

Decoupled mild solutions of deterministic evolution problemswith singular or path-dependent coefficients, represented by backward SDEs

Adrien Barrasso

▶ **To cite this version:**

Adrien Barrasso. Decoupled mild solutions of deterministic evolution problemswith singular or path-dependent coefficients, represented by backward SDEs. Probability [math.PR]. Université Paris Saclay (COmUE), 2018. English. NNT : 2018SACLY009 . tel-01895487

HAL Id: tel-01895487

https://pastel.hal.science/tel-01895487

Submitted on 15 Oct 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

NNT : 2018SACLY009

THÈSE DE DOCTORAT

de

L'UNIVERSITÉ PARIS-SACLAY

École doctorale de mathématiques Hadamard (EDMH, ED 574)

Établissement d'inscription : École nationale supérieure de techniques avancées

Laboratoire d'accueil : Unité de mathématiques appliquées

Spécialité de doctorat: Mathématiques fondamentales

Adrien BARRASSO

Decoupled mild solutions of deterministic evolution problems with singular or path-dependent coefficients, represented by backward SDEs.

Date de soutenance : 17 Septembre 2018

Après avis des rapporteur.e.s :
ZHONGMIN QIAN (University of Oxford)
MARIE-CLAIRE QUENEZ (Université Paris Diderot)

Jury de soutenance :

PIERRE CARDALIAGUET	(Université Paris-Dauphine) Examinateur
ANDREA COSSO	(Università di Bologna) Coencadrant de thèse
FRANÇOIS DELARUE	(Université Nice Sophia Antipolis) Président
MARIE-CLAIRE QUENEZ	(Université Paris Diderot) Rapporteure
FRANCESCO RUSSO	(ENSTA ParisTech) Directeur de thèse
NIZAR TOUZI	(École Polytechnique) Examinateur

In girum imus nocte et consumimur igni

Contents

Introduction **1**

1 A note on time-dependent additive functionals **11**
1.1 Introduction . . . 11
1.2 Preliminaries . . . 12
1.3 Markov classes . . . 13
1.3.1 Definition and basic results . . . 13
1.3.2 Examples of canonical Markov classes . . . 16
1.4 Martingale Additive Functionals . . . 17

2 BSDEs with no driving martingale, Markov processes and associated Pseudo PDEs **31**
2.1 Introduction . . . 31
2.2 Preliminaries . . . 34
2.3 BSDEs without driving martingale . . . 35
2.4 Martingale Problem and Markov classes . . . 43
2.5 Pseudo-PDEs and associated Markovian BSDEs with no driving martingale . . . 49
2.6 Applications . . . 56

Appendices **57**
2.A Markov classes . . . 57
2.B Technicalities related to Section 2.3 . . . 59

3 BSDEs with no driving martingale, Markov processes and associated Pseudo PDEs. Part II: Decoupled mild solutions and Examples **61**
3.1 Introduction . . . 61
3.2 Preliminaries . . . 65
3.3 Decoupled mild solutions of Pseudo-PDEs . . . 69
3.3.1 Definition . . . 70
3.3.2 Existence and uniqueness of a solution . . . 71
3.3.3 Representation of the solution via Markovian BSDEs with no driving martingale 75
3.4 Examples of applications . . . 76
3.4.1 Markovian jump diffusions . . . 76
3.4.2 Pseudo-Differential operators and Fractional Laplacian . . . 77
3.4.3 Parabolic semi-linear PDEs with distributional drift . . . 82
3.4.4 Diffusion equations on differential manifolds . . . 84

Appendices **87**
3.A Markov classes . . . 87
3.B Technicalities concerning homogeneous Markov classes and martingale problems . . 90

4 Martingale driven BSDEs, PDEs and other related deterministic problems **93**
4.1 Introduction . . . 93
4.2 Preliminaries . . . 96
4.3 An alternative formulation of BSDEs driven by a cadlag martingale . . . 97
4.4 Martingale Problem and Markov classes . . . 98
4.5 Pseudo-PDEs and associated Markovian type BSDEs driven by a cadlag martingale . 102
4.5.1 The concepts . . . 102
4.5.2 Decoupled mild solutions of Pseudo-PDEs . . . 105
4.5.3 Existence and uniqueness of a decoupled mild solution . . . 110
4.6 Examples of applications . . . 111
4.6.1 A new approach to Brownian BSDEs and associate semilinear PDEs . . . 112
4.6.2 Parabolic semi-linear PDEs with distributional drift . . . 114

Appendices **117**
4.A Proof of Theorem 4.3.3 and related technicalities . . . 117
4.B Proof of Theorem 4.5.18 . . . 124
4.C Markov classes and Martingale Additive Functionals . . . 125

5 Path-dependent Martingale Problems and Additive Functionals **129**
5.1 Introduction . . . 129
5.2 Preliminaries . . . 131
5.3 Path-dependent canonical classes . . . 132
5.4 Path-dependent Additive Functionals . . . 136
5.5 Path-dependent Martingale problems . . . 139
5.5.1 Abstract Martingale Problems . . . 139
5.5.2 Martingale problem associated to an operator and weak generators . . . 143
5.6 Weak solutions of path-dependent SDEs . . . 145

Appendices **153**
5.A Proofs of Section 5.4 . . . 153

6 Decoupled mild solutions of path-dependent PDEs and IPDEs represented by BSDEs driven by cadlag martingales **159**
6.1 Introduction . . . 159
6.2 Basic vocabulary and Notations . . . 162
6.3 Fundamental tools . . . 162
6.3.1 Path-dependent canonical classes and systems of projectors . . . 162
6.3.2 Path-dependent martingale additive functionals . . . 166
6.4 BSDEs and abstract analytical problem . . . 166
6.4.1 BSDEs driven by a path-dependent MAF . . . 166
6.4.2 Decoupled mild solutions for abstract operators . . . 168
6.5 Decoupled mild solutions of path-dependent PDEs and IPDEs . . . 174
6.5.1 Path-dependent SDEs . . . 174
6.5.2 Dupire's derivatives and path-dependent stochastic calculus . . . 176
6.5.3 Decoupled mild solutions of Path-dependent (I)PDEs . . . 178

Appendices **185**
6.A Some technicalities 185

Introduction

One of the strengths of stochastic analysis has been to discover numerous and various links between stochastic processes and deterministic linear evolution equations. Everything started with the relation established by Albert Einstein in 1905, between the Brownian motion and the heat equation, see for example [41]. Today stochastic analysis has extended that link to the correspondence between different types of stochastic processes and linear deterministic problems for which a non exhaustive list is given below.

1. Solutions to Stochastic Differential Equations (SDEs) and parabolic Partial Differential Equations (PDEs);
2. jump diffusions and Integro Partial Differential Equations (IPDEs);
3. Markov processes and pseudo-differential operators;
4. SDEs and PDEs with distributional drift;
5. SDEs and PDEs in manifolds;
6. path-dependent SDEs and related (I)PDEs etc...

The dynamics in law of each stochastic problem (e.g. SDE) is guided by a family of operators which permits to describe both the evolution in time of the law of the process and the one of the solution of the deterministic problem. When the process is Markovian, those operators are constituted by transition semigroups. In the non-Markovian case, we will see that those can be replaced by a natural system of projectors.

More recently, at the beginning of the 90s, a particular family of stochastic differential equations with terminal condition was introduced. The solution of such an equation is a couple (Y, Z) of adapted processes. They contain a random coefficient f called the *driver* depending pointwise on the solution. They were called Backward Stochastic Differential Equations (BSDEs). When the random dependence of f is expressed through a *forward process* being the solution of a classical SDE (see item 1. above), they constitute the probabilistic representation of some special classes of semilinear PDEs. Later those solutions of SDEs were replaced by more general forward processes, with the idea of extending the stochastic representation of the linear equations mentioned in the above list (see items 2. to 6.) to non-linear equations. For a long time, the notion of viscosity solution was considered as the most adapted notion of solution (although others have been used) to study links between BSDEs and non-linear deterministic problems. In this thesis, we propose a new type of solution for these deterministic equations, which we call *decoupled mild solution*. It is inspired from the usual notion of mild solution and we see it as competitor to the notion of viscosity solution. We prove results of existence and uniqueness of a decoupled mild solution for various types of equations associated to all the situations mentioned above, under often very weak assumptions on the coefficients, and we give arguments in favor of that new notion of solution.

The classical notion of Brownian BSDE was introduced in 1990 by E. Pardoux and S. Peng in [71], after an early work of J.M. Bismut in 1973 in [18]. It is a stochastic differential equation with prescribed terminal condition ξ and driver $\hat{f}$

$$Y_t = \xi + \int_t^T \hat{f}(r, Y_r, Z_r)dr - \int_t^T Z_r dB_r, t \in [0, T], \tag{1}$$

where B is a Brownian motion. We insist on the fact that $\hat{f}$ is also random and in particular progressively measurable for every fixed variable y, z. As mentioned above, the unknown is a couple (Y, Z) of adapted processes. Existence and uniqueness of (1) was established first supposing essentially Lipschitz conditions on f with respect to Y, Z and square integrability-type conditions on $\xi, \hat{f}(\cdot, 0, 0)$. In the sequel those conditions were considerably relaxed, see [74] and references therein. When the randomness of the driver is expressed through a forward diffusion process X and the terminal condition only depends on X_T, the BSDE is often said to be *Markovian*. In order to characterize the link between a Markovian BSDE and a semilinear PDE, one considers the family of forward diffusions $X = X^{s,x}$, where s is the initial time and x the initial position. The corresponding Markovian BSDEs are characterized by the solution $(Y, Z) = (Y^{s,x}, Z^{s,x})$ of the system

$$\left\{ \begin{array}{rcl} X_t^{s,x} & = & x + \int_s^t \beta(r, X_r^{s,x})dr + \int_s^t \sigma(r, X_r^{s,x})dB_r \\ Y_t^{s,x} & = & g(X_T^{s,x}) + \int_t^T f\left(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x}\right) dr - \int_t^T Z_r^{s,x} dB_r, t \in [0, T]. \end{array} \right. \tag{2}$$

Seminal papers on Markovian BSDEs are [76] and [72]. There β and σ are supposed to be Lipschitz (with respect to x), $g, f(\cdot, \cdot, 0, 0)$ to have polynomial growth and f has to be Lipschitz in the variables (y, z). In [76] and in [72], (2) was linked to the semilinear PDE

$$\left\{ \begin{array}{l} \partial_t u + \frac{1}{2} \sum\limits_{i,j \leq d} (\sigma\sigma^\intercal)_{i,j} \partial^2_{x_i x_j} u + \sum\limits_{i \leq d} \beta_i \partial_{x_i} u + f(\cdot, \cdot, u, \sigma\nabla u) = 0 \quad \text{ on } [0, T[\times \mathbb{R}^d \\ u(T, \cdot) = g. \end{array} \right. \tag{3}$$

In particular, if (3) has a classical smooth solution u then $(Y^{s,x}, Z^{s,x}) := (u(\cdot, X_\cdot^{s,x}), \sigma\nabla u(\cdot, X_\cdot^{s,x}))$ solves the second line of (2). Conversely, only under the Lipschitz type conditions mentioned after (2), the solution of the BSDE can be expressed as a function of the forward process, i.e. $(Y^{s,x}, Z^{s,x}) = (u(\cdot, X_\cdot^{s,x}), v(\cdot, X_\cdot^{s,x}))$, see [43]. It was shown in [72] that if moreover f and g are continuous, then u is a viscosity solution of (3). Excepted in the case when u has some minimal differentiability properties in the second variable, see e.g. [52], it is difficult to say something more on v. The analytical identification of v is commonly called the resolution of the *identification problem*. One major contribution of this thesis consists in giving an analytical meaning to v. Since the pioneering work of [72], in the Brownian case, the relations between more general BSDEs and associated deterministic problems have been studied extensively, and innovations have been made in several directions. In [7] the authors introduced a new kind of BSDE including a term with jumps generated by a Poisson measure, where an underlying forward process X solves a jump diffusion equation with Lipschitz type conditions. They associated with it an Integro-Partial Differential Equation (in short IPDE) in which some non-local operators are added to the classical partial differential maps, and proved that, under some continuity and monotonicity conditions on the coefficients, the BSDE provides a viscosity solution of the IPDE. The monotonicity type condition is crucial in [7], indeed, when it does not hold, a counterexemple is given. Such conditions will not be necessary in our work. In Chapter 13 of [8], under some Sobolev-type conditions on the coefficients of the Brownian BSDE (2), it is shown that the function u mentioned above is a solution in the sense of distributions of the parabolic PDE (3). Later, the notion of mild solution of the PDE was used in [4] where the authors tackled diffusion operators generating symmetric Dirichlet forms and associated Markov processes thanks to the theory

of Fukushima Dirichlet forms, see e.g. [53]. Those results were extended to the case of non symmetric Markov processes in [87]. Infinite dimensional setups were considered for example in [52] where an infinite dimensional BSDE could produce the mild solution of a PDE on a Hilbert space. Concerning the study of BSDEs driven by more general processes or random fields than Brownian motion, we have already mentioned BSDEs driven by Poisson measures. In this respect, more recently, BSDEs driven by marked point processes were introduced in [24], see also [5, 6]; in that case the underlying process does not contain any diffusion term. The latter one also includes a diffusive part and it attacks the resolution of a sort of identification problem in the spirit of [52], by making use of the notion of *weak Dirichlet process*.

The BSDEs that we focus on, are situated in the extension of BSDEs driven by a cadlag martingale M. Those were considered by [20] and applied to obtain the celebrated Föllmer-Schweizer decomposition, see e.g. [83]. Instead of being a couple (Y, Z) as in the case of an underlying Brownian filtration, the solution was a triplet (Y, Z, O) where O is a martingale strongly orthogonal to M. When the driver vanishes, this constitutes the generalization of the so called *Kunita-Watanabe* decomposition. Brownian BSDEs involving a supplementary orthogonal term were studied in [43]. More recently BSDEs driven by a martingale were intensively studied by [22]. BSDEs of the same type, but with partial information have been investigated in [23]. Finally, BSDEs in a general filtered space were studied in [67]. The latter appear to be a great extension of the previously existing literature, and the BSDEs that we will consider in this work lie in the scope of this contribution. The general (not necessarily Markovian) BSDE of Pardoux-Peng (1) regained attention recently with the (re)development of path dependent stochastic calculus. Brownian BSDEs of the type

$$Y^{s,\eta} = \xi\left((B_t^{s,\eta})_{t\in[0,T]}\right) + \int_\cdot^T f\left(r, (B_t^{s,\eta})_{t\in[0,r]}, Y_r^{s,\eta}, Z_r^{s,\eta}\right) dr - \int_\cdot^T Z_r^{s,\eta} dB_r, t \in [0,T], \tag{4}$$

where for any $s \in [0,T]$, η belongs to the Skorokhod space of cadlag functions $\mathbb{D}([0,T], \mathbb{R}^d)$, $B^{s,\eta} = \eta(\cdot \wedge s) + (B_{\cdot \vee s} - B_s)$, were associated to the path-dependent semi-linear PDE

$$\begin{cases} D\Phi + \frac{1}{2} Tr(\nabla^2 \Phi) + f(\cdot, \cdot, \Phi, \nabla\Phi) = 0 \quad \text{ on } [0,T[\times\Omega \\ \Phi_T = \xi, \end{cases} \tag{5}$$

where D is the horizontal derivative and ∇ is the vertical gradient intended in the sense of [37, 27]. Path-dependent PDEs of previous type have been investigated by several methods. For instance strict (classical, regular) solutions have been studied in [35, 49, 29] under the point of view of Banach space valued stochastic processes. It was shown for instance in [29, 77] that if the coefficients are regular enough then the mapping $(s,\eta) \longmapsto Y_s^{s,\eta}$ is the unique smooth solution of (5). Another popular approach is the one of *viscosity solutions*, which was considered by several authors. For instance it was shown in [42] that if f is bounded, continuous in t , uniformly continuous in the second variable, and uniformly Lipschitz continuous in (y,z) and if ξ is bounded uniformly continuous, $(s,\eta) \longmapsto Y_s^{s,\eta}$ is a viscosity solution of (5) in some specific sense, where the sense of solutions involves the underlying probability. On another level, [28] considered the so called *strong-viscosity* solutions (based on approximation techniques), which are an analytic concept, the first under non-smoothness conditions. Another interesting approach, probabilistic, but still based on approximation (discretizations) was given by [65]. More recently, [17] produced a viscosity solution to a more general path-dependent (possibly integro)-PDE through dynamic risk measures.

We remark that in [42] and [65] for instance, the underlying *forward process* is the Brownian motion in its path-dependent formulation, i.e. the window Brownian motion. In [29, 28] (resp. [17]) the underlying forward process is a strong (resp. in law) solution of an SDE with functional dependence. In all those cases the solution Φ of (5) was associated to the process $Y^{s,\eta}$ of the solution

couple $(Y^{s,\eta}, Z^{s,\eta})$ of (4) with initial time s and initial condition η. A challenging link to be explored was the link between $Z^{s,\eta}$ and the solution of the path-dependent PDE Φ. For instance in the case when $Y^{s,\eta}$ is of the form $\Phi(t, \mathbb{X}^{s,\eta})$ where $\mathbb{X}^{s,\eta}$ is the solution of functional dependent SDE and Φ is Fréchet $C^{0,1}([0,T] \times C([-T,0])$, [50] has shown that $Z^{s,\eta}$ is closely related to the Radon measure $D_{dx}\Phi(\cdot, \mathbb{X}^{s,\eta}))$ on $x \in [-T, 0]$, evaluated at $\{0\}$. When $\mathbb{X}$ is the window Brownian motion that quantity equals the *vertical* derivative $\nabla\Phi$. This constitutes again a partial resolution of the identification problem in the path-dependent context.

Those path-dependent developments naturally lead to consider a more general path-dependent forward dynamics as the (strong or in law) solution of an SDE with path-dependent coefficients $X^{s,\eta}$ with starting time s and starting path η. Other examples of path-dependent processes are naturally non-Markovian processes as fractional Brownian motion, general Gaussian processes or solutions of Volterra type SDEs. In [86], the authors linked a BSDE which forward process is the solution of a Volterra type SDE to a PDE with Gâteaux type derivatives, involving only a finite number of directions.

Our setup is the following. E is a Polish space, $\Omega := \mathbb{D}(\mathbb{R}_+, E)$ is the Skorokhod space of cadlag functions from $\mathbb{R}_+$ to E which we equip with its Borel σ-field $\mathcal{F}$, its initial filtration $\mathbb{F}^o$ and its (right-continuous) canonical filtration $\mathbb{F}$, see Definition 1.3.1. $(X_t)_{t\in\mathbb{R}_+}$ denotes the canonical process.

We will consider various types of BSDEs with different forward processes, which are solution (in law) of some forward dynamics, in general a (Markovian or not) martingale problem. Those BSDEs will be indexed by the initial time s and point x (or path η) of the forward process. Then we will link those families of BSDEs to non-linear deterministic problems generalizing the usual semilinear PDE in the case that the forward process is a Markov diffusion. As anticipated, our contributions concern both the case of a Markovian forward process and the case of a non-Markovian (or path-dependent) one.

We start discussing our work in the Markovian framework. We consider on $(\Omega, \mathcal{F})$ a *canonical Markov class* $(\mathbb{P}^{s,x})_{(s,x)\in\mathbb{R}_+\times E}$, see Definition 1.3.7. This concept was first introduced by E.B. Dinkin, see [38]. For all (s, x), $\mathbb{P}^{s,x}$ corresponds to the law of the (Markovian) forward process starting in x at time s. That canonical Markov class is assumed to solve a well-posed martingale problem with respect to some linear operator $(\mathcal{D}(a), a)$ (see Definition 2.4.2), where $\mathcal{D}(a)$ is a linear subspace of the space of Borel real valued functions $\mathcal{B}(\mathbb{R}_+ \times E, \mathbb{R})$ defined on $\mathbb{R}_+ \times E$ and $a : \mathcal{D}(a) \longrightarrow \mathcal{B}(\mathbb{R}_+ \times E, \mathbb{R})$. We mean by this that for every (s, x), $\mathbb{P}^{s,x}$ is the unique probability measure such that for all $\phi \in \mathcal{D}(a)$, the process

$$M[\phi]^{s,x} := \phi(\cdot \vee s, X_{\cdot\vee s}) - \int_s^{\cdot\vee s} a(\phi)(X_r)dr \tag{6}$$

introduced in Notation 2.4.4 is, on $[s, +\infty[$, an $(\mathbb{F}, \mathbb{P}^{s,x})$-martingale. The transition kernel (see Definition 1.3.4) of that canonical Markov class will be denoted $(P_{s,t})_{0\le s\le t}$, which means the following: for all real valued bounded Borel ϕ and $s \le t$,

$$\mathbb{E}^{s,x}[\phi(X_t)] = P_{s,t}[\phi](x), \forall x \in E.$$

When $\phi, \psi \in \mathcal{D}(a)$ are such that $\phi\psi \in \mathcal{D}(a)$ we denote

$$\Gamma(\phi, \psi) := a(\phi\psi) - \phi a(\psi) - \psi a(\phi), \tag{7}$$

or shorter $\Gamma(\phi)$ when $\phi = \psi$. Γ is called the **carré du champ operator** and is of great importance in our work. It was first introduced (in the case of time-homogeneous operators) by J.P. Roth in potential analysis (see Chapter III in [79]), and popularized by P.A. Meyer in the study of homogeneous Markov processes, see e.g. [34] Chapter XV Comment 23 or [60] Remark 13.46. It has finally become a fundamental tool in the study of Markov processes and semi-groups, see for instance [3]. A first

approach to face deterministic problems for those equations appears in [64]; that paper also contains an application to obtention of the celebrated Föllmer-Schweizer decomposition.

The first type of BSDE which we study in Chapter 2 is the following. For every (s,x), under the probability $\mathbb{P}^{s,x}$, we consider the BSDE without driving martingale

$$Y_t^{s,x} = g(X_T) + \int_t^T f\left(r, X_r, Y_r^{s,x}, \sqrt{\frac{d\langle M^{s,x}\rangle_r}{dr}}\right) dr - (M_T^{s,x} - M_t^{s,x}), \quad t \in [0,T]. \tag{8}$$

Here the solution is the couple $(Y^{s,x}, M^{s,x})$ where $Y^{s,x}$ is a cadlag adapted square integrable process and $M^{s,x}$ is a square integrable martingale. It is a specific type of BSDE on a filtered space as introduced in [66]. Following the ideas of [43], we can show in Theorem 2.5.15 the existence of a Borel function u and some positive Borel v such that for any $(s,x) \in [0,T] \times E$, the solution of (8) verifies

$$\begin{cases} \forall t \geq s : Y_t^{s,x} = u(t, X_t) \quad \mathbb{P}^{s,x} \text{ a.s.} \\ \frac{d\langle M^{s,x}\rangle_t}{dt} = v^2(t, X_t) \quad dt \otimes d\mathbb{P}^{s,x} \text{ a.e.} \end{cases} \tag{9}$$

In order to prove this technical Theorem 2.5.15 in a very general context (in the sense that the underlying forward Markov process is very general) we extend in Chapter 1 some results concerning homogeneous (Martingale) Additive Functionnals to a time-dependent framework. The theory of (Martingale) Additive Functionnals associated to Markov processes was developped in the '60s, see for instance [38], [68], [19]. A mature version of the homogeneous theory may be found for example in [34], Chapter XV.

In Subsection 2.5 of Chapter 2, we link the BSDEs (8) to the deterministic equation

$$\begin{cases} a(u) + f\left(\cdot, \cdot, u, \Gamma(u)^{\frac{1}{2}}\right) &= 0 \quad \text{on } [0,T] \times E \\ u(T, \cdot) &= g. \end{cases} \tag{10}$$

As in the classical theory of Brownian BSDEs (i.e. the BSDEs with underlying Brownian filtration) and parabolic PDEs, the candidate function to solve (10) is naturally the function u appearing in (9) and which can be expressed as $u(s,x) \equiv Y_s^{s,x}$. For this equation, we introduce in Chapter 3, the notion of **decoupled mild solution**. We will here explain the intuition behind this notion of solution. Later on, that notion will be adapted to other types of equations, but the fundamental idea will be the same and can be sketched below.

The (time-dependent) *semigroup* $(P_{s,t})_{s \leq t}$ associated to a naturally guide the reader to a notion of *mild solution* of the deterministic problem (10). A function u for which $\Gamma(u)$ exists in some sense (classical, weak, as the closure of some operator etc...), is a mild solution of (10) if for all (s,x)

$$u(s,x) = P_{s,T}[g](x) + \int_s^T P_{s,r}[f(\cdot, \cdot, u, \Gamma(u)^{\frac{1}{2}})(r, \cdot)](x) dr. \tag{11}$$

However, in the most general setup, we can only show that the (deterministic) function $(s,x) \longmapsto Y_s^{s,x}$ is Borel, so it is not always possible to define the value of Γ applied to this function. If it were continuous, one could try to make use of a notion of viscosity solution. The concept of viscosity solution has been very popular when (10) reduces to a semilinear PDE, but also in some IPDEs with monotonicity condition and in the case of Hilbert valued equations, see e.g. [45]. Finally as mentioned earlier, a flavor of viscosity solution has been implemented even in the path-dependent case.

We propose an alternative strategy. It relies on the fact that Γ may be expressed with use of the operator a itself. Since $\Gamma(u) = a(u^2) - 2ua(u)$ one can decouple the first line of (10) into the couple of equations

$$\begin{cases} a(u) = -f(\cdot, \cdot, u, v) \\ v^2 = a(u^2) - 2ua(u), \end{cases} \tag{12}$$

where v is a positive real valued auxiliary function. We can then rewrite this as

$$\begin{cases} a(u) = -f(\cdot,\cdot,u,v) \\ a(u^2) = v^2 - 2uf(\cdot,\cdot,u,v). \end{cases} \tag{13}$$

Taking (13) into account, it is now natural to use the semigroup (mild) formulation for this system of equations. A **decoupled mild solution** of (10) will be a Borel function u for which there exists a positive Borel v such that for all (s,x),

$$\begin{cases} u(s,x) &= P_{s,T}[g](x) + \int_s^T P_{s,r}[f(\cdot,\cdot,u,v)(r,\cdot)](x)dr \\ u^2(s,x) &= P_{s,T}[g^2](x) - \int_s^T P_{s,r}[(v^2 - 2uf(\cdot,\cdot,u,v)(r,\cdot)](x)dr. \end{cases} \tag{14}$$

Our first main achievement is to prove in Theorem 3.3.9 that whenever f, g are Borel with classical growth conditions in x and if f is uniformly Lipschitz in y, z, then (10) admits a unique decoupled mild solution which is given by $(s,x) \longmapsto Y_s^{s,x}$. We also showed in Corollary 3.3.10 that a classical solution, if it exists, is a decoupled mild solution (hence is unique), and conversely that if the unique decoupled mild solution belongs to the domain, then it verifies (10) up to a zero potential set (see Definition 2.4.11). An important impact of this approach, is that the component $M^{s,x}$ of the solution $(Y^{s,x}, M^{s,x})$ of the BSDE (8) can be related to the (analytically defined) auxiliary function v appearing in (14). Indeed that function is also the one appearing in (9), meaning that we have $v^2(t, X_t) = \frac{d\langle M^{s,x}\rangle_t}{dt}$ $dt \otimes d\mathbb{P}^{s,x}$ a.e. for all (s,x).

Applications are numerous since the Markov process may be very general. Considering solutions of SDEs possibly with jumps (see [60, 84, 85]), we can tackle in Subsection 3.4.1 (with a slightly different formulation) (I)PDEs of the form

$$\begin{cases} \partial_t\phi + \frac{1}{2}Tr(\sigma\sigma^\intercal\nabla^2\phi) + \beta\nabla\phi + \int_{\mathbb{R}^d}(\phi(\cdot+\gamma(y)) - \phi - \gamma(y)\nabla\phi)F(dy) \\ +f\left(\cdot,\cdot,\phi, \left(\|\sigma\nabla\phi\|^2 + \int_{\mathbb{R}^d}(\phi(\cdot+\gamma(y)) - \phi)^2F(dy)\right)^{\frac{1}{2}}\right) = 0 \\ \phi(T,\cdot) = g, \end{cases} \tag{15}$$

where $\beta : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\sigma : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{M}_d(\mathbb{R})$, $\gamma : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ and F is a finite positive measure not charging 0. IPDE (15) becomes the PDE

$$\begin{cases} \partial_t\phi + \frac{1}{2}Tr(\sigma\sigma^\intercal\nabla^2\phi) + \beta\nabla\phi + f\left(\cdot,\cdot,\phi,\|\sigma\nabla\phi\|\right) = 0, \\ \phi(T,\cdot) = g, \end{cases} \tag{16}$$

when $\gamma \equiv 0$. Considering solutions of SDEs with distributional drift (see [47, 48, 21, 46, 31]) we can address in Subsection 3.4.3 singular non linear PDEs of the form

$$\begin{cases} \partial_t\phi + b'\partial_x\phi + \frac{1}{2}\sigma\partial_x^2\phi + f(\cdot,\cdot,\phi,h'|\sigma\partial_x\phi|) = 0 \\ \phi(T,\cdot) = g, \end{cases} \tag{17}$$

where b is only a continuous function, hence b' is a distribution, and h is an *harmonic* function in the sense that

$$Lh := b'\partial_x h + \frac{1}{2}\sigma\partial_x^2 h = 0, \tag{18}$$

by approximations, in a precise sense.

Considering Markov processes associated to pseudo-differential operators (see [57, 58, 59]) we can discuss in Subsection 3.4.2 pseudo-PDEs such as

$$\begin{cases} \partial_t\phi - (-\Delta)^{\frac{\alpha}{2}}\phi + f\left(\cdot,\cdot,\phi,\left(c_\alpha PV\int_{\mathbb{R}^d}\frac{(\phi(\cdot+y)-\phi)^2}{\|y\|^{d+\alpha}}dy\right)^{\frac{1}{2}}\right) = 0 \\ u(T,\cdot) = g, \end{cases} \tag{19}$$

where $\alpha \in]0,2[$, $(-\Delta)^{\frac{\alpha}{2}} : \phi \longmapsto c_\alpha PV \int_{\mathbb{R}^d} \frac{(\phi(\cdot+y)-\phi)}{\|y\|^{d+\alpha}} dy$ is the fractional Laplace operator, c_α is a fundamental constant and PV stands for principal value. Finally by considering diffusions in manifolds, we can treat in Subsection 3.4.4 the non-linear PDE (16) defined on compact smooth manifold.

The limit of this first group of results, is of course the unidimensionality of $\Gamma(\phi)$. To give an idea, one would like to consider a semilinear PDE, where the driver f should possibly depend on the whole vector $\nabla\phi$ and not just its norm $\|\nabla\phi\|$. This naturally brings us to consider in Chapter 4, another class of BSDEs close to the BSDEs driven by cadlag martingales studied in [22]. Let us consider $\psi_1, \cdots, \psi_d \in \mathcal{D}(a)$ and under each $\mathbb{P}^{s,x}$ the d-dimensional martingale $M[\psi]^{s,x} := (M[\psi_1]^{s,x}, \cdots, M[\psi_d]^{s,x})$ as defined in (6). $M[\psi]^{s,x}$ will be, under $\mathbb{P}^{s,x}$, the driving martingale of the BSDE indexed by (s,x). When the identity belongs to the domain, then X is under each $\mathbb{P}^{s,x}$ a special semimartingale so a natural choice for ψ is to take the identity; the driving martingale is just the martingale part of the canonical process. Under each $\mathbb{P}^{s,x}$ we consider the BSDEs driven by a cadlag martingale

$$Y_t^{s,x} = g(X_T) + \int_t^T f\left(r, X_r, Y_r^{s,x}, \frac{d\langle M^{s,x}, M[\psi]^{s,x}\rangle_r}{dr}\right) dr - (M_T^{s,x} - M_t^{s,x}), \quad t \in [0,T]. \tag{20}$$

Those BSDEs are linked to the deterministic equation

$$\left\{ \begin{array}{rcll} a(u) + f\left(\cdot,\cdot,u,\Gamma(u,\psi)\right) & = & 0 & \text{on } [0,T]\times E \\ u(T,\cdot) & = & g. & \end{array} \right. \tag{21}$$

Reasoning as for (10), we can decompose the first line of (21) into the system

$$\left\{ \begin{array}{l} a(u) = -f(\cdot,\cdot,u,v) \\ v_i = a(u\psi_i) - ua(\psi_i) - \psi_i a(u), \quad 1 \le i \le d, \end{array} \right. \tag{22}$$

and

$$\left\{ \begin{array}{l} a(u) = -f(\cdot,\cdot,u,v) \\ a(u\psi_i) = v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v), \quad 1 \le i \le d, \end{array} \right. \tag{23}$$

where this time v is an $\mathbb{R}^d$-valued auxiliary function. We define a decoupled mild solution of (21) as a Borel function u for which there exists a Borel v such that for all $(s,x) \in [0,T]\times E$,

$$\left\{ \begin{array}{rcl} u(s,x) & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(\cdot,\cdot,u,v\right)(r,\cdot)\right](x) dV_r \\ u\psi_1(s,x) & = & P_{s,T}[g\psi_1(T,\cdot)](x) - \int_s^T P_{s,r}\left[(v_1 + ua(\psi_1) - \psi_1 f\left(\cdot,\cdot,u,v\right))(r,\cdot)\right](x) dr \\ & \cdots & \\ u\psi_d(s,x) & = & P_{s,T}[g\psi_d(T,\cdot)](x) - \int_s^T P_{s,r}\left[(v_d + ua(\psi_d) - \psi_d f\left(\cdot,\cdot,u,v\right))(r,\cdot)\right](x) dr. \end{array} \right. \tag{24}$$

We show in Theorem 4.5.20 that if f, g are Borel with reasonable growth conditions on x and if f is uniformly Lipschitz in (y,z) then (21) admits a unique decoupled mild solution which is given by $(s,x) \longmapsto Y_s^{s,x}$ where this time $Y^{s,x}$ comes from the solution of (20). Again, by Proposition 4.5.17, a classical solution is a decoupled mild solution and whenever the decoupled mild solution belongs to the domain then it verifies (21) up to a zero potential set. The second item of the solution of the BSDE can now be related to v appearing in (24) by $v(t,X_t) = \frac{d\langle M^{s,x}, M[\psi]^{s,x}\rangle_t}{dt}\, dt \otimes d\mathbb{P}^{s,x}$ a.e. for all (s,x). With this extension of our first result, keeping all previous notations, we tackle in Subsection 4.6.2 singular PDEs of the form

$$\left\{ \begin{array}{l} \partial_t\phi + b'\partial_x u + \frac{1}{2}\sigma\partial_x^2 u + f(\cdot,\cdot,u,h'\sigma^2\partial_x u) = 0 \\ \phi(T,\cdot) = g, \end{array} \right. \tag{25}$$

and in Subsection 4.6.1 PDEs of the form

$$\begin{cases} \partial_t \phi + \frac{1}{2} Tr(\sigma\sigma^\intercal \nabla^2 \phi) + \beta \nabla \phi + f\left(\cdot, \cdot, \phi, \sigma\sigma^\intercal \nabla \phi\right) = 0 \\ \phi(T, \cdot) = g, \end{cases} \tag{26}$$

where h is again a function fulfilling (18). We could also tackle easily IPDEs of the form

$$\begin{cases} \partial_t \phi + \frac{1}{2} Tr(\sigma\sigma^\intercal \nabla^2 \phi) + \beta \nabla \phi + \int_{\mathbb{R}^d} (\phi(\cdot + \gamma(y)) - \phi - \gamma(y) \nabla \phi) F(dy) \\ + f\left(\cdot, \cdot, \phi, \sigma\sigma^\intercal \nabla \phi + \int_{\mathbb{R}^d} \gamma(y)(\phi(\cdot + \gamma(y)) - \phi) F(dy)\right) = 0 \\ \phi(T, \cdot) = g, \end{cases} \tag{27}$$

as we will see in the path-dependent framework.

We wish to emphasize that even in the most classical situation of PDE (26), our concepts and results provide a new light to the known literature. Indeed, firstly, we obtain an existence and uniqueness result under very low regularity assumptions on the coefficients. f, g need only be measurable in t, x and β, σ need only to ensure well-posedness of the martingale problem, for example with β Borel and σ continuous invertible, or with β, σ continuous in x at fixed t. Moreover, we provide an analytical meaning for the processes $Z^{s,x} := \frac{d\langle M^{s,x}, M[Id]^{s,x}\rangle_t}{dt}$ coming from the BSDEs, which in the diffusion setup are given by

$$Y_t^{s,x} = g(X_T) + \int_t^T f\left(r, X_r, Y_r^{s,x}, \frac{d\langle M^{s,x}, M[Id]^{s,x}\rangle_r}{dr}\right) dr - (M_T^{s,x} - M_t^{s,x}), \quad t \in [0, T], \tag{28}$$

where $M[Id]^{s,x}$ is the martingale part of the canonical process under $\mathbb{P}^{s,x}$. As stated above, the process $Z^{s,x}$ is strongly related to the function v appearing in (24) which may be interpreted as a generalized gradient of u. This identification of $Z^{s,x}$ is not possible in general when using viscosity solutions.

The next step in our work is to consider path-dependent extensions of our results. We start by extending in Chapter 5, some notions and results of Markov processes theory to the path-dependent setup, since such notions were fundamental in our work. First we introduce the notion of path-dependent canonical class, see Definition 5.3.4. It will be a set of probability measures $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ defined on the canonical space and such that for some fixed (s, η), $\mathbb{P}^{s,\eta}$ models a forward (path-dependent) dynamics in law, with imposed initial path η on the time interval $[0, s]$. More formally, this set of probability measures verifies the following.

1. For every $(s, \eta) \in \mathbb{R}_+ \times \Omega$, $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$;
2. for every $s \in \mathbb{R}_+$ and $G \in \mathcal{F}$, the mapping
$$\begin{array}{ccc} \eta & \longmapsto & \mathbb{P}^{s,\eta}(G) \\ \Omega & \longrightarrow & [0,1] \end{array}$$
is $\mathcal{F}_s^o$-measurable;
3. for every $(s, \eta) \in \mathbb{R}_+ \times \Omega$, $t \geq s$ and $G \in \mathcal{F}$,
$$\mathbb{P}^{s,\eta}(G|\mathcal{F}_t^o)(\omega) = \mathbb{P}^{t,\omega}(G) \text{ for } \mathbb{P}^{s,\eta} \text{ almost all } \omega. \tag{29}$$

It constitutes the natural adaptation to the path-dependent world of the notion of canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in\mathbb{R}_+\times E}$, where in general, $\mathbb{P}^{s,x}$ models the law of some Markov stochastic process, with imposed value x at time s. In particular, (29) is the natural extension of the Markov property $\mathbb{P}^{s,x}(G|X_t) = \mathbb{P}^{t,X_t}(G) \quad \mathbb{P}^{s,x}$ a.s. , see Proposition 1.3.4.

In substitution to the notion a Markov semigroup associated with a canonical Markov class, we introduce the concept of path-dependent system of projectors denoted $(P_s)_{s\in\mathbb{R}_+}$, see Definition 5.3.8. That is a family of operators defined on the set of bounded random variables (in short r.v.) and verifying the following three items.

1. For all $s \in \mathbb{R}_+$, the restriction of P_s to $\mathcal{B}_b^s(\Omega)$ coincides with the identity;
2. for all $s \in \mathbb{R}_+$, P_s maps $\mathcal{B}_b(\Omega)$ into $\mathcal{B}_b^s(\Omega)$;
3. for all $s, t \in \mathbb{R}_+$ with $t \geq s$, $P_s \circ P_t = P_s$,

where $\mathcal{B}_b^s(\Omega)$ denotes the set of bounded $\mathcal{F}_s^o$-measurable r.v. A one-to-one connection between them and path-dependent canonical classes is shown in Corollary 5.3.11. Indeed to any path-dependent canonical class $(\mathbb{P}^{s,x})_{(s,x)\in\mathbb{R}_+\times E}$, one can associate a unique path-dependent system of projectors $(P_s)_{s\in\mathbb{R}_+}$ such that for all bounded r.v. Z and every (s,η), we have

$$\mathbb{E}^{s,\eta}[Z] = P_s[Z](\eta). \tag{30}$$

This steers us towards the notion of **weak generator** $(\mathcal{D}(A), A)$ of $(P_s)_{s\in\mathbb{R}_+}$ which extends the notion of generator of a Markovian semigroup and will permit us to define *mild* type solutions of path-dependent equations. If $(\mathcal{D}(A), A)$ is a linear mapping in the space of $\mathbb{F}^o$-progressively measurable processes, we say that $(\mathcal{D}(A), A)$ is a weak generator of the path-dependent system of projectors $(P_s)_{s\in\mathbb{R}_+}$ if for all $\Phi \in \mathcal{D}(A)$, $(s,\eta) \in \mathbb{R}_+ \times \Omega$ and $t \geq s$, we have

$$P_s[\Phi_t](\eta) = \Phi_s(\eta) + \int_s^t P_s[A(\Phi)_r](\eta)dr, \tag{31}$$

see Definition 5.5.14. We show in Proposition 5.5.16 that $(\mathcal{D}(A), A)$ is a weak generator of the path-dependent system of projectors $(P_s)_{s\in\mathbb{R}_+}$ iff the corresponding (in the sense of (30)) path-dependent canonical class $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ solves the martingale problem associated to $(\mathcal{D}(A), A)$, see Definition 5.5.15.

As in the Markovian set-up, examples of path-dependent canonical classes arise from solutions of a (this time path-dependent) martingale problem as we explain below. Let χ be a set of cadlag processes adapted to the initial filtration $\mathbb{F}^o$. For some given $(s,\eta) \in \mathbb{R}_+ \times \Omega$, we say that a probability measure $\mathbb{P}^{s,\eta}$ on $(\Omega, \mathcal{F})$ **solves the martingale problem with respect to** χ **starting in** (s,η) if

- $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$;
- all elements of χ are on $[s, +\infty[$ $(\mathbb{P}^{s,\eta}, \mathbb{F}^o)$-martingales.

We show in Theorem 5.5.12 that merely under some well-posedness assumption, the set of solutions for varying starting times and paths $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ defines a path-dependent canonical class. This in particularly holds for weak solutions of path-dependent SDEs possibly with jumps when there is existence and uniqueness of a solution, see Theorem 5.6.7.

Once these extensions of Markovian tools are made, in Chapter 6, on a family of path-dependent problems, which naturally extend (21) and are of the type

$$\begin{cases} AY + f(\cdot,\cdot,Y,\Gamma(Y,\Psi)) = 0 \text{ on } [0,T]\times\Omega \\ Y_T = \xi \text{ on } \Omega, \end{cases} \tag{32}$$

where $(\mathcal{D}(A), A)$ is the weak generator of a path-dependent system of projectors $(P_s)_{s\in\mathbb{R}_+}$. $\Psi := (\Psi^1, \cdots, \Psi^d)$ is a given vector of elements of $\mathcal{D}(A)$ and Γ is this time the bilinear map acting on processes by

$$\Gamma(\Phi, \Phi') := A(\Phi\Phi') - \Phi A(\Phi') - \Phi' A(\Phi). \tag{33}$$

A typical example is to consider $\Psi := X$ the canonical process, and a map A given by

$$\begin{array}{rl} (A\Phi)_t(\omega) := & (D\Phi)_t(\omega) + \frac{1}{2}Tr(\sigma_t\sigma_t^\intercal(\nabla^2\Phi)_t(\omega)) + \beta_t(\omega)\cdot(\nabla\Phi)_t(\omega) \\ & + \int(\Phi_t(\omega + \gamma_t(\omega,y)\mathbb{1}_{[t,+\infty[}) - \Phi_t(\omega) - \gamma_t(\omega,y)\cdot(\nabla\Phi)_t(\omega))F(dy), \end{array} \tag{34}$$

where β, σ, γ are bounded (this time path-dependent) predictable coefficients and F is still a bounded positive measure not charging 0. In (34), D is the horizontal derivative and ∇ is the vertical gradient intended in the sense of [37, 27]. In that case, by Proposition 6.5.27 we can evaluate

$$\Gamma(X,\Phi)_t = (\sigma\sigma^\intercal\nabla\Phi)_t + \int_{\mathbb{R}^d} \gamma_t(\cdot,y)(\Phi_t(\cdot+\gamma_t(\cdot,y)\mathbb{1}_{[t,+\infty[}) - \Phi_t)F(dy). \tag{35}$$

If $\gamma \equiv 0$ then (32) becomes the path-dependent PDE

$$\begin{cases} DY + \frac{1}{2}Tr(\sigma\sigma^\intercal\nabla^2 Y) + \beta\cdot\nabla Y + f(\cdot,\cdot,Y,\sigma\sigma^\intercal\nabla Y) = 0 \text{ on } [0,T]\times\Omega \\ Y_T = \xi \text{ on } \Omega. \end{cases} \tag{36}$$

We extend to (32) the notion of decoupled mild solution defining it as a functional Y for which there exists an auxiliary $\mathbb{R}^d$-valued functional $Z := (Z^1, \cdots, Z^d)$ such that for all $(s,\eta) \in [0,T]\times\Omega$ we have

$$\begin{cases} Y_s(\eta) &= P_s[\xi](\eta) + \int_s^T P_s\left[f\left(r,\cdot,Y_r,Z_r\right)\right](\eta)dr \\ Y_s(\eta)\eta^1(s) &= P_s[\xi\Psi_T^1](\eta) - \int_s^T P_s\left[\left(Z_r^1 + Y_r A\Psi_r^1 - \Psi_r^1 f\left(r,\cdot,Y_r,Z_r\right)\right)\right](\eta)dr \\ &\cdots \\ Y_s(\eta)\eta^d(s) &= P_s[\xi\Psi_T^d](\eta) - \int_s^T P_s\left[\left(Z_r^d + Y_r A\Psi_r^d - \Psi_r^d f\left(r,\cdot,Y_r,Z_r\right)\right)\right](\eta)dr. \end{cases} \tag{37}$$

We consider for any (s,η) the BSDE

$$Y_t^{s,\eta} = \xi + \int_t^T f\left(r,\cdot,Y_r^{s,\eta},\frac{d\langle M^{s,\eta}, M[\Psi]^{s,\eta}\rangle_r}{dr}\right)dr - (M_T^{s,\eta} - M_t^{s,\eta}), \quad t\in[0,T], \tag{38}$$

under $\mathbb{P}^{s,\eta}$, where $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ solves a martingale problem associated to $(\mathcal{D}(A), A)$. In (38), $M[\Psi]^{s,\eta}$ is the driving martingale of the BSDE, and is the martingale part of the process Ψ under $\mathbb{P}^{s,\eta}$. These BSDEs are a particular case of those studied in Chapter 4, see Definition 4.3.2. Those BSDEs have this time however a *forward* component which is modeled in law by the fixed family $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$.

An important application for path-dependent (I)PDEs is Theorem 6.5.32 that states the following. Suppose that the path-dependent SDE of coefficients β, σ, γ admits existence and uniqueness in law for every initial condition (s,η), and that β_t, σ_t (resp. $\gamma_t(\cdot,x)$) are continuous for the Skorokhod topology in ω for almost all t (resp. $dt\otimes dF$ a.e.), that $f(\cdot,\cdot,0,0), \xi$ have polynomial growth and that f is Lipschitz in (y,z) uniformly in (t,ω). Then there is a unique decoupled mild solution Y for (32) with $\Psi := X$ and A given in (34). Moreover, both processes Y, Z appearing in (37) can be represented through the associated BSDEs (38). In particular we have $Y : (s,\eta) \longmapsto Y_s^{s,\eta}$ and (37) gives, as in the Markovian case, an analytical meaning to the second process Z obtained through those BSDEs. Indeed, this process may be interpreted as a generalization of the vertical derivative.

Chapter 1

A note on time-dependent additive functionals

This chapter is the object of the paper [12].

Abstract

This note develops shortly the theory of non-homogeneous additive functionals and is a useful support for the analysis of time-dependent Markov processes and related topics. It is a significant tool for the analysis of Markovian BSDEs in law. In particular we extend to a non-homogeneous setup some results concerning the quadratic variation and the angular bracket of Martingale Additive Functionals (in short MAF) associated to a homogeneous Markov processes.

1.1 Introduction

The notion of Additive Functional of a general Markov process is due to E.B Dynkin and has been studied since the early '60s by the Russian, French and American schools of probability, see for example [38], [68], [19]. A mature version of the homogeneous theory may be found for example in [34], Chapter XV. In that context, given an element x in some state space E, $\mathbb{P}^x$ denotes the law of a time-homogeneous Markov process with initial value x.

An **Additive Functional** (AF) is a right-continuous process $(A_t)_{t\geq 0}$ defined on a canonical space, adapted to the canonical filtration such that for any $s \leq t$ and $x \in E$, $A_{s+t} = A_s + A_t \circ \theta_s$ $\mathbb{P}^x$-a.s., where θ is the usual shift operator on the canonical space. If moreover A is under any law $\mathbb{P}^x$ a martingale, then it is called a **Martingale Additive Functional** (MAF). The quadratic variation and angular bracket of a MAF were shown to be AFs in [34]. We extend this type of results to a more general definition of an AF which is closer to the original notion of Additive Functional associated to a stochastic system introduced by E.B. Dynkin, see [39] for instance.

Our setup will be the following. We consider a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ with time index $[0,T]$ and state space E being a Polish space. For any $(s,x) \in [0,T]\times E$, $\mathbb{P}^{s,x}$ corresponds to the probability law (defined on some canonical filtered space $(\Omega, \mathcal{F}, \mathbb{F})$) of a Markov process starting from point x at time s. On $(\Omega, \mathcal{F})$, we define a **non-homogeneous Additive Functional** (shortened by AF) as a real-valued random-field $A := (A^t_u)_{0\leq t\leq u\leq T}$ verifying the two following conditions.

1. For any $0 \leq t \leq u \leq T$, A^t_u is $\mathcal{F}_{t,u}$-measurable;
2. for any $(s,x) \in [0,T] \times E$, there exists a real cadlag $\mathbb{F}^{s,x}$-adapted process $A^{s,x}$ (taken equal to zero on $[0,s]$ by convention) such that for any $x \in E$ and $s \leq t \leq u$, $A^t_u = A^{s,x}_u - A^{s,x}_t$ $\mathbb{P}^{s,x}$ a.s.

$\mathcal{F}_{t,u}$ denotes the σ-field generated by the canonical process between time t and u, and $\mathbb{F}^{s,x}$ is obtained by adding the $\mathbb{P}^{s,x}$ negligible sets to $\mathbb{F}$. $A^{s,x}$ will be called the **cadlag version of** A **under** $\mathbb{P}^{s,x}$. If for any (s,x), $A^{s,x}$ is a $(\mathbb{P}^{s,x},\mathbb{F})$-square integrable martingale then A will be called a square integrable Martingale Additive Functional (in short, square integrable MAF).

The main contributions of the paper are essentially the following. In Section 1.3, we recall the definition and prove some basic results concerning canonical Markov classes. In Section 1.4, we start by defining an AF in Definition 1.4.1. In Proposition 1.4.4, we show that if $(M_u^t)_{0\leq t\leq u\leq T}$ is a square integrable MAF, then there exists an AF $([M]_u^t)_{0\leq t\leq u\leq T}$ which for any $(s,x)\in[0,T]\times E$, has $[M^{s,x}]$ as cadlag version under $\mathbb{P}^{s,x}$. Corollary 1.4.11 states that given two square integrable MAFs $(M_u^t)_{0\leq t\leq u\leq T}$, $(N_u^t)_{0\leq t\leq u\leq T}$, there exists an AF, denoted by $(\langle M,N\rangle_u^t)_{0\leq t\leq u\leq T}$, which has $\langle M^{s,x},N^{s,x}\rangle$ as cadlag version under $\mathbb{P}^{s,x}$. Finally, we prove in Proposition 1.4.17 that if M or N is such that for every (s,x), its cadlag version under $\mathbb{P}^{s,x}$ has its angular bracket absolutely continuous with respect to some continuous non-decreasing function V, then there exists a Borel function v such that for any (s,x), $\langle M^{s,x},N^{s,x}\rangle=\int_s^{\cdot\vee s}v(r,X_r)dV_r$.

The present note constitutes a support for the authors, in the analysis of deterministic problems related to Markovian type backward stochastic differential equations where the forward process is given in law, see e.g. Chapters 2, 3, 4. Indeed, when the forward process of the BSDE does not define a stochastic flow (typically if it is not the strong solution of an SDE but only a weak solution), we cannot exploit the mentioned flow property to show that the solution of the BSDE is a function of the forward process, as it is usually done, see Remark 5.35 (ii) in [74] for instance.

1.2 Preliminaries

The present section is devoted to fix some basic notions, notations and vocabulary. A topological space E will always be considered as a measurable space with its Borel σ-field which shall be denoted $\mathcal{B}(E)$ and if S is another topological space equipped with its Borel σ-field, $\mathcal{B}(E,S)$ (resp. $\mathcal{B}_b(E,S)$, resp. $\mathcal{C}(E,S)$, resp. $\mathcal{C}_b(E,S)$) will denote the set of Borel (resp. bounded Borel, reps. continuous, resp. bounded continuous) functions from E to S. Let $T\in\mathbb{R}_+^*$, $d\in\mathbb{N}^*$, then $\mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d)$ will denote the space of bounded continuous real valued functions on $[0,T]\times\mathbb{R}^d$ which are differentiable in the first variable, twice differentiable in the second with bounded continuous partial derivatives.

Let $(\Omega,\mathcal{F})$, $(E,\mathcal{E})$ be two measurable spaces. A measurable mapping from $(\Omega,\mathcal{F})$ to $(E,\mathcal{E})$ shall often be called a **random variable** (with values in E), or in short r.v. If $\mathbb{T}$ is some set, an indexed set of r.v. with values in E, $(X_t)_{t\in\mathbb{T}}$ will be called a **random field** (indexed by $\mathbb{T}$ with values in E). In particular, if $\mathbb{T}$ is an interval included in $\mathbb{R}_+$, $(X_t)_{t\in\mathbb{T}}$ will be called a **stochastic process** (indexed by $\mathbb{T}$ with values in E). Given a stochastic process, if the mapping

$$\begin{array}{rcl}(t,\omega) & \longmapsto & X_t(\omega)\\ (\mathbb{T}\times\Omega,\mathcal{B}(\mathbb{T})\otimes\mathcal{F}) & \longrightarrow & (E,\mathcal{E})\end{array}$$

is measurable, then the process $(X_t)_{t\in\mathbb{T}}$ will be called a **measurable process** (indexed by $\mathbb{T}$ with values in E).

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a fixed probability space. For any $p\geq 1$, $L^p:=L^p(\mathbb{R})$ will denote the set of real valued random variables with finite p-th moment. Two random fields (or stochastic processes) $(X_t)_{t\in\mathbb{T}}$, $(Y_t)_{t\in\mathbb{T}}$ indexed by the same set and with values in the same space will be said to be **modifications (or versions) of each other** if for every $t\in\mathbb{T}$, $\mathbb{P}(X_t=Y_t)=1$. If the probability space is equipped with a right-continuous filtration $\mathbb{F}=(\mathcal{F}_t)_{t\in\mathbb{T}}$, then $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ will be called **stochastic basis** and will be said to **fulfill the usual conditions** if the probability space is complete and if $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible sets.

Concerning spaces of real valued stochastic processes on the above mentioned stochastic basis, $\mathcal{M}$ will be the space of cadlag martingales. For any $p \in [1,\infty]$ $\mathcal{H}^p$ will denote the subset of $\mathcal{M}$ of elements M such that $\sup_{t\in\mathbb{T}} |M_t| \in L^p$ and in this set we identify indistinguishable elements. $\mathcal{H}^p$ is a Banach space for the norm $\|M\|_{\mathcal{H}^p} = \mathbb{E}[|\sup_{t\in\mathbb{T}} M_t|^p]^{\frac{1}{p}}$, and $\mathcal{H}^p_0$ will denote the Banach subspace of $\mathcal{H}^p$ whose elements start at zero.

A crucial role in the present note, as well as in classical stochastic analysis is played by localization via stopping times. If $\mathbb{T} = [0,T]$ for some $T \in \mathbb{R}^*_+$, a stopping time will be intended as a random variable with values in $[0,T]\cup\{+\infty\}$ such that for any $t \in [0,T]$, $\{\tau \leq t\} \in \mathcal{F}_t$. We define a **localizing sequence of stopping times** as an increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ such that there exists $N \in \mathbb{N}$ for which $\tau_N = +\infty$. Let Y be a process and τ a stopping time, we denote Y^τ the process $t \mapsto Y_{t\wedge\tau}$ which we call **stopped process**. If $\mathcal{C}$ is a set of processes, we define its **localized class** $\mathcal{C}_{loc}$ as the set of processes Y such that there exists a localizing sequence $(\tau_n)_{n\geq 0}$ such that for every n, the stopped process Y^{τ_n} belongs to $\mathcal{C}$.

We say some words about the concept of bracket related to two processes: the square bracket and the angular bracket. They coincide if at least one of the two processes is continuous. For any $M, N \in \mathcal{M}$ $[M,N]$ denotes the **covariation** of M, N. If $M = N$, we write $[M] := [M,N]$. $[M]$ is called **quadratic variation** of M. If $M, N \in \mathcal{H}^2_{loc}$, $\langle M,N\rangle$ (or simply $\langle M\rangle$ if $M = N$) will denote their (predictable) **angular bracket**. $\mathcal{H}^2_0$ will be equipped with scalar product defined by $(M,N)_{\mathcal{H}^2} := \mathbb{E}[M_T N_T] = \mathbb{E}[\langle M,N\rangle_T]$ which makes it a Hilbert space. Two elements M, N of $\mathcal{H}^2_{0,loc}$ will be said to be **strongly orthogonal** if $\langle M,N\rangle = 0$.

If A is an adapted process with bounded variation then $Var(A)$ (resp. $Pos(A)$, $Neg(A)$) will denote its total variation (resp. positive variation, negative variation), see Proposition 3.1, chap. 1 in [61]. In particular for almost all $\omega \in \Omega$, $t \mapsto Var_t(A(\omega))$ is the total variation function of the function $t \mapsto A_t(\omega)$.

For more details concerning these notions, one may consult [78] or [61] for example.

1.3 Markov classes

We recall here some basic definitions and results concerning Markov processes. For a complete study of homogeneous Markov processes, one may consult [34], concerning non-homogeneous Markov classes, our reference was Chapter VI of [40].

1.3.1 Definition and basic results

The first definition refers to the canonical space that one can find in [60], see paragraph 12.63.

Notation 1.3.1. *In the whole section E will be a fixed Polish space (a separable completely metrizable topological space), and $\mathcal{B}(E)$ its Borel σ-field. E will be called the **state space**.*

We consider $T \in \mathbb{R}^*_+$. We denote $\Omega := \mathbb{D}(E)$ the Skorokhod space of functions from $[0,T]$ to E right-continuous with left limits and continuous at time T (e.g. cadlag). For any $t \in [0,T]$ we denote the coordinate mapping $X_t : \omega \mapsto \omega(t)$, and we introduce on Ω the σ-field $\mathcal{F} := \sigma(X_r | r \in [0,T])$.

*On the measurable space $(\Omega,\mathcal{F})$, we introduce the **canonical process***

$$X : \begin{array}{rcl} (t,\omega) & \longmapsto & \omega(t) \\ ([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F}) & \longrightarrow & (E,\mathcal{B}(E)), \end{array} \tag{1.3.1}$$

and the right-continuous filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ where $\mathcal{F}_t := \bigcap_{s\in]t,T]} \sigma(X_r | r \leq s)$ if $t < T$, and $\mathcal{F}_T := \sigma(X_r | r \in [0,T]) = \mathcal{F}$.

*$(\Omega, \mathcal{F}, \mathbb{F})$ will be called the **canonical space** (associated to T and E). For any $t \in [0,T]$ we denote $\mathcal{F}_{t,T} := \sigma(X_r | r \geq t)$, and for any $0 \leq t \leq u < T$ we will denote $\mathcal{F}_{t,u} := \bigcap_{n \geq 0} \sigma(X_r | r \in [t, u + \frac{1}{n}])$.*

Remark 1.3.2. *All the results of the present paper remain valid if Ω is the space of continuous functions from $[0,T]$ to E, and if the time index is equal to $\mathbb{R}_+$.*

We recall that since E is Polish, then $\mathbb{D}(E)$ can be equipped with a Skorokhod distance which makes it a Polish metric space (see Theorem 5.6 in Chapter 3 of [44]), and for which the Borel σ-field is $\mathcal{F}$ (see Proposition 7.1 in Chapter 3 of [44]). This in particular implies that $\mathcal{F}$ is separable, as the Borel σ-field of a separable metric space.

Remark 1.3.3. *The above σ-fields fulfill the properties below.*

1. *For any $0 \leq t \leq u < T$, $\mathcal{F}_{t,u} = \mathcal{F}_u \cap \mathcal{F}_{t,T}$;*
2. *for any $t \geq 0$, $\mathcal{F}_t \vee \mathcal{F}_{t,T} = \mathcal{F}$;*
3. *for any $(s,x) \in [0,T] \times E$, the two first items remain true when considering the $\mathbb{P}^{s,x}$-closures of all the σ-fields;*
4. *for any $t \geq 0$, $\Pi := \{F = F_t \cap F_T^t | (F_t, F_T^t) \in \mathcal{F}_t \times \mathcal{F}_{t,T}\}$ is a π-system generating $\mathcal{F}$, i.e. it is stable with respect to the intersection.*

Definition 1.3.4. *The function*

$$P : \begin{array}{ccc} (s,t,x,A) & \longmapsto & P_{s,t}(x,A) \\ [0,T]^2 \times E \times \mathcal{B}(E) & \longrightarrow & [0,1], \end{array}$$

*will be called **transition kernel** if, for any s,t in $[0,T]$, $x \in E$, $A \in \mathcal{B}(E)$, it verifies the following.*

1. *$P_{s,t}(\cdot, A)$ is Borel,*
2. *$P_{s,t}(x, \cdot)$ is a probability measure on $(E, \mathcal{B}(E))$,*
3. *if $t \leq s$ then $P_{s,t}(x,A) = \mathbb{1}_A(x)$,*
4. *if $s < t$, for any $u > t$, $\int_E P_{s,t}(x,dy) P_{t,u}(y,A) = P_{s,u}(x,A)$.*

The latter statement is the well-known **Chapman-Kolmogorov equation**.

Definition 1.3.5. *A transition kernel P is said to be **measurable in time** if for every $t \in [0,T]$ and $A \in \mathcal{B}(E)$, $(s,x) \longmapsto P_{s,t}(x,A)$ is Borel.*

Remark 1.3.6. *Let P be a transition kernel which is measurable in time, let $\phi \in \mathcal{B}(E, \mathbb{R})$ and $t \in [0,T]$. Assume that for any $(s,x) \in [0,T] \times E$, the integral $\int |\phi|(y) P_{s,t}(x,dy)$ exists and it is finite. Then the mapping $(s,x) \mapsto \int \phi(y) P_{s,t}(x,dy)$ is Borel. This can be easily shown by approximating ϕ by simple functions and using the definition.*

Definition 1.3.7. *A **canonical Markov class** associated to a transition kernel P is a set of probability measures $(\mathbb{P}^{s,x})_{(s,x) \in [0,T] \times E}$ defined on the measurable space $(\Omega, \mathcal{F})$ and verifying for any $t \in [0,T]$ and $A \in \mathcal{B}(E)$*

$$\mathbb{P}^{s,x}(X_t \in A) = P_{s,t}(x,A), \tag{1.3.2}$$

and for any $s \leq t \leq u$

$$\mathbb{P}^{s,x}(X_u \in A | \mathcal{F}_t) = P_{t,u}(X_t, A) \quad \mathbb{P}^{s,x} \text{ a.s.} \tag{1.3.3}$$

The statement below comes Formula 1.7 in Chapter 6 of [40].

Proposition 1.3.8. *For any $(s,x) \in [0,T] \times E$, $t \geq s$ and $F \in \mathcal{F}_{t,T}$ yields*

$$\mathbb{P}^{s,x}(F|\mathcal{F}_t) = \mathbb{P}^{t,X_t}(F) = \mathbb{P}^{s,x}(F|X_t) \ \ \mathbb{P}^{s,x}a.s. \tag{1.3.4}$$

Property (1.3.4) is often called **Markov property**. We recall here the concept of homogeneous canonical Markov classes and its links with Markov classes.

Notation 1.3.9. *A mapping*

$$\tilde{P} : \begin{array}{ccc} E \times [0,T] \times \mathcal{B}(E) & \longrightarrow & [0,1] \\ (t,x,A) & \longmapsto & \tilde{P}_t(x,A), \end{array} \tag{1.3.5}$$

*will be called a **homogeneous transition kernel** if*
$P : (s,t,x,A) \longmapsto \tilde{P}_{t-s}(x,A)\mathbb{1}_{s<t} + \mathbb{1}_A(x)\mathbb{1}_{s\geq t}$ is a transition kernel in the sense of Definition 1.3.4. This in particular implies $\tilde{P} = P_{0,\cdot}(\cdot,\cdot)$.

*A set of probability measures $(\mathbb{P}^x)_{x\in E}$ on the canonical space associated to T and E (see Notation 1.3.1) will be called a **homogeneous canonical Markov class** associated to a homogeneous transition kernel $\tilde{P}$ if*

$$\left\{ \begin{array}{ll} \forall t \in [0,T] \quad \forall A \in \mathcal{B}(E) & , \mathbb{P}^x(X_t \in A) = \tilde{P}_t(x,A) \\ \forall 0 \leq t \leq u \leq T & , \mathbb{P}^x(X_u \in A|\mathcal{F}_t) = \tilde{P}_{u-t}(X_t,A) \quad \mathbb{P}^{s,x}a.s. \end{array} \right. \tag{1.3.6}$$

Given a homogeneous canonical Markov class $(\mathbb{P}^x)_{x\in E}$ associated to a homogeneous transition kernel $\tilde{P}$, one can always consider the canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to the transition kernel $P : (s,x,t,A) \longmapsto \tilde{P}_{t-s}(x,A)\mathbb{1}_{s<t} + \mathbb{1}_A(x)\mathbb{1}_{s\geq t}$. In particular, for any $x \in E$, we have $\mathbb{P}^{0,x} = \mathbb{P}^x$.

For the rest of this section, we are given a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ whose transition kernel is measurable in time. Proposition 3.A.10 in Chapter 3 shows the following.

Proposition 1.3.10. *For any event $F \in \mathcal{F}$, $(s,x) \longmapsto \mathbb{P}^{s,x}(F)$ is Borel. For any random variable Z, if the function $(s,x) \longmapsto \mathbb{E}^{s,x}[Z]$ is well-defined (with possible values in $[-\infty,\infty]$), then it is Borel.*

Definition 1.3.11. *For any $(s,x) \in [0,T] \times E$ we will consider the (s,x)-**completion** $\left(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x} := (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ of the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{s,x})$ by defining $\mathcal{F}^{s,x}$ as the $\mathbb{P}^{s,x}$-completion of $\mathcal{F}$, by extending $\mathbb{P}^{s,x}$ to $\mathcal{F}^{s,x}$ and finally by defining $\mathcal{F}_t^{s,x}$ as the $\mathbb{P}^{s,x}$-closure of $\mathcal{F}_t$, for every $t \in [0,T]$.*

We remark that, for any $(s,x) \in [0,T] \times E$, $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$ is a stochastic basis fulfilling the usual conditions, see 1.4 in [61] Chapter I.

We recall the following simple consequence of Remark 32 in [32] Chapter II.

Proposition 1.3.12. *Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, $\mathbb{P}$ a probability on $(\Omega,\mathcal{F})$ and $\mathcal{G}^{\mathbb{P}}$ the $\mathbb{P}$-closure of $\mathcal{G}$. Let $Z^{\mathbb{P}}$ be a real $\mathcal{G}^{\mathbb{P}}$-measurable random variable. There exists a $\mathcal{G}$-measurable random variable Z such that $Z = Z^{\mathbb{P}}$ $\mathbb{P}$-a.s.*

From this we can deduce the following.

Proposition 1.3.13. *Let $(s,x) \in [0,T]\times E$ be fixed, Z be a random variable and $t \in [s,T]$. Then $\mathbb{E}^{s,x}[Z|\mathcal{F}_t] = \mathbb{E}^{s,x}[Z|\mathcal{F}_t^{s,x}]$ $\mathbb{P}^{s,x}$ a.s.*

Proof. $\mathbb{E}^{s,x}[Z|\mathcal{F}_t]$ is $\mathcal{F}_t$-measurable and therefore $\mathcal{F}_t^{s,x}$-measurable. Moreover, let $G^{s,x} \in \mathcal{F}_t^{s,x}$, by Remark 32 in [32] Chapter II, there exists $G \in \mathcal{F}_t$ such that
$\mathbb{P}^{s,x}(G \cup G^{s,x}) = \mathbb{P}^{s,x}(G \backslash G^{s,x})$ implying $\mathbb{1}_G = \mathbb{1}_{G^{s,x}}$ $\mathbb{P}^{s,x}$ a.s. So

$$\begin{aligned}\mathbb{E}^{s,x}\left[\mathbb{1}_{G^{s,x}}\mathbb{E}^{s,x}[Z|\mathcal{F}_t]\right] &= \mathbb{E}^{s,x}\left[\mathbb{1}_G\mathbb{E}^{s,x}[Z|\mathcal{F}_t]\right] \\ &= \mathbb{E}^{s,x}\left[\mathbb{1}_G Z\right] \\ &= \mathbb{E}^{s,x}\left[\mathbb{1}_{G^{s,x}} Z\right],\end{aligned}$$

where the second equality occurs because of the definition of $\mathbb{E}^{s,x}[Z|\mathcal{F}_t]$. □

In particular, under the probability $\mathbb{P}^{s,x}$, $\mathbb{F}$-martingales and
$\mathbb{F}^{s,x}$-martingales coincide.

We now show that in our setup, a canonical Markov class verifies the **Blumenthal 0-1 law** in the following sense.

Proposition 1.3.14. *Let $(s,x) \in [0,T] \times E$ and $F \in \mathcal{F}_{s,s}$. Then $\mathbb{P}^{s,x}(F)$ is equal to 1 or to 0; In other words, $\mathcal{F}_{s,s}$ is $\mathbb{P}^{s,x}$-trivial.*

Proof. Let $F \in \mathcal{F}_{s,s}$ as introduced in Notation 1.3.1. Since by Remark 1.3.3, $\mathcal{F}_{s,s} = \mathcal{F}_s \cap \mathcal{F}_{s,T}$, then F belongs to $\mathcal{F}_s$ so by conditioning we get

$$\begin{aligned}\mathbb{E}^{s,x}[\mathbb{1}_F] &= \mathbb{E}^{s,x}[\mathbb{1}_F\mathbb{1}_F] \\ &= \mathbb{E}^{s,x}[\mathbb{1}_F\mathbb{E}^{s,x}[\mathbb{1}_F|\mathcal{F}_s]] \\ &= \mathbb{E}^{s,x}[\mathbb{1}_F\mathbb{E}^{s,X_s}[\mathbb{1}_F]],\end{aligned}$$

where the latter equality comes from (1.3.4) because $F \in \mathcal{F}_{s,T}$. But $X_s = x$, $\mathbb{P}^{s,x}$ a.s., so

$$\begin{aligned}\mathbb{E}^{s,x}[\mathbb{1}_F] &= \mathbb{E}^{s,x}[\mathbb{1}_F\mathbb{E}^{s,x}[\mathbb{1}_F]] \\ &= \mathbb{E}^{s,x}[\mathbb{1}_F]^2.\end{aligned}$$

□

1.3.2 Examples of canonical Markov classes

We will list here some well-known examples of canonical Markov classes and some more recent ones.

- Let $E := \mathbb{R}^d$ for some $d \in \mathbb{N}^*$. We are given $\beta \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R}^d)$, $\alpha \in \mathcal{C}_b(\mathbb{R}_+ \times \mathbb{R}^d, S_+^*(\mathbb{R}^d))$ (where $S_+^*(\mathbb{R}^d)$ is the space of symmetric strictly positive definite matrices of size d) and K a Lévy kernel (this means that for every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $K(t,x,\cdot)$ is a σ-finite measure on $\mathbb{R}^d\backslash\{0\}$, $\sup_{t,x} \int \frac{\|y\|^2}{1+\|y\|^2} K(t,x,dy) < \infty$ and for every Borel set $A \in \mathcal{B}(\mathbb{R}^d\backslash\{0\})$, $(t,x) \longmapsto \int_A \frac{\|y\|^2}{1+\|y\|^2} K(t,x,dy)$ is Borel) such that for any $A \in \mathcal{B}(\mathbb{R}^d\backslash\{0\})$, $(t,x) \longmapsto \int_A \frac{y}{1+\|y\|^2} K(t,x,dy)$ is bounded continuous.

  Let a denote the operator defined on some $\phi \in \mathcal{C}_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ by

$$\partial_t\phi + \frac{1}{2}Tr(\alpha\nabla^2\phi) + (\beta, \nabla\phi) + \int \left(\phi(\cdot,\cdot+y) - \phi - \frac{(y,\nabla\phi)}{1+\|y\|^2}\right) K(\cdot,\cdot,dy) \tag{1.3.7}$$

  In [84] (see Theorem 4.3 and the penultimate sentence of its proof), the following is shown. For every $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, there exists a unique probability $\mathbb{P}^{s,x}$ on the canonical space (see Definition 1.3.1) such that $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r)dr$ is a local martingale for every $\phi \in \mathcal{C}_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$ and $\mathbb{P}^{s,x}(X_s = x) = 1$. Moreover $(\mathbb{P}^{s,x})_{(s,x)\in\mathbb{R}_+\times\mathbb{R}^d}$ defines a canonical Markov class and its transition kernel is measurable in time.

- The case $K = 0$ was studied extensively in the celebrated book [85] in which it is also shown that if β, α are bounded and continuous in the second variable, then there exists a canonical Markov class with transition kernel measurable in time $(\mathbb{P}^{s,x})_{(s,x)\in\mathbb{R}_+\times\mathbb{R}^d}$ such that $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r)dr$ is a local martingale for any $\phi \in \mathcal{C}_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$.
- In [80], a canonical Markov class whose transition kernel is the weak fundamental solution of a parabolic PDE in divergence form is exhibited.
- In [55], diffusions on manifolds are studied and shown to define canonical Markov classes.
- Solutions of PDEs with distributional drift are exhibited in [47] and shown to define canonical Markov classes.

Some of previous examples were only studied as homogeneous Markov processes but can easily be shown to fall in the non-homogeneous setup of the present paper as it will be illustrated in Chapter 3.

1.4 Martingale Additive Functionals

We now introduce the notion of non-homogeneous Additive Functional that we use in the paper. This looks to be a good compromise between the notion of Additive Functional associated to a stochastic system introduced by E.B. Dynkin (see for example [39]) and the more popular notion of homogeneous Additive Functional studied extensively, for instance by C. Dellacherie and P.A. Meyer in [34] Chapter XV. This section consists in extending some essential results stated in [34] Chapter XV to our setup.

Our framework is still the canonical space introduced at Notation 1.3.1. In particular X is the canonical process.

Definition 1.4.1. *We denote $\Delta := \{(t,u) \in [0,T]^2 | t \leq u\}$. On $(\Omega, \mathcal{F})$, we define a **non-homogeneous Additive Functional** (shortened AF) as a random-field $A := (A_u^t)_{(t,u)\in\Delta}$ indexed by Δ with values in $\mathbb{R}$, verifying the two following conditions.*

1. *For any $(t,u) \in \Delta$, A_u^t is $\mathcal{F}_{t,u}$-measurable;*
2. *for any $(s,x) \in [0,T] \times E$, there exists a real cadlag $\mathcal{F}^{s,x}$-adapted process $A^{s,x}$ (taken equal to zero on $[0,s]$ by convention) such that for any $x \in E$ and $s \leq t \leq u$, $A_u^t = A_u^{s,x} - A_t^{s,x}$ $\mathbb{P}^{s,x}$ a.s.*

*$A^{s,x}$ will be called the **cadlag version of** A **under** $\mathbb{P}^{s,x}$.*

*An AF will be called a **non-homogeneous square integrable Martingale Additive Functional** (shortened square integrable MAF) if under any $\mathbb{P}^{s,x}$ its cadlag version is a square integrable martingale. More generally an AF will be said to verify a certain property (being non-negative, increasing, of bounded variation, square integrable, having L^1-terminal value) if under any $\mathbb{P}^{s,x}$ its cadlag version verifies it.*

*Finally, given an increasing AF A and an increasing function V, A will be said to be **absolutely continuous with respect to** V if for any $(s,x) \in [0,T] \times E$, $dA^{s,x} \ll dV$ in the sense of stochastic measures.*

Remark 1.4.2. *Let $(\mathbb{P}^x)_{x\in E}$ be a homogeneous canonical Markov class (see Notation 1.3.9). We recall that in the classical literature (see Definition 3 of [34] for instance), an adapted right-continuous process A on the canonical space is called an Additive Functional if for all $0 \leq t \leq u \leq T$ and $x \in E$*

$$A_u = A_t + A_{u-t} \circ \theta_t \quad \mathbb{P}^x \text{ a.s.}, \tag{1.4.1}$$

where $\theta_t : \omega \mapsto \omega\left((t+\cdot)\wedge T\right)$ denotes the shift operator at time t.

Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be the canonical Markov class related to $(\mathbb{P}^x)_{x\in E}$ in the sense of Notation 1.3.9. If for every $0 \leq t \leq u \leq T$, Equation (1.4.1) holds for all ω, then the random field $(t,u) \longmapsto A_u - A_t$ is a non-homogeneous Additive Functional in the sense of Definition 1.4.1.

Example 1.4.3. *Let $\phi \in \mathcal{C}([0,T]\times E,\mathbb{R})$, $\psi \in \mathcal{B}_b([0,T]\times E,\mathbb{R})$ and $V : [0,T] \longmapsto \mathbb{R}$ be right-continuous and non-decreasing function. Then the random field A given by*

$$A_u^t := \phi(u,X_u) - \phi(t,X_t) - \int_t^u \psi(r,X_r)dV_r, \tag{1.4.2}$$

defines a non-homogeneous Additive Functional. Its cadlag version under $\mathbb{P}^{s,x}$ may be given by

$$A^{s,x} = \phi(\cdot\vee s, X_{\cdot\vee s}) - \phi(s,x) - \int_s^{\cdot\vee s} \psi(r,X_r)dV_r. \tag{1.4.3}$$

We now adopt the setup of the first item of Section 1.3.2. We consider some $\phi \in \mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d)$, then the random field M given by

$$M_u^t := \phi(u,X_u) - \phi(t,X_t) - \int_t^u a(\phi)(r,X_r)dr, \tag{1.4.4}$$

defines a square integrable MAF with cadlag version under $\mathbb{P}^{s,x}$ given by

$$M^{s,x} = \phi(\cdot\vee s, X_{\cdot\vee s}) - \phi(s,x) - \int_s^{\cdot\vee s} a(\phi)(r,X_r)dr. \tag{1.4.5}$$

In this section for a given MAF $(M_u^t)_{(t,u)\in\Delta}$ we will be able to exhibit two AF, denoted respectively by $([M]_u^t)_{(t,u)\in\Delta}$ and $(\langle M\rangle_u^t)_{(t,u)\in\Delta}$, which will play respectively the role of a quadratic variation and an angular bracket of it. Moreover we will show that the Radon-Nikodym derivative of the mentioned angular bracket of a MAF with respect to our reference function V is a time-dependent function of the underlying process.

Proposition 1.4.4. *Let $(M_u^t)_{(t,u)\in\Delta}$ be a square integrable MAF, and for any $(s,x)\in[0,T]\times E$, $[M^{s,x}]$ be the quadratic variation of its cadlag version $M^{s,x}$ under $\mathbb{P}^{s,x}$. Then there exists an AF which we will call $([M]_u^t)_{(t,u)\in\Delta}$ and which, for any $(s,x)\in[0,T]\times E$, has $[M^{s,x}]$ as cadlag version under $\mathbb{P}^{s,x}$.*

Proof. We adapt Theorem 16 Chapter XV in [34] to a non homogeneous set-up but the reader must keep in mind that our definition of Additive Functional is different from the one related to the homogeneous case.

