(|E' - E_0|)} \right| = \frac{d}{2}.$$

Ainsi, le nombre de valeurs propres entre  $E_0$  et  $E'$  décroît comme  $e^{-\alpha|E' - E_0|^{-\frac{d}{2}}}$ .

### 1.4.2 La régularité de la mesure limite

Il s'agit d'étudier des propriétés de la mesure  $\mu_H$  définie au Théorème 1.7. Il est évident qu'elles vont dépendre de la forme de la loi aléatoire que l'on choisit pour  $V_\omega$ . Une des propriétés intéressantes est que si  $v(\omega)$  est une variable aléatoire continue alors  $\mu_H$  aussi. Plus précisément nous disposons des deux estimées de Wegner et de Minami [132, 112] qui s'expriment ainsi :

**Proposition 1.10.** (Estimée de Wegner [132]) *Si  $\nu$  est une mesure continue avec une densité borné  $\rho$ , alors*

- $\mathbb{P}(\exists \lambda \in \sigma(H_\Lambda), \lambda \in [a, b]) \leq |\Lambda| \|\rho\|_{L^\infty} |b - a|,$
- $\mathbb{E}(\int_{\mathbb{R}} 1_{[a, b]}(y) \mu_{H, \Lambda}(dy)) \leq \|\rho\|_{L^\infty},$
- $\mu_H$  est absolument continue par rapport à la mesure de Lebesgue avec une densité bornée  $\xi$  telle que  $\|\xi\|_{L^\infty} \leq \|\rho\|_{L^\infty}.$

L'estimée de Wegner ne précise pas comment les valeurs propres s'organisent les unes par rapport aux autres. On dispose cependant de l'estimée de Minami qui a la conséquence suivante : la probabilité que deux valeurs propres se superposent est nulle.

**Proposition 1.11.** (Estimée de Minami [112]) *Avec les mêmes hypothèses que pour l'estimée de Wegner, on a*

$$\mathbb{P}(\exists \lambda_1, \lambda_2 \in \sigma(H_\Lambda) \cap [a, b]) \leq |b - a|^2 \|\rho\|_{L^\infty}^2 |\Lambda|^2.$$

### 1.4.3 Statistiques locales des valeurs propres

On s'intéresse maintenant à la mesure  $\mu_{\Lambda, H}$  dans le détail. On choisit une énergie  $E$  et on zoom vers cette énergie pour observer la répartition microscopique des valeurs propres autour. Plus formellement on définit [112]

**Définition 1.12.** (Mesure locale) Soit  $E \in \mathbb{R}$ , on pose la mesure sur  $\mathbb{R}$  suivante :

$$\varpi_{H, \Lambda, E} = \sum_{\lambda \in \sigma(H_\Lambda)} \delta_{|\Lambda|(\lambda - E)}$$

Comme pour les matrices de Wigner on peut montrer que la loi aléatoire qui donne cette mesure converge lorsque  $\Lambda$  s'étend vers  $\mathbb{Z}^d$ . La question centrale est maintenant de caractériser cette loi limite. En général c'est extrêmement compliqué et c'est encore aujourd'hui une question ouverte. Lorsqu'il n'y a pas localisation d'Anderson, on s'attendrait à ce que les valeurs propres se repoussent entre elles. Plus formellement on pense [57, 96] que

**Conjecture 1.13.** *Lorsqu'il n'y a pas la localisation d'Anderson la loi de  $\varpi_{H, \Lambda, E}$  converge vers la loi  $\text{sine}_\beta$ ,  $\beta = 2$ , qui apparaît pour la loi locale des matrices de Wigner.*

En présence de la localisation d'Anderson par contre c'est plus clair. Les vecteurs propres étant fortement localisés, ils ne dépendent que des réalisations du potentiel aléatoire autour de leur centre de localisation. Ainsi pour deux centres de localisation éloignés l'un de l'autre les valeurs propres associées sont indépendantes. Au final on obtient une mesure de Poisson [112, 65].

**Définition 1.14.** (Statistique locale en présence de la localisation) La loi de la mesure locale  $\varpi_{H,\Lambda,E}$  converge vers la loi de Poisson de paramètre  $\xi(E)$  lorsque  $\Lambda$  s'étend sur  $\mathbb{Z}^d$ .

#### 1.4.4 Quelques illustrations par des simulations numériques

On garde le même modèle que les simulations précédentes.

- Sur la Figure 1.4, on trace la densité des valeurs propres sans et avec le désordre.
- Sur la Figure 1.5, on trace la loi de  $\lambda_i - \lambda_{i+1}$ . Si ces deux valeurs propres ne s'influencent en aucune manière, on devrait obtenir une loi exponentielle. C'est ce que l'on observe avec un fort désordre, lorsqu'il y a localisation.
- Sur la Figure 1.6, on a compté le nombre de valeurs propres dans un intervalle donné, que l'on compare avec ce que donne la loi de Poisson.

Figure 1.4 – Densité d'états intégrée avec et sans désordre.

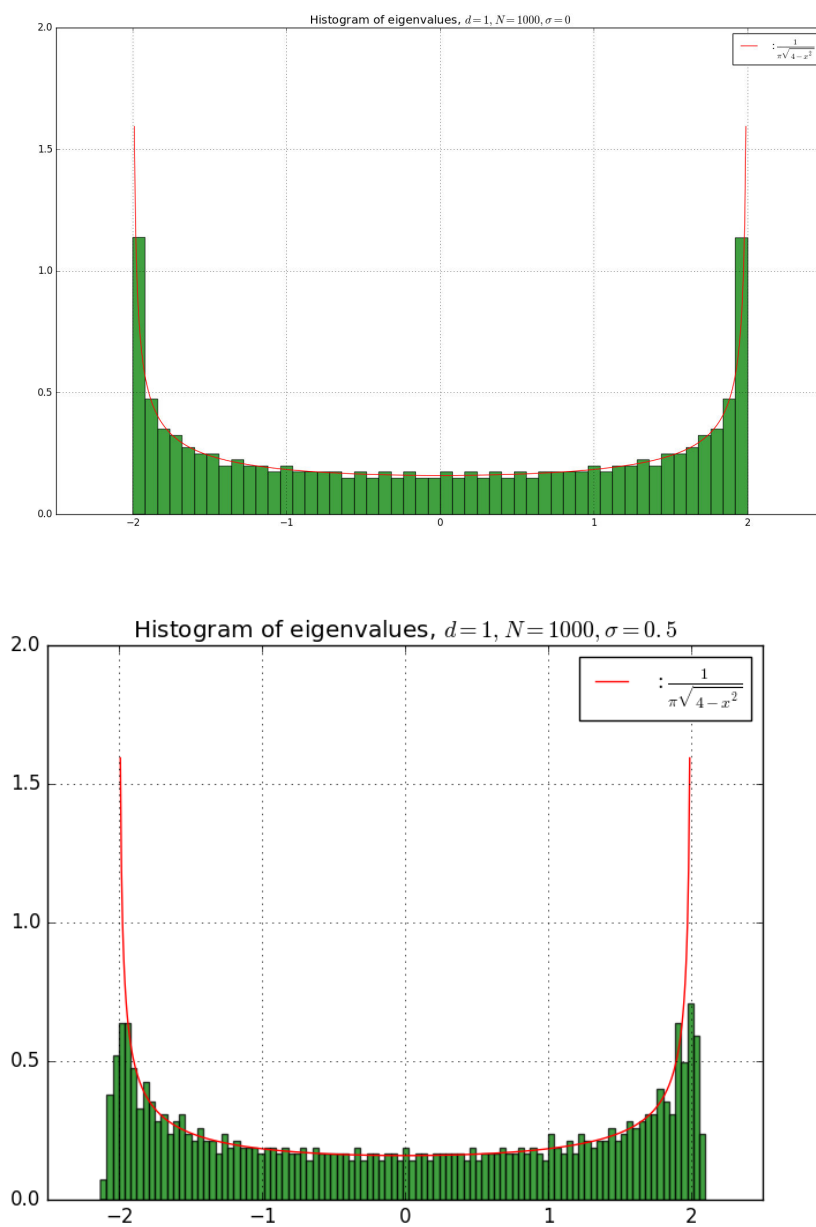




Figure 1.5 – Statistique de la différence entre deux valeurs propres consécutives, pour trois valeurs de  $\sigma$  qui détermine l'amplitude du désordre

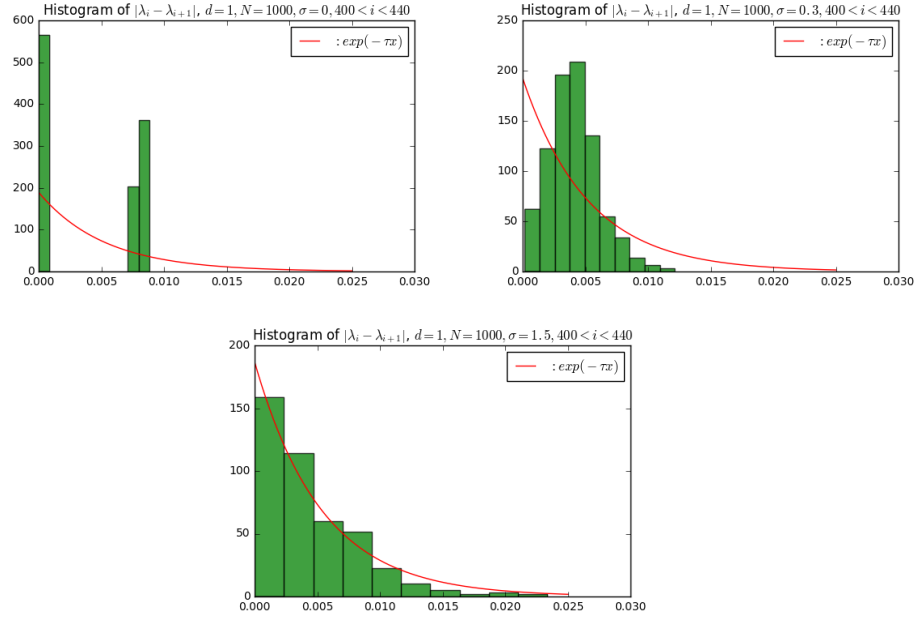
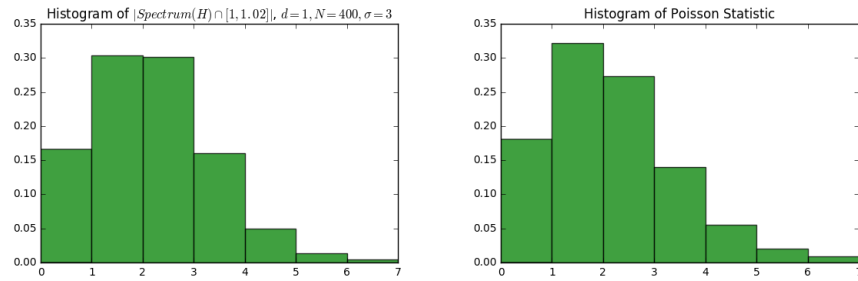


Figure 1.6 – Nombre de vecteurs propres dans un intervalle donné en présence de fort désordre et comparaison avec la loi de Poisson.



## 1.5 Localisation d'Anderson en toute dimension

Les premières techniques mathématiques qui permirent de démontrer l'existence de la localisation d'Anderson pour un système de dimension  $d \geq 2$  furent celles développée par Fröhlich et Spencer [62] et que l'on appelle analyse multi-échelle. Ces techniques sont basées sur l'étude de la résolvante  $(H - z)^{-1}$  pour  $z \in \mathbb{C}$  sur laquelle on prouve des estimées valables avec grande probabilité. Une méthode plus simple fut ensuite développée par Aizenman et Molchanov [8]. Elle permet d'obtenir des estimées sur les moments fractionnaires de  $(H - z)^{-1}$ . Plus particulièrement, il s'agit de montrer

**Définition 1.15.** (Localisation par moment fractionnaire) Il existe  $0 < s < 1$ ,  $\gamma > 0$  et  $C > 0$  tels que pour

$$\mathbb{E}[|(H - z)^{-1}(x, y)|^s] \leq e^{-\gamma\|x-y\|}$$

pour tout  $x, y$  dans  $\mathbb{Z}^d$  et pour tout  $z \in A \subset \mathbb{C}$ .

Les estimées sur la résolvante donnent souvent une bonne idée de la forme des vecteurs propres. En effet si  $z$  converge vers une valeur propre  $\lambda$  et  $\phi_\lambda$  est le vecteur propre correspondant alors  $(z - \lambda)[(H - z)^{-1}(x, y)] \rightarrow \bar{\phi}_\lambda(x)\phi_\lambda(y)$ . À partir de la décroissance exponentielle de la résolvante, on pourra prouver la décroissance exponentielle des vecteurs propres.

Il est possible d'obtenir l'estimée de la définition 1.15 lorsque la densité des valeurs propres est très faible. C'est le cas lorsque  $\nu$  est continue avec une densité  $\rho$  telle que  $\|\rho\|_{L^\infty}$  suffisamment petit, par la Proposition 1.10 ou dans le bas du spectre par la Proposition 1.9. Plus précisément il est possible de prouver le théorème suivant [62, 8, 10, 44, 112, 90, 37, 92] :

**Théorème 1.16.** *Pour  $\nu$  continue avec une densité bornée  $\rho$ , alors*

- *Il existe  $E' > E_0$  tel qu'on a l'ensemble des résultats suivants :*
  - *l'estimée de la localisation par moment fractionnaire pour tout  $z$  avec  $\Re(z) \leq E'$ ,*
  - *sur l'intervalle  $[E_0, E']$  on a la localisation spectrale avec décroissance exponentielle des vecteurs propres,*
  - *sur l'intervalle  $[E_0, E']$  on a la localisation dynamique,*
  - *pour tout  $E \in ]E_0, E'[$ , la statistique locale converge vers un processus ponctuel de Poisson.*
- *De plus, il existe  $\epsilon > 0$  tel que si  $\|\rho\|_{L^\infty} \leq \epsilon$ , alors on a l'ensemble des résultats suivant :*
  - *l'estimée de la localisation par moment fractionnaire est valable pour tout  $z \in \mathbb{C}$ ,*
  - *sur  $\mathbb{R}$ , on a la localisation spectrale avec décroissance exponentielle des vecteurs propres,*
  - *sur  $\mathbb{R}$ , on a la localisation dynamique.*
  - *pour tout  $E \in \text{Supp}(\mu_H)$ , la statistique locale converge vers un processus ponctuel de Poisson.*

Notre travail a consisté à étendre ces résultats à diverses situations (non linéaire, avec potentiel dépendant du temps, etc...).

## 1.6 Résultat du Chapitre 2 : le modèle d'Anderson à une dimension

Le modèle d'Anderson à une dimension est fondamentalement plus simple qu'en dimension quelconque et beaucoup plus de résultats sont connus. C'est sur ce cas ci qu'on a pu construire la première preuve de la localisation d'Anderson [98] et il fut depuis extensivement étudié [35, 37, 127]. La remarque cruciale ici est qu'il est possible d'exprimer explicitement les vecteurs propres avec une formule récursive. En effet, si  $\phi_\lambda$  est un vecteur propre de valeur propre  $\lambda$  alors il satisfait pour tout  $n \geq 1$

$$\phi_\lambda(n+1) = (V_\omega(n) - \lambda)\phi_\lambda(n) - \phi_\lambda(n-1) \quad (1.5)$$

et on peut alors écrire

$$\begin{pmatrix} \phi_\lambda(n+1) \\ \phi_\lambda(n) \end{pmatrix} = T_\lambda(V_\omega(n)) \cdots T_\lambda(V_\omega(2)) T_\lambda(V_\omega(1)) \begin{pmatrix} \phi_\lambda(1) \\ \phi_\lambda(0) \end{pmatrix} \quad (1.6)$$

où on a défini  $T_\lambda(x) = \begin{pmatrix} x - \lambda & -1 \\ 1 & 0 \end{pmatrix}$ . L'étude des vecteurs propres se ramène donc aux propriétés d'un produit de matrices aléatoires  $\prod_{k=1}^n T_\lambda(V_\omega(k)) = T_\lambda(V_\omega(n)) \cdots T_\lambda(V_\omega(1))$  (sauf précision du contraire le produit  $\prod$  se fera toujours de la droite vers la gauche) sur lequel toute une théorie a été développée [64, 63, 103, 27].

### 1.6.1 Lois limites sur la norme d'un produit de matrices aléatoires indépendantes

Les premières motivations pour des produits de matrices aléatoires viennent des systèmes dynamiques. Ils apparaissent naturellement lorsque l'on calcule la Jacobienne d'une évolution d'un modèle ergodique [121].

On écrit  $M_n = \prod_{i=1}^n T_i$  avec  $T_i$  des matrices aléatoires indépendantes et identiquement distribuées et on pose ensuite

$$S_n = \log(\|M_n x\|)$$

On pourra supposer que  $\det T_i = 1$  presque sûrement et utiliser  $\det(M_n) = \prod_{i=1}^n \det(T_i)$  pour le cas général. On supposera également que  $\|T_i\|$  et  $\|T_i^{-1}\|$  sont bornés sur le support de la loi aléatoire. Nous nous renvoyons à [103] pour les hypothèses précises sur les  $T_i$  pour lesquelles les résultats suivants s'appliquent. Nous avons l'heuristique suivante

$$S_n = \log \|M_n x\| = \log \left( \prod_{i=0}^{n-1} \frac{\|M_{i+1} x\|}{\|M_i x\|} \right) = \sum_{i=0}^{n-1} \log \left( \left\| T_{i+1} \frac{M_i x}{\|M_i x\|} \right\| \right)$$

qui fait apparaître une somme de variables aléatoires  $Y_i = \log(\|T_{i+1} \frac{M_i x}{\|M_i x\|}\|)$ . Ces variables ne sont pas indépendantes mais on peut montrer des propriétés de mélange pour  $\frac{M_i x}{\|M_i x\|}$ . Les variables  $Y_i, Y_j$  deviennent alors presque indépendantes pour  $|i - j|$  suffisamment grand et on peut alors montrer les théorèmes limites classiques. Plus précisément, on peut montrer les théorèmes suivants [103] :

**Théorème 1.17.** (Loi forte des grands nombres) Il existe  $\gamma \in \mathbb{R}$  (l'exposant de Lyapunov) tel que pour tout  $x \in \mathbb{S}^n$  on a

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|M_n x\| \rightarrow \gamma$$

presque sûrement.

**Théorème 1.18.** (Théorème central limite) Il existe  $\sigma \in \mathbb{R}$  tel que pour tout  $x \in \mathbb{S}^n$  on a la convergence en loi pour  $n \rightarrow \infty$

$$\frac{1}{\sqrt{n}} (\log \|M_n x\| - \gamma n) \rightarrow \mathcal{N}(0, \sigma^2).$$

Et enfin

**Théorème 1.19.** (Théorème de Donsker) On pose la « marche aléatoire » pour  $t \in [0, T]$

$$X_n(t) = \frac{1}{\sqrt{n}} (\log \|M_{[nt]} x\| - \gamma [nt]).$$

Alors  $X_n(t)$  converge en loi vers  $\sigma B(t)$  où  $B$  est un mouvement Brownien.

### 1.6.2 Localisation d'Anderson pour le modèle à une dimension

On revient maintenant au modèle d'Anderson, où les matrices  $T_i$  sont celles de l'équation (1.6). On a la convergence limite vers l'exposant de Lyapunov

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log (\| \prod_{k=1}^n T_\lambda(V_\omega(k)) \|) = \gamma(\lambda).$$

Pour calculer  $\gamma(\lambda)$ , on dispose des deux formules suivantes [37] :

**Proposition 1.20.** On note  $\zeta$  la mesure sur  $\mathbb{S}^1$  invariante pour le processus de Markov  $\frac{M_n x}{\|M_n x\|}$ . On a alors

$$\gamma(\lambda) = \mathbb{E} \left[ \int_{\mathbb{S}^1} \log \|T_i x\| d\zeta(x) \right].$$

(Formule de Thouless) Avec  $\mu$  la densité d'états intégrée définie dans le théorème 1.7, on a

$$\gamma(\lambda) = \int_{\sigma(H)} \log(|\lambda - y|) d\mu(y).$$

En pratique, ces deux formules sont difficilement utilisables car la mesure invariante  $\zeta$  et la densité d'état intégrée ne sont pas explicites.

On déduit la localisation spectrale à partir de la stricte positivité de l'exposant de Lyapunov et en effet on a le théorème suivant [35, 36, 103]

**Théorème 1.21.** Si la loi aléatoire  $\nu$  n'est pas dégénérée, alors  $\gamma(\lambda) > 0$ .

On a alors [35, 36] le

**Théorème 1.22.** *Si la loi  $\nu$  n'est pas dégénérée, alors on a la localisation spectrale, les vecteurs propres  $\phi_\lambda$  décroissent de manière exponentielle de paramètre  $\gamma(\lambda)$ , la localisation dynamique et la statistique locale des valeurs propres converge vers une loi de Poisson.*

Dans [52] et en utilisant les théorèmes précédents on montre que pour  $n$  loin du centre de localisation  $n_\lambda$  on a

$$\log(|\phi_\lambda(n)|^2 + |\phi_\lambda(n+1)|^2) \approx -\gamma(\lambda)|n - n_\lambda| + \sigma(\lambda)B_{n-n_\lambda}$$

où  $B_t$  est un mouvement brownien sur  $\mathbb{R}$ . Nous y reviendrons dans la partie suivante.

### 1.6.3 Processus avant-arrière pour la construction du vecteur propre

Dans notre article [52], nous avons revisité ce modèle sur un domaine fini  $[0, N]$  avec une nouvelle méthode permettant de retrouver rapidement la loi d'un vecteur propre  $\phi_\lambda$  conditionnellement à ce que  $\lambda$  soit une valeur propre. Comme précédemment on peut définir un processus « vers l'avant » où  $\phi_\lambda^f(k)$  est construit par le produit de

$$T_k \cdots T_2 T_1 \begin{pmatrix} \phi_\lambda(1) \\ \phi_\lambda(0) \end{pmatrix}$$

mais on peut définir également un processus « vers l'arrière » avec  $\phi_\lambda^b(k)$  construit par le produit

$$T_{k+1}^{-1} \cdots T_{N-2}^{-1} T_{N-1}^{-1} \begin{pmatrix} \phi_\lambda(N) \\ \phi_\lambda(N-1) \end{pmatrix}$$

en réécrivant la formule (1.5)

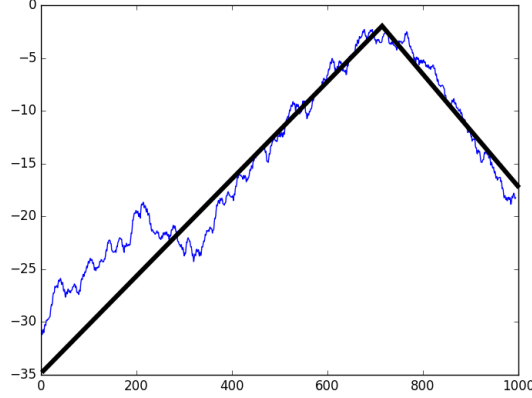
$$\phi_\lambda(n-1) = (V_\omega(n) - \lambda)\phi_\lambda(n) - \phi_\lambda(n+1)$$

et en faisant l'itération en partant de  $N$ . Remarquer que les produits de matrices aléatoires indépendantes donnent des comportements complètement différents selon le sens dans lequel on les réalise. Par exemple, avec les théorèmes précédents on devrait avoir  $|\phi_\lambda^f(k)| \sim c^f e^{\gamma(\lambda)k}$  alors que  $|\phi_\lambda^b(k)| \sim c^b e^{\gamma(N-k)}$  (les dérivées sont inversées). Mais de fait, les matrices  $T_i$  ne sont pas indépendantes puisqu'elles dépendent de la valeur propre  $\lambda$ , qui dépend de tous les  $V_\omega(k)$ . Dans [52], pour construire  $\phi_\lambda$  conditionnellement à  $\lambda$  valeur propre, nous avons remarqué qu'il faut ajouter une « coupure » aléatoire  $n_\lambda$  sur  $[1, N]$  qui de fait s'identifie avec le centre de localisation et  $\phi_\lambda$  est alors donné par

$$\phi_\lambda(k) = \begin{cases} \phi_\lambda^f(k) & \text{si } k \leq n_\lambda \\ \phi_\lambda^b(k) & \text{si } k \geq n_\lambda, \end{cases} \quad (1.7)$$

où on a choisi  $\phi_\lambda(1)$  et  $\phi_\lambda(N-1)$  de telle sorte qu'il n'y ait pas d'incohérence :  $\phi_\lambda^b(n_\lambda) = \phi_\lambda^f(n_\lambda)$  et  $\sum |\phi_\lambda(k)|^2 = 1$ .

Nous avons montré que

FIGURE 1.7 – Tracé des fonctions  $\log(|\phi_\lambda(n)|^2 + |\phi_\lambda(n+1)|^2)$  et de  $\gamma|n - n_\lambda|$ 

**Proposition 1.23.** *On a pour toute fonction lisse  $G : \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$*

$$\begin{aligned} & \mathbb{E} \left[ \frac{1}{N} \sum_{\lambda \in \sigma(H)} G(\lambda, \phi_\lambda) \right] \\ &= \int_{\mathbb{R}} \rho(\lambda) \left( \frac{1}{N} \sum_{n_\lambda=1}^N \mathbb{E}_{\mathcal{P}_{f,1..n_\lambda} \otimes \mathcal{P}_{b,n_\lambda+1,\dots,N}} \left[ G(\lambda, \phi_\lambda) \frac{\delta_{\alpha_{n_\lambda}^f - \alpha_{n_\lambda}^b[\pi]} \sin^2(\alpha_{n_\lambda}^f)}{\rho(\lambda)} \right] \right) d\lambda \end{aligned} \quad (1.8)$$

où  $\phi_\lambda$  sont les vecteurs propres de  $H$ ,  $\rho$  est la densité d'état,  $\mathcal{P}_{f,1..n_\lambda} \otimes \mathcal{P}_{b,n_\lambda+1,\dots,N}$  est la loi aléatoire pour  $\phi_\lambda$  construite par la formule (1.7), et  $\alpha_{n_\lambda}^f$  l'angle que forme le vecteur  $\begin{pmatrix} \phi_\lambda^f(k) \\ \phi_\lambda^f(k-1) \end{pmatrix}$  par rapport à l'axe des abscisses.

Il faut voir les fonctions  $G$  comme des fonctions test dont les espérances décrivent exhaustivement la loi aléatoire d'une fonction propre. L'équation (1.8) se comprend ainsi : la construction donnée par (1.7) conditionnellement au bon recollement des deux processus avant et arrière et avec un facteur correctif en  $\sin^2$  donne exactement la loi aléatoire d'un vecteur propre.

En utilisant cette formule on a pu retrouver les résultats connus sur la localisation en une dimension, et prouver la formule sur les fluctuations browniennes (1.6.2).

Dans la figure 1.7, nous avons tracé les fonctions  $\log(|\phi_\lambda(n)|^2 + |\phi_\lambda(n+1)|^2)$  et  $\gamma|n - n_\lambda|$ , faisant apparaître la coupure en  $n_\lambda$ , l'accroissement linéaire correspondant à l'exposant de Lyapunov et la correction qui est proche du mouvement Brownien.

## 1.7 Résultats du Chapitre 3 : Modèle d'Anderson avec plusieurs particules sous l'approximation de Hartree-Fock

### 1.7.1 L'approximation de Hartree-Fock.

L'approximation de Hartree-Fock est très utilisée en chimie [104]. Nous nous renvoyons à [106], pour une discussion plus mathématique. Nous présentons ici l'approche proposée par Cancès, Deleurence, Lahbabi et Lewin [100, 32]

Considérons un système quantique de  $N$  fermions. Ce système est décrit sur l'ensemble des fonctions antisymétriques à  $N$  variables sur le domaine  $\Omega$ . La dynamique est donnée par l'équation de Schrodinger à  $N$  particules avec le Hamiltonien

$$H_N = \sum_{i=1}^N -\Delta_i + \sum_{i=1}^N V(x_i) + \frac{1}{2} \sum_{i \neq j} w(x_i - x_j)$$

où  $w(x_i - x_j)$  est l'interaction entre la particule située en  $x_i$  et la particule située en  $x_j$ . L'approximation de Hartree Fock consiste à ne considérer que des états quantiques  $\psi$  qui peuvent s'écrire comme le déterminant de Slater

$$\Phi_N(x_1, \dots, x_N) = \frac{1}{\sqrt{N!}} \det(\phi_i(x_j))$$

avec  $\phi_1, \dots, \phi_N$   $N$  fonctions sur  $\Omega$  orthonormées. L'énergie du système s'écrit alors comme

$$\begin{aligned} E_\Phi &= \langle \Phi_N H_N \Phi_N \rangle \\ &= \sum_{i=1}^N \int_{\Omega} |\nabla \phi_i|^2 + \int_{\Omega} V(x) \rho_\Phi + \frac{1}{2} \iint_{\Omega^2} \rho_\Phi(x) \rho_\Phi(y) w(x-y) dx dy \\ &\quad - \frac{1}{2} \iint_{\Omega^2} w(x-y) |\tau_\Phi(x,y)|^2 dx dy \end{aligned}$$

avec  $\rho_\Phi(x) = \sum_i |\phi_i(x)|^2$  et  $\tau_\Phi = \sum \phi_i(x) \bar{\phi}_i(y)$ . Le premier terme est l'énergie cinétique, le deuxième terme est l'influence du potentiel extérieur (il ne dépend que de la densité de particule  $\rho_\Phi$ ) le troisième terme est un terme de champ moyen pour les interactions entre les particules se trouvant en  $x$  et  $y$ , enfin le dernier terme est purement quantique. On l'appelle le terme d'échange et il illustre le fait qu'une particule n'interagit pas avec elle même. En notant  $\gamma$  le projecteur sur l'espace engendré par les  $\phi_i$  on peut réécrire l'énergie sous la forme

$$E_\Phi = \text{Tr}((-\Delta + V)\gamma) + \frac{1}{2} \iint \rho_\gamma(x) \rho_\gamma(y) w(x-y) dx dy - \frac{1}{2} \iint w(x-y) |\tau_\gamma(x,y)|^2 dx dy$$

avec  $\rho_\gamma(x) = \text{Tr}(\delta_x \gamma)$  et  $\tau_\gamma(x,y) = \langle \delta_x, \gamma \delta_y \rangle$ .

Ce qui intéresse le plus les chimistes et les physiciens est l'état qui minimise cette énergie. Une propriété intéressante pour trouver le minimum est la proposition suivante :

**Proposition 1.24.** *Soit  $\gamma$  un minimiseur de  $E_{\Phi}$  alors il existe  $\epsilon_F$  (l'énergie de Fermi) tel que*

$$\begin{cases} \gamma = 1_{H_{\gamma} \leq \epsilon_F} \\ H_{\gamma} = -\Delta + V + A_{\text{eff}}(\gamma) \end{cases} \quad (1.9)$$

où  $A_{\text{eff}}$  est défini par

$$(A_{\text{eff}}(\gamma)\psi)(x) = \psi(x) \int w(x-y)\rho_{\gamma}(y)dy - \int \psi(y)w(y-x)\tau_{\gamma}(x,y)dy$$

pour tout  $\psi \in L^2(\Omega)$ .

### 1.7.2 Localisation d'Anderson pour l'état fondamental dans l'approximation de Hartree-Fock.

Dans notre article [53], nous avons étudié la localisation d'Anderson pour l'état fondamental d'un système de fermions sur  $\mathbb{Z}^d$  dans le modèle de Hartree-Fock. Une hypothèse importante est l'existence d'un trou dans le spectre de  $-\Delta + V$  et qu'il y a autant de particules que de valeurs propres de  $H$  sous ce trou spectral. Dans ce cas on a pu prouver l'unicité du minimum d'énergie. On a

**Proposition 1.25.** *Si il y a un trou dans le spectre de  $-\Delta + V$  suffisamment grand par rapport à  $\|w\|_{\ell^1}$ , alors il existe une unique solution au système d'équation (1.9) avec  $\epsilon_F$  au milieu du trou spectral.*

Cette proposition nous donne un Hamiltonien effectif  $H_{\gamma_{\min}}$ . Cet Hamiltonien se décompose en le terme « linéaire »  $-\Delta + V$  et le terme  $A_{\text{eff}}(\gamma_{\min})$ . Ce terme est le résultat de la minimisation de l'énergie globale et dépend donc de la réalisation aléatoire du potentiel. En particulier, on s'attend à ce qu'il écrante les impuretés et lisse les irrégularités. La problématique est la suivante : ce nouveau terme détruit-il la localisation ? Nous avons pu énoncer des hypothèses sous lesquelles la localisation est conservée. Plus précisément on a le

**Théorème 1.26.** *Si le trou spectral est suffisamment grand par rapport à  $\|w\|_{\ell^1}$  et avec les mêmes hypothèses que le théorème 1.16 on a les mêmes résultats de localisation spectrale.*

A notre connaissance, c'est le premier résultat de localisation pour ce modèle, et l'un des premiers pour les systèmes infinis en interaction.

## 1.8 Chapitre 4 : Modèle d'Anderson avec une perturbation périodique en temps

Lorsque l'on observe la localisation d'Anderson, les électrons ne se propagent pas et la conductivité en courant continu du matériau disparaît alors à température nulle. Pour la conductivité en courant alternatif, on doit s'intéresser à un Hamiltonien qui dépend du temps. On écrit l'équation de Schrödinger

$$i\partial_t\phi = H(t)\phi$$



avec  $H(t) = -\Delta + V_\omega + h(t)$  et  $h(t)$  périodique en temps. À première vue, on s'attendrait à ce que l'oscillation induise des sauts entre les différents états de localisation. Cependant, il n'est pas évident de savoir si pour les temps longs l'onde restera localisée ou diffusera à l'infini.

### 1.8.1 Opérateur dans l'espace de Floquet

L'idée pour traiter le cas périodique en temps est de travailler non pas avec des fonctions de  $L^2(\Omega)$  mais sur un espace étendu  $\mathcal{K} = L^2([0, T] \times \Omega)$  et d'étudier l'opérateur

$$K = i\partial_t - H(t)$$

On réalise la transformée de Fourier sur  $[0, T]$  de cet opérateur pour obtenir  $\hat{K} = k\nu - \hat{H}(k)$  avec  $k \in \mathbb{Z}$  et  $\nu = \frac{2\pi}{T}$  qui agit sur  $L^2(\mathbb{Z} \times \Omega)$ . Pour se convaincre que c'est le bon opérateur à étudier, réalisons la transformée de Fourier sur le dernier intervalle temporel de longueur  $T$

$$\hat{\phi}(x, t, k) = \frac{1}{T} \int_t^{t+T} \phi(x, u) e^{-i\nu k u} du.$$

Alors on peut vérifier que

$$i\partial_t \hat{\phi} = \hat{K} \hat{\phi}.$$

Il s'agit ensuite d'étudier les propriétés spectrales de  $\hat{K}$ .

**Proposition 1.27.** ([128]) *Si  $\hat{K}$  a un spectre purement ponctuel, alors pour tout  $\epsilon > 0$ , il existe  $R > 0$  tel que*

$$\sup_t \sum_{|x| > R} |\phi(x, t)|^2 < \epsilon.$$

Les premiers résultats sur les modèles périodiques et quasi-périodiques sont dus à Bourgain, Soffer et Wang [28, 128] qui étudiaient ce cas comme un modèle simplifié pour comprendre l'influence de l'oscillation d'une poignée d'électrons entre quelques états sur le reste du système. Pour le cas quasi périodique on a  $h(t) = \sum_{i=1}^P \cos(\omega_i t) h_i$ , et il faut augmenter autant l'espace à étudier

$$\mathcal{K}_P = L^2([0, T_1] \times [0, T_2] \times \cdots \times [0, T_P] \times \Omega)$$

avec  $\omega_i T_i = 2\pi$  et on pose

$$K_P = \sum_{j=1}^P i\partial_{\theta_j} - \Delta + V_\omega + \sum_{j=1}^P \cos(\omega_j \theta_j) h_j.$$

Lorsque les  $h_i$  sont localisés dans un petit domaine autour de  $x = 0$ , Bourgain et Wang [28] ont prouvé le théorème suivant

**Theorem 1.28.** [28] *Pour*

$$K_P = \sum_{j=1}^P i\partial_{\theta_j} - \epsilon\Delta + V_\omega + \sum_{j=1}^P \cos(\omega_j \theta_j) h_j.$$

*la probabilité d'avoir la localisation spectrale pour  $K_P$  tend vers 1 lorsque  $\epsilon \rightarrow 0$  (grand désordre).*

### 1.8.2 Absence de diffusion avec une perturbation agissant sur tout l'espace

Nous énonçons ici les résultats que nous avons pu obtenir dans notre article [54].

La principale différence est que l'on reste dans le cas périodique et on retire l'hypothèse que le  $h_i$  agit sur un petit domaine. L'équivalent du  $K_P$  de Bourgain et Wang est notre opérateur

$$\hat{H} = -g\hat{\Delta} + \hat{V}_\omega$$

et nous renvoyons le lecteur à 4.2.1 pour une définition précise.

**Théorème 1.29.** *Si le désordre est suffisamment grand ( $g \leq \epsilon$ ), alors on a les résultats suivants presque sûrement :*

- *Si  $\|h(t)\|$  est dans  $L^2[0, T]$  et  $\frac{1}{T} \geq 1$ , on a la localisation spectrale pour  $\hat{H}$*
- *Si  $\hat{\Delta}$  n'admet qu'un nombre fini de coefficients de Fourier non nuls et  $\frac{1}{T} \geq e^{-g^{-\xi}}$  avec  $\xi > 0$  qui dépend de ce nombre de coefficients, on a la localisation spectrale pour  $\hat{H}$*

La conclusion de ce résultat est que sous les conditions énoncées dans le théorème, la perturbation périodique ne permet pas à l'électron de diffuser à l'infini (corollaire 4.2.1).

## 1.9 Chapitre 5 : Modèle du Jellium inhomogène à une dimension

### 1.9.1 Présentation du modèle

Nous nous intéressons ici à un modèle de particules chargées interagissant les unes avec les autres. Nous plaçons ces particules dans un milieu lui aussi chargé électriquement de sorte que l'ensemble total forme ainsi un système électriquement neutre. Nous souhaiterions comprendre la statistique thermodynamique de ce système et, plus précisément, comment se répartissent les particules les unes par rapport aux autres à une température non nulle. Nous n'étudions ici que le modèle à une dimension. L'interaction de Coulomb à une dimension entre deux particules en  $x$  et  $y$  est donnée par  $-\frac{1}{2}|x - y|$  (la solution de  $-\Delta V = \delta_0$ ). L'énergie totale du système à  $N$  particules de charges  $q_i$  sur le segment  $[-L, L]$  avec une densité de charge  $\rho$  est alors donnée par

$$E(x_1, x_2, \dots, x_N) = -\frac{1}{2} \iint_{[-L, L]^2} \rho(y_1)\rho(y_2)|y_1 - y_2| dy_1 dy_2 - \frac{1}{2} \sum_{i,j} q_i q_j |x_i - x_j| + \sum_i \int_{-L}^L \rho(y) q_i |x_i - y| dy. \quad (1.10)$$

Il s'agit maintenant d'étudier ce système à une température non nulle  $T = \frac{1}{\beta}$  et on se placera dans l'ensemble canonique. La position  $(x_i)_{1 \leq i \leq N}$  des particules est aléatoire et la loi de probabilité de trouver une certaine configuration  $(x_i)_{1 \leq i \leq N}$  est proportionnelle à  $e^{-\beta E(x_1, \dots, x_N)}$ . On étudie en particulier les objets suivant :

— la fonction de partition

$$Z(\beta) = \int \dots \int e^{-\beta E(x_1, \dots, x_N)} \prod_{i=1}^N dx_i,$$

— l'énergie libre par particule

$$f(\beta) = -\frac{1}{N\beta} \log(Z(\beta)),$$

— les fonctions de corrélation à  $k$  points

$$\rho_k(x_1, \dots, x_k) = \frac{1}{Z(\beta)} \int \dots \int e^{-\beta E((x_i)_{1 \leq i \leq N})} \prod_{i=k+1}^N dx_i.$$

### 1.9.2 Le Jellium en milieu homogène

On suppose ici que toutes les charges sont égales  $q_i = q = 1$ , que la densité électrique du milieu est constante  $\rho(x) = \rho = 1$  et que  $L = (N + 1)/2$ . On peut alors se restreindre au cas où les particules sont ordonnées  $-L \leq x_1 \leq x_2 \leq \dots \leq x_N \leq L$ . Il est possible de calculer explicitement la position des particules  $(x_i^{\min})_{1 \leq i \leq N}$  qui minimise l'énergie  $E$ . Lorsque la température tend vers 0, on converge vers le système déterministe  $(x_i)_{1 \leq i \leq N} = (x_i^{\min})_{1 \leq i \leq N}$ . Il existe  $\epsilon_{\min}$  tel que  $E_{\min} \sim N\epsilon_{\min}$  pour  $N$  grand et on a

$$x_i^{\min} = -L - \frac{1}{2} + i$$

pour  $i = 1, \dots, N$ .

Les positions sont régulièrement espacées c'est ce que l'on nomme le cristal de Wigner [134]. On peut montrer que cette structure périodique est conservée même à température non nulle. On pose  $y_i = x_i - x_i^{\min}$  avec la condition  $y_{i+1} \geq y_i - 1$ . Autour de cette position d'équilibre l'énergie s'exprime comme  $E((x_i)_{1 \leq i \leq N}) = E((x_i^{\min})_{1 \leq i \leq N}) + \frac{1}{2} \sum y_i^2$ . La fonction de partition s'écrit alors comme une itération d'intégrale. Plus précisément on a la proposition suivante [97]

**Proposition 1.30.** *On a la formule*

$$Z(\beta) e^{\beta E((x_i^{\min}))} = \int_{-2L+1/2}^{1/2} e^{-\beta \frac{y^2}{2}} [\mathbb{K}^{N-1}(g)](y) dy$$

où  $g(y) = 1_{y \geq -\frac{1}{2}}$  et où l'opérateur  $\mathbb{K} : L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$  est défini par

$$[\mathbb{K}(f)](y) = \int_{-\infty}^{y+1} f(t) e^{-\beta \frac{t^2}{2}} dt.$$

Il se trouve que  $\mathbb{K}$  est un opérateur compact de l'espace  $L^2(\mathbb{R}, e^{-\beta \frac{t^2}{2}})$  dans lui-même. Il n'est pas auto-adjoint mais possède un noyau integral positif. D'après le théorème de Krein Rutmann, il admet donc une unique plus grande valeur propre  $\lambda_0(\beta)$  avec vecteur propre  $\psi_0(\beta)$ . On peut alors énoncer le théorème [7, 97, 30].

**Théorème 1.31.** *On a l'ensemble des résultats suivants :*

— *L'énergie libre limite pour  $N \rightarrow \infty$  est donnée par*

$$f(\beta) = \epsilon_{\min} - \frac{\log(\lambda_0(\beta))}{\beta}.$$

*En particulier, elle est analytique sur  $[0, \infty)$  et il n'y a pas de transition de phase pour ce système.*

— *Pour les particules loin du bord, la densité  $\rho_1(y)$  converge vers  $\psi_0/\|\psi_0\|_{L^1}$  lorsque  $N \rightarrow \infty$ .*

— *Il existe  $\xi, C > 0$  tel que pour tout  $i \neq j$*

$$|\rho_2(y_i, y_j) - \rho_1(y_i)\rho_1(y_j)| \leq Ce^{-\xi|y_i - y_j|}.$$

L'absence de transition de phase est un phénomène omniprésent en dimension 1. L'originalité ici est que le système est un solide pour toute température et non un gaz, ce qui est une conséquence de la très longue portée du potentiel de Coulomb 1D.

### 1.9.3 Le Jellium en milieu inhomogène

On ne suppose plus maintenant que  $\rho(x) = \rho$  mais uniquement que  $0 < \epsilon \leq \rho(x) \leq \frac{1}{\epsilon}$  pour tout  $x \in [-L, L]$ . Ce qui a été fait précédemment dans le cas homogène se généralise également ici avec cependant quelques modifications.

- La position qui minimise l'énergie  $(x_i^{\min})_{1 \leq i \leq N}$  ne satisfait plus  $x_i^{\min} - x_{i-1}^{\min} = 1$  mais satisfait  $\int_{x_{i-1}^{\min}}^{x_i^{\min}} \rho(x) dx = 1$ .
- Autour de la position d'équilibre  $y_i$  on  $E((x_i)_{1 \leq i \leq N}) = E((x_i^{\min})_{1 \leq i \leq N}) + \sum U_i(y_i)$  avec  $U_i$  des potentiels qui ne dépendent que de la densité  $\rho$  autour de  $x_i$ .
- La proposition 1.30 devient

$$Z(\beta)e^{\beta E((x_i^{\min}))} = \int_{-L-x_N^{\min}}^{x_N^{\min}} e^{-\beta U_N(y)} \left[ \prod_{i=1}^{N-1} \mathbb{K}_i(g) \right] (y) dy$$

(produit fait de droite à gauche) avec

$$[\mathbb{K}_i(f)](y) = \int_{-\infty}^{y+x_i^{\min}-x_{i-1}^{\min}} f(t)e^{-\beta U_i(t)} dt.$$

Nous pensons ici particulièrement au cas où  $\rho$  est une perturbation aléatoire d'une densité de référence (constante ou périodique). Cependant nous pouvons obtenir des résultats avec la seule hypothèse  $0 < \epsilon \leq \rho(x) \leq \frac{1}{\epsilon}$  partout sur  $[-L, L]$ . La problématique qui nous a intéressée fut alors de généraliser le Théorème 1.31. Il s'agit d'étudier  $[\prod_{i=1}^{N-1} \mathbb{K}_i(g)]$  mais la méthode spectrale utilisée précédemment ne peut plus s'appliquer ici. On a pu cependant montrer [51] le résultat suivant

**Théorème 1.32.** *On a l'ensemble des résultats suivants :*

- *Lorsque la limite existe, l'énergie libre limite pour  $N \rightarrow \infty$  est analytique réelle sur  $(0, \infty)$  et il n'y a pas de transition de phase pour ce système.*

— Il existe  $\xi, C > 0$  tel que pour tout  $i \neq j$

$$|\rho_2(y_i, y_j) - \rho_1(y_i)\rho_1(y_j)| \leq Ce^{-\xi|y_i - y_j|}$$

On a pu obtenir ces résultats grâce au formalisme de la distance de Hilbert sur des cônes et au théorème de Birkhoff-Hopf que nous expliquons maintenant.

### 1.9.4 Distance de Hilbert sur des cônes et théorème de Birkhoff-Hopf

Dans le cas homogène  $\mathbb{K}^n$  se comporte pour  $n$  grand comme un projecteur sur son plus grand vecteur propre  $\psi_0$  avec un facteur  $\lambda_0(\beta)$ . D'une certaine manière l'image de  $\mathbb{K}^n$  se contracte vers l'espace de dimension un  $\text{Vect}(\psi_0)$ . On présente ici comment il est possible d'obtenir également ce phénomène avec un produit  $\prod_{i=1}^{N-1} \mathbb{K}_i$  grace au théorème de Birkhoff-Hopf.

On appelle cône un sous ensemble  $\mathcal{C}$  d'un espace vectoriel qui est convexe, est stable par multiplication par un scalaire positif ( $x \in \mathcal{C}, t \geq 0 \Rightarrow tx \in \mathcal{C}$ ) et tel que si  $x$  et  $-x$  appartiennent à  $\mathcal{C}$  alors  $x = 0$ .

Un cône définit un ordre partiel :

$$x \leq_{\mathcal{C}} y \Leftrightarrow y - x \in \mathcal{C}$$

On s'intéresse à l'ensemble des opérateurs linéaires  $A$  qui laissent stable le cône ( $x \in \mathcal{C} \Rightarrow A(x) \in \mathcal{C}$ ).

L'exemple le plus simple est le suivant : on définit  $\mathcal{C}$  comme l'ensemble des vecteurs à coefficients positifs. Les matrices ayant des coefficients strictement positifs laissent alors stable ce cône. D'ailleurs, parmi les premières applications du théorème de Birkhoff-Hopf on peut citer une preuve alternative du théorème de Perron Frobenius.

L'idée centrale est de s'intéresser à la distance de Hilbert sur le cône définie de la manière suivante.

