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Instances of uniqueness restoration and equilibria selection in mean field games

Rinel Foguen Tchuendom

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THÈSE DE DOCTORAT

Exemples de restauration d'unicité
et de sélection d'équilibres dans les
jeux à champ moyen

Rinel FOGUEN TCHUENDOM

Laboratoire J.-A. Dieudonné

**Présentée en vue de l'obtention
du grade de docteur en Mathématiques
d'Université Côte d'Azur**

Dirigée par : François DELARUE

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Devant le jury, composé de :

Mme. Mireille BOSSY

M. Luciano CAMPI

M. Jean-François CHASSAGNEUX

M. François DELARUE

M. Romuald ELIE

M. Anthony REVEILLAC

Exemples de restauration d'unicité et de sélection d'équilibres dans les jeux à champ moyen

Jury :

Rapporteurs

M. Jean-François CHASSAGNEUX, Professeur, Université Paris-Diderot
M. Romuald ÉLIE, Professeur, Université Marne-la-Vallée

Examineurs

Mme. Mireille BOSSY, Directrice de Recherche, INRIA Sophia Antipolis - Méditerranée
M. Luciano CAMPI, Professeur, London School Economics
M. François DELARUE, Professeur, Université Nice Sophia Antipolis
M. Anthony RÉVEILLAC, Professeur, INSA Toulouse

Résumé.

Ce manuscrit a pour objectif de présenter plusieurs résultats de restauration d'unicité et de sélection d'équilibres dans les jeux à champ moyen.

La théorie des jeux à champ moyen a été initiée dans les années 2000 par deux groupes de chercheurs, Lasry et Lions en France, et Huang, Caines et Malhamé au Canada. L'objectif de cette théorie est de décrire les équilibres de Nash dans des jeux différentiels stochastiques incluant un grand nombre de joueurs interagissant les uns avec les autres à travers leur mesure empirique commune et présentant suffisamment de symétrie.

Si l'existence d'équilibres dans les jeux à champ moyen est maintenant bien comprise, l'unicité reste connue dans un nombre très limité de cas. A cet égard, la condition la plus connue est celle dite de monotonie, due à Lasry et Lions.

Dans cette thèse, nous démontrons, que pour une certaine classe de jeux à champ moyen, l'unicité peut être rétablie à l'aide d'un forçage aléatoire des dynamiques, communs à tous les joueurs. Un tel forçage est appelé "bruit commun". Nous montrons également que, dans certains cas, il est possible de sélectionner des équilibres en l'absence de bruit commun en faisant tendre le bruit commun vers zéro. Enfin, nous montrons comment ces résultats s'appliquent à des problèmes de type "principal-agents", avec un grand nombre d'agents en interaction.

Mots clés : jeux à champ moyen, bruit commun, restaurer l'unicité, limite zéro bruit.

Abstract.

The purpose of this thesis is to present several results on the restoration of uniqueness and selection of equilibria when uniqueness fails in mean field games.

The theory of mean field games was initiated in the 2000s by two groups of researchers, Lasry and Lions in France, and Huang, Caines, and Malhamé in Canada. The aim of this theory is to describe the Nash equilibria in stochastic differential games involving a large number of players interacting with each other through their common empirical measure, under sufficient symmetry hypothesis.

If the existence of equilibria in mean field games is now well understood, uniqueness remains known in a very limited number of cases. In this respect, the most well-known condition is the monotony hypothesis, due to Lasry and Lions.

In this thesis, we demonstrate that for a certain class of mean field games, uniqueness can be restored by means of a random and common forcing, acting on all the players. Such a forcing is called "common noise". We also show that in some cases it is possible to select equilibria in the absence of common noise by letting the common noise tend towards zero. Finally, we show how these results apply to "principal-agent" problems, with a large number of interacting agents.

Keywords : mean field games, common noise, restoring uniqueness, zero noise limit.

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“Ce qui est difficile prend un peu de temps; ce qui est impossible en prend un peu plus.”

Fridtjof Nansen.

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Pour mes grands-mères ; TEUKAM et PUESSI.

Introduction

1 Grandes populations en interaction : modélisation par des jeux

1.1 Les enjeux de la modélisation

Modéliser de grandes populations en interaction a une importance croissante dans de nombreux domaines. Dans le monde académique, les sciences appliquées en font usage pour accompagner les expérimentations. Et dans le monde économique, la recherche et le développement s'en servent pour accélérer l'innovation.

Les individus de ces populations peuvent être des particules (en physique, biologie etc), des serveurs ou des ordinateurs (en informatique, cryptographie, télécommunications, etc), des animaux ou des plantes (en zoologie, botanique, théorie de l'évolution, etc), ou enfin des personnes humaines (sciences économiques, sciences sociales, sciences politiques, etc). Ainsi, les populations peuvent être des nuées d'oiseaux, des bancs de poissons, des fluides en mouvement, des agents sur un marché de l'énergie ou des matières premières, le public d'un spectacle, les neurones d'un cerveau, des réseaux de serveurs, etc...

On dit d'une population qu'elle est homogène si tous les individus qui la composent sont similaires dans un sens à définir. Dans le cas contraire, on parle d'une population hétérogène. Dans les deux cas, les sources d'interactions sont appelées exogènes quand elles sont situées en dehors des populations, et endogènes dans le cas contraire. Pour une population donnée, on désigne par sa *taille* le nombre d'individus qui la composent.

La plupart des questions essentielles en sciences appliquées ou en recherche et développement ont, à des niveaux variés, un lien avec des grandes populations en interaction. Malgré l'avancée de la technologie et des moyens de calculs, les scientifiques et les ingénieurs font presque toujours face à des difficultés quant à la complexité des problèmes liés à l'analyse de ces grandes populations.

La modélisation, à travers notre intuition, fait partie intégrante de notre compréhension des problèmes auxquels nous nous attaquons. Il s'agit de la rendre aussi fidèle que possible aux phénomènes observés, mais aussi d'être capable d'en étudier les ressorts avec les outils théoriques et numériques à disposition. A cet égard, il est à craindre qu'un problème mal modélisé soit plus difficile à résoudre qu'un problème bien modélisé, car une mauvaise modélisation trahit, bien souvent, une mauvaise compréhension du problème. Pour autant, le souci de rendre les choses aussi accessibles que possible doit conduire, dans l'effort de modélisation, à essayer de réduire la complexité du modèle.

Quoiqu'il en soit, il est nécessaire de conserver une forme de prudence, voire de modestie. Quand il s'agit de modélisation, il est toujours bon de se souvenir de l'adage attribué à Box et Draper: "Tous les modèles sont faux mais certains sont utiles". Ils sont tous faux en raison des hypothèses de simplification nécessaires à la modélisation, et sont utiles lorsqu'ils permettent de mettre en lumière des phénomènes nouveaux, de confirmer ou d'infirmer des intuitions,

ou d'expliquer des résultats d'expériences incohérents avec la théorie en vigueur. Quand la modélisation et l'analyse qui en est faite sont pertinentes, la compréhension du problème réel s'en trouve améliorée et un pas de plus est fait vers sa résolution.

Pour modéliser les grandes populations en interaction, il est courant d'avoir recours aux simplifications suivantes:

- Simplification de l'environnement au sein duquel la population évolue et de la distribution de la population,
- Simplification des sources et des types d'interaction entre les individus de la population,
- Simplification de la dynamique et des objectifs des individus de la population.

1.2 Des modèles champ moyen aux jeux

Pour traiter les grandes population en interaction, une classe de modèles assez répandue est la classe des modèles à champ moyen. Ces modèles supposent que les interactions entre les individus sont régies par une seule donnée sur l'état global de la population (en général une fonction de la mesure empirique de l'état de la population). Cette forme d'interaction est dite faible ou de type champ moyen.

La théorie mathématique supportant ces modèles est vaste et tire son origine de l'étude physique de systèmes de particules liés à la dynamique des fluides [68, 67]. Cette théorie s'est imposée en quelques années et est devenue une des théories mathématiques les plus répandues dans l'étude des grandes populations. Elle est construite sur l'idée que, lorsque la limite est prise sur une infinité (on dit un "continuum") d'individus, les propriétés macroscopiques des modèles champ moyen peuvent être résumées aux propriétés principales d'une seule particule typique, encore appelée particule représentative.

Néanmoins, le reproche est souvent fait aux modèles à champ moyen de se limiter à des situations où les individus n'ont pas d'action propre sur leur dynamique. Ainsi le modèle encode, *a priori*, la réaction des individus face aux aléas (de source interne ou externe) ; dans le cadre d'un "continuum", l'encodage a lieu au niveau de la dynamique de l'individu représentatif. A l'arrivée, il s'agit donc plus une modélisation d'automates/de particules qu'une modélisation d'individus *rationnels*.

La théorie des jeux à champ moyen vise à pallier ce défaut et propose d'échapper à cette contradiction en laissant une capacité de décision à l'individu représentatif. L'individu représentatif est supposé prendre ses décisions de façon rationnelle, ce dont rend compte la théorie des jeux à champ moyen en incluant un critère supplémentaire d'utilité dans la modélisation de la population de taille infinie.

La théorie des jeux a été introduite par Von Neumann et Morgenstern [97] et son traitement avec un continuum de joueurs remonte aux travaux, en économie, de Aumann et Schmeidler [92, 3, 4]. C'est aussi en économie qu'on trouve des travaux de Bergin et Bernhardt sur les jeux avec un continuum de joueurs soumis à un bruit commun [11].

La théorie des jeux à champ moyen exploite le fait que les concepts d'équilibres utilisés dans la théorie des jeux demeurent, sous certaines hypothèses, pertinents dans le cadre d'un continuum de joueurs. De façon plus précise, la théorie des jeux à champ moyen propose une structure de jeu dans laquelle chaque individu se voit assigner un critère d'utilité, qu'il souhaite optimiser, l'utilité dépendant de l'état global de la population. L'interaction globale peut transparaître, soit dans la dynamique des joueurs, soit dans leur critère d'utilité. Afin

d’optimiser le critère d’utilité qui lui est attribué, l’individu peut ajuster sa stratégie (ou son contrôle), dont sa propre dynamique et sa propre utilité dépendent, dans un certain espace de stratégies admissibles. La capacité qui est donnée à l’individu d’optimiser son utilité propre rend compte d’une aptitude à prendre des décisions ; au lieu de parler d’individu, il est de fait bienvenu de parler de joueur (parfois, on emploiera aussi le mot agent). Comme l’interaction est de type champ moyen, les joueurs du continuum ne sont influencés que par l’état macroscopique du système. En particulier, l’influence de chaque joueur, pris de façon isolée, sur les autres est noyée dans le continuum et est de fait infinitésimale ; de plus, la loi statistique de l’état du joueur peut différer de l’état de la population, l’égalité n’étant en fait seulement vérifiée qu’à l’équilibre.

Avec la croissance démographique que le monde connaît, la théorie des jeux à champ moyen a un fort potentiel pour modéliser et proposer des solutions aux problèmes urgents de développement, croissance inclusive, et gestion des ressources alimentaires. En réalité, elle possède un spectre d’applications très large : en économie, comme cela été mentionné, dans la production/consommation de ressources naturelles ou d’énergie, en épidémiologie, vaccinologie et santé publique, pour le dimensionnement d’espaces publics...

1.3 Un bref état de l’art et un court aperçu de la thèse

La théorie des jeux à champs moyen est relativement jeune. Elle a été introduite, simultanément, vers 2006, par Lasry et Lions [75, 76, 77, 78] et Caines, Huang et Malhamé [63, 65, 62, 64]. Depuis, le sujet a attiré énormément d’attention, beaucoup d’efforts ayant été consacrés à l’étude de l’existence d’équilibres.

En l’absence de bruit commun, les premiers résultats d’existence et d’unicité ont fait l’objet de cours de Lions au *Collège de France*, voir [81], qu’on peut également retrouver sous forme écrite dans les notes de Cardaliaguet [18]. Plus tard, en présence du bruit commun, l’existence et l’unicité ont été prouvées *via* l’étude d’une équation dite *maîtresse* (ou “master equation” en anglais), qui est une équation aux dérivées partielles posée sur l’espace des mesures de probabilité et satisfaite par la fonction valeur du jeu, voir par exemple [80], ou, plus récemment, avec des outils mathématiques d’analyse des équations aux dérivés partielles stochastiques couplées et progressives-rétrogrades (dites FBSPDE en anglais), voir par exemple l’article de Cardaliaguet, Delarue, Lasry et Lions [19]. C’est à partir de 2012 que des résultats similaires d’existence et d’unicité ont aussi été prouvés par Carmona et Delarue avec des méthodes probabilistes fondées sur des équations différentielles stochastiques couplées et progressives-rétrogrades (dites FBSDE) de type McKean-Vlasov, voir [22, 21]. L’utilisation des équations “FBSDE” de type McKean-Vlasov dans l’étude des jeux à champ moyen est présentée en détail dans le livre en deux volumes écrit par Carmona et Delarue [23, 24].

On peut aussi trouver dans la littérature des travaux sur la convergence des jeux à N joueurs lorsque la taille de la population tend vers l’infini. Des concepts de convergence faible ont été développés par Fischer et Lacker [46, 72] pour l’étude du comportement asymptotique des équilibres sur les stratégies en boucle ouverte. Dans le cadre des jeux à champs moyen dits linéaires-quadratiques, des résultats de convergence ont aussi été obtenus par Bardi et Priuli [6]. Enfin, l’étude de la master equation [25, 19, 30] a fourni de nouveaux outils pour justifier la convergence dans le cas d’équilibres construits sur des stratégies en boucle markovienne. A cet égard, il faut comprendre que la preuve de la convergence des équilibres tient lieu de justification théorique de la formulation des jeux à champ moyen ; sur un plan mathématique, il s’agit principalement de réécrire le problème de convergence sous une forme proche de celle

rencontrée dans l'étude des modèles de champ moyen sans contrôle. Ceci étant, une autre façon de justifier l'intérêt de la théorie des jeux à champ moyen est de montrer (et c'est en réalité plus facile à faire) que toute solution du jeu, posé sur le continuum, permet de construire un "presque-équilibre" du jeu à N joueurs.

Cette thèse se place dans un cadre où l'existence de solutions aux jeux à champ moyen est garantie mais l'unicité ne l'est peut-être pas. Dans ce cadre, le manuscrit apporte des réponses aux questions suivantes:

1. Existe-t-il des jeux à champ moyen pour lesquels le bruit commun restaure l'unicité ?
2. Peut-on illustrer, via une limite zéro bruit commun, un phénomène de Peano dans les jeux à champ moyen ?
3. Pour les jeux à champ moyen avec de multiples solutions, comment sélectionner une solution ?

Pour ce qui est des deux premières questions, on donne ici des réponses positives. Pour cela, l'idée est de s'inspirer de modèles pratiques qui ont été déjà étudiés et plus précisément de se concentrer sur une classe de modèles de jeux à champ moyen dits linéaires-quadratiques (c'est à dire que les dynamiques sont à coefficients linéaires en le contrôle et la variable d'état et que les coefficients de la fonctionnelle d'utilité sont quadratiques en les mêmes paramètres) avec des interactions via la moyenne de la population (i.e., le premier moment de la distribution de la population). L'avantage est que, pour cette classe de jeux à champ moyen, les équilibres peuvent être assez aisément caractérisés, voire même mis sous une forme semi-explicite. L'analyse asymptotique des équilibres lorsque le bruit commun tend vers 0 n'est menée que dans le cadre unidimensionnel et pour un choix particulier des coefficients. Bien que limitée à une classe d'exemples spécifiques, cette analyse asymptotique donne pour autant une vision claire des techniques de sélection envisageables et surtout des paradoxes qui peuvent émerger selon les règles de sélection qui ont été choisies. Ces questions sont ensuite réabordées, dans le dernier chapitre de la thèse dans le cadre de jeux "principal-agents".

Dans le reste de ce chapitre introductif, on va préciser la forme typique de la dynamique des individus dans une grande population en interaction, avant d'introduire la structure de jeu, aussi bien sur un nombre fini de joueurs que sur un continuum, et de définir les notions de solutions utilisées tout au long du manuscrit. Ensuite, on présentera de façon formelle les outils disponibles pour construire des solutions et on discutera des résultats d'unicité existants dans la littérature. Enfin, on donnera un résumé des résultats de la thèse dans la dernière section.

2 Quelques éléments du jeu à N joueurs

2.1 Dynamiques non-contrôlées

Pour une modélisation dynamique d'un individu (non-contrôlé) au sein d'une grande population en interaction, on utilise ici les outils du calcul différentiel stochastique, en particulier les équations différentielles stochastiques (dites EDS en français ou SDE en anglais).

Pour une taille de population, notée $N \in \mathbb{N} \setminus \{0\}$, on considère N individus qui évoluent de façon dynamique dans un espace d'état. Les états peuvent représenter : la position de l'individu, sa richesse, sa consommation, etc... L'espace des états est souvent un domaine de \mathbb{R}^d (ou même \mathbb{R}^d lui-même), où $d \in \mathbb{N} \setminus \{0\}$.

Pour chaque individu, repéré par un indice $i \in \{1, 2, \dots, N\}$, on désigne la valeur de son état au temps $t \in [0, T]$ par X_t^i . On note ici $T > 0$ l'horizon de temps du modèle sous-jacent (T est parfois infini dans la littérature, mais il sera systématiquement supposé fini dans ce manuscrit). L'évolution en temps continu de la population des N joueurs est décrite par les équations suivantes :

$$\begin{aligned} \forall t \in [0, T], \quad \forall i \in \{1, 2, \dots, N\}, \\ dX_t^i = B(X_t^i, \bar{\mu}_t^N)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i + \sigma_0(X_t^i, \bar{\mu}_t^N)dB_t, \end{aligned} \quad (1)$$

où $\bar{\mu}_t^N$ désigne la distribution empirique

$$\bar{\mu}_t^N := \frac{1}{N} \sum_{k=1}^N \delta_{X_t^k}.$$

De façon générale, les conditions initiales X_0^1, \dots, X_0^N sont supposées indépendantes et de même loi :

$$\lambda_0 := \mathcal{L}(X_0^i), \quad i \in \{1, \dots, N\}.$$

Il s'agit d'une hypothèse simplificatrice de modélisation.

Dans (1), les processus B, W^1, W^2, \dots, W^N sont $N + 1$ mouvements browniens indépendants définis sur un espace de probabilité complet $(\Omega, \mathbb{F}, \mathbb{P})$ muni d'une filtration $(\mathcal{F}_t)_{t \in [0, T]}$ satisfaisant les conditions usuelles. Ces processus stochastiques modélisent, respectivement, le bruit commun à tous les individus et les bruits intrinsèques à chaque individu (parfois aussi appelés bruits idiosyncratiques, propres ou particuliers).

Notons que $\bar{\mu}_t^N$ est une mesure (aléatoire) de probabilité qui décrit l'état global de la population à l'instant t . Il s'agit en fait de la mesure empirique des états de tous les individus à l'instant t et cette mesure empirique régit les interactions entre les individus.

Pour chaque individu, les fonctions B, σ et σ_0 ont pour arguments une variable d'état dans \mathbb{R}^d et une mesure de probabilité sur \mathbb{R}^d . Le plus souvent, l'argument mesure est pris dans l'espace $\mathcal{P}_p(\mathbb{R}^d)$, avec p égal à 1 ou 2, des mesures de probabilité ayant un p -moment fini.

La modélisation de la dynamique des individus (1) peut être simplifiée en supposant, par exemple, que

1. $\lambda_0 = \delta_{x_0}$, pour un certain $x_0 \in \mathbb{R}^d$,
2. σ, σ_0 sont des constantes positives,
3. B est une fonction d'interaction par paire de la forme

$$B(X_t^i, \bar{\mu}_t^N) = \frac{1}{N} \sum_{j=1}^N b(X_t^i, X_t^j),$$

où le domaine de définition de la fonction b est $\mathbb{R}^d \times \mathbb{R}^d$.

Le cas échéant, l'équation du i ème joueur dans (1) devient :

$$dX_t^i = \frac{1}{N} \sum_{j=1}^N b(X_t^i, X_t^j) dt + \sigma dW_t^i + \sigma_0 dB_t, \quad X_0^i = x_0.$$

Lorsque la fonction b est linéaire en la seconde variable, la dépendance champ moyen se réduit à une dépendance par rapport à la moyenne empirique des états des individus de la population. C'est à dire que B se réduit à une fonction de X_t^i et de \bar{X}_t^N , avec

$$\bar{X}_t^N := \int_{\mathbb{R}^d} x d\bar{\mu}_t^N(x) = \frac{1}{N} \sum_{k=1}^N X_t^k, \quad \forall t \in [0, T].$$

L'élément le plus remarquable dans la dynamique des individus (1) est la symétrie imposée au travers du choix des fonctions B , σ et σ_0 , qui sont les mêmes pour tous les joueurs. En particulier, la distribution des individus est invariante par rapport à toutes les permutations possibles : les individus sont dits échangeables. A ce titre, le théorème de De Finetti suggère de considérer la limite de la mesure empirique des individus quand $N \rightarrow \infty$. Asymptotiquement, l'influence d'un individu quelconque, disons X^1 , sur l'état global de la population est en fait de moins en moins forte ; intuitivement, les joueurs ne sont plus corrélés que par la seule réalisation du bruit commun B . En particulier, les joueurs deviennent asymptotiquement indépendants quand $\sigma_0 = 0$: cette propriété est formalisée par le concept de propagation du chaos développé par Sznitman [94] ; le cas échéant, la mesure empirique est identifiée, asymptotiquement, avec la loi théorique de chacun des individus. Lorsque $\sigma_0 \neq 0$, l'indépendance asymptotique n'est vraie que conditionnellement à la réalisation du bruit commun et la mesure empirique est identifiée, dans le régime " $N = \infty$ ", avec la loi conditionnelle de chacun des individus sachant B .

2.2 Modélisation des jeux

En réalité, la modélisation dynamique de grandes populations par des EDS de type (1) est trop simple au regard de certaines applications. En effet, les états des individus sont déterminés *a priori*, alors que, dans la plupart des exemples pratiques et en particulier en sciences économiques ou sociales, les modèles utilisés reconnaissent aux individus la capacité de choisir une stratégie, susceptible d'influencer leur dynamique propre.

De façon générique, on suppose dans le manuscrit que les individus effectuent leurs choix de façon rationnelle, ce dont on rend compte en supposant que le choix des stratégies est soumis, pour chaque individu, à un critère d'optimisation d'utilité. Bien qu'il revienne au même, sur un plan mathématique, de "minimiser" ou de "maximiser", on comprendra, dans la suite de l'introduction, le mot "optimiser" au sens de "minimiser".

Dans la suite, on désigne par \mathbb{A} l'ensemble des stratégies admissibles pour chacun des individus. Pour respecter les propriétés de symétrie discutées dans le paragraphe précédent, \mathbb{A} est supposé commun à tous les individus : les conditions d'admissibilité recouvrent aussi bien des conditions d'observabilité (ou de mesurabilité) que des conditions sur les valeurs prises par les stratégies ; de façon générique, les valeurs admissibles sont identifiées aux éléments d'un domaine A (le plus souvent convexe fermé) d'un espace de dimension finie \mathbb{R}^m . Dans ce cadre, lorsque les individus $i = 1, 2, \dots, N$ choisissent des stratégies $((\alpha_t^i)_{t \in [0, T]}, i = 1, 2, \dots, N)$,

chacune dans \mathbb{A} , l'évolution en temps continu de la population est modélisée à l'aide du système d'EDS : $\forall t \in [0, T]$, et $\forall i \in \{1, 2, \dots, N\}$,

$$dX_t^i = B(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i + \sigma_0(X_t^i, \bar{\mu}_t^N)dB_t, \quad (2)$$

où, comme précédemment, $\bar{\mu}_t^N$ désigne la distribution empirique

$$\bar{\mu}_t^N := \frac{1}{N} \sum_{k=1}^N \delta_{X_t^k},$$

et les conditions initiales X_0^1, \dots, X_0^N sont supposées indépendantes et de même loi :

$$\lambda_0 := \mathcal{L}(X_0^i), \quad i \in \{1, \dots, N\}.$$

Dans un souci de simplicité, on a supposé ici que les choix des individus ne se manifestaient que dans le coefficient de dérive B . En réalité, on aurait pu également autoriser les deux coefficients σ et σ_0 à dépendre du contrôle, mais cela n'aurait pas eu d'intérêt dans notre cadre. Ce qui est important, c'est de comprendre que les choix de chacun des individus influencent la dynamique des autres individus ravers le champ moyen, mais que cette influence diminue au fur et à mesure que grandit la taille de la population.

A chaque individu est associé un critère de coût qu'il souhaite optimiser. On désigne par

$$J^i(\alpha^k, k = 1, 2, \dots, N), \quad \forall i = 1, 2, \dots, N,$$

la fonctionnelle de coût de l'individu i . Comme la notation l'indique (et il s'agit là d'un point fondamental), le critère d'utilité de chaque individu dépend non seulement de sa propre stratégie mais aussi des stratégies des autres individus.

De façon générale, les fonctionnelles $(J^i)_{i=1, \dots, N}$ sont données sous la forme de deux coûts, en l'occurrence : un coût instantané, mis sous la forme d'une fonction F de l'état de l'individu i , de l'état de la population et de la valeur du contrôle du joueur i , et un coût terminal, mis sous la forme d'une fonction G de l'état terminal de l'individu i et de l'état terminal de la population. Plus explicitement,

$$J^i(\alpha^1, \dots, \alpha^i, \dots, \alpha^N) := \mathbb{E} \left[\int_0^T F(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt + G(X_T^i, \bar{\mu}_T^N) \right]. \quad (3)$$

Il est essentiel de remarquer que les fonctions F et G sont les mêmes pour tous les individus !

L'objectif de chacun des individus est de minimiser sa fonctionnelle de coût. Mais, comme les fonctionnelles de chacun dépendent des choix des autres, les individus sont en réalité liés entre eux : la modélisation prend la forme d'un jeu entre les individus et les individus sont désormais des joueurs. Pour autant, en raison de la forme des interactions entre les joueurs, l'influence exercée par un joueur sur un autre est de plus en plus faible lorsque N tend vers l'infini ; on reviendra sur ce point dans la section suivante.

Il nous reste à expliciter le type d'espace dans lequel les joueurs ont le droit de choisir leur stratégies et de quelles informations ils peuvent se servir pour opérer ces choix. Cela nous permettra d'introduire le concept d'équilibre de Nash dans le paragraphe suivant. De

façon générale, \mathbb{A} est la classe des processus à valeurs dans A et progressivement mesurables par rapport à la filtration engendrée par les conditions initiales et par les bruits. Ce point sera précisé dans les chapitres suivants, au cours desquels on se focalisera en réalité sur :

1. des états uni-dimensionnels,
2. des fonctionnelles de coût quadratiques en la variable d'état et en le contrôle,
3. des interactions à travers la seule moyenne empirique du système (au lieu de la distribution empirique).

De fait, on se donne des fonctions réelles f et g de la variable réelle. Pour chaque joueur $i = 1, 2, \dots, N$, la fonctionnelle de coût à optimiser est de la forme :

$$J^i(\alpha^1, \dots, \alpha^i, \dots, \alpha^N) := \mathbb{E} \left[\frac{1}{2} \int_0^T \left((\alpha_t^i)^2 + [X_t^i + f(\bar{X}_t^N)]^2 \right) dt + \frac{1}{2} [X_T^i + g(\bar{X}_T^N)]^2 \right], \quad (4)$$

où

$$\bar{X}_t^N := \frac{1}{N} \sum_{i=1}^N X_t^i.$$

On dira, avec un léger abus de langage, que ces critères sont quadratiques, car les coûts instantanés et terminaux ne dépendent des états propres de chacun des joueurs et de leur contrôle que de façon quadratique. Il s'agit d'un léger abus de langage car, les fonctions f et g étant générales, les coûts dépendent de l'état collectif de façon non-quadratique. On verra dans la suite que cette structure est néanmoins suffisante pour n'avoir à étudier, asymptotiquement, que des problèmes d'optimisation quadratiques.

2.3 Equilibre de Nash

Dans les paragraphes précédents, on a proposé une modélisation d'une grande population sous la forme d'un système d'individus, rebaptisés "joueurs", capables d'agir sur leur propre dynamique. Quelle que soit la taille de la population, la difficulté tient au fait que les joueurs cherchent à minimiser leurs coûts simultanément. Il s'agit de fait de donner une notion de consensus. Pour cela, on s'appuie sur le fait que les individus effectuent des choix de façon non-coopérative. A cet égard, on observe que, sans coopération entre les individus, aucun joueur ne doit pouvoir diminuer son coût propre en déviant de façon unilatérale d'un état consensuel. Ceci correspond au concept d'équilibre de Nash, défini comme suit :

Definition 1 (Equilibre de Nash). *Une collection de stratégies*

$$(\alpha^{*,1}, \dots, \alpha^{*,i}, \dots, \alpha^{*,N}) \in \mathbb{A}^N$$

forme un équilibre de Nash si

$$\forall i = 1, 2, \dots, N, \quad \forall \alpha^i \in \mathbb{A}, \quad J^i(\alpha^{*,1}, \dots, \alpha^{*,i}, \dots, \alpha^{*,N}) \leq J^i(\alpha^{*,1}, \dots, \alpha^i, \dots, \alpha^{*,N}).$$

La définition 1 est en fait la définition des équilibres de Nash sur des stratégies en boucle ouverte. Concrètement, cela signifie que, dans la condition de consensus ci-dessus, les joueurs $j \in \{1, \dots, N\} \setminus \{i\}$ ne réagissent pas au changement de stratégie du joueur i lorsqu'il dévie

de l'équilibre. D'autres notions de stratégie, dites en boucle fermée, sont également utilisées dans la littérature, mais on ne les utilisera pas dans la suite du manuscrit.

En particulier, la notion d'équilibre de Nash peut être reformulée comme suit. Lorsque toutes les stratégies $(\alpha^1, \dots, \alpha^{i-1}, \alpha^{i+1}, \dots, \alpha^N)$ sont fixées, on appelle "meilleure réponse" du joueur i une stratégie $\hat{\alpha}^i$ vérifiant :

$$\forall i = 1, 2, \dots, N, \forall \alpha \in \mathbb{A}, \quad J^i(\alpha^1, \dots, \hat{\alpha}^i, \dots, \alpha^N) \leq J^i(\alpha^1, \dots, \alpha, \dots, \alpha^N).$$

De fait, un équilibre de Nash est un point fixe des fonctions de "meilleure réponse" des joueurs $1, \dots, N$.

Par exemple, si on revient aux critères d'utilité quadratiques explicités ci-dessus et si on remplace, dans la fonctionnelle du joueur i , la moyenne empirique sur tous les joueurs par la moyenne empirique de "tous les autres" joueurs, i.e.

$$\bar{X}_t^{N,-i} := \frac{1}{N-1} \sum_{j \neq i} X_t^j,$$

alors, la stricte convexité assure que, pour chaque joueur, la fonction de meilleure réponse est définie de façon univoque. On reviendra sur ce point dans la suite.

3 Les jeux à champ moyen

3.1 Motivation

On peut résumer l'approche proposée dans la section précédente comme suit :

1. Au lieu d'imposer une dynamique du type (1) pour modéliser la dynamique des individus, une forme de liberté a été octroyée : il s'agit du système (2).
2. Lorsque les joueurs sont rationnels, les états d'équilibre de (2) sont donnés par la définition 1.
3. Une fois déterminés les équilibres de Nash, il est possible de réinjecter les N -uplets $(\alpha^{*,1}, \dots, \alpha^{*,N})$ dans (2) pour obtenir un système du type (1).

Autrement dit, la modélisation s'appuie sur la résolution d'un jeu différentiel stochastique à N joueurs, dont l'analyse passe par la résolution d'une condition de point fixe posée sur les N fonctions de "meilleure réponse".

Malheureusement, les jeux sont connus pour être difficile à résoudre, surtout lorsque le nombre de joueurs est grand : ceci est aisément compréhensible ; le nombre de fonctions de "meilleure réponse" est égal à N et le nombre d'arguments de chacune est également égal à N . La complexité devient donc redoutable lorsque N croît. Pour autant, on peut trouver, dans la littérature de la théorie des jeux différentiels stochastiques et du contrôle optimal, des résultats d'existence de solutions obtenus avec des méthodes d'EDP ou des méthodes EDS progressives-rétrogrades, voir par exemple [20, 8]. Même si les équilibres de Nash ne sont quasiment jamais explicites, il est possible, dans le cas des jeux à critères quadratiques et à dynamique linéaire, dits jeux linéaires-quadratiques, d'en comprendre la structure de façon assez fine ; pour comparaison avec les paragraphes précédents, cela sous-entend, le cas échéant, que f et g dans (4) sont également quadratiques !

C'est précisément ce problème de complexité qui justifie de prêter une attention particulière à la limite $N \rightarrow \infty$. Tout le pari est en effet de parvenir à diminuer la complexité liée à la résolution du jeu en bénéficiant des propriétés de symétrie du modèle pour utiliser des arguments de *propagation du chaos* propres à la théorie des modèles à champ moyen. Le jeu obtenu, suite au passage à la limite $N \rightarrow \infty$, est appelé *jeu à champ moyen*.

Intuitivement, la *propagation du chaos* dit, qu'à limite, le jeu à champ moyen devrait se résumer à l'étude d'un seul joueur représentatif en interaction avec une population à travers la seule description statistique de cette dernière ; le joueur n'a aucune influence sur la population. Cette structure est très proche des jeux avec un continuum d'agents. Parce que chaque joueur est influencé par l'état global de la population et qu'aucun d'entre eux n'a d'influence sur l'état global, le joueur est amené à résoudre un problème d'optimisation standard : il s'agit de minimiser sa fonctionnelle de coût dans l'environnement formé par la population ; le cas échéant, les minima de la fonctionnelle d'optimisation doivent être compris comme les images de la fonction de "meilleure réponse" et un équilibre de Nash n'est rien d'autre qu'un état de la population stable par la fonction de "meilleure réponse".

Pour clarifier les choses, rappelons que: $T > 0$ est le temps terminal du jeu, $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ est un espace de probabilité satisfaisant les conditions usuelles sur lequel sont définis les mouvements Browniens $W = (W_t)_{t \in [0, T]}$ et $B = (B_t)_{t \in [0, T]}$, qui modélisent respectivement le bruit individuel agissant sur la dynamique du joueur représentatif et le bruit commun agissant sur la dynamique du continuum de joueurs. Le plus souvent, \mathbb{F} est donnée par la complétion de la filtration engendrée par \mathcal{F}_0 et par les bruits. Par ailleurs, X_0 est une variable aléatoire de carré intégrable (à valeurs dans l'espace d'état \mathbb{R}^d) et \mathcal{F}_0 -mesurable. Fixons enfin (ne serait-ce que pour illustrer) $\mathbb{A} := \mathcal{H}^2([0, T]; \mathbb{R}^m)$ l'espace des processus \mathbb{F} -progressivement mesurables tels que

$$\mathbb{E} \left[\int_0^T |\alpha_t|^2 dt \right] < \infty.$$

Alors, notre intuition nous conduit à donner la définition suivante d'un équilibre pour un jeu à champ moyen :

Scheme 2 (Jeu à champ moyen). 1. (*Candidat champ moyen.*) On se donne un flot de mesures aléatoires $\mu = (\mu_t)_{t \in [0, T]}$ adapté par rapport à filtration naturelle de B et tel que

$$\mathbb{E} \left[\sup_{t \in [0, T]} \int_{\mathbb{R}^d} |x|^2 d\mu_t(x) \right] < \infty.$$

2. (*Optimisation.*) On considère le problème de contrôle optimal où le joueur représentatif cherche $\alpha^* \in \mathbb{A}$ tel que

$$J(\alpha^*) = \min_{\alpha \in \mathbb{A}} J(\alpha, \mu), \quad J(\alpha, \mu) := \mathbb{E} \left[\int_0^T F(X_t^{\mu, \alpha}, \mu_t, \alpha_t) dt + G(X_T^{\mu, \alpha}, \mu_T) \right], \quad (5)$$

lorsque la dynamique du joueur représentatif est donnée par:

$$\begin{aligned} dX_t^{\mu, \alpha} &= B(X_t^\mu, \mu_t, \alpha_t) dt + \sigma(X_t^\mu, \mu_t) dW_t + \sigma_0(X_t^\mu, \mu_t) dB_t, \\ X_0^{\mu, \alpha} &= X_0. \end{aligned} \quad (6)$$

3. (Condition de Nash.) Enfin, on dit que $(\alpha_t^*, \mu_t)_{t \in [0, T]}$ est un équilibre (ou solution MFG) si

$$\forall t \in [0, T], \mu_t = \mathcal{L}(X_t^{\mu, \alpha^*} | \mathcal{F}_t^B).$$

On termine ce paragraphe en observant que la résolution d'un jeu à champ moyen demande d'abord de résoudre un problème de contrôle stochastique et ensuite un problème de point fixe. L'acronyme "MFG" vient de l'anglais "Mean-Field Game". La condition de point fixe, appelé ci-dessus condition de Nash, est une *condition de consistance de type McKean-Vlasov*.

On notera qu'il existe d'autres types de notions d'équilibre pour des populations de taille infinie en interaction champ moyen. Par exemple, le modèle force parfois tous les joueurs à suivre les mêmes instructions d'un planificateur central ; le cas échéant, l'objectif est de minimiser une fonctionnelle de coût commune à l'ensemble de la population. On parle de *problème de contrôle de type champ moyen* [9, 41, 40, 95].

3.2 Passage à la limite $N \rightarrow \infty$.

Jusqu'ici, la formulation des jeux à champ moyen, telle qu'on l'a donnée dans la sous-section précédente, n'a été obtenue que de façon intuitive. En réalité, se pose la question d'une démonstration rigoureuse d'un lien entre le jeu à N joueurs et du jeu à champ moyen. A cet égard, différents éléments de réponse sont possibles.

La façon la plus couramment utilisée dans la littérature pour faire le lien entre le jeu fini et le jeu limite est de montrer qu'il est possible de construire, à partir des solutions MFG, des presque-équilibres de Nash pour les jeux à N joueurs, au sens de la propriété suivante :

Proposition 3 (Presque-équilibre de Nash). *Soit $(\alpha_t^*, \mu_t)_{t \in [0, T]}$, une solution du jeu à champ moyen décrit dans Schéma 2, $(\alpha_t^*)_{t \in [0, T]}$ s'écrivant sous une forme progressivement mesurable $(\alpha_t^*(X_0, W, B))_{t \in [0, T]}$ de la condition initiale et des bruits propre et commun. Alors, pour chaque $N \in \mathbb{N} \setminus \{0\}$, la collection de stratégies*

$$(\alpha^{*,1}, \dots, \alpha^{*,i}, \dots, \alpha^{*,N}) = \left(\alpha_t^*(X_0^1, W^1, B), \dots, \alpha_t^*(X_0^i, W^i, B), \dots, \alpha_t^*(X_0^N, W^N, B) \right)_{t \in [0, T]},$$

est un presque-équilibre de Nash pour le jeu à N joueurs correspondant, c'est à dire qu'il existe une suite à valeurs positives $(\zeta_N)_{N \geq 1}$ convergeant vers 0 telle que, pour tout $i = 1, 2, \dots, N$ et pour tout autre contrôle admissible α ,

$$J^i(\alpha^{*,1}, \dots, \alpha^{*,i}, \dots, \alpha^{*,N}) \leq J^i(\alpha^{*,1}, \dots, \alpha, \dots, \alpha^{*,N}) + \zeta_N.$$

Cette propriété a un grand intérêt pratique : la stratégie issue de toute solution du jeu à champ moyen induit un presque-équilibre de Nash du jeu à N joueurs. Autrement dit, il suffit de résoudre le jeu limite pour obtenir une presque-solution du jeu à N joueurs, la qualité de l'approximation croissant avec N ; d'une certaine façon, cela permet de surmonter l'obstacle liée à la complexité du jeu à N joueurs. Pour autant, il faut comprendre l'énoncé ci-dessus comme une forme de "méta-énoncé", dont on s'attend à ce qu'il soit vrai de façon générale, mais qui nécessite néanmoins un minimum d'hypothèses (que l'on ne précisera pas ici) pour être démontré.

L'approche par "presque-équilibres" va du jeu limite vers le jeu à N joueurs. On peut citer, entre autres, [17], pour une application de cette approche dans le contexte d'un jeu avec absorption. En fait, se pose aussi la question de l'existence de résultats allant dans le sens

inverse, avoir: existe-t-il une notion de convergence permettant de dire que des équilibres de Nash (ou presque-équilibres de Nash) des jeux à N joueurs convergent vers des solutions du jeu à champ moyen correspondant ?

Il est en réalité beaucoup plus difficile de résoudre ce problème que de démontrer la Proposition 3. Le cadre le plus simple est celui des jeux à champ moyen linéaires-quadratiques, pour lesquels la dynamique est linéaire en l'état du joueur, le contrôle et la moyenne de la population et les fonctions de coût sont quadratiques en les mêmes variables. Ce cas a été étudié par Bardi et Priuli [6] en horizon infini ; Carmona, Fouque et Sun [27] ont également donné un exemple d'application en horizon fini. Pour ce qui est du cas général, plusieurs approches ont été développées : d'une part, les travaux de Fischer [46] et Lacker [72] se sont focalisés sur les limites, au sens de la convergence faible, des équilibres sur des stratégies en boucle ouverte ; d'autre part, la convergence des équilibres sur des stratégies en boucle markovienne a été étudiée par Cardaliaguet, Delarue, Lasry et Lions [19] en utilisant la "master equation" du jeu limite. La "master equation" est une équation aux dérivées partielles, posée sur l'espace des mesures de probabilité et vérifiée par la fonction valeur du jeu à champ moyen (lorsque celui-ci a une unique solution pour toute condition initiale). Cette équation a également été étudiée dans [53, 30, 24]. L'idée développée dans [19] est de tester la fonction valeur du jeu à champ moyen sur les équilibres du jeu à N joueurs pour comparer la valeur des jeux finis et infinis.

3.3 Résultats d'existence

On peut résoudre un jeu à champ moyen par des méthodes probabilistes ou des méthodes analytiques. Les méthodes probabilistes s'appuient, bien souvent, sur le Principe du Maximum Stochastique de Pontryagin (acronymes PMSP en français et PSMP en anglais) pour résoudre le problème de contrôle optimal du joueur représentatif. Les méthodes analytiques utilisent le Principe de la Programmation Dynamique (PPD en français et DPP en anglais) et la théorie des équations de Hamilton-Jacobi-Bellman.

Dans cette sous section, on expose formellement les deux méthodes de résolution et les connexions entre elles.

Pour simplifier la présentation de ces méthodes de résolution, on fait l'hypothèse que les volatilités associées aux deux bruits sont des constantes (i.e., $\sigma(x, \bar{x}) = \sigma \geq 0$ and $\sigma_0(x, \bar{x}) = \sigma_0 \geq 0$), que toutes les grandeurs fini-dimensionnelles sont de dimension 1 (i.e., $d = m = 1$), et que $A = \mathbb{R}$.

Principe de la Programmation Dynamique

Le principe de la programmation dynamique (dit PPD) est un concept fondamental dans l'étude des problèmes de contrôle optimal [52, 51, 87]. Dans ce paragraphe, on présente la stratégie globale de résolution d'un jeu hamp moyen à l'aide du PPD.

L'idée principale est que le PPD permet de décrire la dynamique de la fonction valeur du problème de contrôle optimal comme la solution d'une équation de Hamilton-Jacobi-Bellman. Ceci est bien connu quand les coefficients du problème de contrôle optimal sont déterministes. Lorsqu'ils sont stochastiques (comme c'est le cas lorsque le bruit commun est présent dans le jeu à champ moyen), l'équation de Hamilton-Jacobi-Bellman devient stochastique et son étude est nettement plus compliquée. Cela explique pourquoi, dans la suite de la thèse, on préférera utiliser l'approche probabiliste, fondée sur le PMSP.

Aussi, pour simplifier ce paragraphe, on suppose que $\sigma_0 = 0$ (i.e., il n'y a pas de bruit commun). Le cas échéant, l'environnement $\mu = (\mu_t)_{t \in [0, T]}$ utilisé pour décrire l'état de la population dans Scheme 2 est déterministe et les coefficients du problème de contrôle que doit résoudre le joueur représentatif sont également déterministes. De fait, lorsque μ est donné, on peut chercher à caractériser la dynamique de la fonction valeur $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ du problème de contrôle en environnement μ . Si $(\alpha_t^*)_{t \in [0, T]}$ désigne un contrôle optimal du joueur représentatif en environnement μ et $(X_t^{\mu, \alpha^*})_{t \in [0, T]}$ désigne la trajectoire associée, alors le PPD affirme que, pour tout $(t, x) \in [0, T] \times \mathbb{R}$,

$$u(t, x) = \mathbb{E} \left[\int_t^T F(X_s^{\mu, \alpha^*}, \mu_s, \alpha_s^*) ds + G(X_T^{\mu, \alpha^*}, \mu_T) \middle| X_t^{\mu, \alpha^*} = x \right]. \quad (7)$$

L'objectif est de chercher un lien explicite entre u et $(\alpha_t^*)_{t \in [0, T]}$. Pour cela, on peut introduire le Hamiltonien associé au problème de contrôle optimal du joueur représentatif. Il est défini comme suit :

$$H(x, \mu_t, a, y) := B(x, \mu_t, a)y + F(x, \mu_t, a), \quad \forall (x, a, y) \in \mathbb{R}^3, t \in [0, T]. \quad (8)$$

Le Hamiltonien étant posé, le PPD suggère, qu'à l'instant t et dans l'état x , le joueur représentatif choisit le contrôle optimal en minimisant $H(x, \mu_t, a, \partial_x u(t, x))$ par rapport à a . Autrement dit, si le Hamiltonien admet un unique minimiseur $\hat{a}(t, x, y)$ par rapport à a lorsque les trois autres paramètres sont fixés, le contrôle optimal à l'instant t est de la forme suivante :

$$\alpha_t^* = \hat{a}(X_t^{\mu, \alpha^*}, \mu_t, \partial_x u(t, X_t^{\mu, \alpha^*})), \quad \forall t \in [0, T]. \quad (9)$$

En injectant cette écriture dans le PPD, on dérive l'équation de Hamilton-Jacobi-Bellman suivante :

$$\partial_t u(t, x) + \frac{\sigma^2}{2} \partial_{xx}^2 u(t, x) + \hat{H}(x, \mu_t, \partial_x u(t, x)) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \quad (10)$$

où $u(T, x) = G(x, \mu_T)$, $x \in \mathbb{R}$ et \hat{H} est défini par:

$$\hat{H}(x, \mu_t, y) := \inf_{a \in \mathbb{R}} (B(x, \mu_t, a)y + F(x, \mu_t, a)).$$

A l'aide de la représentation (9) du contrôle optimal, on peut déduire de la condition de consistance (ou d'équilibre de Nash) dans Scheme 2 que $(\mu_t)_{t \in [0, T]}$ est solution du jeu à champ moyen si $(\mu_t)_{t \in [0, T]}$ résout (dans un sens faible) :

$$\partial_t \mu_t + \partial_x (B(x, \mu_t, \hat{a}(x, \mu_t, \partial_x u(t, x))) \mu_t) + \frac{\sigma^2}{2} \partial_{xx}^2 \mu_t = 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \quad (11)$$

avec $\mu_0 = \mathcal{L}(X_0)$ comme condition initiale.

En conclusion, les solutions du jeu à champ moyen sont données par les solutions du système “forward-backward” suivant :

$$\begin{aligned} \partial_t \mu_t + \partial_x (B(x, \mu_t, \hat{a}(x, \mu_t, \partial_x u(t, x))) \mu_t) + \frac{\sigma^2}{2} \partial_{xx}^2 \mu_t &= 0, \\ \partial_t u(t, x) + \frac{\sigma^2}{2} \partial_{xx}^2 u(t, x) + \hat{H}(x, \mu_t, \partial_x u(t, x)) &= 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \end{aligned} \quad (12)$$

où la condition initiale pour $(\mu_t)_{t \in [0, T]}$ est donnée par $\mu_0 = \mathcal{L}(X_0)$, et la condition terminale pour u est donnée par $u(T, x) = G(x, \mu_T)$ pour tout $x \in \mathbb{R}$. Ce système est souvent appelé “système MFG”. La difficulté majeure est qu’il est fortement couplé ! Les coefficients de l’équation pour $(\mu_t)_{t \in [0, T]}$ dépendent de u et réciproquement ; les deux équations étant dirigées dans des directions opposées, il n’y a pas de bonne théorie de Cauchy-Lipschitz, si ce n’est peut-être en temps petit ; en temps arbitraire, existence et unicité sont le plus souvent étudiées séparément.

Naturellement, si on est capable de résoudre (12), la stratégie d’équilibre correspondante s’écrit :

$$\alpha^* = \left(\hat{a}(X_t^{\mu, \alpha^*}, \mu_t, \partial_x u(t, X_t^{\mu, \alpha^*})) \right)_{t \in [0, T]}.$$

Dans le cas où $\sigma_0 = 0$, un état des lieux des travaux effectués sur le système MFG est disponible dans la note de synthèse [54]. On notera en particulier que le système MFG est très utile dans l’étude des jeux à champ moyen à dépendance locale (i.e., lorsque les coefficients dépendent du couple (x, μ) au travers de la valeur de la densité de μ au point x).

L’analogie du système MFG dans le cas $\sigma_0 \neq 0$ est présenté dans [19].

Principe du Maximum Stochastique

Le principe de Pontryagin du maximum stochastique (dit PMSP) est une autre méthode utilisée pour résoudre des problèmes de contrôle optimal, voir par exemple [86, 88, 60]. Dans ce paragraphe, on présente la stratégie globale de résolution d’un jeu à champ moyen à l’aide du PMSP. En comparaison de la méthode s’appuyant sur le PPD, le PMSP s’accommode très facilement de coefficients aléatoires et donc de la présence du bruit commun.

Si on se donne un flot de mesures aléatoires $(\mu_t)_{t \in [0, T]}$ représentant l’évolution de la loi de la population (conditionnellement à la réalisation du bruit commun), on peut introduire, comme précédemment, le Hamiltonien associé au problème de contrôle optimal du joueur représentatif. Il est défini comme dans (3.3), à savoir :

$$H(x, \mu_t, a, y) := B(x, \mu_t, a)y + F(x, \mu_t, a), \quad \forall (x, a, y) \in \mathbb{R}^3, \quad t \in [0, T].$$

On peut ensuite définir, pour tout contrôle $\alpha \in \mathbb{A}$, un processus adjoint $(Y_t)_{t \in [0, T]}$, donné comme la solution de l’équation différentielle stochastique rétrograde : $\forall t \in [0, T]$,

$$dY_t = -\partial_x H(X_t^{\mu, \alpha}, \mu_t, \alpha_t, Y_t) dt + Z_t dW_t + \tilde{Z}_t dB_t, \quad (13)$$

où $Y_T = \partial_x G(X_T^{\mu, \alpha}, \mu_T)$ et $(Z_t)_{t \in [0, T]}$ et $(\tilde{Z}_t)_{t \in [0, T]}$ sont cherchés dans $\mathcal{H}^2([0, T]; \mathbb{R})$.

L’utilisation d’équations différentielles stochastiques rétrogrades dans les problèmes de contrôle optimal remonte aux travaux de Bismut[[12], [13]]. Une condition nécessaire de l’existence d’un contrôle optimal est qu’il minimise le Hamiltonien, lorsque x est remplacé par

$X_t^{\mu, \alpha}$ et y par Y_t . Lorsque le Hamiltonien satisfait des hypothèses de stricte convexité, cette condition devient suffisante, au sens où, s'il existe un quintuplet $(X_t^{\mu, \alpha}, \alpha_t, Y_t, Z_t, \tilde{Z}_t)_{t \in [0, T]}$ tel que :

1. $(X_t^{\mu, \alpha}, \alpha_t)_{t \in [0, T]}$ vérifie la dynamique du joueur représentatif en environnement $(\mu_t)_{t \in [0, T]}$ avec le contrôle $(\alpha_t)_{t \in [0, T]}$,
2. $(X_t^{\mu, \alpha}, \alpha_t, Y_t, Z_t, \tilde{Z}_t)_{t \in [0, T]}$ vérifie (13),
3. α_t minimise $H(X_t^{\mu, \alpha}, \mu_t, a, Y_t)$ par rapport à a ,

alors $(\alpha_t)_{t \in [0, T]}$ est un contrôle optimal et il est unique, voir [89].

On note, au passage, que la stricte convexité du Hamiltonien garantit l'existence d'un unique minimiseur $\hat{a}(t, x, y)$ par rapport à a lorsque les trois autres paramètres sont fixés. On en déduit que, dans le troisième point ci-dessus, le minimiseur est nécessairement de la forme :

$$\alpha_t^* = \hat{a}(X_t, \mu_t, Y_t), \quad \forall t \in [0, T]. \quad (14)$$

De fait, résoudre le problème de contrôle revient à résoudre le problème “forward-backward” en environnement $\mu = (\mu_t)_{t \in [0, T]}$:

$$\begin{aligned} dX_t &= B(X_t, \mu_t, \hat{a}(X_t, \mu_t, Y_t))dt + \sigma dW_t + \sigma_0 dB_t, \\ dY_t &= -\partial_x H(X_t, \mu_t, \hat{a}(X_t, \mu_t, Y_t), Y_t)dt + Z_t dW_t + \tilde{Z}_t dB_t, \end{aligned} \quad (15)$$

avec X_0 comme condition initiale et $Y_T = \partial_x G(X_T, \mu_T)$ comme condition terminale. Le cas échéant, $(\alpha_t^* = \hat{a}(X_t, \mu_t, Y_t))_{t \in [0, T]}$ est le contrôle optimal et $(X_t^{\mu, \alpha^*})_{t \in [0, T]}$ est la trajectoire optimale correspondante.

Pour résoudre le jeu à champ moyen, il ne reste plus qu'à injecter la condition de consistance, voir le dernier point dans Scheme 2 :

$$\mu_t = \mathcal{L}(X_t^{\mu, \alpha^*} | \mathcal{F}_t^B), \quad \forall t \in [0, T].$$

On en déduit que les solutions du jeu à champ moyen sont caractérisées par l'équation différentielle stochastique progressive-rétrograde (dite de type McKean-Vlasov) qui suit:

$$\begin{aligned} dX_t &= \partial_y H(X_t, \mathcal{L}(X_t | \mathcal{F}_t^B), \hat{a}(X_t, \mathcal{L}(X_t | \mathcal{F}_t^B), Y_t), Y_t)dt + \sigma dW_t + \sigma_0 dB_t, \\ dY_t &= -\partial_x H(X_t, \mathcal{L}(X_t | \mathcal{F}_t^B), \hat{a}(X_t, \mathcal{L}(X_t | \mathcal{F}_t^B), Y_t), Y_t)dt + Z_t dW_t + \tilde{Z}_t dB_t, \end{aligned} \quad (16)$$

avec X_0 comme condition initiale et $Y_T = \partial_x G(X_T, \mathcal{L}(X_T | \mathcal{F}_T^B))$ comme condition terminale. Le cas échéant, la stratégie d'équilibre est donnée par :

$$\alpha^* = \left(\hat{a}(X_t, \mathcal{L}(X_t | \mathcal{F}_t^B), Y_t) \right)_{t \in [0, T]}.$$

Le principe du maximum stochastique conduit donc, dans le cadre des jeux à champ moyen, à résoudre un système d'équations différentielles stochastiques progressives-rétrogrades de type McKean-Vlasov, voir par exemple [21, 26]. On retrouve une structure “forward-backward” analogue à celle du système MFG (12), dont la résolution conduit à des difficultés similaires.