For the whole proof $t < u$ will be fixed. We consider a sequence of subdivisions of $[t,u]$: $t = t_1^k < t_2^k < \cdots < t_k^k = u$ such that $\min\limits_{i<k}(t_{i+1}^k - t_i^k) \underset{k\to\infty}{\longrightarrow} 0$. Let $(s,x)\in[0,t]\times E$ with corresponding probability $\mathbb{P}^{s,x}$. For any k, we have $\sum\limits_{i<k}\left(M_{t_{i+1}^k}^{t_i^k}\right)^2 = \sum\limits_{i<k}(M_{t_{i+1}^k}^{s,x} - M_{t_i^k}^{s,x})^2$ $\mathbb{P}^{s,x}$ a.s., so by definition of quadratic variation we know that

$$\sum_{i<k}\left(M_{t_{i+1}^k}^{t_i^k}\right)^2 \underset{k\to\infty}{\overset{\mathbb{P}^{s,x}}{\longrightarrow}} [M^{s,x}]_u - [M^{s,x}]_t. \tag{1.4.6}$$

In the sequel we will construct an $\mathcal{F}_{t,u}$-measurable random variable $[M]_u^t$ such that for any $(s,x)\in[0,t]\times E$, $\sum_{i\leq k}\left(M_{t_{i+1}^k}^{t_i^k}\right)^2 \underset{k\to\infty}{\overset{\mathbb{P}^{s,x}}{\longrightarrow}} [M]_u^t$. In that case $[M]_u^t$ will then be $\mathbb{P}^{s,x}$ a.s. equal to $[M^{s,x}]_u - [M^{s,x}]_t$.

Let $x \in E$. Since M is a MAF, for any k, $\sum_{i<k} \left(M_{t_{i+1}^k}^{t_i^k} \right)^2$ is $\mathcal{F}_{t,u}$-measurable and therefore $\mathcal{F}_{t,u}^{t,x}$-measurable. Since $\mathcal{F}_{t,u}^{t,x}$ is complete, the limit in probability of this sequence, $[M^{t,x}]_u - [M^{t,x}]_t$, is still $\mathcal{F}_{t,u}^{t,x}$-measurable. By Proposition 1.3.12, there is an $\mathcal{F}_{t,u}$-measurable variable which depends on (t,x), that we call $a_t(x,\omega)$ such that

$$a_t(x,\omega) = [M^{t,x}]_u - [M^{t,x}]_t, \mathbb{P}^{t,x} \text{ a.s.} \tag{1.4.7}$$

We will show below that there is a jointly measurable version of $(x,\omega) \mapsto a_t(x,\omega)$. For every integer $n \geq 0$, we set $a_t^n(x,\omega) := n \wedge a_t(x,\omega)$ which is in particular limit in probability of $n \wedge \sum_{i \leq k} \left(M_{t_{i+1}^k}^{t_i^k} \right)^2$ under $\mathbb{P}^{t,x}$. For any integers k, n and any $x \in E$, we define the finite positive measures $\mathbb{Q}^{k,n,x}$, $\mathbb{Q}^{n,x}$ and $\mathbb{Q}^x$ on $(\Omega, \mathcal{F}_{t,u})$ by

1. $\mathbb{Q}^{k,n,x}(F) := \mathbb{E}^{t,x}\left[\mathbb{1}_F \left(n \wedge \sum_{i<k} \left(M_{t_{i+1}^k}^{t_i^k} \right)^2 \right) \right]$;
2. $\mathbb{Q}^{n,x}(F) := \mathbb{E}^{t,x}[\mathbb{1}_F \, (a_t^n(x,\omega))]$;
3. $\mathbb{Q}^x(F) := \mathbb{E}^{t,x}[\mathbb{1}_F \, (a_t(x,\omega))]$.

When k and n are fixed, for any fixed F, by Proposition 1.3.10,
$x \longmapsto \mathbb{E}^{t,x}\left[F \left(n \wedge \sum_{i<k} \left(M_{t_{i+1}^k}^{t_i^k} \right)^2 \right) \right]$, is Borel.

Then $n \wedge \sum_{i<k} \left(M_{t_{i+1}^k}^{t_i^k} \right)^2 \underset{k\to\infty}{\overset{\mathbb{P}^{t,x}}{\longrightarrow}} a_t^n(x,\omega)$, and this sequence is uniformly bounded by the constant n, so the convergence takes place in L^1, therefore $x \longmapsto \mathbb{Q}^{n,x}(F)$ is also Borel as the pointwise limit in k of the functions $x \longmapsto \mathbb{Q}^{k,n,x}(F)$. Similarly, $a_t^n(x,\omega) \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} a_t(x,\omega)$ and is non-decreasing, so by monotone convergence theorem, being a pointwise limit in n of the functions $x \longmapsto \mathbb{Q}^{n,x}(F)$, the function $x \longmapsto \mathbb{Q}^x(F)$ is Borel. We recall that $\mathcal{F}$ is separable.

The just two mentioned properties and the fact that, for any x, we also have (by item 3. above) $\mathbb{Q}^x \ll \mathbb{P}^{t,x}$, allows to show (see Theorem 58 Chapter V in [33]) the existence of a jointly measurable (for $\mathcal{B}(E) \otimes \mathcal{F}_{t,u}$) version of $(x,\omega) \mapsto a_t(x,\omega)$, that we recall to be densities of $\mathbb{Q}^x$ with respect to $\mathbb{P}^{t,x}$. That version will still be denoted by the same symbol.

We can now set $[M]_u^t(\omega) = a_t(X_t(\omega),\omega)$, which is a correctly defined $\mathcal{F}_{t,u}$-measurable random variable. For any x, since $\mathbb{P}^{t,x}(X_t = x) = 1$, we have the equalities

$$[M]_u^t = a_t(x,\cdot) = [M^{t,x}]_u - [M^{t,x}]_t \; \mathbb{P}^{t,x}\text{a.s.} \tag{1.4.8}$$

We will moreover prove that

$$[M]_u^t = [M^{s,x}]_u - [M^{s,x}]_t \;\; \mathbb{P}^{s,x} \text{ a.s.}, \tag{1.4.9}$$

holds for every $(s,x) \in [0,t] \times E$, and not just in the case $s = t$ that we have just established in (1.4.8).

Let us fix $s < t$ and $x \in E$. We show that under any $\mathbb{P}^{s,x}$, $[M]_u^t$ is the limit in probability of $\sum_{i<k} \left(M_{t_{i+1}^k}^{t_i^k} \right)^2$. Indeed, let $\epsilon > 0$: the event

$\left\{\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\right\}$ belongs to $\mathcal{F}_{t,T}$ so by conditioning and using the Markov property (1.3.4) we have

$$\begin{aligned}
&\mathbb{P}^{s,x}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\right)\\
=\;&\mathbb{E}^{s,x}\left[\mathbb{P}^{s,x}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\middle|\mathcal{F}_t\right)\right]\\
=\;&\mathbb{E}^{s,x}\left[\mathbb{P}^{t,X_t}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\right)\right].
\end{aligned}$$

For any fixed y, by (1.4.6) and (1.4.8), $\mathbb{P}^{t,y}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\right)$ tends to zero when k goes to infinity, for every realization ω, it yields that $\mathbb{P}^{t,X_t}\left(\left|\sum_{i<k}\left(M^{t_i^k}_{t_{i+1}^k}\right)^2-[M]^t_u\right|>\epsilon\right)$ tends to zero when k goes to infinity. Since this sequence is dominated by the constant 1, that convergence still holds under the expectation with respect to the probability the probability $\mathbb{P}^{s,x}$, thanks to the dominated convergence theorem.

So we have built an $\mathcal{F}_{t,u}$-measurable variable $[M]^t_u$ such that under any $\mathbb{P}^{s,x}$ with $s\le t$, $[M^{s,x}]_u-[M^{s,x}]_t=[M]^t_u$ a.s. and this concludes the proof. □

We will now extend the result about quadratic variation to the angular bracket of MAFs. The next result can be seen as an extension of Theorem 15 Chapter XV in [34] to a non-homogeneous context.

Proposition 1.4.5. *Let $(B^t_u)_{(t,u)\in\Delta}$ be an increasing AF with L^1-terminal value, for any $(s,x)\in[0,T]\times E$, let $B^{s,x}$ be its cadlag version under $\mathbb{P}^{s,x}$ and let $A^{s,x}$ be the predictable dual projection of $B^{s,x}$ in $(\Omega,\mathcal{F}^{s,x},\mathbb{F}^{s,x},\mathbb{P}^{s,x})$. Then there exists an increasing AF with L^1 terminal value $(A^t_u)_{(t,u)\in\Delta}$ such that under any $\mathbb{P}^{s,x}$, the cadlag version of A is $A^{s,x}$.*

Proof. The first half of the demonstration will consist in showing that

$$\forall(s,x)\in[0,t]\times E,\ (A^{s,x}_u-A^{s,x}_t)\text{ is }\mathcal{F}^{s,x}_{t,u}-\text{measurable}. \tag{1.4.10}$$

We start by recalling a property of the predictable dual projection which we will have to extend slightly. Let us fix (s,x) and the corresponding stochastic basis $(\Omega,\mathcal{F}^{s,x},\mathbb{F}^{s,x},\mathbb{P}^{s,x})$. For any $F\in\mathcal{F}^{s,x}$, let $N^{s,x,F}$ be the cadlag version of the martingale, $r\longmapsto\mathbb{E}^{s,x}[\mathbb{1}_F|\mathcal{F}_r]$. Then for any $0\le t\le u\le T$, the predictable projection of the process $r\mapsto\mathbb{1}_F\mathbb{1}_{[t,u[}(r)$ is $r\mapsto N^{s,x,F}_{r^-}\mathbb{1}_{[t,u[}(r)$, see the proof of Theorem 43 Chapter VI in [33]. Therefore by definition of the dual predictable projection (see Definition 73 Chapter VI in [33]) we have

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F(A^{s,x}_u-A^{s,x}_t)\right]=\mathbb{E}^{s,x}\left[\int_t^u N^{s,x,F}_{r^-}dB^{s,x}_r\right], \tag{1.4.11}$$

for any $F\in\mathcal{F}^{s,x}$.

We will now prove some technical lemmas which in a sense extend this property, and will permit us to operate with a good common version of the random variable $\int_t^u N^{s,x,F}_{r^-}dB^{s,x}_r$ not depending on (s,x).

For the rest of the proof, $0\le t<u\le T$ will be fixed.

Notation 1.4.6. *Let $F\in\mathcal{F}_{t,T}$. We denote for any $r\in[t,T],\omega\in\Omega$, $N^F_r(\omega):=\mathbb{P}^{t,X_t(\omega)}(F)$.*

It is clear that N^F previously introduced is an $(\mathcal{F}_{t,r})_{r\in[t,T]}$-adapted process which does not depend on (s,x), which takes values in $[0,1]$ for all r,ω and by Proposition 1.3.8, for any $(s,x)\in[0,t]\times E$, $N^{s,x,F}$ is, on $[t,T]$, a $\mathbb{P}^{s,x}$-version of N^F.

Lemma 1.4.7. *Let $F\in\mathcal{F}_{t,T}$. There exists an $\mathcal{F}_{t,u}$-measurable random variable which we will denote $\int_t^u N^F_{r^-}dB_r$ such that for any $(s,x)\in[0,t]\times E$,*
$\int_t^u N^F_{r^-}dB_r=\int_t^u N^{s,x,F}_{r^-}dB^{s,x}_r\ \mathbb{P}^{s,x}$ a.s.

Remark 1.4.8. *By definition, the process N^F introduced in Notation 1.4.6 and the r.v. $\int_t^u N^F_{r^-}dB_r$ will not depend on any (s,x).*

Proof. In some sense we wish to integrate $r\mapsto N^F_{r^-}$ against B^t for fixed ω. However first we do not know a priori if the paths $r\mapsto N^F_r$ and $r\mapsto B^t_r$ are measurable, second $r\mapsto N^F_r$ may not have a left limit and B^t may be not of bounded variation. So it is not clear if $\int_t^u N^F_{r^-}dB^t_r$ makes sense for any ω. Moreover under a certain $\mathbb{P}^{s,x}$, $N^{F,s,x}$ and $B^{s,x}_\cdot-B^{s,x}_t$ are only versions of N^F and B^t and not indistinguishable to them. Even if we could compute the aforementioned integral, it would not be clear if $\int_t^u N^F_{r^-}dB^t_r=\int_t^u N^{s,x,F}_{r^-}dB^{s,x}_r\ \mathbb{P}^{s,x}$ a.s.

We start by some considerations about B, setting $W_{tu}:=\{\omega:\sup_{r\in[t,u]\cap\mathbb{Q}}B^t_r<\infty\}$ which is $\mathcal{F}_{t,u}$-measurable, and for $r\in[t,u]$

$$\bar{B}^t_r(\omega):=\begin{cases}\sup\limits_{\substack{t\le v<r\\ v\in\mathbb{Q}}}B^t_v(\omega)\text{ if }\omega\in W_{tu}\\ 0\text{ otherwise.}\end{cases}$$

$\bar{B}^t$ is an increasing, finite (for all ω) process. In general, it is neither a measurable nor an adapted process; however for any $r\in[t,u]$, $\bar{B}^t_r$ is still $\mathcal{F}_{t,u}$-measurable. Since it is increasing, it has right and left limits at each point for every ω, so we can define the process $\tilde{B}^t$ indexed on $[t,u]$ below:

$$\tilde{B}^t_r:=\lim_{\substack{v\downarrow r\\ v\in\mathbb{Q}}}\bar{B}^t_v,r\in[t,u],\tag{1.4.12}$$

when $u\in]t,T[$ and $\tilde{B}^t_T:=B^t_T$ if $u=T$. Therefore $\tilde{B}^t$ is an increasing, cadlag process. It is constituted by $\mathcal{F}_{t,u}$-measurable random variables, and by Theorem 15 Chapter IV of [32], $\tilde{B}^t$ is a also a measurable process (indexed by $[t,u]$).

We can show that $\tilde{B}^t$ is $\mathbb{P}^{s,x}$-indistinguishable from $B^{s,x}_\cdot-B^{s,x}_t$ for any $(s,x)\in[0,t]\times E$. Indeed, let (s,x) be fixed. Since $B^{s,x}_\cdot-B^{s,x}_t$ is a version of B^t and $\mathbb{Q}$ being countable, there exists a $\mathbb{P}^{s,x}$-null set $\mathcal{N}$ such that for all $\omega\in\mathcal{N}^c$ and $r\in\mathbb{Q}\cap[t,u]$, $B^{s,x}_r(\omega)-B^{s,x}_t(\omega)=B^t_r(\omega)$. Therefore for any $\omega\in\mathcal{N}^c$ and $r\in[t,u]$,

$$\begin{aligned}\tilde{B}^t_r(\omega)&=\lim_{\substack{v\downarrow r\\ v\in\mathbb{Q}}}\sup_{\substack{t\le w<v\\ w\in\mathbb{Q}}}B^t_w(\omega)=\lim_{\substack{v\downarrow r\\ v\in\mathbb{Q}}}\sup_{\substack{t\le w<v\\ w\in\mathbb{Q}}}B^{s,x}(\omega)_w-B^{s,x}(\omega)_t\\&=B^{s,x}(\omega)_r-B^{s,x}(\omega)_t,\end{aligned}$$

where the latter equality comes from the fact that $B^{s,x}(\omega)$ is cadlag and increasing. So we have constructed an increasing finite cadlag (for all ω) process and so the path $r\mapsto\tilde{B}^t(\omega)$ is a Lebesgue integrator on $[t,u]$ for each ω.

We fix now $F\in\mathcal{F}_{t,T}$ and we discuss some issues related to N^F . Since it is positive, we can start defining the process $\bar{N}$, for index values $r\in[t,T[$ by $\bar{N}^F_r:=\liminf_{\substack{v\downarrow r\\ v\in\mathbb{Q}}}N^F_v$, and setting $\bar{N}^F_T:=N^F_T$. This

process is (by similar arguments as for $\tilde{B}^t$ defined in (1.4.12)), $\mathbb{P}^{s,x}$-indistinguishable to $N^{s,x,F}$ for all $(s,x) \in [0,t] \times E$. For any $r \in [t,T]$, N_r^F (see Notation 1.4.6) is $\mathcal{F}_{t,r}$-measurable, so $\bar{N}_r^F$ will also be $\mathcal{F}_{t,r}$-measurable for any $r \in [t,T]$ by right-continuity of $\mathcal{F}_{t,\cdot}$ (see Notation 1.3.1). However, $\bar{N}^F$ is not necessarily cadlag for every ω, and also not necessarily a measurable process. We subsequently define

$$W'_{tu} := \{\omega \in \Omega | \text{there is a cadlag function } f \text{ such that } \bar{N}^F(\omega) = f \text{ on } [t,u] \cap \mathbb{Q}\}.$$

By Theorem 18 b) in Chapter IV of [32], W'_{tu} is $\mathcal{F}_{t,u}$-measurable so we can define on $[t,u]$ $\tilde{N}_r^F := \bar{N}_r^F \mathbb{1}_{W'_{tu}}$. $\tilde{N}^F$ is no longer $\mathbb{F}^{s,x}$-adapted, however, it is now cadlag for all ω and therefore a measurable process by Theorem 15 Chapter IV of [32]. The r.v. $\tilde{N}_r^F$ are still $\mathcal{F}_{t,u}$-measurable , and $\tilde{N}^F$ is still $\mathbb{P}^{s,x}$-indistinguishable to $N^{s,x,F}$ on $[t,u]$ for any $(s,x) \in [0,t] \times E$.

Finally we can define $\int_t^u N_{r^-}^F dB_r := \int_t^u \tilde{N}_{r^-}^F d\tilde{B}_r^t$ which is $\mathbb{P}^{s,x}$ a.s. equal to $\int_t^u N_{r^-}^{s,x,F} dB_r^{s,x}$ for any $(s,x) \in [0,t] \times E$. Moreover, since $\tilde{N}^F$ and $\tilde{B}$ are both measurable with respect to $\mathcal{B}([t,u]) \otimes \mathcal{F}_{t,u}$, then $\int_t^u N_{r^-}^F dB_r$ is $\mathcal{F}_{t,u}$-measurable. □

The lemma below is a conditional version of the property (1.4.11).

Lemma 1.4.9. *For any $(s,x) \in [0,t] \times E$ and $F \in \mathcal{F}_{t,T}^{s,x}$ we have $\mathbb{P}^{s,x}$-a.s.*

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_t\right] = \mathbb{E}^{s,x}\left[\int_t^u N_{r^-}^F dB_r \middle| \mathcal{F}_t\right].$$

Proof. Let s, x, F be fixed. By definition of conditional expectation, we need to show that for any $G \in \mathcal{F}_t$ we have

$$\mathbb{E}^{s,x}\left[\mathbb{1}_G \mathbb{1}_F(A_u^{s,x} - A_t^{s,x})\right] = \mathbb{E}^{s,x}\left[\mathbb{1}_G \mathbb{E}^{s,x}\left[\int_t^u N_{r^-}^F dB_r \middle| \mathcal{F}_t\right]\right] \text{ a.s.}$$

For $r \in [t,u]$ we have $\mathbb{E}^{s,x}[\mathbb{1}_{F\cap G}|\mathcal{F}_r] = \mathbb{1}_G \mathbb{E}^{s,x}[\mathbb{1}_F|\mathcal{F}_r]$ a.s. therefore the cadlag versions of those processes are indistinguishable on $[t,u]$ and the random variables $\int_t^u N_{r^-}^{G\cap F} dB_r$ and $\mathbb{1}_G \int_t^u N_{r^-}^F dB_r$ as defined in Lemma 1.4.7 are a.s. equal. So by the non conditional property of dual predictable projection (1.4.11) we have

$$\begin{aligned}
\mathbb{E}^{s,x}\left[\mathbb{1}_G \mathbb{1}_F(A_u^{s,x} - A_t^{s,x})\right] &= \mathbb{E}^{s,x}\left[\int_t^u N_{r^-}^{G\cap F} dB_r\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_G \int_t^u N_{r^-}^F dB_r\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_G \mathbb{E}^{s,x}\left[\int_t^u N_{r^-}^F dB_r \middle| \mathcal{F}_t\right]\right],
\end{aligned}$$

which concludes the proof. □

Lemma 1.4.10. *For any $(s,x) \in [0,t] \times E$ and $F \in \mathcal{F}_{t,T}$ we have $\mathbb{P}^{s,x}$-a.s.,*

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_t\right] = \mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})|X_t\right].$$

Proof. By Lemma 1.4.9 we have

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_t\right] = \mathbb{E}^{s,x}\left[\int_t^u N_{r^-}^F dB_r \middle| \mathcal{F}_t\right].$$

By Lemma 1.4.7, $\int_t^u N_{r^-}^F dB_r$ is $\mathcal{F}_{t,T}$ measurable so the Markov property (1.3.4) implies

$$\mathbb{E}^{s,x}\left[\int_t^u N_{r^-}^F dB_r \middle| \mathcal{F}_t\right] = \mathbb{E}^{s,x}\left[\int_t^u N_{r^-}^F dB_r \middle| X_t\right],$$

therefore $\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_t\right]$ is a.s. equal to a $\sigma(X_t)$-measurable r.v and so is a.s. equal to $\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})|X_t\right]$. □

We are now able to prove (1.4.10) which is the first important issue of the proof of Proposition 1.4.5, which states that By definition, a predictable dual projection is adapted so we already know that $(A_u^{s,x} - A_t^{s,x})$ is $\mathcal{F}_u^{s,x}$-measurable, therefore by Remark 1.3.3, it is enough to show that it is also $\mathcal{F}_{t,T}^{s,x}$-measurable. So we are going to show that

$$A_u^{s,x} - A_t^{s,x} = \mathbb{E}^{s,x}\left[A_u^{s,x} - A_t^{s,x} | \mathcal{F}_{t,T}\right] \mathbb{P}^{s,x} \text{ a.s.} \tag{1.4.13}$$

For this we will show that

$$\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})\right] = \mathbb{E}^{s,x}\left[\mathbb{1}_F \mathbb{E}^{s,x}\left[A_u^{s,x} - A_t^{s,x} | \mathcal{F}_{t,T}\right]\right], \tag{1.4.14}$$

for any $F \in \mathcal{F}$. We will prove (1.4.14) for $F \in \mathcal{F}$ event of the form $F = F_t \cap F_{t,T}$ with $F_t \in \mathcal{F}_t$ and $F_{t,T} \in \mathcal{F}_{t,T}$. By item 4. of Remark 1.3.3, such events form a π-system Π which generates $\mathcal{F}$. Consequently, by the monotone class theorem, (1.4.14) will remain true for any $F \in \mathcal{F}$ and even in $\mathcal{F}^{s,x}$ since $\mathbb{P}^{s,x}$-null set will not impact the equality. This will imply (1.4.13) so that $A_u^{s,x} - A_t^{s,x}$ is $\mathcal{F}_{t,T}^{s,x}$-measurable. At this point, as we have anticipated, we prove (1.4.14) for a fixed $F = F_t \cap F_{t,T} \in \Pi$. By Lemma 1.4.10 we have

$$\begin{aligned}
\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})\right] &= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_t\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x} - A_t^{s,x})|X_t\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_{t,T}\right]|X_t\right]\right],
\end{aligned}$$

where the latter equality holds since $\sigma(X_t) \subset \mathcal{F}_{t,T}$. Now since $\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_{t,T}\right]$ is $\mathcal{F}_{t,T}$-measurable, the Markov property (1.3.4) allows us to substitute the conditional σ-field $\sigma(X_t)$ with $\mathcal{F}_t$ and obtain

$$\begin{aligned}
\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})\right] &= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_{t,T}\right]|\mathcal{F}_t\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{E}^{s,x}\left[\mathbb{1}_{F_{t,T}}(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_{t,T}\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_{F_t}\mathbb{1}_{F_{t,T}}\mathbb{E}^{s,x}\left[(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_{t,T}\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_F\mathbb{E}^{s,x}\left[(A_u^{s,x} - A_t^{s,x})|\mathcal{F}_{t,T}\right]\right].
\end{aligned}$$

This concludes the proof of (1.4.14), therefore (1.4.13) holds so that $A_u^{s,x} - A_t^{s,x}$ is $\mathcal{F}_{t,u}^{s,x}$-measurable and so (1.4.10) is established. This concludes the first part of the proof of Proposition 1.4.5.

We pass to the second part of the proof of Proposition 1.4.5 where we will show that for given $0 < t < u$ there is an $\mathcal{F}_{t,u}$-measurable r.v. A_u^t such that for every $(s,x) \in [0,t] \times E$, $(A_u^{s,x} - A_t^{s,x}) = A_u^t$ $\mathbb{P}^{s,x}$ a.s.

Similarly to what we did with the quadratic variation in Proposition 1.4.4, we start by noticing that for any $x \in E$, since $(A_u^{t,x} - A_t^{t,x})$ is $\mathcal{F}_{t,u}^{t,x}$-measurable, there exists by Proposition 1.3.12 an $\mathcal{F}_{t,u}$-measurable r.v. $a(x,\omega)$ such that

$$a(x,\omega) = A_u^{t,x} - A_t^{t,x} \quad \mathbb{P}^{t,x} \text{ a.s.} \tag{1.4.15}$$

As in the proof of Proposition 1.4.4, we will show the existence of a jointly measurable version of $(x,\omega) \mapsto a(x,\omega)$. For every $x \in E$ we define on $\mathcal{F}_{t,u}$ the positive measure

$$\mathbb{Q}^x : F \longmapsto \mathbb{E}^{t,x}\left[\mathbb{1}_F(A_u^{t,x} - A_t^{t,x})\right] = \mathbb{E}^{t,x}\left[\mathbb{1}_F a(x,\omega)\right]. \tag{1.4.16}$$

By Lemma 1.4.7, and (1.4.11), for every $F \in \mathcal{F}_{t,u}$ we have

$$\mathbb{Q}^x(F) = \mathbb{E}^{t,x}\left[\int_t^u N_{r-}^F dB_r\right], \tag{1.4.17}$$

and we recall that $\int_t^u N_{r^-}^F dB_r$ does not depend on x. So by Proposition 1.3.10 $x \longmapsto \mathbb{Q}^x(F)$ is Borel for any F. Moreover, for any x, $\mathbb{Q}^x \ll \mathbb{P}^{t,x}$. Again by Theorem 58 Chapter V in [33], there exists a version $(x,\omega) \mapsto a(x,\omega)$ measurable for $\mathcal{B}(E) \otimes \mathcal{F}_{t,u}$ of the related Radon-Nikodym densities.

We can now set $A_u^t(\omega) := a(X_t(\omega),\omega)$ which is then an $\mathcal{F}_{t,u}$-measurable r.v. Since $\mathbb{P}^{t,x}(X_t = x) = 1$ and (1.4.15) hold, we have

$$A_u^t = a(X_t,\cdot) = a(x,\cdot) = A_u^{t,x} - A_t^{t,x} \quad \mathbb{P}^{t,x} \text{ a.s.} \tag{1.4.18}$$

We now fix $s < t$ and $x \in E$ and we want to show that we still have
$A_u^t = A_u^{s,x} - A_t^{s,x}$ $\mathbb{P}^{s,x}$ a.s. So, as above, we consider $F \in \mathcal{F}_{t,u}$ and, thanks to (1.4.11) we compute

$$\begin{aligned}
\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})\right] &= \mathbb{E}^{s,x}\left[\int_t^u N_{r^-}^F dB_r\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{E}^{s,x}\left[\int_t^u N_{r^-}^F dB_r | \mathcal{F}_t\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{E}^{t,X_t}\left[\int_t^u N_{r^-}^F dB_r\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{E}^{t,X_t}\left[\mathbb{1}_F A_u^t\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{E}^{s,x}\left[\mathbb{1}_F A_u^t | \mathcal{F}_t\right]\right] \\
&= \mathbb{E}^{s,x}\left[\mathbb{1}_F A_u^t\right].
\end{aligned} \tag{1.4.19}$$

Indeed, concerning the fourth equality we recall that, by (1.4.16), (1.4.17) and (1.4.18), we have $\mathbb{E}^{t,x}\left[\int_t^u N_{r^-}^F dB_r\right] = \mathbb{E}^{t,x}\left[\mathbb{1}_F A_u^t\right]$ for all x, so this equality becomes an equality whatever random variable we plug into x. The third and fifth equalities come from the Markov property (1.3.4) since $\int_t^u N_{r^-}^F dB_r$ and A_u^t are $\mathcal{F}_{t,T}$-measurable. Then, adding $\mathbb{P}^{s,x}$-null sets does not change the validity of (1.4.19), so we have for any $F \in \mathcal{F}_{t,u}^{s,x}$ that $\mathbb{E}^{s,x}\left[\mathbb{1}_F(A_u^{s,x} - A_t^{s,x})\right] = \mathbb{E}^{s,x}\left[\mathbb{1}_F A_u^t\right]$.

Finally, since we had shown in the first half of the proof that $A_u^{s,x} - A_t^{s,x}$ is $\mathcal{F}_{t,u}^{s,x}$-measurable, and since A_u^t also has, by construction, the same measurability property, we can conclude that $A_u^{s,x} - A_t^{s,x} = A_u^t$ $\mathbb{P}^{s,x}$ a.s.

Since this holds for every $t \le u$ and $(s,x) \in [0,t] \times E$, $(A_u^t)_{(t,u)\in\Delta}$ is the desired AF, which ends the proof of Proposition 1.4.5. □

Corollary 1.4.11. *Let M, M' be two square integrable MAFs, let $M^{s,x}$ (respectively $M'^{s,x}$) be the cadlag version of M (respectively M') under $\mathbb{P}^{s,x}$. Then there exists a bounded variation AF with L^1 terminal condition denoted $\langle M, M'\rangle$ such that under any $\mathbb{P}^{s,x}$, the cadlag version of $\langle M, M'\rangle$ is $\langle M^{s,x}, M'^{s,x}\rangle$. If $M = M'$ the AF $\langle M, M'\rangle$ will be denoted $\langle M\rangle$ and is increasing.*

Proof. If $M = M'$, the corollary comes from the combination of Propositions 1.4.4 and 1.4.5, and the fact that the angular bracket of a square integrable martingale is the dual predictable projection of its quadratic variation. Otherwise, it is clear that $M + M'$ and $M - M'$ are square integrable MAFs, so we can consider the increasing MAFs $\langle M - M'\rangle$ and $\langle M + M'\rangle$. We introduce the AF

$$\langle M, M'\rangle = \frac{1}{4}(\langle M + M'\rangle - \langle M - M'\rangle),$$

which by polarization has cadlag version $\langle M^{s,x}, M'^{s,x}\rangle$ under $\mathbb{P}^{s,x}$. $\langle M, M'\rangle$ is therefore a bounded variation AF with L^1 terminal condition. □

We are now going to study the Radon-Nikodym derivative of an increasing continuous AF with respect to some measure. The next result can be seen as an extension of Theorem 13 Chapter XV in [34] in a non-homogeneous setup. We will need the following lemma.

Lemma 1.4.12. *Let $(E,\mathcal{E})$ be a measurable space, let $\mathcal{I}$ be a sub-interval of $\mathbb{R}_+$ and let $f : E \times \mathcal{I} \longrightarrow \mathbb{R}$ be a mapping such that for all $t \in \mathcal{I}$, $x \mapsto f(x,t)$ is measurable with respect to $\mathcal{E}$ and for all $x \in E$, $t \mapsto f(x,t)$ is right-continuous, then f is measurable with respect to $\mathcal{E} \otimes \mathcal{B}(\mathcal{I})$.*

Proof. On $(E,\mathcal{E})$ we introduce the filtration $(\mathcal{E}_t)_{t\in\mathcal{I}}$ where $\mathcal{E}_t = \mathcal{E}$ for all t. In the filtered space $(E,\mathcal{E},(\mathcal{E}_t)_{t\in\mathcal{I}})$, f defines a right-continuous adapted process and is therefore progressively measurable (see Theorem 15 in [32] Chapter IV for instance), and in particular it is measurable. This means that f is measurable with respect to $\mathcal{E}\otimes\mathcal{B}(\mathcal{I})$. □

Proposition 1.4.13. *Let A be a positive, non-decreasing AF absolutely continuous with respect to some continuous non-decreasing function V, and for every $(s,x)\in[0,T[\times E$ let $A^{s,x}$ be the cadlag version of A under $\mathbb{P}^{s,x}$. There exists a Borel function $h\in\mathcal{B}([0,T]\times E,\mathbb{R})$ such that for every $(s,x)\in[0,T]\times E$, $A^{s,x}=\int_s^{\cdot\vee s} h(r,X_r)dV_r$, in the sense of indistinguishability.*

Proof. We set

$$C_u^t = A_u^t + (V_u - V_t) + (u-t), \tag{1.4.20}$$

which is an AF with cadlag versions

$$C_t^{s,x} = A_t^{s,x} + V_t + t, \tag{1.4.21}$$

and we start by showing the statement for A and C instead of A and V. We introduce the intermediary function C so that for any $u>t$ that $\frac{A_u^{s,x}-A_t^{s,x}}{C_u^{s,x}-C_t^{s,x}}\in[0,1]$; that property will be used extensively in connections with the application of dominated convergence theorem.

Since $A^{s,x}$ is non-decreasing for any $(s,x)\in[0,T]\times E$, A can be taken positive (in the sense that $A_u^t(\omega)\geq 0$ for any $(t,u)\in\Delta$ and $\omega\in\Omega$) by considering A^+ (defined by $(A^+)_u^t(\omega) := A_u^t(\omega)^+$) instead of A.

For $t\in[0,T[$ we set

$$\begin{aligned} K_t &:= \liminf_{n\to\infty} \frac{A^t_{t+\frac{1}{n}}}{A^t_{t+\frac{1}{n}} + \frac{1}{n} + (V_{t+\frac{1}{n}} - V_t)} \\ &= \liminf_{n\to\infty}\inf_{p\geq n} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)} \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\min_{n\leq p\leq m} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}. \end{aligned} \tag{1.4.22}$$

By positivity, this liminf always exists and belongs to $[0,1]$ since the sequence belongs to $[0,1]$. For every $(s,x)\in[0,T]\times E$, since for all $t\geq s$ and $n\geq 0$,

$A^t_{t+\frac{1}{n}} = A^{s,x}_{t+\frac{1}{n}} - A^{s,x}_t$ $\mathbb{P}^{s,x}$ a.s., then $K^{s,x}$ defined by $K_t^{s,x} := \liminf_{n\to\infty} \frac{A^{s,x}_{t+\frac{1}{n}}-A^{s,x}_t}{C^{s,x}_{t+\frac{1}{n}}-C^{s,x}_t}$ is a $\mathbb{P}^{s,x}$-version of K, for $t\in[s,T[$. By Lebesgue Differentiation theorem (see Theorem 12 Chapter XV in [34] for a version of the theorem with a general atomless measure), for any (s,x), for $\mathbb{P}^{s,x}$-almost all ω, since $dC^{s,x}(\omega)$ is absolutely continuous with respect to $dA^{s,x}(\omega)$, $K^{s,x}(\omega)$ is a density of $dA^{s,x}(\omega)$ with respect to $dC^{s,x}(\omega)$.

We now show that there exists a Borel function k in $\mathcal{B}([0,T[\times E,\mathbb{R})$ such that under any $\mathbb{P}^{s,x}$, $k(t,X_t)$ is on $[s,T[$ a version of K (and therefore of $K^{s,x}$). For every $t\in[0,T[$, K_t is measurable with respect to $\bigcap_{n\geq 0}\mathcal{F}_{t,t+\frac{1}{n}} = \mathcal{F}_{t,t}$ by construction, taking into account Notation 1.3.1. So for any $(t,x)\in[0,T]\times E$, by Proposition 1.3.14, there exists a constant which we denote $k(t,x)$ such that

$$K_t = k(t,x), \quad \mathbb{P}^{t,x}\text{a.s.} \tag{1.4.23}$$

For any integers (n, m), we define

$$k^{n,m} : (t, x) \mapsto \mathbb{E}^{t,x}\left[\min_{n\le p\le m} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right],$$

and for any n

$$k_n : (t, x) \mapsto \mathbb{E}^{t,x}\left[\inf_{p\ge n} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right], \tag{1.4.24}$$

We start showing that

$$\begin{array}{rccl} \tilde{k}^{n,m} : & (s, x, t) & \longmapsto & \mathbb{E}^{s,x}\left[\min_{n\le p\le m} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right] \mathbb{1}_{s\le t}, \\ & [0, T] \times E \times [0, T[& \longrightarrow & [0, 1], \end{array} \tag{1.4.25}$$

is jointly Borel. In order to do so, we will show that at fixed t, $\tilde{k}^{n,m}(\cdot, \cdot, t)$ is Borel, at fixed (s, x), $\tilde{k}^{n,m}(s, x, \cdot)$ is right-continuous and we will conclude on the joint measurability thanks to Lemma 1.4.12.

If we fix $t \in [0, T[$, then by Proposition 1.3.10

$$(s, x) \longmapsto \mathbb{E}^{s,x}\left[\min_{n\le p\le m} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right]$$

is a Borel map. Since $(s, x) \mapsto \mathbb{1}_{[t,T]}(s)$ is obviously Borel, considering the product of the two previous maps, $\tilde{k}^{n,m}(\cdot, \cdot, t)$ is Borel. We now fix some (s, x) and show that $\tilde{k}^{n,m}(s, x, \cdot)$ is right-continuous. Since that function is equal to zero on $[0, s[$, showing its continuity on $[s, T[$ will be sufficient. We remark that $A^{s,x}$ is continuous $\mathbb{P}^{s,x}$ a.s. V is continuous, and the minimum of a finite number of continuous functions remains continuous. Let $t_q \underset{q\to\infty}{\longrightarrow} t$ be a converging sequence in $[s, T[$. Then $\min_{n\le p\le m} \frac{A^{s,x}_{t_q+\frac{1}{p}} - A^{s,x}_{t_q}}{A^{s,x}_{t_q+\frac{1}{p}} - A^{s,x}_{t_q} + \frac{1}{p} + (V_{t_q+\frac{1}{p}} - V_{t_q})}$ tends a.s. to $\min_{n\le p\le m} \frac{A^{s,x}_{t+\frac{1}{p}} - A^{s,x}_{t}}{A^{s,x}_{t+\frac{1}{p}} - A^{s,x}_{t} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_{t})}$, when q tends to infinity. Since for any $s \le t \le u$, $A^t_u = A^{s,x}_u - A^{s,x}_t$ $\mathbb{P}^{s,x}$ a.s., then $\frac{A^{t_q}_{t_q+\frac{1}{p}}}{A^{t_q}_{t_q+\frac{1}{p}} + \frac{1}{p} + (V_{t_q+\frac{1}{p}} - V_{t_q})}$ tends a.s. to $\frac{A^{t}_{t+\frac{1}{p}}}{A^{t}_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_{t})}$. All those terms belonging to $[0, 1]$, by dominated convergence theorem, the mentioned convergence also holds under the expectation, hence the announced continuity related to $\tilde{k}^{n,m}$ is established and as anticipated, $\tilde{k}^{n,m}$ is jointly measurable in all its variables.

Since $k^{n,m}(t, y) = \tilde{k}^{n,m}(t, t, y)$, by composition we can deduce that for any n, m, $k^{n,m}$ is Borel. By the dominated convergence theorem, $k^{n,m}$ tends pointwise to k^n (which was defined in (1.4.24), when m goes to infinity so k^n are also Borel for every n. Finally, keeping in mind (1.4.22) nd (1.4.23) we have $\mathbb{P}^{t,x}$ a.s.

$$k(t, x) = K_t = \lim_{n\to\infty} \inf_{p\ge n} \frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}.$$

Taking the expectation and again by the dominated convergence theorem, k^n (defined in (1.4.24)) tends pointwise to k when n goes to infinity so k is Borel.

We now show that, for any $(s,x) \in [0,T] \times E$, $k(\cdot, X_\cdot)$ is a $\mathbb{P}^{s,x}$-version of K on $[s,T[$. Since $\mathbb{P}^{t,x}(X_t = x) = 1$, we know that for any $t \in [0,T]$, $x \in E$, we have $K_t = k(t,x) = k(t,X_t)$ $\mathbb{P}^{t,x}$-a.s., and we prove below that for any $t \in [0,T]$, $(s,x) \in [0,t] \times E$, we have $K_t = k(t,X_t)$ $\mathbb{P}^{s,x}$-a.s.

Let $t \in [0,T]$ be fixed. Since A is an AF, for any n, $\frac{A^t_{t+\frac{1}{p}}}{A^t_{t+\frac{1}{p}} + \frac{1}{n} + (V_{t+\frac{1}{n}} - V_t)}$ is $\mathcal{F}_{t,t+\frac{1}{n}}$-measurable. So the event $\left\{ \liminf_{n\to\infty} \frac{A^t_{t+\frac{1}{n}}}{A^t_{t+\frac{1}{n}} + \frac{1}{n} + (V_{t+\frac{1}{n}} - V_t)} = k(t,X_t) \right\}$ belongs to $\mathcal{F}_{t,T}$ and by Markov property (1.3.4), for any $(s,x) \in [0,t] \times E$, we get

$$\begin{aligned} \mathbb{P}^{s,x}(K_t = k(t,X_t)) &= \mathbb{E}^{s,x}[\mathbb{P}^{s,x}(K_t = k(t,X_t)|\mathcal{F}_t)] \\ &= \mathbb{E}^{s,x}[\mathbb{P}^{t,X_t}(K_t = k(t,X_t))] \\ &= 1. \end{aligned}$$

For any (s,x), the process $k(\cdot, X_\cdot)$ is therefore on $[s,T[$ a $\mathbb{P}^{s,x}$-modification of K and therefore of $K^{s,x}$. However it is not yet clear if provides another density of $dA^{s,x}$ with respect to $dC^{s,x}$, which was defined at (1.4.21).

Considering that $(t,u,\omega) \mapsto V_u - V_t$ also defines a positive non-decreasing AF absolutely continuous with respect to C, defined in (1.4.20), we proceed similarly as at the beginning of the proof, replacing the AF A with V.

Let the process K' be defined by

$$K'_t = \liminf_{n\to\infty} \frac{V_{t+\frac{1}{n}} - V_t}{A^t_{t+\frac{1}{n}} + \frac{1}{n} + (V_{t+\frac{1}{n}} - V_t)},$$

and for any(s,x), let $K'^{s,x}$ be defined on $[s,T[$ by

$$K'^{s,x}_t = \liminf_{n\to\infty} \frac{V_{t+\frac{1}{n}} - V_t}{A^{s,x}_{t+\frac{1}{n}} - A^{s,x}_t + \frac{1}{n} + (V_{t+\frac{1}{n}} - V_t)}.$$

Then, for any (s,x), $K'^{s,x}$ on $[s,T[$ is a $\mathbb{P}^{s,x}$-version of K', and it constitutes a density of $dV(\omega)$ with respect to $dC^{s,x}(\omega)$ on $[s,T[$, for almost all ω. One shows then the existence of a Borel function k' such that for any (s,x), $k'(\cdot, X_\cdot)$ is a $\mathbb{P}^{s,x}$-version of K' and a modification of $K'^{s,x}$ on $[s,T[$. So for any (s,x), under $\mathbb{P}^{s,x}$, we can write

$$\left\{ \begin{array}{rcl} A^{s,x} & = & \int_s^{\cdot\vee s} K^{s,x}_r dC^{s,x}_r \\ V_{\cdot\vee s} - V_s & = & \int_s^{\cdot\vee s} K'^{s,x}_r dC^{s,x}_r \end{array} \right.$$

Now since $dA^{s,x} \ll dV$, for a fixed ω, the set $\{r \in [s,T] | K'^{s,x}_r(\omega) = 0\}$ is negligible with respect to dV so also for $dA^{s,x}(\omega)$ and therefore we can write

$$\begin{aligned} A^{s,x} &= \int_s^{\cdot\vee s} K^{s,x}_r dC^{s,x}_r \\ &= \int_s^{\cdot\vee s} \frac{K^{s,x}_r}{K'^{s,x}_r} \mathbb{1}_{\{K'^{s,x}_r \neq 0\}} K'^{s,x}_r dC^{s,x}_r \\ &+ \int_s^{\cdot\vee s} \mathbb{1}_{\{K'^{s,x}_r = 0\}} dA^{s,x}_r \\ &= \int_s^{\cdot\vee s} \frac{K^{s,x}_r}{K'^{s,x}_r} \mathbb{1}_{\{K'^{s,x}_r \neq 0\}} dV_r, \end{aligned}$$

where we use the convention that for any two functions ϕ, ψ then $\frac{\phi}{\psi}\mathbb{1}_{\psi\neq 0}$ is defined by by

$$\frac{\phi}{\psi}\mathbb{1}_{\{\psi\neq 0\}}(x) = \left\{ \begin{array}{l} \frac{\phi(x)}{\psi(x)} \text{ if } \psi(x) \neq 0 \\ 0 \text{ if } \psi(x) = 0. \end{array} \right.$$

We set now $h := \frac{k}{k'}\mathbb{1}_{\{k'_r \neq 0\}}$ which is Borel, and clearly for any (s,x), $h(t,X_t)$ is a $\mathbb{P}^{s,x}$-version of $H^{s,x} := \frac{K^{s,x}}{K'^{s,x}}\mathbb{1}_{\{K'^{s,x}\neq 0\}}$ on $[s,T[$. So by Lemma 2.5.13 in Chapter 2,
$H^{s,x}_t = h(t,X_t)\, dV \otimes d\mathbb{P}^{s,x}$ a.e. and finally we have shown that under any $\mathbb{P}^{s,x}$,
$A^{s,x} = \int_s^{\cdot \vee s} h(r,X_r) dV_r$ on $[0,T[$. Without change of notations we extend h to $[0,T]\times E$ by zero for $t = T$. Since $A^{s,x}$ is continuous $\mathbb{P}^{s,x}$-a.s. previous equality extends to T. □

Proposition 1.4.14. *Let $(A^t_u)_{(t,u)\in\Delta}$ be an AF with bounded variation and taking L^1 values. Then there exists an increasing AF which we denote $(Pos(A)^t_u)_{(t,u)\in\Delta}$ (resp. $(Neg(A)^t_u)_{(t,u)\in\Delta}$) and which, for any $(s,x)\in[0,T]\times E$, has $Pos(A^{s,x})$ (resp. $Neg(A^{s,x})$) as cadlag version under $\mathbb{P}^{s,x}$.*

Proof. By definition of the total variation of a bounded variation function, the following holds. For every $(s,x)\in[0,T]\times E$, $s\le t\le u\le T$ for $\mathbb{P}^{s,x}$ almost all $\omega\in\Omega$, and any sequence of subdivisions of $[t,u]$: $t = t^k_1 < t^k_2 < \cdots < t^k_k = u$ such that $\min\limits_{i<k}(t^k_{i+1} - t^k_i) \underset{k\to\infty}{\longrightarrow} 0$ we have

$$\sum_{i<k} |A^{s,x}_{t^k_{i+1}}(\omega) - A^{s,x}_{t^k_i}(\omega)| \underset{k\to\infty}{\longrightarrow} Var(A^{s,x})_u(\omega) - Var(A^{s,x})_t(\omega), \tag{1.4.26}$$

taking into account the considerations of the end of Section 1.2.

By Proposition 3.3 in [61] Chapter I, we have $Pos(A^{s,x}) = \frac{1}{2}(Var(A^{s,x}) + A^{s,x})$ and $Neg(A^{s,x}) = \frac{1}{2}(Var(A^{s,x}) - A^{s,x})$. Moreover, for any $x\in\mathbb{R}$ we know that $x^+ = \frac{1}{2}(|x|+x)$ and $x^- = \frac{1}{2}(|x|-x)$, so we also have

$$\begin{cases} \sum\limits_{i<k}(A^{s,x}_{t^k_{i+1}}(\omega) - A^{s,x}_{t^k_i}(\omega))^+ & \underset{k\to\infty}{\longrightarrow} & Pos(A^{s,x})_u(\omega) - Pos(A^{s,x})_t(\omega) \\ \sum\limits_{i<k}(A^{s,x}_{t^k_{i+1}}(\omega) - A^{s,x}_{t^k_i}(\omega))^- & \underset{k\to\infty}{\longrightarrow} & Neg(A^{s,x})_u(\omega) - Neg(A^{s,x})_t(\omega), \end{cases} \tag{1.4.27}$$

for $\mathbb{P}^{s,x}$ almost all ω. Since the convergence a.s. implies the convergence in probability, for every $(s,x)\in[0,T]\times E$, $s\le t\le u$ and any sequence of subdivisions of $[t,u]$: $t = t^k_1 < t^k_2 < \cdots < t^k_k = u$ such that $\min\limits_{i<k}(t^k_{i+1} - t^k_i) \underset{k\to\infty}{\longrightarrow} 0$, we have

$$\begin{cases} \sum\limits_{i<k}\left(A^{t^k_i}_{t^k_{i+1}}\right)^+ & \underset{k\to\infty}{\overset{\mathbb{P}^{s,x}}{\longrightarrow}} & Pos(A^{s,x})_u - Pos(A^{s,x})_t \\ \sum\limits_{i<k}\left(A^{t^k_i}_{t^k_{i+1}}\right)^- & \underset{k\to\infty}{\overset{\mathbb{P}^{s,x}}{\longrightarrow}} & Neg(A^{s,x})_u - Neg(A^{s,x})_t. \end{cases} \tag{1.4.28}$$

The proof can now be performed according to the same arguments as in the proof of Proposition 1.4.4, replacing M with A, the quadratic increments with the positive (resp. negative) increments, and the quadratic variation with the positive (resp. negative) variation of an adapted process. □

We assume for now that we are given a fixed stochastic basis fulfilling the usual conditions, and a non-decreasing function V.

Notation 1.4.15. *We denote $\mathcal{H}^{2,V} := \{M\in\mathcal{H}^2_0 | d\langle M\rangle \ll dV\}$ and $\mathcal{H}^{2,\perp V} := \{M\in\mathcal{H}^2_0 | d\langle M\rangle \perp dV\}$.*

Proposition 2.3.5 which proof is postponed to Chapter 2 states the following.

Proposition 1.4.16. *$\mathcal{H}^{2,V}$ and $\mathcal{H}^{2,\perp V}$ are orthogonal sub-Hilbert spaces of $\mathcal{H}^2_0$ and $\mathcal{H}^2_0 = \mathcal{H}^{2,V} \oplus^\perp \mathcal{H}^{2,\perp V}$. Moreover, any element of $\mathcal{H}^{2,V}_{loc}$ is strongly orthogonal to any element of $\mathcal{H}^{2,\perp V}_{loc}$.*

For any $M\in\mathcal{H}^2_0$, we denote by M^V its projection on $\mathcal{H}^{2,V}$.
We can now finally establish the main result of the present note.

Proposition 1.4.17. *Let V be a continuous non-decreasing function. Let M, N be two square integrable MAFs, and assume that the AF $\langle N\rangle$ is absolutely continuous with respect to V. There exists a function $v \in \mathcal{B}([0,T]\times E, \mathbb{R})$ such that for any (s,x), $\langle M^{s,x}, N^{s,x}\rangle = \int_s^{\cdot\vee s} v(r, X_r)dV_r$.*

Proof. By Corollary 1.4.11, there exists a bounded variation AF with L^1 values denoted $\langle M, N\rangle$ such that under any $\mathbb{P}^{s,x}$, the cadlag version of $\langle M, N\rangle$ is
$\langle M^{s,x}, N^{s,x}\rangle$. By Proposition 1.4.14, there exists an increasing AF with L^1 values denoted $Pos(\langle M, N\rangle)$ (resp. $Neg(\langle M, N\rangle)$) such that under any $\mathbb{P}^{s,x}$, the cadlag version of $Pos(\langle M, N\rangle)$ (resp. $Neg(\langle M, N\rangle)$) is $Pos(\langle M^{s,x}, N^{s,x}\rangle)$ (resp. $Neg(\langle M^{s,x}, N^{s,x}\rangle)$). We fix some (s,x) and the associated probability $\mathbb{P}^{s,x}$. Since $\langle N\rangle$ is absolutely continuous with respect to V, comparing Definition 1.4.1 and Notation 1.4.15 we have $N^{s,x} \in \mathcal{H}^{2,V}$. Therefore by Proposition 1.4.16 we have

$$\begin{aligned}\langle M^{s,x}, N^{s,x}\rangle &= \langle (M^{s,x})^V, N^{s,x}\rangle \\ &= \tfrac{1}{4}\langle (M^{s,x})^V + N^{s,x}\rangle - \tfrac{1}{4}\langle (M^{s,x})^V - N^{s,x}\rangle.\end{aligned} \tag{1.4.29}$$

Since both processes $\frac{1}{4}\langle (M^{s,x})^V + N^{s,x}\rangle$, $\frac{1}{4}\langle (M^{s,x})^V - N^{s,x}\rangle$ are increasing and starting at zero, we have $Pos(\langle M^{s,x}, N^{s,x}\rangle) = \frac{1}{4}\langle (M^{s,x})^V + N^{s,x}\rangle$ and

$$Neg(\langle M^{s,x}, N^{s,x}\rangle) = \frac{1}{4}\langle (M^{s,x})^V - N^{s,x}\rangle.$$

Now since $(M^{s,x})^V + N^{s,x}$ and $(M^{s,x})^V - N^{s,x}$ belong to $\mathcal{H}^{2,V}$, we have shown that $dPos(\langle M^{s,x}, N^{s,x}\rangle) \ll dV$ and $dNeg(\langle M^{s,x}, N^{s,x}\rangle) \ll dV$ in the sense of stochastic measures.

Since this holds for all (s,x) Proposition 1.4.13, insures the existence of two functions v_+, v_- in $\mathcal{B}([0,T]\times E, \mathbb{R})$ such that for any (s,x), $Pos(\langle M^{s,x}, N^{s,x}\rangle) = \int_s^{\cdot\vee s} v_+(r, X_r)dV_r$ and $Neg(\langle M^{s,x}, N^{s,x}\rangle) = \int_s^{\cdot\vee s} v_-(r, X_r)dV_r$. The conclusion now follows setting $v = v_+ - v_-$. □

Chapter 2

BSDEs with no driving martingale, Markov processes and associated Pseudo PDEs

This chapter is the object of paper [9].

Abstract

We discuss a class of Backward Stochastic Differential Equations (BSDEs) with no driving martingale. When the randomness of the driver depends on a general Markov process X, those BSDEs are denominated Markovian BSDEs and can be associated to a deterministic problem, called Pseudo-PDE which constitutes the natural generalization of a parabolic semilinear PDE which naturally appears when the underlying filtration is Brownian. We consider two aspects of well-posedness for the Pseudo-PDEs: *classical* and *martingale* solutions.

2.1 Introduction

This paper focuses on a new concept of Backward Stochastic Differential Equation (in short BSDE) with no driving martingale of the form

$$Y_t = \xi + \int_t^T \hat{f}\left(r, \cdot, Y_r, \sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r - (M_T - M_t), \quad t \in [0,T] \tag{2.1.1}$$

defined on a fixed stochastic basis fulfilling the usual conditions. V is a given bounded non-decreasing continuous adapted process, ξ (resp. $\hat{f}$) is a prescribed terminal condition (resp. driver). The unknown will be a couple of cadlag adapted processes (Y, M) where M is a martingale. When $V_t = t$ (2.1.1) is a particular case of the class of BSDEs introduced and studied by [67], for which we bring a new light.

A special case of such BSDEs are the Markovian BSDEs of the form

$$Y_t^{s,x} = g(X_T) + \int_t^T f\left(r, X_r, Y_r^{s,x}, \sqrt{\frac{d\langle M^{s,x}\rangle}{dV}}(r)\right) dV_r - (M_T^{s,x} - M_t^{s,x}), \quad t \in [0,T] \tag{2.1.2}$$

defined in a stochastic basis $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$ where $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ corresponds to the laws (for different starting times s and starting points x) of an underlying forward Markov process with time index $[0,T]$, taking values in a Polish state space E. Indeed this Markov process is supposed to solve a

martingale problem with respect to a given *deterministic* operator a, which is the natural generalization of a stochastic differential equation in law. (2.1.2) will be naturally associated with a deterministic problem involving a, which will be called *Pseudo-PDE*, being of the type

$$\left\{ \begin{array}{rcll} a(u) + f\left(\cdot, \cdot, u, \Gamma(u)^{\frac{1}{2}}\right) & = & 0 & \text{on } [0,T] \times E \\ u(T, \cdot) & = & g, & \end{array} \right. \tag{2.1.3}$$

where $\Gamma(u) = a(u^2) - 2ua(u)$ is a potential theory operator called the *carré du champs operator*. The Markovian BSDE (2.1.2) seems to be appropriated in the case when the forward underlying process X is a general Markov process which does not rely to a fixed reference process or random field as a Brownian motion or a Poisson measure.

The classical notion of Brownian BSDE was introduced in 1990 by E. Pardoux and S. Peng in [71], after an early work of J.M. Bismut in 1973 in [18]. It is a stochastic differential equation with prescribed terminal condition ξ and driver $\hat{f}$; the unknown is a couple (Y, Z) of adapted processes. Of particular interest is the case when the randomness of the driver is expressed through a forward diffusion process X and the terminal condition only depends on X_T. The solution, when it exists, is usually indexed by the starting time s and starting point x of the forward diffusion $X = X^{s,x}$, and it is expressed by

$$\left\{ \begin{array}{rcl} X_t^{s,x} & = & x + \int_s^t \beta(r, X_r^{s,x}) dr + \int_s^t \sigma(r, X_r^{s,x}) dB_r \\ Y_t^{s,x} & = & g(X_T^{s,x}) + \int_t^T f\left(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x}\right) dr - \int_t^T Z_r^{s,x} dB_r, \quad t \in [0,T] \end{array} \right. \tag{2.1.4}$$

where B is a Brownian motion. Existence and uniqueness of (2.1.4) (that we still indicate with BSDE) above was established first supposing essentially Lipschitz conditions on f with respect to the third and fourth variable. β and σ were also supposed to be Lipschitz (with respect to x). In the sequel those conditions were considerably relaxed, see [74] and references therein.
In [76] and in [72] previous BSDE was linked to the semilinear PDE

$$\left\{ \begin{array}{l} \partial_t u + \frac{1}{2} \sum\limits_{i,j \leq d} (\sigma\sigma^\intercal)_{i,j} \partial^2_{x_i x_j} u + \sum\limits_{i \leq d} \beta_i \partial_{x_i} u + f(\cdot, \cdot, u, \sigma \nabla u) = 0 \quad \text{on } [0,T[\times \mathbb{R}^d \\ u(T, \cdot) = g. \end{array} \right. \tag{2.1.5}$$

In particular, if (2.1.5) has a classical smooth solution u then $(Y^{s,x}, Z^{s,x}) := (u(\cdot, X_\cdot^{s,x}), \sigma \nabla u(\cdot, X_\cdot^{s,x}))$ solves the second line of (2.1.4). Conversely, only under the Lipschitz type conditions mentioned after (2.1.4), the solution of the BSDE can be expressed as a function of the forward process $(Y^{s,x}, Z^{s,x}) = (u(\cdot, X_\cdot^{s,x}), v(\cdot, X_\cdot^{s,x}))$, see [43]. When f and g are continuous, u is a viscosity solution of (2.1.5). Excepted in the case when u has some minimal differentiability properties, see e.g. [52], it is difficult to say something more on v. One major contribution of this paper consists in specifying v.

Since the pioneering work of [72], in the Brownian case, the relations between more general BSDEs and associated deterministic problems have been studied extensively, and innovations have been made in several directions.

In [7] the authors introduced a new kind of BSDE including a term with jumps generated by a Poisson measure, where an underlying forward process X solves a jump diffusion equation with Lipschitz type conditions. They associated with it an Integro-Partial Differential Equation (in short IPDE) in which some non-local operators are added to the classical partial differential maps, and proved that, under some continuity conditions on the coefficients, the BSDE provides a viscosity solution of the IPDE. In chapter 13 of [8], under some specific conditions on the coefficients of a Brownian BSDE,

one produces a solution in the sense of distributions of the parabolic PDE. Later, the notion of mild solution of the PDE was used in [4] where the authors tackled diffusion operators generating symmetric Dirichlet forms and associated Markov processes thanks to the theory of Fukushima Dirichlet forms, see e.g. [53]. Those results were extended to the case of non symmetric Markov processes in [87]. Infinite dimensional setups were considered for example in [52] where an infinite dimensional BSDE could produce the mild solution of a PDE on a Hilbert space. Concerning the study of BSDEs driven by more general martingales than Brownian motion, we have already mentioned BSDEs driven by Poisson measures. In this respect, more recently, BSDEs driven by marked point processes were introduced in [24], see also [5]; in that case the underlying process does not contain any diffusion term. Brownian BSDEs involving a supplementary orthogonal term were studied in [43]. We can also mention the study of BSDEs driven by a general martingale in [22]. BSDEs of the same type, but with partial information have been investigated in [23]. A first approach to face deterministic problems for those equations appears in [64]; that paper also contains an application to financial hedging in incomplete markets. Finally, BSDEs in general filtered space were studied in [67] as we have already mentioned.

We come back to the motivations of the paper. Besides introducing and studying the new class of BSDEs (2.1.1), (resp. Markovian BSDEs (2.1.2)), we study the corresponding Pseudo-PDE (2.1.3) and carefully explore their relations in the spirit of the existing links between (2.1.4) and (2.1.5). For the Pseudo-PDE, we analyze well-posedness at two different levels: *classical* solutions, which generalize the $C^{1,2}$-solutions of (2.1.5) and the so called *martingale solutions*. In the following Chapter 3, we also discuss other (analytical) solutions, that we denominate as *decoupled mild* solutions. The main contributions of the paper are essentially the following. In Section 2.3 we introduce the notion of BSDE with no driving martingale (2.1.1). Theorem 2.3.21 states existence and uniqueness of a solution for that BSDE, when the final condition ξ is square integrable and the driver $\hat{f}$ verifies some integrability and Lipschitz conditions. For technical reasons we have decided to provide an independent constructive proof from the one of [67]. Indeed we need that construction for the sequel of the paper. On the other hand, the particular form of our BSDE allows a simple and direct proof.

In Section 2.4, we consider an operator and its domain $(a, \mathcal{D}(a))$; V will be a continuous non-decreasing function. That section is devoted to the formulation of the martingale problem concerning our underlying process X. For each initial time s and initial point x the solution will be a probability $\mathbb{P}^{s,x}$ under which for any $\phi \in \mathcal{D}(a)$,

$$\phi(\cdot, X_\cdot) - \phi(s,x) - \int_s^\cdot a(\phi)(r, X_r) dV_r$$

is a local martingale starting in zero at time s. We will then assume that this martingale problem is well-posed and that its solution $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defines a Markov process. In Proposition 2.4.10, we prove that, under each one of those probabilities, the angular bracket of every square integrable martingale is absolutely continuous with respect to dV. In Definition 2.4.14, we suitably define some extended domains for the operators a and Γ, using some locally convex topology. In Section 2.5 we introduce the Pseudo-PDE (2.1.3) to which we associate the Markovian BSDE (2.1.2), considered under every $\mathbb{P}^{s,x}$. We also introduce the notions of *classical solution* in Definition 2.5.3, and of *martingale solution* in Definition 2.5.18, which is fully probabilistic. Proposition 2.5.20 says the following. Classical solutions of (2.1.3) typically belong to the domain $\mathcal{D}(a)$ and are shown also to be essentially martingale solutions. Conversely a martingale solution belonging to $\mathcal{D}(a)$ is a classical solution, up to so called zero potential sets, see Definition 2.4.11. Proposition 2.5.9 asserts that, given a classical solution $u \in \mathcal{D}(a)$, then for any (s,x) the processes $Y^{s,x} = u(\cdot, X_\cdot)$ and
$M^{s,x} = u(\cdot, X_\cdot) - u(s,x) - \int_s^\cdot f(\cdot, \cdot, u, \Gamma(u)^{\frac{1}{2}})(r, X_r) dV_r$ solve (2.1.2) under the probability $\mathbb{P}^{s,x}$.

Theorem 2.5.15 states that, without any assumptions of regularity, there exist Borel functions u and v such that for any $(s,x) \in [0,T] \times E$, the solution of (2.1.2) verifies

$$\begin{cases} \forall t \geq s : Y_t^{s,x} = u(t, X_t) \quad \mathbb{P}^{s,x} \text{ a.s.} \\ \frac{d\langle M^{s,x} \rangle}{dV}(t) = v^2(t, X_t) \quad dV \otimes d\mathbb{P}^{s,x} \text{ a.e.} \end{cases}$$

Theorems 2.5.21 and 2.5.22 state that the function u mentioned above is the unique martingale solution of (2.1.3). Moreover v is also identified as a function of u through an extension of the carré du champs operator.

In Section 2.6 we list some examples which are developed in Chapter 3. These include Markov processes defined as weak solutions of Stochastic Differential Equations (in short SDEs) including possible jump terms, α-stable Lévy processes associated to fractional Laplace operators, solutions of SDEs with distributional drift and diffusions on compact manifolds.

2.2 Preliminaries

In the whole paper we will use the following notions, notations and vocabulary.

A topological space E will always be considered as a measurable space with its Borel σ-field which shall be denoted $\mathcal{B}(E)$ and if (F, d_F) is a metric space, $\mathcal{C}(E,F)$ (respectively $\mathcal{C}_b(E,F)$, $\mathcal{B}(E,F)$, $\mathcal{B}_b(E,F)$) will denote the set of functions from E to F which are continuous (respectively bounded continuous, Borel, bounded Borel).

On a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for any $p \geq 1$, L^p will denote the set of random variables with finite p-th moment. A measurable space equipped with a right-continuous filtration $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \in \mathbb{T}})$ (where $\mathbb{T}$ is equal to $\mathbb{R}_+$ or to $[0,T]$ for some $T \in \mathbb{R}_+^*$) will be called a **filtered space**. A probability space equipped with a right-continuous filtration $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ will be called called a **stochastic basis** and will be said to **fulfill the usual conditions** if the probability space is complete and if $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible sets. We introduce now some notations and vocabulary about spaces of stochastic processes, on a fixed stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Most of them are taken or adapted from [60] or [61]. A process $(X_t)_{t \in \mathbb{T}}$ is said to be **integrable** if X_t is an integrable r.v. for any t. We will denote $\mathcal{V}$ (resp $\mathcal{V}^+$) the set of adapted, bounded variation (resp non-decreasing) processes starting at 0; $\mathcal{V}^p$ (resp $\mathcal{V}^{p,+}$) the elements of $\mathcal{V}$ (resp $\mathcal{V}^+$) which are predictable, and $\mathcal{V}^c$ (resp $\mathcal{V}^{c,+}$) the elements of $\mathcal{V}$ (resp $\mathcal{V}^+$) which are continuous. If $A \in \mathcal{V}$, we will denote $Pos(A)$ and $Neg(A)$ the positive variation and negative variation parts of A, meaning the unique pair of elements $\mathcal{V}^+$ such that $A = Pos(A) - Neg(A)$ (see Proposition I.3.3 in [61] for their existence) and $Var(A) = Pos(A) + Neg(A)$ its total variation. $\mathcal{M}$ will be the space of cadlag martingales. For any $p \in [1, \infty]$ $\mathcal{H}^p$ will denote the Banach space of elements of $\mathcal{M}$ for which $\|M\|_{\mathcal{H}^p} := \mathbb{E}[|\sup_{t \in \mathbb{T}} M_t|^p]^{\frac{1}{p}} < \infty$ and in this set we identify indistinguishable elements. $\mathcal{H}_0^p$ will denote the Banach subspace of $\mathcal{H}^p$ of elements vanishing at zero.

If $\mathbb{T} = [0,T]$ for some $T \in \mathbb{R}_+^*$, a stopping time will take values in $[0,T] \cup \{+\infty\}$. We define a **localizing sequence of stopping times** as an a.s. increasing sequence of stopping times $(\tau_n)_{n \geq 0}$ such that there a.s. exists $N \in \mathbb{N}$ for which $\tau_N = +\infty$. Let Y be a process and τ a stopping time, we denote by Y^τ the **stopped process** $t \mapsto Y_{t \wedge \tau}$. If $\mathcal{C}$ is a set of processes, we define its **localized class** $\mathcal{C}_{loc}$ as the set of processes Y such that there exists a localizing sequence $(\tau_n)_{n \geq 0}$ such that for every n, the stopped process Y^{τ_n} belongs to $\mathcal{C}$. In particular a process X is said to be locally integrable (resp. locally square integrable) if there is a localizing sequence $(\tau_n)_{n \geq 0}$ such that for every n, $X_t^{\tau_n}$ is integrable (resp. square integrable) for every t.

For any $M \in \mathcal{M}_{loc}$, we denote $[M]$ its **quadratic variation** and if moreover $M \in \mathcal{H}^2_{loc}$, $\langle M \rangle$ will denote its (predictable) **angular bracket**. $\mathcal{H}^2_0$ will be equipped with scalar product defined by $(M,N)_{\mathcal{H}^2_0} = \mathbb{E}[M_T N_T] = \mathbb{E}[\langle M,N\rangle_T]$ which makes it a Hilbert space. Two local martingales M, N will be said to be **strongly orthogonal** if MN is a local martingale starting in 0 at time 0. In $\mathcal{H}^2_{0,loc}$ this notion is equivalent to $\langle M,N\rangle = 0$.

2.3 BSDEs without driving martingale

In the whole present section we are given $T \in \mathbb{R}^*_+$, and a stochastic basis $\left(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ fulfilling the usual conditions. Some proofs and intermediary results of the first part of this section are postponed to Appendix 2.B.

Definition 2.3.1. *Let A and B be in $\mathcal{V}^+$. We will say that dB dominates dA* **in the sense of stochastic measures** *(written $dA \ll dB$) if for almost all ω, $dA(\omega) \ll dB(\omega)$ as Borel measures on $[0,T]$.*

We will say that dB and dA are mutually singular **in the sense of stochastic measures** *(written $dA \perp dB$) if for almost all ω, the Borel measures $dA(\omega)$ and $dB(\omega)$ are mutually singular.*

Let $B \in \mathcal{V}^+$. $dB \otimes d\mathbb{P}$ will denote the positive measure on $(\Omega \times [0,T], \mathcal{F} \otimes \mathcal{B}([0,T]))$ defined for any $F \in \mathcal{F} \otimes \mathcal{B}([0,T])$ by $dB \otimes d\mathbb{P}(F) = \mathbb{E}\left[\int_0^T \mathbb{1}_F(r,\omega) dB_r(\omega)\right]$. A property which holds true everywhere except on a null set for this measure will be said to be true $dB \otimes d\mathbb{P}$ almost everywhere (a.e).

The proof of Proposition below is in Appendix 2.B.

Proposition 2.3.2. *For any A and B in $\mathcal{V}^{p,+}$, there exists a (non-negative $dB \otimes d\mathbb{P}$ a.e.) predictable process $\frac{dA}{dB}$ and a process in $\mathcal{V}^{p,+}$ $A^{\perp B}$ such that*

$$dA^{\perp B} \perp dB \text{ and } A = A^B + A^{\perp B} \text{ a.s.}$$

where $A^B = \int_0^\cdot \frac{dA}{dB}(r) dB_r$. The process $A^{\perp B}$ is unique and the process $\frac{dA}{dB}$ is unique $dB \otimes d\mathbb{P}$ a.e. Moreover, there exists a predictable process K with values in $[0,1]$ (for every (ω,t)), such that $A^B = \int_0^\cdot \mathbb{1}_{\{K_r<1\}} dA_r$ and $A^{\perp B} = \int_0^\cdot \mathbb{1}_{\{K_r=1\}} dA_r$.

The predictable process $\frac{dA}{dB}$ appearing in the statement of Proposition 2.3.2 will beF the **Radon-Nikodym derivative** of A by B.

Remark 2.3.3. *Since for any $s<t$ $A_t - A_s = \int_s^t \frac{dA}{dB}(r) dB_r + A^{\perp B}_t - A^{\perp B}_s$ a.s. where $A^{\perp B}$ is increasing, it is clear that for any $s<t$, $\int_s^t \frac{dA}{dB}(r) dB_r \le A_t - A_s$ a.s. and therefore that for any positive measurable process ϕ we have $\int_0^T \phi_r \frac{dA}{dB}(r) dB_r \le \int_0^T \phi_r dA_r$ a.s.*

If A is in $\mathcal{V}^p$, and $B \in \mathcal{V}^{p,+}$. We set $\frac{dA}{dB} := \frac{dPos(A)}{dB} - \frac{dPos(A)}{dB}$ and $A^{\perp B} := (Pos(A))^{\perp B} - (Neg(A))^{\perp B}$.

Proposition 2.3.4. *Let A_1 and A_2 be in $\mathcal{V}^p$, and $B \in \mathcal{V}^{p,+}$. Then, $\frac{d(A_1+A_2)}{dB} = \frac{dA_1}{dB} + \frac{dA_2}{dB}$ $dV \otimes d\mathbb{P}$ a.e. and $(A_1+A_2)^{\perp B} = A_1^{\perp B} + A_2^{\perp B}$.*

Proof. The proof is an immediate consequence of the uniqueness of the decomposition (2.3.2). □

Let $V \in \mathcal{V}^{p,+}$. We introduce two significant spaces related to V.
$\mathcal{H}^{2,V} := \{M \in \mathcal{H}_0^2 | d\langle M\rangle \ll dV\}$ and $\mathcal{H}^{2,\perp V} := \{M \in \mathcal{H}_0^2 | d\langle M\rangle \perp dV\}$.

The proof of the two propositions below are in Appendix 2.B.

Proposition 2.3.5. *Let $M \in \mathcal{H}_0^2$, and let $V \in \mathcal{V}^{p,+}$. There exists a pair $(M^V, M^{\perp V})$ in $\mathcal{H}^{2,V} \times \mathcal{H}^{2,\perp V}$ such that $M = M^V + M^{\perp V}$ and $\langle M^V, M^{\perp V}\rangle = 0$.*

Moreover, we have $\langle M^V\rangle = \langle M\rangle^V = \int_0^\cdot \frac{d\langle M\rangle}{dV}(r)dV_r$ and $\langle M^{\perp V}\rangle = \langle M\rangle^{\perp V}$ and there exists a predictable process K with values in $[0,1]$ such that
$M^V = \int_0^\cdot \mathbb{1}_{\{K_r<1\}} dM_r$ *and* $M^{\perp V} = \int_0^\cdot \mathbb{1}_{\{K_r=1\}} dM_r$.

Proposition 2.3.6. *$\mathcal{H}^{2,V}$ and $\mathcal{H}^{2,\perp V}$ are orthogonal sub-Hilbert spaces of $\mathcal{H}_0^2$ and $\mathcal{H}_0^2 = \mathcal{H}^{2,V} \oplus^\perp \mathcal{H}^{2,\perp V}$. Moreover, any element of $\mathcal{H}_{loc}^{2,V}$ is strongly orthogonal to any element of $\mathcal{H}_{loc}^{2,\perp V}$.*

Remark 2.3.7. *All previous results extend when the filtration is indexed by $\mathbb{R}_+$.*

We are going to introduce here our Backward Stochastic Differential Equation (BSDE) for which there is no need for having a particular martingale of reference.

We will denote $\mathcal{P}ro$ the σ-field generated by progressively measurable processes defined on $[0,T]\times\Omega$. Given some $V \in \mathcal{V}^{c,+}$, we will indicate by $\mathcal{L}^2(dV\otimes d\mathbb{P})$ (resp. $\mathcal{L}^0(dV\otimes d\mathbb{P})$) the set of (up to indistinguishability) progressively measurable processes ϕ such that $\mathbb{E}[\int_0^T \phi_r^2 dV_r] < \infty$ (resp. $\int_0^T |\phi_r| dV_r < \infty$ $\mathbb{P}$ a.s.) and $L^2(dV\otimes d\mathbb{P})$ the quotient space of $\mathcal{L}^2(dV\otimes d\mathbb{P})$ with respect to the subspace of processes equal to zero $dV\otimes d\mathbb{P}$ a.e. More formally, $L^2(dV\otimes d\mathbb{P})$ corresponds to the classical L^2 space $L^2([0,T]\times\Omega, \mathcal{P}ro, dV\otimes d\mathbb{P})$ and is therefore complete for its usual norm.
$\mathcal{L}^{2,cadlag}(dV\otimes d\mathbb{P})$ (resp. $L^{2,cadlag}(dV\otimes d\mathbb{P})$) will denote the subspace of $\mathcal{L}^2(dV\otimes d\mathbb{P})$ (resp. $L^2(dV\otimes d\mathbb{P})$) of cadlag elements (resp. of elements having a cadlag representative). We emphasize that $L^{2,cadlag}(dV\otimes d\mathbb{P})$ is not a closed subspace of $L^2(dV\otimes d\mathbb{P})$.
The application which associates to a process its corresponding class will be denoted $\phi \mapsto \dot{\phi}$.

The aforementioned BSDE will depend on a triple (V,ξ,f) of coefficients: V is an integrator process, ξ is the **final** condition, f is the **driver**.

Hypothesis 2.3.8.
1. *V is bounded continuous non-decreasing adapted process;*
2. *ξ is a square integrable $\mathcal{F}_T$-measurable r.v.*
3. *$\hat{f} : ([0,T]\times\Omega)\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, measurable with respect to $\mathcal{P}ro\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$.*
4. *$\hat{f}(\cdot,\cdot,0,0) \in \mathcal{L}^2(dV\otimes d\mathbb{P})$.*
5. *There exist positive constants K^Y, K^Z such that, $\mathbb{P}$ a.s. we have for all t, y, y', z, z',*
$$|\hat{f}(t,\cdot,y,z) - \hat{f}(t,\cdot,y',z')| \le K^Y|y-y'| + K^Z|z-z'|. \tag{2.3.1}$$

We start with a lemma.

Lemma 2.3.9. *Let U_1 and U_2 be in $\mathcal{L}^2(dV\otimes d\mathbb{P})$ and such that $\dot{U}_1 = \dot{U}_2$. Let $F : [0,T]\times\Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ be such that $F(\cdot,\cdot,U_1)$ and $F(\cdot,\cdot,U_2)$ are in $\mathcal{L}^0(dV\otimes d\mathbb{P})$, then the processes $\int_0^\cdot F(r,\cdot,U_r^1)dV_r$ and $\int_0^\cdot F(r,\cdot,U_r^2)dV_r$ are indistinguishable.*

Proof. There exists a $\mathbb{P}$-null set $\mathcal{N}$ such that for any $\omega\in\mathcal{N}^c$, $U^1(\omega) = U^2(\omega)$ $dV(\omega)$ a.e. So for any $\omega\in\mathcal{N}^c$, $F(\cdot,\omega,U^1(\omega)) = F(\cdot,\omega,U^2(\omega))$ $dV(\omega)$ a.e. implying
$\int_0^\cdot F(r,\omega,U_r^1(\omega))dV_r(\omega) = \int_0^\cdot F(r,\omega,U_r^2(\omega))dV_r(\omega)$. So $\int_0^\cdot F(r,\cdot,U_r^1)dV_r$ and $\int_0^\cdot F(r,\cdot,U_r^2)dV_r$ are indistinguishable processes. □

In some of the following proofs, we will have to work with classes of processes. According to Lemma 2.3.9, if $\dot{U}$ is an element of $L^2(dV \otimes d\mathbb{P})$, the integral $\int_0^\cdot F(r,\omega,U_r)dV_r$ will not depend on the representantive process U that we have chosen.

We will now give the formulation of our BSDE.

Definition 2.3.10. *We say that a couple $(Y,M) \in \mathcal{L}^{2,cadlag}(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ is a solution of $BSDE(\xi,\hat{f},V)$ if it verifies*

$$Y = \xi + \int_\cdot^T \hat{f}\left(r,\cdot,Y_r,\sqrt{\frac{d\langle M\rangle}{dV}}(r)\right)dV_r - (M_T - M_\cdot) \tag{2.3.2}$$

in the sense of indistinguishability.