**Définition 1.33.** (Distance de Hilbert) Pour  $x, y \in \mathcal{C}$  on pose

$$d(x, y) = \log \frac{\beta_{\min}}{\alpha_{\max}}$$

où  $\beta_{\min} = \min[\beta : \beta x \geq_{\mathcal{C}} y]$  et  $\alpha_{\max} = \max[\alpha : \alpha x \leq_{\mathcal{C}} y]$ .

On a que  $d(x, y) = 0 \Leftrightarrow x = \alpha y$  pour un  $\alpha > 0$ , ce n'est donc une distance que sur le espace projectif. On abusera cependant de l'appellation "distance de Hilbert". La remarque fondamentale est que les opérateurs qui laissent stable le cône sont des contractions pour la distance de Hilbert. En effet on a pour tout  $x, y \in \mathcal{C}$

$$d(A(x), A(y)) \leq d(x, y).$$

Le théorème de Birkhoff-Hopf [22, 79, 58] donne un critère pour que cette contraction soit stricte.

**Théorème 1.34.** (Birkhoff-Hopf [22, 79, 58]) Si le diamètre de  $A(\mathcal{C})$  dans  $\mathcal{C}$  pour la distance de Hilbert est fini ( $\exists M : \forall x, y \in \mathcal{C} : d(A(x), A(y)) \leq M$ ), Alors  $A$  est strictement uniformément contractant pour la distance de Hilbert : c'est à dire il existe  $\kappa < 1$  tel que  $d(A(x), A(y)) \leq \kappa d(x, y)$ .

Dans [51], on a pu montrer que les  $\mathbb{K}_i$  étaient bien contractants sur un certain cône et donc que le produit  $\prod_{i=1}^n \mathbb{K}_i$  se comportait comme un opérateur de rang un pour  $n$  grand de la même manière que dans le modèle homogène. Ceci conduit à la décroissance des fonctions de corrélations. La stricte contraction implique également une certaine stabilité du processus que l'on a pu exploiter pour montrer la régularité de l'énergie libre limite.

### 1.9.5 Le modèle quantique du Jellium

Il est possible d'étudier la variante quantique du problème pour un ensemble de fermions. Il n'est plus possible de négliger l'énergie cinétique et on doit conserver le Hamiltonien

$$H = - \sum_i \Delta_i + E(x_1, \dots, x_N)$$

où  $E$  est l'énergie de Coulomb définie précédemment en (1.10) qui agit ici comme un opérateur de multiplication. La fonction de partition est alors donnée par

$$Z_N(\beta) = \text{Tr}(e^{-\beta H}).$$

On peut cependant se ramener à un modèle probabiliste plus classique grâce à la formule de Feynman-Kac.

**Proposition 1.35.** (*Feynman-Kac*) *On a*

$$Z_N(\beta) = \int_{-L < x_1 < \dots < x_N < L} \mu_{x_1 x_1} \dots \mu_{x_N x_N} (e^{-\int_0^\beta E(\gamma_1(t), \dots, \gamma_N(t)) dt} \mathbf{1}_{(\gamma_1, \dots, \gamma_N) \in W_N}) dx_1 \dots dx_N$$

et

$$\rho(\mathbf{x}; \mathbf{y}) = \frac{1}{Z_N} \mu_{x_1 y_1} \times \mu_{x_2 y_2} \times \dots \times \mu_{x_N y_N} (e^{-\int_0^\beta U(\gamma_1(t), \dots, \gamma_N(t)) dt})$$

où  $W_N$  est l'ensemble

$$W_N(a, b) = \{(\gamma_1, \dots, \gamma_N) | \forall t \in [0, \beta] : a < \gamma_1(t) < \gamma_2(t) < \dots < \gamma_N(t) < b\},$$

$\mu_x$  la probabilité du pont Brownien issue de  $x$  et

$$E(\gamma_1(t), \dots, \gamma_N(t)) = \sum_i U_i(\gamma_i(t)) + E_{min}.$$

Il s'agit maintenant de réaliser une étude du modèle probabiliste non plus sur les positions  $x_i$  mais sur l'ensemble des chemins  $\gamma_i$ . Il se trouve que la méthode spectrale ainsi que la méthode utilisant le théorème de Birkhoff-Hopf peuvent s'adapter à ce cas ci, malgré la complexité apparente du modèle. Le résultat est alors le même que celui du théorème 1.32 mais avec l'hypothèse plus forte que  $\rho$  est quasi-constant :  $\|\rho - \rho_0\|_{L^\infty} \leq \epsilon$  pour  $\epsilon$  assez petit.



# Chapitre 2

## A forward-backward random process for the spectrum of 1D Anderson operators

We present here the work published in [52]

**Abstract.** We give a new expression for the law of the eigenvalues of the discrete Anderson model on the finite interval  $[0, N]$ , in terms of two random processes starting at both ends of the interval. Using this formula, we deduce that the tail of the eigenvectors behaves approximately like  $\exp(\sigma B_{|n-k|} - \gamma \frac{|n-k|}{4})$  where  $B_s$  is the Brownian motion and  $k$  is uniformly chosen in  $[0, N]$  independently of  $B_s$ . A similar result has recently been shown by B. Rifkind and B. Virag in the critical case, that is, when the random potential is multiplied by a factor  $\frac{1}{\sqrt{N}}$ .



We are interested in the one dimensional discrete Anderson model on a finite domain  $[0, N]$ . This model is very classical and has been studied extensively since the 70s. See for example the monograph of Carmona Lacroix [37]. Compared to higher dimensions, it can be considered as a solved problem. However new approaches can always shed new light on this famous system.

The usual approach to tackle this system is the transfer matrix framework. The eigenvectors of the random Schrodinger operator satisfy a recursive relation of order 2,  $u_{n+2} = (V_{n+1} - \lambda)u_{n+1} - u_n$ , which can be written in a matrix form. Using this relation, one can obtain an eigenvector everywhere on  $[0, N]$  from the product of the transfer matrices applied to the boundary values. The advantage of such a formulation is that one can then use the very powerful results for random matrices product and from ergodic theory such as the Oseledets theorem.

In the historical approach of Kunz and Souillard [98] or in the proof from the book [44] a change of variables is used to deal with the conditional probability of the potential  $V$  with a fixed eigenvalue  $\lambda$ . In this short note, we propose another calculation of this conditional probability. We define a random variable  $k$  whose random law is close to the uniform law on  $[0, N]$ . This variable splits the interval  $[0, N]$  into two parts  $[0, k]$  and  $[k, n]$ . On the left part, the matrices product is made from left to right. On the right part, the matrices product is made from right to left. And far from the cut, the laws of the matrices are very close to be independent.

The main interest of our approach is that the connection with the theorems for products of random matrices is more transparent in this setup. From this formula we can recover several known results. Relying on the positivity of Lyapunov exponent, the formula can be used as a new proof of exponential Anderson localization of eigenvectors where the center of localization is uniformly distributed on  $[0, N]$ . Moreover, because it gives an explicit random law we can go beyond the exponential decay of the eigenvectors far from the center of localization and give an explicit law for their tail.

In the first section, we detail the model and we state our result. Then we give some applications of our theorem in the second section. In particular, we write an asymptotic result similar to the result of Rifkind and Virag in [120]. In section 3, we finally give the proof of the theorem.

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## 2.1 Model and main result

We consider the one dimensional Anderson model [12] defined on  $[1, N]$  through the operator.

$$H^{(N)} = -\Delta^{(N)} + V_{\omega}^{(N)}.$$



the condition  $u_{N+1} = 0$  is then satisfied. It will be convenient to denote the vector  $\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}$  as a complex number in the fashion

$$u_{n+1} + iu_n = z_n = r_n e^{i\phi_n}$$

where  $r_n \in \mathbb{R}_+$  and  $\phi_n \in \mathbb{R}/2\pi\mathbb{Z}$ . We also introduce the lifting of  $\phi_k$ , which we denote by  $\theta_k$ . This is just a discrete version of the continuous lifting from  $\mathbb{R}/2\pi\mathbb{Z}$  to  $\mathbb{R}$  into the discrete case. It is defined recursively by

$$\theta_k = \begin{cases} \theta_0 = 0 \\ \phi_k[2\pi] \quad \forall k \in [1, N] \end{cases}$$

and

$$\theta_k - \frac{\pi}{2} \leq \theta_{k+1} < \theta_k + \frac{3\pi}{2}.$$

It can be seen that  $\phi_{k+1}$  does not depend on  $r_{k+1}$  but only on  $\phi_k$  and  $T_\lambda(v_k)$ . Therefore, for simplicity of notation, we use the same notation  $T$  for the (non linear) function  $T_\lambda(v_k)\mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}/2\pi\mathbb{Z}$  :

$$\phi_{k+1} = T_\lambda(v_k)\phi_k$$

Note that it is possible to recover  $r_k$  from  $\phi_0, \phi_1, \dots, \phi_N$  with the formula

$$\frac{r_{k+1}}{r_k} = \frac{r_{k+1}}{u_{k+1}} \frac{u_{k+1}}{r_k} = \frac{\cos \phi_k}{\sin \phi_{k+1}}.$$

For this reason, in the rest of the paper we focus mostly on  $(\phi_k)_{k=0\dots N}$ . We note  $\mathcal{F}(\lambda) = (\phi_k)_{k=0\dots N}$  which has been constructed from the recursive formula  $\phi_{k+1} = T_\lambda(v_k)\phi_k$  and  $\phi_0 = 0$ . And for  $\lambda$  an eigenvalue, we note  $\mathcal{P}h(\lambda) = (\phi_k)_{k=0\dots N}$  the phase of the corresponding eigenvector. Note that it is equal to  $\mathcal{F}(\lambda)$  with the condition  $\phi_N = \frac{\pi}{2}[\pi]$

### 2.1.2 Forward and backward process

In this subsection, we define two natural random laws on the chain  $X = (\phi_k)_{k=0,\dots,N}$ . The first one is the Markov chain starting from  $\phi_0$  with an initial law  $\mu_f$  defined on  $\mathbb{S}^1$  and transition law  $\phi_k \rightarrow \phi_{k+1} = T_\lambda(v_k)\phi_k$  with a random measure  $\nu$  for  $v_k$ . We call it the forward process. The second one is the Markov chain starting from  $\phi_N$  with an initial law  $\mu_b$  and transition law  $\phi_k \rightarrow \phi_{k-1} = T_\lambda^{-1}(v_{k-1})\phi_k$  with a random measure  $\nu$  for  $v_k$  and we call it the backward process. Then we introduce a cut in  $[1, N]$ , and we can define the random law product between these two processes which we call the Forward-backward process.

For a proper definition we use test functions on  $\mathbb{R}^{N+1}$  which are bounded and continuous.

**Definition 2.1.** Forward and backward processes

— The forward process. Let  $\mathcal{P}_f$  be the probability on  $\mathbb{R}^{N+1}$  defined by

$$\mathcal{P}_f(F) = \int \cdots \int d\mu_f(\phi_0) d\nu(v_1) \cdots d\nu(v_n) F(X)$$

for any test function  $F$ .

— The backward process. Let  $\mathcal{P}_b$  be the probability on  $\mathbb{R}^{N+1}$  defined by

$$\mathcal{P}_b(F) = \int \cdots \int d\nu(v_1) \dots d\nu(v_N) d\mu_b(\phi_N) F(X)$$

for any test function  $F$ .

*Remark 2.2.* If we introduce  $\xi_{0,n} : \phi_0 \rightarrow \phi_n^f = \prod_{k=0}^{n-1} T_{\lambda}(v_k) \phi_0$  and if for almost surely any  $v_1, v_2, \dots, v_n, \mu_b$  and the push measure  $\xi(\mu_f)$  are equivalent measures then remark that for any  $F$ :

$$\begin{aligned} \mathcal{P}_b(F) &= \int \cdots \int d\nu(v_1) \dots d\nu(v_n) d\mu_f(X_0) \frac{d\mu_b(X_n)}{d\xi(\mu_f(X_0))} \Big|_{v_1, \dots, v_n} F(X) \\ &= \mathcal{P}_f \left( F \frac{d\mu_b(X_n)}{d\xi(\mu_f(X_0))} \Big|_{v_1, \dots, v_n} \right). \end{aligned}$$

**Definition 2.3.** Forward-Backward process

For  $k \in [0, N]$ , we define  $\mathcal{P}_{f,0..k} \otimes \mathcal{P}_{b,k+1,\dots,N}$  a forward process for  $X^f = \phi_0^f, \phi_1^f, \dots, \phi_k^f$  with  $\phi_0^f = 0, (\mu_f = \delta_0)$  and a backward process for  $X^b = \phi_N^b, \phi_{N-1}^b, \dots, \phi_k^b$ , with  $\phi_N^b = \frac{\pi}{2}$  ( $\mu_b = \delta_{\frac{\pi}{2}}$ ) which are independent from each other.

### 2.1.3 Results

We are now ready to state the main theorem of our paper.

**Theorem 2.4.** (*Law of the spectrum of the 1D Anderson model*) For any test function  $G(\lambda, X)$ , we have

$$\begin{aligned} &\mathbb{E} \left[ \sum_{\lambda \in \sigma(H), X = Ph(\lambda)} G(\lambda, X) \right] \\ &= \int_{\mathbb{R}} d\lambda \sum_{k=1}^N \mathbb{E}_{\mathcal{P}_{f,1..k} \otimes \mathcal{P}_{b,k+1,\dots,N}} \left[ G(\lambda, X) \delta_{\phi_k^f - \phi_k^b[\pi]} \sin^2(\phi_k^f) \right] \end{aligned} \quad (2.2)$$

that we can rewrite

$$\begin{aligned} &\mathbb{E} \left[ \frac{1}{N} \sum_{\lambda \in \sigma(H), X = Ph(\lambda)} G(\lambda, X) \right] \\ &= \int_{\mathbb{R}} \rho(\lambda) \mathbf{1}_{\rho(\lambda) > 0} \left( \frac{1}{N} \sum_{k=1}^N \mathbb{E}_{\mathcal{P}_{f,1..k} \otimes \mathcal{P}_{b,k+1,\dots,N}} \left[ G(\lambda, X) \frac{\delta_{\phi_k^f - \phi_k^b[\pi]} \sin^2(\phi_k^f)}{\rho(\lambda)} \right] \right) d\lambda \end{aligned} \quad (2.3)$$

with  $\rho(\lambda)$  the density of state.

Recall that  $Ph(\lambda)$  is the phase of the eigenvector corresponding of the eigenvalue  $\lambda$ .

This formula is to be understood as follow. One chooses  $k$  randomly in  $[1, N]$  which splits the segment into two parts  $[1, k]$  and  $[k, N]$ . On the left, we obtain a forward process, on the right, we obtain a backward process. The choice of  $k$  is not exactly uniform on  $[0, N]$  because of the condition  $\delta_{\phi_k^f - \phi_k^b} \sin^2(\phi_k)$ . However, for large  $N$ , and for any  $k \leq N$  not too close to 0 or  $N$ , the laws of  $\phi_k^f$  and  $\phi_k^b$  are very close to their invariant measure under action of  $T_\lambda$  and then do not depend of  $k$ . Therefore the law of  $k$  becomes close to the uniform.

There is still a dependence between the two processes at the connection between the forward and backward processes. However, because of the mixing property of the matrix product, the correlations decay exponentially fast outside of the cut  $k$ .

We recall that a stationary process  $X_k$  is called  $(\alpha_n)_{n \in \mathbb{N}}$ -mixing if

$$\forall k, \max_{A, B} |\mathbb{P}(X_k \in A, X_{k+n} \in B) - \mathbb{P}(X_k \in A)\mathbb{P}(X_{k+n} \in B)| \leq \alpha_n$$

The following is a well known result

**Proposition 2.5.** *There exists a constant  $C > 0$  and  $0 < \kappa < 1$  such that the process  $\phi_k$  is  $(C\kappa^n)_{n \in \mathbb{N}}$ -mixing.*

For a proof, see [37] proposition IV.3.12.

## 2.2 Applications

We present here three applications of our result. The first one is a formula for the integrated density of states. The second one is about the form of the tails of the eigenvectors. We then finish with a temperature profile from [49].

### 2.2.1 A formula for the integrated density of states

The following equality can be found as well in [37](proposition VIII.3.10 and problem VIII.6.8).

**Proposition 2.6.** *For  $\lambda \in \mathbb{R}$ , let  $\mu_\lambda(d\phi) = \rho_\lambda(\phi)d\phi$  be the  $T_\lambda$ -invariant measure on  $\mathbb{R}/\mathbb{Z}$ . The density of states*

$$dN(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N} \#\{\sigma(H) \cap [\lambda, \lambda + d\lambda]\}$$

is given by

$$\frac{dN(\lambda)}{d\lambda} = \int_{\mathbb{R}/2\pi\mathbb{Z}} \sin^2(\phi) \rho_\lambda(\phi) \rho_\lambda\left(\frac{\pi}{2} - \phi\right) d\phi.$$

*Proof.* We apply Theorem 2.4, and we choose  $G(s, X) = G(s)$  (that does not depend on  $X$ ) as an approximation of  $1_{s \in [\lambda, \lambda + d\lambda]}$ . Then

$$\begin{aligned} \frac{1}{N} \mathbb{E} \left[ \sum_{\lambda \in \sigma(H_N)} G(\lambda) \right] &= \int G(\lambda) d\lambda \frac{1}{N} \sum_k \mathbb{E}_{\mathcal{P}_{f,1 \dots k} \otimes \mathcal{P}_{b,k+1, \dots, N}} [\delta_{\phi_k^f - \phi_k^b} \sin^2(\phi_k^f)] \\ &= \int G(\lambda) d\lambda \frac{1}{N} \sum_k \int_{\phi} \rho_{k,\lambda}(\phi) \rho_{N-k,\lambda}\left(\frac{\pi}{2} - \phi\right) \sin^2(\phi_k^f) \end{aligned}$$

where  $\rho_{k,\lambda}$  and  $\rho_{N-k,\lambda}$  are the density probabilities of the angles of  $M_k(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $M_{N-k}(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . We can then conclude using that  $\rho_{k,\lambda} \rightarrow \rho_\lambda$  and  $\rho_{N-k,\lambda} \rightarrow \rho_\lambda$  when  $k \rightarrow \infty$  and  $N - k \rightarrow \infty$ .  $\square$

### 2.2.2 Brownian and drift for the eigenvectors

It is well known since the work of Carmona-Klein-Martinelli [36], Goldsheid-Molchanov-Pastur [67] and Kunz-Souillard [98] that the eigenvectors are localized and decay exponentially from the center of localization. An exact form of the eigenvectors has been recently proven in the critical case where  $V$  is replaced by  $\frac{V}{\sqrt{N}}$  in [120]. There Rifkind and Virag proved that the eigenvectors in the bulk have the form  $e^{\sigma \frac{B|t-u|}{2} - \gamma|t-u|}$ . We claim using our formula of Theorem 2.4 that a similar result is universal for the tails of the eigenvector in the non critical case.

For the reader's convenience we recall the heuristic of the following classical results. One can write any product of random matrices  $M_N = \prod_{i=1}^N T_i$  as

$$\log(\|M_N\|) = \log\left(\prod_{i=1}^N \frac{\|M_i\|}{\|M_{i-1}\|}\right) = \sum_{i=1}^N \log\left(\|T_i\left(\frac{M_{i-1}}{\|M_{i-1}\|}\right)\|\right)$$

In the case when  $T_i$  are iid and there are some strong mixing property on  $\frac{M_{i-1}}{\|M_{i-1}\|}$ , the terms  $Y_i = \log\left(\|T_i\left(\frac{M_{i-1}}{\|M_{i-1}\|}\right)\|\right)$  should behave like iid random variables. One can then prove the strong law of large number, the central limit theorem, and Donsker theorem. See the paper of Le Page [103] for this matter. One therefore defines a ‘‘mean’’, a ‘‘variance’’ and a ‘‘random walk’’ as follows.

— The Lyapunov exponent is

$$\gamma(\lambda) := \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}[\log \|M_N(\lambda)\|].$$

— The limit variance is

$$\sigma^2(\lambda) = \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}[(\log \|M_N(\lambda)\| - \gamma(\lambda)N)^2].$$

— The ‘‘Random Walk’’ is

$$S_n = \frac{1}{\sigma(\lambda)} (\log \|M_n(\lambda)\| - \gamma(\lambda)n).$$

and

$$W_N(t) = \frac{1}{\sqrt{N}} S_{\lfloor Nt \rfloor}.$$

Finally, we denote by  $W$  the Wiener measure.

**Theorem 2.7.** *(Limit theorem for products of random matrices) We have the following :*

- $\gamma(\lambda) > 0$  and  $\lim_{N \rightarrow \infty} \frac{1}{N} \log \|M_N(\lambda)\| = \gamma$ , almost surely;
- $\sigma^2(\lambda) > 0$ ;
- $W_N \rightarrow W$  in law.

We refer to [103, Theorems 2 and 3] for the proof of Theorem 2.7.

We recover then the form of a Brownian with drift, both on the right hand side and the left hand side of the cut. For  $\lambda$  an eigenvalue, and  $r_k e^{i\phi_k}$  the corresponding eigenvector, we note  $q_k^\lambda = \log(r_k)$ . For scaling, we set  $q^\lambda(s) = \frac{q_{\lfloor Ns \rfloor}^\lambda}{N}$ .

**Proposition 2.8.** (*Tail of eigenvectors*)

1) *Choosing  $\lambda^{(N)}$  uniformly in  $\sigma(H^{(N)})$ , we have the following convergence in law*

$$(\lambda^{(N)}, q_s^{\lambda^{(N)}}) \rightarrow (\tilde{\lambda}, -|\gamma(\tilde{\lambda})(s-x)|)$$

when  $N \rightarrow \infty$ , where  $\tilde{\lambda}$  is a random variable with law the limiting density of state  $\rho$  and  $x$  an independent variable on  $[0, 1]$  with uniform law.

2) *There exists a sequence of random variables  $\{x^{(N)}\}$  with uniform law on  $[0, 1]$  such that*

$$(\lambda^{(N)}, \sqrt{N}[q_s^{\lambda^{(N)}} + |\gamma(\lambda^{(N)})(s-x^{(N)})|]) \rightarrow (\tilde{\lambda}, \sigma(\tilde{\lambda})W_{s-x})$$

when  $N \rightarrow \infty$ , where  $W$  is the Wiener measure.

The first statement is the very classical result of Anderson localization for the one dimensional model. The eigenvectors decay exponentially from their center of localization and this center is chosen uniformly on the domain. The second statement says that the typical deviation from the decay is the exponential of a Brownian (see Figure 2.1 for an illustration).

Rifkind and Virag [120] studied the eigenvectors in the bulk of the one dimensional Anderson model in the continuous case where the potential is a white noise. It is the limit of the discrete model in the called critical regime where the potential is scaled like  $V_\omega^{(N)} = \frac{1}{\sqrt{N}}V_\omega$ . In this regime, we are not able to speak of localization because the length of the decay is as large as the size of the domain. However they prove the exact law of the form of the eigenvectors

$$q_s^\lambda = -|\gamma(\lambda)(s-x)| + \sigma(\lambda)W_{s-x}$$

To make the connection with our previous proposition, one can actually show that for  $V_\omega = \epsilon v_\omega$ , with  $\mathbb{E}(v_\omega^2) = \sigma^2$ , in the limit  $\epsilon \rightarrow 0$  and  $|\lambda| < 2$ , we have

$$\gamma(\lambda) = \frac{\sigma^2}{4-\lambda^2}\epsilon^2 + o(\epsilon^2)$$

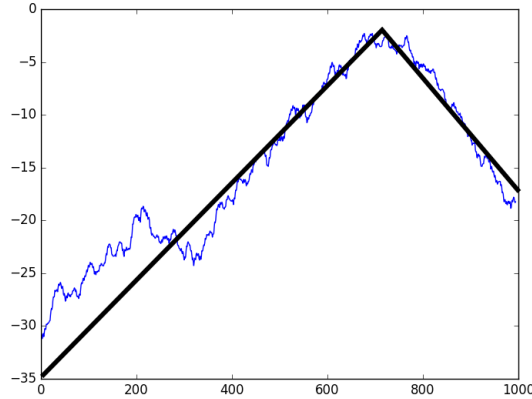
and

$$\frac{\sigma(\lambda)^2}{2} = \frac{\sigma^2}{4-\lambda^2}\epsilon^2 + o(\epsilon^2).$$

*Proof.* (Proposition 2.8)

If in our formula (2.3) the term  $\delta_{\phi_k^f - \phi_k^b}$  were not there, then the forward and the backward process would have been completely independent. Our proposition would have then immediately followed from Theorem 2.7, under the conditions that  $r_k$  obtained by the forward process and the  $r_k$  obtained by the backward process are the same. And that the normalization  $\sum_{n=1}^N |u_n|^2 = 1$  which is replaced at the limit by  $\sup q_s^\lambda = 0$ .

Figure 2.1 – A realization of  $\log \|M_n(\lambda)\|$  for  $N = 1000$ ,  $v_\omega$  uniform on  $[0, 1]$  with Dirichlet boundary conditions. We add a fit of the form  $|\gamma(\lambda)(s - x)|$ .



Therefore we only have to check that the little perturbation around the cut  $k$  has no influence. We fix  $\phi$ . Conditionally of  $\phi_k^b = \phi$  and  $\phi_k^f = \phi$  the F-B processes are independent. The results of Theorem 2.7 are true asymptotically with probability 1. Therefore for any  $\phi$  in a set of full Lebesgue measure on  $\mathbb{S}^1$  the results of Theorem 2.7 are true conditionally of  $\phi_k^b = \phi$  and  $\phi_k^f = \phi$ .  $\square$

### 2.2.3 A temperature profile

We will use our result to explain some numerical observations which have been made in [49]. In this article, the authors are interested in the temperature profile of a disordered chain connected to two thermal baths of temperatures  $T_0$  and  $T_N$  at the boundary 0 and  $N$ . According to [49], the temperature  $T(x)$  at site  $x$  is expected to be given (under certain limiting assumptions for the thermalisation process) by

$$T(x) = \sum_{\lambda \in \sigma(H_N)} |\psi_\lambda(x)|^2 \left( T_0 \frac{|\psi_\lambda(0)|^2}{|\psi_\lambda(0)|^2 + |\psi_\lambda(N)|^2} + T_N \frac{|\psi_\lambda(N)|^2}{|\psi_\lambda(0)|^2 + |\psi_\lambda(N)|^2} \right) \quad (2.4)$$

where  $H_N$  is our one dimensional random Schrödinger operator and  $\psi_\lambda$  are its eigenvectors.

We prove that  $T$  converge to a step function where the transition from  $T_0$  and  $T_N$  happens in a neighbourhood of  $x = \frac{N}{2}$  at a scale  $\sqrt{N}$ . We denote  $x = \lfloor \frac{N}{2} + \sqrt{N}y \rfloor$  and we expect variation of  $T$  with  $y$  of order 1. This has been observed numerically in [49].

**Proposition 2.9.** *We have the following convergence*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[ T \left( \lfloor \sqrt{N}y + \frac{N}{2} \rfloor \right) \right] = T_0 + (T_N - T_0) \int_{\mathbb{R}} \mathbb{P} \left( \mathcal{N}(0, 1) \leq \frac{2\gamma(\lambda)}{\sigma(\lambda)} y \right) dN(\lambda)$$

where  $dN(\lambda)$  is the integrated density of states,  $\gamma(\lambda)$  the Lyapunov exponent and  $\sigma(\lambda)$  the limit variance.



The Lyapunov exponent is positive, continuous and so is bounded from below on the support of  $\sigma(H)$ . The variance  $\sigma(\lambda)$  is bounded, therefore uniformly on  $\lambda$ ,

$$\mathbb{P}(\mathcal{N}(0, 1) \leq \frac{\sigma(\lambda)}{2\gamma(\lambda)}y) \rightarrow 0$$

for  $y \rightarrow -\infty$  and

$$\mathbb{P}(\mathcal{N}(0, 1) \leq \frac{\sigma(\lambda)}{2\gamma(\lambda)}y) \rightarrow 1$$

for  $y \rightarrow \infty$ . We have then  $T(x) \approx T_0$  for  $\frac{N}{2} - x \gg \sqrt{N}$  and  $T(x) = T_N$  for  $x - \frac{N}{2} \gg \sqrt{N}$ , the step function numerically observed.

*Proof.* We use our formula and write:

$$\begin{aligned} \mathbb{E}(T(x)) &= \sum_{k \in [0, N]} \int_{\mathbb{R}} d\lambda \mathbb{E}_{\mathcal{P}_{f, 1 \dots k} \otimes \mathcal{P}_{b, k+1, \dots, N}} \left[ |\psi_\lambda(x)|^2 \right. \\ &\quad \left. \left( T_0 \frac{|\psi_\lambda(0)|^2}{|\psi_\lambda(0)|^2 + |\psi_\lambda(N)|^2} + T_N \frac{|\psi_\lambda(N)|^2}{|\psi_\lambda(0)|^2 + |\psi_\lambda(N)|^2} \right) \times \right. \\ &\quad \left. \times \delta_{\phi_k^f - \phi_k^b} \sin^2(\phi_k) \right]. \end{aligned}$$

With the notation of Proposition 2.8, we write

$$\begin{aligned} &T_0 \frac{|\psi_\lambda(0)|^2}{|\psi_\lambda(0)|^2 + |\psi_\lambda(N)|^2} + T_N \frac{|\psi_\lambda(N)|^2}{|\psi_\lambda(0)|^2 + |\psi_\lambda(N)|^2} \\ &= T_0 \frac{e^{Nq_0^\lambda(N)}}{e^{Nq_0^\lambda(N)} + e^{Nq_1^\lambda(N)}} + T_1 \frac{e^{Nq_1^\lambda(N)}}{e^{Nq_0^\lambda(N)} + e^{Nq_1^\lambda(N)}} \end{aligned}$$

Therefore for  $N$  large, this converges to  $T_0$  for  $q_0^\lambda > q_1^\lambda$  and  $T_1$  for  $q_0^\lambda < q_1^\lambda$ . We have then at the limit a Bernoulli  $T_{int}$  with parameter given by Proposition 2.8:

$$T_{int} = \begin{cases} T_0 & \text{with probability } \mathbb{P}(\mathcal{N}(0, 1) \leq \frac{(2k-N)\gamma(\lambda)}{\sqrt{N}\sigma(\lambda)}) \\ T_N & \text{with probability } \mathbb{P}(\mathcal{N}(0, 1) \geq \frac{(2k-N)\gamma(\lambda)}{\sqrt{N}\sigma(\lambda)}) \end{cases}.$$

In order to conclude, we recall that the whole mass of  $|\psi_\lambda|^2$  is around a few number of sites around  $k$  so

$$\begin{aligned} \mathbb{E}(T(x)) &= \int_{\mathbb{R}} d\lambda \sum_{k \in [x-\alpha(N), x+\alpha(N)]} \mathbb{E}_{\mathcal{P}_{f, 1 \dots k} \otimes \mathcal{P}_{b, k+1, \dots, N}} \left[ |\psi_\lambda(x)|^2 \right. \\ &\quad \left. \left( T_0 \frac{|\psi_\lambda(0)|^2}{|\psi_\lambda(0)|^2 + |\psi_\lambda(N)|^2} + T_N \frac{|\psi_\lambda(N)|^2}{|\psi_\lambda(0)|^2 + |\psi_\lambda(N)|^2} \right) \delta_{\phi_k^f - \phi_k^b} \right. \\ &\quad \left. \sin^2(\phi_k) \right] + O(e^{-\gamma(\lambda)\alpha(N)}) \end{aligned}$$

where we chose  $\alpha(N)$  such that  $\sqrt{N} \gg \alpha(N) \gg 1$ . Moreover for large  $N$ ,

$$\mathbb{P}(\mathcal{N}(0, 1) \geq \frac{(2x-N)\gamma(\lambda)}{\sqrt{N}\sigma(\lambda)}) = \mathbb{P}(\mathcal{N}(0, 1) \geq \frac{(2k-N)\gamma(\lambda)}{\sqrt{N}\sigma(\lambda)}) + o(1),$$

we have then

$$\begin{aligned} \mathbb{E}(T(x)) &= \int_{\mathbb{R}} d\lambda \sum_{k \in [x-\alpha(N), x+\alpha(N)]} \mathbb{E}_{\mathcal{P}_{f,1..k} \otimes \mathcal{P}_{b,k+1,\dots,N}} [|\psi_\lambda(x)|^2 \\ &\quad (T_0 + (T_N - T_0)\mathbb{P}(\mathcal{N}(0,1) \geq \frac{(2x-N)\gamma(\lambda)}{\sqrt{N}\sigma(\lambda)})) \delta_{\phi_k^f - \phi_k^b} \\ &\quad \sin^2(\phi_k)] + o(1). \end{aligned}$$

Finally we use the following formula, for  $x$  not close to the edges ( $\min(x, N-x) \gg 1$ )

$$\sum_{k \in [0, N]} \mathbb{E}_{\mathcal{P}_{f,1..k} \otimes \mathcal{P}_{b,k+1,\dots,N}} [|\psi_\lambda(x)|^2 \delta_{\phi_k^f - \phi_k^b} \sin^2(\phi_k)] = \frac{dN(\lambda)}{d\lambda} + o(1).$$

Indeed, for any  $A$  Borel set of  $\mathbb{R}$ :

$$\begin{aligned} &\int_{\mathbb{R}} 1_A(\lambda) \frac{dN(\lambda)}{d\lambda} d\lambda \\ &= \lim \frac{1}{N} \mathbb{E}(Tr(1_A(H))) \\ &= \frac{1}{N} \sum_x \mathbb{E} \left[ \sum_{\lambda \in A \cap \sigma(H)} |\psi_\lambda(x)|^2 \right] \\ &= \int 1_A(\lambda) d\lambda \frac{1}{N} \sum_x \sum_{k \in [0, N]} \mathbb{E}_{\mathcal{P}_{f,1..k} \otimes \mathcal{P}_{b,k+1,\dots,N}} [|\psi_\lambda(x)|^2 \delta_{\phi_k^f - \phi_k^b} \sin^2(\phi_k)] \end{aligned}$$

We then note that the sums are asymptotically independent of  $x$  for  $x$  not close to the edges. Therefore

$$\begin{aligned} &\int_{\mathbb{R}} 1_A(\lambda) \frac{dN(\lambda)}{d\lambda} d\lambda \\ &= \int 1_A(\lambda) d\lambda \sum_{k \in [0, N]} \mathbb{E}_{\mathcal{P}_{f,1..k} \otimes \mathcal{P}_{b,k+1,\dots,N}} [|\psi_\lambda(x)|^2 \delta_{\phi_k^f - \phi_k^b} \sin^2(\phi_k)] + o(1) \end{aligned}$$

The proposition then follows, namely we have :

$$\mathbb{E}(T(x)) = T_0 + \int_{\mathbb{R}} (T_N - T_0) \mathbb{P} \left( \mathcal{N}(0,1) \geq \frac{(2x-N)\gamma(\lambda)}{\sqrt{N}\sigma(\lambda)} \right) \frac{dN(\lambda)}{d\lambda} d\lambda + o(1)$$

as we wanted.  $\square$

### 2.2.4 Periodic boundary conditions

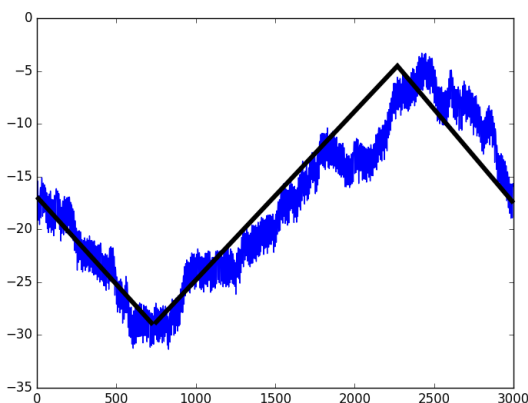
We tried to obtain a similar result for periodic boundary conditions. With the multiscale analysis tools [62], one has the exponential decay from the center

of localization. But it would be also interesting to have an interpretation with forward backward process in this case.

In the critical regime, one would expect the form of the eigenvectors to be like  $e^{F(s)}$ , on  $\mathbb{R}/2\pi\mathbb{Z}$  with  $F(s) = -\gamma \min(|s - u|, |s - u + \pi|) + \sigma \tilde{B}_{s-u}$  with  $u$  uniformly chosen on  $[0, 2\pi]$  and  $\tilde{B}$  a Brownian bridge (see Figure 2.2). So far we have not been able to prove this statement rigorously.

*Remark 2.10.* The condition  $u_{-1} = u_{N+1} = 0$  in the Dirichlet case has to be replaced by  $\text{Tr}(M_N(\lambda)) = 2$ . Indeed, let  $u_n$  be an eigenvector of eigenvalue  $\lambda$  and  $z = \begin{pmatrix} u_1 \\ u_0 \end{pmatrix}$ . Then, periodic boundary conditions mean  $M_N(\lambda)z = z$ . So 1 is an eigenvalue of  $M_N(\lambda)$ . Therefore 1 is a solution of  $x^2 - \text{Tr}(M_N(\lambda))x + 1 = 0$  and so  $\text{Tr}(M_N(\lambda)) = 2$ .

Figure 2.2 – A realization of  $\log \|M_n(\lambda)\|$  with periodic boundary conditions for  $N = 3000$ ,  $v_\omega$  uniform on  $[0, 0.3]$ . We add a fit of the form  $-\gamma \min(|s - u|, |s - u + \pi|)$ .



### 2.3 Proof of Theorem 2.4

*Proof.* We recall that  $\phi_N$  is the angle for the complex  $u_{N+1} + iu_N$ ,  $\theta_N = \phi_N[2\pi]$  and the Dirichlet boundary condition states that  $u_{N+1} = 0$  for an eigenvalue. We have that  $\lambda$  is an eigenvalue if and only if  $\phi_N = \frac{\pi}{2}[\pi]$ , therefore

$$\mathbb{E} \left[ \sum_{\lambda \in \sigma(H), X = \mathcal{P}h(\lambda)} G(\lambda, X) \right] = \mathbb{E} \left[ \sum_{\lambda: \theta_N(\lambda) \in \frac{\pi}{2} + \pi\mathbb{Z}, X = \mathcal{F}(\lambda)} G(\lambda, X) \right].$$

*Remark 2.11.*  $\theta_N : \lambda \rightarrow \theta_N(\lambda)$  is continuous and strictly increasing (see calculation below).

For finite  $N$ , the inverse function  $\theta_N^{-1}$  is continuous, so are  $G$  (continuous function of  $\lambda$  and  $\theta_i$ ),  $X$ . We can therefore write

$$\begin{aligned} & \mathbb{E} \left[ \sum_{\lambda: \theta_N(\lambda) \in \pi\mathbb{Z} + \frac{\pi}{2}, X = \mathcal{F}(\lambda)} G(\lambda, X) \right] \\ &= \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[ \frac{1}{2\epsilon} \sum_{n \in \mathbb{Z}} \int_{\pi n + \pi/2 - \epsilon}^{\pi n + \pi/2 + \epsilon} \sum_{\lambda: \theta_N(\lambda) = s, X = \mathcal{F}(\lambda)} G(\lambda, X) ds \right]. \end{aligned}$$

The rest follows from a change of variables. Let us denote

$$I_\epsilon = \frac{\pi}{2} + \cup_{n \in \mathbb{Z}} [\pi n - \epsilon, \pi n + \epsilon]$$

and

$$\begin{aligned} \mathcal{P}_\epsilon(G) &= \mathbb{E} \left[ \frac{1}{2\epsilon} \sum_{n \in \mathbb{Z}} \int_{2\pi n - \epsilon}^{2\pi n + \epsilon} \sum_{\lambda: \theta_N(\lambda) = s} G(\lambda, \mathcal{F}(\lambda)) ds \right] \\ &= \mathbb{E} \left[ \int_{\mathbb{R}} G(\lambda, \mathcal{F}(\lambda)) \left| \frac{d\theta_N(\lambda)}{d\lambda} \right| \frac{1}{2\epsilon} 1_{\theta_N(\lambda) \in I_\epsilon} d\lambda \right]. \end{aligned}$$

Then

$$\begin{aligned} \frac{d\theta_N(\lambda)}{d\lambda} &= \frac{d\phi_N(\lambda)}{d\lambda} \\ &= \frac{d}{d\lambda} \left[ \prod_{k=1}^N T(v_\omega(k) - \lambda) \phi_0 \right] \\ &= \sum_{k=1}^N \frac{d\phi_N}{d\phi_k} \Big|_{v_\omega(N), \dots, v_\omega(k+1)} \cdot \frac{d}{d\lambda} [T(v_\omega(k) - \lambda)] (\phi_{k-1}). \end{aligned}$$

In this formula appears the term  $\frac{d\phi_N}{d\phi_k} \Big|_{v_\omega(N), \dots, v_\omega(k+1)}$ . It is this term that changes the law from a forward process to a backward process. We then calculate  $\frac{d}{d\lambda} [T(v_\omega(k) - \lambda)] (\phi_{k-1})$  with

$$\begin{pmatrix} u_{k+1} \\ u_k \end{pmatrix} = \begin{pmatrix} (v - \lambda)u_k + u_{k-1} \\ u_k \end{pmatrix},$$

$$\frac{d}{d\lambda} \begin{pmatrix} u_{k+1} \\ u_k \end{pmatrix} = \begin{pmatrix} -u_k \\ 0 \end{pmatrix},$$

and thus

$$\frac{d}{d\lambda} [T(v_\omega(k) - \lambda)] (\phi_{k-1}) = \frac{\begin{pmatrix} u_{k+1} \\ u_k \end{pmatrix} \wedge \begin{pmatrix} -u_k \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} u_{k+1} \\ u_k \end{pmatrix} \right\|^2} = \frac{u_k^2}{u_k^2 + u_{k+1}^2} = \sin^2 \phi_k.$$

We carry on the calculation,

$$\begin{aligned} \mathcal{P}_\epsilon(G) &= \mathbb{E} \left[ \int_{\mathbb{R}} G(\lambda, \mathcal{F}(\lambda)) \left| \frac{d\theta_N(\lambda)}{d\lambda} \right| \frac{1}{2\epsilon} 1_{\theta_N(\lambda) \in I_\epsilon} d\lambda \right] \\ &= \sum_{k=1}^N \int_{\mathbb{R}} d\lambda \left[ \int \dots \int d\nu(v_1) \dots d\nu(v_n) G(\lambda, \mathcal{F}(\lambda)) \frac{d\phi_N}{d\phi_k} \cdot \sin^2(\phi_k) \right] \times \\ &\quad \times \frac{1}{2\epsilon} 1_{\theta_N(\lambda) \in I_\epsilon}. \end{aligned}$$

We artificially add a variable  $\phi$ :

$$\mathcal{P}_\epsilon(G) = \sum_{k=1}^N \int_{\mathbb{R}} d\lambda \left[ \int \dots \int d\nu(v_1) \dots d\nu(v_k) \int_{\mathbb{S}^1} d\phi \delta_{\phi_k}(\phi) \right. \\ \left. \int \dots \int d\nu(v_{k+1}) \dots d\nu(v_N) G(\lambda, \mathcal{F}(\lambda)) \frac{d\phi_N}{d\phi} \cdot \sin^2(\phi_k) \right] \frac{1}{2\epsilon} \mathbf{1}_{\theta_N(\lambda) \in I_\epsilon}.$$

Then we use the remark 2.2 and by taking the limit

$$\frac{1}{2\epsilon} \mathbf{1}_{\phi_N \in I_\epsilon / \pi\mathbb{Z}} d\phi_N \rightarrow \delta_{\phi_N=0[\pi]}$$

we get

$$\mathcal{P}_\epsilon(G) = \sum_{k=0}^N \int_{\mathbb{R}} d\lambda \left[ \int \dots \int d\nu(v_1) \dots d\nu(v_k) \int_{\mathbb{S}^1} \right. \\ \left. \int \dots \int d\phi_N d\nu(v_{k+1}) \dots d\nu(v_N) \delta_{\phi_k}(\phi) G(\lambda, \mathcal{F}(\lambda)) \sin^2(\phi_k) \right] \frac{1}{2\epsilon} \mathbf{1}_{\theta_N(\lambda) \in I_\epsilon} \\ = \sum_{k=0}^N \int_{\mathbb{R}} d\lambda \left[ \mathbb{E}_{\mathcal{P}_{f,1..k} \otimes \mathcal{P}_{b,k+1,\dots,N}^u} [G(\lambda, X) \delta_{\phi_k^f - \phi_k^b} \sin^2(\phi_k) \frac{1}{2\epsilon} \mathbf{1}_{\phi_N \in I_\epsilon / \pi\mathbb{Z}}] \right]$$

with  $\mathcal{P}_{f,1..k} \otimes \mathcal{P}_{b,k+1,\dots,N}^u$  is the forward-backward process with  $\mu_b$  the uniform law on  $\mathbb{S}^1$  and we can then conclude,  $\square$

# Chapitre 3

## Anderson localisation for infinitely many interacting particles in Hartree-Fock theory

We present here the work published in [53]

**Abstract.** We prove the occurrence of Anderson localisation for a system of infinitely many particles interacting with a short range potential, within the ground state Hartree-Fock approximation. We assume that the particles hop on a discrete lattice and that they are submitted to an external periodic potential which creates a gap in the non-interacting one particle Hamiltonian. We also assume that the interaction is weak enough to preserve a gap. We prove that the mean-field operator has exponentially localised eigenvectors, either on its whole spectrum or at the edges of its bands, depending on the strength of the disorder.

### 3.1 Introduction

In 1958, the physicist P.W. Anderson predicted that, in a random medium, diffusion could disappear and waves stay localised [12]. Since then, *Anderson localisation* has played an important role to explain several properties of physical systems.

There are many works on the mathematical side, starting with the simpler one dimensional case in the beginning of the 80s. In dimension 3, the first proof of Anderson localisation was provided by Fröhlich and Spencer [62] who developed a method now called *multiscale analysis*. There are now other approaches which include for instance the *fractional moment method* proposed by Aizenman and Molchanov [8] or the techniques proposed by Imbrie [82].

All these works are restricted to the one wave problem, which is well adapted to optics and acoustics. In condensed matter, the interaction between the particles is expected to play an important role and in this case one speaks of *many-body localisation*. This question is not fully understood and is very actively studied in physics [16, 17, 5, 71]. A typical system of interest is a crystal composed of quantum electrons and classical nuclei placed on a random perturbation of a perfect lattice [23, 100]. This is a very complicated system since the Coulomb interaction is long range and there are of the order of  $10^{23}$  particles, usually mathematically treated as an infinite number. Due to screening effects, the Coulomb potential is often replaced by an effective short range interaction and this is what we are going to do in this work.

*Finite* interacting systems have been recently considered in several works [41, 5, 94, 59] but infinite systems have not been studied thoroughly. One should however mention a series of works on superfluidity and Bose-Einstein condensation in the Lieb-Liniger 1D Bose gas in a strong random potential [125, 95], and the very recent article of Seiringer and Warzel [124] on the 1D Tonks-Girardeau gas.

It is very hard to deal with the exact interacting Schrödinger problem for an infinite system. A useful and widespread approximation is the Hartree-Fock model, where the particles are treated as independent objects but see a field which depends on their own states and is then self-consistently optimised. For random systems this model has been recently introduced by E. Cancès, S. Lahbabi and M. Lewin in [33]. The authors were able to construct a solution of the random nonlinear Hartree-Fock equation for an infinite system of fermions with short range interaction, but the phenomenon of Anderson localisation was only investigated numerically in [99]. We should also mention the papers of Erdos, Salmhofer and Yau [56] and Chen and Rodnianski [38], where the Boltzmann limit of a similar model is studied with a dynamical approach.

The Hartree Fock model and its generalisation including the pairing term are widely used in condensed matter physics [126, 135, 20, 18]. Understanding the interplay between disorder and interactions is indeed a great challenge [123]. For 1D disordered photonic lattices this was for instance studied in [101, 118].

In this work, we complete the program initiated in [33] in the case of a discrete system with a periodic background. In short, we show that the unique solution of the Hartree-Fock equation

$$\begin{cases} H_{min} = -\Delta + V_0 + V_\omega + W * \gamma(x, x) - W(x - y)\gamma(x, y) \\ \gamma = \mathbf{1}_{<\mu}(H_{min}), \end{cases} \quad (3.1)$$

provides a mean-field operator  $H_{min}$  which has exponentially localised eigenvectors, either on its whole spectrum or at the edges of its bands. Our main assumption is that the periodic potential  $V_0$  is sufficiently strong to create a gap for  $-\Delta + V_0$  and that the random part  $V_\omega$  as well as the interaction  $W$  do not alter this gap. We take the chemical potential  $\mu$  in this gap. Our argument is to adapt the well known multiscale analysis in order to include the nonlinear terms.