Connection entre PPD et PMSP

Lorsque $\sigma_0 = 0$, la solution $(X_t, Y_t, Z_t, \tilde{Z}_t)_{t \in [0, T]}$ du système (16) et la solution $(\mu_t, u(t, \cdot))_{t \in [0, T]}$ du système (12) sont reliées par :

$$Y_t = \partial_x u(t, X_t), \quad \forall t \in [0, T].$$

3.4 Résultats d'unicité

Comme expliqué précédemment, des résultats d'existence et d'unicité des solutions du jeu à champ moyen, peuvent être démontrés en temps petit, en utilisant l'approche probabiliste ou l'approche analytique ; en temps petit, des arguments de contraction peuvent en effet être utilisés, voir [65]. En temps arbitraire, les travaux de Larsy et Lions [75] garantissent l'unicité des équilibres du jeu hamp moyen sous une condition supplémentaire de monotonie des coefficients. Cette condition s'applique dans le cas où F dans (5) est à structure séparée, i.e. de la forme $F_0(X_t^{\mu, \alpha}, \mu_t) + F_1(X_t^{\mu, \alpha}, \alpha_t)$ et F_0 et G vérifient la propriété de monotonie :

$$\begin{aligned} \int_{\mathbb{R}} (F_0(x, \mu) - F_0(x', \mu')) d(\mu - \mu')(x) &\geq 0, \\ \int_{\mathbb{R}} (G(x, \mu) - G(x', \mu')) d(\mu - \mu')(x) &\geq 0, \end{aligned}$$

pour toute mesure de probabilité μ et μ' . On peut aussi citer le résultat de [1] qui, pour une certaine classe de jeux à champ moyen, a démontré l'unicité sous une autre hypothèse de monotonie.

De façon générale, en théorie des jeux, l'unicité est plus difficile à démontrer que l'existence d'équilibres. C'est la même chose pour la théorie des jeux hamp moyen. Cette difficulté soulève la question de la sélection des équilibres, lorsqu'il en existe plusieurs. C'est précisément l'objet de cette thèse que d'étudier plusieurs exemples de ce type.

4 Résumé de la thèse

4.1 Restauration de l'unicité par le bruit commun

Au sein de la littérature sur les jeux à champ moyen, il existe une classe privilégiée qui se prête bien à des méthodes de résolution explicite : il s'agit de la classe des problèmes linéaires-quadratiques, voir par exemple [22, 10, 26]. Ceci nous incite à étudier, dans le premier chapitre de la thèse, les questions d'unicité lorsque :

1. la dynamique du joueur représentatif est décrite par une équation différentielle stochastique avec des coefficients de volatilité constants et une dérive linéaire en les variables d'état et d'action,
2. les fonctionnelles de coût sont quadratiques en les variables d'état et d'action.

Précisément, dans la suite du manuscrit, la dynamique du joueur représentatif est de la forme (avec $d = m = 1$) :

$$dX_t = [cX_t + \alpha_t + b(\bar{\mu}_t)] dt + \sigma dW_t + \sigma_0 dB_t,$$

et la fonctionnelle de coût est de la forme :

$$J(\alpha) = \mathbb{E} \left[\int_0^T \frac{1}{2} (\alpha_t^2 + (X_t + f(\bar{\mu}_t))^2) dt + \frac{1}{2} (X_T + g(\bar{\mu}_T))^2 \right],$$

où $\bar{\mu}_t$ est la moyenne (aléatoire) de μ_t . Ci-dessus, σ, σ_0 sont des constantes positives, c est une constante réelle, et b, f et g sont des fonctions bornées lipschitziennes.

On retrouve une structure similaire à (4) (au moins pour ce qui est de la fonctionnelle de coût). **Bien que b, f et g soient non-linéaires, on dira de ces jeux hamp moyen qu'ils sont linéaires-quadratiques** (notés LQ-MFGs) ; il s'agit d'un abus de langage, que l'on répètera à plusieurs reprises dans le manuscrit. Cet abus est justifié par le fait que, lorsque $(\mu_t)_{t \in [0, T]}$ est fixé, le problème d'optimisation du joueur représentatif est un problème de contrôle linéaire-quadratique (au sens usuel). Dans le chapitre 1, on tire parti des méthodes explicites qui existent pour résoudre les problèmes d'optimisation linéaires-quadratiques ; ces méthodes permettent de caractériser de façon très simple les solutions du jeu à champ moyen sous la forme d'un système "forward-backward" de type standard (non McKean-Vlasov) !

Pour clarifier les choses, la résolution du jeu à champ moyen, dans le cadre ci-dessus, obéit à la démarche ci-dessous :

(Jeu à champ moyen de classe LQ.)

1. (Candidat au champ moyen.) On se donne un processus stochastique $(\bar{\mu}_t)_{t \in [0, T]}$ à valeurs dans \mathbb{R} , continu et adapté par rapport à la filtration $(\mathcal{F}_t^B)_{t \in [0, T]}$.
2. (Optimisation.) On résout le problème d'optimisation du joueur représentatif :

$$\inf_{\alpha \in \mathcal{H}^2([0, T]; \mathbb{R})} J(\alpha), \quad J(\alpha) := \frac{1}{2} \mathbb{E} \left[\int_0^T (\alpha_t^2 + (f(\bar{\mu}_t) + X_t)^2) dt + (X_T + g(\bar{\mu}_T))^2 \right],$$

avec

$$dX_t = [cX_t + \alpha_t + b(\bar{\mu}_t)] dt + \sigma dW_t + \sigma_0 dB_t, \quad \forall t \in [0, T],$$

pour une condition initiale X_0 fixée.

3. (Condition de Nash.) On cherche un processus $(\bar{\mu}_t)_{t \in [0, T]}$ tel que :

$$\forall t \in [0, T], \quad \bar{\mu}_t = \mathbb{E}[X_t^{\alpha^*} | \mathcal{F}_t^B].$$

La caractérisation annoncée des équilibres prend la forme suivante (avec $X_0 = \psi$ comme condition initiale) :

Proposition 4 (Caractérisation de solutions LQ-MFG). *Le couple $(\alpha_t, \bar{\mu}_t)_{t \in [0, T]}$ est une solution pour le jeu à champ moyen linéaire-quadratique ci-dessus si et seulement s'il existe un processus stochastique $(\mu_t^{\psi, \sigma_0}, h_t^{\psi, \sigma_0}, Z_t^{\psi, \sigma_0})_{t \in [0, T]}$, adapté par rapport $(F_t^B)_{t \in [0, T]}$, et solution de :*

$$\begin{cases} \forall t \in [0, T], \\ d\mu_t^{\psi, \sigma_0} = [-w_t^{-2} h_t^{\psi, \sigma_0} + w_t^{-1} b(w_t \mu_t^{\psi, \sigma_0})] dt + w_t^{-1} \sigma_0 dB_t, \\ dh_t^{\psi, \sigma_0} = [-w_t f(w_t \mu_t^{\psi, \sigma_0}) - w_t \eta_t b(w_t \mu_t^{\psi, \sigma_0})] dt + Z_t^{\psi, \sigma_0} dB_t, \\ \text{et } \mu_0^{\psi, \sigma_0} = \mathbb{E}[\psi] w_0^{-1}, \quad h_T^{\psi, \sigma_0} = g(\mu_T^{\psi, \sigma_0}). \end{cases} \quad (17)$$

où

$$w_t := \exp \left(\int_t^T (-c + \eta_s) ds \right), \quad \forall t \in [0, T],$$

$$(\eta_t)_{t \in [0, T]} \text{ est l'unique solution de l'équation de Riccati : } \frac{d\eta_t}{dt} = \eta_t^2 - 2c\eta_t - 1, \quad \eta_T = 1.$$

Quand il existe une solution à l'équation différentielle stochastique ci-dessus, on peut construire une solution MFG, $(\alpha_t, \bar{\mu}_t)_{t \in [0, T]}$, à partir de $(\mu_t^{\psi, \sigma_0}, h_t^{\psi, \sigma_0})_{t \in [0, T]}$, (et vice versa), comme suit :

$$\begin{aligned} \bar{\mu}_t &= w_t \mu_t^{\psi, \sigma_0}, \quad \forall t \in [0, T], \\ \alpha_t &= -\eta_t X_t - h_t, \quad \text{où } h_t = w_t^{-1} h_t^{\psi, \sigma_0}, \quad \forall t \in [0, T], \end{aligned}$$

avec $(X_t)_{t \in [0, T]}$ défini implicitement comme la dynamique stochastique du joueur représentatif à l'équilibre au travers de l'EDS :

$$dX_t = [(c - \eta_t)X_t - h_t + b(\bar{\mu}_t)]dt + \sigma dW_t + \sigma_0 dB_t, \quad \forall t \in [0, T].$$

L'intérêt de ce résultat est de caractériser les équilibres au travers d'une équation différentielle stochastique progressive-rétrograde de type standard (i.e., non McKean-Vlasov) vérifiée par la moyenne (conditionnelle) des équilibres. Naturellement, il est beaucoup plus facile de comprendre les problèmes d'unicité pour une équation de type standard que pour une équation de type McKean-Vlasov : une équation progressive-rétrograde de type standard est naturellement associée à une EDP en dimension finie, alors que, comme on l'a déjà écrit, une équation progressive-rétrograde de type McKean-Vlasov est associée à une EDP, dite "master equation", posée sur l'espace des mesures de probabilité.

En particulier (et c'est là le premier résultat de cette thèse), la caractérisation ci-dessus permet de mettre en lumière des nouveaux chemins vers l'unicité des solutions pour les jeux à champ moyen décrits ci-dessus. En effet, la condition de monotonie de Lasry-Lions ne s'applique pas à cette classe de jeux à champ moyen. Pourtant, on démontre dans le chapitre 1 :

Theorem 5 ($\sigma_0 > 0$). *On suppose que $\sigma_0 > 0$, alors les jeux à champ moyen dans notre classe linéaire-quadratique ont une seule solution MFG.*

Ce résultat est en fait un corollaire quasi-immédiat de la Proposition 4 et des résultats de la théorie des équations différentielles stochastiques progressives-rétrogrades. Lorsque $\sigma_0 > 0$, il se trouve en effet que l'équation différentielle stochastique progressive-rétrograde profite de l'effet régularisant du bruit commun pour assurer l'unicité ; ce résultat, non-trivial, est dû à [35].

En réalité, ce résultat prend toute sa mesure au regard du contre-exemple suivant. On peut en effet facilement construire un contre-exemple à l'unicité lorsque $\sigma_0 = 0$:

Proposition 6 ($\sigma_0 = 0$). *Si $f = b = \psi = 0$, et avec*

$$K_t := \int_0^t w_s^{-2} ds, \quad \forall t \in [0, T],$$

choisissons $g : \mathbb{R} \rightarrow \mathbb{R}$ sous la forme :

$$g(x) = \begin{cases} 1 & \text{si } x < -K_T, \\ -x/K_T & \text{si } |x| \leq K_T, \\ -1 & \text{si } x > K_T. \end{cases}$$

Alors, le jeu admet une infinité de solutions MFG, à savoir :

$$\mu_t^* = -Aw_t K_t, \quad \alpha_t^* = -\eta_t X_t - Aw_t^{-1}, \quad \forall t \in [0, T],$$

pour tout $A \in \mathbb{R}$ tel que $|A| \leq 1$.

En résumé, ce chapitre montre que c'est précisément la présence du bruit commun qui garantit l'unicité des équilibres pour la classe LQ de jeux hamp moyen considérée. En ce sens, le résultat obtenu dans le chapitre 1 s'inscrit dans la continuité de résultats antérieurs sur la restauration d'unicité des solutions à des équations différentielles par ajout d'un forçage aléatoire. Ce principe remonte aux travaux précurseurs de Zvonkin [102] ; les notes de cours [47] en donne un panorama assez détaillé.

Le résultat présenté dans le chapitre 1 a donné lieu à une publication dans la revue *Dynamic Games and Applications*.

4.2 Sélection de solutions en l'absence d'unicité

Dans le deuxième chapitre, on se concentre sur une modification du contre-exemple à l'unicité donné dans la Proposition 6. Cette modification repose sur l'introduction d'un temps de choc $\delta \in (0, T)$ dans l'évolution du jeu ; elle a pour effet de réduire le nombre de solutions MFG possibles à trois.

Pour ce jeu à champ moyen particulier (on a toujours $b = f = \psi = 0$), on se pose la question de la sélection des solutions MFG lorsque σ_0 est égal à 0. Pour cela, on propose trois méthodes de sélection et on compare leurs résultats. A travers ce cas particulier, on essaie de mettre en lumière les différences et les similarités qui peuvent exister entre ces approches, susceptibles d'être appliquées à d'autres cas de jeux à champ moyen.

On introduit le temps de choc $\delta \in (0, T)$ en modifiant, dans l'énoncé de la Proposition 6, la fonction g comme suit:

$$g(x) = \begin{cases} 1 & \text{if } x < -r_\delta, \\ -x/r & \text{if } |x| \leq r_\delta, \\ -1 & \text{if } x > r_\delta, \end{cases} \quad (18)$$

où $r = \int_\delta^T w_s^{-2} ds$. Cette opération réduit l'ensemble des solutions MFG à trois éléments.

Proposition 7 (Choc à $\delta \in (0, T)$, $b = f = \psi = 0$). *Il n'existe que trois solutions à l'équation différentielle stochastique progressive-rétrograde (17) avec $b = f = \psi = 0$ et avec la fonction g définie ci-dessus. Ces solutions sont données comme suit :*

$$(\mu_t^{0,0}, h_t^{0,0}, Z_t^{0,0})_{t \in [0, T]} = \left(-A \int_0^t w_s^{-2} ds, A, 0 \right)_{t \in [0, T]} \quad \text{pour } A \in \{-1, 0, 1\}.$$

Face à ces trois solutions MFG possibles, on propose les trois approches de sélection suivantes :

1. (Sélection par coût minimal.) On choisit, parmi les trois équilibres (ou solutions MFG), les équilibres qui assurent au joueur représentatif le plus petit coût;
2. (Sélection par limite zéro bruit commun.) On considère la solution MFG unique au même jeu à champ moyen, mais en présence du bruit commun (i.e $\sigma_0 > 0$), et on cherche sa limite quand $\sigma_0 \rightarrow 0$. Lorsque cette limite existe (au sens faible), on choisit les équilibres qui sont inclus dans son support.
3. (Sélection par limite N -joueurs.) On considère le jeu à N -joueurs associé au jeu à champ moyen sans bruit commun. On suppose que la suite admet au moins un équilibre de Nash pour tout N , et on étudie la limite des équilibres quand $N \rightarrow +\infty$. Lorsque cette limite existe (au sens faible), on choisit les équilibres qui sont inclus dans son support.

Le résultat principal de ce chapitre est :

Theorem 8 (Paradoxe dans la sélection). *Pour le jeu à champ moyen sans bruit commun décrit ci-dessus (i.e., $\sigma_0 = 0, \delta \in (0, T)$, $b = f = \psi = 0$ et g donnée par (18)), la sélection par coût minimal choisit l'équilibre associé à $A = 0$, alors que les approches de sélection par limite zéro bruit commun et par limite N -joueurs choisissent aléatoirement les équilibres associés à $A = 1$ et $A = -1$, chacun avec probabilité $1/2$.*

Ce résultat met en lumière une contradiction entre les trois méthodes de sélection : la méthode de sélection par coût minimal est parfois utilisée dans le cas d'un jeu avec un continuum de joueurs ; dans le cas présent, elle donne un résultat différent des deux autres méthodes, qui peuvent être considérées comme naturelles du point de vue de la modélisation.

Sélection par coût minimal. La première méthode s'appuie sur l'idée que le joueur représentatif reste rationnel et pourrait donc "choisir" parmi les équilibres (ou solutions MFG) possibles celui (ou ceux) pour lequel son coût est minimal. Le résultat précis, dont la preuve repose sur des calculs explicites, s'écrit :

Proposition 9 (Sélection par coût minimal). *Parmi les trois solutions possibles données dans la Proposition 7, l'approche de sélection par coût minimal désigne la solution correspondant à $A = 0$, i.e.,*

$$\left(\mu_t = 0, \alpha_t = -\eta_t X_t \right)_{t \in [0, T]} \quad \text{où} \quad X_t = w_t \int_0^t \sigma w_s^{-1} dW_t, \quad \forall t \in [0, T].$$

Sélection par limite zéro bruit commun La seconde méthode est inspirée des travaux précurseurs de Bafico et Baldi [5], qui ont été repris par la suite dans des problèmes de modélisation issus de la physique [37, 47, 48]. Cette méthode vise à tirer profit du fait que l'ajout du bruit commun restaure l'unicité de la solution MFG (chapitre 1). L'objectif est de fait d'étudier le comportement des solutions MFG lorsque la viscosité σ_0 est évanescence. A cet égard, le résultat obtenu dans le chapitre 2 est le suivant :

Theorem 10. *La suite de mesures de probabilité (avec les mêmes notations que dans la Proposition 4) $(\mathbb{P} \circ (\mu_t^{0, \sigma_0})_{0 \leq t \leq T}^{-1})_{\sigma_0 \in (0, 1)}$ converge au sens faible (sur l'espace des fonctions continues réelles sur $[0, T]$) vers*

$$\frac{1}{2} \delta_{(k_t)_{t \in [0, T]}} + \frac{1}{2} \delta_{(-k_t)_{t \in [0, T]}}$$

où $(k_t := \int_0^t w_s^{-2} ds)_{t \in [0, T]}$.

La preuve du Théorème 10 s'appuie sur le fait que la solution de l'équation progressive-rétrograde (17) (avec 0 comme condition initiale) peut être découplée, au sens où

$$h_t = \theta^{\sigma_0}(t, \mu_t^{0, \sigma_0}), \quad t \in [0, T],$$

pour une fonction $\theta^{\sigma_0} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$. Cette propriété est à la base de la méthode de résolution dite “four-step scheme” pour les équations progressives-rétrogrades, voir [82]. Ici, θ^{σ_0} est appelé “champ découplant” et est solution d'une équation aux dérivées partielles parabolique :

$$\begin{cases} \forall (t, x) \in [0, T] \times \mathbb{R}, \\ \partial_t \theta^{\sigma_0}(t, x) - w_t^{-2} \theta^{\sigma_0}(t, x) \partial_x \theta^{\sigma_0}(t, x) + \frac{1}{2} \sigma_0^2 w_t^{-2} \partial_{xx}^2 \theta^{\sigma_0}(t, x) = 0, \\ \theta^{\sigma_0}(T, x) = g(x), \quad \forall x \in \mathbb{R}, \end{cases} \quad (19)$$

Cette équation aux dérivées partielles est une équation de type Burgers. Dans le cadre des jeux à champ moyen, elle peut être comprise comme une forme réduite de la “master equation”. De façon essentielle, la propriété de découplage permet de réécrire la dynamique de $(\mu_t^{0, \sigma_0})_{t \in [0, T]}$ sous la forme d'une EDS classique :

$$\mu_t^{0, \sigma_0} = - \int_0^t w_s^{-2} \theta^{\sigma_0}(s, \mu_s^{0, \sigma_0}) ds + \sigma_0 \int_0^t w_s^{-1} dB_s, \quad \forall t \in [0, T].$$

Une partie de la démonstration du Théorème 10 repose sur le fait que, lorsque σ_0 tend vers 0, θ^{σ_0} converge vers la solution “entropique” de l'EDP à viscosité nulle. Cette solution entropique est donnée par

$$\theta(t, x) = \begin{cases} -\text{sign}(x) & \text{si } t \leq \delta, \quad x \in \mathbb{R}, \\ -\text{sign}(x) & \text{si } t \geq \delta, \quad |x| \geq r_\delta - r_t, \\ -\frac{x}{r_\delta - r_t} & \text{si } t > \delta, \quad |x| < r_\delta - r_t, \end{cases}$$

où $r_t = \int_t^T w_s^{-2} ds$. Lorsque σ_0 est petit, on s'attend de fait à ce que μ_t^{0, σ_0} se comporte, au moins en temps petit, comme la solution d'une EDS dirigée par un mouvement brownien d'intensité σ_0 et par la dérive $-w_t^{-2} \theta(t, \cdot)$ (ou $w_t^{-2} \text{sign}$) :

$$d\tilde{\mu}_t = w_t^{-2} \text{sign}(t, \tilde{\mu}_t) dt + \sigma_0 dB_t, \quad (20)$$

avec $\tilde{\mu}_0 = 0$ comme condition initiale. La solution de (20) tombe précisément dans le cadre étudié par Bafico et Baldi [5] : les solutions de (20) convergent au sens faible vers la même limite que celle de l'énoncé du Théorème 10.

Pour autant, il n'est pas possible d'appliquer directement le résultat de [5] pour démontrer le Théorème 10. Il est nécessaire d'avoir des estimations fines de la différence $\theta^{\sigma_0}(t, x) - \theta(t, x)$ au voisinage de $x = 0$, ce que l'on obtient dans le chapitre 2 en appliquant une méthode d'analyse asymptotique de type Laplace à la représentation de Cole-Hopf de θ^{σ_0} :

$$\theta^{\sigma_0}(t, x) = \frac{\int_{\mathbb{R}} \left(\frac{x-y}{r_t} \right) \exp\left(\sigma_0^{-2} \left(- \int_0^y g(v) dv - \frac{(x-y)^2}{2r_t} \right) \right) dy}{\int_{\mathbb{R}} \exp\left(\sigma_0^{-2} \left(- \int_0^y g(v) dv - \frac{(x-y)^2}{2r_t} \right) \right) dy}.$$

La difficulté principale est d'estimer la différence $\theta^{\sigma_0}(t, x) - \theta(t, x)$ au voisinage de $x = 0$: pour $t < \delta$, la convergence ne peut pas être uniforme autour de $x = 0$ alors que, loin de 0, la différence $\theta^{\sigma_0}(t, \cdot) - \theta(t, \cdot)$ converge uniformément vers 0.

Une fois la différence $\theta^{\sigma_0} - \theta$ estimée de façon fine, le reste de la preuve consiste à suivre la relecture, donnée dans [36], des travaux de Bafico et Baldi [5]. L'idée de [36] est d'étudier, dans la limite $\sigma_0 \rightarrow 0$, une fenêtre espace-temps au sein de laquelle le bruit et la dérive rentrent en compétition. De façon générale, cette fenêtre est assimilée à un point espace-temps, appelé "point de transition". Schématiquement, un point de transition est un point espace-temps $(\pm\epsilon_0, t_0)$, qui converge vers $(0, 0)$ lorsque $\sigma_0 \rightarrow 0$, et satisfaisant les propriétés suivantes :

1. Pour tout $|x| \leq \epsilon_0$, la probabilité que le processus $(\mu_t^{x, \sigma_0})_{t \in [0, T]}$ atteigne $\pm\epsilon_0$ en temps proche de t_0 tend vers 1, lorsque $\sigma_0 \rightarrow 0$. C'est à dire que, pour de très petites valeurs de σ_0 , le processus $(\mu_t^{x, \sigma_0})_{t \in [0, T]}$, initialisé en $x \in [-\epsilon_0, \epsilon_0]$, a de très grandes chances d'atteindre le niveau de transition ϵ_0 en temps t_0 ,
2. La probabilité que le processus $(\mu_t^{x, \sigma_0})_{t \in (0, T]}$, initialisé en $x \in [-\epsilon_0, \epsilon_0]$, s'éloigne de 0 tend vers 1, lorsque $\sigma_0 \rightarrow 0$.

Intuitivement l'idée est la suivante :

- en temps plus petit que t_0 , le bruit domine la dérive et le processus $(\mu_t^{0, \sigma_0})_{t \in [0, T]}$ ne fait que fluctuer autour de 0 ;
- au fur et à mesure que le temps s'approche de t_0 , le bruit et la dérive deviennent comparables ;
- au voisinage de t_0 , le bruit fait suffisamment osciller le processus pour qu'il atteigne le niveau ϵ_0 ;
- après t_0 , la dérive commence à l'emporter sur le bruit ; comme elle pointe vers l'extérieur de 0, elle emmène le processus loin de 0 à une vitesse proche de 1 (en valeur absolue).

Explicitement, on montre dans le chapitre 2 est qu'il est possible d'utiliser le point de transition suivant :

$$(\epsilon_0, t_0) := (\sigma_0^2 |\ln \sigma_0|^2, \sigma_0^2 |\ln \sigma_0|^4).$$

Sélection par limite N -joueurs La troisième et dernière méthode de sélection étudiée dans le chapitre 2 est certainement la plus en phase avec l'approche particulière des jeux à champ moyen. Il s'agit d'étudier le jeu à N joueurs correspondant, sans bruit commun, et de déterminer le comportement asymptotique des équilibres quand N tend vers l'infini.

Pour tout $i = 1, \dots, N$, la dynamique du joueur i est modélisée par le processus stochastique $(X_t^i)_{t \in [0, T]}$, à valeurs réelles, suivant :

$$\begin{cases} dX_t^i = [cX_t^i + \alpha_t^i]dt + \sigma dW_t^i, & \forall t \in [0, T], \\ X_0^i = 0, \end{cases}$$

où $(\alpha_t^i)_{t \in [0, T]}$ est un contrôle admissible vivant dans l'espace des processus stochastiques progressivement-mesurables par rapport à la filtration engendrée par les bruits (indépendants)

W^1, \dots, W^N et vérifiant la condition d'intégrabilité :

$$\mathbb{E} \left[\int_0^T |\alpha_s|^2 ds \right] < \infty.$$

De façon à ce que la fonction de "meilleure réponse" soit aussi simple que possible, on modélise l'interaction champ moyen au travers de la moyenne empirique "des autres joueurs" :

$$\bar{X}_t^{i,N} = \frac{1}{N-1} \sum_{j \neq i}^N X_t^j, \quad \forall t \in [0, T].$$

Intuitivement, le fait que le joueur i soit exclu de cette moyenne empirique est négligeable quand N tend vers l'infini.

Si on se donne une collection de contrôles $(\alpha_t^1, \dots, \alpha_t^N)_{t \in [0, T]}$ pour tous les joueurs, alors le coût inimiser pour le joueur i est :

$$J^i(\alpha^1, \dots, \alpha^i, \dots, \alpha^N) := \mathbb{E} \left[\int_0^T \frac{1}{2} (\alpha_t^i)^2 dt + \frac{1}{2} \left(X_T^i + g(\bar{X}_T^{i,N}) \right)^2 \right]. \quad (21)$$

Dans le chapitre 2, on ne démontre pas l'existence d'équilibres pour le jeu à N joueurs. En revanche, on donne une condition de premier ordre que tout équilibre doit vérifier et on étudie le comportement asymptotique des solutions de cette condition de premier ordre.

Cette condition de premier ordre s'écrit sous la forme d'un principe de Pontryagin (PMSP) pour les jeux. Si $(X_t^i ; i = 1, 2, \dots, N)_{t \in [0, T]}$ sont les trajectoires d'un équilibre de Nash, alors il existe des processus stochastiques $(V_t^i, z_t^{i,k} ; i, k = 1, 2, \dots, N)_{t \in [0, T]}$, tels que la collection $(X_t^i, V_t^i, z_t^{i,k} ; i, k = 1, 2, \dots, N)_{t \in [0, T]}$ soit solution du système d'équations différentielles stochastiques progressives-rétrogrades :

$$\begin{cases} dX_t^i = [(c - \eta_t)X_t^i - V_t^i] dt + \sigma dW_t^i, & \forall t \in [0, T], \quad X_0^i = 0. \\ dV_t^i = -(c - \eta_t)V_t^i dt + \sum_{k=1}^N z_t^{i,k} dW_t^k, & \forall t \in [0, T], \quad V_T^i = g(\mu_T^{i,N}), \\ i \in \{1, 2, \dots, N\}, \\ \dot{\eta}_t = \eta_t^2 - 2c\eta_t - 1, & \forall t \in [0, T], \quad \eta_T = 1. \end{cases}$$

En rappelant que

$$w_t = \exp \left(- \int_t^T (c - \eta_s) ds \right),$$

et en utilisant les changements de variables :

$$\tilde{\mu}_t^N = \frac{w_t^{-1}}{N} \sum_{i=1}^N \bar{X}_t^{i,N}, \quad \tilde{\mu}_t^{i,N} = w_t^{-1} \bar{X}_t^{i,N}, \quad v_t^N = \frac{w_t}{N} \sum_{i=1}^N V_t^i, \quad \text{pour tout } t \in [0, T],$$

on obtient que le processus $(\tilde{\mu}_t^N, v_t^N)_{t \in [0, T]}$ est nécessairement solution de:

$$\begin{cases} \forall t \in [0, T], \\ d\tilde{\mu}_t^N = -w_t^{-2}v_t^N dt + \frac{\sigma}{N} \sum_{i=1}^N w_t^{-1} dW_t^i, \quad \tilde{\mu}_0^N = 0, \\ (v_t^N)_{t \in [0, T]} \text{ est une martingale continue, } v_T^N = \frac{1}{N} \sum_{i=1}^N g(\tilde{\mu}_T^{i, N}). \end{cases} \quad (22)$$

Ce système (et c'est là le noeud du problème) peut être compris, au moins intuitivement, comme une version approchée de l'équation (17).

Le résultat principal de ce paragraphe est l'analogie du Théorème 10 :

Theorem 11. *La suite de mesures de probabilités $(\mathbb{P} \circ (\tilde{\mu}_t^N)_{0 \leq t \leq T}^{-1})_{N \geq 1}$ converge au sens faible, lorsque $N \rightarrow \infty$, vers*

$$\frac{1}{2} \delta_{(k_t)_{t \in [0, T]}} + \frac{1}{2} \delta_{(-k_t)_{t \in [0, T]}}.$$

Ce résultat conduit au même principe de sélection que celui énoncé dans le Théorème 10. Intuitivement, le lien entre les deux approches repose sur le fait que, dans (22), la diffusion dirigeant l'équation pour $(\tilde{\mu}_t)_{t \in [0, T]}$ peut être réécrite en introduisant un nouveau mouvement brownien :

$$B_t = \frac{1}{\sqrt{N}} \sum_{i=1}^N W_t^i, \quad t \in [0, T].$$

Le terme de bruit dans (22) se lit alors $\sigma_0 dB_t$ avec $\sigma_0 = \sigma N^{-1/2}$! En remplaçant par ailleurs $\tilde{\mu}_T^{i, N}$ par $\tilde{\mu}_T^N$, on retrouve exactement le système (17), avec $b = f = 0$.

Si ce raisonnement intuitif explique pourquoi le résultat doit être le même que dans le Théorème 10, la mise en oeuvre est plus difficile. La grosse difficulté est que l'équation (22) n'est pas fermée, puisque la condition terminale n'est pas donnée par une fonction de $\tilde{\mu}_T^N$ mais une fonction des $(\tilde{\mu}_T^{i, N})_{i=1, \dots, N}$. Autrement dit, il n'est pas possible de construire un champ de découplage sur le modèle de ce qui a été exposé dans la preuve du Théorème 10. Dans la suite du manuscrit, on propose de construire un champ de découplage approché, que l'on appelle parfois un "presque champ de découplage". En fait, l'idée est simplement de comparer le processus v_t^N avec une fonction de $\tilde{\mu}_t^N$. Une solution naïve serait de comparer v_t^N avec $\theta^N(t, \tilde{\mu}_t^N)$, où $\theta^N := \theta^{\sigma/\sqrt{N}}$, mais cette approche ne semble pas aboutir. En effet, il faut rappeler que la version limite de l'équation (11), obtenue en posant formellement $N = \infty$, admet plusieurs solutions ; on ne peut donc pas espérer des propriétés robustes de stabilité pour cette équation et, par conséquent, on ne peut pas non plus espérer que des propriétés robustes de stabilité pour (11) puissent rester vraies le long de la limite $N \rightarrow \infty$.

Pour pallier ce problème, l'idée utilisée dans la thèse est d'obtenir des propriétés faibles de stabilité à travers des propriétés de monotonie pour le système (11). Ceci est fait en deux étapes.

Pour expliquer la première étape, on rappelle que $r_\delta = \int_\delta^T w_s^{-2} ds$. On se donne par ailleurs une suite $(\gamma_N)_{N \geq 1} \subseteq (0, r_\delta/2)$ telle que $\gamma_N \rightarrow 0$ lorsque $N \rightarrow +\infty$. On utilise ces

deux quantités pour définir la fonction :

$$\tilde{g}(x) := \begin{cases} g(x), & \text{si } x \leq r_\delta - 2\gamma_N, \\ g(r_\delta - 2\gamma_N) & \text{si } r_\delta - 2\gamma_N \leq x \leq r_\delta - \gamma_N, \\ g(x - \gamma_N) & \text{si } r_\delta - \gamma_N \leq x \leq r_\delta + \gamma_N, \\ g(x) & \text{si } x \geq r_\delta + \gamma_N. \end{cases} \quad (23)$$

On associe à la fonction \tilde{g} la solution $\tilde{\theta}^N$ de l'équation (19), mais avec $\sigma_0 = \sigma/\sqrt{N}$ comme volatilité et avec \tilde{g} comme condition au bord :

$$\begin{cases} \partial_t \tilde{\theta}_t^N - w_t^{-2} \tilde{\theta}^N \partial_x \tilde{\theta}^N + \frac{1}{2} \left(\frac{\sigma^2}{N}\right) w_t^{-2} \partial_{xx}^2 \tilde{\theta}_t^N = 0, & \forall t \in [0, T], \quad \forall x \in \mathbb{R}, \\ \tilde{\theta}^N(T, x) = \tilde{g}(x), & \forall x \in \mathbb{R}. \end{cases}$$

On est capable de comparer cette fonction $\tilde{\theta}^N$ à θ^N , et de fait à la solution entropique θ de l'équation (19) avec g comme condition au bord. Il s'agit de la conclusion de la première étape.

La deuxième étape est de comparer $(v_t^N)_{t \in [0, T]}$ et $(-\tilde{\theta}^N(t, \tilde{\mu}_t^N))_{t \in [0, T]}$. Par des arguments de monotonie/convexité (dans lesquels la forme de \tilde{g} intervient explicitement), on prouve dans le chapitre 2 que :

Lemma 12. *Supposons qu'on choisisse la suite $(\gamma_N)_{N \geq 1}$ telle que*

$$\gamma_N \sqrt{N} \rightarrow +\infty \quad \text{lorsque } N \rightarrow +\infty.$$

Alors, avec probabilité 1,

$$-v_t^N \geq -\tilde{\theta}^N(t, \tilde{\mu}_t^N) + \Delta_t^N, \quad \text{for all } t \in [0, T],$$

où

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left[\sup_{t \in [0, T]} |\Delta_t^N|^2 \right] = 0.$$

Ce lemme justifie le rôle de champ “presque-découplant” joué par la fonction $\tilde{\theta}^N$. Il dit par ailleurs, que lorsque $(\mu_t^N)_{t \in [0, T]}$ est dans le demi-plan supérieur, le processus $(-v_t^N)_{t \in [0, T]}$ pousse le processus $(\tilde{\mu}_t^N)_{t \in [0, T]}$ à s'éloigner de 0 à un rythme au moins comparable à une diffusion dirigée par la seule dérive $\tilde{\theta}^N$. En utilisant le fait que $\tilde{\theta}^N$ et θ^N sont proches, on retrouve un phénomène de fuite comparable à celui utilisé dans la sélection par viscosité évanescence. Le reste de la preuve consiste à symétriser l'argument pour traiter les instants où le processus $(\tilde{\mu}_t^N)_{t \in [0, T]}$ traverse le demi-plan inférieur et à appliquer un argument de type point de transition.

4.3 Illustration dans un problème Principal-Agents

Dans le dernier chapitre, on étudie un jeu à champ moyen de type “principal-agents”. Dans cette forme particulière de jeu à champ moyen, il y a à la fois un “principal” et un continuum de joueurs, à qui le principal délègue une tâche en contrepartie d'une rétribution ; à ce titre, les joueurs deviennent donc des “agents”. Comme dans les jeux à champ moyen classiques, les agents s'organisent selon un consensus (équilibre de Nash), une fois que le principal a fixé

le niveau de la rétribution ; le principal optimise la rétribution pour maximiser le bénéfice à tirer du travail de ses agents. L'équilibre qui en résulte (s'il existe) est appelé équilibre de Stackelberg ; le cas échéant, on dit que le jeu à champ moyen de type "principal-agents" admet une solution.

Un exemple pratique

Du point de vue de la modélisation, on peut penser au principal comme à un gouvernement qui souhaite inciter des citoyens, qui sont également des producteurs/consommateurs, accroître leurs échanges (économiques) avec l'extérieur. Pour rendre compte de la multitude de citoyens qui composent la société dont le gouvernement a la responsabilité, il est raisonnable d'assimiler l'ensemble des citoyens-producteurs/consommateurs à un continuum d'agents. Le principal et les agents appartiennent à une économie dans laquelle l'activité est liée à la production/consommation de richesses exportées ou importées. Dans une telle économie, la production de biens vendus à l'étranger conduit à l'augmentation de la richesse collective ; par ailleurs, la consommation de biens achetés à l'étranger, si elle contribue à une fuite ponctuelle de capitaux, permet de faire venir des produits de technologie avancée et potentiellement d'améliorer l'appareil de production. A cet égard, il paraît souhaitable, pour un gouvernement, d'éviter que l'économie ne soit trop fermée et, de fait, de favoriser, si besoin, les exportations et les importations.

Dans ce cadre, il est raisonnable de supposer que le principal ne peut observer que la richesse moyenne échangée avec l'extérieur. En supposant que T désigne un horizon de temps pour sa politique, le gouvernement peut, de fait, avoir pour objectif de favoriser les échanges avec l'extérieur au cas où ceux-ci seraient trop faibles. Pour autant, le gouvernement ne peut pas directement contrôler la production et la consommation (ou plus précisément les importations et les exportations) des citoyens ; il peut simplement proposer des incitations, par exemple au travers d'orientations fiscales. Les agents, quant à eux, sont supposés être individualistes (i.e., ils ne s'intéressent qu'eux propres situations) et grégaires, au sens où ils ont tendance à imiter le comportement global de la société. Ce phénomène d'agrégation est à l'origine d'effets de rebond ; de là, il peut avoir des effets vertueux d'économies d'échelle et permettre à la production d'augmenter. Les agents bénéficient des incitations mises en place par le gouvernement à l'issue de la période T . L'asymétrie qui existe entre le principal et les agents s'apparente à un "aléa moral". On utilise, dans la suite, les outils développés par Elie et ses co-auteurs [44, 45] pour l'analyse de ce problème.

Modélisation mathématique

La durée d'une période sur laquelle les exportations/importations sont mesurées est notée T . Pour un agent représentatif donné, on désigne par X_t , pour $t \in [0, T]$, sa "balance commerciale personnelle" avec l'extérieur à l'instant t . Pour simplifier, on suppose que $X_0 = 0$.

On modélise l'évolution de $(X_t)_{t \in [0, T]}$ à l'aide d'un processus d'Ornstein-Uhlenbeck, dont la dynamique est prescrite par l'équation différentielle stochastique suivante :

$$dX_t = \kappa(m_t - X_t)dt + dW_t + \epsilon dB_t, \quad \forall t \in [0, T],$$

avec (comme on l'a dit) $X_0 = 0$. Ci-dessus, κ et ϵ sont des constantes positives ; la constante positive κ représente un facteur d'agrégation : plus κ est grand, plus l'agent est sensible aux comportements des autres joueurs. Par ailleurs, $(B, W) := (B_t, W_t)_{t \in [0, T]}$ sont des processus

stochastiques construits sur un espace de probabilité $(\Omega, \mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$, \mathbb{F} désignant la complétion de la filtration engendrée par B et W . Ces processus B et W représentent, respectivement, les fluctuations systémiques de l'économie et les fluctuations intrinsèques de la balance de l'agent. Sur un plan pratique, la probabilité \mathbb{P} rend compte du comportement de (B, W) en l'absence d'incitations gouvernementales ; dans la suite, ce régime "sans incitations" est appelé BAU (pour "business as usual" en anglais). En régime BAU, ou, de façon équivalente, sous la probabilité \mathbb{P} , $(B_t)_{t \in [0, T]}$ et $(W_t)_{t \in [0, T]}$ sont supposés être des mouvements browniens indépendants. Le cas échéant, m_t est compris comme la moyenne conditionnelle de X_t sachant B , c'est-à-dire $m_t := \mathbb{E}[X_t | B]$ (lorsque $\epsilon = 0$, m_t est simplement compris comme la moyenne de X_t , c'est-à-dire $m_t := \mathbb{E}[X_t]$).

En particulier, en régime BAU, la balance commerciale moyenne (conditionnellement au bruit commun) est égale à $(m_t = \epsilon B_t)_{t \in [0, T]}$: en l'absence de bruit commun, elle n'évolue pas ; il n'y a, en moyenne, pas d'échange avec l'extérieur.

Les contrats que le principal peut proposer à l'agent représentatif sont des variables aléatoires strictement positives et \mathcal{F}_T -mesurables (lorsque $\epsilon = 0$, les contrats sont simplement mesurables par rapport à la tribu engendrée par W). On désigne un contrat par ξ et on note \mathcal{C} la collection de ces contrats : dans le chapitre 3, plusieurs hypothèses d'intégrabilité sont formulées sur ξ ; il s'agit essentiellement d'hypothèses d'intégrabilité exponentielle.

Une fois proposé un contrat ξ , l'agent représentatif contrôle sa balance commerciale en choisissant un processus $\alpha = (\alpha_t)_{t \in [0, T]}$, là encore avec des propriétés d'intégrabilité suffisantes (pour permettre d'appliquer le théorème de Girsanov, voir le chapitre 3 pour les détails) : si α_t est positif, l'agent exporte à l'instant t ; si α_t est négatif, il importe. Lorsque le processus α est non-nul, il n'est plus raisonnable de modéliser les perturbations intrinsèques $(W_t)_{t \in [0, T]}$ par un mouvement brownien ; pour rendre compte du contrôle exercé par l'agent, on écrit $(W_t)_{t \in [0, T]}$ sous la forme

$$W_t =: W_t^\alpha + \int_0^t \alpha_s ds, \quad t \in [0, T].$$

La richesse de l'agent est anticipée sous une nouvelle mesure de probabilité \mathbb{P}^α , définie par rapport à \mathbb{P} par la densité :

$$\frac{d\mathbb{P}^\alpha}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = L_{0t}(\alpha) := \exp \left(\int_0^t \alpha_s dW_s - \frac{1}{2} \int_0^t \alpha_s^2 ds \right), \quad \forall t \in [0, T].$$

Avec ces notations, la dynamique de $(X_t)_{t \in [0, T]}$ se met sous la forme :

$$dX_t = \kappa(m_t - X_t)dt + (\alpha_t dt + dW_t^\alpha) + \epsilon dB_t, \quad X_0 = 0, \quad \forall t \in [0, T],$$

où $(W_t^\alpha)_{t \in [0, T]}$ est un mouvement Brownien sous la probabilité \mathbb{P}^α . Et, pour chaque incitation (ou contrat) $\xi \in \mathcal{C}$ et chaque contrôle α , le gain espéré de l'agent représentatif est :

$$J^A(\xi, \alpha) = \mathbb{E} \left[L_{0T}(\alpha) \left(\xi - \frac{1}{2} \int_0^T \alpha_s^2 ds \right) \right].$$

La coût quadratique dans la définition de J^A peut être compris comme un coût de production lorsque α_s est positif ; lorsqu'il est négatif, on peut le concevoir comme le coût de la formation nécessaire à l'utilisation d'une nouvelle technologie, importée de l'extérieur.

On ajoute, dans ce cadre, une contrainte supplémentaire. Pour un contrat donné, chaque agent cherche un contrôle optimal tel que le gain espéré en utilisant ce contrôle soit supérieur

ou égal ne constante $R \geq 0$, appelée utilité de réservation (ou contrainte de participation). Cette constante a la justification suivante : si l'agent ne peut pas espérer un gain au moins égal à R , alors il ne participe pas à l'effort souhaité par le principal, autrement dit, il refuse le contrat ξ .

Equilibre entre les agents

Le jeu à champ moyen à résoudre par l'agent représentatif est donné par :

1. (Candidat au champ moyen.) On se donne un processus à valeurs réelles $(m_t)_{t \in [0, T]}$ et un contrat $\xi \in \mathcal{C}$.
2. (Optimisation.) On cherche des contrôles $\alpha^*(\xi)$ (dans la classe des contrôles admissibles \mathbb{A}) tels que

$$V^A(\xi) = \sup_{\alpha \in \mathbb{A}} J^A(\xi, \alpha),$$

$$\text{avec } V^A(\xi) = J^A(\xi, \alpha^*(\xi)) \geq R.$$

3. (Condition de Nash.) On cherche $(m_t)_{t \in [0, T]}$ vérifiant la condition de point fixe :

$$m_t = \mathbb{E} [L_{0T}(\alpha^*(\xi)) X_t^* | B], \quad \forall t \in [0, T],$$

où

$$dX_t^* = \kappa(m_t - X_t)dt + \alpha_t^*(\xi)dt + dW_t^{\alpha^*} + \epsilon dB_t, \quad X_0 = 0, \quad \forall t \in [0, T].$$

On notera que la condition de Nash est calculée sous la mesure de probabilité modélisant l'action de l'agent. En l'absence de bruit commun, la moyenne conditionnelle dans la condition de Nash est réduite à une simple moyenne.

La résolution du problème d'optimisation ci-dessus s'appuie sur le résultat suivant :

Proposition 13. *Etant donné $\xi \in \mathcal{C}$, il existe un processus $(Y_t, \hat{Z}_t, \hat{\zeta}_t)_{t \in [0, T]}$, solution de l'équation différentielle stochastique rétrograde (dite BSDE en anglais) :*

$$dY_t = -\frac{1}{2} \hat{Z}_t^2 dt + \hat{Z}_t dW_t + \hat{\zeta}_t dB_t, \quad \forall t \in [0, T], \quad Y_T = \xi,$$

telle que :

1. $(\hat{Z}_t)_{t \in [0, T]} \in \mathbb{A}$;
2. $\forall \alpha \in \mathbb{A}, \quad J^A(\xi, \alpha) \leq J^A(\xi, \hat{Z})$;
3. $\alpha_t^* = \hat{Z}_t, \quad \forall t \in [0, T]$.

En l'absence de bruit commun, le processus $\hat{\zeta}$ est nul.

Cette proposition permet de définir les contrats ξ applicables. Etant donnée la contrainte de participation $R \geq 0$, $\xi \in \mathcal{C}$ est applicable s'il existe un réel $\hat{Y} \geq R$, un processus \hat{Z} admissible (c'est-à-dire dans \mathbb{A}) et un processus progressivement-mesurable (et suffisamment intégrable) $\hat{\zeta}$ tels que :

$$\xi = \hat{Y} - \frac{1}{2} \int_0^T \hat{Z}_s^2 ds + \int_0^T \hat{Z}_s dW_s + \int_0^T \hat{\zeta}_s dB_s.$$

Lorsque $\epsilon = 0$, $\hat{\zeta}$ est nul.

On peut facilement montrer que l'ensemble des contrats applicables est non vide. On en déduit le résultat suivant pour les solutions du jeu à champ moyen, lorsque ξ est donné :

Proposition 14 (Solution MFG). *Etant donné un contrat applicable ξ , le jeu à champ moyen a une unique solution, donnée par $(m_t, \hat{Z}_t)_{t \in [0, T]}$, où $(m_t, Y_t, \hat{Z}_t, \hat{\zeta}_t)_{t \in [0, T]}$ est solution de l'équation différentielle stochastique progressive-rétrograde qui suit :*

$$\begin{cases} m_t = \int_0^t \mathbb{E}[L_{0T}(\hat{Z})\hat{Z}_s|B] ds + \epsilon B_t, & \forall t \in [0, T], \\ Y_t = \xi + \frac{1}{2} \int_t^T \hat{Z}_s^2 ds - \int_t^T \hat{Z}_s dW_s - \int_t^T \hat{\zeta}_s dB_s, & \forall t \in [0, T]. \end{cases}$$

Lorsque $\epsilon = 0$ et ξ est mesurable par rapport à la tribu engendrée par W , le processus \hat{Z} est progressivement-mesurable par rapport à la filtration engendrée par W et le processus $\hat{\zeta}$ est nul.

Optimisation du principal

Le principal souhaite inciter la population à sortir d'une situation dans laquelle il juge les échanges avec l'extérieur trop faibles ; cette situation s'apparente à un piège dont le principal souhaite sortir. Il s'agit de fait de mesurer la distance de m_T à 0. Pour cela, on introduit la fonction de perte $G : \mathbb{R} \rightarrow \mathbb{R}$, définie par :

$$G(x) := \frac{-x^2}{2(T-\delta)} \mathbb{1}_{|x| \leq T-\delta} + \left(\frac{T-\delta}{2} - \text{sign}(x)x \right) \mathbb{1}_{|x| > T-\delta},$$

où $\delta \in [0, T)$. Il est facile de voir que G est une fonction concave ayant son unique maximum en 0 ; la concavité est stricte au voisinage de 0 et est d'autant plus marquée que δ est proche de T . L'objectif du principal est de maximiser le gain moyen :

$$J^P(\xi) := \mathbb{E}^{\hat{Z}} \left[-G(m_T) - \xi \right],$$

où \hat{Z} est la stratégie de l'agent représentatif et m_T est l'état terminal de la population dans l'équilibre de Nash correspondant à la rétribution ξ . La subtilité ici tient au fait que $-G$ est convexe ! La maximisation de J^P vise donc à pousser m_T loin de 0 ; plus δ est proche de T , plus G pénalise les profils pour lesquels m_T est proche de 0.

Naturellement, le choix de G est lié aux résultats des chapitres précédents. Ici, $G'(x) = g(x)$ avec :

$$g(x) = \begin{cases} 1 & \text{si } x < -(T-\delta), \\ -x/(T-\delta) & \text{si } |x| \leq T-\delta, \\ -1 & \text{si } x > T-\delta. \end{cases}$$

Le principal doit donc résoudre le problème d'optimisation suivant :

1. (Optimisation.) On cherche, parmi les contrats applicables, un contrat ξ^* , tel que, pour tout autre contrat applicable ξ ,

$$J^P(\xi) \leq J^P(\xi^*) = \mathbb{E}^{\hat{Z}^*} \left[-G(m_T^*) - \xi^* \right],$$

où $(m_t^*, \hat{Z}_t^*)_{t \in [0, T]}$ correspond à la stratégie et à l'état de la population à l'équilibre lorsque le joueur représentatif accepte le contrat ξ^* .

Lorsque ce problème d'optimisation a une solution, le triplet $(\xi^*, m_t^*, \hat{Z}_t^*)_{t \in [0, T]}$ constitue un équilibre de Stackelberg entre le principal et l'agent représentatif. Autrement dit, le triplet $(\xi^*, m_t^*, \hat{Z}_t^*)_{t \in [0, T]}$ est une solution au jeu à champ moyen de type "principal-agents". On peut alors démontrer que $(\hat{Z}_t^*)_{t \in [0, T]}$ est nécessairement adapté à la filtration engendrée par le bruit commun B (au moins si $\epsilon > 0$; lorsque $\epsilon = 0$, le processus $(\hat{Z}_t^*)_{t \in [0, T]}$ est nécessairement déterministe) ; de là, on en déduit, par des arguments classiques de contrôle stochastique, le résultat suivant :

Theorem 15 (Caractérisation des équilibres). *Lorsque $\epsilon > 0$, le triplet $(\xi^*, m_t^*, \hat{Z}_t^*)_{t \in [0, T]}$ est une solution au jeu à champ moyen de type "principal-agents" si et seulement si $(m_t^*, \hat{Z}_t^*)_{t \in [0, T]}$ s'écrit $(m_t^*, \hat{Z}_t^*)_{t \in [0, T]} = (x_t, -y_t)_{t \in [0, T]}$ où*

$$\begin{cases} x_t = - \int_0^t y_s ds + \epsilon B_t, \\ y_t = \partial_x G(x_T) - \int_t^T z_s dB_s, \quad \forall t \in [0, T]. \end{cases} \quad (24)$$

Le cas échéant, le contrat ξ^* s'écrit sous la forme :

$$\xi^* = R - \frac{1}{2} \int_0^T y_s^2 ds + \int_0^T y_s dW_s + \int_0^T \zeta_s dB_s,$$

pour un processus $(\zeta_s)_{s \in [0, T]}$ suffisamment intégrable ; ce processus $(\zeta_s)_{s \in [0, T]}$ n'est pas uniquement déterminé.

Lorsque $\epsilon = 0$ et $\delta = 0$, la caractérisation ci-dessus reste vraie au sens où $(\xi^*, m_t^*, \hat{Z}_t^*)_{t \in [0, T]}$ est une solution au jeu à champ moyen si et seulement si $(m_t^*, \hat{Z}_t^*)_{t \in [0, T]} = (x_t, -y_t)_{t \in [0, T]}$ où

$$\begin{cases} x_t = - \int_0^t y_s ds, \\ y_t = \partial_x G(x_T), \quad \forall t \in [0, T]. \end{cases} \quad (25)$$

Le cas échéant, le contrat ξ^* s'écrit sous la forme :

$$\xi^* = R - \frac{1}{2} \int_0^T y_s^2 ds + \int_0^T y_s dW_s, \quad \forall t \in [0, T],$$

Lorsque $\epsilon = 0$ et $\delta > 0$, la caractérisation ci-dessus n'est plus vraie. Précisément, $(\xi^*, m_t^*, \hat{Z}_t^*)_{t \in [0, T]}$ est une solution au jeu à champ moyen si et seulement si $(m_t^*, \hat{Z}_t^*)_{t \in [0, T]} = (x_t, -y_t)_{t \in [0, T]}$ où $(x_t, -y_t)_{t \in [0, T]}$ est une solution **non-nulle** de (25). Le cas échéant, le contrat ξ^* s'écrit comme dans le cas $\delta = 0$.

On retrouve, dans (24), l'équation (17) avec $b = f = \psi = 0$! Lorsque $\epsilon > 0$, cette équation a une unique solution ! Naturellement, (25) est la version déterministe de (24) : lorsque $\delta = 0$, cette équation a une infinité de solutions ; lorsque $\delta > 0$, elle a trois solutions, dont une triviale ; de façon remarquable, la solution triviale est automatiquement exclue. Au regard du vocabulaire introduit précédemment, on pourrait dire qu'elle n'est pas "sélectionnée" par le principal : en fait, on verra dans le Chapitre 3 que la solution triviale de (24) est, dans le cas $\delta > 0$, un minimum d'utilité du principal ; elle peut donc être comprise comme un "équilibre instable".

On en déduit le théorème de “restauration d’unicité” suivant :

Theorem 16 (Existence et unicité). *Lorsque $\epsilon > 0$, il existe une unique solution au jeu à champ moyen de type “principal-agents” (l’unicité de la forme du contrat étant au processus ζ près). La dynamique de la population est de la forme :*

$$m_t^* = - \int_0^t \theta^\epsilon(s, m_s^*) ds + \epsilon B_t, \quad t \in [0, T],$$

où θ^ϵ est la solution classique de l’EDP :

$$\begin{cases} \partial_t \theta^\epsilon(t, x) - \theta^\epsilon(t, x) \partial_x \theta^\epsilon(t, x) + \frac{\epsilon^2}{2} \partial_{xx} \theta^\epsilon(t, x) = 0, & \forall t \in [0, T], \quad \forall x \in \mathbb{R}, \\ \theta^\epsilon(T, x) = \partial_x G(x), & \forall x \in \mathbb{R}. \end{cases}$$

Le cas échéant,

$$\begin{aligned} \xi^* &= R - \frac{1}{2} \int_0^T (\theta^\epsilon(t, m_t^*))^2 dt + \int_0^T (-\theta^\epsilon(t, m_t)) dW_t + \int_0^T \zeta_s dB_s, \\ \hat{Z}_t^* &= -\theta^\epsilon(t, m_t^*), \end{aligned}$$

pour un processus $(\zeta_t)_{t \in [0, T]}$ suffisamment intégrable.

Lorsque $\epsilon = 0$ et $\delta = 0$, il y a une infinité de solutions. Comme dans la Proposition 6, elles sont données par

$$(\xi^*, m_t^*, \hat{Z}_t^*)_{t \in [0, T]} = \left(R + \frac{A^2 T}{2} + AW_T^A, At, A \right)_{t \in [0, T]},$$

avec A variant dans $[-1, 1]$. Toutes ces solutions conduisent à une utilité du principale égale à $-R$.

Lorsque $\epsilon = 0$ et $\delta > 0$, il y a deux solutions, données par

$$(\xi^*, m_t^*, \hat{Z}_t^*)_{t \in [0, T]} = \left(R + \frac{A^2 T}{2} + AW_T^A, At, A \right)_{t \in [0, T]},$$

avec A variant dans $\{-1, 1\}$. L’utilité du principal est égale à $-R + \delta A^2$.