Proposition 2.3.11. *If (Y,M) solves $BSDE(\xi,\hat{f},V)$, and if we denote $\hat{f}\left(r,\cdot,Y_r,\sqrt{\frac{d\langle M\rangle}{dV}}(r)\right)$ by $\hat{f}_r$, then for any $t \in [0,T]$, a.s. we have*

$$\begin{cases} Y_t &= \mathbb{E}\left[\xi + \int_t^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right] \\ M_t &= \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_0\right]. \end{cases} \tag{2.3.3}$$

Proof. Since $Y_t = \xi + \int_t^T \hat{f}_r dV_r - (M_T - M_t)$ a.s., Y being an adapted process and M a martingale, taking the expectation in (2.3.2) at time t, we directly get $Y_t = \mathbb{E}\left[\xi + \int_t^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right]$ and in particular that $Y_0 = \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_0\right]$. Since $M_0 = 0$, looking at the BSDE at time 0 we get $M_T = \xi + \int_0^T \hat{f}_r dV_r - Y_0 = \xi + \int_0^T \hat{f}_r dV_r - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_0\right]$. Taking the expectation with respect to $\mathcal{F}_t$ in the above inequality gives the second line of (2.3.3).

□

We will proceed showing that $BSDE(\xi,\hat{f},V)$ has a unique solution. At this point we introduce a significant map Φ which will map $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ into its subspace $L^{2,cadlag}(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$. From now on, until Notation 2.3.15, we fix a couple $(\dot{U},N) \in L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ to which we will associate $(\dot{Y},M)$ which, as we will show, will belong to $L^{2,cadlag}(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$. We will show that $(\dot{U},N) \mapsto (\dot{Y},M)$ is a contraction for a certain norm. In all the proofs below, $\dot{U}$ will only appear in integrals driven by dV through a representative U.

Proposition 2.3.12. *For any $t \in [0,T]$, $\int_t^T \hat{f}^2\left(r,\cdot,U_r,\sqrt{\frac{d\langle N\rangle}{dV}}(r)\right)dV_r$ is in L^1 and $\left(\int_t^T \hat{f}\left(r,\cdot,U_r,\sqrt{\frac{d\langle N\rangle}{dV}}(r)\right)dV_r\right)$ is in L^2.*

Proof. By Cauchy-Schwarz inequality and thanks to the boundedness of V together the Lipschitz conditions on f in Hypothesis 2.3.8, there exist a positive constant C such that, for any $t \in [0,T]$, we have

$$\begin{aligned} &\left(\int_t^T \hat{f}\left(r,\cdot,U_r,\sqrt{\tfrac{d\langle N\rangle}{dV}}(r)\right)dV_r\right)^2 \leq V_T^2 \int_t^T \hat{f}^2\left(r,\cdot,U_r,\sqrt{\tfrac{d\langle N\rangle}{dV}}(r)\right)dV_r \\ &\leq C\left(\int_t^T \hat{f}^2\left(r,\cdot,0,0\right)dV_r + \int_t^T U_r^2 dV_r + \int_t^T \tfrac{d\langle N\rangle}{dV}(r)dV_r\right). \end{aligned} \tag{2.3.4}$$

The three terms on the right are in L^1. Indeed, by Remark 2.3.3 $\int_t^T \frac{d\langle N\rangle}{dV}(r)dV_r \leq (\langle N\rangle_T - \langle N\rangle_t)$ which belongs to L^1 since N is taken in $\mathcal{H}^2$. By Hypothesis 2.3.8, $f(\cdot,\cdot,0,0)$ is in $\mathcal{L}^2(dV \otimes d\mathbb{P})$, and $\dot{U}$ was also taken in $L^2(dV \otimes d\mathbb{P})$. This concludes the proof. □

We can therefore state the following definition.

Definition 2.3.13. *Setting* $\hat{f}_r = \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}}(r)\right)$*, we define* M *as the cadlag version of the martingale* $t \mapsto \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_0\right]$.

It admits a cadlag version taking into account Theorem 4 in Chapter IV of [33], since the stochastic basis fulfills the usual conditions. We denote by Y *the cadlag process defined by* $Y_t = \xi + \int_t^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}}(r)\right) dV_r - (M_T - M_t)$*. This will be called the* **cadlag reference process** *and we will often omit its dependence to* $(\dot{U}, N)$*.*

According to previous definition, it is not clear whether Y is adapted, however, we have the almost sure equalities

$$\begin{aligned} Y_t &= \xi + \int_t^T \hat{f}_r dV_r - (M_T - M_t) \\ &= \xi + \int_t^T \hat{f}_r dV_r - \left(\xi + \int_0^T \hat{f}_r dV_r - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right]\right) \\ &= \mathbb{E}\left[\xi + \int_0^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right] - \int_0^t \hat{f}_r dV_r \\ &= \mathbb{E}\left[\xi + \int_t^T \hat{f}_r dV_r \middle| \mathcal{F}_t\right]. \end{aligned} \tag{2.3.5}$$

Since Y is cadlag and adapted, by Theorem 15 Chapter IV of [32], it is progressively measurable.

Proposition 2.3.14. *M belongs to $\mathcal{H}_0^2$ and $\sup_{t\in[0,T]} |Y_t| \in L^2$.*

Proof. M is square integrable and vanishes at 0 by Definition 2.3.13 and Proposition 2.3.12. A consequence of Definition 2.3.13, of Cauchy-Schwarz inequality and of the boundedness of V is the existence of some $C, C' > 0$ such that, a.s.,

$$\begin{aligned} \sup_{t\in[0,T]} Y_t^2 &\leq C\left(\xi^2 + \sup_{t\in[0,T]}\left(\int_t^T \hat{f}_r dV_r\right)^2 + \sup_{t\in[0,T]}(M_T - M_t)^2\right) \\ &\leq C'\left(\xi^2 + \int_0^T \hat{f}_r^2 dV_r + \sup_{t\in[0,T]} M_t^2\right) \end{aligned} \tag{2.3.6}$$

which belongs to L^1 by Proposition 2.3.12 and the fact that ξ and M are square integrable. □

Since Y is cadlag progressively measurable, $\sup_{t\in[0,T]} |Y_t| \in L^2$ and since V is bounded, it is clear that $Y \in \mathcal{L}^{2,cadlag}(dV \otimes d\mathbb{P})$ and the corresponding class $\dot{Y}$ belongs to $L^{2,cadlag}(dV \otimes d\mathbb{P})$.

Notation 2.3.15. *We denote by* Φ *the operator which associates to a couple* $(\dot{U}, N)$ *the couple* $(\dot{Y}, M)$*.*

$$\Phi : \begin{array}{ccc} L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2 & \longrightarrow & L^{2,cadlag}(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2 \\ (\dot{U}, N) & \longmapsto & (\dot{Y}, M). \end{array}$$

Proposition 2.3.16. *The mapping* $(Y, M) \longmapsto (\dot{Y}, M)$ *induces a bijection between the set of solutions of* $BSDE(\xi, \hat{f}, V)$ *and the set of fixed points of* Φ*.*

Proof. First, let (U, N) be a solution of $BSDE(\xi, \hat{f}, V)$, let $(\dot{Y}, M) := \Phi(\dot{U}, N)$ and let Y be the reference cadlag process associated to U as in Definition 2.3.13. By this same definition, M is the cadlag version of

$t \mapsto \mathbb{E}\left[\xi + \int_0^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}}(r)\right) dV_r \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\xi + \int_0^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}}(r)\right) dV_r \middle| \mathcal{F}_0\right]$, but by Proposition 2.3.11, so is N, meaning $M = N$. Again by Definition 2.3.13, $Y = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}}(r)\right) dV_r - (N_T - N_\cdot)$ which is equal to U thanks to (2.3.2), so $Y = U$ in the sense of indistinguishability, and in particular, $\dot{U} = \dot{Y}$, implying $(\dot{U}, N) = (\dot{Y}, M) = \Phi(\dot{U}, N)$. The mapping $(Y, M) \longmapsto (\dot{Y}, M)$ therefore does indeed map the set of solutions of $BSDE(\xi, \hat{f}, V)$ into the set of fix points of Φ.

The map is surjective. Indeed let $(\dot{U}, N)$ be a fixed point of Φ, the couple (Y, M) of Definition 2.3.13 verifies $Y = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}}(r)\right) dV_r - (M_T - M_\cdot)$ in the sense of indistinguishability, and $(\dot{Y}, M) = \Phi(\dot{U}, N) = (\dot{U}, N)$, so by Lemma 2.3.9, $\int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r$ and $\int_\cdot^T \hat{f}\left(r, \cdot, U_r, \sqrt{\frac{d\langle N\rangle}{dV}}(r)\right) dV_r$ are indistinguishable and $Y = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r - (M_T - M_\cdot)$, meaning that (Y, M) solves $BSDE(\xi, \hat{f}, V)$.

We finally show that it is injective. Let us consider two solutions (Y^1, M) and (Y^2, M) of $BSDE(\xi, \hat{f}, V)$ with $\dot{Y^1} = \dot{Y^2}$. By Lemma 2.3.9, the processes $\int_\cdot^T \hat{f}\left(r, \cdot, Y^1_r, \sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r$ and $\int_\cdot^T \hat{f}\left(r, \cdot, Y^2_r, \sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r$ are indistinguishable, so taking (2.3.2) into account, we have $Y^1 = Y^2$. □

From now on, if $(\dot{Y}, M)$ is the image by Φ of a couple $(\dot{U}, N) \in L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}^2_0$, by default, we will always refer to the cadlag reference process Y of $\dot{Y}$ defined in Definition 2.3.13.

Lemma 2.3.17. *Let Y be a cadlag adapted process satisfying $\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^2\right] < \infty$ and M be a square integrable martingale. Then there exists a constant $C > 0$ such that for any $\epsilon > 0$ we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t Y_{r^-} dM_r\right|\right] \leq C\left(\frac{\epsilon}{2}\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^2\right] + \frac{1}{2\epsilon}\mathbb{E}\left[[M]_T\right]\right).$$

In particular, $\int_0^\cdot Y_{r^-} dM_r$ is a uniformly integrable martingale.

Proof. By Burkholder-Davis-Gundy and Cauchy-Schwarz inequalities, there exists $C > 0$ such that

$$\begin{array}{rcl} \mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t Y_{r^-} dM_r\right|\right] & \leq & C\mathbb{E}\left[\sqrt{\int_0^T Y^2_{r^-} d[M]_r}\right] \\ \leq C\mathbb{E}\left[\sqrt{\sup_{t\in[0,T]} Y_t^2 [M]_T}\right] & \leq & C\sqrt{\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^2\right]\mathbb{E}[[M]]_T} \\ \leq C\left(\frac{\epsilon}{2}\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^2\right] + \frac{1}{2\epsilon}\mathbb{E}\left[[M]_T\right]\right) & < & +\infty. \end{array}$$

So $\int_0^\cdot Y_{r^-} dM_r$ is a uniformly integrable local martingale, and therefore a martingale. □

Lemma 2.3.18. *Let Y be a cadlag adapted process and $M \in \mathcal{H}^2$. Assume the existence of a constant $C > 0$ and an L^1-random variable Z such that for any $t \in [0,T]$, $Y_t^2 \leq C\left(Z + \left|\int_0^t Y_{r-} dM_r\right|\right)$. Then $\sup_{t\in[0,T]} |Y_t| \in L^2$.*

Proof. For any stopping time τ we have

$$\sup_{t\in[0,\tau]} Y_t^2 \leq C\left(Z + \sup_{t\in[0,\tau]} \left|\int_0^t Y_{r-} dM_r\right|\right). \tag{2.3.7}$$

Since Y_{t-} is caglad and therefore locally bounded, (see Definition p164 in [78]) we define $\tau_n = \inf\{t > 0 : Y_{t-} \geq n\}$. It yields $\int_0^{\cdot\wedge\tau_n} Y_{r-} dM_r$ is in $\mathcal{H}^2$ since its angular bracket is equal to $\int_0^{\cdot\wedge\tau_n} Y_{r-}^2 d\langle M\rangle_r$ which is inferior to $n^2\langle M\rangle_T \in L^1$. By Doob's inequality we know that $\sup_{t\in[0,\tau_n]} \left|\int_0^t Y_{r-} dM_r\right|$ is L^2 and using (2.3.7), we get that $\sup_{t\in[0,\tau_n]} Y_t^2$ is L^1. By (2.3.7) applied with τ_n and taking expectation, we get $\mathbb{E}\left[\sup_{t\in[0,\tau_n]} Y_t^2\right] \leq C'\left(1 + \mathbb{E}\left[\sup_{t\in[0,\tau_n]} \left|\int_0^t Y_{r-} dM_r\right|\right]\right)$, for some C' which does not depend on n. By Lemma 2.3.17 applied to (Y^{τ_n}, M) there exists $C'' > 0$ such that for any $n \in \mathbb{N}^*$ and $\epsilon > 0$,
$\mathbb{E}\left[\sup_{t\in[0,\tau_n]} Y_t^2\right] \leq C''\left(1 + \frac{\epsilon}{2}\mathbb{E}\left[\sup_{t\in[0,\tau_n]} Y_t^2\right] + \frac{1}{2\epsilon}\mathbb{E}\left[[M]_T\right]\right)$. Choosing $\epsilon = \frac{1}{C''}$, it follows that there exists $C_3 > 0$ such that for any $n > 0$,
$\frac{1}{2}\mathbb{E}\left[\sup_{t\in[0,\tau_n]} Y_t^2\right] \leq C_3\left(1 + \mathbb{E}\left[[M]_T\right]\right) < \infty$. By monotone convergence theorem, taking the limit in n we get the result. □

Proposition 2.3.19. *Let $\lambda \in \mathbb{R}$, let $(\dot{U}, N)$, $(\dot{U}', N')$ be in $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$, let $(\dot{Y}, M)$, $(\dot{Y}', M')$ be their images by Φ and let Y, Y' be the cadlag representatives of $\dot{Y}$, $\dot{Y}'$ introduced in Definition 2.3.13. Then $\int_0^\cdot e^{\lambda V_r} Y_{r-} dM_r$, $\int_0^\cdot e^{\lambda V_r} Y'_{r-} dM'_r$, $\int_0^\cdot e^{\lambda V_r} Y_{r-} dM'_r$ and $\int_0^\cdot e^{\lambda V_r} Y'_{r-} dM_r$ are martingales.*

Proof. Thanks to Proposition 2.3.14 we know that $\sup_{t\in[0,T]} |Y_t|$ and $\sup_{t\in[0,T]} |Y'_t|$ are L^2. Moreover since M and M' are square integrable, the statement yields therefore as a consequence of Lemma 2.3.17 and the fact that V is bounded. □

We will now show that Φ is a contraction for a certain norm. This will imply that it has a unique fixed point in $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ since this space is complete and therefore that $BSDE(\xi, \hat{f}, V)$ has a unique solution thanks to Proposition 2.3.16.
For any $\lambda > 0$, on $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ we define the norm
$\|(\dot{Y}, M)\|_\lambda^2 := \mathbb{E}\left[\int_0^T e^{\lambda V_r} Y_r^2 dV_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda V_r} d\langle M\rangle_r\right]$. Since V is bounded, these norms are all equivalent to the usual one of this space, which corresponds to $\lambda = 0$.

Proposition 2.3.20. *There exists $\lambda > 0$ such that for any $(\dot{U}, N) \in L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$, $\left\|\Phi(\dot{U}, N)\right\|_\lambda^2 \leq \frac{1}{2}\left\|(\dot{U}, N)\right\|_\lambda^2$. In particular, Φ is a contraction in $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ for the norm $\|\cdot\|_\lambda$.*

Proof. Let $(\dot{U}, N)$ and $(\dot{U}', N')$ be two couples of $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$, let $(\dot{Y}, M)$ and $(\dot{Y}', M')$ be their images via Φ and let Y, Y' be the cadlag reference process of $\dot{Y}$, $\dot{Y}'$ introduced in Definition 2.3.13. We will write $\bar{Y}$ for $Y - Y'$ and we adopt a similar notation for other processes. We will also write

$$\bar{f}_t := \hat{f}\left(t,\cdot,U_t,\sqrt{\tfrac{d\langle N\rangle}{dV}}(t)\right) - \hat{f}\left(t,\cdot,U'_t,\sqrt{\tfrac{d\langle N'\rangle}{dV}}(t)\right).$$

By additivity, we have $d\bar{Y}_t = -\bar{f}_t dV_t + d\bar{M}_t$. Since $\bar{Y}_T = \xi - \xi = 0$, applying the integration by parts formula to $\bar{Y}_t^2 e^{\lambda V_t}$ between 0 and T we get

$$\bar{Y}_0^2 - 2\int_0^T e^{\lambda V_r}\bar{Y}_r\bar{f}_r dV_r + 2\int_0^T e^{\lambda V_r}\bar{Y}_{r-}d\bar{M}_r + \int_0^T e^{\lambda V_r}d[\bar{M}]_r + \lambda\int_0^T e^{\lambda V_r}\bar{Y}_r^2 dV_r = 0.$$

Since, by Proposition 2.3.19, the stochastic integral with respect to $\bar{M}$ is a real martingale, by taking the expectations we get

$$\mathbb{E}\left[\bar{Y}_0^2\right] - 2\mathbb{E}\left[\int_0^T e^{\lambda V_r}\bar{Y}_r\bar{f}_r dV_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda V_r}d\langle\bar{M}\rangle_r\right] + \lambda\mathbb{E}\left[\int_0^T e^{\lambda V_r}\bar{Y}_r^2 dV_r\right] = 0.$$

So by re-arranging and by using the Lipschitz condition on f stated in Hypothesis 2.3.8, we get

$$\begin{aligned}
& \lambda\mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}\bar{Y}_r^2 dV_r\right] + \mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}d\langle\bar{M}\rangle_r\right] \\
\leq\ & 2K^Y\mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}|\bar{Y}_r||\bar{U}_r|dV_r\right] \\
& +2K^Z\mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}|\bar{Y}_r|\left|\sqrt{\frac{d\langle N\rangle}{dV}}(r) - \sqrt{\frac{d\langle N'\rangle}{dV}}(r)\right|dV_r\right] \\
\leq\ & (K^Y\alpha + K^Z\beta)\mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}|\bar{Y}_r|^2 dV_r\right] + \frac{K^Y}{\alpha}\mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}|\bar{U}_r|^2 dV_r\right] \\
& +\frac{K^Z}{\beta}\mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}\left|\sqrt{\frac{d\langle N\rangle}{dV}}(r) - \sqrt{\frac{d\langle N'\rangle}{dV}}(r)\right|^2 dV_r\right],
\end{aligned}$$

for any positive α and β. Then we pick $\alpha = 2K^Y$ and $\beta = 2K^Z$, which gives us

$$\begin{aligned}
& \lambda\mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}\bar{Y}_r^2 dV_r\right] + \mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}d\langle\bar{M}\rangle_r\right] \\
\leq\ & 2((K^Y)^2 + (K^Z)^2)\mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}|\bar{Y}_r|^2 dV_r\right] \\
+\ & \tfrac{1}{2}\mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}|\bar{U}_r|^2 dV_r\right] + \tfrac{1}{2}\mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}\left|\sqrt{\frac{d\langle N\rangle}{dV}}(r) - \sqrt{\frac{d\langle N'\rangle}{dV}}(r)\right|^2 dV_r\right].
\end{aligned}$$

We choose now $\lambda = 1 + 2((K^Y)^2 + (K^Z)^2)$ and we get

$$\begin{aligned}
& \mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}\bar{Y}_r^2 dV_r\right] + \mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}d\langle\bar{M}\rangle_r\right] \\
\leq\ & \tfrac{1}{2}\mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}|\bar{U}_r|^2 dV_r\right] + \tfrac{1}{2}\mathbb{E}\left[\textstyle\int_0^T e^{\lambda V_r}\left|\sqrt{\frac{d\langle N\rangle}{dV}}(r) - \sqrt{\frac{d\langle N'\rangle}{dV}}(r)\right|^2 dV_r\right].
\end{aligned} \tag{2.3.8}$$

On the other hand, since by Proposition 2.B.1 we know that $\frac{d\langle N\rangle}{dV}\frac{d\langle N'\rangle}{dV} - \left(\frac{d\langle N,N'\rangle}{dV}\right)^2$ is a positive process, we have

$$\begin{aligned}
\left|\sqrt{\tfrac{d\langle N\rangle}{dV}} - \sqrt{\tfrac{d\langle N'\rangle}{dV}}\right|^2 &= \tfrac{d\langle N\rangle}{dV} - 2\sqrt{\tfrac{d\langle N\rangle}{dV}}\sqrt{\tfrac{d\langle N'\rangle}{dV}} + \tfrac{d\langle N'\rangle}{dV} \\
&\leq \tfrac{d\langle N\rangle}{dV} - 2\tfrac{d\langle N,N'\rangle}{dV} + \tfrac{d\langle N'\rangle}{dV} \\
&= \tfrac{d\langle\bar{N}\rangle}{dV}\ dV\otimes d\mathbb{P}\text{ a.e.}
\end{aligned} \tag{2.3.9}$$

Therefore, since by Remark 2.3.3 we have $\int_0^\cdot e^{\lambda V_r}\frac{d\langle\bar{N}\rangle}{dV}(r)dV_r \leq \int_0^\cdot e^{\lambda V_r}d\langle\bar{N}\rangle_r$, then expression (2.3.8) implies
$\mathbb{E}\left[\int_0^T e^{\lambda V_r}\bar{Y}_r^2 dV_r + \int_0^T e^{\lambda V_r}d\langle\bar{M}\rangle_r\right] \leq \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda V_r}|\bar{U}_r|^2 dV_r + \int_0^T e^{\lambda V_r}d\langle\bar{N}\rangle_r\right]$, which proves the contraction for the norm $\|\cdot\|_\lambda$. □

Theorem 2.3.21. *If $(\xi, \hat{f})$ verifies Hypothesis 2.3.8 then $BSDE(\xi, \hat{f}, V)$ has a unique solution.*

Proof. The space $L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ is complete and Φ defines on it a contraction for the norm $\|(\cdot,\cdot)\|_\lambda$ for some $\lambda > 0$, so Φ has a unique fixed point in
$L^2(dV \otimes d\mathbb{P}) \times \mathcal{H}_0^2$. Then by Proposition 2.3.16, $BSDE(\xi, \hat{f}, V)$ has a unique solution. □

Remark 2.3.22. *Let (Y, M) be the solution of $BSDE(\xi, \hat{f}, V)$ and $\dot{Y}$ the class of Y in $L^2(dV \otimes d\mathbb{P})$. Thanks to Proposition 2.3.16, we know that*
$(\dot{Y}, M) = \Phi(\dot{Y}, M)$ and therefore by Propositions 2.3.14 and 2.3.19 that $\sup_{t\in[0,T]} |Y_t|$ is L^2 and that $\int_0^\cdot Y_{r-} dM_r$ is a real martingale.

Remark 2.3.23. *Let $(\xi, \hat{f}, V)$ satisfying Hypothesis 2.3.8. Until now we have considered the related BSDE on the interval $[0, T]$. Without restriction of generality we can consider a BSDE on a restricted interval $[s, T]$ for some $s \in [0, T[$. The results and comments of this section immediately extend to this case. In particular there exists a unique couple of processes (Y^s, M^s), indexed by $[s, T]$ such that Y^s is adapted, cadlag and verifies $\mathbb{E}[\int_s^T (Y_r^s)^2 dV_r] < \infty$, such that M^s is a martingale starting at 0 in s and such that $Y_\cdot^s = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y_r^s, \sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r - (M_T^s - M_\cdot^s)$ in the sense of indistinguishability on $[s, T]$.*
Moreover, if (Y, M) denotes the solution of $BSDE(\xi, \hat{f}, V)$ then $(Y, M_\cdot - M_s)$ and (Y^s, M^s) coincide on $[s, T]$. This follows by the uniqueness argument for the restricted BSDE to $[s, T]$.

The lemma below shows that, in order to verify that a couple (Y, M) is the solution of $BSDE(\xi, \hat{f}, V)$, it is not necessary to verify the square integrability of Y since it will be automatically fulfilled.

Lemma 2.3.24. *Let $(\xi, \hat{f}, V)$ verify Hypothesis 2.3.8 and consider $BSDE(\xi, \hat{f}, V)$ defined in Definition 2.3.10. Assume that there exists a cadlag adapted process Y with $Y_0 \in L^2$, and $M \in \mathcal{H}_0^2$ such that*

$$Y = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r - (M_T - M_\cdot), \tag{2.3.10}$$

in the sense of indistinguishability. Then $\sup_{t\in[0,T]} |Y_t| \in L^2$. In particular,
$Y \in \mathcal{L}^2(dV \otimes d\mathbb{P})$ and (Y, M) is the unique solution of $BSDE(\xi, \hat{f}, V)$.

On the other hand if (Y, M) verifies (2.3.10) *on $[s, T]$ with $s < T$, if $Y_s \in L^2$, $M_s = 0$ and if we denote (U, N) the unique solution of $BSDE(\xi, \hat{f}, V)$, then (Y, M) and $(U, N_\cdot - N_s)$ are indistinguishable on $[s, T]$.*

Proof. Let $\lambda > 0$ and $t \in [0, T]$. By integration by parts formula applied to $Y^2 e^{-\lambda V}$ between 0 and t we get

$$\begin{aligned} Y_t^2 e^{-\lambda V_t} - Y_0^2 &= -2\int_0^t e^{-\lambda V_r} Y_r \hat{f}\left(r, \cdot, Y_r, \sqrt{\tfrac{d\langle M\rangle}{dV}}(r)\right) dV_r + 2\int_0^t e^{-\lambda V_r} Y_{r-} dM_r \\ &\quad + \int_0^t e^{-\lambda V_r} d[M]_r - \lambda \int_0^t e^{-\lambda V_r} Y_r^2 dM_r. \end{aligned}$$

By re-arranging the terms and using the Lipschitz conditions in Hypothesis 2.3.8, we get

$$\begin{aligned} & Y_t^2 e^{-\lambda V_t} + \lambda \int_0^t e^{-\lambda V_r} Y_r^2 dV_r \\ \leq\ & Y_0^2 + 2\int_0^t e^{-\lambda V_r} |Y_r| |\hat{f}| \left(r, \cdot, Y_r, \sqrt{\tfrac{d\langle M\rangle}{dV}}(r)\right) dV_r + 2\left|\int_0^t e^{-\lambda V_r} Y_{r-} dM_r\right| \\ & + \int_0^t e^{-\lambda V_r} d[M]_r \\ \leq\ & Y_0^2 + \int_0^t e^{-\lambda V_r} |\hat{f}|^2(r, \cdot, 0, 0) dV_r + (2K^Y + 1 + K^Z)\int_0^t e^{-\lambda V_r} |Y_r|^2 dV_r \\ & + 2\left|\int_0^t e^{-\lambda V_r} Y_{r-} dM_r\right| + \int_0^t e^{-\lambda V_r} d[M]_r. \end{aligned}$$

Choosing $\lambda = 2K^Y + 1 + K^Z$ this gives

$$\begin{aligned} Y_t^2 e^{-\lambda V_t} &\leq Y_0^2 + \int_0^t e^{-\lambda V_r} |\hat{f}|^2(r,\cdot,0,0) dV_r + K^Z \int_0^t e^{-\lambda V_r} \frac{d\langle M\rangle}{dV}(r) dV_r \\ &\quad + 2\left| \int_0^t e^{-\lambda V_r} Y_{r-} dM_r \right| + \int_0^t e^{-\lambda V_r} d[M]_r. \end{aligned}$$

Since V is bounded, there is a constant $C > 0$, such that for any $t \in [0,T]$

$$Y_t^2 \leq C \left(Y_0^2 + \int_0^T |\hat{f}|^2(r,\cdot,0,0) dV_r + \int_0^T \frac{d\langle M\rangle}{dV}(r) dV_r + [M]_T + \left| \int_0^t Y_{r-} dM_r \right| \right).$$

By Hypothesis 2.3.8 and since we assumed $Y_0 \in L^2$ and $M \in \mathcal{H}^2$, the first four terms on the right hand side are integrable and we can conclude by Lemma 2.3.18.

An analogous proof also holds on the interval $[s,T]$ taking into account Remark 2.3.23. □

If the underlying filtration is Brownian and $V_t = t$, we can identify the solution of the BSDE with no driving martingale to the solution of a Brownian BSDE.

Let B be a 1-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $T \in \mathbb{R}_+^*$ and for any $t \in [0,T]$, let $\mathcal{F}_t^B$ denote the σ-field $\sigma(B_r | r \in [0,t])$ augmented with the $\mathbb{P}$-negligible sets.
In the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}^B, \mathbb{P})$, let $V_t = t$ and $(\xi, \hat{f})$ satisfy Hypothesis 2.3.8. Let (Y, M) be the unique solution of $BSDE(\xi, \hat{f}, V)$, see Theorem 2.3.21.

Proposition 2.3.25. *We have $Y = U$, $M = \int_0^\cdot Z_r dB_r$, where (U,Z) is the unique solution of the Brownian BSDE*

$$U = \xi + \int_\cdot^T \hat{f}(r,\cdot,U_r,|Z_r|)\, dr - \int_\cdot^T Z_r dB_r. \tag{2.3.11}$$

Proof. By Theorem 1.2 in [70], (2.3.11) admits a unique solution (U,Z) of progressively measurable processes such that $Z \in L^2(dt \otimes d\mathbb{P})$. It is known that $\sup_{t\in[0,T]} |U_t| \in L^2$ and therefore that $U \in \mathcal{L}^2(dt \otimes d\mathbb{P})$, see Proposition 1.1 in [70] for instance. We define $N = \int_0^\cdot Z_r dB_r$. The couple (U,N) belongs to $\mathcal{L}^2(dt \otimes d\mathbb{P}) \times \mathcal{H}_0^2$. N verifies $\frac{d\langle N\rangle_r}{dr} = Z_r^2\ dt \otimes d\mathbb{P}$ a.e. So by (2.3.11), the couple (U,N) verifies $U = \xi + \int_\cdot^T \hat{f}\left(r,\cdot,U_r,\sqrt{\frac{d\langle N\rangle_r}{dr}}\right) dr - (N_T - N_\cdot)$ in the sense of indistinguishability. It therefore solves $BSDE(\xi,\hat{f},V)$ and the assertion yields by uniqueness of the solution. □

2.4 Martingale Problem and Markov classes

In this section, we introduce the Markov process which will later be the forward process which will be coupled to a BSDE in order to constitute Markovian BSDEs with no driving martingales. For details about the exact mathematical background that we use to define our Markov process, one can consult the Section 2.A of the Appendix. We also introduce the martingale problem related to this Markov process.

Let E be a Polish space and $T \in \mathbb{R}_+^*$ be a fixed horizon. From now on, $(\Omega, \mathcal{F}, \mathbb{F})$ denotes the canonical space and $(X_t)_{t\in[0,T]}$ the canonical process defined in Definition 2.A.1. We consider a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to a transition kernel measurable in time as defined in Definitions 2.A.5 and 2.A.4, and for any $(s,x) \in [0,T] \times E$, $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$ will denote the stochastic basis introduced in Definition 2.A.9 and which fulfills the usual conditions.

The following notion of Martingale Problem comes from [60] Chapter XI.

Definition 2.4.1. *Let χ be a family of stochastic processes defined on a filtered space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}})$. We say that a probability measure $\mathbb{P}$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ solves the **martingale problem** associated to χ if under $\mathbb{P}$ all elements of χ are in $\mathcal{M}_{loc}$. We denote $\mathcal{MP}(\chi)$ the set of probability measures solving this martingale problem. $\mathbb{P}$ in $\mathcal{MP}(\chi)$ is said to be **extremal** if there can not exist distinct probability measures $\mathbb{Q}, \mathbb{Q}'$ in $\mathcal{MP}(\chi)$ and $\alpha \in]0,1[$ such that $\mathbb{P} = \alpha\mathbb{Q} + (1-\alpha)\mathbb{Q}'$.*

We now introduce a Martingale problem associated to an operator, following closely the formalism of D.W. Stroock and S.R.S Varadhan in [85]. We will see in Remark 2.4.3 that both Definitions 2.4.1 and 2.4.2 are closely related.

Definition 2.4.2. *Let us consider a domain $\mathcal{D}(a) \subset \mathcal{B}([0,T] \times E, \mathbb{R})$ which is a linear algebra; a linear operator $a : \mathcal{D}(a) \longrightarrow \mathcal{B}([0,T] \times E, \mathbb{R})$ and a non-decreasing continuous function $V : [0,T] \to \mathbb{R}_+$ starting at 0.*
We say that a set of probability measures $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defined on $(\Omega, \mathcal{F})$ solves the **martingale problem associated to** *$(\mathcal{D}(a), a, V)$ if, for any*
$(s,x) \in [0,T] \times E$, $\mathbb{P}^{s,x}$ verifies

(a) $\mathbb{P}^{s,x}(\forall t \in [0,s], X_t = x) = 1$;

(b) for every $\phi \in \mathcal{D}(a)$, $\left(t \longmapsto \phi(t, X_t) - \phi(s,x) - \int_s^t a(\phi)(r, X_r)dV_r\right)$, $t \in [s,T]$, is a cadlag $(\mathbb{P}^{s,x}, \mathbb{F})$-local martingale.

*We say that the Martingale Problem is **well-posed** if for any $(s,x) \in [0,T] \times E$, $\mathbb{P}^{s,x}$ is the only probability measure satisfying those two properties.*

Remark 2.4.3. *In other words, $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ solves the martingale problem associated to $(\mathcal{D}(a), a, V)$ if and only if, for any $(s,x) \in [0,T] \times E$,*
$\mathbb{P}^{s,x} \in \mathcal{MP}(\chi^{s,x})$ (see Definition 2.4.1), where $\chi^{s,x}$ is the family of processes
$\left\{t \mapsto \mathbb{1}_{[s,T]}(t)\left(\phi(t,X_t) - \phi(s,x) - \int_s^t a(\phi)(r,X_r)dV_r\right) \middle| \phi \in \mathcal{D}(a)\right\}$, together with processes
$\left\{t \mapsto \mathbb{1}_{\{r\}}(t)(X_t - x) \middle| r \in [0,s]\right\}$.

Notation 2.4.4. *For every $(s,x) \in [0,T] \times E$ and $\phi \in \mathcal{D}(a)$, the process*
$t \mapsto \mathbb{1}_{[s,T]}(t)\left(\phi(t,X_t) - \phi(s,x) - \int_s^t a(\phi)(r,X_r)dV_r\right)$ will be denoted $M[\phi]^{s,x}$.

$M[\phi]^{s,x}$ is a cadlag $(\mathbb{P}^{s,x}, \mathbb{F})$-local martingale which is equal to 0 on $[0,s]$, and by Proposition 2.A.10, it is also a $(\mathbb{P}^{s,x}, \mathbb{F}^{s,x})$-local martingale.

The following Hypothesis 2.4.5 is assumed for the rest of this section.

Hypothesis 2.4.5. *The Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ solves a well-posed Martingale Problem associated to a triplet $(\mathcal{D}(a), a, V)$ in the sense of Definition 2.4.2.*

The bilinear operator below was introduced (in the case of time-homogeneous operators) by J.P. Roth in potential analysis (see Chapter III in [79]), and popularized by P.A. Meyer in the study of homogeneous Markov processes, see e.g. [34] Chapter XV Comment 23 or [60] Remark 13.46. It has finally become a fundamental tool in the study of Markov processes and semi-groups, see for instance [3]. It will be central in our work.

Definition 2.4.6. *We set*

$$\Gamma : \begin{array}{ccc} \mathcal{D}(a) \times \mathcal{D}(a) & \to & \mathcal{B}([0,T] \times E) \\ (\phi, \psi) & \mapsto & a(\phi\psi) - \phi a(\psi) - \psi a(\phi). \end{array} \tag{2.4.1}$$

*When $\phi = \psi$, $\Gamma(\phi, \phi)$ will be denoted $\Gamma(\phi)$. The operator Γ is called the **carré du champs operator**.*

This operator will appear in the expression of the angular bracket of the local martingales that we have defined.

Proposition 2.4.7. *For any $\phi \in \mathcal{D}(a)$ and $(s,x) \in [0,T] \times E$, $M[\phi]^{s,x}$ belongs to $\mathcal{H}^2_{0,loc}$. Moreover, for any $(\phi,\psi) \in \mathcal{D}(a) \times \mathcal{D}(a)$ and $(s,x) \in [0,T] \times E$, we have*

$$\langle M[\phi]^{s,x}, M[\psi]^{s,x} \rangle = \int_s^\cdot \Gamma(\phi,\psi)(r,X_r) dV_r,$$

on the interval $[s,T]$, in the stochastic basis $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$.

Proof. We fix some $(s,x) \in [0,T] \times E$ and the associated probability $\mathbb{P}^{s,x}$. For any ϕ, ψ in $\mathcal{D}(a)$, by integration by parts on $[s,T]$ we have

$$\begin{array}{rl} & M[\phi]^{s,x} M[\psi]^{s,x} \\ = & \int_s^\cdot M[\phi]^{s,x}_{r^-} dM[\psi]^{s,x}_r + \int_s^\cdot M[\psi]^{s,x}_{r^-} dM[\phi]^{s,x}_r + [M[\phi]^{s,x}, M[\psi]^{s,x}] \\ = & \int_s^\cdot M[\phi]^{s,x}_{r^-} dM[\psi]^{s,x}_r + \int_s^\cdot M[\psi]^{s,x}_{r^-} dM[\phi]^{s,x}_r + [\phi(\cdot,X_\cdot), \psi(\cdot,X_\cdot)] \\ = & \int_s^\cdot M[\phi]^{s,x}_{r^-} dM[\psi]^{s,x}_r + \int_s^\cdot M[\psi]^{s,x}_{r^-} dM[\phi]^{s,x}_r + \phi\psi(\cdot,X_\cdot) \\ & -\phi\psi(s,x) - \int_s^\cdot \phi(r^-,X_{r^-}) d\psi(r,X_r) - \int_s^\cdot \psi(r^-,X_{r^-}) d\phi(r,X_r). \end{array}$$

Since $\phi\psi$ belongs to $\mathcal{D}(a)$, we can use the decomposition of $\phi\psi(\cdot,X_\cdot)$ given by (b) in Definition 2.4.2 and

$$\begin{array}{rl} & M[\phi]^{s,x} M[\psi]^{s,x} \\ = & \int_s^\cdot M[\phi]^{s,x}_{r^-} dM[\psi]^{s,x}_r + \int_s^\cdot M[\psi]^{s,x}_{r^-} dM[\phi]^{s,x}_r + \int_s^\cdot a(\phi\psi)(r,X_r) dV_r \\ & + M^{s,x}[\phi\psi] - \int_s^\cdot \phi a(\psi)(r,X_r) dV_r - \int_s^\cdot \psi a(\phi)(r,X_r) dV_r \\ & - \int_s^\cdot \phi(r^-,X_{r^-}) dM^{s,x}[\psi]_r - \int_s^\cdot \psi(r^-,X_{r^-}) dM^{s,x}[\phi]_r \\ = & \int_s^\cdot \Gamma(\phi,\psi)(r,X_r) dV_r + \int_s^\cdot M[\phi]^{s,x}_{r^-} dM[\psi]^{s,x}_r + \int_s^\cdot M[\psi]^{s,x}_{r^-} dM[\phi]^{s,x}_r \\ & + M^{s,x}[\phi\psi] - \int_s^\cdot \phi(r^-,X_{r^-}) dM^{s,x}[\psi]_r - \int_s^\cdot \psi(r^-,X_{r^-}) dM^{s,x}[\phi]_r. \end{array} \tag{2.4.2}$$

Since V is continuous, this implies that $M[\phi]^{s,x} M[\psi]^{s,x}$ is a special semimartingale with bounded variation predictable part $\int_s^\cdot \Gamma(\phi,\psi)(r,X_r) dV_r$. In particular taking $\phi = \psi$, we have on $[s,T]$ that $(M[\phi]^{s,x})^2 = \int_s^\cdot \Gamma(\phi)(r,X_r) dV_r + N^{s,x}$, where $N^{s,x}$ is some local martingale. The first element in previous sum is locally bounded since it is a continuous process. The second one is locally integrable as every local martingale. Finally $(M[\phi]^{s,x})^2$ is locally integrable, implying that $M[\phi]^{s,x}$ is in $\mathcal{H}^2_{0,loc}$.

Let us come back to two given $\phi, \psi \in \mathcal{D}(a)$. Since we know that $M[\phi]^{s,x}$, $M[\psi]^{s,x}$ belong to $\mathcal{H}^2_{0,loc}$ we can consider $\langle M[\phi]^{s,x}, M[\psi]^{s,x} \rangle$ which, by definition, is the unique predictable process with bounded variation such that
$M[\phi]^{s,x} M[\psi]^{s,x} - \langle M[\phi]^{s,x}, M[\psi]^{s,x} \rangle$ is a local martingale. So necessarily, taking (2.4.2) into account, $\langle M[\phi]^{s,x}, M[\psi]^{s,x} \rangle = \int_s^\cdot \Gamma(\phi,\psi)(r,X_r) dV_r$. □

Taking $\phi = \psi$ in Proposition 2.4.7, yields the following.

Corollary 2.4.8. *For any $(s,x) \in [0,T] \times E$ and $\phi \in \mathcal{D}(a)$, $M[\phi]^{s,x} \in \mathcal{H}^{2,V}_{loc}$.*

Proposition 2.4.9. *Let $(s,x) \in [0,T] \times E$ and $\mathbb{P}^{s,x}$ be fixed. If $N \in \mathcal{H}^\infty_{0,loc}$ is strongly orthogonal to $M[\phi]^{s,x}$ for all $\phi \in \mathcal{D}(a)$ then it is necessarily equal to 0.*

Proof. In Hypothesis 2.4.5, for any $(s,x) \in [0,T] \times E$ we have assumed that $\mathbb{P}^{s,x}$ was the unique element of $\mathcal{MP}(\chi^{s,x})$, where $\chi^{s,x}$ was introduced in Remark 2.4.3. Therefore $\mathbb{P}^{s,x}$ is extremal in $\mathcal{MP}(\chi^{s,x})$. So thanks to the Jacod-Yor Theorem (see e.g. Theorem 11.2 in [60]), we know that if an element N of $\mathcal{H}^\infty_{0,loc}$ is strongly orthogonal to all the $M[\phi]^{s,x}$ then it is equal to zero. □

Proposition 2.4.10. *If Hypothesis 2.4.5 is verified then for any $(s,x) \in [0,T] \times E$, in the stochastic basis $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$, we have $\mathcal{H}_0^2 = \mathcal{H}^{2,V}$.*

Proof. We fix $(s,x) \in [0,T] \times E$. It is enough to show the inclusion $\mathcal{H}_0^2 \subset \mathcal{H}^{2,V}$. We start considering a bounded martingale $N \in \mathcal{H}_0^\infty$ and showing that it belongs to $\mathcal{H}^{2,V}$. Since N belongs to $\mathcal{H}_0^2$, we can consider the corresponding $N^V, N^{\perp V}$ in $\mathcal{H}_0^2$, appearing in the statement of Proposition 2.3.5. We show below that N^V and $N^{\perp V}$ are locally bounded, which will permit us to use Jacod-Yor theorem on $N^{\perp V}$.

Indeed, by Proposition 2.3.5 there exists a predictable process K such that $N^V = \int_s^\cdot \mathbb{1}_{\{K_r<1\}} dN_r$ and $N^{\perp V} = \int_s^\cdot \mathbb{1}_{\{K_r=1\}} dN_r$. So if N is bounded then it has bounded jumps; by previous way of characterizing N^V and $N^{\perp V}$, their jumps can be expressed $(\Delta N^V)_t = \mathbb{1}_{\{K_t<1\}} \Delta N_t$ and $(\Delta N^{\perp V})_t = \mathbb{1}_{\{K_t=1\}} \Delta N_t$ (see Theorem 8 Chapter IV.3 in [78]), so they also have bounded jumps which implies that they are locally bounded, see (2.4) in [60].

So $N^{\perp V}$ is in $\mathcal{H}^\infty_{0,loc}$ and by construction it belongs to $\mathcal{H}^{2,\perp V}$. Since by Corollary 2.4.8, all the $M[\phi]^{s,x}$ belong to $\mathcal{H}^{2,V}_{loc}$, then, by Proposition 2.3.6, $N^{\perp V}$ is strongly orthogonal to all the $M[\phi]^{s,x}$. Consequently, by Proposition 2.4.9, $N^{\perp V}$ is equal to zero. This shows that $N = N^V$ which by construction belongs to $\mathcal{H}^{2,V}$, and consequently that $\mathcal{H}_0^\infty \subset \mathcal{H}^{2,V}$, which concludes the proof when N is a bounded martingale.

We can conclude by density arguments as follows. Let $M \in \mathcal{H}_0^2$. For any integer $n \in \mathbb{N}^*$, we denote by M^n the martingale in $\mathcal{H}_0^\infty$ defined as the cadlag version of $t \mapsto \mathbb{E}^{s,x}[((-n) \vee M_T \wedge n)|\mathcal{F}_t]$. Now $(M_T^n - M_T)^2 \underset{n\to\infty}{\longrightarrow} 0$ a.s. and this sequence is bounded by $4M_T^2$ which is an integrable r.v. So by the dominated convergence theorem $\mathbb{E}^{s,x}\left[(M_T^n - M_T)^2\right] \underset{n\to\infty}{\longrightarrow} 0$. Then by Doob's inequality, $\sup_{t\in[0,T]} (M_t^n - M_t) \xrightarrow[n\to\infty]{L^2} 0$ meaning that $M^n \xrightarrow[n\to\infty]{\mathcal{H}^2} M$. Since $\mathcal{H}_0^\infty \subset \mathcal{H}^{2,V}$, then M^n belongs to $\mathcal{H}^{2,V}$ for any $n \geq 0$. Moreover $\mathcal{H}^{2,V}$ is closed in $\mathcal{H}^2$, since by Proposition 2.3.6, it is a sub-Hilbert space. Finally we have shown that $M \in \mathcal{H}^{2,V}$. □

Since V is continuous, it follows in particular that every $(\mathbb{P}^{s,x}, \mathbb{F}^{s,x})$-square integrable martingale has a continuous angular bracket. By localization, the same assertion holds for local square integrable martingales.

We will now be interested in extending the domain $\mathcal{D}(a)$.

For any $(s,x) \in [0,T] \times E$ we define the positive bounded **potential measure** $U(s,x,\cdot)$ on $([0,T] \times E, \mathcal{B}([0,T]) \otimes \mathcal{B}(E))$ by

$$U(s,x,\cdot) : \begin{array}{ccc} \mathcal{B}([0,T]) \otimes \mathcal{B}(E) & \longrightarrow & [0, V_T] \\ A & \longmapsto & \mathbb{E}^{s,x}\left[\int_s^T \mathbb{1}_{\{(t,X_t)\in A\}} dV_t\right]. \end{array}$$

Definition 2.4.11. *A Borel set $A \subset [0,T]\times E$ will be said to be* **of zero potential** *if, for any $(s,x) \in [0,T]\times E$ we have $U(s,x,A) = 0$.*

Notation 2.4.12. *Let $p > 0$. We introduce*
$\mathcal{L}^p_{s,x} := \mathcal{L}^p(U(s,x,\cdot)) = \left\{ f \in \mathcal{B}([0,T] \times E, \mathbb{R}) : \mathbb{E}^{s,x}\left[\int_s^T |f|^p(r, X_r) dV_r\right] < \infty \right\}$.
That classical $\mathcal{L}^p$-space is equipped with the seminorm
$\|\cdot\|_{p,s,x} : f \mapsto \left(\mathbb{E}^{s,x}\left[\int_s^T |f(r,X_r)|^p dV_r\right]\right)^{\frac{1}{p}}$. *We also introduce*

$$\mathcal{L}^0_{s,x} := \mathcal{L}^0(U(s,x,\cdot)) = \left\{ f \in \mathcal{B}([0,T] \times E, \mathbb{R}) : \int_s^T |f|(r, X_r) dV_r < \infty \quad \mathbb{P}^{s,x} \text{ a.s.} \right\}.$$

We then denote for any $p \in \mathbb{N}$

$$\mathcal{L}_X^p = \bigcap_{(s,x)\in[0,T]\times E} \mathcal{L}_{s,x}^p. \tag{2.4.3}$$

Let $\mathcal{N}$ be the linear sub-space of $\mathcal{B}([0,T]\times E,\mathbb{R})$ containing all functions which are equal to 0, $U(s,x,\cdot)$ a.e. for every (s,x).

For any $p \in \mathbb{N}$, we define the quotient space $L_X^p = \mathcal{L}_X^p/\mathcal{N}$.

If $p \in \mathbb{N}^*$, L_X^p can be equipped with the topology generated by the family of semi-norms $(\|\cdot\|_{p,s,x})_{(s,x)\in[0,T]\times E}$ which makes it a separate locally convex topological vector space, see Theorem 5.76 in [1].

Proposition 2.4.13. *Let f and g be in $\mathcal{L}_X^0$. Then f and g are equal up to a set of zero potential if and only if for any $(s,x)\in[0,T]\times E$, the processes $\int_s^\cdot f(r,X_r)dV_r$ and $\int_s^\cdot g(r,X_r)dV_r$ are indistinguishable under $\mathbb{P}^{s,x}$. Of course in this case f and g correspond to the same element of L_X^0.*

Proof. Let $\mathbb{P}^{s,x}$ be fixed. Evaluating the total variation of $\int_s^\cdot (f-g)(r,X_r)dV_r$ yields that $\int_s^\cdot f(r,X_r)dV_r$ and $\int_s^\cdot g(r,X_r)dV_r$ are indistinguishable if and only if $\int_s^T |f-g|(r,X_r)dV_r = 0$ a.s. Since that r.v. is non-negative, this is true if and only if $\mathbb{E}^{s,x}\left[\int_s^T |f-g|(r,X_r)dV_r\right] = 0$ and therefore if and only if $U(s,x,N)=0$, where N is the Borel subset of $[0,T]\times E$, defined by $\{(t,y) : f(t,y)\neq g(t,y)\}$. This concludes the proof of Proposition 2.4.13.

□

We can now define our notion of **extended generator**.

Definition 2.4.14. *We first define the **extended domain** $\mathcal{D}(\mathfrak{a})$ as the set functions $\phi\in\mathcal{B}([0,T]\times E,\mathbb{R})$ for which there exists*
$\psi\in\mathcal{B}([0,T]\times E,\mathbb{R})$ such that under any $\mathbb{P}^{s,x}$ the process

$$\mathbb{1}_{[s,T]}\left(\phi(\cdot,X_\cdot)-\phi(s,x)-\int_s^\cdot \psi(r,X_r)dV_r\right) \tag{2.4.4}$$

(which is not necessarily cadlag) has a cadlag modification in $\mathcal{H}_0^2$.

Proposition 2.4.15. *Let $\phi\in\mathcal{B}([0,T]\times E,\mathbb{R})$. There is at most one (up to zero potential sets) $\psi\in\mathcal{B}([0,T]\times E,\mathbb{R})$ such that under any $\mathbb{P}^{s,x}$, the process defined in* (2.4.4) *has a modification which belongs to $\mathcal{M}_{loc}$.*
If moreover $\phi\in\mathcal{D}(a)$, then $a(\phi)=\psi$ up to zero potential sets. In this case, according to Notation 2.4.4, for every $(s,x)\in[0,T]\times E$, $M[\phi]^{s,x}$ is the $\mathbb{P}^{s,x}$ cadlag modification in $\mathcal{H}_0^2$ of
$\mathbb{1}_{[s,T]}\left(\phi(\cdot,X_\cdot)-\phi(s,x)-\int_s^\cdot\psi(r,X_r)dV_r\right)$.

Proof. Let ψ^1 and ψ^2 be two functions such that for any $\mathbb{P}^{s,x}$,
$\mathbb{1}_{[s,T]}\left(\phi(\cdot,X_\cdot)-\phi(s,x)-\int_s^\cdot\psi^i(r,X_r)dV_r\right),\quad i=1,2$, admits a cadlag modification which is a local martingale.

Then, under a fixed $\mathbb{P}^{s,x}$, $\phi(\cdot,X_\cdot)$ has two cadlag modifications which are therefore indistinguishable, and by uniqueness of the decomposition of special semimartingales, $\int_s^\cdot\psi^1(r,X_r)dV_r$ and $\int_s^\cdot\psi^2(r,X_r)dV_r$ are indistinguishable on $[s,T]$. Since this is true under any $\mathbb{P}^{s,x}$, the two functions are equal up to a zero-potential set because of Proposition 2.4.13.

Concerning the second part of the statement, let $\phi\in\mathcal{D}(a)\cap\mathcal{D}(\mathfrak{a})$. The result follows by Definition 2.4.2 and the uniqueness of the function ϕ established just before. □

Definition 2.4.16. *Let $\phi \in \mathcal{D}(\mathfrak{a})$ as in Definition 2.4.14. We denote again by $M[\phi]^{s,x}$, the unique cadlag version of the process* (2.4.4) *in $\mathcal{H}_0^2$. Taking Proposition 2.4.13 into account, this will not generate any ambiguity with respect to Notation 2.4.4. Proposition 2.4.13, also permits to define without ambiguity the operator*

$$\mathfrak{a}: \begin{array}{ccc} \mathcal{D}(\mathfrak{a}) & \longrightarrow & L_X^0 \\ \phi & \longmapsto & \psi. \end{array}$$

*$\mathfrak{a}$ will be called the **extended generator**.*

We now want to extend the carré du champs operator $\Gamma(\cdot,\cdot)$ to $\mathcal{D}(\mathfrak{a}) \times \mathcal{D}(\mathfrak{a})$.

Proposition 2.4.17. *Let ϕ_1, ϕ_2 be in $\mathcal{D}(\mathfrak{a})$. There exists a (unique up to zero-potential sets) function in $\mathcal{B}([0,T] \times E, \mathbb{R})$ which we will denote $\mathfrak{G}(\phi_1, \phi_2)$ such that under any $\mathbb{P}^{s,x}$,*

$$\langle M[\phi_1]^{s,x}, M[\phi_2]^{s,x} \rangle = \int_s^{\cdot} \mathfrak{G}(\phi_1, \phi_2)(r, X_r) dV_r, [s,T],$$

up to indistinguishability.

If moreover ϕ_1 and ϕ_2 belong to $\mathcal{D}(a)$, then $\Gamma(\phi_1, \phi_2) = \mathfrak{G}(\phi_1, \phi_2)$ up to zero potential sets.

Proof. Let $\phi_1, \phi_2 \in \mathcal{D}(\mathfrak{a})$ according to Definition 2.4.16. We take some representative of the classes $\mathfrak{a}(\phi_i)$ for $i = 1, 2$, still denoted by the same symbol and define the square integrable MAFs (see Definition 2.A.11) $M[\phi_i]$ by

$$M[\phi_i]_u^t(\omega) = \begin{cases} \phi_i(u, X_u(\omega)) - \phi_i(t, X_t(\omega)) - \int_t^u \mathfrak{a}(\phi_i)(r, X_r(\omega)) dV_r \\ \qquad \text{if } \int_t^u |\mathfrak{a}(\phi_i)|(r, X_r(\omega)) dV_r < +\infty \\ 0 \text{ otherwise.} \end{cases} \tag{2.4.5}$$

Indeed, for every $(s,x) \in [0,T] \times E$, $M[\phi_i]^{s,x}$ is the cadlag version under $\mathbb{P}^{s,x}$.

The existence of $\mathfrak{G}(\phi_1, \phi_2)$ therefore derives from Proposition 2.A.12. By Proposition 2.4.13 that function is determined up to a zero-potential set.

The second statement holds because of Proposition 2.4.7. □

Definition 2.4.18. *The bilinear operator $\mathfrak{G} : \mathcal{D}(\mathfrak{a}) \times \mathcal{D}(\mathfrak{a}) \longmapsto L_X^0$ will be called the **extended carré du champs operator**.*

When $\phi_1 = \phi_2$, $\mathfrak{G}(\phi_1, \phi_1)$ will be denoted $\mathfrak{G}(\phi_1)$.

According to Definition 2.4.14, we do not have necessarily $\mathcal{D}(a) \subset \mathcal{D}(\mathfrak{a})$, however we have the following.

Corollary 2.4.19. *If $\phi \in \mathcal{D}(a)$ and $\Gamma(\phi) \in \mathcal{L}_X^1$, then $\phi \in \mathcal{D}(\mathfrak{a})$ and $(a(\phi), \Gamma(\phi)) = (\mathfrak{a}(\phi), \mathfrak{G}(\phi))$ up to zero potential sets.*

Proof. Given some $\phi \in \mathcal{D}(a)$, by Definition 2.4.14, if for every $(s,x) \in [0,T] \times E$, $M[\phi]^{s,x}$ is square integrable, then $\phi \in \mathcal{D}(\mathfrak{a})$. By Proposition 2.4.7, for every $(s,x) \in [0,T] \times E$ $M[\phi]^{s,x}$ is a $\mathbb{P}^{s,x}$ square integrable if and only if $\Gamma(\phi) \in \mathcal{L}_X^1$. So the statement holds because of Propositions 2.4.15 and 2.4.17. □

2.5 Pseudo-PDEs and associated Markovian BSDEs with no driving martingale

In this section, we still consider $T \in \mathbb{R}_+^*$, a Polish space E and the associated canonical space $(\Omega, \mathcal{F}, \mathbb{F})$ and canonical process $(X_t)_{t\in[0,T]}$, see Definition 2.A.1. We also consider a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ and assume the following for the rest of the Section.

Hypothesis 2.5.1. *The transition kernel of $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ is measurable in time (see Definitions 2.A.5 and 2.A.4) and $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ solves a well-posed martingale problem associated to a triplet $(\mathcal{D}(a), a, V)$, see Definition 2.4.2 and Hypothesis 2.4.5.*

We will investigate here a specific type of BSDE with no driving martingale $BSDE(\xi, \hat{f}, V)$ which we will call **of Markovian type**, or **Markovian** BSDE, in the following sense. The process V will be the (deterministic) function V introduced in Definition 2.4.2, the final condition ξ will only depend on the final value of the canonical process X_T and the randomness of the driver $\hat{f}$ at time t will only appear via the value at time t of the forward process X. Given a function $f : [0,T] \times E \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we will set $\hat{f}(t,\omega,y,z) = f(t, X_t(\omega), y, z)$ for $t \in [0,T], \omega \in \Omega, y, z \in \mathbb{R}$.

That BSDE will be connected with the deterministic problem below.

Notation 2.5.2. *From now on, we fix some $g \in \mathcal{B}(E, \mathbb{R})$ and $f \in \mathcal{B}([0,T] \times E \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.*

Definition 2.5.3. *We will call **Pseudo-Partial Differential Equation** (in short Pseudo-PDE) the following equation with final condition:*

$$\begin{cases} a(u)(t,x) + f\left(t, x, u(t,x), \Gamma(u)^{\frac{1}{2}}(t,x)\right) = 0 \quad \text{on } [0,T]\times E \\ u(T,\cdot) = g. \end{cases} \tag{2.5.1}$$

*We will say that u is a **classical solution** of the Pseudo-PDE if it belongs to $\mathcal{D}(a)$ and verifies* (2.5.1).

Notation 2.5.4. *Equation* (2.5.1) *will be denoted $Pseudo - PDE(f, g)$.*

For the rest of this section, we will also assume that f, g verify the following.

Hypothesis 2.5.5.
- $g(X_T)$ *is* L^2 *under* $\mathbb{P}^{s,x}$ *for every* $(s,x) \in [0,T] \times E$;
- $t \longmapsto f(t, X_t, 0, 0) \in \mathcal{L}^2_X$;
- *there exist* $K^Y, K^Z > 0$ *such that for all* (t, x, y, y', z, z'),

$$|f(t,x,y,z) - f(t,x,y',z')| \leq K^Y |y - y'| + K^Z |z - z'|. \tag{2.5.2}$$

With the equation $Pseudo - PDE(f,g)$, we will associate the family of BSDEs with no driving martingale indexed by $(s,x) \in [0,T] \times E$ and defined on the interval $[0,T]$ and in the stochastic basis $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$, given by

$$Y_t^{s,x} = g(X_T) + \int_t^T f\left(r, X_r, Y_r^{s,x}, \sqrt{\frac{d\langle M^{s,x}\rangle}{dV}}(r)\right) dV_r - (M_T^{s,x} - M_t^{s,x}). \tag{2.5.3}$$

Notation 2.5.6. *Equation* (2.5.3) *will be denoted* $BSDE^{s,x}(f,g)$. *It corresponds to* $BSDE(g(X_T), \hat{f}, V)$ *with* $\mathbb{P} := \mathbb{P}^{s,x}$.

Remark 2.5.7. .

1. *If Hypothesis 2.5.5 is verified then Hypothesis 2.3.8 is verified for* (2.5.3). *By Theorem 2.3.21, for any* (s,x), $BSDE^{s,x}(f,g)$ *has a unique solution, in the sense of Definition 2.3.10.*
2. *Even if the underlying process* X *admits no (even generalized) moments, given a couple* (f,g) *such that* $f(\cdot,\cdot,0,0)$ *and* g *are bounded, the considerations of this section still apply. In particular the connection between the* $BSDE^{s,x}(f,g)$ *and the corresponding* $Pseudo - PDE(f,g)$ *still exists.*

Notation 2.5.8. *From now on,* $(Y^{s,x}, M^{s,x})$ *will always denote the (unique) solution of* $BSDE^{s,x}(f,g)$.

The goal of our work is to understand if and how the solutions of equations $BSDE^{s,x}(f,g)$ produce a solution of $Pseudo - PDE(f,g)$ and reciprocally.

We will start by showing that if $Pseudo - PDE(f,g)$ has a classical solution, then this one provides solutions to the associated $BSDE^{s,x}(f,g)$.

Proposition 2.5.9. *Let* u *be a classical solution of* $Pseudo - PDE(f,g)$ *verifying* $\Gamma(u) \in \mathcal{L}^1_X$. *Then, for any* $(s,x) \in [0,T] \times E$, *if* $M[u]^{s,x}$ *is as in Notation 2.4.4, we have that* $(u(\cdot, X_\cdot), M[u]^{s,x})$ *and* $(Y^{s,x}, M_\cdot^{s,x} - M_s^{s,x})$ *are* $\mathbb{P}^{s,x}$*-indistinguishable on* $[s,T]$.

Proof. Let (s,x) be fixed. Since $u \in \mathcal{D}(a)$, the martingale problem in the sense of Definition 2.4.2 and (2.5.1) imply that, on $[s,T]$, under $\mathbb{P}^{s,x}$

$$\begin{aligned}
& u(\cdot, X_\cdot) \\
= \;& u(T, X_T) - \int_\cdot^T a(u)(r, X_r) dV_r - (M[u]_T^{s,x} - M[u]_\cdot^{s,x}) \\
= \;& g(X_T) + \int_\cdot^T f\left(r, X_r, u(r, X_r), \Gamma(u)^{\frac{1}{2}}(r, X_r)\right) - (M[u]_T^{s,x} - M[u]_\cdot^{s,x}) \\
= \;& g(X_T) + \int_\cdot^T f\left(r, X_r, Y_r, \sqrt{\tfrac{d\langle M[u]^{s,x}\rangle}{dV}}(r)\right) dV_r - (M[u]_T^{s,x} - M[u]_\cdot^{s,x}),
\end{aligned}$$

where the latter equality comes from Proposition 2.4.7. Since $\Gamma(u) \in \mathcal{L}^1_X$ it follows that $\mathbb{E}^{s,x}\left[\langle M[u]^{s,x}\rangle_T\right] = \mathbb{E}^{s,x}\left[\int_s^T \Gamma(u)(r, X_r) dV_r\right] < \infty$. This means that $M[u]^{s,x} \in \mathcal{H}^2_0$, so by Lemma 2.3.24 $(u(\cdot, X_\cdot), M[u]_\cdot^{s,x})$ and $(Y^{s,x}, M_\cdot^{s,x} - M_s^{s,x})$ are indistinguishable on $[s,T]$. □

We will now adopt the converse point of view, starting with the solutions of the equations $BSDE^{s,x}(f,g)$. Below we will show that there exist Borel functions u and $v \geq 0$ such that for any $(s,x) \in [0,T] \times E$, for all $t \in [s,T]$, $Y_t^{s,x} = u(t,X_t)$ $\mathbb{P}^{s,x}$-a.s., and $\frac{d\langle M^{s,x}\rangle}{dV} = v^2(\cdot, X_\cdot) \quad dV \otimes d\mathbb{P}^{s,x}$ a.e. on $[s,T]$. This will be the object of Theorem 2.5.15, whose an analogous formulation exists in the Brownian framework, see e.g. Theorem 4.1 in [43]. We start with a lemma.

Lemma 2.5.10. *Let $\tilde{f} \in \mathcal{L}_X^2$. Let, for any $(s,x) \in [0,T] \times E$, $(\tilde{Y}^{s,x}, \tilde{M}^{s,x})$ be the (unique by Theorem 2.3.21 and Remark 2.3.23) solution of*

$$\tilde{Y}_t^{s,x} = g(X_T) + \int_t^T \tilde{f}\,(r, X_r)\, dV_r - (\tilde{M}_T^{s,x} - \tilde{M}_t^{s,x}), \quad t \in [s,T],$$

in $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$. Then there exist two functions u and $v \geq 0$ in $\mathcal{B}([0,T] \times E, \mathbb{R})$ such that for any $(s,x) \in [0,T] \times E$

$$\begin{cases} \forall t \in [s,T] : \tilde{Y}_t^{s,x} = u(t, X_t) \quad \mathbb{P}^{s,x}\text{a.s.} \\ \frac{d\langle \tilde{M}^{s,x}\rangle}{dV} = v^2(\cdot, X_\cdot) \quad dV \otimes d\mathbb{P}^{s,x} \text{ a.e. on } [s,T]. \end{cases}$$

Proof. We set $u : (s,x) \mapsto \mathbb{E}^{s,x}\left[g(X_T) + \int_s^T \tilde{f}\,(r, X_r)\, dV_r\right]$ which is Borel by Proposition 2.A.7 and Lemma 2.A.8. Therefore by (2.A.3) in Remark 2.A.6, for a fixed $t \in [s,T]$ $\mathbb{P}^{s,x}$- a.s. we have

$$\begin{aligned} u(t,X_t) &= \mathbb{E}^{t,X_t}\left[g(X_T) + \int_t^T \tilde{f}\,(r, X_r)\, dV_r\right] \\ &= \mathbb{E}^{s,x}\left[g(X_T) + \int_t^T \tilde{f}\,(r, X_r)\, dV_r \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}^{s,x}\left[\tilde{Y}_t^{s,x} + (\tilde{M}_T^{s,x} - \tilde{M}_t^{s,x}) | \mathcal{F}_t\right] \\ &= \tilde{Y}_t^{s,x}, \end{aligned}$$

since $\tilde{M}^{s,x}$ is a martingale and $\tilde{Y}^{s,x}$ is adapted. Then the square integrable AF (see Definition 2.A.11) defined by

$$M_{t'}^t(\omega) = \begin{cases} u(t', X_{t'}) - u(t, X_t) + \int_t^{t'} \tilde{f}(r, X_r(\omega)) dV_r \\ \qquad \text{if } \int_t^{t'} |\tilde{f}|(r, X_r(\omega)) dV_r < +\infty \\ 0 \text{ otherwise} \end{cases} \tag{2.5.4}$$

is a MAF whose cadlag version under $\mathbb{P}^{s,x}$ is $\tilde{M}^{s,x}$. The existence of the function v follows setting $v = \sqrt{k}$ in Proposition 2.A.12.

□

We now define the Picard iterations associated to the contraction defining the solution of the BSDE associated with $BSDE^{s,x}(f,g)$.

Notation 2.5.11. *For a fixed $(s,x) \in [0,T] \times E$, $\Phi^{s,x}$ will denote the contraction $\Phi : L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}_0^2$ introduced in Notation 2.3.15 with respect to the BSDE associated with $BSDE^{s,x}(f,g)$, see Notation 2.5.8 In the sequel we will not distinguish between a couple $(\dot{Y}, M)$ in $L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}_0^2$ and (Y, M), where Y is the reference cadlag process of $\dot{Y}$, according to Definition 2.3.13. We then convene the following.*

1. $(Y^{0,s,x}, M^{0,s,x}) := (0,0)$;
2. $\forall k \in \mathbb{N}^* : (Y^{k,s,x}, M^{k,s,x}) := \Phi^{s,x}(Y^{k-1,s,x}, M^{k-1,s,x})$,

meaning that for $k \in \mathbb{N}^*$, $(Y^{k,s,x}, M^{k,s,x})$ is the solution of the BSDE

$$Y^{k,s,x} = g(X_T) + \int_\cdot^T f\left(r, X_r, Y^{k-1,s,x}, \sqrt{\frac{d\langle M^{k-1,s,x}\rangle}{dV}}(r)\right) dV_r - (M_T^{k,s,x} - M_\cdot^{k,s,x}). \tag{2.5.5}$$

The processes $(Y^{k,s,x}, M^{k,s,x})$ will be called the **Picard iterations** of $BSDE^{s,x}(f,g)$

Proposition 2.5.12. *For each $k \in \mathbb{N}$, there exist functions u_k and $v_k \geq 0$ in $\mathcal{B}([0,T]\times E, \mathbb{R})$ such that for every $(s,x) \in [0,T]\times E$*

$$\begin{cases} \forall t \in [s,T] : Y_t^{k,s,x} = u_k(t, X_t) \quad \mathbb{P}^{s,x} a.s. \\ \frac{d\langle M^{k,s,x}\rangle}{dV} = v_k^2(\cdot, X_\cdot) \quad dV \otimes d\mathbb{P}^{s,x} \text{ a.e. on } [s,T]. \end{cases} \tag{2.5.6}$$

Lemma 2.5.13. *Let $(s,x) \in [0,T]\times E$ be fixed and let ϕ, ψ be two measurable processes. If ϕ and ψ are $\mathbb{P}^{s,x}$-modifications of each other, then they are equal $dV \otimes d\mathbb{P}^{s,x}$ a.e.*

Proof. Since for any $t \in [0,T]$, $\phi_t = \psi_t$ $\mathbb{P}^{s,x}$ a.s. we can write by Fubini's theorem $\mathbb{E}^{s,x}\left[\int_0^T \mathbb{1}_{\phi_t \neq \psi_t} dV_t\right] = \int_0^T \mathbb{P}^{s,x}(\phi_t \neq \psi_t) dV_t = 0$. □

Proof of Proposition 2.5.12.
We proceed by induction on k. It is clear that $(u_0, v_0) = (0,0)$ verifies the assertion for $k = 0$.
Now let us assume that functions u_{k-1}, v_{k-1} exist, for some integer $k \geq 1$, verifying (2.5.6) for k replaced with $k-1$.
We fix $(s,x) \in [0,T]\times E$. By Lemma 2.5.13, $(Y^{k-1,s,x}, Z^{k-1,s,x}) = (u_{k-1}, v_{k-1})(\cdot, X_\cdot)$ $dV \otimes \mathbb{P}^{s,x}$ a.e. on [s,T]. Therefore by (2.5.5), on $[s,T]$
$Y^{k,s,x} = g(X_T) + \int_\cdot^T f\left(r, X_r, u_{k-1}(r,X_r), v_{k-1}(r,X_r)\right) dV_r - (M_T^{k,s,x} - M_\cdot^{k,s,x})$.
Since $\Phi^{s,x}$ maps $L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}_0^2$ into itself (see Definition 2.3.15), obviously all the Picard iterations belong to $L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}_0^2$. In particular, $Y^{k-1,s,x}$ and $\sqrt{\frac{d\langle M^{k-1,s,x}\rangle}{dV}}$ belong to $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$. So, by recurrence assumption on u_{k-1} and v_{k-1}, it follows that Therefore, using the assumptions f in Hypothesis 2.5.5, $f(\cdot,\cdot,u_{k-1},v_{k-1}) \in \mathcal{L}_X^2$. The existence of u_k and v_k now comes from Lemma 2.5.10 applied to $\tilde{f} := f(\cdot,\cdot,u_{k-1},v_{k-1})$. This establishes the induction step for a general k and allows to conclude the proof.

□

Now we intend to pass to the limit in k. For any $(s,x) \in [0,T]\times E$, we have seen in Proposition 2.3.20 that $\Phi^{s,x}$ is a contraction in $\left(L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}_0^2, \|\cdot\|_\lambda\right)$ for some $\lambda > 0$, so we know that the sequence $(Y^{k,s,x}, M^{k,s,x})$ converges to $(Y^{s,x}, M^{s,x})$ in this topology.
The proposition below also shows an a.e. corresponding convergence, adapting the techniques of Corollary 2.1 in [43].

Proposition 2.5.14. *For any $(s,x) \in [0,T]\times E$, $Y^{k,s,x} \underset{k\to\infty}{\longrightarrow} Y^{s,x}$ $dV \otimes d\mathbb{P}^{s,x}$ a.e. and $\sqrt{\frac{d\langle M^{k,s,x}\rangle}{dV}} \underset{k\to\infty}{\longrightarrow} \sqrt{\frac{d\langle M^{s,x}\rangle}{dV}}$ $dV \otimes d\mathbb{P}^{s,x}$ a.e.*

Proof. We fix (s,x) and the associated probability. In this proof, all superscripts s,x are dropped. We set $Z^k = \sqrt{\frac{d\langle M^k\rangle}{dV}}$ and $Z = \sqrt{\frac{d\langle M\rangle}{dV}}$. By Proposition 2.3.20, there exists $\lambda > 0$ such that for any $k \in \mathbb{N}^*$

$$\begin{aligned} & \mathbb{E}\left[\int_0^T e^{-\lambda V_r} |Y_r^{k+1} - Y_r^k|^2 dV_r + \int_0^T e^{-\lambda V_r} d\langle M^{k+1} - M^k\rangle_r\right] \\ \leq\ & \tfrac{1}{2}\mathbb{E}\left[\int_0^T e^{-\lambda V_r} |Y_r^k - Y_r^{k-1}|^2 dV_r + \int_0^T e^{-\lambda V_r} d\langle M^k - M^{k-1}\rangle_r\right], \end{aligned}$$

therefore

$$\begin{aligned} & \sum_{k\geq 0} \mathbb{E}\left[\int_0^T e^{-\lambda V_r} |Y_r^{k+1} - Y_r^k|^2 dV_r\right] + \mathbb{E}\left[\int_0^T e^{-\lambda V_r} d\langle M^{k+1} - M^k\rangle_r\right] \\ \leq\ & \sum_{k\geq 0} \tfrac{1}{2^k}\left(\mathbb{E}\left[\int_0^T e^{-\lambda V_r} |Y_r^1|^2 dV_r\right] + \mathbb{E}\left[\int_0^T e^{-\lambda V_r} d\langle M^1\rangle_r\right]\right) \\ <\ & \infty. \end{aligned} \tag{2.5.7}$$

Thanks to (2.3.9) and (2.5.7) we have
$\sum_{k\geq 0}\left(\mathbb{E}\left[\int_0^T e^{-\lambda V_r}|Y_r^{k+1}-Y_r^k|^2 dV_r\right]+\mathbb{E}\left[\int_0^T e^{-\lambda V_r}|Z_r^{k+1}-Z_r^k|^2 dV_r\right]\right)<\infty$. So by Fubini's theorem we have
$\mathbb{E}\left[\int_0^T e^{-\lambda V_r}\left(\sum_{k\geq 0}(|Y_r^{k+1}-Y_r^k|^2+|Z_r^{k+1}-Z_r^k|^2)\right)dV_r\right]<\infty$.

Consequently the sum $\sum_{k\geq 0}\left(|Y_r^{k+1}(\omega)-Y_r^k(\omega)|^2+|Z_r^{k+1}(\omega)-Z_r^k(\omega)|^2\right)$ is finite on a set of full $dV\otimes d\mathbb{P}$ measure. So on this set of full measure, the sequence $(Y_t^{k+1}(\omega),Z_t^{k+1}(\omega))$ converges, and the limit is necessarily equal to $(Y_t(\omega),Z_t(\omega))$ $dV\otimes d\mathbb{P}$ a.e. because of the $L^2(dV\otimes d\mathbb{P})$ convergence that we have mentioned in the lines before the statement of the present Proposition 2.5.14.

□

Theorem 2.5.15. *There exist two functions u and $v\geq 0$ in $\mathcal{B}([0,T]\times E,\mathbb{R})$ such that for every $(s,x)\in[0,T]\times E$,*

$$\begin{cases}\forall t\in[s,T]: Y_t^{s,x}=u(t,X_t) \quad \mathbb{P}^{s,x}\text{ a.s.}\\ \frac{d\langle M^{s,x}\rangle}{dV}=v^2(\cdot,X_\cdot) \quad dV\otimes d\mathbb{P}^{s,x}\text{ a.e. on }[s,T].\end{cases}\tag{2.5.8}$$

Proof. We set $\bar{u}:=\limsup_{k\in\mathbb{N}} u_k$, in the sense that for any $(s,x)\in[0,T]\times E$, $\bar{u}(s,x)=\limsup_{k\in\mathbb{N}} u_k(s,x)$ and $v:=\limsup_{k\in\mathbb{N}} v_k$. $\bar{u}$ and v are Borel functions. We know by Propositions 2.5.12, 2.5.14 and Lemma 2.5.13 that for every $(s,x)\in[0,T]\times E$

$$\begin{cases}u_k(\cdot,X_\cdot) \underset{k\to\infty}{\longrightarrow} Y^{s,x} \quad dV\otimes d\mathbb{P}^{s,x}\text{ a.e. on }[s,T]\\ v_k(\cdot,X_\cdot) \underset{k\to\infty}{\longrightarrow} Z^{s,x} \quad dV\otimes d\mathbb{P}^{s,x}\text{ a.e. on }[s,T],\end{cases}$$

where $Z^{s,x}:=\sqrt{\frac{d\langle M^{s,x}\rangle}{dV}}$. Therefore, for some fixed $(s,x)\in[0,T]\times E$ and on the set of full $dV\otimes d\mathbb{P}^{s,x}$ measure on which these convergences hold we have

$$\begin{cases}\bar{u}(t,X_t(\omega))=\limsup_{k\in\mathbb{N}} u_k(t,X_t(\omega))=\lim_{k\in\mathbb{N}} u_k(t,X_t(\omega))=Y_t^{s,x}(\omega)\\ v(t,X_t(\omega))=\limsup_{k\in\mathbb{N}} v_k(t,X_t(\omega))=\lim_{k\in\mathbb{N}} v_k(t,X_t(\omega))=Z_t^{s,x}(\omega).\end{cases}\tag{2.5.9}$$

This shows in particular the existence of v and the validity of the second line of (2.5.8).

It remains to show the existence of u so that the first line of (2.5.8) holds. Thanks to the $dV\otimes d\mathbb{P}^{s,x}$ equalities concerning v and $\bar{u}$ stated in (2.5.9), under every $\mathbb{P}^{s,x}$ we actually have

$$Y^{s,x}=g(X_T)+\int_\cdot^T f\left(r,X_r,\bar{u}(r,X_r),v(r,X_r)\right)dV_r-(M_T^{s,x}-M_\cdot^{s,x}).\tag{2.5.10}$$

Now (2.5.10) can be considered as a BSDE where the driver does not depend on y and z. For any $(s,x)\in[0,T]\times E$, $Y^{s,x}$ and $Z^{s,x}$ belong to $\mathcal{L}^2(dV\otimes d\mathbb{P}^{s,x})$, then by (2.5.9), so do $\bar{u}(\cdot,X_\cdot)\mathbb{1}_{[s,T]}$ and $v(\cdot,X_\cdot)\mathbb{1}_{[s,T]}$, meaning that $\bar{u}$ and v belong to $\mathcal{L}_X^2$. Using the two assumptions made on f in Hypothesis 2.5.5, $f(\cdot,\cdot,\bar{u},v)$ also belongs to $\mathcal{L}_X^2$. We can therefore apply Lemma 2.5.10 to $\tilde{f}=f(\cdot,\cdot,\bar{u},v)$, and conclude to the existence of a Borel function u such that for every $(s,x)\in[0,T]\times E$, $Y^{s,x}$ is on $[s,T]$ a $\mathbb{P}^{s,x}$-version of $u(\cdot,X_\cdot)$.