The paper is organised as follows. In the next section, we properly define the discrete Hartree-Fock model and state our main localisation results. Sections 3.3 to 3.6 are devoted to the proof of our results. Then in section 3.7 we present some simple one dimensional numerical simulations in order to illustrate our findings. We also look at some cases which are not covered by our theorems and find, for instance, an interesting delocalisation phenomenon at the Fermi level in the absence of a gap, which deserves further investigations.

## 3.2 Model and results

We consider a system of fermions on a subset  $\Lambda$  of the lattice  $\mathbb{Z}^d$  and which are submitted to an external potential  $V$ . Without interactions the system is described by the one-particle Hamiltonian

$$H^\Lambda = -\Delta + V. \quad (3.2)$$

In the canonical basis  $(\delta_x)_{x \in \Lambda}$  of  $\ell^2(\Lambda)$ , the operator  $H^\Lambda$  is defined by

$$\forall x, y \in \Lambda, \quad (\delta_x, H^\Lambda \delta_y) = \begin{cases} V(x) & \text{if } x = y, \\ 1 & \text{if } |x - y| = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (3.3)$$

When  $\Lambda$  is the whole lattice  $\mathbb{Z}^d$  we will use the simpler notation  $H = H^{\mathbb{Z}^d}$ . Our theorems will actually hold for finite as well as for infinite domains. Since  $H$  will later be perturbed by a non linear term describing the interactions between the particles, we will often call  $H$  the linear part.

The potential  $V$  describes a crystal lattice which is randomly perturbed. It is therefore assumed to be of the form

$$V = V_0 + V_\omega, \quad (3.4)$$

where  $V_0$  is a periodic potential (of an arbitrary period) and  $V_\omega$  is a random potential. We use the Anderson tight binding model where the value of  $V_\omega$  is chosen independently at each site of  $\Lambda$  with the same random probability law  $\mathbb{P}$ :

$$V_\omega = \sum_{i \in \Lambda} v_i(\omega) \delta_i. \quad (3.5)$$

Here the  $v_i$  are iid random variables.

Under suitable regularity assumptions on  $\mathbb{P}$ , it is well known that  $H$  displays Anderson localisation. This means that there exist small intervals at the edges of the spectral bands where the spectrum is pure point with exponentially decaying eigenvectors (“Lifshitz tail”). Moreover, if the random potential  $V_\omega$  is strong



enough (compared to  $-\Delta$ ) then the whole spectrum is pure point with localised eigenvectors.

We will show that if the interactions between the particles are small enough then the same holds for the interacting model. Before introducing interactions, we first discuss the precise assumptions that we will use for  $\mathbb{P}$  and  $V_0$ .

**(A1) Regularity of  $\mathbb{P}$ .** We assume that  $\mathbb{P}$  has a bounded support and that it has a density  $\rho$  with respect to the Lebesgue measure. Moreover we assume that there exist  $d_1 < d_2 < \dots < d_k$  such that  $\text{Supp}(\mathbb{P}) \subset [d_1, d_k]$  and  $\rho$  is Lipschitz on the intervals  $]d_i, d_i + 1[$ ,  $i = 1, \dots, (k - 1)$ .

We think that this assumption can be weakened in several possible ways but we will not discuss this for the sake of simplicity. With no loss of generality, we can assume that  $0 \in \text{Supp}(\mathbb{P})$ .

**(A2) Gap.** We assume that  $H$  has a gap  $[a, b]$  in its (deterministic [91]) spectrum.

This assumption implies that the periodic potential  $V_0$  is not constant and strong enough. Indeed the spectrum of  $H$  is almost surely equal to

$$\sigma(H) = \sigma(-\Delta + V_0) + \text{supp}(\mathbb{P}),$$

when the support of  $\mathbb{P}$  is an interval. In particular it is enough to assume that  $-\Delta + V_0$  has a gap of size  $G_0 > 2|\text{supp}(\mathbb{P})|$ . When the domain  $\Lambda$  is large enough then  $H^\Lambda$  will have a gap as well. We call the size of the gap  $G = b - a$ , and choose a chemical potential  $\mu$  in the middle  $\mu = (a + b)/2$ .

Now we turn to the definition of the interacting model. We assume that the interaction is translation-invariant, symmetric and decays fast enough.

**(A3) Short range interaction.** We assume that there exists  $\nu > 0$  such that

$$|W(x - y)| \leq C e^{-\nu|x-y|} \quad (3.6)$$

In the Hartree-Fock model (see for example [13, 15, 107, 110, 129, 14]) the system in the domain  $\Lambda$  is completely described by its one-particle density matrix which is an orthogonal projection  $\gamma$  on  $\ell^2(\Lambda)$ . More precisely, for a finite system, the rank- $N$  projection  $\gamma = \sum_{i=1}^N |\phi_i\rangle\langle\phi_i|$  corresponds to the  $N$ -particle wave function  $\Psi$ ,

$$\Psi(x_1, x_2, \dots, x_N) = \frac{1}{\sqrt{N!}} \det(\phi_i(x_j)), \quad (3.7)$$

called a Slater determinant. In a finite domain  $\Lambda$ , the many-body energy of this state is

$$\begin{aligned} & \left( \Psi, \left[ \sum_i^N -\Delta_i + V(x_i) + \frac{1}{2} \sum_{i \neq j} W(x_i - x_j) \right] \Psi \right) \\ &= \text{Tr}(H^\Lambda \gamma) + \frac{1}{2} \sum_{x, y \in \Lambda} W(x - y) \gamma(x, x) \gamma(y, y) - \frac{1}{2} \sum_{x, y \in \Lambda} W(x - y) |\gamma(x, y)|^2. \end{aligned} \quad (3.8)$$

We define the effective interaction  $A_{\text{eff}}$  by

$$A_{\text{eff}}(\gamma)(x, y) = \left[ \sum_n W(n-y)\gamma(n, n)\delta_{x=y} \right] - W(x-y)\gamma(x, y) \quad (3.9)$$

for any  $\gamma$  positive bounded operator. It is the derivative of the interaction part of the energy.

Any minimiser  $\gamma$  of the energy, with fixed rank  $N$ , is a solution of the non-linear equation which involve an effective Hamiltonian.

$$\begin{cases} H_{\min} = -\Delta + V + A_{\text{eff}}(\gamma) \\ \gamma = \mathbb{1}_{<\mu'}(H_{\min}), \end{cases} \quad (3.10)$$

In other words  $\gamma$  is the projection on the space spanned by the  $N$  eigenfunctions of  $H_{\min}$  of lower eigenvalue ( $\lambda_N(H_{\min}) \leq \mu' < \lambda_{N+1}(H_{\min})$ ) and  $H_{\min}$  itself depends of  $\gamma$ . To be more precise the equation (3.10) holds under the condition that  $\lambda_{N+1}(H_{\min}) > \lambda_N(H_{\min})$  which is known to be automatically satisfied when  $W > 0$  [14, 15].

We will first show that our problem is well defined for a bounded domain  $\Lambda$  as well as for the infinite domain  $\Lambda = \mathbb{Z}^d$ .

**Theorem 3.2.1** (HF ground states for infinitely many particles). Let  $V'$  be a bounded function such that  $-\Delta + V'$ , defined on a subset  $\Lambda$  of  $\mathbb{Z}^d$ , has a gap  $[a', b']$  in its spectrum, with  $G' = b' - a'$  and  $\mu' = (b' + a')/2$ . If  $\|W\|_{\ell^1} < G'/6$ , then there exists a unique solution of the system  $\gamma = \mathbb{1}_{\leq\mu'}(-\Delta + V' + A_{\text{eff}}(\gamma))$ . If  $\Lambda$  is finite, the trace is preserved:

$$\text{Tr}(\mathbb{1}_{\leq\mu'}(-\Delta + V' + A_{\text{eff}}(\gamma))) = \text{Tr}(\mathbb{1}_{\leq\mu'}(-\Delta + V'))$$

and, because of uniqueness, our solution is as well the unique minimiser of the energy among all Hartree-Fock states with fixed particles number  $N = \text{Tr}(\mathbb{1}_{\leq\mu'}(-\Delta + V'))$ .

This result will be shown in Section 3.3. The continuous case is more complicated and has been studied in [100]. When  $\Lambda$  is a large cube and  $V' = V = V_0 + V_\omega$  then the number of particles  $\text{Tr}(\mathbb{1}_{\leq\mu'}(-\Delta + V'))$  is proportional to the volume, which is the interesting physical case. For  $\Lambda = \mathbb{Z}^d$ ,  $H_{\min}(\omega)$  exists and its spectrum, as in the linear model, does not depend on  $\omega$  almost surely. The reason is that  $H_{\min}$  is stationary with respect to space translations, which follows from the uniqueness in the theorem.

The main result of our paper is the following:

**Theorem 3.2.2** (Anderson localisation in Hartree-Fock theory). Under the assumptions (A1), (A2) and (A3), then the following holds:

1. There is  $\epsilon$  such that if  $\|W\|_{\ell^1} < \epsilon$  then there are small intervals at the edges of the bands of the spectrum of  $H_{\min}$ , where the spectrum is pure point with exponentially decaying eigenvectors.
2. There exists  $\epsilon$  which only depends on  $C$  and  $\nu$  in (3.6) such that if  $\|\rho\|_\infty + \|\rho'\|_\infty < \epsilon$  and  $\|W\|_{\ell^1} < G/6$ , then the whole spectrum of  $H_{\min}$  is pure point and its eigenvectors are exponentially decaying in space.

We remark that because of **(A2)**, the second part of the statement can only be applied in cases where  $V_0$  is also very large. This theorem will be shown by using the multiscale analysis on  $H_{\min}$ . We will proceed in three steps.

**Step 1.** We show that because of the gap,  $\gamma_{min}$  only depends locally on the potential. From a practical point of view, in order to know how  $\gamma_{min}$  looks like in a box of size  $L$  after solving the minimising problem for  $\mathbb{Z}^d$ , solving the minimising problem for a box of size  $2L$  will be enough to have a very good approximation. In a more mathematical formulation:

**Theorem 3.2.3** (Locality). We assume that (A2) and (A3) hold. There exist  $a > 0$ ,  $\nu > 0$  and  $C > 0$  such that, if  $\|W\|_{\ell^1} < aG$ , then for any modification of the potential  $V_\omega \rightarrow V_\omega + \delta V$  so that it does not bridge the gap and the gap stays large, the change induced to the minimising projector given by Theorem 3.2.1 satisfies

$$\sup_{y_2 \in \Lambda} |\gamma_{min}(V + \delta V)(y_1, y_2) - \gamma_{min}(V)(y_1, y_2)| \leq \|\delta V\| C e^{-\nu d(y_1, \text{supp}(\delta V))}. \quad (3.11)$$

Here  $d(y_1, \text{supp}(\delta V))$  is the distance between  $y_1$  and the support of the perturbation  $\delta V$ . And  $\|\delta V\|$  is the norm of  $\delta V$  seen as a multiplicative operator. The constants do not depend on the choice of the domain  $\Lambda$ .

In practice, we will use this result when we change the random realisation of the potential:  $V_\omega + \delta V$  belongs to  $\text{supp}(\mathbb{P})$

This theorem will be proved in Section 3.4.

**Step 2.** We then show in section 3.5 a kind of Wegner estimate. We denote by:  $(H_{min})|_\Lambda$  the submatrices  $\mathbb{1}_\Lambda H_{min} \mathbb{1}_\Lambda$ , restricted to  $\ell^2(\Lambda)$  where  $\Lambda$  is a finite cube in  $\mathbb{Z}^d$ .

**Theorem 3.2.4** (Wegner estimate). Assuming (A1), (A2), (A3), there exists  $a > 0$  such that if  $\|W\|_{\ell^1} < aG$  then there exists a constant  $C$  so that

$$\begin{aligned} & \mathbb{P}[d(\sigma[(H_{min})|_\Lambda], \lambda) < \epsilon] \\ & \leq C|\Lambda|\sqrt{\epsilon} \left( |\text{Supp}(\rho)|^{-1/2} + \|\rho\|_\infty |\text{Supp}(\rho)|^{1/2} + \|\rho'\|_\infty |\text{Supp}(\rho)|^{3/2} \right), \end{aligned} \quad (3.12)$$

for any  $\lambda \in \mathbb{C}$ .

This result says that there is no arbitrary small interval where we can find for sure an eigenvalue of  $(H_{min})|_\Lambda$ .

**Step 3.** In the last step we perform the multiscale analysis. This is explained in Section 3.6.

### 3.3 Construction of the mean-field Hamiltonian: proof of Theorem 3.2.1

The aim of this section is to prove that our operator  $H_{min}$  is well defined. We will show that finding the unique solution  $\gamma_{min}$  in presence of a gap can be done using a fixed point lemma.

In this subsection we solve our system

$$\begin{cases} H_{min} = -\Delta + V' + A_{\text{eff}}(\gamma) \\ \gamma = \mathbb{1}_{<\mu'}(H_{min}) \end{cases} \quad (3.13)$$

under the assumption that  $\mu'$  is inside a gap  $[a', b']$  in the spectrum, where  $\mu' = (a' + b')/2$ . We introduce  $G' = b' - a'$ .

Let  $\mathcal{C}$  be a loop in the complex plane surrounding a part  $I$  of the spectrum of an operator  $H$ . We make the assumption that the loop does not cross the spectrum (which implies that there exist gaps above and below  $I$ ). Then

$$\mathbb{1}_I(H) = \frac{1}{2i\pi} \oint_{\mathcal{C}} (H - z)^{-1} dz \quad (3.14)$$

is the projector on the spectral subspace associated with  $I$ .

Let us define an application that gives this projector.

**Definition 3.3.1** (Fixed point map). Let  $\mathcal{C}$  be a fixed loop in the complex plane. For all  $\gamma$  orthogonal projector and  $H_{\text{eff}}(\gamma) = -\Delta + V' + A_{\text{eff}}(\gamma)$ , we define

$$F(\gamma) = \frac{1}{2i\pi} \oint_{\mathcal{C}} (H_{\text{eff}}(\gamma) - z)^{-1} dz. \quad (3.15)$$

This application enable us to reformulate our system (3.13) as

$$F(\gamma) = \gamma, \quad (3.16)$$

where the loop  $\mathcal{C}$  crosses the real axis at  $\mu'$ . Recall that  $H_{\text{min}}$  is bounded so that we can always enclose all of its spectrum below  $\mu'$ . Because  $\|A_{\text{eff}}\|$  is bounded by  $2\|W\|_{\ell^1}$ , we always have

$$d(\sigma(H_{\text{eff}}(\gamma)), \mu') \geq \frac{b' - a'}{2} - 2\|W\|_{\ell^1}. \quad (3.17)$$

So  $(b' - a')/2 > 2\|W\|_{\ell^1}$  is enough to ensure that  $\mathcal{C}$  never crosses the spectrum of  $H_{\text{eff}}$  and  $F$  is always well defined. In order to solve (3.16) we will show that if  $\|W\|_{\ell^1}$  is small enough then  $F$  is a contraction.

*Proof of Theorem 3.2.1.* Let  $\gamma_1$  and  $\gamma_2$  be two orthogonal projectors. Then we have

$$\begin{aligned} & F(\gamma_1) - F(\gamma_2) \\ &= -\frac{1}{2i\pi} \oint_{\mathcal{C}} (H_{\text{eff}}(\gamma_2) - z)^{-1} (H_{\text{eff}}(\gamma_1) - H_{\text{eff}}(\gamma_2)) (H_{\text{eff}}(\gamma_1) - z)^{-1} dz \\ &= -\frac{1}{2i\pi} \oint_{\mathcal{C}} (H_{\text{eff}}(\gamma_2) - z)^{-1} (A_{\text{eff}}(\gamma_1) - A_{\text{eff}}(\gamma_2)) (H_{\text{eff}}(\gamma_1) - z)^{-1} dz. \end{aligned}$$

The map  $F$  does not depend on the choice of the surrounding loop provided it encloses the appropriate part of the spectrum. Expanding it continuously to infinity, we can replace it in this formula by  $\mathcal{C} = \mu' + i\mathbb{R}$  and write  $z = \mu' + is$ . We estimate  $A_{\text{eff}}(\gamma_1) - A_{\text{eff}}(\gamma_2) = A_{\text{eff}}(\gamma_1 - \gamma_2)$  with  $2\|W\|_{\ell^1}\|\gamma_1 - \gamma_2\|$  and  $(H_{\text{eff}}(\gamma_2) - \mu' - is)^{-1} \leq ((G'/2 - 2\|W\|_{\ell^1})^2 + s^2)^{-1/2}$ . Therefore

$$\|F(\gamma_1) - F(\gamma_2)\| \leq \frac{\|W\|_{\ell^1}\|\gamma_1 - \gamma_2\|}{\pi} \int_{\mathbb{R}} ((G'/2 - 2\|W\|_{\ell^1})^2 + s^2)^{-1} ds, \quad (3.18)$$

so that

$$\|F(\gamma_1) - F(\gamma_2)\| \leq \frac{\|W\|_{\ell^1}\|\gamma_1 - \gamma_2\|}{(G'/2 - 2\|W\|_{\ell^1})}. \quad (3.19)$$

Now, if  $\|W\|_{L^1}$  is smaller than  $G'/6$ , then  $F$  is contracting, so it has a unique fixed point.  $\square$

This concludes the first section, we have shown that in the presence of a gap, (3.13) has always a unique solution.

### 3.4 Local influence: proof of Theorem 3.2.3

The aim of this section is to show that under hypothesis (A2) and (A3) the random potential in a domain  $\Lambda_2$  will only have a very small influence on  $A_{\text{eff}}$  in  $\Lambda_1$  if  $\Lambda_2$  is far enough from  $\Lambda_1$ . This implies a weak form of independence between the submatrices  $(H_{\text{min}})_{|\Lambda_1}$  and  $(H_{\text{min}})_{|\Lambda_2}$  which is necessary for the multiscale analysis. The key tool is a Combes-Thomas estimate.

#### 3.4.1 Combes-Thomas estimate

Because we want to use it for more general operators than just the Laplacian, we have written again the details of the proof.

**Definition 3.4.1** (Exponential off-diagonal decay operator). We will say that an operator  $K$  on  $L^2(\Lambda)$  has exponential off-diagonal decay if there exist a rate  $M > 0$ , and a constant  $C$  so that  $(\delta_x, K\delta_y) \leq C \exp(-M|x-y|)$  for all  $x, y \in \Lambda$ .

In our case, we note that  $-\Delta + V + A_{\text{eff}}$  is an exponential decay operator if  $W(x-y)$  decays exponentially.

**Lemma 3.4.1** (Combes-Thomas estimate). Let  $K$  be an exponential off-diagonal decay operator and  $\Sigma$  be its spectrum. Let  $\lambda \in \mathbb{C}$  so that  $d(\lambda, \Sigma) > 0$ . Then there exist  $\nu > 0$  and  $C > 0$  such that

$$(\delta_x, (K - \lambda)^{-1}\delta_y) \leq Ce^{-\nu|x-y|}. \quad (3.20)$$

*Proof.* Let  $f(z) := e^{-\nu|x-z|}$  we have

$$\begin{aligned} (\delta_x, (K - \lambda)^{-1}\delta_y) &= (\delta_x, f(z)^{-1}f(z)(K - \lambda)^{-1}f(z)^{-1}f(z)\delta_y) \\ &= f(x)^{-1}f(y)(\delta_x, f(z)(K - \lambda)^{-1}f(z)^{-1}\delta_y) \\ &= e^{-\nu|x-y|}(\delta_x, (f(z)Kf(z)^{-1} - \lambda)^{-1}\delta_y). \end{aligned} \quad (3.21)$$

Then we have

$$((f(z)Kf(z)^{-1} - K)u)(n) = \sum_{m \in \Lambda} (e^{-\nu(|x-n|-|x-m|)} - 1)K(m, n)u(m), \quad (3.22)$$

so

$$\|(f(z)Kf(z)^{-1} - K)u\| \leq C \sum_{m \in \Lambda} (e^{\nu(|n-m|)} - 1)e^{-M(n-m)}|u(m)| < \infty,$$

and therefore

$$\|(f(z)Kf(z)^{-1} - H)\| \leq C\|(e^{\nu|x|} - 1)e^{-M|x}\|_{\ell^1}.$$

Because of the dominated convergence theorem, this converges to 0 with  $\nu$  going to 0. So there exist  $\nu > 0$  so that  $\|(f(z)Kf(z)^{-1} - H)\| < d(\lambda, \Sigma)$  and

$$d(\lambda, \Sigma(f(z)Kf(z)^{-1})) \geq d(\lambda, \Sigma) - \|(f(z)Kf(z)^{-1} - H)\| > 0.$$

So there exists  $C'$  such that

$$\|(f(z)Kf(z)^{-1} - \lambda)^{-1}\| \leq \frac{1}{d(\lambda, \Sigma(f(z)Kf(z)^{-1}))} = C'$$

and using (3.21) we find

$$(\delta_x, (K - \lambda)^{-1}\delta_y) \leq C' e^{-\nu|x-y|}.$$

□

### 3.4.2 Local influence

We prove here Theorem 3.2.3. We use again the map defined in Section 3.3.

$$F_V(\gamma) = \frac{1}{2i\pi} \oint_{\mathcal{C}} (-\Delta + V + A_{\text{eff}}(\gamma) - z)^{-1} dz, \quad (3.23)$$

where  $\mathcal{C}$  is the loop enclosing the whole part of the spectrum below the middle of the gap  $\mu$ . We will denote by  $\gamma_{\min}(V)$  the solution of the system given by Theorem 3.2.1 and recall that  $F_V(\gamma_{\min}(V)) = \gamma_{\min}(V)$ .

*Proof of Theorem 3.2.3.* Because we can write  $\delta V = \sum_{k=1}^K \delta V / K$  and apply our theorem  $K$  times we can suppose  $\delta V$  arbitrary small. We are looking for the new fixed point for  $F_{V+\delta V}$  which is the limit of  $(F_{V+\delta V})^n(\gamma)$ . We start from  $\gamma = \gamma_{\min}(V)$  and remark that

$$\begin{aligned} (\gamma_{\min}(V + \delta V) - \gamma) &= \lim_{n \rightarrow \infty} \left( (F_{V+\delta V})^n(\gamma) - \gamma \right) \\ &= \sum_{n=0}^{\infty} \left( (F_{V+\delta V})^{n+1}(\gamma) - (F_{V+\delta V})^n(\gamma) \right). \end{aligned} \quad (3.24)$$

Before starting the calculation, we note that because of the proof of Theorem 3.2.1, we already have that  $F_{V+\delta V}$  is a contracting operator. Then we have

$$\|(F_{V+\delta V})^{n+1}(\gamma) - (F_{V+\delta V})^n(\gamma)\| \leq C\tau^n.$$

with  $\tau < 1$ . We now prove that this finite perturbation is concentrated around  $\Omega$ .

**Step 1.** We evaluate the first term of the sum  $F_{V+\delta V}(\gamma) - \gamma = F_{V+\delta V}(\gamma) - F_V(\gamma)$ , as follows

$$\begin{aligned} &F_{V+\delta V}(\gamma) - F_V(\gamma) \\ &= \frac{1}{2i\pi} \oint_{\mathcal{C}} \frac{1}{H_{\text{eff}}(\gamma) + \delta V - z} - \frac{1}{H_{\text{eff}}(\gamma) - z} dz \\ &= \frac{1}{2i\pi} \oint_{\mathcal{C}} \frac{1}{H_{\text{eff}}(\gamma) - z} \left( \delta V \sum_{k \geq 0} ((H_{\text{eff}}(\gamma) - z)^{-1} \delta V)^k \right) \frac{1}{H_{\text{eff}}(\gamma) - z} dz \\ &= \frac{1}{2i\pi} \oint_{\mathcal{C}} \frac{1}{H_{\text{eff}}(\gamma) - z} B(\delta V) \frac{1}{H_{\text{eff}}(\gamma) - z} dz, \end{aligned}$$

where

$$B(\delta V) = \delta V \sum_{k \geq 0} ((H_{\text{eff}}(\gamma) - z)^{-1} \delta V)^k. \quad (3.25)$$

This sum converges as soon as  $\|\delta V\| < \frac{G}{6}$ . Note that if the support of  $\delta V$  has a bounded support  $\Omega$  so has  $B(\delta V)$ . We will use the Combes-Thomas estimate (3.20) for  $y_1, y_2$  outside  $\Omega$ . Remark that if we choose the loop  $\mathcal{C}$  correctly,  $\nu$  does not depend on  $z$  but only on the size on the gap. We find

$$\begin{aligned} & |(F_{V+\delta V}(\gamma))(y_1, y_2) - (F_V(\gamma))(y_1, y_2)| \\ &= \oint_{\mathcal{C}} (y_1, \frac{1}{H_{\text{eff}}(\gamma) - z} B(\delta V) \frac{1}{H_{\text{eff}}(\gamma) - z} y_2) dz \\ &\leq \frac{1}{2i\pi} \oint_{\mathcal{C}} C^2 \sum_{x, x' \in \Omega} e^{-\nu|y_1-x|} (x, B(\delta V)x') e^{-\nu|x'-y_2|} dz \\ &\leq |\mathcal{C}| C^2 \sum_{|x-y_1| \geq d(y_1, \Omega)} \sum_{x' \in \Omega} e^{-\nu|y_1-x|} (x, B(\delta V)x') e^{-\nu|x'-y_2|} dz \end{aligned}$$

We can now use that

$$\sum_{|x-y_1| \geq d(y_1, \Omega)} e^{-\nu|y_1-x|} \leq C|y_1 - x|^{d-1} e^{-\nu d(y_1, \Omega)} \leq C' e^{-\nu' d(y_1, \Omega)}$$

with  $0 < \nu' < \nu$ .

$$\begin{aligned} & |(F_{V+\delta V}(\gamma))(y_1, y_2) - (F_V(\gamma))(y_1, y_2)| \\ &\leq C^2 |\mathcal{C}| e^{-\nu' d(y_1, \Omega)} \sum_{x' \in \Omega} \max_{x \in \mathbb{Z}^d} |(x, B(\delta V)x') e^{-\nu|x'-y_2|}| \\ &\leq C'' \|\delta V\| e^{-\nu' d(y_1, \Omega)} \end{aligned}$$

where  $C''$  is just a constant. With  $\|W\|_{\ell^1}$  small enough, there exists  $\tau < 1$  so that

$$[A_{\text{eff}}(F_{V+\delta V}(\gamma)) - A_{\text{eff}}(\gamma)](y_1, y_2) \leq \tau \|\delta V\| e^{-\nu' d(y_1, \Omega)}. \quad (3.26)$$

**Step 2.** We evaluate the remainder of the sum. We repeat the previous argument with

$$\delta A_{\text{eff}} = A_{\text{eff}}(F_{V+\delta V}^{n-1}(\gamma)) - A_{\text{eff}}(F_{V+\delta V}^{n-2}(\gamma)) \quad (3.27)$$

instead of  $\delta V$ .

There is only one little difference:  $B$  does not have a bounded support any more but still has an off-diagonal exponential decay (proved by iteration with constant  $\nu > \nu' > 0$ ). We just check this does not bring more difficulties:

We carry on the calculation

$$\begin{aligned}
& |(F_{V+\delta V})^n(\gamma)(y_1, y_2) - (F_{V+\delta V})^{n-1}(\gamma)(y_1, y_2)| \\
& \leq C^2 \sum_{x, x' \in \Omega} e^{-\nu|y_1-x|} (x, B(\delta A_{\text{eff}})x') e^{-\nu|x'-y_2|} dy \\
& \leq C^2 \|\delta A_{\text{eff}}\| \sum_{x, x' \in \mathbb{Z}^2} e^{-\nu|y_1-x|} e^{-\nu'd(x, \Omega)} e^{-\nu'd(x', \Omega)} e^{-\nu|x'-y_2|} \\
& \leq C_2 \|\delta A_{\text{eff}}\| e^{-\nu'd(y_1, \Omega)} \sum_{x, x' \in \mathbb{Z}^2} e^{\nu'd(y_1, \Omega) - \nu'|y_1-x| - \nu'd(x, \Omega)} e^{-(\nu-\nu')|y_1-x|} \\
& \quad \times e^{-\nu d(x', \Omega)} e^{-\nu|x'-y_2|} \\
& \leq C'_2 \|(F_{V+\delta V}^{n-1}(\gamma)) - F_{V+\delta V}^{n-2}(\gamma)\| e^{-\nu'd(y_1, \Omega)},
\end{aligned}$$

where  $C'_2$  is just another constant. We can conclude by iteration that

$$|A_{\text{eff}}(F_{V+\delta V}^n(\gamma))(y_1, y_2) - A_{\text{eff}}(F_V^{n-1}(\gamma))(y_1, y_2)| \leq \|\delta V\| \tau^n e^{-\nu'd(y_1, \Omega)}, \quad (3.28)$$

and so

$$\begin{aligned}
& |(A_{\text{eff}}(\gamma_{\min}(V + \delta V)) - A_{\text{eff}}(\gamma_{\min}(V)))(y_1, y_2)| \\
& = \lim_{n \rightarrow \infty} |A_{\text{eff}}(F_{V+\delta V}^n(\gamma))(y_1, y_2) - A_{\text{eff}}(\gamma)(y_1, y_2)| \\
& \leq \|\delta V\| \frac{1}{1-\tau} e^{-\nu'd(y_1, \text{supp}(\delta V))},
\end{aligned}$$

as we wanted.  $\square$

### 3.5 Wegner estimate: proof of Theorem 3.2.4

In this section, we prove the Wegner-type estimate in Theorem 3.2.4.

The idea of the proof is the following. At first sight we do not have any idea of how  $A_{\text{eff}}$  looks like. But because of the gap, if

$$\gamma = \mathbf{1}_{\leq \mu}(-\Delta + V + A_{\text{eff}}(\gamma)) \quad (3.29)$$

then

$$\gamma = \mathbf{1}_{\leq (\mu+\alpha)}(-\Delta + V + A_{\text{eff}}(\gamma)) = \mathbf{1}_{\leq (\mu)}(-\Delta + V - \alpha \mathbf{1}_{\mathbb{Z}^d} + A_{\text{eff}}(\gamma)) \quad (3.30)$$

for any  $\alpha \in \mathbb{R}$  with  $2|\alpha|$  smaller than the gap. So if we could add  $\alpha$  to the random potential with  $\alpha$  a smooth random variable, in this case every eigenvalue of  $H_{\min}$  would just be offset by  $\alpha$  no matter what the non linear part is and we are done. In our case, we will make the change of variable  $(V_\omega(x)) \rightarrow (\alpha = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} V_\omega(x), V_\omega(x) - \alpha)$  for  $x \in \Lambda$  and we expect that the conditional density of  $\alpha$  is smooth enough and that the change induced to  $\gamma$  is small.

*Proof of Theorem 3.2.4.* Let  $\Lambda = \Lambda_L(n)$  be the cube in  $\mathbb{Z}^d$  of size  $L$  with its



center in  $n$  and  $\Lambda_{2L}(n)$  the cube twice bigger. Because of (3.30) and (3.11)

$$\begin{aligned}
& \left\| \frac{d}{d\alpha} ((\gamma_{min})(V + \alpha \mathbf{1}_{\Lambda_{2L}(n)})_{|\Lambda_L(n)}) \right\| \\
&= \left\| \frac{d}{d\alpha} ((\gamma_{min})_{|\Lambda_L(n)}(V + \alpha \mathbf{1}_{\mathbb{Z}^d} - \alpha \mathbf{1}_{\Lambda_{2L}(n)^c}) \right\| \\
&= \left\| \frac{d}{d\alpha} ((\gamma_{min})_{|\Lambda_L(n)}(V - \alpha \mathbf{1}_{\Lambda_{2L}(n)^c}) \right\| \\
&\leq C e^{-\nu d(\Lambda_L(n), \Lambda_{2L}(n)^c)} \\
&\leq C e^{-\nu L}.
\end{aligned}$$

We suppose that  $L$  is large enough so that  $2\|W\|_{\ell^1} C e^{-\nu L} \leq 1/2$  and we obtain that

$$\alpha \rightarrow \|\mathbf{1}_{\Lambda_L(n)} A_{\text{eff}}(V + \alpha \mathbf{1}_{\Lambda_{2L}(n)}) \mathbf{1}_{\Lambda_L(n)}\| \quad (3.31)$$

is  $1/2$  Lipschitz. Under this hypothesis, for any  $\lambda_i(\alpha)$  eigenvalue of

$$\left( -\Delta + V + \alpha \mathbf{1}_{\Lambda_{2L}(n)} + A_{\text{eff}}(\gamma_{min}(V + \alpha \mathbf{1}_{\Lambda_{2L}(n)})) \right)_{|\Lambda_L(n)},$$

we have

$$\frac{d}{d\alpha} \lambda_i(\alpha) \geq 1 - \left\| \frac{d}{d\alpha} (\mathbf{1}_{\Lambda_L(n)} A_{\text{eff}}(V + \alpha \mathbf{1}_{\Lambda_{2L}(n)}) \mathbf{1}_{\Lambda_L(n)}) \right\| \geq \frac{1}{2}. \quad (3.32)$$

Let  $\lambda \in \mathbb{R}$  and  $\epsilon > 0$ . Let  $D_0 = \{d_1 < d_2 < \dots < d_k\}$  be so that  $\rho(s)$  is Lipschitz on  $]d_n, d_{n+1}[$ . Let  $f$  and  $\delta$  be two positive functions that will be chosen later. We define the following events

$$O_x := \{\omega : \forall y \text{ such that } |y - V_\omega(x)| < \delta(\epsilon) : \rho(y) > f(\epsilon), \text{ and } d(V_\omega(x), D_0) > \delta(\epsilon)\} \quad (3.33)$$

for any  $x \in \Lambda_{2L}(n)$ . We now estimate

$$\begin{aligned}
& \mathbb{P}\left(d(\sigma[(H_{min})_{|\Lambda_L(n)}], \lambda) < \epsilon\right) \\
&\leq \mathbb{P}\left(\cup_{x \in \Lambda_{2L}(n)} O_x^c\right) + \mathbb{P}\left(\cap_{x \in \Lambda_{2L}(n)} O_x \cap d(\sigma[(H_{min})_{|\Lambda_L(n)}], \lambda) < \epsilon\right).
\end{aligned} \quad (3.34)$$

We will deal with each term separately. Starting with the left term, we erase the indices because the probability does not depend of the position and argue as follows:

$$\begin{aligned}
& \mathbb{P}\left(\cup_{x \in \Lambda_{2L}(n)} O_x^c\right) \\
&\leq |\Lambda_{2L}(n)| \mathbb{P}(O_x^c) \\
&= 2^d |\Lambda| \mathbb{P}(O^c) \\
&\leq 2^d |\Lambda| (\mathbb{P}(d(V_\omega, D_0) < \delta(\epsilon)) + \mathbb{P}(V_\omega : \exists y : |y - V_\omega| < \delta(\epsilon), \rho(y) < f(\epsilon))) \\
&\leq 2^d |\Lambda| (\mathbb{P}(d(V_\omega, D_0) < \delta(\epsilon)) + \mathbb{P}(V_\omega : \rho(V_\omega) < f(\epsilon) + \delta(\epsilon) \|\rho'\|_\infty)) \\
&\leq 2^d |\Lambda| \left( \|\rho\|_\infty 2\delta(\epsilon) \#|D_0| + (\delta(\epsilon) \|\rho'\|_\infty + f(\epsilon)) |\text{supp}(\rho)| \right).
\end{aligned} \quad (3.35)$$

The right term in (3.34) can be estimated by introducing the mean and the resolvent, using

$$\mathbb{1}_{[\lambda-\epsilon, \lambda+\epsilon]}(\alpha) \leq \frac{2\epsilon^2}{((\lambda-\alpha)^2 + \epsilon^2)} = 2\epsilon \Im\left(\frac{1}{\lambda-\alpha+i\epsilon}\right). \quad (3.36)$$

We simplify a bit the notation using  $\cap_x O_x$  instead of  $\cap_{x \in \Lambda_{2L}(n)} O_x$ . We get

$$\begin{aligned} & \mathbb{P}\left(\cap_x O_x \cap d(\sigma[(H_{min})|_{\Lambda_L(n)}], \lambda) < \epsilon\right) \\ &= \mathbb{E}\left[\mathbb{1}_{d(\sigma(H_{min}^\Lambda), \lambda) < \epsilon} \mathbb{1}_{\cap_x O_x}\right] \\ &\leq 2\epsilon \mathbb{E}\left[\Im\left(\text{Tr}[(H_{min})|_{\Lambda_L(n)} - \lambda + i\epsilon]^{-1}\right) \mathbb{1}_{\cap_x O_x}\right]. \end{aligned}$$

We now make a change of variable for  $V_\omega(x) \in \Lambda_{2L}(n)$  :

$$(V_\omega(x)) \rightarrow \left(\alpha = \frac{1}{2^d |\Lambda|} \sum_{x \in \Lambda_{2L}(n)} V_\omega(x), V_\omega(x) - \alpha\right). \quad (3.37)$$

We write  $\tilde{V}_\omega(x) = V_\omega(x) - \alpha$  and  $\xi_{\tilde{V}}(\alpha)$  for the conditional random density of the mean knowing  $\tilde{V}$ . We first integrate over  $\alpha$ , then over  $\tilde{V}$  (we denote the expectation by  $\mathbb{E}_{\tilde{V}}$ ):

$$\begin{aligned} & \mathbb{P}\left(\cap_x O_x \cap d(\sigma[(H_{min})|_{\Lambda_L(n)}], \lambda) < \epsilon\right) \\ &\leq \mathbb{E}_{\tilde{V}}\left[2\epsilon \int \Im\left(\text{Tr}[(H_{min})|_{\Lambda_L} - \lambda + i\epsilon]^{-1}\right) \mathbb{1}_{\cap_x O_x} \xi_{\tilde{V}_i}(\alpha) d\alpha\right] \\ &\leq 2|\epsilon| \mathbb{E}_{\tilde{V}}\left[\sum_{\lambda_i \in \sigma((H_{min})|_{\Lambda_L})} \int \Im\left(\frac{1}{(\lambda_i(\alpha) - \lambda + i\epsilon)}\right) \mathbb{1}_{\cap_x O_x} \xi_{\tilde{V}_i}(\alpha) d\alpha\right]. \end{aligned}$$

We estimate the integral with a change of variable  $\alpha' = (\lambda_i(\alpha) - \lambda)$ ,  $d\alpha' = \left(\frac{d}{d\alpha} \lambda_i\right) d\alpha$ . Recall that

$$\int \frac{\epsilon}{(\alpha')^2 + \epsilon^2} \frac{\xi_{\tilde{V}}(\alpha) \mathbb{1}_{\cap_x O_x}}{\frac{d}{d\alpha} \lambda_i} d\alpha' \leq \pi \sup\left[\frac{|\xi_{\tilde{V}}(\alpha) \mathbb{1}_{\cap_x O_x}|}{\frac{d}{d\alpha} \lambda_i}\right]. \quad (3.38)$$

So we have

$$\mathbb{P}\left(\cap_x O_x \cap d(\sigma[(H_{min})|_{\Lambda_L(n)}], \lambda) < \epsilon\right) \leq 2\pi |\epsilon| |\Lambda_L| \mathbb{E}_{\tilde{V}}\left[\sup\left[\frac{|\xi_{\tilde{V}}(\alpha) \mathbb{1}_{\cap_x O_x}|}{\frac{d}{d\alpha} \lambda_i}\right]\right].$$

Finally, because of (3.32),  $\frac{d}{d\alpha} \lambda_i > 1/2$  and we get

$$\mathbb{P}\left(\cap_x O_x \cap d(\sigma[(H_{min})|_{\Lambda_L(n)}], \lambda) < \epsilon\right) \leq 4\pi |\epsilon| |\Lambda_L| \sup[\xi_{\tilde{V}_i} \mathbb{1}_{\cap_x O_x}]. \quad (3.39)$$

From now on, it is enough to have an estimate on  $\xi_{\tilde{V}} \mathbb{1}_{\cap_x O_x}$ . A computation gives

$$\xi_{\tilde{V}_i}(\alpha) d\alpha = \mathbb{P}\left(\sum_x V_\omega(x) \in [\alpha, \alpha + d\alpha] | \tilde{V}\right) = \frac{\prod_{i \in \Lambda} \rho(\tilde{V}_i + \alpha)}{\int \prod_{i \in \Lambda} \rho(\tilde{V}_i + \alpha') d\alpha'} d\alpha. \quad (3.40)$$

Let  $\alpha_0 \in \mathbb{R}$ . If we do not have  $\tilde{V}_\omega(x) + \alpha_0 \in O_x$  for all  $x$ , then  $\xi_{\tilde{V}}(\alpha_0)\mathbb{1}_{\cap_x O_x} = 0$  and we have nothing else to do. So we can assume that  $\forall \alpha, |\alpha - \alpha_0| < \delta(\epsilon) \Rightarrow \rho(\tilde{V}(x) + \alpha) > f(\epsilon)$  for all  $x \in \Lambda_{2L}(n)$  and all  $|\alpha - \alpha_0| < \delta(\epsilon)$ . So we have

$$\frac{\frac{d}{d\alpha} \prod_{x \in \Lambda_{2L}} \rho(\tilde{V} + \alpha)}{\prod_{x \in \Lambda_{2L}} \rho(\tilde{V} + \alpha)} = \sum_{x \in \Lambda_{2L}} \frac{\rho'(\tilde{V}_\omega(x) + \alpha)}{\rho(\tilde{V}_\omega(x) + \alpha)} \leq 2^d |\Lambda_L| \frac{\|\rho'\|_\infty}{f(\epsilon)}. \quad (3.41)$$

From this differential equation we get

$$\prod_{x \in \Lambda_{2L}} \rho(\tilde{V}_\omega(x) + \alpha) \geq \exp\left(-(|\alpha - \alpha_0|)2^d |\Lambda_L| \frac{\|\rho'\|_\infty}{f(\epsilon)}\right) \prod_{x \in \Lambda_{2L}} \rho(\tilde{V}_\omega(x) + \alpha_0)$$

and, after integrating,

$$\begin{aligned} & \int_{\alpha_0 - \delta(\epsilon)}^{\alpha_0 + \delta(\epsilon)} \prod \rho(\tilde{V}_\omega(x) + \alpha) d\alpha \\ & \geq [1 - \exp(-|\delta(\epsilon)|2^d |\Lambda_L| \frac{\|\rho'\|_\infty}{f(\epsilon)})] \frac{f(\epsilon)}{2^d |\Lambda_L| \|\rho'\|_\infty} \prod \rho(\tilde{V}_\omega(x) + \alpha_0). \end{aligned}$$

Therefore

$$\frac{\prod \rho(\tilde{V}_\omega(x) + \alpha_0)}{\int \prod \rho(\tilde{V}_\omega(x) + \alpha) d\alpha} \leq \frac{2^d |\Lambda_L| \|\rho'\|_\infty}{f(\epsilon)(1 - \exp(-(|\delta(\epsilon)|)2^d |\Lambda_L| \frac{\|\rho'\|_\infty}{f(\epsilon)})}$$

and hence we have

$$\frac{\prod \rho(\tilde{V}_\omega(x) + \alpha_0)}{\int \prod \rho(\tilde{V}_\omega(x) + \alpha) d\alpha} \leq 2 \max\left(\frac{2^d |\Lambda_L| \|\rho'\|_\infty}{f(\epsilon)}, \frac{1}{\delta(\epsilon)}\right). \quad (3.42)$$

We finally obtain

$$\xi_{\tilde{V}_i} \mathbb{1}_{\cap_x O_x} \leq 2 \max\left(\frac{2^d |\Lambda_L| \|\rho'\|_\infty}{f(\epsilon)}, \frac{1}{\delta(\epsilon)}\right), \quad (3.43)$$

for all  $\hat{V}$  and all  $\alpha_0$ .

To conclude, putting (3.34), (3.35), (3.39) and (3.43) together, we have

$$\begin{aligned} & \mathbb{P}\left(d(\sigma[(H_{min})|_{\Lambda_L(n)}], \lambda) < \epsilon\right) \\ & \leq 2^d |\Lambda| \left[ \|\rho\|_\infty \delta(\epsilon) \# D_0 + (\delta(\epsilon) \|\rho'\|_\infty + f(\epsilon)) |\text{supp}(\rho)| \right. \\ & \quad \left. + 8\pi\epsilon \cdot \max\left(\frac{|\Delta| \|\rho'\|_\infty}{f(\epsilon)}, \frac{1}{\delta(\epsilon)}\right) \right], \end{aligned}$$

from which we can conclude (3.12) choosing  $f(\epsilon) = \sqrt{\epsilon}/|\text{supp}(\rho)|^{-3/2}$  and  $\delta(\epsilon) = \sqrt{\epsilon}|\text{supp}(\rho)|^{1/2}$ . Hence

$$\begin{aligned} & \mathbb{P}\left(d(\sigma[(H_{min})|_{\Lambda_L(n)}], \lambda) < \epsilon\right) \\ & \leq C |\Lambda| \left[ |\text{supp}(\rho)|^{-1/2} + \|\rho\|_\infty |\text{supp}(\rho)|^{1/2} + \|\rho'\|_\infty |\text{supp}(\rho)|^{3/2} \right], \end{aligned}$$

where the constant  $C$  only depends on the cardinal of  $D_0$ .  $\square$

In particular, with the change  $V \rightarrow \ell V$ , we deduce that

$$\mathbb{P}\left(d(\sigma[(H_{min})|_{\Lambda_L(n)}], \lambda) < \epsilon\right) \rightarrow 0 \quad (3.44)$$

when  $\ell \rightarrow \infty$ .

### 3.6 Multiscale analysis

We will now start the proof of the multiscale analysis. There will be very little differences with the proof we can find in [90, Part 10] and we will follow the method exposed there step by step. But because it is a more general case, we have written the proof again.

#### 3.6.1 The setting

For any operator  $K$  with off-diagonal exponential decay and  $\Lambda \subset \mathbb{Z}^d$ , we define a border operator  $\Gamma$  by

$$\Gamma_{K,\Lambda}(x, y) = \begin{cases} K(x, y) & \text{if } (x \in \Lambda \text{ and } y \notin \Lambda) \text{ or } (y \in \Lambda \text{ and } x \notin \Lambda) \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition is a form of the Schur complement formula.

**Proposition 3.6.1.** Let  $K$  with off-diagonal exponential decay,  $\Lambda$  a box of size  $L$ , and  $\lambda \in \mathbb{C} \setminus \mathbb{R}$ . Then

$$(K - \lambda)^{-1}(x, y) = - \sum_{u \in \Lambda, v \notin \Lambda} (K^\Lambda - \lambda)^{-1}(x, u) \Gamma_{K,\Lambda}(u, v) (K - \lambda)^{-1}(v, y) \quad (3.45)$$

for any  $x \in \Lambda$  and any  $y \notin \Lambda$ , where

$$K^\Lambda(x, y) = \begin{cases} K(x, y) & \text{if } x \in \Lambda \text{ and } y \in \Lambda \\ 0 & \text{otherwise} \end{cases}$$

is the restriction of  $K$  to  $\Lambda$ .

*Proof.* We can divide  $K$  into the following three parts

$$K = K^\Lambda + \Gamma_{K,\Lambda} + K^{\Lambda^c} \quad (3.46)$$

where  $\Lambda^c$  is the complement of  $\Lambda$ . We here use the resolvent formula

$$(K - \lambda)^{-1} = (K^\Lambda - \lambda)^{-1} + (K^{\Lambda^c} - \lambda)^{-1} - ((K^\Lambda - \lambda)^{-1} + (K^{\Lambda^c} - \lambda)^{-1}) \Gamma_{K,\Lambda} (K - \lambda)^{-1}. \quad (3.47)$$

Just remark now that  $(K^\Lambda - \lambda)^{-1}(x, y) = 0$  and  $(K^{\Lambda^c} - \lambda)^{-1}(x, y) = 0$  if  $x \in \Lambda$  and  $y \notin \Lambda$ .  $\square$

We now apply the multiscale method. Let  $\Lambda_L(n)$  be the box of side length  $2L + 1$  centered at  $n \in \mathbb{Z}^d$ . We replace the random potential  $V_\omega$  by an arbitrary constant outside the box  $\Lambda_{2L}(n)$  in order to make the mean field Hamiltonian inside  $\Lambda_L(n)$  independent of what is happening outside  $\Lambda_{2L}(n)$ :

$$\hat{V}_\omega^{\Lambda_L(n)}(x) = \begin{cases} V_\omega(x) & \text{if } x \in \Lambda_{2L}(n), \\ 0 & \text{otherwise.} \end{cases}$$

Recall that  $0 \in \text{supp}(\mathbb{P})$ . From this potential we can obtain with Theorem 3.2.1 the minimiser  $\gamma_{min}(\hat{V}_\omega^{\Lambda_L(n)})$  and the mean-field Hamiltonian  $H_{min}(\hat{V}_\omega^{\Lambda_L(n)})$ . We denote its restriction to  $\Lambda_L(n)$  by

$$\hat{H}(n, L) := \left( H_{min}(\hat{V}_\omega^{\Lambda_L(n)}) \right)_{|\Lambda_L(n)}.$$

We introduce this Hamiltonian because of two properties. First it is independent of what is happening outside  $\Lambda_{2L}(n)$ . Second, it is a good approximation of  $(H_{min})|_{\Lambda_L(n)}$ . Indeed, from Theorem 3.2.3 we have

$$\|(H_{min})|_{\Lambda_L(n)} - \hat{H}(n, L)\| < De^{-\nu L} \quad (3.48)$$

where  $D$  does not depend on  $n$  and  $L$ .