Sélection d’équilibres

Le Théorème 16 nous incite à utiliser les méthodes du chapitre 2 pour sélectionner les solutions dans le cas $\epsilon = 0$ (au moins dans le cas $\delta > 0$; le cas $\delta = 0$ est plus compliqué). On démontre, dans le chapitre 3, les deux résultats suivants :

Theorem 17 (Sélection par limite petit bruit). *Lorsque $\epsilon \rightarrow 0$, la loi de $(m_t^*)_{t \in [0, T]}$ sous \mathbb{P} (la viscosité du bruit commun étant comprise comme égale à ϵ) converge vers*

$$\frac{1}{2} \delta_{(t)_{t \in [0, T]}} + \frac{1}{2} \delta_{(-t)_{t \in [0, T]}}.$$

Pour ce résultat, on observera que la loi de $(m_t^*)_{t \in [0, T]}$ sous \mathbb{P} est en réalité la même que la loi de $(m_t^*)_{t \in [0, T]}$ sous $\mathbb{P}^{\hat{Z}^*}$, avec $(\hat{Z}_t^*)_{t \in [0, T]}$ comme dans le Théorème 16. Ceci découle du fait que $(m_t^*)_{t \in [0, T]}$ est adapté à la filtration engendrée par B .

Une autre méthode de sélection consiste à sélectionner les solutions qui maximisent le critère du principal. Mais, en réalité, cette approche est précisément celle que retient le principal pour choisir un contrat.

De façon remarquable (et c'est ici la différence avec le chapitre 2), la sélection à laquelle procède le principal est ici en adéquation avec le critère de sélection par viscosité évanescence. En effet, lorsque $\epsilon = 0$ et $\delta > 0$, les solutions (qui maximisent le critère du principal) coïncident avec celles obtenues en faisant tendre le bruit commun vers 0.

En apparence, le paradoxe observé dans le chapitre 2 a disparu, puisque la sélection par viscosité évanescence donne le même résultat que la sélection par optimisation du coût. En réalité, on observe toujours un paradoxe, mais au niveau des agents : pour $A \in \{-1, 1\}$, le gain espéré des agents est égal à R ; ceci est aussi le gain espéré pour le contrat trivial $\xi = R$ (qui correspond formellement à l'équilibre instable obtenu en posant $A = 0$ dans l'énoncé du Théorème 16). Mais, un calcul simple montre que, pour les deux solutions au jeu à champ moyen, la variance du gain de l'agent est égale à T , alors que cette variance est nulle pour le contrat trivial $\xi = R$. Autrement dit, le principal impose sa sélection aux agents. quand bien même ceux-ci préféreraient, sous une hypothèse raisonnable d'aversion au risque la situation instable $A = 0$.

Chapter 1

Uniqueness for Linear-Quadratic Mean Field Games with common noise

The purpose of this chapter is to show that a common noise may restore uniqueness in mean-field games. To this end, we focus on a class of examples driven by linear dynamics and quadratic cost functions, but with a possibly general nonlinear dependence upon the mean of the population. Given these linear quadratic mean-field games, we prove existence and uniqueness of solutions in presence of a common noise and construct a counter-example for which uniqueness fails in absence of common noise. This illustrates the principle, already observed in dynamical systems like ODEs, that introducing an appropriate noise may restore uniqueness.

1.1 Introduction

A short reminder on MFGs (for those who skipped Chapter 1)

As explained in the previous chapter, the theory of Mean Field Games (MFGs for short) is concerned with the study of asymptotic Nash equilibria for stochastic differential games with an infinite number of players subject to a mean field interaction (i.e each player is affected by the other players only through the empirical distribution of the system). In this regard, it is worth recalling that a Nash equilibrium constitutes a consensus (or compromise) between all the players from which no player has unilateral incentive to deviate.

As the number of players (which we denote by the upper case N throughout the chapter) of the stochastic differential game increases, finding Nash equilibria becomes an increasingly complex problem as it typically involves a system of N PDEs set on a space of dimension N . The motivation for studying the asymptotic regime is to reduce the underlying complexity. At least in the case where the players are driven by independent noises, the hope is indeed to take benefit from the theory of propagation of chaos for mean-field interacting systems (see for example [94]) in order to reduce the analysis of the whole system to the analysis of a single representative player. We made clear this fact in the introductory chapter.

In the analysis of the limiting MFGs, the representative player aims at minimizing a cost functional while interacting with an environment described by a flow of distributions. Finding Nash equilibria thus consists in finding optimal states whose flow of marginal distributions matches exactly the flow of distributions describing the environment. This is a constraint

of McKean-Vlasov type which requires to solve a fixed point problem over the set of time-dependent paths with values in the space of probability measures.

MFGs were introduced independently and simultaneously by Lasry and Lions [77] and by Caines, Huang and Malhamé [65] (who used the name of Nash Certainty Equivalence). We refer to the notes written by Cardaliaguet for a very good introduction to the subject ([18]). We also refer to the works of Carmona and Delarue, who studied MFGs with a probabilistic approach, see for instance [22, 26] together with the recent two-volume monograph [23, 24]. Many other authors have contributed to the rapid development of the theory, see Chapter 1 if needed. Under regularity conditions on the cost functional, existence of Nash equilibria has been proved in the above works (typically by using Schauder's fixed point argument). Further monotonicity conditions introduced by Lasry and Lions provide uniqueness of Nash equilibrium.

Purpose of the chapter

In this chapter, we investigate a class of Linear-Quadratic Mean Field Games (LQ-MFGs) in which the representative player at equilibrium interacts with the mean of its distribution. Here and below, we call LQ-MFG a mean-field game whose cost functionals are quadratic in the state and control variables and whose dynamics is linear in the state and control variables: **Still, the coefficients may depend in a more general fashion upon the distribution of the population; this is contrast with earlier works on mean-field games, in which the coefficients of LQ-MFGs are also required to be linear or quadratic with respect to the mean of the population.**

Inspired by earlier works in that direction, we suppose further that, in addition to the independent noises, the N players in the finite game are also subject to a common (or systemic) noise, such a modeling being motivated by practical applications. We refer for instance to the review by Guéant, Lasry and Lions [57] for earlier examples of mean-fields games with a common noise. We also refer to the linear-quadratic model (including linear-quadratic coefficients with respect to the mean of the population) introduced by Carmona, Fouque and Sun [27] for another example involving a common noise. In the latter article, the objective is to provide a detailed analysis of systemic risk in an interbanking borrowing and lending model: In this regard, the key feature is that the common noise affects the whole system and thus acts globally. In fact, financial market models often consider a kind of common market noise affecting the agents: For instance, in [29], demand could be seen as a common noise. Actually, mean field games with common noise are also related with mean-field games with a major agent, as introduced by Huang et al. ([61],[85]). The difference is that mean-field games with a major agent include a so-called major agent, whose actions impact the whole system; this is contrast with mean-field games with a common noise, in which all the players are usually regarded as small. Carmona and Zhu [28] provided a probabilistic approach to MFGs with a major agent, see also Chapter 7 in [24].

We proceed to find Nash equilibria through Carmona and Delarue's scheme, based on the theory of forward backward stochastic differential equations (FBSDE for short) of the McKean-Vlasov type. The major change is that, due to the presence of common noise, the representative player at equilibrium feels the mean-field interaction through its conditional expectation given the common noise. The environment is thus described by a stochastic process whose randomness comes only from the common noise.

The strategy is to characterize the environment as the forward component of an auxiliary FBSDE driven by the common noise only. Thanks to the common noise, this FBSDE is non-degenerate and thus satisfies an existence and uniqueness theorem proved by Delarue in [35]. This establishes the existence and uniqueness of a Nash equilibrium for this class of LQ-MFGs.

Restoration of uniqueness

The fact that, in the presence of a common noise, the FBSDE that characterizes the equilibria is uniquely solvable is the key result of this chapter. It should be regarded as an example of restoration of uniqueness by randomization of the equilibria.

In order to stress the fact that the common noise really helps for uniqueness, we present afterwards a counter-example to uniqueness of Nash equilibria for a mean-field game in the same class of LQ-MFGs but in the absence of common noise. This provides a concrete example when common noise restores uniqueness.

In fact, restoration of uniqueness by addition of a noise has had a long history in probability theory, but to the best of our knowledge, the example proposed in this chapter is the first one in the literature for MFGs. Several situations outside of the MFGs framework in which noise restores uniqueness are presented in the monograph by Flandoli (see [47]). Generally speaking, the idea of restoring uniqueness to ODEs by means of a random forcing goes back to the earlier work of Zvonkin [102] on the solvability of one-dimensional stochastic differential equations driven by non-Lipschitz continuous drifts. Several authors also contributed to the subject and addressed the higher dimensional framework, among which Veretennikov [96], Flandoli, Russo and Wolf [49, 50], Krylov and Röckner [71], Davie [34]... Similar questions have been also addressed in the framework of infinite dimensional stochastic differential equations, see for instance Flandoli, Gubinelli and Priola [48] and the aforementioned monograph by Flandoli [47]. We also refer to Duboscq and Réveillac [43] for similar questions when the drift in the dynamics is stochastic.

For expository purposes the work presented in this chapter involves one dimensional equations with prescribed coefficients but the results remain valid for higher dimensions.

1.2 A class of Linear-Quadratic N -players Games

We consider stochastic differential games with fixed terminal time $T > 0$ and $N \in \mathbb{N}$ players. Let $B = (B_t)_{t \in [0, T]}, (W_t^i)_{t \in [0, T]}, i = 1, \dots, N$, be $N + 1$ independent one dimensional Brownian motions defined on a complete filtered probability space $(\Omega, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t \in [0, T]}, \mathbb{P})$ satisfying the usual conditions. Let $\psi^i, i = 1, \dots, N$, be independent and identically distributed $\hat{\mathcal{F}}_0$ -measurable random variables taking values in \mathbb{R} and independent of all the Brownian motions.

Let σ, σ_0 be non-negative constants, $c \in \mathbb{R}$, and $b, f, g : \mathbb{R} \rightarrow \mathbb{R}$ be given Lipschitz continuous and bounded functions.

Description of the game

Given the above materials, we consider the following linear quadratic stochastic differential game with N players.

For all $i = 1, \dots, N$, the i^{th} player's state process during the game is given by $(X_t^i)_{t \in [0, T]}$ and takes values in \mathbb{R} . We consider a mean-field interaction given by an average of all players' states, as given by the empirical mean:

$$\mu_t^N = \frac{1}{N} \sum_{j=1}^N X_t^j, \quad \forall t \in [0, T].$$

With each player, we associate the the cost function and stochastic dynamics below:

$$\begin{aligned} & J(\alpha^1, \dots, \alpha^i, \dots, \alpha^N) \\ & := \mathbb{E} \left[\int_0^T \frac{1}{2} \left[(\alpha_t^i)^2 + \left(f(\mu_t^N) + X_t^i \right)^2 \right] dt + \frac{1}{2} \left(X_T^i + g(\mu_T^N) \right)^2 \right], \end{aligned} \quad (1.1)$$

where the dynamics $(X_t^i)_{t \in [0, T]}$ are given by

$$\begin{cases} dX_t^i = [cX_t^i + \alpha_t^i + b(\mu_t^N)] dt + \sigma dW_t^i + \sigma_0 dB_t, & \forall t \in [0, T], \\ X_0^i = \psi^i. \end{cases} \quad (1.2)$$

The cost function of each player depends on the strategies of the other players through the mean field process $(\mu_t^N)_{t \in [0, T]}$. In fact, each player controls her/his own state process by choosing a control process $\alpha^i = (\alpha_t^i)_{t \in [0, T]} \in \hat{\mathcal{H}}^2$, where $\hat{\mathcal{H}}^2$ is the set of $(\hat{\mathcal{F}}_t)_{t \in [0, T]}$ -progressively measurable processes $(\alpha_t)_{t \in [0, T]}$ satisfying

$$\mathbb{E} \left[\int_0^T |\alpha_s|^2 ds \right] < \infty.$$

When $\sigma_0 > 0$, expectation includes integration with respect to B , which is common to all the players. We say that we are in presence of common noise. It is important to note that the players are strongly correlated under the action of B . When $\sigma_0 = 0$, B is not added to any player's state, hence the players are weakly correlated as they depend upon one another through the empirical mean only. In that case, we say that we are in absence of common noise. We call B the common (or systemic) noise and σ_0 its intensity. In both cases the players are exchangeable and we can study the asymptotic regime of this game.

Finding Nash equilibria consists in finding sets of (consensual) controls between the players that minimize the cost functional of any player when all the others use the consensual controls, see the introductory chapter. This is a complex problem when N is large. The strategy proposed by the MFG theory in order to reduce the complexity is to find Nash equilibria for the asymptotic regime of the game ($N = \infty$) and to use the equilibria strategies in the regime $N = \infty$ as approximate-Nash equilibria for the N -player LQ-Games.

1.2.1 The asymptotic regime: A class of LQ-MFGs

By taking the limit as N tends to infinity in the above class of LQ-Games, we obtain a class of LQ-MFGs for which the representative player's state process, denoted by $(X_t)_{t \in [0, T]}$ (taking values in \mathbb{R}), interacts with its own expectation (in absence of common noise) or with its conditional expectation given B (in the presence of common noise). Generally speaking, this

follows from a propagation of chaos (or conditional propagation of chaos in the presence of common noise) property.

We now consider two Brownian motions $B = (B_t)_{t \in [0, T]}$, $(W_t)_{t \in [0, T]}$ defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ satisfying the usual conditions. Let the representative player's initial state be given by $\psi \in \mathcal{L}_{\mathcal{F}_0}^2$, the latter denoting the space of square integrable, \mathcal{F}_0 -measurable random variables. We suppose that the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ corresponds to the natural filtration generated by ψ, W, B augmented with \mathbb{P} -null sets. Also, we denote by $(\mathcal{F}_t^B)_{t \in [0, T]}$ the filtration generated by B only and augmented with \mathbb{P} -null sets. Lastly, we call \mathcal{H}^2 the space of $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable processes satisfying

$$\mathbb{E} \left[\int_0^T |\alpha_s|^2 ds \right] < \infty.$$

Finding Nash equilibria for this class of LQ-MFGs is possible through the scheme proposed by Carmona and Delarue [22]. In our situation, this scheme reads as follows:

Scheme 1.1. (*MFGs-solution scheme.*)

1. (*Mean field Input.*) Consider a continuous $(\mathcal{F}_t^B)_{t \in [0, T]}$ -adapted process $(\mu_t)_{t \in [0, T]}$ taking values in \mathbb{R} . This process is intended to match the representative player's flow of conditional expectations given B at equilibrium.
2. (*Cost Minimization.*) Solve the following stochastic optimal control problem for the representative player:
Find $\alpha^* \in \mathcal{H}^2$, satisfying

$$J(\alpha^*) = \min_{\alpha \in \mathcal{H}^2} J(\alpha) \tag{1.3}$$

$$\begin{aligned} &:= \min_{\alpha \in \mathcal{H}^2} \mathbb{E} \left[\int_0^T \frac{1}{2} [\alpha_t^2 + (f(\mu_t) + X_t)^2] dt \right. \\ &\quad \left. + \frac{1}{2} (X_T + g(\mu_T))^2 \right], \end{aligned} \tag{1.4}$$

under the stochastic dynamics:

$$\begin{cases} dX_t = [cX_t + \alpha_t + b(\mu_t)] dt + \sigma dW_t + \sigma_0 dB_t, & \forall t \in [0, T], \\ X_0 = \psi. \end{cases} \tag{1.5}$$

3. (*McKean-Vlasov constraint.*) Find $(\mu_t)_{t \in [0, T]}$ such that,

$$\forall t \in [0, T], \quad \mu_t = \mathbb{E}[X_t^{\alpha^*} | \mathcal{F}_t^B].$$

Remark 1.2. It is possible to show that, for all $t \in [0, T]$, $\mathbb{E}[X_t^{\alpha^*} | \mathcal{F}_t^B] = \mathbb{E}[X_t^{\alpha^*} | \mathcal{F}_T^B]$. Indeed, if $\mathcal{F}_{t, T}^B$ denotes the σ -field generated by the increments of B on $(t, T]$ augmented with \mathbb{P} -null sets, then, for all $t \in [0, T]$

$$\mathcal{F}_T^B = \mathcal{F}_t^B \vee \mathcal{F}_{t, T}^B.$$

So, for all $t \in [0, T]$, $\mathbb{E}[X_t^{\alpha^*} | \mathcal{F}_T^B] = \mathbb{E}[X_t^{\alpha^*} | \mathcal{F}_t^B \vee \mathcal{F}_{t,T}^B]$. Now, \mathcal{F}_t^B and $\mathcal{F}_{t,T}^B$ are independent and since $X_t^{\alpha^*}$ is measurable with respect to \mathcal{F}_t^B , we have

$$\mathbb{E}[X_t^{\alpha^*} | \mathcal{F}_t^B \vee \mathcal{F}_{t,T}^B] = \mathbb{E}[X_t^{\alpha^*} | \mathcal{F}_t^B].$$

With this observation, the McKean-Vlasov constraint (iii) below now reads:

(iii') (McKean-Vlasov constraint.) Find $(\mu_t)_{t \in [0, T]}$ such that

$$\forall t \in [0, T], \quad \mu_t = \mathbb{E}[X_t^{\alpha^*} | \mathcal{F}_T^B]. \quad (1.6)$$

Moreover, notice that, since the paths $t \mapsto X_t^{\alpha^*}$ are continuous, one can show that there exist continuous versions of the paths $t \mapsto \mathbb{E}[X_t^{\alpha^*} | \mathcal{F}_T^B]$.

1.3 Solvability of Scheme 1.1.

1.3.1 Stochastic Maximum Principle

We solve the problem (1.3-1.5) using Pontryagin's Stochastic Maximum Principle which yields a stochastic Hamiltonian system. For a review of this principle, see for example [89] and [101].

Definition 1.3.1. The problem (1.3-1.5) admits the following Hamiltonian:

$$H(t, a, x, y, u) = y[cx + a + b(u)] + \frac{1}{2}a^2 + \frac{1}{2}(x + f(u))^2 \quad (1.7)$$

for all $t \in [0, T]$, $a, x, y, u \in \mathbb{R}$.

Proposition 1.3. Suppose that we are given a continuous $(\mathcal{F}_t^B)_{t \in [0, T]}$ -adapted process $(\mu_t)_{t \in [0, T]}$ taking values in \mathbb{R} . Let H be the Hamiltonian for the problem (1.3-1.5) and $\alpha^* = (\alpha_t^*)_{t \in [0, T]} \in \mathcal{H}^2$. Then $(\alpha_t^*)_{t \in [0, T]}$ is a solution to the problem (1.3-1.5) if and only if there exists an adapted solution $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ to the adjoint FBSDE:

$$\begin{cases} dX_t = \partial_y H(t, \alpha_t^*, X_t, Y_t, \mu_t) dt + \sigma dW_t + \sigma_0 dB_t, & \forall t \in [0, T] \\ dY_t = -\partial_x H(t, \alpha_t^*, X_t, Y_t, \mu_t) dt + Z_t dW_t + Z_t^0 dB_t, & \forall t \in [0, T] \\ X_0 = \psi, \quad Y_T = X_T + g(\mu_T). \end{cases} \quad (1.8)$$

subject to

$$H(t, \alpha_t^*, X_t, Y_t, \mu_t) = \min_{a \in \mathbb{R}} H(t, a, X_t, Y_t, \mu_t), \quad \forall t \in [0, T], \quad \mathbb{P} - a.s.$$

Proof. The proposition follows from the Pontryagin stochastic maximum principle with the fact that for all $(t, y, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, we have convexity of the map $(a, x) \mapsto H(t, a, x, y, u)$. (See Theorem 5.4.6 in [89]) \square

Remark 1.4. 1. We say that $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ is an adapted solution to the adjoint FBSDE, if X, Y are $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted processes and Z, Z^0 are $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable processes satisfying

$$\mathbb{E} \left[\sup_{t \in [0, T]} [|X_t|^2 + |Y_t|^2] + \int_0^T [|Z_t|^2 + |Z_t^0|^2] dt \right] < \infty,$$

and solving system (7) \mathbb{P} -almost surely.

2. Observe that for all $(t, y, u) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, the map $(a, x) \mapsto H(t, a, x, y, u)$ is strictly convex. Hence, for all $(t, x, y, u) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, there is a unique $a^* = a^*(t, x, y, u) \in \mathbb{R}$ such that

$$H(t, a^*(t, x, y, u), x, y, u) = \min_{a \in \mathbb{R}} H(t, a, x, y, u).$$

Thanks to the strict convexity of H , we know that the zeros of $\partial_a H$ are the minimizers of H . In our situation the unique minimizer is given by $a^*(t, x, y, u) = -y$.

Therefore, for all $t \in [0, T]$, α_t^* in the previous proposition is uniquely defined as a function of (t, X_t, Y_t, μ_t) , precisely:

$$\forall t \in [0, T], \quad \alpha_t^* = -Y_t. \quad (1.9)$$

Proposition 1.5. *Suppose that we are given a continuous $(\mathcal{F}_t^B)_{t \in [0, T]}$ -adapted process $(\mu_t)_{t \in [0, T]}$ taking values in \mathbb{R} . The process $\alpha^* = -Y \in \mathcal{H}^2$ is a solution to the stochastic optimal control problem (1.3-1.5) if and only if $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ is an adapted solution to the FBSDE:*

$$\begin{cases} dX_t = [cX_t - Y_t + b(\mu_t)]dt + \sigma dW_t + \sigma_0 dB_t, & \forall t \in [0, T], \\ dY_t = [-X_t - cY_t - f(\mu_t)]dt + Z_t dW_t + Z_t^0 dB_t, & \forall t \in [0, T], \\ X_0 = \psi, \quad Y_T = X_T + g(\mu_T). \end{cases} \quad (1.10)$$

Proof. The proposition follows immediately from Proposition 3.15 and Remark 1.4. \square

In order to solve Scheme 1.1, we have to find solutions to FBSDE (1.10), subject to the constraint that the given process μ satisfies:

$$\forall t \in [0, T], \quad \mu_t = \mathbb{E}[X_t | \mathcal{F}_T^B]. \quad (1.11)$$

In this probabilistic approach, solutions to the MFG problem can be used to construct solutions to the conditional McKean-Vlasov FBSDE below (and vice versa)

$$\begin{cases} dX_t = [cX_t - Y_t + b(\mathbb{E}[X_t | \mathcal{F}_T^B])]dt + \sigma dW_t + \sigma_0 dB_t, & \forall t \in [0, T], \\ dY_t = [-X_t - cY_t - f(\mathbb{E}[X_t | \mathcal{F}_T^B])]dt + Z_t dW_t + Z_t^0 dB_t, & \forall t \in [0, T], \\ X_0 = \psi, \quad Y_T = X_T + g(\mathbb{E}[X_T | \mathcal{F}_T^B]). \end{cases} \quad (1.12)$$

1.3.2 Solvability of (1.10)

In this subsection, we show that given a continuous $(\mathcal{F}_t^B)_{t \in [0, T]}$ -adapted process $(\mu_t)_{t \in [0, T]}$, the problem (1.10) is uniquely solvable. Then we derive a characterization of the solution using an appropriate Ansatz.

Proposition 1.6. *Suppose that we are given a continuous $(\mathcal{F}_t^B)_{t \in [0, T]}$ -adapted process $(\mu_t)_{t \in [0, T]}$ taking values in \mathbb{R} . Then, there exist a unique adapted solution $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ to the*

FBSDE (1.10)

$$\begin{cases} dX_t = [cX_t - Y_t + b(\mu_t)]dt + \sigma dW_t + \sigma_0 dB_t, & \forall t \in [0, T] \\ dY_t = [-X_t - cY_t - f(\mu_t)]dt + Z_t dW_t + Z_t^0 dB_t, & \forall t \in [0, T] \\ X_0 = \psi, \quad Y_T = X_T + g(\mu_T). \end{cases}$$

Proof. Using the changes of variables

$$\bar{X}_t = X_t - \psi, \quad \bar{Y}_t = Y_t - X_t, \quad \bar{Z}_t = Z_t - \sigma, \quad \bar{Z}_t^0 = Z_t^0 - \sigma_0, \quad \forall t \in [0, T],$$

we get that solutions to problem (1.10) can be used to construct solution to FBSDE below (and vice versa)

$$\begin{cases} d\bar{X}_t = [(-1+c)\bar{X}_t - \bar{Y}_t + (-1+c)\psi + b(\mu_t)]dt + \sigma dW_t + \sigma_0 dB_t, & \forall t \in [0, T], \\ d\bar{Y}_t = [(1-c)\bar{Y}_t - 2c\bar{X}_t - 2c\psi - b(\mu_t) - f(\mu_t)]dt + \bar{Z}_t dW_t + \bar{Z}_t^0 dB_t, & \forall t \in [0, T], \\ X_0 = 0, \quad \bar{Y}_T = g(\mu_T). \end{cases}$$

Following the article of Yong (see [100]) on linear FBSDE, solutions to FBSDE above can be used to construct solutions to the reduced FBSDE (1.13) below (and vice versa)

$$\begin{cases} d\tilde{X}_t = [(-1+c)\tilde{X}_t - \tilde{Y}_t]dt, & \forall t \in [0, T] \\ d\tilde{Y}_t = [-2c\tilde{X}_t + (1-c)\tilde{Y}_t]dt + \tilde{Z}_t dW_t + \tilde{Z}_t^0 dB_t, & \forall t \in [0, T] \\ \tilde{X}_0 = 0, \quad \tilde{Y}_T = m, \end{cases} \quad (1.13)$$

where m is \mathcal{F}_T -measurable.

Now, using Theorem 6.1 in [100] in this situation, we conclude that the reduced FBSDE (1.13) has a unique solution if and only if

$$(0, 1) \exp(\mathcal{A}t)(0, 1)' > 0, \quad \forall t \in [0, T].$$

where $(0, 1)'$ denotes the transpose $(0, 1)$ and

$$\mathcal{A} = \begin{bmatrix} A & B \\ \hat{A} & \hat{B} \end{bmatrix} = \begin{bmatrix} (-1+c) & -1 \\ -2c & (1-c) \end{bmatrix}.$$

After some computations, we obtain

$$(0, 1) \exp(\mathcal{A}t)(0, 1)' = \frac{(1-c+\sqrt{1+c^2}) \exp(2t\sqrt{1+c^2}) - (1-c-\sqrt{1+c^2})}{2\sqrt{1+c^2} \exp(t\sqrt{1+c^2})}$$

It is enough to check the sign of

$$(1-c+\sqrt{1+c^2}) \exp(2t\sqrt{1+c^2}) - (1-c-\sqrt{1+c^2}).$$

The expression above has its minimum at $t = 0$, given by $(1-c+\sqrt{1+c^2}) - (1-c-\sqrt{1+c^2}) = 2\sqrt{1+c^2} > 0$.

So, $(1-c+\sqrt{1+c^2}) \exp(2t\sqrt{1+c^2}) - (1-c-\sqrt{1+c^2}) > 0, \quad \forall t \in [0, T]$. This implies $(0, 1) \exp(\mathcal{A}t)(0, 1)' > 0$ for all $t \in [0, T]$, and FBSDE (1.13) has a unique solution. \square

In fact, Theorem 6.1 in [100] gives more than a uniqueness result. It also provides an existence and uniqueness result for the Riccati ODE

$$\frac{dP_t}{dt} = P_t^2 + 2(1-c)P_t - 2c, \quad P_T = 0.$$

And, it states that the unique adapted solution to the reduced FBSDE (1.13) satisfies $\tilde{Y}_t = P_t \tilde{X}_t + p_t$, $\forall t \in [0, T]$, where $(p_t)_{t \in [0, T]}$ solves an associated BSDE.

Now, we want to characterize the solution of FBSDE (1.10). Inspired by the previous discussion and more generally by earlier studies on linear FBSDE (see for example [100, 99]), we make the following Ansatz.

Ansatz: We want the solution of (1.10) to satisfy the condition that Y has the following ‘linear’ form:

$$Y_t = \eta_t X_t + h_t, \quad \forall t \in [0, T], \quad (1.14)$$

where $(\eta_t)_{t \in [0, T]} \in \mathcal{C}^1$ is the unique solution to the Riccati ODE

$$\frac{d\eta_t}{dt} = \eta_t^2 - 2c\eta_t - 1, \quad \eta_T = 1, \quad (1.15)$$

The existence and uniqueness of a solution to (1.15) follow easily from the uniqueness of $(P_t)_{t \in [0, T]}$ by using the change of variable $\eta_t = P_t + 1$, for all $t \in [0, T]$.

And, $h = (h_t)_{t \in [0, T]}$ in (1.14) is required to be an $(\mathcal{F}_t^B)_{t \in [0, T]}$ -adapted process whose randomness comes only from the common noise and satisfies the BSDE:

$$\begin{cases} dh_t = [(-c + \eta_t)h_t - f(\mu_t) - \eta_t b(\mu_t)]dt + Z_t^1 dB_t, & \forall t \in [0, T], \\ h_T = g(\mu_T). \end{cases} \quad (1.16)$$

The following statement makes the conjecture clear:

Proposition 1.7. *Suppose that we are given a continuous $(\mathcal{F}_t^B)_{t \in [0, T]}$ -adapted process $(\mu_t)_{t \in [0, T]}$ taking values in \mathbb{R} . Then the solution, $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$, to problem (1.10) satisfies (1.14) with $h = (h_t)_{t \in [0, T]}$ satisfying BSDE (1.16).*

Proof. Let $(\eta_t)_{t \in [0, T]}$ be the solution to (1.15) and $(h_t, Z_t^1)_{t \in [0, T]}$ be the solution to the problem (1.16) (which is obviously uniquely solvable). We want to show that the unique solution $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ to the problem (1.10) satisfies (1.14). We do this by construction.

Let $(X_t)_{t \in [0, T]}$ be the solution of the forward SDE

$$\begin{cases} dX_t = [-(-c + \eta_t)X_t - h_t + b(\mu_t)]dt + \sigma dW_t + \sigma_0 dB_t, & \forall t \in [0, T], \\ X_0 = \psi. \end{cases} \quad (1.17)$$

Let $Y_t = \eta_t X_t + h_t$, $\forall t \in [0, T]$. By Itô’s formula

$$dY_t = \frac{d\eta_t}{dt} X_t dt + \eta_t dX_t + dh_t.$$

Then by substituting, in the above expression, (1.15), (1.16), (1.24) and putting $Z_t^0 = Z_t^1 + \eta_t \sigma_0$, $Z_t = \eta_t \sigma$, for all $t \in [0, T]$, we see that $(Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ solves the following backward SDE:

$$\begin{cases} dY_t = [-X_t - c\eta_t X_t - ch_t - f(\mu_t)]dt + Z_t dW_t + Z_t^0 dB_t, \quad \forall t \in [0, T], \\ Y_T = X_T + g(\mu_T). \end{cases}$$

Therefore $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ solves the following FBSDE

$$\begin{cases} dX_t = [cX_t - Y_t + b(\mu_t)]dt + \sigma dW_t + \sigma_0 dB_t, \quad \forall t \in [0, T], \\ dY_t = [-X_t - cY_t - f(\mu_t)]dt + Z_t dW_t + Z_t^0 dB_t, \quad \forall t \in [0, T], \\ X_0 = \psi, \quad Y_T = X_T + g(\mu_T). \end{cases} \quad (1.18)$$

By uniqueness, we identify the solution we have constructed with the solution of (1.10). It satisfies (1.14), since $Y_t = \eta_t X_t + h_t$, $\forall t \in [0, T]$, with $(\eta_t)_{t \in [0, T]}$ the solution to (1.15) and $(h_t, Z_t^1)_{t \in [0, T]}$ solution to (1.16). This concludes the proof. \square

1.3.3 Solvability of the matching problem (1.6)

Proposition 1.8. *Let $(\mu_t)_{t \in [0, T]}$ be an $(\mathcal{F}_t^B)_{t \in [0, T]}$ -adapted process with values in \mathbb{R} and then $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ be the unique solution to problem (1.10). Then, $\mu_t = \mathbb{E}[X_t | \mathcal{F}_T^B]$, for all $t \in [0, T]$, if and only if*

$$\begin{cases} d\mu_t = [-(-c + \eta_t)\mu_t - h_t + b(\mu_t)]dt + \sigma_0 dB_t, \quad \forall t \in [0, T], \\ \mu_0 = \mathbb{E}[\psi]. \end{cases} \quad (1.19)$$

Proof. First step. Consider a process $(\mu_t)_{t \in [0, T]}$, $(\mathcal{F}_t^B)_{t \in [0, T]}$ -adapted with values in \mathbb{R} , and call $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ the unique solution to problem (1.10). By the previous proposition $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ satisfies ansatz (1.14).

For all $t \in [0, T]$, taking conditional expectation of X_t given \mathcal{F}_T^B yields,

$$\mathbb{E}[X_t | \mathcal{F}_T^B] = \mathbb{E} \left[\psi + \int_0^t (cX_s - Y_s + b(\mu_s))ds + \int_0^t \sigma dW_s + \int_0^t \sigma_0 dB_s \mid \mathcal{F}_T^B \right].$$

Since $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ satisfies (1.14) and $(h_t)_{t \in [0, T]}$ is \mathcal{F}_T^B -measurable,

$$\begin{aligned} \mathbb{E}[X_t | \mathcal{F}_T^B] &= \mathbb{E}[\psi] + \int_0^t \mathbb{E} \left[-(-c + \eta_s)X_s - h_s + b(\mu_s) \mid \mathcal{F}_T^B \right] ds + \int_0^t \sigma_0 dB_s \\ &= \mathbb{E}[\psi] + \int_0^t \left(-(-c + \eta_s)\mathbb{E}[X_s | \mathcal{F}_T^B] - h_s + b(\mu_s) \right) ds + \int_0^t \sigma_0 dB_s. \end{aligned}$$

Suppose that $\mu_t = \mathbb{E}[X_t | \mathcal{F}_T^B]$, $\forall t \in [0, T]$. Then,

$$\mu_t = \mathbb{E}[\psi] + \int_0^t \left(-(-c + \eta_s)\mu_s - h_s + b(\mu_s) \right) ds + \int_0^t \sigma_0 dB_s, \quad \forall t \in [0, T].$$

Hence,

$$\begin{cases} d\mu_t = [-(-c + \eta_t)\mu_t - h_t + b(\mu_t)]dt + \sigma_0 dB_t, \quad \forall t \in [0, T], \\ \mu_0 = \mathbb{E}[\psi]. \end{cases}$$

Second step. Consider a process $(\mu_t)_{t \in [0, T]}$, $(\mathcal{F}_t^B)_{t \in [0, T]}$ -adapted with values in \mathbb{R} , and call $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ the unique solution to problem (1.10). By the previous proposition $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ satisfies ansatz (1.14).

Taking the conditional expectation of X_t given \mathcal{F}_T^B and proceeding as in the first step, we get

$$\mathbb{E}[X_t | \mathcal{F}_T^B] = \mathbb{E}[\psi] + \int_0^t [-(-c + \eta_s) \mathbb{E}[X_s | \mathcal{F}_T^B] - h_s + b(\mu_s)] ds + \int_0^t \sigma_0 dB_s. \quad (1.20)$$

Suppose that μ is a solution to (1.19). Then,

$$\mu_t = \mathbb{E}[\psi] + \int_0^t (-(-c + \eta_s) \mu_s - h_s + b(\mu_s)) ds + \int_0^t \sigma_0 dB_s, \quad \forall t \in [0, T]. \quad (1.21)$$

Subtracting (1.20) from (1.21) gives:

$$(\mu_t - \mathbb{E}[X_t | \mathcal{F}_T^B]) + \int_0^t (-c + \eta_s) (\mu_s - \mathbb{E}[X_s | \mathcal{F}_T^B]) ds = 0, \quad \forall t \in [0, T]. \quad (1.22)$$

We thus have a linear ordinary differential equation with initial value zero. It follows that

$$\mu_t - \mathbb{E}[X_t | \mathcal{F}_T^B] = 0, \quad \forall t \in [0, T].$$

The proof is complete. \square

Here is now the main result of this section:

Proposition 1.9. *There exists $(\alpha_t^*, \mu_t)_{t \in [0, T]}$ an MFGs-solution if and only if there exists $(\mu_t, h_t, Z_t^1)_{t \in [0, T]}$ an adapted solution to the FBSDE:*

$$\begin{cases} d\mu_t = [-(-c + \eta_t) \mu_t - h_t + b(\mu_t)] dt + \sigma_0 dB_t, & \forall t \in [0, T], \\ dh_t = [(-c + \eta_t) h_t - f(\mu_t) - \eta_t b(\mu_t)] dt + Z_t^1 dB_t, & \forall t \in [0, T], \\ h_T = g(\mu_T), \quad \mu_0 = \mathbb{E}[\psi]. \end{cases} \quad (1.23)$$

Moreover, the optimal feedback is given by:

$$\alpha_t^* = -\eta_t X_t - h_t, \quad \forall t \in [0, T].$$

Proof. First step. Suppose that $(\alpha_t^*, \mu_t)_{t \in [0, T]}$ is an MFGs-solution.

Since $(\mu_t)_{t \in [0, T]}$ is $(\mathcal{F}_t^B)_{t \in [0, T]}$ -adapted with values in \mathbb{R} , by Proposition 1.5, there exists $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ solution to (1.10) and $(\alpha_t^* = -Y_t)_{t \in [0, T]}$ solves the stochastic optimal control problem (1.3-1.5).

By Proposition 1.7, $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ solution to (1.10) satisfies ansatz (1.14). Therefore, we have $Y_t = \eta_t X_t + h_t$, $\forall t \in [0, T]$, with $(\eta_t)_{t \in [0, T]}$ the solution to (1.15) and $(h_t, Z_t^1)_{t \in [0, T]}$ solution to (1.16).

Also, since $(\alpha_t^*, \mu_t)_{t \in [0, T]}$ is an MFGs-solution, $(\mu_t)_{t \in [0, T]}$ must verify the condition $\mu_t = \mathbb{E}[X_t | \mathcal{F}_T^B]$, $\forall t \in [0, T]$. And by Proposition 1.8, it follows that $(\mu_t)_{t \in [0, T]}$ is a solution to (1.19).

Hence, $(\mu_t, h_t, Z_t^1)_{t \in [0, T]}$ is a solution to (1.23).

Second step. Suppose that we are given $(\mu_t, h_t, Z_t^1)_{t \in [0, T]}$ solution to (1.23). Clearly, $(\mu_t)_{t \in [0, T]}$ is $(\mathcal{F}_t^B)_{t \in [0, T]}$ -adapted with values in \mathbb{R} .

Let $(X_t)_{t \in [0, T]}$ be the solution of the forward SDE

$$\begin{cases} dX_t = [-(-c + \eta_t)X_t - h_t + b(\mu_t)] dt + \sigma dW_t + \sigma_0 dB_t, & \forall t \in [0, T], \\ X_0 = \psi. \end{cases} \quad (1.24)$$

Let also $Z_t^0 := Z_t^1 + \eta_t \sigma_0$, $Z_t := \eta_t \sigma$ and $Y_t := \eta_t X_t + h_t$, $\forall t \in [0, T]$.

By Propositions 1.5, 1.6 and 1.7, $(X_t, Y_t, Z_t, Z_t^0)_{t \in [0, T]}$ solves problem (1.10) and $(\alpha_t^* = -Y_t)_{t \in [0, T]}$ solves the stochastic optimal control problem (1.3-1.5).

Finally, it remains to check that the given $(\mu_t)_{t \in [0, T]}$ satisfies the McKean-Vlasov constraint; $\mu_t = \mathbb{E}[X_t | \mathcal{F}_T^B]$, $\forall t \in [0, T]$. This follows from Proposition 1.8.

Hence, $(\alpha_t^* = -Y_t, \mu_t)_{t \in [0, T]}$ is an MFGs-solution. \square

1.4 Unique solvability and Common noise

The next proposition shows that in presence of the common noise we have a unique Nash equilibrium for this class of LQ-MFGs.

This is possible thanks to the previous proposition which makes the equivalence between the solvability of the class of LQ-MFGs considered and the solvability of the auxiliary FBSDE (1.23). In presence of common noise, the system (1.23) is said to be non-degenerate. For an insight on the solvability of such FBSDE, see for example [82].

1.4.1 Unique solvability ($\sigma_0 > 0$)

Proposition 1.10. *Suppose that $\sigma_0 > 0$, then there is a unique Nash equilibrium for the class of LQ-MFGs under study.*

Proof. To prove this proposition, it is enough to show that there exists a unique adapted solution $(\mu_t, h_t, Z_t^1)_{t \in [0, T]}$ to the problem (1.23).

Let us define the following smooth, invertible and bounded function:

$$w_t = \exp \left(\int_t^T (-c + \eta_s) ds \right), \quad \forall t \in [0, T].$$

We now consider the transformations

$$\mu_t^* = w_t^{-1} \mu_t, \quad \forall t \in [0, T], \quad (1.25)$$

$$h_t^* = w_t h_t, \quad \forall t \in [0, T]. \quad (1.26)$$

Using these transformations, it follows immediately that $(\mu_t, h_t, Z_t^1)_{t \in [0, T]}$ is an adapted solution to the problem (1.23) if and only if $(\mu_t^*, h_t^*, Z_t^2)_{t \in [0, T]}$ is an adapted solution to

$$\begin{cases} d\mu_t^* = [-w_t^{-2} h_t^* + w_t^{-1} b(w_t \mu_t^*)] dt + w_t^{-1} \sigma_0 dB_t, & \forall t \in [0, T], \\ dh_t^* = [-w_t f(w_t \mu_t^*) - w_t \eta_t b(w_t \mu_t^*)] dt + Z_t^2 dB_t, & \forall t \in [0, T], \\ \mu_0^* = \mathbb{E}[\psi] w_0^{-1}, \quad h_T^* = g(\mu_T^*). \end{cases} \quad (1.27)$$

Finally, since f, b, g are given bounded and Lipschitz continuous functions, $w_t^{-1} > 0 \forall t \in [0, T]$ and $\sigma_0 > 0$, the problem (1.27) satisfies the hypothesis of the existence and uniqueness theorem of Delarue [35, Theorem 2.6].

Therefore, the system of FBSDE (1.23) admits a unique adapted solution and the proof is complete. \square

1.4.2 Non-uniqueness ($\sigma_0 = 0$)

Finding Nash equilibria for LQ-MFGs under study in the absence of common noise is equivalent to solving the FBSDE (1.23) with $\sigma_0 = 0$.

Thanks to the transformations (1.25-1.26) in the previous proof, the solvability of the system of FBSDEs (1.23) with $\sigma_0 = 0$ is equivalent to the solvability of the following problem:

Find $(\mu_t^*, h_t^*, Z_t^2)_{t \in [0, T]}$ an adapted solution to:

$$\begin{cases} d\mu_t^* = [-w_t^{-2}h_t^* + w_t^{-1}b(w_t\mu_t^*)]dt, & \forall t \in [0, T], \\ dh_t^* = [-w_t f(w_t\mu_t^*) - w_t\eta_t b(w_t\mu_t^*)]dt + Z_t^2 dB_t, & \forall t \in [0, T], \\ \mu_0^* = \mathbb{E}[\psi]w_0^{-1}, h_T^* = g(\mu_T^*). \end{cases} \quad (1.28)$$

Counter-Example to uniqueness: To construct a counter-example to uniqueness, we choose $f = b = \psi = 0$. We set $K_t = \int_0^t w_s^{-2} ds$, $\forall t \in [0, T]$, so that $K_T > 0$. Since the terminal time $T > 0$ is fixed, $R = K_T > 0$ is a constant.

Now, let us define $g : \mathbb{R} \rightarrow \mathbb{R}$ as follows;

$$g(x) = \begin{cases} 1 & \text{if } x < -R, \\ -x/R & \text{if } |x| \leq R, \\ -1 & \text{if } x > R. \end{cases}$$

For the specified LQ-MFG above, the problem (1.28) reads as follows:

Find a triple $(\mu_t^*, h_t^*, Z_t^2)_{t \in [0, T]}$, an adapted solution to

$$\begin{cases} d\mu_t^* = -w_t^{-2}h_t^* dt, & \forall t \in [0, T], \\ dh_t^* = Z_t^2 dB_t, & \forall t \in [0, T], \\ \mu_0^* = 0, h_T^* = g(\mu_T^*). \end{cases} \quad (1.29)$$

For all $A \in \mathbb{R}$, such that $|A| \leq 1$, the processes $(-AK_t, A, 0)_{t \in [0, T]}$ are adapted solutions to (1.29).

Hence, we found infinitely many Nash equilibria for this LQ-MFG without common noise. The corresponding optimal feedbacks are given by

$$\alpha_t^* = -\eta_t X_t - Aw_t^{-1}, \quad \forall t \in [0, T],$$

for all $A \in \mathbb{R}$, such that $|A| \leq 1$.

1.5 Conclusion and prospects

The results exposed in this note illustrate the power of adding common noise as an hypothesis in the study of MFGs from a mathematical perspective. For the class of LQ-MFGs studied here, the uniqueness of the Nash equilibrium is obtained from the common noise hypothesis. In particular, no monotonicity hypothesis is required! These results are in line with the general idea that adding a noise to a dynamical system can help to achieve uniqueness.

In connection with this chapter, we propose the following prospects for further research:

1. An interesting question concerns the zero noise limit of the Nash equilibria of the LQ-MFGs with common noise when uniqueness fails for the situation without common noise. Are weak limits unique? Which equilibria do the weak limits select?

In the next chapter, we will address this question in a specific subclass of the LQ-MFGs we have considered in this chapter.

2. Another prospect is to find other examples of restoration of uniqueness. The key fact we used in this chapter is that the solutions to the class of LQ-MFGs we addressed have a Gaussian distribution, conditional on the realization of the common noise (and on the initial condition when the latter is random). And, in fact, only the mean of the equilibria depends on the common noise, the variance being entirely prescribed in terms of the data of the game. In other words, equilibria belong to a one-dimensional parametric model, the (conditional) mean of the equilibria playing the role of the parameter. In this regard, our strategy for restoring uniqueness consists in randomizing the dynamics satisfied by the parameter. Of course, the fact that the parameter is of finite dimension (in fact it is just of dimension one in the framework we addressed above) is absolutely crucial.

So, we may wonder about other examples of a parametric model for which our randomization method could be applied. In this regard, it should be interesting to consider more general models admitting Gaussian (or conditionally Gaussian) equilibria but with a variance that explicitly depends on the common noise. In this framework, it is worth mentioning that the LQ structure used in this chapter forces the optimal feedback to be affine, see (1.14), and thus the equilibria to be Gaussian. So, for instance, we could think of a similar model to the one considered in this chapter but with a volatility coefficient that depends on the conditional mean μ of the population.

We refer to Guéant, Lasry and Lions [57], for another example of an MFG admitting parametric equilibria with a Pareto distribution.

3. We emphasize the fact that, in [24, Chapter 3], the authors extended the model addressed in this chapter by allowing the coefficients b , f and g to depend on the whole distribution of the population (and not only on the mean). In fact, this is well understood: Since the equilibria have a conditional Gaussian distribution with a prescribed variance, the dependence upon the general law of the population may be reformulated, in equilibrium, as a mere dependence upon the mean of the population.
4. Lastly, we emphasize the fact that similar questions should be addressed for models with non-parametric equilibria at the cost of forcing the whole dynamics by an infinite dimensional noise. Indeed, in the general non-parametric case, the (conditional) distribution of the population is expected to live in an infinite dimensional space; as the

forcing noise is usually required to have a non-degenerate structure, it should be then needed to use a noise of infinite dimension. Still, this raises many questions, including the form of the infinite dimensional noise that should equip the space of probability measures.

Chapter 2

On selection in linear quadratic mean field games

In this chapter, we pursue our discussion on uniquely solvable mean-field games with a common noise whose corresponding counterpart without common noise have many equilibria. We study the selection problem for those mean field games without common noise via three approaches.

A common approach is to select, amongst all the equilibria, those yielding the minimal cost for the representative player. Another one is to select equilibria that are included in the support of the zero noise limit of the mean-field game with common noise. A last one is to select equilibria supported by the limit of the mean-field component of the corresponding N -player game as the number of players goes to infinity. The contribution of this chapter is to provide an example in which the last two approaches select the same equilibria, but the first approach selects another one.

2.1 Introduction

In the previous chapter, we presented a class of linear quadratic mean field games (LQ-MFGs for short) with a common noise for which equilibria are characterized by a one dimensional usual FBSDE. In this standard FBSDE, the forward part describes the mean of the players' states and the backward one accounts for the affine part of the feedback controls. (As already explained, we call those mean-field games LQ because the coefficients entering the optimization problem are linear-quadratic: still the dependence upon the measure argument can be in the form of a general non-linear function of the mean.)

2.1.1 Description of the mean field game and related selection of equilibria

For the sake of clarity, we recall the class of the LQ-MFGs addressed in Chapter 2 and the corresponding characterization of equilibria through FBSDEs.

We are given two Brownian motions $B = (B_t)_{t \in [0, T]}$ and $W = (W_t)_{t \in [0, T]}$ defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ satisfying the usual conditions. The representative player's initial state is given in the form of a random variable $\psi \in \mathcal{L}^2_{\mathcal{F}_0}, \mathcal{L}^2_{\mathcal{F}_0}$ standing for the collection of square integrable \mathcal{F}_0 -measurable random variables. We suppose (mostly for convenience) that the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ corresponds to the natural filtration generated by ψ, W, B augmented with \mathbb{P} -null sets. Also, we let $(\mathcal{F}_t^B)_{t \in [0, T]}$ be the filtration generated by B only and augmented with \mathbb{P} -null sets.

Throughout the chapter, we consider controls $\alpha := (\alpha_t)_{t \in [0, T]} \in \mathcal{H}^2$, where \mathcal{H}^2 is the space of $(\mathcal{F}_t)_{t \in [0, T]}$ -progressively measurable processes satisfying

$$\mathbb{E} \left[\int_0^T |\alpha_t|^2 dt \right] < +\infty.$$

Finally, we consider a constant $c \in \mathbb{R}$ and three bounded and Lipschitz continuous functions $f, b, g : \mathbb{R} \mapsto \mathbb{R}$. The MFG problem considered in the previous chapter reads:

Scheme 2.1. (MFG-problem)

1. (Mean field Input) Consider a continuous $(\mathcal{F}_t^B)_{t \in [0, T]}$ -adapted process $(\mu_t)_{t \in [0, T]}$ taking values in \mathbb{R} .
2. (Cost Minimization) Find $\alpha^* \in \mathcal{H}^2$, satisfying

$$J(\alpha^*) = \min_{\alpha \in \mathcal{H}^2} J(\alpha) := \min_{\alpha \in \mathcal{H}^2} \mathbb{E} \left[\int_0^T \frac{1}{2} [\alpha_t^2 + (f(\mu_t) + X_t)^2] dt + \frac{1}{2} (X_T + g(\mu_T))^2 \right]$$

under the stochastic dynamics

$$\begin{cases} dX_t = [cX_t + \alpha_t + b(\mu_t)]dt + \sigma dW_t + \sigma_0 dB_t, & \forall t \in [0, T], \\ X_0 = \psi. \end{cases} \quad (2.1)$$

3. (McKean-Vlasov constraint) Find $(\mu_t)_{t \in [0, T]}$ such that:

$$\forall t \in [0, T], \quad \mu_t = \mathbb{E}[X_t^{\alpha^*} | \mathcal{F}_t^B].$$

We recall from Chapter 2 that one can characterize the solutions of this MFG-problem through FBSDEs as in the following proposition.

Proposition 2.2. *There exists $(\alpha_t, \mu_t)_{t \in [0, T]}$ an MFG-solution if and only if there exists an $(\mathcal{F}_t^B)_{t \in [0, T]}$ adapted solution $(\mu_t^{\psi, \sigma_0}, h_t^{\psi, \sigma_0}, Z_t^{\psi, \sigma_0})_{t \in [0, T]}$ to the FBSDE:*

$$\begin{cases} \forall t \in [0, T], \\ d\mu_t^{\psi, \sigma_0} = [-w_t^{-2} h_t^{\psi, \sigma_0} + w_t^{-1} b(w_t \mu_t^{\psi, \sigma_0})] dt + w_t^{-1} \sigma_0 dB_t, \\ dh_t^{\psi, \sigma_0} = [-w_t f(w_t \mu_t^{\psi, \sigma_0}) - w_t \eta_t b(w_t \mu_t^{\psi, \sigma_0})] dt + Z_t^{\psi, \sigma_0} dB_t, \\ \text{and } \mu_0^{\psi, \sigma_0} = \mathbb{E}[\psi] w_0^{-1}, \quad h_T^{\psi, \sigma_0} = g(\mu_T^{\psi, \sigma_0}). \end{cases} \quad (2.2)$$

where

$$w_t := \exp \left(\int_t^T (-c + \eta_s) ds \right), \quad \forall t \in [0, T],$$

$$\eta := (\eta_t)_{t \in [0, T]} \text{ is the unique solution to the Riccati ODE: } \frac{d\eta_t}{dt} = \eta_t^2 - 2c\eta_t - 1, \quad \eta_T = 1.$$

When FBSDE (2.2) is solvable, $(\alpha_t, \mu_t)_{t \in [0, T]}$ and $(\mu_t^{\psi, \sigma_0}, h_t^{\psi, \sigma_0})_{t \in [0, T]}$ are connected by the following relationships:

$$\begin{aligned} \mu_t &= w_t \mu_t^{\psi, \sigma_0}, \quad \forall t \in [0, T], \\ \alpha_t &= -\eta_t X_t - h_t, \quad \text{where } h_t = w_t^{-1} h_t^{\psi, \sigma_0}, \quad \forall t \in [0, T], \end{aligned}$$

$(X_t)_{t \in [0, T]}$ being implicitly defined as the solution of the forward equation (2.1).

We know from Chapter 2 that we can find mild solutions under which:

1. In the presence of common noise (i.e $\sigma_0 > 0$), FBSDE (2.2) is uniquely solvable, in which case there is a unique equilibrium to the LQ-MFG;
2. In the absence of common noise (i.e $\sigma_0 = 0$), FBSDE (2.2) is solvable, but it admits multiple solutions. In that case, which we call *non-degenerate*, there may be multiple equilibria.

In the latter (degenerate) case, a relevant question is which equilibria should we select? To answer this question, we consider three methods of selection, as described below:

1. (minimal cost selection) Select equilibria that minimize the cost of the representative player in the LQ-MFG without common noise;
2. (zero noise limit selection) Consider the unique stochastic equilibrium to LQ-MFG with common noise and find its weak limit as $\sigma_0 \rightarrow 0$. If this limit exists, we select the equilibria that are included in the support of the limit.
3. (N -player limit selection) Solve the N -player game without common noise and find the weak limit of the equilibrium as $N \rightarrow +\infty$. If this limit exists, we select the equilibria that are included in the support of the limit.

Whilst the first method is directly connected with the optimization structure underpinning the LQ-MFG, the second approach is in fact much more general. Indeed, the idea of restoring uniqueness by means of a random forcing has been extensively studied in probability theory. It goes back to the earlier work of Zvonkin [102] on the solvability of one-dimensional stochastic differential equations driven by non-Lipschitz continuous drifts. Several authors also contributed to the subject and addressed the higher dimensional framework, among which Veretennikov [96], Flandoli, Russo and Wolf [49, 50], Krylov and Röckner [71], Davie [34]... Similar questions have been also addressed in the framework of infinite dimensional stochastic differential equations, see for instance Flandoli, Gubinelli and Priola [48] and the monograph by Flandoli [47]. We also refer to Duboscq and Réveillac [43] for similar questions when the drift in the dynamics is stochastic. Still, although restoration of uniqueness has been investigated in various frameworks, including, as we just mentioned, infinite dimensional ones, finding the zero-noise limit when the corresponding deterministic or ordinary equation has multiple solutions is a challenging question, for which fewer results are known. The earlier result in this direction is due to Bafico and Baldi, see [5]; it provides a rather complete picture of the selection procedure for one-dimensional dynamics with isolated singularities. Examples treated in [5] will serve us as a benchmark example throughout this chapter, but, in fact, we will mostly follow another approach to these examples, due to Delarue and Flandoli [36]. In [36], the authors not only address the zero noise limit but also make explicit the typical time

at which selection occurs; our strategy is to do the same below. We refer the reader to [55] for another point of view on zero noise limit for one-dimensional dynamics, based on large deviations, and to [2, 37, 66] for examples of selection in higher (possibly infinite) dimension.