□

Remark 2.5.16. *Since $\bar{u}(\cdot, X_\cdot) = Y^{s,x} = u(\cdot, X_\cdot)$ $dV \otimes d\mathbb{P}^{s,x}$ a.e. for every $(s,x) \in [0,T] \times E$, one can remark that $u = \bar{u}$ up to a zero potential set, and in particular that $u \in \mathcal{L}^2_X$ since $\bar{u}$ does.*
Moreover, for any $(s,x) \in [0,T] \times E$, the stochastic convergence

$$(Y^{k,s,x}, M^{k,s,x}) \xrightarrow[k\to\infty]{L^2(dV\otimes d\mathbb{P}^{s,x})\times\mathcal{H}^2} (Y^{s,x}, M^{s,x})$$ *now has the functional counterpart* $$\begin{cases} u_k & \xrightarrow{\|\cdot\|_{2,s,x}} & u \\ v_k & \xrightarrow{\|\cdot\|_{2,s,x}} & v, \end{cases}$$

which yields $$\begin{cases} u_k & \xrightarrow{L^2_X} & u \\ v_k & \xrightarrow{L^2_X} & v, \end{cases}$$ *where we recall that the locally convex topological space L^2_X was introduced in Notation 2.4.12.*

Corollary 2.5.17. *For any $(s,x) \in [0,T] \times E$ and for any $t \in [s,T]$, the couple of functions (u,v) obtained in Theorem 2.5.15 verifies $\mathbb{P}^{s,x}$ a.s.*

$$u(t, X_t) = g(X_T) + \int_t^T f\left(r, X_r, u(r, X_r), v(r, X_r)\right) dV_r - (M_T^{s,x} - M_t^{s,x}),$$

where $M^{s,x}$ denotes the martingale part of the unique solution of $BSDE^{s,x}(f,g)$.

Proof. The corollary follows from Theorem 2.5.15 and Lemma 2.5.13. □

We now introduce now a probabilistic notion of solution for $Pseudo - PDE(f,g)$.

Definition 2.5.18. *A function $u : [0,T] \times E \to \mathbb{R}$ will be said to be a* **martingale solution** *of $Pseudo - PDE(f,g)$ if $u \in \mathcal{D}(\mathfrak{a})$ and*

$$\begin{cases} \mathfrak{a}(u) &= -f(\cdot, \cdot, u, \mathfrak{G}(u)^{\frac{1}{2}}) \\ u(T, \cdot) &= g. \end{cases} \tag{2.5.11}$$

Remark 2.5.19. *The first equation of* (2.5.11) *holds in L^0_X, hence up to a zero potential set. The second one is a pointwise equality.*

Proposition 2.5.20. *A classical solution u of $Pseudo - PDE(f,g)$ such that $\Gamma(u) \in \mathcal{L}^1_X$, is also a martingale solution.*
Conversely, if u is a martingale solution of $Pseudo - PDE(f,g)$ belonging to $\mathcal{D}(a)$, then u is a classical solution of $Pseudo - PDE(f,g)$ up to a zero-potential set, meaning that the first equality of (2.5.1) *holds up to a set of zero potential.*

Proof. Let u be a classical solution of $Pseudo - PDE(f,g)$ verifying $\Gamma(u) \in \mathcal{L}^1_X$, Definition 2.5.3 and Corollary 2.4.19 imply that $u \in \mathcal{D}(\mathfrak{a})$, $u(T,\cdot) = g$, and the equalities up to zero potential sets

$$\mathfrak{a}(u) = a(u) = -f(\cdot, \cdot, u, \Gamma(u)) = -f(\cdot, \cdot, u, \mathfrak{G}(u)), \tag{2.5.12}$$

which shows that u is a martingale solution.
Similarly, the second statement follows by Definition 2.5.18 and again Corollary 2.4.19. □

Theorem 2.5.21. *Assume Hypothesis 2.5.1 and 2.5.5 and let (u,v) be the functions defined in Theorem 2.5.15. Then $u \in \mathcal{D}(\mathfrak{a})$, $v^2 = \mathfrak{G}(u)$ and u is a martingale solution of $Pseudo - PDE(f,g)$.*

Proof. For any $(s,x) \in [0,T] \times E$, by Corollary 2.5.17, for $t \in [s,T]$, we have $u(t,X_t) - u(s,x) = -\int_s^t f(r, X_r, u(r,X_r), v(r,X_r))dV_r + (M_t^{s,x} - M_s^{s,x})$ $\mathbb{P}^{s,x}$ a.s. so by Definition 2.4.16, $u \in \mathcal{D}(\mathfrak{a})$, $\mathfrak{a}(u) = -f(\cdot,\cdot,u,v)$ and

$M[u]^{s,x} = M_\cdot^{s,x} - M_s^{s,x}$.
Moreover by Theorem 2.5.15 we have $\frac{d\langle M^{s,x}\rangle}{dV} = v^2(\cdot, X_\cdot)\, dV \otimes d\mathbb{P}^{s,x}$ a.e. on $[s,T]$, so by Proposition 2.4.17 it follows $v^2 = \mathfrak{G}(u)$ and therefore, the L_X^2 equality
$\mathfrak{a}(u) = -f(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}})$, which establishes the first line of (2.5.11).

Concerning the second line, we have for any $x \in E$,
$u(T,x) = u(T,X_T) = g(X_T) = g(x)\ \mathbb{P}^{T,x}$ a.s. so $u(T,\cdot) = g$ (in the deterministic pointwise sense). □

We conclude the section with Theorem 2.5.22 which states that the previously constructed martingale solution of $Pseudo - PDE(f,g)$ is unique.

Theorem 2.5.22. *Under Hypothesis 2.5.1 and 2.5.5, $Pseudo - PDE(f,g)$ admits a unique martingale solution.*

Proof. Existence has been the object of Theorem 2.5.21.

Let u and u' be two elements of $\mathcal{D}(\mathfrak{a})$ solving (2.5.11) and let $(s,x) \in [0,T] \times E$ be fixed. By Definition 2.4.14 and Remark 2.3.23, the process $u(\cdot, X_\cdot)$ (respectively $u'(\cdot, X_\cdot)$) under $\mathbb{P}^{s,x}$ admits a cadlag modification $U^{s,x}$ (respectively $U'^{s,x}$) on $[s,T]$, which is a special semimartingale with decomposition

$$\begin{array}{rcl} U^{s,x} & = & u(s,x) + \int_s^\cdot \mathfrak{a}(u)(r,X_r)dV_r + M[u]^{s,x} \\ & = & u(s,x) - \int_s^\cdot f\left(r, X_r, u(r,X_r), \mathfrak{G}(u)^{\frac{1}{2}}(r,X_r)\right) dV_r + M[u]^{s,x} \\ & = & u(s,x) - \int_s^\cdot f\left(r, X_r, U^{s,x}, \mathfrak{G}(u)^{\frac{1}{2}}(r,X_r)\right) dV_r + M[u]^{s,x}, \end{array} \tag{2.5.13}$$

where the third equality of (2.5.13) comes from Lemma 2.5.13. Similarly we have $U'^{s,x} = u'(s,x) - \int_s^\cdot f\left(r, X_r, U'^{s,x}, \mathfrak{G}(u')^{\frac{1}{2}}(r,X_r)\right) dV_r + M[u']^{s,x}$).

The processes $M[u]^{s,x}$ and $M[u']^{s,x}$ (introduced at Definition 2.4.16) belong to $\mathcal{H}_0^2$; by Proposition 2.4.17, $\langle M[u]^{s,x}\rangle = \int_s^\cdot \mathfrak{G}(u)(r,X_r)dV_r$ (respectively $\langle M[u']^{s,x}\rangle = \int_s^\cdot \mathfrak{G}(u',u')(r,X_r)dV_r$). Moreover since $u(T,\cdot) = u'(T,\cdot) = g$, then $u(T,X_T) = u'(T,X_T) = g(X_T)$ a.s. then the couples $(U^{s,x}, M[u]^{s,x})$ and $(U'^{s,x}, M[u']^{s,x})$ both verify the equation (with respect to $\mathbb{P}^{s,x}$).

$$Y_\cdot = g(X_T) + \int_\cdot^T f\left(r, X_r, Y_r, \sqrt{\frac{d\langle M\rangle}{dV}}(r)\right) dV_r - (M_T - M_\cdot) \tag{2.5.14}$$

on $[s,T]$.

Even though we do not have a priori information on the square integrability of $U^{s,x}$ and $U'^{s,x}$, we know that $M[u]^{s,x}$ and $M[u']^{s,x}$ are in $\mathcal{H}^2$ and equal to zero at time s, and that $U_s^{s,x}$ and $U_s'^{s,x}$ are deterministic so L^2. By Lemma 2.3.24 and the fact that $(U^{s,x}, M[u]^{s,x})$ and $(U'^{s,x}, M[u']^{s,x})$ solve the BSDE in the weaker sense (2.5.14), it is sufficient to conclude that both solve $BSDE^{s,x}(f,g)$ on $[s,T]$. By Theorem 2.3.21 and Remark 2.3.23 the two couples are $\mathbb{P}^{s,x}$-indistinguishable. This implies that $u(\cdot,X_\cdot)$ and $u'(\cdot,X_\cdot)$ are $\mathbb{P}^{s,x}$-modifications one of the other on $[s,T]$. In particular, considering their values at time s, we have $u(s,x) = u'(s,x)$. We therefore have $u' = u$.

□

Corollary 2.5.23. *There is at most one classical solution u of $Pseudo - PDE(f,g)$ such that $\Gamma(u) \in \mathcal{L}_X^1$.*

Proof. The proof follows from Proposition 2.5.20 and Theorem 2.5.22. □

2.6 Applications

In Chapter 3 which is the continuation of the present chapter, several examples are studied. The examples below fit in the framework of Section 2.4.
A first typical example is the setup of jump diffusions as in the formalism D.W. Stroock in [84]. These are Markov processes which solve a Martingale problem associated to an operator of the type

$$\begin{aligned} a(\phi) &= \partial_t\phi + \tfrac{1}{2}\sum_{i,j\leq d}(\sigma\sigma^\intercal)_{i,j}\partial^2_{x_ix_j}\phi + \sum_{i\leq d}\beta_i\partial_{x_i}\phi \\ &\quad + \int\left(\phi(\cdot,\cdot+y)-\phi(\cdot,y)-\tfrac{1}{1+\|y\|^2}\sum_{i\leq d}y_i\partial_{x_i}\phi\right)K(\cdot,\cdot,dy), \end{aligned}$$

where β is a bounded Borel function with values in $\mathbb{R}^d$ and σ is a continuousBorel function with values in $GL_d(\mathbb{R})$, the set of invertible matrices of size d. K is a Lévy kernel.

We also study Markov processes associated to a large class of pseudo-differential operators with the formalism of N. Jacob in [59]. A typical example of equation considered is

$$\begin{cases} \partial_t u - (-\Delta)^{\frac{\alpha}{2}}u = f(\cdot,\cdot,u,\Gamma^\alpha(u)^{\frac{1}{2}}) \text{ on } [0,T]\times\mathbb{R}^d \\ u(T,\cdot) = g. \end{cases} \tag{2.6.1}$$

Here, the fractional Laplace operator $(-\Delta)^{\frac{\alpha}{2}}$ is given for some $\alpha\in]0,2[$ by $\phi\longmapsto c_\alpha PV\int_{\mathbb{R}^d}\frac{(\phi(\cdot+y)-\phi)}{\|y\|^{d+\alpha}}dy$ where c_α is some positive constant and PV denotes the principal value operator.

$$\Gamma^\alpha(\phi) = c_\alpha PV\int_{\mathbb{R}^d}\frac{(\phi(\cdot,\cdot+y)-\phi)^2}{\|y\|^{d+\alpha}}dy \tag{2.6.2}$$

is the corresponding Carré du champ. The forward process of the corresponding BSDEs is the α-stable Levy process.

Another example of application is given by solutions of SDEs with distributional drift, which are studied in [47]. These permit to tackle semilinear parabolic PDEs with distributional drift of type

$$\begin{cases} \partial_t u + \frac{1}{2}\sigma^2\partial^2_x u + b'\partial_x u + f(\cdot,\cdot,u,\sigma|\partial_x u|) = 0 \quad \text{ on } [0,T]\times\mathbb{R} \\ u(T,\cdot) = g, \end{cases} \tag{2.6.3}$$

where b is only a continuous function, hence b' is a distribution.

Finally, examples in non Euclidean state spaces are given with the study of diffusions in a compact differential manifold M. A typical example is the Brownian motion in a Riemannian manifold. The equation considered is then of type

$$\begin{cases} \partial_t u + \Delta_M u + f(\cdot,\cdot,u,\|\nabla_M u\|_2) = 0 \quad \text{ on } [0,T]\times M \\ u(T,\cdot) = g, \end{cases} \tag{2.6.4}$$

where Δ_M is the Laplace-Beltrami operator and ∇_M is the gradient in local coordinates. More general equations are considered in Chapter 3.

Appendix

2.A Markov classes

We recall in this Appendix some basic definitions and results concerning Markov processes. For a complete study of homogeneous Markov processes, one may consult [34], concerning non-homogeneous Markov classes, our reference was chapter VI of [40]. Some results are proven in Chapters 1 and 3.

The first definition refers to the canonical space that one can find in [60], see paragraph 12.63.

Notation 2.A.1. *In the whole section E will be a fixed Polish space (a separable completely metrizable topological space), and $\mathcal{B}(E)$ its Borel σ-field. E will be called the **state space**.*

We consider $T \in \mathbb{R}_+^*$. We denote $\Omega := \mathbb{D}(E)$ the Skorokhod space of functions from $[0,T]$ to E right-continuous with left limits and continuous at time T (e.g. cadlag). For any $t \in [0,T]$ we denote the coordinate mapping $X_t : \omega \mapsto \omega(t)$, and we introduce on Ω the σ-field $\mathcal{F} := \sigma(X_r | r \in [0,T])$.

*On the measurable space $(\Omega, \mathcal{F})$, we introduce the measurable **canonical process***

$$X : \begin{array}{rcl} (t,\omega) & \longmapsto & \omega(t) \\ ([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F}) & \longrightarrow & (E, \mathcal{B}(E)), \end{array}$$

and the right-continuous filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ where $\mathcal{F}_t := \bigcap_{s\in]t,T]} \sigma(X_r | r \leq s)$ if $t < T$, and $\mathcal{F}_T := \sigma(X_r | r \in [0,T]) = \mathcal{F}$.

*$(\Omega, \mathcal{F}, \mathbb{F})$ will be called the **canonical space** (associated to T and E).*

For any $t \in [0,T]$ we denote $\mathcal{F}_{t,T} := \sigma(X_r | r \geq t)$, and for any $0 \leq t \leq u < T$ we will denote $\mathcal{F}_{t,u} := \bigcap_{n\geq 0} \sigma(X_r | r \in [t, u + \frac{1}{n}])$.

Remark 2.A.2. *Previous definitions and all the notions of this Appendix, extend to a time interval equal to $\mathbb{R}_+$ or replacing the Skorokhod space with the Wiener space of continuous functions from $[0,T]$ (or $\mathbb{R}_+$) to E.*

Definition 2.A.3. *The function*

$$P : \begin{array}{rcl} (s,t,x,A) & \longmapsto & P_{s,t}(x,A) \\ [0,T]^2 \times E \times \mathcal{B}(E) & \longrightarrow & [0,1], \end{array}$$

*will be called **transition kernel** if, for any s, t in $[0,T]$, $x \in E$, $A \in \mathcal{B}(E)$, it verifies*

1. *$x \mapsto P_{s,t}(x,A)$ is Borel,*
2. *$B \mapsto P_{s,t}(x,B)$ is a probability measure on $(E, \mathcal{B}(E))$,*
3. *if $t \leq s$ then $P_{s,t}(x,A) = \mathbb{1}_A(x)$,*
4. *if $s < t$, for any $u > t$, $\int_E P_{s,t}(x,dy) P_{t,u}(y,A) = P_{s,u}(x,A)$.*

The latter statement is the well-known **Chapman-Kolmogorov equation**.

Definition 2.A.4. *A transition kernel P for which the first item is reinforced supposing that $(s,x) \longmapsto P_{s,t}(x,A)$ is Borel for any t, A, will be said **measurable in time**.*

Definition 2.A.5. *A **canonical Markov class** associated to a transition kernel P is a set of probability measures $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defined on the measurable space $(\Omega,\mathcal{F})$ and verifying for any $t \in [0,T]$ and $A \in \mathcal{B}(E)$*

$$\mathbb{P}^{s,x}(X_t \in A) = P_{s,t}(x,A), \tag{2.A.1}$$

and for any $s \leq t \leq u$

$$\mathbb{P}^{s,x}(X_u \in A|\mathcal{F}_t) = P_{s,u}(X_t,A) \quad \mathbb{P}^{s,x} \text{ a.s.} \tag{2.A.2}$$

Remark 2.A.6. *Formula 1.7 in Chapter 6 of [40] states that for any $F \in \mathcal{F}_{t,T}$ yields*

$$\mathbb{P}^{s,x}(F|\mathcal{F}_t) = \mathbb{P}^{t,X_t}(F) = \mathbb{P}^{s,x}(F|X_t) \ \ \mathbb{P}^{s,x}\text{a.s.} \tag{2.A.3}$$

Property (2.A.3) *will be called **Markov property**.*

For the rest of this section, we are given a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ which transition kernel is measurable in time.

Proposition 2.A.7. *For any event $F \in \mathcal{F}$, $(s,x) \longmapsto \mathbb{P}^{s,x}(F)$ is Borel. For any random variable Z, if the function $(s,x) \longmapsto \mathbb{E}^{s,x}[Z]$ is well-defined (with possible values in $[-\infty,\infty]$), then it is Borel.*

Lemma 2.A.8. *Let V be a continuous non-decreasing function on $[0,T]$ and $f \in \mathcal{B}([0,T]\times E)$ be such that for every (s,x), $\mathbb{E}^{s,x}[\int_s^T |f(r,X_r)|dV_r] < \infty$, then $(s,x) \longmapsto \mathbb{E}^{s,x}[\int_s^T f(r,X_r)dV_r]$ is Borel.*

Definition 2.A.9. *For any $(s,x) \in [0,T]\times E$ we will consider the (s,x)-**completion** $\left(\Omega,\mathcal{F}^{s,x},\mathbb{F}^{s,x} := (\mathcal{F}_t^{s,x})_{t\in[0,T]},\mathbb{P}^{s,x}\right)$ of the stochastic basis $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}^{s,x})$ by defining $\mathcal{F}^{s,x}$ as the $\mathbb{P}^{s,x}$-completion of $\mathcal{F}$, by extending $\mathbb{P}^{s,x}$ to $\mathcal{F}^{s,x}$ and finally by defining $\mathcal{F}_t^{s,x}$ as the $\mathbb{P}^{s,x}$-closure of $\mathcal{F}_t$ for every $t \in [0,T]$.*

We remark that, for any $(s,x) \in [0,T]\times E$, $(\Omega,\mathcal{F}^{s,x},\mathbb{F}^{s,x},\mathbb{P}^{s,x})$ is a stochastic basis fulfilling the usual conditions.

Proposition 1.3.13 in Chapter 1 states the following.

Proposition 2.A.10. *Let $(s,x) \in [0,T]\times E$ be fixed, Z be a random variable and $t \in [s,T]$, then $\mathbb{E}^{s,x}[Z|\mathcal{F}_t] = \mathbb{E}^{s,x}[Z|\mathcal{F}_t^{s,x}]$ $\mathbb{P}^{s,x}$ a.s.*

We recall here Definition 1.4.1 in Chapter 1.

Definition 2.A.11. *We denote $\Delta := \{(t,u) \in [0,T]^2 | t \leq u\}$. On $(\Omega,\mathcal{F})$, we define a **non-homogeneous Additive Functional** (shortened AF) as a random-field indexed by Δ $A := (A_u^t)_{(t,u)\in\Delta}$, with values in $\mathbb{R}$, verifying the two following conditions.*

1. *For any $(t,u) \in \Delta$, A_u^t is $\mathcal{F}_{t,u}$-measurable;*
2. *for any $(s,x) \in [0,T]\times E$, there exists a real cadlag $\mathcal{F}^{s,x}$-adapted process $A^{s,x}$ (taken equal to zero on $[0,s]$ by convention) such that for any $x \in E$ and $s \leq t \leq u$, $A_u^t = A_u^{s,x} - A_t^{s,x}$ $\mathbb{P}^{s,x}$ a.s.*

*$A^{s,x}$ will be called the **cadlag version of** A **under** $\mathbb{P}^{s,x}$.*

*An AF will be called a **non-homogeneous square integrable Martingale Additive Functional** (shortened square integrable MAF) if under any $\mathbb{P}^{s,x}$ its cadlag version is a square integrable martingale.*

A immediate consequence of Proposition 1.4.17 in Chapter 1 is the following.

Proposition 2.A.12. *Given an increasing continuous function V, if in every stochastic basis $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$, we have $\mathcal{H}_0^2 = \mathcal{H}^{2,V}$, then we can state the following.*
Let M and M' be two square integrable MAFs and let $M^{s,x}$ (respectively $M'^{s,x}$) be the cadlag version of M (respectively M') under a fixed $\mathbb{P}^{s,x}$. There exists a Borel function $k \in \mathcal{B}([0,T] \times E, \mathbb{R})$ such that for any $(s,x) \in [0,T] \times E$, $\langle M^{s,x}, M'^{s,x} \rangle = \int_s^{\cdot} k(r, X_r) dV_r$.
In particular if M is a square integrable MAF and $M^{s,x}$ its cadlag version under a fixed $\mathbb{P}^{s,x}$, there exists a Borel function $k \in \mathcal{B}([0,T] \times E, \mathbb{R})$ (which can be taken positive) such that for any $(s,x) \in [0,T] \times E$, $\langle M^{s,x} \rangle = \int_s^{\cdot} k(r, X_r) dV_r$.

2.B Technicalities related to Section 2.3

Proof of Proposition 2.3.2. Since we have $dA \ll dA + dB$ in the sense of stochastic measures with A, B predictable, there exists a predictable positive process K such that $A = \int_0^{\cdot} K_s dA_s + \int_0^{\cdot} K_s dB_s$ up to indistinguishability, see Proposition I.3.13 in [61]. Now there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that for any $\omega \in \mathcal{N}^c$ we have $0 \leq \int_0^{\cdot} K_s(\omega) dB_s(\omega) = \int_0^{\cdot} (1 - K_s(\omega)) dA_s(\omega)$, so $K(\omega) \leq 1$ $dA(\omega)$ a.e. on $\mathcal{N}^c$. Therefore if we set $E(\omega) = \{t : K_t(\omega) = 1\}$ and $F(\omega) = \{t : K_t(\omega) < 1\}$ then $E(\omega)$ and $F(\omega)$ are disjoint Borel sets and $dA(\omega)$ has all its mass in $E(\omega) \cup F(\omega)$ so we can decompose $dA(\omega)$ within these two sets.

We therefore define the processes $A^{\perp B} = \int_0^{\cdot} \mathbb{1}_{\{K_s = 1\}} dA_s$ and ; $A^B = \int_0^{\cdot} \mathbb{1}_{\{K_s < 1\}} dA_s$. $A^{\perp B}$ and A^B are both in $\mathcal{V}^{p,+}$, and $A = A^{\perp B} + A^B$. In particular the (stochastic) measures $dA^{\perp B}$ and dA^B fulfill $dA^{\perp B}(\omega)(G) = dA(\omega)(E(\omega) \cap G)$ and $dA^B(\omega)(G) = dA(\omega)(F(\omega) \cap G)$.
We remark $dA^{\perp B} \perp dB$ in the sense of stochastic measures. Indeed, fixing $\omega \in \mathcal{N}^c$, for $t \in E(\omega)$, $K_t(\omega) = 1$, so $\int_{E(\omega)} dA(\omega) = \int_{E(\omega)} dA(\omega) + \int_{E(\omega)} dB(\omega)$ implying that $\int_{E(\omega)} dB(\omega) = 0$. Since for any $\omega \in \mathcal{N}^c$, $dB(\omega)\,(E(\omega)) = 0$ while $dA^{\perp B}(\omega)$ has all its mass in $E(\omega)$, which gives this first result.

Now let us prove $dA^B \ll dB$ in the sense of stochastic measure.
Let $\omega \in \mathcal{N}^c$, and let $G \in \mathcal{B}([0,T])$, such that $\int_G dB(\omega) = 0$. Then

$$\begin{aligned} \int_G dA^B(\omega) &= \int_{G \cap F(\omega)} dA(\omega) \\ &= \int_{G \cap F(\omega)} K(\omega) dA(\omega) + \int_{G \cap F(\omega)} K(\omega) dB(\omega) \\ &= \int_{G \cap F(\omega)} K(\omega) dA(\omega). \end{aligned}$$

So $\int_{G \cap F(\omega)} (1 - K(\omega)) dA(\omega) = 0$, but $(1 - K(\omega)) > 0$ on $F(\omega)$.
So $dA^B(\omega)(G) = 0$. Consequently for every $\omega \in \mathcal{N}^c$, $dA^B(\omega) \ll dB(\omega)$ and so that $dA^B \ll dB$.
Now, since K is positive and $K(\omega) \leq 1$ $dA(\omega)$ a.e. for almost all ω, we can replace K by $K \wedge 1$ which is still positive predictable, without changing the associated stochastic measures $dA^B, dA^{\perp B}$; therefore we can consider that $K_t(\omega) \in [0,1]$ for all (ω, t).
We remark that for $\mathbb{P}$ almost all ω the decomposition $A^{\perp B}$ and A^B is unique because of the corresponding uniqueness of the decomposition in the Lebesgue-Radon-Nikodym theorem for each fixed $\omega \in \mathcal{N}^c$.

Since $dA^B \ll dB$, again by Proposition I.3.13 in [61], there exists a predictable positive process that we will call $\frac{dA}{dB}$ such that $A^B = \int_0^{\cdot} \frac{dA}{dB} dB$ and which is only unique up to $dB \otimes d\mathbb{P}$ null sets. □

Proposition 2.B.1. *Let M and M' be two local martingales in $\mathcal{H}^2_{loc}$ and let $V \in \mathcal{V}^{p,+}$. We have $\frac{d\langle M \rangle}{dV} \frac{d\langle M' \rangle}{dV} - \left(\frac{d\langle M, M' \rangle}{dV} \right)^2 \geq 0 \quad dV \otimes d\mathbb{P}$ a.e.*

Proof. Let $x \in \mathbb{Q}$. Since $\langle M + xM' \rangle$ is an increasing process starting at zero, then by Proposition 2.3.2, we have $\frac{d\langle M+xM' \rangle}{dV} \geq 0 \quad dV \otimes d\mathbb{P}$ a.e.

By the linearity property stated in Proposition 2.3.4, we have
$0 \leq \frac{d\langle M+xM' \rangle}{dV} = \frac{d\langle M \rangle}{dV} + 2x\frac{d\langle M,M' \rangle}{dV} + x^2\frac{d\langle M' \rangle}{dV}$ $dV \otimes d\mathbb{P}$ a.e. Since $\mathbb{Q}$ is countable, there exists a $dV \otimes d\mathbb{P}$-null set $\mathcal{N}$ such that for $(\omega, t) \notin \mathcal{N}$ and $x \in \mathbb{Q}$,
$\frac{d\langle M \rangle}{dV}(\omega,t) + 2x\frac{d\langle M,M' \rangle}{dV}(\omega,t) + x^2\frac{d\langle M' \rangle}{dV}(\omega,t) \geq 0$. By continuity of polynomes, this holds for any $x \in \mathbb{R}$. Expressing the discriminant of this polynome, we deduce that
$4\left(\frac{d\langle M,M' \rangle}{dV}(\omega,t)\right)^2 - 4\frac{d\langle M \rangle}{dV}(\omega,t)\frac{d\langle M' \rangle}{dV}(\omega,t) \leq 0$ for all $(\omega, t) \notin \mathcal{N}$. □

Proof of Proposition 2.3.5. Since the angular bracket $\langle M \rangle$ of a square integrable martingale M always belongs to $\mathcal{V}^{p,+}$, by Proposition 2.3.2, we can consider the processes $\langle M \rangle^V$ and $\langle M \rangle^{\perp V}$; in particular there exists a predictable process K with values in $[0,1]$ such that $\langle M \rangle^V = \int_0^\cdot \mathbb{1}_{\{K_s<1\}} d\langle M \rangle_s$ and $\langle M \rangle^{\perp V} = \int_0^\cdot \mathbb{1}_{\{K_s=1\}} d\langle M \rangle_s$.

We can then set $M^V = \int_0^\cdot \mathbb{1}_{\{K_s<1\}} dM_s$ and $M^{\perp V} = \int_0^\cdot \mathbb{1}_{\{K_s=1\}} dM_s$ which are well-defined because K is predictable, and therefore $\mathbb{1}_{\{K_t<1\}}$ and $\mathbb{1}_{\{K_t=1\}}$ are also predictable. $M^V, M^{\perp V}$ belong to $\mathcal{H}_0^2$ because their angular brackets are both bounded by $\langle M \rangle_T \in L^1$. Since K takes values in $[0,1]$, we have
$M^V + M^{\perp V} = \int_0^\cdot \mathbb{1}_{\{K_s<1\}} dM_s + \int_0^\cdot \mathbb{1}_{\{K_s=1\}} dM_s = M$;
$\langle M^V \rangle = \int_0^\cdot \mathbb{1}_{\{K_s<1\}} d\langle M \rangle_s = \langle M \rangle^V$; $\langle M^{\perp V} \rangle = \int_0^\cdot \mathbb{1}_{\{K_s=1\}} d\langle M \rangle_s = \langle M \rangle^{\perp V}$
and $\langle M^V, M^{\perp V} \rangle = \int_0^\cdot \mathbb{1}_{\{K_s<1\}} \mathbb{1}_{\{K_s=1\}} d\langle M \rangle_s = 0$. □

Proof of Proposition 2.3.6. We start by remarking that for any M_1, M_2 in $\mathcal{H}_0^2$, a consequence of Kunita-Watanabe's decomposition (see Theorem 4.27 in [61]) is that $dVar(\langle M_1, M_2 \rangle) \ll d\langle M_1 \rangle$ and $dVar(\langle M_1, M_2 \rangle) \ll d\langle M_2 \rangle$.

Now, let M_1 and M_2 be in $\mathcal{H}^{2,V}$. We have $dVar(\langle M_1, M_2 \rangle) \ll d\langle M_1 \rangle \ll dV$. So since $\langle M_1 + M_2 \rangle = \langle M_1 \rangle + 2\langle M_1, M_2 \rangle + \langle M_2 \rangle$, then $d\langle M_1 + M_2 \rangle \ll dV$ which shows that $\mathcal{H}^{2,V}$ is a vector space.

If M_1 and M_2 are in $\mathcal{H}^{2,\perp V}$, then since $dVar(\langle M_1, M_2 \rangle) \ll d\langle M_1 \rangle$ we can write $Var(\langle M_1, M_2 \rangle) = \int_0^\cdot \frac{dVar(\langle M_1,M_2 \rangle)}{d\langle M_1 \rangle} d\langle M_1 \rangle$ which is almost surely singular with respect to dV since M_1 belongs to $\mathcal{H}^{2,\perp V}$. So, by the bilinearity of the angular bracket $\mathcal{H}^{2,\perp V}$ is also a vector space.

Finally if $M_1 \in \mathcal{H}^{2,V}$ and $M_2 \in \mathcal{H}^{2,\perp V}$ then $dVar(\langle M_1, M_2 \rangle) \ll d\langle M_1 \rangle \ll dV$ but we also have seen that if $d\langle M_2 \rangle$ is singular to dV then so is $dVar(\langle M_1, M_2 \rangle) \ll d\langle M_2 \rangle$.

For fixed ω, a measure being simultaneously dominated and singular with respect to to $dV(\omega)$ is necessarily the null measure, so $dVar(\langle M_1, M_2 \rangle) = 0$ as a stochastic measure. Therefore M_1 and M_2 are strongly orthogonal, which implies in particular that M_1 and M_2 are orthogonal in $\mathcal{H}_0^2$.

So we have shown that $\mathcal{H}^{2,V}$ and $\mathcal{H}^{2,\perp V}$ are orthogonal sublinear-spaces of $\mathcal{H}_0^2$ but we also know that $\mathcal{H}_0^2 = \mathcal{H}^{2,V} + \mathcal{H}^{2,\perp V}$ thanks to Proposition 2.3.5, therefore $\mathcal{H}_0^2 = \mathcal{H}^{2,V} \oplus^\perp \mathcal{H}^{2,\perp V}$. This implies that $\mathcal{H}^{2,V} = (\mathcal{H}^{2,\perp V})^\perp$ and $\mathcal{H}^{2,\perp V} = (\mathcal{H}^{2,V})^\perp$ and therefore that these spaces are closed. So they are sub-Hilbert spaces. We also have shown that they were strongly orthogonal spaces, in the sense that any $M^1 \in \mathcal{H}^{2,V}$, $M^2 \in \mathcal{H}^{2,\perp V}$ are strongly orthogonal. By localization the strong orthogonality property also extends to $M^1 \in \mathcal{H}_{loc}^{2,V}$, $M^2 \in \mathcal{H}_{loc}^{2,\perp V}$. □

Chapter 3

BSDEs with no driving martingale, Markov processes and associated Pseudo PDEs. Part II: Decoupled mild solutions and Examples

This chapter is the object of paper [10].

Abstract

Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be a family of probability measures, where E is a Polish space, defined on the canonical probability space $\mathbb{D}([0,T],E)$ of E-valued cadlag functions. We suppose that a martingale problem with respect to a time-inhomogeneous generator a is well-posed. We consider also an associated semilinear *Pseudo-PDE* for which we introduce a notion of so called *decoupled mild* solution and study the equivalence with the notion of martingale solution introduced in the previous chapter. We also investigate well-posedness for decoupled mild solutions and their relations with a special class of BSDEs without driving martingale. The notion of decoupled mild solution is a good candidate to replace the notion of viscosity solution which is not always suitable when the map a is not a PDE operator.

3.1 Introduction

The framework of this paper is the canonical space $\Omega = \mathbb{D}([0,T],E)$ of cadlag functions defined on the interval $[0,T]$ with values in a Polish space E. This space will be equipped with a family $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ of probability measures indexed by an initial time $s \in [0,T]$ and a starting point $x \in E$. For each $(s,x) \in [0,T]\times E$, $\mathbb{P}^{s,x}$ corresponds to the law of an underlying forward Markov process with time index $[0,T]$, taking values in the Polish state space E which is characterized as the solution of a well-posed martingale problem related to a certain operator $(\mathcal{D}(a),a)$ and an increasing continuous function $V : [0,T] \to \mathbb{R}$. In the previous Chapter 2 we have introduced a semilinear equation generated by $(\mathcal{D}(a),a)$, called *Pseudo-PDE* of the type

$$\left\{\begin{array}{rcll} a(u) + f\left(\cdot,\cdot,u,\Gamma(u)^{\frac{1}{2}}\right) & = & 0 & \text{on } [0,T]\times E \\ u(T,\cdot) & = & g, & \end{array}\right. \tag{3.1.1}$$

where $\Gamma(u) = a(u^2) - 2ua(u)$ is a potential theory operator called the *carré du champs operator*. A classical solution of (3.1.1) is defined as an element of $\mathcal{D}(a)$ verifying (3.1.1). In Chapter 2 we have also

defined the notion of *martingale solution* of (3.1.1), see Definition 3.2.22. A function u is a martingale solution if (3.1.1) holds replacing the map a (resp. Γ) with an extended operator $\mathfrak{a}$ (resp. $\mathfrak{G}$) which is introduced in Definition 3.2.15 (resp. 3.2.17). The martingale solution extends the (analytical) notion of classical solution, however it is a probabilistic concept. The objectives of the present paper are essentially three.

1. To introduce an alternative notion of (this time analytical) solution, that we call *decoupled mild*, since it makes use of the time-dependent transition kernel associated with a. This new type of solution will be shown to be essentially equivalent to the martingale one.

2. To show existence and uniqueness of decoupled mild solutions.

3. To emphasize the link with solutions of Markovian BSDEs without driving martingale introduced in Chapter 2.

The aforementioned Markovian BSDEs are of the form

$$Y_t^{s,x} = g(X_T) + \int_t^T f\left(r, X_r, Y_r^{s,x}, \sqrt{\frac{d\langle M^{s,x}\rangle}{dV}}(r)\right) dV_r - (M_T^{s,x} - M_t^{s,x}), \quad t \in [0,T], \quad t \in [0,T], \tag{3.1.2}$$

in a stochastic basis $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$ which depends on (s,x). Under suitable conditions, the solution of this Markovian BSDE is a couple $(Y^{s,x}, M^{s,x})$ of cadlag stochastic processes where $M^{s,x}$ is a martingale. This was introduced and studied in a more general setting in Chapter 2, see [66] for a similar formulation.

We refer to the introduction and reference list of previous chapter for an extensive description of contributions to non-Brownian type BSDEs.

The classical Markovian BSDE, which is driven by a Brownian motion is of the form

$$\left\{ \begin{array}{rcl} X_t^{s,x} & = & x + \int_s^t \beta(r, X_r^{s,x}) dr + \int_s^t \sigma(r, X_r^{s,x}) dB_r \\ Y_t^{s,x} & = & g(X_T^{s,x}) + \int_t^T f\left(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x}\right) dr - \int_t^T Z_r^{s,x} dB_r, \end{array} \right. \tag{3.1.3}$$

where B is a Brownian motion. Existence and uniqueness for (3.1.3) was established first supposing mainly Lipschitz conditions on f with respect to the third and fourth variable. β and σ were also assumed to be Lipschitz (with respect to x) and to have linear growth. In the sequel those conditions were considerably relaxed, see [74] and references therein. This is a particular case of a more general (non-Markovian) Brownian BSDE introduced in 1990 by E. Pardoux and S. Peng in [71], after an early work of J.M. Bismut in 1973 in [18].

Equation (3.1.3) was a probabilistic representation of a semilinear partial differential equation of parabolic type with terminal condition:

$$\left\{ \begin{array}{l} \partial_t u + \frac{1}{2} \sum\limits_{i,j \le d} (\sigma\sigma^\intercal)_{i,j} \partial^2_{x_i x_j} u + \sum\limits_{i \le d} \beta_i \partial_{x_i} u + f(\cdot, \cdot, u, \sigma \nabla u) = 0 \quad \text{ on } [0,T[\times \mathbb{R}^d \\ u(T, \cdot) = g. \end{array} \right. \tag{3.1.4}$$

Given, for every (s,x), a solution $(Y^{s,x}, Z^{s,x})$ of the Markovian BSDE (3.1.3), under some continuity assumptions on the coefficients, see e.g. [72], it was proved that the function $u(s,x) := Y_s^{s,x}$ is a viscosity solution of (3.1.4), see also [76, 72, 76, 43], for related work.

We prolong this idea in a general case where the Markovian BSDE is (3.1.2) with solution $(Y^{s,x}, M^{s,x})$. In that case $u(s,x) := Y_s^{s,x}$ will be the decoupled mild solution of (3.1.1), see Theorem 3.3.15; in that general context the decoupled mild solution replaces the one of viscosity, for reasons that we will

explain below. One celebrated problem in the case of Brownian Markovian BSDEs is the characterization of $Z^{s,x}$ through a deterministic function v. This is what we will call the *identification problem*. In general the link between v and u is not always analytically established, excepted when u has some suitable differentiability property, see e.g. [8]: in that case v is closely related to the gradient of u. In our case, the notion of decoupled mild solution allows to identify (u, v) as the analytical solution of a deterministic problem. In the literature, the notion of mild solution of PDEs was used in finite dimension in [4], where the authors tackled diffusion operators generating symmetric Dirichlet forms and associated Markov processes thanks to the theory of Fukushima Dirichlet forms, see e.g. [53]. A partial extension to the case of non-symmetric Dirichlet forms is performed in [63]. Infinite dimensional setups were considered for example in [51] where an infinite dimensional BSDE could produce the mild solution of a PDE on a Hilbert space.

Let $(B, \|\cdot\|)$ be a functional Banach space of real Borel functions defined on E and A be an unbounded operator on $(B, \|\cdot\|)$. In the theory of evolution equations one often considers systems of the type

$$\left\{ \begin{array}{rcl} \partial_t u + Au & = & l \text{ on } [0,T] \times \mathbb{R}^d \\ u(T,\cdot) & = & g, \end{array} \right. \tag{3.1.5}$$

where $l : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ are such that $l(t,\cdot)$ and g belong to B for every $t \in [0,T]$. The idea of mild solutions consists to consider $-A$ (when possible) as the infinitesimal generator of a semigroup of operators $(P_t)_{t\geq 0}$ on $(B, \|\cdot\|)$, in the following sense. There is $\mathcal{D}(A) \subset B$, a dense subset on which $-Af = \lim_{t\to 0^+} \frac{1}{t}(P_t f - f)$. In particular one may think of $(P_t)_{t\geq 0}$ as the heat kernel semi-group and A as $\frac{1}{2}\Delta$. The approach of mild solutions is also very popular in the framework of stochastic PDEs see e. g. [30].

When A is a local operator, one solution (in the sense of distributions, or in the sense of evaluation against test functions) to the linear evolution problem with terminal condition (3.1.5) is the so called *mild solution*

$$u(s,\cdot) = P_{T-s}[g] - \int_s^T P_{r-s}[l(r,\cdot)]dr. \tag{3.1.6}$$

If l is explicitly a function of u then (3.1.6) becomes itself an equation and a mild solution would consist in finding a fixed point of (3.1.6). Let us now suppose the existence of a map $S : \mathcal{D}(S) \subset B \to B$, typically S being the gradient, when (P_t) is the heat kernel semigroup. The natural question is what would be a natural replacement for a *mild solution* of

$$\left\{ \begin{array}{rcl} \partial_t u + Au & = & f(s,\cdot,u,Su) \text{ on } [0,T] \times \mathbb{R}^d \\ u(T,\cdot) & = & g. \end{array} \right. \tag{3.1.7}$$

If the domain of S is B, then it is not difficult to extend the notion of mild solution to this case. One novelty of our approach consists is considering the case of solutions $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ for which $Su(t,\cdot)$ is not defined.

1. Suppose one expects a solution not to be classical, i.e. such that $u(r,\cdot)$ should not belong to the domain of $\mathcal{D}(A)$ but to be in the domain of S. In the case when Pseudo-PDEs are usual PDEs, one think to possible solutions which are not $C^{1,2}$ but admitting a gradient, typically viscosity solutions which are differentiable in x. In that case the usual idea of mild solutions theory applies to equations of type (3.1.7).

   In this setup, inspired by (3.1.6) a mild solution of the equation is naturally defined as a solution of the integral equation

$$u(s,\cdot) = P_{T-s}[g] + \int_s^T P_{r-s}[f(r,\cdot,u(r,\cdot),Su(r,\cdot))]dr. \tag{3.1.8}$$

2. However, there may be reasons for which the candidate solution u is such that $u(t,\cdot)$ does not even belong to $\mathcal{D}(S)$. In the case of PDEs it is often the case for viscosity solutions of PDEs which do not admit a gradient. In that case the idea is to replace (3.1.8) with

$$u(s,\cdot) = P_{T-s}[g] + \int_s^T P_{r-s}[f(r,\cdot,u(r,\cdot),v(r,\cdot))]dr. \tag{3.1.9}$$

and to add a second equality which expresses in a *mild* form the equality $v(r,\cdot) = Su(r,\cdot)$.

We will work out previous methodology for the $Pseudo-PDE(f,g)$. In that case S will be given by the mapping $u \longmapsto \Gamma(u)^{\frac{1}{2}}$. If A is the laplacian for instance one would have $\Gamma(u) = \|\nabla u\|^2$. For pedagogical purposes, one can first consider an operator a of type $\partial_t + A$ when $-A$ is the generator of a Markovian (time-homogeneous) semigroup. In this case,

$$\begin{aligned}\Gamma(u) &= \partial_t(u^2) + A(u^2) - 2u\partial_t u - 2uAu \\ &= A(u^2) - 2uAu.\end{aligned}$$

Equation

$$\partial_t u + Au + f\left(\cdot,\cdot,u,\Gamma(u)^{\frac{1}{2}}\right) = 0, \tag{3.1.10}$$

could therefore be decoupled into the system

$$\begin{cases} \partial_t u + Au + f(\cdot,\cdot,u,v) = 0 \\ v^2 = \partial_t(u^2) + A(u^2) - 2u(\partial_t u + Au), \end{cases} \tag{3.1.11}$$

which furthermore can be expressed as

$$\begin{cases} \partial_t u + Au &= -f(\cdot,\cdot,u,v) \\ \partial_t(u^2) + A(u^2) &= v^2 - 2uf(\cdot,\cdot,u,v) \end{cases} \tag{3.1.12}$$

Taking into account the existing notions of mild solution (3.1.6) (resp. (3.1.8)), for corresponding equations (3.1.5) (resp. (3.1.7)), one is naturally tempted to define a decoupled mild solution of (3.1.1) as a function u for which there exist $v \geq 0$ such that

$$\begin{cases} u(s,\cdot) &= P_{T-s}[g] + \int_s^T P_{r-s}[f(r,\cdot,u(r,\cdot),v(r,\cdot))]dr \\ u^2(s,\cdot) &= P_{T-s}[g^2] - \int_s^T P_{r-s}[v^2(r,\cdot) - 2u(r,\cdot)f(r,\cdot,u(r,\cdot),v(r,\cdot))]dr. \end{cases} \tag{3.1.13}$$

As we mentioned before, our approach is alternative to a possible notion of viscosity solution for the $Pseudo-PDE(f,g)$. That notion will be the object of a subsequent paper, at least in the case when the driver do not depend on the last variable. In the general case the notion of viscosity solution does not fit well because of lack of suitable comparison theorems. On the other hand, even in the recent literature (see [7]) in order to show existence of viscosity solutions specific conditions exist on the driver. In our opinion our approach of decoupled mild solutions for $Pseudo-PDE(f,g)$ constitutes an interesting novelty even in the case of semilinear parabolic PDEs.

The main contributions of the paper are essentially the following. In Section 3.3.1, Definition 3.3.4 introduces our notion of decoupled mild solution of (3.1.1) in the general setup. In Section 3.3.2, Proposition 3.3.7 states that under a square integrability type condition, every martingale solution is a decoupled mild solution of (3.1.1). Conversely, Proposition 3.3.8 shows that every decoupled mild solution is a martingale solution. In Theorem 3.3.9 we prove existence and uniqueness of a decoupled mild solution for (3.1.1). In Section 3.3.3, we show how the unique decoupled mild solution of (3.1.1) can be represented via the Markovian BSDEs (3.1.2). In Section 3.4 we develop examples of Markov

processes and corresponding operators a falling into our abstract setup. In Section 3.4.1, we work in the setup of [85], the Markov process is a diffusion with jumps and the corresponding operator is of diffusion type with an additional non-local operator. In Section 3.4.2 we consider Markov processes associated to pseudo-differential operators (typically the fractional Laplacian) as in [59]. In Section 3.4.3 we study a semilinear parabolic PDE with distributional drift, and the corresponding process is the solution an SDE with distributional drift as defined in [47]. Finally, in Section 3.4.4, we are interested with diffusions on differential manifolds and associated diffusion operators, an example being the Brownian motion in a Riemannian manifold associated to the Laplace-Beltrami operator.

3.2 Preliminaries

In this section we will recall the notations, notions and results of previous Chapter 2, which will be used here.

Notation 3.2.1. *In the whole paper, concerning functional spaces we will use the following notations.*
A topological space E will always be considered as a measurable space with its Borel σ-field which shall be denoted $\mathcal{B}(E)$. Given two topological spaces, E, F, then $\mathcal{C}(E,F)$ (respectively $\mathcal{B}(E,F)$) will denote the set of functions from E to F which are continuous (respectively Borel) and if F is a metric space, $\mathcal{C}_b(E,F)$ (respectively $\mathcal{B}_b(E,F)$) will denote the set of functions from E to F which are bounded continuous (respectively bounded Borel). For any $p \in [1,\infty]$, $d \in \mathbb{N}^*$, $(L^p(\mathbb{R}^d), \|\cdot\|_p)$ will denote the usual Lebesgue space equipped with its usual norm.

On a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for any $p \in \mathbb{N}^*$, L^p will denote the set of random variables with finite p-th moment.

*A probability space equipped with a right-continuous filtration $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$ (where $\mathbb{T}$ is equal to $\mathbb{R}_+$ or to $[0,T]$ for some $T \in \mathbb{R}_+^*$) will be called called a **stochastic basis** and will be said to **fulfill the usual conditions** if the probability space is complete and if $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible sets. When a stochastic basis is fixed, $\mathcal{P}ro$ denotes the **progressive σ-field** on $\mathbb{T}\times\Omega$.*

On a fixed stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, we will use the following notations and vocabulary, concerning spaces of stochastic processes, most of them being taken or adapted from [60] or [61]. $\mathcal{V}$ (resp $\mathcal{V}^+$) will denote the set of adapted, bounded variation (resp non-decreasing) processes starting at 0; $\mathcal{V}^p$ (resp $\mathcal{V}^{p,+}$) the elements of $\mathcal{V}$ (resp $\mathcal{V}^+$) which are predictable, and $\mathcal{V}^c$ (resp $\mathcal{V}^{c,+}$) the elements of $\mathcal{V}$ (resp $\mathcal{V}^+$) which are continuous.
$\mathcal{M}$ will be the space of cadlag martingales. For any $p \in [1,\infty]$ $\mathcal{H}^p$ will denote the subset of $\mathcal{M}$ of elements M such that $\sup_{t\in\mathbb{T}} |M_t| \in L^p$ and in this set we identify indistinguishable elements. It is a Banach space for the norm $\|M\|_{\mathcal{H}^p} = \mathbb{E}[|\sup_{t\in\mathbb{T}} M_t|^p]^{\frac{1}{p}}$, and $\mathcal{H}_0^p$ will denote the Banach subspace of $\mathcal{H}^p$ containing the elements starting at zero.

*If $\mathbb{T} = [0,T]$ for some $T \in \mathbb{R}_+^*$, a stopping time will be considered as a random variable with values in $[0,T]\cup\{+\infty\}$. We define a **localizing sequence of stopping times** as an increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ such that there exists $N \in \mathbb{N}$ for which $\tau_N = +\infty$. Let Y be a process and τ a stopping time, we denote Y^τ the process $t \mapsto Y_{t\wedge\tau}$ which we call **stopped process**. If $\mathcal{C}$ is a set of processes, we define its **localized class** $\mathcal{C}_{loc}$ as the set of processes Y such that there exist a localizing sequence $(\tau_n)_{n\geq 0}$ such that for every n, the stopped process Y^{τ_n} belongs to $\mathcal{C}$.*

*For any $M \in \mathcal{M}_{loc}$, we denote $[M]$ its **quadratic variation** and if moreover $M \in \mathcal{H}^2_{loc}$, $\langle M\rangle$ will denote its (predictable) **angular bracket**. $\mathcal{H}_0^2$ will be equipped with scalar product defined by $(M,N)_{\mathcal{H}^2} = \mathbb{E}[M_T N_T] = \mathbb{E}[\langle M,N\rangle_T]$ which makes it a Hilbert space. Two local martingales M, N will be said to be **strongly orthogonal** if MN is a local martingale starting in 0 at time 0. In $\mathcal{H}^2_{0,loc}$ this notion is equivalent to $\langle M,N\rangle = 0$.*

Concerning the following definitions and results, we are given some $T \in \mathbb{R}_+^*$, and a stochastic basis $\left(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ fulfilling the usual conditions.

Definition 3.2.2. *Let A and B be in $\mathcal{V}^+$. We will say that dB dominates dA* **in the sense of stochastic measures** *(written $dA \ll dB$) if for almost all ω, $dA(\omega) \ll dB(\omega)$ as Borel measures on $[0,T]$.*

Let $B \in \mathcal{V}^+$. $dB \otimes d\mathbb{P}$ will denote the positive measure on $(\Omega \times [0,T], \mathcal{F} \otimes \mathcal{B}([0,T]))$ defined for any $F \in \mathcal{F} \otimes \mathcal{B}([0,T])$ by $dB \otimes d\mathbb{P}(F) = \mathbb{E}\left[\int_0^T \mathbb{1}_F(t,\omega) dB_t(\omega)\right]$. A property which holds true everywhere except on a null set for this measure will be said to be true $dB \otimes d\mathbb{P}$ almost everywhere (a.e.).

We recall that given two processes A, B in $\mathcal{V}^{p,+}$, if $dA \ll dB$, there exists a predictable process which we will denote $\frac{dA}{dB}$ and call **Radon-Nikodym derivative** of A by B, verifying $A = \int_0^\cdot \frac{dA}{dB}(r) dB_r$, see Proposition I.3.13 in [61].

As in previous Chapter 2, we will be interested in a Markov process which is the solution of a martingale problem which we now recall below. For definitions and results concerning Markov processes, the reader may refer to Appendix 3.A. In particular, let E be a Polish space and $T \in \mathbb{R}_+$ be a finite value we now consider $(\Omega, \mathcal{F}, \mathbb{F})$ the canonical space and $(X_t)_{t\in[0,T]}$ the canonical process which are introduced in Notation 3.A.1, and a canonical Markov class measurable in time $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$, see Definitions 3.A.6 and 3.A.4. We will also consider the completed stochastic basis $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$, see Definition 3.A.8.

We now recall what the notion of martingale problem associated to an operator introduced in Section 2.4 of Chapter 2, see Definition 2.4.2.

Definition 3.2.3. *Given a linear algebra $\mathcal{D}(a) \subset \mathcal{B}([0,T]\times E, \mathbb{R})$, a linear operator a mapping $\mathcal{D}(a)$ into $\mathcal{B}([0,T]\times E, \mathbb{R})$ and a non-decreasing continuous function $V : [0,T] \to \mathbb{R}_+$ starting at 0, we say that a set of probability measures $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defined on $(\Omega, \mathcal{F})$ solves the* **Martingale Problem associated to** *$(\mathcal{D}(a), a, V)$ if, for any $(s,x) \in [0,T]\times E$, $\mathbb{P}^{s,x}$ verifies*

(a) $\mathbb{P}^{s,x}(\forall t \in [0,s], X_t = x) = 1$;

(b) for every $\phi \in \mathcal{D}(a)$, the process $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r) dV_r$, $t \in [s,T]$ is a cadlag $(\mathbb{P}^{s,x}, \mathbb{F})$-local martingale.

We say that the **Martingale Problem is well-posed** *if for any $(s,x) \in [0,T]\times E$, $\mathbb{P}^{s,x}$ is the only probability measure satisfying the properties (a) and (b).*

As for Chapter 2, in the sequel of the paper we will assume the following.

Hypothesis 3.2.4. *The Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ solves a well-posed Martingale Problem associated to a triplet $(\mathcal{D}(a), a, V)$ in the sense of Definition 3.2.3.*

Notation 3.2.5. *For every $(s,x) \in [0,T]\times E$ and $\phi \in \mathcal{D}(a)$, the process $t \mapsto \mathbb{1}_{[s,T]}(t)\left(\phi(t, X_t) - \phi(s,x) - \int_s^t a(\phi)(r, X_r) dV_r\right)$ will be denoted $M[\phi]^{s,x}$.*

$M[\phi]^{s,x}$ is a cadlag $(\mathbb{P}^{s,x}, \mathbb{F})$-local martingale equal to 0 on $[0,s]$, and by Proposition 3.A.9, it is also a $(\mathbb{P}^{s,x}, \mathbb{F}^{s,x})$-local martingale.

The bilinear operator below was introduced (in the case of time-homogeneous operators) by J.P. Roth in potential analysis (see Chapter III in [79]), and popularized by P.A. Meyer and others in the study of homogeneous Markov processes (see for example Exposé II: L'opérateur carré du champs in [69] or 13.46 in [60]).

Definition 3.2.6. *We introduce the bilinear operator*

$$\Gamma : \begin{array}{ccc} \mathcal{D}(a)\times\mathcal{D}(a) & \to & \mathcal{B}([0,T]\times E) \\ (\phi,\psi) & \mapsto & a(\phi\psi)-\phi a(\psi)-\psi a(\phi). \end{array} \tag{3.2.1}$$

*When $\phi=\psi$, $\Gamma(\phi,\phi)$ will be denoted $\Gamma(\phi)$. The operator Γ is called the **carré du champs operator**.*

The angular bracket of the martingales introduced in Notation 3.2.5 are expressed via the operator Γ. Proposition 2.4.7 of Chapter 2, tells the following.

Proposition 3.2.7. *For any $\phi\in\mathcal{D}(a)$ and $(s,x)\in[0,T]\times E$, $M[\phi]^{s,x}$ is in $\mathcal{H}^2_{0,loc}$. Moreover, for any $(\phi,\psi)\in\mathcal{D}(a)\times\mathcal{D}(a)$ and $(s,x)\in[0,T]\times E$ we have in $(\Omega,\mathcal{F}^{s,x},\mathbb{F}^{s,x},\mathbb{P}^{s,x})$ and on the interval $[s,T]$*

$$\langle M[\phi]^{s,x},M[\psi]^{s,x}\rangle=\int_s^{\cdot}\Gamma(\phi,\psi)(r,X_r)dV_r. \tag{3.2.2}$$

We introduce some significant spaces related to V.

Notation 3.2.8. $\mathcal{H}^{2,V}:=\{M\in\mathcal{H}^2_0|d\langle M\rangle\ll dV\}$.
We will also denote $\mathcal{L}^2(dV\otimes d\mathbb{P})$ the set of (up to indistinguishability) progressively measurable processes ϕ such that $\mathbb{E}[\int_0^T\phi_r^2dV_r]<\infty$.

Proposition 2.4.10 of Chapter 2 says the following.

Proposition 3.2.9. *If Hypothesis 3.2.4 is verified then under any $\mathbb{P}^{s,x}$,*
$\mathcal{H}^2_0=\mathcal{H}^{2,V}$.

In the sequel, several functional equations will hold up to a **zero potential** set that we recall below.

Definition 3.2.10. *For any $(s,x)\in[0,T]\times E$ we define the **potential measure** $U(s,x,\cdot)$ on $\mathcal{B}([0,T]\times E)$ by $U(s,x,A):=\mathbb{E}^{s,x}\left[\int_s^T\mathbb{1}_{\{(t,X_t)\in A\}}dV_t\right]$.*

A Borel set $A\in\mathcal{B}([0,T]\times E)$ will be said to be **of zero potential** *if, for any $(s,x)\in[0,T]\times E$ we have $U(s,x,A)=0$.*

Notation 3.2.11. *Let $p>0$, we define*
$\mathcal{L}^p_{s,x}:=\left\{f\in\mathcal{B}([0,T]\times E,\mathbb{R}):\mathbb{E}^{s,x}\left[\int_s^T|f|^p(r,X_r)dV_r\right]<\infty\right\}$ *on which we introduce the usual semi-norm* $\|\cdot\|_{p,s,x}:f\mapsto\left(\mathbb{E}^{s,x}\left[\int_s^T|f(r,X_r)|^pdV_r\right]\right)^{\frac{1}{p}}$ *We also denote*
$\mathcal{L}^0_{s,x}:=\left\{f\in\mathcal{B}([0,T]\times E,\mathbb{R}):\int_s^T|f|(r,X_r)dV_r<\infty\,\mathbb{P}^{s,x}\text{ a.s.}\right\}$.
For any $p\geq0$, we then define an intersection of these spaces, i.e. $\mathcal{L}^p_X:=\bigcap_{(s,x)\in[0,T]\times E}\mathcal{L}^p_{s,x}$.
Finally, let $\mathcal{N}$ the linear subspace of $\mathcal{B}([0,T]\times E,\mathbb{R})$ containing all functions which are equal to 0 $U(s,x,\cdot)$ a.e. for every (s,x). For any $p\in\mathbb{N}$, we define the quotient space $L^p_X:=\mathcal{L}^p_X/\mathcal{N}$. If $p\geq1$, L^p_X can be equipped with the topology generated by the family of semi-norms $(\|\cdot\|_{p,s,x})_{(s,x)\in[0,T]\times E}$ which makes it into a separable locally convex topological vector space.

The statement below was stated in Proposition 2.4.13 of Chapter 2.

Proposition 3.2.12. *Let f and g be in $\mathcal{B}([0,T]\times E,\mathbb{R})$ such that the processes $\int_s^{\cdot}f(r,X_r)dV_r$ and $\int_s^{\cdot}g(r,X_r)dV_r$ are finite $\mathbb{P}^{s,x}$ a.s. for any $(s,x)\in[0,T]\times E$. Then f and g are equal up a zero potential set if and only if $\int_s^{\cdot}f(r,X_r)dV_r$ and $\int_s^{\cdot}g(r,X_r)dV_r$ are indistinguishable under $\mathbb{P}^{s,x}$ for any $(s,x)\in[0,T]\times E$.*

We recall that if two functions f, g differ only on a zero potential set then they represent the same element of L^p_X.

We recall our notion of **extended generator**.

Definition 3.2.13. *We first define the **extended domain** $\mathcal{D}(\mathfrak{a})$ as the set functions $\phi \in \mathcal{B}([0,T]\times E,\mathbb{R})$ for which there exists $\psi \in \mathcal{B}([0,T]\times E,\mathbb{R})$ such that under any $\mathbb{P}^{s,x}$ the process*

$$\mathbb{1}_{[s,T]}\left(\phi(\cdot,X_\cdot)-\phi(s,x)-\int_s^\cdot \psi(r,X_r)dV_r\right) \tag{3.2.3}$$

(which is not necessarily cadlag) has a cadlag modification in $\mathcal{H}^2_0$.

Proposition 2.4.15 in Chapter 2 states the following.

Proposition 3.2.14. *Let $\phi \in \mathcal{B}([0,T]\times E,\mathbb{R})$. There is at most one (up to zero potential sets) $\psi \in \mathcal{B}([0,T]\times E,\mathbb{R})$ such that under any $\mathbb{P}^{s,x}$, the process defined in* (3.2.3) *has a modification which belongs to $\mathcal{M}_{loc}$.*
If moreover $\phi \in \mathcal{D}(a)$, then $a(\phi)=\psi$ up to zero potential sets. In this case, according to Notation 3.2.5, for every $(s,x)\in[0,T]\times E$, $M[\phi]^{s,x}$ is the $\mathbb{P}^{s,x}$ cadlag modification in $\mathcal{H}^2_0$ of
$\mathbb{1}_{[s,T]}\left(\phi(\cdot,X_\cdot)-\phi(s,x)-\int_s^\cdot \psi(r,X_r)dV_r\right)$.

Definition 3.2.15. *Let $\phi \in \mathcal{D}(\mathfrak{a})$ as in Definition 3.2.13. We denote again by $M[\phi]^{s,x}$, the unique cadlag version of the process* (3.2.3) *in $\mathcal{H}^2_0$. Taking Proposition 3.2.12 into account, this will not generate any ambiguity with respect to Notation 3.2.5. Proposition 3.2.12, also permits to define without ambiguity the operator*

$$\mathfrak{a}: \begin{array}{ccc} \mathcal{D}(\mathfrak{a}) & \longrightarrow & L^0_X \\ \phi & \longmapsto & \psi. \end{array}$$

*$\mathfrak{a}$ will be called the **extended generator**.*

We also extend the carré du champs operator $\Gamma(\cdot,\cdot)$ to $\mathcal{D}(\mathfrak{a})\times\mathcal{D}(\mathfrak{a})$.

Proposition 2.4.17 in Chapter 2 states the following.

Proposition 3.2.16. *Let ϕ and ψ be in $\mathcal{D}(\mathfrak{a})$, there exists a (unique up to zero-potential sets) function in $\mathcal{B}([0,T]\times E,\mathbb{R})$ which we will denote $\mathfrak{G}(\phi,\psi)$ such that under any $\mathbb{P}^{s,x}$,*
$\langle M[\phi]^{s,x}, M[\psi]^{s,x}\rangle = \int_s^\cdot \mathfrak{G}(\phi,\psi)(r,X_r)dV_r$ on $[s,T]$, up to indistinguishability. If moreover ϕ and ψ belong to $\mathcal{D}(a)$, then $\Gamma(\phi,\psi)=\mathfrak{G}(\phi,\psi)$ up to zero potential sets.

Definition 3.2.17. *The bilinear operator $\mathfrak{G}: \mathcal{D}(\mathfrak{a})\times\mathcal{D}(\mathfrak{a}) \longmapsto L^0_X$ will be called the **extended carré du champs operator**. When $\phi=\psi$, $\mathfrak{G}(\phi,\phi)$ will be denoted $\mathfrak{G}(\phi)$.*

According to Definition 3.2.13, we do not have necessarily $\mathcal{D}(a)\subset\mathcal{D}(\mathfrak{a})$, however we have the following.

Corollary 3.2.18. *If $\phi\in\mathcal{D}(a)$ and $\Gamma(\phi)\in\mathcal{L}^1_X$, then $\phi\in\mathcal{D}(\mathfrak{a})$ and $(a(\phi),\Gamma(\phi))=(\mathfrak{a}(\phi),\mathfrak{G}(\phi))$ up to zero potential sets.*

We also recall Lemma 2.5.13 of Chapter 2.

Lemma 3.2.19. *Let $(s,x)\in[0,T]\times E$ be fixed and let ϕ,ψ be two measurable processes. If ϕ and ψ are $\mathbb{P}^{s,x}$-modifications of each other, then they are equal $dV\otimes d\mathbb{P}^{s,x}$ a.e.*

We now keep in mind the Pseudo-Partial Differential Equation (in short Pseudo-PDE), with final condition, that we have introduced in Chapter 2.
Let us consider the following data.

1. A measurable final condition $g \in \mathcal{B}(E, \mathbb{R})$;
2. a measurable nonlinear function $f \in \mathcal{B}([0,T] \times E \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

The equation is

$$\left\{ \begin{array}{rcl} a(u) + f\left(\cdot, \cdot, u, \Gamma(u)^{\frac{1}{2}}\right) & = & 0 \quad \text{on } [0,T] \times E \\ u(T, \cdot) & = & g. \end{array} \right. \tag{3.2.4}$$

Notation 3.2.20. *Equation* (3.2.4) *will be denoted* $Pseudo - PDE(f, g)$.

Definition 3.2.21. *We will say that* u *is a* ***classical solution*** *of* $Pseudo - PDE(f, g)$ *if it belongs to* $\mathcal{D}(a)$ *and verifies* (3.2.4).

Definition 3.2.22. *A function* $u : [0,T] \times E \to \mathbb{R}$ *will be said to be a* **martingale solution** *of* $Pseudo - PDE(f, g)$ *if* $u \in \mathcal{D}(\mathfrak{a})$ *and*

$$\left\{ \begin{array}{rcl} \mathfrak{a}(u) & = & -f(\cdot, \cdot, u, \mathfrak{G}(u)^{\frac{1}{2}}) \\ u(T, \cdot) & = & g. \end{array} \right. \tag{3.2.5}$$

Until the end of these preliminaries, we will assume some growth conditions on the functions (f, g).

Hypothesis 3.2.23. *A couple of functions*
$f \in \mathcal{B}([0,T] \times E \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and* $g \in \mathcal{B}(E, \mathbb{R})$ *will be said* ***to verify*** H^{lip} *if there exist positive constants* K^Y, K^Z *such that*

1. $g(X_T)$ *is* L^2 *under* $\mathbb{P}^{s,x}$ *for every* $(s,x) \in [0,T] \times E$;
2. $t \longmapsto f(t, X_t, 0, 0) \in \mathcal{L}^2_X$;
3. $\forall (t, x, y, y', z, z') : \quad |f(t,x,y,z) - f(t,x,y',z)| \leq K^Y |y - y'| + K^Z |z - z'|$.

We conclude these preliminaries by stating the Theorem of existence and uniqueness of a martingale solution for $Pseudo - PDE(f, g)$. It was the object of Theorem 2.5.21 of Chapter 2.

Theorem 3.2.24. *Let* $(\mathbb{P}^{s,x})_{(s,x) \in [0,T] \times E}$ *be a Markov class associated to a transition kernel measurable in time (see Definitions 3.A.6 and 3.A.4) which fulfills Hypothesis 3.2.4, i.e. it is a solution of a well-posed Martingale Problem associated with the triplet* $(\mathcal{D}(a), a, V)$. *Let* (f, g) *be a couple verifying* H^{lip}, *see Hypothesis 3.2.23.*
Then $Pseudo - PDE(f, g)$ *has a unique martingale solution.*

We also had shown (see Proposition 2.5.20 in Chapter 2) that the unique martingale solution is the only possible classical solution if there is one, as stated below.

Proposition 3.2.25. *Under the conditions of previous Theorem 3.2.24, a classical solution* u *of* $Pseudo - PDE(f, g)$ *such that* $\Gamma(u) \in \mathcal{L}^1_X$, *is also a martingale solution.*
Conversely, if u *is a martingale solution of* $Pseudo - PDE(f, g)$ *belonging to* $\mathcal{D}(a)$, *then* u *is a classical solution of* $Pseudo - PDE(f, g)$ *up to a zero-potential set, meaning that the first equality of* (3.2.4) *holds up to a set of zero potential.*

3.3 Decoupled mild solutions of Pseudo-PDEs

All along this section we will consider a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x) \in [0,T] \times E}$ associated to a transition kernel P measurable in time (see Definitions 3.A.6, 3.A.4) verifying Hypothesis 3.2.4 for a certain $(\mathcal{D}(a), a, V)$. We are also given a couple of functions $f \in \mathcal{B}([0,T] \times E \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $g \in \mathcal{B}(E, \mathbb{R})$.

3.3.1 Definition

As mentioned in the introduction, in this section we introduce a notion of solution of our $Pseudo - PDE(f,g)$ that we will denominate *decoupled mild*, which is a generalization of the mild solution concept for partial differential equation. We will show that such solution exists and is unique. Indeed, that function will be the one appearing in Theorem 3.3.13.

A function u will be a *decoupled mild* solution of $Pseudo - PDE(f,g)$ if there is a function v such that the couple (u,v) is a (decoupled mild) solution of the identification problem $IP(f,g)$. In this section we first go through a notion of *decoupled mild solution* for the identification problem, which has particular interest in itself.

We will be interested in functions (f,g) which satisfy weaker conditions than those of type H^{lip} (see Hypothesis 3.2.23) namely the following ones.

Hypothesis 3.3.1. *A couple of functions*
$f \in \mathcal{B}([0,T]\times E\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ *and* $g\in\mathcal{B}(E,\mathbb{R})$ *will be said* ***to verify*** H^{growth} *if there exists a positive constant* C *such that*

1. $g(X_T)$ *is* L^2 *under* $\mathbb{P}^{s,x}$ *for every* $(s,x)\in[0,T]\times E$*;*
2. $t\longmapsto f(t,X_t,0,0)\in\mathcal{L}^2_X$*;*
3. $\forall(t,x,y,z):\quad |f(t,x,y,z)|\leq C(f(t,x,0,0)+|y|+|z|)$.

Notation 3.3.2. *Let* s,t *in* $[0,T]$ *with* $s\leq t$, $x\in E$ *and* $\phi\in\mathcal{B}(E,\mathbb{R})$*, if the expectation* $\mathbb{E}^{s,x}[|\phi(X_t)|]$ *is finite, then* $P_{s,t}[\phi](x)$ *will denote* $\int_E\phi(y)P_{s,t}(x,dy)$ *or equivalently* $\mathbb{E}^{s,x}[\phi(X_t)]$ *.*

We recall two important measurability properties.

Remark 3.3.3. *Let* $\phi\in\mathcal{B}(E,\mathbb{R})$.

- *Suppose that for any* (s,x,t), $\mathbb{E}^{s,x}[|\phi(X_t)|]<\infty$ *then by Proposition 3.A.12,* $(s,x,t)\longmapsto P_{s,t}[\phi](x)$ *is Borel.*
- *Suppose that for every* (s,x), $\mathbb{E}^{s,x}[\int_s^T|\phi(X_r)|dV_r]<\infty$*. Then by Lemma 3.A.11,* $(s,x)\longmapsto\int_s^T P_{s,r}[\phi](x)dVr$ *is Borel.*

In our general setup, considering some operator a, the equation

$$a(u)+f\left(\cdot,\cdot,u,\Gamma(u)^{\frac{1}{2}}\right)=0, \tag{3.3.1}$$

can be naturally decoupled into

$$\left\{\begin{array}{rcl} a(u) &=& -f(\cdot,\cdot,u,v)\\ \Gamma(u) &=& v^2.\end{array}\right. \tag{3.3.2}$$

Since $\Gamma(u)=a(u^2)-2ua(u)$, this system of equation will be rewritten as

$$\left\{\begin{array}{rcl} a(u) &=& -f(\cdot,\cdot,u,v)\\ a(u^2) &=& v^2-2uf(\cdot,\cdot,u,v).\end{array}\right. \tag{3.3.3}$$

On the other hand our Markov process X is time non-homogeneous and V_t can be more general than t, which leads us to the following definition of a decoupled mild solution.

Definition 3.3.4. *Let* (f,g) *be a couple verifying* H^{growth}*.*
Let $u,v\in\mathcal{B}([0,T]\times E,\mathbb{R})$ *be two Borel functions with* $v\geq 0$*.*

1. *The couple (u,v) will be called* **solution of the identification problem determined by** (f,g) *or simply* **solution of** $IP(f,g)$ *if u and v belong to $\mathcal{L}^2_X$ and if for every $(s,x)\in[0,T]\times E$,*

$$\left\{\begin{array}{rcl} u(s,x) & = & P_{s,T}[g](x)+\int_s^T P_{s,r}\left[f\left(r,\cdot,u(r,\cdot),v(r,\cdot)\right)\right](x)dV_r \\ u^2(s,x) & = & P_{s,T}[g^2](x)-\int_s^T P_{s,r}\left[v^2(r,\cdot)-2uf\left(r,\cdot,u(r,\cdot),v(r,\cdot)\right)\right](x)dV_r. \end{array}\right. \tag{3.3.4}$$

2. *The function u will be called* **decoupled mild solution** *of $Pseudo-PDE(f,g)$ if there is a function v such that the couple (u,v) is a solution of $IP(f,g)$.*

Lemma 3.3.5. *Let $u,v\in\mathcal{L}^2_X$, and let f be a Borel function satisfying items 2 and 3 of H^{growth}, then $f(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^2_X$ and $uf(\cdot,\cdot,u,v)$ to $\mathcal{L}^1_X$.*

Proof. Thanks to the growth condition on f in H^{growth}, there exists a constant $C>0$ such that for any $(s,x)\in[0,T]\times E$,

$$\begin{array}{rl} & \mathbb{E}^{s,x}\left[\int_t^T f^2(r,X_r,u(r,X_r),v(r,X_r))dV_r\right] \\ \leq & C\mathbb{E}^{s,x}\left[\int_t^T(f^2(r,X_r,0,0)+u^2(r,X_r)+v^2(r,X_r))dV_r\right]<\infty, \end{array} \tag{3.3.5}$$

since we have assumed that u^2,v^2 belong to $\mathcal{L}^1_X$, and since we have made Hypothesis H^{growth}. This means that $f^2(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^1_X$. Since $2|uf(\cdot,\cdot,u,v)|\leq u^2+f^2(\cdot,\cdot,u,v)$ then $uf(\cdot,\cdot,u,v)$ also belongs to $\mathcal{L}^1_X$. □

Remark 3.3.6. *Consequently, under the assumptions of Lemma 3.3.5 all the terms in* (3.3.4) *make sense.*

3.3.2 Existence and uniqueness of a solution

Proposition 3.3.7. *Assume that (f,g) verifies H^{growth} (see Hypothesis 3.3.1) and let $u\in\mathcal{L}^2_X$ be a martingale solution of $Pseudo-PDE(f,g)$. Then $(u,\mathfrak{G}(u)^{\frac{1}{2}})$ is a solution of $IP(f,g)$ and in particular, u is a decoupled mild solution of $Pseudo-PDE(f,g)$.*

Proof. Let $u\in\mathcal{L}^2_X$ be a martingale solution of $Pseudo-PDE(f,g)$. We emphasize that, taking Definition 3.2.13 and Proposition 3.2.16 into account, $\mathfrak{G}(u)$ belongs to $\mathcal{L}^1_X$, or equivalently that $\mathfrak{G}(u)^{\frac{1}{2}}$ belongs to $\mathcal{L}^2_X$. By Lemma 3.3.5, it follows that $f\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)\in\mathcal{L}^2_X$ and $uf\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)\in\mathcal{L}^1_X$. We fix some $(s,x)\in[0,T]\times E$ and the corresponding probability $\mathbb{P}^{s,x}$. We are going to show that

$$\left\{\begin{array}{rcl} u(s,x) & = & P_{s,T}[g](x)+\int_s^T P_{s,r}\left[f\left(r,\cdot,u(r,\cdot),\mathfrak{G}(u)^{\frac{1}{2}}(r,\cdot)\right)\right](x)dV_r \\ u^2(s,x) & = & P_{s,T}[g^2](x)-\int_s^T P_{s,r}\left[\mathfrak{G}(u)(r,\cdot)-2uf\left(r,\cdot,u(r,\cdot),\mathfrak{G}(u)^{\frac{1}{2}}(r,\cdot)\right)\right](x)dV_r. \end{array}\right. \tag{3.3.6}$$

Combining Definitions 3.2.13, 3.2.15, 3.2.22, we know that on $[s,T]$, the process $u(\cdot,X_\cdot)$ has a cadlag modification which we denote $U^{s,x}$ which is a special semimartingale with decomposition

$$U^{s,x}=u(s,x)-\int_s^\cdot f\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dV_r+M[u]^{s,x}, \tag{3.3.7}$$

where $M[u]^{s,x}\in\mathcal{H}^2_0$. Definition 3.2.22 also states that $u(T,\cdot)=g$, implying that

$$u(s,x)=g(X_T)+\int_s^T f\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dV_r-M[u]^{s,x}_T \text{ a.s.} \tag{3.3.8}$$

Taking the expectation, by Fubini's theorem we get

$$\begin{array}{rcl} u(s,x) & = & \mathbb{E}^{s,x}\left[g(X_T)+\int_s^T f\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dV_r\right] \\ & = & P_{s,T}[g](x)+\int_s^T P_{s,r}\left[f\left(r,\cdot,u(r,\cdot),\mathfrak{G}(u)^{\frac{1}{2}}(r,\cdot)\right)\right](x)dV_r. \end{array} \tag{3.3.9}$$

By integration by parts, we obtain

$$d(U^{s,x})_t^2 = -2U_t^{s,x} f\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(t,X_t)dV_t + 2U_{t^-}^{s,x} dM[u]_t^{s,x} + d[M[u]^{s,x}]_t, \tag{3.3.10}$$

so integrating from s to T, we get

$$\begin{array}{rcl} & & u^2(s,x) \\ & = & g^2(X_T)+2\int_s^T U_r^{s,x} f\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dV_r - 2\int_s^T U_{r^-}^{s,x} dM[u]_r^{s,x} - [M[u]^{s,x}]_T \\ & = & g^2(X_T)+2\int_s^T uf\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dV_r - 2\int_s^T U_{r^-}^{s,x} dM[u]_r^{s,x} - [M[u]^{s,x}]_T, \end{array} \tag{3.3.11}$$

where the latter line is a consequence of Lemma 3.2.19. The next step will consist in taking the expectation in equation (3.3.11), but before, we will check that $\int_s^\cdot U_{r^-}^{s,x} dM[u]_r^{s,x}$ is a martingale. Thanks to (3.3.7) and Jensen's inequality, there exists a constant $C>0$ such that

$$\sup_{t\in[s,T]}(U_t^{s,x})^2 \le C\left(\int_s^T f^2\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dV_r + \sup_{t\in[s,T]}(M[u]_t^{s,x})^2\right). \tag{3.3.12}$$

Since $M[u]^{s,x}\in\mathcal{H}_0^2$ and $f\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)\in\mathcal{L}_X^2$, it follows that $\sup_{t\in[s,T]}(U_t^{s,x})^2\in L^1$ and Lemma 2.3.17 in Chapter 2 states that $\int_s^\cdot U_{r^-}^{s,x} dM[u]_r^{s,x}$ is a martingale. Taking the expectation in (3.3.11), we now obtain

$$\begin{array}{rcl} u^2(s,x) & = & \mathbb{E}^{s,x}\left[g^2(X_T)+\int_s^T 2uf\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dV_r - [M[u]^{s,x}]_T\right] \\ & = & \mathbb{E}^{s,x}\left[g^2(X_T)+\int_s^T 2uf\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)(r,X_r)dV_r - \langle M[u]^{s,x}\rangle_T\right] \\ & = & \mathbb{E}^{s,x}\left[g^2(X_T)\right] - \mathbb{E}^{s,x}\left[\int_s^T\left(\mathfrak{G}(u)-2uf\left(\cdot,\cdot,u,\mathfrak{G}(u)^{\frac{1}{2}}\right)\right)(r,X_r)dV_r\right] \\ & = & P_{s,T}[g^2](x)-\int_s^T P_{s,r}\left[\mathfrak{G}(u)(r,\cdot)-2u(r,\cdot)f\left(r,\cdot,u(r,\cdot),\mathfrak{G}(u)^{\frac{1}{2}}(r,\cdot)\right)\right](x)dV_r, \end{array} \tag{3.3.13}$$

where the third equality derives from Proposition 3.2.16 and the fourth from Fubini's theorem. This concludes the proof. □

We now show the converse result of Proposition 3.3.7.