**Definition 3.6.1** (L-resonance). A number  $\lambda \in \mathbb{R}$  is called L-resonant for the box  $\Lambda_L(n)$  if there exists  $A_c$  with  $\|A_c\| \leq 2D \exp(-\nu L)$  and

$$d(\lambda, \sigma[\hat{H}(n, L) + A_c]) \leq \exp(-\sqrt{L}), \quad (3.49)$$

where  $D$  is the constant defined in (3.48)

Remark that our definition of non-resonance is equivalent to

$$d(\lambda, \sigma[\hat{H}(n, L)]) > \exp(-\sqrt{L}) - 2D \exp(-\nu L). \quad (3.50)$$

We have added the operator  $A_c$  in the above definition to handle the difference between  $(H_{min})|_{\Lambda_L(n)}$  and  $\hat{H}(n, L)$ . This corresponds to Definition 9.1 in [90].

**Definition 3.6.2** ( $(L, \zeta, \lambda)$ -good box). The box  $\Lambda_L(n)$  is called an  $(L, \zeta, \lambda)$ -good box if

1. it is not L-resonant;
2. for any  $x \in \Lambda_{\sqrt{L}}(n)$ ,  $y \notin \Lambda_L(n)$  and  $A_c$  with  $\|A_c\| \leq 2D \exp(-\nu L)$ ,

$$\sum_v |\hat{H}(n, L) + A_c - \lambda|^{-1}(x, v)| |\Gamma_{\hat{H}(n, L) + A_c, \Lambda_L(n)}(v, y)| \leq \exp(-\zeta|y - x|). \quad (3.51)$$

### 3.6.2 From a scale to another

Let  $L_0$  be not too small and set  $L_k = L_0^{\alpha^k}$  with  $1 < \alpha < 2$ . In this subsection, we prove the following proposition.

**Proposition 3.6.2.** If the following conditions are satisfied

1. for any 4 boxes of side length  $L_k$  in  $\Lambda_{L_{k+1}}(n)$ , separated from each other by a distance of at least  $2L_k$ , there is at least one which is  $(L_k, \zeta, \lambda)$ -good with  $\zeta > 20/\sqrt{L_k}$ ,
2. no box in  $\Lambda_{L_{k+1}}(n)$  of side length  $4L_k, 12L_k, 20L_k$  is  $L_k$ -resonant;
3. the domain  $\Lambda_{L_{k+1}}(n)$  is not  $L_{k+1}$ -resonant,

then the cube  $\Lambda_{L_{k+1}}(n)$  is  $(L_{k+1}, \zeta_{k+1}, \lambda)$ -good with a decay satisfying  $\zeta_{k+1} > 20/\sqrt{L_{k+1}}$ .

This proposition corresponds to Theorem 10.20 in [90].

*Proof of Proposition 3.6.2.* Let  $A_c$  be so that  $\|A_c\| \leq 2De^{-\nu L_{k+1}}$ . Let

$$(\hat{H}(n, L_{k+1}))|_{\Lambda_{L_k}(m)}$$

be the restriction to the box  $\Lambda_{L_k}(m)$  of  $\hat{H}(n, L_{k+1})$ , with  $\Lambda_k(m) \subset \Lambda_{k+1}(n)$ . Because of Theorem 3.2.3, we have

$$\|(\hat{H}(n, L_{k+1}))|_{\Lambda_{L_k}(m)} - \hat{H}(m, L_k)\| \leq De^{-\nu L_k}$$

and so

$$\|(\hat{H}(n, L_{k+1}) + A_c)|_{\Lambda_{L_k}(m)} - \hat{H}(m, L)\| \leq 2De^{-\nu L_k}$$

for  $L_k$  big enough.

Let  $K = \hat{H}(n, L_{k+1}) + A_c$  and for simplicity we will just write  $\Gamma_\Lambda$  instead of  $\Gamma_{K,\Lambda}$ . Because of what we have just said, if  $\Lambda_{L_k}(m)$  is  $(L_k, \zeta, E)$  good then

$$\sum_v (K^{\Lambda_{L_k}(m)} - \lambda)^{-1}(x, v) |\Gamma(v, y)| \leq \exp(-\zeta|y - x|) \quad (3.52)$$

and

$$\|(K^{\Lambda_{jL_k}(m)} - \lambda)^{-1}\| \leq \exp(\sqrt{jL_k}) \quad (3.53)$$

if  $\Lambda_{jL_k}(m)$  is not  $jL_k$ -resonant for  $j = 4, 12, 20$ .

The idea is to use equation (3.45) as many times as we want. For any  $v$  appearing in the equation (3.45), we can define another box  $\Lambda(v)$  with  $y \notin \Lambda(v)$  and repeat the formula with  $v$  instead of  $x$ . Proceeding this way again and again, we get after iteration

$$\begin{aligned} & (K - \lambda)^{-1}(x, y) \\ &= \sum_{(u_i, v_i)_{i=1..n}} (K^{\Lambda_1} - \lambda)^{-1}(x, u_1) \Gamma_{\Lambda_1}(u_1, v_1) (K^{\Lambda_2} - \lambda)^{-1}(v_1, u_2) \end{aligned} \quad (3.54)$$

$$\times \Gamma_{\Lambda_2}(u_2, v_2) \dots (K - \lambda)^{-1}(v_n, y). \quad (3.55)$$

We will write the indices of the sum as a tree  $\mathcal{T}$  of chains  $\mathcal{X} = (u_i, v_i, \Lambda_i)_{i \leq n}$  with  $v_i \in \Lambda_i$ ,  $u_{i+1} \in \Lambda_i$ ,  $v_{i+1} \notin \Lambda_i$  and  $y \notin \Lambda_i$ . We first sum over the  $u_i$ 's so as to reduce our chains to  $\mathcal{X} = (v_i, \Lambda_i)_{i \leq n}$  and we introduce an upper bound  $R_{\mathcal{X}}$  such that

$$\begin{aligned} R_{\mathcal{X}} \geq & \sum_{(u_i)_{i=1..n}} |(K^{\Lambda_1} - \lambda)^{-1}(x, u_1) \Gamma_{\Lambda_1}(u_1, v_1) (K^{\Lambda_2} - \lambda)^{-1}(v_1, u_2) \\ & \times \Gamma_{\Lambda_2}(u_2, v_2) \dots \Gamma(u_n, v_n)|. \end{aligned} \quad (3.56)$$

Then, Equation (3.54) gives

$$|(K - \lambda)^{-1}(x, y)| \leq \|(K - \lambda)^{-1}\| \sum_{\mathcal{X} \text{ leaves of } \mathcal{T}} R_{\mathcal{X}}. \quad (3.57)$$

This formula is very general and is valid for any expansion. Different choices for the construction of the tree exit in the literature and we will follow the one from [90]. The goal is to get at least one good box at each step. The choice of the  $\Lambda_i$  and the construction of the tree  $\mathcal{T}$  of  $\mathcal{X}$  and  $R_{\mathcal{X}}$  are made according to the following algorithm.

We start from  $x$  so we define  $v_0 = x$  and  $R = 1$ . The choice of  $\Lambda_{i+1}$  will depend on  $v_i$ .

- If we get close to the boundary or to  $y$ , ( $d(v_i, \partial\Lambda_{L_{k+1}}(n)) < L_k$  or  $d(v_i, y) < L_k$ ) then we stop. The construction of this chain is over and we carry on with the other branches of the tree.
- Otherwise
  - if  $\Lambda_{L_k}(v_i)$  is an  $(L_k, \zeta, E)$ -good box then

$$\begin{aligned}
& R_{\mathcal{X}}(K - \lambda)^{-1}(v_i, y) \\
& \leq R_{\mathcal{X}} \sum_{u_{i+1}, v_{i+1}} (K^{\Lambda_i} - \lambda)^{-1}(v_i, u_{i+1}) \Gamma_{\Lambda_i}(u_{i+1}, v_{i+1}) \\
& \quad \times (K - \lambda)^{-1}(v_{i+1}, y) \\
& \leq \sum_{v_{i+1} \notin \Lambda_{L'}(v_i)} R_{\mathcal{X}} \exp(-\zeta|v_{i+1} - v_i|) (K - \lambda)^{-1}(v_{i+1}, y)
\end{aligned}$$

so for each  $v_{i+1}$  outside  $\Lambda_L(v_i)$ , we set

$$R_{\mathcal{X}+v_{i+1}} = R_{\mathcal{X}} \exp(-\zeta|v_{i+1} - v_i|) \quad (3.58)$$

and carry on the algorithm with the new chain  $\mathcal{X} + v_{i+1}$ ;

- else if  $\Lambda_{L_k}(u_i)$  is not a good box, choose  $j = 4$  or  $12$  or  $L_k$  such that for every  $v$  in  $\Lambda_{2jL_k} \setminus \Lambda_{jL_k}$ ,  $\Lambda_{L_k}(v)$  is a good box and  $\Lambda_{jL_k}$  is not resonant. It is always possible to do this because of the following remark: Either 3 boxes are far away from each other then there are 3 boxes  $M_1, M_2, M_3$  of size  $4L_k$  separated by at least  $2L_k$  so that every cube  $\Lambda_{L_k}(m) \subset \Lambda_{L_{k+1}}(m)$  whose center is not included in  $\cup_{i=1,2,3} M_i$  are  $(L_k, \zeta, \lambda)$ -good. Or two of them are close and the other is far away then there are two boxes  $M_1$  of size  $12L_k$  and  $M_2$  of size  $4L_k$  separated by at least  $2L_k$  so that every cube in  $\Lambda_{L_k}(m) \subset \Lambda_{L_{k+1}}(m)$  whose center is not included in  $M_i$  are  $(L_k, \zeta, \lambda)$ -good. Or the three of them are together then there exist one box  $M_1$  of size  $20L_k$  so that every cube  $\Lambda_{L_k}(m) \subset \Lambda_{L_{k+1}}(m)$  whose center is not included in  $M_1$  are  $(L_k, \zeta, \lambda)$ -good. We can assume that the good box decay is smaller than the off diagonal decay parameter  $\nu$ ,  $\zeta < \nu$ . We then have

$$\begin{aligned}
& R_{\mathcal{X}}(K - \lambda)^{-1}(v_i, y) \\
& \leq R_{\mathcal{X}} \sum_{u_{i+1}, v_{i+1}} (K^{\Lambda_{i+1}} - \lambda)^{-1}(v_i, u_{i+1}) \Gamma_{\Lambda_{i+1}}(u_{i+1}, v_{i+1}) \\
& \quad \times (K - \lambda)^{-1}(v_{i+1}, y) \\
& \leq R_{\mathcal{X}} \sum_{u_{i+1}, v_{i+1} \in \Lambda_{2jL_k}(v_i)} (K^{\Lambda_{i+1}} - \lambda)^{-1}(v_i, u_{i+1}) \Gamma_{\Lambda_{i+1}}(u_{i+1}, v_{i+1}) \\
& \quad \times (K - \lambda)^{-1}(v_{i+1}, y) + \sum_{u_{i+1}, v_{i+1} \notin \Lambda_{2jL_k}(u_i)} (K^{\Lambda_i} - \lambda)^{-1}(v_i, u_{i+1}) \\
& \quad \times \Gamma_{\Lambda_{i+1}}(u_{i+1}, v_{i+1}) (K - \lambda)^{-1}(v_{i+1}, y)
\end{aligned}$$

and then

$$\begin{aligned}
 & R_{\mathcal{X}}(K - \lambda)^{-1}(v_i, y) \\
 & \leq R_{\mathcal{X}} \sum_{\substack{u_{i+1}, v_{i+1} \in \Lambda_{2jL_k}(v_i) \\ u'_{i+1}, v'_{i+1} \notin \Lambda_{L_k}(v_{i+1})}} (K^{\Lambda_{i+1}} - \lambda)^{-1}(v_i, u_{i+1}) \Gamma(u_{i+1}, v_{i+1}) \times \\
 & \quad \times (K^{\Lambda_{L_k}(v_{i+1})} - \lambda)^{-1}(v_{i+1}, u'_{i+1}) \Gamma_{\Lambda_{L_k}(v_{i+1})}(u'_{i+1}, v'_{i+1}) \\
 & \quad \times (K - \lambda)^{-1}(v'_{i+1}, y) + \sum_{v_{i+1} \notin \Lambda_{2jL_k}(u_i)} (jL_k)^d C \exp(\sqrt{jL_k}) \\
 & \quad \times \exp(-\nu(|v_{i+1} - v_i| - jL_k))(K - \lambda)^{-1}(v_{i+1}, y) \\
 & \leq R_{\mathcal{X}} \sum_{v_{i+1}} (2jL_k)^d \exp(\sqrt{jL_k}) \exp(-\zeta \max(|v_{i+1} - v_i| - 2jL_k, L_k)) \\
 & \quad \times (K - \lambda)^{-1}(v_{i+1}, y) + \sum_{v_{i+1} \notin \Lambda_{2jL_k}(u_i)} (jL_k)^d C \exp(\sqrt{jL_k}) \\
 & \quad \exp(-\nu(|v_{i+1} - v_i| - jL_k))(K - \lambda)^{-1}(v_{i+1}, y).
 \end{aligned}$$

Therefore we can set

$$R_{\mathcal{X}+v_{i+1}} = R_{\mathcal{X}} 2C (2jL_k)^d \exp(\sqrt{jL_k}) \exp(-\zeta \max(L_k, |v_{i+1} - v_i| - 2jL_k)) \tag{3.59}$$

and as previously, we carry on with the new chain  $\mathcal{X} + v_{i+1}$ .

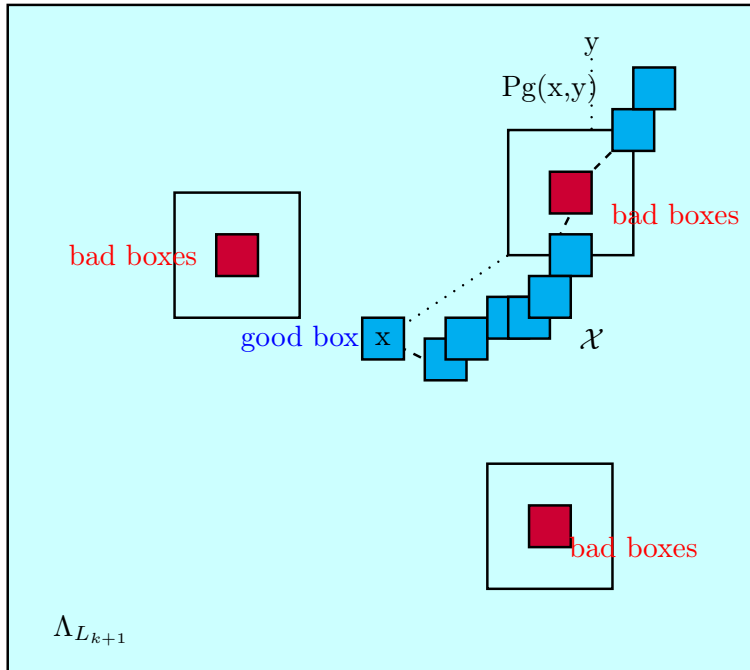


Figure 3.1 – Schematic representation a typical chain  $\mathcal{X}$  used in the proof of Proposition 3.6.2

We have finished the description of the algorithm. We define the "good



path length"  $P_g(x, v)$  as the minimum length between  $x$  and  $v$  when any cube  $M = \Lambda_{2^j L_k}(v)$  containing bad boxes defined in the procedure can be crossed for free. We easily check that (3.58) and (3.59) imply that

$$R_{\mathcal{X}} \leq \exp(-\zeta \max(\text{length}(\mathcal{X})L_k, P_g(x, v_i))). \quad (3.60)$$

for every chain  $\mathcal{X}$ . From this, our algorithm gives us the following estimate:

**Proposition 3.6.3.**

$$\sum_{\mathcal{X}} R_{\mathcal{X}} \leq \frac{1}{1 - (L_{k+1})^d \exp(-\zeta L_k)} (L_{k+1})^{d \frac{L_{k+1}}{L_k}} \exp(-\zeta(P_g(x, y))). \quad (3.61)$$

*Proof.* We have

$$\begin{aligned} \sum_{(\mathcal{X})} R_{\mathcal{X}} &= \sum_{N=1}^{\infty} \sum_{\mathcal{X}, N=\text{length}(\mathcal{X})} R_{\mathcal{X}} \\ &\leq \sum_{N=1}^{\infty} \sum_{\mathcal{X}, N=\text{length}(\mathcal{X})} \exp(-\zeta \max(NL_k, P_g(x, y))) \\ &\leq \sum_{N=1}^{\infty} \sum_{\mathcal{X}, N=\text{length}(\mathcal{X})} \exp(-\zeta(P_g(x, y) + \max(0, N - \frac{L_{k+1}}{L_k})L_k)) \\ &\leq (L_{k+1})^{d \frac{L_{k+1}}{L_k}} \exp(-\zeta(P_g(x, y))) \sum_{N=0}^{\infty} L_{k+1}^{Nd} \exp(-\zeta NL_k) \\ &\leq \frac{1}{1 - (L_{k+1})^d \exp(-\zeta L_k)} (L_{k+1})^{d \frac{L_{k+1}}{L_k}} \exp(-\zeta(P_g(x, y))), \end{aligned}$$

because at each step  $i$  there are only  $L_{k+1}^d$  possible  $v_i$ 's.  $\square$

The fact that there are only 3 bad boxes implies that the good path length is close to the usual distance  $P_g(x, y) \geq |x - y| - 20L_k$ . We can now conclude. Let  $y$  be outside  $\Lambda_{L_{k+1}}$ . Then

$$\begin{aligned} &\sum_v |(\hat{H}(n, L+1) + A_c - \lambda)^{-1}(x, v)| |\Gamma(v, y)| \\ &\leq \sum_v \|((\hat{H}(n, L+1) + A_c - \lambda)^{-1})\| \frac{1}{1 - (L_{k+1})^d \exp(-\zeta L_k)} (L_{k+1})^{L_{k+1}/L_k} \\ &\quad \times \exp(-\zeta(|x - v| - 20)) C \exp(-\nu|v - y|) \\ &\leq C' (L_{k+1})^d \exp(\log(L_k) L_k^{\alpha-1}) \exp(\sqrt{L_{k+1}}) \exp(-\zeta(|x - y| - 20)) \end{aligned}$$

In order to conclude, we just remark that  $\sqrt{L_k} + \log(L_k) L_k^{\alpha-1} + d \log(L_k) = o(|x - y|)$ . So we can choose

$$\zeta_{k+1} = \zeta_k - \frac{\sqrt{L_k} + \log(L_k) L_k^{\alpha-1} + d \log(L_k)}{L_k}, \quad (3.62)$$

which finishes the argument.  $\square$

### 3.6.3 The multiscale

**Theorem 3.6.1.** Assume that there exists a gap big enough in the spectrum, that the law of potential has a density Lipschitz by part and that  $\|W\|_{L^1}$  is small enough compared to the gap. Then, let  $L_0$  be large enough,  $\lambda \in \mathbb{R}$   $\zeta > 1/\sqrt{L_0}$ ,  $p > 2d$  and  $1 < \alpha < 2p/(p + 2d)$ . If for any cubes  $\Lambda_{L_0}(n_0)$ ,  $\Lambda_{L_0}(m_0)$  separated by at least  $2L_0$ ,

$$\mathbb{P}(\exists \lambda \in I : \Lambda_{L_0}(n) \text{ and } \Lambda_{L_0}(m) \text{ are not } (L_0, \zeta, \lambda)\text{-good}) \leq \frac{1}{L_0^{2p}}, \quad (3.63)$$

then for any  $k$ , and any cubes  $\Lambda_{L_k}(n_k)$ ,  $\Lambda_{L_k}(m_k)$  separated by at least  $2L_k$ , we have

$$\mathbb{P}(\exists \lambda \in I : \Lambda_{L_k}(n_k) \text{ and } \Lambda_{L_k}(m_k) \text{ are not } (L_k, \zeta, \lambda) \text{ good}) \leq \frac{1}{L_k^{2p}}, \quad (3.64)$$

where  $L_{k+1} = L_k^\alpha$

This theorem is similar to Theorem 10.22 of [90].

*Proof.* The demonstration is done by iteration using Proposition 3.6.2. Suppose there exists  $\lambda \in I$  such that  $\Lambda_{L_{k+1}}(n)$  and  $\Lambda_{L_{k+1}}(m)$  are not  $(L_{k+1}, \zeta, \lambda)$ -good. Then for each box one of the hypothesis of Proposition 3.6.2 fails. So either one of the boxes admits 4 separated bad sub-boxes, or there exists  $\lambda$  such that for the two boxes, one of their sub-boxes shows a resonance at  $\lambda$ . The probability that the hypothesis over the existence of 4 bad boxes is not true can be estimated by iteration. Indeed, 4 cubes means 2 pairs. Because of independence, the 4-cubes probability will be the 2-cubes probability squared and because there are only  $L^d$  cubes, hence  $L^{4d}$  4 cubes combinations we can estimate this probability by

$$\begin{aligned} &\leq CL^{4d} \left( \frac{1}{(L_k)^{2p}} \right)^2 \\ &\leq C \frac{1}{(L_{k+1})^{4\frac{p}{\alpha} - 4d}} \\ &\leq \frac{1}{4} \frac{1}{L_{k+1}^{2p}}. \end{aligned}$$

The last inequality is true for  $L_k$  big enough because  $2p < 4\frac{p}{\alpha} - 4d$  so  $\alpha < \frac{2p}{p+2d}$ .

The probability of the non resonance hypothesis is controlled by Wegner estimate (3.12) as it is done Theorem 10.22 of [90] with  $\mathcal{O}(L^d \sqrt{e^{-\sqrt{L_k}}}) = o(L^{-2p})$ .  $\square$

From this and (3.12) we can deduce the following corollaries. The proof can be found again in [90, Part 9 and Part 11].

**Corollary 3.6.1.** In presence of strong disorder, meaning

$$\left( |\text{supp}(\rho)|^{-1/2} + \|\rho\|_\infty |\text{supp}(\rho)|^{1/2} + \|\rho'\|_\infty |\text{supp}(\rho)|^{3/2} \right)$$

small enough, then  $H_{min}$  has pure point spectrum and its eigenvectors are localised in space.

Furthermore the Lifshitz tail is not modified too much for  $\|W\|$  very small because

$$\mathbb{P}[d(\sigma[(H_{min})|_\Lambda], \lambda) < \epsilon] \leq \mathbb{P}[d(\sigma[H^\Lambda], \lambda) < \epsilon + 2\|W\|_{\ell^1}], \quad (3.65)$$

so we also get the following

**Corollary 3.6.2.** There is  $\epsilon$  such that if  $\|W\|_{\ell^1} < \epsilon$  there are small intervals at the edges of the bands of the spectrum of  $H_{min}$ , where the spectrum is pure point with exponentially decaying eigenvectors.

The two results conclude the proof of Theorem 3.2.2.

### 3.7 Numerical simulations

In this section, we present some simple numerical simulations in order to illustrate our theorems. Due to the computational cost we restrict ourselves to the one dimensional case, which however is known to present stronger localisation effects than in higher dimensions. It would be interesting to generalise our simulation to dimension 2 and 3.

We take  $\Lambda = [0, L]$  with  $L$  ranging from two hundred to a few thousands lattice sites,  $200 \leq L \leq 2000$ . The discrete one-dimensional Laplacian is defined in (3.3). The deterministic potential  $V_0$  is 2-periodic:

$$V_0(n) = \begin{cases} \xi & \text{if } n \text{ is even} \\ -\xi & \text{if } n \text{ is odd,} \end{cases} \quad (3.66)$$

where  $\xi > 0$  is a parameter. The probability of the random potential  $V_\omega$  is the uniform law over the interval  $[0, \zeta]$  where  $\zeta$  is another parameter. For  $W(x-y)$ , we use a simple next-to-nearest neighbour interaction of the form

$$W(x-y) = \begin{cases} q & \text{if } x-y=0 \\ q/2 & \text{if } |x-y|=1 \\ q/4 & \text{if } |x-y|=2 \text{ or } 3 \\ 0 & \text{otherwise,} \end{cases} \quad (3.67)$$

where  $q$  is another parameter. Our model depends therefore on three parameters  $\xi$ ,  $\zeta$  and  $q$ . When  $\xi > 2 + \zeta$  the spectrum of the linear Hamiltonian  $-\Delta + V$  is composed of two distinct intervals. We then choose  $q$  such as to keep a gap in the spectrum and ensure that the map  $F$  is contracting (Theorem 3.2.1). In this model the particles fill half of the energy states, that is, there are  $N = L/2$  particles.

#### 3.7.1 Illustration of Theorems 3.2.1, 3.2.3, 3.2.4 and 3.2.2

In order to construct the solution  $\gamma_{min}$  and the associated mean-field Hamiltonian  $H_{min}$ , we use the fixed point algorithm employed in the proof of Theorem 3.2.1.

We have tested that adding a dirac at the site 250 to the potential  $V$  induces a perturbation in the non linear minimiser, which decays exponentially fast

(Theorem 3.2.3). In Figure 3.2 we plot the relative density  $\gamma_{min}(V+\delta_{250})(x, x) - \gamma_{min}(V)(x, x)$ .

We conclude the illustration of our results with the case of Theorem 3.2.2. Since we deal with a one-dimensional system all the eigenvectors are localised, even in a regime of parameters which is not covered by the second part of Theorem 3.2.2. This is shown in Figure 3.3. It is an interesting open problem to prove a stronger localisation result in the one-dimensional Hartree-Fock model.

### 3.7.2 Closing the gap: insulators and metals

We have increased the intensity  $q$  of the interaction up to the point where the gap closes. For a large interaction the fixed point algorithm used to construct the solution in Theorem 3.2.1 does not work. Instead we have used the optimal damping algorithm of [34] which works perfectly. In general, we have observed that the eigenvectors are less localised except at the edges of the spectrum. There is no sign of a phase transition. The situation is more interesting if we erase the gap by choosing  $\xi = 0$ . A small delocalisation phenomenon seems to appear at the Fermi energy  $\mu$ , see Figure 3.4. We hope to be able to understand these phenomena rigorously in the future.

**Acknowledgement** I would like to thank Mathieu Lewin for his help and all the time he gave me during this work.

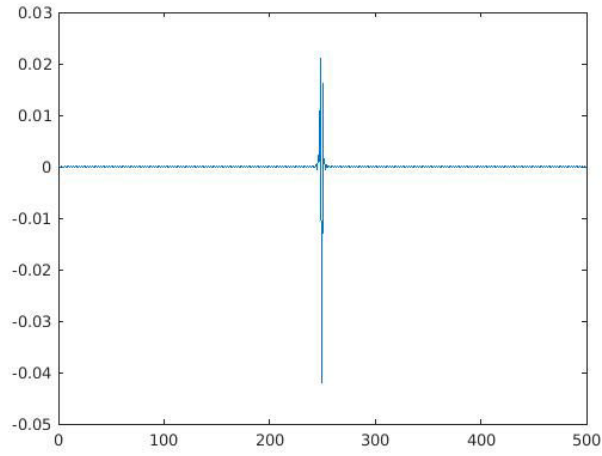


Figure 3.2 – The relative density  $\gamma_{min}(V+\delta_{250})(x,x)-\gamma_{min}(V)(x,x)$  with  $\xi = 1$ ,  $\zeta = 1$   $q = 2$ ,  $L = 500$ .

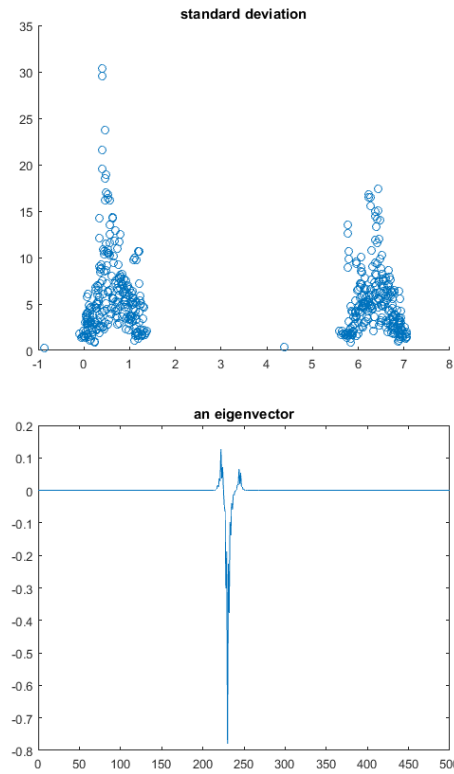


Figure 3.3 – Left: Standard deviation of all the eigenvectors of the mean field operator  $H_{min}$  in terms of their eigenvalue. Here the size of the domain is  $L = 1000$ , hence a value of 10 shows localisation. Right: an eigenvector chosen at random. The values of the parameters are  $\xi = 1$ ,  $\zeta = 1$  and  $q = 2$ .

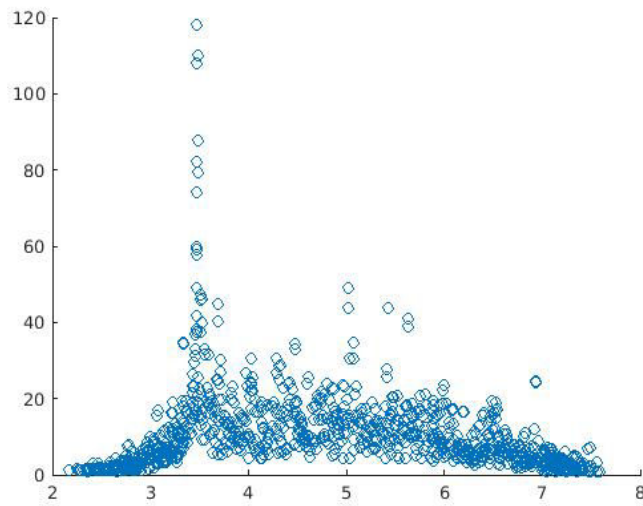


Figure 3.4 – Standard deviation of the nonlinear eigenvectors in terms of their eigenvalue, with the periodic potential dropped out ( $\xi = 0$ ). Here there are only  $L/4$  particles. The Fermi level is  $\mu = 3.5$  and there is no gap. A small delocalisation seems to appear at the Fermi level. The other parameters are  $\zeta = 4$  and  $q = 4$ .



# Chapitre 4

## Anderson localisation for periodically driven systems

We present here the work published in [54] in collaboration with François Huvneers.

**Abstract.** We study the persistence of localization for a strongly disordered tight-binding Anderson model on the lattice  $\mathbb{Z}^d$ , periodically driven on each site. Under two different sets of conditions, we show that Anderson localization survives if the driving frequency is higher than some threshold value that we determine. We discuss the implication of our results for recent development in condensed matter physics, we compare them with the predictions issuing from adiabatic theory, and we comment on the connexion with Mott's law, derived within the linear response formalism.



## 4.1 Introduction

In this paper, we study the fate of Anderson localization in periodically driven systems. Let  $H_0$  be the tight-binding Anderson Hamiltonian on the lattice  $\mathbb{Z}^d$ . At strong enough disorder, it is well known that all eigenstates of  $H_0$  are exponentially localized (see [12][62][8] as well as [90] for more references). Let us then consider a periodic time-dependent Hamiltonian of the form

$$H(t) = H_0 + gH_1(t) \quad (4.1)$$

with  $H_1(t) = H_1(t+T)$  for some period  $T$ , and with  $g$  some coupling constant. We assume that  $H_1(t)$  acts everywhere locally : there exists  $R$  such that

$$|\langle x|H_1(t)|y\rangle| = 0$$

for all  $x, y \in \mathbb{Z}^d$ , and all time  $t$ , as soon as  $|x - y| > R$ .

The time-evolution of an initial wave function  $\psi(0)$  is governed by the time-dependent Schrödinger equation :

$$i \frac{d\phi(t)}{dt} = H(t)\phi(t).$$

The long time properties of the solutions of this equation are best understood through the Floquet eigenstates of  $H(t)$ . The question addressed in this paper can then be rephrased as follows : Under suitable regularity conditions on the time-dependence of  $H_1(t)$ , is there a range of values for  $g$  and  $T$  such that the structure of the eigenfunctions of  $H_0$  is only weakly affected by the periodic potential  $H_1(t)$ , so that the the Floquet eigenstates of  $H(t)$  are themselves localized? We answer this question positively in Theorem 4.2.1 below, for two different regularity conditions on  $H_1$ , leading to different allowed values for  $g$  and  $T$ .

**Localization and Floquet physics.** The above question has already received some attention in the mathematical physics community. The connection with the discrete non-linear Schrödinger equation (DNLS) constituted a first motivation, see [28][128]. In this context, the more general case of a quasi-periodic driving shows up naturally : In a first approximation, the non-linearity in the DNLS equation can be replaced by a quasi-periodic perturbation. On the other hand, in this perspective, it is natural to restrict oneself to spatially localized perturbations ( $\langle x|H_1(t)|y\rangle$  decays fast as  $x$  or  $y$  goes to infinity and not only as  $|x - y|$  goes to infinity as we consider) ; indeed, stability results for the DNLS equation all deal with originally localized wave packets.

More recently, periodically driven Hamiltonian systems have been studied intensively in condensed matter theory. For two reasons at least :

First, from a theoretical perspective, driven systems constitute the first examples of dynamics out-of-equilibrium systems, lacking even energy conservation. The natural question that arises is whether the system will absorb energy until it reaches an infinite temperature state (i.e. a state with maximal entropy), as it would be the case for a chaotic system, or whether some extensive effectively conserved quantity emerges, forbidding energy absorption after some transient regime [45][46][3][4][1][2][50][31][87]. For non-interacting particles on a lattice,

as we consider in this paper, this issue becomes trivial and fully independent of the issue of Anderson localization, once the driving frequency becomes higher than the bandwidth of individual particles, see [3][1]. Nevertheless, thanks to the Anderson localization phenomenon, our results guarantee the existence of an effective extensive conserved quantity for frequencies much below this trivial threshold, see Proposition 4.2.2 below.

Second, from a more practical point of view, driven systems furnish a way to engineer topological states of matter [115][109]. Though this possibility is not a priori related to the phenomenon of Anderson localization, it turns out that, for interacting many-body systems, localization makes it possible to “lift” phase transitions from the ground state to the full spectrum [81]. This observation is at the heart of very recent investigations of new phases of matter inside the many-body localized phase [88][130].

Hence, in view of the increasing role played by localized Floquet systems in modern condensed matter physics, it appeared useful to bring some firm mathematical foundations to the theory of Anderson localization in periodically driven systems, even though the need for mathematical rigor forces us to restrict the setup to non-interacting particles. Results in this direction already appeared in [76], where the localization for some random unitary operators is established; this question is directly related to ours since the long time evolution of a periodically system is governed by the spectral properties of the unitary  $U(T)$ , where  $U(t)$  solves  $idU(t)/dt = H(t)U(t)$ . However, for a Hamiltonian as in (4.1), we do not recover the particular form for  $U$  studied in [76].

Before stating our results, we now introduce two more specific aspects that deserved clarification and motivated the present article.

**Adiabatic Theory.** Time-dependent Hamiltonian systems varying smoothly and slowly enough with time can be described through the use of adiabatic theory. Here, adapting the analysis from [4], we argue that localization emerges when level crossings in the system become typically non-adiabatic, and we determine the threshold frequency above which this happens.

Let us first remind the theory of the Landau-Zener effect for a time-dependent two-levels Hamiltonian  $G(t)$  [102][136]. To make the connection with our problem, let us assume that  $G(t)$  is of the form  $G(t) = PH(t)P$  where  $P$  projects on two eigenstates of  $H_0$ . Moreover, we assume that  $G(t)$  varies smoothly on the scale of one period, i.e. we can write  $G(t) = \tilde{G}(\nu t)$  for some smooth  $2\pi$ -periodic function  $\tilde{G}$  and  $\nu = \frac{2\pi}{T}$ . It is then convenient to move to the basis of the eigenstates of  $H_0$ , i.e. the basis where  $P$  is diagonal, and to decompose

$$G(t) = G_{\text{dia}}(t) + G_{\text{off}}(t),$$

as a sum of the diagonal and off-diagonal part. We notice that the time-dependent part of  $G_{\text{dia}}(t)$  is of order  $g$ . We set  $\langle 1|G_{\text{off}}(t)|2 \rangle =: g'$ , where  $g'$  depends mainly on the distance between the two localization centers of the two states projected on by  $P$ , and is typically much smaller than  $g$ . Finally we assume that the two levels of  $PH_0P$  are close enough ( $g$ -close in fact) to each others so that the system undergoes an avoided crossing as time evolves : At some time, the levels of  $G_{\text{dia}}(t)$  cross, while  $G_{\text{off}}(t)$  induces level repulsion, leading to an avoided crossing for  $G(t)$ .

If the system is initially (i.e. before the crossing) prepared in an eigenstate of  $G_{\text{dia}}(t)$ , Landau-Zener theory tells us that, after the crossing, the state in which the systems ends up depends on the value of

$$\frac{|\langle 1|G_{\text{off}}|2\rangle|^2}{v_{12}} \sim \frac{(g')^2}{g\nu}, \quad (4.2)$$

where  $v_{12}$  is the rate of change in the energy of  $H_{\text{dia}}(t)$  at the crossing. At high frequency, when this value is much smaller than 1, the crossing is non-adiabatic and the system remains in the original state; at intermediate frequency, when this value is of order 1, the system ends up in a superposition of the eigenstates of  $G_{\text{dia}}(t)$ ; and finally at low frequency, when this value is much smaller than 1, the crossing is adiabatic and the system ends up in the other eigenstate of  $G_{\text{dia}}(t)$ .

The above scenario, valid for a two level systems, may be seen as a caricature of the localization-delocalization transition : non-adiabatic crossings do not entail hybridization of unperturbed eigenstates, while intermediate and adiabatic crossings, present at low enough frequency, allow the system to move from one state to the other, and constitute a possible mechanism for delocalization. Based on this picture, let us try to determine a critical value of  $\nu$  above which localization survives. Let us fix  $g$  in (4.1) as well as  $W$  characterizing the strength of the disorder. Let us then pick a point  $a \in \mathbb{Z}^d$ . We first determine a minimal length  $L^*$  so that there is typically at least one crossing between the state centered around  $a$  and an other state with localization center in a ball of radius  $L^*$  around  $a$ . Since the probability of finding a crossing in a ball of radius  $L$  is of the order of  $L^d \frac{g}{W}$ , we find

$$L^* \sim \left(\frac{W}{g}\right)^{1/d}.$$

The effective coupling between a state centered around  $a$  and a state at a distance  $L$  of  $a$ , corresponding to  $g'$  in (4.2) is of the order of

$$g' \sim g e^{-L/\xi},$$

where  $\xi$  is the localization length of  $H_0$ . Hence, from (4.2), we find that localization will survive if

$$\frac{g^2 e^{-2L^*/\xi}}{g\nu} \ll 1 \quad \Leftrightarrow \quad \frac{\nu}{g} \gg e^{-\frac{2}{\xi}(\frac{W}{g})^{1/d}}. \quad (4.3)$$

In Theorem 4.2.1 below, for a smooth driving (condition (C1)), we prove localization for  $\nu$  larger than some threshold value comparable to what we obtain in (4.3). We notice that the Landau Zener theory proceeds through non-perturbative arguments. Instead, our proof is based on the multi-scale analysis developed in [62], which is mainly a perturbative approach. It is thus somehow remarkable that the same upshot can be recovered in two a priori very different ways.

Finally we notice that the approach through adiabatic theory outlined above is only expected to work for  $H_1(t)$  depending smoothly on time. Unfortunately, both in theoretical and experimental physics works, it is a common protocol to just shift between two Hamiltonians periodically. This leads obviously to a non-smooth time-dependence. As we wanted to cover this case as well, we also

derived a result for  $H_1$  being only in square-integrable in time; see Theorem 4.2.1 below with the condition (C2). The lack of smoothness forced us to increase significantly the threshold on  $\nu$  with respect to (4.3).

**Mott’s law.** Mott’s law asserts that the ac-conductivity of an Anderson insulator behaves as

$$\sigma(\nu) \sim \nu^2 (\log(1/\nu))^{d+1} \quad \text{as } \nu \rightarrow 0$$

([113], see also [68] for the case of interacting electrons). An upper bound on  $\sigma(\nu)$  was rigorously established in [93] (with  $d + 1$  replaced by  $d + 2$ ). The conductivity  $\sigma(\nu)$  is derived within the linear response (LR) formalism; in our set-up, this corresponds to fixing  $\nu$  and taking the limit  $g \rightarrow 0$  while observing the dynamics over a time of order  $\nu/g^2$ . In such a regime, the hypotheses of Theorem 4.2.1 below are satisfied (we consider a monochromatic perturbation with frequency  $\nu$  so that condition (C1) holds) : The dynamics is localized for  $g$  small enough once  $\nu$  has been fixed.<sup>1</sup> It may thus come as a surprise that still  $\sigma(\nu) > 0$ .

This puzzling behavior was recently analyzed in details for many-body systems in [69]. As it was pointed out to us by [48], the conductivity  $\sigma(\nu)$  is computed for a system in equilibrium at zero or finite temperature. Moreover, as can be expected from its definition, for  $g > 0$ , LR should in general furnish only an accurate description of the dynamics for a transient regime in time of order  $\nu/g^2$ . It is true though that, for “generic” or “ergodic” systems, it is reasonable to think that the predictions from LR remain valid for much longer time scales : While heating, the system remains approximately in equilibrium and LR can be applied iteratively until the infinite temperature state is reached. This is manifestly not true for localized systems as long as  $g$  is small enough compared to  $\nu$  : The conductivity  $\sigma(\nu) > 0$  represents mainly the Rabi oscillation of rare resonant spots (“cat states”) in the Hamiltonian  $H_0$ , but these oscillations do not need to entail delocalization on the longer time scales described by the Floquet physics.

**Organisation of the paper.** The precise definition of the model studied in this paper together with our results are presented in Section 4.2. The main steps of the proof of our main theorem are contained in Section 4.3, while some more technical intermediate results are shown in Sections 4.4 to 4.6. The two corollaries are shown in Section 4.7. In several places, the proof of our results proceeds through a straightforward adaptation of delicate but well-known methods; as much as possible, we choose to describe in details only the steps where some significant amount of new material was required.

**Acknowledgments.** We are especially grateful to W. De Roeck for enlightening discussions on Mott’s law as well as previous collaborations on this topic. We thank D. Abanin, W.-W. Ho and M. Knap for previous collaboration and/or useful discussions.

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1. Strictly speaking, our model does not coincide with that studied in e.g. [93], as we do not explicitly include an electric field. However, it could be incorporated without affecting our conclusions.

## 4.2 Models and results

### 4.2.1 The models

We consider a lattice model on  $\mathbb{Z}^d$  and we note  $|x| = \sup_{i=1..d} |x_i|$ . Our results could be of course extended to more general lattices. We are interested in the long time behavior of the Schrödinger equation :

$$i \frac{d}{dt} \phi(t) = H(t) \phi(t), \quad (4.4)$$

where the function  $\phi(t)$  is defined on  $L^2(\mathbb{Z}^d)$  for any  $t$ , and the Hamiltonian  $H(t)$  is a periodic function with frequency  $\nu = 2\pi/T$ . The operator  $H(t)$  is an idealized version of (4.1) : We move to the basis where  $H_0$  is diagonal and we replace it by an uncorrelated random potential  $V_\omega$ , while we assume that  $H_1(t)$  is still a nearest-neighbor hopping (Anderson model) :

$$H(t) = -g\Delta(t) + V_\omega. \quad (4.5)$$

Here  $-\Delta(t)$  is hermitian operator for any  $t$  such that  $-\Delta(t)(x, y) = 0$  if  $|x - y| > 1$  and

$$\|-\Delta(t)(x, y)\|_{L^2([0;T])} \leq 1 \quad (4.6)$$

for any  $x, y$ . We use the notation  $-\Delta$  because in the usual time-independent Anderson model,  $-\Delta(t)$  is the usual discrete Laplacian on  $\ell(\mathbb{Z}^d)$

$$-\Delta\phi(x) = \frac{1}{2d} \sum_{|y-x|=1} \phi(y),$$

There exists a unitary operator  $U(t)$ , with  $U(0) = Id$  such that  $\phi(t) = U(t)\phi(0)$  and satisfying

$$i \frac{d}{dt} U(t) = H(t)U(t), \quad (4.7)$$

Existence and uniqueness of solution of (4.4) and (4.7) can be proved using a usual fixed point technique.

**(RP) Potential regularity.** We assume the following form for the random potential which are widely used in the literature :

$$V_\omega = \sum_{x \in \mathbb{Z}^d} v_x \delta_x \quad (4.8)$$

where  $v_x$  are i.i.d. random variables, with a bounded density  $\rho$ , such that  $\|\rho\|_\infty < \infty$  defined on a bounded support  $[-M; M]$ . We choose units such that  $\|\rho\|_\infty = 1$ . Furthermore we will assume that the density  $\rho$  is piecewise  $\mathcal{C}^1$ .

The time-dependent term  $-g\Delta(t)$  is considered to be a perturbation of order  $g \ll 1$ , usually referred to as the strong disorder regime. We treat this model in two particular cases.

**(C1) Smooth driving.** We suppose that  $-\Delta(t)(x, y)$  is a monochromatic signal : For any  $x$  and  $y$ ,

$$-\Delta(t)(x, y) = a_{x,y} + b_{x,y} \cos(\nu t) + b'_{x,y} \sin(\nu t) \quad (4.9)$$

with  $a_{x,y} = a_{y,x}$ ,  $b'_{x,y} = b'_{y,x}$  and  $b_{x,y} = b_{y,x}$ . In this regime, we are able to prove localization for frequencies  $\nu$  up to a threshold comparable to the one given in (4.3). Moreover, we claim that the result can then be extended to a hopping  $-\Delta(t)$  with Fourier coefficients that decay fast enough, but we focus on the case of single Fourier mode for simplicity.

**(C2)  $L^2$  driving.** We only assume (4.6). In this case, a much larger threshold value for  $\nu$  is needed, actually  $\nu \geq 1$ . We refer to [4] for the optimality of this condition.

**Remark 1.** Between these two extreme cases, one could obviously consider intermediate regularity cases, depending on the decay of the Fourier coefficients of  $-\Delta(t)$ . This should lead to other conditions on  $\nu$  that are not investigated in this paper.

## 4.2.2 The Floquet operator

We will work in the Fourier space instead of the time-domain, and we denote by  $\hat{x} = (x, k)$  a point of  $\mathbb{Z}^d \times \mathbb{Z}$ . Let's introduce the central object of our paper :

**Definition 4.2.1.** Let

$$\hat{H} = -g\hat{\Delta} + \hat{V}_\omega \quad (4.10)$$

be a Hamiltonian on  $\ell^2(\mathbb{Z}^d \times \mathbb{Z})$ , with

$$-\hat{\Delta}\hat{\psi}(x, k) = - \sum_{|y-x|\leq 1} \sum_{k'} \hat{\Delta}_{x,y}(k') \hat{\psi}(y, k - k') \quad (4.11)$$

where  $\hat{\Delta}_{x,y}(k) = \frac{1}{T} \int_0^T \Delta_{x,y}(t) e^{-i\nu kt} dt$  and

$$\hat{V}_\omega = V_\omega + k\nu. \quad (4.12)$$

In the mono-chromatic case (C1), the Laplacian  $-\hat{\Delta}$  is explicitly given by

$$\begin{aligned} & -\hat{\Delta}\hat{\psi}(x, k) \\ &= \sum_{|y-x|\leq 1} \left[ a_{x,y} \hat{\psi}(y, k) + \frac{b_{x,y} + ib'_{x,y}}{2} \hat{\psi}(y, k+1) + \frac{b_{x,y} - ib'_{x,y}}{2} \hat{\psi}(y, k-1) \right] \end{aligned}$$

We remark that it is a local operator, meaning it connect only sites  $\hat{x}, \hat{y}$  such that  $|\hat{x} - \hat{y}| = 1$  in the space-Fourier graph  $\mathbb{Z}^d \times \mathbb{Z}$ . In the general  $L^2$  case (C2), this is no longer true. Indeed, points  $(x, k), (y, k')$  could be connected with  $|k - k'|$  arbitrary large.