As for the third method, it is directly connected with the fact that mean-field games are understood as limiting versions of games with finitely many players. In this regard, it is a rather challenging question to show that equilibria to the finite player games do converge to a solution of a mean-field game. In fact, the approach to this question depends on the nature of the equilibria: In the finite player system, equilibria may be searched in an open or closed loop form. As for open loop equilibria, as we have considered in Chapter 2, weak compactness methods have been studied by Fischer [46] and Lacker [72]. Generally speaking, the point therein is to prove that the support of any weak limit of the laws of the empirical distributions of the finite player game equilibria is included in the set of solutions to the limiting mean-field game. Still, to the best of our knowledge, nothing has been said so far on the exact shape of this support: This is the question we want to address below in the particular example specified in the next paragraph. As for equilibria in closed loop form, the only main general result on the convergence of equilibria is due to Cardaliaguet, Delarue, Lasry and Lions [19] and is based on the so-called master equation for mean field games, which is a PDE set on the space of probability measures: In our case, this PDE should be understood (up to correction terms) as the decoupling field of (2.2). We will go back to this point next.

2.1.2 A particular case

We are going to illustrate the aforementioned three approaches in a particular LQ-MFG. Throughout the chapter, we consider the particular case when $f = b = \psi = 0$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$g(x) := -\frac{x}{r_\delta} \mathbb{1}_{|x| \leq r_\delta} - \text{sign}(x) \mathbb{1}_{|x| > r_\delta}, \quad (2.3)$$

where for a fixed time $\delta \in (0, T)$, $r_\delta := \int_\delta^T w_s^{-2} ds > 0$.

Proposition 2.2 states that, in order to find an equilibrium to this particular LQ-MFG, it is sufficient (and in fact necessary as well) to find a continuous, $(\mathcal{F}_t^B)_{t \in [0, T]}$ adapted, solution $(\mu_t^{0, \sigma_0}, h_t^{0, \sigma_0}, Z_t^{0, \sigma_0})_{t \in [0, T]}$ to the FBSDE

$$\begin{cases} d\mu_t^{0, \sigma_0} = [-w_t^{-2} h_t^{0, \sigma_0}] dt + w_t^{-1} \sigma_0 dB_t, & \forall t \in [0, T] \\ dh_t^{0, \sigma_0} = Z_t^{0, \sigma_0} dB_t, & \forall t \in [0, T] \\ \mu_0^{0, \sigma_0} = 0, \quad h_T^{0, \sigma_0} = g(\mu_T^{0, \sigma_0}). \end{cases} \quad (2.4)$$

In the presence of common noise (i.e $\sigma_0 > 0$), the FBSDE (2.4) has a unique solution. Thus there exists a unique equilibrium $(w_t \mu_t^{0, \sigma_0}, \alpha_t = -\eta_t X_t - w_t^{-1} h_t^{0, \sigma_0})_{t \in [0, T]}$ whose randomness depends only on the common noise B .

In absence of common noise (i.e $\sigma_0 = 0$), the system (2.4) becomes

$$\begin{cases} d\mu_t^{0, 0} = [-w_t^{-2} h_t^{0, 0}] dt, & \forall t \in [0, T] \\ dh_t^{0, 0} = 0, & \forall t \in [0, T] \\ \mu_0^{0, 0} = 0, \quad h_T^* = g(\mu_T^{0, 0}). \end{cases} \quad (2.5)$$

Our analysis is based upon the following observation that (2.5) has multiple solutions:

Proposition 2.3. *There exist three solutions to (2.5), that are*

$$(\mu_t^{0,0}, h_t^{0,0}, Z_t^{0,0})_{t \in [0, T]} = \left(-A \int_0^t w_s^{-2} ds, A, 0 \right)_{t \in [0, T]} \quad \text{for } A \in \{-1, 0, 1\}. \quad (2.6)$$

Proof. The first point is to check that the functions given in the statement are indeed solutions to the equation. In fact, the only difficult point is to check the boundary condition. When $A = 0$, there is no difficulty. When $A = 1$, we observe that $|\mu_T^{0,0}| \geq r_\delta$. Hence, $g(\mu_T^{0,0}) = 1$, which is indeed equal to 1. The case $A = -1$ is treated in the same way.

It then remains to check that there are no other solutions. In fact, whatever the solution, the process $(h_t^{0,0})_{t \in [0, T]}$ must be constant, hence it must be equal to some $A \in \mathbb{R}$. Then, $\mu_T^{0,0} = -A \int_0^T w_t^{-2} dt$.

If $|A \int_0^T w_t^{-2} dt| \leq r_\delta$, then the terminal boundary condition writes $A = -A \int_0^T w_t^{-2} dt / r_\delta$, which yields $A = 0$. If $|A \int_0^T w_t^{-2} dt| > r_\delta$, the boundary condition is in $\{-1, 1\}$ and we get $A \in \{-1, 1\}$. □

2.1.3 Main result and organization of the chapter

The main result of this chapter can be summarized as follows.

Theorem 2.4. *As for the example introduced in Subsection 2.1.2, the minimal cost selection selects, in the regime $\sigma_0 = 0$, the equilibrium corresponding to $A = 0$, whilst the zero-noise limit and the N -player game approaches select a randomized equilibrium, as given by the equilibrium $A = 1$ with probability $1/2$ and by the equilibrium $A = -1$ with probability $1/2$.*

So, the striking fact of this chapter is to show that the minimal cost selection does not yield the same result as the other two approaches!

The chapter is organized as follows. We implement the first method, which we call minimal cost selection, in Section 2.2. In Section 2.3, we make clear what is the notion of master equation in our setting. It plays a key role in the subsequent analysis of the zero-noise limit and of the convergence of the N -player equilibria. Section 2.4 is dedicated to the analysis of the zero-noise limit, whilst we focus on the limit of the N -player equilibria in Section 2.4. Further computations, that are used in the text, are detailed in Appendix.

2.2 Minimal cost selection

Keep in mind the particular LQ-MFG (2.4) and focus more specifically on the case without common noise (i.e $\sigma_0 = 0$), see (2.5). As stated in Proposition 2.3, one can construct three distinct equilibria to the LQ-MFG. The representative player is thus faced with a choice of the equilibrium to implement in the LQ-MFG. Recall that the representative player's aim is to minimize a cost functional. Because being at equilibrium does not imply minimization of the cost functional, a way to choose an equilibrium among the three available ones is to find which one(s) yield the minimal cost. This is what we call below **the cost minimization approach**.

In our specific framework, we have the following result:

Proposition 2.5. *With the notations of Proposition 2.3, the cost minimization approach selects the equilibrium corresponding to $A = 0$, i.e*

$$(\mu_t = 0, \alpha_t = -\eta_t X_t)_{t \in [0, T]} \quad \text{where} \quad X_t = w_t \int_0^t \sigma w_s^{-1} dW_t, \quad \forall t \in [0, T].$$

Proof. Given, $A \in \{-1, 0, 1\}$, the dynamics of the representative player, $(X_t)_{t \in [0, T]}$, and the controls, $(\alpha_t)_{t \in [0, T]}$, at equilibrium are given by

$$\begin{cases} X_0 = 0, \\ dX_t = [(c - \eta_t)X_t - Aw_t^{-1}]dt + \sigma dW_t, \quad \forall t \in [0, T] \\ \alpha_t = -\eta_t X_t - Aw_t^{-1} \quad \forall t \in [0, T]. \end{cases}$$

We recall that the cost functional is given by

$$J(\alpha) = \mathbb{E} \left[\int_0^T \frac{1}{2} [\alpha_t^2 + (X_t)^2] dt + \frac{1}{2} (X_T + g(\mu_T))^2 \right], \quad \alpha \in \mathcal{H}^2.$$

By replacing the control at equilibrium in the cost functional, we get (with an obvious notation for J_A)

$$J_A = \mathbb{E} \left[\int_0^T \frac{1}{2} [(-\eta_t X_t - w_t^{-1} A)^2 + (X_t)^2] dt + \frac{1}{2} (X_T + A)^2 \right], \quad A \in \{-1, 0, 1\}.$$

In order to expand J_A , we recall that

$$\begin{aligned} \mathbb{E}[X_t] &= \mu_t = -Aw_t \int_0^t w_s^{-2} ds, \\ \mathbb{V}[X_t] &= \mathbb{E}[(X_t - \mu_t)^2] = w_t^2 \sigma^2 \int_0^t w_s^{-2} ds, \quad \forall t \in [0, T]. \end{aligned}$$

We then expand J_A as follows

$$\begin{aligned} J_A &= \mathbb{E} \left[\int_0^T \frac{1}{2} [(-\eta_t X_t - w_t^{-1} A)^2 + (X_t)^2] dt + \frac{1}{2} (X_T + A)^2 \right], \\ &= \frac{1}{2} \mathbb{E} \left[\int_0^T \left((1 + \eta_t^2) X_t^2 + 2A\eta_t w_t^{-1} X_t + A^2 w_t^{-2} \right) dt \right] + \frac{1}{2} \mathbb{E} [(X_T + A)^2] \\ &= \frac{1}{2} \int_0^T \left((1 + \eta_t^2) \mathbb{E} [(X_t - \mu_t)^2] + (1 + \eta_t^2) \mu_t^2 + 2A\eta_t w_t^{-1} \mu_t + A^2 w_t^{-2} \right) dt \\ &\quad + \frac{1}{2} \mathbb{E} [(X_T + A)^2] \end{aligned}$$

$$\begin{aligned}
J_A &= \frac{A^2}{2} \left[\int_0^T \left((1 + \eta_t^2) w_t^2 \left(\int_0^t w_s^{-2} ds \right)^2 - 2\eta_t \left(\int_0^t w_s^{-2} ds \right) + w_t^{-2} \right) dt \right] \\
&\quad + \frac{A^2}{2} \left(1 - w_T \int_0^T w_s^{-2} ds \right)^2 \\
&\quad + \frac{1}{2} w_T^2 \int_0^T \sigma^2 w_t^{-2} dt + \frac{1}{2} \int_0^T (1 + \eta_t^2) \sigma^2 w_t^{-2} \left(\int_0^t w_s^{-2} ds \right) dt,
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
\mathbb{E}[(X_T + A)^2] &= (A + \mu_T)^2 + \mathbb{E}[(X_T - \mu_T)^2] \\
&= A^2 \left(1 - w_T \int_0^T w_s^{-2} ds \right)^2 + w_T^2 \int_0^T \sigma^2 w_t^{-2} dt \quad \forall t \in [0, T]
\end{aligned}$$

In order to conclude, it remains to take into account the fact that

$$\begin{aligned}
&(1 + \eta_t^2) w_t^2 \left(\int_0^t w_s^{-2} ds \right)^2 - 2\eta_t \left(\int_0^t w_s^{-2} ds \right) + w_t^{-2} \\
&= w_t^2 \left(\int_0^t w_s^{-2} ds \right)^2 + \left(\eta_t w_t \int_0^t w_s^{-2} ds - w_t^{-1} \right)^2 > 0, \quad \forall t \in [0, T].
\end{aligned}$$

One concludes that J_A , for $A \in \{-1, 0, 1\}$, is minimal when $A = 0$, and $J_{\pm 1} > J_0$, which completes the proof. \square

Although this rule of selection is sometimes met in the literature, the purpose of this chapter is precisely to prove that it leads in fact to contradictory results with the other rules of selection we addressed in the presentation of this chapter.

Remark 2.6. *The fact that the method based upon minimal cost selection does not yield sharp results should not come as a surprise. Indeed, it is worth mentioning that, in Scheme 2.1, we can add any function of μ_T to the terminal cost entering the definition of J . Obviously, this should not change the minimizers of J since the value of μ_T is kept frozen in the optimization procedure. Still, this may certainly modify the output of the selection method.*

In the analysis performed above, such a procedure would consist in adding a function of A to the cost J . By tuning for free this additional function, we could modify the result of the above analysis. This strongly suggests that the minimal cost selection method is of a limited scope.

2.3 Master equation and related PDE estimates

In this section, we consider the case $\sigma_0 \in (0, 1)$ in our benchmark example. By Chapter 2, we know that there is a unique equilibrium to the LQ-MFG, which is described by FBSDE (2.4).

Because we know that FBSDE (2.4) is uniquely solvable, we can use Ma-Protter-Yong's four-step-scheme [82] to represent the solution, see also [83]. The four-step-scheme provides a so-called decoupling field that decouples the two forward and backward equations of the FBSDE, meaning that it permits to represent the backward component of the solution in terms of the forward one. Due to the diffusive effect of the Brownian motion $(B_t)_{t \in [0, T]}$, such

a decoupling field is smooth. Through the Cole-Hopf transformation (which we make clear below), it can be represented explicitly and then inserted into the FBSDE (2.4): This allows to read the forward component of (2.4) as a standard SDE.

2.3.1 Master equation

The first step is to make the connection between the aforementioned decoupling field and the notion of master equation.

The concept of master equation was introduced by Lions [81] in his lectures on mean field games at *Collège de France*. Generally speaking, the master equation is an equation for the value of the mean-field game. It is regarded as a function of the initial conditions of the game, which include: Initial time, initial state of the representative player and initial state of the population. To ensure that the value function indeed makes sense, uniqueness of equilibria must be true.

In our case, $\sigma_0 \in (0, 1)$ and the LQ-MFG has a unique equilibrium. Still, we prefer to write down an equation for the (optimal) feedback function of the LQ-MFG instead of an equation for the value function. In fact both are related with one another through a standard minimization argument of the Hamiltonian and, in our framework, the (optimal) feedback function is given by the opposite of the derivative of the value function, the derivative being taken with respect to the private state of the representative player. We refer to [23, Chapters 3 and 4] for details.

In fact, we know from Proposition 2.2 that the equilibrium strategy of the LQ-MFG must be of the form

$$\alpha_t = -\eta_t X_t - w_t^{-1} h_t^{\psi, \sigma_0}, \quad \forall t \in [0, T],$$

where h^{ψ, σ_0} solves the backward equation in (2.4) (with a general initial condition ψ instead of 0 for the forward process).

Now, as we recalled right above, it is a standard fact from FBSDE theory, see for instance [35, 82], that the backward process can be put in the form

$$h_t^{\psi, \sigma_0} = \theta^{\sigma_0}(t, \mu_t^{\psi, \sigma_0}), \quad \forall t \in [0, T].$$

Here, θ^{σ_0} is called the decoupling field of the FBSDE (2.4). At the end of the day, the function

$$[0, T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \ni (t, x, m) \mapsto -\eta_t x - w_t^{-1} \theta^{\sigma_0}(t, x, \bar{m}),$$

is the right candidate for solving the master equation (for the feedback function). Here $\mathcal{P}_1(\mathbb{R})$ is the space of probability measures on \mathbb{R} with a finite first moment and \bar{m} stands for the mean of m when $m \in \mathcal{P}_1(\mathbb{R})$. In fact, instead of writing down the full master equation (which is a difficult object, see [19, 30, 53] and [24, Chapter 5]), we just write down the equation for θ^{σ_0} , which is enough for our own purpose. To do so, notice from [35, 38], see also the book [73], that θ^{σ_0} belongs to $\mathcal{C}^{1,2}([0, T] \times \mathbb{R}; \mathbb{R}) \cap \mathcal{C}([0, T] \times \mathbb{R}; \mathbb{R})$ and is a classical solution to the following quasilinear parabolic PDE with terminal condition :

$$\begin{cases} \forall (t, x) \in [0, T] \times \mathbb{R}, \\ \partial_t \theta^{\sigma_0}(t, x) - w_t^{-2} \theta^{\sigma_0}(t, x) \partial_x \theta^{\sigma_0}(t, x) + \frac{1}{2} \sigma_0^2 w_t^{-2} \partial_{xx}^2 \theta^{\sigma_0}(t, x) = 0, \\ \theta^{\sigma_0}(T, x) = g(x), \quad \forall x \in \mathbb{R}. \end{cases} \quad (2.7)$$

This PDE is well-known in the literature: it is a Burgers type PDE. For a theoretical and numerical entry points on the analysis of Burgers type PDEs, we refer to the textbook [79], and to the article [15]. It is uniquely solvable and a representation of its solution is obtained through the Cole-Hopf transformation. For every $t \in [0, T)$, we write $r_t = \int_t^T w_s^{-2} ds$. This representation reads as follows:

$$\theta^{\sigma_0}(t, x) = \frac{\int_{\mathbb{R}} \left(\frac{x-y}{r_t}\right) \exp(\sigma_0^{-2}(-\int_0^y g(v)dv - \frac{(x-y)^2}{2r_t})) dy}{\int_{\mathbb{R}} \exp(\sigma_0^{-2}(-\int_0^y g(v)dv - \frac{(x-y)^2}{2r_t})) dy}. \quad (2.8)$$

2.3.2 A priori bounds

We here collect several key estimates for θ^{σ_0} . The first one is

Lemma 2.7. *The function $|\theta^{\sigma_0}|$ is bounded by 1, for any $\sigma_0 > 0$.*

Proof. The proof follows from the fact that $|g|$ is bounded by 1 and that θ^{σ_0} is obtained by transporting g along the forward component of (2.4). \square

Things are much worse for the first-order derivative (and in fact this is the reason why the analysis of the case $\sigma_0 = 0$ is so difficult). In fact, by standard results in the theory of nonlinear parabolic equations, see for instance the monograph [73], see also [35, 38, 82] for a probabilistic point of view, θ^{σ_0} is Lipschitz continuous in space, uniformly in time, but the **Lipschitz constant depends on σ_0 !** Still, we have the following bound that gives a bound on the rate of explosion as σ_0 tends to 0.

Lemma 2.8. *There exists a constant C such that*

$$|\partial_x \theta^{\sigma_0}(t, x)| \leq \frac{C}{\sigma_0^2}, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

Proof. We perform the change of variable:

$$\hat{\theta}^{\sigma_0}(t, x) = \theta^{\sigma_0}(\sigma_0^2 t, \sigma_0^2 x), \quad \forall (t, x) \in [0, \frac{T}{\sigma_0^2}) \times \mathbb{R}.$$

Then,

$$\begin{cases} \partial_t \hat{\theta}^{\sigma_0}(t, x) - w_t^{-2} \hat{\theta}^{\sigma_0}(t, x) \partial_x \hat{\theta}^{\sigma_0}(t, x) + \frac{1}{2} w_t^{-2} \partial_{xx}^2 \hat{\theta}^{\sigma_0}(t, x) = 0, & \forall (t, x) \in [0, \frac{T}{\sigma_0^2}) \times \mathbb{R}, \\ \hat{\theta}^{\sigma_0}(\frac{T}{\sigma_0^2}, x) = g(\sigma_0^2 x), & \forall x \in \mathbb{R}. \end{cases}$$

Now, the result follows from standard PDE estimates for uniformly parabolic equation, see the same references as before: [73] for PDE arguments and [35, 38, 82] for the probabilistic point of view. \square

Although the gradient may blow up, we have in fact an upper bound for it.

Lemma 2.9. *For any $\sigma_0 \in (0, 1)$, the function θ^{σ_0} is non-increasing in x .*

Proof. As θ^{σ_0} is the decoupling field of (2.4), we have:

$$\theta^{\sigma_0}(t, x) = \mathbb{E}[g(\mu_T^{t,x,\sigma_0})],$$

where

$$d\mu_s^{t,x,\sigma_0} = -w_s^{-2}\theta^{\sigma_0}(s, \mu_s^{t,x,\sigma_0})ds + \sigma_0 w_s^{-1}dB_s, \quad s \in [t, T],$$

Using standard results for one-dimensional SDEs driven by Lipschitz coefficients, we know that $x \leq y$ implies $\mu_T^{t,x,\sigma_0} \leq \mu_T^{t,y,\sigma_0}$ with probability 1. Since g is non-increasing, we complete the proof. \square

2.3.3 Zero-noise limit of the decoupling field

It is a well-known fact that, as σ_0 tends to 0, θ^{σ_0} converges (in a sense that is made clear below) to the so-called *entropy solution* of the inviscid version of (2.7). Again, we refer to [79]. The limit is given by the field θ , whose definition is as follows. For all $(t, x) \in ([0, T] \times \mathbb{R})$, we let:

$$\theta(t, x) = \begin{cases} -\text{sign}(x) & \text{if } t \leq \delta, \quad x \in \mathbb{R}, \\ -\text{sign}(x) & \text{if } t \geq \delta, \quad |x| \geq r_\delta - r_t, \\ -\frac{x}{r_\delta - r_t} & \text{if } t > \delta, \quad |x| < r_\delta - r_t, \end{cases} \quad (2.9)$$

where we recall the definition of δ in (2.3).

Most of our analysis for the zero-noise limit of the LQ-MFG with common noise as σ_0 tends to 0 is based upon sharp estimates of the difference between the fields θ^{σ_0} and θ . In this regard, we have the following bounds on the difference $\theta^{\sigma_0}(t, x) - \theta(t, x)$, for $\sigma_0 \in (0, 1)$ and for some $(t, x) \in [0, T] \times \mathbb{R}$.

Proposition 2.10. *Consider $\Psi(t, x, \sigma_0) := \theta^{\sigma_0}(t, x) - \theta(t, x)$ defined for all $(t, x, \sigma_0) \in [0, T] \times \mathbb{R} \times (0, 1)$.*

1. *For every $(t, x) \in [0, \delta] \times ((-\infty, r_\delta - r_t) \cup (r_t - r_\delta, +\infty))$ such that $|x| \geq 2\sigma_0\sqrt{r_0|\ln \sigma_0|}$, it holds that*

$$|\Psi(t, x, \sigma_0)| \leq \sqrt{\frac{2}{\pi r_\delta}} \left[2(1 + |x| + 2r_0) + \sqrt{\frac{\pi r_0}{2}} \left(2 + \frac{|x|}{r_\delta} \right) \right] \sigma_0.$$

2. *For every $(t, x) \in [0, \delta] \times (r_\delta - r_t, r_t - r_\delta)$ such that $|x| \geq 2\sigma_0\sqrt{r_0|\ln \sigma_0|}$, it holds that*

$$|\Psi(t, x, \sigma_0)| \leq \left(4 + 2\sqrt{2} + \frac{2\sigma_0}{\sqrt{\pi r_\delta}} \right) e\sigma_0^{4\sqrt{2r_0}} + \left(\frac{2\sigma_0}{\sqrt{\pi r_\delta}} + 2\sqrt{2} \right) \sigma_0^2.$$

- 2'. *For $\eta \in (0, \delta)$, $(t, x) \in [0, \delta - \eta] \times ((r_\delta - r_t)/2, (r_t - r_\delta)/2)$, it holds that*

$$|\Psi(t, x, \sigma_0)| \leq \left(4 + 2\sqrt{2} + \frac{2\sigma_0}{\sqrt{\pi r_\delta}} \right) \exp\left(-\frac{2x}{\sigma_0^2}\right) + \frac{2\sigma_0}{\sqrt{\pi r_\delta}} + 2\sqrt{2} \exp\left(-\frac{(r_\delta - \eta - r_\delta)^2}{2r_0\sigma_0^2}\right).$$

3. *For every $(t, x) \in [\delta, T] \times ((-\infty, r_t - r_\delta - 2\sigma_0\sqrt{r_0|\ln \sigma_0|}) \cup [r_\delta - r_t + 2\sigma_0\sqrt{r_0|\ln \sigma_0|}, +\infty))$, it holds that*

$$|\Psi(t, x, \sigma_0)| \leq \sqrt{\frac{2}{\pi r_t}} \left[2 + 2r_\delta \left(1 + \frac{r_\delta + |x|}{r_t} \right) + \sqrt{\frac{\pi r_\delta}{2}} \left(1 + \frac{r_\delta + |x|}{r_t} \right) \right] \sigma_0.$$

Proof. The proof consists in applying Laplace asymptotic analysis techniques to the Cole-Hopf representation of $\Psi(t, x, \sigma_0)$. For the ease of exposition, it is postponed to the appendix. \square

Corollary 2.11. *Call \mathcal{D}_{σ_0} the domain*

$$\mathcal{D}_{\sigma_0} := \{(t, x) \in [0, T] \times \mathbb{R} : x \geq (r_\delta - r_t)_+ + \sigma_0^2 |\ln \sigma_0|\}.$$

Then,

$$\lim_{\sigma_0 \rightarrow 0} \sup_{(t, x) \in \mathcal{D}_{\sigma_0}} |\Psi(t, x, \sigma_0)| = 0.$$

Proof. We know from Proposition 2.10 that

$$\lim_{\sigma_0 \rightarrow 0} \sup_{(t, x) \in \mathcal{D}_{\sigma_0} : |x| \leq r_\delta} |\Psi(t, x, \sigma_0)| = 0.$$

Also, since $\theta^{\sigma_0}(t, \cdot)$ is bounded by 1 and non-increasing, we have, for all $x > r_\delta$,

$$\Psi(t, x, \sigma_0) = 1 + \theta^{\sigma_0}(t, x) \geq 0,$$

and

$$\Psi(t, x, \sigma_0) = 1 + \theta^{\sigma_0}(t, x) \leq 1 + \theta^{\sigma_0}(t, r_\delta) = \Psi(t, r_\delta, \sigma_0).$$

Therefore,

$$\sup_{t \in [0, T]} \sup_{|x| \geq r_\delta} |\Psi(t, x, \sigma_0)| \leq \sup_{t \in [0, T]} |\Psi(t, r_\delta, \sigma_0)|.$$

The last term tends to 0 with σ_0 since $(t, r_\delta) \in \mathcal{D}_{\sigma_0}$ for σ_0 small enough (uniformly in t). Proceeding similarly with $x < 0$, we complete the proof. \square

2.3.4 L^1 stability

In the analysis, we shall make use of the following lemma, which is a key property of scalar conservation laws, see for instance [79].

Lemma 2.12. *Consider a Lipschitz continuous bounded function $\tilde{g} \in \mathcal{C}(\mathbb{R}; \mathbb{R})$ such that $\tilde{g} \geq g$, and, for $\sigma_0 \in (0, 1)$, call $\tilde{\theta}^{\sigma_0}(t, x)$ the classical solution to*

$$\begin{cases} \forall t \in [0, T] \times \mathbb{R}, \\ \partial_t \tilde{\theta}^{\sigma_0}(t, x) - w_t^{-2} \tilde{\theta}^{\sigma_0}(t, x) \partial_x \tilde{\theta}^{\sigma_0}(t, x) + \frac{1}{2} \sigma_0^2 w_t^{-2} \partial_{xx}^2 \tilde{\theta}^{\sigma_0}(t, x) = 0, \\ \tilde{\theta}^{\sigma_0}(T, x) = \tilde{g}(x), \quad \forall x \in \mathbb{R}. \end{cases} \quad (2.10)$$

Then, it holds that the sign of the difference $\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0}$ is preserved that is

$$(\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0})(t, x) \geq 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}.$$

Moreover, if \tilde{g} and g coincide outside a compact subset of \mathbb{R} , then the space integral of $|\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0}|$ is also preserved, i.e

$$\int_{-\infty}^{+\infty} |(\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0})(t, x)| dx = \int_{-\infty}^{+\infty} |(\tilde{g} - g)(x)| dx, \quad \text{for all } \forall t \in [0, T].$$

Proof. First step. The fact that (2.10) is well-posed is a standard fact in the theory of nonlinear parabolic equations, see for instance [73], see also [35, 38, 82] for the probabilistic interpretation. Importantly, since \tilde{g} is Lipschitz continuous, $\tilde{\theta}^{\sigma_0}$ is also Lipschitz continuous in space, uniformly in time.

We then observe that the difference $\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0}$ is the solution of

$$\partial_t(\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0}) - w_t^{-2}\tilde{\theta}^{\sigma_0}\partial_x(\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0}) + \frac{1}{2}\sigma_0^2w_t^{-2}\partial_{xx}^2(\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0}) - w_t^{-2}\partial_x\theta^{\sigma_0}(\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0}) = 0.$$

Consider now the solution of the stochastic differential equation:

$$d\tilde{X}_s = -w_s^{-2}\tilde{\theta}^{\sigma_0}(s, \tilde{X}_s)ds + \sigma_0dB_s, \quad \forall s \in [t, T], \quad (2.11)$$

with $\tilde{X}_t = x$ as initial condition. Then, Feynman-Kac formula yields the following representation:

$$(\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0})(t, x) = \mathbb{E}\left[(\tilde{g} - g)(\tilde{X}_T) \exp\left(-\int_t^T w_s^{-2}\partial_x\theta^{\sigma_0}(s, \tilde{X}_s)ds\right)\right]. \quad (2.12)$$

Notice that, by the same argument as above, the gradient is bounded in the right-hand side. Since $\tilde{g} \geq g$, we deduce that $\tilde{\theta}^{\sigma_0} \geq \theta^{\sigma_0}$.

Second step. Next, we turn to the preservation of the L^1 norm. By the first step, it suffices to check that

$$\int_{-\infty}^{+\infty} (\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0})(t, x)dx = \int_{-\infty}^{+\infty} (\tilde{g} - g)(x)dx, \quad \text{for all } t \in [0, T].$$

Intuitively, to prove that the integrals are preserved, it is enough to show that

$$\partial_t \int_{-\infty}^{+\infty} (\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0})(t, x)dx = 0 \quad \forall t \in [0, T].$$

The key fact is to check that all these integrals are well-defined. To do so, we notice from the fact that \tilde{g} is bounded that $\tilde{\theta}^{\sigma_0}$ is also bounded. Therefore, we can find a constant $C > 0$ such that, for any $(t, x) \in [0, T] \times \mathbb{R}$ and any $R > 0$,

$$\mathbb{P}(|\tilde{X}_T| \leq R) \leq \mathbb{P}\left(|x| \leq R + CT + C \sup_{s \in [t, T]} |B_s - B_t|\right),$$

where $\tilde{X}_t = x$, with \tilde{X} as in (2.11). Hence, for $|x| > R + CT$,

$$\mathbb{P}(|\tilde{X}_T| \leq R) \leq \exp\left(-c|x| - (R + CT)\right)^2.$$

Choosing R such that the support of $\tilde{g} - g$ is included in $[-R, R]$, we deduce from (2.12) that

$$\sup_{0 \leq t \leq T} |(\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0})(t, x)| \leq \exp(-cx^2), \quad \forall x \in \mathbb{R}, \quad (2.13)$$

for a possibly new value of c , which suffices to prove that $\int_{-\infty}^{+\infty} (\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0})(t, x)dx$ is well-defined.

We now write the PDE satisfied by $(\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0})$ in the form

$$\partial_t(\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0}) - \frac{1}{2}w_t^{-2}\partial_x((\tilde{\theta}^{\sigma_0})^2 - (\theta^{\sigma_0})^2) + \frac{1}{2}\sigma_0^2w_t^{-2}\partial_{xx}^2(\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0}) = 0.$$

For a smooth compactly supported function $\eta : \mathbb{R} \rightarrow \mathbb{R}$, we get

$$\begin{aligned} \int_{-\infty}^{+\infty} (\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0})(t, x)\eta(x)dx &= \int_{-\infty}^{+\infty} (\tilde{g} - g)(x)\eta(x)dx \\ &+ \frac{1}{2}w_t^{-2} \int_{-\infty}^{+\infty} ((\tilde{\theta}^{\sigma_0})^2 - (\theta^{\sigma_0})^2)(t, x)\eta'(x)dx \\ &+ \frac{1}{2}\sigma_0^2w_t^{-2} \int_{-\infty}^{+\infty} (\tilde{\theta}^{\sigma_0} - \theta^{\sigma_0})\eta''(x)dx. \end{aligned}$$

Take now η as being equal to 1 on $[-R, R]$, to 0 outside $[-(R+1), R+1]$ and such that $|\eta'|$ and $|\eta''|$ are bounded by 2. Then, by (2.13), we observe that the last two terms in the above equality tends to 0 as R tends to ∞ . The result easily follows. \square

2.4 Zero noise limit selection

The zero noise limit problem for our benchmark example requires to find the limit of the unique equilibrium (when $\sigma_0 > 0$) as $\sigma_0 \rightarrow 0$ (i.e as the common noise B influence on the players vanishes). Through Proposition 2.2, it is equivalent to study the limit of the unique solution of FBSDE (2.4) as $\sigma_0 \rightarrow 0$. In this regard, the previous section allows us to reduce the problem to the analysis of the zero-noise limit of the forward SDE

$$dX_t = -w_t^{-2}\theta^{\sigma_0}(t, X_t)dt + \sigma_0w_t^{-1}dB_t, \quad (2.14)$$

which we derived from the four-step-scheme. This prompts us to use the asymptotic form of the decoupling field as σ_0 tends to 0, as discussed in Proposition 2.10, in order to study the asymptotic behaviour of (2.14). The difficulty to do so comes from the fact that the limit of the decoupling field is discontinuous at $x = 0$ and for t close to 0. In particular, similar to the famous Peano example for differential equations (the situation is even worse since the limiting drift is in fact non-continuous whilst Peano example is for an ODE with a continuous drift), the zero-noise limit of the forward SDE is not uniquely solvable. Taking benefit of the fact that this SDE is set in dimension 1, we manage to adapt the techniques from [36] to determine the solutions of the asymptotic forward SDE that are selected in the limit. Precisely, we show that this approach selects the extremal equilibria (i.e $A \in \{1, -1\}$) in Proposition 2.3. Using the same terminology as in [36], we also exhibit a transition space-time point, namely a space-time point with the following two features:

1. the probability that the solution (2.14), when starting from the singularity 0 (at time 0) reaches the transition space-point in a time of the same order as the transition time-point stays away from zero, when σ_0 tends to 0,
2. the probability that the solution to (2.14), once it has is reached the transition space-point, has a probability converging to 1 (as σ_0 tends to 0) to stay away from the singularity 0.

2.4.1 Main result

Thanks to the four-step-scheme, the solution to FBSDE (2.4) can now be identified with the solution of (2.14) with 0 as initial condition at time 0. More precisely, thanks to Proposition 2.10, we can write $(\mu_t^{x,\sigma_0})_{t \in [0,T]}$ as the solution of the SDE below with initial condition $x = 0$:

$$\begin{cases} \mu_t^{x,\sigma_0} &= x - \int_0^t w_s^{-2} \theta^{\sigma_0}(s, \mu_s^{x,\sigma_0}) dt + \int_0^t w_s^{-1} \sigma_0 dB_s \\ &= x - \int_0^t w_s^{-2} (\theta(s, \mu_s^{x,\sigma_0}) + \Psi(s, \mu_s^{x,\sigma_0}, \sigma_0)) dt + \int_0^t w_s^{-1} \sigma_0 dB_s, \quad \forall t \in [0, T]. \end{cases} \quad (2.15)$$

The solution $(\mu_t^{x,\sigma_0})_{t \in (0,T]}$ satisfies the strong Markov property and we can study its behaviour independently at different starting points and different time slots.

As explained before, our objective is to find a transition point for the process $(\mu_t^{x,\sigma_0})_{t \in (0,T]}$. A transition point in this setting is in fact a pair of two space-time points $(\pm \epsilon_0, t_0)$, which get closer and closer to $(0, 0)$ as σ_0 tends to 0. The transition point has the following two properties:

1. For all $|x| \leq \epsilon_0$, the probability that the process $(\mu_t^{x,\sigma_0})_{t \in [0,T]}$ reaches $\pm \epsilon_0$, in a time of the same scale as t_0 , tends to 1, as $\sigma_0 \rightarrow 0$. Thus, for very small σ_0 , the process $(\mu_t^{x,\sigma_0})_{t \in [0,T]}$, starting at x such that $|x| \leq \epsilon_0$, has a large probability of reaching the transition point,
2. The probability that the process $(\mu_t^{x,\sigma_0})_{t \in (0,T]}$, with $|x| \geq \epsilon_0$, escapes away from 0 tends to 1 as $\sigma_0 \rightarrow 0$.

For the latter item, we compare the trajectories of $(\mu_t^{x,\sigma_0})_{t \in (0,T]}$ with the trajectory of a deterministic function.

Finally, as $\sigma_0 \rightarrow 0$, the effect of the common noise vanishes and outside of null events, the trajectories of $(\mu_t^{x,\sigma_0})_{t \in (0,T]}$ starting from $x = 0$ will concentrate on the equilibria $A \in \{-1, 1\}$ in Proposition 2.3. For symmetry reasons, the two equilibria will be charged with the same probability. To make it clear, here is the main result of this section:

Theorem 2.13. *Consider $(k_t := \int_0^t w_s^{-2} ds)_{t \in [0,T]} \in \mathcal{C}([0, T]; \mathbb{R})$. Then, the sequence $(\mathbb{P} \circ (\mu_t^{\sigma_0})_{0 \leq t \leq T}^{-1})_{\sigma_0 \in (0,1)}$ converges, as $\sigma_0 \rightarrow 0$, to*

$$\frac{1}{2} \delta_{(k_t)_{t \in [0,T]}} + \frac{1}{2} \delta_{(-k_t)_{t \in [0,T]}}.$$

A technical tool

Throughout the proof, we will use the following property of θ^{σ_0} , which basically says that, the velocity field θ^{σ_0} points away from the singular point. Of course, this is a key fact in our analysis.

Lemma 2.14. *Given $(t, x, \sigma_0) \in [0, T] \times \mathbb{R} \times (0, 1)$, it holds that $-x(\theta^{\sigma_0}(t, x)) \geq 0$.*

Proof. Observe that $-x\theta^{\sigma_0}(t, x) \geq 0 \iff -xN(t, x) \geq 0$ where

$$N(t, x) = \int_{\mathbb{R}} \left(\frac{x-y}{r_t} \right) \exp \left(\sigma_0^{-2} \left(- \int_0^y g(r) dr - \frac{(x-y)^2}{2r_t} \right) \right) dy.$$

Let us define $f(y) = \exp\left(-\sigma_0^{-2} \int_0^y g(r) dr\right)$, for all $(\sigma_0, y) \in (0, 1) \times \mathbb{R}$. Since g is odd, f is even, namely $f(y) = f(-y)$, for $y \in \mathbb{R}$. Moreover, f is non-decreasing on $(0, +\infty)$ and non-increasing on $(-\infty, 0)$.

Applying the change of variable $z = x - y$ and splitting the integral in the definition of N into two pieces, we get

$$N(t, x) = \int_0^{+\infty} \frac{z}{r_t} \exp\left(-\frac{z^2}{2\sigma_0^2 r_t}\right) \left(f(x-z) - f(x+z)\right) dz.$$

Therefore, since f is even, we deduce that, for all $x \in \mathbb{R}$,

$$N(t, x) = - \int_0^{+\infty} \frac{z}{r_t} \exp\left(-\frac{z^2}{2\sigma_0^2 r_t}\right) \left(f(-x-z) - f(-x+z)\right) dz = -N(t, -x). \quad (2.16)$$

Case 1: Let $x \geq 0$.

If $z \leq x$, it holds that $0 \leq x-z \leq x+z$ and $f(x-z) - f(x+z) \leq 0$. On the other hand, if $z > x$, it holds that $0 \leq z-x \leq x+z$ and $f(x-z) - f(x+z) = f(z-x) - f(x+z) \leq 0$. Therefore, $f(x-z) - f(x+z) \leq 0$ for all $z \in [0, +\infty)$, and it follows that $N(t, x) \leq 0$ and $-xN(t, x) \geq 0$.

Case 2: Let $x \leq 0$.

From (2.16) and Case 1, it holds that $N(t, x) = -N(t, -x) \geq 0$. Therefore $-xN(t, x) \geq 0$. The proof is complete. \square

2.4.2 Reaching the transition point

We define our transition point as

$$(\epsilon_0, t_0) := (\sigma_0^2 |\ln \sigma_0|^2, \sigma_0^2 |\ln \sigma_0|^4). \quad (2.17)$$

We will regard ϵ_0 and t_0 as functions of σ_0 . We call them infinitesimal functions of σ_0 in the sense that they tend to 0 with σ_0 .

With the transition point, we associate the following hitting time: For all $\psi \in \mathcal{C}([0, T], \mathbb{R})$, consider

$$\tau_{\epsilon_0}(\psi) := \inf\{t \in [0, T) : |\psi_t| > \epsilon_0\}, \quad (2.18)$$

with the convention that $\tau_{r_0}(\psi_t) := T$ when the set in the right-hand side is empty.

Proposition 2.15. (Transition point). *Consider \tilde{t}_0 , a positive infinitesimal function of σ_0 , such that $\lim_{\sigma_0 \rightarrow 0} \tilde{t}_0/t_0 = +\infty$. Then,*

$$\forall x \in \mathbb{R} \quad \mathbb{P}(\tau_{\epsilon_0}(\mu^{x, \sigma_0}) > \tilde{t}_0) \rightarrow 0 \quad \text{as} \quad \sigma_0 \rightarrow 0.$$

Proof. Consider $(\mu^{x, \sigma_0})_{t \in [0, T]}$ the forward process starting at $x \in \mathbb{R}$. By Itô's formula, it holds that

$$\begin{aligned} d(\mu_t^{x, \sigma_0})^2 &= -2\mu_t^{x, \sigma_0} \theta^{\sigma_0}(t, \mu_t^{x, \sigma_0}) w_t^{-2} dt + \sigma_0^2 w_t^{-2} dt + 2\sigma_0 \mu_t^{x, \sigma_0} w_t^{-1} dB_t \\ &\geq \sigma_0^2 w_t^{-2} dt + 2\sigma_0 \mu_t^{x, \sigma_0} w_t^{-1} dB_t, \quad \forall t \in [0, T], \end{aligned}$$

since $-2\mu_t^{x,\sigma_0}\theta^{\sigma_0}(t,\mu_t^{x,\sigma_0}) \geq 0$ for all $t \in [0, T]$ as a consequence of Lemma 2.14.

Therefore, for all $t \in [0, T]$, the following inequalities hold

$$\mathbb{E}(\epsilon_0^2) \geq \mathbb{E}(|\mu_{t \wedge \tau_{\epsilon_0}}^{x,\sigma_0}|^2) \geq \mathbb{E}(x^2) + \mathbb{E}\left(\sigma_0^2 \int_0^{t \wedge \tau_{\epsilon_0}(\mu^{x,\sigma_0})} w_s^{-2} ds\right).$$

In particular,

$$\mathbb{E}(\tau_{\epsilon_0}(\mu^{x,\sigma_0})) = \mathbb{E}(T \wedge \tau_{\epsilon_0}(\mu^{x,\sigma_0})) \leq \sigma_0^{-2} \epsilon_0^2 \left(\min_{t \in [0, T]} w_t^{-2}\right)^{-1} = t_0 \times \max_{t \in [0, T]} w_t^2.$$

By hypothesis, there exists a function $l(\sigma_0) > 0$ such that $\tilde{t}_0 = l(\sigma_0)t_0$ and $l(\sigma_0) \rightarrow \infty$ as $\sigma_0 \rightarrow 0$. By Markov inequality,

$$\begin{aligned} \mathbb{P}(\tau_{\epsilon_0}(\mu^{x,\sigma_0}) > \tilde{t}_0) &\leq \frac{1}{l(\sigma_0)t_0} \mathbb{E}(\tau_{\epsilon_0}(\mu^{x,\sigma_0})) \\ &\leq \frac{1}{l(\sigma_0)} \times \max_{t \in [0, T]} w_t^2 \rightarrow 0, \quad \text{as } \sigma_0 \rightarrow 0. \end{aligned}$$

The proof is complete. \square

2.4.3 Restarting from the transition point

In order to investigate what happens when we restart from the transition point, we prove first the following lemma:

Lemma 2.16. *Consider $\gamma \in (0, 1)$ and a positive infinitesimal function \tilde{t}_0 of σ_0 . For δ as before, define*

$$\begin{aligned} B(s, \sigma_0, \gamma) &:= (1 - \gamma) \int_{\tilde{t}_0}^s w_r^{-2} dr, \quad s \geq \delta, \\ C(s) &:= \int_{\delta}^s w_r^{-2} dr = r_s - r_\delta, \quad s \geq \delta. \end{aligned}$$

There exist positive constants $\epsilon_\delta > 0$ and $c_\delta \in (0, 1)$, such that for all $\sigma_0 \in (0, \epsilon_\delta)$ and $\gamma \in (0, c_\delta)$, the set

$$\{s \in [\delta, T] : B(s, \sigma_0, \gamma) = C(s)\}$$

is empty.

Proof. Since \tilde{t}_0 is an infinitesimal function of σ_0 , there exists $\epsilon_\delta > 0$ such that $\tilde{t}_0 \in (0, \delta/2)$ whenever $\sigma_0 \in (0, \epsilon_\delta)$.

Define also c_δ as follows

$$0 < c_\delta := \frac{\int_{\delta/2}^{\delta} w_r^{-2} dr}{\int_0^T w_r^{-2} dr} < 1.$$

Observe that, whenever $\sigma_0 \in (0, \epsilon_\delta)$ and $\gamma \in (0, c_\delta)$,

$$\gamma \int_{\tilde{t}_0}^T w_r^{-2} dr < \int_{\tilde{t}_0}^{\delta} w_r^{-2} dr. \quad (2.19)$$

Assume now that $\{s \in [\delta, T] : B(s, \sigma_0, \gamma) = C(s)\}$ is not empty and call \bar{t} its infimum. It holds that

$$\begin{aligned} B(\bar{t}, \sigma_0, \gamma) = C(\bar{t}) &\implies (1 - \gamma) \int_{\bar{t}_0}^{\delta} w_r^{-2} dr + (1 - \gamma) \int_{\delta}^{\bar{t}} w_r^{-2} dr = \int_{\delta}^{\bar{t}} w_r^{-2} dr \\ &\implies \int_{\bar{t}_0}^{\delta} w_r^{-2} dr = \gamma \int_{\bar{t}_0}^{\bar{t}} w_r^{-2} dr. \end{aligned}$$

Then, by (2.19), we deduce

$$\begin{aligned} \gamma \in (0, c_\delta) &\implies \gamma \int_{\bar{t}_0}^T w_r^{-2} dr < \int_{\bar{t}_0}^{\delta} w_r^{-2} dr \\ &\implies \gamma \int_{\bar{t}_0}^T w_r^{-2} dr < \gamma \int_{\bar{t}_0}^{\bar{t}} w_r^{-2} dr, \end{aligned}$$

which is a contradiction with the fact $\bar{t} \leq T$. This completes the proof. \square

The above lemma says that, if σ_0 is sufficiently small, we can find $\gamma > 0$ sufficiently small such that $B(s, \sigma_0, \gamma)$ and $C(s)$ do not intersect on $s \in [\delta, T]$. Therefore, $B(s, \sigma_0, \gamma) \geq C(s)$, for $s \in [\delta, T]$, $\sigma_0 \in (0, \epsilon_\delta)$ and $\gamma \in (0, c_\delta)$. This prompts us to introduce

$$\tau_\gamma^+(\psi) := \inf \left\{ t \in (0, T) : \psi_t < \sigma_0^2 |\ln \sigma_0| + (1 - \gamma) \int_0^t w_s^{-2} ds \right\}, \quad (2.20)$$

for $\psi \in \mathcal{C}([0, T], \mathbb{R})$, with the convention that $\tau_\gamma^+(\psi) := T$ if the set is empty. Similarly, we let

$$\tau_\gamma^-(\psi) = \inf \left\{ t \in (0, T) : \psi_t > -\sigma_0^2 |\ln \sigma_0| - (1 - \gamma) \int_0^t w_s^{-2} ds \right\}. \quad (2.21)$$

Proposition 2.17. *For $\gamma \in (0, c_\delta)$, it holds that*

$$\inf_{x \geq \epsilon_0} \mathbb{P}(\tau_\gamma^+(\mu^{x, \sigma_0}) = T) \rightarrow 1, \quad \text{as } \sigma_0 \rightarrow 0,$$

and

$$\inf_{x \leq -\epsilon_0} \mathbb{P}(\tau_\gamma^-(\mu^{x, \sigma_0}) = T) \rightarrow 1, \quad \text{as } \sigma_0 \rightarrow 0.$$

Proof. Consider $\sigma_0 \in (0, 1)$ and take $x > 0$.

First step. We suppose that $\tau_\gamma^+(\mu^{x, \sigma_0}) < T$. So, it holds that, for all $t \in [0, \tau_\gamma^+(\mu^{x, \sigma_0})]$,

$$\begin{aligned} \mu_t^{x, \sigma_0} &\geq \sigma_0^2 |\ln \sigma_0| + (1 - \gamma) \int_0^t w_s^{-2} ds \\ &\geq \sigma_0^2 |\ln \sigma_0| + (1 - \gamma) \int_{t_0}^t w_s^{-2} ds. \end{aligned}$$

Consider $t \in [\delta, \tau_\gamma^+(\mu^{x, \sigma_0})]$, when this interval is not empty. Since $\gamma \in (0, c_\delta)$, it follows from Lemma 2.16 that, for sufficiently small $\sigma_0 > 0$ (i.e $\sigma_0 \in (0, \epsilon_\delta)$),

$$\mu_t^{x, \sigma_0} \geq \sigma_0^2 |\ln \sigma_0| + (1 - \gamma) \int_{t_0}^t w_s^{-2} ds > \sigma_0^2 |\ln \sigma_0| + r_\delta - r_t.$$

We deduce the following two things. First, if $\tau_\gamma^+(\mu^{x,\sigma_0}) < T$, we obviously have

$$\theta(t, \mu_t^{x,\sigma_0}) = -\text{sign}(\mu_t^{x,\sigma_0}) = -1, \quad \forall t \in [0, \tau_\gamma^+(\mu^{x,\sigma_0})]. \quad (2.22)$$

Moreover, for all $t \in [0, \tau_\gamma^+(\mu^{x,\sigma_0})]$, the point μ_t^{x,σ_0} lies in the domain of application of Corollary 2.11, which yields

$$|\Psi(t, \mu_t^{x,\sigma_0}, \sigma_0)| \leq \gamma/2, \quad (2.23)$$

whenever $\sigma_0 > 0$ is sufficiently small (uniformly in t).

Hence, for sufficiently small $\sigma_0 > 0$ such that claims (2.22) and (2.23) both hold, we observe that, for all $t \in [0, \tau_\gamma^+(\mu^{x,\sigma_0})]$,

$$\begin{aligned} \mu_t^{x,\sigma_0} &= x - \int_0^t \theta(s, \mu_s^{x,\sigma_0}) w_s^{-2} ds - \int_0^t \Psi(s, \mu_s^{x,\sigma_0}, \sigma_0) w_s^{-2} ds + \sigma_0 \int_0^t w_s^{-1} dB_s \\ &= x + \int_0^t w_s^{-2} ds - \int_0^t \Psi(s, \mu_s^{x,\sigma_0}, \sigma_0) w_s^{-2} ds + \sigma_0 \int_0^t w_s^{-1} dB_s \\ &\geq x + (1 - \gamma/2) \int_0^t w_s^{-2} ds + \sigma_0 \int_0^t w_s^{-1} dB_s. \end{aligned}$$

Second step. To simplify notation, let us write $\tau = \tau_\gamma^+(\mu^{x,\sigma_0})$ and observe that (recall that we assumed $\tau < T$)

$$\mu_\tau^{x,\sigma_0} = \sigma_0^2 |\ln \sigma_0| + (1 - \gamma) \int_0^\tau w_s^{-2} ds.$$

By the conclusion of the previous step, we deduce that there exists $r \in (0, T)$ such that

$$\sigma_0^2 |\ln \sigma_0| + (1 - \gamma) \int_0^r w_s^{-2} ds \geq x + (1 - \gamma/2) \int_0^r w_s^{-2} ds + \sigma_0 \int_0^r w_s^{-1} dB_s.$$

Therefore, whenever $x \geq \epsilon_0 = \sigma_0^2 |\ln \sigma_0|^2$ and $\sigma_0 > 0$ is sufficiently small, on the event $\{\tau_\gamma^+(\mu^{x,\sigma_0}) < T\}$, there exists $r \in (0, T)$ such that

$$\sigma_0^{-1} \int_0^r w_s^{-1} dB_s + \sigma_0^{-2} (\gamma/2) \int_0^r w_s^{-2} ds \leq |\ln \sigma_0| (1 - |\ln \sigma_0|) =: v(\sigma_0).$$

Then, we can find a Brownian motion $(\tilde{B}_u)_{u \geq 0}$ such that

$$\int_0^t w_s^{-1} dB_s = \tilde{B}_{k_t}, \quad t \in [0, T],$$

where

$$k_t = \int_0^t w_s^{-2} ds.$$

This implies that, whenever $x \geq \epsilon_0 = \sigma_0^2 |\ln \sigma_0|^2$ and $\sigma_0 > 0$ is sufficiently small, on the event $\{\tau_\gamma^+(\mu^{x,\sigma_0}) < T\}$, there exists $r \in (0, k_T)$ such that

$$\sigma_0^{-1} \tilde{B}_r + \sigma_0^{-2} (\gamma/2) r \leq v(\sigma_0).$$

Introducing the rescaled Brownian motion

$$(\hat{B}_s := \sigma_0^{-1} \tilde{B}_{\sigma_0^2 s})_{s \geq 0},$$

we deduce that, whenever $x \geq \epsilon_0 = \sigma_0^2 |\ln \sigma_0|^2$ and $\sigma_0 > 0$ is sufficiently small,

$$\tau_\gamma^+(\mu^{x, \sigma_0}) < T \implies \exists s \in (0, \sigma_0^{-2} k_T) \text{ such that } \hat{B}_s + (\gamma/2)s \leq v(\sigma_0).$$

Third step. Next, we define $(Z_s := \hat{B}_s + (\gamma/2)s)_{s \in [0, \sigma_0^{-2} k_T]}$ and $\zeta := \inf\{s \in [0, \sigma_0^{-2} k_T] : Z_s \leq v(\sigma_0)\}$, with the convention that $\zeta = \sigma_0^{-2} k_T$ if the set is empty.

Consider \mathbb{Q} a probability measure under which $(Z_s)_{s \in [0, \sigma_0^{-2} k_T]}$ is a Brownian motion. By Girsanov's theorem, such a probability measure may be defined through :

$$\frac{d\mathbb{P}}{d\mathbb{Q}} \Big|_{\mathcal{F}_{\sigma_0^{-2} k_T}} = \exp \left(\frac{\gamma}{2} Z_{\sigma_0^{-2} k_T} - \frac{\gamma^2}{8} \sigma_0^{-2} k_T \right).$$

Then,

$$\begin{aligned} \mathbb{P}(\tau_\gamma^+(\mu^{x, \sigma_0}) < T) &\leq \mathbb{P}(\exists s \in (0, \sigma_0^{-2} k_T) : Z_s \leq v(\sigma_0)) \\ &= \mathbb{E}^{\mathbb{P}}(\mathbb{1}_{\zeta < \sigma_0^{-2} k_T}) \\ &= \mathbb{E}^{\mathbb{Q}} \left(\exp \left(\frac{\gamma}{2} Z_\zeta - \frac{\gamma^2}{8} \zeta \right) \mathbb{1}_{\zeta < \sigma_0^{-2} k_T} \right) \\ &\leq \exp((\gamma/2)v(\sigma_0)). \end{aligned}$$

Therefore, for all $x \geq \epsilon_0$,

$$\mathbb{P}(\tau_\gamma^+(\mu^{x, \sigma_0}) = T) = 1 - \mathbb{P}(\tau_\gamma^+(\mu^{x, \sigma_0}) < T) \geq 1 - \exp((\gamma/2)v(\sigma_0)).$$

Taking the supremum over $x \geq \epsilon_0$ and then the limit as $\sigma_0 \rightarrow 0$, we get

$$1 \geq \inf_{x \geq \epsilon_0} \mathbb{P}(\tau_\gamma^+(\mu^{x, \sigma_0}) = T) \geq 1 - \exp((\gamma/2)v(\sigma_0)) \xrightarrow{\sigma_0 \rightarrow 0^+} 1,$$

where we used the fact that $v(\sigma_0) \rightarrow -\infty$. The proof is complete, the case when $x < 0$ being handled in a similar way. \square

2.4.4 Conclusion of the proof of Theorem 2.13

Proof. Let (ϵ_0, t_0) be the transition point described above and given by (2.17). Let $\tau_{\epsilon_0}, \tau_\gamma^-, \tau_\gamma^+$ be the stopping times defined in (2.18), (2.20) and (2.21). Moreover, consider \tilde{t}_0 , an infinitesimal function of σ_0 , such that $\frac{\tilde{t}_0}{t_0} \rightarrow +\infty$ as $\sigma_0 \rightarrow 0$.

To prove that the solution to SDE (2.15) with initial point $x = 0$ concentrates on the curves k^+ and k^- as $\sigma_0 \rightarrow 0$, we consider the event

$$\begin{aligned} B^{\sigma_0} = & \left\{ \mu_{\tilde{t}_0+t}^{0, \sigma_0} \geq \sigma_0^2 |\ln \sigma_0| + (1 - \gamma)k_t, \forall t \in [0, T] \right\} \\ & \cup \left\{ \mu_{\tilde{t}_0+t}^{0, \sigma_0} \leq -\sigma_0^2 |\ln \sigma_0| + (1 - \gamma)(-k)_t, \forall t \in [0, T] \right\}, \end{aligned}$$

and show that $\mathbb{P}(B^{\sigma_0}) \rightarrow 1$ as $\sigma_0 \rightarrow 0$.