Proposition 3.3.8. *Assume that (f,g) verifies H^{growth}, see Hypothesis 3.3.1. Every decoupled mild solution of $Pseudo-PDE(f,g)$ is a also a martingale solution. Moreover, if (u,v) solves $IP(f,g)$, then $v^2=\mathfrak{G}(u)$ (up to zero potential sets).*

Proof. Let u and $v\ge 0$ be a couple of functions in $\mathcal{L}_X^2$ verifying (3.3.4). We first note that, the first line of (3.3.4) with $s=T$, gives $u(T,\cdot)=g$.
We fix $(s,x)\in[0,T]\times E$ and the associated probability $\mathbb{P}^{s,x}$, and on $[s,T]$, we set $U_t:=u(t,X_t)$ and $N_t:=u(t,X_t)-u(s,x)+\int_s^t f(r,X_r,u(r,X_r),v(r,X_r))dV_r$.

Combining the first line of (3.3.4) applied in $(s,x) = (t, X_t)$ and the Markov property (3.A.3), and since $f(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^2_X$ (see Lemma 3.3.5) we get the a.s. equalities

$$\begin{array}{rcl} U_t &=& u(t,X_t) \\ &=& P_{t,T}[g](X_t) + \int_t^T P_{t,r}\left[f\left(r,\cdot,u(r,\cdot),v(r,\cdot)\right)\right](X_t)dV_r \\ &=& \mathbb{E}^{t,X_t}\left[g(X_T) + \int_t^T f(r,X_r,u(r,X_r),v(r,X_r))dV_r\right] \\ &=& \mathbb{E}^{s,x}\left[g(X_T) + \int_t^T f(r,X_r,u(r,X_r),v(r,X_r))dV_r|\mathcal{F}_t\right], \end{array} \tag{3.3.14}$$

from which we deduce that $N_t = \mathbb{E}^{s,x}\left[g(X_T) + \int_s^T f(r,X_r,u(r,X_r),v(r,X_r))dV_r|\mathcal{F}_t\right] - u(s,x)$ a.s. So N is a martingale. We can therefore consider on $[s,T]$ and under $\mathbb{P}^{s,x}$, $N^{s,x}$ the cadlag version of N, and the special semimartingale
$U^{s,x} := u(s,x) - \int_s^\cdot f(r,X_r,u(r,X_r),v(r,X_r))dV_r + N^{s,x}$ which is a cadlag version of U. By Jensen's inequality for both expectation and conditional expectation, we have

$$\begin{array}{rcl} \mathbb{E}^{s,x}[(N^{s,x})^2_t] &=& \mathbb{E}^{s,x}\left[\left(\mathbb{E}^{s,x}\left[g(X_T) + \int_s^T f(r,X_r,u(r,X_r),v(r,X_r))dV_r|\mathcal{F}_t\right] - u(s,x)\right)^2\right] \\ &\leq& 3u^2(s,x) + 3\mathbb{E}^{s,x}[g^2(X_T)] + 3\mathbb{E}^{s,x}\left[\int_s^T f^2(r,X_r,u(r,X_r),v(r,X_r))dV_r\right] \\ &<& \infty, \end{array} \tag{3.3.15}$$

where the second term is finite because of H^{growth}, and the same also holds for the third one because $f(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}^2_X$, see Lemma 3.3.5. So $N^{s,x}$ is square integrable. We have therefore shown that under any $\mathbb{P}^{s,x}$, the process $u(\cdot,X_\cdot) - u(s,x) + \int_s^\cdot f(r,X_r,u(r,X_r),v(r,X_r))dV_r$ has on $[s,T]$ a modification in $\mathcal{H}^2_0$. Definitions 3.2.13 and 3.2.15, justify that $u \in \mathcal{D}(\mathfrak{a})$, $\mathfrak{a}(u) = -f(\cdot,\cdot,u,v)$ and that for any $(s,x) \in [0,T]\times E$, $M[u]^{s,x} = N^{s,x}$.

To conclude that u is a martingale solution of $Pseudo - PDE(f,g)$, there is left to show that $\mathfrak{G}(u) = v^2$, up to zero potential sets. By Proposition 3.2.16, this is equivalent to show that for every $(s,x) \in [0,T]\times E$, $\langle N^{s,x}\rangle = \int_s^\cdot v^2(r,X_r)dV_r$, in the sense of indistinguishability.

We fix again $(s,x) \in [0,T]\times E$ and the associated probability, and now set

$$N'_t := u^2(t,X_t) - u^2(s,x) - \int_s^t (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dV_r.$$

Combining the second line of (3.3.4) applied in $(s,x) = (t,X_t)$ and the Markov property (3.A.3), and since $v^2, uf(\cdot,\cdot,u,v)$ belong to $\mathcal{L}^1_X$ (see Lemma 3.3.5) we get the a.s. equalities

$$\begin{array}{rcl} u^2(t,X_t) &=& P_{t,T}[g^2](X_t) - \int_t^T P_{t,r}\left[(v^2(r,\cdot) - 2u(r,\cdot)f\left(r,\cdot,u(r,\cdot),v(r,\cdot)\right))\right](X_t)dV_r \\ &=& \mathbb{E}^{t,X_t}\left[g^2(X_T) - \int_t^T (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dV_r\right] \\ &=& \mathbb{E}^{s,x}\left[g^2(X_T) - \int_t^T (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dV_r|\mathcal{F}_t\right], \end{array} \tag{3.3.16}$$

from which we deduce that for any $t \in [s,T]$,

$$N'_t = \mathbb{E}^{s,x}\left[g^2(X_T) - \int_s^T (v^2 - uf(\cdot,\cdot,u,v))(r,X_r)dV_r|\mathcal{F}_t\right] - u^2(s,x) \text{ a.s.}$$

So N' is a martingale. We can therefore consider on $[s,T]$ and under $\mathbb{P}^{s,x}$, $N'^{s,x}$ the cadlag version of N'.

The process $u^2(s,x) + \int_s^\cdot (v^2 - uf(\cdot,\cdot,u,v))(r,X_r)dV_r + N'^{s,x}$ is therefore a cadlag special semimartingale which is a $\mathbb{P}^{s,x}$-version of $u^2(\cdot,X)$ on $[s,T]$. But we also had shown that

$$U^{s,x} = u(s,x) - \int_s^\cdot f(r,X_r,u(r,X_r),v(r,X_r))dV_r + N^{s,x}$$

is a version of $u(\cdot,X)$, which by integration by part implies that

$$u^2(s,x) - 2\int_s^\cdot U_r^{s,x} f(\cdot,\cdot,u,v)(r,X_r)dV_r + 2\int_s^\cdot U_{r^-}^{s,x} dN_r^{s,x} + [N^{s,x}]$$

is another cadlag semimartingale which is a $\mathbb{P}^{s,x}$-version of $u^2(\cdot,X)$ on $[s,T]$.
$\int_s^\cdot (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dV_r + N'^{s,x}$ is therefore indistinguishable from
$-2\int_s^\cdot U_r^{s,x} f(\cdot,\cdot,u,v)(r,X_r)dV_r + 2\int_s^\cdot U_{r^-}^{s,x} dN_r^{s,x} + [N^{s,x}]$ which can be written

$$\langle N^{s,x}\rangle - 2\int_s^\cdot U_r^{s,x} f(\cdot,\cdot,u,v)(r,X_r)dV_r + 2\int_s^\cdot U_{r^-}^{s,x} dN_r^{s,x} + ([N^{s,x}] - \langle N^{s,x}\rangle),$$

where $\langle N^{s,x}\rangle - 2\int_s^\cdot U_r^{s,x} f(\cdot,\cdot,u,v)(r,X_r)dV_r$ is predictable with bounded variation and $2\int_s^\cdot U_{r^-}^{s,x} dN_r^{s,x} + ([N^{s,x}] - \langle N^{s,x}\rangle)$ is a local martingale. By uniqueness of the decomposition of a special semimartingale, we have

$$\int_s^\cdot (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dV_r = \langle N^{s,x}\rangle - 2\int_s^\cdot U_r^{s,x} f(\cdot,\cdot,u,v)(r,X_r)dV_r,$$

and by Lemma 3.2.19,

$$\int_s^\cdot (v^2 - 2uf(\cdot,\cdot,u,v))(r,X_r)dV_r = \langle N^{s,x}\rangle - 2\int_s^\cdot uf(\cdot,\cdot,u,v)(r,X_r)dV_r,$$

which finally yields $\langle N^{s,x}\rangle = \int_s^\cdot v^2(r,X_r)dV_r$ as desired. □

We recall that $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ is a Markov class associated to a transition kernel measurable in time (see Definitions 3.A.6 and 3.A.4) which fulfills Hypothesis 3.2.4, i.e. it is a solution of a well-posed Martingale Problem associated with the triplet $(\mathcal{D}(a),a,V)$.

Theorem 3.3.9. *Let (f,g) be a couple verifying H^{lip}, see Hypothesis 3.2.23. Then $Pseudo-PDE(f,g)$ has a unique decoupled mild solution.*

Proof. This derives from Theorem 3.2.24 and Propositions 3.3.7, 3.3.8. □

Corollary 3.3.10. *Assume that (f,g) verifies H^{lip}, see Hypothesis 3.2.23. A classical solution u of $Pseudo-PDE(f,g)$ such that $\Gamma(u)\in\mathcal{L}_X^1$, is also a decoupled mild solution.*
Conversely, if u is a decoupled mild solution of $Pseudo-PDE(f,g)$ belonging to $\mathcal{D}(a)$, then u is a classical solution of $Pseudo-PDE(f,g)$ up to a zero-potential set, meaning that the first equality of (3.2.4) *holds up to a set of zero potential.*

Proof. The statement holds by Proposition 3.3.8 and Proposition 3.2.25. □

3.3.3 Representation of the solution via Markovian BSDEs with no driving martingale

In Chapter 2, the following family of Markovian BSDEs with no driving martingale indexed by $(s,x) \in [0,T] \times E$ was introduced.

Definition 3.3.11. *Let $(s,x) \in [0,T] \times E$ and the associated stochastic basis $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$ be fixed. A couple*
$(Y^{s,x}, M^{s,x}) \in \mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}_0^2$ will be said to solve $BSDE^{s,x}(f,g)$ if it verifies on $[0,T]$, in the sense of indistinguishability

$$Y^{s,x} = g(X_T) + \int_\cdot^T f\left(r, X_r, Y_r^{s,x}, \sqrt{\frac{d\langle M^{s,x}\rangle}{dV}}(r)\right) dV_r - (M_T^{s,x} - M_\cdot^{s,x}). \tag{3.3.17}$$

If (3.3.17) is only satisfied on a smaller interval $[t_0,T]$, with $0 < t_0 < T$, we say that $(Y^{s,x}, M^{s,x})$ solves $BSDE^{s,x}(f,g)$ on $[t_0,T]$.

The following result follows from Theorem 2.3.21 in Chapter 2.

Theorem 3.3.12. *Assume that (f,g) verifies H^{lip}, see Hypothesis 3.2.23. Then for any $(s,x) \in [0,T] \times E$, $BSDE^{s,x}(f,g)$ has a unique solution.*

In the following theorem, we summarize the links between the $BSDE^{s,x}(f,g)$ and the notion of martingale solution of $Pseudo - PDE(f,g)$. These are shown in Theorem 2.5.15, Remark 2.5.16, Theorem 2.5.21 and Theorem 2.5.22 of Chapter 2.

Theorem 3.3.13. *Assume that (f,g) verifies H^{lip} (see Hypothesis 3.2.23) and let $(Y^{s,x}, M^{s,x})$ denote the (unique) solution of $BSDE^{s,x}(f,g)$ for fixed (s,x). Let u be the unique martingale solution of $Pseudo - PDE(f,g)$. For every $(s,x) \in [0,T] \times E$, on the interval $[s,T]$,*

- *$Y^{s,x}$ and $u(\cdot, X_\cdot)$ are $\mathbb{P}^{s,x}$-modifications, and equal $dV \otimes d\mathbb{P}^{s,x}$ a.e.;*
- *$M^{s,x}$ and $M[u]^{s,x}$ are $\mathbb{P}^{s,x}$-indistinguishable.*

Moreover u belongs to $\mathcal{L}_X^2$ and for any $(s,x) \in [0,T] \times E$, we have $\frac{d\langle M^{s,x}\rangle}{dV} = \mathfrak{G}(u)(\cdot, X_\cdot)\ dV \otimes d\mathbb{P}^{s,x}$ a.e.

Remark 3.3.14. *The martingale solution u of $Pseudo - PDE$ exists and is unique by Theorem 3.2.24.*

We can therefore represent the unique decoupled mild solution of $Pseudo - PDE(f,g)$ via the stochastic equations $BSDE^{s,x}(f,g)$ as follows.

Theorem 3.3.15. *Assume that (f,g) verifies H^{lip} (see Hypothesis 3.2.23) and let $(Y^{s,x}, M^{s,x})$ denote the (unique) solution of $BSDE^{s,x}(f,g)$ for fixed (s,x).*

Then for any $(s,x) \in [0,T] \times E$, the random variable $Y_s^{s,x}$ is $\mathbb{P}^{s,x}$ a.s. equal to a constant (which we still denote $Y_s^{s,x}$), and the function

$$u : (s,x) \longmapsto Y_s^{s,x} \tag{3.3.18}$$

is the unique decoupled mild solution of $Pseudo - PDE(f,g)$.

Proof. By Theorem 3.3.13, there exists a Borel function u such that for every $(s,x) \in [0,T] \times E$, $Y_s^{s,x} = u(s, X_s) = u(s,x)$ $\mathbb{P}^{s,x}$ a.s. and u is the unique martingale solution of $Pseudo - PDE(f,g)$. By Proposition 3.3.7, it is also its unique decoupled mild solution. □

Remark 3.3.16. *The function v such that (u,v) is the unique solution of the identification problem $IP(f,g)$ also has a stochastic representation since it verifies for every $(s,x) \in [0,T] \times E$, on the interval $[s,T]$,*
$\frac{d\langle M^{s,x}\rangle}{dV} = v^2(\cdot, X_\cdot)\ dV \otimes d\mathbb{P}^{s,x}$ a.e. where $M^{s,x}$ is the martingale part of the solution of $BSDE^{s,x}$.

Conversely, under the weaker condition H^{growth} if one knows the solution of $IP(f,g)$, one can (for every (s,x)) produce a version of a solution of $BSDE^{s,x}(f,g)$ as follows. This is only possible with the notion of decoupled mild solution: even in the case of Brownian BSDEs the knowledge of the viscosity solution of the related PDE would (in general) not be sufficient to reconstruct the family of solutions of the BSDEs.

Proposition 3.3.17. *Assume that (f,g) verifies H^{growth}, see Hypothesis 3.3.1. Suppose the existence of a solution (u,v) to $IP(f,g)$, and let $(s,x) \in [0,T] \times E$ be fixed. Then*

$$\left(u(\cdot,X), \quad u(\cdot,X) - u(s,x) + \int_s^{\cdot} f(\cdot,\cdot,u,v)(r,X_r)dV_r \right) \tag{3.3.19}$$

admits on $[s,T]$ a $\mathbb{P}^{s,x}$-version $(Y^{s,x}, M^{s,x})$ which solves $BSDE^{s,x}$ on $[s,T]$.

Proof. By Proposition 3.3.8, u is a martingale solution of $Pseudo-PDE(f,g)$ and $v^2 = \mathfrak{G}(u)$. We now fix $(s,x) \in [0,T] \times E$. Combining Definitions 3.2.15, 3.2.17 and 3.2.22, we know that $u(T,\cdot) = g$ and that on $[s,T]$, $u(\cdot,X)$ has a $\mathbb{P}^{s,x}$-version $U^{s,x}$ with decomposition $U^{s,x} = u(s,x) - \int_s^{\cdot} f(\cdot,\cdot,u,v)(r,X_r)dV_r + M[u]^{s,x}$, where $M[u]^{s,x}$ is an element of $\mathcal{H}_0^2$ of angular bracket $\int_s^{\cdot} v^2(r,X_r)dV_r$ and is a version of $u(\cdot,X) - u(s,x) + \int_s^{\cdot} f(\cdot,\cdot,u,v)(r,X_r)dV_r$. By Lemma 3.2.19, taking into account $u(T,\cdot) = g$, the couple $(U^{s,x}, M[u]^{s,x})$ verifies on $[s,T]$, in the sense of indistinguishability

$$U^{s,x} = g(X_T) + \int_{\cdot}^{T} f\left(r, X_r, U_r^{s,x}, \sqrt{\frac{d\langle M[u]^{s,x}\rangle}{dV}}(r) \right) dV_r - (M[u]_T^{s,x} - M[u]_{\cdot}^{s,x}) \tag{3.3.20}$$

with $M[u]^{s,x} \in \mathcal{H}_0^2$ verifying $M[u]_s^{s,x} = 0$ (see Definition 3.2.15) and $U_s^{s,x}$ is deterministic so in particular is a square integrable r.v. Following a slight adaptation of the proof of Lemma 2.3.24 in Chapter 2 (see Remark 3.3.18 below), this implies that $U^{s,x} \in \mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$ and therefore that $(U^{s,x}, M[u]^{s,x})$ is a solution of $BSDE^{s,x}(f,g)$ on $[s,T]$. □

Remark 3.3.18. *Indeed Lemma 2.3.24 in Chapter 2 , taking into account Notation 5.5 ibidem, can be applied rigorously only under H^{lip} for (f,g). However, the same proof easily allows an extension to our framework H^{growth}.*

3.4 Examples of applications

We now develop some examples. Some of the applications that we are interested in involve operators which only act on the space variable, and we will extend them to time-dependent functions. The reader may consult Appendix 3.B, concerning details about such extensions. In all the items below there will be a canonical Markov class with transition kernel measurable in time which is solution of a well-posed Martingale Problem associated to some triplet $(\mathcal{D}(a), a, V)$ as introduced in Definition 3.2.3. Therefore all the results of this paper will apply to all the examples below, namely Theorem 3.2.24, Propositions 3.2.25, 3.3.7 and 3.3.8, Theorem 3.3.9, Corollaries 3.3.10 and 3.3.10, Theorems 3.3.12, 3.3.13 and 3.3.15 and Proposition 3.3.17. In particular, Theorem 3.3.9 states in all the cases, under suitable Lipschitz type conditions for the driver f, that the corresponding Pseudo-PDE admits a unique decoupled mild solution. In all the examples $T \in \mathbb{R}_+^*$ will be fixed.

3.4.1 Markovian jump diffusions

In this subsection, the state space will be $E := \mathbb{R}^d$ for some $d \in \mathbb{N}^*$. We are given $\beta \in \mathcal{B}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, $\alpha \in \mathcal{B}([0,T] \times \mathbb{R}^d, S_+^*(\mathbb{R}^d))$ (where $S_+^*(\mathbb{R}^d)$ is the space of symmetric strictly positive definite

matrices of size d) and K a Lévy kernel: this means that for every $(t,x) \in [0,T] \times \mathbb{R}^d$, $K(t,x,\cdot)$ is a σ-finite measure on $\mathbb{R}^d \backslash \{0\}$, $\sup_{t,x} \int \frac{\|y\|^2}{1+\|y\|^2} K(t,x,dy) < \infty$ and for every Borel set $A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$, $(t,x) \longmapsto \int_A \frac{\|y\|^2}{1+\|y\|^2} K(t,x,dy)$ is Borel. We will consider the operator a defined by

$$\partial_t \phi + \frac{1}{2} Tr(\alpha \nabla^2 \phi) + (\beta, \nabla \phi) + \int \left(\phi(\cdot, \cdot + y) - \phi(\cdot, y) - \frac{(y, \nabla \phi)}{1 + \|y\|^2} \right) K(\cdot, \cdot, dy), \tag{3.4.1}$$

on the domain $\mathcal{D}(a)$ which is here the linear algebra $\mathcal{C}_b^{1,2}([0,T] \times \mathbb{R}^d, \mathbb{R})$ of real continuous bounded functions on $[0,T] \times \mathbb{R}^d$ which are continuously differentiable in the first variable with bounded derivative, and twice continuously differentiable in the second variable with bounded derivatives.

Concerning martingale problems associated to parabolic PDE operators, one may consult [85]. Since we want to include integral operators, we will adopt the formalism of D.W. Stroock in [84]. Its Theorem 4.3 and the penultimate sentence of its proof states the following.

Theorem 3.4.1. *Suppose that β is bounded, that α is bounded continuous and that for any $A \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$, $(t,x) \longmapsto \int_A \frac{y}{1+\|y\|^2} K(t,x,dy)$ is bounded continuous. Then, for every (s,x), there exists a unique probability $\mathbb{P}^{s,x}$ on the canonical space (see Definition 3.A.1) such that $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r) dr$ is a local martingale for any $\phi \in \mathcal{D}(a)$ and $\mathbb{P}^{s,x}(X_s = x) = 1$. Moreover $(\mathbb{P}^{s,x})_{(s,x) \in [0,T] \times \mathbb{R}^d}$ defines a Markov class and its transition kernel is measurable in time.*

The Martingale Problem associated to $(\mathcal{D}(a), a, V_t \equiv t)$ in the sense of Definition 3.2.3 is therefore well-posed and solved by $(\mathbb{P}^{s,x})_{(s,x) \in [0,T] \times \mathbb{R}^d}$.

In this context, $\mathcal{D}(a)$ is an algebra and for ϕ, ψ in $\mathcal{D}(a)$, the carré du champs operator is given by

$$\Gamma(\phi, \psi) = \sum_{i,j \leq d} \alpha_{i,j} \partial_{x_i} \phi \partial_{x_j} \psi + \int_{\mathbb{R}^d \backslash \{0\}} (\phi(\cdot, \cdot + y) - \phi)(\psi(\cdot, \cdot + y) - \psi) K(\cdot, \cdot, dy).$$

We will consider a couple (f,g) satisfying H^{lip} (its items 1 and 2 hold for example if g and $f(\cdot, \cdot, 0, 0)$ are bounded).

Proposition 3.4.2. *Under the assumptions of Theorem 3.4.1, and if (f,g) verify H^{lip} (see Hypothesis 3.2.23), $Pseudo - PDE(f,g)$ admits a unique decoupled mild solution in the sense of Definition 3.3.4.*

Proof. $\mathcal{D}(a)$ is an algebra. Moreover $(\mathbb{P}^{s,x})_{(s,x) \in [0,T] \times \mathbb{R}^d}$ is a Markov class which is measurable in time, and it solves the well-posed Martingale Problem associated to $(\mathcal{D}(a), a, V_t \equiv t)$. Therefore our Theorem 3.3.9 applies. □

3.4.2 Pseudo-Differential operators and Fractional Laplacian

This section concerns pseudo-differential operators with negative definite symbol, see [58] for an extensive description. A typical example of such operators will be the fractional Laplacian $\Delta^{\frac{\alpha}{2}}$ with $\alpha \in]0,2[$, see Chapter 3 in [36] for a detailed study of this operator. We will mainly use the notations and vocabulary of N. Jacob in [57], [58] and [59], some results being attributed to W. Hoh [54]. We fix $d \in \mathbb{N}^*$. $\mathcal{C}_c^\infty(\mathbb{R}^d)$ will denote the space of real functions defined on $\mathbb{R}^d$ which are infinitely continuously differentiable with compact support and $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of fast decreasing real smooth functions also defined on $\mathbb{R}^d$. $\mathcal{F}u$ will denote the Fourier transform of a function u whenever it is well-defined. For $u \in L^1(\mathbb{R}^d)$ we use the convention $\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i(x,\xi)} u(x) dx$.

Definition 3.4.3. *A function $\psi \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$ will be said **negative definite** if for any $k \in \mathbb{N}$, $\xi_1, \cdots, \xi_k \in \mathbb{R}^d$, the matrix $(\psi(\xi^j) + \psi(\xi^l) - \psi(\xi^j - \xi^l))_{j,l=1,\cdots,k}$ is symmetric positive definite.*

*A function $q \in \mathcal{C}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ will be called a **continuous negative definite symbol** if for any $x \in \mathbb{R}^d$, $q(x, \cdot)$ is continuous negative definite*

In this case we introduce the pseudo-differential operator $q(\cdot, D)$ defined by

$$q(\cdot, D)(u)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,\xi)} q(x, \xi) \mathcal{F}u(\xi) d\xi. \tag{3.4.2}$$

Remark 3.4.4. *By Theorem 4.5.7 in [57], $q(\cdot, D)$ maps the space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ of smooth functions with compact support into itself. In particular $q(\cdot, D)$ will be defined on $\mathcal{C}_c^\infty(\mathbb{R}^d)$. However, the proof of this Theorem 4.5.7 only uses the fact that if $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ then $\mathcal{F}\phi \in \mathcal{S}(\mathbb{R}^d)$ and this still holds for every $\phi \in \mathcal{S}(\mathbb{R}^d)$. Therefore $q(\cdot, D)$ is well-defined on $\mathcal{S}(\mathbb{R}^d)$ and maps it into $\mathcal{C}(\mathbb{R}^d, \mathbb{R})$.*

A typical example of such pseudo-differential operators is the fractional Laplacian defined for some fixed $\alpha \in]0, 2[$ on $\mathcal{S}(\mathbb{R}^d)$ by

$$(-\Delta)^{\frac{\alpha}{2}}(u)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,\xi)} \|\xi\|^\alpha \mathcal{F}u(\xi) d\xi. \tag{3.4.3}$$

Its symbol has no dependence in x and is the continuous negative definite function $\xi \mapsto \|\xi\|^\alpha$. Combining Theorem 4.5.12 and 4.6.6 in [59], one can state the following.

Theorem 3.4.5. *Let ψ be a continuous negative definite function satisfying for some $r_0, c_0 > 0$: $\psi(\xi) \geq c_0 \|\xi\|^{r_0}$ if $\|\xi\| \geq 1$. Let M be the smallest integer strictly superior to $(\frac{d}{r_0} \vee 2) + d$. Let q be a continuous negative symbol verifying, for some $c, c' > 0$ and $\gamma : \mathbb{R}^d \to \mathbb{R}_+^*$, the following items.*

- *$q(\cdot, 0) = 0$ and $\sup_{x \in \mathbb{R}^d} |q(x, \xi)| \underset{\xi \to 0}{\longrightarrow} 0$;*
- *q is $\mathcal{C}^{2M+1-d}$ in the first variable and for any $\beta \in \mathbb{N}^d$ with $\|\beta\| \leq 2M + 1 - d$, $\|\partial_x^\beta q\| \leq c(1 + \psi)$;*
- *$q(x, \xi) \geq \gamma(x)(1 + \psi(x))$ if $x \in \mathbb{R}^d$, $\|\xi\| \geq 1$;*
- *$q(x, \xi) \leq c'(1 + \|\xi\|^2)$ for every (x, ξ).*

Then the homogeneous Martingale Problem associated to $(-q(\cdot, D), \mathcal{S}(\mathbb{R}^d))$ is well-posed (see Definition 3.B.3) and its solution $(\mathbb{P}^x)_{x \in \mathbb{R}^d}$ defines a homogeneous Markov class, see Notation 3.B.1.

We will now introduce the time-inhomogeneous domain which will be used to extend $\mathcal{D}(-q(\cdot, D)) = \mathcal{S}(\mathbb{R}^d)$.

Definition 3.4.6. *Let τ be a Hausdorff topological linear space. We will denote by $\mathcal{C}^1([0, T], \tau)$ the set of functions $\phi \in \mathcal{C}([0, T], \tau)$ such that there exists a function $\partial_t \phi \in \mathcal{C}([0, T], \tau)$ verifying the following. For every $t_0 \in [0, T]$ we have $\frac{1}{(t - t_0)}(\phi(t) - \phi(t_0)) \xrightarrow[t \to t_0]{\tau} \partial_t \phi(t_0)$.*

We recall that a topological algebra is a topological space equipped with a structure of linear algebra such that addition, multiplication and multiplication by a scalar are continuous.

Lemma 3.4.7. *Let $\mathcal{A}$ be a (Hausdorff) topological algebra, then $\mathcal{C}^1([0, T], \mathcal{A})$ is a linear algebra, and for any $\phi, \psi \in \mathcal{C}^1([0, T], \mathcal{A})$, we have $\partial_t(\phi\psi) = \psi \partial_t \phi + \phi \partial_t \psi$.*

Proof. The proof is very close to the one of $\mathbb{R}$. □

Remark 3.4.8. *Classical examples of topological algebras are $\mathcal{C}_c^\infty(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{C}^\infty(\mathbb{R}^d)$, $\mathcal{C}^m(\mathbb{R}^d)$ for some $m \in \mathbb{N}$ (equipped with their usual topologies), or $W^{k,p}(\mathbb{R}^d) \bigcap W^{k,\infty}(\mathbb{R}^d)$ for any $k \in \mathbb{N}^*, p \geq 1$, where $W^{k,p}(\mathbb{R}^d)$ denotes the usual Sobolev space of parameters k, p. Those are all Fréchet algebras except for $\mathcal{C}_c^\infty(\mathbb{R}^d)$ which is only locally convex one.*

Notation 3.4.9. *We set $\mathcal{D}(\partial_t - q(\cdot, D)) := \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$.*

Elements in $\mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$ will also be seen as functions of two variables, and since convergence in $\mathcal{S}(\mathbb{R}^d)$ implies pointwise convergence, the usual notion of partial derivative coincides with the notation ∂_t introduced in Definition 3.4.6. Any $\phi \in \mathcal{D}(\partial_t - q(\cdot, D))$ clearly verifies

- $\forall t \in [0,T]$, $\phi(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)$ and $\forall x \in \mathbb{R}^d$, $\phi(\cdot, x) \in \mathcal{C}^1([0,T], \mathbb{R})$;
- $\forall t \in [0,T]$, $\partial_t \phi(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)$.

Our goal now is to show that $\mathcal{D}(\partial_t - q(\cdot, D)$ also verifies the other items needed to be included in $\mathcal{D}^{max}(\partial_t - q(\cdot, D))$ (see Notation 3.B.5) and therefore that Corollary 3.B.8 applies with this domain.

Notation 3.4.10. *Let $\alpha, \beta \in \mathbb{N}^d$ be multi-indices, we introduce the semi-norm*

$$\begin{array}{rccc} & \mathcal{S}(\mathbb{R}^d) & \longrightarrow & \mathbb{R} \\ \|\cdot\|_{\alpha,\beta} : & \phi & \longmapsto & \sup\limits_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta \phi(x)|. \end{array} \tag{3.4.4}$$

$\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space whose topology is determined by the family of seminorms $\|\cdot\|_{\alpha,\beta}$. In particular those seminorms are continuous.
In what follows, $\mathcal{F}_x$ will denote the Fourier transform taken in the space variable.

Proposition 3.4.11. *Let $\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$. Then $\mathcal{F}_x\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$. Moreover if $\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$, then $\mathcal{F}_x\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$ and*
$\partial_t \mathcal{F}_x\phi = \mathcal{F}_x \partial_t \phi$.

Proof. $\mathcal{F}_x : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d)$ is continuous, so $\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d)$ implies $\mathcal{F}_x\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$. If $\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$ then $\partial_t\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d)$ so $\mathcal{F}_x\partial_t\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d)$. Then for any $t_0 \in [0,T]$, the convergence
$\frac{1}{t-t_0}(\phi(t,\cdot) - \phi(t_0,\cdot)) \overset{\mathcal{S}(\mathbb{R}^d)}{\underset{t \to t_0}{\longrightarrow}} \partial_t\phi(t_0,\cdot)$ is preserved by the continuous mapping $\mathcal{F}_x$ meaning that (by linearity)
$\frac{1}{t-t_0}(\mathcal{F}_x\phi(t,\cdot) - \mathcal{F}_x\phi(t_0,\cdot)) \overset{\mathcal{S}(\mathbb{R}^d)}{\underset{t \to t_0}{\longrightarrow}} \mathcal{F}_x\partial_t\phi(t_0,\cdot)$. Since $\mathcal{F}_x\partial_t\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$, we have shown that $\mathcal{F}_x\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$ and $\partial_t\mathcal{F}_x\phi = \mathcal{F}_x\partial_t\phi$. □

Proposition 3.4.12. *If $\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$, then for any $\alpha, \beta \in \mathbb{N}^d$,*
$(t,x) \longmapsto x^\alpha \partial_x^\beta \phi(t,x)$ is bounded.

Proof. Let α, β be fixed. Since the maps $\|\cdot\|_{\alpha,\beta} : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ are continuous, for every $\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$, the application $t \mapsto \|\phi(t,\cdot)\|_{\alpha,\beta}$ is continuous on the compact interval $[0,T]$ and therefore bounded, which yields the result. □

Proposition 3.4.13. *If $\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$ and $\alpha, \beta \in \mathbb{N}^d$, then there exist non-negative functions $\psi_{\alpha,\beta} \in L^1(\mathbb{R}^d)$ such that for every $(t,x) \in [0,T] \times \mathbb{R}^d$, $|x^\alpha \partial_x^\beta \phi(t,x)| \leq \psi_{\alpha,\beta}(x)$.*

Proof. We decompose

$$\begin{aligned} |x^\alpha \partial_x^\beta \phi(t,x)| &= |x^\alpha \partial_x^\beta \phi(t,x)| \mathbb{1}_{[-1,1]^d}(x) + |x^{\alpha+(2,\cdots,2)} \partial_x^\beta \phi(t,x)| \frac{1}{\prod_{i\leq d} x_i^2} \mathbb{1}_{\mathbb{R}^d \setminus [-1,1]^d}(x) \\ &\leq C(\mathbb{1}_{[-1,1]^d}(x) + \frac{1}{\prod_{i\leq d} x_i^2} \mathbb{1}_{\mathbb{R}^d \setminus [-1,1]^d}(x)), \end{aligned} \tag{3.4.5}$$

where C is some constant which exists thanks to Proposition 3.4.12. □

Proposition 3.4.14. *Let q be a continuous negative definite symbol verifying the assumptions of Theorem 3.4.5 and let $\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$. Then for any $x \in \mathbb{R}^d$, $t \mapsto q(\cdot, D)\phi(t,x) \in \mathcal{C}^1([0,T], \mathbb{R})$ and $\partial_t q(\cdot, D)\phi = q(\cdot, D)\partial_t \phi$.*

Proof. We fix $\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$, and $x \in \mathbb{R}^d$. We wish to show that for any $\xi \in \mathbb{R}^d$, $t \longmapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,\xi)} q(x,\xi) \mathcal{F}_x \phi(t,\xi) d\xi$ is $\mathcal{C}^1$ with derivative

$$t \longmapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,\xi)} q(x,\xi) \mathcal{F}_x \partial_t \phi(t,\xi) d\xi.$$

Since $\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$, then $\partial_t \phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$ and by Proposition 3.4.11, $\mathcal{F}_x \partial_t \phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$. Moreover since q verifies the assumptions of Theorem 3.4.5, then $|q(x,\xi)|$ is bounded by $c'(1 + \|\xi\|^2)$ for some constant c'. Therefore by Proposition 3.4.13, there exists a non-negative $\psi \in L^1(\mathbb{R}^d)$ such that for every t, ξ, $|q(x,\xi)\mathcal{F}_x \partial_t \phi(t,\xi)| \leq \psi(\xi)$. Since by Proposition 3.4.11, $\mathcal{F}_x \partial_t \phi = \partial_t \mathcal{F}_x \phi$, this implies that for any (t,ξ), $|\partial_t e^{i(x,\xi)} q(x,\xi) \mathcal{F}_x \phi(t,\xi)| \leq \psi(\xi)$. So by the theorem about the differentiation of integrals depending on a parameter, for any $\xi \in \mathbb{R}^d$, $t \longmapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,\xi)} q(x,\xi) \mathcal{F}_x \phi(t,\xi) d\xi$ is of class $\mathcal{C}^1$ with derivative $t \longmapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,\xi)} q(x,\xi) \mathcal{F}_x \partial_t \phi(t,\xi) d\xi$.

□

Proposition 3.4.15. *Let q be a continuous negative definite symbol verifying the assumptions of Theorem 3.4.5 and let $\phi \in \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$. Then ϕ, $\partial_t \phi$, $q(\cdot, D)\phi$ and $q(\cdot, D)\partial_t \phi$ are bounded.*

Proof. Proposition 3.4.12 implies that any element of $\mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$ is bounded, so we immediately deduce that ϕ and $\partial_t \phi$ are bounded.

Since q verifies the assumptions of Theorem 3.4.5, for any fixed $(t,x) \in [0,T] \times \mathbb{R}^d$, we have

$$\begin{aligned} |q(\cdot, D)\phi(t,x)| &= \left| \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i(x,\xi)} q(x,\xi) \mathcal{F}_x \phi(t,\xi) d\xi \right| \\ &\leq C \int_{\mathbb{R}^d} (1 + \|\xi\|^2) |\mathcal{F}_x \phi(t,\xi)| d\xi, \end{aligned} \tag{3.4.6}$$

for some constant C. Since $\phi \in \mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$ then, by Proposition 3.4.11, $\mathcal{F}_x \phi$ also belongs to $\mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$, and by Proposition 3.4.12, there exists a positive $\psi \in L^1(\mathbb{R}^d)$ such that for any (t,ξ), $(1 + \|\xi\|^2)|\mathcal{F}_x \phi(t,\xi)| \leq \psi(\xi)$, so for any (t,x), $|q(\cdot, D)\phi(t,x)| \leq \|\psi\|_1$.

Similar arguments hold replacing ϕ with $\partial_t \phi$ since it also belongs to $\mathcal{C}([0,T], \mathcal{S}(\mathbb{R}^d))$. □

Remark 3.4.16. *$\mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$ seems to be a domain which is particularly appropriate for time-dependent Fourier analysis and it fits well for our framework. On the other hand it is not so fundamental to require such regularity for classical solutions for Pseudo-PDEs, so that we could consider a larger domain. For example the Fréchet algebra $\mathcal{S}(\mathbb{R}^d)$ could be replaced with the Banach algebra $W^{d+3,1}(\mathbb{R}^d) \bigcap W^{d+3,\infty}(\mathbb{R}^d)$ in all the previous proofs.*

Even bigger domains are certainly possible, we will however not insist on such refinements.

Corollary 3.4.17. *Let q be a continuous negative definite symbol verifying the hypotheses of Theorem 3.4.5. Then $\mathcal{D}(\partial_t - q(\cdot, D))$ is a linear algebra included in $\mathcal{D}^{max}(\partial_t - q(\cdot, D))$ as defined in Notation 3.B.5.*

Proof. We recall that, according to Notation 3.4.9 $\mathcal{D}(\partial_t - q(\cdot, D)) = \mathcal{C}^1([0,T], \mathcal{S}(\mathbb{R}^d))$. The proof follows from Lemma 3.4.7, Propositions 3.4.14 and 3.4.15, and the comments under Notation 3.4.9. □

Corollary 3.4.18. *Let q be a continuous negative definite symbol verifying the hypotheses of Theorem 3.4.5, let $(\mathbb{P}^x)_{x\in\mathbb{R}^d}$ be the corresponding homogeneous Markov class exhibited in Theorem 3.4.5, let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ be the corresponding Markov class (see Notation 3.B.1), let $(\mathcal{D}(\partial_t - q(\cdot, D)), \partial_t - q(\cdot, D))$ be as in Notation 3.4.9. Then*

- *$(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ solves the well-posed Martingale Problem associated to $(\mathcal{D}(\partial_t - q(\cdot, D)), \partial_t - q(\cdot, D), V_t \equiv t)$;*
- *its transition kernel is measurable in time.*

Proof. The first statement directly comes from Theorem 3.4.5 and Corollaries 3.4.17 3.B.8, and the second from Proposition 3.B.2. □

Remark 3.4.19. *The symbol of the fractional Laplacian $q : (x, \xi) \mapsto \|\xi\|^\alpha$ trivially verifies the assumptions of Theorem 3.4.5. Indeed, it has no dependence in x, so it is enough to set $\psi : \xi \mapsto \|\xi\|^\alpha$, $c_0 = c = c' = 1$, $r_0 = \alpha$ and $\gamma = \frac{1}{2}$.*

The Pseudo-PDE that we focus on is the following.

$$\begin{cases} \partial_t u - q(\cdot, D)u = f(\cdot, \cdot, u, \Gamma(u)^{\frac{1}{2}}) \text{ on } [0,T] \times \mathbb{R}^d \\ u(T, \cdot) = g, \end{cases} \tag{3.4.7}$$

where q is a continuous negative definite symbol verifying the assumptions of Theorem 3.4.5 and Γ is the associated carré du champs operator, see Definition 3.2.6.

Remark 3.4.20. *By Proposition 3.3 in [36], for any $\alpha \in]0, 2[$, there exists a constant c_α such that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$,*

$$(-\Delta)^{\frac{\alpha}{2}}\phi = c_\alpha PV \int_{\mathbb{R}^d} \frac{(\phi(\cdot + y) - \phi)}{\|y\|^{d+\alpha}} dy, \tag{3.4.8}$$

where PV is a notation for principal value, see (3.1) in [36]. Therefore in the particular case of the fractional Laplace operator, the carré du champs operator Γ^α associated to $(-\Delta)^{\frac{\alpha}{2}}$ is given by

$$\begin{array}{rl} & \Gamma^\alpha(\phi) \\ = & c_\alpha PV \int_{\mathbb{R}^d} \frac{(\phi^2(\cdot,\cdot+y)-\phi^2)}{\|y\|^{d+\alpha}} dy - 2\phi c_\alpha PV \int_{\mathbb{R}^d} \frac{(\phi(\cdot,\cdot+y)-\phi)}{\|y\|^{d+\alpha}} dy \\ = & c_\alpha PV \int_{\mathbb{R}^d} \frac{(\phi(\cdot,\cdot+y)-\phi)^2}{\|y\|^{d+\alpha}} dy. \end{array} \tag{3.4.9}$$

Proposition 3.4.21. *Let q be a continuous negative symbol verifying the assumptions of Theorem 3.4.5, let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ be the Markov class which by Corollary 3.4.18 solves the well-posed Martingale Problem associated to $(\mathcal{D}(\partial_t - q(\cdot, D)), \partial_t - q(\cdot, D), V_t \equiv t)$.*

For any (f, g) verifying H^{lip} (see Hypothesis 3.2.23), $Pseudo - PDE(f, g)$ admits a unique decoupled mild solution in the sense of Definition 3.3.4.

Proof. The assertion comes from Corollary 3.4.18 and Theorem 3.3.9. □

3.4.3 Parabolic semi-linear PDEs with distributional drift

In this section we will use the formalism and results obtained in [47] and [48], see also [81], [31] for more recent developments. In particular the latter paper treats interesting applications to polymers. Those papers introduced a suitable framework of Martingale Problem related to a PDE operator containing a distributional drift b' which is the derivative of a continuous function. [46] established a first work in the n-dimensional setting.

Let $b, \sigma \in \mathcal{C}^0(\mathbb{R})$ such that $\sigma > 0$. By mollifier, we intend a function $\Phi \in \mathcal{S}(\mathbb{R})$ with $\int \Phi(x)dx = 1$. We denote $\Phi_n(x) = n\Phi(nx)$, $\sigma_n^2 = \sigma^2 * \Phi_n$, $b_n = b * \Phi_n$.
We then define $L_n g = \frac{\sigma_n^2}{2} g'' + b_n' g'$. $f \in \mathcal{C}^1(\mathbb{R})$ is said to be a solution to $Lf = \dot{l}$ where $\dot{l} \in \mathcal{C}^0$, if for any mollifier Φ, there are sequences (f_n) in $\mathcal{C}^2$, $(\dot{l}_n)$ in $\mathcal{C}^0$ such that $L_n f_n = (\dot{l}_n)$, $f_n \xrightarrow{\mathcal{C}^1} f$, $\dot{l}_n \xrightarrow{\mathcal{C}^0} \dot{l}$. We will assume that $\Sigma(x) = \lim_{n\to\infty} 2\int_0^x \frac{b_n'}{\sigma_n^2}(y)dy$ exists in $\mathcal{C}^0$ independently from the mollifier.

By Proposition 2.3 in [47] there exists a solution $h \in \mathcal{C}^1$ to $Lh = 0$, $h(0) = 0$, $h'(0) = 1$. Moreover it verifies $h' = e^{-\Sigma}$. Moreover by Remark 2.4 in [47], for any $\dot{l} \in \mathcal{C}^0$, $x_0, x_1 \in \mathbb{R}$, there exists a unique solution of

$$Lf(x) = \dot{l},\ f \in \mathcal{C}^1,\ f(0) = x_0,\ f'(0) = x_1. \tag{3.4.10}$$

$\mathcal{D}_L$ is defined as the set of $f \in \mathcal{C}^1$ such that there exists some $\dot{l} \in \mathcal{C}^0$ with $Lf = \dot{l}$. And by Lemma 2.9 in [47] it is equal to the set of $f \in \mathcal{C}^1$ such that $\frac{f'}{h'} \in \mathcal{C}^1$. So it is clearly an algebra.
h is strictly increasing, I will denote its image. Let L^0 be the classical differential operator defined by $L^0\phi = \frac{\sigma_0^2}{2}\phi''$, where

$$\sigma_0(y) = \begin{cases} (\sigma h')(h^{-1}(y)) & : \ y \in I \\ 0 & : \ y \in I^c. \end{cases} \tag{3.4.11}$$

Let v be the unique solution to $Lv = 1$, $v(0) = v'(0) = 0$, we will assume that

$$v(-\infty) = v(+\infty) = +\infty, \tag{3.4.12}$$

which represents a non-explosion condition. In this case, Proposition 3.13 in [47] states that the Martingale Problem associated to $(\mathcal{D}_L, L, V_t \equiv t)$ is well-posed. Its solution will be denoted $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$. By Proposition 2.13, $\mathcal{D}_{L^0} = C^2(I)$. and by Proposition 3.2 in [47], the Martingale Problem associated to $(\mathcal{D}_{L^0}, L^0, V_t \equiv t)$ is also well-posed, we will call $(\mathbb{Q}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ its solution. Moreover under any $\mathbb{P}^{s,x}$ the canonical process is a Dirichlet process, and $h^{-1}(X)$ is a semimartingale that we call Y solving the SDE $Y_t = h(x) + \int_s^t \sigma_0(Y_s)dW_s$ in law, where the law of Y is $\mathbb{Q}^{s,x}$. X_t is a $\mathbb{P}^{s,x}$-Dirichlet process whose martingale component is $\int_s^{\cdot} \sigma(X_r)dW_r$. $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ and $(\mathbb{Q}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ both define Markov classes.

We introduce now the domain that we will indeed use.

Definition 3.4.22. *We set*

$$\mathcal{D}(a) = \left\{ \phi \in \mathcal{C}^{1,1}([0,T]\times\mathbb{R}) : \frac{\partial_x \phi}{h'} \in \mathcal{C}^{1,1}([0,T]\times\mathbb{R}) \right\}, \tag{3.4.13}$$

which clearly is a linear algebra.

On $\mathcal{D}(a)$, we set $L\phi := \frac{\sigma^2 h'}{2}\partial_x(\frac{\partial_x\phi}{h'})$ and $a(\phi) := \partial_t\phi + L\phi$.

Proposition 3.4.23. *Let Γ denote the carré du champ operator associated to a, let ϕ, ψ be in $\mathcal{D}(a)$, then $\Gamma(\phi,\psi) = \sigma^2\partial_x\phi\partial_x\psi$.*

Proof. We fix ϕ, ψ in $\mathcal{D}(a)$. We write

$$\begin{aligned}\Gamma(\phi,\psi) &= (\partial_t + L)(\phi\psi) - \phi(\partial_t + L)(\psi) - \psi(\partial_t + L)(\phi) \\ &= \frac{\sigma^2 h'}{2}\left(\partial_x\left(\frac{\partial_x\phi\psi}{h'}\right) - \phi\partial_x\left(\frac{\partial_x\psi}{h'}\right) - \psi\partial_x\left(\frac{\partial_x\phi}{h'}\right)\right) \\ &= \sigma^2\partial_x\phi\partial_x\psi.\end{aligned} \tag{3.4.14}$$

□

Emphasizing that b' is a distribution, the equation that we will study in this section is therefore given by

$$\begin{cases} \partial_t u + \frac{1}{2}\sigma^2\partial_x^2 u + b'\partial_x u + f(\cdot,\cdot,u,\sigma|\partial_x u|) = 0 \quad \text{on } [0,T]\times\mathbb{R} \\ u(T,\cdot) = g. \end{cases} \tag{3.4.15}$$

Proposition 3.4.24. $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ *solves the Martingale Problem associated to* $(a, \mathcal{D}(a), V_t \equiv t)$.

Proof. $(t,y) \mapsto \phi(t, h^{-1}(y))$ is of class $\mathcal{C}^{1,2}$; moreover $\partial_x\left(\phi(r,\cdot)\circ h^{-1}\right) = \frac{\partial_x\phi}{h'}\circ h^{-1}$ and $\partial_x^2\left(\phi(r,\cdot)\circ h^{-1}\right) = \frac{2L\phi}{\sigma^2 h'^2}\circ h^{-1} = \frac{2L\phi}{\sigma_0^2}\circ h^{-1}$. By Itô formula we have

$$\begin{aligned}\phi(t,X_t) &= \phi(t,h^{-1}(Y_t)) \\ &= \phi(s,x) + \int_s^t \left(\partial_t\phi(r,h^{-1}(Y_r)) + \frac{1}{2}\sigma_0^2(Y_r)\partial_x^2\left(\phi(r,\cdot)\circ h^{-1}\right)(Y_r)\right)dr \\ &\quad + \int_s^t \sigma_0(r,h^{-1}(Y_r))\partial_x\left(\phi(r,\cdot)\circ h^{-1}\right)(Y_r)dW_r \\ &= \phi(s,x) + \int_s^t \left(\partial_t\phi(r,h^{-1}(Y_r)) + L\phi(r,h^{-1}(Y_r))\right)dr \\ &\quad + \int_s^t \sigma_0(r,h^{-1}(Y_r))\frac{\partial_x\phi(r,h^{-1}(Y_r))}{h'(Y_r)}dW_r \\ &= \phi(s,x) + \int_s^t \left(\partial_t\phi(r,X_r) + l(r,X_r)\right)dr + \int_s^t \sigma(r,X_r)\partial_x\phi(r,X_r)dW_r.\end{aligned} \tag{3.4.16}$$

Therefore $\phi(t,X_t) - \phi(s,x) - \int_s^t a(\phi)(r,X_r)dr = \int_s^t \sigma(r,X_r)\partial_x\phi(r,X_r)dW_r$ is a local martingale. □

In order to consider the $BSDE^{s,x}(f,g)$ for functions (f,g) having polynomial growth in x we will show the following result. We formulate here the supplementary assumption, called (TA) in [47]. This means the existence of strictly positive constants c_1, C_1 such that

$$c_1 \le \frac{e^{\Sigma}}{\sigma} \le C_1. \tag{3.4.17}$$

Proposition 3.4.25. *We suppose that (TA) is fulfilled and σ has linear growth. Then, for any $p > 0$ and $(s,x) \in [0,T]\times\mathbb{R}$, $\mathbb{E}^{s,x}[|X_T|^p] < \infty$ and $\mathbb{E}^{s,x}[\int_s^T |X_r|^p dr] < \infty$. In other words, if g and $f(\cdot,\cdot,0,0)$ have polynomial growth in x uniformly in t, then (f,g) verify the first two items of H^{growth} or equivalently the first two items of H^{lip}.*

Proof. We start by proving the proposition in the divergence form case, meaning that $b = \frac{\sigma^2}{2}$.
Let (s,x) and $t \in [s,T]$ be fixed. Thanks to the Aronson estimates, see e.g. [2] and also Section 5. of [47], there is a constant $M > 0$ such that

$$\begin{aligned}\mathbb{E}^{s,x}[|X_t|^p] &= \int_{\mathbb{R}} |y|^p p_{t-s}(x,y)dy \\ &\le \frac{M}{\sqrt{t-s}}\int_{\mathbb{R}} |y|^p e^{-\frac{|x-y|^2}{M(t-s)}}dz \\ &= M^{\frac{3}{2}}\int_{\mathbb{R}} |x + z\sqrt{M(t-s)}|^p e^{-z^2}dz \\ &\le \sum_{k=0}^p M^{\frac{3+k}{2}}\binom{p}{k}|x|^k|t-s|^{\frac{p-k}{2}}\int_{\mathbb{R}} |z|^{p-k}e^{-z^2}dz,\end{aligned} \tag{3.4.18}$$

which (for fixed (s,x)) is bounded in $t \in [s,T]$ and therefore Lebesgue integrable in t on $[s,T]$. This in particular shows that $\mathbb{E}^{s,x}[|X_T|^p]$ and $\mathbb{E}^{s,x}[\int_s^T |X_r|^p dr] (= \int_s^T \mathbb{E}^{s,x}[|X_r|^p]dr)$ are finite.

Now we will consider the case in which X only verifies (3.4.17) and we will add the hypothesis that σ has linear growth.
Then there exists a process Z (see Lemma 5.6 in [47]) solving an SDE with distributional drift of divergence form generator, and a function k of class $\mathcal{C}^1$ such that $X = k^{-1}(Z)$. The (3.4.17) condition implies that there exist two constants such that $0 < c \leq k'\sigma \leq C$ implying that for any x, $(k^{-1})'(x) = \frac{1}{k'ok^{-1}(x)} \leq \frac{\sigma ok^{-1}(x)}{c} \leq C_2(1+|k^{-1}(x)|)$, for a positive constant C_2. So by Gronwall Lemma there exists $C_3 > 0$ such that $k^{-1}(x) \leq C_3 e^{C_2|x|}$, $\forall x \in \mathbb{R}$.
Now thank to the Aronson estimates on the transition function p^Z of Z, for every $p > 0$, we have

$$\begin{array}{rcl} \mathbb{E}^{s,x}[|X|_t^p] & \leq & C_3 \int e^{C_2 p|z|} p^Z_{t-s}(k(x), z)dz \\ & \leq & \int e^{C_2 p|z|} \frac{M}{\sqrt{t}} e^{-\frac{|k(x)-z|^2}{Mt}} dz \\ & \leq & M^{\frac{3}{2}} \int e^{C_2 p(\sqrt{Mt}|y|+k(x))} e^{-y^2} dy \\ & \leq & Ae^{Bk(x)}, \end{array} \tag{3.4.19}$$

where A, B are two constants depending on p and M. This implies that $\mathbb{E}^{s,x}[|X_T|^p] < \infty$ and $\mathbb{E}^{s,x}[\int_s^T |X_r|^p dr] < \infty$. □

We can now state the main result of this section.

Proposition 3.4.26. *Assume that the non-explosion condition* (3.4.12) *is verified and the validity of one of the two following items.*

- *the (TA) condition* (3.4.17) *is fulfilled, σ has linear growth, $f(\cdot,\cdot,0,0), g$ have polynomial growth in x and f verifies item 3 of H^{lip} (see Hypothesis 3.2.23)*
- *$f(\cdot,\cdot,0,0), g$ are bounded and f verifies item 3 of H^{lip}.*

Then (3.4.15) *has a unique decoupled mild solution u in the sense of Definition 3.3.4.*

Proof. The assertion comes from Theorem 3.3.9 which applies thanks to Propositions 3.4.24, 3.4.25 and 3.B.2. □

Remark 3.4.27. 1. *A first analysis linking PDEs (in fact second order elliptic differential equations) with distributional drift and BSDEs was performed by [82]. In those BSDEs the final horizon was a stopping time.*

2. *In [56], the authors have considered a class of BSDEs involving particular distributions.*

3.4.4 Diffusion equations on differential manifolds

In this section, we will provide an example of application in a non Euclidean space. We consider a compact connected smooth differential manifold M of dimension n. We denote by $\mathcal{C}^\infty(M)$ the linear algebra of smooth functions from M to $\mathbb{R}$, and $(U_i, \phi_i)_{i\in I}$ its atlas. The reader may consult [62] for an extensive introduction to the study of differential manifolds, and [55] concerning diffusions on differential manifolds.

Lemma 3.4.28. *M is Polish.*

Proof. By Theorem 1.4.1 in [62] M may be equipped with a Riemannian metric, that we denote by g and its topology may be metricized by the associated distance which we denote by d. As any compact metric space, (M, d) is separable and complete so that M is a Polish space. □

We denote by $(\Omega, \mathcal{F}, \mathbb{F})$ the canonical space associated to M and T, and $(X_t)_{t\in[0,T]}$ the canonical process, see Definition 3.A.1.

Definition 3.4.29. *An operator $L : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ will be called a **smooth second order elliptic non degenerate differential operator on** M if for any chart $\phi : U \longrightarrow \mathbb{R}^n$ there exist smooth $\beta : \phi(U) \longrightarrow \mathbb{R}^n$ and $\alpha : \phi(U) \longrightarrow S_+^*(\mathbb{R}^n)$ such that on $\phi(U)$ for any $f \in \mathcal{C}^\infty(M)$ we have*

$$Lf(\phi^{-1}(x)) = \frac{1}{2}\sum_{i,j=1}^{n} \alpha^{i,j}(x)\partial_{x_i x_j}(f\circ\phi^{-1})(x) + \sum_{i=1}^{n}\beta^i(x)\partial_{x_i}(f\circ\phi^{-1})(x). \tag{3.4.20}$$

α and β depend on the chart ϕ but this dependence will be sometimes omitted and we will say that for some given local coordinates,
$Lf = \frac{1}{2}\sum\limits_{i,j=1}^{n} \alpha^{i,j}\partial_{x_i x_j} f + \sum\limits_{i=1}^{n}\beta^i\partial_{x_i} f$.
The following definition comes from [55], see Definition 1.3.1.

Definition 3.4.30. *Let L denote a smooth second order elliptic non degenerate differential operator on M. Let $x \in M$. A probability measure $\mathbb{P}^x$ on $(\Omega, \mathcal{F})$ will be called an L**-diffusion starting in** x if*

- $\mathbb{P}^x(X_0 = x) = 1$;
- *for every $f \in \mathcal{C}^\infty(M)$, $f(X) - \int_0^\cdot Lf(X_r)dr$ is a $(\mathbb{P}^x, \mathbb{F})$ local martingale.*

Remark 3.4.31. *No explosion can occur for continuous stochastic processes with values in a compact space, so there is no need to consider paths in the compactification of M as in Definition 1.1.4 in [55].*
Theorems 1.3.4 and 1.3.5 in [55] state that for any $x \in M$, there exists a unique L-diffusion starting in x. Theorem 1.3.7 in [55] implies that those probability measures $(\mathbb{P}^x)_{x\in M}$ define a homogeneous Markov class.

For a given operator L, the carré du champs operator Γ is given (in local coordinates) by $\Gamma(\phi,\psi) = \sum\limits_{i,j=1}^{n}\alpha^{i,j}\partial_{x_i}\phi\partial_{x_j}\phi$, see equation (1.3.3) in [55]. We wish to emphasize here that the carré du champs operator has recently become a powerful tool in the study of geometrical properties of Riemannian manifolds. The reader may refer e.g. to [3].

Definition 3.4.32. *$(\mathbb{P}^x)_{x\in M}$ will be called the L**-diffusion**. If M is equipped with a specific Riemannian metric g and L is chosen to be equal to $\frac{1}{2}\Delta_g$ where Δ_g the Laplace-Beltrami operator associated to g, then $(\mathbb{P}^x)_{x\in M}$ will be called the **Brownian motion associated to** g, see [55] Chapter 3 for details.*

We now fix some smooth second order elliptic non degenerate differential operator L and the L-diffusion $(\mathbb{P}^x)_{x\in M}$. We introduce the associated Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times M}$ as described in Notation 3.B.1, which by Proposition 3.B.2 is measurable in time.

Notation 3.4.33. *We define $\mathcal{D}(\partial_t + L)$ the set of functions $u : [0,T]\times M \longrightarrow \mathbb{R}$ such that, for any chart $\phi : U \longrightarrow \mathbb{R}^n$, the mapping*

$$\begin{array}{rcl} [0,T]\times\phi(U) & \longrightarrow & \mathbb{R} \\ (t,x) & \longmapsto & u(t,\phi^{-1}(x)) \end{array} \tag{3.4.21}$$

belongs to $\mathcal{C}^\infty([0,T]\times\phi(U),\mathbb{R})$, the set of infinitely continuously differentiable functions in the usual Euclidean setup.

Lemma 3.4.34. *$\mathcal{D}(\partial_t + L)$ is a linear algebra included in $\mathcal{D}^{max}(\partial_t + L)$ as defined in Notation 3.B.5.*

Proof. For some fixed chart $\phi : U \longrightarrow \mathbb{R}^n$, $\mathcal{C}^\infty([0,T] \times \phi(U), \mathbb{R})$ is an algebra, so it is immediate that $\mathcal{D}(\partial_t + L)$ is an algebra.

Moreover, if $u \in \mathcal{D}(\partial_t + L)$, it is clear that

- $\forall x \in M$, $u(\cdot, x) \in \mathcal{C}^1([0,T], \mathbb{R})$ and $\forall t \in [0,T]$, $u(t,\cdot) \in \mathcal{C}^\infty(M)$;
- $\forall t \in [0,T]$, $\partial_t u(t,\cdot) \in \mathcal{C}^\infty(M)$ and $\forall x \in M$, $Lu(\cdot, x) \in \mathcal{C}^1([0,T], \mathbb{R})$.

Given a chart $\phi : U \longrightarrow \mathbb{R}^n$, by the Schwarz Theorem allowing the commutation of partial derivatives (in the classical Euclidean setup), we have for $x \in \phi(U)$

$$\begin{aligned}
\partial_t \circ L(u)(t, \phi^{-1}(x)) &= \tfrac{1}{2}\sum_{i,j=1}^n \alpha^{i,j}(x)\partial_t \partial_{x_i x_j}(u(\cdot, \phi^{-1}(\cdot))(t,x) + \sum_{i=1}^n \beta^i(x)\partial_t \partial_{x_i}(u(\cdot, \phi^{-1}(\cdot))(t,x) \\
&= \tfrac{1}{2}\sum_{i,j=1}^n \alpha^{i,j}(x)\partial_{x_i x_j} \partial_t(u(\cdot, \phi^{-1}(\cdot))(t,x) + \sum_{i=1}^n \beta^i(x)\partial_{x_i} \partial_t(u(\cdot, \phi^{-1}(\cdot))(t,x) \\
&= L \circ \partial_t(u)(t, \phi^{-1}(x)).
\end{aligned} \tag{3.4.22}$$

So $\partial_t \circ Lu = L \circ \partial_t u$. Finally $\partial_t u$, Lu and $\partial_t \circ Lu$ are continuous (since they are continuous on all the sets $[0,T] \times U$ where U is the domain of a chart) and are therefore bounded as continuous functions on the compact set $[0,T] \times M$. This concludes the proof. □

Corollary 3.4.35. *$(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times M}$ solves the well-posed Martingale Problem associated to $(\partial_t + L, \mathcal{D}(\partial_t + L), V_t \equiv t)$ in the sense of Definition 3.2.3.*

Proof. The corollary derives from Lemma 3.4.34 and Corollary 3.B.8. □

We fix a couple (f, g) verifying H^{lip} (see Hypothesis 3.2.23) and consider the PDE

$$\begin{cases} \partial_t u + Lu + f(\cdot, \cdot, u, \Gamma(u)^{\frac{1}{2}}) = 0 \quad \text{on } [0,T] \times M \\ u(T, \cdot) = g. \end{cases} \tag{3.4.23}$$

Since Theorem 3.3.9 applies, we have the following result.

Proposition 3.4.36. *Equation* (3.4.23) *admits a unique decoupled mild solution u in the sense of Definition 3.3.4.*

Remark 3.4.37. *Since M is compact, we emphasize that if g is continuous and f is continuous in t, x Lipschitz in y, z then $f(\cdot, \cdot, 0, 0), g$ are bounded so (f, g) verifies H^{lip}.*

Appendix

3.A Markov classes

In this Appendix we recall some basic definitions and results concerning Markov processes. For a complete study of homogeneous Markov processes, one may consult [34], concerning non-homogeneous Markov classes, our reference was chapter VI of [40]. The first definition refers to the canonical space that one can find in [60], see paragraph 12.63.

Notation 3.A.1. *In the whole section E will be a fixed Polish space (a separable completely metrizable topological space). E will be called the* ***state space****.*

We consider $T \in \mathbb{R}_+^*$. We denote $\Omega := \mathbb{D}([0,T],E)$ the space of functions from $[0,T]$ to E right-continuous with left limits and continuous at time T, e.g. cadlag. For any $t \in [0,T]$ we denote the coordinate mapping $X_t : \omega \mapsto \omega(t)$, and we introduce on Ω the σ-field $\mathcal{F} := \sigma(X_r | r \in [0,T])$.

On the measurable space $(\Omega, \mathcal{F})$, we introduce the measurable ***canonical process***

$$X : \begin{array}{rcl} (t,\omega) & \longmapsto & \omega(t) \\ ([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F}) & \longrightarrow & (E,\mathcal{B}(E)), \end{array}$$

and the right-continuous filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ where $\mathcal{F}_t := \bigcap_{s\in]t,T]} \sigma(X_r | r \leq s)$ if $t < T$, and $\mathcal{F}_T := \sigma(X_r | r \in [0,T]) = \mathcal{F}$.
$(\Omega, \mathcal{F}, \mathbb{F})$ will be called the ***canonical space*** *(associated to T and E).*
For any $t \in [0,T]$ we denote $\mathcal{F}_{t,T} := \sigma(X_r | r \geq t)$, and for any $0 \leq t \leq u < T$ we will denote $\mathcal{F}_{t,u} := \bigcap_{n\geq 0} \sigma(X_r | r \in [t, u+\frac{1}{n}])$.

Since E is Polish, we recall that $\mathbb{D}([0,T],E)$ can be equipped with a Skorokhod distance which makes it a Polish metric space (see Theorem 5.6 in chapter 3 of [44], and for which the Borel σ-field is $\mathcal{F}$, see Proposition 7.1 in Chapter 3 of [44]. This in particular implies that $\mathcal{F}$ is separable, as the Borel σ-field of a separable metric space.

Remark 3.A.2. *Previous definitions and all the notions of this Appendix, extend to a time interval equal to $\mathbb{R}_+$ or replacing the Skorokhod space with the Wiener space of continuous functions from $[0,T]$ (or $\mathbb{R}_+$) to E.*

Definition 3.A.3. *The function*

$$P : \begin{array}{rcl} (s,t,x,A) & \longmapsto & P_{s,t}(x,A) \\ [0,T]^2\times E\times\mathcal{B}(E) & \longrightarrow & [0,1], \end{array}$$

will be called ***transition kernel*** *if, for any s,t in $[0,T]$, $x_0 \in E$, $A \in \mathcal{B}(E)$, it verifies*

1. *$x \mapsto P_{s,t}(x,A)$ is Borel,*

2. *$B \mapsto P_{s,t}(x_0, B)$ is a probability measure on $(E, \mathcal{B}(E))$,*

3. *if $t \leq s$ then $P_{s,t}(x_0, A) = \mathbb{1}_A(x_0)$,*

4. *if $s < t$, for any $u > t$, $\int_E P_{s,t}(x_0, dy) P_{t,u}(y, A) = P_{s,u}(x_0, A)$.*

The latter statement is the well-known **Chapman-Kolmogorov equation**.

Definition 3.A.4. *A transition kernel P for which the first item is reinforced supposing that $(s, x) \longmapsto P_{s,t}(x, A)$ is Borel for any t, A, will be said **measurable in time**.*

Remark 3.A.5. *Let P be a transition kernel which is measurable in time. By approximation by step functions, one can easily show that, for any Borel function ϕ from E to $\bar{\mathbb{R}}$ then $(s, x) \mapsto \int_E \phi(y) P_{s,t}(x, dy)$ is Borel, provided ϕ is quasi integrable for every (s, x).*

Definition 3.A.6. *A **canonical Markov class** associated to a transition kernel P is a set of probability measures $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defined on the measurable space $(\Omega, \mathcal{F})$ and verifying for any $t \in [0, T]$ and $A \in \mathcal{B}(E)$*

$$\mathbb{P}^{s,x}(X_t \in A) = P_{s,t}(x, A), \tag{3.A.1}$$

and for any $s \leq t \leq u$

$$\mathbb{P}^{s,x}(X_u \in A | \mathcal{F}_t) = P_{t,u}(X_t, A) \quad \mathbb{P}^{s,x} \text{ a.s.} \tag{3.A.2}$$

Remark 3.A.7. *Formula 1.7 in Chapter 6 of [40] states that for any $F \in \mathcal{F}_{t,T}$ yields*

$$\mathbb{P}^{s,x}(F|\mathcal{F}_t) = \mathbb{P}^{t,X_t}(F) = \mathbb{P}^{s,x}(F|X_t) \ \ \mathbb{P}^{s,x} a.s. \tag{3.A.3}$$

Property (3.A.3) *will be called **Markov property**.*

For the rest of this section, we are given a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ with transition kernel P.
We will complete the σ-fields $\mathcal{F}_t$ of the canonical filtration by $\mathbb{P}^{s,x}$ as follows.

Definition 3.A.8. *For any $(s, x) \in [0, T] \times E$ we will consider the (s, x)-**completion** $\left(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x} := (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ of the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{s,x})$ by defining $\mathcal{F}^{s,x}$ as the $\mathbb{P}^{s,x}$-completion of $\mathcal{F}$, by extending $\mathbb{P}^{s,x}$ to $\mathcal{F}^{s,x}$ and finally by defining $\mathcal{F}_t^{s,x}$ as the $\mathbb{P}^{s,x}$-closure of $\mathcal{F}_t$ (meaning $\mathcal{F}_t$ augmented with the $\mathbb{P}^{s,x}$-negligible sets) for every $t \in [0, T]$.*

We remark that, for any $(s, x) \in [0, T] \times E$, $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$ is a stochastic basis fulfilling the usual conditions, see (1.4) in chapter I of [61]). We recall that considering a conditional expectation with respect to a σ-field augmented with the negligible sets or not, does not change the result. In particular we have the following.

Proposition 3.A.9. *Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be a canonical Markov class. Let $(s, x) \in [0, T] \times E$ be fixed, Z be a random variable and $t \in [s, T]$, then*
$\mathbb{E}^{s,x}[Z|\mathcal{F}_t] = \mathbb{E}^{s,x}[Z|\mathcal{F}_t^{s,x}] \ \mathbb{P}^{s,x}$ *a.s.*

Proposition 3.A.10. *Let Z be a random variable. If the function $(s, x) \longmapsto \mathbb{E}^{s,x}[Z]$ is well-defined (with possible values in $[-\infty, \infty]$), then at fixed $s \in [0, T]$, $x \longmapsto \mathbb{E}^{s,x}[Z]$ is Borel. If moreover the transition kernel P is measurable in time then, $(s, x) \longmapsto \mathbb{E}^{s,x}[Z]$ is Borel.*

In particular if $F \in \mathcal{F}$ be fixed, then at fixed $s \in [0, T]$, $x \longmapsto \mathbb{P}^{s,x}(F)$ is Borel. If the transition kernel P is measurable in time then, $(s, x) \longmapsto \mathbb{P}^{s,x}(F)$ is Borel.

Proof. We will only deal with the case of a measurable in time transition kernel since the other case is proven in a very similar way.

We consider first the case $Z = 1_F$ where $F \in \mathcal{F}$. We start by assuming that F is of the form $\bigcap_{i\leq n}\{X_{t_i} \in A_i\}$, where $n \in \mathbb{N}^*$, $0 = t_0 \leq t_1 < \cdots < t_n \leq T$ and $A_1, \cdots, A_n$ are Borel sets of E, and we denote by Π the set of such events.

In this proof we will make use of monotone class arguments, see for instance Section 4.3 in [1] for the definitions of π-systems and λ-systems and for the presently used version of the monotone class theorem, also called the Dynkin's lemma.

We remark that Π is a π-system (see Definition 4.9 in [1]) generating $\mathcal{F}$. For such events we can explicitly compute $\mathbb{P}^{s,x}(F)$. We compute this when (s,x) belongs to $[t_{i^*-1}-1, t_{i^*}[\times E$ for some $0 < i^* \leq n$. On $[t_n, T] \times E$, a similar computation can be performed. We will show below that those restricted functions are Borel, the general result will follow by concatenation. We have

$$\begin{aligned}
& \mathbb{P}^{s,x}(F) \\
= & \prod_{i=1}^{i^*-1} \mathbb{1}_{A_i}(x)\mathbb{E}^{s,x}\left[\prod_{j=i^*}^{n} \mathbb{1}_{A_i}(X_{t_i})\right] \\
= & \prod_{i=1}^{i^*-1} \mathbb{1}_{A_i}(x)\mathbb{E}^{s,x}\left[\prod_{j=i^*}^{n-1} \mathbb{1}_{A_i}(X_{t_i})\mathbb{E}^{s,x}[\mathbb{1}_{A_n}(X_{t_n})|\mathcal{F}_{t_{n-1}}]\right] \\
= & \prod_{i=1}^{i^*-1} \mathbb{1}_{A_i}(x)\mathbb{E}^{s,x}\left[\prod_{j=i^*}^{n-1} \mathbb{1}_{A_i}(X_{t_i})P_{t_{n-1},t_n}(X_{t_{n-1}}, A_n)\right] \\
= & \cdots \\
= & \prod_{i=1}^{i^*-1} \mathbb{1}_{A_i}(x) \int \left(\prod_{j=i^*+1}^{n} \mathbb{1}_{A_j}(x_j)P_{t_{j-1},t_j}(x_{j-1}, dx_j)\right) \mathbb{1}_{A_{i^*}}(x_{i^*})P_{s,t_{i^*}}(x, dx_{i^*}),
\end{aligned}$$

which indeed is Borel in (s,x) thank to Definition 3.A.4 and Remark 3.A.5.

We can extend this result to any event F by the monotone class theorem. Indeed, let Λ be the set of elements F of $\mathcal{F}$ such that $(s,x) \mapsto \mathbb{P}^{s,x}(F)$ is Borel. For any two events F^1, F^2, in Λ with $F^1 \subset F^2$, since for any (s,x),
$\mathbb{P}^{s,x}(F^2 \backslash F^1) = \mathbb{P}^{s,x}(F^2) - \mathbb{P}^{s,x}(F^1)$, $(s,x) \mapsto \mathbb{P}^{s,x}(F^2 \backslash F^1)$ is still Borel. For any increasing sequence $(F^n)_{n\geq 0}$ of elements of Λ, $\mathbb{P}^{s,x}(\bigcup_{n\in\mathbb{N}} F^n) = \lim_{n\to\infty} \mathbb{P}^{s,x}(F^n)$ so $(s,x) \mapsto \mathbb{P}^{s,x}(\bigcup_{n\in\mathbb{N}} F^n)$ is still Borel, therefore Λ is a λ-system containing the π-system Π which generates $\mathcal{F}$. So by the monotone class theorem, $\Lambda = \mathcal{F}$, which shows the case $Z = 1_F$.

We go on with the proof when Z is a general r.v. If $Z \geq 0$, there exists an increasing sequence $(Z_n)_{n\geq 0}$ of simple functions on Ω converging pointwise to Z, and thank to the first statement of the Proposition, for each of these functions, $(s,x) \mapsto \mathbb{E}^{s,x}[Z_n]$ is Borel. Therefore since by monotonic convergence, $\mathbb{E}^{s,x}[Z_n] \underset{n\to\infty}{\longrightarrow} \mathbb{E}^{s,x}[Z]$, then $(s,x) \mapsto \mathbb{E}^{s,x}[Z]$ is Borel as the pointwise limit of Borel functions. For a general Z one just has to consider its decomposition $Z = Z^+ - Z^-$ where Z^+ and Z^- are positive. □

Lemma 3.A.11. *Assume that the transition kernel of the canonical Markov class is measurable in time.*

Let V be a continuous non-decreasing function on $[0,T]$ and $f \in \mathcal{B}([0,T] \times E, \mathbb{R})$ be such that for every $(s,x) \in [0,T] \times E$, $\mathbb{E}^{s,x}[\int_s^T |f(r, X_r)|dV_r] < \infty$. Then $(s,x) \longmapsto \mathbb{E}^{s,x}[\int_s^T f(r, X_r)dV_r]$ is Borel.

Proof. We will start by showing that on $([0,T] \times E) \times [0,T]$, the function
$k^n : (s,x,t) \mapsto \mathbb{E}^{s,x}[\int_t^T ((-n) \vee f(r, X_r) \wedge n)dV_r]$ is Borel, where $n \in \mathbb{N}$.

Let $t \in [0,T]$ be fixed. Then by Proposition 3.A.10, $(s,x) \mapsto \mathbb{E}^{s,x}[\int_t^T ((-n) \vee f(r, X_r) \wedge n)dV_r]$ is Borel. Let $(s,x) \in [0,T] \times E$ be fixed and $t_m \underset{m\to\infty}{\longrightarrow} t$ be a converging sequence in $[0,T]$. Since V is

continuous, $\int_{t_m}^T((-n)\vee f(r,X_r)\wedge n)dV_r \underset{m\to\infty}{\longrightarrow} \int_t^T((-n)\vee f(r,X_r)\wedge n)dV_r$ a.s. Since this sequence is uniformly bounded, by dominated convergence theorem, the same convergence holds under the expectation. This implies that $t\mapsto \mathbb{E}^{s,x}[\int_t^T((-n)\vee f(r,X_r)\wedge n)dV_r]$ is continuous. By Lemma 4.51 in [1], k^n is therefore jointly Borel.

By composing with $(s,x,t)\mapsto(s,x,s)$, we also have that for any $n\geq 0$, $\tilde{k}^n:(s,x)\longmapsto \mathbb{E}^{s,x}[\int_s^T((-n)\vee f(r,X_r)\wedge n)dV_r]$ is Borel. Then by letting n tend to infinity, $(-n)\vee f(r,X_r)\wedge n$ tends $dV\otimes d\mathbb{P}^{s,x}$ a.e. to $f(r,X_r)$ and since we assumed $\mathbb{E}^{s,x}[\int_s^T|f(r,X_r)|dV_r]<\infty$, by dominated convergence, $\mathbb{E}^{s,x}[\int_s^T((-n)\vee f(r,X_r)\wedge n)dV_r]$ tends to $\mathbb{E}^{s,x}[\int_s^T f(r,X_r)dV_r]$.
$(s,x)\longmapsto \mathbb{E}^{s,x}[\int_s^T f(r,X_r)dV_r]$ is therefore Borel as the pointwise limit of the $\tilde{k}^n$ which are Borel. □

Proposition 3.A.12. *Let $f\in\mathcal{B}([0,T]\times E,\mathbb{R})$ be such that for any (s,x,t), $\mathbb{E}^{s,x}[|f(t,X_t)|]<\infty$ then at fixed $s\in[0,T]$, $(x,t)\longmapsto\mathbb{E}^{s,x}[f(t,X_t)]$ is Borel. If moreover the transition kernel P is measurable in time, then $(s,x,t)\longmapsto\mathbb{E}^{s,x}[f(t,X_t)]$ is Borel.*

Proof. We will only show the case in which p is measurable in time since the other case is proven very similarly.