The new Hamiltonien  $\hat{H}$  gives the evolution of the "finite time Fourier series" of  $\phi(t)$  defined as follows

$$\check{\phi}(x, k, t) = \frac{1}{T} \int_t^{t+T} \phi(x, u) e^{-i\nu ku} du. \quad (4.13)$$

We get formally a time-independent Schrödinger equation governed by the Hamiltonian  $\hat{H}$  :

**Proposition 4.2.1.**

$$i\partial_t\check{\phi}(x, k, t) = \hat{H}\check{\phi}(x, k, t) \quad (4.14)$$

*Proof.*

$$\begin{aligned} i\partial_t\check{\phi}(x, k, t) &= \frac{1}{T} \int_t^{t+T} i\partial_u [\phi(x, u)e^{-i\nu ku}] du \\ &= \frac{1}{T} \int_t^{t+T} (k\nu + H(u))\phi(x, u)e^{-i\nu ku} du \\ &= \frac{1}{T} \int_t^{t+T} (k\nu + V_\omega)\phi(x, u)e^{-i\nu k u} du \\ &\quad + g \sum_{|y-x|\leq 1} \sum_{k'} (-\hat{\Delta}_{x,y}(k'))\phi(y, u)e^{-i\nu(k-k')u} du \\ &= (V_\omega + k\nu)\check{\phi}(x, k, t) + g \sum_{|y-x|\leq 1} \sum_{k'} (-\hat{\Delta}_{x,y}(k'))\check{\phi}(y, k-k', t) \\ &= \hat{H}\check{\phi}(x, k, t). \end{aligned}$$

□

The time evolution of  $\check{\phi}$  is deduced from the eigenvectors of  $\hat{H}$  :

$$\bar{\lambda}\hat{\psi} = (-g\hat{\Delta} + \hat{V}_\omega)\hat{\psi} \quad (4.15)$$

Looking for the eigenvectors of  $\hat{H}$  is equivalent to the search of solution of the form  $\phi(t) = e^{i\bar{\lambda}t}\psi(t)$  with  $\psi$  a  $T$ -periodic function (Floquet theory). Indeed, in the Fourier variables, (4.4) is equivalent to (4.15). In particular, as we will see, localization for  $\hat{H}$  implies the absence of diffusion for  $\phi$ .

**Remark 2.** Because  $\psi(t)e^{i\bar{\lambda}t} = \psi(t)e^{-i\nu t}e^{i(n\nu + \bar{\lambda})t}$ , if  $(\psi, \bar{\lambda})$  is a solution then  $(\psi(t)e^{-i\nu t}, n\nu + \bar{\lambda})$  is a solution as well for any  $n \in \mathbb{Z}$ . Hence it is enough to consider the case  $\bar{\lambda} \in [0; \nu)$ .

### 4.2.3 Results

Our main theorem states Anderson localisation for  $\hat{H}$ .

**Theorem 4.2.1.** There exists  $\epsilon > 0$  such that, if  $g < \epsilon$ , and if  $\nu \geq e^{-g^{-\frac{1}{4p+8d}}}$  for some  $p > 2d$  under the condition (C1), or if  $\nu \geq 1$  under the condition (C2), then  $\hat{H}$  exhibits localization :Its spectrum is pure point and its eigenvectors decay exponentially in space,  $\mathbb{P}$  a.s.

**Remark 3.** Under (C1), we will see that the eigenvectors are also deterministically exponentially localized along the frequency axis.

The two following corollaries do not logically follow from Theorem 4.2.1, but rather from a refinement of its proof. The first one shows the absence of diffusion for solutions of (4.4) (dynamical localization) :

**Corollary 4.2.1.** There exist  $\epsilon > 0$  and  $q > 0$  (and one may take  $q \rightarrow \infty$  as  $\epsilon \rightarrow 0$ ) such that, if  $g < \epsilon$  and  $\nu \geq e^{-g^{-\frac{1}{4p+8d}}}$  for some  $p > 2d$  under (C1), or  $\nu \leq 1$  under (C2), then

$$\mathbb{E} \left( \sup_{t>0} \sum_{x \in \mathbb{Z}^d} |x|^q |\phi(x, t)|^2 \right) < \infty \quad (4.16)$$

for any solution  $\phi(x, t)$  of (4.4) with initial condition  $\phi(x, 0)$  defined on a bounded support.

The second one deals with the existence of a local effective Hamiltonian, i.e. an Hamiltonian  $H_{eff}$  such that

$$U(T) = e^{-iTH_{eff}}$$

and such that  $H_{eff}(x, y)$  decays fast as  $|x - y| \rightarrow \infty$ . Under the conditions of Theorem 4.2.1, given  $\bar{\lambda} \in [0, \nu[$  and a corresponding eigenfunction  $\hat{\psi}_{\bar{\lambda}}(k, x)$  of  $\hat{H}$ , and given  $t \in \mathbb{R}$ , let us denote by  $P_{\psi_{\bar{\lambda}}(\cdot, t)}$  the rank one operator

$$L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d), f \mapsto (\psi_{\bar{\lambda}}(\cdot, 0), f) \psi_{\bar{\lambda}}(\cdot, t).$$

The representation

$$U(t) = \sum_{\bar{\lambda} \in [0, \nu[} e^{-i\bar{\lambda}t} P_{\psi_{\bar{\lambda}}(\cdot, t)}$$

holds. Hence, since the functions  $\psi_{\bar{\lambda}}(\cdot, t)$  are  $T$ -periodic in time, we may set

$$H_{eff} = \sum_{\bar{\lambda} \in [0, \nu[} \bar{\lambda} P_{\psi_{\bar{\lambda}}(\cdot, 0)}, \quad (4.17)$$

which defines an operator on  $L^2(\mathbb{Z}^d)$ . Under condition (C1), we have a more<sup>2</sup>

**Corollary 4.2.2.** There exist  $\epsilon > 0$  and  $q > 0$  (and one may take  $q \rightarrow \infty$  as  $\epsilon \rightarrow 0$ ) such that, if  $g < \epsilon$ ,  $\nu \geq e^{-g^{-\frac{1}{4p+8d}}}$  for some  $p > 2d$ , and under condition (C1), then

$$\mathbb{E}(|x - y|^q |H_{eff}(x, y)|) < \infty.$$

with  $H_{eff}$  as defined by (4.17).

### 4.3 Proof of Theorem 4.2.1

We will prove that the Hamiltonian  $\hat{H}$  reveals localisation by applying the classical tools of the multi-scale analysis (MSA). Thanks to the huge literature on MSA, it will be enough for us to prove a probability estimate, usually referred to as Wegner estimate, and the initialization of the MSA to show the localisation (as well as some extra technical results when dealing with the  $L^2$  case, i.e. under assumption (C2)).

We start with the Wegner estimate. Below we call columns sets of the form  $\Lambda_0 \times I \subset \mathbb{Z}^d \times \mathbb{Z}$ , for some finite spatial box  $\Lambda_0$  and some frequency interval  $I$ . Given  $\Lambda \subset \mathbb{Z}^d \times \mathbb{Z}$  and given  $H$  acting on  $L^2(\mathbb{Z}^d \times \mathbb{Z})$ , we denote by  $H|_{\Lambda}$  the operator acting on  $L^2(\Lambda)$  such that  $H|_{\Lambda}(\hat{x}, \hat{y}) = H(\hat{x}, \hat{y})$  for all  $\hat{x}, \hat{y} \in \Lambda$ .

2. The result would be of little interest under condition (C2), since at high frequency, the existence of a local effective Hamiltonian follows from much more general considerations, see [1].



**Proposition 4.3.1** (Wegner Estimate). Let  $\Lambda_0 \subset \mathbb{Z}^d$  be finite. Then

1. (The finite column case) For any  $K \in \mathbb{N}$ ,  $k_0 \in \mathbb{Z}$  so that  $\Lambda_0 \times [k_0 - K; k_0 + K] \subset \mathbb{Z}^d \times \mathbb{Z}$ , we have

$$\begin{aligned} \forall E, \mathbb{P}(\exists \bar{\lambda} \text{ eigenvalue of } \hat{H}_{|\Lambda_0 \times [k_0 - K; k_0 + K]} : \bar{\lambda} \in [E - \epsilon, E + \epsilon]) \\ \leq 2\pi\epsilon(2K + 1)|\Lambda_0| \|\rho\|_\infty. \end{aligned} \quad (4.18)$$

2. (The infinite column case) There exists a constant  $C$  which depends only on  $\|\rho\|_{L^\infty}$  and  $\|\rho'\|_{L^\infty}$ , such that for  $\Lambda_0 \times \mathbb{Z} \subset \mathbb{Z}^d \times \mathbb{Z}$ , we also have

$$\begin{aligned} \mathbb{P}(\exists \bar{\lambda} \text{ eigenvalue of } \hat{H}_{|\Lambda_0 \times \mathbb{Z}} : \bar{\lambda} \in [E - \epsilon, E + \epsilon]) \\ \leq 2\pi\sqrt{\epsilon}|\Lambda_0| \|\rho\|_\infty \max(1, \frac{M}{\nu}). \end{aligned} \quad (4.19)$$

The proof of this proposition will be carried out in section 4.4. Part 1. will be needed to establish Theorem 4.2.1 under the assumption (C1) and part 2. under the assumption (C2). The crucial property that allows to show the second part of this proposition is contained in Remark 2 : If  $\hat{\psi}(x, k)$  is an eigenvector with eigenvalue  $\bar{\lambda}$  of  $\hat{H}_{|\Lambda_0 \times \mathbb{Z}}$ , then  $\hat{\psi}(x, k - k_0)$  is also an eigenvector with eigenvalue  $\bar{\lambda} + \nu k_0$  for any  $k_0 \in \mathbb{Z}$ . Therefore the eigenvalue are of the form  $\{\bar{\lambda}_i : i = 1, \dots, |\Lambda_0|\} + \nu\mathbb{Z}$ , allowing to use  $|\Lambda_0|$  in the rhs of (4.19) instead of the cardinal of the column which in this case is infinite.

The second ingredient in the MSA consists in proving the exponential decay of the resolvent  $(\hat{H} - \lambda)^{-1}$  with high probability for a given  $\lambda \in \mathbb{R}$ . We will follow [90]. To initialize the MSA, we need to show that, given a point  $\hat{x} \in \mathbb{Z}^d \times \mathbb{Z}$ , there exists with high probability a finite domain around  $\hat{x}$ , called "good box", where the resolvent decay exponentially. From now on we fix some  $\lambda \in [0, \nu]$ . Indeed, it is enough to consider values of  $\lambda$  in this interval, because of the symmetry described in Remark 2.

For  $\Lambda \subset \mathbb{Z}^d \times \mathbb{Z}$ , we will write

$$\partial^{in} \Lambda = \{\hat{x} \in \Lambda : \exists \hat{y} \notin \Lambda, \hat{\Delta}(\hat{x}, \hat{y}) \neq 0\} \quad (4.20)$$

$$\partial^{ext} \Lambda = \{\hat{x} \notin \Lambda : \exists \hat{y} \in \Lambda, \hat{\Delta}(\hat{x}, \hat{y}) \neq 0\} \quad (4.21)$$

### 4.3.1 Smooth driving (C1)

**Definition 4.3.1** (Good box). Under the assumption (C1), we say that  $(x + [-L, L]^d) \times [k_0 - K, k_0 + K]$  is a  $\mu$ -good box, for some  $\mu > 0$ , if, for any  $(y, k) \in \partial^{in}((x + [-L, L]^d) \times [k_0 - K, k_0 + K])$ ,

$$|((x, k_0), (\hat{H}_{|(x + [-L, L]^d) \times [k_0 - K, k_0 + K]} - \lambda)^{-1}(y, k))| \leq e^{-\mu|(x, k_0) - (y, k)|} \quad (4.22)$$

where  $|(x, k_0) - (y, k)| = |k_0 - k| + \sum_{i=1}^d |x_i - y_i|$ .

The difference between our model and the classical Anderson model is the absence of independence along the frequency axis. However we have the following proposition.

**Proposition 4.3.2.** If  $|k_0| > \frac{M + \sqrt{g}}{\nu} + K$  then for any  $\Lambda_0 \subset \mathbb{Z}^d$ ,  $\Lambda_0 \times [k_0 - K; k_0 + K]$  is a  $-\ln(2(d+1)g)$  good box.

The proof of this proposition will appear as a simple case of the proof of Proposition 4.3.3 below (see Section 4.5 after the proof of Proposition 4.5.1). Thanks to this proposition, it is now enough then to study boxes close to the  $k = 0$  axis. Once we restrict ourselves to such boxes, non-intersecting boxes are stochastically independent, and we can proceed with the usual MSA approach. So the idea of the proof is to show initialization of the MSA for boxes like  $\Lambda_0 \times [-\frac{2(M+\sqrt{g})}{\nu}; \frac{2(M+\sqrt{g})}{\nu}]$ .

**Remark 4.** For any  $x \in \mathbb{Z}^d$ , there exists  $k$  such that  $|\hat{V}(x, k) - \lambda| \leq \nu$

Hence, there is no way avoiding a resonance of order  $\nu$  for all  $x$ , and we cannot look for good boxes as free of any resonances. Nevertheless, we prove that good boxes appears with high probability when  $g \ll 1$ . Let  $p > d$ .

**Proposition 4.3.3** (Initialisation of the MSA under the assumption (C1)). Assume that (C1) holds. For any  $\mu > 0$ ,  $L^* \in \mathbb{N}$ , there exist  $\epsilon > 0$  and  $L \geq L^*$  such that for any  $g < \epsilon$ , such that if  $\nu > \exp(-\frac{1}{g^{8d+4p}})$  then

$$\mathbb{P}(B_L(x) \text{ is a } \mu\text{-good box}) > 1 - \frac{1}{L^{2p}} \quad (4.23)$$

where  $B_L(x) = x + [-L; L]^d \times [-\frac{M}{\nu}; \frac{M}{\nu}]$ .

The proof of this proposition will be carried over in Section 4.5. For the usual Anderson model, Theorem 4.2.1 would follow from (see Theorem 8.3 in [90]) :

1. MSA initialisation (Theorem 11.1 in [90]),
2. Wegner estimate (Theorem 5.23 in [90]),
3. Independence of these two properties for two distinct boxes (obvious in the usual model).

As already said, the only peculiarity of our model under assumption (C1) is the special form of the potential. In our case, it will thus be enough to prove

1. MSA initialization : Proposition 4.3.3,
2. Wegner estimate : Eq. (4.18) in Proposition 4.3.1,
3. Independence : Proposition 4.3.2.

Theorem 4.2.1 is then obtained as Theorem 8.3 in [90].

### 4.3.2 $L^2$ driving (C2)

A new problem appears here : For which distance on  $\mathbb{Z}^d \times \mathbb{Z}$  should we prove the exponential decay ? In the smooth case,  $\hat{\Delta}$  was a local operator, so the usual distance works fine. But because  $g(k' - k)$  is non-zero for  $k - k'$  large if the driving is only in  $L^2([0, T])$ , the operator  $\hat{\Delta}$  connects now points  $(\hat{x}, \hat{y})$  that are not close to each other in  $\mathbb{Z}^d \times \mathbb{Z}$  and there is no exponential decay along the frequency  $k$ . In order to prove exponential decay on  $\mathbb{Z}^d$ , we introduce a new decay function on  $\mathbb{Z}^d \times \mathbb{Z}$ , which can actually easily be used in the "random walk expansion" that appears in the MSA.

**Definition 4.3.2.** Let  $G : (\mathbb{Z}^d \times \mathbb{Z})^2 \rightarrow \mathbb{R}$  such that for all any  $\hat{x}_0 \in \mathbb{Z}^d \times \mathbb{Z}$ ,  $G(\hat{x}_0, \cdot) \in L^1(\mathbb{Z}^d \times \mathbb{Z})$  with  $\|G(\hat{x}_0, \cdot)\|_{L^1} < 1/2$ . We define the decay function  $d_G$  by

$$d_G(\hat{x}, \hat{y}) = -\ln \left( \sum_{\mathcal{C}(\hat{x} \rightarrow \hat{y})} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})| \right) \quad (4.24)$$

for any  $\hat{x}, \hat{y} \in \mathbb{Z}^d \times \mathbb{Z}$  if  $\hat{x} \neq \hat{y}$  and 0 otherwise, where  $\mathcal{C}(\hat{x} \rightarrow \hat{y})$  is the set of all finite sequences of the type  $(\hat{x} = \hat{z}_0, \hat{z}_1, \hat{z}_2, \dots, \hat{z}_k = \hat{y})$  (or “paths” from  $\hat{x}$  to  $\hat{y}$ ).

Let  $P : \mathbb{Z}^d \times \mathbb{Z} \rightarrow \mathbb{R}$  be defined by

$$P((x, k)) = \begin{cases} 1/\sqrt{g} & \text{if } k\nu \in [-M - \sqrt{g}, M + \sqrt{g}], \\ \frac{1}{\nu(|k|-1) - M} & \text{if } k\nu \notin [-M - \sqrt{g}, M + \sqrt{g}]. \end{cases}$$

We say that  $\hat{x}$  is a resonant site if  $|\hat{V}_\omega(\hat{x}) - \lambda| < \sqrt{g}$ . We have defined the function  $P(\hat{x})$  such that if there is no resonant site on  $x \times \mathbb{Z}$ , then  $P(\hat{x}) > \frac{1}{|\hat{V}_\omega(\hat{x}) - \lambda|}$ .

**Definition 4.3.3.** Under assumption (C2), we say that  $C_L(x) = (x + [-L, L]^d) \times \mathbb{Z} \subset \mathbb{Z}^d \times \mathbb{Z}$  is a  $\mu$ -good column if there exists a decay function  $\tilde{d}_G$  such that

$$|\langle \hat{x}, (\hat{H}_{|C_L(x)} - \lambda)^{-1} \hat{y} \rangle| \leq P(\hat{x}) e^{-\tilde{d}_G(\hat{x}, \hat{y})}$$

for all  $\hat{y} \in \partial^{in} C_L(x)$ , and such that

$$\sum_{\hat{y} \in \partial^{in} C_L(x)} e^{-\tilde{d}_G(\hat{x}, \hat{y})} < e^{-\mu L}.$$

**Proposition 4.3.4** (Initialisation of the MSA under the assumption (C2)). Assume that (C2) holds. For any  $\mu > 0$ ,  $L^* \in \mathbb{N}$ , there exist  $\epsilon > 0$  and  $L \geq L^*$  such that for any  $g < \epsilon$ , such that if  $\nu > 1$  then

$$\mathbb{P}(C_L(x) \text{ is a } \mu\text{-good column}) > 1 - \frac{1}{L^{2p}}. \quad (4.25)$$

As in the smooth case (C1), Theorem 4.2.1 will follow from the Wegner estimate (Eq. (4.19) in Proposition 4.3.1) the initialization of the MSA (Proposition 4.3.4), and the stochastic independence of distinct columns (obvious here). But there is still one difference : the MSA has to be performed with infinite columns. This issue will be addressed in Section 4.6.4, where we explain the technicals adaptations to perform in the proof in [90].

## 4.4 Wegner Estimate

In this Section, we prove Proposition 4.3.1 (Wegner estimate). For (4.18) (finite column), we closely follow [132], while for (4.19) (infinite column), we follow [53] (see also [40]). Thanks to the resolvent formula, we have the Schur formula : for any  $P$  projector and  $B = PBP$ , then

$$P(A + B)^{-1}P = ((PA^{-1}P)^{-1} + B)^{-1} \quad (4.26)$$

Where the two last “ $\cdot^{-1}$ ” in the right hand side correspond to the inverse for operators restricted to  $\text{range}(P)$ .

*Proof of (4.18).* We follow the proof from [90]. Let  $\Lambda \subset \mathbb{Z}^d \times \mathbb{Z}$ ,  $E \in \mathbb{R}$ . Let  $P_x$ ,  $x \in \mathbb{Z}^d$  the projectors on the subspace  $\{x\} \times [k_0 - K, k_0 + K]$  and  $\Lambda_0 \subset \mathbb{Z}^d$  the projection of  $\Lambda$  on its first parameters.

$$\begin{aligned}
& \mathbb{P}(\exists \bar{\lambda} \text{ eigenvalue of } \hat{H}_{|\Lambda} : \bar{\lambda} \in [E - \epsilon, E + \epsilon]) \\
& \leq \mathbb{E}(\text{Tr}(1_{[E-\epsilon, E+\epsilon]}(\hat{H}_{|\Lambda}))) \\
& \leq \mathbb{E}(2\epsilon \Im(\text{Tr}(\hat{H}_{|\Lambda} - E - i\epsilon)^{-1})) \\
& = \mathbb{E}\left[2\epsilon \Im\left(\sum_{x \in \Lambda_0} \text{Tr}(P_x(\hat{H}_{|\Lambda} - E - i\epsilon)^{-1}P_x)\right)\right] \\
& = 2\epsilon \Im\left(\sum_{x \in \Lambda_0} \mathbb{E}\left[\text{Tr}\left(\left((P_x(\hat{H}_{|\Lambda} - v_x P_x - E - i\epsilon)^{-1}P_x\right)^{-1} + v_x P_x\right)^{-1}\right)\right]\right) \\
& = 2\epsilon \sum_{x \in \Lambda_0} \mathbb{E}_{V_y: y \neq x} \left[ \int \Im\left(\text{Tr}\left(\left((P_x(\hat{H}_{|\Lambda} - v_x P_x - E - i\epsilon)^{-1}P_x\right)^{-1} + v_x P_x\right)^{-1}\right)\rho(x)dv_x\right) \right] \\
& = 2\epsilon \sum_{x \in \Lambda_0} \mathbb{E}_{V_y: y \neq x} \left[ \int \left( \sum_{\mu_i \in \sigma((P_x(\hat{H}_{|\Lambda} - v_x P_x - E - i\epsilon)^{-1}P_x)^{-1})} \Im((\mu_i + v_x)^{-1}) \right) \rho(x)dv_x \right] \\
& \leq 2\epsilon \sum_{x \in \Lambda_0} \mathbb{E}_{V_y: y \neq x} (\pi \|\rho\|_\infty (2K + 1)) \\
& \leq 2\pi\epsilon(2K + 1)|\Lambda_0| \|\rho\|_\infty,
\end{aligned}$$

where, to get the last equality, we used that  $P_x$  acts as the identity on the subspace generated by  $P_x$ .  $\square$

*Proof of (4.19).* Let  $\Lambda_0$  be a finite subset of  $\mathbb{Z}^d$ . We make a change of variable for the potential  $\alpha = \frac{1}{|\Lambda_0|} \sum_{x \in \Lambda_0} V_\omega(x)$ . As in [53] (see also [40]), the conditional probability of  $\alpha$  knowing  $\tilde{V}(x) = V_\omega(x) - \alpha$  for all  $x \in \Lambda_0$ , admits a density  $\xi_{\tilde{V}}(\alpha)$  and there exists a constant  $C$  such that, on a set  $U$  belonging to the sigma-algebra generated by  $\tilde{V}(x)$  for all  $x \in \Lambda_0$ , and with probability larger than  $1 - C\sqrt{\epsilon}$ ,

$$\|\xi_{\tilde{V}}\|_\infty \leq \frac{C}{\sqrt{\epsilon}} ((2M)^{1/2} \|\rho\|_\infty + (2M)^{3/2} \|\rho'\|_\infty) \quad (4.27)$$

Because of the symmetry described in Remark 2, for any realization  $(\tilde{V}, \alpha_0)$ , there exist  $\bar{\lambda}_1, \dots, \bar{\lambda}_{|\Lambda_0|} \in [0, \nu]$  such that  $\sigma(\hat{H}_{\tilde{V}, \alpha_0}) = \{\bar{\lambda}_1, \dots, \bar{\lambda}_{|\Lambda_0|}\} + \nu\mathbb{Z}$ . Now, keeping  $\tilde{V}$  fixed and changing  $\alpha$ , one gets  $\sigma(\hat{H}_{\tilde{V}, \alpha}) = \{\bar{\lambda}_1 + (\alpha - \alpha_0), \dots, \bar{\lambda}_{|\Lambda_0|} + (\alpha - \alpha_0)\} + \nu\mathbb{Z}$ . Then, for any  $E \in \mathbb{R}$ ,

$$\begin{aligned}
& \mathbb{P}(d(\sigma(\hat{H}), E) < \epsilon) \\
& \leq C\sqrt{\epsilon} + \mathbb{P}(\{d(\sigma(\hat{H}), E) < \epsilon\} \cap U) \\
& \leq C\sqrt{\epsilon} + \mathbb{E}_{\tilde{V}} \left( \mathbb{1}_U \sum_{i=1}^{|\Lambda_0|} \sum_{k \in \mathbb{Z}} \int \mathbf{1}(|\bar{\lambda}_i + k\nu + (\alpha - \alpha_0) - E| < \epsilon) \xi_{\tilde{V}}(\alpha) d\alpha \right) \\
& \leq C\sqrt{\epsilon} + 2\epsilon \frac{1}{\sqrt{\epsilon}} C ((2M)^{1/2} \|\rho\|_\infty + (2M)^{3/2} \|\rho'\|_\infty) K_0
\end{aligned}$$

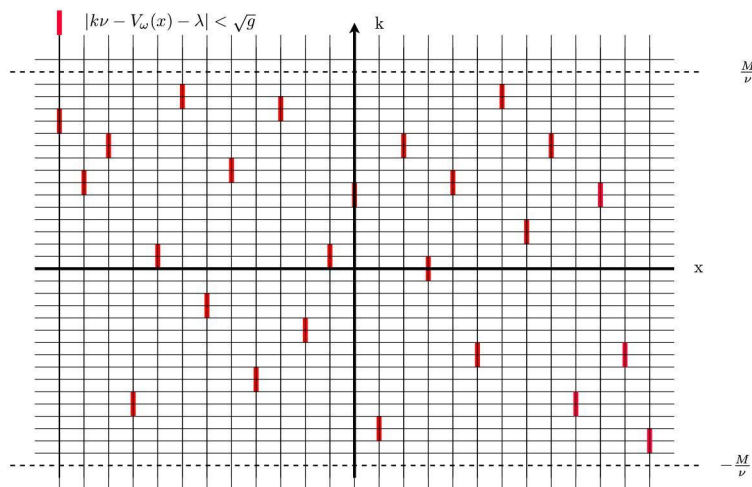


FIGURE 4.1 – resonance sites

where  $K_0$  is the maximum number of eigenvalue  $\bar{\lambda}$  in  $\sigma(\hat{H}_{\bar{V}, \alpha_0})$  such that there exists  $\alpha \in [-M, M]$  such that  $|\bar{\lambda} + \alpha - \alpha_0 - E| < \epsilon$  with non-zero probability. In particular we have  $K_0 \leq 2|\Lambda_0| \frac{M}{\nu} + 1$ .  $\square$

## 4.5 Smooth driving (C1)

*Proof of Proposition 4.3.3.* Proposition 4.3.3 is deduced from Proposition 4.5.1 and Proposition 4.5.2 below.  $\square$

The key tool for the MSA is the following formula :

$$(\hat{v}_0, (\hat{H} - \lambda)^{-1} \hat{z}) = \sum_{\hat{u} \in \partial^{in} \Lambda, \hat{v} \in \partial^{ext} \Lambda} (\hat{v}_0, (\hat{H}_{|\Lambda} - \lambda)^{-1} \hat{u}) (\hat{u}, g \hat{\Delta} \hat{v}) (\hat{v}, (\hat{H} - \lambda)^{-1} \hat{z}) \quad (4.28)$$

for any  $\hat{v}_0 \in \Lambda$  and  $\hat{z} \notin \Lambda$ , and  $\Lambda \subset \mathbb{Z}^d \times \mathbb{Z}$  with  $z \notin \Lambda$ , which is a direct application of the well known resolvent formula. We will repeat it as many times as we can, replacing  $v$  for  $v_0$  and choosing correctly the new  $\Lambda$ . The next subsection deals with this question.

### 4.5.1 Resonant sites, security box and propagation decay

Remind that  $\hat{v} = (x, k) \in \mathbb{Z}^d \times \mathbb{Z}$  is a resonant site if  $|\hat{V}_\omega(\hat{v}) - \lambda| = |v_x + \nu k - \lambda| < \sqrt{g}$ . Obviously, for any  $x$  there exists a segment  $K_x \subset \mathbb{Z}$  so that  $(x, k)$  is a resonant site for  $k \in K_x$ , where  $K_x$  is of the form  $K_x = \mathbb{Z} \cap [k_0 - \sqrt{g}/\nu, k_0 + \sqrt{g}/\nu]$  for some  $k_0$  that depends on  $V_\omega(x)$  (Figure 4.5.1). Around each segment of resonant sites  $K_x$ , we define a security box  $\Lambda_{K_x} = \{z \in \mathbb{Z}^d \times \mathbb{Z} : d(z, K_x) < N\}$ , where  $N$  is an integer that will be defined later, and  $d$  is the usual graph distance on  $\mathbb{Z}^d \times \mathbb{Z}$ .

We will say that a set of the form  $\Lambda_0 \times I \subset \mathbb{Z}^d \times \mathbb{Z}$  is not strongly resonant if  $d(\sigma(\hat{H}|_{\Lambda_0 \times I}), \lambda) > \nu^2 \alpha(g)$ , where  $\alpha(g)$  is a function which will be defined at the end of the proof of Proposition 4.5.2 below.

**Proposition 4.5.1.** Let  $L \in \mathbb{N}$ . If no security boxes intersect, if no security box is strongly resonant, and if  $(x + [-L, L]^d) \times \mathbb{Z}$  is not strongly resonant, then for any  $y \in \partial^{in}(x + [-L, L]^d)$ ,  $k_1, k_2 \in \mathbb{Z}$ ,

$$((x, k_1), (\hat{H}|_{(x+[-L, L]^d) \times [k_0-K, k_0+K]} - \lambda)^{-1}(y, k_2)) \leq \frac{2(\sqrt{g}^{\frac{N}{2}})^{n_0}}{(\nu^2 \alpha(g))^2} \quad (4.29)$$

where  $n_0 = \lfloor \frac{d((x, k_1), (y, k_2))}{2N} \rfloor$ .

In particular this proposition implies that  $(x + [-L, L]^d) \times [k_0 - K, k_0 + K]$  is a  $\mu$ -good box with

$$\mu = -\left(\frac{\ln(g)}{4} - \frac{2 \ln(\nu^2 \alpha(g))}{L}\right).$$

*Proof.* For this proof, we work inside the the space  $L^2((x + [-L, L]^d) \times [k_0 - K, k_0 + K])$  and we write simply  $\hat{H}$  instead of  $\hat{H}|_{(x+[-L, L]^d) \times [k_0-K, k_0+K]}$ .

Iterating (4.28), we obtain the usual random walk expansion for the resolvent (see e.g. [90]) : Given  $\hat{u}_0, \hat{z} \in \mathbb{Z}^d \times \mathbb{Z}$ ,

$$\begin{aligned} (\hat{u}_0, (\hat{H} - \lambda)^{-1} \hat{z}) &= \sum_{\hat{v}_i \in \partial_i^{in} \Lambda(\hat{v}_{i-1}), \hat{v}_i \in \partial^{ext} \Lambda(\hat{v}_{i-1})} (\hat{u}_0, (\hat{H}|_{\Lambda(\hat{v}_0)} - \lambda)^{-1}, \hat{u}_1)(\hat{u}_1, g \hat{\Delta} \hat{v}_1) \\ &\quad (\hat{v}_1, (\hat{H}|_{\Lambda(\hat{v}_1)} - \lambda)^{-1} \hat{u}_2)(\hat{u}_2, g \hat{\Delta} \hat{v}_2) \dots (\hat{v}_n, (\hat{H} - \lambda)^{-1} \hat{z}). \end{aligned} \quad (4.30)$$

In this writing, we need to specify when we stop iterating (4.28) and how  $\Lambda(\hat{v}_{i-1})$  is defined. The following choice will guarantee the desired exponential decay :

1. If  $|\hat{v} - \hat{z}| \leq N$  we stop iterating (4.28).
2. if  $\hat{v} \in \mathbb{Z}^d \times \mathbb{Z}$  is not a resonant site, we choose  $\Lambda(\hat{v}) = \{\hat{v}\}$ . There are then at most  $6d + 2$  points in  $\partial^{ext} \Lambda(\hat{v})$ .
3. if  $\hat{v} = (x, k)$  is a resonance site, we choose  $\Lambda(\hat{v}) = \Lambda_{K_x}$ . There are at most  $CdN^{d-1}(N + \sqrt{g}/\nu)$  points in  $\partial^{ext} \Lambda(\hat{v})$  for some numerical constant  $C > 0$ .

See Figure 4.5.1 is a typical chain.

From (4.30), we obtain

$$\begin{aligned} |(\hat{u}_0, (\hat{H} - \lambda)^{-1} \hat{z})| &\leq \\ &\sum |(\hat{u}_0, (\hat{H}|_{\Lambda(\hat{v}_0)} - \lambda)^{-1}, \hat{u}_1)(\hat{u}_1, g \hat{\Delta} \hat{v}_1)| |(\hat{v}_1, (\hat{H}|_{\Lambda(\hat{v}_1)} - \lambda)^{-1} \hat{u}_2) \times \\ &\quad \times (\hat{u}_2, g \hat{\Delta} \hat{v}_2)| \dots \|(\hat{H} - \lambda)^{-1}\|. \end{aligned} \quad (4.31)$$

The factors in each term in this sum are bounded in two different ways, depending on whether they are resonant or not :

1. If  $\hat{v}_i = (x, k)$  is not a resonant site, then  $(\hat{H}|_{\Lambda} - \lambda) = (v_x + k\nu - \lambda)\delta_{(x, k)}$  so that

$$\begin{aligned} |((x, k), (\hat{H}|_{\Lambda(\hat{v}_i)} - \lambda)^{-1}(x, k))((x, k), g \hat{\Delta}(x', k'))| &\leq \frac{|((x, k), g \hat{\Delta}(x', k'))|}{\sqrt{g}} \\ &\leq \sqrt{g}. \end{aligned} \quad (4.32)$$

2. If  $\hat{v}_i = (x, k)$  belongs to  $K_x$ , then

$$|((x, k), (\hat{H}_{|\Lambda(\hat{v}_i)} - \lambda)^{-1}(x', k'))((x', k'), g\hat{\Delta}(x'', k''))| \leq \frac{g}{d(\sigma(\hat{H}_{|\Lambda_{K_x}}, \lambda))}. \quad (4.33)$$

The sum in (4.31) will be small, if for every path joining  $\hat{u}_0$  to  $\hat{z}$ , the number  $n$  of non resonant sites is large enough to dominate the resonant terms (indexed by  $J$ ), i.e.

$$(2(d+1)\sqrt{g})^n \ll \prod_{j \in J} d(\sigma(\hat{H}_{|\Lambda_j}), \lambda) \quad (4.34)$$

We can now understand the reason why we have introduced the security boxes : Assuming that no security boxes intersect one to another, then  $u_i$  is a resonant site implies that  $u_{i+1}$  is not resonant and its distance to any resonant sites is at least larger than  $N$ . From this we can deduce that for any path joining  $\hat{u}_0$  to  $\hat{z}$ , every resonant term is followed by at least  $N$  non resonant ones. Let  $N \in \mathbb{N}$  such that

$$\frac{N^{d-1}((2N + \frac{\sqrt{g}}{\nu}))(2d+2)^{N+1}(\sqrt{g})^{\frac{N-1}{2}}}{\nu^2 \alpha(g)} < 1 \quad (4.35)$$

Then, if  $\hat{u}_i$  is resonant, and assuming that, there is no strongly resonant security box, and no intersecting security boxes, we find that the following product of  $N+1$  consecutive factors can be bounded as

$$\begin{aligned} & |(\hat{v}_i, (\hat{H}_{|\Lambda(\hat{v}_i)} - \lambda)^{-1}\hat{u}_{i+1})(\hat{u}_{i+1}, g\hat{\Delta}v_{i+1})| \dots \\ & |(\hat{v}_{i+N}, (\hat{H}_{|\Lambda(\hat{v}_{i+N})} - \lambda)^{-1}\hat{u}_{i+N+1})(\hat{u}_{i+N+1}, g\hat{\Delta}v_{i+N+1})| \\ & \leq \frac{(\sqrt{g})^N}{d(\sigma(\hat{H}_{|\Lambda_{K_x}}, \lambda))} \\ & \leq \frac{(\sqrt{g})^{\frac{N+1}{2}}}{N^{d-1}((2N + \frac{\sqrt{g}}{\nu}))(2(d+1))^N}. \end{aligned}$$

And for any  $l = k(N+1) + s$  with  $s < N+1$ .

$$\begin{aligned} & |(\hat{v}_i, (\hat{H}_{|\Lambda(\hat{v}_i)} - \lambda)^{-1}\hat{u}_{i+1})(\hat{u}_{i+1}, g\hat{\Delta}v_{i+1})| \dots \\ & |(\hat{v}_{i+l-1}, (\hat{H}_{|\Lambda(\hat{v}_{i+l-1})} - \lambda)^{-1}\hat{u}_{i+N+1})(\hat{u}_{i+l}, g\hat{\Delta}v_{i+l})| \\ & \leq \left( \frac{(\sqrt{g})^{\frac{N+1}{2}}}{N^{d-1}((2N + \frac{\sqrt{g}}{\nu}))(2(d+1))^N} \right)^k \frac{(\sqrt{g})^{s-1}}{\nu^2 \alpha(g)}. \end{aligned}$$

We can now conclude the proof. Indeed, any path connecting  $\hat{v}_0$  to  $\hat{z}$  contains at least  $(d((x, k_1), (y, k_2)) - N)/2$  steps. Denoting by  $A_l$  the set of paths connecting

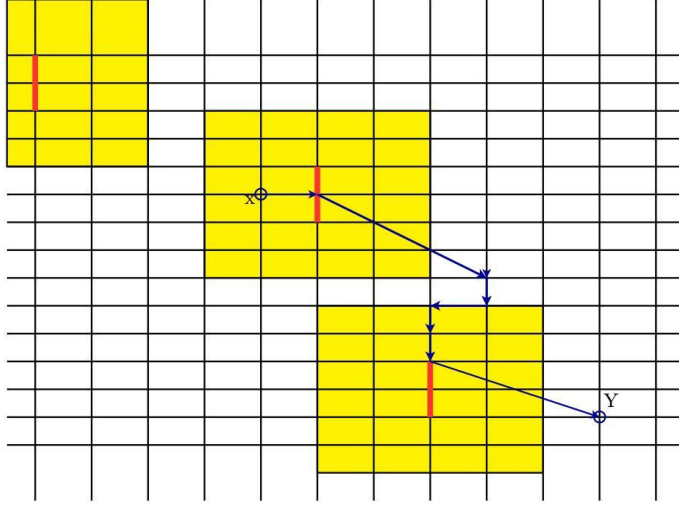


FIGURE 4.2 – A typical chain from  $X$  to  $Y$ . In red the resonance sites and in yellow the security boxes with  $N = 2$ .

$\hat{v}_0$  to  $\hat{z}$  in  $l$  steps, we find

$$\begin{aligned}
|(\hat{u}_0, (\hat{H} - \lambda)^{-1} \hat{z})| &\leq \sum_{l=(d((x,k_1),(y,k_2))-N)/2}^{\infty} |A_l| \left( \frac{(\sqrt{g})^{\frac{N+1}{2}}}{N^{d-1}((2N + \frac{\sqrt{g}}{\nu})) (2(d+1))^N} \right)^k \\
&\quad \times \frac{(\sqrt{g})^{s-1}}{\nu^2 \alpha(g)} \frac{1}{\nu^2 \alpha(g)} \\
&\leq \sum_{l=(d((x,k_1),(y,k_2))-N)/2}^{\infty} \sqrt{g}^{l/2} \frac{1}{(\nu^2 \alpha(g))^2} \\
&\leq \frac{(\sqrt{g}^{\frac{N}{2}})^{n_0}}{(1 - \sqrt{g})(\nu^2 \alpha(g))^2}
\end{aligned}$$

□

*Proof of Proposition 4.3.2.* For any  $\hat{x} \in \Lambda_0 \times [k_0 - K; k_0 + K]$ ,  $|\hat{V}(\hat{x}) - \lambda| \geq \sqrt{g}$ . One can now do the random walk development as previously with no resonance terms. □

**Proposition 4.5.2.** The probability of the event “there is no strongly resonant security box, and no intersecting security boxes” is smaller than  $1/L^{2d}$  when  $g$  goes to 0 assuming  $N = \mathcal{O}(\frac{\ln(\nu)}{\ln(g)})$ ,  $L = m_1 N$ , with  $m_1$  a fixed large integer and  $|\ln(\nu)| \leq g^{-\frac{1}{8d+4p}}$ .



*Proof.* To deal with the strongly resonant boxes, we use the Wegner type estimate (4.18) with  $\epsilon = \nu^2 \alpha(g)$  :

$$\begin{aligned}
& \mathbb{P}(\Lambda_{K_x} \text{ is strongly resonant}) \\
& \leq \sum_{k_0=-M/\nu}^{M/\nu} \mathbb{P}(\Lambda_{K_x} \text{ is strongly resonant and} \\
& \quad K_x = \mathbb{Z} \cap [k_0 - 1/(\nu\sqrt{g}), k_0 + 1/(\nu\sqrt{g})]) \\
& \leq \sum_{k_0=-M/\nu}^{M/\nu} \mathbb{P}(\Lambda_{\mathbb{Z} \cap [k_0 - 1/(\sqrt{g}\nu), k_0 + 1/(\sqrt{g}\nu)]} \text{ is strongly resonant}) \\
& \leq \frac{2M}{\nu} 2\pi\nu^2 \alpha(g) (N^d (\frac{2\sqrt{g}}{\nu} + 2N)) \|\rho\|_\infty \\
& \leq 4M (N^d (\frac{2\sqrt{g}}{\nu} + 2N)) \nu \alpha(g)
\end{aligned} \tag{4.36}$$

$$\leq 4M (N^d (\frac{2\sqrt{g}}{\nu} + 2N)) \nu \alpha(g) \tag{4.37}$$

We deal now with the probability of non intersecting security boxes : For any  $x, y \in [-L, L]^d$ ,  $\Lambda_{K_x} \cap \Lambda_{K_y} = \emptyset$ . This will be satisfied if there is no  $|k| \leq 2N$  such that  $|v_x - v_y + k\nu| \leq \sqrt{g}$ . If  $\nu \leq \sqrt{g}$ , the probability  $P$  of intersecting security boxes is bounded by :

$$\begin{aligned}
P & \leq \frac{(2L)^d ((2L)^d - 1)}{2} \mathbb{P}(|v_x - v_y| < 2(N\nu + \sqrt{g})) \\
& \leq 2(2L)^d ((2L)^d - 1) (N\nu + \sqrt{g}) \|\rho\|_\infty
\end{aligned} \tag{4.38}$$

and in any case (when  $\nu > \sqrt{g}$ ) by

$$P \leq 2(2L)^d ((2L)^d - 1) (N + 1) \sqrt{g} \|\rho\|_\infty \tag{4.39}$$

From (4.38) (or (4.39)) and Proposition 4.5.1 we conclude the proof of our theorem. We need :

$$\begin{cases} 4M(N^d (\frac{2\sqrt{g}}{\nu} + 2N)) \nu \alpha(g) \leq \frac{1}{2L^{2p}} \\ 2(2L)^d ((2L)^d - 1) (N\nu + \sqrt{g}) \|\rho\|_\infty \leq \frac{1}{2L^{2p}} \\ -(\frac{\ln(g)}{4} - \frac{2\ln(\nu^2 \alpha(g))}{L}) > \mu \end{cases} \tag{4.40}$$

or (when  $\nu > \sqrt{g}$ )

$$\begin{cases} 4M(N^d (\frac{2\sqrt{g}}{\nu} + 2N)) \nu \alpha(g) \leq \frac{1}{2L^{2p}} \\ 2(2L)^d ((2L)^d - 1) (N + 1) \sqrt{g} \|\rho\|_\infty \leq \frac{1}{2L^{2p}} \\ -(\frac{\ln(g)}{4} - \frac{2\ln(\nu^2 \alpha(g))}{L}) > \mu \end{cases} \tag{4.41}$$

and (4.35). We set  $\alpha(g) = 1$  in case of  $\nu < \sqrt{g}$  and  $\alpha(g) = g$  in case of  $\nu > \sqrt{g}$ .

1.  $N = n_1 \frac{\ln(\nu)}{\sqrt{g}}$  with  $n_1 > 7$ .
2.  $L = m_1 N$  with  $m_1$  a large enough integer.

We have then  $-(\frac{\ln(g)}{4} - \frac{\ln(\nu^2 \alpha(g))}{L}) > |\ln(g)| (\frac{1}{4} - \frac{1}{m_1})$ . Then assume  $|\ln(\nu)| \leq g^{-\frac{1}{8d+4p}}$ . So we get  $L^{4d+2p} \sqrt{g} = O(g^{1/4})$ . Finally the three conditions of (4.40) are satisfied in the limit  $g \rightarrow 0$  and this is the end of the proof of 4.3.3.  $\square$

## 4.6 $L^2$ driving (C2)

We now consider the case of an  $L^2$  driving. In this set up, we will work on infinite columns  $C_L(x) = (x + [-L, L]^d) \times \mathbb{Z}$ , so that distinct column are independent with respect to the disorder. Instead, one should be careful in the random walk expansion since infinite sums appear. That this is not a problem comes from the decay of the Green function at the large frequencies :

### 4.6.1 Decay of the Green function along the frequency axes

**Proposition 4.6.1.** Let  $\hat{\phi}$  be an eigenfunction of  $\hat{H}$  with eigenvalue  $\bar{\lambda}$ . Then

$$\sum_{x,k} \|k\nu - \lambda|\hat{\phi}(x, k)|^2 \leq (g + M)^2. \quad (4.42)$$

In particular

$$|\hat{\phi}(x, k)| \leq \frac{1 + M + g}{1 + |k\nu - \bar{\lambda}|} \quad (4.43)$$

for any  $x$ .

*Proof.* We use the time representation of  $\hat{\phi}$ . Recall that  $\hat{\phi}(t) = e^{i\lambda t}\psi(t)$  with  $\psi$  solution of (4.4). Since the evolution is unitary, for all  $t \in [0, T]$ ,  $\|\hat{\phi}(t)\| = \|\psi(t)\| = \|\psi(0)\| = \|\phi(0)\|$ . So

$$\begin{aligned} \sum_{x,k} \|k\nu - \lambda|\hat{\phi}(x, k)|^2 &= \frac{1}{T} \int_0^T \|(i\partial_t - \bar{\lambda})\phi(t)\|^2 dt \\ &= \frac{1}{T} \int_0^T \|(-g\Delta(t) + V)\phi(t)\|^2 dt \\ &\leq \frac{1}{T} \int_0^T \|(g\Delta(t) + V)\|^2 dt \\ &\leq g^2 + 2M \frac{1}{T} \int_0^T \|(g\Delta(t))\| dt + M^2 \\ &\leq (g + M)^2, \end{aligned}$$

and we deduce that  $(1 + (|k\nu - \lambda|))\hat{\phi}(x, k)$  is square integrable.  $\square$

From this we can deduce an estimate for the resolvent :

**Proposition 4.6.2.** There exist a constant  $C$  depending only on  $\nu$  so that we have

$$|(\hat{z}, (\hat{H}|_{C_L(x)} - \lambda)^{-1}\hat{y})| \leq \frac{(2L + 1)^{d/2}(2 + M)P(\hat{x})}{1 + |k_z - k_y|} \left( \sup_i \frac{1}{|\lambda - \bar{\lambda}_i|} + C \right)$$

for any  $\hat{z} = (z, k_z)$ ,  $\hat{y} = (y, k_y) \in C_L(x)$ , where  $\bar{\lambda}_i$  are the eigenvalue of  $\hat{H}|_{C_L(x)}$ .