Indeed, thanks to the Markov property, it holds that

$$1 \geq \mathbb{P}(B^{\sigma_0}) \geq \mathbb{P}(\tau_{\epsilon_0}(\mu^{0,\sigma_0}) < \tilde{t}_0) \inf_{j \in \{+,-\}} \mathbb{P}(\tau_\gamma^j(\mu^{j\epsilon_0,\sigma_0}) = T),$$

and Propositions 2.15 and 2.17 show that the right-hand side tends to 1 (provided that γ is small enough).

If we call \mathbb{Q} a weak limit (as σ_0 tends to 0) of the law of $(\mu_t^{0,\sigma_0})_{t \in [0,T]}$ on $\mathcal{C}([0,T]; \mathbb{R})$ (under \mathbb{P}) (such a limit exists as the sequence is obviously tight), we get that \mathbb{Q} only charges curves $(\psi_t)_{t \in [0,T]}$ that satisfy either

$$\psi_t \geq (1 - \gamma)k_t, \quad \forall t \in [0, T],$$

or

$$\psi_t \leq -(1 - \gamma)k_t, \quad \forall t \in [0, T],$$

for all $\gamma \in (0, c_\delta)$. Hence, \mathbb{Q} only charges curves $(\psi_t)_{0 \leq t \leq T}$ that satisfy either

$$\psi_t \geq k_t, \quad \forall t \in [0, T],$$

or

$$\psi_t \leq -k_t, \quad \forall t \in [0, T],$$

Now, since $|\theta^{\sigma_0}|$ is bounded by 1, see Lemma 2.7, it is clear \mathbb{Q} only charges curves $(\psi_t)_{0 \leq t \leq T}$ that satisfy

$$|\psi_t| \geq k_t, \quad \forall t \in [0, T],$$

which shows that \mathbb{Q} only charges $(k_t)_{t \in [0,T]}$ and $(-k_t)_{t \in [0,T]}$. By symmetry, the weight of each is exactly 1/2. \square

2.5 N -player limit selection

We now come to the last method of selection. As shown in Subsection 2.5.1, we can indeed associate with our particular LQ-MFG a game with a finite number of players and then address the asymptotic form of the equilibria (if any) as the number of players tends to ∞ . In fact, the connection between mean-field games and games with finitely many players is a major question in the theory of mean-field games, see for instance the references [80, 18, 23].

Below, we prove that, for a finite number N of players, Nash equilibria (if any) solve a forward-backward stochastic particle system. The goal is thus to address the asymptotic form, under the limit $N \rightarrow \infty$, of the solution to this particle system and to see which equilibria of the LQ-MFG (2.5) are charged by the weak limits (if any).

Basically, the main result that we show in this section is that the equilibria that are selected in this way are the same as those obtained in the previous section.

2.5.1 The associated N -players games

In this paragraph, we formulate the version with finitely many players of the mean-field game we have been considering so far. As already explained in introduction, the fact that the mean field game has a counterpart in the form of a stochastic differential game with a finite number

of players is not a big surprise: This connection is pretty standard and, in fact, it is the basis of the whole theory of mean-field games, see the aforementioned references.

The striking fact in the game with finitely many players we address below is that each player is driven by its own Brownian motion. In other words, noises are independent; they are said to be idiosyncratic.

So, for the description of the game, we consider an integer $N \in \mathbb{N}^*$, which stands for the number of players in the game. Then, for the same time horizon as before, we call $(W_t^i ; i = 1, \dots, N)_{t \in [0, T]}$ a collection of N independent one dimensional Brownian motions defined on a (common) complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We call $(\hat{\mathcal{F}}_t)_{t \in [0, T]}$ the usual augmentation of the filtration generated by $((W_t^i)_{t \in [0, T]} ; i = 1, \dots, N)$.

Also $\sigma > 0$ and $c \in \mathbb{R}$ are the same constants as in the system (2.1), and $g : \mathbb{R} \rightarrow \mathbb{R}$ is the Lipschitz continuous and bounded function we defined earlier. Importantly, σ_0 is 0 in this paragraph: There is no common noise; but somehow, we show below that there is an *intrinsic common noise* of variance $\sigma_0 = N^{-1/2}$ in the system. We will insist repeatedly on this fact which will serve us as a guideline.

Formulation of the game

For all $i = 1, \dots, N$, the evolution of i^{th} player's state during the game is described by the real-valued process $(X_t^i)_{t \in [0, T]}$. Noticeably, player i sees the other players through an aggregate quantity, which is here given by the average of the states of all these other players, namely

$$\mu_t^{i, N} = \frac{1}{N-1} \sum_{j \neq i}^N X_t^j, \quad \forall t \in [0, T].$$

Of course, the fact that interactions are designed in such a way is the cornerstone for the mean-field structure we addressed in Scheme 2.1.

Remark 2.18. *In some of the articles on the subject, authors include in the definition of the empirical measure the own state of player i , in which case $\mu_t^{i, N}$ becomes independent of i and writes*

$$\mu_t^N = \frac{1}{N} \sum_{j=1}^N X_t^j.$$

As explained in [24, Chapter 6], the limiting game should be the same. Still, we here work with the first form of the empirical measure as it is more convenient for our own purposes.

Player i has the following dynamics:

$$\begin{cases} dX_t^i = [cX_t^i + \alpha_t^i]dt + \sigma dW_t^i, & \forall t \in [0, T], \\ X_0^i = 0, \end{cases} \quad (2.24)$$

where $(\alpha_t^i)_{t \in [0, T]}$ is a control process belonging to the space $\hat{\mathcal{H}}^2$ of $(\hat{\mathcal{F}}_t)_{t \in [0, T]}$ -progressively measurable processes satisfying

$$\mathbb{E} \left[\int_0^T |\alpha_s|^2 ds \right] < \infty.$$

Given the tuple of controls $(\alpha_t^1, \dots, \alpha_t^N)_{t \in [0, T]}$, we associate with player i the following cost functional:

$$J^i(\alpha^1, \dots, \alpha^i, \dots, \alpha^N) := \mathbb{E} \left[\int_0^T \frac{1}{2} \left[(\alpha_t^i)^2 + (X_t^i)^2 \right] dt + \frac{1}{2} \left(X_T^i + g(\mu_T^{i, N}) \right)^2 \right]. \quad (2.25)$$

We then recall the following standard definition:

Definition 2.19. We call $(\alpha_t^1, \dots, \alpha_t^N)_{t \in [0, T]}$ a Nash equilibrium if, for any $i \in \{1, \dots, N\}$ and any other process $(\beta_t)_{t \in [0, T]}$,

$$J^i(\alpha^1, \dots, \alpha^{i-1}, \beta, \alpha^{i+1}, \dots, \alpha^N) \geq J^i(\alpha^1, \dots, \alpha^{i-1}, \alpha^i, \alpha^{i+1}, \dots, \alpha^N).$$

In other words, a Nash equilibrium is a consensus between the players: None of them can be better off by deviating unilaterally from the consensus.

It must be emphasized that the definition given above is restricted to so-called equilibria over controls in open-loop form: When player i changes her/his own strategy, the others keep playing the same realizations of $((\alpha_t^j)_{t \in [0, T]}; j \neq i)$. This is contrast with equilibria over controls in Markovian closed loop form, which are addressed in the PDE literature: Equilibria over controls in Markovian closed loop form are in the form $(\bar{\alpha}^j(t, X_t^1, \dots, X_t^N))_{t \in [0, T]}$, for $j = 1, \dots, N$, for functions $\bar{\alpha}^j : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$; whenever player i deviates, she/he chooses another feedback function $\bar{\beta} : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ instead of $\bar{\alpha}^i$ while the others keep using $\bar{\alpha}^j$; still, as the values of the state process $(X_t^i)_{t \in [0, T]}$ change, the realizations of the control processes $((\bar{\alpha}^j(t, X_t^1, \dots, X_t^N))_{t \in [0, T]}; j \neq i)$ change as well. We refer to [23, Chapter 2] for a review.

We shall not address the case of equilibria over controls in Markovian closed loop form in the text, but this could make sense as well.

First order condition

Similar to (2.4) (for mean-field games), we can write down a first order condition for the Nash equilibria of the game (2.24)–(2.25) in the form of a system of forward-backward stochastic differential equations. This is the cornerstone of our selection result. Again, we refer to [23, Chapter 2] for details on the derivation of this forward-backward system.

The first order condition writes as follows. Any Nash equilibrium to the associated N -player game (2.24)–(2.25) is in the set of solutions of the following system of forward-backward SDEs:

$$\begin{cases} dX_t^i = [cX_t^i - Y_t^{i,i}]dt + \sigma dW_t^i, & \forall t \in [0, T], & X_0^i = 0. \\ dY_t^{i,i} = [-X_t^i - cY_t^{i,i}]dt + \sum_{k=1}^N Z_t^{i,i,k} dW_t^k, & \forall t \in [0, T], & Y_T^{i,i} = X_T^i + g(\mu_T^{i, N}), \\ i \in \{1, 2, \dots, N\}. \end{cases} \quad (2.26)$$

In other words, the state processes of any Nash equilibrium must coincide with the forward paths of some solution to the above system.

Below, we do not discuss whether (2.26) is a sufficient condition or not. Usually, it is known to be sufficient in the case when the coefficients of the cost functional (2.25) are convex, but the latter is not true here.

Following our strategy for solving (2.4), we search for a solution $(X_t^i, Y_t^{i,i}, Z_t^{i,i,k}; i, k = 1, 2, \dots, N)_{t \in [0, T]}$ to the FBSDE (2.26) in the form $Y_t^{i,i} = \eta_t X_t^i + V_t^i$ for all $i = 1, 2, \dots, N$ and $t \in [0, T]$, where

$$dV_t^i = \chi_t^i dt + \sum_{k=1}^N z_t^{i,k} dW_t^k, \quad V_T^i = g(\mu_T^{i,N}).$$

In fact, we can show that there is a solution $(X_t^i, Y_t^{i,i}, Z_t^{i,i,k}; i, k = 1, 2, \dots, N)_{t \in [0, T]}$ to FBSDE (2.26) if and only if we can construct a solution $(X_t^i, V_t^i, z_t^{i,k}; i, k = 1, 2, \dots, N)_{t \in [0, T]}$ to FBSDE (2.27) below:

$$\begin{cases} dX_t^i = [(c - \eta_t)X_t^i - V_t^i] dt + \sigma dW_t^i, & \forall t \in [0, T], \quad X_0^i = 0. \\ dV_t^i = -(c - \eta_t)V_t^i dt + \sum_{k=1}^N z_t^{i,k} dW_t^k, & \forall t \in [0, T], \quad V_T^i = g(\mu_T^{i,N}), \\ i \in \{1, 2, \dots, N\}, \\ \dot{\eta}_t = \eta_t^2 - 2c\eta_t - 1, & \forall t \in [0, T], \quad \eta_T = 1. \end{cases} \quad (2.27)$$

Recall that $w_t = \exp(-\int_t^T (c - \eta_s) ds)$ and define the rescaled average players' states of all the players:

$$\tilde{\mu}_t^N = \frac{w_t^{-1}}{N} \sum_{i=1}^N X_t^i, \quad v_t^N = \frac{w_t}{N} \sum_{i=1}^N V_t^i, \quad \text{for all } t \in [0, T].$$

Therefore, when there exists a Nash equilibrium to the associated N -player game, the process $(\tilde{\mu}_t^N, v_t^N)_{t \in [0, T]}$ is a solution to (use the fact that $w_T^{-1} = 1$)

$$\begin{cases} \forall t \in [0, T], \\ d\tilde{\mu}_t^N = -w_t^{-2} v_t^N dt + \frac{\sigma}{N} \sum_{i=1}^N w_t^{-1} dW_t^i, & \tilde{\mu}_0^N = 0, \\ (v_t^N)_{t \in [0, T]} \text{ is a continuous martingale,} & v_T^N = \frac{1}{N} \sum_{i=1}^N g(\tilde{\mu}_T^{i,N}), \end{cases} \quad (2.28)$$

where

$$\tilde{\mu}_t^{i,N} = \frac{w_t^{-1}}{N-1} \sum_{j \neq i} X_t^j.$$

Main statement

Here is now our main statement:

Theorem 2.20. *Consider $(k_t := \int_0^t w_s^{-2} ds)_{t \in [0, T]} \in \mathcal{C}([0, T]; \mathbb{R})$. The sequence $(\mathbb{P}_\circ(\tilde{\mu}_t^N)_{0 \leq t \leq T}^{-1})_{N \geq 1}$ converges, as $N \rightarrow \infty$, to*

$$\frac{1}{2} \delta_{(k_t)_{t \in [0, T]}} + \frac{1}{2} \delta_{(-k_t)_{t \in [0, T]}}.$$

Of course, it says that the equilibria that are selected in Proposition 2.3 are the same as those selected in Theorem 2.13.

2.5.2 Approximate decoupling field

The problem with (2.28) is that the terminal condition is not in the form of a function of μ_T^N . Still, what we expect is that the solution to (2.28) should get closer and closer (as N tends

to ∞) to the solution of the same system but with the terminal boundary condition

$$v_T^N = \frac{1}{N} \sum_{i=1}^N g(\tilde{\mu}_T^N) = g(\tilde{\mu}_T^N),$$

namely (we put a hat on the symbols to distinguish from (2.28))

$$\begin{cases} \forall t \in [0, T], \\ d\hat{\mu}_t^N = -w_t^{-2} \hat{v}_t^N dt + \frac{\sigma}{N} \sum_{i=1}^N w_t^{-1} dW_t^i, & \hat{\mu}_0^N = 0, \\ (\hat{v}_t^N)_{t \in [0, T]} \text{ is a continuous martingale,} & \hat{v}_T^N = g(\hat{\mu}_T^N). \end{cases} \quad (2.29)$$

Letting

$$B_t = \frac{1}{\sqrt{N}} \sum_{i=1}^N W_t^i, \quad t \in [0, T], \quad (2.30)$$

we then recover (2.4) and $(B_t)_{t \in [0, T]}$, which is a Brownian motion with respect to $(\hat{\mathcal{F}}_t)_{t \in [0, T]}$, plays the role of a common noise with intensity $\sigma_0 = N^{-1/2}$!

However, this does not work so easily since (2.29) and (2.28) do not coincide. So, we pay some price below to estimate the distance between the solutions of the two systems.

Comparison argument

The main difficulty to compare (2.29) and (2.28) is the fact that, as N tends to ∞ , the system becomes ill-posed. So, we cannot expect for robust stability properties, uniformly in the parameter N , to estimate the difference between the solutions of the two equations.

The strategy we use below is based upon a comparison principle. As shown by Lemma 2.12, a form of comparison should be indeed in force independently of the value of N .

In order to put things in order, we recall that $r_\delta = \int_\delta^T w_s^{-2} ds$ and we consider a sequence of positive real numbers $(\gamma_N)_{N \geq 1} \subseteq (0, r_\delta/2)$ such that $\gamma_N \rightarrow 0$ as $N \rightarrow +\infty$. We then define the function

$$\tilde{g}(x) := \begin{cases} g(x), & \text{if } x \leq r_\delta - 2\gamma_N, \\ g(r_\delta - 2\gamma_N) & \text{if } r_\delta - 2\gamma_N \leq x \leq r_\delta - \gamma_N, \\ g(x - \gamma_N) & \text{if } r_\delta - \gamma_N \leq x \leq r_\delta + \gamma_N, \\ g(x) & \text{if } x \geq r_\delta + \gamma_N. \end{cases} \quad (2.31)$$

We then have the following lemma:

Lemma 2.21. *The functions g and \tilde{g} satisfy*

$$\tilde{g} \geq g \quad \text{and} \quad \int_{-\infty}^{+\infty} (\tilde{g} - g)(x) dx = 2 \frac{\gamma_N^2}{r_\delta}.$$

Proof. The inequality $\tilde{g} \geq g$ is a consequence of the fact that g is non-increasing. As for the second part of the statement, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} (\tilde{g} - g)(x) dx &= \int_{r_\delta - 2\gamma_N}^{r_\delta + \gamma_N} (\tilde{g} - g)(x) dx \\ &= \int_{r_\delta - 2\gamma_N}^{r_\delta - \gamma_N} \frac{x - r_\delta + 2\gamma_N}{r_\delta} dx + \int_{r_\delta - \gamma_N}^{r_\delta} \frac{\gamma_N}{r_\delta} dx + \int_{r_\delta}^{r_\delta + \gamma_N} \frac{r_\delta - x + \gamma_N}{r_\delta} dx \\ &= 2 \frac{\gamma_N^2}{r_\delta}. \end{aligned}$$

□

Following (2.7), we can associate with g and \tilde{g} the functions θ^N and $\tilde{\theta}^N$ in $C^{1,2}([0, T]; \mathbb{R}) \cap C([0, T] \times \mathbb{R}; \mathbb{R})$, classical solutions to

$$\begin{cases} \partial_t \tilde{\theta}_t^N - w_t^{-2} \tilde{\theta}_t^N \partial_x \tilde{\theta}_t^N + \frac{1}{2} \left(\frac{\sigma^2}{N} \right) w_t^{-2} \partial_{xx}^2 \tilde{\theta}_t^N = 0, & \forall t \in [0, T], \quad \forall x \in \mathbb{R}, \\ \partial_t \theta_t^N - w_t^{-2} \theta_t^N \partial_x \theta_t^N + \frac{1}{2} \left(\frac{\sigma^2}{N} \right) w_t^{-2} \partial_{xx}^2 \theta_t^N = 0, & \forall t \in [0, T], \quad \forall x \in \mathbb{R}, \\ \tilde{\theta}^N(T, x) = \tilde{g}(x), \quad \theta^N(T, x) = g(x), & \forall x \in \mathbb{R}. \end{cases} \quad (2.32)$$

Then, thanks to Lemma 2.12, we have $\tilde{\theta}^N \geq \theta^N$ and

$$0 \leq \int_{-\infty}^{+\infty} (\tilde{\theta}^N - \theta^N)(t, x) dx = \int_{-\infty}^{+\infty} (\tilde{g} - g)(t, x) dx = \frac{2\gamma_N^2}{r_\delta}, \quad \text{for all } t \in [0, T]. \quad (2.33)$$

Comparison between v^N and $\tilde{\theta}^N$

Recall the process $(k_t = \int_0^t w_s^{-2} ds)_{t \in [0, T]}$. Here is our main comparison result:

Lemma 2.22. *Choose $(\gamma_N)_{N \geq 1}$ in (2.31) such that*

$$\gamma_N \sqrt{N} \rightarrow +\infty \quad \text{as } N \rightarrow +\infty.$$

Then, with probability 1,

$$-v_t^N \geq -\tilde{\theta}^N(t, \mu_t^N) + \Delta_t^N, \quad \text{for all } t \in [0, T],$$

where

$$\lim_{N \rightarrow +\infty} \mathbb{E} \left[\sup_{t \in [0, T]} |\Delta_t^N|^2 \right] = 0.$$

Proof. First step. For all $i = 1, 2, \dots, N$, we can quantify the distance between $\tilde{\mu}_T^N$ and $\tilde{\mu}_T^{i, N}$ as follows:

$$|\tilde{\mu}_T^N - \tilde{\mu}_T^{i, N}| = \frac{1}{N} |\tilde{X}_T^i - \tilde{\mu}_T^{i, N}|,$$

where $(\tilde{X}_t^i := w_t^{-1} X_t^i, \tilde{V}_t^i := w_t V_t^i)_{t \in [0, T]}$ solves

$$d\tilde{X}_t^i = -w_t^{-2} \tilde{V}_t^i dt + \sigma w_t^{-1} dW_t^i.$$

Therefore,

$$d(\tilde{X}_t^i - \tilde{\mu}_t^{i,N}) = -w_t^{-2} \left(\tilde{V}_t^i - \frac{1}{N-1} \sum_{j \neq i} \tilde{V}_t^j \right) dt + \sigma \left(w_t^{-1} dW_t^i - \frac{1}{N-1} \sum_{j \neq i} w_t^{-1} dW_t^j \right).$$

Obviously, $(\tilde{V}_t^i)_{t \in [0, T]}$ is a martingale with a terminal boundary condition that belongs to $[-1, 1]$. Therefore,

$$\begin{aligned} |\tilde{\mu}_T^N - \tilde{\mu}_T^{i,N}| &\leq \frac{1}{N} \left[2k_T + \sigma \left(\left| \int_0^T w_s^{-1} dW_s^i \right| + \frac{1}{N-1} \sum_{j \neq i} \left| \int_0^T w_s^{-1} dW_s^j \right| \right) \right] \\ &\leq \frac{2}{N} \left[k_T + \sigma \max_{i \in \{1, \dots, N\}} \left| \int_0^T w_s^{-1} dW_s^i \right| \right]. \end{aligned}$$

Second step. By a straightforward application of Itô's formula, we obtain

$$d(v_t^N - \tilde{\theta}^N(t, \tilde{\mu}_t^N)) = w_t^{-2} \partial_x \tilde{\theta}^N(t, \tilde{\mu}_t^N) (v_t^N - \tilde{\theta}^N(t, \tilde{\mu}_t^N)) dt + dM_t,$$

where $(M_t)_{t \in [0, T]}$ is a square integrable martingale. Therefore,

$$v_t^N - \tilde{\theta}^N(t, \tilde{\mu}_t^N) = \mathbb{E} \left[\exp \left(- \int_t^T w_s^{-2} \partial_x \tilde{\theta}^N(s, \tilde{\mu}_s^N) ds \right) \left[v_T^N - \tilde{g}(\tilde{\mu}_T^N) \right] \middle| \hat{\mathcal{F}}_t \right] \quad \text{for all } t \in [0, T].$$

Now, by Lemma 2.8, we know that there exists a positive constant C , independent of N , such that

$$|\partial_x \tilde{\theta}^N(s, \tilde{\mu}_s^N)| \leq CN \quad \text{for all } s \in [0, T].$$

Therefore, by letting $A_N := \{ \max_{i \in \{1, \dots, N\}} \left| \int_0^T w_s^{-1} dW_s^i \right| \leq C\sqrt{N} \}$, we get

$$-v_t^N \geq -\tilde{\theta}^N(t, \tilde{\mu}_t^N) + \Delta_t^N + \mathbb{E} \left[\exp(CN) [\tilde{g}(\tilde{\mu}_T^N) - v_T^N] \mathbb{1}_{A_N} \middle| \hat{\mathcal{F}}_t \right] \quad \text{for all } t \in [0, T], \quad (2.34)$$

with

$$\begin{aligned} |\Delta_t^N| &= \mathbb{E} \left[\exp(CN) [v_T^N - \tilde{g}(\tilde{\mu}_T^N)] \mathbb{1}_{(A_N)^c} \middle| \hat{\mathcal{F}}_t \right] \\ &\leq 2 \exp(CN) \mathbb{P} \left\{ \max_{i \in \{1, \dots, N\}} \left| \int_0^T w_s^{-1} dW_s^i \right| > C\sqrt{N} \middle| \hat{\mathcal{F}}_t \right\}. \end{aligned}$$

By standard estimates for the supremum of a Brownian motion, we notice that, for a constant c depending on k_T , but independent of C ,

$$\max_{i \in \{1, \dots, N\}} \mathbb{P} \left\{ \left| \int_0^T w_s^{-1} dW_s^i \right| > C\sqrt{N} \right\} \leq \frac{1}{c} \exp\left(-\frac{C^2 N}{c}\right),$$

and then

$$\mathbb{P}\left\{\max_{i \in \{1, \dots, N\}} \left| \int_0^T w_s^{-1} dW_s^i \right| > C\sqrt{N}\right\} \leq \frac{N}{c} \exp\left(-\frac{C^2 N}{c}\right).$$

Hence, modifying the value of c , we get, for any $p \geq 1$

$$\exp(CpN) \mathbb{P}\left\{\max_{i \in \{1, \dots, N\}} \left| \int_0^T w_s^{-1} dW_s^i \right| > C\sqrt{N}\right\} \leq \frac{c}{C^2} \exp\left(CN\left(p - \frac{C}{c}\right)\right).$$

Choosing C large enough, we deduce from Doob's inequality that

$$\lim_{N \rightarrow \infty} \mathbb{E}\left[\sup_{t \in [0, T]} |\Delta_t^N|^2\right] = 0.$$

Third step. From inequality (2.34) above, it is sufficient to show that $[\tilde{g}(\tilde{\mu}_T^N) - v_T^N] \geq 0$ on the event A_N .

We thus return to the conclusion of the first step and we notice that, on the event A_N , for all $i = 1, 2, \dots, N$,

$$|\tilde{\mu}_T^N - \tilde{\mu}_T^{i, N}| \leq \frac{2}{N} \left[k_T + \sigma \max_{i \in \{1, \dots, N\}} \left| \int_0^T w_s^{-1} dW_s^i \right| \right] \leq \frac{cC}{\sqrt{N}},$$

for a new value of the constant c .

Recall now γ_N from (2.31) that $\gamma_N \in (0, r_\delta/2)$.

Case a. Suppose that $\tilde{\mu}_T^N < r_\delta - \frac{cC}{\sqrt{N}}$. Then, $\tilde{\mu}_T^{i, N} < r_\delta$ (for all $i \in \{1, \dots, N\}$) and by concavity of g for values less than or equal to r_δ , we obtain

$$v_T^N = \frac{1}{N} \sum_{i=1}^N g(\tilde{\mu}_T^{i, N}) \leq g(\tilde{\mu}_T^N) \leq \tilde{g}(\tilde{\mu}_T^N).$$

Case b. Suppose that $\tilde{\mu}_T^N \geq r_\delta + \frac{cC}{\sqrt{N}}$. Then, $\tilde{\mu}_T^{i, N} \geq r_\delta$ (for all $i \in \{1, \dots, N\}$), and we obtain

$$v_T^N = \frac{1}{N} \sum_{i=1}^N g(\tilde{\mu}_T^{i, N}) = -1 = g(\tilde{\mu}_T^N) \leq \tilde{g}(\tilde{\mu}_T^N).$$

Case c. Suppose that $r_\delta - \frac{cC}{\sqrt{N}} \leq \tilde{\mu}_T^N \leq r_\delta + \frac{cC}{\sqrt{N}}$. Then, $\tilde{\mu}_T^N - \frac{2cC}{\sqrt{N}} \leq \tilde{\mu}_T^{i, N} \leq \tilde{\mu}_T^N + \frac{2cC}{\sqrt{N}}$ (for all $i \in \{1, \dots, N\}$), since g is non-increasing, we obtain, if $\gamma_N \geq 2cC/\sqrt{N}$,

$$\begin{aligned} 0 &= \tilde{g}(\tilde{\mu}_T^N) - g(\tilde{\mu}_T^N - \gamma_N) \leq \tilde{g}(\tilde{\mu}_T^N) - g\left(\tilde{\mu}_T^N - \frac{2cC}{\sqrt{N}}\right) \\ &\leq \tilde{g}(\tilde{\mu}_T^N) - \frac{1}{N} \sum_{i=1}^N g(\tilde{\mu}_T^{i, N}) = \tilde{g}(\tilde{\mu}_T^N) - v_T^N. \end{aligned}$$

The proof is complete. \square

Upper bound for $\tilde{\theta}^N$

Similar to Lemma 2.9, the first point is to notice that $\tilde{\theta}^N$ is non-increasing and that θ^N is also non-increasing.

Lemma 2.23. *For any fixed $N \geq 1$, the functions θ^N and $\tilde{\theta}^N$ are non-increasing in the space argument.*

We make use of the non-increasing property to get a relevant upper bound for $\tilde{\theta}$.

Lemma 2.24. *Choose $(\gamma_N)_{N \geq 1}$ in (2.31) such that*

$$\gamma_N \sqrt{N} \rightarrow +\infty \quad \text{as } N \rightarrow +\infty.$$

Let $(a_N)_{N \geq 1}$ be a sequence of positive real numbers converging to $+\infty$. Then, there exists another sequence $(\varepsilon_N)_{N \geq 1}$ converging to 0 such that:

$$\forall t \in [0, \delta - \eta], \quad -\tilde{\theta}^N(t, a_N \gamma_N^2) \geq 1 - \varepsilon_N.$$

Proof. By (2.33), we know that for any $t \in [0, T]$ and any two reals $a < b$,

$$-\int_a^b \tilde{\theta}^N(t, x) dx \geq -\int_a^b \theta^N(t, x) dx - \frac{2\gamma_N^2}{r_\delta}.$$

Hence, by Lemma 2.23,

$$-\tilde{\theta}^N(t, b) \geq -\theta^N(t, a) - \frac{2\gamma_N^2}{r_\delta(b-a)}. \quad (2.35)$$

We now recall the estimate proven in Proposition 2.10. For η as in the statement, N large enough and $t \in [0, \delta - \eta]$,

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, \delta - \eta]} |1 + \theta^N(t, \gamma_N^2)| = 0.$$

Therefore, choosing $a = \gamma_N^2$ and $b = a_N \gamma_N^2$ in (2.35), we complete the proof. \square

2.5.3 Staying away from the singularity

We now proceed as in the previous section, except that we first investigate how the process $(\tilde{\mu}_t^N)_{t \in [0, T]}$ stays away from the singularity.

To do so, we summarize the results of the last two sections in the following form:

$$\begin{aligned} \tilde{\mu}_t^N &\geq \tilde{\mu}_s^N + \int_s^t w_r^{-2} (-\tilde{\theta}^N(r, \mu_r^N) + \Delta_t^N) dr + \frac{\sigma}{\sqrt{N}} \int_s^t w_r^{-1} dB_r, \quad \forall 0 \leq s \leq t \leq T, \\ \tilde{\mu}_t^N &\leq \tilde{\mu}_s^N + \int_s^t w_r^{-2} dr + \frac{\sigma}{\sqrt{N}} \int_s^t w_r^{-1} dB_r, \quad \forall 0 \leq s \leq t \leq T, \end{aligned}$$

with $(B_t)_{t \in [0, T]}$ as in (2.30).

Lemma 2.25. *Assume that $\gamma_N \sqrt{N} \rightarrow +\infty$ and let $(a_N)_{N \geq 1}$ and $(b_N)_{N \geq 1}$ be two sequences of real numbers such that $\lim_{N \rightarrow +\infty} a_N = \lim_{N \rightarrow +\infty} b_N = \lim_{N \rightarrow +\infty} (b_N/a_N) = +\infty$. For a*

sequence of stopping times $(\varrho_N)_{N \geq 1}$ such that $\tilde{\mu}_{\varrho_N}^N \geq b_N \gamma_N^2$ if $\varrho_N \leq \delta/2$, let

$$\tau_N = \inf\{s \geq \varrho_N : \tilde{\mu}_s^N \leq a_N \gamma_N^2\}.$$

Then, for any $\eta \in (0, \delta/2)$, the following convergence holds true in \mathbb{P} -probability,

$$\lim_{N \rightarrow \infty} \left[\mathbb{P}(\tau_N \leq \delta - \eta \mid \hat{\mathcal{F}}_{\varrho_N}) \mathbf{1}_{\{\varrho_N \leq \delta/2\}} \right] = 0.$$

Proof. Observe that for all $s \in (\varrho_N, \tau_N \wedge (\delta - \eta))$, it holds that $\tilde{\mu}_s^N \geq b_N \gamma_N^2 \geq a_N \gamma_N^2$ and by Lemmas 2.23 and 2.24,

$$-\tilde{\theta}^N(s, \tilde{\mu}_s^N) \geq 1 - \varepsilon_N,$$

for a sequence $(\varepsilon_N)_{N \geq 1}$ that tends to 0 as N tends to $+\infty$ (and which only depends on η , $(\gamma_N)_{N \geq 1}$ and $(a_N)_{N \geq 1}$).

Therefore, on the event $\{\varrho_N \leq \delta/2\} \cap \{\sup_{t \in [0, T]} |\Delta_t^N| \leq 1/4\}$,

$$\tilde{\mu}_s^N \geq \mu_{\varrho_N}^N + \left(\frac{3}{4} - \varepsilon_N\right)(k_s - k_t) + \frac{\sigma}{\sqrt{N}}(\hat{B}_{k_s} - \hat{B}_{k_{\varrho_N}}), \quad s \in (\varrho_N, \tau_N \wedge (\delta - \eta)],$$

where $(\hat{B}_t)_{t \in [0, k_T]}$ is a Brownian motion with respect to the filtration $(\hat{\mathcal{F}}_{k_t^{\circ-1}})_{t \in [0, k_T]}$, with $k_{k_t}^{\circ-1} = t$ for $t \in [0, T]$, such that

$$\int_0^t w_s^{-2} dB_s = \hat{B}_{k_t}, \quad \forall t \in [0, T].$$

Hence, on the same event as before, for $\varepsilon_N \leq 1/4$,

$$\tilde{\mu}_s^N \geq b_N \gamma_N^2 + \frac{1}{2}(k_s - k_{\varrho_N}) + \frac{\sigma}{\sqrt{N}}(\hat{B}_{k_s} - \hat{B}_{k_{\varrho_N}}), \quad s \in (\varrho_N, \tau_N \wedge (\delta - \eta)].$$

Therefore, if $\tau_N \leq \delta - \eta$, then at $\hat{\tau}_N := \tau_N \wedge (\delta - \eta)$ it holds that (at least for N large enough such that $N \leq \gamma_N^2$ and $\varepsilon_N \leq 1/4$)

$$-\frac{(b_N - a_N)}{N} \geq -(b_N - a_N) \gamma_N^2 \geq \frac{\sigma}{\sqrt{N}} \left[\frac{\sqrt{N}}{2\sigma} (k_{\hat{\tau}_N} - k_{\varrho_N}) + (\hat{B}_{k_{\hat{\tau}_N}} - \hat{B}_{k_{\varrho_N}}) \right].$$

Define

$$\beta_s = \frac{\sqrt{N}}{2\sigma} \int_{k_{\varrho_N}}^{k_{\hat{\tau}_N} \wedge s} dr + \hat{B}_s, \quad s \in [0, k_T].$$

We know known that $(\beta_s)_{s \in [0, k_T]}$ is a Wiener process under the probability measure \mathbb{Q} defined by:

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp \left(\frac{\sqrt{N}}{2\sigma} (\beta_{k_{\hat{\tau}_N}} - \beta_{k_{\varrho_N}}) - \frac{N}{8\sigma^2} (k_{\hat{\tau}_N} - k_{\varrho_N}) \right).$$

Now,

$$\begin{aligned}
& \mathbb{P}\left(\{\tau_N \leq \delta - \eta\} \cap \{\varrho_N \leq \delta/2\} \cap \left\{\sup_{t \in [0, T]} |\Delta_t^N| \leq 1/4\right\}\right) \\
& \leq \mathbb{E}^{\mathbb{Q}} \left[\exp\left(\frac{\sqrt{N}}{2\sigma}(\beta_{k_{\hat{\tau}_N}} - \beta_{k_{\varrho_N}}) - \frac{N}{8\sigma^2}(k_{\hat{\tau}_N} - k_{\varrho_N})\right) \mathbf{1}_{\{\beta_{k_{\hat{\tau}_N}} - \beta_{k_{\varrho_N}} \leq \frac{-(b_N - a_N)}{\sigma\sqrt{N}}\}} \right] \\
& \leq \mathbb{E}^{\mathbb{Q}} \left[\exp\left(-\frac{b_N - a_N}{2\sigma^2} - \frac{N}{8\sigma^2}(k_{\hat{\tau}_N} - k_{\varrho_N})\right) \mathbf{1}_{\{\beta_{k_{\hat{\tau}_N}} - \beta_{k_{\varrho_N}} \leq \frac{-(b_N - a_N)}{\sigma\sqrt{N}}\}} \right] \\
& \leq \exp\left(-\frac{b_N - a_N}{2\sigma^2}\right) \rightarrow 0 \quad \text{as } N \rightarrow +\infty.
\end{aligned}$$

Together with the last line in the statement of Lemma 2.22, we deduce that

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(\{\tau_N \leq \delta - \eta\} \cap \{\varrho_N \leq \delta/2\}\right) = 0,$$

which may be rewritten in the form

$$\lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{P}\left(\{\tau_N \leq \delta - \eta\} \mid \hat{\mathcal{F}}_{\varrho_N}\right) \mathbf{1}_{\{\varrho_N \leq \delta/2\}} \right] = 0,$$

from which the result follows. \square

We deduce the following lemma, which is the analogue of the results proven in Subsection 2.4.3.

Lemma 2.26. *Assume that $\gamma_N \sqrt{N} \rightarrow +\infty$ and let $(b_N)_{N \geq 1}$ be a sequence of real numbers such that $\lim_{N \rightarrow +\infty} b_N = +\infty$. For a given $\epsilon > 0$ and a given sequence of stopping times $(\varrho_N)_{N \geq 1}$ such that $\tilde{\mu}_{\varrho_N}^N \geq b_N \gamma_N^2$ if $\varrho_N \leq \delta/2$, let*

$$\tau'_N = \inf\{t \geq \varrho_N : \tilde{\mu}_t^N \leq \tilde{\mu}_{\varrho_N}^N + k_t - k_{\varrho_N} - \epsilon\}.$$

Then, for any $\eta \in (0, \delta/2)$, the following convergence holds true in \mathbb{P} -probability,

$$\lim_{N \rightarrow \infty} \left[\mathbb{P}(\tau'_N < \delta - \eta \mid \hat{\mathcal{F}}_{\varrho_N}) \mathbf{1}_{\{\varrho_N \leq \delta/2\}} \right] = 0.$$

Proof. With the same notation as in Lemma 2.25, we work on the event $\{\varrho_N \leq \delta/2\} \cap \{\sup_{t \in [0, T]} |\Delta_t^N| \leq \epsilon^2/2\}$. Following the proof of Lemma 2.25 with $(a_N = \sqrt{b_N})_{N \geq 1}$ and with τ_N as therein, we deduce that, for $\varepsilon_N < \epsilon^2/2$,

$$\tilde{\mu}_s^N \geq b_N \gamma_N^2 + (1 - \epsilon^2)(k_s - k_{\varrho_N}) + \frac{\sigma}{\sqrt{N}}(\hat{B}_{k_s} - \hat{B}_{k_{\varrho_N}}), \quad s \in (\varrho_N, \tau_N \wedge (\delta - \eta)).$$

Hence,

$$\tilde{\mu}_s^N \geq k_s - k_{\varrho_N} - \epsilon^2 k_T - \frac{2\sigma}{\sqrt{N}} \sup_{s \in [0, T]} |\hat{B}_{k_s}|, \quad s \in (\varrho_N, \tau_N \wedge (\delta - \eta)).$$

and then, on the supplementary event $\{\epsilon^2 k_T + \frac{2\sigma}{\sqrt{N}} \sup_{s \in [0, T]} |\hat{B}_{k_s}| \leq \epsilon\}$, we have $\tau'_N \geq \tau_N \wedge (\delta - \eta)$.

So,

$$\begin{aligned} \mathbb{P}(\tau'_N < \delta - \eta \mid \hat{\mathcal{F}}_{\varrho_N}) \mathbf{1}_{\{\varrho_N \leq \delta/2\}} &\leq \mathbb{P}(\tau_N < \delta - \eta \mid \hat{\mathcal{F}}_{\varrho_N}) \mathbf{1}_{\{\varrho_N \leq \delta/2\}} \\ &+ \mathbb{P}\left(\sup_{t \in [0, T]} |\Delta_t^N| \leq \frac{\epsilon^2}{2} \mid \hat{\mathcal{F}}_{\varrho_N}\right) \mathbf{1}_{\{\varrho_N \leq \delta/2\}} \\ &+ \mathbb{P}\left(\epsilon^2 k_T + \frac{2\sigma}{\sqrt{N}} \sup_{s \in [0, T]} |\hat{B}_{k_s}| > \epsilon \mid \hat{\mathcal{F}}_{\varrho_N}\right) \mathbf{1}_{\{\varrho_N \leq \delta/2\}}. \end{aligned}$$

Without any loss of generality, we can assume that $\epsilon^2 k_T < \epsilon/2$. It is then pretty straightforward to prove that

$$\lim_{N \rightarrow +\infty} \mathbb{P}\left(\epsilon^2 k_T + \frac{2\sigma}{\sqrt{N}} \sup_{s \in [0, T]} |\hat{B}_{k_s}| > \epsilon\right) = 0.$$

We easily deduce that

$$\lim_{N \rightarrow +\infty} \left[\mathbb{P}\left(\epsilon^2 k_T + \frac{2\sigma}{\sqrt{N}} \sup_{s \in [0, T]} |\hat{B}_{k_s}| > \epsilon \mid \hat{\mathcal{F}}_{\varrho_N}\right) \mathbf{1}_{\{\varrho_N \leq \delta/2\}} \right] = 0.$$

It remains to recall from Lemma 2.25 that, in \mathbb{P} -probability,

$$\lim_{N \rightarrow \infty} \left[\mathbb{P}(\tau_N < \delta - \eta \mid \hat{\mathcal{F}}_{\varrho_N}) \mathbf{1}_{\{\varrho_N \leq \delta/2\}} \right] = 0,$$

and to recall from Lemma 2.22 that, in \mathbb{P} -probability also,

$$\lim_{N \rightarrow \infty} \left[\mathbb{P}\left(\sup_{t \in [0, T]} |\Delta_t^N| \leq \frac{\epsilon^2}{2} \mid \hat{\mathcal{F}}_{\varrho_N}\right) \mathbf{1}_{\{\varrho_N \leq \delta/2\}} \right] = 0.$$

The result easily follows. □

2.5.4 Reaching the transition area

We start this third step with the following general lemma.

Lemma 2.27. *For a positive continuous (deterministic) path $(\check{w}_t)_{t \in [0, T]}$, let $(\check{X}_t)_{t \in [0, T]}$ be a one-dimensional Itô process of the form*

$$d\check{X}_t = -\check{w}_t^{-2} \check{Y}_t ds + \check{w}_t^{-1} d\check{B}_t, \quad t \in [0, T],$$

where $(\check{B}_t)_{t \in [0, T]}$ is a Brownian motion with respect to some filtration $(\check{\mathcal{F}}_t)_{t \in [0, T]}$ and $(\check{Y}_t)_{t \in [0, T]}$ is a $[-1, 1]$ -valued adapted process. For a real $a \geq 1$ and some stopping time ϱ with respect to the filtration $(\hat{\mathcal{F}}_t)_{t \in [0, T]}$, let

$$\tau = \inf\{t \geq \varrho : |\check{X}_t| \geq a\}.$$

Then, on the event $\{|\check{X}_\varrho| < a\}$, we have

$$\mathbb{P}(\tau \leq T \mid \check{\mathcal{F}}_\varrho) \geq \frac{1}{c} \exp(-c(\check{k}_T - \check{k}_\varrho)) \exp\left(-\frac{ca^2}{(\check{k}_T - \check{k}_\varrho)}\right),$$

where c is a strictly positive constant only depending on T and

$$\check{k}_t = \int_0^t \check{w}_s^{-2} ds, \quad t \in [0, T].$$

Proof. Without any loss of generality, we can assume that $\rho = 0$. We then let \mathbb{Q} be the probability measure defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(\int_0^T \check{Y}_s \check{w}_s^{-1} d\check{B}_s - \frac{1}{2} \int_0^T |\check{Y}_s|^2 \check{w}_s^{-2} ds\right).$$

Under \mathbb{Q} , the process

$$\check{W}_t = \check{B}_t - \int_0^t \check{w}_s^{-1} \check{Y}_s ds, \quad t \in [0, T],$$

is a Brownian motion with respect to the filtration $(\check{F}_t)_{t \in [0, T]}$. Moreover,

$$\check{X}_t - \check{X}_0 = \int_0^t \check{w}_s^{-1} d\check{W}_s, \quad t \in [0, T].$$

We then obtain

$$\begin{aligned} \mathbb{Q}\left(\sup_{t \in [0, T]} (\check{X}_t - \check{X}_0) \geq a \mid \check{\mathcal{F}}_0\right) &= \mathbb{Q}\left(\sup_{t \in [0, T]} \left[\int_0^t \check{w}_s^{-1} d\check{W}_s\right] \geq a \mid \check{\mathcal{F}}_0\right) \\ &= \mathbb{Q}\left(\sup_{t \in [0, T]} \left[\int_0^t \check{w}_s^{-1} d\check{W}_s\right] \geq a\right). \end{aligned}$$

Then, by Gaussian estimates, we obtain

$$\begin{aligned} \mathbb{Q}\left(\sup_{t \in [0, T]} (\check{X}_t - \check{X}_0) \geq a\right) &\geq \mathbb{Q}\left(\int_0^T \check{w}_s^{-1} d\check{W}_s \geq a\right) \\ &\geq \frac{1}{\sqrt{2\pi\check{k}_T}} \int_a^{a+1} \exp\left(-\frac{x^2}{2\check{k}_T}\right) dx \geq \frac{1}{\sqrt{2\pi\check{k}_T}} \exp\left(-\frac{2a^2}{\check{k}_T}\right). \end{aligned}$$

Now,

$$\begin{aligned} \mathbb{Q}\left(\sup_{t \in [0, T]} (\check{X}_t - \check{X}_0) \geq a \mid \check{\mathcal{F}}_0\right) &= \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mathbf{1}_{\{\sup_{t \in [0, T]} (\check{X}_t - \check{X}_0) \geq a\}} \mid \check{\mathcal{F}}_0\right] \\ &\leq \mathbb{E}^{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^2 \mid \check{\mathcal{F}}_0\right]^{1/2} \mathbb{P}\left(\sup_{t \in [0, T]} (\check{X}_t - \check{X}_0) \geq a \mid \check{\mathcal{F}}_0\right)^{1/2}. \end{aligned}$$

It is completely standard to prove that

$$\mathbb{E}^{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^2 \mid \check{\mathcal{F}}_0\right] \leq \exp(2\check{k}_T).$$

So, we end up with

$$\mathbb{P}\left(\sup_{t \in [0, T]} (\check{X}_t - \check{X}_0) \geq a \mid \check{\mathcal{F}}_0\right)^{1/2} \geq \frac{1}{\sqrt{2\pi\check{k}_T}} \exp(-\check{k}_T) \exp\left(-\frac{2a^2}{\check{k}_T}\right).$$

which completes the proof on the event $\check{X}_0 \geq 0$. Changing $(\check{X}_t)_{t \in [0, T]}$ into $(-\check{X}_t)_{t \in [0, T]}$, we easily tackle the case when $\check{X}_0 \leq 0$. \square

We use Lemma 2.27 to get:

Lemma 2.28. *There exist a sequence of positive reals $(\lambda_N)_{N \geq 1}$ converging to $+\infty$, with*

$$\lim_{N \rightarrow +\infty} \lambda_N / \sqrt{N} = 0,$$

and another sequence $(t_N)_{N \geq 1}$ converging to 0 such that

$$\lim_{N \rightarrow +\infty} \mathbb{P}(\sigma_N \leq t_N) = 1,$$

where, for all $N \geq 1$,

$$\sigma_N = \inf \left\{ t \geq 0 : |\tilde{\mu}_t^N| \geq \frac{\lambda_N \sigma^2}{N} \right\},$$

with $(\tilde{\mu}_t^N)_{t \in [0, T]}$ as in (2.28).

Proof. Recall from (2.28) and (2.30) that there exists a martingale $(v_t^N)_{t \in [0, T]}$, bounded by 1, such that $(\tilde{\mu}_t^N)_{t \in [0, T]}$ satisfies the SDE

$$d\tilde{\mu}_t^N = -w_t^{-2} v_t^N dt + \frac{\sigma w_t^{-1}}{\sqrt{N}} dB_t, \quad \forall t \in [0, T].$$

By using the change of variables,

$$\check{X}_t := \frac{N}{\sigma^2} \tilde{\mu}_{\frac{\sigma^2 t}{N}}^N, \quad \check{Y}_t := v_{\frac{\sigma^2 t}{N}}^N, \quad \check{w}_t := w_{\frac{\sigma^2 t}{N}}, \quad \check{B}_t := \frac{\sqrt{N}}{\sigma} B_{\frac{\sigma^2 t}{N}}, \quad \forall t \in [0, T].$$

It is well checked that

$$d\check{X}_t = -\check{w}_t^{-2} \check{Y}_t dt + \check{w}_t^{-1} d\check{B}_t, \quad \check{X}_0 = 0, \quad t \in [0, \frac{N}{\sigma^2} T]. \quad (2.36)$$

Obviously $(\check{B}_t)_{t \in [0, \frac{N}{\sigma^2} T]}$ is a Brownian motion and $(\check{X}_t)_{t \in [0, \frac{N}{\sigma^2} T]}$ is an Itô process for a common rescaled filtration $(\check{\mathcal{F}}_t)_{t \in [0, \frac{N}{\sigma^2} T]}$. We also let

$$\check{\sigma}_N := \inf \left\{ t \geq 0 : |\check{X}_t| \geq \lambda_N \right\} = \frac{N}{\sigma^2} \sigma_N,$$

where $\lambda_N \geq 1$ will be fixed later on.

We claim that for the same constant $c > 0$ as in the statement of Lemma 2.27, it holds, for any integer $i \in \{0, \dots, N\}$,

$$\begin{aligned} & \mathbb{P} \left(\check{\sigma}_N \geq \frac{(i+1)T}{\sigma^2} \middle| \check{\sigma}_N > \frac{iT}{\sigma^2} \right) \\ & \leq 1 - \frac{1}{c} \exp(-cN(k_{\frac{(i+1)T}{\sigma^2}} - k_{\frac{iT}{\sigma^2}})) \exp \left(-\frac{c\sigma^2 \lambda_N^2}{N(k_{\frac{(i+1)T}{\sigma^2}} - k_{\frac{iT}{\sigma^2}})} \right). \end{aligned} \quad (2.37)$$

The proof works as follows. We consider the process $(\check{X}_t)_{t \in [\check{\sigma}_N \wedge \frac{iT}{\sigma^2}, \frac{NT}{\sigma^2}]}$ and we apply Lemma 2.27 with $\tau = \check{\sigma}_N \wedge \frac{iT}{\sigma^2}$ and $a = \lambda_N$. On the event $\{\check{\sigma}_N \geq \frac{iT}{\sigma^2}\}$, we get

$$\mathbb{P}\left(\check{\sigma}_N \leq \frac{(i+1)T}{\sigma^2} \mid \check{\mathcal{F}}_{\check{\sigma}_N \wedge \frac{iT}{\sigma^2}}\right) \geq \frac{1}{c} \exp(-c(\check{k}_{\frac{(i+1)T}{\sigma^2}} - \check{k}_{\frac{iT}{\sigma^2}})) \exp\left(-\frac{c\lambda_N^2}{(\check{k}_{\frac{(i+1)T}{\sigma^2}} - \check{k}_{\frac{iT}{\sigma^2}})}\right),$$

with

$$\check{k}_t = \int_0^t \check{w}_s^{-2} ds = \int_0^t w_{\frac{\sigma^2 s}{N}}^{-2} ds = \frac{N}{\sigma^2} k_{\frac{\sigma^2 t}{N}},$$

from which we get (2.37).

By iterating (2.37) and by modifying the value of the constant c , we deduce that

$$\mathbb{P}\left(\check{\sigma}_N \geq \frac{i}{\sigma^2} T\right) \leq \left(1 - \frac{1}{c} \exp(-c\lambda_N^2)\right)^i.$$

Recall that $\ln(1-u) \leq -u$ for $u \in [0, 1)$. Thus,

$$\mathbb{P}\left(\check{\sigma}_N \geq \frac{i}{\sigma^2} T\right) \leq \exp\left(i \ln\left(1 - \frac{1}{c} \exp(-c\lambda_N^2)\right)\right) \leq \exp\left(-\frac{i}{c} \exp(-c\lambda_N^2)\right). \quad (2.38)$$

Choose a sequence $(t_N)_{N \geq 1}$ converging to 0 as N tends to $+\infty$ such that the sequence $(Nt_N/T)_{N \geq 1}$ is integer valued and converges to $+\infty$. Define λ_N as

$$\sqrt{Nt_N} \exp\left(-c\lambda_N^2\right) = 1,$$

that is

$$\lambda_N = \sqrt{\frac{1}{2c} \ln(Nt_N)}.$$

Letting $i = Nt_N/T$ in (2.38) and assuming without any loss of generality that $\lambda_N \geq 1$ for all $N \geq 1$, we deduce that

$$\mathbb{P}(\sigma_N \geq t_N) = \mathbb{P}\left(\check{\sigma}_N \geq \frac{N}{\sigma^2} t_N\right) \leq \exp\left[-\frac{Nt_N}{cT} \exp(-c\lambda_N^2)\right] = \exp\left[-\frac{1}{cT} \sqrt{Nt_N}\right],$$

which completes the proof. \square

2.5.5 Conclusion

By combining Lemmas 2.26 and 2.27, we obtain:

Lemma 2.29. *There exists a sequence of stopping times $(\sigma_N)_{N \geq 1}$, converging to 0 in probability, such that, for any $\epsilon > 0$, letting*

$$\begin{aligned} \vartheta_N^+ &:= \inf\left\{t \geq \sigma_N : \tilde{\mu}_t^N \leq \tilde{\mu}_{\sigma_N}^N + k_t - k_{\sigma_N} - \epsilon\right\}, \\ \vartheta_N^- &:= \inf\left\{t \geq \sigma_N : \tilde{\mu}_t^N \geq \tilde{\mu}_{\sigma_N}^N - (k_t - k_{\sigma_N}) + \epsilon\right\}, \end{aligned}$$

it holds, for any $\eta \in (0, \delta/2)$,

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(\left\{\vartheta_N^+ < \delta - \eta, \tilde{\mu}_{\sigma_N}^N \geq 0\right\} \cup \left\{\vartheta_N^- < \delta - \eta, \tilde{\mu}_{\sigma_N}^N \leq 0\right\}\right) = 0.$$

Proof. We choose the sequence $(\sigma_N)_{N \geq 1}$ given by Lemma 2.28, for the same sequence $(\lambda_N)_{N \geq 1}$ as therein. We then let $(b_N := \sigma \sqrt{\lambda_N})_{N \geq 1}$ and $(\gamma_N := \sqrt{b_N/N})_{N \geq 1}$. Obviously, $\lim_{N \rightarrow +\infty} b_N = +\infty$, $\lim_{N \rightarrow +\infty} N\gamma_N^2 = +\infty$ and $\lim_{N \rightarrow +\infty} \gamma_N = 0$. Moreover, letting

$$\sigma_N^+ = \inf \left\{ t \geq 0 : \tilde{\mu}_t^N \geq \frac{\lambda_N \sigma^2}{N} \right\},$$

we have

$$\tilde{\mu}_{\sigma_N^+}^N = \frac{\lambda_N \sigma^2}{N} = \frac{b_N^2}{N} = b_N \gamma_N^2 \quad \text{if } \sigma_N^+ \leq \frac{\delta}{2},$$

Therefore, applying Lemma 2.26 with $(\varrho_N = \sigma_N^+)_{N \geq 1}$, we deduce that, with $(\tau'_N)_{N \geq 1}$ defined as in the statement of Lemma 2.26,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\tau'_N < \delta - \eta, \sigma_N^+ \leq \frac{\delta}{2} \right) = 0.$$

In particular,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\tau'_N < \delta - \eta, \sigma_N = \sigma_N^+, \sigma_N \leq \frac{\delta}{2} \right) = 0.$$

On the event $\{\sigma_N = \sigma_N^+\}$, τ'_N is equal to ϑ_N^+ . Therefore,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\vartheta_N^+ < \delta - \eta, \sigma_N = \sigma_N^+, \sigma_N \leq \frac{\delta}{2} \right) = 0,$$

and by Lemma 2.27,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\vartheta_N^+ < \delta - \eta, \sigma_N = \sigma_N^+ \right) = 0.$$

By symmetry, we also have

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\vartheta_N^- < \delta - \eta, \sigma_N = \sigma_N^- \right) = 0,$$

with

$$\sigma_N^- = \inf \left\{ t \geq 0 : \tilde{\mu}_t^N \leq -\frac{\lambda_N \sigma^2}{N} \right\}.$$

The result easily follows. \square

In order to proceed further, we need a new notation. For any $\eta \in (0, \delta/2)$ and $t \in [0, \delta/2]$, we let $F(t, \eta) = \{\gamma \in \mathcal{C}([t, \delta - \eta]; \mathbb{R}) : \gamma_s \geq k_s - k_t, s \in [t, \delta - \eta]\}$ and, for any $\epsilon > 0$, $F^\epsilon(t, \eta) = \{\gamma \in \mathcal{C}([t, \delta - \eta]; \mathbb{R}) : \gamma_s \geq k_s - k_t - \epsilon, s \in [t, \delta - \eta]\}$. Obviously, $F(t, \eta)$ and $F^\epsilon(t, \eta)$ are closed subsets of $\mathcal{C}([t, \delta - \eta]; \mathbb{R})$.