We start by showing the statement for $f\in\mathcal{C}_b([0,T]\times E,\mathbb{R})$. X is cadlag so $t\longmapsto f(t,X_t)$ also is. So for any fixed $(s,x)\in[0,T]\times E$ if we take a converging sequence $t_n\underset{n\to\infty}{\longrightarrow}t^+$(resp. t^-), an easy application of the Lebesgue dominated convergence theorem shows that $t\longmapsto\mathbb{E}^{s,x}[f(t,X_t)]$ is cadlag. On the other hand, by Proposition 3.A.10, for a fixed t, $(s,x)\longmapsto\mathbb{E}^{s,x}[f(t,X_t)]$ is Borel. Therefore by Theorem 15 Chapter IV of [32], $(s,x,t)\longmapsto\mathbb{E}^{s,x}[f(t,X_t)]$ is jointly Borel.

In order to extend the result to any $f\in\mathcal{B}_b([0,T]\times E,\mathbb{R})$, we consider the subset $\mathcal{I}$ of functions $f\in\mathcal{B}_b([0,T]\times E,\mathbb{R})$ such that $(s,x,t)\longmapsto\mathbb{E}^{s,x}[f(t,X_t)]$ is Borel. Then $\mathcal{I}$ is a linear space stable by uniform convergence and by monotone convergence and containing $\mathcal{C}_b([0,T]\times E,\mathbb{R})$ which is stable by multiplication and generates the Borel σ-field $\mathcal{B}([0,T])\otimes\mathcal{B}(E)$. So by Theorem 21 in Chapter I of [32], $\mathcal{I}=\mathcal{B}_b([0,T]\times E,\mathbb{R})$. This theorem is sometimes called the functional monotone class theorem. Now for any positive Borel function f, we can set $f_n=f\wedge n$ which is bounded Borel. Since by monotonic convergence, $\mathbb{E}^{s,x}[f_n(t,X_t)]$ tends to $\mathbb{E}^{s,x}[f(t,X_t)]$, then $(s,x,t)\longmapsto\mathbb{E}^{s,x}[f(t,X_t)]$ is Borel as the pointwise limit of Borel functions. Finally for a general f it is enough to decompose it into $f=f^+-f^-$ where f^+,f^- are positive functions. □

3.B Technicalities concerning homogeneous Markov classes and martingale problems

We start by introducing homogeneous Markov classes. In this section, we are given a Polish space E and some $T\in\mathbb{R}^*$.

Notation 3.B.1. *A mapping $\tilde{P}:[0,T]\times E\times\mathcal{B}(E)\longmapsto[0,1]$ will be called a **homogeneous transition kernel** if $P:(s,t,x,A)\longmapsto\tilde{P}_{t-s}(x,A)\mathbb{1}_{s<t}+\mathbb{1}_A(x)\mathbb{1}_{s\geq t}$ is a transition kernel in the sense of Definition 3.A.3. This in particular implies $\tilde{P}=P_{0,\cdot}(\cdot,\cdot)$.*
*A set of probability measures $(\mathbb{P}^x)_{x\in E}$ on the canonical space associated to T and E (see Notation 3.A.1) will be called a **homogeneous Markov class** associated to a homogeneous transition kernel $\tilde{P}$ if*

$$\begin{cases}\forall t\in[0,T]\quad\forall A\in\mathcal{B}(E)\quad,\mathbb{P}^x(X_t\in A)=\tilde{P}_t(x,A)\\ \forall 0\leq t\leq u\leq T\quad,\mathbb{P}^x(X_u\in A|\mathcal{F}_t)=\tilde{P}_{u-t}(X_t,A)\quad\mathbb{P}^{s,x}a.s.\end{cases}\tag{3.B.1}$$

Given a homogeneous Markov class $(\mathbb{P}^x)_{x\in E}$ associated to a homogeneous transition kernel $\tilde{P}$, one can always consider the Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ associated to the transition kernel $P:(s,t,x,A)\longmapsto\tilde{P}_{t-s}(x,A)\mathbb{1}_{s<t}+\mathbb{1}_A(x)\mathbb{1}_{s\geq t}$. In particular, for any $x\in E$, we have $\mathbb{P}^{0,x}=\mathbb{P}^x$.

We show that a homogeneous transition kernel necessarily produces a measurable in time non homogeneous transition kernel.

Proposition 3.B.2. *Let $\tilde{P}$ be a homogeneous transition kernel and let P be the associated non homogeneous transition kernel as described in Notation 3.B.1. Then P is measurable in time in the sense of Definition 3.A.4.*

Proof. Given that $P : (s,t,x,A) \longmapsto \tilde{P}_{t-s}(x,A)\mathbb{1}_{s<t} + \mathbb{1}_A(x)\mathbb{1}_{s\geq t}$, it is actually enough to show that $\tilde{P}_\cdot(\cdot,A)$ is Borel for any $A \in \mathcal{B}(E)$. We can also write $\tilde{P} = P_{0,\cdot}(\cdot,\cdot)$, so P is measurable in time if $P_{0,\cdot}(\cdot,A)$ is Borel for any $A \in \mathcal{B}(E)$, and this holds thanks to Proposition 3.A.12 applied to $f := \mathbb{1}_A$. □

We then introduce below the notion of homogeneous martingale problems.

Definition 3.B.3. *Given A an operator mapping a linear algebra $\mathcal{D}(A) \subset \mathcal{B}_b(E,\mathbb{R})$ into $\mathcal{B}_b(E,\mathbb{R})$, we say that a set of probability measures $(\mathbb{P}^x)_{x\in E}$ on the measurable space $(\Omega,\mathcal{F})$ (see Notation 3.A.1) solves the* ***homogeneous Martingale Problem*** *associated to $(\mathcal{D}(A),A)$ if for any $x \in E$, $\mathbb{P}^x$ satisfies*

- *for every $\phi \in \mathcal{D}(A)$, $\phi(X_\cdot) - \int_0^\cdot A\phi(X_r)dr$ is a $(\mathbb{P}^x,\mathbb{F})$-local martingale;*
- *$\mathbb{P}^x(X_0 = x) = 1$.*

We say that this **homogeneous Martingale Problem is well-posed** *if for any $x \in E$, $\mathbb{P}^x$ is the only probability measure on $(\Omega,\mathcal{F})$ verifying those two items.*

Remark 3.B.4. *If $(\mathbb{P}^x)_{x\in E}$ is a homogeneous Markov class solving the homogeneous Martingale Problem associated to some $(\mathcal{D}(A),A)$, then the corresponding $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ (see Notation 3.B.1) solves the Martingale Problem associated to $(\mathcal{D}(A),A,V_t \equiv t)$ in the sense of Definition 3.2.3. Moreover if the homogeneous Martingale Problem is well-posed, so is the latter one.*

So a homogeneous Markov process solving a homogeneous martingale problem falls into our setup. We will now see how we can pass from an operator A which only acts on time-independent functions to an evolution operator $\partial_t + A$, and see how our Markov class still solves the corresponding martingale problem.

Notation 3.B.5. *Let E be a Polish space and let A be an operator mapping a linear algebra $\mathcal{D}(A) \subset \mathcal{B}_b(E,\mathbb{R})$ into $\mathcal{B}_b(E,\mathbb{R})$.*
If $\phi \in \mathcal{B}([0,T]\times E,\mathbb{R})$ is such that for every $t \in [0,T]$, $\phi(t,\cdot) \in \mathcal{D}(A)$, then $A\phi$ will denote the mapping $(t,x) \longmapsto A(\phi(t,\cdot))(x)$.

We now introduce the time-inhomogeneous domain associated to A which we denote $\mathcal{D}^{max}(\partial_t + A)$ and which consists in functions $\phi \in \mathcal{B}_b([0,T]\times E,\mathbb{R})$ verifying the following conditions:

- *$\forall x \in E$, $\phi(\cdot,x) \in \mathcal{C}^1([0,T],\mathbb{R})$ and $\forall t \in [0,T]$, $\phi(t,\cdot) \in \mathcal{D}(A)$;*
- *$\forall t \in [0,T]$, $\partial_t\phi(t,\cdot) \in \mathcal{D}(A)$ and $\forall x \in E$, $A\phi(\cdot,x) \in \mathcal{C}^1([0,T],\mathbb{R})$;*
- *$\partial_t \circ A\phi = A \circ \partial_t\phi$;*
- *$\partial_t\phi$, $A\phi$ and $\partial_t \circ A\phi$ belong to $\mathcal{B}_b([0,T]\times E,\mathbb{R})$.*

On $\mathcal{D}^{max}(\partial_t + A)$ we will consider the operator $\partial_t + A$.

Remark 3.B.6. *With these notations, it is clear that $\mathcal{D}^{max}(\partial_t + A)$ is a sub-linear space of $\mathcal{B}_b([0,T]\times E,\mathbb{R})$. It is in general not a linear algebra, but always contains $\mathcal{D}(A)$, and even $\mathcal{C}^1([0,T],\mathbb{R}) \otimes \mathcal{D}(A)$, the linear algebra of functions which can be written $\sum_{k\leq N} \lambda_k\psi_k\phi_k$ where $N \in \mathbb{N}$, and for any k, $\lambda_k \in \mathbb{R}$, $\psi_k \in \mathcal{C}^1([0,T],\mathbb{R})$, $\phi_k \in \mathcal{D}(A)$. We also notice that $\partial_t + A$ maps $\mathcal{D}^{max}(\partial_t + A)$ into $\mathcal{B}_b([0,T]\times E,\mathbb{R})$.*

Lemma 3.B.7. *Let us consider the same notations and under the same assumptions as in Notation 3.B.5. Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be a Markov class solving the well-posed Martingale Problem associated to $(A, \mathcal{D}(A), V_t \equiv t)$ in the sense of Definition 3.2.3. Then it also solves the well-posed martingale problem associated to $(\partial_t + A, \mathcal{A}, V_t \equiv t)$ for any linear algebra $\mathcal{A}$ included in $\mathcal{D}^{max}(\partial_t + A)$.*

Proof. We start by noticing that since $\mathcal{D}(A) \subset \mathcal{B}_b(E, \mathbb{R})$ and is mapped into $\mathcal{B}_b(E, \mathbb{R})$, then for any $(s, x) \in [0, T] \times E$ and $\phi \in \mathcal{D}(A)$, $M^{s,x}[\phi]$ is bounded and is therefore a martingale.

We fix $(s, x) \in [0, T] \times E$, $\phi \in \mathcal{D}^{max}(\partial_t + A)$ and $s \le t \le u \le T$ and we will show that

$$\mathbb{E}^{s,x}\left[\phi(u, X_u) - \phi(t, X_t) - \int_t^u (\partial_t + A)\phi(r, X_r)dr \middle| \mathcal{F}_t\right] = 0, \tag{3.B.2}$$

which implies that $\phi(\cdot, X_\cdot) - \int_s^\cdot (\partial_t + A)\phi(r, X_r)dr$, $t \in [s, T]$ is a $\mathbb{P}^{s,x}$-martingale. We have

$$\begin{array}{rl} & \mathbb{E}^{s,x}[\phi(u, X_u) - \phi(t, X_t)|\mathcal{F}_t] \\ = & \mathbb{E}^{s,x}[(\phi(u, X_t) - \phi(t, X_t)) + (\phi(u, X_u) - \phi(u, X_t))|\mathcal{F}_t] \\ = & \mathbb{E}^{s,x}\left[\int_t^u \partial_t\phi(r, X_t)dr + \left(\int_t^u A\phi(u, X_r)dr + (M^{s,x}[\phi(u, \cdot)]_u - M^{s,x}[\phi(u, \cdot)]_t)\right)|\mathcal{F}_t\right] \\ = & \mathbb{E}^{s,x}\left[\int_t^u \partial_t\phi(r, X_t)dr + \int_t^u A\phi(u, X_r)dr|\mathcal{F}_t\right] \\ = & I_0 - I_1 + I_2, \end{array}$$

where $I_0 = \mathbb{E}^{s,x}\left[\int_t^u \partial_t\phi(r, X_r)dr + \int_t^u A\phi(r, X_r)dr|\mathcal{F}_t\right]$; $I_1 = \mathbb{E}^{s,x}\left[\int_t^u (\partial_t\phi(r, X_r) - \partial_t\phi(r, X_t))dr|\mathcal{F}_t\right]$ $I_2 = \mathbb{E}^{s,x}\left[\int_t^u (A\phi(u, X_r) - A\phi(r, X_r))dr|\mathcal{F}_t\right]$. (3.B.2) will be established if one proves that $I_1 = I_2$. We do this below.

At fixed r and ω, $v \longmapsto A\phi(v, X_r(\omega))$ is $\mathcal{C}^1$, therefore $A\phi(u, X_r(\omega)) - A\phi(r, X_r(\omega)) = \int_r^u \partial_t A\phi(v, X_r(\omega))dv$ and $I_2 = \mathbb{E}^{s,x}\left[\int_t^u \int_r^u \partial_t A\phi(v, X_r)dvdr|\mathcal{F}_t\right]$. Then
$I_1 = \mathbb{E}^{s,x}\left[\int_t^u \int_t^r A\partial_t\phi(r, X_v)dvdr|\mathcal{F}_t\right] + \mathbb{E}^{s,x}\left[\int_t^u (M^{s,x}[\partial_t\phi(r, \cdot)]_r - M^{s,x}[\partial_t\phi(r, \cdot)]_t)dr|\mathcal{F}_t\right]$.
Since $\partial_t\phi$ and $A\partial_t\phi$ are bounded, $M^{s,x}[\partial_t\phi(r, \cdot)]_r(\omega)$ is uniformly bounded in (r, ω), so by Fubini's theorem for conditional expectations we have

$$\begin{array}{rl} & \mathbb{E}^{s,x}[\int_t^u (M^{s,x}[\partial_t\phi(r, \cdot)]_r - M^{s,x}[\partial_t\phi(r, \cdot)]_t)dr|\mathcal{F}_t] \\ = & \int_t^u \mathbb{E}^{s,x}[M^{s,x}[\partial_t\phi(r, \cdot)]_r - M^{s,x}[\partial_t\phi(r, \cdot)]_t|\mathcal{F}_t]dr \\ = & 0. \end{array} \tag{3.B.3}$$

Finally since $\partial_t A\phi = A\partial_t\phi$ and again by Fubini's theorem for conditional expectations, we have $\mathbb{E}^{s,x}\left[\int_t^u \int_r^u \partial_t A\phi(v, X_r)dvdr|\mathcal{F}_t\right] = \mathbb{E}^{s,x}\left[\int_t^u \int_t^r A\partial_t\phi(r, X_v)dvdr|\mathcal{F}_t\right]$ so $I_1 = I_2$ which concludes the proof. □

In conclusion we can state the following.

Corollary 3.B.8. *Given a homogeneous Markov class $(\mathbb{P}^x)_{x\in E}$ solving a well-posed homogeneous Martingale Problem associated to some $(\mathcal{D}(A), A)$, there exists a Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ which transition kernel is measurable in time and such that for any algebra $\mathcal{A}$ included in $\mathcal{D}^{max}(\partial_t + A)$, $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ solves the well-posed Martingale Problem associated to $(\partial_t + A, \mathcal{A}, V_t \equiv t)$ in the sense of Definition 3.2.3.*

Chapter 4

Martingale driven BSDEs, PDEs and other related deterministic problems

This chapter is the object of paper [11].

Abstract

We focus on a class of BSDEs driven by a cadlag martingale and corresponding Markov type BSDEs which arise when the randomness of the driver appears through a Markov process. To those BSDEs we associate a deterministic problem which, when the Markov process is a Brownian diffusion, is nothing else but a parabolic type PDE. The solution of the deterministic problem is intended as *decoupled mild solution*, and it is formulated with the help of a time-inhomogeneous semigroup.

4.1 Introduction

Markovian backward stochastic differential equations (BSDEs) are BSDEs in the sense of [71] involving a forward dynamics described by a Markov (often a diffusion) process X. Those are naturally linked to a parabolic PDE, which constitutes a particular deterministic problem. In particular, under reasonable conditions, which among others ensure well-posedness, the solutions of BSDEs produce *viscosity* type solutions for the mentioned PDE. In this paper we focus on *Pseudo-PDEs* which are the corresponding deterministic problems associated to the case of a Markovian BSDE when this is driven by a cadlag martingale and when the underlying forward process is a general Markov process. In that case the concept of viscosity solution (based on comparison theorems) is not completely appropriated. For this we propose a new type of solution called *decoupled mild* which extends the usual notion of mild solution which is very familiar to the experts of PDEs. We establish an existence and uniqueness theorem in the class of Borel functions having a certain growth condition.

In the Brownian framework, BSDEs were introduced first by E. Pardoux and S. Peng in [71]. An interesting particular case appears when the random dependence of the driver generally denoted by f comes through a diffusion process X and the terminal condition only depends on X_T. The solution, when it exists, is usually indexed by the starting time s and starting point x of the forward diffusion $X = X^{s,x}$, and it is expressed by

$$\left\{ \begin{array}{rcl} X_t^{s,x} & = & x + \int_s^t \beta(r, X_r^{s,x})dr + \int_s^t \sigma(r, X_r^{s,x})dB_r, \quad t \in [0,T] \\ Y_t^{s,x} & = & g(X_T^{s,x}) + \int_t^T f\left(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x}\right) dr - \int_t^T Z_r^{s,x} dB_r, \quad t \in [0,T], \end{array} \right. \tag{4.1.1}$$

where B is a Brownian motion. In [76] and in [72] previous Markovian BSDE was linked to the semilinear PDE

$$\begin{cases} \partial_t u + \frac{1}{2}\sum_{i,j\leq d}(\sigma\sigma^\intercal)_{i,j}\partial^2_{x_ix_j}u + \sum_{i\leq d}\beta_i\partial_{x_i}u + f((\cdot,\cdot),u,\sigma\nabla u) = 0 \quad \text{on } [0,T[\times\mathbb{R}^d \\ u(T,\cdot) = g. \end{cases} \tag{4.1.2}$$

In particular, if (4.1.2) has a classical smooth solution u then $(Y^{s,x},Z^{s,x}) := (u(\cdot,X^{s,x}_\cdot),\sigma\nabla u(\cdot,X^{s,x}_\cdot))$ solves the second line of (4.1.1). Conversely, only under the Lipschitz type conditions on β,σ,f,g, the solution of the BSDE can be expressed as a function (u,v) of the forward process, i.e. $(Y^{s,x},Z^{s,x}) = (u(\cdot,X^{s,x}_\cdot),v(\cdot,X^{s,x}_\cdot))$, see [43]. When f and g are continuous, u is a viscosity solution of (4.1.2). In chapter 13 of [8], under some specific conditions on the coefficients of a Brownian BSDE, one produces a solution in the sense of distributions of the parabolic PDE. Later, a first notion of mild solution of the PDE was used in [4]. In [52] v was associated with a generalized form of $\sigma\nabla u$. Excepted in the case when previous u has some minimal differentiability properties, it is difficult to say something more on v. To express v in the general case, for instance when u is only a viscosity solution of the PDE, is not an easy task. Some authors call this the *identification problem*.

In [7] the authors introduced a new kind of Markovian BSDE including a term with jumps generated by a Poisson measure, where an underlying forward process X solves a jump diffusion equation with Lipschitz type conditions. They associated with it an Integro-Partial Differential Equation (in short IPDE) in which some non-local operators are added to the classical partial differential maps, and proved that, under some continuity and monotonicity conditions on the coefficients, the BSDE provides a viscosity solution of the IPDE. Concerning the study of BSDEs driven by more general martingales than Brownian motion, we have already mentioned BSDEs driven by Poisson measures. In this respect, more recently, BSDEs driven by marked point processes were introduced in [24], see also [5]; in that case the underlying process does not contain any diffusion term. Brownian BSDEs involving a supplementary orthogonal term were studied in [43]. A notion of BSDE driven by a martingale also involving a supplementary orthogonal martingale has appeared, see for instance [20], [22] and references therein.

In this paper, we consider a BSDE whose given data are a continuous increasing process $\hat{V}$, a square integrable martingale $\hat{M}$, a terminal condition ξ and a driver $\hat{f}$. A solution will be a couple (Y,M) verifying

$$Y = \xi + \int_\cdot^T \hat{f}\left(r,\cdot,Y_r,\frac{d\langle M,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r - (M_T - M_\cdot), \tag{4.1.3}$$

where Y is cadlag adapted and M is a square integrable martingale. We show existence and uniqueness of a solution for (4.1.3).

We will then be interested in a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ with time interval $[0,T]$ and state space E being a Polish space. This will be supposed to be a solution of a martingale problem related to an operator $(\mathcal{D}(a),a)$ and a non-decreasing function V, meaning that for any $\phi\in\mathcal{D}(a)$, and $(s,x)\in[0,T]\times E$, $M[\phi]^{s,x} := \mathbb{1}_{[s,T]}\left(\phi(\cdot,X_\cdot) - \phi(s,x) - \int_s^\cdot a(\phi)(r,X_r)dV_r\right)$ is a $\mathbb{P}^{s,x}$-square integrable martingale. We will fix some function $\psi := (\psi_1,\cdots,\psi_d)\in\mathcal{D}(a)^d$ and at Notation 4.5.6 we will introduce some special BSDEs driven by a martingale which we will call Markovian BSDEs.

Those will be indexed by some $(s,x)\in[0,T]\times E$, defined in some stochastic basis $(\Omega,\mathcal{F}^{s,x},\mathbb{F}^{s,x},\mathbb{P}^{s,x})$ and will have the form

$$Y^{s,x} = g(X_T) + \int_\cdot^T f\left(r,X_r,Y^{s,x}_r,\frac{d\langle M^{s,x},M[\psi]^{s,x}\rangle}{dV}(r)\right)dV_r - (M^{s,x}_T - M^{s,x}_\cdot), \tag{4.1.4}$$

where X is the canonical process, g is a Borel function with some growth condition and f is a Borel function with some growth condition with respect to the second variable, and Lipschitz conditions

with respect to the third and fourth variables. Those Markovian BSDEs will be linked to the Pseudo-PDE

$$\left\{ \begin{array}{rcll} a(u) + f\left(\cdot, \cdot, u, \Gamma^{\psi}(u)\right) & = & 0 & \text{on } [0,T] \times E \\ u(T, \cdot) & = & g, & \end{array} \right. \tag{4.1.5}$$

where $\Gamma^{\psi}(u) := (a(u\psi_i) - ua(\psi_i) - \psi_i a(u))_{i \in [\![1;d]\!]}$, see Definition 4.5.3. We introduce the notion of *classical* solution which is an element of $\mathcal{D}(a)$ fulfilling (4.1.5). We call Γ^{ψ} the *ψ-generalized gradient*, due to the fact that when $E = \mathbb{R}^d$, $a = \partial_t + \frac{1}{2}\Delta$ and $\psi_i : (t,x) \longmapsto x_i$ for all $i \in [\![1,d]\!]$ then $\Gamma^{\psi}(u) = \nabla u$. In this particular setup, the forward Markov process is of course the Brownian motion. In that case the space $\mathcal{D}(a)$ where classical solutions are defined is $C^{1,2}([0,T] \times \mathbb{R}^d)$.

We show the existence of a Borel function u in some extended domain $\mathcal{D}(\mathfrak{a})$ such that for every $(s,x) \in [0,T] \times E$, $Y^{s,x}$ is, on $[s,T]$, a $\mathbb{P}^{s,x}$-modification of $u(\cdot, X_\cdot)$. At Definition 4.5.8 we will introduce the notion of *martingale solution* for the Pseudo-PDE (4.1.5). We then show that previous u is the unique martingale solution of (4.1.5), which means that it solves (4.1.5) where the maps a and Γ^{ψ} are respectively replaced with some extended operators $\mathfrak{a}$ and $\mathfrak{G}^{\psi}$. We also show that previous u is the unique *decoupled mild solution* of the same equation. We explain below that notion of solution which is introduced at Definition 4.5.13.

A Borel function u will be called decoupled mild solution if there exists an $\mathbb{R}^d$-valued Borel function $v := (v_1, \cdots, v_d)$ such that for every (s,x),

$$\left\{ \begin{array}{rcl} u(s,x) & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(\cdot,\cdot,u,v\right)(r,\cdot)\right](x) dV_r \\ u\psi_1(s,x) & = & P_{s,T}[g\psi_1(T,\cdot)](x) - \int_s^T P_{s,r}\left[(v_1 + ua(\psi_1) - \psi_1 f\left(\cdot,\cdot,u,v\right))(r,\cdot)\right](x) dV_r \\ & \cdots & \\ u\psi_d(s,x) & = & P_{s,T}[g\psi_d(T,\cdot)](x) - \int_s^T P_{s,r}\left[(v_d + ua(\psi_d) - \psi_d f\left(\cdot,\cdot,u,v\right))(r,\cdot)\right](x) dV_r, \end{array} \right. \tag{4.1.6}$$

where P is the time-dependent transition kernel associated to the Markov class and to the operator a, see Notation 4.5.11. v coincides with $\mathfrak{G}^{\psi}(u)$ and the couple (u,v) will be called solution to the *identification problem*, see Definition 4.5.13. The intuition behind this notion of solution relies to the fact that the equation $a(u) = -f(\cdot,\cdot,u,\Gamma^{\psi}(u))$ can be decoupled into the system

$$\left\{ \begin{array}{rcl} a(u) & = & -f(\cdot,\cdot,u,v) \\ v_i & = & \Gamma^{\psi_i}(u), \quad i \in [\![1;d]\!], \end{array} \right. \tag{4.1.7}$$

which may be rewritten

$$\left\{ \begin{array}{rcl} a(u) & = & -f(\cdot,\cdot,u,v) \\ a(u\psi_i) & = & v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v), \quad i \in [\![1;d]\!]. \end{array} \right. \tag{4.1.8}$$

Martingale solutions were introduced in Chapter 2 and decoupled mild solutions in Chapter 3, in relation to a specific type of Pseudo-PDE, for which v was one-dimensional and which did not include the usual parabolic PDE related to classical BSDEs. A first approach to classical solutions to a general deterministic problem, associated with forward BSDEs with applications to the so called *Föllmer-Schweizer decomposition* was performed by [64].

The paper is organized as follows. In Section 4.3 we introduce an alternative formulation (4.1.3) for BSDEs driven by cadlag martingales discussed in [22]: we formulate in Theorem 4.3.3 existence and uniqueness for such equations. In Section 4.4, we introduce a canonical Markov class and the martingale problem which it is assumed to solve. We also define the extended domain $\mathcal{D}(\mathfrak{a})$ in Definition 4.4.11 and the extended operator $\mathfrak{a}$ (resp. $\mathfrak{G}^{\psi}$) in Definition 4.4.13 (resp. Notation 4.4.16). In Section 4.5, we introduce the Pseudo-PDE (4.1.5) (see Definition 4.5.3), the associated Markovian BSDEs

(4.1.4), see Notation 4.5.6. We introduce the notion of martingale solution of the Pseudo-PDE in (4.5.8) and of decoupled mild solution in Definition 4.5.13. Propositions 4.5.15 and 4.5.16 show the equivalence between martingale solutions and decoupled mild solutions. Proposition 4.5.17 states that any classical solution is a decoupled mild solution and conversely that any decoupled mild solution belonging to $\mathcal{D}(\Gamma^\psi)$ is a classical solution up to what we call a zero potential set. Let $(Y^{s,x}, M^{s,x})$ denote the unique solution of the associated BSDE (4.1.4), denoted by $BSDE^{s,x}(f,g)$. In Theorem 4.5.18 we show the existence of some $u \in \mathcal{D}(\mathfrak{a})$ such that for every $(s,x) \in [0,T] \times E$, $Y^{s,x}$ is a $\mathbb{P}^{s,x}$-modification of $u(\cdot, X_\cdot)$ on $[s,T]$. Theorem 4.5.20 states that the function $(s,x) \longmapsto Y_s^{s,x}$ is the unique decoupled mild solution of (4.1.5). Proposition 4.5.23 states that if the functions (u,v) verify (4.1.6), then for any (s,x), the couple of processes $\Big(u(t,X_t), \quad u(t,X_t) - u(s,x) + \int_s^t f(\cdot,\cdot,u,v)(r,X_r)dV_r\Big)_{t\in[s,T]}$ has a $\mathbb{P}^{s,x}$-version which solves $BSDE^{s,x}(f,g)$ on $[s,T]$. Finally in Section 4.6 we study some examples of applications. In section 4.6.1 we deal with parabolic semi-linear PDEs and in Section 4.6.2 with parabolic semi-linear PDEs with distributional drift.

4.2 Preliminaries

In the whole paper we will use the following notions, notations and vocabulary.
For any integers $k \leq n$, $[\![k;n]\!]$ will denote the set of integers i verifying $k \leq i \leq n$.
A topological space E will always be considered as a measurable space with its Borel σ-field which shall be denoted $\mathcal{B}(E)$. If (F,d_F) is a metric space, $\mathcal{C}(E,F)$ (respectively $\mathcal{C}_b(E,F)$, $\mathcal{B}(E,F)$, $\mathcal{B}_b(E,F)$) will denote the set of functions from E to F which are continuous (respectively bounded continuous, Borel, bounded Borel). $\mathbb{T}$ will stand for a real interval, of type $[0,T]$ with $T \in \mathbb{R}_+^*$ or $\mathbb{R}_+$.

On a fixed probability space $(\Omega,\mathcal{F},\mathbb{P})$, for any $p>0$, L^p will denote the set of random variables with finite p-th moment. A probability space equipped with a right-continuous filtration $(\Omega,\mathcal{F},\mathbb{F}:=(\mathcal{F}_t)_{t\in\mathbb{T}},\mathbb{P})$ will be called called a **stochastic basis** and will be said to **fulfill the usual conditions** if the probability space is complete and if $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible sets. We introduce now some notations and vocabulary about spaces of stochastic processes, on a fixed stochastic basis $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$. Most of them are taken or adapted from [60] or [61]. We will denote $\mathcal{V}$ (resp $\mathcal{V}^+$) the set of adapted, bounded variation (resp non-decreasing) processes vanishing at 0; $\mathcal{V}^p$ (resp $\mathcal{V}^{p,+}$) the elements of $\mathcal{V}$ (resp $\mathcal{V}^+$) which are predictable, and $\mathcal{V}^c$ (resp $\mathcal{V}^{c,+}$) the elements of $\mathcal{V}$ (resp $\mathcal{V}^+$) which are continuous; $\mathcal{M}$ will be the space of cadlag martingales. For any $p \in [1,\infty]$, $\mathcal{H}^p$ will denote the Banach space of elements of $\mathcal{M}$ for which $\|M\|_{\mathcal{H}^p} := \mathbb{E}[|\sup_{t\in\mathbb{T}} M_t|^p]^{\frac{1}{p}} < \infty$ and in this set we identify indistinguishable elements. $\mathcal{H}_0^p$ will denote the Banach subspace of $\mathcal{H}^p$ of elements vanishing at zero.
If $\mathbb{T}=[0,T]$ for some $T \in \mathbb{R}_+^*$, a stopping time will take values in $[0,T]\cup\{+\infty\}$. Let Y be a process and τ a stopping time, we denote by Y^τ the **stopped process** $t \mapsto Y_{t\wedge\tau}$. If $\mathcal{C}$ is a set of processes, we define its **localized class** $\mathcal{C}_{loc}$ as the set of processes Y such that there exist a localizing sequence $(\tau_n)_{n\geq 0}$ such that for every n, the stopped process Y^{τ_n} belongs to $\mathcal{C}$. By **localizing sequence of stopping times** we mean an increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ such that a.s. there exists $N \in \mathbb{N}$ for which $\tau_N = +\infty$.
For any $M,N \in \mathcal{M}_{loc}$, we denote $[M,N]$ their **quadratic covariation** and simply $[M]$ if $M=N$ and if moreover $M,N \in \mathcal{H}^2_{loc}$, $\langle M,N\rangle$ will denote their (predictable) **angular bracket**, or simply $\langle M\rangle$ if $M=N$.
$\mathcal{P}ro$ will denote the σ-field generated by progressively measurable processes defined on $[0,T]\times\Omega$.

From now on, we are given $T \in \mathbb{R}_+^*$. Until the end of Section 4.3, we also fix a stochastic basis

$\big(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\big)$ fulfilling the usual conditions.

Definition 4.2.1. *Let A and B be in $\mathcal{V}^+$. We will say that dB dominates dA* **in the sense of stochastic measures** *(written $dA \ll dB$) if for almost all ω, $dA(\omega) \ll dB(\omega)$ as Borel measures on $[0,T]$.*

We will say that dB and dA are mutually singular **in the sense of stochastic measures** *(written $dA \perp dB$) if for almost all ω, the Borel measures $dA(\omega)$ and $dB(\omega)$ are mutually singular.*

Let $B \in \mathcal{V}^+$. $dB \otimes d\mathbb{P}$ will denote the positive measure on $(\Omega \times [0,T], \mathcal{F} \otimes \mathcal{B}([0,T]))$ defined for any $F \in \mathcal{F} \otimes \mathcal{B}([0,T])$ by $dB \otimes d\mathbb{P}(F) = \mathbb{E}\left[\int_0^T \mathbb{1}_F(\omega, r) dB_r(\omega)\right]$. A property which holds true everywhere except on a null set for this measure will be said to be true $dB \otimes d\mathbb{P}$ almost everywhere (a.e.).

The proposition below was the object of Proposition 2.3.2 in Chapter 2.

Proposition 4.2.2. *For any A and B in $\mathcal{V}^{p,+}$, there exists a (non-negative $dB \otimes d\mathbb{P}$ a.e.) predictable process $\frac{dA}{dB}$ and a process in $\mathcal{V}^{p,+}$ $A^{\perp B}$ such that*

$$dA^{\perp B} \perp dB \text{ and } A = A^B + A^{\perp B} \text{ a.s.},$$

where $A^B = \int_0^\cdot \frac{dA}{dB}(r) dB_r$. The process $A^{\perp B}$ is unique up to indistinguishability and the process $\frac{dA}{dB}$ is unique $dB \otimes d\mathbb{P}$ a.e.

The predictable process $\frac{dA}{dB}$ appearing in the statement of Proposition 4.2.2 will be called the **Radon-Nikodym derivative** of A by B.

If A belongs to $\mathcal{V}$, we will denote by $Var(A)$ (resp. $Pos(A)$, resp $Neg(A)$) the total (resp. positive, resp. negative) variation of A, meaning the unique pair of elements $\mathcal{V}^+$ such that $A = Pos(A) - Neg(A)$, see Proposition I.3.3 in [61] for their existence. If A is in $\mathcal{V}^p$, and $B \in \mathcal{V}^{p,+}$. We set $\frac{dA}{dB} := \frac{dPos(A)}{dB} - \frac{dNeg(A)}{dB}$ and $A^{\perp B} := Pos(A)^{\perp B} - Neg(A)^{\perp B}$.
Below we restate Proposition 2.3.4 in Chapter 2.

Proposition 4.2.3. *Let A_1 and A_2 be in $\mathcal{V}^p$, and $B \in \mathcal{V}^{p,+}$. Then, $\frac{d(A_1+A_2)}{dB} = \frac{dA_1}{dB} + \frac{dA_2}{dB}$ $dB \otimes d\mathbb{P}$ a.e. and $(A_1 + A_2)^{\perp B} = A_1^{\perp B} + A_2^{\perp B}$.*

The following lemma was the object of Lemma 2.5.13 in Chapter 2.

Lemma 4.2.4. *Let V be a non-decreasing function. If two measurable processes are $\mathbb{P}$-modifications of each other, then they are also equal $dV \otimes d\mathbb{P}$ a.e.*

4.3 An alternative formulation of BSDEs driven by a cadlag martingale

We are now going to introduce here an alternative formulation for Backward Stochastic Differential Equations driven by a general cadlag martingale investigated for instance by [22].

Given some $\hat{V} \in \mathcal{V}^{c,+}$, we will indicate by $\mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$ the set of (up to indistinguishability) progressively measurable processes ϕ such that $\mathbb{E}[\int_0^T \phi_r^2 d\hat{V}_r] < \infty$.
$\mathcal{L}^{2,cadlag}(d\hat{V} \otimes d\mathbb{P})$ will denote the subspace of cadlag elements of $\mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$.

We will now fix a bounded process $\hat{V} \in \mathcal{V}^{c,+}$, an $\mathcal{F}_T$-measurable random variable ξ called the **final condition**, a square integrable **reference martingale** $\hat{M} := (\hat{M}^1, \cdots, \hat{M}^d)$ taking values in $\mathbb{R}^d$ for some $d \in \mathbb{N}^*$, and a **driver** $\hat{f} : ([0,T] \times \Omega) \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$, measurable with respect to $\mathcal{P}ro \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$. We will assume that $(\xi, \hat{f}, \hat{M})$ verify the following hypothesis.

Hypothesis 4.3.1.

1. $\xi \in L^2$;
2. $\hat{f}(\cdot, \cdot, 0, 0) \in \mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$;
3. *There exist positive constants K^Y, K^Z such that, $\mathbb{P}$ a.s. for all t, y, y', z, z', we have*

$$|\hat{f}(t, \cdot, y, z) - \hat{f}(t, \cdot, y', z')| \leq K^Y |y - y'| + K^Z \|z - z'\|; \tag{4.3.1}$$

4. $\frac{d\langle \hat{M} \rangle}{d\hat{V}}$ *is bounded.*

We will now formulate precisely our BSDE.

Definition 4.3.2. *We say that a couple $(Y, M) \in \mathcal{L}^{2,cadlag}(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}^2_0$ is a solution of $BSDE(\xi, \hat{f}, V, \hat{M})$ if it verifies*

$$Y = \xi + \int_{\cdot}^{T} \hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M} \rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M_T - M_\cdot) \tag{4.3.2}$$

in the sense of indistinguishability.

The proof of the theorem below is very similar to the one of Theorem 2.3.21 in Chapter 2. For the convenience of the reader, it is therefore postponed to Appendix 4.A.

Theorem 4.3.3. *If $(\xi, \hat{f}, \hat{V}, \hat{M})$ verifies Hypothesis 4.3.1 then $BSDE(\xi, \hat{f}, \hat{V}, \hat{M})$ has a unique solution.*

Remark 4.3.4. *Let $(\xi, \hat{f}, \hat{V}, \hat{M})$ satisfying Hypothesis 4.3.1. We can consider a BSDE on a restricted interval $[s, T]$ for some $s \in [0, T[$. Previous discussion and Theorem 4.3.3 extend easily to this case. In particular there exists a unique couple of processes (Y^s, M^s), indexed by $[s, T]$ such that Y^s is adapted, cadlag and verifies $\mathbb{E}[\int_s^T (Y^s_r)^2 d\hat{V}_r] < \infty$, such that M^s is a martingale vanishing in s and such that $Y^s = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y^s_r, \frac{d\langle M^s, \hat{M} \rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M^s_T - M^s_\cdot)$ in the sense of indistinguishability on $[s, T]$.*
Moreover, if (Y, M) denotes the solution of $BSDE(\xi, \hat{f}, \hat{V}, \hat{M})$ then $(Y, M_\cdot - M_s)$ and (Y^s, M^s) coincide on $[s, T]$. This follows by an uniqueness argument resulting by Theorem 4.3.3 on time interval $[s, T]$.

4.4 Martingale Problem and Markov classes

In this section, we introduce the Markov process which will later explain the random dependence of the driver $\hat{f}$ of our BSDE driven by a cadlag martingale. For that reason that BSDE will be called Markovian.

For details about the exact mathematical background necessary for our Markov process, one can consult Section 4.C of the Appendix. That process will be supposed to solve a martingale problem described below.

Let E be a Polish space. From now on, $(\Omega, \mathcal{F}, \mathbb{F})$ denotes the canonical space and $(X_t)_{t \in [0,T]}$ the canonical process defined in Definition 4.C.1. We consider a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x) \in [0,T] \times E}$

associated to a transition kernel measurable in time as defined in Definitions 4.C.5 and 4.C.4, and for any $(s,x) \in [0,T] \times E$, $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$ will denote the stochastic basis introduced in Definition 4.C.7 and which fulfills the usual conditions.

Our Martingale problem will be associated to an operator, in a close formalism to the one of D.W. Stroock and S.R.S Varadhan in [85].

Definition 4.4.1. *Let $V : [0,T] \to \mathbb{R}_+$ be a non-decreasing continuous function vanishing at 0.*
Let us consider a linear operator $a : \mathcal{D}(a) \subset \mathcal{B}([0,T] \times E, \mathbb{R}) \longrightarrow \mathcal{B}([0,T] \times E, \mathbb{R})$, where the domain $\mathcal{D}(a)$ is a linear space.

We say that a family of probability measures $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ defined on $(\Omega, \mathcal{F})$ solves the **martingale problem associated to** *$(\mathcal{D}(a), a, V)$ if, for any*
$(s,x) \in [0,T] \times E$, $\mathbb{P}^{s,x}$ verifies the following.

(a) $\mathbb{P}^{s,x}(\forall t \in [0,s], X_t = x) = 1$;

(b) for every $\phi \in \mathcal{D}(a)$, $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r)dV_r$, $t \in [s,T]$, is a cadlag $(\mathbb{P}^{s,x}, \mathbb{F})$ square integrable martingale.

We say that the Martingale Problem is ***well-posed*** *if for any $(s,x) \in [0,T] \times E$, $\mathbb{P}^{s,x}$ is the only probability measure satisfying those two properties.*

We anticipate that well-posedness for the martingale problem will not be an hypothesis in the sequel.

Notation 4.4.2. *For every $(s,x) \in [0,T] \times E$ and $\phi \in \mathcal{D}(a)$, the process*
$t \mapsto \mathbb{1}_{[s,T]}(t) \left(\phi(t, X_t) - \phi(s,x) - \int_s^t a(\phi)(r, X_r)dV_r\right)$ will be denoted $M[\phi]^{s,x}$.

$M[\phi]^{s,x}$ is a cadlag $(\mathbb{P}^{s,x}, \mathbb{F})$ square integrable martingale equal to 0 on $[0,s]$, and by Proposition 4.C.8, it is also a $(\mathbb{P}^{s,x}, \mathbb{F}^{s,x})$ square integrable martingale.

Notation 4.4.3. *Let $\phi \in \mathcal{D}(a)$. We set, for $0 \le t \le u \le T$*

$$M[\phi]_u^t := \begin{cases} \phi(u, X_u) - \phi(t, X_t) - \int_t^u a(\phi)(r, X_r)dV_r \text{ if } \int_t^u |a(\phi)|(r, X_r)dV_r < \infty, \\ 0 \text{ otherwise.} \end{cases} \tag{4.4.1}$$

$M[\phi]$ is a square integrable Martingale Additive Functional (in short MAF), see Definition 4.C.9, whose cadlag version under $\mathbb{P}^{s,x}$ for every $(s,x) \in [0,T] \times E$, is $M[\phi]^{s,x}$.

From now on we fix some $d \in \mathbb{N}^*$ and a vector $\psi = (\psi_1, \dots, \psi_d) \in \mathcal{D}(a)^d$.
For any $(s,x) \in [0,T] \times E$, the $\mathbb{R}^d$-valued martingale $(M[\psi_1]^{s,x}, \cdots, M[\psi_d]^{s,x})$ will be denoted $M[\psi]^{s,x}$.

Definition 4.4.4. *For any $\phi_1, \phi_2 \in \mathcal{D}(a)$ such that $\phi_1\phi_2 \in \mathcal{D}(a)$ we set $\Gamma(\phi_1, \phi_2) := a(\phi_1\phi_2) - \phi_1 a(\phi_2) - \phi_2 a(\phi_1)$. Γ will be called the* **carré du champs operator**. *We set $\mathcal{D}(\Gamma^\psi) := \left\{\phi \in \mathcal{D}(a) : \forall i \in [\![1;d]\!], \phi\psi^i \in \mathcal{D}(a)\right\}$.*
We define the linear operator $\Gamma^\psi : \mathcal{D}(\Gamma^\psi) \longrightarrow \mathcal{B}([0,T] \times E, \mathbb{R}^d)$ by

$$\Gamma^\psi(\phi) := \left(\Gamma^{\psi_i}(\phi)\right)_{i\in[\![1;d]\!]} := \left(a(\phi\psi_i) - \phi a(\psi_i) - \psi_i a(\phi)\right)_{i\in[\![1;d]\!]}. \tag{4.4.2}$$

*Γ^ψ will be called the ψ-**generalized gradient operator**.*

This operator appears in the expression of the angular bracket of the local martingales that we have defined.

Proposition 4.4.5. *If $\phi \in \mathcal{D}(\Gamma^\psi)$, then for any $(s,x) \in [0,T] \times E$ and $i \in [\![1;d]\!]$ we have*

$$\langle M[\phi]^{s,x}, M[\psi_i]^{s,x} \rangle = \int_s^{\cdot \vee s} \Gamma^{\psi_i}(\phi)(r, X_r) dV_r, \tag{4.4.3}$$

in the stochastic basis $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$.

Proof. The result follows from a slight modification of the proof of Proposition 2.4.7 of Chapter 2 in which $\mathcal{D}(a)$ was assumed to be stable by multiplication and $M[\phi]^{s,x}$ could potentially be a local martingale which is not a martingale. □

We will later need the following assumption.

Hypothesis 4.4.6. *For every $i \in [\![1;d]\!]$, the Additive Functional $\langle M[\psi_i] \rangle$ (see Proposition 4.C.10) is absolutely continuous with respect to dV, see Definition 4.C.9.*

Taking $\phi = \psi_i$ for some $i \in [\![1;d]\!]$ in Proposition 4.4.5, yields the following.

Corollary 4.4.7. *If $\psi_i^2 \in \mathcal{D}(a)$ for all $i \in [\![1;d]\!]$, then Hypothesis 4.4.6 is fulfilled.*

We will now consider a suitable extension of the domain $\mathcal{D}(a)$.

For any $(s,x) \in [0,T] \times E$ we define the positive bounded **potential measure** $U(s,x,\cdot)$ on $([0,T] \times E, \mathcal{B}([0,T]) \otimes \mathcal{B}(E))$ by

$$U(s,x,\cdot) : \begin{array}{ccc} \mathcal{B}([0,T]) \otimes \mathcal{B}(E) & \longrightarrow & [0, V_T] \\ A & \longmapsto & \mathbb{E}^{s,x}\left[\int_s^T \mathbb{1}_{\{(t,X_t) \in A\}} dV_t\right]. \end{array}$$

Definition 4.4.8. *A Borel set $A \subset [0,T] \times E$ will be said to be* **of zero potential** *if, for any $(s,x) \in [0,T] \times E$ we have $U(s,x,A) = 0$.*

Notation 4.4.9. *Let $p > 0$. We introduce*
$\mathcal{L}^p_{s,x} := \mathcal{L}^p(U(s,x,\cdot)) = \left\{ f \in \mathcal{B}([0,T] \times E, \mathbb{R}) : \mathbb{E}^{s,x}\left[\int_s^T |f|^p(r, X_r) dV_r\right] < \infty \right\}$.
For $p \geq 1$, that classical $\mathcal{L}^p$-space is equipped with the seminorm
$\| \cdot \|_{p,s,x} : f \mapsto \left(\mathbb{E}^{s,x}\left[\int_s^T |f(r, X_r)|^p dV_r\right]\right)^{\frac{1}{p}}$. *We also introduce*
$\mathcal{L}^0_{s,x} := \mathcal{L}^0(U(s,x,\cdot)) = \left\{ f \in \mathcal{B}([0,T] \times E, \mathbb{R}) : \int_s^T |f|(r, X_r) dV_r < \infty \quad \mathbb{P}^{s,x} \text{ a.s.} \right\}$.
For any $p \geq 0$ we set

$$\mathcal{L}^p_X = \bigcap_{(s,x) \in [0,T] \times E} \mathcal{L}^p_{s,x}. \tag{4.4.4}$$

Let $\mathcal{N}$ be the linear sub-space of $\mathcal{B}([0,T] \times E, \mathbb{R})$ containing all functions which are equal to 0, $U(s,x,\cdot)$ a.e. for every (s,x).
For any $p \geq 0$, we define the quotient space $L^p_X = \mathcal{L}^p_X / \mathcal{N}$.
If $p \geq 1$, L^p_X can be equipped with the topology generated by the family of semi-norms $(\| \cdot \|_{p,s,x})_{(s,x) \in [0,T] \times E}$ which makes it a separate locally convex topological vector space, see Theorem 5.76 in [1].

We recall that Proposition 2.4.13 in Chapter 2 states the following.

Proposition 4.4.10. *Let f and g be in $\mathcal{L}_X^0$. Then f and g are equal up to a set of zero potential if and only if for any $(s,x) \in [0,T] \times E$, the processes $\int_s^{\cdot} f(r,X_r)dV_r$ and $\int_s^{\cdot} g(r,X_r)dV_r$ are indistinguishable under $\mathbb{P}^{s,x}$. Of course in this case f and g correspond to the same element of L_X^0.*

We introduce now our notion of **extended generator** starting from its domain.

Definition 4.4.11. *We first define the **extended domain** $\mathcal{D}(\mathfrak{a})$ as the set of functions $\phi \in \mathcal{B}([0,T] \times E, \mathbb{R})$ for which there exists*
$\chi \in \mathcal{L}_X^0$ such that under any $\mathbb{P}^{s,x}$ the process

$$\mathbb{1}_{[s,T]} \left(\phi(\cdot, X_\cdot) - \phi(s,x) - \int_s^{\cdot} \chi(r, X_r)dV_r \right) \tag{4.4.5}$$

(which is not necessarily cadlag) has a cadlag modification in $\mathcal{H}_0^2$.

A direct consequence of Proposition 2.4.15 in Chapter 2 is the following.

Proposition 4.4.12. *Let $\phi \in \mathcal{B}([0,T] \times E, \mathbb{R})$. There is at most one (up to zero potential sets) $\chi \in \mathcal{L}_X^0$ such that under any $\mathbb{P}^{s,x}$, the process defined in* (4.4.5) *has a modification which belongs to $\mathcal{H}^2$.*
If moreover $\phi \in \mathcal{D}(a)$, then $a(\phi) = \chi$ up to zero potential sets. In this case, according to Notation 4.4.2, for every $(s,x) \in [0,T] \times E$, $M[\phi]^{s,x}$ is the $\mathbb{P}^{s,x}$ cadlag modification in $\mathcal{H}_0^2$ of $\mathbb{1}_{[s,T]} \left(\phi(\cdot, X_\cdot) - \phi(s,x) - \int_s^{\cdot} \chi(r, X_r)dV_r \right)$.

Definition 4.4.13. *Let $\phi \in \mathcal{D}(\mathfrak{a})$ as in Definition 4.4.11. We denote again by $M[\phi]^{s,x}$, the unique cadlag version of the process* (4.4.5) *in $\mathcal{H}_0^2$. Taking Proposition 4.4.10 into account, this will not generate any ambiguity with respect to Notation 4.4.2. Proposition 4.4.10, also permits to define without ambiguity the operator*

$$\mathfrak{a} : \begin{array}{ccc} \mathcal{D}(\mathfrak{a}) & \longrightarrow & L_X^0 \\ \phi & \longmapsto & \chi. \end{array}$$

*$\mathfrak{a}$ will be called the **extended generator**.*

Remark 4.4.14. *$\mathfrak{a}$ extends a in the sense that $\mathcal{D}(a) \subset \mathcal{D}(\mathfrak{a})$ (comparing Definitions 4.4.11 and 4.4.1) and if $\phi \in \mathcal{D}(a)$ then $a(\phi)$ is an element of the class $\mathfrak{a}(\phi)$, see Proposition 4.4.12.*

We also introduce an extended ψ-generalized gradient.

Proposition 4.4.15. *Assume the validity of Hypothesis 4.4.6. Let $\phi \in \mathcal{D}(\mathfrak{a})$ and $i \in [\![1;d]\!]$. There exists a (unique up to zero-potential sets) function in $\mathcal{B}([0,T] \times E, \mathbb{R})$ which we will denote $\mathfrak{G}^{\psi_i}(\phi)$ such that under any $\mathbb{P}^{s,x}$, $\langle M[\phi]^{s,x}, M[\psi_i]^{s,x} \rangle = \int_s^{\cdot \vee s} \mathfrak{G}^{\psi_i}(\phi)(r, X_r)dV_r$ up to indistinguishability.*

Proof. We fix $i \in [\![1;d]\!]$. Let $M[\psi_i]$ be the square integrable MAF (see Definition 4.C.9) presented in Notation 4.4.3. We introduce the random field $M[\phi] = (M[\phi]_u^t)_{(0 \le t \le u \le T)}$ as follows. We fix some χ in the class $\mathfrak{a}(\phi)$ and set

$$M[\phi]_u^t := \begin{cases} \phi(u, X_u) - \phi(t, X_t) - \int_t^u \chi(r, X_r)dV_r \text{ if } \int_t^u |\chi|(r, X_r)dV_r < \infty, t \le u, \\ 0 \text{ elsewhere,} \end{cases} \tag{4.4.6}$$

We emphasize that, a priori, the function χ is only in $\mathcal{L}_X^0$ implying that at fixed $t \le u$, $\int_t^u |\chi|(r, X_r(\omega))dV_r$ is not finite for every $\omega \in \Omega$, but only on a set which is $\mathbb{P}^{s,x}$-negligible for all $(s,x) \in [0,t] \times E$.
According to Definition 4.C.9 $M[\phi]$ is an AF whose cadlag version under $\mathbb{P}^{s,x}$ is $M[\phi]^{s,x}$. Of course $M[\psi_i]^{s,x}$ is the cadlag version of $M[\psi_i]$ under $\mathbb{P}^{s,x}$.
By Definition 4.4.13, since $\phi \in \mathcal{D}(\mathfrak{a})$, $M[\phi]^{s,x}$ is a square integrable martingale for every (s,x), so $M[\phi]$ is a square integrable MAF. Then by Corollary 4.4.7, the AF $\langle M[\psi_i] \rangle$ is absolutely continuous with respect to dV. The existence of $\mathfrak{G}^{\psi_i}(\phi)$ now follows from Proposition 4.C.11 and the uniqueness follows by Proposition 4.4.10. □

Notation 4.4.16. *If 4.4.6 holds, we can introduce the linear operator*

$$\mathfrak{G}^{\psi} : \begin{array}{ccc} \mathcal{D}(\mathfrak{a}) & \longrightarrow & (L^0_X)^d \\ \phi & \longmapsto & (\mathfrak{G}^{\psi_1}(\phi), \cdots, \mathfrak{G}^{\psi_d}(\phi)), \end{array} \tag{4.4.7}$$

which will be called the ***extended ψ-generalized gradient****.*

Corollary 4.4.17. *Let $\phi \in \mathcal{D}(\Gamma^{\psi})$. Then $\Gamma^{\psi}(\phi) = \mathfrak{G}^{\psi}(\phi)$ up to zero potential sets.*

Proof. Comparing Propositions 4.4.5 and 4.4.15, for every $(s,x) \in [0,T] \times E$ and $i \in [\![1;d]\!]$, $\int_s^{\cdot \vee s} \Gamma^{\psi_i}(\phi)(r, X_r) dV_r$ and $\int_s^{\cdot \vee s} \mathfrak{G}^{\psi_i}(\phi)(r, X_r) dV_r$ are $\mathbb{P}^{s,x}$-indistinguishable. We can conclude by Proposition 4.4.10. □

$\mathfrak{G}^{\psi}$ therefore extends Γ^{ψ} as well as $\mathfrak{a}$ extends a, see Remark 4.4.14.

4.5 Pseudo-PDEs and associated Markovian type BSDEs driven by a cadlag martingale

4.5.1 The concepts

In this section, we still consider $T \in \mathbb{R}^*_+$, a Polish space E and the associated canonical space $(\Omega, \mathcal{F}, \mathbb{F})$ and canonical process $(X_t)_{t \in [0,T]}$, see Definition 4.C.1. We also consider a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x) \in [0,T] \times E}$ associated to a transition kernel measurable in time (see Definitions 4.C.5 and 4.C.4) which solves a martingale problem associated to a triplet $(\mathcal{D}(a), a, V)$, see Definition 4.4.1.

We will investigate here a specific type of BSDE driven by a cadlag martingale, denoted by $BSDE(\xi, \hat{f}, \hat{V}, \hat{M})$ which we will call **of Markovian type**, or **Markovian** BSDE, in the following sense. The process $\hat{V}$ will be the (deterministic) function V introduced in Definition 4.4.1, the final condition ξ will only depend on the final value of the canonical process X_T and the randomness of the driver $\hat{f}$ at time t will only appear via the value at time t of the forward process X. Given a function $f : [0,T] \times E \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, we will set $\hat{f}(t, \omega, y, z) = f(t, X_t(\omega), y, z)$ for $t \in [0,T], \omega \in \Omega, y \in \mathbb{R}, z \in \mathbb{R}^d$.
Given d functions $\psi_1, \cdots, \psi_d$ in $\mathcal{D}(a)$, we will set $\hat{M} := (M[\psi_1]^{s,x}, \cdots, M[\psi_d]^{s,x})$.

That BSDE will be connected with the deterministic problem in Definition 4.5.3.

We fix an integer $d \in \mathbb{N}^*$ and some functions $\psi_1, \cdots, \psi_d \in \mathcal{D}(a)$ which in the sequel, will verify the following hypothesis.

Hypothesis 4.5.1. *For any $i \in [\![1;d]\!]$ we have the following*

- *Hypothesis 4.4.6 holds;*
- *$a(\psi_i) \in \mathcal{L}^2_X$;*
- *$\mathfrak{G}^{\psi_i}(\psi_i)$ is bounded.*

In particular, for every $i \in [\![1;d]\!]$, previous hypothesis implies the following.

Proposition 4.5.2.

- *For any $(s,x) \in [0,T] \times E$, $\hat{M} := M[\psi]^{s,x}$ verifies item 4. of Hypothesis 4.3.1 with respect to $\hat{V} := V$.*

- *for every* $(s,x) \in [0,T] \times E$, $\sup_{t\in[s,T]} |\psi_i(t,X_t)|^2$ *belongs to* L^1 *under* $\mathbb{P}^{s,x}$;
- $\psi_i \in \mathcal{L}^2_X$.

Proof. The first item follows form the fact that for any $(s,x) \in [0,T]\times E$, $\langle M[\psi]^{s,x}\rangle = \int_s^{\cdot\vee s} \mathfrak{G}^{\psi_i}(\psi_i)(r,X_r)dV_r$ (see Proposition 4.4.15), and the fact that $\mathfrak{G}^{\psi_i}(\psi_i)$ is bounded. Concerning the second item, for any $(s,x) \in [0,T]\times E$, the martingale problem gives $\psi(\cdot,X) = \psi_i(s,x) + \int_s^{\cdot} a(\psi_i)(r,X_r)dV_r + M[\psi_i]^{s,x}$, see Definition 4.4.1. By Jensen's inequality, we have $\sup_{t\in[s,T]} |\psi_i(t,X_t)|^2 \leq C(\psi_i^2(s,x) + \int_s^T a^2(\psi_i)(r,X_r)dV_r + \sup_{t\in[s,T]} (M[\psi_i]_t^{s,x})^2)$ for some $C>0$. It is therefore L^1 since $a(\psi_i) \in \mathcal{L}^2_X$ and $M[\psi_i]^{s,x} \in \mathcal{H}^2$. The last item is a direct consequence of the second one. □

Definition 4.5.3. *Let us consider some* $g \in \mathcal{B}(E,\mathbb{R})$ *and* $f \in \mathcal{B}([0,T]\times E\times\mathbb{R}\times\mathbb{R},\mathbb{R}^d)$.
We will call ***Pseudo-Partial Differential Equation*** *related to* (f,g) *(in short* $Pseudo-PDE(f,g)$*) the following equation with final condition:*

$$\left\{\begin{array}{rcll} a(u) + f\left(\cdot,\cdot,u,\Gamma^{\psi}(u)\right) & = & 0 & \text{on } [0,T]\times E \\ u(T,\cdot) & = & g. & \end{array}\right. \tag{4.5.1}$$

We will say that u *is a* ***classical solution*** *of* $Pseudo-PDE(f,g)$ *if* $u, u\psi_i, i \in [\![1;d]\!]$ *belong to* $\mathcal{D}(a)$ *and if* u *verifies* (4.5.1).

The connection between a Markovian BSDE and a $Pseudo-PDE(f,g)$, will be possible under some growth conditions on the functions (f,g).

Hypothesis 4.5.4. *A couple of functions* $f \in \mathcal{B}([0,T]\times E\times\mathbb{R}\times\mathbb{R}^d,\mathbb{R})$ *and* $g \in \mathcal{B}(E,\mathbb{R})$ *will be said* ***to verify*** H^{lip} *if there exist positive constants* K^Y, K^Z *such that*

1. $g(X_T)$ *is* L^2 *under* $\mathbb{P}^{s,x}$ *for every* $(s,x) \in [0,T]\times E$;
2. $t \longmapsto f(t,X_t,0,0) \in \mathcal{L}^2_X$;
3. $\forall (t,x,y,y',z,z') : \quad |f(t,x,y,z) - f(t,x,y',z')| \leq K^Y|y-y'| + K^Z\|z-z'\|$.

(f,g) *will be said* ***to verify*** H^{growth} *if the following lighter Hypothesis hold. There exist a positive constant* C *such that*

1. $g(X_T)$ *is* L^2 *under* $\mathbb{P}^{s,x}$ *for every* $(s,x) \in [0,T]\times E$;
2. $t \longmapsto f(t,X_t,0,0) \in \mathcal{L}^2_X$;
3. $\forall (t,x,y,z) : \quad |f(t,x,y,z)| \leq C(f(t,x,0,0) + |y| + \|z\|)$.

Remark 4.5.5. *We fix for now a couple* (f,g) *verifying* H^{lip}. *For any* $(s,x) \in [0,T]\times E$, *in the stochastic basis* $(\Omega,\mathcal{F}^{s,x},\mathbb{F}^{s,x},\mathbb{P}^{s,x})$ *and setting* $\hat{V} := V$, *the triplet* $\xi := g(X_T)$, $\hat{f} : (t,\omega,y,z) \longmapsto f(t,X_t(\omega),y,z)$, $\hat{M} := M[\psi]^{s,x}$ *verifies Hypothesis 4.3.1.*

With the equation $Pseudo-PDE(f,g)$, we will associate the following family of BSDEs indexed by $(s,x) \in [0,T]\times E$, driven by a cadlag martingale.

Notation 4.5.6. *For any $(s,x) \in [0,T] \times E$, we consider in the stochastic basis $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$ and on the interval $[0,T]$ the $BSDE(\xi, \hat{f}, V, \hat{M})$ where $\xi = g(X_T)$, $\hat{f} : (t,\omega,y,z) \longmapsto f(t, X_t(\omega), y, z)$, $\hat{M} = M[\psi]^{s,x}$).*
This BSDE will from now on be denoted $BSDE^{s,x}(f,g)$ and its unique solution (see Theorem 4.3.3 and Remark 4.5.5) will be denoted $(Y^{s,x}, M^{s,x})$.

If H^{lip} is fulfilled by (f,g), then $(Y^{s,x}, M^{s,x})$ is therefore the unique couple in $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}_0^2$ verifying

$$Y_\cdot^{s,x} = g(X_T) + \int_\cdot^T f\left(r, X_r, Y_r^{s,x}, \frac{d\langle M^{s,x}, M[\psi]^{s,x}\rangle}{dV}(r)\right) dV_r - (M_T^{s,x} - M_\cdot^{s,x}). \tag{4.5.2}$$

Remark 4.5.7. *Even if the underlying process X admits no generalized moments, given a couple (f,g) such that $f(\cdot,\cdot,0,0)$ and g are bounded, the considerations of this section still apply. In particular the connections that we will establish between the $BSDE^{s,x}(f,g)$ and the corresponding $Pseudo-PDE(f,g)$ still take place.*

The goal of our work is to emphasize the precise link under general enough conditions between the solutions of equations $BSDE^{s,x}(f,g)$ and of $Pseudo-PDE(f,g)$. In particular we will emphasize that a solution of $BSDE^{s,x}(f,g)$ produces a solution of $Pseudo - PDE(f,g)$ and reciprocally.

We now introduce a probabilistic notion of solution for $Pseudo - PDE(f,g)$.

Definition 4.5.8. *A Borel function $u : [0,T] \times E \to \mathbb{R}$ will be said to be a* **martingale solution** *of $Pseudo - PDE(f,g)$ if $u \in \mathcal{D}(\mathfrak{a})$ and*

$$\left\{ \begin{array}{rcl} \mathfrak{a}(u) & = & -f(\cdot,\cdot,u,\mathfrak{G}^\psi(u)) \\ u(T,\cdot) & = & g. \end{array} \right. \tag{4.5.3}$$

Remark 4.5.9. *The first equation of* (4.5.3) *holds in L_X^0, hence up to a zero potential set. The second one is a pointwise equality.*

Proposition 4.5.10. *Let (f,g) verify H^{growth}. Let u be a martingale solution of $Pseudo - PDE(f,g)$. Then for any $(s,x) \in [0,T] \times E$, the couple of processes*

$$\left(u(t,X_t), \quad u(t,X_t) - u(s,x) + \int_s^t f(\cdot,\cdot,u,\mathfrak{G}^\psi(u))(r,X_r) dV_r\right)_{t\in[s,T]} \tag{4.5.4}$$

has a $\mathbb{P}^{s,x}$-version which is a solution on $[s,T]$ of $BSDE^{s,x}(f,g)$, see Remark 4.3.4.

Moreover, $u \in \mathcal{L}_X^2$.

Proof. Let $u \in \mathcal{D}(\mathfrak{a})$ be a solution of (4.5.3) and let $(s,x) \in [0,T] \times E$ be fixed. By Definition 4.4.11 and Remark 4.3.4, the process $u(\cdot, X_\cdot)$ under $\mathbb{P}^{s,x}$ admits a cadlag modification $U^{s,x}$ on $[s,T]$, which is a special semimartingale with decomposition

$$\begin{array}{rcl} U^{s,x} & = & u(s,x) + \int_s^\cdot \mathfrak{a}(u)(r,X_r) dV_r + M[u]^{s,x} \\ & = & u(s,x) - \int_s^\cdot f\left(r, X_r, u(r,X_r), \mathfrak{G}^\psi(u)(r,X_r)\right) dV_r + M[u]^{s,x} \\ & = & u(s,x) - \int_s^\cdot f\left(r, X_r, U^{s,x}, \frac{d\langle M[u]^{s,x}, M[\psi]^{s,x}\rangle}{dV}\right) dV_r + M[u]^{s,x}, \end{array} \tag{4.5.5}$$

where the third equality of (4.5.5) comes from Lemma 4.2.4 and Proposition 4.4.15. Moreover since $u(T,\cdot) = g$, then $U_T^{s,x} = u(T,X_T) = g(X_T)$ a.s. so the couple $(U^{s,x}, M[u]^{s,x})$ verifies the following equation on $[s,T]$ (with respect to $\mathbb{P}^{s,x}$):

$$U_\cdot^{s,x} = g(X_T) + \int_\cdot^T f\left(r, X_r, U_r^{s,x}, \frac{d\langle M[u]^{s,x}, M[\psi]^{s,x}\rangle}{dV}(r)\right) dV_r - (M[u]_T^{s,x} - M[u]_\cdot^{s,x}). \tag{4.5.6}$$

$M[u]^{s,x}$ (introduced at Definition 4.4.13) belongs to $\mathcal{H}_0^2$ but we do not have a priori information on the square integrability of $U^{s,x}$. However we know that $M[u]^{s,x}$ is equal to zero at time s, and that $U_s^{s,x}$ is deterministic so square integrable. We can therefore apply Lemma 4.A.12 which implies that $(U^{s,x}, M[u]^{s,x})$ solves $BSDE^{s,x}(f,g)$ on $[s,T]$. In particular, $U^{s,x}$ belongs to $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$ for every (s,x), so by Lemma 4.2.4 and Definition 4.4.9, $u \in \mathcal{L}_X^2$. □

4.5.2 Decoupled mild solutions of Pseudo-PDEs

In this section we introduce an analytical notion of solution of our $Pseudo - PDE(f,g)$ that we will denominate *decoupled mild* since it inspired by the mild solution of partial differential equation. We will show that it is equivalent to the notion of martingale solution introduced in Definition 4.5.8.

Notation 4.5.11. *Let P denote the transition kernel of the canonical class (see Definition 4.C.3). Let s,t in $[0,T]$ with $s \leq t$, $x \in E$ and $\phi \in \mathcal{B}(E,\mathbb{R})$, if ϕ is integrable with respect to $P_{s,t}(x,\cdot)$, then $P_{s,t}[\phi](x)$ will denote its integral.*

We recall two important measurability properties.

Remark 4.5.12.

- *Let $\phi \in \mathcal{B}(E,\mathbb{R})$ be such that for any (s,x,t), $\mathbb{E}^{s,x}[|\phi(X_t)|] < \infty$, then $(s,x,t) \longmapsto P_{s,t}[\phi](x)$ is Borel, see Proposition 3.A.12 in Chapter 3.*
- *Let $\phi \in \mathcal{L}_X^1$, then $(s,x) \longmapsto \int_s^T P_{s,r}[\phi](x)dV_r$ is Borel, see Lemma 2.A.8 in Chapter 3.*

Our notion of decoupled mild solution relies on the fact that the equation $a(u)+f\left(\cdot,\cdot,u,\Gamma^{\psi}(u)\right) = 0$ can be naturally decoupled into

$$\left\{\begin{array}{rcl} a(u) & = & -f(\cdot,\cdot,u,v) \\ v_i & = & \Gamma^{\psi_i}(u), \quad i \in [\![1;d]\!]. \end{array}\right. \tag{4.5.7}$$

Then, by definition of the carré du champ operator (see Definition 4.4.4), we formally have $i \in [\![1;d]\!]$, $a(u\psi_i) = \Gamma^{\psi_i}(u) + ua(\psi_i) + \psi_i a(u)$. So the system of equations (4.5.7) can be rewritten as

$$\left\{\begin{array}{rcl} a(u) & = & -f(\cdot,\cdot,u,v) \\ a(u\psi_i) & = & v_i + ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v), \quad i \in [\![1;d]\!]. \end{array}\right. \tag{4.5.8}$$

Inspired by the usual notions of mild solution, this naturally leads us to the following definition of a mild solution.

Definition 4.5.13. *Let (f,g) be a couple verifying H^{growth}. Let $u \in \mathcal{B}([0,T]\times E,\mathbb{R})$ and $v \in \mathcal{B}([0,T]\times E,\mathbb{R}^d)$.*

1. *The couple (u,v) will be called* **solution of the identification problem determined by** *(f,g) or simply* **solution of** *$IP(f,g)$ if $u, v_1, \cdots, v_d$ belong to $\mathcal{L}_X^2$ and if for every $(s,x) \in [0,T]\times E$,*

$$\left\{\begin{array}{rcl} u(s,x) & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(\cdot,\cdot,u,v\right)(r,\cdot)\right](x)dV_r \\ u\psi_1(s,x) & = & P_{s,T}[g\psi_1(T,\cdot)](x) - \int_s^T P_{s,r}\left[\left(v_1 + ua(\psi_1) - \psi_1 f\left(\cdot,\cdot,u,v\right)\right)(r,\cdot)\right](x)dV_r \\ & \cdots & \\ u\psi_d(s,x) & = & P_{s,T}[g\psi_d(T,\cdot)](x) - \int_s^T P_{s,r}\left[\left(v_d + ua(\psi_d) - \psi_d f\left(\cdot,\cdot,u,v\right)\right)(r,\cdot)\right](x)dV_r. \end{array}\right. \tag{4.5.9}$$

2. *The function u will be called* **decoupled mild solution** *of $Pseudo-PDE(f,g)$ if there exist a function v such that the couple (u,v) is a solution of $IP(f,g)$.*

Lemma 4.5.14. *Let $u, v_1, \cdots, v_d \in \mathcal{L}_X^2$, let (f,g) be a couple satisfying H^{growth} and let $\psi_1, \cdots, \psi_d$ verify Hypothesis 4.5.1. Then $f(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}_X^2$ and for every $i \in [\![1;d]\!]$, $\psi_i f(\cdot,\cdot,u,v)$, and $ua(\psi_i)$, belong to $\mathcal{L}_X^1$. For any $(s,x) \in [0,T] \times E$, $i \in [\![1;d]\!]$, $g(X_T)\psi_i(T,X_T)$ belongs to L^1 under $\mathbb{P}^{s,x}$. In particular, all terms in* (4.5.9) *make sense.*

Proof. Thanks to the growth condition on f in H^{growth}, there exists a constant $C > 0$ such that for any $(s,x) \in [0,T] \times E$,

$$\begin{array}{rcl} & & \mathbb{E}^{s,x}\left[\int_t^T f^2(r,X_r,u(r,X_r),v(r,X_r))dV_r\right] \\ &\leq& C\mathbb{E}^{s,x}\left[\int_t^T (f^2(r,X_r,0,0) + u^2(r,X_r) + \|v\|^2(r,X_r))dV_r\right]. \end{array} \tag{4.5.10}$$

Previous quantity is finite since we have assumed that $u, v_1, \cdots, v_d$ belong to $\mathcal{L}_X^2$, taking into account Hypothesis H^{growth}. This means that $f^2(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}_X^1$. All the other assertions are easily obtained taking into account Hypothesis 4.5.1, H^{growth} and the classical inequality $2|ab| \leq a^2+b^2$. □

Proposition 4.5.15. *Let (f,g) verify H^{growth}. Let u be a martingale solution of $Pseudo-PDE(f,g)$, then $(u, \mathfrak{G}^\psi(u))$ is a solution of $IP(f,g)$ and in particular, u is a decoupled mild solution of $Pseudo-PDE(f,g)$.*

Proof. Let u be a martingale solution of $Pseudo-PDE(f,g)$. By Proposition 4.5.10, $u \in \mathcal{L}_X^2$. Taking into account Definition 4.4.13, for every (s,x), $M[u]^{s,x} \in \mathcal{H}_0^2$ under $\mathbb{P}^{s,x}$. So by Lemma 4.A.2, for any $i \in [\![1;d]\!]$, $\frac{d\langle M[u]^{s,x}, M[\psi^i]^{s,x}\rangle}{dV}$ belongs to $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$. Taking Proposition 4.4.15 into account, this means that $\mathfrak{G}^{\psi_i}(u) \in \mathcal{L}_X^2$ for every i. By Lemma 4.5.14, it follows that $f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)$ belongs to $\mathcal{L}_X^2$ and so for any $i \in [\![1;d]\!]$, $\psi_i f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)$ and $ua(\psi_i)$, belong to $\mathcal{L}_X^1$.
We fix some $(s,x) \in [0,T] \times E$ and the correspondent probability $\mathbb{P}^{s,x}$, and we are going to show that

$$\left\{\begin{array}{rcl} u(s,x) &=& P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)(r,\cdot)\right](x)dV_r \\ u\psi_1(s,x) &=& P_{s,T}[g\psi_1(T,\cdot)](x) - \int_s^T P_{s,r}\left[\left(\mathfrak{G}(u,\psi_1) + ua(\psi_1) - \psi_1 f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)\right)(r,\cdot)\right](x)dV_r \\ & \cdots & \\ u\psi_d(s,x) &=& P_{s,T}[g\psi_d(T,\cdot)](x) - \int_s^T P_{s,r}\left[\left(\mathfrak{G}(u,\psi_d) + ua(\psi_d) - \psi_d f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)\right)(r,\cdot)\right](x)dV_r. \end{array}\right. \tag{4.5.11}$$

Combining Definitions 4.4.11, 4.4.13, 4.5.8, we know that on $[s,T]$, the process $u(\cdot,X_\cdot)$ has a modification which we denote $U^{s,x}$ which is a special semimartingale with decomposition

$$U^{s,x} = u(s,x) - \int_s^\cdot f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)(r,X_r)dV_r + M[u]^{s,x}, \tag{4.5.12}$$

and $M[u]^{s,x} \in \mathcal{H}_0^2$.
Definition 4.5.8 also states that $u(T,\cdot) = g$, implying that

$$u(s,x) = g(X_T) + \int_s^T f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)(r,X_r)dV_r - M[u]_T^{s,x} \text{ a.s.} \tag{4.5.13}$$

Taking the expectation, by Fubini's theorem we get

$$\begin{array}{rcl} u(s,x) &=& \mathbb{E}^{s,x}\left[g(X_T) + \int_s^T f\left(\cdot,\cdot,u,\mathfrak{G}^\psi(u)\right)(r,X_r)dV_r\right] \\ &=& P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(r,\cdot,u(r,\cdot),\mathfrak{G}^\psi(u)(r,\cdot)\right)\right](x)dV_r. \end{array} \tag{4.5.14}$$

We now fix $i \in [\![1;d]\!]$. By integration by parts, taking (4.5.12) and Definition 4.4.1 into account, we obtain

$$\begin{aligned} d(U_t^{s,x}\psi_i(t,X_t)) &= -\psi_i(t,X_t)f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)(t,X_t)dV_t + \psi_i(t^-,X_{t^-})dM[u]_t^{s,x} \\ &\quad + U_t^{s,x}a(\psi_i)(t,X_t)dV_t + U_{t^-}^{s,x}dM[\psi_i]_t^{s,x} + d[M[u]^{s,x},M[\psi_i]^{s,x}]_t, \end{aligned} \tag{4.5.15}$$

so integrating from s to T, we get

$$\begin{aligned} & u\psi_i(s,x) \\ = \;& g(X_T)\psi_i(T,X_T) + \int_s^T \psi_i(t,X_t)f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)(r,X_r)dV_r - \int_s^T \psi_i(r^-,X_{r^-})dM[u]_r^{s,x} \\ & - \int_s^T U_t^{s,x}a(\psi_i)(r,X_r)dV_r - \int_s^T U_{r^-}^{s,x}dM[\psi_i]_r^{s,x} - [M[u]^{s,x},M[\psi_i]^{s,x}]_T \\ = \;& g(X_T)\psi_i(T,X_T) - \int_s^T \left(ua(\psi_i) - \psi_i f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)\right)(r,X_r)dV_r - \int_s^T \psi_i(r^-,X_{r^-})dM[u]_r^{s,x} \\ & - \int_s^T U_{r^-}^{s,x}dM[\psi_i]_r^{s,x} - [M[u]^{s,x},M[\psi_i]^{s,x}]_T, \end{aligned} \tag{4.5.16}$$

where the latter equality is a consequence of Lemma 4.2.4.
The next step will consist in taking the expectation in equation (4.5.16), but before, we will check that $\int_s^{\cdot} U_{r^-}^{s,x}dM[\psi_i]_r^{s,x}$ and $\int_s^{\cdot} \psi_i(r^-,X_{r^-})dM[u]_r^{s,x}$ are martingales.
By Proposition 4.4.15, $\langle M[\psi_i]^{s,x}\rangle = \int_s^{\cdot\vee s} \mathfrak{G}^{\psi_i}(\psi_i)(r,X_r)dV_r$. So the angular bracket of $\int_s^{\cdot} U_{r^-}^{s,x}dM[\psi_i]_r^{s,x}$ at time T is equal to $\int_s^T u^2\mathfrak{G}^{\psi_i}(\psi_i)(r,X_r)dV_r$ which is an integrable r.v. since $\mathfrak{G}^{\psi_i}(\psi_i)$ is bounded and $u \in \mathcal{L}_X^2$. Therefore $\int_s^{\cdot} U_{r^-}^{s,x}dM[\psi_i]_r^{s,x}$ is a square integrable martingale.
Then, by Hypothesis 4.5.1 and Proposition 4.5.2, $\sup_{t\in[s,T]} |\psi_i(t,X_t)|^2 \in L^1$, and by Definition 4.4.13, $M[u]^{s,x} \in \mathcal{H}^2$ so by Lemma 2.3.17 in Chapter 2, $\int_s^{\cdot} \psi_i(r^-,X_{r^-})dM[u]_r^{s,x}$ is a martingale.