*Proof.* We decompose  $\hat{H}_{|C_L(x)}$  into its eigenvectors and we apply Cauchy Schwartz. The eigenvalues of  $\hat{H}_{|C_L(x)}$  are all of the form  $\bar{\lambda}_i + k\nu$ , where we can assume that  $\bar{\lambda}_i$  are such that  $|\bar{\lambda}_i + k\nu - \lambda| \geq \nu/2$  if  $k \neq 0$ . Then

$$\begin{aligned}
& (\hat{z}, (\hat{H}_{|C_L(x)} - \lambda)^{-1} \hat{y}) \\
&= \sum_{i=1}^{|\Lambda|} \sum_{k \in \mathbb{Z}} \frac{1}{\bar{\lambda}_i + \nu k - \lambda} \phi_{\bar{\lambda}_i + \nu k}(\hat{z}) \phi_{\bar{\lambda}_i + \nu k}(\hat{y}) \\
&\leq \left( \sum_{i=1}^{|\Lambda|} \sum_{k \in \mathbb{Z}} (1 + |\bar{\lambda}_i + \nu(k - k_z)|)^2 |\phi_{\bar{\lambda}_i + \nu k}(\hat{z})|^2 \right)^{1/2} \\
&\quad \left( \sum_{i=1}^{|\Lambda|} \sum_{k \in \mathbb{Z}} \frac{1}{|\bar{\lambda}_i + \nu k - \lambda|^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu(k - k_z)|)^2} |\phi_{\bar{\lambda}_i + \nu k}(\hat{y})|^2 \right)^{1/2} \\
&= \left( \sum_{i=1}^{|\Lambda|} \sum_{k \in \mathbb{Z}} (1 + |\bar{\lambda}_i + \nu(k - k_z)|)^2 |\phi_{\bar{\lambda}_i}(z, k_z - k)|^2 \right)^{1/2} \\
&\quad \left( \sum_{i=1}^{|\Lambda|} \sum_{k \in \mathbb{Z}} \frac{1}{|\bar{\lambda}_i + \nu k - \lambda|^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu(k - k_z)|)^2} |\phi_{\bar{\lambda}_i + \nu k}(\hat{y})|^2 \right)^{1/2}
\end{aligned}$$

We use now (4.42) to control the first factor, and (4.43) to get an estimate on  $|\phi_{\bar{\lambda}_i + \nu k}(\hat{y})|$  in the second one :

$$\begin{aligned}
& (\hat{z}, (\hat{H}_{|C_L(x)} - \lambda)^{-1} \hat{y}) \\
&\leq (1 + M + g)^2 \left( \sum_{i=1}^{|\Lambda|} \sum_{k \in \mathbb{Z}} \frac{1}{|\bar{\lambda}_i + \nu k - \lambda|^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu(k - k_z)|)^2} \times \right. \\
&\quad \left. \times \frac{1}{(1 + |\bar{\lambda}_i + \nu(k - k_y)|)^2} \right)^{1/2} \\
&= (1 + M + g)^2 \left( \sum_{i=1}^{|\Lambda|} \frac{1}{|\bar{\lambda}_i - \lambda|^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu(k - k_z)|)^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu k_y|)^2} \right. \\
&\quad \left. + \sum_{i=1}^{|\Lambda|} \sum_{k \in \mathbb{Z}^*} \frac{1}{|\bar{\lambda}_i + \nu k - \lambda|^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu(k - k_z)|)^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu(k - k_y)|)^2} \right)^{1/2} \\
&\leq |\Lambda|^{1/2} (1 + M + g)^2 \left( \sup_i \frac{1}{|\lambda - \bar{\lambda}_i|} + C \right) P(\hat{z}) \frac{1}{(1 + |k_z - k_y|)},
\end{aligned}$$

where the last inequality comes from the estimate of the integral

$$\int dk \frac{1}{1 + k^2} \frac{1}{1 + (k - k_z)^2} \frac{1}{1 + (k - k_y)^2} \sim \frac{1}{(1 + |k_z|)^2} \frac{1}{(1 + |k_z - k_y|)^2}.$$

□

**Definition 4.6.1.** We say that  $C_L(x)$  is not strongly resonant if

$$\inf_{\bar{\lambda}_i \in \sigma(\hat{H}_{|C_L(x)})} \{|\bar{\lambda}_i - \lambda|\} > e^{-\sqrt{L}}. \quad (4.44)$$

In particular, if  $C_L(x)$  is not strongly resonant, we have

$$|(\hat{z}, (\hat{H}|_{C_L(x)} - \lambda)^{-1}\hat{y})| \leq \frac{CL^{d/2}P(\hat{z})}{1 + |k_z - k_y|} e^{\sqrt{L}}$$

where  $C$  is a constant.

### 4.6.2 The decay function

If Anderson localization is most of the time studied over  $\mathbb{Z}^d$ , the problem could be raised on any set of point  $X$ . It is indeed easy to define a random potential  $V(x)$ ,  $x \in X$  and a "Laplacian"  $\Delta(x_1, x_2)$  without assuming a particular geometry of the system. But to recover the decay, one should then first define a decay function, and  $\Delta$  is the only object that we can use to construct such a decay function. We first give a general definition.

**Definition 4.6.2.** Let  $G : X \times X \rightarrow \mathbb{R}_+$ , for any  $\hat{x}, \hat{y} \in X$ ,

$$d_G(\hat{x}, \hat{y}) = -\ln \left( \sum_{\mathcal{C}(\hat{x} \rightarrow \hat{y})} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})| \right) \quad (4.45)$$

if  $\hat{x} \neq \hat{y}$  and 0 otherwise, where  $\mathcal{C}(\hat{x} \rightarrow \hat{y})$  is the set of all paths  $\hat{x} = \hat{z}_0, \hat{z}_1, \hat{z}_2, \dots, \hat{z}_k = \hat{y}$  from  $\hat{x}$  to  $\hat{y}$ .

**Proposition 4.6.3.** If for any  $z \in X$ ,  $\sum_{z_1} |G(z, z_1)| < 1/2$ , then  $d_G$  is positive and satisfies the triangle inequality.

*Proof.* We first check that  $d_G$  is positive. Let  $\hat{x}, \hat{y}$

$$\begin{aligned} & \sum_{\mathcal{C}(\hat{x} \rightarrow \hat{y})} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})| \\ & \leq \sum_{\hat{y}'} \sum_{\mathcal{C}(\hat{x} \rightarrow \hat{y}')} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})| \\ & \leq \sum_{n>0} \prod_{i=0}^n \left( \max_{\hat{z}_i} \sum_{\hat{z}_{i+1} \in X} |G(\hat{z}_i, \hat{z}_{i+1})| \right) \\ & = \sum_{n>0} \left( \max_{\hat{x}} \sum_{\hat{y} \in X} |G(\hat{x}, \hat{y})| \right)^n \\ & = \frac{\left( \max_{\hat{x}} \sum_{\hat{y} \in X} |G(\hat{x}, \hat{y})| \right)}{1 - \left( \max_{\hat{x}} \sum_{\hat{y} \in X} |G(\hat{x}, \hat{y})| \right)} \\ & < 1. \end{aligned}$$

We now check the triangle inequality. Let  $\hat{z}$  be another point in  $X$ .

$$\begin{aligned}
& d_G(\hat{x}, \hat{y}) + d_G(\hat{y}, \hat{z}) \\
&= -\ln \left( \sum_{C(\hat{x} \rightarrow \hat{y})} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})| \right) - \ln \left( \sum_{C(\hat{y} \rightarrow \hat{z})} \prod_j |G(\hat{z}_j, \hat{z}_{j+1})| \right) \\
&= -\ln \left( \sum_{C(\hat{x} \rightarrow \hat{y})} \sum_{C(\hat{y} \rightarrow \hat{z})} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})| \prod_j |G(\hat{z}_j, \hat{z}_{j+1})| \right) \\
&\geq -\ln \left( \sum_{C(\hat{x} \rightarrow \hat{z})} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})| \right) \\
&= d_G(\hat{x}, \hat{z}).
\end{aligned}$$

□

### 4.6.3 initialisation of the multiscale

*Proof of Proposition 4.3.4.* Proposition 4.3.4 follows from Propositions 4.6.5 and 4.6.6 below. □

**Definition 4.6.3.** We will use  $d_G$  with  $X = \mathbb{Z}^d \times \mathbb{Z}$  and

$$G(\hat{x}, \hat{y}) = g |\hat{\Delta}(\hat{x}, \hat{y}) P(\hat{y})| \quad (4.46)$$

Remark that we also have  $\Delta(\hat{z}, \cdot) P(\cdot) \in L^1$  because  $\Delta(\hat{z}, \cdot) \in L^2$  and  $P(\cdot) \in L^2$ . We will write  $\|G\|_{\ell^1_{max}} = \sup_x \sum_y G(x, y)$ . This quantity goes to zero as  $g \rightarrow 0$ . The decay function is related to usual distance on  $\mathbb{Z}^d$  through the following proposition :

**Proposition 4.6.4.** For any  $\hat{x} = (x, k_x)$ ,

$$\sum_{z:|x-z|=L} \sum_k e^{-d_G((x, k_x), (z, k))} \leq e^{L \ln(\|G\|_{\ell^1_{max}}) - \ln(1 - \|G\|_{\ell^1_{max}})} \quad (4.47)$$

in particular  $\hat{z} = (z, k_z)$ ,  $|x - z| > L$ .

$$d_G(\hat{x}, \hat{z}) \geq L(-\ln(\|G\|_{\ell^1_{max}})) + \ln(1 - \|G\|_{\ell^1_{max}}) \quad (4.48)$$

*Proof.* Because no path of length smaller than  $L$  connect  $\hat{x}$  with the boundary of  $\{(z, k) : |x - z| > L\}$ ,

$$\sum_{C(\hat{x} \rightarrow \hat{z})} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})| \leq \sum_{n>L} \|G\|_{\ell^1_{max}}^n \leq \frac{\|G\|_{\ell^1_{max}}^L}{1 - \|G\|_{\ell^1_{max}}}. \quad (4.49)$$

So

$$d_G(\hat{x}, \hat{z}) \geq -L \ln(\|G\|_{\ell^1_{max}}) + \ln(1 - \|G\|_{\ell^1_{max}}).$$

□

**Proposition 4.6.5.** If there is no resonant site at all in  $C_L(x)$ , and if  $\hat{H}|_{C_L(x)}$  has no eigenvalue  $\bar{\lambda}_i$  with  $|\bar{\lambda}_i - \lambda| \leq \sqrt{g}$ , then  $C_L(x)$  is a  $(\mu', \tilde{d}_G)$  good column

*Proof.* We use here again the resolvent formula :

$$(\hat{x}, (\hat{H}_{|C_L(x)} - \lambda)^{-1} \hat{y}) = \sum_{\hat{z}} \frac{g \hat{\Delta}(\hat{x}, \hat{z})}{\hat{V}(\hat{x}) - \lambda} (\hat{z}, (\hat{H}_{|C_L(x)} - \lambda)^{-1} \hat{y}).$$

Applying it several times yields the usual random walk expansion :

$$\begin{aligned} & (\hat{x}, (\hat{H}_{|C_L(x)} - \lambda)^{-1} \hat{y}) \\ &= \sum_{\hat{z}, \hat{z}_1, \hat{z}_2, \dots, \hat{z}_n} \frac{g \hat{\Delta}(\hat{x}, \hat{z}_1)}{\hat{V}(\hat{x}) - \lambda} \frac{g \hat{\Delta}(\hat{z}_1, \hat{z}_2)}{\hat{V}(\hat{z}_1) - \lambda} \dots \frac{g \hat{\Delta}(\hat{z}_{n-1}, \hat{z}_n)}{\hat{V}(\hat{z}_{n-1}) - \lambda} (\hat{z}_n, (\hat{H}_{|C_L(x)} - \lambda)^{-1} \hat{y}) \end{aligned}$$

Because there is no resonant site,  $\frac{1}{\hat{V}(\hat{z}) - \lambda} \leq P(\hat{z})$  for any  $\hat{z} \in C_L(x)$ . So

$$\begin{aligned} & |(\hat{x}, (\hat{H}_{|C_L(x)} - \lambda)^{-1} \hat{y})| \\ &= P(\hat{x}) \sum_{\hat{z}, \hat{z}_1, \hat{z}_2, \dots, \hat{z}_n} |g \hat{\Delta}(\hat{x}, \hat{z}_1) P(\hat{z}_1) \\ &\quad \times g \hat{\Delta}(\hat{z}_1, \hat{z}_2) \dots P(\hat{z}_{n-1}) g \hat{\Delta}(\hat{z}_{n-1}, \hat{z}_n) (\hat{z}_n, (\hat{H}_{|C_L(x)} - \lambda)^{-1} \hat{y})| \\ &\leq CP(\hat{x}) \sum_{\hat{z}, \hat{z}_1, \hat{z}_2, \dots, \hat{z}_n} |g \hat{\Delta}(\hat{x}, \hat{z}_1) P(\hat{z}_1) g \hat{\Delta}(\hat{z}_1, \hat{z}_2) \dots P(\hat{z}_{n-1}) \\ &\quad \times g \hat{\Delta}(\hat{z}_{n-1}, \hat{z}_n) P(\hat{z}_n)| \frac{L^{d/2}}{\sqrt{g}} \\ &\leq CL^{d/2} \frac{P(\hat{x})}{\sqrt{g}} \sum_{C(x \rightarrow y)} \prod_i |g \hat{\Delta}(\hat{z}_i, \hat{z}_{i+1})| P(\hat{z}_{i+1}) \end{aligned}$$

where the first inequality is obtained through Proposition 4.6.2 and the hypothesis on the eigenvalues  $\bar{\lambda}_i$ . So one has

$$|(\hat{x}, (\hat{H}_{|C_L(x)} - \lambda)^{-1} \hat{y})| \leq CL^{d/2} \frac{P(\hat{x})}{\sqrt{g}} e^{-\bar{d}_G(\hat{x}, \hat{y})}$$

□

**Proposition 4.6.6.** The probability of the event “there is no resonant site at all in  $C_L(x)$ , and  $\hat{H}_{|C_L(x)}$  has no eigenvalue  $\lambda_i$  with  $|\lambda_i - \lambda| \leq \sqrt{g}$ ” goes to 0 with  $g \rightarrow 0$ .

*Proof.* First,

$$\mathbb{P}(\text{there is no resonant site in } C_L(x)) \leq \|\rho\|_\infty \frac{2M}{\nu} (2L+1)^d \sqrt{2g}. \quad (4.50)$$

Next, thanks to Wegner estimate,

$$\mathbb{P}(C_L(x) \text{ is not strongly resonant}) \leq \|\rho\|_\infty \frac{2M}{\nu} (2L+1)^d \sqrt{2g}. \quad (4.51)$$

This gives the proposition for  $g \rightarrow 0$ . □

#### 4.6.4 Technical results for the iteration of the MSA

We have proved that for a fixed  $L$ ,  $C_L(c)$  is a good column with high probability. MSA induces that the property is valid for all  $L_k$  with  $L_{k+1} = L_k^\alpha$ ,  $L_0 = L$ . We change the technical detail 10.42 or 10.81 in [90] by an estimate (4.53). The rest of the MSA will follow. To illustrate this, we prove Proposition 4.6.8 which is the equivalent of Theorem 10.14 in [90].

The estimate of Green function are used for the following remark.

**Proposition 4.6.7.**

$$\sup_{\hat{x}, \hat{y}} \sum_{k_y} \sum_{\hat{z}} \frac{1}{1 + |k_x - k_y|} |\Delta(\hat{y}, \hat{z})P(\hat{z})| < \infty \quad (4.52)$$

In particular  $G(\hat{x}, \cdot) = \sum_{k_y} \frac{1}{1 + |k_x - k_y|} |\Delta(\hat{y}, \cdot)P(\cdot)|$  is in  $L^1$  uniformly in  $x$ .

*Proof.* We have  $\sqrt{|\Delta(\hat{y}, \cdot)|} \in L^4$ , with a norm that can be bounded uniformly in  $\hat{y}$ ,  $\frac{1}{1+|\cdot|} \in L^{\frac{4}{3}}$  and  $P(\cdot) \in L^{\frac{4}{3}}$ .

$$\begin{aligned} & \sup_{\hat{x}, \hat{y}} \sum_{k_y} \sum_{\hat{z}} \frac{1}{1 + |k_x - k_y|} |\Delta(\hat{y}, \hat{z})P(\hat{z})| \\ & \leq \left( \sup_{\hat{x}, \hat{y}, \hat{z}} \sum_{k_y} \frac{1}{1 + |k_x - k_y|} \sqrt{|\Delta(\hat{y}, \hat{z})|} \right) \left( \sup_{\hat{y}} \sum_{\hat{z}} \sqrt{|\Delta(\hat{y}, \hat{z})|} |P(\hat{z})| \right) \\ & \leq \left( \left\| \frac{1}{1 + |\cdot|} \right\|_{L^{\frac{4}{3}}} \left\| \sqrt{|\Delta(\hat{y}, \cdot)|} \right\|_{L^4} \right) \left( \left\| \sqrt{|\Delta(\hat{y}, \cdot)|} \right\|_{L^4} \|P(\cdot)\|_{L^{\frac{4}{3}}} \right) \\ & < \infty \end{aligned}$$

□

**Proposition 4.6.8.** If there is no two distinct small scale columns  $C_{L_k}(y) \subset C_{L_{k+1}}(x)$  which are not  $\mu$ -good, and there is no columns  $C_{2L_k}(y') \subset C_{L_{k+1}}(x)$  that are strongly resonant and  $C_{L_{k+1}}(x)$  is not strongly resonant, then  $C_{L_{k+1}}(x)$  is  $\mu'$  good with  $\mu' > \mu - \frac{3L_k}{L_{k+1}}$ .

*Proof.* Let  $d_G$  the decay function used for the small scale good boxes. In the case of  $C_{L_k}$  is a bad column, we use the resolvent development twice

$$\begin{aligned} & |(\hat{x}, (\hat{H}_{|C_{L_{k+1}}(x)} - \lambda)^{-1} \hat{y})| \\ & \leq \sum_{\substack{\hat{z}_1 \in \partial^{in} C_{L_{2k}}(x), \\ \hat{z}_2 \in \partial^{ext} C_{L_{2k}}(x)}} |(\hat{x}, (\hat{H}_{|C_{2L_k}(x)} - \lambda)^{-1} \hat{z}_1) g \hat{\Delta}(\hat{z}_1, \hat{z}_2) (\hat{z}_2, (\hat{H}_{|C_{L_{k+1}}(x)} - \lambda)^{-1} \hat{y})| \\ & \leq \sum_{\substack{\hat{z}_1 \in \partial^{in} C_{L_{2k}}(x) \\ \hat{z}_2 \in \partial^{ext} C_{L_{2k}}(x)}} \sum_{\substack{\hat{z}_3 \in \partial^{in} C_{L_k}(z_2) \\ \hat{z}_4 \in \partial^{ext} C_{L_k}(z_2)}} |(\hat{x}, (\hat{H}_{|C_{2L_k}(x)} - \lambda)^{-1} \hat{z}_1) g \hat{\Delta}(\hat{z}_1, \hat{z}_2) \\ & \quad (\hat{z}_2, (\hat{H}_{|C_{L_k}(x)} - \lambda)^{-1} \hat{z}_3) g \hat{\Delta}(\hat{z}_3, \hat{z}_4) (\hat{z}_4, (\hat{H}_{|C_{L_{k+1}}(x)} - \lambda)^{-1} \hat{y})| \\ & \leq P(\hat{x}) \sum_{\substack{\hat{z}_1 \in \partial^{in} C_{L_{2k}}(x) \\ \hat{z}_2 \in \partial^{ext} C_{L_{2k}}(x)}} \sum_{\substack{\hat{z}_3 \in \partial^{in} C_{L_k}(z_2) \\ \hat{z}_4 \in \partial^{ext} C_{L_k}(z_2)}} e^{-\sqrt{L_k}} \frac{C(2L_k)^{d/2}}{1 + |k_{\hat{x}} - k_{\hat{z}_1}|} |g \hat{\Delta}(\hat{z}_1, \hat{z}_2)| \\ & \quad P(\hat{z}_2) e^{-d_G(\hat{z}_2, \hat{z}_3)} |g \hat{\Delta}(\hat{z}_3, \hat{z}_4) (\hat{z}_4, (\hat{H}_{|C_{L_k}(x)} - \lambda)^{-1} \hat{y})| \end{aligned}$$

So let us define  $G'$  as follows :

$$G'(\hat{x}, \hat{y}) = e^{-d_G(\hat{x}, \hat{y})}$$

if  $C_{L_k}(x)$  is a  $\mu$  good box and  $\hat{y} \in \partial^{ext} C_{L_k}(x)$ , and

$$G'(\hat{x}, \hat{y}) = \sum_{\substack{\hat{z}_1 \in \partial^{in} C_{L_{2k}}(x) \\ \hat{z}_2 \in \partial^{ext} C_{L_{2k}}(x) \\ \hat{z}_3 \in \partial^{in} C_{L_k}(z_2)}} e^{\sqrt{L_k}} \frac{C(2L_k)^{d/2}}{1 + |k_{\hat{x}} - k_{\hat{z}_1}|} |g\hat{\Delta}(\hat{z}_1, \hat{z}_2)| P(\hat{z}_2) e^{-d_G(\hat{z}_2, \hat{z}_3)} \\ \times g\hat{\Delta}(\hat{z}_3, \hat{y}) P(\hat{y}) \quad (4.53)$$

if  $C_L(x)$  is a bad box.

Because of the previous remark, there is a constant  $C$  independent of  $L_k$  such that for the second case :  $\|G'\|_{L^1} \leq C' e^{2\sqrt{L_k}} e^{-\mu L_k}$ . We can then recover the usual tools, using that  $e^{-\mu L_k}$  dominate the other terms for  $L_k$  large. In particular because for any path from  $x$  to  $\partial C_L(x)$  there is at least  $(\frac{L_{k+1}}{L_k} - 3) \mu$  good boxes. So, with the same argument of Proposition 4.6.4,

$$\sum_{\hat{y} \in \partial^{in} C_L(x)} e^{-d_{G'}(\hat{x}, \hat{y})} \leq e^{-\mu(L_{k+1} - 3L_k) - \ln(1 - \|G'\|_{\ell^1_{max}})}$$

□

## 4.7 Proof of the corollaries

As said, Corollaries 4.2.1 and 4.2.2 do not follow logically from Theorem 4.2.1; instead one should go through the MSA once again and refine several estimates. This work has been carried over in [47], and one indicates here only the main steps as well as the few needed extra adaptations.

Let us start with Corollary 4.2.1.

**Proposition 4.7.1.** there exist  $p > 0$  such that :

$$\mathbb{E}(\sup_{t>0} \sum_{x \in \mathbb{Z}^d} \sum_k |x|^p |\check{\phi}(x, k, t)|^2) < \infty \quad (4.54)$$

*Proof.* Thanks to the MSA carried over in this paper, one can check that the results of [47] holds; in particular the assumptions of Theorem 3.1 in [47] are satisfied. □

In order to recover  $\phi$  from  $\check{\phi}$  we use the following proposition. Remind that, thanks to (4.6), we have  $\|H(t)\|_{L^1[0;T]} \leq \sqrt{T} \|H(t)\|_{L^2[0;T]}$ .

**Proposition 4.7.2.** Let  $\psi(t) \in L^2(\mathbb{Z}^d)$  satisfying  $\|\psi(t)\|_{L^2} = 1$  for all  $t \in \mathbb{R}$  be a solution of

$$i\partial_t \psi(t) = A(t)\psi(t) \quad (4.55)$$

where for any  $t$   $A(t)$  is hermitian,  $C = \|A(\cdot)\|_{L^1([0;T])} < \infty$  and  $(x, A(t)y) = 0$  if  $|x - y| > 1$ . For any  $t \in [0, T]$  and any  $x_0 \in \mathbb{Z}^d$ , we have

$$\sum_{|z-x_0|<R} |\psi(z, t)|^2 \geq |\psi(x_0, 0)|^2 \left(1 - \left(e^C \sum_{k \geq R} \frac{(2dC)^k}{k!}\right)^2\right) - e^C \sum_{k \geq R} \frac{(2dC)^k}{k!} |\psi(x_0, 0)|. \quad (4.56)$$



*Proof.* Let's separate  $\psi(0) = \mathbb{1}_{x=x_0}\psi(0) + \mathbb{1}_{x \neq x_0}\psi(0)$ . Because the  $A(t)$  is hermitian, there exists  $U(t)$  unitary such that

$$\psi(t) = U(t)\psi(0) = U(t)(\mathbb{1}_{x=x_0}\psi(0)) + U(t)(\mathbb{1}_{x \neq x_0}\psi(0)) \quad (4.57)$$

Calling  $\psi_1 = U(t)(\mathbb{1}_{x=x_0}\psi(0))$ ,  $\psi_2 = U(t)(\mathbb{1}_{x \neq x_0}\psi(0))$  we have  $(\psi_1, \psi_2) = 0$  and  $\|\psi_1\|^2 + \|\psi_2\|^2 = 1$ . Because  $\mathbb{1}_{|z-x_0| < R}$  is a projector,

$$\begin{aligned} (\psi_1 + \psi_2, \mathbb{1}_{|z-x_0| < R}(\psi_1 + \psi_2)) &= (\psi_1, \mathbb{1}_{|z-x_0| < R}\psi_1) + (\psi_2, \mathbb{1}_{|z-x_0| < R}\psi_2) \\ &\quad + 2\mathcal{R}(\psi_1, \mathbb{1}_{|z-x_0| < R}\psi_2) \\ &\geq (\psi_1, \mathbb{1}_{|z-x_0| < R}\psi_1) - 2|(\psi_2, \mathbb{1}_{|z-x_0| \geq R}\psi_1)| \\ &\geq \|\psi_1\|^2 - \|\mathbb{1}_{|z-x_0| \geq R}\psi_1\|^2 - 2|(\psi_2, \mathbb{1}_{|z-x_0| \geq R}\psi_1)| \\ &\geq \|\psi_1\|^2 - \|\mathbb{1}_{|z-x_0| \geq R}\psi_1\|^2 - 2\|\mathbb{1}_{|z-x_0| \geq R}\psi_1\| \end{aligned}$$

We now proof that the locality of  $A(t)$  implies that  $\|\mathbb{1}_{|z-x_0| \geq R}\psi_1\|^2$  is small.

$$i \frac{d}{dt} \psi_1(y, t) = A(t)\psi_1(y, t) = \sum_{|y'-y| \leq 1} A_{y,y'}(t)\psi_1(y', t).$$

Hence

$$\frac{d}{dt} |\psi_1(y, t)| \leq \sum_{|y'-y| \leq 1} |A_{y,y'}(t)| |\psi_1(y', t)| \leq \|A(t)\| \sum_{|y'-y| \leq 1} |\psi_1(y', t)|$$

Let now  $a(y, t)$  solution of the system

$$\begin{cases} \frac{d}{dt} a(y, t) = \|A(t)\| \sum_{|y'-y| \leq 1} a(y', t) \\ a(y, 0) = |\psi_1(x_0, 0)| \mathbb{1}_{y=x_0} \end{cases} \quad (4.58)$$

We have then for any  $(y, t)$

$$|\psi_1(y, t)| \leq a(y, t) \quad (4.59)$$

We can evaluate  $a$  with the following remark : Let  $X(t)$  be the classical markovian random walk on  $\mathbb{Z}$  of variable rate  $\|A(t)\|$  and starting at point  $x_0$ . Its generator is

$$\frac{d}{dt} \mathbb{P}_{x_0}(X(t) = y) = \|A(t)\| \sum_{|y'-y|} (\mathbb{P}_{x_0}(X(t) = y') - \mathbb{P}_{x_0}(X(t) = y)) \quad (4.60)$$

and then we have

$$e^{-(2d+1) \int_0^t \|A(u)\| du} a(y, t) = a(x_0, 0) \mathbb{P}_{x_0}(X(t) = y) \quad (4.61)$$

We can then deduce

$$\sum_{y \geq R} a(y, t) \leq a(x_0, 0) e^{(2d+1) \int_0^t \|A(u)\| du} \mathbb{P}(N_{2d \int_0^t \|A(u)\| du} \geq R) \quad (4.62)$$

where  $N_{2d \int_0^t \|A(u)\| du}$  is the Poisson process of parameter  $2d \int_0^t \|A(u)\| du$ . So for any  $t \leq T$

$$\sum_{y \geq R} a(y, t) \leq a(x_0, 0) e^C \sum_{k \geq R} \frac{(2dC)^k}{k!} \quad (4.63)$$

We can now conclude

$$\begin{aligned}
\sum_{|z-x_0|<R} |\psi(z,t)|^2 &= (\psi_1 + \psi_2, \mathbf{1}_{|z-x_0|<R}(\psi_1 + \psi_2)) \\
&\geq \|\psi_1\|^2 - \|\mathbf{1}_{|z-x_0|\geq R}\psi_1\|^2 - \|\mathbf{1}_{|z-x_0|\geq R}\psi_1\| \\
&\geq |\psi(x_0,0)|^2 - |\psi(x_0,0)|^2 (e^C \sum_{k\geq R} \frac{(2dC)^k}{k!})^2 \\
&\quad - |\psi(x_0,0)| (e^C \sum_{k\geq R} \frac{(2dC)^k}{k!})
\end{aligned}$$

□

The above proposition and the dynamical localisation of  $\check{\phi}$  enable us to conclude :

**Proposition 4.7.3.** For any  $\epsilon > 0$ , there exist some constants  $C_\epsilon, D_\epsilon$  such that

$$C_\epsilon \sum_{x \in \mathbb{Z}^d} \sum_k |x|^p |\check{\phi}(x, k, t)|^2 + D_\epsilon \geq \sum_{x_0 \in \mathbb{Z}^d} |x_0|^{p-\epsilon} |\phi(x_0, t)|^2 \quad (4.64)$$

*Proof.* Let  $\epsilon > 0$ . Let now  $x_0 \mapsto R(x_0)$  be such that

$$\sum_{x_0 \in \mathbb{Z}^d} |x_0|^p \sum_{k \geq R(x_0)} \frac{(2dC)^k}{k!} < \infty \quad (4.65)$$

and such that, for all  $x_0 \in \mathbb{Z}^d$ ,

$$e^C \sum_{k \geq R(x_0)} \frac{(2dC)^k}{k!} < \frac{1}{2} \quad (4.66)$$

moreover that  $|x - x_0| < R(x_0)$  then  $|x - x_0| < (1 + \epsilon)R(x)$ , and such there is constant  $C_\epsilon$  such that

$$\sum_{|x-x_0| \leq (1+\epsilon)R(x)} |x_0|^{p-\epsilon} \leq C_\epsilon |x|^p \quad (4.67)$$

for  $|x_0| > 1$ . For example we could have chosen  $R(x) = \ln(|x|)^2$  for large  $x$ .

$$\begin{aligned}
& \mathbb{E}(\sup_{t>0} \sum_{x \in \mathbb{Z}^d} \sum_k |x|^p |\check{\phi}(x, k, t)|^2) \\
&= \mathbb{E}(\sup_{t>0} \sum_{x \in \mathbb{Z}^d} |x|^p \frac{1}{T} \int_t^{t+T} |\phi(x, u)|^2 du) \\
&\geq \frac{1}{C_\epsilon} \mathbb{E}(\sup_{t>0} \sum_{x \in \mathbb{Z}^d} \sum_{|x-x_0| \leq (1+\epsilon)R(x)} \frac{1}{T} \int_t^{t+T} |x_0|^{p-\epsilon} |\phi(x, u)|^2 du) \\
&\geq \frac{1}{C_\epsilon} \mathbb{E}(\sup_{t>0} \sum_{x_0 \in \mathbb{Z}^d} |x_0|^{p-\epsilon} \frac{1}{T} \int_t^{t+T} \sum_{|x-x_0| \leq R(x_0)} |\phi(x, u)|^2 du) \\
&\geq \frac{1}{C_\epsilon} \mathbb{E}(\sup_{t>0} \sum_{x_0 \in \mathbb{Z}^d} |x_0|^{p-\epsilon} \frac{1}{T} \int_t^{t+T} |\psi(x_0, t)|^2 - |\psi(x_0, t)|^2 (e^C \sum_{k \geq R} \frac{(2dC)^k}{k!})^2 \\
&\quad - |\psi(x_0, t)| (e^C \sum_{k \geq R} \frac{(2dC)^k}{k!}) du) \\
&\geq \frac{1}{2C_\epsilon} \mathbb{E}(\sup_{t>0} \sum_{x_0 \in \mathbb{Z}^d} |x_0|^{p-\epsilon} |\psi(x_0, t)|^2) - e^C \frac{1}{C_\epsilon} \sum_{x_0 \in \mathbb{Z}^d} |x_0|^p \sum_{k \geq R(x_0)} \frac{(2dC)^k}{k!}
\end{aligned}$$

So

$$\mathbb{E}(\sup_{t>0} \sum_{x_0 \in \mathbb{Z}^d} |x_0|^{p-\epsilon} |\psi(x_0, t)|^2) < \infty \quad (4.68)$$

□

Let us now come to Corollary 4.2.2 :

*Proof of Corollary 4.2.2.* Since

$$\psi_{\bar{\lambda}}(\cdot, 0) = \sum_{k \in \mathbb{Z}} \hat{\psi}(\cdot, k),$$

we can write

$$H_{eff}(x, y) = \sum_{(k,l) \in \mathbb{Z}^2} \sum_{\bar{\lambda} \in [0, \nu[} \bar{\lambda} \psi_{\bar{\lambda}}(x, k) \bar{\psi}_{\bar{\lambda}}(y, l) = \sum_{k,l} ((x, k), \eta(\hat{H}))(y, k)$$

with

$$\eta : \mathbb{R} \rightarrow \mathbb{R}, s \mapsto \eta(s) = 1_{[0, \nu[}(s)s.$$

Again, thanks to the MSA shown in this paper, and the deterministic exponential decay along the frequency axis of the eigenfunctions under Assumption (C1), it can again be checked that the hypotheses of Theorem 3.1 in [47] are satisfied. □

# Chapitre 5

## Analysis of the one dimensional inhomogeneous Jellium model with the Birkhoff-Hopf Theorem

We present here the work published in [51]

**Abstract.** We use the Hilbert distance on cones and the Birkhoff-Hopf Theorem to prove decay of correlation, analyticity of the free energy and a central limit theorem in the one dimensional Jellium model with non constant density charge background, both in the classical and quantum cases.

## 5.1 Introduction

The Jellium model describes a system of electrons interacting with each other in a continuous background of opposite charge. It is a very fundamental system in quantum chemistry and condensed matter physics [66, 116, 29]. The model has been initially introduced by Wigner [134]. In (quasi-)one dimension it has then been rigorously studied when the background is uniform by Kunz [97], Brascamp-Lieb [30], Aizenman-Martin [7] and many others [24, 19, 39, 73, 85, 6, 84, 105]. This model is known to reveal a symmetry breaking called the Wigner crystal. One major difficulty is that the Coulomb potential is long range. In dimension one, the interaction is like  $-|x - y|$  and therefore the force between two particles does not depend on their mutual distance which simplifies a lot the problem.

In this paper we study the inhomogeneous Jellium model in which the background is not constant. The inhomogeneous case is very important for applications, at least in three dimensions [72, 78, 111]. The Wigner crystal still appears for a periodic background, provided that the charge in one period is equal to the charge of the particles. Here we consider any background in one dimension and the system will not necessarily be crystallized.

One of the most important properties of the constant background model is that the partition function can be written in the form

$$\mathcal{Z}_N(\beta) = \langle a, T(\beta)^N b \rangle$$

where  $T(\beta)$  is a compact operator with positive kernel in some  $L^2$  space, which depends smoothly on the inverse temperature  $\beta$ . By the Krein-Rutmann Theorem [11],  $T(\beta)$  has a unique largest eigenvalue  $\lambda(\beta) > 0$  which is always non degenerate, hence is also a smooth function of  $\beta$ . As a consequence, the free energy behaves as

$$f_N(\beta) = -\frac{1}{N\beta} \log(\mathcal{Z}_N(\beta)) = -\frac{1}{\beta} \log(\lambda(\beta)) - \frac{1}{\beta N} \log(\langle a, v \rangle \langle v, b \rangle) + O(\kappa^N)$$

where  $v$  is the unique positive eigenvector associated with  $\lambda(\beta)$  and  $\kappa < 1$ . In fact,  $T(\beta)^N / \lambda(\beta)^N$  is close to the rank-one projection on  $v$  and this can also be used to prove the decay of correlations.

In the inhomogeneous case, the partition function takes the form

$$\mathcal{Z}_N(\beta) = \left\langle a, \prod_{0 \leq i \leq N-1} T_i(\beta) b \right\rangle \quad (5.1)$$

where the product is from right to left  $\prod_{0 \leq i \leq N-1} T_i(\beta) = T_{N-1}(\beta) \cdots T_0(\beta)$  and the operators  $T_i(\beta)$  are no longer equal to each other. Our goal is to generalize the results proved in the homogeneous case to the inhomogeneous case. For this we will replace the spectral approach by the Birkhoff-Hopf theorem [22, 79]. The main idea behind this method is to quantify how a product of many operators with positive kernels can be well approximated by a rank-one operator. A main tool is the so-called Hilbert distance on cones, a concept which will be discussed at length later on.

Using these tools we will prove the decay of correlations and the smoothness of the free energy in the inhomogeneous Jellium model. In the classical case, we

can essentially handle any background, but in the quantum case we require it to be close to a constant. Our method is general and can be applied to other one dimensional inhomogeneous systems in statistical physics like the Ising model. It could also be useful for log gases [55, 61]. For this reason, we will present the theory in the abstract framework of cones on any Banach spaces, in a form which is more adapted to the setting of statistical physics.

Our paper is organized as follows. We first describe in Section 5.2 the Jellium model and state our main results both for the classical and the quantum cases. We then introduce in Section 5.3.1 the framework required to state the Birkhoff-Hopf Theorem. Afterward, we suggest a new formulation of weak ergodicity using rank-one operators and prove it in Section 5.3.3. As it is shown in Section 5.3.4, the rank-one approximation implies that the  $k$ -particle marginals are well approximated by (independent) products of the 1-particle marginals. In Section 5.3.5 we prove the regularity of the abstract free energy. Finally we deal with the inhomogeneous Jellium model. Section 5.4.1 and Section 5.4.2 are dedicated to the proof of the classical and quantum cases, respectively. The main result of these two sections is the construction of an appropriate cone such that the theorems of the previous sections can be applied.

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## 5.2 The Jellium model

In this section, we present the Jellium model and state all our results. The proofs will be given in Section 5.4.1 and Section 5.4.2.

### 5.2.1 The classical Jellium model

#### Mathematical formalism

We consider  $N$  particles of negative charges  $-q_1, \dots, -q_N$  placed on a line  $-L < x_1 < x_2 < \dots < x_N < L$  in an inhomogeneous fixed density of charge  $\rho \in L^1([-L; L])$  such that  $\int_{-L}^L \rho(s) ds = \sum_{i=1}^N q_i$ . The one dimensional solution of  $u'' = \delta_0$  is  $u(x) = -\frac{1}{2}|x|$ , which gives us the total energy of the system

$$E(x_1, x_2, \dots, x_N) = -\frac{1}{2} \iint_{[-L, L]^2} \rho(y_1) \rho(y_2) |y_1 - y_2| dy_1 dy_2 \\ - \frac{1}{2} \sum_{1 \leq i, j \leq N} q_i q_j |x_i - x_j| + \sum_{i=1}^N q_i \int_{-L}^L \rho(y) |x_i - y| dy.$$

The first term is the background-background interaction, the second term accounts for the electron-electron interaction and the third term for the background-electron interaction.

Let us first calculate the state of minimum energy. For each particle  $i$  the position  $\tilde{x}_i$  which minimizes the energy is such that

$$\int_{-L}^{\tilde{x}_i} \rho(y) dy = \sum_{1 \leq j < i} q_j + \frac{q_i}{2}.$$

It is the condition that for each particle there is the same amount of charge on its right side and on its left side, such that the particle is at equilibrium. In the homogeneous case, we have for any  $i$ ,  $\rho|\tilde{x}_{i+1} - \tilde{x}_i| = q$ . Therefore at  $T = 0$ , the electrons are located on  $\frac{q}{\rho}\mathbb{Z}$  (the Wigner cristal). But for a general background the lattice is not necessarily a solution.

We subtract the minimum of the energy and rewrite

$$E(x_1, \dots, x_N) = E(\tilde{x}_1, \dots, \tilde{x}_N) - 2 \sum_{i=1}^N q_i \int_{\tilde{x}_i}^{x_i} \rho(y)(y - x_i) dy.$$

We denote by

$$U_i(s) = -2q_i \int_{\tilde{x}_i}^{\tilde{x}_i+s} (y - \tilde{x}_i - s) \rho(y) dy$$

the potential felt by the  $i^{\text{th}}$ -particle around its stable position.

We are interested in the canonical model at positive temperature. The position of the particles  $x_i$  are now random and the probability of a set of positions  $(x_i)_{i=1, \dots, N}$  is proportional to  $e^{-\beta E(x_1, \dots, x_N)}$  (Gibbs measure).

The relevant physical properties of the system are obtained from the partition function  $Z$  given by

$$\mathcal{Z}_N(\beta) = e^{-\beta E(\tilde{x}_1, \dots, \tilde{x}_N)} \int \dots \int_{-L < x_1 < x_2 < \dots < x_N < L} \prod_{i=1}^N e^{-2\beta q_i \int_{\tilde{x}_i}^{x_i} \rho(y)(y-x_i) dy} dx_i$$

and its free energy per particle

$$f_N(\beta) = -\frac{1}{N\beta} \log(\mathcal{Z}_N(\beta)).$$

We also introduce the marginals  $\rho_I(x_{i_1}, x_{i_2}, \dots, x_{i_k})$ , for the probability of the positions of the  $k$  particles of the subset  $I = \{i_1, i_2, \dots, i_k\} \subset \{1, \dots, N\}$ . More rigorously, it is the unique function such that for all test functions  $g \in L^\infty([-L, L]^k)$ ,

$$\begin{aligned} & \iiint g(x_{i_1}, x_{i_2}, \dots, x_{i_k}) \rho_I(x_{i_1}, \dots, x_{i_k}) dx_{i_1} \dots dx_{i_k} \\ &= \frac{e^{-\beta E(\tilde{x}_1, \dots, \tilde{x}_N)}}{\mathcal{Z}_N(\beta)} \iint_{-L < x_1 < x_2 < \dots < x_N < L} g(x_{i_1}, \dots, x_{i_k}) \times \\ & \quad \times \prod_{k=1}^N e^{-2\beta q_k \int_{\tilde{x}_k}^{x_k} \rho(y)(y-x_k) dy} dx_i. \end{aligned}$$

Our main interest is to know whether the particles are strongly correlated or not. This can be quantified by looking at the truncated correlation functions, which we introduce below.

**Definition 5.1.** (Cluster property) We say that

— the particles are *independent* if

$$\rho_{\{1, \dots, N\}}(x_1, \dots, x_N) = \prod_{i=1}^N \rho(x_i),$$

— the particles satisfy a *cluster property* if there exists  $I \cup J = \{1, \dots, N\}$ ,  $I \cap J = \emptyset$  such that

$$\rho_{\{1, \dots, N\}}(x_1, \dots, x_N) = \rho_I((x_i)_{i \in I}) \rho_J((x_j)_{j \in J}).$$

In order to characterize the “clusters” we introduce the truncated marginal:

**Definition 5.2.** (Truncated marginal) The *truncated marginals*  $\rho_J^T$ ,  $k = |J|$  are defined recursively as follows:

$$\rho_J^T(x_{j_1}, \dots, x_{j_k}) = \rho_J(x_{j_1}, \dots, x_{j_k}) - \sum_{I_1 \cup I_2 \cup \dots \cup I_r = J} \prod_{l=1}^r \rho_{I_l}^T((x_i)_{i \in I_l}).$$

The truncated marginals appear to be the good indicator for clustering properties. Indeed we have the following proposition.

**Proposition 5.3.** *If  $\rho_n(x_1, \dots, x_n) = \rho_I((x_i)_{i \in I}) \rho_J((x_j)_{j \in J})$  then for all  $I'$  such that  $I' \cap I \neq \emptyset$  and  $I' \cap J \neq \emptyset$  then  $\rho_{I'}^T((x_i)_{i \in I'}) = 0$ .*

For the reader’s convenience we have written the proof of Proposition 5.3 in Appendix A. We are now ready to state our main results.

### Main results

In the classical case we make the following assumptions:

- (H1) There exist  $q, Q > 0$  such that for all  $i$ ,  $0 < q \leq q_i \leq Q$ .
- (H2) There exists  $0 < m < M$  such that for all  $t \in [-L, L]$ ,  $m \leq \rho(t) \leq M$ .

These assumptions (H1,H2) imply the following bounds for the potential:

$$U_i(s) \geq qms^2$$

and

$$\left| \frac{d}{ds} U_i(s) \right| \geq |s|mq.$$

Our first result is to be understood as follows: If we consider particles which are far away from each other (meaning that there are a lot of other particles between them) then the marginal is exponentially close to the independent marginal. We also get the cluster property: if groups of particles are far from each others, then the marginal is exponentially close to the independent cluster marginal.

**Theorem 5.4.** *For any  $\beta > 0$ , there exists  $\kappa < 1$  such that for any  $I \subset \{0, \dots, N\}$ ,  $|I| = k$  we have*

$$\left| \rho_I(x_{i_1}, x_{i_2}, \dots, x_{i_k}) - \prod_{i_l \in I} \rho_{\{i_l\}}(x_{i_l}) \right| \leq C_k \kappa^d$$



for some  $C_k > 0$ , provided that between any two consecutive particle in  $I$  there are at least  $d$  others particles (in practice take  $d = \inf |i_l - i_{l+1}| - 1$ ). Also we have

$$|\rho_I^T(x_{i_1}, x_{i_2}, \dots, x_{i_k})| \leq C_k \kappa^D$$

when there exist two consecutive particles in  $I$  with at least  $D$  others particles between them (in practice take  $D = \max |i_l - i_{l+1}| - 1$ ).

Our next result concerns the regularity of the free energy, which is a fundamental property for one-dimensional systems in statistical physics.

**Theorem 5.5.** *For any  $\beta_0 > 0$ , there exists  $\Delta\beta > 0$  such that the free energy is smooth on  $[\beta_0 - \Delta\beta, \beta_0 + \Delta\beta]$  uniformly on  $N$ . More precisely we have*

$$\left| \frac{d^k}{d\beta^k} f_N \right| \leq M_k,$$

for all  $k$  with  $M_k > 0$  independent of  $N$  and for all  $\beta \in [\beta_0 - \Delta\beta, \beta_0 + \Delta\beta]$ .

In the proof we will show the following estimate on  $M_k$ :

$$M_k \leq k! c^k \tag{5.2}$$

for some  $c > 0$ . From this bound we obtain the analyticity of the limiting free energy.

**Theorem 5.6.** *For any  $\beta_0 > 0$ , there exists  $\Delta\beta > 0$  such that if there exists  $f$  such that  $f_N \rightarrow f$  on  $[\beta_0 - \Delta\beta, \beta_0 + \Delta\beta]$  for a subsequence  $N \rightarrow \infty$ , then  $f$  is real analytic on  $[\beta_0 - \Delta\beta, \beta_0 + \Delta\beta]$ .*

As a corollary, the system will not reveal any phase transition for  $\beta \neq \infty$ .

**Corollary 5.7.** *If the charge background is periodic or if it is constructed randomly with an process, there exists a limiting function  $f$  such that  $f_N \rightarrow f$  (almost surely in the ergodic case) and  $f$  is real analytic.*

This generalizes the results of Kunz [97].

## 5.2.2 The quantum model

### Mathematical formalism

We now give our results for the quantum problem. Physically we should consider indistinguishable fermions, with of course all the same charge  $q_i = q$ . Here we will rather consider distinguishable particles with possibly different charges, but whose positions are constrained to be ordered as  $x_1 < \dots < x_N$ . This is more general mathematically.

In the classical case, we neglect the kinetic energy because in phase space momentum and position are independent for the Gibbs measure. This is no longer true in the quantum case and we have to consider the whole  $N$ -particle Hamiltonian

$$H = -\frac{1}{2} \sum_{i=1}^N \partial_{x_i}^2 + E(x_1, \dots, x_N).$$

defined on  $H^2(\mathcal{X})$  with  $\mathcal{X} = \{(x_1, \dots, x_N) : -L < x_1 < \dots < x_N < L\}$  with Dirichlet boundary conditions at  $x_i = x_j$  and at the two ends  $\mp L$ . The quantum canonical function is

$$\mathcal{Z}_N^Q(\beta) = \text{Tr}(\exp(-\beta H))$$

and the free energy is

$$f_N^Q(\beta) = -\frac{1}{\beta N} \log(\mathcal{Z}_N^Q(\beta)).$$

We have the following Feynman-Kac formula [86] for the partition function  $\mathcal{Z}_N^Q(\beta)$ .