Here is now the main claim:

Proposition 2.30. *Let \mathbb{P}^∞ be a limit point of the sequence $(\mathbb{P} \circ (\tilde{\mu}_s^N)_{s \in [0, T]}^{-1})_{N \geq 1}$. Then,*

$$\mathbb{P}^\infty \left(\left\{ \gamma \in \mathcal{C}([0, T]; \mathbb{R}) : \gamma_{[0, \delta - \eta]} \in F(0, \eta) \right\} \cup \left\{ \gamma \in \mathcal{C}([0, T]; \mathbb{R}) : -\gamma_{[0, \delta - \eta]} \in F(0, \eta) \right\} \right) = 1.$$

In the proof, we shall merely write $F(0, \eta) \cup (-F(0, \eta))$ for $\{\gamma \in \mathcal{C}([0, T]; \mathbb{R}) : \gamma_{[0, \delta - \eta]} \in F(0, \eta)\} \cup \{\gamma \in \mathcal{C}([0, T]; \mathbb{R}) : -\gamma_{[0, \delta - \eta]} \in F(0, \eta)\}$.

Proof. We apply Lemma 2.29. For a given $\epsilon > 0$, it says that

$$\lim_{N \rightarrow +\infty} \mathbb{P} \left(\{\vartheta_N^+ \geq \delta - \eta, \tilde{\mu}_{\sigma_N}^N \geq 0\} \cap \{\vartheta_N^- \geq \delta - \eta, \tilde{\mu}_{\sigma_N}^N \leq 0\} \right) = 1.$$

Take $t \in (0, \delta/2)$. On the event $\{\sigma_N \leq t\} \cap \{\vartheta_N^+ \geq \delta - \eta, \tilde{\mu}_{\sigma_N}^N \geq 0\}$, we have

$$\tilde{\mu}_s^N \geq \tilde{\mu}_{\sigma_N}^N + k_s - k_{\sigma_N} - \epsilon \geq k_s - k_t + k_t - k_{\sigma_N} - \epsilon \geq k_s - k_t - \epsilon.$$

Therefore, on the event $\{\sigma_N \leq t\} \cap \{\vartheta_N^+ \geq \delta - \eta, \tilde{\mu}_{\sigma_N}^N \geq 0\}$, $(\tilde{\mu}_s^N)_{s \in [0, T]} \in F^\epsilon(t, \eta)$, if we regard $F^\epsilon(t, \eta)$ as a subset of $\mathcal{C}([0, T]; \mathbb{R})$.

In fact, $F^\epsilon(t, \eta) \cup (-F^\epsilon(t, \eta))$ may be also regarded as a subset of $\mathcal{C}([0, T]; \mathbb{R})$. Then, Lemma 2.29 says that

$$\lim_{N \rightarrow \infty} \mathbb{P} \left((\tilde{\mu}_s^N)_{s \in [0, T]} \in F^\epsilon(t, \eta) \cup (-F^\epsilon(t, \eta)) \right) = 1.$$

Since $F^\epsilon(t, \eta) \cup (-F^\epsilon(t, \eta))$ is closed, we get, by the portmanteau theorem, that, for all $t \in [0, \delta/2]$ and $\epsilon > 0$,

$$\mathbb{P}^\infty(F^\epsilon(t, \eta) \cup (-F^\epsilon(t, \eta))) = 1.$$

Intersecting over all the positive and rational reals ϵ , we get:

$$\mathbb{P}^\infty(F(t, \eta) \cup (-F(t, \eta))) = 1.$$

Intersecting over all the rational reals $t \in [0, \delta/2]$, we complete the proof. \square

Here is the complete statement.

Theorem 2.31. *The sequence $(\mathbb{P} \circ (\mu_t^N)_{t \in [0, T]}^{-1})_{N \geq 1}$ converges to*

$$\frac{1}{2} \delta_{(k_t)_{t \in [0, T]}} + \frac{1}{2} \delta_{(-k_t)_{t \in [0, T]}}.$$

Proof. First step. It is easily checked that the sequence $(\mathbb{P} \circ (\mu_t^N)_{t \in [0, T]}^{-1})_{N \geq 1}$ is tight on $\mathcal{C}([0, T]; \mathbb{R})$. Also, recalling that $(v_t^N)_{t \in [0, T]}$ in (2.28) is bounded by 1, we deduce that, for any limiting point \mathbb{P}^∞ , the canonical process $(\gamma_t)_{t \in [0, T]}$ on $\mathcal{C}([0, T]; \mathbb{R})$ satisfies $|\gamma_t| \leq k_t$ with probability 1 under \mathbb{P}^∞ .

Second step. By Proposition 2.30 and with the same notation as therein, the canonical process $(\gamma_t)_{t \in [0, T]}$ on $\mathcal{C}([0, T]; \mathbb{R})$ must satisfy

$$\mathbb{P}^\infty \left(\{\gamma_t = k_t, \quad t \in [0, \delta]\} \cup \{\gamma_t = -k_t, \quad t \in [0, \delta]\} \right) = 1.$$

Obviously, γ and $-\gamma$ have the same law under \mathbb{P}^∞ . We deduce that

$$\mathbb{P}^\infty(\gamma_t = k_t, \quad t \in [0, \delta]) = \mathbb{P}^\infty(\gamma_t = -k_t, \quad t \in [0, \delta]) = \frac{1}{2}.$$

Third Step. It now remains to extend the result to the whole $[\delta, T]$. To do so, we make use of the conclusion of the second step. It says that, for any $\varepsilon \in (0, \delta)$,

$$\lim_{N \rightarrow +\infty} \mathbb{P}(|\tilde{\mu}_\delta^N| > k_\delta - k_\varepsilon/2) = 1.$$

By recalling that $(v_t^N)_{t \in [0, T]}$ is bounded by 1, it is pretty easy to deduce that, for any $\varepsilon > 0$, we can find $\delta' > \delta$ such that

$$\lim_{N \rightarrow +\infty} \mathbb{P}\left(\inf_{t \in [\delta, \delta']} |\tilde{\mu}_t^N| > k_\delta - k_\varepsilon\right) = 1. \quad (2.39)$$

We shall conclude by using the sole fact that $\delta' - \delta > 0$. To do so, we shall prove that

$$\lim_{N \rightarrow \infty} \mathbb{E}\left[\sup_{t \in [\delta', T]} |\tilde{\mu}_t^N - (\tilde{\mu}_{\delta'}^N + k_t - k_{\delta'})|^2 \mathbb{1}_{\tilde{\mu}_{\delta'}^N > k_\delta - k_\varepsilon}\right] = 0, \quad (2.40)$$

and similarly,

$$\lim_{N \rightarrow \infty} \mathbb{E}\left[\sup_{t \in [\delta', T]} |\tilde{\mu}_t^N - (\tilde{\mu}_{\delta'}^N + k_t - k_{\delta'})|^2 \mathbb{1}_{\tilde{\mu}_{\delta'}^N < -k_\delta + k_\varepsilon}\right] = 0, \quad (2.41)$$

which is obviously sufficient to conclude. Indeed, by combining (2.39), (2.40) and (2.41), we get, for any $\varepsilon > 0$, for any $\delta' > \delta$,

$$\lim_{N \rightarrow \infty} \mathbb{P}\left(\{\forall t \in [\delta, T], \tilde{\mu}_t^N \geq k_t + k_\delta - k_{\delta'} - 2k_\varepsilon\} \cup \{\forall t \in [\delta, T], \tilde{\mu}_t^N \leq -k_t - k_\delta + k_{\delta'} + 2k_\varepsilon\}\right) = 1.$$

Arguing as in the second step, we then complete the proof.

It thus remains to prove (2.40) and (2.41). We prove the first of the two claims only, as the second one may be proved in the same way. We observe that

$$\mathbb{E}\left[\sup_{t \in [\delta', T]} |\tilde{\mu}_t^N - (\tilde{\mu}_{\delta'}^N + k_t - k_{\delta'})|^2 \mid \hat{\mathcal{F}}_{\delta'}\right]^{1/2} \leq (k_T - k_{\delta'}) \sup_{t \in [\delta', T]} \mathbb{E}[|v_t^N + 1|^2 \mid \hat{\mathcal{F}}_{\delta'}]^{1/2} + \frac{C}{\sqrt{N}}.$$

Recalling that $(v_t^N + 1)_{t \in [\delta', T]}$ is a martingale w.r.t. $(\hat{\mathcal{F}}_t)_{t \in [\delta', T]}$, we get, for a constant C independent of N and δ' ,

$$\begin{aligned} \mathbb{E}\left[\sup_{t \in [\delta', T]} |\tilde{\mu}_t^N - (\tilde{\mu}_{\delta'}^N + k_t - k_{\delta'})|^2 \mid \hat{\mathcal{F}}_{\delta'}\right]^{1/2} &\leq (k_T - k_{\delta'}) \mathbb{E}[|v_T^N + 1|^2 \mid \hat{\mathcal{F}}_{\delta'}]^{1/2} + \frac{C}{\sqrt{N}} \\ &\leq (k_T - k_{\delta'}) \mathbb{E}[|g(\tilde{\mu}_T^N) + 1|^2 \mid \hat{\mathcal{F}}_{\delta'}]^{1/2} + \frac{C}{\sqrt{N}}. \end{aligned}$$

On the event $\{\tilde{\mu}_{\delta'}^N > k_\delta - k_\varepsilon\}$, $g(\tilde{\mu}_{\delta'}^N + k_T - k_{\delta'}) \leq g(k_T - k_{\delta'} + k_\delta - k_\varepsilon)$; for $\delta' - \delta$ and ε small enough, $k_T - k_{\delta'} + k_\delta - k_\varepsilon \geq r_\delta$ and then $g(\tilde{\mu}_{\delta'}^N + k_T - k_{\delta'}) = -1$. Therefore, since g is $\frac{1}{r_\delta}$ Lipschitz,

$$\begin{aligned} &\mathbb{E}\left[\sup_{t \in [\delta', T]} |\tilde{\mu}_t^N - (\tilde{\mu}_{\delta'}^N + k_t - k_{\delta'})|^2 \mid \hat{\mathcal{F}}_{\delta'}\right]^{1/2} \\ &\leq \frac{k_T - k_{\delta'}}{r_\delta} \mathbb{E}[|\tilde{\mu}_T^N - (\tilde{\mu}_{\delta'}^N + k_T - k_{\delta'})|^2 \mid \hat{\mathcal{F}}_{\delta'}]^{1/2} + \frac{C}{\sqrt{N}}. \end{aligned}$$

So,

$$(r_\delta - k_T + k_{\delta'}) \mathbb{E} \left[\sup_{t \in [\delta', T]} |\tilde{\mu}_t^N - (\tilde{\mu}_{\delta'}^N + k_t - k_{\delta'})|^2 \mid \hat{\mathcal{F}}_{\delta'} \right]^{1/2} \leq \frac{C}{\sqrt{N}}.$$

Now, $r_\delta - k_T + k_{\delta'} = k_{\delta'} - k_\delta > 0$. This shows (2.40). \square

2.6 Appendix: Proof of Proposition 2.10

Consider $\sigma_0 \in (0, 1)$, we wish to find estimates for the residue $\Psi(t, x, \sigma_0) = \theta^{\sigma_0}(t, x) - \theta(t, x)$, for different configurations of $(t, x) \in [0, T] \times \mathbb{R}$. In fact, it is easily checked (using a change of variable) that θ^{σ_0} is odd in x and, thus, that Ψ is also odd in x . Therefore, we can just focus on $\Psi(t, x, \sigma_0)$ for $x > 0$.

To simplify notation, we write $\lambda := \sigma_0^{-2} \in (1, +\infty)$, so that obtaining asymptotic expressions as $\sigma_0 \rightarrow 0$ is equivalent to obtaining asymptotic expressions as $\lambda \rightarrow +\infty$. We also use the definitions:

$$\begin{aligned} g(y) &:= -\frac{y}{r_\delta} \mathbb{1}_{|y| \leq r_\delta} - \text{sign}(y) \mathbb{1}_{|y| > r_\delta}, \\ h(y) &:= \left(-\int_0^y g(v) dv - \frac{(x-y)^2}{2r_t} \right), \\ \text{erf}(y) &:= \frac{2}{\sqrt{\pi}} \int_0^y \exp(-v^2) dv, \end{aligned}$$

for $(t, y) \in [0, T] \times \mathbb{R}$.

Last, throughout the proof, we use the generic notation $Q(y)$ for a polynomial function of y of degree less than or equal to 1. Possibly, Q may depend on (t, x) .

Case 1: We say that $(t, x) \in \mathcal{D}_1$ if and only if $t \leq \delta$ and $x > r_t - r_\delta > 0$.

Preliminary computation: For every (t, x) in \mathcal{D}_1 , we define $y^* := x + r_t \in (r_\delta, +\infty)$. We claim that y^* is a global maximum of h . Indeed, h is (strictly) decreasing after y^* and (strictly) increasing before, as checked in the computations below:

$$\begin{aligned} h'(y) &= 1 + \frac{x-y}{r_t} < 1 + \frac{x-y^*}{r_t} = 0, \quad \text{for all } y \in (y^*, +\infty), \\ h'(y) &= 1 + \frac{x-y}{r_t} > 1 + \frac{x-y^*}{r_t} = 0, \quad \text{for all } y \in (r_\delta, y^*), \\ h'(y) &= y \left(\frac{r_t - r_\delta}{r_t r_\delta} \right) + \frac{x}{r_t} > -r_\delta \left(\frac{r_t - r_\delta}{r_t r_\delta} \right) + \frac{r_t - r_\delta}{r_t} = 0, \quad \text{for all } y \in (-r_\delta, r_\delta), \\ h'(y) &= -1 + \frac{x-y}{r_t} > -1 + \frac{x+r_\delta}{r_t} > -1 + \frac{r_t - r_\delta + r_\delta}{r_t} = 0, \quad \text{for all } y \in (-\infty, -r_\delta). \end{aligned} \tag{2.42}$$

Due to the different regimes of the function $h(y)$, we propose a representation of the basic integral present in the residue. This representation is useful to obtain asymptotic expressions

through Laplace's method (see for instance [39] for standard use in probability).

$$\begin{aligned}
\int_{-\infty}^{\infty} Q(y) \exp(\lambda h(y)) dy &= \int_{-\infty}^{r_\delta} Q(y) \exp(\lambda h(y)) dy + \int_{r_\delta}^{\infty} Q(y) \exp(\lambda h(y)) dy \\
&= \int_{-\infty}^{r_\delta} Q(y) \exp(\lambda h(y)) dy + \int_{r_\delta - y^*}^{\infty} Q(u + y^*) \exp(\lambda h(u + y^*)) du
\end{aligned} \tag{2.43}$$

Observe that, for all $u \in (r_\delta - y^*, +\infty)$,

$$\begin{aligned}
h(u + y^*) - h(y^*) &= - \int_{y^*}^{u+y^*} g(v) dv - \frac{(x - (y^* + u))^2}{2r_t} + \frac{(x - y^*)^2}{2r_t} \\
&= u - \frac{u^2}{2r_t} + \frac{(x - y^*)u}{r_t} = -\frac{u^2}{2r_t} < 0.
\end{aligned}$$

In order to compute the second integral in (2.43), we perform the following change of variable

$$s = \frac{u}{\sqrt{2r_t}},$$

and we use the following notation

$$B = B(y^*, r_\delta) := \frac{r_\delta - y^*}{\sqrt{2r_t}} < 0.$$

Then, recalling that Q is a polynomial function of order 1, we get

$$\begin{aligned}
&\int_{r_\delta - y^*}^{\infty} Q(u + y^*) \exp(\lambda h(u + y^*)) du \\
&= \sqrt{2r_t} \int_B^{+\infty} \left(Q(y^*) + Q'(y^*) \sqrt{2r_t} s \right) \exp(\lambda(h(y^*) - s^2)) ds \\
&= \sqrt{2r_t} \exp(\lambda h(y^*)) \left[Q(y^*) \int_B^{+\infty} \exp(-\lambda s^2) ds + Q'(y^*) \sqrt{2r_t} \int_B^{+\infty} s \exp(-\lambda s^2) ds \right] \\
&= \exp(\lambda h(y^*)) \left[Q(y^*) \sqrt{\frac{\pi r_t}{2}} \frac{(1 - \operatorname{erf}(\sqrt{\lambda} B))}{\sqrt{\lambda}} + Q'(y^*) r_t \frac{\exp(-\lambda B^2)}{\lambda} \right].
\end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} Q(y) \exp(\lambda h(y)) dy = \exp(\lambda h(y^*)) \left[Q(y^*) \sqrt{\frac{\pi r_t}{2}} \frac{(1 - \operatorname{erf}(\sqrt{\lambda} B))}{\sqrt{\lambda}} + \mathcal{R}(t, x, \lambda) \right]. \tag{2.44}$$

The remainder term $\mathcal{R}(t, x, \lambda)$ is explicitly given by

$$\begin{aligned} \mathcal{R}(t, x, \lambda) &:= \exp\left(\lambda \int_{y^*}^{-r_t} h'(z) dz\right) \int_{-\infty}^{-r_t} Q(y) \exp\left(\lambda \int_{-r_t}^y h'(z) dz\right) dy \\ &\quad + \exp\left(\lambda \int_{y^*}^{r_\delta} h'(z) dz\right) \int_{-r_t}^{r_\delta} Q(y) \exp\left(\lambda \int_{r_\delta}^y h'(z) dz\right) dy \\ &\quad + Q'(y^*) r_t \frac{\exp(-\lambda B^2)}{\lambda}. \end{aligned}$$

Using Cole-Hopf formula: Define now $q(y) = \frac{x-y}{r_t} + \text{sign}(x) = \frac{x-y}{r_t} + 1$, thus $q(y^*) = 0$. Using (2.8), we deduce that

$$\Psi(t, x, \sigma_0) = \theta^{\sigma_0}(t, x) - \theta(t, x) = \frac{\int_{-\infty}^{+\infty} q(y) \exp(\lambda h(y)) dy}{\int_{-\infty}^{+\infty} \exp(\lambda h(y)) dy}.$$

Since $y^* > 0$, $h(y^*) \leq 0$ and then $\exp(\lambda h(y^*)) \leq 1$. Hence, the above asymptotic expansion (2.44), with $Q = q$ and $Q = 1$, implies

$$|\Psi(t, x, \sigma_0)| \leq \frac{\left| \int_{-\infty}^{+\infty} q(y) \exp(\lambda h(y)) dy \right|}{\int_{-\infty}^{+\infty} \exp(\lambda h(y)) dy} \leq \frac{q(y^*) \sqrt{\frac{\pi r_t}{2}} \left(\frac{1 - \text{erf}(\sqrt{\lambda} B)}{\sqrt{\lambda}} \right) + |\mathcal{R}(t, x, \lambda)|}{\sqrt{\frac{\pi r_t}{2}} \left(\frac{1 - \text{erf}(\sqrt{\lambda} B)}{\sqrt{\lambda}} \right) + \int_{-\infty}^{r_\delta} \exp\left(\lambda \int_{y^*}^y h'(z) dz\right) dy},$$

the remainder \mathcal{R} being computed with $Q = q$. Since $q(y^*) = 0$, it follows that

$$\begin{aligned} &|\Psi(t, x, \sigma_0)| \\ &\leq \sqrt{\frac{2}{\pi r_t}} \left(\frac{\sqrt{\lambda}}{1 - \text{erf}(\sqrt{\lambda} B)} \right) |\mathcal{R}(t, x, \lambda)| \leq \sqrt{\frac{2}{\pi r_t}} \sqrt{\lambda} [\mathcal{I}_1(t, x, \lambda) + \mathcal{I}_2(t, x, \lambda) + \mathcal{I}_3(t, x, \lambda)], \end{aligned} \tag{2.45}$$

where

$$\begin{aligned} \mathcal{I}_1(t, x, \lambda) &:= \left| r_t q'(y^*) \frac{\exp(-\lambda B^2)}{\lambda} \right| \leq \frac{1}{\lambda}, \text{ since } q'(y^*) = -\frac{1}{r_t}, \\ \mathcal{I}_2(t, x, \lambda) &:= \left| \exp\left(\lambda \int_{y^*}^{r_\delta} h'(z) dz\right) \int_{-r_t}^{r_\delta} q(y) \exp\left(\lambda \int_{r_\delta}^y h'(z) dz\right) dy \right|, \\ \mathcal{I}_3(t, x, \lambda) &:= \left| \exp\left(\lambda \int_{y^*}^{-r_t} h'(z) dz\right) \int_{-\infty}^{-r_t} q(y) \exp\left(\lambda \int_{-r_t}^y h'(z) dz\right) dy \right|. \end{aligned}$$

Handling \mathcal{I}_2 : Observing from (2.42) that

$$\int_{r_\delta}^{y^*} h'(z) dz = \frac{1}{r_t} \int_{r_\delta}^{y^*} (y^* - z) dr = \frac{1}{2r_t} (y^* - r_\delta)^2 \geq \frac{x^2}{2r_t},$$

and that $h' > 0$ on $(-\infty, -r_\delta)$ and on $(-r_\delta, r_\delta)$, we get

$$\begin{aligned}
\mathcal{I}_2(t, x, \lambda) &\leq \exp\left(\lambda \int_{y^*}^{r_\delta} h'(z) dz\right) \int_{-r_t}^{r_\delta} |q(y)| \exp\left(\lambda \int_{r_\delta}^y h'(z) dz\right) dy \\
&= \exp\left(\frac{-\lambda(y^* - r_\delta)^2}{2r_t}\right) \int_{-r_t}^{r_\delta} |q(y)| \exp\left(\lambda \int_{r_\delta}^y h'(z) dz\right) dy \\
&\leq \exp\left(\frac{-\lambda x^2}{2r_t}\right) \int_{-r_t}^{r_\delta} |q(y)| \exp\left(\lambda \int_{r_\delta}^y h'(z) dz\right) dy \\
&\leq \exp\left(\frac{-\lambda x^2}{2r_0}\right) \int_{-r_t}^{r_\delta} |q(y)| \exp\left(\lambda \int_{r_\delta}^y h'(z) dz\right) dy \\
&\leq \exp\left(\frac{-\lambda x^2}{2r_0}\right) \int_{-r_t}^{r_\delta} (|q(y)| \times 1) dy \\
&\leq \exp\left(\frac{-\lambda x^2}{2r_0}\right) (2r_t) \max_{|y| \leq r_t} |q(y)| \leq \exp\left(\frac{-\lambda x^2}{2r_0}\right) 2(x + 2r_t).
\end{aligned}$$

Handling \mathcal{I}_3 : As for \mathcal{I}_3 , we use the fact that $q(y) \geq 0$, for $y \leq -r_t$. Using the fact that $h' > 0$ on $(-\infty, -r_t)$ and proceeding as in the previous step and then using the fact that $h'(y) > -(r_t + y)/r_t$ for $y < -r_t$, we get

$$\begin{aligned}
\mathcal{I}_3(t, x, \lambda) &= \exp\left(\lambda \int_{y^*}^{-r_t} h'(z) dz\right) \int_{-\infty}^{-r_t} q(y) \exp\left(\lambda \int_{-r_t}^y h'(z) dz\right) dy \\
&\leq \exp\left(\lambda \int_{y^*}^{r_\delta} h'(z) dz\right) \int_{-\infty}^{-r_t} q(y) \exp\left(\lambda \int_{-r_t}^y h'(z) dz\right) dy \\
&\leq \exp\left(\frac{-\lambda x^2}{2r_0}\right) \int_{-\infty}^{-r_t} q(y) \exp\left(\lambda \int_{-r_t}^y h'(z) dz\right) dy \\
&\leq \exp\left(\frac{-\lambda x^2}{2r_0}\right) \int_{-\infty}^{-r_t} q(y) \exp\left(\frac{-\lambda(y + r_t)^2}{2r_t}\right) dy \\
&= \exp\left(\frac{-\lambda x^2}{2r_0}\right) \int_{-\infty}^{-r_t} [q(-r_t) + (y + r_t)q'(-r_t)] \exp\left(\frac{-\lambda(y + r_t)^2}{2r_t}\right) dy \\
&\leq \exp\left(\frac{-\lambda x^2}{2r_0}\right) \left[\frac{q(-r_t)}{\sqrt{\lambda}} \int_{-\infty}^0 \exp\left(\frac{-u^2}{2r_t}\right) du + \frac{q'(-r_t)}{\lambda} \int_{-\infty}^0 u \exp\left(\frac{-u^2}{2r_t}\right) du \right] \\
&\leq \exp\left(\frac{-\lambda x^2}{2r_0}\right) \left[\frac{1}{\sqrt{\lambda}} \sqrt{\frac{\pi r_t}{2}} \left(2 + \frac{x}{r_t}\right) + \frac{1}{\lambda} \right]
\end{aligned}$$

Conclusion: Now consider $(t, x) \in \mathcal{D}_1$ such that $x^2 \geq 2r_0 \ln \lambda / \lambda$. Then, by the last two steps,

$$\mathcal{I}_2(t, x, \lambda) + \mathcal{I}_3(t, x, \lambda) \leq \frac{1}{\lambda} \left[2(x + 2r_t) + \frac{1}{\sqrt{\lambda}} \sqrt{\frac{\pi r_t}{2}} \left(2 + \frac{x}{r_t}\right) + \frac{1}{\lambda} \right].$$

Therefore, by (2.45), it holds, for all $(t, x) \in \mathcal{D}_1$ with $x^2 \geq 2r_0 \ln \lambda / \lambda = (4r_0)\sigma_0^2 |\ln \sigma_0|$,

$$\begin{aligned} |\Psi(t, x, \sigma_0)| &\leq \sqrt{\frac{2}{\pi r_t}} \frac{1}{\sqrt{\lambda}} \left[2 + 2(x + 2r_t) + \sqrt{\frac{\pi r_t}{2}} \left(2 + \frac{x}{r_t} \right) \right] \\ &\leq \sqrt{\frac{2}{\pi r_\delta}} \frac{1}{\sqrt{\lambda}} \left[2(1 + |x| + 2r_0) + \sqrt{\frac{\pi r_0}{2}} \left(2 + \frac{|x|}{r_\delta} \right) \right] \\ &\leq C(x) \frac{1}{\sqrt{\lambda}} = C(x) \sigma_0. \end{aligned}$$

By symmetry, the same holds if $t < \delta$ and $x < r_\delta - r_t < 0$.

Case 2: We say that $(t, x) \in \mathcal{D}_2$ if and only if $t \leq \delta$ and $r_\delta - r_t < x < r_t - r_\delta$.

Preliminary computation: For every $(t, x) \in \mathcal{D}_2$, we define $\bar{y} := \frac{-xr_\delta}{r_t - r_\delta}$, $y_1^* := x - r_t$, $y_2^* := x + r_t$. It holds that y_1^* is a global maximum of h on $(-\infty, -\bar{y})$ and y_2^* is a global maximum of h on $(\bar{y}, +\infty)$. Indeed,

$$\begin{aligned} h'(y) &= -1 + \frac{x - y}{r_t} > -1 + \frac{x - y_1^*}{r_t} = 0, \quad \text{for all } y \in (-\infty, y_1^*), \\ h'(y) &= -1 + \frac{x - y}{r_t} < -1 + \frac{x - y_1^*}{r_t} = 0, \quad \text{for all } y \in (y_1^*, -r_\delta), \\ h'(y) &= y \left(\frac{r_t - r_\delta}{r_t r_\delta} \right) + \frac{x}{r_t} < \bar{y} \left(\frac{r_t - r_\delta}{r_t r_\delta} \right) + \frac{x}{r_t} = 0, \quad \text{for all } y \in (-r_\delta, \bar{y}), \\ h'(y) &= y \left(\frac{r_t - r_\delta}{r_t r_\delta} \right) + \frac{x}{r_t} > \bar{y} \left(\frac{r_t - r_\delta}{r_t r_\delta} \right) + \frac{r_\delta - r_t}{r_t} = 0, \quad \text{for all } y \in (\bar{y}, r_\delta), \\ h'(y) &= 1 + \frac{x - y}{r_t} > 1 + \frac{x - y_2^*}{r_t} = 0, \quad \text{for all } y \in (r_\delta, y_2^*), \\ h'(y) &= 1 + \frac{x - y}{r_t} < 1 + \frac{x - y_2^*}{r_t} = 0, \quad \text{for all } y \in (y_2^*, +\infty). \end{aligned}$$

We then proceed as in Case 1. Namely, for a polynomial function of order less than or equal to 1, we compute

$$\begin{aligned} &\int_{-\infty}^{\infty} Q(y) \exp(\lambda h(y)) dy \\ &= \int_{-\infty}^{\bar{y}} Q(y) \exp(\lambda h(y)) dy + \int_{\bar{y}}^{+\infty} Q(y) \exp(\lambda h(y)) dy \\ &= \int_{-\infty}^{-r_\delta} Q(y) \exp(\lambda h(y)) dy + \int_{-r_\delta}^{r_\delta} Q(y) \exp(\lambda h(y)) dy + \int_{r_\delta}^{+\infty} Q(y) \exp(\lambda h(y)) dy \\ &= \int_{-\infty}^{-r_\delta - y_1^*} Q(u + y_1^*) \exp(\lambda h(u + y_1^*)) du + \int_{-r_\delta}^{r_\delta} Q(y) \exp(\lambda h(y)) dy \\ &\quad + \int_{r_\delta - y_2^*}^{+\infty} Q(u + y_2^*) \exp(\lambda h(u + y_2^*)) du. \end{aligned}$$

We have

$$\begin{aligned}\forall u \in (-\infty, -r_\delta - y_1^*), \quad h(u + y_1^*) - h(y_1^*) &= -\frac{u^2}{2r_t} < 0, \\ \forall u \in (r_\delta - y_2^*, +\infty), \quad h(u + y_2^*) - h(y_2^*) &= -\frac{u^2}{2r_t} < 0.\end{aligned}$$

Now, letting

$$B_1 = -\frac{r_\delta + y_1^*}{\sqrt{2r_t}},$$

we have

$$\begin{aligned}& \int_{-\infty}^{-r_\delta - y_1^*} Q(u + y_1^*) \exp(\lambda h(u + y_1^*)) du \\ &= \exp(\lambda h(y_1^*)) \int_{-\infty}^{-r_\delta - y_1^*} (Q(y_1^*) + Q'(y_1^*)u) \exp(-\frac{\lambda u^2}{2r_t}) du \\ &= \sqrt{2r_t} \exp(\lambda h(y_1^*)) \int_{-\infty}^{B_1} (Q(y_1^*) + \sqrt{2r_t} Q'(y_1^*)s) \exp(-\lambda s^2) ds \\ &= \sqrt{2r_t} \exp(\lambda h(y_1^*)) \left(Q(y_1^*) \sqrt{\frac{\pi}{4}} \frac{\operatorname{erf}(\sqrt{\lambda} B_1) + 1}{\sqrt{\lambda}} + Q'(y_1^*) \frac{\sqrt{2r_t}}{2\lambda} \exp(-\lambda B_1^2) \right) \\ &= \exp(\lambda h(y_1^*)) Q(y_1^*) \sqrt{\frac{\pi r_t}{2}} \frac{\operatorname{erf}(\sqrt{\lambda} B_1) + 1}{\sqrt{\lambda}} + Q'(y_1^*) \frac{r_t}{\lambda} \exp(-\lambda B_1^2).\end{aligned}$$

Similarly, letting

$$B_2 = \frac{r_\delta - y_2^*}{\sqrt{2r_t}},$$

we have

$$\begin{aligned}& \int_{r_\delta - y_2^*}^{+\infty} Q(u + y_2^*) \exp(\lambda h(u + y_2^*)) du \\ &= \exp(\lambda h(y_2^*)) Q(y_2^*) \sqrt{\frac{\pi r_t}{2}} \frac{\operatorname{erf}(1 - \sqrt{\lambda} B_2)}{\sqrt{\lambda}} + Q'(y_2^*) \frac{r_t}{\lambda} \exp(-\lambda B_2^2).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} Q(y) \exp(\lambda h(y)) dy &= \exp(\lambda h(y_1^*)) Q(y_1^*) \sqrt{\frac{r_t \pi}{2}} \frac{(\operatorname{erf}(\sqrt{\lambda} B_1) + 1)}{\sqrt{\lambda}} \\ &\quad + \exp(\lambda h(y_2^*)) Q(y_2^*) \sqrt{\frac{r_t \pi}{2}} \frac{(1 - \operatorname{erf}(\sqrt{\lambda} B_2))}{\sqrt{\lambda}} + \mathcal{R}(t, x, \lambda),\end{aligned}\tag{2.46}$$

with

$$\begin{aligned}\mathcal{R}(t, x, \lambda) &= \exp(\lambda h(y_1^*)) Q'(y_1^*) r_t \frac{\exp(-\lambda B_1^2)}{\lambda} \\ &\quad + \exp(\lambda h(y_2^*)) Q'(y_2^*) r_t \frac{\exp(-\lambda B_2^2)}{\lambda} + \int_{-r_\delta}^{r_\delta} Q(y) \exp(\lambda h(y)) dy.\end{aligned}$$

Using Cole-Hopf formula: Define now $q(y) = \frac{x-y}{r_t} + \text{sign}(x)$, and recall that

$$\Psi(t, x, \sigma_0) = \theta^{\sigma_0}(t, x) - \theta(t, x) = \frac{\int_{-\infty}^{+\infty} q(y) \exp(\lambda h(y)) dy}{\int_{-\infty}^{+\infty} \exp(\lambda h(y)) dy}.$$

Assume first that $0 \leq x < r_t - r_\delta$. Then, $q(y) = \frac{x-y}{r_t} + 1$, and then $q(y_1^*) = 2$ and $q(y_2^*) = 0$. Therefore, By (2.46) with $Q(y) = q(y)$ and with $Q(y) = 1$, we obtain

$$|\Psi(t, x, \sigma_0)| \leq \frac{2\sqrt{\pi r_t} \exp(\lambda h(y_1^*)) \left(\frac{1 + \text{erf}(\sqrt{\lambda} B_1)}{\sqrt{\lambda}} \right) + 2|\mathcal{R}(t, x, \lambda)|}{\sqrt{\pi r_t} \exp(\lambda h(y_1^*)) \left(\frac{1 + \text{erf}(\sqrt{\lambda} B_1)}{\sqrt{\lambda}} \right) + \sqrt{\pi r_t} \exp(\lambda h(y_2^*)) \left(\frac{1 - \text{erf}(\sqrt{\lambda} B_2)}{\sqrt{\lambda}} \right)},$$

the remainder \mathcal{R} being computed with $Q = q$. Therefore,

$$\begin{aligned} |\Psi(t, x, \sigma_0)| &\leq \frac{2\sqrt{\pi r_t} \exp(\lambda h(y_1^*)) \left(\frac{1 + \text{erf}(\sqrt{\lambda} B_1)}{\sqrt{\lambda}} \right)}{\sqrt{\pi r_t} \exp(\lambda h(y_2^*)) \left(\frac{1 - \text{erf}(\sqrt{\lambda} B_2)}{\sqrt{\lambda}} \right)} + \mathcal{I}_1(t, x, \lambda) + \mathcal{I}_2(t, x, \lambda) \\ &= 2 \exp(\lambda[h(y_1^*) - h(y_2^*)]) \left(\frac{1 + \text{erf}(\sqrt{\lambda} B_1)}{1 - \text{erf}(\sqrt{\lambda} B_2)} \right) + 2\mathcal{I}_1(t, x, \lambda) + 2\mathcal{I}_2(t, x, \lambda), \end{aligned}$$

where

$$\begin{aligned} \mathcal{I}_1(t, x, \lambda) &= \frac{\exp(\lambda h(y_1^*)) \frac{\exp(-\lambda B_1^2)}{\lambda} + \exp(\lambda h(y_2^*)) \frac{\exp(-\lambda B_2^2)}{\lambda}}{\sqrt{\pi r_t} \exp(\lambda h(y_2^*)) \left(\frac{1 - \text{erf}(\sqrt{\lambda} B_2)}{\sqrt{\lambda}} \right)}, \\ \mathcal{I}_2(t, x, \lambda) &= \frac{\int_{-r_\delta}^{r_\delta} |q(y)| \exp(\lambda h(y)) dy}{\sqrt{\pi r_t} \exp(\lambda h(y_2^*)) \left(\frac{1 - \text{erf}(\sqrt{\lambda} B_2)}{\sqrt{\lambda}} \right)}. \end{aligned}$$

Notice now the following key facts:

$$\begin{aligned} h(y_1^*) - h(y_2^*) &= -2x, \\ B_2 < 0, \quad B_2^2 &= \frac{(x + r_t - r_\delta)^2}{2r_t} \geq \frac{x^2}{2r_t}. \end{aligned} \tag{2.47}$$

In particular,

$$2 \exp(\lambda[h(y_1^*) - h(y_2^*)]) \left(\frac{1 + \text{erf}(\sqrt{\lambda} B_1)}{1 - \text{erf}(\sqrt{\lambda} B_2)} \right) \leq 4 \exp(-2\lambda x),$$

and

$$|\mathcal{I}_1(t, x, \lambda)| \leq \frac{1}{\sqrt{\lambda \pi r_t}} (\exp(-2\lambda x) + \exp(-\lambda x^2)).$$

Handling \mathcal{I}_2 : To handle $\mathcal{I}_2(t, x, \lambda)$, we notice that $|q(y)| \leq 1$ for $y \in (-r_\delta, r_\delta)$. Also, for $y \in (\bar{y}, r_\delta)$

$$h(r_\delta) - h(y) = \int_y^{r_\delta} \frac{z - \bar{y}}{r_\delta} dz \geq \frac{(r_\delta - y)^2}{2r_\delta}.$$

Hence,

$$\int_{\bar{y}}^{r_\delta} |q(y)| \exp(\lambda h(y)) dy \leq \exp(\lambda h(r_\delta)) \int_{\bar{y}}^{r_\delta} \exp\left(-\lambda \frac{(r_\delta - y)^2}{2r_\delta}\right) dy \leq \exp(\lambda h(r_\delta)) \sqrt{\frac{2\pi r_\delta}{\lambda}}.$$

By the same argument, but using in addition the fact that h is decreasing on (y_1^*, \bar{y}) ,

$$\int_{-r_\delta}^{\bar{y}} |q(y)| \exp(\lambda h(y)) dy \leq \exp(\lambda h(-r_\delta)) \sqrt{\frac{2\pi r_\delta}{\lambda}} \leq \exp(\lambda h(y_1^*)) \sqrt{\frac{2\pi r_\delta}{\lambda}}.$$

Now,

$$h(y_1^*) = h(y_2^*) - 2x, \quad h(r_\delta) = h(y_2^*) - \frac{1}{2r_t}(r_\delta - y_2^*)^2 = h(y_2^*) - B_2^2 \leq h(y_2^*) - \frac{x^2}{2r_t}.$$

Hence,

$$\int_{-r_\delta}^{r_\delta} |q(y)| \exp(\lambda h(y)) dy \leq \sqrt{\frac{2\pi r_\delta}{\lambda}} \exp(\lambda h(y_2^*)) \left(\exp(-2\lambda x) + \exp\left(-\frac{\lambda x^2}{2r_t}\right) \right).$$

Therefore,

$$\begin{aligned} |\mathcal{I}_2(t, x, \lambda)| &\leq \sqrt{\frac{2r_\delta}{r_t}} \left(\exp(-2\lambda x) + \exp\left(-\frac{\lambda x^2}{2r_t}\right) \right) \\ &\leq \sqrt{2} \left(\exp(-2\lambda x) + \exp\left(-\frac{\lambda x^2}{2r_t}\right) \right). \end{aligned}$$

Conclusion: Collecting the various terms, we obtain

$$|\Psi(t, x, \sigma_0)| \leq \left(4 + 2\sqrt{2} + \frac{2}{\sqrt{\lambda\pi r_\delta}}\right) \exp(-2\lambda x) + \left(\frac{2}{\sqrt{\lambda\pi r_\delta}} + 2\sqrt{2}\right) \exp\left(-\frac{\lambda x^2}{2r_0}\right).$$

Assume now that $x > \sqrt{\frac{2r_0 \ln \lambda}{\lambda}}$. Then,

$$\begin{aligned} \exp(-2\lambda x) &< \frac{1}{\lambda^{2\sqrt{2r_0}}}, \\ \exp\left(\frac{\lambda x^2}{2r_0}\right) &< \frac{1}{\lambda}, \end{aligned}$$

Thus, it holds that

$$|\Psi(t, x, \sigma_0)| \leq \left(4 + 2\sqrt{2} + \frac{2\sigma_0}{\sqrt{\pi r_\delta}}\right) \sigma_0^{4\sqrt{2r_0}} + \left(\frac{2\sigma_0}{\sqrt{\pi r_\delta}} + 2\sqrt{2}\right) \sigma_0^2,$$

whenever

$$x^2 \geq 2r_0 \left(\frac{\ln \lambda}{\lambda} \right) = (4r_0)\sigma_0^2 |\ln \sigma_0|.$$

By symmetry, we obtain a similar result when $x \in (r_\delta - r_t, 0)$.

Case 2': When $t < \delta - \eta$ and $|x| \leq (r_t - r_\delta)/2$, we can improve the above bounds. Indeed, returning to (2.47), we have $B_2^2 \geq (r_{\delta-\eta} - r_\delta)^2/(2r_0)$, and as a result

$$|\mathcal{I}_2(t, x, \lambda)| \leq \sqrt{2} \left(\exp(-2\lambda x) + \exp\left(-\frac{\lambda(r_{\delta-\eta} - r_\delta)^2}{2r_0}\right) \right),$$

and then

$$|\Psi(t, x, \sigma_0)| \leq \left(4 + 2\sqrt{2} + \frac{2}{\sqrt{\lambda\pi r_\delta}}\right) \exp(-2\lambda x) + \frac{2}{\sqrt{\lambda\pi r_\delta}} + 2\sqrt{2} \exp\left(-\frac{\lambda(r_{\delta-\eta} - r_\delta)^2}{2r_0}\right).$$

Case 3: We say that $(t, x) \in \mathcal{D}_3$ if and only if $t \geq \delta$ and $x \leq r_t - r_\delta - \sqrt{\frac{2r_0 \ln \lambda}{\lambda}} < r_t - r_\delta < 0$.

Preliminary computation: For every $(t, x) \in \mathcal{D}_3$, we define $y^* := x - r_t \in (-\infty, -r_\delta)$. Then, y^* is a global maximum of h , as shown by the computations below:

$$h'(y) = -1 + \frac{x - y}{r_t} > -1 + \frac{x - y^*}{r_t} = 0, \text{ for all } y \in (-\infty, y^*),$$

$$h'(y) = -1 + \frac{x - y}{r_t} < -1 + \frac{x - y^*}{r_t} = 0, \text{ for all } y \in (y^*, -r_\delta),$$

$$h'(y) = y \left(\frac{r_t - r_\delta}{r_t r_\delta} \right) + \frac{x}{r_t} < -r_\delta \left(\frac{r_t - r_\delta}{r_t r_\delta} \right) + \frac{r_t - r_\delta}{r_t} = 0, \text{ for all } y \in (-r_\delta, r_\delta),$$

$$h'(y) = 1 + \frac{x - y}{r_t} < 1 + \frac{x - r_\delta}{r_t} < 1 + \frac{x - r_t}{r_t} < 1 + \frac{r_t - r_\delta - r_t}{r_t} = 0, \text{ for all } y \in (r_\delta, +\infty).$$

Proceeding as before, we claim

$$\begin{aligned} \int_{-\infty}^{\infty} Q(y) \exp(\lambda h(y)) dy &= \int_{-\infty}^{-r_\delta} Q(y) \exp(\lambda h(y)) dy + \int_{-r_\delta}^{+\infty} Q(y) \exp(\lambda h(y)) dy \\ &= \int_{-\infty}^{-r_\delta - y^*} Q(u + y^*) \exp(\lambda h(u + y^*)) du + \int_{-r_\delta}^{+\infty} Q(y) \exp(\lambda h(y)) dy \end{aligned}$$

We then observe that

$$\forall u \in (-\infty, -r_\delta - y^*), \quad h(u + y^*) - h(y^*) = -\frac{u^2}{2r_t} < 0.$$

Hence, letting $B := (-r_\delta - y^*)/\sqrt{2r_t} > 0$, we get

$$\begin{aligned}
& \int_{-\infty}^{-r_\delta - y^*} Q(u + y^*) \exp(\lambda h(u + y^*)) du \\
&= \sqrt{2r_t} \int_{-\infty}^B (Q(y^*) + Q'(y^*)\sqrt{2r_t}s) \exp(\lambda(h(y^*) - s^2)) ds \\
&= \sqrt{2r_t} \exp(\lambda h(y^*)) \left[Q(y^*) \int_{-\infty}^B \exp(-\lambda s^2) ds + Q'(y^*)\sqrt{2r_t} \int_{-\infty}^B s \exp(-\lambda s^2) ds \right] \\
&= \exp(\lambda h(y^*)) \left[\sqrt{\frac{\pi}{2r_t}} Q(y^*) \frac{(\operatorname{erf}(1 + \sqrt{\lambda}B))}{\sqrt{\lambda}} + Q'(y^*)r_t \frac{\exp(-\lambda B^2)}{\lambda} \right].
\end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} Q(y) \exp(\lambda h(y)) dy = \exp(\lambda h(y^*)) \left[\sqrt{\frac{\pi}{2r_t}} Q(y^*) \frac{(1 + \operatorname{erf}(\sqrt{\lambda}B))}{\sqrt{\lambda}} + \mathcal{R}(t, x, \lambda) \right]$$

The remainder term $\mathcal{R}(t, x, \lambda)$ is explicitly given by

$$\begin{aligned}
\mathcal{R}(t, x, \lambda) &= \exp\left(\lambda \int_{y^*}^{r_\delta} h'(z) dz\right) \int_{r_\delta}^{+\infty} Q(y) \exp\left(\lambda \int_{r_\delta}^y h'(z) dz\right) dy \\
&\quad + \exp\left(\lambda \int_{y^*}^{-r_\delta} h'(z) dz\right) \int_{-r_\delta}^{r_\delta} Q(y) \exp\left(\lambda \int_{-r_\delta}^y h'(z) dz\right) dy \\
&\quad + Q'(y^*)r_t \left(\frac{\exp(-\lambda B^2)}{\lambda}\right).
\end{aligned}$$

Using Cole-Hopf formula: Define now $q(y) = \frac{x-y}{r_t} - 1$, so that $q(y^*) = 0$, and recall once again that

$$\Psi(t, x, \sigma_0) = \theta^{\sigma_0}(t, x) - \theta(t, x) = \frac{\int_{-\infty}^{+\infty} q(y) \exp(\lambda h(y)) dy}{\int_{-\infty}^{+\infty} \exp(\lambda h(y)) dy}.$$

Hence, the above asymptotic expressions implies

$$|\Psi(t, x, \sigma_0)| \leq \frac{\left| \int_{-\infty}^{+\infty} q(y) \exp(\lambda h(y)) dy \right|}{\int_{-\infty}^{+\infty} \exp(\lambda h(y)) dy} \leq \frac{|\mathcal{R}(t, x, \lambda)|}{\sqrt{\frac{\pi r_t}{2}} \left(\frac{1 + \operatorname{erf}(\sqrt{\lambda}B)}{\sqrt{\lambda}} \right) + \int_{-r_\delta}^{+\infty} \exp\left(\lambda \int_{y^*}^y h'(z) dz\right) dy},$$

the remainder \mathcal{R} being computed with $Q = q$.

It follows that

$$|\Psi(t, x, \sigma_0)| \leq \sqrt{\frac{2}{\pi r_t}} \left(\frac{\sqrt{\lambda}}{1 + \operatorname{erf}(\sqrt{\lambda}B)} \right) |\mathcal{R}(t, x, \lambda)| \leq \sqrt{\frac{2\lambda}{\pi r_t}} [\mathcal{I}_1(t, x, \lambda) + \mathcal{I}_2(t, x, \lambda) + \mathcal{I}_3(t, x, \lambda)],$$

where, using the fact that $q'(y^*) = -\frac{1}{r_t}$,

$$\begin{aligned}\mathcal{I}_1(t, x, \lambda) &:= \left| r_t q'(y^*) \frac{\exp(-\lambda B^2)}{\lambda} \right| \leq \frac{1}{\lambda}, \\ \mathcal{I}_2(t, x, \lambda) &:= \left| \exp \left(\lambda \int_{y^*}^{-r_\delta} h'(z) dz \right) \int_{-r_\delta}^{r_\delta} q(y) \exp \left(\lambda \int_{-r_\delta}^y h'(z) dz \right) dy \right|, \\ \mathcal{I}_3(t, x, \lambda) &:= \left| \exp \left(\lambda \int_{y^*}^{r_\delta} h'(z) dz \right) \int_{r_\delta}^{+\infty} q(y) \exp \left(\lambda \int_{r_\delta}^y h'(z) dz \right) dy \right|.\end{aligned}$$

Handling \mathcal{I}_2 : Recalling that $h'(z) = (y^* - z)/r_t$ for $z \in (y^*, -r_\delta)$ and that $h'(z) < 0$ for $z \in (-r_\delta, r_\delta)$, we deduce that

$$\begin{aligned}\mathcal{I}_2(t, x, \lambda) &\leq \exp \left(\lambda \int_{y^*}^{-r_\delta} h'(z) dz \right) \int_{-r_\delta}^{r_\delta} |q(y)| \exp \left(\lambda \int_{-r_\delta}^y h'(z) dz \right) dy \\ &\leq \exp \left(\frac{-\lambda(y^* + r_\delta)^2}{2r_t} \right) \int_{-r_\delta}^{r_\delta} |q(y)| \exp \left(\lambda \int_{-r_\delta}^y h'(z) dz \right) dy \\ &\leq \exp \left(\frac{-r_0 \ln \lambda}{r_t} \right) \int_{-r_\delta}^{r_\delta} |q(y)| \exp \left(\lambda \int_{-r_\delta}^y h'(z) dz \right) dy \\ &\leq \exp \left(-\ln \lambda \right) \int_{-r_\delta}^{r_\delta} |q(y)| \exp \left(\lambda \int_{-r_\delta}^y h'(z) dz \right) dy \\ &\leq \frac{1}{\lambda} \int_{-r_\delta}^{r_\delta} (|q(y)| \times 1) dy \\ &\leq \frac{2r_\delta}{\lambda} \max_{|y| \leq r_\delta} |q(y)| \leq \frac{2r_\delta}{\lambda} \left(\frac{r_\delta - x}{r_t} + 1 \right).\end{aligned}$$

Handling \mathcal{I}_3 : Since $q(z) \leq 0$, for all $z \geq r_\delta$, $h'(z) < 0$ for $z \in (y^*, -r_\delta) \cup (-r_\delta, r_\delta)$, and $h'(z) < r_\delta - z$, for $z \in (r_\delta, +\infty)$, we have

$$\begin{aligned}\mathcal{I}_3(t, x, \lambda) &\leq \exp \left(\lambda \int_{y^*}^{r_\delta} h'(z) dz \right) \int_{r_\delta}^{+\infty} (-q(y)) \exp \left(\lambda \int_{r_\delta}^y h'(z) dz \right) dy \\ &\leq \frac{1}{\lambda} \int_{r_\delta}^{+\infty} (-q(y)) \exp \left(\lambda \int_{r_\delta}^y h'(z) dz \right) dy \\ &\leq \frac{1}{\lambda} \int_{r_\delta}^{+\infty} (-q(y)) \exp \left(\frac{-\lambda(y - r_\delta)^2}{2r_t} \right) dy \\ &\leq \frac{1}{\lambda} \int_{r_\delta}^{+\infty} -[q(r_\delta) + (y - r_\delta)q'(r_\delta)] \exp \left(\frac{-\lambda(y - r_\delta)^2}{2r_t} \right) dy \\ &\leq \frac{1}{\lambda} \left[\frac{-q(r_\delta)}{\sqrt{\lambda}} \int_0^{+\infty} \exp \left(\frac{-u^2}{2r_t} \right) du + \frac{-q'(r_\delta)}{\lambda} \int_0^{+\infty} u \exp \left(\frac{-u^2}{2r_t} \right) du \right] \\ &\leq \frac{1}{\lambda} \left[\sqrt{\frac{\pi r_t}{2\lambda}} \left(1 + \frac{r_\delta - x}{r_t} \right) + \frac{1}{\lambda} \right] \leq \frac{1}{\lambda} \left[\sqrt{\frac{\pi r_\delta}{2}} \left(1 + \frac{r_\delta - x}{r_t} \right) + 1 \right].\end{aligned}$$

Conclusion: Therefore, for all $(t, x) \in \mathcal{D}_4$, it holds that

$$\begin{aligned} |\Psi(t, x, \sigma_0)| &\leq \sqrt{\frac{2}{\pi r_t \lambda}} \left[2 + 2r_\delta \left(1 + \frac{r_\delta + |x|}{r_t} \right) + \sqrt{\frac{\pi r_\delta}{2}} \left(1 + \frac{r_\delta + |x|}{r_t} \right) \right] \\ &= \sigma_0 \sqrt{\frac{2}{\pi r_t}} \left[2 + 2r_\delta \left(1 + \frac{r_\delta + |x|}{r_t} \right) + \sqrt{\frac{\pi r_\delta}{2}} \left(1 + \frac{r_\delta + |x|}{r_t} \right) \right]. \end{aligned}$$

By symmetry, the same holds when $x > r_\delta - r_t + 2\sigma_0\sqrt{r_0|\ln \sigma_0|}$, which completes the proof.

2.7 Prospects

In connection with this chapter, we propose the following prospects for further research:

1. A first question is to generalize the results obtained in this chapter to a larger class of models. This question could be addressed in three different steps:

- (a) The first one would be to consider the same model but for a more general non-decreasing terminal function g or even a more general terminal condition (that would be non-monotone).

A related question is to choose $\delta = 0$ instead of $\delta \in (0, T)$ in (2.3).

- (b) A second step is to investigate an LQ model with more general coefficients, including a drift b and a running cost f that depend on the mean of the population, as we discussed in Chapter 2. As made clear in Chapter 2, the nice feature is that, despite the fact that the coefficients have a more general form, equilibria remain of Gaussian type and are still described by an autonomous one-dimensional FBSDE. The benchmark example we addressed in this chapter provides a peculiar instance of the kind of autonomous one-dimensional FBSDEs we may face with such an approach.

- (c) Lastly, we could think of non LQ models, that is of models for which the cost coefficients in the definition of J are no longer quadratic in x and/or the drift in the dynamics is no longer linear of x . Since zero-noise limits are of a greater difficulty in dimension greater than or equal to 2, the models that should be addressed first should be one-dimensional statistical parametric models only, in the sense that that the distributions of the equilibria should be parametrized by a one-dimensional variable.

2. Another question is to validate numerically the results obtained in this chapter, and especially the selection method based upon the zero-noise limit.

For sure, this is an interesting issue, but it should certainly require a modicum of care. The numerical method should indeed capture properly the fact that equilibria do escape from the singularity; so, it should be chosen accordingly.

Once a numerical method has been chosen, it would be interesting to address numerically the form of the transition point and to compare with the theoretical analysis we have provided.

3. Another objective would be to recast the mean-field games addressed in this chapter in the form of potential mean-field games and, in such a case, to select the equilibria

that minimize the cost functional underpinning the corresponding mean-field control problem. We refer to [23, Chapter 6] for an overview on potential mean-field games.

4. Lastly, we can think of embedding the results obtained in this chapter in wider models. In fact, we give a concrete example in the next chapter.

Chapter 3

On restoration and selection for “principal-agent” MFG: an LQ model.

In this chapter, we consider a mean-field game of “principal-agent” type. This type of mean-field games consists of a continuum of players (called agents) who are incentivized by the principal to engage in a mean-field game. We say that the game is solved if there exists, at least, an MFG solution to the corresponding mean-field game and a Stackelberg equilibrium between the principal and the representative agent implementing an MFG solution.