We can now take the expectation in (4.5.16), to get

$$\begin{aligned} & u\psi_i(s,x) \\ = \;& \mathbb{E}^{s,x}\left[g(X_T)\psi_i(T,X_T) - \int_s^T \left(ua(\psi_i) - \psi_i f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)\right)(r,X_r)dV_r - [M[u]^{s,x},M[\psi_i]^{s,x}]_T\right] \\ = \;& \mathbb{E}^{s,x}\left[g(X_T)\psi_i(T,X_T) - \int_s^T \left(ua(\psi_i) + \mathfrak{G}^{\psi_i}(u) - \psi_i f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)\right)(r,X_r)dV_r\right], \end{aligned} \tag{4.5.17}$$

since u and ψ_i belong to $\mathcal{D}(\mathfrak{a})$. Indeed the second equality follows from the fact $[M[u]^{s,x},M[\psi_i]^{s,x}] - \langle M[u]^{s,x},M[\psi_i]^{s,x}\rangle$ is a martingale and Proposition 4.4.15.

Since we have assumed that $u \in \mathcal{L}_X^2$, Lemma 4.5.14 says that $f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right) \in \mathcal{L}_X^2$, Hypothesis 4.5.1 implies that ψ_i and $a(\psi_i)$ are in $\mathcal{L}_X^2$, so all terms in the integral inside the expectation in the third line belong to $\mathcal{L}_X^1$. We can therefore apply Fubini's theorem to get

$$u\psi_i(s,x) = P_{s,T}[g\psi_i(T,\cdot)](x) - \int_s^T P_{s,r}\left[\left(ua(\psi_i) + \mathfrak{G}^{\psi_i}(u) - \psi_i f\left(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)\right)\right)(r,\cdot)\right](x)dV_r. \tag{4.5.18}$$

This concludes the proof. □

Proposition 4.5.15 admits a converse implication.

Proposition 4.5.16. *Let (f,g) verify H^{growth}. Every decoupled mild solution of $Pseudo-PDE(f,g)$ is a also a martingale solution. Moreover, if (u,v) solves $IP(f,g)$, then $v = \mathfrak{G}^{\psi}(u)$, up to zero potential sets.*

Proof. Let u and v_i, $i \in [\![1;d]\!]$ in $\mathcal{L}_X^2$ verify (4.5.9). We observe that the first line of (4.5.9) with $s = T$, gives $u(T,\cdot) = g$.

We fix $(s,x) \in [0,T] \times E$ and the associated probability $\mathbb{P}^{s,x}$. On $[s,T]$, we set $U := u(\cdot, X)$ and $N := u(\cdot, X) - u(s,x) + \int_s^{\cdot} f(r, X_r, u(r, X_r), v(r, X_r))dV_r$.

For some $t \in [s,T]$, we combine the first line of (4.5.9) applied in $(s,x) = (t, X_t)$ and the Markov property (4.C.3). Since $f(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}_X^2$ (see Lemma 4.5.14) we get the a.s. equalities

$$\begin{array}{rcl} U_t & = & u(t, X_t) \\ & = & P_{t,T}[g](X_t) + \int_t^T P_{t,r}\left[f\left(r, \cdot, u(r,\cdot), v(r,\cdot)\right)\right](X_t)dV_r \\ & = & \mathbb{E}^{t,X_t}\left[g(X_T) + \int_t^T f(r, X_r, u(r, X_r), v(r, X_r))dV_r\right] \\ & = & \mathbb{E}^{s,x}\left[g(X_T) + \int_t^T f(r, X_r, u(r, X_r), v(r, X_r))dV_r | \mathcal{F}_t\right], \end{array} \tag{4.5.19}$$

from which we deduce that $N_t = \mathbb{E}^{s,x}\left[g(X_T) + \int_s^T f(r, X_r, u(r, X_r), v(r, X_r))dV_r | \mathcal{F}_t\right] - u(s,x)$ a.s. and so N is a martingale. We can therefore consider on $[s,T]$ and under $\mathbb{P}^{s,x}$, the cadlag version $N^{s,x}$ of N. We extend now $N_t^{s,x}$, to $t \in [0,T]$, putting its value equal to zero on $[0,s]$, and consider the special semimartingale

$$U^{s,x} := u(s,x) - \int_s^{\cdot} f(r, X_r, u(r, X_r), v(r, X_r))dV_r + N^{s,x}, \tag{4.5.20}$$

indexed on $[s,T]$ which is obviously a cadlag version of U.

By Jensen's inequality, we have

$$\begin{array}{rcl} \mathbb{E}^{s,x}[(N_T^{s,x})^2] & = & \mathbb{E}^{s,x}\left[\left(g(X_T) + \int_s^T f(r, X_r, u(r, X_r), v(r, X_r))dV_r - u(s,x)\right)^2\right] \\ & \leq & 3u^2(s,x) + 3\mathbb{E}^{s,x}[g^2(X_T)] + 3\mathbb{E}^{s,x}\left[\int_s^T f^2(r, X_r, u(r, X_r), v(r, X_r))dV_r\right] \\ & < & \infty, \end{array} \tag{4.5.21}$$

where the second term is finite because of H^{growth}, and the third one because $f(\cdot,\cdot,u,v)$ belongs to $\mathcal{L}_X^2$, see Lemma 4.5.14. So $N^{s,x}$ is square integrable. We have therefore shown that under any $\mathbb{P}^{s,x}$, the process $\left(u(\cdot, X_\cdot) - u(s,x) + \int_s^{\cdot} f(r, X_r, u(r, X_r), v(r, X_r))dV_r\right)\mathbb{1}_{[s,T]}$ has a modification in $\mathcal{H}_0^2$. According to Definitions 4.4.11 and 4.4.13 we have $u \in \mathcal{D}(\mathfrak{a})$, $\mathfrak{a}(u) = -f(\cdot,\cdot,u,v)$ and for any $(s,x) \in [0,T] \times E$, $M[u]^{s,x} = N^{s,x}$ in the sense of $P^{s,x}$-indistinguishability.
So to conclude that u is a martingale solution of $Pseudo-PDE(f,g)$, there is left to show $\mathfrak{G}^{\psi}(u) = v$, up to zero potential sets. By Proposition 4.4.15, this is equivalent to show that for every $(s,x) \in [0,T] \times E$ and $i \in [\![1;d]\!]$,

$$\langle M^{s,x}[u], M^{s,x}[\psi_i]\rangle = \int_s^{\cdot \vee s} v_i(r, X_r)dV_r, \tag{4.5.22}$$

in the sense of indistinguishability.

We fix again $(s,x) \in [0,T] \times E$, the associated probability, and some $i \in [\![1;d]\!]$. Combining the $(i+1)$th line of (4.5.9) applied in $(s,x) = (t, X_t)$ and the Markov property (4.C.3), taking into account the fact that all terms belong to $\mathcal{L}_X^1$ (see Lemma 4.5.14, Hypothesis 4.5.1) we get the a.s. equalities

$$\begin{array}{rcl} u\psi_i(t, X_t) & = & P_{t,T}[g\psi_i(T,\cdot)](X_t) - \int_t^T P_{t,r}\left[\left(v_i + ua(\psi_i) - \psi_i f\left(\cdot,\cdot,u,v\right)\right)(r,\cdot)\right](X_t)dV_r \\ & = & \mathbb{E}^{t,X_t}\left[g(X_T)\psi_i(T, X_T) - \int_t^T \left(v_i + ua(\psi_i) - \psi_i f\left(\cdot,\cdot,u,v\right)\right)(r, X_r)dV_r\right] \\ & = & \mathbb{E}^{s,x}\left[g(X_T)\psi_i(T, X_T) - \int_t^T \left(v_i + ua(\psi_i) - \psi_i f\left(\cdot,\cdot,u,v\right)\right)(r, X_r)dV_r | \mathcal{F}_t\right]. \end{array} \tag{4.5.23}$$

Setting, for $t \in [s,T]$, $N_t^i := u\psi_i(t,X_t) - \int_s^t (v_i + ua(\psi)^i - \psi_i f(\cdot,\cdot,u,v))(r,X_r)dV_r$, from (4.5.23) we deduce that, for any $t \in [s,T]$,

$$N_t^i = \mathbb{E}^{s,x}\left[g(X_T)\psi_i(T,X_T) - \int_s^T \left(v_i + ua(\psi_i) - \psi_i f\left(\cdot,\cdot,u,v\right)\right)(r,X_r)dV_r \middle| \mathcal{F}_t \right]$$

a.s. So N^i is a martingale. We can therefore consider on $[s,T]$ and under $\mathbb{P}^{s,x}$, the cadlag version $N^{i,s,x}$ of N^i.

The process

$$\int_s^\cdot \left(v_i + ua(\psi_i) - \psi_i f\left(\cdot,\cdot,u,v\right)\right)(r,X_r)dV_r + N^{i,s,x}, \tag{4.5.24}$$

is therefore a cadlag special semimartingale which is a $\mathbb{P}^{s,x}$-version of $u\psi_i(\cdot,X)$ on $[s,T]$. But we also had shown, see (4.5.20), that
$U^{s,x} = u(s,x) - \int_s^\cdot f(r,X_r,u(r,X_r),v(r,X_r))dV_r + N^{s,x}$ is a version of $u(\cdot,X)$ which by integration by parts on the process $U^{s,x}\psi_i(\cdot,X_\cdot)$ implies that

$$\begin{array}{l} u\psi_i(s,x) + \int_s^\cdot U_r^{s,x} a(\psi_i)(r,X_r)dV_r + \int_s^\cdot U_{r^-}^{s,x} dM^{s,x}[\psi_i]_r \\ - \int_s^\cdot \psi_i f(\cdot,\cdot,u,v)(r,X_r)dV_r + \int_s^\cdot \psi_i(r^-,X_{r^-})dM^{s,x}[u]_r + [M^{s,x}[u], M^{s,x}[\psi_i]] \end{array} \tag{4.5.25}$$

is another cadlag semimartingale which is a $\mathbb{P}^{s,x}$-version of $u\psi_i(\cdot,X)$ on $[s,T]$. Now (4.5.25) equals

$$\mathcal{M}^i + \mathcal{V}^i, \tag{4.5.26}$$

where

$$\begin{array}{rcl} \mathcal{M}_t^i & = & u\psi_i(s,x) + \displaystyle\int_s^t U_{r^-}^{s,x} dM^{s,x}[\psi_i]_r + \int_s^t \psi_i(r^-,X_{r^-})dM^{s,x}[u]_r \\ & + & ([M^{s,x}[u], M^{s,x}[\psi_i]]_t - \langle M^{s,x}[u], M^{s,x}[\psi_i]\rangle_t, \end{array}$$

is a local martingale and

$$\mathcal{V}_t^i = \langle M^{s,x}[u], M^{s,x}[\psi_i]\rangle_t + \int_s^t U_r^{s,x} a(\psi_i)(r,X_r)dV_r - \int_s^t \psi_i f(\cdot,\cdot,u,v)(r,X_r)dV_r,$$

is a predictable with bounded variation vanishing at zero process. Now (4.5.26) and (4.5.24) are two cadlag versions of $u\psi_i(\cdot,X)$ on $[s,T]$.

By the uniqueness of the decomposition of a special semimartingale, identifying the bounded variation predictable components and using Lemma 4.2.4 we get

$$\begin{array}{rcl} \displaystyle\int_s^\cdot (v_i & + & ua(\psi_i) - \psi_i f(\cdot,\cdot,u,v))(r,X_r)dV_r \\ & = & \langle M^{s,x}[u], M^{s,x}[\psi_i]\rangle + \displaystyle\int_s^\cdot ua(\psi_i)(r,X_r)dV_r - \int_s^\cdot \psi_i f(\cdot,\cdot,u,v)(r,X_r)dV_r. \end{array}$$

This yields $\langle M^{s,x}[u], M^{s,x}[\psi_i]\rangle = \int_s^{\cdot\vee s} v_i(r,X_r)dV_r$ as desired, which implies (4.5.22).

□

Proposition 4.5.17. *Let (f,g) verify H^{growth}. A classical solution of $Pseudo{-}PDE(f,g)$ is also a decoupled mild solution.*

Conversely, a decoupled mild solution of $Pseudo - PDE(f,g)$ belonging to $\mathcal{D}(\Gamma^\psi)$ is a classical solution of $Pseudo - PDE(f,g)$ up to a zero-potential set, meaning that it verifies the first equality of (4.5.1) *up to a set of zero potential.*

Proof. Let u be a classical solution of $Pseudo-PDE(f,g)$. Definition 4.5.3 and Corollary 4.4.17 imply that $u(T,\cdot)=g$, and the equalities up to zero potential sets

$$\mathfrak{a}(u)=a(u)=-f(\cdot,\cdot,u,\Gamma^{\psi}(u))=-f(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u)), \tag{4.5.27}$$

which shows that u is a martingale solution and by Proposition 4.5.15 it is also a decoupled mild solution.

Similarly, the second statement follows by Proposition 4.5.16, Definition 4.5.8, and again Corollary 4.4.17. □

4.5.3 Existence and uniqueness of a decoupled mild solution

In this subsection, the functions (f,g) appearing in $Pseudo-PDE(f,g)$ are fixed and verify H^{lip}.

Using arguments which are very close to those developed in Chapter 2, one can show the following theorem. For the convenience of the reader, we postpone the adapted proof to Appendix 4.B.

Let $(Y^{s,x},M^{s,x})$ be for any $(s,x)\in[0,T]\times E$ the unique solution of (4.5.2), see Notation 4.5.6.

Theorem 4.5.18. *Let (f,g) verify H^{lip}. There exists $u\in\mathcal{D}(\mathfrak{a})$ such that for any $(s,x)\in[0,T]\times E$*

$$\left\{\begin{array}{rcl}\forall t\in[s,T]: Y_t^{s,x} &=& u(t,X_t)\quad \mathbb{P}^{s,x}\text{a.s.}\\ M^{s,x} &=& M[u]^{s,x},\end{array}\right.$$

and in particular $\frac{d\langle M^{s,x},M[\psi]^{s,x}\rangle}{dV}=\mathfrak{G}^{\psi}(u)(\cdot,X.)\,dV\otimes d\mathbb{P}^{s,x}$ a.e. on $[s,T]$. Moreover, for every (s,x), $Y_s^{s,x}$ is $\mathbb{P}^{s,x}$ a.s. equal to a constant (which we shall still denote $Y_s^{s,x}$) and $u(s,x)=Y_s^{s,x}$ for every $(s,x)\in[0,T]\times E$.

Corollary 4.5.19. *Let (f,g) verify H^{lip}. For any $(s,x)\in[0,T]\times E$, the functions u obtained in Theorem 4.5.18 verifies $\mathbb{P}^{s,x}$ a.s. on $[s,T]$*

$$u(t,X_t)=g(X_T)+\int_t^T f\left(r,X_r,u(r,X_r),\mathfrak{G}^{\psi}(u)(r,X_r)\right)dV_r-(M[u]_T^{s,x}-M[u]_t^{s,x}),$$

and in particular, $\mathfrak{a}(u)=-f(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u))$.

Proof. The corollary follows from Theorem 4.5.18 and Lemma 4.2.4. □

Theorem 4.5.20. *Let $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ be a Markov class associated to a transition kernel measurable in time (see Definitions 4.C.5 and 4.C.4) which solves a martingale problem associated with the triplet $(\mathcal{D}(a),a,V)$. Let (f,g) be a couple verifying H^{lip}.*

Then $Pseudo-PDE(f,g)$ has a unique decoupled mild solution given by

$$u:\begin{array}{ccc}[0,T]\times E &\longrightarrow& \mathbb{R}\\ (s,x) &\longmapsto& Y_s^{s,x},\end{array} \tag{4.5.28}$$

where $(Y^{s,x},M^{s,x})$ denotes the (unique) solution of $BSDE^{s,x}(f,g)$ for fixed (s,x).

Proof. Let u be the function exhibited in Theorem 4.5.18. In order to show that u is a decoupled mild solution of $Pseudo-PDE(f,g)$, it is enough by Proposition 4.5.15 to show that it is a martingale solution.
In Corollary 4.5.19, we have already seen that $\mathfrak{a}(u)=-f(\cdot,\cdot,u,\mathfrak{G}^{\psi}(u))$.
Concerning the second line of (4.5.3), for any $x\in E$, we have

$u(T,x) = u(T,X_T) = g(X_T) = g(x)$ $\mathbb{P}^{T,x}$ a.s., so $u(T,\cdot) = g$, in the deterministic pointwise sense.
We now show uniqueness. By Proposition 4.5.16, it is enough to show that $Pseudo - PDE(f,g)$ admits at most one martingale solution. Let u, u' be two martingale solutions of $Pseudo - PDE(f,g)$. We fix $(s,x) \in [0,T] \times E$. By Proposition 4.5.10, both couples, indexed by $[s,T]$,
$\big(u(\cdot,X), \quad u(\cdot,X) - u(s,x) + \int_s^\cdot f(\cdot,\cdot,u,\mathfrak{G}^\psi(u))(r,X_r)dV_r\big)$ and
$\big(u'(\cdot,X), \quad u'(\cdot,X) - u'(s,x) + \int_s^\cdot f(\cdot,\cdot,u',\mathfrak{G}^\psi(u'))(r,X_r)dV_r\big)$ admit a $\mathbb{P}^{s,x}$-version which solves $BSDE^{s,x}(f,g)$ on $[s,T]$. By Theorem 4.3.3 and Remark 4.3.4, $BSDE^{s,x}(f,g)$ admits a unique solution, so $u(\cdot,X_\cdot)$ and $u'(\cdot,X_\cdot)$ are $\mathbb{P}^{s,x}$-modifications one of the other on $[s,T]$. In particular, considering their values at time s, we have $u(s,x) = u'(s,x)$. We therefore have $u' = u$. □

Corollary 4.5.21. *Let (f,g) verify H^{lip}. There is at most one classical solution of $Pseudo - PDE(f,g)$ and this only possible classical solution is the unique decoupled mild solution $(s,x) \longmapsto Y_s^{s,x}$, where $(Y^{s,x}, M^{s,x})$ denotes the (unique) solution of $BSDE^{s,x}(f,g)$ for fixed (s,x).*

Proof. The proof follows from Proposition 4.5.17 and Theorem 4.5.20. □

Remark 4.5.22. *The function v such that (u,v) is the unique solution of the identification problem $IP(f,g)$ also has a stochastic representation since it verifies for every $(s,x) \in [0,T] \times E$, on the interval $[s,T]$, $\frac{d\langle M^{s,x}, M^{s,x}[\psi]\rangle}{dV} = v(\cdot,X_\cdot)\, dV \otimes d\mathbb{P}^{s,x}$ a.e. where $M^{s,x}$ is the martingale part of the solution of $BSDE^{s,x}(f,g)$.*

The existence of a decoupled mild solution of $Pseudo - PDE(f,g)$ provides in fact an existence theorem for $BSDE^{s,x}(f,g)$ for any (s,x). The following constitutes the converse of Theorem 4.5.20.

Proposition 4.5.23. *Assume (f,g) verifies H^{growth}. Let (u,v) be a solution of $IP(f,g)$, let $(s,x) \in [0,T] \times E$ and the associated probability $\mathbb{P}^{s,x}$ be fixed. The couple*

$$\left(u(t,X_t), \quad u(t,X_t) - u(s,x) + \int_s^t f(\cdot,\cdot,u,v)(r,X_r)dV_r\right)_{t\in[s,T]} \tag{4.5.29}$$

has a $\mathbb{P}^{s,x}$-version which solves $BSDE^{s,x}(f,g)$ on $[s,T]$.

In particular if (f,g) verifies the stronger hypothesis H^{lip} and (u,v) is the unique solution of $IP(f,g)$, then for any $(s,x) \in [0,T] \times E$,
$\Big(u(t,X_t), \quad u(t,X_t) - u(s,x) + \int_s^t f(\cdot,\cdot,u,v)(r,X_r)dV_r\Big)_{t\in[s,T]}$ is a $\mathbb{P}^{s,x}$ modification of the unique solution of $BSDE^{s,x}(f,g)$ on $[s,T]$.

Proof. It follows from Propositions 4.5.16, and 4.5.10.

□

4.6 Examples of applications

We now develop some examples. In all the items below there will be a canonical Markov class with transition kernel being measurable in time which is solution of a Martingale Problem associated to some triplet $(\mathcal{D}(a), a, V)$ as introduced in Definition 4.4.1. Therefore all the results of this chapter will apply to all the examples below. In particular, Propositions 4.5.16, 4.5.17, Theorem 4.5.20, Corollary 4.5.21 and Proposition 4.5.23 will apply but we will mainly emphasize Theorem 4.5.20 and Corollary 4.5.21.

In all the examples $T > 0$ will be fixed.

4.6.1 A new approach to Brownian BSDEs and associate semilinear PDEs

In this subsection, the state space will be $E := \mathbb{R}^d$ for some $d \in \mathbb{N}^*$.

Notation 4.6.1. *A function $\phi \in \mathcal{B}([0,T] \times \mathbb{R}^d, \mathbb{R})$ will be said to have **polynomial growth** if there exists $p \in \mathbb{N}$ and $C > 0$ such that for every $(t,x) \in [0,T] \times \mathbb{R}^d$, $|\phi(t,x)| \leq C(1 + \|x\|^p)$. For any $k, p \in \mathbb{N}$, $\mathcal{C}^{k,p}([0,T] \times \mathbb{R}^d)$ (resp. $\mathcal{C}_b^{k,p}([0,T] \times \mathbb{R}^d)$, resp. $\mathcal{C}_{pol}^{k,p}([0,T] \times \mathbb{R}^d)$) will denote the sublinear algebra of $\mathcal{C}([0,T] \times \mathbb{R}^d, \mathbb{R})$ of functions admitting continuous (resp. bounded continuous, resp. continuous with polynomial growth) derivatives up to order k in the first variable and order p in the second.*

We consider bounded Borel functions $\beta \in \mathcal{B}_b([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ and $\alpha \in \mathcal{B}_b([0,T] \times \mathbb{R}^d, S_d^+(\mathbb{R}))$ where $S_d^+(\mathbb{R})$ is the space of symmetric non-negative $d \times d$ real matrices. We define for $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$ the operator a by

$$a(\phi) = \partial_t \phi + \frac{1}{2} \sum_{i,j \leq d} \alpha_{i,j} \partial^2_{x_i x_j} \phi + \sum_{i \leq d} \beta_i \partial_{x_i} \phi. \tag{4.6.1}$$

We will assume the following.

Hypothesis 4.6.2. *There exists a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x) \in [0,T] \times \mathbb{R}^d}$ which solves the Martingale Problem associated to $(\mathcal{C}_b^{1,2}([0,T] \times \mathbb{R}^d), a, V_t \equiv t)$ in the sense of Definition 4.4.1.*

We now recall a non-exhaustive list of sets of conditions on β, α under which Hypothesis 4.6.2 is satisfied.

1. α is continuous non-degenerate, in the sense that for any t, x, $\alpha(t,x)$ is invertible, see Theorem 4.2 in [84];
2. β and α are continuous in the second variable, see Exercise 12.4.1 in [85];
3. $d = 1$ and α is uniformly positive on compact sets, see Exercise 7.3.3 in [85].

Remark 4.6.3. *When the first or third item above is verified, the mentioned Markov class is unique, but if the second one holds, uniqueness may not hold. We therefore fix a Markov class which solves the martingale problem. We wish to emphasize that given a fixed Markov class, we will obtain some uniqueness results concerning the martingale solution and the decoupled mild solution of an associated PDE, but that for every Markov class solving the martingale problem may correspond a different solution.*

In this context, for ϕ, ψ in $\mathcal{D}(a)$, the carré du champs operator (see Definition 4.4.4) is given by $\Gamma(\phi, \psi) = \sum_{i,j \leq d} \alpha_{i,j} \partial_{x_i} \phi \partial_{x_j} \psi$.

Remark 4.6.4. *By a localization procedure, it is also clear that for every $(s,x) \in [0,T] \times \mathbb{R}^d$, $\mathbb{P}^{s,x}$ verifies that for any $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$, $\phi(\cdot, X_\cdot) - \int_s^\cdot a(\phi)(r, X_r) dr \in \mathcal{H}^2_{loc}$ and that Proposition 4.4.5 extends to all $\phi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^d)$.*

We set now $\mathcal{D}(a) = \mathcal{C}_{pol}^{1,2}([0,T] \times \mathbb{R}^d)$.
For any $i \in [\![1; d]\!]$, the function Id_i denotes $(t,x) \longmapsto x_i$ which belongs to $\mathcal{D}(a)$ and $Id := (Id_1, \cdots, Id_d)$. It is clear that for any i, $a(Id_i) = \beta_i$, and for any i, j, $Id_i Id_j \in \mathcal{D}(a)$ and $\Gamma(Id_i, Id_j) = \alpha_{i,j}$. In particular, by Corollary 4.4.7, $(Id_1, \cdots, Id_d)$ verify Hypothesis 4.4.6 and since β, α are bounded, they verify Hypothesis 4.5.1.

For any i we can therefore consider the MAF $M[Id_i] : (t,u) \mapsto X^i_u - X^i_t - \int_t^u \beta_i(r, X_r) dr$ whose cadlag version under $\mathbb{P}^{s,x}$ for every $(s,x) \in [0,T] \times \mathbb{R}^d$ is $M[Id_i]^{s,x} = \mathbb{1}_{[s,T]} \left(X^i - x_i - \int_s^\cdot \beta_i(r, X_r) dr \right)$ and we have for any i, j $\langle M[Id_i]^{s,x}, M[Id_j]^{s,x} \rangle = \int_s^{\cdot \vee s} \alpha_{i,j}(r, X_r) dr$.

Lemma 4.6.5. *Let $(s,x) \in [0,T] \times \mathbb{R}^d$ and associated probability $\mathbb{P}^{s,x}$, $i \in [\![1;d]\!]$ and $p \in [1,+\infty[$ be fixed. Then $\sup_{t\in[s,T]} |X_t^i|^p \in L^1$.*

Proof. We have $X^i = x_i + \int_s^\cdot \beta_i(r,X_r)dr + M[Id_i]^{s,x}$ where β_i is bounded so it is enough to show that $\sup_{t\in[s,T]} |M[Id_i]_t^{s,x}|^p \in L^1$. Since $\langle M[Id_i]^{s,x}\rangle = \int_s^{\cdot\vee s} \alpha_{i,i}(r,X_r)dr$, which is bounded, the result holds by Burkholder-Davis-Gundy inequality. □

Corollary 4.6.6. *$(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ solves the Martingale Problem associated to $(\mathcal{C}^{1,2}_{pol}([0,T]\times\mathbb{R}^d), a, V_t \equiv t)$ in the sense of Definition 4.4.1.*

Proof. By Remark 4.6.4, for any $\phi \in \mathcal{C}^{1,2}_{pol}([0,T]\times\mathbb{R}^d)$ and $(s,x)\in[0,T]\times\mathbb{R}^d$, $\phi(\cdot,X_\cdot) - \int_s^\cdot a(\phi)(r,X_r)dr$ is a $\mathbb{P}^{s,x}$-local martingale. Since ϕ and $a(\phi)$ have polynomial growth, Lemma 4.6.5 and Jensen's inequality imply that it is also a square integrable martingale. □

Let $g : \mathbb{R}^d \longmapsto \mathbb{R}$ an $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d \longmapsto \mathbb{R}$. We consider the PDE

$$\begin{cases} \partial_t u + \frac{1}{2}\sum_{i,j\le d} \alpha_{i,j}\partial^2_{x_ix_j}u + \sum_{i\le d}\beta_i\partial_{x_i}u + f(\cdot,\cdot,u,\alpha\nabla u) = 0 \\ u(T,\cdot) = g. \end{cases} \tag{4.6.2}$$

We emphasize that for $u \in \mathcal{C}^{1,2}_{pol}([0,T]\times\mathbb{R}^d)$, $\alpha\nabla u = \Gamma^{Id}(u)$. The associated decoupled mild equation is given by

$$\begin{cases} u(s,x) &= P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(\cdot,\cdot,u,v\right)(r,\cdot)\right](x)dr \\ u(s,x)x_i &= P_{s,T}[gId_i](x) - \int_s^T P_{s,r}\left[(v_i + u\beta_i - Id_i f\left(\cdot,\cdot,u,v\right))(r,\cdot)\right](x)dr, i \in [\![1;d]\!], \end{cases} \tag{4.6.3}$$

$(s,x)\in[0,T]\times\mathbb{R}^d$, where P is the transition kernel of the Markov class.

Proposition 4.6.7. *Assume that Hypothesis 4.6.2 is verified, that $f(\cdot,\cdot,0,0), g$ have polynomial growth in x uniformly in t and that f verifies item 3 of H^{lip}.*

Then equation (4.6.2) *has a unique decoupled mild solution u.*

Moreover it has at most one classical solution which (when it exists) equals this function u.

Proof. $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ verifies a martingale problem in the sense of Definition 4.4.1 and has a transition kernel which is measurable in time. Moreover $(Id_1,\cdots,Id_d)$ verify Hypothesis 4.5.1. By Lemma 4.6.5, the polynomial growth of f,g imply that they verify items 1 and 2 of H^{lip}, which third item is also assumed to hold. So Theorem 4.5.20 and Corollary 4.5.21 apply. □

Remark 4.6.8. *The unique decoupled mild solution mentioned in the previous proposition admits the probabilistic representation given in Theorem 4.5.20.*

Remark 4.6.9. *In the classical literature, a Brownian BSDE is linked to a slightly different type of parabolic PDE, see the introduction of the present paper, or [72] for more details.*

The PDE which is generally considered is of the type

$$\begin{cases} \partial_t u + \frac{1}{2}\sum_{i,j\le d} (\sigma\sigma^\intercal)_{i,j}\partial^2_{x_ix_j}u + \sum_{i\le d}\beta_i\partial_{x_i}u + f(\cdot,\cdot,u,\sigma\nabla u) = 0 \\ u(T,\cdot) = g, \end{cases} \tag{4.6.4}$$

(where $\sigma = \sqrt{\alpha}$ in the sense of non-negative symmetric matrices) rather than (4.6.2)*. In fact, the only difference is that the term $\sigma\nabla u$ replaces $\alpha\nabla u$ in the fourth argument of the driver f.*

We recall that the Markovian BSDE was given in (2)*.*

Under the probability $\mathbb{P}^{s,x}$ (for some fixed (s,x)), one can introduce the square integrable martingale $\tilde{M}[Id]^{s,x} := \int_s^{\cdot}(\sigma^\intercal)^+(r,X_r)dM[Id]_r^{s,x}$ where $A \mapsto A^+$ denotes the Moore-Penrose pseudo-inverse operator, see [15] chapter 1. The Brownian BSDE (2) *can then be reexpressed here as*

$$Y_t^{s,x} = g(X_T) + \int_t^T f\left(r, X_r, Y_r^{s,x}, \frac{d\langle M^{s,x}, \tilde{M}[Id]^{s,x}\rangle_r}{dr}\right) dr - (M_T^{s,x} - M_t^{s,x}). \tag{4.6.5}$$

Under the assumptions of Proposition 4.6.7 where $\alpha = \sigma\sigma^\intercal$), it is possible to show that (4.6.5) *constitutes the probabilistic representation of* (4.6.4) *performing similar arguments as in our approach for* (4.6.2)*. In particular we can show existence and uniqueness of a function $u \in \mathcal{L}_X^2$ for which there exists $v_1, \cdots, v_d \in \mathcal{L}_X^2$ such that for all $(s,x) \in [0,T]\times\mathbb{R}^d$,*

$$\left\{\begin{array}{rcl} u(s,x) & = & P_{s,T}[g](x) + \int_s^T P_{s,r}\left[f\left(\cdot,\cdot,u,(\sigma^\intercal)^+v\right)(r,\cdot)\right](x)dr \\ u(s,x)x_i & = & P_{s,T}[gId_i](x) - \int_s^T P_{s,r}\left[\left(v_i + u\beta_i - Id_i f\left(\cdot,\cdot,u,(\sigma^\intercal)^+v\right)\right)(r,\cdot)\right](x)dr, i \in [\![1;d]\!], \end{array}\right. \tag{4.6.6}$$

and that this function u is the only possible classical solution of (4.6.4) *in $\mathcal{C}_{pol}^{1,2}([0,T]\times\mathbb{R}^d)$.* (4.6.6) *constitutes the "good" version of decoupled mild solution for the* (4.6.4)*.*

This technique is however technically more complicated and for purpose of illustration we prefer to remain in our set up (which is by the way close to (4.6.4)*) to keep our notion of decoupled-mild solution more comprehensible.*

Remark 4.6.10. *It is also possible to treat jump diffusions instead of continuous diffusions (see [84]), and under suitable conditions on the coefficients, it is also possible to prove existence and uniqueness of a decoupled mild solution for equations of type*

$$\left\{\begin{array}{l} \partial_t u + \frac{1}{2}Tr(\alpha\nabla^2 u) + (\beta,\nabla u) + \int \left(u(\cdot,\cdot+\gamma(\cdot,\cdot,y)) - u - (\gamma(\cdot,\cdot,y),\nabla u)\right)F(dy) + f(\cdot,\cdot,u,\Gamma^{Id}(u)) = 0 \\ u(T,\cdot) = g, \end{array}\right. \tag{4.6.7}$$

where $\gamma : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \longmapsto \mathbb{R}^d$ and F is a bounded positive measure not charging 0. In that framework we have

$$\Gamma^{Id} : \phi \longmapsto \alpha\nabla\phi + \int \gamma(\cdot,\cdot,y)(\phi(\cdot,\cdot+\gamma(\cdot,\cdot,y)) - \phi(\cdot,\cdot))F(dy). \tag{4.6.8}$$

4.6.2 Parabolic semi-linear PDEs with distributional drift

In this subsection we will use the formalism and results obtained by in [47] and [48], see also [81] and [31] for more recent developments in dimension 1. Those authors make reference to stochastic differential equations with distributional drift whose solution are possibly non-semimartingales. Those papers introduced a suitable framework of Martingale Problem related to a PDE operator involving a distributional drift b' which is the derivative of a continuous function. [46] established a first work in the n-dimensional setting, later developments were discussed by [21]. Other Markov processes associated to diffusion operators which are not semimartingales were produced when the diffusion operator is in divergence form, see e.g. [80] or Markov processes associated to singular PDEs involving paracontrolled distributions introduced in [21].

Let $b, \sigma \in \mathcal{C}^0(\mathbb{R})$ such that $\sigma > 0$. By a mollifier, we intend a function $\Phi \in \mathcal{S}(\mathbb{R})$ with $\int \Phi(x)dx = 1$. We denote $\Phi_n(x) = n\Phi(nx)$, $\sigma_n^2 = \sigma^2 * \Phi_n$, $b_n = b * \Phi_n$.

We then define $L_n g = \frac{\sigma_n^2}{2} g'' + b_n' g'$. $f \in \mathcal{C}^1(\mathbb{R})$ is said to be a solution to $Lf = \dot{l}$ where $\dot{l} \in \mathcal{C}^0(\mathbb{R})$, if for any mollifier Φ, there are sequences (f_n) in $\mathcal{C}^2(\mathbb{R})$, $(\dot{l}_n)$ in $\mathcal{C}^0(\mathbb{R})$ such that

$$L_n f_n = (\dot{l}_n),\ f_n \overset{\mathcal{C}^1(\mathbb{R})}{\longrightarrow} f,\ \dot{l}_n \overset{\mathcal{C}^0(\mathbb{R})}{\longrightarrow} \dot{l}. \tag{4.6.9}$$

We will assume that $\Sigma(x) = \lim_{n\to\infty} 2\int_0^x \frac{b_n'}{\sigma_n^2}(y)dy$ exists in $\mathcal{C}^0(\mathbb{R})$ independently from the mollifier.

By Proposition 2.3 in [47] there exists a solution $h \in \mathcal{C}^1(\mathbb{R})$ to $Lh = 0, h(0) = 0, h'(0) = 1$. Moreover it verifies $h' = e^{-\Sigma}$.

$\mathcal{D}_L$ is defined as the set of $f \in \mathcal{C}^1(\mathbb{R})$ such that there exists some $\dot{l} \in \mathcal{C}^0(\mathbb{R})$ with $Lf = \dot{l}$ and it is a linear algebra.

Let v be the unique solution to $Lv = 1, v(0) = v'(0) = 0$ (see Remark 2.4 in [47]), we will assume that

$$v(-\infty) = v(+\infty) = +\infty, \tag{4.6.10}$$

which represents a non-explosion condition. In this case, Proposition 3.13 in [47] states that a certain martingale problem associated to $(\mathcal{D}_L, L)$ is well-posed. Its solution will be denoted $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$.

X is a $\mathbb{P}^{s,x}$-Dirichlet process for every (s,x) and $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ defines a Markov class and Proposition 3.B.2 in Chapter 3 implies that its transition kernel is measurable in time.

We introduce the domain that we will indeed use.

Definition 4.6.11. *We set*

$$\mathcal{D}^{max}(a) = \left\{ \phi \in \mathcal{C}^{1,1}([0,T]\times\mathbb{R}) : \frac{\partial_x \phi}{h'} \in \mathcal{C}^{1,1}([0,T]\times\mathbb{R}) \right\} \tag{4.6.11}$$

On $\mathcal{D}^{max}(a)$, we set $L\phi := \frac{\sigma^2 h'}{2}\partial_x(\frac{\partial_x\phi}{h'})$ and $a(\phi) := \partial_t\phi + L\phi$. We then define the smaller domain

$$\mathcal{D}(a) = \left\{ \phi \in \mathcal{D}^{max}(a) : \sigma\partial_x\phi \in \mathcal{C}^{0,0}_{pol}([0,T]\times\mathbb{R}) \right\}. \tag{4.6.12}$$

We formulate here some supplementary assumptions that we will make, the first one being called (TA) in [47].

Hypothesis 4.6.12.

- *There exists $c_1, C_1 > 0$ such that $c_1 \le \sigma h' \le C_1$;*
- *σ has linear growth.*

The first item states in particular that $\sigma h'$ is bounded so $h \in \mathcal{D}(a)$. Proposition 3.2 in [47] states that for every (s,x), $\langle M[h]^{s,x}\rangle = \int_s^{\cdot\vee s}(\sigma h')^2(X_r)dr$. Moreover the AF $\langle M[h]\rangle_u^t = \int_t^u (\sigma h')^2(X_r)dr$ is absolutely continuous with respect to $V_t \equiv t$. Therefore Hypothesis 4.4.6 is verified (for $\psi = h$) and $\mathfrak{G}^h(h) = (\sigma h')^2$. Since this function is bounded and clearly $a(h) = 0$ then h verifies Hypothesis 4.5.1.

We will therefore consider Γ^h the h-generalized gradient associated to a, and Proposition 3.4.23 in Chapter 3 implies the following.

Proposition 4.6.13. *Let $\phi \in \mathcal{D}(\Gamma^h)$, then $\Gamma^h(\phi) = \sigma^2 h' \partial_x \phi$.*

Remarking that b' is a distribution, the equation that we will study in this section is the following.

$$\begin{cases} \partial_t u + \frac{1}{2}\sigma^2\partial_x^2 u + b'\partial_x u + f(\cdot,\cdot,u,\sigma^2 h'\partial_x u) = 0 \quad \text{on } [0,T]\times\mathbb{R} \\ u(T,\cdot) = g. \end{cases} \tag{4.6.13}$$

The associated PDE in the decoupled mild sense is given by

$$\left\{\begin{array}{rcl} u(s,x) & = & P_{T-s}[g](x)+\int_s^T P_{r-s}\left[f\left(\cdot,\cdot,u,v\right)(r,\cdot)\right](x)dr \\ u(s,x)h(x) & = & P_{T-s}[gh](x)-\int_s^T P_{r-s}\left[\left(v-hf\left(\cdot,\cdot,u,v\right)\right)(r,\cdot)\right](x)dr, \end{array}\right. \tag{4.6.14}$$

$(s,x)\in[0,T]\times\mathbb{R}$, where P is the (homogeneous) transition kernel of the Markov class.

In order to consider the $BSDE^{s,x}(f,g)$ for functions (f,g) having polynomial growth in x, we had shown in Chapter 3 the following result, stated as Proposition 3.4.25.

Proposition 4.6.14. *We suppose that Hypothesis 4.6.12 is fulfilled. Then, for any $p\in\mathbb{N}$ and $(s,x)\in[0,T]\times\mathbb{R}$, $\mathbb{E}^{s,x}[|X_T|^p]<\infty$ and $\mathbb{E}^{s,x}[\int_s^T|X_r|^p dr]<\infty$. In other words, if $f(\cdot,\cdot,0,0),g$ have polynomial growth in x uniformly in t then they verify items 1 and 2 of H^{growth} or H^{lip}.*

Proposition 4.6.15. *We suppose that Hypothesis 4.6.12 is fulfilled. Then $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times\mathbb{R}^d}$ solves the Martingale Problem associated to $(a,\mathcal{D}(a),V_t\equiv t)$ in the sense of Definition 4.4.1.*

Proof. Let $(s,x)\in[0,T]\times\mathbb{R}$ be fixed. Proposition 3.4.24 in Chapter 3 implies that for any $\phi\in\mathcal{D}(a)$, $\phi(\cdot,X_\cdot)-\int_s^\cdot a(\phi)(r,X_r)dr$ is a (continuous) $\mathbb{P}^{s,x}$-local martingale, so taking Definition 4.4.1 into account, it is enough to show that this local martingale is a square integrable martingale. Considering Definition 3.4.22, Proposition 3.4.24 and Proposition 3.2.7 in Chapter 3, we know that the angular bracket of this local martingale is given by $\int_s^\cdot(\sigma\partial_x\phi)^2(r,X_r)dr$. Since $\phi\in\mathcal{D}(a)$ then $\sigma\partial_x\phi$ has polynomial growth, so by Proposition 4.6.14, $\int_s^T(\sigma\partial_x\phi)^2(r,X_r)dr\in L^1$ and this implies that the overmentioned local martingale is a square integrable martingale. □

We can now state the main result of this section.

Proposition 4.6.16. *Assume that the non-explosion condition* (4.6.10) *is verified, that Hypothesis 4.6.12 is fulfilled, that $f(\cdot,\cdot,0,0),g$ have polynomial growth in x uniformly in t and that f verifies item 3. of H^{lip}, see Hypothesis 4.5.4. Then equation* (4.6.13) *has a unique decoupled mild solution u. It has at most one classical solution which can only be equal to u.*

Remark 4.6.17. *The unique decoupled mild solution u can be of course represented by* (4.5.28)*, Theorem 4.5.20.*

Proof. The assertions come from Theorem 4.5.20 and Corollary 4.5.21 which applies thanks to Propositions 4.6.15, 4.6.14, and the fact that h verifies Hypothesis 4.5.1. □

Remark 4.6.18.

1. *[82] has made a first analysis linking elliptic PDEs (in fact second order ODEs) with distributional drift and BSDEs. In those BSDEs the final horizon was a stopping time.*
2. *[56] have considered a class of BSDEs involving distributions in their setting.*

Appendix

4.A Proof of Theorem 4.3.3 and related technicalities

We adopt here the same notations as at the beginning of Section 4.3. We will denote $L^2(d\hat{V} \otimes d\mathbb{P})$ the quotient space of $\mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$ with respect to the subspace of processes equal to zero $d\hat{V} \otimes d\mathbb{P}$ a.e. $L^2(d\hat{V} \otimes d\mathbb{P})$ is a Hilbert space equipped with its usual norm. $L^{2,cadlag}(d\hat{V} \otimes d\mathbb{P}))$ will denote the subspace of $L^2(d\hat{V} \otimes d\mathbb{P}))$ of elements having a cadlag representative. We emphasize that $L^{2,cadlag}(d\hat{V} \otimes d\mathbb{P})$ is not a closed subspace of $L^2(d\hat{V} \otimes d\mathbb{P})$.
The application which to a process associate its class will be denoted $\phi \mapsto \dot{\phi}$.

Proposition 4.A.1. *If (Y, M) solves $BSDE(\xi, \hat{f}, V, \hat{M})$, and if we denote $\hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right)$ by $\hat{f}_r$, then for any $t \in [0, T]$, a.s. we have*

$$\left\{ \begin{array}{rcl} Y_t & = & \mathbb{E}\left[\xi + \int_t^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_t\right] \\ M_t & = & \mathbb{E}\left[\xi + \int_0^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_0\right]. \end{array} \right. \tag{4.A.1}$$

Proof. Since $Y_t = \xi + \int_t^T \hat{f}_r d\hat{V}_r - (M_T - M_t)$ a.s., Y being an adapted process and M a martingale, taking the expectation in (4.3.2) at time t, we directly get $Y_t = \mathbb{E}\left[\xi + \int_t^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_t\right]$ and in particular that $Y_0 = \mathbb{E}\left[\xi + \int_0^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_0\right]$. Since $M_0 = 0$, looking at the BSDE at time 0 we get

$$M_T = \xi + \int_0^T \hat{f}_r d\hat{V}_r - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_0\right].$$

Taking the expectation with respect to $\mathcal{F}_t$ in the above inequality gives the second line of (4.A.1). □

Lemma 4.A.2. *Let $M \in \mathcal{H}^2$ and ϕ be a bounded positive process. Then there exists a constant $C > 0$ such that for any $i \in [\![1; d]\!]$,*
$\int_0^T \phi_r \left(\frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r \leq C \int_0^T \phi_r d\langle M\rangle_r$. In particular, $\frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}}$ belongs to $L^2(d\hat{V} \otimes d\mathbb{P})$.

Proof. We fix $i \in [\![1; d]\!]$. By Hypothesis 4.3.1 $\frac{d\langle \hat{M}^i\rangle}{d\hat{V}}$ is bounded; using Proposition 2.B.1 and Remark 2.3.3 in Chapter 2, we show the existence of $C > 0$ such that

$$\begin{array}{rcl} \int_0^T \phi_r \left(\frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r & \leq & \int_0^T \phi_r \frac{d\langle \hat{M}^i\rangle}{d\hat{V}}(r) \frac{d\langle M\rangle}{d\hat{V}}(r) d\hat{V}_r \\ & \leq & C \int_0^T \phi_r \frac{d\langle M\rangle}{d\hat{V}}(r) d\hat{V}_r \\ & \leq & C \int_0^T \phi_r d\langle M\rangle_r. \end{array} \tag{4.A.2}$$

In particular, setting $\phi = 1$, we have $\int_0^T \left(\frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r \leq C\langle M\rangle_T$ which belongs to L^1 since $M \in \mathcal{H}_0^2$. □

We fix for now a couple $(\dot{U}, N) \in L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ and we consider a representative U of $\dot{U}$. Until Proposition 4.A.6 included, we will use the notation $\hat{f}_r := \hat{f}\left(r, \cdot, U_r, \frac{d\langle N, \hat{M}\rangle}{d\hat{V}}(r)\right)$.

Proposition 4.A.3. *For any $t \in [0,T]$, $\int_t^T \hat{f}_r^2 d\hat{V}_r$ belongs to L^1 and $\left(\int_t^T \hat{f}_r d\hat{V}_r\right)$ is in L^2.*

Proof. By Jensen's inequality, thanks to the Lipschitz conditions on $\hat{f}$ in Hypothesis 4.3.1 and by Lemma 4.A.2 there exist positive constants $C, C', C"$ such that, for any $t \in [0,T]$, we have

$$\begin{aligned}\left(\int_t^T \hat{f}_r d\hat{V}_r\right)^2 &\leq C\int_t^T \hat{f}_r^2 d\hat{V}_r \\ &\leq C'\left(\int_t^T \hat{f}^2(r,\cdot,0,0)\, d\hat{V}_r + \int_t^T U_r^2 d\hat{V}_r + \sum_{i=1}^d \int_t^T \left(\frac{d\langle N, \hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r\right) \\ &\leq C"\left(\int_t^T \hat{f}^2(r,\cdot,0,0)\, d\hat{V}_r + \int_t^T U_r^2 d\hat{V}_r + (\langle N\rangle_T - \langle N\rangle_t)\right).\end{aligned} \tag{4.A.3}$$

All terms on the right-hand side are in L^1. Indeed, N is taken in $\mathcal{H}^2$, $\dot{U}$ in $L^2(d\hat{V} \otimes d\mathbb{P})$ and by Hypothesis 4.3.1, $f(\cdot,\cdot,0,0)$ is in $\mathcal{L}^2(d\hat{V} \otimes d\mathbb{P})$. This concludes the proof. □

We can therefore state the following definition.

Definition 4.A.4. *Let M be the cadlag version of the martingale $t \mapsto \mathbb{E}\left[\xi + \int_0^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\xi + \int_0^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_0\right]$.*

M is square integrable by Proposition 4.A.3. It admits a cadlag version taking into account Theorem 4 in Chapter IV of [33], since the stochastic basis fulfills the usual conditions. We denote by Y the cadlag process defined by
$Y_t = \xi + \int_t^T \hat{f}_r d\hat{V}_r - (M_T - M_t)$. This will be called the **cadlag reference process** *and we omit its dependence to $(\dot{U}, N)$.*

Proposition 4.A.5. *Y and M take square integrable values.*

Proof. We already know that M is a square integrable martingale. As we have seen in Proposition 4.A.3, $\int_t^T \hat{f}_r d\hat{V}_r$ belongs to L^2 for any $t \in [0,T]$ and by Hypothesis 4.3.1, $\xi \in L^2$. So by (4.A.1) and Jensen's inequality for conditional expectation we have

$$\begin{aligned}\mathbb{E}\left[Y_t^2\right] &= \mathbb{E}\left[\mathbb{E}\left[\xi + \int_t^T \hat{f}_r d\hat{V}_r \middle| \mathcal{F}_t\right]^2\right] \\ &\leq \mathbb{E}\left[\mathbb{E}\left[\left(\xi + \int_t^T \hat{f}_r d\hat{V}_r\right)^2 \middle| \mathcal{F}_t\right]\right] \\ &\leq \mathbb{E}\left[2\xi^2 + 2\int_t^T \hat{f}_r^2 d\hat{V}_r\right],\end{aligned}$$

which is finite. □

Proposition 4.A.6. $\sup_{t\in[0,T]} |Y_t| \in L^2$ *and in particular, $Y \in \mathcal{L}^{2,cadlag}(d\hat{V} \otimes \mathbb{P})$.*

Proof. Since $dY_r = -\hat{f}_r d\hat{V}_r + dM_r$, by integration by parts formula we get

$$d(Y_r^2 e^{-\hat{V}_r}) = -2e^{-\hat{V}_r} Y_r \hat{f}_r d\hat{V}_r + 2e^{-\hat{V}_r} Y_{r^-} dM_r + e^{-\hat{V}_r} d[M]_r - e^{-\hat{V}_r} Y_r^2 d\hat{V}_r.$$

So integrating from 0 to some $t \in [0,T]$, we get

$$\begin{aligned} Y_t^2 e^{-\hat{V}_t} &= Y_0^2 - 2\int_0^t e^{-\hat{V}_r} Y_r \hat{f}_r d\hat{V}_r + 2\int_0^t e^{-\hat{V}_r} Y_{r^-} dM_r \\ &\quad + \int_0^t e^{-\hat{V}_r} d[M]_r - \int_0^t e^{-\hat{V}_r} Y_r^2 d\hat{V}_r \\ &\leq Y_0^2 + \int_0^t e^{-\hat{V}_r} Y_r^2 d\hat{V}_r + \int_0^t e^{-\hat{V}_r} \hat{f}_r^2 d\hat{V}_r \\ &\quad + 2\left|\int_0^t e^{-\hat{V}_r} Y_{r^-} dM_r\right| + \int_0^t e^{-\hat{V}_r} d[M]_r - \int_0^t e^{-\hat{V}_r} Y_r^2 d\hat{V}_r \\ &\leq Z + 2\left|\int_0^t e^{-\hat{V}_r} Y_{r^-} dM_r\right|, \end{aligned}$$

where $Z = Y_0^2 + \int_0^T e^{-\hat{V}_r} \hat{f}_r^2 d\hat{V}_r + \int_0^T e^{-\hat{V}_r} d[M]_r$. Therefore, for any $t \in [0,T]$ we have $(Y_t e^{-\hat{V}_t})^2 \leq Y_t^2 e^{-\hat{V}_t} \leq Z + 2\left|\int_0^t e^{-\hat{V}_r} Y_{r^-} dM_r\right|$. Thanks to Propositions 4.A.3 and 4.A.5, Z is integrable, so we can conclude by Lemma 2.3.18 in Chapter 2 applied to the process $Y e^{-\hat{V}}$, and the fact that $\hat{V}$ is bounded.

Since Y is cadlag progressively measurable, $\sup_{t\in[0,T]} |Y_t| \in L^2$ and since $\hat{V}$ is bounded, it is clear that $Y \in \mathcal{L}^{2,cadlag}(d\hat{V} \otimes d\mathbb{P})$ and the corresponding class $\dot{Y}$ belongs to $L^{2,cadlag}(d\hat{V} \otimes d\mathbb{P})$. □

Thanks to Propositions 4.A.5 and 4.A.6, we are allowed to introduce the following operator.

Notation 4.A.7. *We denote by Φ the operator which associates to a couple $(\dot{U}, N)$ the couple $(\dot{Y}, M)$.*

$$\begin{array}{rccc} \Phi: & L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2 & \longrightarrow & L^{2,cadlag}(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2 \\ & (\dot{U}, N) & \longmapsto & (\dot{Y}, M). \end{array}$$

Proposition 4.A.8. *The mapping $(Y, M) \longmapsto (\dot{Y}, M)$ induces a bijection between the set of solutions of $BSDE(\xi, \hat{f}, \hat{V}, \hat{M})$ and the set of fixed points of Φ.*

Proof. First, let (U, N) be a solution of $BSDE(\xi, \hat{f}, V, \hat{M})$, let $(\dot{Y}, M) := \Phi(\dot{U}, N)$ and let Y be the reference cadlag process associated to U as in Definition 4.A.4. By this same definition, M is the cadlag version of $t \mapsto \mathbb{E}\left[\xi + \int_0^T \hat{f}\left(r, \cdot, U_r, \frac{d\langle N, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\xi + \int_0^T \hat{f}\left(r, \cdot, U_r, \frac{d\langle N, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r \middle| \mathcal{F}_0\right]$, but by Proposition 4.A.1, so is N, meaning $M = N$. Again by Definition 4.A.4, $Y = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, U_r, \frac{d\langle N, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (N_T - N_\cdot)$ which is equal to U thanks to (4.3.2), so $Y = U$ in the sense of indistinguishability. In particular, $\dot{U} = \dot{Y}$, implying $(\dot{U}, N) = (\dot{Y}, M) = \Phi(\dot{U}, N)$. Therefore, the mapping $(Y, M) \longmapsto (\dot{Y}, M)$ does indeed map the set of solutions of $BSDE(\xi, \hat{f}, V, \hat{M})$ into the set of fixed points of Φ.

The map Φ is surjective. Indeed let $(\dot{U}, N)$ be a fixed point of Φ, the couple (Y, M) of Definition 4.A.4 verifies $Y = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, U_r, \frac{d\langle N, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M_T - M_\cdot)$ in the sense of indistinguishability, and $(\dot{Y}, M) = \Phi(\dot{U}, N) = (\dot{U}, N)$, so by Lemma 2.3.9 in Chapter 2, $\int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r$ and $\int_\cdot^T \hat{f}\left(r, \cdot, U_r, \frac{d\langle N, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r$ are indistinguishable and $Y = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M_T - M_\cdot)$, meaning that (Y, M) (which is a preimage of $(\dot{U}, N)$) solves $BSDE(\xi, \hat{f}, V, \hat{M})$.

We finally show that it is injective. Let us consider two solutions (Y, M) and (Y', M) of $BSDE(\xi, \hat{f}, V, \hat{M})$ with $\dot{Y} = \dot{Y}'$. By Lemma 2.3.9 in Chapter 2 the processes $\int_\cdot^T \hat{f}\left(r, \cdot, Y_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r$ and $\int_\cdot^T \hat{f}\left(r, \cdot, Y'_r, \frac{d\langle M, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r$ are indistinguishable, so taking (4.3.2) into account, we have $Y = Y'$. □

Proposition 4.A.9. *Let $\lambda \in \mathbb{R}$, let $(\dot{U}, N)$, $(\dot{U}', N') \in L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$, let $(\dot{Y}, M)$, $(\dot{Y}', M')$ be their images through Φ and let Y, Y' be the cadlag representatives of $\dot{Y}$, $\dot{Y}'$ introduced in Definition 4.A.4. Then $\int_0^\cdot e^{\lambda \hat{V}_r} Y_{r-} dM_r$, $\int_0^\cdot e^{\lambda \hat{V}_r} Y'_{r-} dM'_r$, $\int_0^\cdot e^{\lambda \hat{V}_r} Y_{r-} dM'_r$ and $\int_0^\cdot e^{\lambda \hat{V}_r} Y'_{r-} dM_r$ are martingales.*

Proof. $\hat{V}$ is bounded and thanks to Proposition 4.A.6 we know that $\sup_{t\in[0,T]} |Y_t|$ and $\sup_{t\in[0,T]} |Y'_t|$ are L^2. Moreover, since M and M' are square integrable, the statement yields therefore as a consequence Lemma 2.3.17 in Chapter 2. □

Starting from now, if $(\dot{Y}, M)$ is the image by Φ of some $(\dot{U}, N) \in L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$, by default, we will always refer to the cadlag reference process Y of $\dot{Y}$ defined in Definition 4.A.4.

For any $\lambda \geq 0$, on $L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ we define the norm $\|(\dot{Y}, M)\|_\lambda^2 := \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} Y_r^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} d\langle M\rangle_r\right]$. Since $\hat{V}$ is bounded, these norms are all equivalent.

Proposition 4.A.10. *There exists $\lambda > 0$ such that for any $(\dot{U}, N) \in L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$, $\left\|\Phi(\dot{U}, N)\right\|_\lambda^2 \leq \frac{1}{2}\left\|(\dot{U}, N)\right\|_\lambda^2$. In particular, Φ is a contraction in $L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$ for the norm $\|\cdot\|_\lambda$.*

Proof. Let $(\dot{U}, N)$ and $(\dot{U}', N')$ be two couples belonging to $L^2(d\hat{V} \otimes d\mathbb{P}) \times \mathcal{H}_0^2$, let $(\dot{Y}, M)$ and $(\dot{Y}', M')$ be their images via Φ and let Y, Y' be the cadlag reference process of $\dot{Y}$, $\dot{Y}'$ introduced in Definition 4.A.4. We will write $\bar{Y}$ for $Y - Y'$ and we adopt a similar notation for other processes. We will also write

$$\bar{f}_t := \hat{f}\left(t, \cdot, U_t, \frac{d\langle N, \hat{M}\rangle}{d\hat{V}}(t)\right) - \hat{f}\left(t, \cdot, U'_t, \frac{d\langle N', \hat{M}\rangle}{d\hat{V}}(t)\right).$$

By additivity, we have $d\bar{Y}_t = -\bar{f}_t d\hat{V}_t + d\bar{M}_t$. Since $\bar{Y}_T = \xi - \xi = 0$, applying the integration by parts formula to $\bar{Y}_t^2 e^{\lambda \hat{V}_t}$ between 0 and T we get

$$\bar{Y}_0^2 - 2\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r \bar{f}_r d\hat{V}_r + 2\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_{r-} d\bar{M}_r + \int_0^T e^{\lambda \hat{V}_r} d[\bar{M}]_r + \lambda \int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r^2 d\hat{V}_r = 0.$$

Since, by Proposition 4.A.9, the stochastic integral with respect to $\bar{M}$ is a real martingale, by taking the expectations we get

$$\mathbb{E}\left[\bar{Y}_0^2\right] - 2\mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r \bar{f}_r d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} d\langle \bar{M}\rangle_r\right] + \lambda \mathbb{E}\left[\int_0^T e^{\lambda \hat{V}_r} \bar{Y}_r^2 d\hat{V}_r\right] = 0.$$

So by re-arranging previous expression, by the Lipschitz condition on $\hat{f}$ stated in Hypothesis 4.3.1, by the linearity of the Radon-Nikodym derivative stated in Proposition 4.2.3 and by Lemma 4.A.2, we get

$$
\begin{aligned}
& \lambda\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{Y}_r^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r} d\langle\bar{M}\rangle_r\right] \\
\leq\ & 2\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}|\bar{Y}_r||\bar{f}_r| d\hat{V}_r\right] \\
\leq\ & 2K^Y\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}|\bar{Y}_r||\bar{U}_r| d\hat{V}_r\right] + 2K^Z\sum_{i=1}^d \mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}|\bar{Y}_r|\left|\tfrac{d\langle\bar{N},\hat{M}^i\rangle}{d\hat{V}}(r)\right| d\hat{V}_r\right] \\
\leq\ & (K^Y\alpha + dK^Z\beta)\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{Y}_r^2 d\hat{V}_r\right] + \tfrac{K^Y}{\alpha}\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{U}_r^2 d\hat{V}_r\right] \\
& + \tfrac{K^Z}{\beta}\sum_{i=1}^d \mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\left(\tfrac{d\langle\bar{N},\hat{M}^i\rangle}{d\hat{V}}(r)\right)^2 d\hat{V}_r\right] \\
\leq\ & (K^Y\alpha + dK^Z\beta)\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{Y}_r^2 d\hat{V}_r\right] + \tfrac{K^Y}{\alpha}\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{U}_r^2 d\hat{V}_r\right] \\
& + \tfrac{CdK^Z}{\beta}\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r} d\langle\bar{N}\rangle_r\right],
\end{aligned}
$$

for some positive C and any positive α and β. The latter equality holds by Hypothesis 4.3.1 4. Then we pick $\alpha = 2K^Y$ and $\beta = 2CdK^Z$, which gives us

$$
\begin{aligned}
& \lambda\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{Y}_r^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r} d\langle\bar{M}\rangle_r\right] \\
\leq\ & 2((K^Y)^2 + C(dK^Z)^2)\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{Y}_r^2 d\hat{V}_r\right] \\
+\ & \tfrac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{U}_r^2 d\hat{V}_r\right] + \tfrac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r} d\langle\bar{N}\rangle_r\right].
\end{aligned}
$$

We choose now $\lambda = 1 + 2((K^Y)^2 + C(dK^Z)^2)$ and we get

$$
\begin{aligned}
& \mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{Y}_r^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r} d\langle\bar{M}\rangle_r\right] \\
\leq\ & \tfrac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r}\bar{U}_r^2 d\hat{V}_r\right] + \tfrac{1}{2}\mathbb{E}\left[\int_0^T e^{\lambda\hat{V}_r} d\langle\bar{N}\rangle_r\right],
\end{aligned}
\tag{4.A.4}
$$

which proves the contraction for the norm $\|\cdot\|_\lambda$. □

Proof of Theorem 4.3.3.

The space $L^2(d\hat{V}\otimes d\mathbb{P})\times\mathcal{H}_0^2$ is complete and Φ defines on it a contraction for the norm $\|(\cdot,\cdot)\|_\lambda$ for some $\lambda > 0$, so Φ has a unique fixed point in
$L^2(d\hat{V}\otimes d\mathbb{P})\times\mathcal{H}_0^2$. Then by Proposition 4.A.8, $BSDE(\xi,\hat{f},V,\hat{M})$ has a unique solution. □

Remark 4.A.11. *Let (Y,M) be the solution of $BSDE(\xi,\hat{f},V,\hat{M})$ and $\dot{Y}$ the class of Y in $L^2(d\hat{V}\otimes d\mathbb{P})$. Thanks to Proposition 4.A.8, we know that $(\dot{Y},M) = \Phi(\dot{Y},M)$ and therefore by Propositions 4.A.6 and 4.A.9 that $\sup_{t\in[0,T]}|Y_t|$ is L^2 and that $\int_0^\cdot Y_{r-}dM_r$ is a real martingale.*

The lemma below shows that, in order to verify that a couple (Y,M) is the solution of $BSDE(\xi,\hat{f},V,\hat{M})$, it is not necessary to verify the square integrability of Y since it will be automatically fulfilled.

Lemma 4.A.12. *We consider $(\xi,\hat{f},V,\hat{M})$ such that $\xi,\hat{M}$ verify items 1., 2. of Hypothesis 4.3.1 but where item 3. is replaced by the weaker following hypothesis on $\hat{f}$. There exists $C>0$ such that $\mathbb{P}$ a.s., for all t,y,z,*

$$
|\hat{f}(t,\omega,y,z)| \leq C(1+|y|+\|z\|). \tag{4.A.5}
$$

Assume that there exists a cadlag adapted process Y with $Y_0\in L^2$, and $M\in\mathcal{H}_0^2$ such that

$$
Y = \xi + \int_\cdot^T \hat{f}\left(r,\cdot,Y_r,\frac{d\langle M,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r - (M_T - M_\cdot), \tag{4.A.6}
$$

in the sense of indistinguishability. Then $\sup_{t\in[0,T]}|Y_t|$ *is* L^2*. In particular,* $Y \in \mathcal{L}^2(d\hat{V}\otimes d\mathbb{P})$ *and if* $(\xi,\hat{f},V,\hat{M})$ *verify Hypothesis 4.3.1* (Y,M) *is the unique solution of* $BSDE(\xi,\hat{f},V,\hat{M})$ *in the sense of Definition 4.3.2.*

On the other hand if (Y,M) *verifies* (4.A.6) *on* $[s,T]$ *with* $s<T$*,* $Y_s\in L^2$ *and* $M_s=0$ *then* $\sup_{t\in[s,T]}|Y_t|$ *is* L^2*. In particular if* $(\xi,\hat{f},V,\hat{M})$ *verify Hypothesis 4.3.1 and if we denote* (U,N) *the unique solution of* $BSDE(\xi,\hat{f},V,\hat{M})$*, then* (Y,M) *and* $(U,N.-N_s)$ *are indistinguishable on* $[s,T]$*.*

Proof. Let $\lambda>0$ and $t\in[0,T]$. By integration by parts formula applied to $Y^2e^{-\lambda\hat{V}}$ between 0 and t we get

$$\begin{aligned} Y_t^2e^{-\lambda\hat{V}_t}-Y_0^2 &= -2\int_0^t e^{-\lambda\hat{V}_r}Y_r\hat{f}\left(r,\cdot,Y_r,\tfrac{d\langle M,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r+2\int_0^t e^{-\lambda\hat{V}_r}Y_{r^-}dM_r\\ &\quad+\int_0^t e^{-\lambda\hat{V}_r}d[M]_r-\lambda\int_0^t e^{-\lambda\hat{V}_r}Y_r^2d\hat{V}_r. \end{aligned}$$

By re-arranging the terms and using the Lipschitz conditions item 3. of in Hypothesis 4.3.1, we get

$$\begin{aligned} &Y_t^2e^{-\lambda\hat{V}_t}+\lambda\int_0^t e^{-\lambda\hat{V}_r}Y_r^2d\hat{V}_r\\ \le\ & Y_0^2+2\int_0^t e^{-\lambda\hat{V}_r}|Y_r||\hat{f}|\left(r,\cdot,Y_r,\tfrac{d\langle M,\hat{M}\rangle}{d\hat{V}}(r)\right)d\hat{V}_r+2\left|\int_0^t e^{-\lambda\hat{V}_r}Y_{r^-}dM_r\right|\\ &+\int_0^t e^{-\lambda\hat{V}_r}d[M]_r\\ \le\ & Y_0^2+\int_0^t e^{-\lambda\hat{V}_r}\hat{f}^2(r,\cdot,0,0)d\hat{V}_r+(2K^Y+1+K^Z)\int_0^t e^{-\lambda\hat{V}_r}Y_r^2d\hat{V}_r\\ &+K^Z\sum_{i=1}^d\int_0^t e^{-\lambda\hat{V}_r}\left(\tfrac{d\langle M,\hat{M}^i\rangle}{d\hat{V}}(r)\right)^2d\hat{V}_r+2\left|\int_0^t e^{-\lambda\hat{V}_r}Y_{r^-}dM_r\right|+\int_0^t e^{-\lambda\hat{V}_r}d[M]_r. \end{aligned}$$

Picking $\lambda=2K^Y+1+K^Z$ and using Lemma 4.A.2, this gives

$$\begin{aligned} Y_t^2e^{-\lambda\hat{V}_t} &\le Y_0^2+\int_0^t e^{-\lambda\hat{V}_r}|\hat{f}|^2(r,\cdot,0,0)d\hat{V}_r+K^ZC\int_0^t e^{-\lambda\hat{V}_r}d\langle M\rangle_r\\ &\quad+2\left|\int_0^t e^{-\lambda\hat{V}_r}Y_{r^-}dM_r\right|+\int_0^t e^{-\lambda\hat{V}_r}d[M]_r, \end{aligned}$$

for some $C>0$. Since $\hat{V}$ is bounded, there is a constant $C'>0$, such that for any $t\in[0,T]$

$$Y_t^2\le C'\left(Y_0^2+\int_0^T|\hat{f}|^2(r,\cdot,0,0)d\hat{V}_r+\langle M\rangle_T+[M]_T+\left|\int_0^t Y_{r^-}dM_r\right|\right).$$

By Hypothesis 4.3.1 and since we assumed $Y_0\in L^2$ and $M\in\mathcal{H}^2$, the first four terms on the right-hand side are integrable and we can conclude by Lemma 2.3.18 in Chapter 2.

An analogous proof also holds on the interval $[s,T]$ taking into account Remark 4.3.4. In particular, if (U,N) is the unique solution of $BSDE(\xi,\hat{f},V,\hat{M})$ then $(U,N-N_s)$ is a solution on $[s,T]$. The final statement result follows by the uniqueness argument of Remark 4.3.4. □

In the sequel we will not distinguish between a couple $(\dot{Y},M)$ in $L^2(d\hat{V}\otimes d\mathbb{P})\times\mathcal{H}_0^2$ and (Y,M), where Y is the reference cadlag process of $\dot{Y}$, according to Definition 4.A.4.

Notation 4.A.13. *Let* $\Phi: L^2(d\hat{V}\otimes d\mathbb{P})\times\mathcal{H}_0^2$ *be the operator introduced in Notation 4.A.7. We then convene the following.*

1. $(Y^0,M^0):=(0,0)$*;*
2. $\forall k\in\mathbb{N}^*:(Y^k,M^k):=\Phi(Y^{k-1},M^{k-1})$*,*

meaning that for $k \in \mathbb{N}^*$, (Y^k, M^k) is the solution of the BSDE

$$Y^k = \xi + \int_\cdot^T \hat{f}\left(r, \cdot, Y^{k-1}, \frac{d\langle M^{k-1}, \hat{M}\rangle}{d\hat{V}}(r)\right) d\hat{V}_r - (M_T^k - M_\cdot^k). \tag{4.A.7}$$

Definition 4.A.14. *The processes $(Y^k, M^k)_{k\in\mathbb{N}}$ will be called the Picard iterations associated to $BSDE(\xi, \hat{f}, \hat{V}, \hat{M})$.*

We know that Φ is a contraction in $L^2(d\hat{V}\otimes d\mathbb{P}^{s,x}) \times \mathcal{H}_0^2$ for a certain norm, so that (Y^k, M^k) tends to (Y, M) in this topology. The proposition below also shows an a.e. corresponding convergence, adapting the techniques of Corollary 2.1 in [43].

Proposition 4.A.15. $Y^k \underset{k\to\infty}{\longrightarrow} Y \quad d\hat{V}\otimes d\mathbb{P}$ *a.e. and for any* $i \in [\![1; d]\!]$,
$\frac{d\langle M^k, \hat{M}^i\rangle}{d\hat{V}} \underset{k\to\infty}{\longrightarrow} \frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}} \quad d\hat{V}\otimes d\mathbb{P}$ *a.e.*

Proof of Proposition 4.A.15.

For any $i \in [\![1; d]\!]$ and $k \in \mathbb{N}$ we set $Z^{i,k} := \frac{d\langle M^k, \hat{M}^i\rangle}{d\hat{V}}$ and $Z^i := \frac{d\langle M, \hat{M}^i\rangle}{d\hat{V}}$. By Proposition 4.A.10, there exists $\lambda > 0$ such that for any $k \in \mathbb{N}^*$

$$\begin{aligned} & \mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y_r^{k+1} - Y_r^k|^2 d\hat{V}_r + \int_0^T e^{-\lambda\hat{V}_r} d\langle M^{k+1} - M^k\rangle_r\right] \\ \leq\ & \tfrac{1}{2}\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y_r^k - Y_r^{k-1}|^2 d\hat{V}_r + \int_0^T e^{-\lambda\hat{V}_r} d\langle M^k - M^{k-1}\rangle_r\right], \end{aligned}$$

therefore

$$\begin{aligned} & \sum_{k\geq 0}\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y_r^{k+1} - Y_r^k|^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r} d\langle M^{k+1} - M^k\rangle_r\right] \\ \leq\ & \sum_{k\geq 0}\tfrac{1}{2^k}\left(\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y_r^1|^2 d\hat{V}_r\right] + \mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r} d\langle M^1\rangle_r\right]\right) \\ <\ & \infty. \end{aligned} \tag{4.A.8}$$

For every fixed (i, k), the linearity property stated in Proposition 4.2.3) says that $Z_r^{i,k+1} - Z_r^{i,k} = \frac{d\langle M^{k+1} - M^k, \hat{M}^i\rangle}{d\hat{V}}$. Therefore combining equation (4.A.8) and Lemma 4.A.2, we get $\sum_{k\geq 0}\left(\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Y_r^{k+1} - Y_r^k|^2 d\hat{V}_r\right] + \sum_{i=1}^d \mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Z_r^{i,k+1} - Z_r^{i,k}|^2 d\hat{V}_r\right]\right) < \infty$. So by Fubini's theorem we have

$$\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}\left(\sum_{k\geq 0}\left(|Y_r^{k+1} - Y_r^k|^2 + \sum_{i=1}^d |Z_r^{i,k+1} - Z_r^{i,k}|^2\right)\right) d\hat{V}_r\right] < \infty.$$

Consequently the sum $\sum_{k\geq 0}\left(|Y_r^{k+1}(\omega) - Y_r^k(\omega)|^2 + \sum_{i=1}^d |Z_r^{i,k+1}(\omega) - Z_r^{i,k}(\omega)|^2\right)$ is finite on a set of full $d\hat{V}\otimes d\mathbb{P}$ measure. So on this set of full measure, the sequence $(Y_t^k(\omega), (Z_t^{i,k}(\omega))_{i\in[\![1;d]\!]})$ converges, and the limit is necessarily equal to $(Y_t(\omega), (Z_t^i(\omega))_{i\in[\![1;d]\!]})$ $d\hat{V}\otimes d\mathbb{P}$ a.e. Indeed, as we have mentioned in the lines before the statement of the present Proposition 4.A.15, we already know that Y^k converges to Y in $L^2(d\hat{V}\otimes d\mathbb{P})$. Since by Lemma 4.A.2, $\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r}|Z_r^{i,k} - Z_r^i|^2 d\hat{V}_r\right] \leq C\mathbb{E}\left[\int_0^T e^{-\lambda\hat{V}_r} d\langle M^k - M\rangle_r\right]$, for every (i, k), where C is a positive constant which does not depend on (i, k), the convergence of M^k to M in $\mathcal{H}_0^2$ also implies the convergence of $Z^{i,k}$ to Z^i in $L^2(d\hat{V}\otimes d\mathbb{P})$.

□

4.B Proof of Theorem 4.5.18

Lemma 4.B.1. *Let $\tilde{f} \in \mathcal{L}^2_X$. For every $(s,x) \in [0,T] \times E$, let $(\tilde{Y}^{s,x}, \tilde{M}^{s,x})$ be the unique (by Theorem 4.3.3 and Remark 4.3.4) solution of*

$$\tilde{Y}^{s,x}_\cdot = g(X_T) + \int_\cdot^T \mathbb{1}_{[s,T]}(r)\tilde{f}\,(r, X_r)\, dV_r - (\tilde{M}^{s,x}_T - \tilde{M}^{s,x}_\cdot) \tag{4.B.1}$$

in $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$. Then there exist $\tilde{u} \in \mathcal{D}(\mathfrak{a})$ such that for any $(s,x) \in [0,T] \times E$

$$\left\{ \begin{array}{rcl} \forall t \in [s,T] : \tilde{Y}^{s,x}_t & = & \tilde{u}(t, X_t) \quad \mathbb{P}^{s,x}a.s. \\ \tilde{M}^{s,x} & = & M[\tilde{u}]^{s,x}, \end{array} \right.$$

and in particular $\frac{d\langle \tilde{M}^{s,x}, M[\psi]^{s,x}\rangle}{dV} = \mathfrak{G}^\psi(\tilde{u})(\cdot, X_\cdot)\, dV \otimes d\mathbb{P}^{s,x}$ a.e. on $[s,T]$.

Proof. We set $\tilde{u} : (s,x) \mapsto \mathbb{E}^{s,x}\left[g(X_T) + \int_s^T \tilde{f}\,(r, X_r)\, dV_r\right]$ which is Borel by Proposition 3.A.10 and Lemma 3.A.11 in Chapter 3. Therefore by (4.C.3) in Remark 4.C.6, for every fixed $t \in [s,T]$ we have $\mathbb{P}^{s,x}$- a.s.

$$\begin{array}{rcl} \tilde{u}(t, X_t) & = & \mathbb{E}^{t,X_t}\left[g(X_T) + \int_t^T \tilde{f}\,(r, X_r)\, dV_r\right] \\ & = & \mathbb{E}^{s,x}\left[g(X_T) + \int_t^T \tilde{f}\,(r, X_r)\, dV_r \middle| \mathcal{F}_t\right] \\ & = & \mathbb{E}^{s,x}\left[\tilde{Y}^{s,x}_t + (\tilde{M}^{s,x}_T - \tilde{M}^{s,x}_t) | \mathcal{F}_t\right] \\ & = & \tilde{Y}^{s,x}_t. \end{array}$$

By (4.B.1) we have $d\tilde{Y}^{s,x}_t = -\tilde{f}(t, X_t)dV_t + d\tilde{M}^{s,x}_t$, so for every fixed $t \in [s,T]$, $\tilde{u}(t, X_t) = \tilde{u}(s,x) - \int_s^t \tilde{f}(r, X_r)dV_r - \tilde{M}^{s,x}_t$ $\mathbb{P}^{s,x}$- a.s. Since $\tilde{M}^{s,x}$ is square integrable and since previous relation holds for any (s,x) and t, Definition 4.4.13 implies that $\tilde{u} \in \mathcal{D}(\mathfrak{a})$, $\mathfrak{a}(\tilde{u}) = -\tilde{f}$ and $\tilde{M}^{s,x} = M[\tilde{u}]^{s,x}$ for every (s,x), hence the announced results. □

Notation 4.B.2. *For a fixed $(s,x) \in [0,T] \times E$, we will denote by $(Y^{k,s,x}, M^{k,s,x})_{k\in\mathbb{N}}$ the Picard iterations associated to $BSDE^{s,x}(f,g)$.*

Proposition 4.B.3. *For each $k \in \mathbb{N}$, there exists $u_k \in \mathcal{D}(\mathfrak{a})$, such that for every $(s,x) \in [0,T] \times E$*

$$\left\{ \begin{array}{rcl} \forall t \in [s,T] : Y^{k,s,x}_t & = & u_k(t, X_t) \quad \mathbb{P}^{s,x}a.s. \\ M^{k,s,x} & = & M[u_k]^{s,x}. \end{array} \right. \tag{4.B.2}$$

Remark 4.B.4. *In particular,* (4.B.2) *implies that $\frac{d\langle M^{k,s,x}, M[\psi]^{s,x}\rangle}{dV} = \mathfrak{G}^\psi(u_k)(\cdot, X_\cdot)\, dV \otimes d\mathbb{P}^{s,x}$ a.e. on $[s,T]$.*

Proof. We proceed by induction on k. It is clear that $u_0 = 0$ verifies the assertion for $k = 0$.
Now let us assume that the function u_{k-1} exists, for some integer $k \geq 1$, verifying (4.B.2) and in particular Remark 4.B.4, for k replaced with $k-1$.
We fix $(s,x) \in [0,T] \times E$. By Lemma 4.2.4, $(Y^{k-1,s,x}, \frac{d\langle M^{k-1,s,x}, M[\psi]^{s,x}\rangle}{dV}) = (u_{k-1}, \mathfrak{G}^\psi(u_{k-1}))(\cdot, X_\cdot)$ $dV \otimes \mathbb{P}^{s,x}$ a.e. on $[s,T]$. Therefore by (4.A.7), on $[s,T]$
$Y^{k,s,x}_\cdot = g(X_T) + \int_\cdot^T f\left(r, X_r, u_{k-1}(r, X_r), \mathfrak{G}^\psi(u_{k-1})(r, X_r)\right) dV_r - (M^{k,s,x}_T - M^{k,s,x}_\cdot)$.
Since $\Phi^{s,x}$ maps $L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}^2_0$ into itself (see Definition 4.A.7), obviously all the Picard iterations belong to $L^2(dV \otimes d\mathbb{P}^{s,x}) \times \mathcal{H}^2_0$. In particular, by Lemma 4.A.2 $Y^{k-1,s,x}$ and for every $i \in [\![1;d]\!]$, $\frac{d\langle M^{k-1,s,x}, M[\psi_i]^{s,x}\rangle}{dV}$ belong to $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,x})$. So, by recurrence assumption on u_{k-1}, it follows that u_{k-1} and for any $i \in [\![1;d]\!]$, $\mathfrak{G}^{\psi_i}(u_{k-1})$ belong to $\mathcal{L}^2_X$. By H^{lip} (see Hypotheses 4.5.4), $f(\cdot,\cdot,u_{k-1}, \mathfrak{G}^\psi(u_{k-1})) \in \mathcal{L}^2_X$. The existence of u_k now comes from Lemma 4.B.1 applied to $\tilde{f} :=$

$f(\cdot,\cdot,u_{k-1},\mathfrak{G}^\psi(u_{k-1}))$, which establishes the induction step for a general k and allows to conclude the proof.