**Proposition 5.8.** (*Feynman Kac formula*) *We have*

$$\mathcal{Z}_N^Q(\beta) = \int_{\mathcal{X}} \mu_{x_1 x_1} \dots \mu_{x_N x_N} (e^{-\int_0^\beta E(\gamma_1(t), \dots, \gamma_N(t)) dt} \mathbf{1}_{(\gamma_1, \dots, \gamma_N) \in W_N}) dx_1 \dots dx_N \quad (5.3)$$

and

$$\rho(\mathbf{x}; \mathbf{y}) = \frac{1}{Z_N} \mu_{x_1 y_1} \times \mu_{x_2 y_2} \times \dots \times \mu_{x_N y_N} (e^{-\int_0^\beta E(\gamma_1(t), \dots, \gamma_N(t)) dt})$$

where  $\mathcal{X} = \{(x_1, \dots, x_N) : -L < x_1 < \dots < x_N < L\}$ ,

$$W_N = \{(\gamma_1, \dots, \gamma_N) | \forall t \in [0, \beta] : -L < \gamma_1(t) < \gamma_2(t) < \dots < \gamma_N(t) < L\}$$

is the Weyl chamber and  $\mu_{x,y}$  are the probability measures of a Brownian bridge from  $x$  to  $y$  of length  $\beta$ .

The random system we study in the quantum model is no longer the positions  $(x_i)_{i \leq N}$  but rather the paths  $(\gamma_i)_{i \leq N}$ . We define the extended marginals on the set of paths  $\rho^\Gamma(\gamma_1, \dots, \gamma_N)$  and we are able to apply the theorems of Section 5.3.4 in this set up. However, for simplicity we will only states the results on the position marginals  $\rho_k(x_{i_1}, \dots, x_{i_k})$  which satisfy, for any bounded function  $g : [-L, L]^N \rightarrow \mathbb{R}$ ,

$$\begin{aligned} & \int \dots \int \rho_k(x_{i_1}, \dots, x_{i_k}) g(x_{i_1}, \dots, x_{i_k}) dx_1 \dots dx_k \\ &= \frac{1}{Z_N(\beta)} \int_{-L < x_1 < \dots < x_N < L} \mu_{x_1 x_1} \dots \mu_{x_N x_N} (e^{-\int_0^\beta E(\gamma_1(t), \dots, \gamma_N(t)) dt} \mathbf{1}_{(\gamma_1, \dots, \gamma_N) \in W_N}) \\ & \quad g(x_{i_1}, \dots, x_{i_k}) dx_1 \dots dx_N \end{aligned}$$

### Main results

Unfortunately, in the quantum case we are only able to prove a result in a perturbation regime where  $\rho$  and the  $q_i$  are almost constant. We therefore make the following assumptions :

- (HQ1)  $q(1 - \epsilon) \leq q_i \leq q(1 + \epsilon)$  for all  $i$ .
- (HQ2)  $\rho(1 - \epsilon) \leq \rho(t) \leq \rho(1 + \epsilon)$  for all  $t$ .

for some  $\epsilon > 0$  that will be assumed small enough.

**Theorem 5.9.** *For any  $\beta > 0$ , under condition (HQ1-2) for  $\epsilon > 0$  small enough, there exists  $\kappa < 1$ , such that for all  $I \subset \{1, \dots, N\}$ ,  $|I| = k$ ,*

$$|\rho_I(x_{i_1}, x_{i_2}, \dots, x_{i_k}) - \prod_{i_l \in I} \rho_{\{i_l\}}(x_{i_l})| \leq C_k \kappa^d$$

for some  $C_k > 0$ , if between any two consecutive particle in  $I$  there are at least  $d$  others particles (in practice take  $d = \inf |i_l - i_{l+1}| - 1$ ). On the other hand,

$$|\rho_I^T(x_{i_1}, x_{i_2}, \dots, x_{i_k})| \leq C_k \kappa^D$$

if there exists two consecutive particles in  $I$  with at least  $D$  others particles between them (in practice take  $D = \max |i_l - i_{l+1}| - 1$ ).

As in the classical case we obtain the regularity of the partition function  $f_N^Q$ .

**Theorem 5.10.** *For any  $\beta_0 > 0$ , there exists  $\Delta\beta > 0$  such that under condition (HQ1-2) with  $\epsilon > 0$  small enough, the free energy is  $C^\infty$  on  $[\beta_0 - \Delta\beta, \beta_0 + \Delta\beta]$  and for all  $k$  we have*

$$\left| \frac{d^k}{d\beta^k} f_N \right| \leq M_k$$

with  $M_k$  independent of  $N$ .

Finally we can prove analyticity of the free energy with the same estimate as (5.2).

**Theorem 5.11.** *For any  $\beta_0 > 0$ , there exists  $\Delta\beta > 0$  such that under condition (HQ1-2), for  $\epsilon > 0$  small enough, if  $f_N$  admits a limit  $f$  for a subsequence  $N \rightarrow \infty$ , then  $f$  is real analytic on  $[\beta_0 - \Delta\beta, \beta_0 + \Delta\beta]$ .*

**Corollary 5.12.** *We make the same assumptions as in Theorem 5.9. If the charge background is periodic or if it is constructed randomly with an ergodic process, there exists a limiting function  $f$  such that  $f_N \rightarrow f$  (almost surely in the ergodic case) and  $f$  is real analytic.*

All these results (Theorem 5.9, 5.10, 5.11 and Corollary 5.12) are presented in [86] in the homogeneous case and rely on the Krein-Rutmann Theorem. Here we also use the Krein-Rutmann theorem in order to construct the cone.

### 5.3 General theory to apply the Birkhoff-Hopf theorem

The Birkhoff-Hopf theorem has been used for instance to study non linear integrable equations, weak ergodic theorems, or the so-called *DAD* problem [26].

We first introduce the notion of cone and the Hilbert distance. In this set up we can state the Birkhoff-Hopf theorem. Then we prove Theorem 5.4 and Theorem 5.5 with the extra assumption of strictly contracting operators.

### 5.3.1 Framework and Birkhoff-Hopf theorem.

We follow [58] for the notation and we refer to this paper for a proof of the Birkhoff Hopf theorem 5.18. Let  $E$  be a real linear Banach space.

**Definition 5.13.** (Abstract cone)  $\mathcal{C} \subset E$  is called a *cone* if

1.  $\mathcal{C}$  is convex,
2.  $\lambda\mathcal{C} \subset \mathcal{C}$  for any  $\lambda \geq 0$ ,
3.  $\mathcal{C} \cap -\mathcal{C} = \{0\}$ .

Using  $\mathcal{C}$  we define a partial order on  $E$

**Definition 5.14.** (Partial order) For any  $x, y \in E$ , we write  $x \leq_{\mathcal{C}} y$  if  $y - x \in \mathcal{C}$ .

For clarity we will use  $\leq$  instead of  $\leq_{\mathcal{C}}$  if there is no confusion about the cone.

**Definition 5.15.** If  $\mathcal{C}$  is a cone, we define the dual cone  $\mathcal{C}^*$  by

$$\mathcal{C}^* := \{f \in E^* : \forall x \in \mathcal{C}, (f, x) \geq 0\}.$$

The set  $\mathcal{C}^*$  is a cone if  $\mathcal{C} - \mathcal{C}$  is dense and in particular if  $\mathcal{C}$  has nonempty interior. We say that  $x, y \in \mathcal{C}$  are *comparable* and write  $x \sim y$  if there exist  $\alpha, \beta > 0$  such that  $\alpha x \leq y \leq \beta x$ . This defines an equivalence relation. We say that  $\mathcal{C}$  is *normal* if there exists  $\gamma > 0$  such that  $\forall x, y \in \mathcal{C}, 0 \leq x \leq y \Rightarrow \|x\| \leq \gamma \|y\|$ .

**Definition 5.16.** For any  $x, y \in \mathcal{C}$  comparable, we define the Hilbert metric by

$$d_{\mathcal{C}}(x, y) = \log \frac{\beta_{\min}(x, y)}{\alpha_{\max}(x, y)}$$

where

$$\alpha_{\max}(x, y) = \sup \{\alpha > 0 : \alpha x \leq y\}$$

and

$$\beta_{\min}(x, y) = \inf \{\beta > 0 : y \leq \beta x\}$$

The Hilbert metric is a metric on the projective space of  $\mathcal{C}$ .

We say that  $T : E \rightarrow E$  is *order-preserving* if  $x \leq y \Rightarrow T(x) \leq T(y)$ . If  $T$  is a linear operator (the only case we will consider here) this is equivalent to  $T(\mathcal{C}) \subset \mathcal{C}$ . In this paper, we also denote them by *increasing* operators or *positive* operators because of the example of matrices with positive entries (there will be no symmetric matrices in this paper).

*Remark 5.17.* If  $T$  is order-preserving then  $T$  is non-expanding for the Hilbert metric. Indeed  $\alpha x \leq y \leq \beta x$  implies  $\alpha T(x) \leq T(y) \leq \beta T(x)$ .

We introduce the projective diameter

$$\Delta(T) = \sup \{d_{\mathcal{C}}(T(x), T(y)) : x, y \in \mathcal{C}, T(x) \sim T(y)\}$$

and the contracting ratio

$$\kappa(T) = \inf \{c > 0 : \forall x, y \in \mathcal{C}, d_{\mathcal{C}}(T(x), T(y)) \leq cd_{\mathcal{C}}(x, y), T(x) \sim T(y)\}$$

Here is the main theorem we will use :

**Theorem 5.18.** (Birkhoff-Hopf [22, 79]) *If  $T$  is order-preserving then*

$$\kappa(T) = \tanh\left(\frac{\Delta(T)}{4}\right).$$

The result has to be understood as follows: if the image of the cone of the order preserving operator is strictly inside the cone ( $\Delta(T) < \infty$ ), then the operator is strictly contracting ( $\kappa(T) < 1$ ) for the Hilbert metric.

### 5.3.2 Reformulation of one dimensional statistical physics models using the cones and positive operators framework

We now use the previous formalism to study the partition function and the marginals from statistical physics in the framework of positive operators.

**Definition 5.19.** (Density function) Let  $u \in \mathcal{C}^*$  and  $v \in \mathcal{C}$ , and let  $(T_i)_{i \leq N}$ ,  $X, Y, (X_i)_{i \leq N}$  be positive bounded operators. We define

— the partition function by

$$\mathcal{Z} = (u, T_N \dots T_0 v),$$

— the one-point density function by

$$\rho_{K_1}(X) = \frac{1}{\mathcal{Z}}(u, T_N \dots T_{K_1+1} X T_{K_1} \dots T_0 v),$$

— the pair correlation function by

$$\rho_{K_2, K_1}(Y, X) = \frac{1}{\mathcal{Z}}(u, T_N \dots T_{K_2+1} Y T_{K_2} \dots T_{K_1+1} X T_{K_1} \dots T_0 v),$$

— the  $k$ -point correlation function by

$$\rho_{K_k, \dots, K_2, K_1}(X_k, \dots, X_1) = \frac{1}{\mathcal{Z}}(u, T_N \dots T_{K_k+1} X_k T_{K_k} \dots T_{K_1+1} X_1 T_{K_1} \dots T_0 v).$$

The operators  $X, Y, (X_i)$  should be thought of as test functions acting on the position of the  $K_i^{\text{th}}$  particle.

*Remark 5.20.* The simplest model that can be written in this formalism is the one dimensional Ising model [117]. All the results stated above for Jellium can be easily adapted to the inhomogeneous one dimensional Ising model.

We also think of Markov processes on a finite or compact set, in which case  $T_i$  is the transitive kernel from  $X_i$  to  $X_{i+1}$ .

#### Decay of correlations

The following theorem states the exponential decay of the correlation functions.

**Theorem 5.21.** (Decay of correlations) Let  $(T_i)_{i=1,\dots,N}$  be positive operators such that

$$\Delta(T_i) \leq M < \infty$$

for any  $i$ . Then there exists  $c > 0$  which depends only on  $k$ , such that for  $\min |K_{j+1} - K_j|$  large enough, we have

$$\begin{aligned} & (1 - c\kappa^{\min_j |K_{j+1} - K_j|}) \prod_{i=1}^k \rho_{K_i}(X_i) \\ & \leq \rho_{K_k, \dots, K_1}(X_k, \dots, X_1) \leq (1 + c\kappa^{\min_j |K_{j+1} - K_j|}) \prod_{i=1}^k \rho_{K_i}(X_i) \end{aligned}$$

with  $\kappa = \tanh(\frac{M}{4})$ .

The decay of correlations is an important concept in statistical physics and it is ubiquitous in one-dimensional systems [122].

The proof of Theorem 5.21 is provided below in Section 5.3.4.

### Regularity of the free energy

The second theorem states that the partition function depending on a parameter is smooth, if the operators are smooth enough. In order to express the “regularity” of the operator in the framework of a cone and the Hilbert distance, one has to construct the following norm. The following results says that the distance is close to being a norm in the neighborhood of any point  $x_0$ .

**Proposition 5.22.** Let  $x_0 \in \mathcal{C}$ . For any  $\epsilon > 0$ , there exists  $r > 0$ , a function  $f$  and a norm  $\mathcal{N}_{x_0}$  defined on the projective space such that  $d_{\mathcal{C}}$  can be written as follows

$$d_{\mathcal{C}}(x, y) = f(x, y) \mathcal{N}_{x_0}(y - x)$$

for all  $x, y$  such that  $d_{\mathcal{C}}(x, x_0) < r$  and  $d_{\mathcal{C}}(y, x_0) < r$ , with  $|f(x, y) - 1| < \epsilon$ .

We can now state our second main result

**Theorem 5.23.** Let  $\mathcal{C}$  be a cone,  $a \in \mathcal{C}^*$ ,  $b \in \mathcal{C}$ , and let  $T_i(\beta)$  be a one parameter family of bounded operators for  $\beta$  in the neighborhood  $[\beta_0 - \delta, \beta_0 + \delta]$  of  $\beta_0$ , which are contracting of parameter  $\kappa < 1$ , uniformly in  $\beta$  and  $i$ . For all  $i$ , we denote by  $\mathcal{N}_i$  the norms defined in Proposition 5.22 around  $x_i = \prod_{j=0}^{i-1} T_j(\beta)b$ . Assume that the derivatives in  $\beta$  of the operators are uniformly bounded for these norms, that is,

$$\forall i, \quad \left\| \frac{d^k T_i}{d\beta^k} \right\|_{\mathcal{N}_i \rightarrow \mathcal{N}_{i+1}} \leq C'_k$$

for some constant  $C'_k$  independent of  $i$  and of  $\beta \in [\beta_0 - \delta, \beta_0 + \delta]$ . Then

$$f_N(\beta) = \frac{1}{N} \log \left\langle a, \prod_{i=0}^{N-1} T_i(\beta)b \right\rangle$$

is uniformly smooth, meaning there is a constant  $C$  which depends only on  $\kappa$  and  $(M_k)_{k \leq n}$  such that :

$$\left| \frac{d^n f_N(\beta)}{d\beta^n} \right| \leq C(\kappa, (M_k)_{k \leq n})$$

where  $M_k = \sup_{i, \beta \in [\beta_0 - \delta, \beta_0 + \delta]} \left\| \frac{d^k T_i}{d\beta^k} \right\|_{\mathcal{N}_i \rightarrow \mathcal{N}_{i+1}}$ .  
Moreover if the following limit exists

$$f(\beta) = \lim_{k \rightarrow \infty} \frac{1}{N_k} \log \left\langle a, \prod_{i=0}^{N_k-1} T_i(\beta) b \right\rangle$$

for a sequence  $N_k \rightarrow \infty$ , then it is smooth on a neighborhood of  $\beta_0$ :

$$\left| \frac{d^n f(\beta)}{d\beta^n} \right| \leq C(\kappa, (M_k)_{k \leq n}).$$

If the positive operator appears to be uniformly analytic for the constructed norm then the free energy is analytic. More precisely we have the following theorem

**Theorem 5.24.** *With the same assumptions as in Theorem 5.23, if there exists  $r \geq 0$  such that*

$$\frac{\|\partial_\beta^n T_i\|_{\mathcal{N}_i \rightarrow \mathcal{N}_{i+1}}}{n!} \leq r^n$$

for all  $n$ , then  $f$  is real analytic around  $\beta$  with radius of convergence at least equal to  $(1 - \kappa)/r$ .

The two theorems of this section are proved later in Section 5.3.5.

### Central Limit Theorem

We consider the particular case where the space is  $L^1(\Lambda)$ , with  $\Lambda$  a measurable set and the cone is  $\mathcal{C} = \{f \in L^1(\Lambda) : f \geq 0\}$ . We construct the canonical random process  $y_i$  as follows. Let  $A_1, \dots, A_N \subset \Lambda$ , and take as test functions  $1_{A_1}, 1_{A_2}, \dots, 1_{A_N}$ . Then we define

$$\mathbb{P}(y_1 \in A_1, y_2 \in A_2, \dots, y_N \in A_N) = \rho_{1, \dots, N}(1_{A_1}, \dots, 1_{A_N}) = \frac{1}{Z} \left\langle a \prod_{i=1}^N T_i 1_{A_i} b \right\rangle. \quad (5.4)$$

for some  $a \in \mathcal{C}^*$  and  $b \in \mathcal{C}$ .

The decay of correlation in Theorem 5.21 is the mixing property of the process  $(y_i)$ .

The Central Limit Theorem has been proved for a huge number of random processes like martingales [75], Markov processes [77, 74] or random products of matrices [103]. One of the classical proofs of the central limit theorem uses the regularity of the Laplace transform, this is what we adapt here.

**Theorem 5.25.** *(Central Limit) Let  $h_i : \Lambda \rightarrow \mathbb{R}$  be such that  $\mathbb{E}(\exp(h_i(y_i))) < \infty$  where the mean is on the probability (5.4). Let  $T_i(\beta) = e^{\beta h_i(y_i)} T_i$ . If the  $T_i(\beta)$  satisfy the assumptions of Theorem 5.23, then we have*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \frac{1}{\sqrt{N}} \sum (h_i(y_i) - \mathbb{E}(h_i(y_i))) \geq x \right) - \mathbb{P}(\mathcal{N}(0, \sigma_N^2) \geq x) \right| = O \left( \frac{1}{\sqrt{N}} \right)$$

where  $\sigma_N^2$  is the second derivative of the free energy

$$\sigma_N^2 = \frac{d^2}{d\beta^2} \left[ \frac{1}{N} \log \left\langle a, \prod_{i=0}^{N-1} T_i(\beta)b \right\rangle \right].$$

This Theorem is proved in Section 5.3.8.

### 5.3.3 Rank-one operator approximation

One of the first historical applications of the Birkhoff-Hopf Theorem was in population demography [43]. An age structure diagram  $f$  evolves due to birth and death, with death and birth rates not constant in time and one can calculate its time evolution. It appears that even if  $f$  does not converge to an equilibrium, the long time evolution is independent of the initial age structure  $f(0)$ . Namely this is a weak ergodicity property: if  $f_1$  and  $f_2$  are two solutions of the evolution with different initial data,  $\|f_1(t)/\|f_1(t)\| - f_2(t)/\|f_2(t)\|\| \rightarrow 0$ .

In this section, we formulate weak ergodicity in term of a rank-one operator approximation and we give a construction and an estimate of such an approximation in case where several contracting operators are composed one after another.

#### The cone of order preserving operators

We state here some simple results about the set of order preserving operators.

**Lemma 5.26.** *Let  $\mathcal{C}$  with  $\mathcal{C}$ - $\mathcal{C}$  dense. The set of corresponding order-preserving operators is a cone.*

We denote by  $\mathcal{P}_{\mathcal{C}}$  this cone and only  $\mathcal{P}$  if there is no confusion.

*Proof.* We check every point of the definition.

1. If  $A, B$  are order preserving operators then  $A + B$  is an order preserving operator. Indeed  $(A + B)(x) \in \mathcal{C}$  for all  $x \in \mathcal{C}$ .
2. The set of order preserving operators is invariant by product of strictly positive scalars.
3. Let  $A \in \mathcal{P} \cap -\mathcal{P}$ , then  $(f, Ax) = 0$  for all  $x \in \mathcal{C}$  and all  $f \in \mathcal{C}^*$ . Therefore  $(f_1 - f_2, A(x_1 - x_2)) = 0$  for all  $x_1, x_2 \in \mathcal{C}$  and all  $f_1, f_2 \in \mathcal{C}^*$ . Therefore  $A = 0$  since  $\mathcal{C}^* - \mathcal{C}^*$  and  $\mathcal{C} - \mathcal{C}$  are dense.

□

**Example 5.27.** One can think of  $\mathcal{C}$  the positive vectors in  $\mathbb{R}^n$  and the set of matrices  $\mathcal{M}_n(\mathbb{R})$  with positive coefficients.

We have the following order on the set of operators :

$$B \geq_{\mathcal{P}} A \Leftrightarrow (B - A)(\mathcal{C}) \subset \mathcal{C}$$

and the corresponding Hilbert distance

$$d_{\mathcal{P}}(A, B) = \min \left( \log \left( \frac{\beta}{\alpha} \right) : \alpha A \leq B \leq \beta A \right).$$



*Remark 5.28.* Let  $A, B, C, D$  be positive operator. If  $A \leq_{\mathcal{P}} B$  and  $C \leq_{\mathcal{P}} D$  then  $AC \leq_{\mathcal{P}} BD$ . Indeed  $(B - A)C(\mathcal{C}) \subset (B - A)(\mathcal{C}) \subset \mathcal{C}$ , and  $B(D - C)(\mathcal{C}) \subset B(\mathcal{C}) \subset \mathcal{C}$ . Therefore  $BD \geq_{\mathcal{P}} BC \geq_{\mathcal{P}} AC$ .

Unfortunately that  $T$  is contracting in the cone  $\mathcal{C}$  does not imply that  $\tilde{T} : A \rightarrow TA$  is contracting as well in the cone  $\mathcal{P}$ . One can take for example:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

in which case  $d_{\mathcal{P}}(A, B) = \log\left(\frac{2}{(1/2)}\right) = \log(4)$  and  $d_{\mathcal{P}}(TA, TB) = \log(4)$  as well.

**Lemma 5.29.** *Let  $A, B, C, D$  be increasing operators. Then*

$$d_{\mathcal{P}}(AB, CD) \leq d_{\mathcal{P}}(A, C) + d_{\mathcal{P}}(B, D).$$

*Proof.* Let  $\alpha_1, \beta_1, \alpha_2, \beta_2$  such that  $\alpha_1 A \leq C \leq \beta_1 A$ ,  $\alpha_2 B \leq D \leq \beta_2 B$  and  $\log\left(\frac{\beta_1}{\alpha_1}\right) - d_{\mathcal{P}}(A, C) \leq \epsilon$ ,  $\log\left(\frac{\beta_2}{\alpha_2}\right) - d_{\mathcal{P}}(B, D) \leq \epsilon$ . Then  $\alpha_1 \alpha_2 AC \leq BD \leq \beta_1 \beta_2 AC$  and we have  $d_{\mathcal{P}}(AB, CD) \leq \log\left(\frac{\beta_1 \beta_2}{\alpha_1 \alpha_2}\right) \leq d_{\mathcal{P}}(A, C) + d_{\mathcal{P}}(B, D) + 2\epsilon$ .  $\square$

Now we construct a rank-one operator  $L = z \cdot l$ , with a vector  $z \in E$ , and a linear form  $l \in E^*$  to approximate a contracting operator  $T$ . It is natural to choose  $z \in T(\mathcal{C})$ . We construct  $l$  in the following subsection.

### Rank-one operator construction

We construct here the rank-one operator close to a contracting operator.

**Lemma 5.30.** *Let  $a : C \rightarrow \mathbb{R}_+$  and  $b : C \rightarrow \mathbb{R}_+$  be such that*

1. *there exist  $M_1, M_2 < \infty$ , for all  $x$ ,  $a(x) \leq M_1 \|x\|$  and  $b(x) \leq M_2 \|x\|$ ,*
2. *for all  $\lambda \geq 0$  and  $x \in C$ ,  $a(\lambda x) = \lambda a(x)$  and  $b(\lambda x) = \lambda b(x)$ ,*
3. *for all  $x \in C$   $a(x) \leq b(x)$ ,*
4. *for all  $x, y \in C$   $a(x + y) \geq a(x) + a(y)$  and  $b(x + y) \leq b(x) + b(y)$ .*

*Then there exists a linear form  $l \in C^*$  such that, for any  $x \in C$ ,*

$$a(x) \leq l(x) \leq b(x).$$

*Proof.* We check that  $b$  is a convex function,

$$b(tx + (1 - t)y) \leq b(tx) + b((1 - t)y) = tb(x) + (1 - t)b(y),$$

and that  $a$  is a concave function,

$$a(tx + (1 - t)y) \geq a(tx) + a((1 - t)y) = ta(x) + (1 - t)a(y).$$

Let us define two sets :  $B = \{(x, s) \in C \times \mathbb{R} : b(x) \leq s\}$  and  $A = \{(x, s) \in C \times \mathbb{R} : a(x) > s\}$ . Then  $A \cap B = \emptyset$ ,  $A$  and  $B$  are convex. Because of the Hahn-Banach separation theorem, there exists  $l \neq 0$  a linear form on  $E \times \mathbb{R}$  such that for all  $(x, s) \in B$ ,  $l(x, s) \geq 0$  and all  $(x, s) \in A$ ,  $l(x, s) \leq 0$ . We have  $l(x, s) = l_1(x) + \alpha s$  with  $l_1 \in E^*$  and  $\alpha \in \mathbb{R}$ .

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We then prove that  $\alpha > 0$ . For any  $s_0 > 0$  we have  $(0, s_0) \in B$ ,  $(0, -s_0) \in A$ ,  $\alpha s_0 \geq -\alpha s_0$  and as a conclusion  $\alpha \geq 0$ . If  $\alpha = 0$ , because for any  $x \in C$  there exist  $s_1$  and  $s_2$  such that  $(x, s_1) \in B$  and  $(x, s_2) \in A$ , we have  $0 \geq l(x, s_2) = l_1(x) = l(x, s_1) \geq 0$  and therefore  $l_1(x) = 0$ . Let  $x_0 \in \overset{\circ}{C}$ , and  $V_\epsilon(x_0) \subset C$  a small ball with center  $x_0$  and radius  $\epsilon$ . Then for all  $y$  with  $\|y\| < \epsilon$ , we have  $l_1(y) = l_1(y + x_0) = 0$ . As a conclusion  $l = l_1 = 0$  which is absurd, so  $\alpha \neq 0$ .

Let  $l_0 = -\frac{l_1}{\alpha}$ . Since  $(x, b(x)) \in B$ ,  $-l_0(x) + b(x) \geq 0$  we have  $l_0(x) \leq b(x)$ . Moreover for  $\epsilon > 0$ ,  $(x, a(x) - \epsilon) \in A$   $-l_0(x) + a(x) - \epsilon \leq 0$  and therefore  $l_0(x) \geq a(x)$ .  $\square$

**Corollary 5.31.** *There exists a rank one operator  $L_T = z \cdot l$  with  $z \in C$  and  $l \in C^*$  such that  $d_{\mathcal{P}}(T, L_T) \leq 2\Delta(T)$ .*

*Proof.* Let  $z = T(y_0) \in T(C)$  and define  $a$  and  $b$  as follows: for any  $x \in C$

$$a(x) =_{def} \max \{ \alpha : \alpha T(y_0) \leq_C T(x) \}$$

and

$$b(x) =_{def} \min \{ \beta : T(x) \leq_C \beta T(y_0) \}.$$

It is possible to check the hypothesis of Lemma 5.30. Indeed we have that

$$\begin{cases} \alpha_1 T(y_0) \leq_C T(x_1) \leq_C \beta_1 T(y_0), \\ \alpha_2 T(y_0) \leq_C T(x_2) \leq_C \beta_2 T(y_0) \end{cases}$$

implies

$$(\alpha_1 + \alpha_2)T(y_0) \leq_C T(x_1) + T(x_2) \leq_C (\beta_1 + \beta_2)T(y_0)$$

and therefore  $a(x_1 + x_2) \geq a(x_1) + a(x_2)$  and  $b(x_1 + x_2) \leq b(x_1) + b(x_2)$ . We also have

$$a(\lambda x) = \lambda a(x) \text{ and } b(\lambda x) = \lambda b(x)$$

for all  $\lambda \geq 0$  and  $x \in C$ . We can then apply Lemma 5.30: there exists a linear form  $l$  with  $a(x) \leq l(x) \leq b(x)$ . Moreover  $\log \frac{b(x)}{a(x)} \leq \Delta(T)$  for all  $x \in C$ . We then have for all  $x$   $\frac{l(x)}{a(x)} \leq e^{-\Delta(T)}$  and  $\frac{b(x)}{l(x)} \leq e^{\Delta(T)}$  and therefore

$$e^{-\Delta(T)}T(x) \leq_C T(y_0) \cdot l(x) \leq_C e^{\Delta(T)}T(x).$$

As a conclusion  $d_{\mathcal{P}}(T(y_0) \cdot l, T) \leq \log(e^{2\Delta(T)}) \leq 2\Delta(T)$ .  $\square$

**Corollary 5.32.** *Let  $(T_i)_{i=0, \dots, N}$  be positive operators. If*

1.  $\Delta(T_0(C)) \leq R < \infty$ ,
2.  $T_i$   $i = 1, \dots, N$  are uniformly contracting of parameter  $\kappa < 1$ ,

*then there exists a linear form  $l$ ,  $z_0 \in C$ ,  $\|z_0\| = 1$  and  $l \in C^*$  such that*

$$d_{\mathcal{P}}((T_N \dots T_0), z_0 \cdot l) \leq 2\kappa^N R.$$

*Proof.* We have  $\Delta(T_N \dots T_0) \leq \kappa^N R$  and the result follows from the previous corollary.  $\square$

### 5.3.4 Decay of correlation function

Here we prove Theorem 5.21. The idea is to replace the product of contracting operator between the  $k$  points of measure by a rank-one operator. We will do so for  $k = 2$  and for  $k > 2$  this will be exactly the same. More precisely we will prove the following

**Theorem 5.33.** *(Theorem 5.21 in the case  $k = 2$ ) Let  $(T_i)_{i=1, \dots, N}$  be positive operators such that  $\Delta(T_i(C)) \leq M < \infty$  for any  $i$ , and let  $K_1, K_2 \in \mathbb{N}$  be such that  $1 \leq K_1 \leq K_2 \leq N$ . Let  $u \in \mathcal{C}^*, v \in \mathcal{C}$ , and  $X, Y$  be two positive operators. Let  $Z, \rho_{K_1}, \rho_{K_2}, \rho_{K_1, K_2}$  constructed as in Definition 5.19 with  $u, v$  and  $(T_i)_{i=1, \dots, N}$ . Then:*

$$\begin{aligned} e^{-8R(\kappa^{K_1} + \kappa^{K_2 - K_1} + \kappa^{N - K_2})} \rho_{K_1}(X) \rho_{K_2}(Y) \\ \leq \rho_{K_2, K_1}(Y, X) \leq e^{8R(\kappa^{K_1} + \kappa^{K_2 - K_1} + \kappa^{N - K_2})} \rho_{K_1}(X) \rho_{K_2}(Y). \end{aligned}$$

One can use this theorem for  $K_1, K_2 - K_1, N - K_2$  large. In this case, the Taylor expansion of  $e^x$  gives

$$|\rho_{K_1}(X) \rho_{K_2}(Y) - \rho_{K_1, K_2}(X, Y)| \leq 16R(\kappa^{K_1} + \kappa^{K_2 - K_1} + \kappa^{N - K_2}) \|X\| \|Y\|,$$

which decays exponentially.

*Proof.* Let us introduce  $L_{K_1 0} = z_{K_1 0} l_{K_1 0}$ ,  $L_{K_2 K_1} = z_{K_2 K_1} l_{K_2 K_1}$  and  $L_{NK_2} = z_{NK_2} l_{NK_2}$  which are rank-one operators such that

$$\begin{cases} d_{\mathcal{P}}((T_{K_1} \dots T_0), L_{K_1 0}) \leq 2\kappa^{K_1} R, \\ d_{\mathcal{P}}((T_{K_2} \dots T_{K_1}), L_{K_2 K_1}) \leq 2\kappa^{K_2 - K_1} R, \\ d_{\mathcal{P}}((T_N \dots T_{K_2}), L_{NK_2}) \leq 2\kappa^{N - K_2} R. \end{cases}$$

And we can assume that (just multiply by a constant)

$$\begin{cases} L_{K_1 0} \leq_{\mathcal{P}} T_{K_1} \dots T_0 \leq_{\mathcal{P}} L_{K_1 0} e^{2\kappa^{K_1} R}, \\ L_{K_2 K_1} \leq_{\mathcal{P}} T_{K_2} \dots T_{K_1} \leq_{\mathcal{P}} L_{K_2 K_1} e^{2\kappa^{K_2 - K_1} R}, \\ L_{NK_2} \leq_{\mathcal{P}} T_N \dots T_{K_2} \leq_{\mathcal{P}} L_{NK_2} e^{2\kappa^{N - K_2} R}. \end{cases}$$

We then use Proposition 5.29, to obtain the inequality for the partition function,

$$d_{\mathcal{P}}(T_N \dots T_0, L_{NK_2} L_{K_2 K_1} L_{K_1 0}) \leq 2R(\kappa^{N - K_2} + \kappa^{K_2 - K_1} + \kappa^{K_1}),$$

for the density function

$$\begin{aligned} d_{\mathcal{P}}(T_N \dots T_{K_1+1} X T_{K_1} \dots T_0, L_{NK_2} L_{K_2 K_1} X L_{K_1 0}) &\leq 2R(\kappa^{N - K_2} + \kappa^{K_2 - K_1} + \kappa^{K_1}) \\ d_{\mathcal{P}}(T_N \dots T_{K_2+1} Y T_{K_2} \dots T_0, L_{NK_2} Y L_{K_2 K_1} L_{K_1 0}) &\leq 2R(\kappa^{N - K_2} + \kappa^{K_2 - K_1} + \kappa^{K_1}), \end{aligned}$$

and the pair correlation function

$$\begin{aligned} d_{\mathcal{P}}(T_N \dots T_{K_2+1} Y T_{K_2} \dots T_{K_1+1} X T_{K_1} \dots T_0, L_{NK_2} Y L_{K_2 K_1} X L_{K_1 0}) \\ \leq 2R(\kappa^{N - K_2} + \kappa^{K_2 - K_1} + \kappa^{K_1}). \end{aligned}$$

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Moreover we have

$$\begin{aligned}
& (u, L_{NK_2} L_{K_2 K_1} L_{K_1 0} v) \cdot (u, L_{NK_2} Y L_{K_2 K_1} X L_{K_1 0} v) \\
&= (u, z_{NK_2}) (l_{NK_2} z_{K_2 K_1}) (l_{K_2 K_1} z_{K_1 0}) (l_{K_1 0} v) (u, z_{NK_2}) (l_{NK_2} Y (z_{K_2 K_1})) \\
&\quad (l_{K_2 K_1} X (z_{K_1 0})) (l_{K_1 0} v) \\
&= (u, z_{NK_2}) (l_{NK_2} z_{K_2 K_1}) (l_{K_2 K_1} X (z_{K_1 0})) (l_{K_1 0} v) (u, z_{NK_2}) (l_{NK_2} Y (z_{K_2 K_1})) \\
&\quad (l_{K_2 K_1} z_{K_1 0}) (l_{K_1 0} v) \\
&= (u, L_{NK_2} L_{K_2 K_1} X L_{K_1 0} v) \cdot (u, L_{NK_2} Y L_{K_2 K_1} L_{K_1 0} v).
\end{aligned}$$

We recall that

$$\begin{aligned}
L_{K_1 0} &\leq_{\mathcal{P}} T_{K_1} \dots T_0 \leq_{\mathcal{P}} L_{K_1 0} e^{2\kappa^{K_1} R} \\
&\Leftrightarrow \forall v \in \mathcal{C}, u \in \mathcal{C}^* \quad (u, L_{K_1 0} v) \leq (u, T_{K_1} \dots T_0 v) \leq (u, L_{K_1 0} v) e^{2\kappa^{K_1} R}
\end{aligned}$$

(and the same for  $L_{K_2 K_1}, L_{NK_2}$ ) and this allows us to conclude that

$$\begin{aligned}
& Z^2 \rho_{K_1}(X) \rho_{K_2}(Y) \\
&\leq (u, L_{NK_2} L_{K_2 K_1} X L_{K_1 0} v) \cdot (u, L_{NK_2} Y L_{K_2 K_1} L_{K_1 0} v) e^{4R(\kappa^N - \kappa^2 + \kappa^{K_2 - K_1} + \kappa^{K_1})} \\
&\leq (u, T_N \dots T_{K_2+1} X T_{K_2} \dots T_{K_1+1} X T_{K_1} \dots T_0 v) \cdot (u, T_N \dots T_0 v) \\
&\quad \times e^{8R(\kappa^N - \kappa^2 + \kappa^{K_2 - K_1} + \kappa^{K_1})} \\
&\leq Z^2 \rho_{K_2, K_1}(Y, X) e^{8R(\kappa^N - \kappa^2 + \kappa^{K_2 - K_1} + \kappa^{K_1})}.
\end{aligned}$$

We also have by a similar manipulation

$$Z^2 \rho_{K_1}(X) \rho_{K_2}(Y) \geq Z^2 \rho_{K_2, K_1}(Y, X) e^{-8R(\kappa^N - \kappa^2 + \kappa^{K_2 - K_1} + \kappa^{K_1})}.$$

□

The proof of the decay of the cluster correlation is the same. One just has to replace  $X_i$  by  $X_i = T_{i+1} Y_{i, l-1} \dots T_{i+1} Y_{i, 2} T_i Y_{i, 1}$ , which are positive operators.

#### 5.3.5 Smoothness of the free energy

In this section, we prove Proposition 5.22 and Theorem 5.23.

##### Proof of Proposition 5.22

*Proof.* Let  $H$  a hyperplane such that  $E = \text{Vect}(\{x_0\}, H)$ . The projective space is locally isomorph to  $H$ . Let  $B$  be the convex set containing all the  $s \in H$  for which there exist  $(\alpha_+, \alpha_-, \beta_-, \beta_+)$  satisfying  $\alpha_+ x_0 \leq x_0 + s \leq \beta_+ x_0$ , with  $\beta_+ - \alpha_+ \leq r$ , and  $\alpha_- x_0 \leq x_0 - s \leq \beta_- x_0$ , with  $\beta_- - \alpha_- \leq r$ , for some  $r$  small enough. This set is symmetric with respect to the transformation  $s \rightarrow -s$ . Therefore, it is the ball of the norm  $\|s\| = r \cdot \inf(\lambda \in \mathbb{R}, \frac{s}{\lambda} \in B)$ . Let us check that this norm is close to the distance. Let  $s_1, s_2 \in H$  be such that  $d(x_0 + s_1, x_0) < r$  and  $d(x_0 + s_2, x_0) < r$ . We have then  $\alpha_1 x_0 \leq x_0 + s_1 \leq \beta_1 x_0$  and  $\alpha_2 x_0 \leq x_0 + s_2 \leq \beta_2 x_0$  and because  $r$  is very small, we can write  $\alpha_1 = 1 + \delta\alpha_1$ ,  $\alpha_2 = 1 + \delta\alpha_2$ ,  $\beta_1 = 1 + \delta\beta_1$ ,  $\beta_2 = 1 + \delta\beta_2$ . At first order we have  $d(x_0 + s_1, x_0) = \delta\beta_1 - \delta\alpha_1 + o(|\delta\beta_1|, |\delta\alpha_1|)$  and  $d(x_0 + s_2, x_0) = \delta\beta_2 - \delta\alpha_2 + o(|\delta\beta_2|, |\delta\alpha_2|)$ .

We now check that  $d(x_0 + s_1, x_0) = (1 + O(r))\|s_1\|$ . First we have

$$d(x_0 + s_1, x_0) \geq (1 + O(r))\|s_1\|.$$

Indeed, for any  $\lambda \in \mathbb{R}$ , we have  $\lambda(1 + \delta\alpha_1)x_0 + (1 - \lambda)x_0 \leq x_0 + \lambda s \leq \lambda(1 + \delta\beta_1)x_0 + (1 - \lambda)x_0$  and  $x_0 + \lambda(\delta\alpha_1)x_0 \leq x_0 + \lambda s \leq x_0 + \lambda\delta\beta_1x_0$ . With  $\lambda = \frac{r}{\delta\beta_1 - \delta\alpha_1}$ , we obtain

$$x_0 + \frac{r}{\delta\beta_1 - \delta\alpha_1}(\delta\alpha_1)x_0 \leq x_0 + \frac{r}{\delta\beta_1 - \delta\alpha_1}s \leq x_0 + \frac{r}{\delta\beta_1 - \delta\alpha_1}\delta\beta_1x_0$$

Therefore

$$\|s\| \leq \delta\beta_1 - \delta\alpha_1 = (1 + O(r))d(x_0, x_0 + s).$$

Then we claim that

$$d(x_0 + s_1, x_0) \leq (1 + O(r))\|s_1\|.$$

Indeed let  $\lambda$  be such that for any  $\alpha, \beta$   $\alpha x_0 \leq x_0 + \frac{s}{\lambda} \leq \beta x_0 \Rightarrow \beta - \alpha \geq r$ . Then for any  $\alpha, \beta$   $\lambda\alpha x_0 + (1 - \lambda)x_0 \leq x_0 + s \leq \lambda\beta x_0 + (1 - \lambda)x_0 \Rightarrow \beta - \alpha \geq r$ . Therefore

$$\begin{aligned} d(x_0, x_0 + s) &\leq \log \frac{1 - \lambda + \lambda\beta}{1 - \lambda + \lambda\alpha} = \log \frac{1 + \lambda\delta\beta}{1 + \lambda\delta\alpha} \\ &= (\lambda(\delta\beta - \delta\alpha))(1 + O(r)) \leq \|s\|(1 + O(r)). \end{aligned}$$

We finally check that  $d(x_0 + s_1, x_0 + s_2) = (1 + O(r))d(x_0 + s_1 - s_3, x_0 + s_2 - s_3)$  for any  $s_1, s_2, s_3 \in B$ . We have,  $\alpha_3 x_0 \leq x_0 + s_3 \leq \beta_3 x_0$ , and  $d(x_0 + s_1, x_0 + s_2) = (\delta\beta - \delta\alpha)(1 + O(r))$  with  $(1 + \delta\alpha)(x_0 + s_1) \leq x_0 + s_2 \leq (1 + \delta\beta)(x_0 + s_1)$ . Then

$$(1 + \delta\alpha + O(r)\delta\alpha)(x_0 + s_1 + s_3) \leq (1 + \delta\alpha)(x_0 + s_1 + s_3) - \delta\alpha s_3 \leq x_0 + s_2 + s_3$$

and

$$x_0 + s_2 + s_3 \leq (1 + \delta\beta)(x_0 + s_1 + s_3) - \delta\alpha s_3 \leq (1 + \delta\beta + O(r))(x_0 + s_1 + s_3).$$

We conclude that  $d(x_0 + s_1, x_0 + s_2) = \|s_2 - s_1\|(1 + O(r))$ .  $\square$

**Example 5.34.** Let  $\mathcal{C}$  be the cone of positive vectors in  $\mathbb{R}^n$  and let  $x_0 = (x^1, \dots, x^n)$  and  $H = \{s : \sum_i s^i = 0\}$ . Then in a neighborhood of  $x_0$ , we have

$$\alpha x_0 \leq x_0 + s \leq \beta x_0$$

with

$$\alpha = \max_{\alpha} \alpha x^i \leq x^i + s^i = 1 + \min \frac{s^i}{x^i}$$

and

$$\beta = \min_{\beta} \beta x^i \geq x^i + s^i = 1 + \max \frac{s^i}{x^i}.$$

In addition

$$d(x_0, x_0 + s) = \log \frac{1 + \max \frac{s^i}{x^i}}{1 + \min \frac{s^i}{x^i}} \approx \max \frac{s^i}{x^i} - \min \frac{s^i}{x^i} = \max \frac{s^i}{x^i} + \max \frac{-s^i}{x^i}.$$

Finally the constructed norm is then:

$$\|s_1 - s_2\| = \max \frac{s_1^i - s_2^i}{x^i} + \max \frac{s_2^i - s_1^i}{x^i}.$$

### 5.3.6 Proof of Theorem 5.23

In order to prove Theorem 5.23 we will need the following lemma.

**Lemma 5.35.** *Let  $E_n$  be Banach spaces with norms  $\|\cdot\|_n$ . Consider the functions  $u_n : \mathbb{R} \rightarrow E_n$  iteratively defined by*

$$u_{n+1}(s) = g_n(s, u_n(s)),$$

with  $g_n(s, 0) = 0$  which are assumed to be uniformly contracting,  $\|\partial_2 g_n\|_{\|\cdot\|_n \rightarrow \|\cdot\|_{n+1}} \leq \kappa$  with  $\kappa < 1$ . If the  $g_n$  are uniformly  $C^k$  then the  $u_n$  are uniformly  $C^k$ .

*Proof.* We prove by induction that there exists constant a  $C_k$  such that for all  $n$ ,  $\|\frac{d^k}{ds^k} u_n\|_n \leq C_k$ . Computing the derivative gives

$$\frac{d^k}{ds^k} u_{n+1} = \partial_2 g_n \cdot \frac{d^k}{ds^k} u_n + Q\left(g_n, (\partial_1^r \partial_2^p g_n), \left(\frac{d^i}{ds^i} u_n\right)_{i < k}\right)$$

where  $Q$  is a polynomial involving lower order derivatives of  $u_n$  and the derivatives of  $g_n$ . Because of the induction hypothesis, there exists  $C_{k-1}$  such that for all  $n$  and all  $i < k$ ,  $\|\frac{d^i}{ds^i} u_n\| \leq C_{k-1}$ . Therefore  $Q$  can be uniformly bounded by a constant  $\tilde{C}_k$  which depends only on  $\sup_{n \in \mathbb{N}} \|\partial_1^r \partial_2^p g_n\|$  and  $C_{k-1}$ . We have therefore

$$\left\| \frac{d^k}{ds^k} u_{n+1} \right\|_{n+1} \leq \kappa \left\| \frac{d^k}{ds^k} u_n \right\|_n + \tilde{C}_k$$

and we can then set

$$C_k = \frac{1}{1 - \kappa} \tilde{C}_k. \quad (5.5)$$

We can now conclude because if  $g$  is contracting for  $d$  then it is contracting for  $\|\cdot\|$ .  $\square$

**Example 5.36.** Consider  $T(\beta) = \begin{pmatrix} \beta & \epsilon \\ \epsilon & 1 \end{pmatrix}$ , with largest eigenvalue

$$\lambda(\beta) = \frac{(\beta + 1) + \sqrt{(\beta - 1)^2 + 4\epsilon^2}}{2}$$

and  $\log(\lambda(\beta))$  for  $\beta$  around 1. In the usual positive cone,

$$\Delta(T(\beta)(\mathcal{C})) = d_{\mathcal{C}} \left( \begin{pmatrix} 1 \\ \epsilon \end{pmatrix}, \begin{pmatrix} \epsilon \\ 1 \end{pmatrix} \right) = |2 \log(\epsilon)|.$$

The Birkhoff-Hopf theorem gives  $\kappa = \tanh(\log(\epsilon)/2) \approx 1 - 2\epsilon$ . Around the point  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ , the norm  $\mathcal{N}$  is equal to the norm  $\|\cdot\|_{\infty}$  (see example 5.34). The iterative formula (5.5) gives a constant behave like  $C_k \approx (2\epsilon)^{(1-k)}$  and this is what we get with the exact calculation of  $\frac{d}{d\beta^k} [\log(\lambda(\beta))]$ .

*Remark 5.37.* If  $T$  is contracting for the distance  $d$ , then  $T$  is locally contracting for  $\mathcal{N}$ .

We can now finish the proof of Theorem 5.23.