The game considered in this chapter appears naturally when modelling an economy in which a government (i.e., a principal) wants to incentivize the economic agents (i.e., business managers) to move away from a “Business As Usual” (BAU, for short) paradigm that is regarded, by the government, as a trap for the economy. The government can only help to find a way out from the trap by proposing rewards to the business managers, those rewards being paid at the end of a given economic cycle of period $T > 0$. In this framework, the government is unable to monitor/control directly the behavior of the representative agent. Thus, the whole forms a finite time horizon game and the principal has to solve a second best (or moral hazard) problem.

By investigating the form of the rewards (or contracts) that the principal can offer and, in this respect, by following the framework of Elie et al, see [45, 44], we show below that the solutions of this game can be characterized through the solutions of a classical stochastic control problem.

As in the previous chapters, one can do so both with and without common noise. In the specific context that is addressed below, we show that uniqueness may only fail for the game with a continuum of agents without common noise. Similar to Chapter 2, this shows that common noise may restore uniqueness for “principal-agent” mean-field games. When uniqueness fails, we select certain solutions by implementing two of the three selection methods introduced for standard mean-field games. In contrast with the results obtained in the previous chapter, we show that these two selection approaches do agree in this new context.

3.1 Introduction

It is a contemporary consensus amongst economists (behavioral and classical) that introducing well-thought incentives and/or nudges in any human activity has the desirable consequence of improving the performance of the people involved in the activity. In fact, most of the questions involving human interaction and performance are, at their core, a matter of nudges and/or incentives. For example, which bonuses shareholders should propose to the

managers to induce desired profitability? How much can the introduction of a “speed camera lottery”, charging higher fines from drivers exceeding the speed limit and inviting all drivers respecting the speed limit to a lottery, help to increase the ratio of drivers respecting the speed limit? Or what kind of contract can an insurance company propose to its customers, in order to reduce the human effect on the occurrence of insured disasters? In general, a good understanding of nudges and how they work is useful in environment design for most human activities. And a good understanding of incentives allows for efficient contract design amongst people engaged in most human activities. With regard to the above examples, we can distinguish between two entities: a Principal, who builds and proposes (or implements) a contract (or nudge) to the second entity, the Agents (or the users).

Dealing with a continuum of exchangeable agents in mean-field type interaction, the second best problem for the principal is formulated as a mean-field game of “principal-agent” type. In these games, the representative agent can, in exchange of some effort, accept a contract proposed by the principal. Often, the contracts that are proposed involve payments of a salary (or reward), either as a lump-sum at final time, say $T > 0$, or in the form of payments at a series of times included in $[0, T]$. Most of the time, the principal just has partial information on the actions of the representative agent. In the literature, this asymmetry of information is referred to as *moral hazard* and principal’s problem of designing a contract which will be accepted by the representative agent at equilibrium, while maximizing its own utility, is referred to as the second best problem. Clearly, the second best problem consists in the search of a Stackelberg equilibrium between the principal and the representative agent in the mean-field game.

To the best knowledge of the author, Stackelberg equilibria in mean-field games have been studied through aggregation with delays in [7] and the first rigorous treatment of mean-field game of “principal-agent” type was provided by Elie, Mastrolia and Possamai [45, 44]. In this framework, the relation between the representative agent and the principal is similar to the relation between a principal and an agent in the so-called *contract theory*. Contract theory can be seen as a reformulation of *mechanism design theory*, which is mathematically elegant and important for problems concerning micro-management, specifically for the identification of best business practices across collaborative markets. The principal searches for a solution to the second best problem, under a contract theory whose features are:

- Only the Principal can design and propose a contract to the agent;
- The agent can accept or refuse the contract (i.e., there are participation constraints);
- Once the agent accepts a contract it cannot be renegotiated later on;
- The principal is able to compute the agent’s best reaction to each of its contracts.

Since the Nobel Prize in Economic Sciences was awarded in 2016 to Oliver Hart and Bengt Holmstrom for their fundamental contributions to contract theory, the theory has known a renewed surge of interest. For a general introduction to contract theory, we refer to Bolton and Dewatripont [14], Çvitanic and Zhang [33], and many examples can be found in Laffont and Martimort [74]. The static or discrete time problems may easily become of a high complexity, see among others, Rogerson [98] and Spear and Srivastava [93]. In the continuous time setting, the main result can be found in Holmstrom and Milgrom [59], where more realistic contracts are treated –including contracts that are linear in the output– and are shown to be solutions to suitable (but intricate) versions of the dynamic programming

principle. Among others, these results were improved by Sannikov [90, 91], and more recently a generalization was provided by Cvitanic, Possamai and Touzi [32, 31], who used, to do so, so-called second order Backward Stochastic Differential Equations (2-BSDEs for short).

As for games with finitely many (“ $N < \infty$ ”) exchangeable agents in mean-field type interaction, the aforementioned general approach given by Cvitanic, Possamai and Touzi [32, 31] to the principal N -agent second-best problem is especially relevant. In this context, all the agents can, in exchange for some effort, accept a contract proposed by the principal. Contracts proposed by the principal consist of an N -tuple of contracts proposed to each of the N Agents. As before, the contract proposed to each agent involves payments of a salary (or reward), either as a lump-sum at final time, say $T > 0$, or in the form of payments at a series of times included in $[0, T]$. In the literature, this situation is known as multi-agents models. In the discrete time setting, we refer (among others) to the early works of Green and Stockey [56] and of Harris, Kriebel and Raviv [58]. In the continuous time setting, we refer to Koo, Shim and Sung [70]. Recent improvements may found in the article of Elie and Possamai [45], in which the search of a Nash equilibrium (once the N agents have accepted principal’s contract) is characterized by the study of a system of N dimensional (often quadratic) BSDEs that permits to reduce principal’s second-best problem to a classical control problem. In this framework, as in the multi-agent literature, it is often assumed that the principal chooses which Nash equilibrium (between the N agents) prevails when there are several of them. However, it has been demonstrated by Mookherjee [84] that there may be a Nash equilibrium that is better for the agents but is not considered by the principal. Among other similar observations, this result shows that the question of contract selection for second best problems is still largely open. This chapter must be regarded as one (small) step forward in this direction.

The chapter is organized as follows. In the first section, we introduce an illustrative model of “principal-agent” mean-field game and state, for this model, the main selection results when uniqueness fails and common noise restore uniqueness. In the second section, we prove the solvability results for the “principal-agent” mean-field game (both with and without common noise). Finally, the third (and last) section provides a proof of the main selection results announced in the first section.

3.1.1 Notations and Model Set Up

Notations. Given a fixed terminal time, $T > 0$, we consider positive constants $\kappa, \epsilon \geq 0$, and a standard 2-dimensional Brownian motion $(B, W) := (B_t, W_t)_{t \in [0, T]}$ defined on a complete probability space $(\Omega, \mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ satisfying the usual conditions. The filtration is chosen as follows: Denoting by \mathcal{N} the set of all \mathbb{P} -null events, we let $\mathcal{F}_t := \sigma(B_s, W_s; s \leq t) \vee \mathcal{N}$; similarly, we also let: $\mathcal{F}_t^B := \sigma(B_s; s \leq t) \vee \mathcal{N}$.

In the sequel, we call respectively $B := (B_t)_{t \in [0, T]}$ and $W := (W_t)_{t \in [0, T]}$ the common (systemic) and idiosyncratic (unsystematic) noises driving the dynamics of representative agent’s state.

We consider the normed space $(\mathbb{R}, |\cdot|)$ and denote the set of \mathbb{R} -valued and progressively-measurable processes by $\mathcal{P}(\mathbb{R})$.

Subspaces $\mathbf{H}^2(\mathbb{R})$ and $\mathbf{S}^2(\mathbb{R})$. We also define the following spaces:

$$\begin{aligned}\mathbf{H}^2(\mathbb{R}) &:= \left\{ X \in \mathcal{P}(\mathbb{R}) \mid \mathbb{E} \left[\int_0^T |X_t|^2 dt \right] < +\infty \right\}, \\ \mathbf{S}^2(\mathbb{R}) &:= \left\{ X \in \mathcal{P}(\mathbb{R}) \mid \mathbb{P}(X \in \mathcal{C}([0, T]; \mathbb{R})) = 1 \quad \text{and} \quad \mathbb{E} \left[\sup_{t \in [0, T]} |X_t|^2 \right] < \infty \right\},\end{aligned}$$

where, in the second line, $\mathcal{C}([0, T]; \mathbb{R})$ is the space of continuous functions from $[0, T]$ into \mathbb{R} .

Subspace $\mathcal{A}(\mathbb{R})$. We also call $\mathcal{A}(\mathbb{R})$ the collection of processes $X \in \mathcal{P}(\mathbb{R})$ for which there exists a (finite) constant c such that, for any stopping time τ with respect to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$, the following holds true

$$\mathbb{P} \left(\mathbb{E} \left[\int_\tau^T |X_s|^2 ds \mid \mathcal{F}_\tau \right] \leq c \right) = 1.$$

It implies that the martingale $(M_t := \int_0^t X_s dW_s)_{t \in [0, T]}$ is of Bounded Mean Oscillation (BMO for short), see [69]. In particular, the exponential local martingale

$$\left(L_{0t} := \exp \left(\int_0^t X_s dW_s - \frac{1}{2} \int_0^t (X_s)^2 ds \right) \right)_{t \in [0, T]}$$

is a uniformly integrable martingale and there exist two constants $\epsilon, c' > 0$, only depending on c , such that the following reverse Hölder inequality holds true with probability 1 for all $t \in [0, T]$:

$$\mathbb{E}[L_{0T}(X)^{1+\epsilon} \mid \mathcal{F}_t] \leq c' L_{0t}(X)^{1+\epsilon}.$$

In particular, $\mathbb{E}[\sup_{t \in [0, T]} L_{0t}(X)^{1+\epsilon}] < \infty$. Moreover, we also have

$$\forall p \geq 1, \quad \mathbb{E} \left[\left(\int_0^T |X_t|^2 dt \right)^p \right] < \infty,$$

see Corollary 2.1 in [69].

Class \mathcal{C} . Moreover, we call \mathcal{C} the collection of random variables $\xi : \Omega \rightarrow \mathbb{R}$ that are \mathcal{F}_T -measurable and for which we can find a constant c such that, in the martingale decomposition,

$$\xi = \mathbb{E}[\xi] + \int_0^T Z_t dW_t + \int_0^T \zeta_t dB_t, \quad (3.1)$$

with $(Z_t)_{t \in [0, T]} \in \mathbf{H}^2(\mathbb{R})$ and $(\zeta_t)_{t \in [0, T]} \in \mathbf{H}^2(\mathbb{R})$, the process $(Z_t)_{t \in [0, T]}$ satisfies, $d\mathbb{P} \otimes dt$ almost everywhere,

$$|Z_t| \leq c,$$

and the process $(\zeta_t)_{t \in [0, T]}$ satisfies

$$\forall \lambda > 0, \quad \mathbb{E} \left[\exp \left(\lambda \sup_{t \in [0, T]} \left| \int_0^t \zeta_s dB_s \right| \right) \right] < \infty.$$

In particular, any $\xi \in \mathcal{C}$ satisfies the exponential integrability condition

$$\forall \lambda > 0, \quad \mathbb{E} \left[\exp(\lambda |\xi|) \right] < \infty,$$

Subspaces $\mathcal{D}(\mathbb{R}^2)$ and $\mathcal{D}_0(\mathbb{R})$. Lastly, we denote by $\mathcal{D}(\mathbb{R}^2)$ the collection of pairs $(\hat{Z}, \hat{\zeta}) \in \mathcal{A}(\mathbb{R}) \times \mathbf{H}^2(\mathbb{R})$, with

$$\forall \lambda > 0, \quad \mathbb{E} \left[\exp \left(\lambda \sup_{t \in [0, T]} \left| \int_0^t \hat{\zeta}_s dB_s \right| \right) \right] < \infty,$$

such that

$$-\frac{1}{2} \int_0^T \hat{Z}_s^2 ds + \int_0^T \hat{Z}_s dW_s + \int_0^T \hat{\zeta}_s dB_s \in \mathcal{C}.$$

Then, we denote by $\mathcal{D}_0(\mathbb{R})$ the collection of processes $\hat{Z} \in \mathcal{A}(\mathbb{R})$ such that $(\hat{Z}, 0) \in \mathcal{D}(\mathbb{R}^2)$. Obviously, $(\hat{Z}, \hat{\zeta}) \in \mathcal{D}(\mathbb{R}^2)$ if and only if $\hat{Z} \in \mathcal{D}_0(\mathbb{R})$ and $\hat{\zeta} \in \mathbf{H}^2(\mathbb{R})$ satisfies

$$\forall \lambda > 0, \quad \mathbb{E} \left[\exp \left(\lambda \sup_{t \in [0, T]} \left| \int_0^t \hat{\zeta}_s dB_s \right| \right) \right] < \infty.$$

The model set-up. In the model addressed in the chapter, we consider both a principal (the government) and a representative agent (a business manager). The principal wants to influence the representative agent to make it deviate from the BAU paradigm (both the principal and the agent are gender-neutral.).

We suppose that the wealth of the representative agent is dynamically modelled by the process, $X := (X_t)_{t \in [0, T]} \in \mathcal{P}(\mathbb{R})$, which solves the stochastic differential equation (SDE for short) below:

$$dX_t = \kappa(m_t - X_t)dt + dW_t + \epsilon dB_t, \quad X_0 = 0, \quad \forall t \in [0, T]. \quad (3.2)$$

In the SDE above, when $\epsilon > 0$, we say that the common noise is present and, when $\epsilon = 0$, we say that the common noise is absent. The choice of an Ornstein-Uhlenbeck process for $(X_t)_{t \in [0, T]}$ features the fact that representative agent's wealth is bound to synchronize with the process $m := (m_t)_{t \in [0, T]} \in \mathcal{P}(\mathbb{R})$ at a rate given by $\kappa \geq 0$. This process $(m_t)_{t \in [0, T]}$ is the so-called mean-field process. It describes the conditional expected wealth of the continuum of agents given the common noise when the latter is present and the expected wealth of the continuum of agents when there is no common noise: To make it clear, the mean-field process is deterministic when $\epsilon = 0$ and is just adapted to the filtration generated by the common noise when $\epsilon > 0$. In words, in the BAU dynamics, one has $(m_t = \mathbb{E}[X_t | B])_{t \in [0, T]}$ when $\epsilon > 0$ and $(m_t = \mathbb{E}[X_t])_{t \in [0, T]}$ when $\epsilon = 0$; the above equation is thus a McKean-Vlasov equation (of the conditional type when $\epsilon > 0$).

It is easy to observe that, when representative agent's wealth follows the BAU dynamics, the process $(m_t)_{t \in [0, T]}$ is equal to $(\epsilon B_t)_{t \in [0, T]}$. That is, the expected wealth of the agents is completely determined by the common noise, which leaves little room for the economy to grow. This constitutes a trap for the economy and justifies the principal's intention to get the economy out of this trap via a mean-field game of "principal-agent" type.

The incentive to exit from the trap takes the form of a contract. In this framework, the final reward perceived by the representative agent depends upon its own effort for escaping

the trap. The key point is that this effort reads as a control, which affects the dynamics of its own wealth. We formulate the deviation of the wealth from the BAU regime through a change of variable and a related change of measure. Suppose indeed that the representative agent only chooses a strategy, $(\alpha_t)_{t \in [0, T]} \in \mathcal{A}(\mathbb{R})$, to control its wealth after accepting a contract from the principal, then the evolution of its wealth is rewritten in the form:

$$dX_t = \kappa(m_t - X_t)dt + (\alpha_t dt + dW_t^\alpha) + \epsilon dB_t, \quad X_0 = 0, \quad \forall t \in [0, T], \quad (3.3)$$

where

$$\alpha_t dt + dW_t^\alpha = dW_t.$$

A direct application of Girsanov’s change of probability theorem (which is licit by the very definition of the class $\mathcal{A}(\mathbb{R})$) allows to define a probability measure \mathbb{P}^α under which $(W_t^\alpha)_{t \in [0, T]}$ is a Brownian motion, namely:

$$\frac{d\mathbb{P}^\alpha}{d\mathbb{P}} | \mathcal{F}_t = L_{0t}(\alpha) := \exp \left(\int_0^t \alpha_s dW_s - \frac{1}{2} \int_0^t \alpha_s^2 ds \right), \quad \forall t \in [0, T].$$

To emphasize the fact that the statistical behavior of the agent is no longer the same when a non-zero strategy is implemented, we will then evaluate the law of the dynamics of the wealth $(X_t)_{t \in [0, T]}$ under the new probability \mathbb{P}^α .

The contracts the principal wishes to propose as incentives to the representative agent, $\xi \in \mathcal{C}$, are of reward type (tax cut, cash deposit, debt reevaluation, etc) at final time $T > 0$. The agent is promised a reward at final time T for its effort for deviating from “Business As Usual”. When $\epsilon = 0$, elements ξ of \mathcal{C} will be implicitly required to be \mathcal{F}_T^W measurable; equivalently, ζ is null in (3.1) and Z is progressively-measurable with respect to the augmentation of the filtration generated by W .

However, before accepting to deviate from the BAU dynamics, the representative agent requires that, at equilibrium, its expected utility be greater than or equal to a given reservation utility, say $R \geq 0$. This is a participation constraint given by the continuum of agents to the government. It can be seen as the minimal motivation required by the agents to deviate from “Business As Usual”. Once the representative agent agrees, the contract cannot be changed and the principal cannot monitor agent’s choice of deviating strategy. These are the terms of the contract binding them.

Though the principal cannot monitor agent’s strategy, it assumes from the theory of rational expectations that the representative agent, once the reservation utility is guaranteed, aims at maximizing its own expected utility.

With this assumption, the principal can then achieve descriptions of representative agent’s best reaction to the contracts it proposes. The class of contracts for which representative agent’s best reaction can be obtained is called *applicable contracts*. Finally, the principal chooses amongst the applicable contracts those that maximize its own expected utility.

In the next subsection, we formulate the mean-field game of “principal-agent” type underlying our model and give existence (and uniqueness) results for the solution to this game.

3.1.2 The Game Formulation And Solvability Result

The definition of agent’s reward relies on the following auxiliary function G . For $R \geq 0$ and $\delta \in [0, T)$, we define the function $G : \mathbb{R} \rightarrow \mathbb{R}$ as:

$$G(x) := \frac{-x^2}{2(T-\delta)} \mathbb{1}_{|x| \leq (T-\delta)} + \left(\frac{(T-\delta)}{2} - \text{sign}(x)x \right) \mathbb{1}_{|x| > (T-\delta)}.$$

This function is chosen so that $G'(x) = g(x), \forall x \in \mathbb{R}$, where the function g is defined analogously to (2.3) in Chapter 2. To make it clear, we have:

$$g(x) := -\frac{x}{(T-\delta)} \mathbb{1}_{|x| \leq (T-\delta)} - \text{sign}(x) \mathbb{1}_{|x| > (T-\delta)}.$$

Clearly, the function $-G$ is convex and has its minimum at $x = 0$. When the agents follow the BAU dynamics, the expected wealth is given by $m_T = 0$ in absence of common noise. This is the aforementioned trap. Mathematically speaking, escaping from this trap is the same as pushing m_T away from zero or, equivalently, as maximizing $-G(m_T)$.

This leads us to consider the following mean-field game of “principal-agent” type:

Scheme 3.1 (MFG of “principal-agents” type).

1. (Agent’s MFG problem) Given an arbitrary contract $\xi \in \mathcal{C}$, representative agent’s expected utility reads:

$$J^A(\xi, \alpha) = \mathbb{E} \left[L_{0T}(\alpha) \left(\xi - \frac{1}{2} \int_0^T \alpha_s^2 ds \right) \right].$$

Then, the representative agent finds a solution $(\alpha_t^*(\xi))_{t \in [0, T]}$ to the optimal control problem

$$\sup_{\alpha \in \mathcal{A}(\mathbb{R})} J^A(\xi, \alpha),$$

with the participation constraint $V^A(\xi) := J^A(\xi, \alpha^*(\xi)) \geq R$. We then solve the corresponding mean-field game by looking for a solution $(m_t^*(\xi))_{t \in [0, T]}$ to the fixed point condition:

$$m_t = \mathbb{E} [L_{0T}(\alpha^*(\xi)) X_t^* | \mathcal{F}_T^B], \quad \forall t \in [0, T],$$

where

$$dX_t^* = \kappa(m_t - X_t)dt + \alpha_t^*(\xi)dt + dW_t^{\alpha^*} + \epsilon dB_t, \quad X_0 = 0, \quad \forall t \in [0, T].$$

2. (Principal’s second best problem) Given a class of applicable contracts, principal’s expected utility is given by:

$$J^P(\xi) = \mathbb{E}^{\alpha^*(\xi)} \left[-G(m_T^*(\xi)) - \xi \right].$$

Then, the principal finds contracts ξ^* amongst the applicable contracts, maximizing its expected utility. In the right-hand side, the process $(m_t^*(\xi^*), \alpha_t^*(\xi^*))_{t \in [0, T]}$ constitutes a solution to representative agent’s mean-field game, as defined in Step (1).

We say that $(\xi^*, m_t^*(\xi^*), \alpha_t^*(\xi^*))_{t \in [0, T]}$ is a solution to the mean-field game of “principal-agent” type.

Observe that, for $\alpha \in \mathcal{A}(\mathbb{R})$ and X with finite moments of any order, $\mathbb{E}[L_{0T}(\alpha)X | \mathcal{F}_T^B]$ is also equal to $\mathbb{E}^\alpha[X | \mathcal{F}_T^B]$, which we also denote by $\mathbb{E}^\alpha[X | B]$.

The main solvability result for this mean-field game of “principal-agent” type is given by the next theorem.

Theorem 3.2. *Assume that $\epsilon > 0$. Then, the mean-field game of “principal-agent” type described in scheme (3.1) admits $(\xi^*, m_t^*(\xi^*), \alpha_t^*(\xi^*))_{t \in [0, T]}$ as a solution if and only if there exists a pair $(\hat{Z}^*, \hat{\zeta}^*) := (\hat{Z}_t^*, \hat{\zeta}_t^*)_{t \in [0, T]} \in \mathcal{D}(\mathbb{R}^2)$ such that:*

$$\mathbb{E}^{\hat{Z}^*} \left[\left(-G(m_T^*(\xi^*)) - \frac{1}{2} \int_0^T (\hat{Z}_t^*)^2 dt \right) \right] \geq \mathbb{E}^{\hat{Z}} \left[\left(-G(m_T(\hat{Z})) - \frac{1}{2} \int_0^T \hat{Z}_t^2 dt \right) \right], \quad \forall \hat{Z} \in \mathcal{D}_0(\mathbb{R}),$$

where, in the right-hand side,

$$m_t(\hat{Z}) = \int_0^t \mathbb{E}^{\hat{Z}} [\hat{Z}_s | \mathcal{F}_T^B] ds + \epsilon B_t, \quad \forall t \in [0, T], \quad (3.4)$$

and where, in the left-hand side,

$$m_t^*(\xi^*) = m_t(\hat{Z}^*), \quad \forall t \in [0, T]. \quad (3.5)$$

Moreover, in that case,

$$\begin{aligned} \alpha_t^*(\xi^*) &= \hat{Z}_t^*, \quad \forall t \in [0, T], \\ \xi^* &= R - \frac{1}{2} \int_0^T (\hat{Z}_s^*)^2 ds + \int_0^T \hat{Z}_s^* dW_s + \int_0^T \hat{\zeta}_s^* dB_s, \quad \forall t \in [0, T]. \end{aligned} \quad (3.6)$$

The result remains true in the case $\epsilon = 0$, except for the facts that $\mathbb{E}^{\hat{Z}}[\hat{Z}_s | \mathcal{F}_T^B]$ in (3.4) becomes $\mathbb{E}^{\hat{Z}}[\hat{Z}_s]$, that \hat{Z}^* is progressively-measurable with respect to the augmentation of the filtration generated by W and that $\hat{\zeta}^*$ is necessarily zero.

Theorem 3.2 will be used to characterize the state of the population under a Stackelberg equilibrium. In this regard, Theorem 3.2 says the following:

1. The shape of the equilibrium is characterized by the process \hat{Z}^* . Meanwhile, the process $\hat{\zeta}^*$ has no role in the determination of the equilibrium; it only affects the shape of the contract, but it has no influence on the mean utility of the agent nor on the mean utility of the principal.

In fact, the reward of the agent reads:

$$\xi^* - \frac{1}{2} \int_0^T (\hat{Z}_s^*)^2 ds = R + \int_0^T \hat{Z}_s^* dW_s^{\hat{Z}^*} + \int_0^T \hat{\zeta}_s^* dB_s,$$

and it has a (for a given value of \hat{Z}^*) a lower variance under $\mathbb{P}^{\hat{Z}^*}$ when $\hat{\zeta}^* = 0$.

2. If \hat{Z}^* drives an equilibrium, then it is necessarily progressively-measurable with respect to the augmentation of the filtration generated by B , at least when $\epsilon > 0$. When $\epsilon = 0$, \hat{Z}^* is in fact deterministic. We make these two facts clear in Lemma 3.3 below.

In particular, when $\epsilon > 0$, we can write the reward of the principal as

$$-G(m_T(\hat{Z}^*)) - \xi^* = -G(m_T(\hat{Z}^*)) - R - \frac{1}{2} \int_0^T (\hat{Z}_s^*)^2 ds - \int_0^T \hat{Z}_s^* dW_s^{\hat{Z}^*} - \int_0^T \hat{\zeta}_s^* dB_s.$$

Then, by representing $G(m_T(\hat{Z}^*)) + R + \frac{1}{2} \int_0^T (\hat{Z}_s^*)^2 ds$ as a stochastic integral with respect to B (up to the mean), we see that we can choose $\hat{\zeta}^*$ (non-zero) in such a way that

$$-G(m_T(\hat{Z}^*)) - \xi^* = J^P(\xi^*) - \int_0^T \hat{Z}_s^* dW_s^{\hat{Z}^*},$$

in which case the variance of the reward of the principal is minimal.

3. The first and second points right above show that the selection of $\hat{\zeta}^*$ is not entirely clear (when $\epsilon > 0$). As we are just interested in the statistical distribution of the population, this has no real consequence for us.

3.1.3 Restoration of uniqueness

We use Theorem 3.2 to prove that, as in Chapter 2, uniqueness fails in the absence of common noise (i.e., $\epsilon = 0$) while it is preserved in the presence of common noise (i.e., $\epsilon > 0$).

The argument is based upon the following lemma, which we already accounted for in the previous subsection:

Lemma 3.3. *Assume $\epsilon > 0$. For any solution $(\xi^*, (m_t^*(\xi^*), \alpha_t^*(\xi^*))_{t \in [0, T]})$ to the mean-field game of “principal-agent” type, as given by Theorem 3.2, the process $\hat{Z}^* := (\hat{Z}_t^*)_{t \in [0, T]}$ has a version that is $(\mathcal{F}_t^B)_{t \in [0, T]}$ progressively measurable.*

When $\epsilon = 0$, the process $\hat{Z}^ := (\hat{Z}_t^*)_{t \in [0, T]}$ has a deterministic version.*

Proof. We just prove the case $\epsilon > 0$ (the other case is treated in a similar way). The result follows from a convexity argument. Indeed,

$$\mathbb{E}^{\hat{Z}^*} \left[\left(-G(m_T^*(\xi)) - \frac{1}{2} \int_0^T (\hat{Z}_t^*)^2 dt \right) \right] \leq \mathbb{E}^{\hat{Z}^*} \left[\left(-G(m_T^*(\xi)) - \frac{1}{2} \int_0^T \mathbb{E}^{\hat{Z}^*} [\hat{Z}_t^* | \mathcal{F}_t^B]^2 dt \right) \right], \quad (3.7)$$

with equality if and only if, $d\mathbb{P}^{\hat{Z}^*} \otimes dt$ almost everywhere, \hat{Z}_t^* is equal to $\tilde{Z}_t^* := \mathbb{E}^{\hat{Z}^*} [\hat{Z}_t^* | \mathcal{F}_t^B]$. Now, since the variables $(\tilde{Z}_t^*)_{t \in [0, T]}$ are \mathcal{F}_t^B -measurable, the right-hand side is also equal to:

$$\mathbb{E}^{\hat{Z}^*} \left[\left(-G(m_T^*(\xi)) - \frac{1}{2} \int_0^T \mathbb{E}^{\hat{Z}^*} [\hat{Z}_t^* | \mathcal{F}_t^B]^2 dt \right) \right] = \mathbb{E} \left[\left(-G(m_T^*(\xi)) - \frac{1}{2} \int_0^T (\tilde{Z}_t^*)^2 dt \right) \right].$$

Thanks to (3.5), we have in fact $m_T^*(\xi) = m_T(\hat{Z}^*) = m_T(\tilde{Z}^*)$. This says that:

$$\mathbb{E}^{\hat{Z}^*} \left[\left(-G(m_T^*(\xi)) - \frac{1}{2} \int_0^T (\hat{Z}_t^*)^2 dt \right) \right] \leq \mathbb{E} \left[\left(-G(m_T(\tilde{Z}^*)) - \frac{1}{2} \int_0^T (\tilde{Z}_t^*)^2 dt \right) \right],$$

with equality if and only if, $d\mathbb{P}^{\hat{Z}^*} \otimes dt$ almost everywhere, \hat{Z}_t^* is equal to \tilde{Z}_t^* . Now, if \tilde{Z}^* belongs to \mathcal{D}_0 , Theorem 3.2 says that:

$$\begin{aligned} \mathbb{E}^{\hat{Z}^*} \left[\left(-G(m_T^*(\xi)) - \frac{1}{2} \int_0^T (\hat{Z}_t^*)^2 dt \right) \right] &\geq \mathbb{E}^{\tilde{Z}^*} \left[\left(-G(m_T(\tilde{Z}^*)) - \frac{1}{2} \int_0^T (\tilde{Z}_t^*)^2 dt \right) \right] \\ &= \mathbb{E} \left[\left(-G(m_T(\tilde{Z}^*)) - \frac{1}{2} \int_0^T (\tilde{Z}_t^*)^2 dt \right) \right]. \end{aligned}$$

In fact, this remains true even if \tilde{Z}^* does not belong to \mathcal{D}_0 : It suffices to approximate \tilde{Z}^* in $\mathbb{H}^2(\mathbb{R})$ by a bounded sequence, to apply Theorem 3.2 to the approximated sequence and to pass to the limit.

We deduce that any $(\hat{Z}_t^*)_{t \in [0, T]}$ as in the statement of Theorem 3.2 must satisfy, $d\mathbb{P}^{\hat{Z}^*} \otimes dt$ almost everywhere, $\hat{Z}_t^* = \mathbb{E}^{\hat{Z}^*}[\hat{Z}_t^* | \mathcal{F}_t^B]$. In particular, for almost every $t \in [0, T]$, \hat{Z}_t^* is \mathcal{F}_t^B -measurable. Since \hat{Z}_t^* is already known to be \mathcal{F}_t -measurable, it is independent of $\sigma(B_s - B_t, s \in [t, T])$ under $\mathbb{P}^{\hat{Z}^*}$, from which we deduce that, $d\mathbb{P}^{\hat{Z}^*} \otimes dt$ almost everywhere, $\hat{Z}_t^* = \mathbb{E}[\hat{Z}_t^* | \mathcal{F}_t^B]$. Equivalently, \hat{Z}^* coincides with its optional projection on $(\mathcal{F}^B)_{t \in [0, T]}$. \square

In fact, the same argument as the one used in the proof of Lemma 3.3 shows that, in the statement of Theorem 3.2, we can reduce the inequality

$$\mathbb{E}^{\hat{Z}^*} \left[\left(-G(m_T^*(\xi^*)) - \frac{1}{2} \int_0^T (\hat{Z}_t^*)^2 dt \right) \right] \geq \mathbb{E} \left[\left(-G(m_T(\hat{Z})) - \frac{1}{2} \int_0^T \hat{Z}_t^2 dt \right) \right]$$

to the processes \hat{Z} that belong to $\mathbb{H}^2(\mathbb{R})$ and that are $(\mathcal{F}_t^B)_{t \in [0, T]}$ -progressively measurable if $\epsilon > 0$ and that are deterministic if $\epsilon = 0$. In particular, solving the principal-agent mean-field game is the same as solving the standard optimal control problem:

$$\sup_{\hat{Z}} \mathbb{E} \left[\left(-G(m_T(\hat{Z})) - \frac{1}{2} \int_0^T \hat{Z}_t^2 dt \right) \right], \quad (3.8)$$

the supremum being taken over processes $\hat{Z} \in \mathbb{H}^2(\mathbb{R})$ that are progressively measurable with respect to the filtration $(\mathcal{F}_t^B)_{t \in [0, T]}$ (or deterministic if $\epsilon = 0$). If \hat{Z}^* is a maximizer and is bounded, then it belongs to $\mathcal{D}_0(\mathbb{R})$ and a solution to the principal-agent mean field game is obtained by implementing (3.6). If this maximizer is unique, then the mean-field game of “principal-agent” is uniquely solvable in the sense that law of the equilibrium regime of the agent is uniquely determined (but ζ^* is not uniquely determined).

As G is concave, we cannot use the stochastic maximum principle to characterize the solutions to (3.8). Still, by a direct analysis of the optimal control problem, we manage to prove the following statement:

Theorem 3.4. *When $\epsilon > 0$, solutions to the mean-field game of “principal-agent” type are given by $(m_t^*(\xi^*), \alpha_t^*(\xi^*))_{t \in [0, T]} = (x_t, -h_t)_{t \in [0, T]}$ and*

$$\xi^* = R - \frac{1}{2} \int_0^T h_s^2 ds + \int_0^T (-h_s) dW_s + \int_0^T \zeta_s^* dB_s, \quad \forall t \in [0, T],$$

where $(0, \zeta^* := (\zeta_t^*)_{t \in [0, T]}) \in \mathcal{D}(\mathbb{R})$ and where $(x_t, h_t, z_t)_{t \in [0, T]}$ is the unique solution to the FBSDE

$$\begin{cases} x_t = - \int_0^t h_s ds + \epsilon B_t, \quad \forall t \in [0, T], \\ h_t = \partial_x G(x_T) - \int_t^T z_s dB_s, \quad \forall t \in [0, T]. \end{cases} \quad (3.9)$$

When $\epsilon = 0$ and $\delta \neq 0$ (see the definition of G), solutions to the mean-field game of “principal-agent” type are given by $(m_t^*(\xi^*), \alpha_t^*(\xi^*))_{t \in [0, T]} = (x_t, -h_t)_{t \in [0, T]}$ and

$$\xi^* = R - \frac{1}{2} \int_0^T h_s^2 ds + \int_0^T (-h_s) dW_s, \quad \forall t \in [0, T],$$

where $(x_t, h_t)_{t \in [0, T]}$ is one of the two non-zero solutions of the FBODE

$$\begin{cases} x_t = - \int_0^t h_s ds, & \forall t \in [0, T], \\ h_t = \partial_x G(x_T), & \forall t \in [0, T]. \end{cases} \quad (3.10)$$

When $\epsilon = 0$ and $\delta = 0$, solutions to the mean-field game are given by $(m_t^*(\xi^*), \alpha_t^*(\xi^*))_{t \in [0, T]} = (x_t, -h_t)_{t \in [0, T]}$, where $(x_t, h_t)_{t \in [0, T]}$ is one of the infinitely many solutions of the FBODE (3.10).

We directly obtain the following three cases for existence and uniqueness:

Case 1: When there is common noise (i.e., $\epsilon > 0$), the mean-field game of “principal-agent” is uniquely solvable in the sense that the law of the agent in the equilibrium regime is uniquely determined (but ζ^* is not uniquely determined).

Case 2: When there is no common noise (i.e., $\epsilon = 0$) and $\delta \neq 0$, the mean-field game of “principal-agent” type under study has two solutions. The analysis performed in Subsection 3.2.2 shows that these two solutions are given by:

$$\left(\xi^A = R + \frac{A^2 T}{2} + AW_T^A, \quad m_t^* = At, \quad \hat{Z}_t^* = A \right)_{t \in [0, T]}, \quad A \in \{-1, 1\}.$$

In fact, the analysis below also shows that there exists an “unstable” equilibrium, obtained by letting $A = 0$ in the above formula. Precisely, we prove below the following second-order expansion for principal’s utility around $\xi^0 = R$:

Lemma 3.5. *For any other contract ξ satisfying the participation constraint $V^A(\xi) = R$, it holds that:*

$$J^P(\xi) = J^P(R) + O(\mathbb{E}^{\alpha^*(\xi)}[|\xi - R|^2]),$$

where $O(\cdot)$ is the Landau symbol.

Here, the terminology “unstable equilibrium” must be understood as in physics: Principal’s utility has a zero derivative at $\xi^0 = R$ (at least in certain directions), but this critical point is unstable because it minimizes locally the utility along contracts of the form ξ^A , for $A \in \mathbb{R}$, see Subsection 3.2.2.

Case 3: When there is no common noise (i.e., $\epsilon = 0$) and $\delta = 0$, the mean-field game of “principal-agent” type has infinitely many solutions. The analysis performed in Subsection 3.2.2 shows that these solutions are given by:

$$\left(\xi^A = R + \frac{A^2 T}{2} + AW_T^A, \quad m_t^* = At, \quad \hat{Z}_t^* = A \right)_{t \in [0, T]}, \quad A \in [-1, 1].$$

3.1.4 Selection of equilibria

In the case when there are several solutions, a relevant question is: Which solution should we select? To answer this question, we may introduce methods analogous to those addressed in Chapter 2, namely:

1. (Principal’s selection) A first method consists in selecting the solutions with the greatest value for the expected utility of the principal, which is similar to the first selection method addressed in Chapter 2. Here, this method exactly coincides with the second step in our formulation of the mean-field game; so, all the solutions of the mean-field game should be selected.
2. (Zero noise limit selection) Another method is to consider the unique stochastic solution when $\epsilon > 0$ (say with $\zeta^* = 0$, i.e. minimal variance for the representative agent), and find its weak limit as $\epsilon \rightarrow 0$. If this limit exists, we select the solutions that are included in the support of the limit.

We can now state the selection results when $\delta > 0$ in the definition of the function G (see Subsection 3.1.2).

Theorem 3.6. *Assume that $\delta > 0$. Then, as for the mean-field game of “principal-agent” type described in scheme (3.1) without common noise, principal’s selection approach selects any of the two solutions corresponding to $A \in \{-1, +1\}$, whilst the zero-noise limit selection approach selects a randomized solution, as given by the solution $A = 1$ with probability $1/2$ and by the solution $A = -1$ with probability $1/2$.*

In particular, none of the two approaches selects the “unstable” equilibria $\xi = \xi^0$ (i.e., $A = 0$). Still, in the latter regime, the variance of the reward of the agent is minimal.

In contrast with Chapter 2, the two selection approaches yield here similar results. What happens is that the common noise forces the equilibrium to leave the unstable configuration (corresponding to $A = 0$ in the statement).

3.2 Solving the mean-field game of “principal-agent” type

3.2.1 Solving representative agent’s MFG problem

Given an arbitrary contract, we recall representative agent’s expected utility:

$$J^A(\xi, \alpha) = \mathbb{E} \left[L_{0T}(\alpha) \left(\xi - \frac{1}{2} \int_0^T \alpha_s^2 ds \right) \right].$$

The representative agent aims at solving the particular MFG problem below:

1. (Input Step)
Consider an arbitrary contract $\xi \in \mathcal{C}$ together with an arbitrary $(\mathcal{F}_t^B)_{t \in [0, T]}$ -progressively measurable process $(m_t)_{t \in [0, T]}$ such that $\sup_{t \in [0, T]} |m_t|$ has finite moments of any order. (If $\epsilon = 0$, ξ is required to be \mathcal{F}_T -measurable and $(m_t)_{t \in [0, T]}$ is required to be constant.)
2. (Optimization Step)
Find a strategy $\alpha^*(\xi) \in \mathcal{A}(\mathbb{R})$ such that

$$V^A(\xi) = \sup_{\alpha \in \mathcal{A}(\mathbb{R})} J^A(\xi, \alpha),$$

where $V^A(\xi) := J^A(\xi, \alpha^*(\xi)) \geq R$, and

$$dX_t = \kappa(m_t - X_t)dt + \alpha_t dt + dW_t^\alpha + \epsilon dB_t, \quad X_0 = 0, \quad \forall t \in [0, T].$$

3. (Matching Step)

Find the mean-field process solution of the matching problem:

$$m_t = \mathbb{E} [L_{0T}(\alpha^*(\xi))X_t^* | \mathcal{F}_t^B], \quad \forall t \in [0, T],$$

where

$$dX_t^* = \kappa(m_t - X_t)dt + \alpha_t^*(\xi)dt + dW_t^{\alpha^*(\xi)} + \epsilon dB_t, \quad X_0 = 0, \quad \forall t \in [0, T].$$

(If $\epsilon = 0$, the conditional expectation in the definition of m_t reduces to a mere expectation.)

Remark 3.7. For every $\xi \in \mathcal{C}$, thanks to the strict concavity of representative agent’s Hamiltonian function, the MFG problem described above admits at most one solution. We make it clear below.

The Continuation Utility Approach

Next, given $\xi \in \mathcal{C}$ and $\alpha \in \mathcal{A}(\mathbb{R})$, we introduce the continuation utility of the representative agent. This process evaluates representative agent’s expected gain on the time interval $[t, T]$ when it deviates from BAU with the strategy α under the contract ξ . The continuation utility of the representative agent is defined as follows:

$$Y_t(\alpha) = \mathbb{E} \left[L_{tT}(\alpha) \left(\xi - \frac{1}{2} \int_t^T \alpha_s^2 ds \right) | \mathcal{F}_t \right], \quad \forall t \in [0, T],$$

with

$$L_{tT}(\alpha) := \exp \left(\int_t^T \alpha_s dW_s - \frac{1}{2} \int_t^T \alpha_s^2 ds \right).$$

Notice that, for simplicity, we do not specify in the notation the fact that the process $Y(\alpha)$ depends on ξ ; but, obviously, it does.

The proposition below provides the semi-martingale expansion of agent’s continuation utility.

Proposition 3.8. For all $(\alpha, \xi) \in \mathcal{A}(\mathbb{R}) \times \mathcal{C}$, there exist two processes $(Z_t(\alpha))_{t \in [0, T]}$ and $(\zeta_t(\alpha))_{t \in [0, T]}$ in $\mathbf{H}^2(\mathbb{R})$ such that $(Y_t(\alpha), Z_t(\alpha), \zeta_t(\alpha))_{t \in [0, T]} \in \mathbf{S}^2(\mathbb{R}) \times \mathbf{H}^2(\mathbb{R}) \times \mathbf{H}^2(\mathbb{R})$ satisfies the BSDE below:

$$\begin{cases} dY_t(\alpha) = \left(\frac{1}{2} \alpha_t^2 - \alpha_t Z_t(\alpha) \right) dt + Z_t(\alpha) dW_t + \zeta_t(\alpha) dB_t, & \forall t \in [0, T] \\ Y_T(\alpha) = \xi \quad \text{and} \quad \mathbb{E}[Y_0(\alpha)] = J^A(\xi, \alpha). \end{cases} \quad (3.11)$$

Moreover, for any $p \geq 1$,

$$\mathbb{E} \left[\sup_{t \in [0, T]} |Y_t(\alpha)|^p + \left(\int_0^T [|Z_t(\alpha)|^2 + |\zeta_t(\alpha)|^2] dt \right)^{p/2} \right] < \infty.$$

Proof. First step. We first check that $(Y_t(\alpha))_{t \in [0, T]} \in \mathbf{S}^2(\mathbb{R})$. To do so, we use conditional Cauchy-Schwarz inequality in the following way:

$$\begin{aligned} |Y_t(\alpha)|^2 &= \mathbb{E} \left[L_{tT}(\alpha) \left(\xi - \frac{1}{2} \int_t^T \alpha_s^2 ds \right) \middle| \mathcal{F}_t \right]^2 \\ &\leq \mathbb{E} [L_{tT}(\alpha) \middle| \mathcal{F}_t] \mathbb{E} \left[L_{tT}(\alpha) \left(\xi - \frac{1}{2} \int_t^T \alpha_s^2 ds \right)^2 \middle| \mathcal{F}_t \right] \\ &= \mathbb{E} \left[L_{tT}(\alpha) \left(\xi - \frac{1}{2} \int_t^T \alpha_s^2 ds \right)^2 \middle| \mathcal{F}_t \right]. \end{aligned} \quad (3.12)$$

Now, we observe that $L_{tT}(\alpha) = L_{0T}(\alpha)L_{0t}(\alpha)^{-1}$. In particular, by the reverse Hölder inequality satisfied by elements belonging to the class $\mathcal{A}(\mathbb{R})$, we can find $\epsilon, c > 0$ such that

$$\mathbb{E} [L_{tT}(\alpha)^{1+\epsilon} \middle| \mathcal{F}_t] \leq c.$$

Injecting this bound in (3.12) and applying conditional Hölder inequality, we get:

$$\begin{aligned} |Y_t(\alpha)|^2 &\leq c^{1/(1+\epsilon)} \mathbb{E} \left[\left| \xi - \frac{1}{2} \int_t^T \alpha_s^2 ds \right|^{2(1+\epsilon)/\epsilon} \middle| \mathcal{F}_t \right]^{\epsilon/(1+\epsilon)} \\ &\leq c^{1/(1+\epsilon)} \mathbb{E} \left[\left(|\xi| + \frac{1}{2} \int_0^T \alpha_s^2 ds \right)^{2(1+\epsilon)/\epsilon} \middle| \mathcal{F}_t \right]^{\epsilon/(1+\epsilon)}. \end{aligned}$$

The end of the proof follows from Doob’s maximal inequality, together with the fact that, by definition of the two classes $\mathcal{A}(\mathbb{R})$ and \mathcal{C} , it holds:

$$\begin{aligned} \mathbb{E} [\exp(|\xi|)] &< \infty, \\ \forall p \geq 1, \quad \mathbb{E} \left[\left(\int_0^T |\alpha_t|^2 dt \right)^p \right] &< \infty. \end{aligned}$$

Observe that, in fact, we have the stronger result:

$$\forall p \geq 1, \quad \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t(\alpha)|^p \right] < \infty.$$

Also, by definition of the continuation utility, notice that $\mathbb{E}[Y_0(\alpha)] = J^A(\xi, \alpha)$.

Second step. Next, we consider the \mathbb{P} -martingale stochastic process, $(M_t)_{t \in [0, T]}$, given by:

$$M_t := L_{0t}(\alpha) \left(Y_t(\alpha) - \frac{1}{2} \int_0^t \alpha_r^2 dr \right).$$

By the first step, the process $(M_t)_{t \in [0, T]}$ satisfies $\mathbb{E}[\sup_{0 \leq t \leq T} |M_t|^{1+\epsilon}] < \infty$ for some $\epsilon > 0$. The fact that it is a martingale may be proved as follows:

$$\begin{aligned}
& \mathbb{E} \left[M_t | \mathcal{F}_s \right] \\
&= \mathbb{E} \left[L_{0t}(\alpha) Y_t(\alpha) | \mathcal{F}_s \right] - \frac{1}{2} \mathbb{E} \left[L_{0t}(\alpha) \int_0^t \alpha_r^2 dr | \mathcal{F}_s \right] \\
&= \mathbb{E} \left[L_{0t}(\alpha) \mathbb{E} \left[L_{sT}(\alpha) \left(\xi - \frac{1}{2} \int_t^T \alpha_s^2 ds \right) | \mathcal{F}_t \right] | \mathcal{F}_s \right] - \frac{1}{2} \mathbb{E} \left[L_{0t}(\alpha) \int_0^t \alpha_r^2 dr | \mathcal{F}_s \right] \\
&= L_{0s}(\alpha) \left(\mathbb{E} \left[L_{sT}(\alpha) \left(\xi - \frac{1}{2} \int_t^T \alpha_s^2 ds \right) | \mathcal{F}_s \right] - \frac{1}{2} \mathbb{E} \left[L_{sT}(\alpha) \int_s^t \alpha_r^2 dr | \mathcal{F}_s \right] - \frac{1}{2} \int_0^s \alpha_r^2 dr \right) \\
&= L_{0s}(\alpha) \left(Y_s(\alpha) - \frac{1}{2} \int_0^s \alpha_r^2 dr \right) = M_s, \quad \forall s \leq t.
\end{aligned}$$

By the martingale representation theorem, there exist two processes $(Z_t)_{t \in [0, T]}$ and $(\zeta_t)_{t \in [0, T]}$ in $\mathcal{P}(\mathbb{R})$ such that:

$$\mathbb{E} \left[\left(\int_0^T (|Z_t|^2 + |\zeta_t|^2) dt \right)^{(1+\epsilon)/2} \right] < \infty,$$

and

$$dM_t = Z_t dW_t + \zeta_t dB_t, \quad \forall t \in [0, T].$$

Then, we obtain from a little stochastic calculus the expression:

$$\begin{aligned}
& dY_t(\alpha) \\
&= \frac{1}{2} \alpha_t^2 dt + d(L_{0t}^{-1}(\alpha) M_t), \quad \forall t \in [0, T] \\
&= \frac{1}{2} \alpha_t^2 dt + (-\alpha_t Z_t(\alpha) dt + Z_t(\alpha) dW_t + \zeta_t(\alpha) dB_t), \quad \text{with } \begin{cases} Z_t(\alpha) := L_{0t}^{-1}(\alpha) (Z_t - \alpha_t M_t), \\ \zeta_t(\alpha) := L_{0t}^{-1}(\alpha) \zeta_t, \end{cases} \\
&= \left(\frac{1}{2} \alpha_t^2 - \alpha_t Z_t(\alpha) \right) dt + Z_t(\alpha) dW_t + \zeta_t(\alpha) dB_t, \quad \forall t \in [0, T], \quad \text{and } Y_T(\alpha) = \xi.
\end{aligned}$$

Third step. It remains to show that the stochastic processes $(Y_t(\alpha), Z_t(\alpha), \zeta_t(\alpha))_{t \in [0, T]}$ built above do satisfy the required integrability conditions.

We already proved that:

$$\forall p \geq 1, \quad \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t(\alpha)|^p \right] < \infty. \tag{3.13}$$

To show that $(Z_t(\alpha))_{t \in [0, T]}$ and $(\zeta_t(\alpha))_{t \in [0, T]}$ belong to $\mathbf{H}^2(\mathbb{R})$, we apply Itô's formula to $(Y_t(\alpha)^2)_{t \in [0, T]}$. For any stopping time τ such that:

$$\mathbb{E} \left[\int_0^\tau (|Z_t(\alpha)|^2 + |\zeta_t(\alpha)|^2) dt \right] < \infty,$$

we get:

$$\mathbb{E} \left[\int_0^\tau [|Z_t(\alpha)|^2 + |\zeta_t(\alpha)|^2] dt \right] \leq \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t(\alpha)|^2 + \int_0^\tau |Y_t(\alpha)| (\alpha_t^2 + 2|\alpha_t| |Z_t(\alpha)|) dt \right].$$

Recalling (3.13) and the fact that $\alpha \in \mathcal{A}(\mathbb{R})$, we deduce that there exists a constant C , independent of τ and the value of which value may vary from line to line, such that:

$$\mathbb{E} \left[\int_0^\tau [|Z_t(\alpha)|^2 + |\zeta_t(\alpha)|^2] dt \right] \leq C + C \mathbb{E} \left[\sup_{t \in [0, T]} |Y_t(\alpha)|^2 \int_0^\tau |\alpha_t|^2 dt \right]^{1/2} \mathbb{E} \left[\int_0^\tau |Z_t(\alpha)|^2 dt \right]^{1/2}.$$

By Young’s inequality, we deduce that:

$$\mathbb{E} \left[\int_0^\tau [|Z_t(\alpha)|^2 + |\zeta_t(\alpha)|^2] dt \right] \leq C.$$

Letting τ tend to T , we deduce that $(Z_t(\alpha))_{t \in [0, T]} \in \mathbf{H}^2(\mathbb{R})$ and $(\zeta_t(\alpha))_{t \in [0, T]} \in \mathbf{H}^2(\mathbb{R})$.

In fact, a similar argument permits to show that:

$$\forall p \geq 1, \quad \mathbb{E} \left[\left(\int_0^T [|Z_t(\alpha)|^2 + |\zeta_t(\alpha)|^2] dt \right)^p \right] < \infty.$$

Indeed, by taking a stopping time τ such that:

$$\mathbb{E} \left[\left(\int_0^\tau [|Z_t(\alpha)|^2 + |\zeta_t(\alpha)|^2] dt \right)^p \right] < \infty,$$

and taking power p in the expansion:

$$\begin{aligned} & |Y_\tau(\alpha)|^2 - |Y_0(\alpha)|^2 \\ & - \int_0^\tau Y_t(\alpha) (\alpha_t^2 - 2\alpha_t Z_t(\alpha)) dt - 2 \int_0^\tau Y_t(\alpha) Z_t(\alpha) dW_t - 2 \int_0^\tau Y_t(\alpha) \zeta_t(\alpha) dB_t \\ & = \int_0^\tau (|Z_t(\alpha)|^2 + |\zeta_t(\alpha)|^2) dt, \end{aligned}$$

we get:

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^\tau (|Z_t(\alpha)|^2 + |\zeta_t(\alpha)|^2) dt \right)^p \right] \\ & \leq C + C \mathbb{E} \left[\left(\int_0^\tau |Y_t(\alpha)| (|\alpha_t| |Z_t(\alpha)|) dt \right)^p \right] \\ & \quad + C \mathbb{E} \left[\left(\int_0^\tau |Y_t(\alpha)|^2 |Z_t(\alpha)|^2 dt \right)^{p/2} + \left(\int_0^\tau |Y_t(\alpha)|^2 |\zeta_t(\alpha)|^2 dt \right)^{p/2} \right], \end{aligned}$$

and we easily conclude as before. \square

Building on the previous proposition, and its dynamic description of representative agent’s continuation utility, the next proposition allows us to propose a candidate for the optimal strategy to representative agent’s optimization problem.

Proposition 3.9. *Given $\xi \in \mathcal{C}$, there exists a unique triple process $(\hat{Y}_t, \hat{Z}_t, \hat{\zeta}_t)_{t \in [0, T]} \in \mathbf{S}^2(\mathbb{R}) \times \mathbf{H}^2(\mathbb{R}) \times \mathbf{H}^2(\mathbb{R})$ solution to the BSDE:*

$$d\hat{Y}_t = -\frac{1}{2}\hat{Z}_t^2 dt + \hat{Z}_t dW_t + \hat{\zeta}_t dB_t, \quad \forall t \in [0, T], \quad Y_T = \xi. \quad (3.14)$$

It satisfies:

1. $(\hat{Z}_t)_{t \in [0, T]} \in \mathcal{A}(\mathbb{R})$;
2. $\forall \alpha \in \mathcal{A}(\mathbb{R}), \quad J^A(\xi, \alpha) \leq J^A(\xi, \hat{Z})$.

Moreover,

$$\forall \lambda > 1, \quad \mathbb{E}[\exp(\lambda \sup_{t \in [0, T]} |\hat{Y}_t|)] < \infty,$$

and

$$\forall p \geq 1, \quad \mathbb{E}\left[\left(\int_0^T |\hat{\zeta}_t|^2 dt\right)^p\right] < \infty.$$

Of course, if $\epsilon = 0$ and ξ is required to be \mathcal{F}_T^W -measurable, then \hat{Z} is $(\mathcal{F}_t^W)_{t \in [0, T]}$ -progressively measurable and the process $\hat{\zeta}$ is null.

Proof. First step. Given $\xi \in \mathcal{C}$, the existence a solution $(\hat{Y}_t, \hat{Z}_t, \hat{\zeta}_t)_{t \in [0, T]} \in \mathbf{S}^2(\mathbb{R}) \times \mathbf{H}^2(\mathbb{R}) \times \mathbf{H}^2(\mathbb{R})$ to the BSDE (3.14) follows from Briand and Hu’s results on quadratic BSDEs with unbounded terminal values, see [16]. In particular, thanks to Proposition 3 therein, we deduce the following bound:

$$\forall t \in [0, T], \quad |\hat{Y}_t| \leq \ln \mathbb{E}[\exp(|\xi|)|\mathcal{F}_t].$$

Moreover,

$$\forall p > 1, \quad \mathbb{E}\left[\left(\int_0^T (|\hat{Z}_t|^2 + |\hat{\zeta}_t|^2) dt\right)^p\right] < \infty.$$

Also, recalling that ξ expands:

$$\xi = \mathbb{E}[\xi|\mathcal{F}_T] + \int_t^T Z_s dW_s + \int_t^T \zeta_s dB_s,$$

with $(Z_s)_{s \in [0, T]}$ and $(\zeta_s)_{s \in [0, T]}$ as in (3.1), we get:

$$\hat{Y}_t = \mathbb{E}[\xi|\mathcal{F}_t] + \frac{1}{2} \int_t^T |\hat{Z}_s|^2 ds - \int_t^T (\hat{Z}_s - Z_s) dW_s - \int_t^T (\hat{\zeta}_s - \zeta_s) dB_s, \quad t \in [0, T],$$

and then $(\bar{Y}_t = \hat{Y}_t - \mathbb{E}[\xi|\mathcal{F}_t], \bar{Z}_t = \hat{Z}_t - Z_t, \bar{\zeta}_t = \hat{\zeta}_t - \zeta_t)_{t \in [0, T]}$ is a solution of the BSDE:

$$\bar{Y}_t = \frac{1}{2} \int_t^T (|Z_s|^2 + 2Z_s \bar{Z}_s + |\bar{Z}_s|^2) ds - \int_t^T \bar{Z}_s dW_s - \int_t^T \bar{\zeta}_s dB_s, \quad t \in [0, T].$$

By construction, this solution satisfies:

$$\forall \lambda > 0, \quad \mathbb{E}[\exp(\lambda \sup_{t \in [0, T]} |\bar{Y}_t|)] < \infty,$$

and

$$\forall p \geq 1, \quad \mathbb{E} \left[\left(\int_0^T (|\bar{Z}_t|^2 + |\bar{\zeta}_t|^2) dt \right)^p \right] \leq 1,$$

and, in fact, it is the unique one to satisfy these two bounds, see again [16]. Now, since the process $(Z_t)_{t \in [0, T]}$ is (almost everywhere bounded), the theory of quadratic BSDEs says that the unique solution must satisfy the following two properties (see for instance [42]):

1. The process $(\bar{Y}_t)_{t \in [0, T]}$ is bounded by a deterministic constant;
2. The process $(\bar{Z}_t)_{t \in [0, T]}$ belongs to $\mathcal{A}(\mathbb{R})$.

Of course, we deduce that the process $(\hat{Z}_t)_{t \in [0, T]}$ belongs to $\mathcal{A}(\mathbb{R})$.

Second step. For all $\alpha \in \mathcal{A}(\mathbb{R})$, Proposition 3.8 provides processes, $(Y_t(\alpha), Z_t(\alpha), \zeta_t(\alpha))_{t \in [0, T]}$, solution to:

$$dY_t(\alpha) = \left(\frac{1}{2} \alpha_t^2 - \alpha_t Z_t(\alpha) \right) dt + Z_t(\alpha) dW_t + \zeta_t(\alpha) dB_t, \quad \forall t \in [0, T] \quad Y_T(\alpha) = \xi. \quad (3.15)$$

We want to compare $(Y_t(\alpha))_{t \in [0, T]}$ and $(\hat{Y}_t)_{t \in [0, T]}$. We first do so when the process $(\alpha_t)_{t \in [0, T]}$ is bounded by a deterministic constant. By Proposition 3 in [16], we deduce that $\sup_{t \in [0, T]} |Y_t(\alpha)|$ has exponential moments of any order. Also,

$$\forall p \geq 1, \quad \mathbb{E} \left[\left(\int_0^T [|Z_t(\alpha)|^2 + |\zeta_t(\alpha)|^2] dt \right)^p \right] < \infty.$$

We then rewrite the drift in the form:

$$\frac{1}{2} \alpha_t^2 - \alpha_t Z_t(\alpha) = -\frac{1}{2} |Z_t(\alpha)|^2 + \frac{1}{2} |\alpha_t - Z_t(\alpha)|^2 \geq -\frac{1}{2} |Z_t(\alpha)|^2.$$

Therefore, we can invoke Theorem 5 in [16] to compare (3.15) and (3.14). This directly yields:

$$\forall t \in [0, T] \quad Y_t(\alpha) \leq \hat{Y}_t \quad \mathbb{P} - a.s. \quad .$$

In particular,

$$J^A(\xi, \alpha) = \mathbb{E}[Y_0(\alpha)] \leq \mathbb{E}[Y_0].$$

Third step. We now identify Y_0 with $J^A(\xi, \hat{Z})$. To do so, we notice that the equation for $(\hat{Y}_t, \hat{Z}_t, \hat{\zeta}_t)_{t \in [0, T]}$ may be rewritten in the form:

$$d\hat{Y}_t = \left(\frac{1}{2} \hat{Z}_t^2 - \hat{Z}_t \hat{Z}_t \right) dt + \hat{Z}_t dW_t + \hat{\zeta}_t dB_t, \quad \forall t \in [0, T] \quad \hat{Y}_T = \xi.$$

Moreover, since $(\hat{Z}_t)_{t \in [0, T]} \in \mathcal{A}(\mathbb{R})$, we deduce from Proposition 3.8 that there exists

$$(Y_t(\hat{Z}), Z_t(\hat{Z}), \zeta_t(\hat{Z}))_{t \in [0, T]} \in \mathbf{S}^2(\mathbb{R}) \times \mathbf{H}^2(\mathbb{R}) \times \mathbf{H}^2(\mathbb{R}),$$

satisfying:

$$dY_t(\hat{Z}) = \left(\frac{1}{2} \hat{Z}_t^2 - Z_t(\hat{Z}) \hat{Z}_t \right) dt + Z_t(\hat{Z}) dW_t + \zeta_t(\hat{Z}) dB_t, \quad \forall t \in [0, T] \quad Y_T(\hat{Z}) = \xi.$$

We observe that $Y_T(\hat{Z}) - Y_T = 0$ and by putting $\tilde{Z}_t := (Z_t(\hat{Z}) - \hat{Z}_t)$, $\forall t \in [0, T]$,

$$\begin{aligned} d(Y_t(\hat{Z}) - Y_t) &= -\hat{Z}_t \tilde{Z}_t dt + \tilde{Z}_t dW_t, \\ &= \tilde{Z}_t d\hat{W}_t, \quad \text{where } \hat{W}_t = W_t - \int_0^t \hat{Z}_s ds, \quad \forall t \in [0, T]. \end{aligned}$$

Assume for a while $\left(\int_0^t \tilde{Z}_s d\hat{W}_s \right)_{t \in [0, T]}$ is a $\mathbb{P}^{\hat{Z}}$ -martingale. Then,

$$\mathbb{E}[Y(\hat{Z})_0 - Y_0] = \mathbb{E}[L_{0T}(\hat{Z})(Y(\hat{Z})_0 - Y_0)] = \mathbb{E}[L_{0T}(\hat{Z})(Y_T(\hat{Z}) - Y_T)] = 0.$$

The fact that $\left(\int_0^t \tilde{Z}_s d\hat{W}_s \right)_{t \in [0, T]}$ is a $\mathbb{P}^{\hat{Z}}$ -martingale follows from the following two things:

1. For some $\epsilon > 0$, $\mathbb{E}[L_{0T}(\hat{Z})^{1+\epsilon}] < \infty$.

2. For any $p \geq 1$,

$$\mathbb{E} \left[\left(\int_0^T |\tilde{Z}_t|^2 dt \right)^p \right] < \infty,$$

see again Proposition 3.8.

This suffices to prove that:

$$\mathbb{E}^{\hat{Z}} \left[\int_0^T |\tilde{Z}_t|^2 dt \right] < \infty.$$

Fourth step. So far, we have proven that, if $\alpha \in \mathcal{A}(\mathbb{R})$ is a bounded process, then

$$J^A(\xi, \alpha) \leq J^A(\xi, \hat{Z}).$$

In order to prove the inequality for any $\alpha \in \mathcal{A}(\mathbb{R})$, we use an approximation argument. Indeed, for $\alpha \in \mathcal{A}(\mathbb{R})$, we may call, for any integer $n \geq 1$, $\alpha^n \in \mathcal{A}(\mathbb{R})$ the process obtained by projecting α onto the interval $[-n, n]$.

Clearly, if we call c a constant satisfying

$$\mathbb{P} \left(\mathbb{E} \left[\int_\tau^T |\alpha_s|^2 ds \mid \mathcal{F}_\tau \right] \leq c \right) = 1,$$

for any stopping time τ , then, for any $n \geq 1$ and for the same constant c ,

$$\mathbb{P} \left(\mathbb{E} \left[\int_\tau^T |\alpha_s^n|^2 ds \mid \mathcal{F}_\tau \right] \leq c \right) = 1,$$

which shows that there exists $\epsilon > 0$ such that:

$$\sup_{n \geq 1} \mathbb{E}[L_{0T}(\alpha^n)^{1+\epsilon}] < \infty.$$

It is then pretty easy to prove that:

$$\lim_{n \rightarrow \infty} J^A(\xi, \alpha_n) = J^A(\xi, \alpha).$$

Passing to the limit in the inequality

$$J^A(\xi, \alpha_n) \leq J^A(\xi, \hat{Z}), \quad n \geq 1,$$

we complete the proof. \square

The next step is to prove that the optimal strategy identified in the statement of Proposition 3.9 is the unique one.

Proposition 3.10. *Given $\xi \in \mathcal{C}$, the triple process $(\hat{Y}_t, \hat{Z}_t, \hat{\zeta}_t)_{t \in [0, T]} \in \mathbf{S}^2(\mathbb{R}) \times \mathbf{H}^2(\mathbb{R}) \times \mathbf{H}^2(\mathbb{R})$ identified in Proposition 3.9 satisfies $\alpha_t = \hat{Z}_t \, d\mathbb{P} \otimes dt$ almost everywhere, for all $\alpha \in \mathcal{A}(\mathbb{R})$ such that $J^A(\xi, \alpha) = J^A(\xi, \hat{Z})$.*

Proof. First step. With $\alpha \in \mathcal{A}(\mathbb{R})$, we associate the triple $(Y_t(\alpha), Z_t(\alpha), \zeta_t(\alpha))_{t \in [0, T]}$, as given by Proposition 3.8. The first point is to check that $(Z_t(\alpha))_{t \in [0, T]} \in \mathcal{A}(\mathbb{R})$. To do so, we recall that ξ has the form:

$$\xi = \mathbb{E}[\xi | \mathcal{F}_t] + \int_t^T Z_s dW_s + \int_t^T \zeta_s dB_s,$$

where $(Z_t)_{t \in [0, T]}$ is a bounded process and

$$\forall \lambda > 0, \quad \mathbb{E} \left[\exp \left(\lambda \sup_{t \in [0, T]} \left| \int_0^t \zeta_s dB_s \right| \right) \right] < \infty.$$

Hence, the process $(\bar{Y}_t(\alpha) := Y_t(\alpha) - \mathbb{E}[\xi | \mathcal{F}_t], \bar{Z}_t(\alpha) := Z_t(\alpha) - Z_t, \bar{\zeta}_t(\alpha) := \zeta_t(\alpha) - \zeta_t)_{t \in [0, T]}$ satisfies

$$\bar{Y}_t(\alpha) = - \int_t^T \left(\frac{1}{2} \alpha_s^2 - \alpha_s Z_s - \alpha_s \bar{Z}_s(\alpha) \right) ds - \int_t^T \bar{Z}_s(\alpha) dW_s - \int_t^T \bar{\zeta}_s(\alpha) dB_s.$$

Therefore,

$$\bar{Y}_t(\alpha) = - \mathbb{E}^\alpha \left[\int_t^T \left(\frac{1}{2} \alpha_s^2 - \alpha_s Z_s \right) ds \mid \mathcal{F}_t \right],$$

the right-hand side being obviously well-defined. In particular, recalling that $(Z_t)_{t \in [0, T]}$ is a bounded process, we have

$$|\bar{Y}_t(\alpha)| \leq C + C \mathbb{E}^\alpha \left[\int_t^T \alpha_s^2 ds \mid \mathcal{F}_t \right],$$

for a non-negative deterministic constant C (independent of t). In fact, by Theorem 3.3 in [69], the right-hand side is also bounded by a deterministic constant. Allowing C to vary from line to line, we deduce that

$$|\bar{Y}_t(\alpha)| \leq C.$$

Applying Itô’s formula to the process $(\bar{Y}_t(\alpha))_{t \in [0, T]}$, we deduce that there exists a constant c such that, for any stopping time τ ,

$$\mathbb{P} \left(\mathbb{E} \left[\int_\tau^T |\bar{Z}_s(\alpha)|^2 ds \mid \mathcal{F}_t \right] \leq c \right) = 1,$$

which shows that $(\bar{Z}_t(\alpha))_{t \in [0, T]}$ belongs to $\mathcal{A}(\mathbb{R})$. Hence, $(Z_t(\alpha))_{t \in [0, T]}$ also belongs to $\mathcal{A}(\mathbb{R})$.

Second step. Now, we write:

$$Y_t(\alpha) = \xi - \int_t^T \frac{1}{2} (|Z_s(\alpha) - \alpha_s|^2 - |Z_s(\alpha)|^2) ds - \int_t^T Z_s(\alpha) dW_s - \int_t^T \zeta_s(\alpha) dB_s,$$

which can be rewritten in the form:

$$Y_0(\alpha) + \frac{1}{2} \int_0^T |Z_t(\alpha) - \alpha_t|^2 dt = \xi - \frac{1}{2} \int_0^T |Z_t(\alpha)|^2 dt - \int_0^T Z_t(\alpha) dW_t^{Z(\alpha)} - \int_0^T \zeta_t(\alpha) dB_t.$$

Taking expectations under $\mathbb{P}^{Z(\alpha)}$, we deduce that:

$$J(\xi, \alpha) + \frac{1}{2} \mathbb{E}^{Z(\alpha)} \int_0^T |Z_t(\alpha) - \alpha_t|^2 dt \leq J(\xi, Z(\alpha)).$$

Since $J(\xi, \alpha) = J(\xi, \hat{Z})$ and $J(\xi, Z(\alpha)) \leq J(\xi, \hat{Z})$, we deduce that:

$$\frac{1}{2} \mathbb{E}^{Z(\alpha)} \int_0^T |Z_t(\alpha) - \alpha_t|^2 dt = 0,$$

which shows that $(Y(\alpha), Z(\alpha), \zeta(\alpha))$ solves the same BSDE as $(\hat{Y}_t, \hat{Z}_t, \hat{\zeta}_t(\alpha))_{t \in [0, T]}$. It is easy to check the integrability conditions that are required in the statement of the uniqueness Theorem 5 in [16]. This shows that $Z(\alpha)$ and \hat{Z} coincide. \square

Applicable contracts and MFG solution

With the information provided by the previous results on the best reaction of the representative agent to an arbitrary contract, we can define a set of applicable contracts.

Definition 3.2.1 (Applicable contracts). *Given $R \geq 0$ and $\xi \in \mathcal{C}$, we say that ξ is an applicable contract if there exists a real constant $\hat{Y} \geq R$ and two processes $\hat{Z} \in \mathcal{A}(\mathbb{R})$ and $\hat{\zeta} \in \mathbf{H}^2(\mathbb{R})$, with*

$$\forall p \geq 1, \quad \mathbb{E} \left[\left(\int_0^T |\hat{\zeta}_t|^2 dt \right)^p \right] < \infty,$$

such that

$$\xi = \hat{Y} - \frac{1}{2} \int_0^T \hat{Z}_s^2 ds + \int_0^T \hat{Z}_s dW_s + \int_0^T \hat{\zeta}_s dB_s.$$

Remark 3.11. *The set of applicable contracts defined above is obviously non-empty. In fact, if $\xi \in \mathcal{C}$ satisfies*

$$\mathbb{E}[\exp(-|\xi|)] \leq \exp(-R),$$

then ξ is applicable. Indeed, by Proposition 3.9, there exists an \mathcal{F}_0 -measurable random variable \hat{Y} and two processes $\hat{Z} \in \mathcal{A}(\mathbb{R})$ and $\hat{\zeta} \in \mathbf{H}^2(\mathbb{R})$, with

$$\forall p \geq 1, \quad \mathbb{E} \left[\left(\int_0^T |\hat{\zeta}_t|^2 dt \right)^p \right] < \infty,$$

such that

$$\xi = \hat{Y} - \frac{1}{2} \int_0^T \hat{Z}_s^2 ds + \int_0^T \hat{Z}_s dW_s + \int_0^T \hat{\zeta}_t dB_t.$$

Thanks to the proof of Proposition 3.9, the following estimate holds:

$$\hat{Y} \geq -\ln \mathbb{E}[\exp(-|\xi|)].$$

Since $\mathbb{E}[\exp(-|\xi|)] \leq \exp(-R)$ implies $-\ln \mathbb{E}[\exp(-|\xi|)] \geq R$, we deduce that $\hat{Y} \geq R$.

For all $R \geq 0$, this prompts us to define the set

$$\mathcal{C}(R) = \{\xi \in \mathcal{C} \mid \mathbb{E}[\exp(-|\xi|)] \leq \exp(-R)\} \subset \mathcal{C},$$

then it holds that $\mathcal{C}(R) \neq \emptyset$ and all $\xi \in \mathcal{C}(R)$ are applicable contracts.

It remains to solve representative agent’s MFG problem when the principal proposes the applicable contracts. The next proposition characterizes the solutions.

Proposition 3.12. *Given an applicable contract ξ , the unique solution to representative agent’s MFG problem is given by $(m_t^*(\xi), \alpha_t^*(\xi))_{t \in [0, T]} = (m_t, \hat{Z}_t)_{t \in [0, T]}$, $(m_t, Y_t, \hat{Z}_t, \hat{\zeta}_t)_{t \in [0, T]}$ solving the FBSDE below:*

$$\begin{cases} m_t = \int_0^t \mathbb{E}[L_{0T}(\hat{Z}) \hat{Z}_s | \mathcal{F}_T^B] ds + \epsilon B_t, & \forall t \in [0, T], \\ Y_t = \xi + \frac{1}{2} \int_t^T \hat{Z}_s^2 ds - \int_t^T \hat{Z}_s dW_s - \int_t^T \hat{\zeta}_s dB_s, & \forall t \in [0, T]. \end{cases} \quad (3.16)$$

If $\epsilon = 0$ and ξ is \mathcal{F}_T^W -measurable, then the conditional expectation in the definition of $(m_t)_{t \in [0, T]}$ reduces to a mere expectation; moreover, the process $(\hat{Z}_t)_{t \in [0, T]}$ is $(\mathcal{F}_t^W)_{t \in [0, T]}$ -progressively measurable and $\hat{\zeta}$ is null.

Proof. First step. Clearly, given an applicable contract ξ , FBSDE (3.16) is decoupled. Thanks to Proposition 3.9, we know that the process $(\hat{Z}_t)_{t \in [0, T]}$ is an optimal strategy for the representative agent. Also, by Proposition 3.10, such an optimal strategy is unique.

Second step. We show that $(m_t)_{t \in [0, T]}$, as defined in the statement, does satisfy the matching condition. Consider the optimally managed project dynamics below:

$$X_t^{\hat{Z}} = \int_0^t [\kappa(m_s - X_s^{\hat{Z}}) + \hat{Z}_s] ds + W_t^{\hat{Z}} + \epsilon B_t, \quad \forall t \in [0, T].$$

Multiplying by $L_{0T}(\hat{Z})$, taking condition expectation and using the fact that $\mathcal{F}_T^{W^{\hat{Z}}}$ and \mathcal{F}_T^B are independent under $\mathbb{P}^{\hat{Z}}$, we obtain:

$$\mathbb{E}[L_{0T}(\hat{Z}) X_t^{\hat{Z}} | \mathcal{F}_T^B] = \kappa \int_0^t (m_s - \mathbb{E}[L_{0T}(\hat{Z}) X_s^{\hat{Z}} | \mathcal{F}_T^B]) ds + \int_0^t \mathbb{E}[L_{0T}(\hat{Z}) \hat{Z}_s | \mathcal{F}_T^B] ds + \epsilon B_t,$$

for all $t \in [0, T]$. Making the difference with the equation for $(m_t)_{t \in [0, T]}$, we get:

$$(m_t - \mathbb{E}[L_{0T}(\hat{Z}) X_t^{\hat{Z}} | \mathcal{F}_T^B]) = -\kappa \int_0^t (m_s - \mathbb{E}[L_{0T}(\hat{Z}) X_s^{\hat{Z}} | \mathcal{F}_T^B]) ds, \quad \forall t \in [0, T],$$

and the left-hand side has to be zero.

It is easily checked that any other solution to the matching problem has the same form as $(m_t)_{t \in [0, T]}$.

Third step. Finally, for all applicable contracts ξ , the participation constraint of the representative agent is satisfied. Indeed,

$$V^A(\xi) := J^A(\xi, \hat{Z}) = \mathbb{E}[Y_0(\hat{Z})] = \hat{Y} \geq R,$$

which completes the proof. \square

3.2.2 Solving principal’s second best problem

Remark 3.13. 1) Taking into account the fact that ξ is an applicable contract, the previous proposition characterizes representative agent’s MFG solution through solutions of the 2-dimensional SDE:

$$\begin{bmatrix} m_t \\ Y_t \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{Y} \end{bmatrix} + \int_0^t \begin{pmatrix} \mathbb{E}^{\hat{Z}}[\hat{Z}_s | B] \\ \frac{1}{2} \hat{Z}_s^2 \end{pmatrix} ds + \int_0^t \begin{pmatrix} \epsilon & 0 \\ \hat{\zeta}_s & \hat{Z}_s \end{pmatrix} \begin{pmatrix} dB_s \\ dW_s^{\hat{Z}} \end{pmatrix}, \quad \forall t \in [0, T]. \quad (3.17)$$

2) Given $R \geq 0$, an applicable contract ξ and the unique solution to the above agent MFG problem, given by $(m_t, \hat{Z}_t, \hat{\zeta}_t)_{t \in [0, T]}$, principal’s expected utility writes:

$$J^P(\xi) = \mathbb{E}^{\hat{Z}} \left[-G(m_T) - \xi \right],$$

where we recall that, for a given $\delta \in [0, T)$, $G : \mathbb{R} \rightarrow \mathbb{R}$ reads:

$$G(x) := \frac{-x^2}{2(T-\delta)} \mathbb{1}_{|x| \leq (T-\delta)} + \left(\frac{(T-\delta)}{2} - \text{sign}(x)x \right) \mathbb{1}_{|x| > (T-\delta)}.$$

Since the principal has access to a set of applicable contracts for which it has a good understanding of the best reaction of the representative agent at equilibrium, it is left with the following optimization problem.

1. (Optimization Step)

Find an applicable contract ξ^* , such that, for any other applicable contract ξ ,

$$J^P(\xi) \leq J^P(\xi^*) = \mathbb{E}^{\hat{Z}^*} \left[-G(m_T^*) - \xi^* \right],$$

where $(m_t^*, \hat{Z}_t^*, \hat{\zeta}_t^*)_{t \in [0, T]}$ is representative agent’s MFG solution under the applicable contract ξ^* .

Characterization of principal’s optimization problem

The next statement covers Theorem 3.2.

Proposition 3.14. Assume that $\epsilon > 0$. Then, principal’s optimization problem has a solution ξ^* , amongst applicable contracts, if and only if there exists a pair $(\hat{Z}^*, \hat{\zeta}^*) := (\hat{Z}_t^*, \hat{\zeta}_t^*)_{t \in [0, T]}$

in $\mathcal{D}(\mathbb{R}^2)$ such that

$$\mathbb{E}^{\hat{Z}^*} \left[\left(-G(m_T^*) - \frac{1}{2} \int_0^T (\hat{Z}_t^*)^2 dt \right) \right] \geq \mathbb{E}^{\hat{Z}} \left[\left(-G(m_T(\hat{Z})) - \frac{1}{2} \int_0^T \hat{Z}_t^2 dt \right) \right], \quad \forall \hat{Z} \in \mathcal{D}_0,$$

where

$$m_t(\hat{Z}) = \int_0^t \mathbb{E}^{\hat{Z}} [\hat{Z}_s | B] ds + \epsilon B_t, \quad \forall t \in [0, T],$$

and $(m_t^* := m_t(\hat{Z}^*))_{t \in [0, T]}$.

In that case, the solution $(\xi^*, m_t^*(\xi^*), \alpha_t^*(\xi^*))_{t \in [0, T]}$ to the mean-field game of “principal-agent” type is given by:

$$\begin{aligned} \alpha_t^*(\xi^*) &= \hat{Z}_t^*, \quad \forall t \in [0, T], \\ m_t^*(\xi) &= \int_0^t \mathbb{E}^{\hat{Z}^*} [\hat{Z}_s^* | B] ds + \epsilon B_t, \quad \forall t \in [0, T], \\ \xi^* &= R - \frac{1}{2} \int_0^T (\hat{Z}_s^*)^2 ds + \int_0^T \hat{Z}_s^* dW_s + \int_0^T \hat{\zeta}_s^* dB_s, \quad \forall t \in [0, T]. \end{aligned}$$

The result remains true in the case $\epsilon = 0$, except for the facts that $\mathbb{E}^{\hat{Z}}[\hat{Z}_s | B]$ becomes $\mathbb{E}^{\hat{Z}}[\hat{Z}_s]$, that \hat{Z}^* is progressively-measurable with respect to $(\mathcal{F}_t^W)_{t \in [0, T]}$ and that $\hat{\zeta}^*$ is necessarily zero.

Proof. First step. In this particular model, all the applicable contracts ξ are completely characterized by triples $(\hat{Y}, \hat{Z}, \hat{\zeta})$, where $\hat{Y} \in [R, +\infty)$ and $(\hat{Z}, \hat{\zeta}) \in \mathcal{D}(\mathbb{R}^2)$. To make it clear, we have:

$$\begin{aligned} &\sup_{\xi} J^P(\xi) \\ &= \sup_{\hat{Y} \in [R, +\infty), (\hat{Z}, \hat{\zeta}) \in \mathcal{D}(\mathbb{R}^2)} \mathbb{E}^{\hat{Z}} \left[-G(m_T(\hat{Z})) - \left(\hat{Y} - \frac{1}{2} \int_0^T \hat{Z}_s^2 ds + \int_0^T \hat{Z}_s dW_s + \int_0^T \hat{\zeta}_s dB_s \right) \right]. \end{aligned}$$

This quantity can be further reduced as follows:

$$\begin{aligned} \sup_{\xi} J^P(\xi) &= \sup_{\hat{Y} \in [R, +\infty), (\hat{Z}, \hat{\zeta}) \in \mathcal{D}(\mathbb{R}^2)} \mathbb{E}^{\hat{Z}} \left[-G(m_T(\hat{Z})) - \left(\hat{Y} - \frac{1}{2} \int_0^T \hat{Z}_s^2 ds + \int_0^T \hat{Z}_s dW_s \right) \right] \\ &= \sup_{\hat{Y} \in [R, +\infty), (\hat{Z}, \hat{\zeta}) \in \mathcal{D}(\mathbb{R}^2)} \mathbb{E}^{\hat{Z}} \left[-G(m_T(\hat{Z})) - \left(\hat{Y} + \frac{1}{2} \int_0^T \hat{Z}_s^2 ds + \int_0^T \hat{Z}_s dW_s^{\hat{Z}} \right) \right] \\ &= \sup_{\hat{Y} \in [R, +\infty), (\hat{Z}, \hat{\zeta}) \in \mathcal{D}(\mathbb{R}^2)} \mathbb{E}^{\hat{Z}} \left[-G(m_T(\hat{Z})) - \left(\hat{Y} + \frac{1}{2} \int_0^T \hat{Z}_s^2 ds \right) \right] \\ &= \sup_{\hat{Y} \in [R, +\infty), (\hat{Z}, \hat{\zeta}) \in \mathcal{D}(\mathbb{R}^2)} \mathbb{E}^{\hat{Z}} \left[\left(-G(m_T(\hat{Z})) - \frac{1}{2} \int_0^T \hat{Z}_s^2 ds \right) - \hat{Y} \right], \end{aligned}$$

from which we deduce:

$$\begin{aligned} \sup_{\xi} J^P(\xi) &= \sup_{(\hat{Z}, \hat{\zeta}) \in \mathcal{D}(\mathbb{R}^2)} \mathbb{E}^{\hat{Z}} \left[\left(-G(m_T(\hat{Z})) - \frac{1}{2} \int_0^T \hat{Z}_s^2 ds \right) \right] + \sup_{\hat{Y} \in [R, +\infty)} \mathbb{E} \left[-\hat{Y} \right] \\ &= \sup_{(\hat{Z}, \hat{\zeta}) \in \mathcal{D}(\mathbb{R}^2)} \mathbb{E}^{\hat{Z}} \left[\left(-G(m_T(\hat{Z})) - \frac{1}{2} \int_0^T \hat{Z}_s^2 ds \right) \right] - R. \end{aligned}$$

Clearly, if (ξ^*) solves principal’s optimization problem, then the pair $(m_t^*(\xi^*), \alpha_t^*(\xi^*))_{t \in [0, T]} = (m_t^*, \hat{Z}_t^*)_{t \in [0, T]}$ is the corresponding MFG solution to the representative agent’s MFG problem, and it holds that:

$$\mathbb{E}^{\hat{Z}^*} \left[\left(-G(m_T^*) - \frac{1}{2} \int_0^T (\hat{Z}_t^*)^2 dt \right) \right] \geq \mathbb{E}^{\hat{Z}} \left[\left(-G(m_T(\hat{Z})) - \frac{1}{2} \int_0^T \hat{Z}_t^2 dt \right) \right], \quad \forall \hat{Z} \in \mathcal{D}_0.$$

Second step. Conversely, if $\hat{Z}^* \in \mathcal{D}_0$ maximizes $\mathbb{E}^{\hat{Z}} \left[\left(-G(m_T(\hat{Z})) - \frac{1}{2} \int_0^T \hat{Z}_t^2 dt \right) \right]$ over $\hat{Z} \in \mathcal{D}_0$ and if $(\hat{Z}^*, \hat{\zeta}^*) \in \mathcal{D}(\mathbb{R}^2)$, put:

$$\xi^* = R - \frac{1}{2} \int_0^T (\hat{Z}_s^*)^2 ds + \int_0^T \hat{Z}_s^* dW_s + \int_0^T \hat{\zeta}_s^* dB_s, \quad \forall t \in [0, T].$$

Obviously, ξ^* is an applicable contract. It follows that the process $(m_t^*(\xi^*), \alpha_t^*(\xi^*))_{t \in [0, T]} = (m_t(\hat{Z}^*), \hat{Z}_t^*)_{t \in [0, T]}$ is the corresponding MFG solution to representative agent’s MFG problem and thus ξ^* solves principal’s optimization problem. □

Stochastic control problem

By Lemma 3.3, we know that solving the principal-agent mean-field game is the same as solving the standard optimal control problem

$$\begin{aligned} &\sup_{\hat{Z}} \mathbb{E} \left[\left(-G(m_T(\hat{Z})) - \frac{1}{2} \int_0^T (\hat{Z}_t)^2 dt \right) \right] \\ &= - \inf_{\hat{Z}} \mathbb{E} \left[\left(G(m_T(\hat{Z})) + \frac{1}{2} \int_0^T (\hat{Z}_t)^2 dt \right) \right], \end{aligned} \tag{3.18}$$

with

$$m_t(\hat{Z}) = \int_0^t \hat{Z}_s ds + \epsilon B_t, \quad \forall t \in [0, T], \tag{3.19}$$

the supremum being taken over processes $\hat{Z} \in \mathbb{H}^2(\mathbb{R})$ that are progressively measurable with respect to the filtration $(\mathcal{F}_t^B)_{t \in [0, T]}$ (or deterministic is $\epsilon = 0$). Namely, if \hat{Z}^* is a maximizer and is bounded, then it belongs to $\mathcal{D}_0(\mathbb{R})$ and a solution to the principal-agent mean field game is obtained by implementing (3.6). If this maximizer is unique, then the mean-field game of “principal-agent” is uniquely solvable in the sense that the law of the equilibrium state is uniquely determined (but $\hat{\zeta}^*$ is not uniquely determined).

With this reformulation in mind, we can define the following Hamiltonian for problem (3.18-3.19):

Definition 3.2.2. *The problem (3.18-3.19) admits the following Hamiltonian:*

$$H(\hat{z}, h) = h\hat{z} + \frac{1}{2}\hat{z}^2, \quad \forall h, \hat{z} \in \mathbb{R} \times \mathbb{R}. \quad (3.20)$$

Proposition 3.15. *Let H be the Hamiltonian for the problem (3.18-3.19). If an $(\mathcal{F}_t^B)_{t \in [0, T]}$ -progressively measurable (deterministic if $\epsilon = 0$) process $(\hat{Z}_t^*)_{t \in [0, T]} \in \mathbf{H}^2(\mathbb{R})$ is a solution to the problem (3.18-3.19), then there exists an $(\mathcal{F}_t^B)_{t \in [0, T]}$ -progressively measurable (deterministic if $\epsilon = 0$) triple $(m_t, h_t, z_t)_{t \in [0, T]}$, solution to the adjoint FBSDE:*

$$\begin{cases} dm_t = \partial_h H(\hat{Z}_t^*, h_t) dt + \epsilon dB_t, & \forall t \in [0, T], \\ dh_t = -\partial_m H(\hat{Z}_t^*, h_t) dt + z_t dB_t, & \forall t \in [0, T], \\ m_0 = 0, \quad h_T = \partial_x G(m_T), \end{cases} \quad (3.21)$$

subject to

$$H(\hat{Z}_t^*, h_t) = \min_{\hat{z} \in \mathbb{R}} H(\hat{z}, h_t), \quad \forall t \in [0, T], \quad \mathbb{P} - a.s. \quad .$$

(If $\epsilon = 0$, then z is required to be 0.)

Proof. The proposition follows from the necessary condition in the Pontryagin stochastic maximum principle. \square

Pay attention that the Pontryagin stochastic maximum principle does not provide a sufficient condition because G is concave (or equivalently $-G$ is convex). Returning to the original dynamics (3.2-3.3), we understand that the fact that H is convex but G is concave is the key point to force the regime without common noise (i.e., $\epsilon = 0$) to admit several equilibria (see the analysis right below).

Although the Pontryagin stochastic maximum principle does not apply in its sufficient form, we succeed to prove the following characterization in the case $\epsilon > 0$:

Proposition 3.16. *Assume $\epsilon > 0$. Then, an $(\mathcal{F}_t^B)_{t \in [0, T]}$ -progressively measurable process $(\hat{Z}_t^*)_{t \in [0, T]} \in \mathbf{H}^2(\mathbb{R})$ is a solution to problem (3.18-3.19) if and only if there exists an $(\mathcal{F}_t^B)_{t \in [0, T]}$ -progressively measurable triple $(x_t, h_t, z_t)_{t \in [0, T]}$ satisfying $(\hat{Z}_t^* = -h_t)_{t \in [0, T]}$ and solving the FBSDE:*

$$\begin{cases} x_t = -\int_0^t h_s ds + \epsilon B_t, & \forall t \in [0, T], \\ h_t = \partial_x G(x_T) - \int_t^T z_s dB_s, & \forall t \in [0, T]. \end{cases} \quad (3.22)$$

Moreover, the latter FBSDE is uniquely solvable and the corresponding process $(h_t)_{t \in [0, T]}$ is bounded (by a deterministic constant).

Proof. First step. The necessary condition in the statement follows immediately from Proposition 3.15 together with the fact that, for all $h \in \mathbb{R}$, the map $\hat{z} \mapsto H(\hat{z}, h)$ is strictly convex. This allows for the minimizers $\hat{z}^* \in \mathbb{R}$ to be defined by as function of $h \in \mathbb{R}$ as follows:

$$\partial_{\hat{z}} H(\hat{z}^*, h) = 0 \iff (h + \hat{z}^*) = 0 \iff \hat{z}^* = -h.$$

Substituting accordingly in the FBSDE system (3.21), we get (3.22).

Second step. In order to prove the converse, we use the assumption that $\epsilon > 0$. In that case, the FBSDE (3.22) is uniquely solvable. We already accounted for this fact in Chapter 2. Basically, existence and uniqueness of a solution $(x_t, h_t, z_t)_{t \in [0, T]}$ are guaranteed by Theorem 2.6 in [35] on non-degenerate FBSDEs. The solution can be expressed through the four-step scheme as follows. Call $\theta^\epsilon \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}; \mathbb{R}) \cap \mathcal{C}([0, T] \times \mathbb{R}; \mathbb{R})$ the classical solution to the PDE:

$$\begin{cases} \partial_t \theta^\epsilon(t, x) - \theta^\epsilon(t, x) \partial_x \theta^\epsilon(t, x) + \frac{\epsilon^2}{2} \partial_{xx} \theta^\epsilon(t, x) = 0, & \forall t \in [0, T], \quad \forall x \in \mathbb{R}, \\ \theta^\epsilon(T, x) = \partial_x G(x), & \forall x \in \mathbb{R}. \end{cases} \quad (3.23)$$

Then,

$$h_t = \theta^\epsilon(t, x_t), \quad \text{and} \quad z_t = \epsilon \partial_x \theta^\epsilon(t, x_t), \quad \forall t \in [0, T],$$

where

$$x_t = - \int_0^t \theta^\epsilon(s, x_s) ds + \epsilon B_t, \quad \forall t \in [0, T].$$

With θ^ϵ , we associate the value function:

$$\Theta^\epsilon(t, x) = \int_0^x \theta^\epsilon(t, y) dy + \frac{\epsilon^2}{2} \int_t^T \partial_x \theta^\epsilon(s, 0) ds.$$

By integrating (3.23) (noticing that $\theta^\epsilon(t, 0) = 0$ because θ^ϵ is odd in x), we see that Θ^ϵ satisfies the Hamilton-Jacobi-Bellman equation:

$$\begin{cases} \partial_t \Theta^\epsilon(t, x) - \frac{1}{2} |\partial_x \Theta^\epsilon(t, x)|^2 + \frac{\epsilon^2}{2} \partial_{xx} \Theta^\epsilon(t, x) = 0, & \forall t \in [0, T], \quad \forall x \in \mathbb{R}, \\ \Theta^\epsilon(T, x) = G(x), & \forall x \in \mathbb{R}. \end{cases} \quad (3.24)$$

Recalling that θ^ϵ is bounded and Lipschitz continuous in x (uniformly in time), we deduce that $\Theta^\epsilon \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}; \mathbb{R}) \cap \mathcal{C}^{0,1}([0, T] \times \mathbb{R}; \mathbb{R})$, with a bounded x -derivative. This fact permits us to implement a standard verification argument, which we make clear for reader's convenience. For an $(\mathcal{F}_t^B)_{t \in [0, T]}$ -progressively measurable process $(\hat{Z}_t)_{t \in [0, T]} \in \mathbf{H}^2(\mathbb{R})$, we expand $(\Theta^\epsilon(t, m_t(\hat{Z})))_{t \in [0, T]}$ by Itô's formula. We get:

$$\begin{aligned} d\Theta^\epsilon(t, m_t(\hat{Z})) &= [\partial_t \Theta^\epsilon(t, m_t(\hat{Z})) + \hat{Z}_t \theta^\epsilon(t, m_t(\hat{Z})) + \frac{\epsilon^2}{2} \partial_{xx} \Theta^\epsilon(t, m_t(\hat{Z}))] dt \\ &\quad + \epsilon \theta^\epsilon(t, m_t(\hat{Z})) dB_t, \end{aligned}$$

where we used the fact that $\partial_x \Theta^\epsilon = \theta^\epsilon$. By (3.24), we obtain:

$$d\Theta^\epsilon(t, m_t(\hat{Z})) = [\frac{1}{2} |\theta^\epsilon(t, m_t(\hat{Z}))|^2 + \hat{Z}_t \theta^\epsilon(t, m_t(\hat{Z}))] dt + \epsilon \theta^\epsilon(t, m_t(\hat{Z})) dB_t,$$

Since θ^ϵ is bounded, Θ^ϵ is at most of linear growth in x . Hence, we can easily integrate from 0 to T and then take expectations. We deduce that:

$$\mathbb{E} \left[\Theta^\epsilon(T, m_T(\hat{Z})) + \frac{1}{2} \int_0^T |\hat{Z}_t|^2 dt \right] = \Theta^\epsilon(0, 0) + \frac{1}{2} \mathbb{E} \int_0^T |\hat{Z}_t + \theta^\epsilon(t, m_t(\hat{Z}))|^2 dt.$$

By definition of the boundary condition in (3.24), we deduce that

$$\mathbb{E} \left[G(m_T(\hat{Z})) + \frac{1}{2} \int_0^T |\hat{Z}_t|^2 dt \right] = \Theta^\epsilon(0, 0) + \frac{1}{2} \mathbb{E} \int_0^T |\hat{Z}_t + \theta^\epsilon(t, m_t(\hat{Z}))|^2 dt.$$

When $(\hat{Z}_t = -h_t)_{t \in [0, T]}$, the right-hand side reduces to $\Theta^\epsilon(0, 0)$. Obviously, this is the unique optimal strategy to (3.18-3.19). \square

The case $\epsilon = 0$ is more subtle.

Proposition 3.17. *Assume $\epsilon = 0$.*

If $\delta \neq 0$, the problem (3.18-3.19) has exactly two solutions, given by $(\hat{Z}_t^ = A)_{t \in [0, T]}$, for $A \in \{-1, 1\}$. For the two of them, there exists a (unique deterministic) pair $(x_t, h_t)_{t \in [0, T]}$ satisfying $(\hat{Z}_t^* = -h_t)_{t \in [0, T]}$ and solving the FBODE:*

$$\begin{cases} x_t = - \int_0^t h_s ds, & \forall t \in [0, T], \\ h_t = \partial_x G(x_T), & \forall t \in [0, T], \end{cases} \quad (3.25)$$

but the latter FBODE has three (and not two!) solutions, given by $((x_t, h_t) = (At, -A))_{t \in [0, T]}$, with $A \in \{-1, 0, 1\}$.

If $\delta = 0$, the problem (3.18-3.19) has infinitely many solutions, given by $(\hat{Z}_t^ = A)_{t \in [0, T]}$, for $A \in [-1, 1]$. For any of them, there exists a (unique deterministic) pair $(x_t, h_t)_{t \in [0, T]}$ satisfying $(\hat{Z}_t^* = -h_t)_{t \in [0, T]}$ and solving the FBODE:*

$$\begin{cases} x_t = - \int_0^t h_s ds, & \forall t \in [0, T], \\ h_t = \partial_x G(x_T), & \forall t \in [0, T]. \end{cases} \quad (3.26)$$

The latter FBODE has exactly $((x_t, h_t) = (At, -A))_{t \in [0, T]}$, with $A \in [-1, 1]$, as solutions.

In a nutshell, the statement says that, when $\delta \neq 0$, the system (3.26) does not characterize the solutions to (3.18-3.19). We will see next that the solution $((x_t, h_t) = (0, 0))_{t \in [0, T]}$ corresponds to an unstable Stackelberg equilibrium for the original “principal-agent” mean-field game.

Proof. When $\epsilon = 0$, equation (3.23) degenerates and we can no longer use the PDE argument. So, we perform a case by case analysis.

First step. When $\delta \neq 0$, we know from Chapter 2 that the system (3.26) has exactly three solutions, given by

$$(x_t, h_t)_{t \in [0, T]} = (At, -A)_{t \in [0, T]}, \quad A \in \{-1, 0, 1\}. \quad (3.27)$$

We check by hand whether they are solutions to the optimal control problem (3.18-3.19). To do so, we recall that we are looking for deterministic optimal trajectories. We then start with the following preliminary observation. Given a square-integrable deterministic path $(\hat{Z}_t)_{t \in [0, T]}$, we may define:

$$[\hat{Z}] = \frac{1}{T} \int_0^T \hat{Z}_t dt.$$

Then, regarding $[\hat{Z}]$ as a constant path, we have:

$$G(m_T(\hat{Z})) = G(m_T([\hat{Z}])).$$

In particular, by a standard convexity argument,

$$G(m_T(\hat{Z})) + \frac{1}{2} \int_0^T |\hat{Z}_t|^2 dt \geq G(m_T([\hat{Z}])) + \frac{T}{2} [\hat{Z}]^2,$$

with equality if and only if \hat{Z} is almost everywhere constant. So, this says that the solutions to (3.18-3.19) must be constant. In other words, solving (3.18-3.19) is the same as solving

$$\inf_{A \in \mathbb{R}} \left[G(TA) + \frac{T}{2} A^2 \right].$$

Now, for any A with $|A|T > T - \delta$,

$$G(TA) + \frac{T}{2} A^2 = \frac{T - \delta}{2} - T|A| + \frac{T}{2} A^2 = -\frac{\delta}{2} + \frac{T}{2} (|A| - 1)^2. \quad (3.28)$$

When $|A|T \leq T - \delta$,

$$G(TA) + \frac{T}{2} A^2 = -\frac{T^2 A^2}{2(T - \delta)} + \frac{T}{2} A^2 = -\frac{\delta T A^2}{2(T - \delta)} \geq -\frac{\delta(T - \delta)}{2T}. \quad (3.29)$$

Hence, we deduce that, in (3.27), only the paths associated to $A = \pm 1$ are solutions to (3.18-3.19).

Third step. When $\delta = 0$, we recover the counter-example studied in Chapter 1. The formulas (3.28) and (3.29) become

$$G(TA) + \frac{T}{2} A^2 = \frac{T}{2} (|A| - 1)^2, \quad \text{if } |A| \geq 1,$$

and

$$G(TA) + \frac{T}{2} A^2 = 0, \quad \text{if } |A| \leq 1,$$

proving that there are infinitely many solutions to (3.18-3.19), given by $(\hat{Z}_t^* = A)_{t \in [0, T]}$ for $A \in [-1, 1]$.

Meanwhile, it is easy to see that solutions to (3.26) are necessarily of the form

$$(x_t, h_t) = (At, -A), \quad t \in [0, T],$$

for some $A \in \mathbb{R}$. In fact, those paths are solutions if and only if $g(AT) = -A$, which is equivalent to $A \in [-1, 1]$. So, the solutions to (3.18-3.19) are exactly given by the solutions to (3.26). \square

Returning to the “principal-agent” MFG

The key point is that, for all the solutions to (3.18-3.19), as given by Propositions 3.16 and 3.17, the process $(Z_t^*)_{t \in [0, T]}$ is bounded. So, following our discussion right after (3.18-3.19), these solutions are exactly the solutions to the “principal-agent” mean-field game. This

completes the proof of Theorem 3.4.

When $\epsilon = 0$, the computations performed in the proof of Proposition 3.17 show that, for all $A \in \mathbb{R}$, the cost associated to $\xi^A := R - \frac{A^2 T}{2} + AW_T$, is

$$\begin{aligned} J^P(\xi^A) &= \mathbb{E}^A \left[-G(-AT) - \left(R + \frac{A^2 T}{2} + AW_T^A \right) \right] \\ &= -R + \begin{cases} \frac{\delta}{2} - \frac{T}{2} (|A| - 1)^2 & \text{if } |A|T > T - \delta, \\ \frac{\delta(T - \delta)A^2}{2T} & \text{if } |A|T \leq T - \delta, \end{cases} \end{aligned} \quad (3.30)$$

which shows that $A = 0$ is a local minimum of the map $A \mapsto J^P(\xi^A)$. Together with Lemma 3.5, the proof of which is given right below, this shows that, although it is not a solution to the MFG problem, $\xi^0 = R$ should be regarded as an “unstable” equilibrium.

Proof of Lemma 3.5. Take a contract of the form

$$\xi = \hat{Y} - \frac{1}{2} \int_0^T \hat{Z}_s^2 ds + \int_0^T \hat{Z}_s dW_s = \hat{Y} + \frac{1}{2} \int_0^T \hat{Z}_s^2 ds + \int_0^T \hat{Z}_s dW_s^{\hat{Z}},$$

where $\hat{Z} \in \mathcal{A}(\mathbb{R})$ is progressively-measurable with respect to the filtration generated by W .

Then, by Jensen’s inequality and since $\hat{Y} \geq R$,

$$\begin{aligned} \mathbb{E}^{\hat{Z}} [|\xi - R|^2] &= \mathbb{E}^{\hat{Z}} \left[\left| \hat{Y} - R + \frac{1}{2} \int_0^T \hat{Z}_s^2 ds + \int_0^T \hat{Z}_s dW_s^{\hat{Z}} \right|^2 \right] \\ &\geq \left| \hat{Y} - R + \frac{1}{2} \int_0^T \mathbb{E}^{\hat{Z}}(\hat{Z}_s^2) ds \right|^2 \\ &\geq |\hat{Y} - R|^2 + \frac{1}{4} \left(\int_0^T \mathbb{E}^{\hat{Z}}(\hat{Z}_s^2) ds \right)^2. \end{aligned}$$

Write now

$$\int_0^T \hat{Z}_s dW_s^{\hat{Z}} = \xi - R - (\hat{Y} - R) - \frac{1}{2} \int_0^T \hat{Z}_s^2 ds,$$

and take the L^2 norm under $\mathbb{P}^{\hat{Z}}$. We deduce that, for some universal constant $c > 0$,

$$\mathbb{E}^{\hat{Z}} \int_0^T (\hat{Z}_s^2) ds \leq c \mathbb{E}^{\hat{Z}} [|\xi - R|^2].$$

Now, by Proposition 3.12,

$$\begin{aligned} |J^P(\xi) - J^P(R)| &= \mathbb{E}^{\hat{Z}} \left[G \left(\int_0^T \mathbb{E}^{\hat{Z}}[Z_s] ds \right) - G(0) - (\xi - R) \right] \\ &= \mathbb{E}^{\hat{Z}} \left[G \left(\int_0^T \mathbb{E}^{\hat{Z}}[Z_s] ds \right) - G(0) - (\hat{Y} - R) - \frac{1}{2} \int_0^T \hat{Z}_s^2 ds \right]. \end{aligned}$$

Recalling that G is Lipschitz-continuous, the result easily follows when $\hat{Y} = R$. \square

3.3 Proof of selection results

In this section, we give the proof of Theorem 3.6. Here, we consider the mean-field game of “principal-agent” type without common noise (i.e $\epsilon = 0$) studied in Section 3.2. Also, we assume $\delta > 0$.

Principal’s selection: ($\epsilon = 0$)

Under this method, we allow the principal to select amongst all the possible solutions the solutions associated with the greatest expected utility. In fact, this is exactly what the mean-field game does. So, in Proposition 3.17, we select the two equilibria associated with $A = 1$ and $A = -1$.

Zero noise limit selection: ($\epsilon \rightarrow 0$)

Under this method, we consider the unique stochastic solution when $\epsilon > 0$, and find its weak limit as $\epsilon \rightarrow 0$. When this limit exists, we select the solutions that are included in the support of the limit.

Recall that the unique solution to the mean-field game of “principal-agent” type under study when $\epsilon > 0$, is given by (at least, if we choose the contract that minimizes the variance of the reward of the agent)

$$\begin{aligned}\xi^* &= R - \frac{1}{2} \int_0^T (\theta^\epsilon(t, x_t))^2 dt + \int_0^T (-\theta^\epsilon(t, x_t)) dW_t, \\ m_t^*(\xi^*) &= - \int_0^t \theta^\epsilon(s, x_s) ds + \epsilon B_t, \quad \forall t \in [0, T], \\ \alpha_t^*(\xi^*) &= -\theta^\epsilon(t, x_t), \quad \forall t \in [0, T],\end{aligned}$$

where θ^ϵ is the solution to the PDE:

$$\begin{cases} \partial_t \theta^\epsilon - \theta^\epsilon \partial_x \theta^\epsilon + \frac{\epsilon^2}{2} \partial_{xx} \theta^\epsilon = 0, & \forall t \in [0, T], \quad \forall x \in \mathbb{R}, \\ \theta^\epsilon(T, x) = \partial_x G(x), & \forall x \in \mathbb{R}. \end{cases}$$

Mimicking the zero-noise limit proof of Chapter 2, we obtain the following result:

Proposition 3.18. *As $\epsilon \rightarrow 0$, the sequence of probability laws,*

$$\left(\mathbb{P}^{\alpha^*(\xi)} \circ (m_t^*(\xi^*))_{t \in [0, t]}^{-1} \right)_{\epsilon \in (0, 1)}$$

converges to

$$\frac{1}{2} \delta_{(t)_{t \in [0, T]}} + \frac{1}{2} \delta_{(-t)_{t \in [0, T]}}.$$

Finally, in regard to the behavior of the agents, we make the following remark.

Remark 3.19. *From representative’s agent point of view, the equilibria (including the so-called unstable one) corresponding to $A = 0, -1, 1$ in the statement of Proposition 3.17 allow*

for the same expected utility with different variances. Indeed, for $A \in \{-1, 0, 1\}$,

$$\begin{aligned}\mathbb{E}^A \left[\left(R + \frac{A^2 T}{2} + AW_T^A \right) - \frac{A^2 T}{2} \right] &= R, \\ \mathbb{V}^A \left[\left(R - \frac{A^2 T}{2} + AW_T^A \right) - \frac{A^2 T}{2} \right] &= \mathbb{V}^A [AW_T^A] = A^2 T.\end{aligned}$$

Hence, whilst the expected utility is the same for all the equilibria, the equilibria that yields the smallest (actually zero) variance is the unstable one, namely $A = 0$.

Though the risk-averse agents would prefer the equilibria associated with $A = 0$, none of the selection approaches selects the best for the agents. This observation is coherent with the spirit of our model. Indeed, the situation $A = 0$ corresponds to the “Business As Usual” paradigm, which for the agents simply means gaining a straight $R > 0$ value from the contract without making any effort to deviate from their BAU practices.

3.4 Prospects

In connection with this chapter, we propose the following prospects for further research:

1. Comparing this chapter and Chapter 2, an immediate question to ask is: Can we obtain solutions to the MFG of “principal-agent” type by studying the weak limit of a sequence of N -agent games? In other words, can we implement an N -player limit selection method for our particular MFG of “principal-agent” type?

A natural way of doing would be to consider a game with one principal and N exchangeable agents. Each would feel two noises: a proper one (independent of the others) and (possibly) a common one. The principal would give N contracts: each contract would depend upon the proper noise of the agent and upon the common noise as well.

For sure, for such an N -player game, we should be able to provide a necessary condition for any Nash equilibrium in the form of a system of forward-backward equations. In fact, since the N agents are individualistic (as per their utility), it appears that finding a Nash equilibrium would be tantamount to finding a collection of optimizers for each of the agents, which is to say that the aforementioned system of forward-backward equations should consist in N somewhat decoupled equations. Strategies of the agents would only aggregate in the mean state of the economy.

As for the principal, its own utility would depend upon the terminal mean state of the economy and upon the empirical mean of all the contracts.

Other formulations should be conceivable and are left for further prospects.

2. When $\epsilon = 0$ and $\delta \neq 0$, the two solutions of the “principal-agent” game (i.e., $A = \pm 1$ with the previous notations) yield the same expected utility (i.e., R) for the agent. This remains true if we include the “unstable equilibrium” (i.e., $A = 0$ with the previous notation), which coincides somehow with the BAU regime. As for principal’s expected utility, it is given by $-R + \frac{A^2 \delta}{2}$.

If we sum up these expected utilities, we obtain a global expected utility. It is given by $R - R + \frac{A^2 \delta}{2} = \frac{A^2 \delta}{2}$. Amongst all these solutions, the less risky one for the agents is

$A = 0$, since, in that case, the reward of the agent has a zero variance. This solution should be regarded as the “free choice” solution for the agents. Still, as the principal aims at maximizing its own utility, it selects the solutions associated to $A = \pm 1$. It is then interesting to compare the ratio between the two global expected utilities when $A = 0$ and when $A = \pm 1$. Here, this ratio is equal to 0! We should call it the price of anarchy of principal’s choice over the agents free choice.

Generally speaking, the price of anarchy is the ratio between the global expected utility when the choice is given to the agent and when the choice is given to a different entity (or obtained through some “selection method”). When the price of anarchy is equal to 1, the choice is indifferent. When the price of anarchy is greater than 1, then it is better to use agents’ choice; and when it is less than 1, it is better to use principal’s choice.

In this chapter, the selection method based upon the zero noise limit selects the same solutions as those chosen by the price of anarchy. Still, we learned from the two last chapters that there could be other instances of MFG for which this is not the case and for which a paradox could exist. So, a natural question is to ask whether similar paradoxes could be cooked up within the framework of Chapter 3.

More generally, as we know that game theory is rigged with paradoxes, we could wonder about the behavior of the zero noise selection method in cases when there are several solutions with the same global expected utility, which is for instance what happens when $\delta = 0$. Of course, this would ask for additional mathematical work: For, instance it should be needed to define rigorously what is a global expected utility in the context of games with a continuum of players.

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