*Proof.* We denote

$$u_n(\beta) = \frac{\prod_{i=0}^{n-1} T_i(\beta)b}{\|\prod_{i=0}^{n-1} T_i(\beta)\|}$$

and we decompose the log of the product as

$$f_N(\beta) = \frac{1}{N} \log \langle a, \prod_{i=0}^{N-1} T_i(\beta)b \rangle = \frac{1}{N} \log \langle a, u_N(\beta) \rangle + \frac{1}{N} \sum \log(\|T_i(\beta)u_i(\beta)\|).$$

Because the  $T_i$  are smooth we only have to make sure that the  $u_i$  are smooth as well. This follows from the previous lemma. The function  $u_i(\beta)$  is smooth for the constructed norm  $\|\cdot\|_i$ . But because the cone is normal,

$$\alpha x \leq y \leq \beta x \Rightarrow \|y - \frac{\beta + \alpha}{2}x\| \leq \frac{\beta - \alpha}{2}\|x\|,$$

we have then that  $\|x\| \cdot \|x - y\|_i \geq \|x - y\|$  and we conclude because  $\|u_n(\beta)\| = 1$ .  $\square$

### 5.3.7 Proof of Theorem 5.24

**Proposition 5.38.** *Let  $g_1, g_2, \dots, g_n \dots$ , be analytic functions such that for any  $k$ ,  $g_k(x) = \sum_i b_{k,n}x^n$  with  $|b_{k,n}| \leq r^n$  and  $|b_{k,0}| \leq 1$ . Let  $f_0 = 1$  and  $f_{k+1} = (1 + \kappa g_{k+1}f_k)$ . Then for any  $k$ ,*

$$f_k = \sum c_{k,n}x^n$$

with  $c_{k,n} \leq d_n$  where  $d_n$  are the coefficient of the Taylor expansion of  $\frac{1-rx}{(1-\kappa)-rx}$ . In particular, if  $f_n$  admits a limit  $f_\infty$ , then  $f_\infty$  is analytic.

*Proof.* We can assume  $b_{k,n} = r^n$  for any  $k, n$ . Indeed another configuration would give a smaller  $c_{k,n}$ . We expand  $f_k$  and have:  $f_k = \sum_{i=0}^k \left(\frac{\kappa}{1-rx}\right)^i$  whose coefficients are then smaller than those of  $\sum_{i=0}^{\infty} \left(\frac{\kappa}{1-rx}\right)^i = \frac{1-rx}{(1-\kappa)-rx}$ .  $\square$

**Corollary 5.39.** *Let  $g_n$  and  $u_n$  be as in Proposition 5.35 with  $g_n$  affine in the second variable. Suppose that there exists  $r \geq 0$  such that  $\frac{\|\partial_s^i g\|}{i!} \leq r^i$  for all  $i$ , then  $u_n$  are analytic with coefficients of its Taylor series bounded by that of  $\frac{1-rx}{(1-\kappa)-rx}$ . In particular if  $u_n$  admits a limit then it is analytic with convergence radius  $\frac{1-\kappa}{r}$ .*

*Proof.* This follows from the fact that

$$u_n(s) - u_n(0) = g_n(s, u_{n-1}(s) - u_{n-1}(0)) + g_n(s, u_{n-1}(0)) - u_n(0).$$

$\square$

Theorem 5.24 then follows replacing the  $g$  by the  $T$ .

### 5.3.8 Proof of Theorem 5.25

We now prove the central limit theorem 5.25 from the regularity of the Laplace transform.

*Proof.* By Theorem 5.24 we have that

$$f_N(\beta) = \frac{1}{N} \log \left\langle a, \prod_{i=0}^{N-1} e^{\beta h_i T_i b} \right\rangle = \frac{1}{N} \log \mathbb{E}[e^{\beta \sum_{i=0}^{N-1} h_i}]$$

is smooth with first derivatives bounded by  $C$ . Therefore

$$f_N(\alpha) := N f_N\left(\frac{\alpha}{\sqrt{N}}\right)$$

satisfies  $\partial_\alpha f_N|_{\alpha=0} = \gamma_N \sqrt{N} \leq C\sqrt{N}$ ,  $\partial_\alpha^2 [f_N - \gamma_N \sqrt{N} \alpha]|_{\alpha=0} = \sigma_N^2 \leq C$  and  $\partial_\alpha^3 [f_N - \gamma_N \sqrt{N} \alpha] \leq \frac{C}{\sqrt{N}}$ . Then  $(f_N(\alpha) - \gamma_N \sqrt{N} \alpha) = 1 + \frac{(\sigma_N \alpha)^2}{2} + O(\frac{1}{\sqrt{N}})$ . Therefore the Laplace transform is close to the one of a Gaussian and we can conclude with the usual Berry Essen inequality.  $\square$

## 5.4 Proofs for the Jellium model

### 5.4.1 Proof for the classical Jellium model

We first write the partition function in the form of products of operators. Recall that  $U_i(s) = -2q_i \int_{\tilde{x}_i}^{\tilde{x}_i+s} (y - \tilde{x}_i) \rho(y) dy$  with  $\tilde{x}_i$  the equilibrium position of the particle  $i$ . We note  $\delta = \frac{1}{2} \min(|\tilde{a}_i - \tilde{a}_{i+1}|)$ .

**Definition 5.40.** (Iterative operator) Let  $T_i$  be the operator defined for any function  $f$  in  $L^1$  or  $L^\infty$  by

$$T_i f(x) = \int_{s=x-\tilde{x}_{i+1}+\tilde{x}_i}^{\infty} e^{-\beta U(s)} f(s) ds.$$

In particular, we can rewrite the partition function as

$$\mathcal{Z}_N(\beta) = e^{-\beta E(\tilde{x}_1, \dots, \tilde{x}_N)} \left\langle \mathbf{1}_{x_N < L - \tilde{x}_N} e^{-\beta U(x_N)}, \left( \prod_{i=1}^{N-1} T_i \right) \mathbf{1}_{x_1 > -L - \tilde{x}_1} \right\rangle$$

We are then in the setting of Section 5.3.4.

### Construction of a uniform invariant cone

We first notice that we cannot directly apply the Birkhoff-Hopf Theorem with the cone of positive functions  $\mathcal{C}_0$ . Indeed we have the

*Remark 5.41.* For any  $T_i$ ,  $\Delta_{\mathcal{C}_0}(T_i) = \infty$ . For example  $\text{Supp}[T_i(1_{[0,1]})] = (-\infty; 1 + (\tilde{x}_{i+1} - \tilde{x}_i)]$  and  $\text{Supp}(T_i(1_{[2,3]})) = (-\infty, 3 + (\tilde{x}_{i+1} - \tilde{x}_i)]$  and then we have for any  $\alpha > 0$  and  $1 + (\tilde{x}_{i+1} - \tilde{x}_i) < t < 3 + (\tilde{x}_{i+1} - \tilde{x}_i)$ ,  $(T_i(1_{[0,1]}) - \alpha T_i(1_{[2,3]}))(t) < 0$  so  $\alpha_{\min} = 0$ .



The solution is to construct another cone. If we were restricted to a bounded interval, then the simplest solution would be to consider finite products of  $T_i$ , instead of one by one. The kernel of  $\prod_{i=n}^{n+k-1} T_i$  is strictly positive on  $\{(x, y), y \geq x - 2n\delta\}$ , and therefore with  $n$  such that  $2n\delta > 2A$ , the kernel is strictly positive.  $\prod T_i$  are then contracting for the cone  $\{f \geq 0\}$ .

In our case, because of the multiplication by  $e^{-U_i(s)}$ , we will be able to neglect the influence of  $f e^{-U_i(s)}$  outside  $(-A, A)$ . We choose  $A$  such that

$$\int_A^\infty e^{-U_i(s)} ds \leq \frac{\delta}{2} e^{-U_i(A)}$$

(for example, because of  $\frac{d}{ds} U(s) \geq qms$ , we can choose  $A = \frac{2}{\delta qm}$ ). In Proposition 5.42 we will define a cone such that  $f$  can be slightly negative for  $\{x : x \geq A\}$ . We also make it so that  $T_i$  are contracting and not only  $\prod_k^{k+n} T_i$ . The price to pay is more restrictions. Intuitively, it is how  $\prod T_i f$  looks like for  $f \geq 0$ .

Let us divide the interval  $[-A, A]$  in small intervals with  $I_k = [k\delta/2, (k+1)\delta/2]$  with  $k \in \mathbb{Z}$  and  $-2\frac{A}{\delta} - 1 = k_{min} \leq k \leq k_{max} = 2\frac{A}{\delta} + 1$ . We suppose  $\frac{A}{\delta} \in \mathbb{N}$  to simplify the notation. We note

$$I_k(f) = \int_{I_k} f(x) dx$$

**Proposition 5.42.** *There exist  $(\epsilon_k)_{-2\frac{A}{\delta} \leq k \leq 2\frac{A}{\delta}}$  such that the cone  $\mathcal{C}$  defined by*

$$f \in \mathcal{C} \Leftrightarrow \begin{cases} \forall t \geq A & f(t) + \epsilon_{k_{max}}(f) \geq 0, \\ \forall t \leq -A & f(t) \geq 0, \\ \text{on } -A \leq t \leq A & f \text{ is decreasing,} \\ \forall t \leq -A & f(t) \geq f(-A), \\ \forall k \in [-2\frac{A}{\delta}, 2\frac{A}{\delta}] & I_{k-1}(f) \leq \frac{1}{\epsilon_k} I_k(f), \\ \forall t \leq -A & f(t) \leq \frac{1}{\epsilon'} I_{k_{min}}(f), \end{cases}$$

satisfies that, for any  $i$ ,  $T_i$  is  $d_C$  contracting.

We use only inequalities implying linear forms acting on  $f$ , therefore  $\mathcal{C}$  is convex and  $\lambda\mathcal{C} = \mathcal{C}$  for  $\lambda > 0$ . If  $f \in -\mathcal{C} \cap \mathcal{C}$  then  $f(t) = 0$  for all  $t \leq A$  and then  $f(t) = 0$  for all  $t \geq A$ . Finally  $\mathcal{C} - \mathcal{C}$  is dense. Indeed one can find a function  $f \in \mathcal{C}$  strictly decreasing which satisfies all the inequalities strictly. Therefore for any compact set there is a small ball  $B$  of Lipschitz functions with support on this set such that  $f + B \subset \mathcal{C}$ . Then  $B \subset \mathcal{C} - \mathcal{C}$ .

This cone may seem a bit artificial, however it behaves nicely with respect to the iteration of  $T_i$ . For the proof we need the following Lemma

**Lemma 5.43.** *Let  $y, x > 0$ ,  $K$  linear and  $a, b, u, v \geq 0$  such that  $Kx \geq ax + uy$  and  $Ky \leq by + vx$ . If  $a > b$  or  $u > 0$ , then there exist  $\epsilon > 0$  such that if  $\frac{1}{\epsilon}x \geq y$  then  $\frac{1}{\epsilon}Kx > Ky$ .*

*Proof.* If  $a > b$ , then for  $\epsilon$  small enough,  $\frac{b}{\epsilon} + v < \frac{a}{\epsilon}$  and we have  $Ky \leq by + vx \leq (\frac{b}{\epsilon} + v)x < \frac{a}{\epsilon}x \leq \frac{1}{\epsilon}Kx$ . If  $u > 0$ , then we have  $\frac{1}{\epsilon}Kx - Ky \geq (\frac{a}{\epsilon} - v)x - (b - \frac{u}{\epsilon})y > 0$  for  $\epsilon$  small enough.  $\square$

We can now carry on the proof of Proposition 5.42.

*Proof.* We construct the  $\epsilon_k$  recursively. Because  $f$  is decreasing and  $e^{-U_i}$  are uniformly integrable, there exists  $u$  such that  $\int_{-\alpha}^{\infty} f e^{-U_i(s)} ds \leq u I_{k_{min}}(f)$ . We then have

$$\sup_{\mathbb{R}} T_i f = \int_{\mathbb{R}} e^{-U_i(s)} f(s) ds \leq \sup f \cdot \int_{-\infty}^{-A} e^{-U_i(s)} ds + u I_{k_{min}}(f).$$

Moreover  $I_{k_{min}}(Tf) \geq \delta e^{-U_i(-A)} I_{k_{min}}(Tf)$  and then  $\int_{-\infty}^{-A} e^{-U_i(s)} ds < \delta e^{-U_i(-A)}$ . By Lemma 5.43 there exists  $\epsilon'$  such that for any  $i$  and  $t$ ,  $T_i f(t) < \frac{1}{\epsilon'} I_{k_{min}}(T_i f)$ .

Suppose we have constructed every  $\epsilon_k$  up to  $k = k_0$ , and let us construct  $\epsilon_{k_0+1}$ . Because of the induction hypothesis there exist  $b_{k_0}$  such that  $\sup_t f(t) \leq b_{k_0} I_{k_0}(f)$  so there exists  $b'_{k_0}$  such that  $I_{k_0}(T_i f) \leq b'_{k_0} I_{k_0}(f)$ . Moreover

$$\begin{aligned} \forall a \in I_{k_0+1}, T_i f(a) &= \int_{a-\tilde{x}_{i+1}-\tilde{x}_i}^{\infty} f(s) e^{-U_i(s)} ds \\ &\geq \int_{I_{k_0}} f(s) e^{-\max_{s \in I_{k_0}} U_i(s)} ds \\ &\geq e^{-\max_{s \in I_{k_0}} U_i(s)} I_{k_0}(f). \end{aligned}$$

So thanks to Lemma 5.43, there exists  $\epsilon_{k_0+1}$  such that if for all  $k \leq k_0$ ,  $I_k(f) \leq \frac{1}{\epsilon_{k+1}} I_{k+1}(f)$  then for all  $k \leq k_0 + 1$ ,  $I_k(T_i f) < \frac{1}{\epsilon_{k+1}} I_{k_0}(T_i f)$ . We also have for any  $a \leq A$

$$\begin{aligned} T_i f(a) &= \int_{a-\tilde{x}_{i+1}-\tilde{x}_i}^{\infty} f(s) e^{-U(s)} ds \\ &\geq \int_{A-\delta}^A f(s) e^{-U_i(s)} ds - \epsilon \int_A^{\infty} e^{-U_i(s)} ds \cdot I_{k_{max}}(f) \\ &\geq e^{-U_i(A)} (1 - \epsilon \frac{\delta}{2}) I_{k_{max}}(f). \end{aligned}$$

In particular  $T_i f(a) \geq 0$ . Moreover for any  $a > A$ , we have

$$\begin{aligned} T_i f(a) &\geq -\epsilon \int_A^{\infty} e^{-U_i(s)} ds \cdot I_{k_{max}}(f) \geq -\epsilon \frac{\delta}{2} e^{-U_i(A)} I_{k_{max}}(f) \\ &\geq -\epsilon \frac{1}{1 - \epsilon \frac{\delta}{2}} I_{k_{max}}(Kf). \end{aligned}$$

Because  $f \geq 0$  on  $(-\infty, A]$ ,  $Tf$  is decreasing on  $]-\infty, A+\delta]$ . To conclude, it will be enough to compare  $T_i f$  with  $T_i g$  for  $f, g \in \mathcal{C}$  and  $I_{k_{max}}(f) = I_{k_{max}}(g) = 1$ . Because all the inequalities become strict, there exists  $\epsilon''$  such that for any  $f \in \mathcal{C}$  with  $I_{k_{max}}(f) = 1$ , if  $\|g\|_{L^\infty} \leq \epsilon''$  then  $T_i(f - g) \in \mathcal{C}$ . Moreover for any  $g \in \mathcal{C}$  with  $I_{k_{max}}(g) = 1$ ,  $\|g\|_{L^\infty} \leq \prod \frac{1}{\epsilon_k}$ . So  $T_i(f - \epsilon'' \prod \epsilon_k g) \in \mathcal{C}$ . And this concludes the proof because then  $\Delta \leq 2 \log(\epsilon'' \prod \epsilon_k)$ .  $\square$

*Remark 5.44.* If we denote by  $\mathcal{H}_k$  the assertion " $I_{k-1}(f) \leq \frac{1}{\epsilon_k} I_k(f)$ ", we have actually proved that if  $f$  satisfies all the condition of  $\mathcal{C}$  except  $(\mathcal{H}_i)_{i=r, \dots, k_{max}}$ , then  $T_i f$  satisfies all the condition of  $\mathcal{C}$  except  $(\mathcal{H}_i)_{i=r+1, \dots, k_{max}}$ . This implies that if  $f \geq 0$  and  $\text{supp}(f) \subset [-A, A]$ , then  $\prod_{i=k}^{k+n} T_i f \in \mathcal{C}$  is as wanted.

**Decay of correlation.**

Here we prove Theorem 5.4.

We can carry on with the construction of conditions like  $I_{k-1}(f) \leq \frac{1}{\epsilon_k} I_k(f)$  after  $k_{max}$  in Proposition 5.42. We denote by  $\mathcal{C}_m$  this more specified cone replacing  $\forall k \in [-2\frac{A}{\delta}, 2\frac{A}{\delta}] I_{k-1}(f) \leq \frac{1}{\epsilon_k} I_k(f)$  by  $\forall k \in [-2\frac{A}{\delta}, 2\frac{A}{\delta} + m] I_{k-1}(f) \leq \frac{1}{\epsilon_k} I_k(f)$ . We have then the

**Proposition 5.45.** *If  $g \in \mathcal{C}$  then  $\prod_{i=k}^{k+n} T_i g \in \mathcal{C}_i$ .*

*Proof.* By Proposition 5.42,  $\prod_{i=k}^{k+n} T_i g \in \mathcal{C}$ . Therefore it is enough to prove the remaining conditions. Let  $g_i \in \mathcal{C}_i$ . As previously,

$$T_i g(a) \geq -\epsilon \frac{\delta}{2} e^{-U_i(A+\delta i)} I_{k_{max}}(f) \geq -\epsilon \frac{1}{1 - \epsilon \frac{\delta}{2}} I_{k_{max}+i}(Tf).$$

As a consequence we have that for  $f \geq 0$  and  $\text{supp}(f) \in (-\infty, A + n\delta]$  then  $f \prod_{i=k}^{k+n} T_i g \geq 0$  for all  $g \in \mathcal{C}$ .  $\square$

*Remark 5.46.* If  $f \geq 0$  and  $\text{supp}(f) \in [A, \infty)$  then  $T_i f \in \mathcal{C}$ . Therefore for  $f$ ,  $\text{supp}(f) \in [a - dl, a - dl]$   $dl \leq \delta$  and  $A \leq a \leq A + n\delta$  then  $T_{n+k+1} F \prod_{i=k}^{k+n} T_i$  is order preserving for the cone  $\mathcal{C}$ .

**Proposition 5.47.** *For  $f \geq 0$  with  $\text{supp}(f) \in [a - dl, a + dl]$   $dl \leq \delta$  and  $a \geq A + n\delta$ , we have*

$$\left\langle 1, \prod_{i=k}^{k+n} T_i f \right\rangle \leq e^{-n(|\frac{a}{\delta}|^2 - c)} \langle 1, f \rangle \left\langle 1, \prod_{i=k}^{k+n} T_i 1 \right\rangle.$$

By iteration  $\text{supp}(\prod_i T_i f) \in [a - n \max(\bar{a}_i - \bar{a}_{i+1}), \infty[$ . Therefore

$$\begin{aligned} \|\prod T_i f\|_\infty &\leq \langle 1, f \rangle e^{-\sum \min_i (U_i(a - k(\max(\bar{a}_i - \bar{a}_{i+1}))))} \\ &= \langle 1, f \rangle e^{-\gamma \min \sum (a - k(\max(\bar{a}_i - \bar{a}_{i+1}))^2}. \end{aligned}$$

*Proof.* There exists  $\lambda > 0$  such that  $\langle 1, \prod_{i=k}^{k+n-1} T_i 1 \rangle \geq \lambda^n$ , and we set  $c = \log \lambda$ . The result follows.  $\square$

We are now ready to prove the decay of the correlation functions. Recall that for  $i_1, i_2, \dots, i_k$ , we have the  $k$ -th marginal defined by

$$\begin{aligned} \rho_k(x_{i_1}, x_{i_2}, \dots, x_{i_k}) &= \frac{1}{\mathcal{Z}_N(\beta)} e^{-\beta E(\bar{x}_1, \dots, \bar{x}_N)} \\ &\int \dots \int_{-L < x_1 < x_2 < \dots < x_N < L} \prod_{i \neq i_1, i_2, \dots, i_k} e^{-2\beta q_i \int_{\bar{x}_i}^{x_i} \rho(y)(y - x_i) dy} \prod dx_i. \end{aligned}$$

Note that there exists  $r$  such that for  $\rho_k(x_{i_1}, x_{i_2}, \dots, x_{i_k}) \leq e^{-r \max |x|^3}$ .

**Corollary 5.48.** *There exists  $\kappa < 1$  and  $C_k > 0$  such that*

$$|\rho_k(x_{i_1}, x_{i_2}, \dots, x_{i_k}) - \prod_{l=1}^k \rho_1(x_{i_l})| \leq C_k \kappa^{\inf |i_l - i_{l+1}|}.$$

*Proof.* Let  $x_{i_1}, \dots, x_{i_k} \in \mathbb{R}^k$  and let  $\delta^{(n)}$  be an approximation of the Dirac  $\delta_0$ . We evaluate

$$\begin{aligned} & \iiint [\rho_k(y_{i_1}, y_{i_2}, \dots, y_{i_k}) - \prod \rho_1(y_{i_1})] \prod \delta^{(n)}(y_{i_k} - x_{i_k}) dy_{i_k} \\ &= \iiint [\rho_k(y_{i_1}, y_{i_2}, \dots, y_{i_k}) - \prod \rho_1(y_{i_1})] \prod \delta_{x_{i_k}}^{(n)}(y_{i_k}) dy_{i_k} \end{aligned}$$

which, in our formalism, is equal to

$$\begin{aligned} & \frac{1}{\mathcal{Z}_N(\beta)} (u, T_N \dots T_{K_k+1} \delta_{x_k}^{(n)} T_{K_k} \dots T_{K_1+1} \delta_1^{(n)} T_{K_1} \dots T_0 v) \\ & - \prod \frac{1}{Z} (u, T_N \dots T_{K_1+1} \delta_{x_k}^{(n)} T_{K_1} \dots T_0 v). \end{aligned}$$

To begin with, assume  $-A < x_1, \dots, x_k < A$ . Because of Remark 5.46, for any  $i \in [1, k]$ ,  $T_{K_k+m} \dots T_{K_k+1} \delta_{x_k}^{(n)} T_{K_k}$  is a positive operator. Changing  $C_k$ , we can suppose  $\inf(|i_l - i_{l+1}|) > m$ . Therefore, denoting  $X_l = T_{K_l+m} \dots T_{K_l+1} \delta_{x_l}^{(n)} T_{K_l}$ , we can apply Theorem 5.33 and we obtain :

$$\begin{aligned} & \left| \frac{1}{\mathcal{Z}_N(\beta)} (u, T_N \dots T_{K_k+1} \delta_{x_k}^{(n)} T_{K_k} \dots T_{K_1+1} \delta_1^{(n)} T_{K_1} \dots T_0 v) \right. \\ & \left. - \prod \frac{1}{Z} (u, T_N \dots T_{K_1+1} \delta_{x_k}^{(n)} T_{K_1} \dots T_0 v) \right| \\ & \leq C_k \kappa^{\inf(i_l - i_{l+1})}, \end{aligned}$$

where  $C_k = 2k(2k+1)R\kappa^{-m}$  if  $\inf(i_l - i_{l+1})$  is larger than a constant  $c$ . Suppose that there exist  $|x_i| > \epsilon \inf(|i_l - i_{l+1}| \delta)$  then  $\rho_k(x_{i_1}, x_{i_2}, \dots, x_{i_k}) \leq e^{-r(\epsilon\delta)^2 \inf_l |i_l - i_{l+1}|^2}$  and we are done. If for all  $i$ ,  $|x_i| \leq \epsilon \inf(|i_l - i_{l+1}| \delta)$ , then as previously  $T_{K_k+m} \dots T_{K_k+1} \delta_{x_k}^{(n)} T_{K_k}$  is positive for  $m = \epsilon \inf |i_l - i_{l+1}|$ , hence

$$\begin{aligned} & \left| \frac{1}{\mathcal{Z}_N(\beta)} (u, T_N \dots T_{K_k+1} \delta_{x_k}^{(n)} T_{K_k} \dots T_{K_1+1} \delta_1^{(n)} T_{K_1} \dots T_0 v) \right. \\ & \left. - \prod \frac{1}{Z} (u, T_N \dots T_{K_1+1} \delta_{x_k}^{(n)} T_{K_1} \dots T_0 v) \right| \\ & \leq C'_k \kappa^{(1-\epsilon) \inf(i_l - i_{l+1})} \end{aligned}$$

with  $C_k = 2k(2k+1)R$ . We can conclude replacing  $\kappa$  by  $\kappa^{(1-\epsilon)}$ .  $\square$

### Smoothness of the free energy for the classical Jellium model.

Now we use Theorem 5.23 to prove the smoothness of the free energy. We first have to check its hypothesis. This is the aim of the following proposition

**Proposition 5.49.** *Let  $u_0 \in T_i(\mathcal{C})$ . Then*

$$\|[\partial_\beta^n T_i(\beta)]\|_{\mathcal{N}_i \rightarrow \mathcal{N}_{i+1}} < \infty$$

for all  $n$ , where  $\mathcal{N}_i$  and  $\mathcal{N}_{i+1}$  are the norm constructed in Proposition 5.22 around  $u_0$  and  $T_i(\beta)u_0$  respectively, and

$$\mathcal{N}(\partial_\beta^n T_i(\beta)u_0) < \infty.$$

To simplify the calculation, we introduce an approximating norm. We define

$$\begin{aligned}
a_1(s) &= \sup_{t \geq A} \frac{s(t) + \epsilon I_{k_{max}}(s)}{\nu}, \\
a_2(s) &= \sup_{t \leq A} \frac{s(t)}{\nu}, \\
a_3(s) &= \sup_{-A \leq t \leq A} \frac{s'(t)}{\nu}, \\
a_4(s) &= \sup_{t \leq -A} \frac{s(t) - s(-A)}{\nu}, \\
a_5(s) &= \sup_k \frac{\frac{1}{\epsilon_k} I_k(s) - I_{k-1}(s)}{\nu}, \\
a_6(s) &= \sup_{t \leq -A} \frac{s(t) - \frac{1}{\epsilon'} I_{k_{min}}(s)}{\nu},
\end{aligned}$$

and we set  $A_\nu(s) = \max(a_1(s), a_2(s), a_3(s), a_4(s), a_5(s), a_6(s))$ . Finally we define  $\|s\|_\nu = \max(A(-s), A(s))$ .

**Proposition 5.50.** *For  $u_0$  in  $T_i(\mathcal{C})$ , there exists  $\nu > 0$  such that*

$$\|\cdot\|_{i+1} \leq \|\cdot\|_\nu.$$

*Proof.* Let  $u_0 \in T_i(\mathcal{C})$  We calculate the norm of Proposition 5.22. Let  $s$

$$\alpha u_0 \leq u_0 + s \leq \beta u_0$$

with

$$\alpha = \max_\alpha \begin{cases} \forall t \geq A & s(t) + \epsilon I_{k_{max}}(s) \geq (\alpha - 1)[u_0(t) + \epsilon I_{k_{max}}(t)] \\ \forall t \leq A & s(t) \geq (\alpha - 1)u_0(t) \\ \text{on } -A \leq t \leq A & s'(t) \leq (\alpha - 1)u_0'(t) \\ \forall t \leq -A & s(t) - s(-A) \geq (\alpha - 1)[u_0(t) - u_0(-A)] \\ \forall k \in [-2\frac{A}{\delta}, 2\frac{A}{\delta}] & I_{k-1}(s) - \frac{1}{\epsilon_k} I_k(s) \leq (1 - \alpha)[I_{k-1}(u_0) - \frac{1}{\epsilon_k} I_k(u_0)] \\ \forall t \leq -A & s(t) - \frac{1}{\epsilon'} I_{k_{min}}(s) \leq (1 - \alpha)[u_0(t) - \frac{1}{\epsilon'} I_{k_{min}}(u_0)] \end{cases}$$

Because  $u_0 \in T_i(\mathcal{C})$ , there exists  $\epsilon_0 > 0$  such that all the functions depending on  $u_0$  on the right side of the equation ( $[u_0(t) + \epsilon I_{k_{max}}(t)], u_0(t), \dots$ ) can be bounded by  $\epsilon_0$ . We obtain

$$\alpha' = \max_{\alpha'} \begin{cases} \forall t \geq A & s(t) + \epsilon I_{k_{max}}(s) \geq (\alpha' - 1)\epsilon_0 \\ \forall t \leq A & s(t) \geq (\alpha' - 1)\epsilon_0 \\ \text{on } -A \leq t \leq A & s'(t) \leq (\alpha' - 1)\epsilon_0 \\ \forall t \leq -A & s(t) - s(-A) \geq (\alpha' - 1)\epsilon_0 \\ \forall k \in [-2\frac{A}{\delta}, 2\frac{A}{\delta}] & I_{k-1}(s) - \frac{1}{\epsilon_k} I_k(s) \leq (1 - \alpha')\epsilon_0 \\ \forall t \leq -A & s(t) - \frac{1}{\epsilon'} I_{k_{min}}(s) \leq (1 - \alpha')\epsilon_0 \end{cases}$$

and we then have  $\alpha' \leq \alpha$ . We have constructed then  $\|\cdot\|_\nu$  with  $\nu = \epsilon_0$ .  $\square$

We finish the proof of Proposition 5.49.

*Proof.* We can now calculate  $\|\partial_\beta^n T_i(\beta)\|_{\|\cdot\|_i \rightarrow \|\cdot\|_\nu}$ . Let  $w$  with  $\|w\|_i \leq 1$ , In particular, there exists  $r > 0$  such that  $d(u_0, u_0 + r.w) \leq 2r$ . Therefore

$$(1 - 2r)u_0(t) \leq u_0 + rw(t) \leq (1 + 2r)u_0(t)$$

for all  $t < A$ . Hence  $|I_{k_{max}}(r.w)| \leq 2r.I_{k_{max}}(u_0)$  and for all  $t > A$ :

$$(1 - 2r)[u_0(t)] - 4r.I_{k_{max}}(u_0) \leq u_0(t) + rw(t) \leq (1 + 2r)u_0(t) + 4r.I_{k_{max}}(u_0).$$

Therefore :

$$[\partial_\beta^n T_i(\beta)w](x) = \int_{x-\tilde{x}_i-\tilde{x}_{i+1}}^{\infty} (-U_i(y))^n e^{-\beta U_i(y)} w(y) dy$$

and there exist  $c_1, c_2$  and  $c_3$  such that

$$\|[\partial_\beta^n T_i(\beta)w]\|_{L^\infty} \leq c_1 \|u_0\|_{L^\infty}$$

and

$$\|[\partial_\beta^n e^{-\beta U(x_n)} T_i(\beta)w]\|_{L^1} \leq c_2 \|u_0\|_{L^\infty}$$

and

$$\|[\partial_\beta^n T_i(\beta)w]'\|_{[-A, A]} \leq c_3 \|u_0\|_{L^\infty}.$$

There exists then  $c'$  such that  $\|[\partial_\beta^n T_i(\beta)w]\|_\nu \leq c' \|u_0\|_{L^\infty}$ . Then we have shown that  $\|[\partial_\beta^n T_i(\beta)w]\|_{L^\infty}$  is a uniformly bounded operator for  $\|\cdot\|_i \rightarrow \|\cdot\|_\nu$  and then for  $\|\cdot\|_i \rightarrow \|\cdot\|_{i+1}$ . We can now conclude the proof of Theorem 5.5. Moreover,

$$\begin{aligned} \int_{x-\tilde{x}_i-\tilde{x}_{i+1}}^{\infty} U_i(y)^n e^{-\beta U_i(y)} w(y) dy &\leq \int_{x-\tilde{x}_i-\tilde{x}_{i+1}}^{\infty} (Ay^2)^n e^{-\beta ay^2} w(y) dy \\ &\leq \|w\|_{L^\infty} \left(\frac{A}{\beta a}\right)^n \Gamma\left(n - \frac{1}{2}\right) \end{aligned}$$

We can then apply Theorem 5.24 which ends the proof of analyticity of the free energy of the classical Jellium model.  $\square$

## 5.4.2 Proof for the quantum Jellium model

### Decay of correlations, smoothness of the free energy.

**Definition 5.51.** For any  $f : \mathcal{C}([0, \beta]) \rightarrow \mathbb{R}$ , we define  $T_i : (\mathcal{C}([0, \beta]) \rightarrow \mathbb{R}) \rightarrow (\mathcal{C}([0, \beta]) \rightarrow \mathbb{R})$

$$T_i f(\gamma) = \int_{\mathcal{C}([0, \beta])} \mathbf{1}_{\forall t, \gamma(t) < \eta(t) + \delta_i} f(\eta) \nu_i(d\eta),$$

where  $\nu_i = \frac{1}{c} \int_{\mathbb{R}} \nu_{i,xx} dx$  with Radon Nikodym density  $\frac{d\nu_{i,xx}}{d\mu_{xx}}(\gamma) = e^{-\int_0^\beta U_i(\gamma(t)) dt}$ .

We recall the result concerning the homogeneous case that  $T_i$  is a compact operator on  $L^2(\mathcal{C}[0, \beta], \nu)$  with a unique largest eigenvalue  $\lambda_M > 0$ . Indeed,  $T$  is Hilbert-Schmidt:

$$\iint_{\mathcal{C}([0, \beta]) \times \mathcal{C}([0, \beta])} e^{-2 \int U(\gamma(t)) dt} \mathbf{1}_{\forall t, \gamma(t) \leq \eta(t) + \delta_i} d\nu(\gamma) d\nu(\eta) < \infty.$$

We have to check that  $\nu_i$  are bounded measures. For large  $y$  and  $B$  a brownian motion we have  $\mathbb{P}(\exists |B_t| > \frac{y}{2}) \leq 2e^{-|y|^2}$ . Then

$$\begin{aligned} \iint_{\mathbb{R} \times \Gamma} e^{-\int U_i(\gamma) dt} d\mu_{xx}(\gamma) dx &\leq \int_{\mathbb{R}} [\mu_{xx}(\inf_t \gamma(t) < \frac{x}{2})] + e^{-\beta U_i(\frac{x}{2})} dx \\ &\leq \int_{\mathbb{R}} c[e^{-(\frac{x}{2})^2} + e^{-\beta U(\frac{x}{2})}] dx < \infty \end{aligned}$$

The largest eigenvalue is unique because the operator is irreducible and we can apply the Krein Rutmann Theorem.

**Theorem 5.52.** *Let  $T$  be a bounded operator on a real Banach space whose spectral radius  $\rho(T)$  is a non degenerate eigenvalue with eigenvector  $v$ . Assume in addition that  $T' = T - \rho(T)vv^*$  has a spectral radius  $\rho(T') < \rho(T)$ . Then there exists a cone  $\mathcal{C}$  such that the operator is  $\mathcal{C}$ -positive and strictly contracting for  $d_{\mathcal{C}}$ .*

*Proof.* We can suppose that the larger eigenvalue is 1 and let  $u_0$  be its eigenvector. We construct

$$\mathcal{C} = \text{positive linear combinations of } \cup_n T^n \left( B(u_0, \epsilon(1 - \epsilon_n)) \right)$$

with  $\epsilon_n$  a strictly decreasing sequence. Because there exists  $N$  such that  $(T - u_0 u_0^*)^n$  is contracting for  $n \geq N$ ,

$$\mathcal{C} = \text{positive linear combinations of } \cup_{n \leq N_1} T^n \left( B(u_0, \epsilon(1 - \epsilon_n)) \right).$$

Indeed, let  $x \in B(u_0, \epsilon(1 - \epsilon_{N_1+1}))$ ,  $x = u_0 + y + s u_0$  with  $u_0^* y = 0$  and  $\|y\| \leq c\|\epsilon\|$  and  $s \leq c\|\epsilon\|$ . There exists  $N_1$  such that  $\|(T - u_0 u_0^*)^{N_1+1}\| \leq \frac{(1-\epsilon_0)}{2c}$ . Then

$$\begin{aligned} T^{N_1+1}(x) &= (u_0 u_0^*) x + (T - u_0 u_0^*)^{N_1+1}(x) \\ &= (1+s)u_0 + (T - u_0 u_0^*)^{N_1+1}(y) \in B((1+s)u_0, (1+s)\epsilon(1 - \epsilon_0)), \end{aligned}$$

because  $\|(T - u_0 u_0^*)^{N_1+1}(y)\| \leq \frac{\epsilon c(1-\epsilon_0)}{2c} \leq (1+s)\epsilon(1 - \epsilon_0)$ .

Let  $x \in \mathcal{C}$ , with  $u_0^* x = 1$ . Then  $x = \sum_{i=0}^{N_1} a_i x_i$ ,  $a_i \geq 0$  with  $x_i \in T^i(B(u_0, \epsilon(1 - \epsilon_i)))$ . Let us construct  $\alpha$  and  $\beta$  such that  $\alpha u_0 \leq T(x) \leq \beta u_0$ .

First because  $B(u_0, \epsilon(1 - \epsilon_0)) \subset \mathcal{C}$ , we choose  $\beta \leq \frac{\|T(x)\|}{\epsilon(1-\epsilon_0)}$ , and we immediatly have  $u_0 - \frac{1}{\beta} T(x) \geq 0$ .

Second for all  $i$ ,  $T(x_i) \in T^{i+1}(B(u_0, \epsilon(1 - \epsilon_i)))$  and therefore  $T(\frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)} x_i) \in T^{i+1}(B(\frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)} u_0, \epsilon(1 - \epsilon_{i+1})))$ . We have then

$$T\left(\frac{(1 - \epsilon_{i+1})}{(1 - \epsilon_i)} x_i\right) - \left(1 - \frac{(1 - \epsilon_{i+1})}{(1 - \epsilon_i)}\right) u_0 \in T^{i+1}(B(u_0, \epsilon(1 - \epsilon_{i+1})))$$

and also

$$T\left(\frac{(1 - \epsilon_{i+1})}{(1 - \epsilon_i)} x_i\right) \geq \left(1 - \frac{(1 - \epsilon_{i+1})}{(1 - \epsilon_i)}\right) u_0.$$

So with  $M = \max \frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)}$  and  $m = \min \frac{(1-\epsilon_{i+1})}{(1-\epsilon_i)}$  we have

$$MT(x) \geq m\left(\sum a_i\right)u_0$$

and we can conclude that

$$\frac{m \sum a_i}{M} u_0 \leq T(x).$$

□

Such a construction is stable by small compact perturbations.

**Proposition 5.53.** *There exists  $\delta_0 > 0$  such that for  $\delta T$  compact operator with  $\|\delta T\| \leq \delta_0$ .*

$$\Delta_{\mathcal{C}}((T + \delta T)(\mathcal{C})) < 2\Delta_{\mathcal{C}}(T(\mathcal{C}))$$

where  $\mathcal{C}$  is the cone constructed in Theorem 5.52

*In particular  $T + \delta T$  is a positive contracting operator for  $d_{\mathcal{C}}$ .*

*Proof.* We rewrite the proof of Theorem 5.52: We keep  $u_0$  because  $(T + \delta T)(u_0) \in B(u_0, (\epsilon(1 - \epsilon_0)))$ . First, it is enough to change  $\beta \leq \frac{\|(T + \delta T)(x)\|}{\epsilon(1 - \epsilon_0)}$  and we have  $u_0 - \frac{1}{\beta}(T + \delta T)(x) \geq 0$ . Second, we also have

$$\begin{aligned} & (T + \delta T)\left(\frac{(1 - \epsilon_{i+1})}{(1 - \epsilon_i)}x_i\right) - \delta T\left(\frac{(1 - \epsilon_{i+1})}{(1 - \epsilon_i)}x_i\right) \\ & \geq (T + \delta T)\left(\frac{(1 - \epsilon_{i+1})}{(1 - \epsilon_i)}x_i\right) - u_0 \frac{\|\delta T\left(\frac{(1 - \epsilon_{i+1})}{(1 - \epsilon_i)}x_i\right)\|}{\epsilon(1 - \epsilon_0)} \\ & \geq \left[1 - \frac{(1 - \epsilon_{i+1})}{(1 - \epsilon_i)}\right] - \frac{\delta_0}{\epsilon(1 - \epsilon_0)} \Big] u_0 \end{aligned}$$

and we can finish the proof as previously for  $\delta_0$  small enough. □

*Remark 5.54.* Actually we have that  $T(x) \in \overset{\circ}{\mathcal{C}}$  for all  $x \in \mathcal{C}$ . Indeed, as previously

$$\frac{m \sum a_i}{M} u_0 \leq T(x)$$

and then for any  $\|y\| \leq \frac{\epsilon(1 - \epsilon_0)}{M} m \sum a_i$  we have  $T(x) + y \geq \frac{m \sum a_i}{M} u_0 + y \geq 0$ .

The construction of the cone is simple enough that we can calculate the norm of Theorem 5.22. Because of the previous remark it is equivalent to the space norm.

**Proposition 5.55.** *There exists  $c > 0$  such that for any  $y$  in the projected space,*

$$c\|y\| \leq \|y\|_N \leq \frac{1}{c}\|y\|$$

where  $\|\cdot\|_N$  is the norm constructed in Proposition 5.22 for the cone  $\mathcal{C}$  in a neighborhood of  $T(x)$ .



*Proof.* Because of the previous remark,  $T(x) \in \mathring{\mathcal{C}}$ . Therefore there exists  $r > 0$  such that  $B(T(x), r) \subset \mathcal{C}$ . For any  $y$  we have

$$T(x) \left(1 - \frac{s\|y\|}{r}\right) \leq T(x) + sy \leq T(x) \left(1 + \frac{s\|y\|}{r}\right)$$

and we obtain  $d_{\mathcal{C}}(T(x) + sy, T(x)) \leq 2\frac{s\|y\|}{r} + o(\frac{s\|y\|}{r})$ . For the other direction,  $\mathcal{C} \subset \text{cone from } B(u_0, \frac{1}{2})$  (for  $\epsilon$  small). In addition  $T(x) + y \notin \mathcal{C}$  for  $\|y\| = 1$  and then  $\alpha T(x) \leq T(x) + sy$  implies  $\alpha \leq (1 - s)$  and so  $\|y\|_N \leq \|y\|$ .  $\square$

We can now finish the proof of Theorem 5.10

*Theorem 5.10.*  $T(\beta)$  is  $C^\infty$  for the norm  $\|\cdot\|$  and thanks to the previous proposition also for any norm constructed  $\|\cdot\|_N$  around  $T(\beta)(x)$ . We can then apply Theorem 5.23. The analyticity follows as well: if  $T(\beta)$  is analytic with coefficient bounded by  $r^n$  for the norm  $\|\cdot\|$  then the coefficients are bounded by  $cr^n$  for the norm  $\mathcal{N}$ .  $\square$

We focus now on the decay of correlation and the proof of Theorem 5.9. Recall that we want to prove that there exists  $\kappa < 1$  such that

$$|\rho_k(x_{i_1}, x_{i_2}, \dots, x_{i_k}) - \prod \rho_1(x_{i_i})| \leq C_k \kappa^{\min |i_i - i_{i+1}|}$$

*Theorem 5.9.* There exists  $\epsilon_0 > 0$  such that for  $f$  with  $\|f\|_{L^1} \leq \epsilon_0$ ,  $T_{i+1}(1+f)T_i$  are positive operator for the cone  $\mathcal{C}$ .

Indeed, let  $u \in \mathcal{C}$ , then  $T_i(u) \in L^\infty(\Gamma)$ , therefore  $fT_i(u) \in L^1$  with  $\|fT_i(u)\|_{L^1} \leq c\delta_0\|u\|$ , and finally  $\|T_{i+1}fT_i(u)\|_{L^\infty} \leq c'\delta\|u\|$ . Because  $T(\mathcal{C})$  is compact on the projected space and thanks to Remark 5.54, there exists  $r > 0$  such that for all  $i$ ,  $u \in \mathcal{C}$ ,  $B(T_{i+1}T_i(u), r\|u\|) \subset \mathcal{C}$ . As a conclusion  $T_{i+1}(1+f)T_i(u) = T_{i+1}T_i(u) + T_{i+1}fT_i(u) \in \mathcal{C}$ . We can now apply Theorem 5.33 with the same notation. We find

$$\begin{aligned} & e^{-2(k+1)R(\sum_{j=0}^k \kappa^{K_{j+1}-K_j})} \prod_{i=1}^k \rho_{K_i}(T_{i+1}(1+f_i)T_i) \\ & \leq \rho_{K_k, \dots, K_1}(T_{k+1}(1+f_k)T_k, \dots, T_1(1+f_1)T_1) \end{aligned}$$

and

$$\begin{aligned} & \rho_{K_k, \dots, K_1}(T_{k+1}(1+f_k)T_k, \dots, T_1(1+f_1)T_1) \\ & \leq e^{2(k+1)R(\sum_{j=0}^k \kappa^{K_{j+1}-K_j})} \prod_{i=1}^k \rho_{K_i}(T_{i+1}(1+f_i)T_i). \end{aligned}$$

The rest follows from a induction on  $k$  and this concludes the proof of Theorem 5.9.  $\square$

### 5.5 Proof of Proposition 5.3

If  $I' = \{i_1, i_2\}$ , then  $x_{i_1}$  and  $x_{i_2}$  are independent,  $\rho_2(x_{i_1}, x_{i_2}) = \rho(x_{i_1})\rho(x_{i_2})$  and then  $\rho^T(x_{i_1}, x_{i_2}) = 0$ . For larger a  $I'$ , we have

$$\begin{aligned}
& \sum_{I_1 \cup I_2 \cup \dots \cup I_r = I', l=1}^r \prod \rho_{I_l}^T((x_i)_{i \in I_l}) \\
&= \sum_{\substack{I_1 \cup I_2 \cup \dots \cup I_r = \{1, \dots, n\} \\ I_l \subset I \text{ or } I_l \subset J}} \prod_{l=1}^r \rho_{I_l}^T((x_i)_{i \in I_l}) \\
&= \left( \sum_{I_1 \cup I_2 \cup \dots \cup I_{r_1} = I \cap I'} \prod_{l=1}^{r_1} \rho_{I_l}^T((x_i)_{i \in I_l}) \right) \left( \sum_{I_1 \cup I_2 \cup \dots \cup I_{r_2} = J \cap I'} \prod_{l=1}^{r_2} \rho_{I_l}^T((x_i)_{i \in I_l}) \right) \\
&= \rho_{I' \cap I}((x_i)_{i \in I \cap I'}) \rho_{I' \cap J}((x_i)_{i \in J \cap I'})
\end{aligned}$$

and therefore

$$\begin{aligned}
& \rho_{I'}^T((x_i)_{i \in I'}) \\
&= \rho_{I' \cap I}((x_i)_{i \in I \cap I'}) \rho_{I' \cap J}((x_i)_{i \in J \cap I'}) - \sum_{I_1 \cup I_2 \cup \dots \cup I_r = I', l=1}^r \prod \rho_{I_l}^T((x_i)_{i \in I_l}) \\
&= 0.
\end{aligned}$$



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## Résumé

Cette thèse est consacrée à l'étude mathématique de divers systèmes de particules classiques et quantiques, en milieu désordonné. Elle comprend quatre travaux publiés ou soumis. Dans le premier nous fournissons une nouvelle formule permettant de prouver la localisation d'Anderson en une dimension d'espace et de caractériser la décroissance des fonctions propres à l'infini. Le second contient l'une des premières preuves de la localisation pour une infinité de particules en interaction, dans l'approximation d'Hartree-Fock. Le troisième est dédié au modèle d'Anderson soumis à une perturbation périodique en temps. Sous certaines conditions sur la fréquence d'oscillation nous prouvons l'absence de diffusion. Dans le dernier travail nous montrons la décroissance des corrélations pour le modèle du Jellium en une dimension dans un fond inhomogène, en utilisant la distance de Hilbert sur les cônes et le théorème de Birkhoff-Hopf.

## Mots Clés

Localisation d'Anderson, analyse multi-échelle, produit de matrices aléatoires, modèle de Hartree-Fock, Jellium inhomogène, théorème de Birkhoff-Hopf

## Abstract

This thesis is devoted to the mathematical study of some systems of classical and quantum particles, in a disordered medium. It comprises four published or submitted works. In the first one we provide a new formula allowing to prove Anderson localisation in one space dimension and to characterise the decay at infinity of the eigenfunctions. The second contains one of the first proofs of localisation for infinitely many particles in interaction, in the Hartree-Fock approximation. The third work is dedicated to the Anderson model in a time-periodic perturbation. Under certain conditions on the oscillation frequency we prove the absence of diffusion. In the last work we show the decay of correlations for the one-dimensional Jellium model in an inhomogeneous background, using the Hilbert distance on cones and the Birkhoff-Hopf theorem.

## Keywords

Anderson localisation, multi-scale analysis, product of random matrices, Hartree-Fock model, inhomogeneous Jellium, Birkhoff-Hopf theorem