□

Proof of Theorem 4.5.18. We set $\bar{u} := \limsup_{k\in\mathbb{N}} u_k$, in the sense that for any $(s,x)\in[0,T]\times E$, $\bar{u}(s,x) = \limsup_{k\in\mathbb{N}} u_k(s,x)$ and $v := \limsup_{k\in\mathbb{N}} v_k$. $\bar{u}$ and v are Borel functions. Let us fix now $(s,x)\in[0,T]\times E$. We know by Propositions 4.B.3, 4.A.15 and Lemma 4.2.4 that

$$\left\{\begin{array}{rcll} u_k(\cdot,X_\cdot) & \underset{k\to\infty}{\longrightarrow} & Y^{s,x} & dV\otimes d\mathbb{P}^{s,x} \text{ a.e. on } [s,T] \\ \mathfrak{G}^\psi(u_k)(\cdot,X_\cdot) & \underset{k\to\infty}{\longrightarrow} & Z^{s,x} & dV\otimes d\mathbb{P}^{s,x} \text{ a.e. on } [s,T], \end{array}\right.$$

where $Z^{s,x} := \frac{d\langle M^{s,x},M[\psi]^{s,x}\rangle}{dV}$. Therefore, and on the subset of $[s,T]\times E$ of full $dV\otimes d\mathbb{P}^{s,x}$ measure on which these convergences hold we have

$$\left\{\begin{array}{rcccccl} \bar{u}(t,X_t(\omega)) & = & \limsup\limits_{k\in\mathbb{N}} u_k(t,X_t(\omega)) & = & \lim\limits_{k\in\mathbb{N}} u_k(t,X_t(\omega)) & = & Y_t^{s,x}(\omega) \\ v(t,X_t(\omega)) & = & \limsup\limits_{k\in\mathbb{N}} \mathfrak{G}^\psi(u_k)(t,X_t(\omega)) & = & \lim\limits_{k\in\mathbb{N}} \mathfrak{G}^\psi(u_k)(t,X_t(\omega)) & = & Z_t^{s,x}(\omega). \end{array}\right. \tag{4.B.3}$$

Thanks to the $dV\otimes d\mathbb{P}^{s,x}$ equalities concerning v and $\bar{u}$ stated in (4.B.3), under $\mathbb{P}^{s,x}$ we actually have

$$Y^{s,x} = g(X_T) + \int_\cdot^T f\left(r,X_r,\bar{u}(r,X_r),v(r,X_r)\right)dV_r - (M_T^{s,x} - M_\cdot^{s,x}). \tag{4.B.4}$$

Now (4.B.4) can be considered as a BSDE where the driver does not depend on y and z. $Y^{s,x}$ and $Z^{s,x}$ belong to $\mathcal{L}^2(dV\otimes d\mathbb{P}^{s,x})$ (see Lemma 4.A.2), then by (4.B.3), so do $\bar{u}(\cdot,X_\cdot)\mathbb{1}_{[s,T]}$ and $v(\cdot,X_\cdot)\mathbb{1}_{[s,T]}$, meaning that $\bar{u}$ and v belong to $\mathcal{L}^2_X$. By H^{lip}, $f(\cdot,\cdot,\bar{u},v)$ also belongs to $\mathcal{L}^2_X$. We can therefore apply Lemma 4.B.1 to $\tilde{f} := f(\cdot,\cdot,\bar{u},v)$, and conclude.

Concerning the last statement of the theorem, for any $(s,x)\in[0,T]\times E$, we have $Y_s^{s,x} = u(s,X_s) = u(s,x)$ $\mathbb{P}^{s,x}$ a.s. so $Y_s^{s,x}$ is $\mathbb{P}^{s,x}$ a.s. equal to a constant and u is the mapping $(s,x)\longmapsto Y_s^{s,x}$. □

4.C Markov classes and Martingale Additive Functionals

We recall in this Appendix section some basic definitions and results concerning Markov processes. For a complete study of homogeneous Markov processes, one may consult [34], concerning non-homogeneous Markov classes, our reference was chapter VI of [40]. Some results are only stated, they were howether carefully proven in Chapter 1.

The first definition refers to the canonical space that one can find in [60], see paragraph 12.63.

Notation 4.C.1. *In the whole section E will be a fixed Polish space (a separable completely metrizable topological space). It will be called the **state space**.*

We fix $T\in\mathbb{R}_+^*$. We denote $\Omega := \mathbb{D}(E)$ the Skorokhod space of functions from $[0,T]$ to E right-continuous with left limits and continuous at time T (for which we also use the french acronym cadlag). For any $t\in[0,T]$ we denote the coordinate mapping $X_t : \omega\mapsto\omega(t)$, and we introduce on Ω the σ-field $\mathcal{F} := \sigma(X_r | r\in[0,T])$.

*On the measurable space $(\Omega,\mathcal{F})$, we introduce the **canonical process***

$$X : \begin{array}{rcl} (t,\omega) & \longmapsto & \omega(t) \\ ([0,T]\times\Omega,\mathcal{B}([0,T])\otimes\mathcal{F}) & \longrightarrow & (E,\mathcal{B}(E)), \end{array}$$

and the right-continuous filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ *where* $\mathcal{F}_t := \bigcap_{s\in]t,T]} \sigma(X_r|r \leq s)$ *if* $t < T$*, and* $\mathcal{F}_T := \sigma(X_r|r \in [0,T]) = \mathcal{F}$.

$(\Omega, \mathcal{F}, \mathbb{F})$ *will be called the* ***canonical space*** *(associated to* T *and* E*).*

For any $t \in [0,T]$ *we denote* $\mathcal{F}_{t,T} := \sigma(X_r|r \geq t)$*, and for any* $0 \leq t \leq u < T$ *we will denote* $\mathcal{F}_{t,u} := \bigcap_{n\geq 0} \sigma(X_r|r \in [t, u + \frac{1}{n}])$.

Remark 4.C.2. *Previous definitions and all the notions of this Appendix, extend to a time interval equal to* $\mathbb{R}_+$ *or replacing the Skorokhod space with the space of continuous functions from* $[0,T]$ *(or* $\mathbb{R}_+$*) to* E*. but since our goal is to work on a finite time interval, we will not consider this situation.*

Definition 4.C.3. *The function*

$$P : \begin{array}{rcl} (s,t,x,A) & \longmapsto & P_{s,t}(x,A) \\ [0,T]^2 \times E \times \mathcal{B}(E) & \longrightarrow & [0,1], \end{array}$$

will be called ***transition kernel*** *if, for any* s, t *in* $[0,T]$*,* $x \in E$*,* $A \in \mathcal{B}(E)$*, it verifies the following.*

1. $P_{s,t}(\cdot, A)$ *is Borel,*
2. $P_{s,t}(x, \cdot)$ *is a probability measure on* $(E, \mathcal{B}(E))$*,*
3. *if* $t \leq s$ *then* $P_{s,t}(x, A) = \mathbb{1}_A(x)$*,*
4. *if* $s < t$*, for any* $u > t$*,* $\int_E P_{s,t}(x, dy) P_{t,u}(y, A) = P_{s,u}(x, A)$.

The latter statement is the well-known **Chapman-Kolmogorov equation**.

Definition 4.C.4. *A transition kernel* P *for which the first item is reinforced supposing that* $(s,x) \longmapsto P_{s,t}(x,A)$ *is Borel for any* t, A*, will be said* ***measurable in time****.*

Definition 4.C.5. *A* ***canonical Markov class*** *associated to a transition kernel* P *is a set of probability measures* $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ *defined on the measurable space* $(\Omega, \mathcal{F})$ *and verifying for any* $t \in [0,T]$ *and* $A \in \mathcal{B}(E)$

$$\mathbb{P}^{s,x}(X_t \in A) = P_{s,t}(x, A), \tag{4.C.1}$$

and for any $s \leq t \leq u$

$$\mathbb{P}^{s,x}(X_u \in A|\mathcal{F}_t) = P_{t,u}(X_t, A) \quad \mathbb{P}^{s,x} \text{ a.s.} \tag{4.C.2}$$

Remark 4.C.6. *Formula 1.7 in Chapter 6 of [40] states that for any* $F \in \mathcal{F}_{t,T}$ *yields*

$$\mathbb{P}^{s,x}(F|\mathcal{F}_t) = \mathbb{P}^{t,X_t}(F) = \mathbb{P}^{s,x}(F|X_t) \;\; \mathbb{P}^{s,x} a.s. \tag{4.C.3}$$

Property (4.C.3) *will be called* ***Markov property****.*

For the rest of this section, we are given a canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in[0,T]\times E}$ which transition kernel is measurable in time.

Definition 4.C.7. *For any* $(s,x) \in [0,T] \times E$ *we will consider the* (s,x)***-completion*** $\left(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x} := (\mathcal{F}_t^{s,x})_{t\in[0,T]}, \mathbb{P}^{s,x}\right)$ *of the stochastic basis* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{s,x})$ *by defining* $\mathcal{F}^{s,x}$ *as the* $\mathbb{P}^{s,x}$*-completion of* $\mathcal{F}$ *, by extending* $\mathbb{P}^{s,x}$ *to* $\mathcal{F}^{s,x}$ *and finally by defining* $\mathcal{F}_t^{s,x}$ *as the* $\mathbb{P}^{s,x}$*-closure of* $\mathcal{F}_t$ *for every* $t \in [0,T]$.

We remark that, for any $(s,x) \in [0,T] \times E$, $(\Omega, \mathcal{F}^{s,x}, \mathbb{F}^{s,x}, \mathbb{P}^{s,x})$ is a stochastic basis fulfilling the usual conditions.

We recall the following simple consequence of Remark 32 in [32] Chapter II.

Proposition 4.C.8. *Let $(s,x) \in [0,T] \times E$ be fixed, Z be a random variable and $t \in [s,T]$, then $\mathbb{E}^{s,x}[Z|\mathcal{F}_t] = \mathbb{E}^{s,x}[Z|\mathcal{F}_t^{s,x}]$ $\mathbb{P}^{s,x}$ a.s.*

We now introduce the notion of non-homogeneous Additive Functional that we use in the paper.

Definition 4.C.9. *We denote $\Delta := \{(t,u) \in [0,T]^2 | t \leq u\}$. On $(\Omega, \mathcal{F})$, we define a **non-homogeneous Additive Functional** (shortened AF) as a random-field $A := (A_u^t)_{(t,u)\in\Delta}$ with values in $\mathbb{R}$ verifying the two following conditions.*

1. *For any $(t,u) \in \Delta$, A_u^t is $\mathcal{F}_{t,u}$-measurable;*
2. *for any $(s,x) \in [0,T] \times E$, there exists a real cadlag $\mathbb{F}^{s,x}$-adapted process $A^{s,x}$ (taken equal to zero on $[0,s]$ by convention) such that for any $x \in E$ and $s \leq t \leq u$, $A_u^t = A_u^{s,x} - A_t^{s,x}$ $\mathbb{P}^{s,x}$ a.s.*

*$A^{s,x}$ will be called the **cadlag version of A under** $\mathbb{P}^{s,x}$.*

*An AF will be called a **non-homogeneous square integrable Martingale Additive Functional** (shortened square integrable MAF) if under any $\mathbb{P}^{s,x}$ its cadlag version is a square integrable martingale. More generally an AF will be said to verify a certain property (being non-negative, increasing, of bounded variation, square integrable, having L^1 terminal value) if under any $\mathbb{P}^{s,x}$ its cadlag version verifies it.*

*Finally, given an increasing AF A and an increasing function V, A will be said to be **absolutely continuous with respect to** V if for any $(s,x) \in [0,T] \times E$, $dA^{s,x} \ll dV$ in the sense of stochastic measures.*

The two following results are proven in Chapter 1.

Proposition 4.C.10. *Let M, M' be two square integrable MAFs, let $M^{s,x}$ (respectively $M'^{s,x}$) be the cadlag version of M (respectively M') under $\mathbb{P}^{s,x}$. Then there exists a bounded variation AF with L^1 terminal condition denoted $\langle M, M' \rangle$ such that under any $\mathbb{P}^{s,x}$, the cadlag version of $\langle M, M' \rangle$ is $\langle M^{s,x}, M'^{s,x} \rangle$. If $M = M'$ the AF $\langle M, M' \rangle$ will be denoted $\langle M \rangle$ and is increasing.*

Proposition 4.C.11. *Let V be a continuous non-decreasing function. Let M, N be two square integrable MAFs, and assume that the AF $\langle N \rangle$ is absolutely continuous with respect to V. There exists a function $v \in \mathcal{B}([0,T] \times E, \mathbb{R})$ such that for any (s,x), $\langle M^{s,x}, N^{s,x} \rangle = \int_s^{\cdot \vee s} v(r, X_r) V_r$.*

Chapter 5

Path-dependent Martingale Problems and Additive Functionals

This chapter is the object of paper [14].

Abstract

The paper introduces and investigates the natural extension to the path-dependent setup of the usual concept of canonical Markov class introduced by Dynkin and which is at the basis of the theory of Markov processes. That extension, indexed by starting paths rather than starting points will be called path-dependent canonical class. Associated with this is the generalization of the notions of semi-group and of additive functionals to the path-dependent framework. A typical example of such family is constituted by the laws $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$, where for fixed time s and fixed path η defined on $[0,s]$, $\mathbb{P}^{s,\eta}$ is the (unique) solution of a path-dependent martingale problem or more specifically a weak solution of a path-dependent SDE with jumps, with initial path η. In the following Chapter 6, we apply those results to study path-dependent analysis problems associated with BSDEs.

5.1 Introduction

In this paper we extend some aspects of the theory of Markov processes to the (non-Markovian) path-dependent case. The crucial object of canonical Markov class introduced by Dynkin is replaced with the one of *path-dependent canonical class*. The associated notion of Markov semigroup is extended to the notion of *path-dependent system of projectors*. The classical Markovian concept of (Martingale) Additive Functional is generalized to the one of *path-dependent (Martingale) Additive Functional*. We then study some general path-dependent martingale problems with applications to weak solutions of path-dependent SDEs (possibly) with jumps and show that, under well-posedness, the solution of the martingale problem provides a path-dependent canonical class. The following Chapter 6 will exploit these results to extend the links between BSDEs and (possibly Integro) PDEs obtained in Chapter 4, to a path-dependent framework.

The theory of Additive Functionals associated to a Markov process was initiated during the early '60s, see the historical papers [38], [68], [19] and see [34] for a complete theory in the homogeneous setup. The strong links between martingale problems and Markov processes were first observed for the study of weak solutions of SDEs in [85], and more generally in [44] or [60] for example. Weak solutions of path-dependent SDEs possibly with jumps were studied in [60], where the author shows their equivalence to some path-dependent martingale problems and proves existence and uniqueness of a solution under Lipschitz conditions. More recent results concerning path-dependent martingale

problems may be found in [17]. However, at our knowledge, the structure of the set of solutions for different starting paths was not yet studied.

The setup of this paper is the canonical space $(\Omega, \mathcal{F})$ where $\Omega := \mathbb{D}(\mathbb{R}_+, E)$ is the Skorokhod space of cadlag functions from $\mathbb{R}_+$ into a Polish space E and $\mathcal{F}$ is its Borel σ-field. $X = (X_t)_{t\in\mathbb{R}_+}$ denotes the canonical process and the initial filtration $\mathbb{F}^o$ is defined by $\mathcal{F}^o_t := \sigma(X_r | r \in [0,t])$ for all $t \geq 0$.

A path-dependent canonical class will be a set of probability measures $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ defined on the canonical space and such that, for some fixed (s,η), $\mathbb{P}^{s,\eta}$ models a forward (path-dependent) dynamics in law, with imposed initial path η on the time interval $[0,s]$. As already mentioned, it constitutes the natural adaptation to the path-dependent world of the notion of canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in\mathbb{R}_+\times E}$, where in general, $\mathbb{P}^{s,x}$ models the law of some Markov stochastic process, with imposed value x at time s. $\mathbb{F}^{s,\eta}$ is the augmented initial filtration fulfilling the usual conditions.

In substitution of a Markov semigroup associated with a canonical Markov class, we introduce a path-dependent system of projectors denoted $(P_s)_{s\in\mathbb{R}_+}$ and a one-to-one connection between them and path-dependent canonical classes. Each projector P_s acts on the space of bounded random variables. This brings us to introduce the notion of **weak generator** $(\mathcal{D}(A), A)$ of $(P_s)_{s\in\mathbb{R}_+}$ which will permit us in Chapter 6 to define *mild* type solutions of path-dependent PDEs of the form

$$\begin{cases} D\Phi + \frac{1}{2}Tr(\sigma\sigma^\intercal\nabla^2\Phi) + \beta\nabla\Phi + f(\cdot,\cdot,\Phi,\sigma\sigma^\intercal\nabla\Phi) = 0 \text{ on } [0,T]\times\Omega \\ \Phi_T = \xi \text{ on } \Omega, \end{cases} \tag{5.1.1}$$

where D is the horizontal derivative and ∇ the vertical gradient in the sense of [37, 26] and β, σ are progressively measurable path-dependent coefficients.

As mentioned earlier, given a path-dependent canonical class we also introduce the notion of path-dependent Additive Functional (resp. path-dependent square integrable Martingale Additive Functional), which is a real-valued random-field $M := (M_{t,u})_{0\leq t\leq u<+\infty}$ such that for any $(s,\eta) \in \mathbb{R}_+\times\Omega$, there exists a real cadlag $\mathbb{F}^{s,\eta}$-adapted process (resp. $\mathbb{F}^{s,\eta}$-square integrable martingale) $M^{s,\eta}$ called the **cadlag version of** M **under** $\mathbb{P}^{s,\eta}$, and verifying for all $s \leq t \leq u$ that $M_{t,u} = M^{s,\eta}_u - M^{s,\eta}_t$ $\mathbb{P}^{s,\eta}$ a.s. Under some reasonable measurability assumptions on the path-dependent canonical class, we extend to our path-dependent setup some classical results of Markov processes theory concerning the quadratic covariation and the angular bracket of square integrable MAFs. As in the Markovian set-up, examples of path-dependent canonical classes arise from solutions of a (this time path-dependent) martingale problem as we explain below. Let χ be a set of cadlag processes adapted to the initial filtration $\mathbb{F}^o$. For some given $(s,\eta) \in \mathbb{R}_+\times\Omega$, we say that a probability measure $\mathbb{P}^{s,\eta}$ on $(\Omega,\mathcal{F})$ **solves the martingale problem with respect to** χ **starting in** (s,η) if

- $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$;
- all elements of χ are on $[s,+\infty[$ $(\mathbb{P}^{s,\eta}, \mathbb{F}^o)$-martingales.

We show that merely under some well-posedness assumptions, the set of solutions for varying starting times and paths $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ defines a path-dependent canonical class. This in particularly holds for weak solutions of path-dependent SDEs possibly with jumps.

The paper is organized as follows. In Section 5.3, we introduce the notion of path-dependent canonical class in Definition 5.3.4 and of path-dependent system of projectors in Definition 5.3.8 and prove a one-to-one correspondence between those two concepts in Corollary 5.3.11. In Section 5.4, we introduce the notion of path-dependent Additive Functional, in short AF (resp. Martingale Additive Functional, in short MAF). We state in Proposition 5.4.6 and Corollary 5.4.9 that for a given square integrable path-dependent MAF $(M_{t,u})_{(t,u)\in\Delta}$, we can exhibit two non-decreasing path-dependent AFs with $\mathcal{L}^1$-terminal value, denoted respectively by $([M]_{t,u})_{(t,u)\in\Delta}$ and $(\langle M\rangle_{t,u})_{(t,u)\in\Delta}$, which will

play respectively the role of a quadratic variation and an angular bracket of it. Then in Corollary 5.4.12, we state that the Radon-Nikodym derivative of the mentioned angular bracket of a square integrable path-dependent MAF with respect to a reference function V, is a progressively measurable process which does not depend on the probability. In Section 5.5, we introduce what we mean by path-dependent martingale problem with respect to a set of processes χ, to a time s and a starting path η, see Definition 5.5.4. Suppose that χ is a countable set of cadlag $\mathbb{F}^o$-adapted processes which are uniformly bounded on each interval $[0,T]$; in Proposition 5.5.12, we state that, whenever the martingale problem with respect to χ is well-posed, then the solution $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ defines a path-dependent canonical class. In Subsection 5.5.2, Definition 5.5.14 introduces the notion of weak generator of a path-dependent system of projectors, and Definition 5.5.15 that of martingale problem associated to a path-dependent operator $(D(A),A)$. Suppose now that for any (s,η) the martingale problem associated with $(D(A),A)$ is well-posed, and let $(P_s)_{s\in\mathbb{R}_+}$ be the system of projectors associated to the canonical class constituted by the solutions $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$. Then $(D(A),A)$ is a weak generator of $(P_s)_{s\in\mathbb{R}_+}$, and $(P_s)_{s\in\mathbb{R}_+}$ is the unique system of projectors such that this holds. In other words, $(P_s)_{s\in\mathbb{R}_+}$ can be analytically associated to $(D(A),A)$ without ambiguity. Finally, in Section 5.6, we consider path-dependent SDEs with jumps, whose coefficients are denoted by β,σ,γ. If for any couple (s,η), the SDE has a unique weak solution, then Theorem 5.6.7 ensures that the set of solutions $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ defines a path-dependent canonical class. Under the additional assumptions that β,σ,γ are bounded and continuous in ω for fixed other variables, then Proposition 5.6.13 states that $(s,\eta)\longmapsto\mathbb{P}^{s,\eta}$ is continuous for the topology of weak convergence.

5.2 Preliminaries

In the whole paper we will use the following notions, notations and vocabulary.

A topological space E will always be considered as a measurable space with its Borel σ-field which shall be denoted $\mathcal{B}(E)$ and if S is another topological space equipped with its Borel σ-field, $\mathcal{B}(E,S)$ will denote the set of Borel functions from E to S. For some fixed $d\in\mathbb{N}^*$, $\mathcal{C}_c^\infty(\mathbb{R}^d)$ will denote the set of smooth functions with compact support. For fixed $d,k\in\mathbb{N}^*$, $\mathcal{C}^k(\mathbb{R}^d)$, (resp. $\mathcal{C}_b^k(\mathbb{R}^d)$) will denote the set of functions k times differentiable with continuous (resp. bounded continuous) derivatives.

Let $(\Omega,\mathcal{F})$, $(E,\mathcal{E})$ be two measurable spaces. A measurable mapping from $(\Omega,\mathcal{F})$ to $(E,\mathcal{E})$ shall often be called a **random variable** (with values in E), or in short r.v. If $\mathbb{T}$ is the indices set, a family $(X_t)_{t\in\mathbb{T}}$ of r.v. with values in E, will be called a **random field** (indexed by $\mathbb{T}$ with values in E). In the particular case when $\mathbb{T}$ is a subinterval of $\mathbb{R}_+$, $(X_t)_{t\in\mathbb{T}}$ will be called a **stochastic process** (indexed by $\mathbb{T}$ with values in E). If the mapping $\begin{array}{ccc}(t,\omega)&\longmapsto&X_t(\omega)\\(\mathbb{T}\times\Omega,\mathcal{B}(\mathbb{T})\otimes\mathcal{F})&\longrightarrow&(E,\mathcal{E})\end{array}$ is measurable, then the process (or random field) $(X_t)_{t\in\mathbb{T}}$ will be said to be **measurable** (indexed by $\mathbb{T}$ with values in E).

On a fixed probability space $(\Omega,\mathcal{F},\mathbb{P})$, for any $p\geq 1$, $\mathcal{L}^p$ will denote the set of real-valued random variables with finite p-th moment. Two random fields (or stochastic processes) $(X_t)_{t\in\mathbb{T}}$, $(Y_t)_{t\in\mathbb{T}}$ indexed by the same set and with values in the same space will be said to be **modifications (or versions) of each other** if for every $t\in\mathbb{T}$, $\mathbb{P}(X_t=Y_t)=1$. A filtered probability space $\big(\Omega,\mathcal{F},\mathbb{F}:=(\mathcal{F}_t)_{t\in\mathbb{R}_+},\mathbb{P}\big)$ will be called called **stochastic basis** and will be said to **fulfill the usual conditions** if the filtration is right-continuous, if the probability space is complete and if $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible sets. Let us fix a stochastic basis $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$. If $Y=(Y_t)_{t\in\mathbb{R}_+}$ is a stochastic process and τ is a stopping time, we denote Y^τ the process $t\mapsto Y_{t\wedge\tau}$ which we call **stopped process** (by τ). If $\mathcal{C}$ is a set of processes, we will say that Y is locally in $\mathcal{C}$ (resp. locally verifies some property) if there exist an a.s. increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ tending a.s. to infinity such that for every n, the stopped process

Y^{τ_n} belongs to $\mathcal{C}$ (resp. verifies this property).

Given two martingales M, N, we denote by $[M]$ (resp. $[M, N]$) the **quadratic variation** of M (resp. **covariation** of M, N). If M, N are locally square integrable martingales, $\langle M, N\rangle$ (or simply $\langle M\rangle$ if $M = N$) will denote their (predictable) **angular bracket**. Two locally square integrable martingales vanishing at zero M, N will be said to be **strongly orthogonal** if $\langle M, N\rangle = 0$.
If A is an adapted process with bounded variation then $Var(A)$ (resp. $Pos(A)$, $Neg(A)$) will denote its total variation (resp. positive variation, negative variation), see Proposition 3.1, chap. 1 in [61]. In particular for almost all $\omega \in \Omega$, $t \mapsto Var_t(A(\omega))$ is the total variation function of the function $t \mapsto A_t(\omega)$.

5.3 Path-dependent canonical classes

We will introduce here an abstract context which is relevant for the study of path-dependent stochastic equations. The definitions and results which will be presented here are inspired from the theory of Markov processes and of additive functionals which one can find for example in [34].

The first definition refers to the canonical space that one can find in [60], see paragraph 12.63.

Notation 5.3.1. *In the whole section, E will be a fixed Polish space, i.e. a separable completely metrizable topological space, that we call the **state space**.*

*Ω will denote $\mathbb{D}(\mathbb{R}_+, E)$ the space of functions from $\mathbb{R}_+$ to E being right-continuous with left limits (e.g. cadlag). For every $t \in \mathbb{R}_+$ we denote the coordinate mapping $X_t : \omega \mapsto \omega(t)$ and we define on Ω the σ-field $\mathcal{F} := \sigma(X_r | r \in \mathbb{R}_+)$. On the measurable space $(\Omega, \mathcal{F})$, we introduce **initial filtration** $\mathbb{F}^o := (\mathcal{F}^o_t)_{t\in\mathbb{R}_+}$, where $\mathcal{F}^o_t := \sigma(X_r | r \in [0, t])$, and the (right-continuous) **canonical filtration** $\mathbb{F} := (\mathcal{F}_t)_{t\in\mathbb{R}_+}$, where $\mathcal{F}_t := \bigcap_{s>t} \mathcal{F}^o_s$. $(\Omega, \mathcal{F}, \mathbb{F})$ will be called the **canonical space** (associated to E). On $\mathbb{R}_+ \times \Omega$, we will denote by $\mathcal{P}ro^o$ (resp. $\mathcal{P}re^o$) the $\mathbb{F}^o$-progressive (resp. $\mathbb{F}^o$-predictable) σ-field. Ω will be equipped with the Skorokhod topology which is Polish since E is Polish (see Theorem 5.6 in chapter 3 of [44]), and for which the Borel σ-field is $\mathcal{F}$, see Proposition 7.1 in chapter 3 of [44]. This in particular implies that $\mathcal{F}$ is separable, being the Borel σ-field of a separable metric space.*

$\mathcal{P}(\Omega)$ will denote the set of probability measures on Ω and will be equipped with the topology of weak convergence of measures which also makes it a Polish space since Ω is Polish (see Theorems 1.7 and 3.1 in [44] chapter 3). It will also be equipped with the associated Borel σ-field.

Notation 5.3.2. *For any $\omega \in \Omega$ and $t \in \mathbb{R}_+$, the path ω stopped at time t: $r \mapsto \omega(r \wedge t)$ will be denoted ω^t.*

Remark 5.3.3. *In Sections 5.3,5.4 and Subsections 5.5.1, 5.5.2, all notions and results can easily be adapted to different canonical spaces Ω: for instance, $\mathcal{C}(\mathbb{R}_+, E)$, the space of continuous functions from $\mathbb{R}_+$ to E; $\mathcal{C}([0, T], E)$ (resp. $\mathbb{D}([0, T], E)$) the space of continuous (resp. cadlag) functions from $[0, T]$ to E, for some $T > 0$; fixing $x \in E$, $\mathcal{C}_x(\mathbb{R}_+, E)$ (resp. $\mathcal{C}_x([0, T], E)$) the space of continuous functions from $\mathbb{R}_+$ (resp. $[0, T]$) to E starting at x .*

Definition 5.3.4. *A **path-dependent canonical class** will be a family $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ of probability measures defined on the canonical space $(\Omega, \mathcal{F})$, which verifies the three following items.*

1. *For every $(s, \eta) \in \mathbb{R}_+ \times \Omega$, $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$;*

2. *for every $s \in \mathbb{R}_+$ and $F \in \mathcal{F}$, the mapping*
$$\begin{array}{ccc} \eta & \longmapsto & \mathbb{P}^{s,\eta}(F) \\ \Omega & \longrightarrow & [0, 1] \end{array}$$
is $\mathcal{F}^o_s$-measurable;

3. *for every* $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $t \geq s$ *and* $F \in \mathcal{F}$,

$$\mathbb{P}^{s,\eta}(F|\mathcal{F}_t^o)(\omega) = \mathbb{P}^{t,\omega}(F) \text{ for } \mathbb{P}^{s,\eta} \text{ almost all } \omega. \tag{5.3.1}$$

This implies in particular that for every $(s,\eta) \in \mathbb{R}_+ \times \Omega$ *and* $t \geq s$, *then* $(\mathbb{P}^{t,\omega})_{\omega\in\Omega}$ *is a regular conditional expectation of* $\mathbb{P}^{s,\eta}$ *by* $\mathcal{F}_t^o$, *see the Definition above Theorem 1.1.6 in [85] for instance.*

A path-dependent canonical class $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ *will be said to be* ***progressive*** *if for every* $F \in \mathcal{F}$, *the mapping* $(t,\omega) \longmapsto \mathbb{P}^{t,\omega}(F)$ *is* $\mathbb{F}^o$*-progressively measurable.*

In concrete examples, path-dependent canonical classes will always verify the following important hypothesis which is a reinforcement of (5.3.1).

Hypothesis 5.3.5. *For every* $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $t \geq s$ *and* $F \in \mathcal{F}$,

$$\mathbb{P}^{s,\eta}(F|\mathcal{F}_t)(\omega) = \mathbb{P}^{t,\omega}(F) \text{ for } \mathbb{P}^{s,\eta} \text{ almost all } \omega. \tag{5.3.2}$$

Remark 5.3.6. *By approximation through simple functions, one can easily show the following. Let* Z *be a random variable.*

- *Let* $s \geq 0$. *The functional* $\eta \longmapsto \mathbb{E}^{s,\eta}[Z]$ *is* $\mathcal{F}_s^o$*-measurable and for every* $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $t \geq s$, $\mathbb{E}^{s,\eta}[Z|\mathcal{F}_t^o](\omega) = \mathbb{E}^{t,\omega}[Z]$ *for* $\mathbb{P}^{s,\eta}$ *almost all* ω, *provided previous expectations are finite;*
- *if the path-dependent canonical class is progressive,* $(t,\omega) \longmapsto \mathbb{E}^{t,\omega}[Z]$ *is* $\mathbb{F}^o$*-progressively measurable, provided previous expectations are finite.*

Notation 5.3.7.

- $\mathcal{B}_b(\Omega)$ *(resp.* $\mathcal{B}_b^+(\Omega)$*) will denote the space of measurable (resp. non-negative measurable) bounded r.v.*
- *Let* $s \geq 0$. $\mathcal{B}_b^s(\Omega)$ *will denote the space of* $\mathcal{F}_s^o$*-measurable bounded r.v.*

Definition 5.3.8.

1. *A linear map* $Q : \mathcal{B}_b(\Omega) \to \mathcal{B}_b(\Omega)$ *is said* **positivity preserving monotonic** *if for every* $\phi \in \mathcal{B}_b^+(\Omega)$ *then* $Q[\phi] \in \mathcal{B}_b^+(\Omega)$ *and for every increasing converging (in the pointwise sense) sequence* $f_n \underset{n}{\longrightarrow} f$ *we have that* $Q[f_n] \underset{n}{\longrightarrow} Q[f]$ *in the pointwise sense.*
2. *A family* $(P_s)_{s\in\mathbb{R}_+}$ *of positivity preserving monotonic linear operators on* $\mathcal{B}_b(\Omega)$ *will be called a* ***path-dependent system of projectors*** *if it verifies the three following properties.*
   - *For all* $s \in \mathbb{R}_+$, *the restriction of* P_s *to* $\mathcal{B}_b^s(\Omega)$ *coincides with the identity;*
   - *for all* $s \in \mathbb{R}_+$, P_s *maps* $\mathcal{B}_b(\Omega)$ *into* $\mathcal{B}_b^s(\Omega)$;
   - *for all* $s,t \in \mathbb{R}_+$ *with* $t \geq s$, $P_s \circ P_t = P_s$.

Proposition 5.3.9. *Let* $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ *be a path-dependent canonical class. For every* $s \in \mathbb{R}_+$, *we define* $P_s : \phi \longmapsto (\eta \mapsto \mathbb{E}^{s,\eta}[\phi])$. *Then* $(P_s)_{s\in\mathbb{R}_+}$ *defines a path-dependent system of projectors.*

Proof. For every $s \geq 0$ each map P_s is linear, positivity preserving and monotonic using the usual properties of the expectation under a given probability. The rest follows taking into account Definitions 5.3.4, 5.3.8 and Remark 5.3.6.

□

Proposition 5.3.10. *Let $(P_s)_{s\in\mathbb{R}_+}$ be a path-dependent system of projectors. For any $(s,\eta)\in\mathbb{R}_+\times\Omega$, we set*

$$\mathbb{P}^{s,\eta}:\left(\begin{array}{ccc} F & \longmapsto & P_s[\mathbb{1}_F](\eta) \\ \mathcal{F} & \longrightarrow & \mathbb{R} \end{array}\right). \tag{5.3.3}$$

Then for all (s,η), $\mathbb{P}^{s,\eta}$ defines a probability measure and $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ is a path-dependent canonical class.

Proof. We fix s and η. Since $\emptyset,\Omega\in\mathcal{F}_s^o$, then by the first item of Definition 5.3.8, $P_s[\mathbb{1}_\emptyset]=\mathbb{1}_\emptyset$ and $P_s[\mathbb{1}_\Omega]=\mathbb{1}_\Omega$, so $\mathbb{P}^{s,\eta}(\emptyset)=0$ and $\mathbb{P}^{s,\eta}(\Omega)=1$. For any $F\in\mathcal{F}$, since P_s is positivity preserving and $\mathbb{1}_\emptyset\le\mathbb{1}_F\le\mathbb{1}_\Omega$ then $\mathbb{1}_\emptyset\le P_s[\mathbb{1}_F]\le\mathbb{1}_\Omega$ so, $\mathbb{P}^{s,\eta}$ takes values in $[0,1]$. If $(F_n)_n$ is a sequence of pairwise disjoint elements of $\mathcal{F}$ then the increasing sequence $\sum_{k=0}^N\mathbb{1}_{F_k}$ converges pointwise to $\mathbb{1}_{\bigcup_n F_n}$. Since the P_s are linear and monotonic then $\sum_n P_s[\mathbb{1}_{F_n}]=P_s[\mathbb{1}_{\bigcup_n F_n}]$, hence $\sum_n\mathbb{P}^{s,\eta}(F_n)=\mathbb{P}^{s,\eta}\left(\bigcup_n F_n\right)$. So for every (s,η), $\mathbb{P}^{s,\eta}$, is σ-additive, positive, vanishing in $\emptyset$ and takes value 1 in Ω hence is a probability measure.

Then, for any (s,η) we have $\mathbb{P}^{s,\eta}(\omega^s=\eta^s)=P_s[\mathbb{1}_{\{\omega^s=\eta^s\}}](\eta)=\mathbb{1}_{\{\omega^s=\eta^s\}}(\eta)=1$ since $\{\omega^s=\eta^s\}\in\mathcal{F}_s^o$, so item 1. of Definition 5.3.4 is satisfied. Concerning item 2., at fixed $s\in\mathbb{R}_+$ and $F\in\mathcal{F}$, we have $(\eta\mapsto\mathbb{P}^{s,\eta}(F))=P_s[\mathbb{1}_F]$ which is $\mathcal{F}_s^o$-measurable since P_s has its range in $\mathcal{B}_b^s(\Omega)$, see Definition 5.3.8.

It remains to show item 3. We now fix $(s,\eta)\in\mathbb{R}_+\times\Omega$, $t\ge s$ and $F\in\mathcal{F}$ and show that (5.3.1) holds. Let $G\in\mathcal{F}_t^o$. We need to show that $\mathbb{E}^{s,\eta}[\mathbb{1}_G\mathbb{1}_F]=\mathbb{E}^{s,\eta}[\mathbb{1}_G(\zeta)\mathbb{E}^{t,\zeta}[\mathbb{1}_F]]$. We have

$$\begin{aligned}\mathbb{E}^{s,\eta}[\mathbb{1}_G\mathbb{1}_F] &= \mathbb{E}^{s,\eta}[\mathbb{E}^{t,\zeta}[\mathbb{1}_G(\omega)\mathbb{1}_F(\omega)]] \\ &= \mathbb{E}^{s,\eta}[\mathbb{E}^{t,\zeta}[\mathbb{1}_G(\zeta)\mathbb{1}_F(\omega)]] \\ &= \mathbb{E}^{s,\eta}[\mathbb{1}_G(\zeta)\mathbb{E}^{t,\zeta}[\mathbb{1}_F(\omega)]],\end{aligned}$$

where the first equality comes from the fact that $P_s=P_s\circ P_t$ and the second from the fact that $G\in\mathcal{F}_t^o$ and $\mathbb{P}^{t,\zeta}(\omega^t=\zeta^t)=1$ so $\mathbb{1}_G=\mathbb{1}_G(\zeta)\ \mathbb{P}^{t,\zeta}$ a.s. □

Corollary 5.3.11. *The mapping*

$$\Phi:(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}\longmapsto(Z\longmapsto(\eta\mapsto\mathbb{E}^{s,\eta}[Z]))_{s\in\mathbb{R}_+}, \tag{5.3.4}$$

is a bijection between the set of path-dependent canonical classes and the set of path-dependent system of projectors, whose reciprocal map is given by

$$\Phi^{-1}:(P_s)_{s\in\mathbb{R}_+}\longmapsto(F\mapsto P_s[\mathbb{1}_F](\eta))_{(s,\eta)\in\mathbb{R}_+\times\Omega}. \tag{5.3.5}$$

Proof. Φ is by Proposition 5.3.9 well-defined. Moreover it is injective since if $\mathbb{P}^1$ and $\mathbb{P}^2$ are two probabilities such that respective expectations of all the bounded r.v. are the same then $\mathbb{P}^1=\mathbb{P}^2$. Then given a path-dependent system of projectors $(P_s)_{s\in\mathbb{R}_+}$, by Proposition 5.3.10
$(\mathbb{P}^{s,\eta}:F\mapsto P_s[\mathbb{1}_F](\eta))_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ is a path-dependent canonical class. It is then enough to show that the image through Φ of that path-dependent canonical class is indeed $(P_s)_{s\in\mathbb{R}_+}$. Let $(Q_s)_{s\in\mathbb{R}_+}$ denote its image by Φ, in order to conclude we are left to show that $Q_s=P_s$ for all s.
We fix s. For every $F\in\mathcal{F},\eta\in\Omega$ we have $Q_s[\mathbb{1}_F](\eta)=\mathbb{P}^{s,\eta}(F)=P_s[\mathbb{1}_F](\eta)$ so Q_s and P_s coincide on the indicator functions, hence on the simple functions by linearity, and everywhere by monotonicity and the fact that every bounded Borel function is the limit of an increasing sequence of simple functions. □

Definition 5.3.12. *From now on, two elements mapped by the previous bijection will be said to be **associated**.*

Remark 5.3.13. *Path-dependent canonical classes naturally extend canonical Markov classes (see Definition 4.C.5 in Chapter 4 for instance) as follows.*

Let $(\mathbb{P}^{s,x})_{(s,x)\in\mathbb{R}_+\times E}$ be a canonical Markov class with state space E and let $(P_{s,t})_{0\leq s\leq t}$ denote its transition kernel, see Definition 4.C.3 in Chapter 4.

For all $(s,\eta)\in\mathbb{R}_+\times\Omega$, let $\mathbb{P}^{s,\eta}$ be the unique probability measure on $(\Omega,\mathcal{F})$ such that $\mathbb{P}^{s,\eta}(\omega^s=\eta^s)$ and $\mathbb{P}^{s,\eta}$ coincides on $\sigma(X_r|r\geq s)$ with $\mathbb{P}^{s,\eta(s)}$. Then $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ is a path-dependent canonical class. Let $(P_s)_{s\in\mathbb{R}_+}$ denote the associated path-dependent system of projectors. Then for all bounded Borel $\phi:E\mapsto\mathbb{R}$, $\eta\in\Omega$ and $0\leq s\leq t$ we have

$$P_s[\phi\circ X_t](\eta)=\mathbb{E}^{s,\eta}[\phi(X_t)]=\mathbb{E}^{s,\eta(s)}[\phi(X_t)]=P_{s,t}[\phi](\eta(s)). \tag{5.3.6}$$

Notation 5.3.14. *For the rest of this section, we are given a path-dependent canonical class $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ and $(P_s)_{s\in\mathbb{R}_+}$ denotes the associated path-dependent system of projectors.*

Definition 5.3.15. *Let $\mathbb{P}$ be a probability on $(\Omega,\mathcal{F})$. If $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, we call $\mathbb{P}$-**closure** of $\mathcal{G}$ the σ-field generated by $\mathcal{G}$ and the set of $\mathbb{P}$-negligible sets. We denote it $\mathcal{G}^{\mathbb{P}}$. In the particular case $\mathcal{G}=\mathcal{F}$, we call $\mathcal{F}^{\mathbb{P}}$ $\mathbb{P}$-**completion** of $\mathcal{F}$.*

Remark 5.3.16. *Thanks to Remark 32.b) in Chapter II of [32], we have an equivalent definition of the $\mathbb{P}$-closure of some sub-σ-field $\mathcal{G}$ of $\mathcal{F}$ which can be characterized by the following property: $B\in\mathcal{G}^{\mathbb{P}}$ if and only if there exist $F\in\mathcal{G}$ such that $\mathbb{1}_B=\mathbb{1}_F$ $\mathbb{P}$ a.s.*

Moreover, $\mathbb{P}$ can be extended to a probability on $\mathcal{G}^{\mathbb{P}}$ by setting $\mathbb{P}(B):=\mathbb{P}(F)$ for such events.

Notation 5.3.17. *For any $(s,\eta)\in\mathbb{R}_+\times\Omega$ we will consider the stochastic basis $\left(\Omega,\mathcal{F}^{s,\eta},\mathbb{F}^{s,\eta}:=(\mathcal{F}_t^{s,\eta})_{t\in\mathbb{R}_+},\mathbb{P}^{s,\eta}\right)$ where $\mathcal{F}^{s,\eta}$ is the $\mathbb{P}^{s,\eta}$-completion of $\mathcal{F}$, $\mathbb{P}^{s,\eta}$ is extended to $\mathcal{F}^{s,\eta}$ and $\mathcal{F}_t^{s,\eta}$ is the $\mathbb{P}^{s,\eta}$-closure of $\mathcal{F}_t$ for every $t\in\mathbb{R}_+$.*

We remark that, for any $(s,\eta)\in\mathbb{R}_+\times\Omega$, $(\Omega,\mathcal{F}^{s,\eta},\mathbb{F}^{s,\eta},\mathbb{P}^{s,\eta})$ is a stochastic basis fulfilling the usual conditions, see 1.4 in [61] Chapter I.

A direct consequence of Remark 32.b) in Chapter II of [32] is the following.

Proposition 5.3.18. *Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, $\mathbb{P}$ a probability on $(\Omega,\mathcal{F})$ and $\mathcal{G}^{\mathbb{P}}$ the $\mathbb{P}$-closure of $\mathcal{G}$. Let $Z^{\mathbb{P}}$ be a real $\mathcal{G}^{\mathbb{P}}$-measurable random variable. There exists a $\mathcal{G}$-measurable random variable Z such that $Z=Z^{\mathbb{P}}$ $\mathbb{P}$-a.s.*

Proposition 5.3.18 yields the following.

Proposition 5.3.19. *Let $\mathbb{P}$ be a probability measure on $(\Omega,\mathcal{F})$, let $\mathbb{G}:=(\mathcal{G}_t)_{t\in\mathbb{R}_+}$ be a filtration and $\mathbb{G}^{\mathbb{P}}$ denote $(\mathcal{G}_t^{\mathbb{P}})_{t\in\mathbb{R}_+}$. Let Z be a positive or $\mathcal{L}^1$-random variable and $t\in\mathbb{R}_+$. Then $\mathbb{E}[Z|\mathcal{G}_t]=\mathbb{E}[Z|\mathcal{G}_t^{\mathbb{P}}]$ $\mathbb{P}$ a.s. In particular, $(\mathbb{P},\mathbb{G})$-martingales are also $(\mathbb{P},\mathbb{G}^{\mathbb{P}})$-martingales.*

According to Proposition 5.3.19 for $\mathbb{P}=\mathbb{P}^{s,\eta}$, the related conditional expectations with respect to $\mathcal{F}_t^{s,\eta}$ coincide with conditional expectations with respect to $\mathcal{F}_t$. For that reason we will only use the notation $\mathbb{E}^{s,\eta}[\,\cdot\,|\mathcal{F}_t]$ omitting the (s,η)-superscript over $\mathcal{F}_t$.

In the next proposition, $\mathcal{F}_t^{o,s,\eta}$ will denote for any $(s,\eta)\in\mathbb{R}_+\times\Omega$ and $t\geq s$ the $\mathbb{P}^{s,\eta}$-closure of $\mathcal{F}_t^o$.

Proposition 5.3.20. *Assume that Hypothesis 5.3.5 holds. For any $(s,\eta)\in\mathbb{R}_+\times\Omega$ and $t\geq s$, $\mathcal{F}_t^{o,s,\eta}=\mathcal{F}_t^{s,\eta}$.*

Proof. We fix s, η, t. Since inclusion $\mathcal{F}_t^{o,s,\eta} \subset \mathcal{F}_t^{s,\eta}$ is obvious, we show the converse inclusion.
Let $F^{s,\eta} \in \mathcal{F}_t^{s,\eta}$. By Remark 5.3.16, there exists $F \in \mathcal{F}_t$, such that $\mathbb{1}_{F^{s,\eta}} = \mathbb{1}_F \ \mathbb{P}^{s,\eta}$ a.s. It is therefore sufficient to prove the existence of some $F^o \in \mathcal{F}_t^o$ such that $\mathbb{1}_{F^o} = \mathbb{1}_F \ \mathbb{P}^{s,\eta}$ a.s. (and therefore $\mathbb{1}_{F^o} = \mathbb{1}_{F^{s,\eta}} \ \mathbb{P}^{s,\eta}$ a.s.) to conclude that $F^{s,\eta} \in \mathcal{F}_t^{o,s,\eta}$.

We set $Z : \begin{array}{ccc} \omega & \longmapsto & \mathbb{P}^{t,\omega}(F) \\ \Omega & \longrightarrow & [0,1] \end{array}$. By (5.3.2) and the fact that $F \in \mathcal{F}_t$, we have

$$Z(\omega) = \mathbb{P}^{t,\omega}(F) = \mathbb{E}^{s,\eta}[\mathbb{1}_F|\mathcal{F}_t](\omega) = \mathbb{1}_F(\omega) \quad \mathbb{P}^{s,\eta}\text{a.s.} \tag{5.3.7}$$

By Definition 5.3.4, Z is $\mathcal{F}_t^o$-measurable, so $F^o := Z^{-1}(\{1\})$ belongs to $\mathcal{F}_t^o$, and we will proceed showing that $\mathbb{1}_{F^o} = \mathbb{1}_F \ \mathbb{P}^{s,\eta}$ a.s.

By construction, $\mathbb{1}_{F^o}(\omega) = 1$ iff $\mathbb{P}^{t,\omega}(F) = 1$ and $\mathbb{1}_{F^o}(\omega) = 0$ iff $\mathbb{P}^{t,\omega}(F) \in [0,1[$. So

$$\begin{array}{rl} & \{\omega : \mathbb{1}_{F^o}(\omega) \neq \mathbb{1}_F(\omega)\} \\ = & \{\omega : \mathbb{1}_{F^o}(\omega) = 1 \text{ and } \mathbb{1}_F(\omega) = 0\} \bigcup \{\omega : \mathbb{1}_{F^o}(\omega) = 0 \text{ and } \mathbb{1}_F(\omega) = 1\} \\ = & \{\omega : \mathbb{P}^{t,\omega}(F) = 1 \text{ and } \mathbb{1}_F(\omega) = 0\} \bigcup \{\omega : \mathbb{P}^{t,\omega}(F) \in [0,1[\text{ and } \mathbb{1}_F(\omega) = 1\} \\ \subset & \{\omega : \mathbb{P}^{t,\omega}(F) \neq \mathbb{1}_F(\omega)\}, \end{array} \tag{5.3.8}$$

where the latter set is $\mathbb{P}^{s,\eta}$-negligible by (5.3.7). □

Combining Propositions 5.3.18 and 5.3.20, we have the following.

Corollary 5.3.21. *Assume that Hypothesis 5.3.5 holds and let us fix $(s,\eta) \in \mathbb{R}_+ \times \Omega$ and $t \geq s$. Given an $\mathcal{F}_t^{s,\eta}$-measurable r.v. $Z^{s,\eta}$, there exists an $\mathcal{F}_t^o$-measurable r.v. Z^o such that $Z^{s,\eta} = Z^o \ \mathbb{P}^{s,\eta}$ a.s.*

Definition 5.3.22. *If $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is a probability space and $\mathcal{G}$ is a sub-σ-field of $\tilde{\mathcal{F}}$, we say that $\mathcal{G}$ is $\mathbb{P}$-**trivial** if for any element G of $\mathcal{G}$, then $\mathbb{P}(G) \in \{0,1\}$.*

Corollary 5.3.23. *Assume that Hypothesis 5.3.5 holds. For every $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $\mathcal{F}_s^o$ and $\mathcal{F}_s$ are $\mathbb{P}^{s,\eta}$-trivial.*

Proof. We fix $(s,\eta) \in \mathbb{R}_+ \times \Omega$. We start by showing that $\mathcal{F}_s^o$ is $\mathbb{P}^{s,\eta}$-trivial. For every $B \in \mathcal{F}_s^o$ and ω we have $\mathbb{1}_B(\omega) = \mathbb{1}_B(\omega^s)$, and since $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$, we have $\mathbb{1}_B(\omega^s) = \mathbb{1}_B(\eta^s) \ \mathbb{P}^{s,\eta}$ a.s. So $\mathbb{P}^{s,\eta}(B) = \mathbb{E}^{s,\eta}[\mathbb{1}_B(\omega)] = \mathbb{1}_B(\eta^s) \in \{0,1\}$. Then, it is clear that adding $\mathbb{P}^{s,\eta}$-negligible sets does not change the fact of being $\mathbb{P}^{s,\eta}$-trivial, so $\mathcal{F}_s^{o,s,\eta}$ (which by Proposition 5.3.20 is equal to $\mathcal{F}_s^{s,\eta}$) is $\mathbb{P}^{s,\eta}$-trivial and therefore so is $\mathcal{F}_s \subset \mathcal{F}_s^{s,\eta}$.

□

5.4 Path-dependent Additive Functionals

In this section, we introduce the notion of Path-dependent Additive Functionals that we use in the chapter. As already anticipated, this can be interpreted as a path-dependent extension of the notion of non-homogeneous Additive Functionals of a canonical Markov class developed in Chapter 1. For that reason, several proofs of this section are very similar to those of Chapter 1 and are inspired from [34] Chapter XV, which treats the time-homogeneous case.

We keep on using Notation 5.3.1 and we fix a path-dependent canonical class $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ and assume the following for the whole section.

Hypothesis 5.4.1. *$(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ is progressive and verifies Hypothesis 5.3.5.*

We will use the notation $\Delta := \{(t,u) \in \mathbb{R}_+^2 | t \leq u\}$.

Definition 5.4.2. *On $(\Omega, \mathcal{F})$, a **path-dependent Additive Functional** (in short path-dependent AF) will be a random-field $A := (A_{t,u})_{(t,u)\in\Delta}$ with values in $\mathbb{R}$ verifying the two following conditions.*

1. *For any $(t,u) \in \Delta$, $A_{t,u}$ is $\mathcal{F}^o_u$-measurable;*
2. *for any $(s,\eta) \in \mathbb{R}_+ \times \Omega$, there exists a real cadlag $\mathbb{F}^{s,\eta}$-adapted process $A^{s,\eta}$ (taken equal to zero on $[0,s]$ by convention) such that for any $\eta \in \Omega$ and $s \le t \le u$,*

$$A_{t,u} = A^{s,\eta}_u - A^{s,\eta}_t \quad \mathbb{P}^{s,\eta} \text{ a.s.}$$

*We denote by A^t the ($\mathbb{F}^o$-adapted) process $u \mapsto A_{t,u}$ indexed by $[t,+\infty[$. For any $(s,\eta) \in [0,t]\times\Omega$, $A^{s,\eta}_\cdot - A^{s,\eta}_t$ is a $\mathbb{P}^{s,\eta}$-version of A^t on $[t,+\infty[$. $A^{s,\eta}$ will be called the **cadlag version of** A **under** $\mathbb{P}^{s,\eta}$.*

*A path-dependent Additive Functional will be called a **path-dependent Martingale Additive Functional** (in short path-dependent MAF) if under any $\mathbb{P}^{s,\eta}$ its cadlag version is a martingale.*

More generally, a path-dependent AF will be said to verify a certain property (being non-decreasing, of bounded variation, square integrable, having $\mathcal{L}^1$-terminal value) if under any $\mathbb{P}^{s,\eta}$ its cadlag version verifies it.

*Finally, given two increasing path-dependent AFs A and B, A will be said to be **absolutely continuous with respect to** B if for any $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $dA^{s,\eta} \ll dB^{s,\eta}$ in the sense of stochastic measures. This means that $dA^{s,\eta}(\omega)$ is absolutely continuous with respect to $dB^{s,\eta}(\omega)$ for $\mathbb{P}^{s,\eta}$ almost all ω.*

Remark 5.4.3. *The set of path-dependent AFs (resp. path-dependent AFs with bounded variation, path-dependent AFs with $\mathcal{L}^1$-terminal value, path-dependent MAFs, square integrable path-dependent MAFs) is a linear space.*

Lemma 5.4.4. *Let M be an $\mathbb{F}^o$-adapted process such that for all (s,η), on $[s,+\infty[$, M is a $(\mathbb{P}^{s,\eta}, \mathbb{F}^o)$-martingale.*

Then, for all (s,η), $M_{\cdot\vee s} - M_s$ admits a $\mathbb{P}^{s,\eta}$-version which is a $(\mathbb{P}^{s,\eta}, \mathbb{F}^{s,\eta})$ cadlag martingale $M^{s,\eta}$ vanishing in $[0,s]$. In particular $M_{t,u}(\omega) := M_u(\omega) - M_t(\omega)$ defines a path-dependent MAF with cadlag version $M^{s,\eta}$ under $\mathbb{P}^{s,\eta}$.

Proof. By Propositions 5.3.19 and 5.3.20, M is also on $[s,+\infty[$ a $(\mathbb{P}^{s,\eta}, \mathbb{F}^{s,\eta})$-martingale hence $M_{\cdot\vee s} - M_s$ is on $\mathbb{R}_+$ a $(\mathbb{P}^{s,\eta}, \mathbb{F}^{s,\eta})$-martingale and vanishes on $[0,s]$. Since $\mathbb{F}^{s,\eta}$ satisfies the usual conditions, then $M_{\cdot\vee s} - M_s$ admits a cadlag $\mathbb{P}^{s,\eta}$-modification $M^{s,\eta}$ which also is a $(\mathbb{P}^{s,\eta}, \mathbb{F}^{s,\eta})$-martingale vanishing in $[0,s]$. It clearly verifies that $M_{t,u} = M_u - M_t = M^{s,\eta}_u - M^{s,\eta}_t$ $\mathbb{P}^{s,\eta}$-a.s. for all $s \le t \le u$. □

Example 5.4.5. *Let Z be an $\mathcal{F}$-measurable bounded r.v. A typical example of process verifying the conditions of previous Lemma 5.4.4 is given by $M^Z : (t,\omega) \longmapsto \mathbb{E}^{t,\omega}[Z]$, see Remark 5.3.6.*

The following results state that, for a given square integrable path-dependent MAF $(M_{t,u})_{(t,u)\in\Delta}$ we can exhibit two non-decreasing path-dependent AFs with $\mathcal{L}^1$-terminal value, denoted respectively by $([M]_{t,u})_{(t,u)\in\Delta}$ and $(\langle M\rangle_{t,u})_{(t,u)\in\Delta}$, which will play respectively the role of a quadratic variation and an angular bracket of it. Moreover we will show that the Radon-Nikodym derivative of the mentioned angular bracket of a square integrable path-dependent MAF with respect to a reference function V is a progressively measurable process which does not depend on the probability.

The proof of the proposition below is postponed to the appendix.

Proposition 5.4.6. *Let $(M_{t,u})_{(t,u)\in\Delta}$ be a square integrable path-dependent MAF, and for any $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $[M^{s,\eta}]$ denote the quadratic variation of its cadlag version $M^{s,\eta}$ under $\mathbb{P}^{s,\eta}$. Then there exists a non-decreasing path-dependent AF with $\mathcal{L}^1$-terminal value which we will call $([M]_{t,u})_{(t,u)\in\Delta}$ and which, for any $(s,\eta) \in \mathbb{R}_+ \times \Omega$, has $[M^{s,\eta}]$ as cadlag version under $\mathbb{P}^{s,\eta}$.*

The next result can be seen as an extension of Theorem 15 Chapter XV in [34] to a path-dependent context and will be needed to show that the result above also holds for the angular bracket. Its proof is also postponed to the appendix.

Proposition 5.4.7. *Let $(B_{t,u})_{(t,u)\in\Delta}$ be a non-decreasing path-dependent AF with $\mathcal{L}^1$- terminal value. For any $(s,\eta)\in\mathbb{R}_+\times\Omega$, let $B^{s,\eta}$ be its cadlag version under $\mathbb{P}^{s,\eta}$ and let $A^{s,\eta}$ be the predictable dual projection of $B^{s,\eta}$ in $(\Omega,\mathcal{F}^{s,\eta},\mathbb{F}^{s,\eta},\mathbb{P}^{s,\eta})$. Then there exists a non-decreasing path-dependent AF with $\mathcal{L}^1$-terminal value $(A_{t,u})_{(t,u)\in\Delta}$ such that under any $\mathbb{P}^{s,\eta}$, the cadlag version of A is $A^{s,\eta}$.*

Remark 5.4.8.

1. *About the notion of dual predictable projection (also called compensator) related to some stochastic basis we refer to Theorem 3.17 in Chapter I of [61].*

2. *We recall that, whenever M,N are two local martingales, the angle bracket $\langle M,N\rangle$ is the dual predictable projection of $[M,N]$, see Proposition 4.50 b) in Chapter I of [61].*

Corollary 5.4.9. *Let $(M_{t,u})_{(t,u)\in\Delta}$, $(N_{t,u})_{(t,u)\in\Delta}$ be two square integrable path-dependent MAFs, let $M^{s,\eta}$ (respectively $N^{s,\eta}$) be the cadlag version of M (respectively N) under $\mathbb{P}^{s,\eta}$. Then there exists a bounded variation path-dependent AF with $\mathcal{L}^1$-terminal value, denoted $(\langle M,N\rangle_{t,u})_{(t,u)\in\Delta}$, such that under any $\mathbb{P}^{s,\eta}$, the cadlag version of $\langle M,N\rangle$ is $\langle M^{s,\eta},N^{s,\eta}\rangle$. If $M=N$ the path-dependent AF $\langle M,N\rangle$ will be denoted $\langle M\rangle$ and is non-decreasing.*

Proof. This can be proved as for Corollary 1.4.11 in Chapter 1, replacing parameter (s,x) with (s,η). □

The result below concerns the Radon-Nikodym derivative of a non-decreasing continuous path-dependent AF with respect to some reference measure dV. Its proof is postponed to the Appendix.

Proposition 5.4.10. *Let $V:\mathbb{R}_+\longrightarrow\mathbb{R}$ be a non-decreasing continuous function. Let A be a non-negative, non-decreasing path-dependent AF absolutely continuous with respect to V, and for any $(s,\eta)\in\mathbb{R}_+\times\Omega$ let $A^{s,\eta}$ be the cadlag version of A under $\mathbb{P}^{s,\eta}$. There exists an $\mathbb{F}^o$-progressively measurable process h such that for any $(s,\eta)\in\mathbb{R}_+\times\Omega$, $A^{s,\eta}=\int_s^{\cdot\vee s}h_r dV_r$, in the sense of indistinguishability.*

Proposition 5.4.11. *Let $(A_{t,u})_{(t,u)\in\Delta}$ be a path-dependent AF with bounded variation, taking $\mathcal{L}^1$-terminal value. Then there exists an increasing path-dependent AF that we denote $(Pos(A)_{t,u})_{(t,u)\in\Delta}$ (resp. $(Neg(A)_{t,u})_{(t,u)\in\Delta}$), which, for any $(s,\eta)\in\mathbb{R}_+\times\Omega$, has $Pos(A^{s,\eta})$ (resp. $Neg(A^{s,\eta})$)) as cadlag version under $\mathbb{P}^{s,\eta}$.*

Proof. This can be proved similarly as for Proposition 1.4.14 in Chapter 1, replacing parameter (s,x) with (s,η). □

Corollary 5.4.12. *Let V be a continuous non-decreasing function. Let M and N be two square integrable path-dependent MAFs and let $M^{s,\eta}$ (respectively $N^{s,\eta}$) be the cadlag version of M (respectively N) under a fixed $\mathbb{P}^{s,\eta}$. Assume that $\langle N\rangle$ is absolutely continuous with respect to dV. There exists an $\mathbb{F}^o$-progressively measurable process k such that for any $(s,\eta)\in\mathbb{R}_+\times\Omega$, $\langle M^{s,\eta},N^{s,\eta}\rangle=\int_s^{\cdot\vee s}k_r dV_r$.*

Proof. The proof follows the same lines as the one of Proposition 1.4.17 in Chapter 1 replacing parameter (s,x) by (s,η) and Borel functions of (t,X_t) with $\mathbb{F}^o$-progressively measurable processes. We make use of Corollary 5.4.9, Propositions 5.4.11 and 5.4.10, respectively in substitution of Corollary 1.4.11 an Propositions 1.4.14 and 1.4.13. □

Corollary 5.4.13. *Let V be a continuous non-decreasing function. Let M (resp. N) be an $\mathbb{F}^o$-adapted process such that for all (s,η), M (resp. N) is on $[s,+\infty[$ a $(\mathbb{P}^{s,\eta},\mathbb{F}^o)$-square integrable martingale. For any (s,η), let $M^{s,\eta}$ (resp. $N^{s,\eta}$) denote its $\mathbb{P}^{s,\eta}$-cadlag version. Assume that for all (s,η), $d\langle N^{s,\eta}\rangle \ll dV$.*
Then there exists an $\mathbb{F}^o$-progressively measurable process k such that for any $(s,\eta)\in\mathbb{R}_+\times\Omega$, $\langle M^{s,\eta},N^{s,\eta}\rangle = \int_s^{\cdot\vee s} k_r dV_r$.

Proof. The mentioned cadlag versions exist because of Lemma 5.4.4. The statement follows by the same Lemma 5.4.4 and Corollary 5.4.12.

□

5.5 Path-dependent Martingale problems

5.5.1 Abstract Martingale Problems

In this section we show that, whenever a (path-dependent) martingale problem is well-posed, then its solution is a path-dependent canonical class verifying Hypothesis 5.3.5. This relies on the same mathematical tools than those used by D.S Stroock and S.R.S Varadhan in the context of Markovian diffusions in [85]. Indeed it was already known that the ideas of [85] could be used in any type of Markovian setup and not just for martingale problems associated to diffusions, see [44] for example. One of the interests of the following lines is to show that their scope goes beyond the Markovian framework. First we prove that $\eta\mapsto\mathbb{P}^{s,\eta}$ is measurable, using well-posedness arguments and the celebrated Kuratowsky Theorem. Then we show in Proposition 5.5.12 that the solution of the martingale problem verifies (5.3.2), which is the analogous formulation of Markov property, through the theory of regular conditional expectations and again the fact that the martingale problem is well-posed.

Notation 5.5.1. *For every $t\in\mathbb{R}_+$, $\Omega^t := \{\omega\in\Omega : \omega=\omega^t\}$ will denote the set of constant paths after time t. We also denote $\Lambda := \{(s,\eta)\in\mathbb{R}_+\times\Omega : \eta\in\Omega^s\}$.*

Proposition 5.5.2.

1. *Λ is a closed subspace of $\mathbb{R}_+\times\Omega$, hence a Polish space when equipped with the induced topology.*
2. *For any $t\in\mathbb{R}_+$, Ω^t is also a closed subspace of Ω.*

Proof. We will only show the first statement since the proof of the second one is similar but simpler. Let $(s_n,\eta_n)_n$ be a sequence in Λ. Let $(s,\eta)\in\mathbb{R}_+\times\Omega$ and assume that $s_n\to s$ and that η_n tends to η for the Skorokhod topology. Then η_n tends to η Lebesgue a.e. Let $\epsilon>0$. There is a subsequence (s_{n_k}) such that $|s_{n_k}-s|\le\epsilon$, implying that for all k, η_{n_k} is constantly equal to $\eta_{n_k}(s_{n_k})$ on $[s+\epsilon,+\infty[$. Since η_n tends to η Lebesgue a.e., then necessarily, $\eta_{n_k}(s_{n_k})$ tends to some $c\in E$ and η takes value c a.e. on $[s+\epsilon,+\infty[$. This holds for every ϵ, and η is cadlag, so η is constantly equal to c on $[s,+\infty[$, implying that $(s,\eta)\in\Lambda$. □

From now on, Λ, introduced in Notation 5.5.1, is equipped with the trace topology.

Proposition 5.5.3. *The Borel σ-field $\mathcal{B}(\Lambda)$ is equal to the trace σ-field $\Lambda\cap\mathcal{P}ro^o$. For any $t\in\mathbb{R}_+$, the Borel σ-field $\mathcal{B}(\Omega^t)$ is equal to the trace σ-field $\Omega^t\cap\mathcal{F}_t^o$.*

Proof. Again we only show the first statement since the proof of the second one is similar. By definition of the topology on Λ, it is clear that $\mathcal{B}(\Lambda)=\Lambda\cap\mathcal{B}(\mathbb{R}_+\times\Omega)=\Lambda\cap(\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F})$ contains $\Lambda\cap\mathcal{P}ro^o$. We show the converse inclusion. The sets $\Lambda\cap([s,u]\times\{\omega(r)\in A\})$ for $s,u,r\in\mathbb{R}_+$ with $s\le u$, $A\in\mathcal{B}(E)$ generate $\Lambda\cap(\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F})$ so it is enough to show that these sets belong to $\Lambda\cap\mathcal{P}ro^o$.

We fix $s \leq u$ and r in $\mathbb{R}_+$, and $A \in \mathcal{B}(E)$. We have

$$\begin{aligned} \Lambda \bigcap ([s,u] \times \{\omega(r) \in A\}) &= \left\{ (t,\omega) : \left\{ \begin{array}{l} t \in [s,u] \\ \omega = \omega^t \\ \omega(r) \in A \end{array} \right. \right\} \\ &= \left\{ (t,\omega) : \left\{ \begin{array}{l} t \in [s,u] \\ \omega = \omega^t \\ \omega(r \wedge t) \in A \end{array} \right. \right\} \\ &= \Lambda \bigcap \left\{ (t,\omega) : \left\{ \begin{array}{l} t \in [s,u] \\ \omega(r \wedge t) \in A. \end{array} \right. \right\} \end{aligned} \quad . \tag{5.5.1}$$

We are left to show that $\left\{ (t,\omega) : \left\{ \begin{array}{l} t \in [s,u] \\ \omega(r \wedge t) \in A \end{array} \right. \right\} \in \mathcal{P}ro^o$, or equivalently that

$$t \mapsto \mathbb{1}_{[s,u]}(t) \mathbb{1}_A(X_{r \wedge t}) \text{ is } \mathbb{F}^o - \text{progressively measurable.} \tag{5.5.2}$$

Now $t \mapsto X_{r \wedge t}$ is right-continuous and $\mathbb{F}^o$-adapted so it is an E-valued $\mathbb{F}^o$-progressively measurable process, see Theorem 15 in [32] Chapter IV. By composition with a Borel function, $t \mapsto \mathbb{1}_A(X_{r \wedge t})$ is a real-valued $\mathbb{F}^o$-progressively measurable process; (5.5.2) follows since $t \mapsto \mathbb{1}_{[s,u]}(t)$ is $\mathbb{F}^o$-progressively measurable and the product of the two $\mathbb{F}^o$-progressively measurable processes remains $\mathbb{F}^o$-progressively measurable. □

Definition 5.5.4. *Let $(s,\eta) \in \Lambda$ and χ be a set of $\mathbb{F}^o$-adapted processes. We say that a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ **solves the martingale problem with respect to χ starting in** (s,η) if*

- $\mathbb{P}(\omega^s = \eta^s) = 1$,
- *all elements of χ are on $[s, +\infty[$ $(\mathbb{P}, \mathbb{F}^o)$-martingales.*

Remark 5.5.5. *We insist on the following important fact. If $M \in \chi$ is cadlag and $\mathbb{P}$ solves the martingale problem associated to χ, then by Theorem 3 in [33] Chapter VI, M is also on $[s, +\infty[$ a $(\mathbb{P}, \mathbb{F})$-martingale.*

Notation 5.5.6. *For fixed $(s,\eta) \in \Lambda$ and χ, the set of probability measures solving the martingale problem with respect to χ starting in (s,η) will be denoted $MP^{s,\eta}(\chi)$.*

Definition 5.5.7. *Let us consider a set χ of processes. If for every $(s,\eta) \in \Lambda$, $MP^{s,\eta}(\chi)$ is reduced to a single element $\mathbb{P}^{s,\eta}$, we will say that the martingale problem associated to χ is **well-posed**. In this case we will always extend the mapping*

$$\begin{array}{ccc} (s,\eta) & \longmapsto & \mathbb{P}^{s,\eta} \\ \Lambda & \longrightarrow & \mathcal{P}(\Omega) \end{array} \tag{5.5.3}$$

to $\mathbb{R}_+ \times \Omega$ by setting for all $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $\mathbb{P}^{s,\eta} := \mathbb{P}^{s,\eta^s}$.

Notation 5.5.8. *We fix a dense sequence $(x_n)_{n \geq 0}$ of elements of E. For any $s \in \mathbb{R}_+$, we will denote by Π_s the set of elements of $\mathcal{F}_s^o$ of type $\{\omega(t_1) \in B(x_{i_1}, r_1), \cdots, \omega(t_N) \in B(x_{i_N}, r_N)\}$ where $N \in \mathbb{N}$, $t_1, \cdots, t_N \in [0,s] \cap \mathbb{Q}$, $i_1, \cdots, i_N \in \mathbb{N}$, $r_1, \cdots, r_N \in \mathbb{Q}_+$ and where $B(x,r)$ denotes the open ball centered in x and of radius r.*

It is easy to show that for any $s \in \mathbb{R}_+$, Π_s is a countable π-system generating $\mathcal{F}_s^o$, see [1] Definition 4.9 for the notions of π-system and λ-system.

Below we consider the set $\mathcal{A}_s$ of probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F})$ for which there exists $\eta \in \Omega$ such that $\mathbb{P}$ solves the martingale problem with respect to χ starting at (s,η).

Proposition 5.5.9. *We fix a countable set χ of cadlag $\mathbb{F}^o$-adapted processes which are uniformly bounded on each interval $[0,T]$, and some $s \in \mathbb{R}_+$. Let $\mathcal{A}_s := \bigcup_{\eta\in\Omega} MP^{s,\eta}(\chi)$. Then $\mathcal{A}_s$ is a Borel set of $\mathcal{P}(\Omega)$.*

For the proof of this proposition we need a technical lemma.

Lemma 5.5.10. *We fix $s \in \mathbb{R}_+$. An element $\mathbb{P}$ of $\mathcal{P}(\Omega)$ belongs to $\mathcal{A}_s$ if and only if it verifies the following conditions:*

1. *$\mathbb{P}(F) \in \{0,1\}$ for all $F \in \Pi_s$;*
2. *$\mathbb{E}^{\mathbb{P}}[(M_u - M_t)\mathbb{1}_F] = 0$ for all $M \in \chi$, $t,u \in [s,+\infty[\cap\mathbb{Q}$ such that $t \le u$, $F \in \Pi_t$.*

Proof. By definition of $\mathcal{A}_s$, an element $\mathbb{P}$ of $\mathcal{P}(\Omega)$ belongs to $\mathcal{A}_s$ iff

a) there exists $\eta \in \Omega$ such that $\mathbb{P}(\omega^s = \eta^s) = 1$;

b) for all $M \in \chi$, $(M_t)_{t\in[s,+\infty[}$ is a $(\mathbb{P},\mathbb{F}^o)$-martingale.

Item a) above is equivalent to saying that $\mathcal{F}_s^o$ is $\mathbb{P}$-trivial which is equivalent to item 1. of the Lemma statement by Dynkin's Lemma (see 4.11 in [1]), since Π_s is a π-system generating $\mathcal{F}_s^o$ and since the sets $F \in \mathcal{F}_s^o$ such that $\mathbb{P}(F) \in \{0,1\}$ form a λ-system. On the other hand, it is clear that item b) above implies item 2. in the statement of the Lemma.

Conversely, assume that $M \in \chi$ satisfies item 2. of the statement. We fix $s \le t \le u$. Let $(t_n)_n, (u_n)_n$ be two sequences of rational numbers which converge respectively to t,u strictly from the right and such that $t_n \le u_n$ for all n. For every fixed n, we have $\mathbb{E}^{\mathbb{P}}[(M_{u_n} - M_{t_n})\mathbb{1}_G] = 0$ for all $G \in \Pi_t$. We then pass to the limit in n using the fact that M is right-continuous at fixed ω, and the dominated convergence theorem and taking into account the fact that M is bounded on compact intervals; this yields $\mathbb{E}^{\mathbb{P}}[(M_u - M_t)\mathbb{1}_G] = 0$ for all $G \in \Pi_t$. Since sets $G \in \mathcal{F}_t^o$ verifying this property form a λ-system and since Π_t is a π-system generating $\mathcal{F}_t^o$, then by Dynkin's lemma (see 4.11 in [1]), $\mathbb{E}^{\mathbb{P}}[(M_u - M_t)\mathbb{1}_G] = 0$ for all $G \in \mathcal{F}_t^o$. This implies that $(M_t)_{t\in[s,+\infty[}$ is a $(\mathbb{P},\mathbb{F}^o)$-martingale which concludes the proof of Lemma 5.5.10. □

Proof of Proposition 5.5.9.
We fix $s \in \mathbb{R}_+$. We recall that for any bounded random variable ϕ, $\mathbb{P} \mapsto \mathbb{E}^{\mathbb{P}}[\phi]$ is Borel. In particular for all $F \in \Pi_s$, $\mathbb{P} \longmapsto \mathbb{P}(F)$ and for all $M \in \chi$, $t,u \in [s,+\infty[\cap\mathbb{Q}$, $F \in \Pi_t$, $\mathbb{P} \longmapsto \mathbb{E}^{\mathbb{P}}[(M_u - M_t)\mathbb{1}_F]$ are Borel maps. The result follows by Lemma 5.5.10, taking into account the fact Π_t is countable for any t, and χ and the rational number set $\mathbb{Q}$ are also countable. Indeed since $\{0\}$ and $\{0,1\}$ are Borel sets, $\mathcal{A}_s$ is Borel being a countable intersection of preimages of Borel sets by Borel functions. □

Proposition 5.5.11. *Let χ be a countable set of cadlag $\mathbb{F}^o$-adapted processes which are uniformly bounded on each interval $[0,T]$. We assume that the martingale problem associated to χ is well-posed, see Definition 5.5.7. Let $s \in \mathbb{R}_+$. Then $\Phi_s : \begin{pmatrix} \eta & \longmapsto & \mathbb{P}^{s,\eta} \\ \Omega^s & \longrightarrow & \mathcal{P}(\Omega) \end{pmatrix}$ is Borel. Moreover, $\begin{pmatrix} (s,\eta) & \longmapsto & \mathbb{P}^{s,\eta} \\ \mathbb{R}_+\times\Omega & \longrightarrow & \mathcal{P}(\Omega) \end{pmatrix}$ is $\mathbb{F}^o$-adapted.*

Proof. We fix $s \in \mathbb{R}_+$ and set

$$\Phi_s : \begin{matrix} \eta & \longmapsto & \mathbb{P}^{s,\eta} \\ \Omega^s & \longrightarrow & \mathcal{A}_s, \end{matrix} \tag{5.5.4}$$

where $\mathcal{A}_s$ is defined as in Proposition 5.5.9. Φ_s is surjective by construction. It is also injective. Indeed, if $\eta_1,\eta_2 \in \Omega^s$ are different, there exists $t \in [0,s]$ such that $\eta_1(t) \neq \eta_2(t)$ and we have $\mathbb{P}^{s,\eta_1}(\omega(t) = \eta_1(t)) = 1$ and $\mathbb{P}^{s,\eta_2}(\omega(t) = \eta_2(t)) = 1$ so clearly $\mathbb{P}^{s,\eta_1} \neq \mathbb{P}^{s,\eta_2}$.

We can therefore introduce the reciprocal mapping

$$\Phi_s^{-1} : \begin{array}{ccc} \mathbb{P}^{s,\eta} & \longmapsto & \eta \\ \mathcal{A}_s & \longrightarrow & \Omega^s, \end{array} \tag{5.5.5}$$

which is a bijection. We wish to show that it is Borel. Since the Borel σ-algebra of Ω^s is generated by the sets of type $\{\omega(r \wedge s) \in A\}$ where $r \in \mathbb{R}_+$ and $A \in \mathcal{B}(E)$, it is enough to show that $\Phi_s(\{\omega(r \wedge s) \in A\})$ is for any r, A a Borel subset of $\mathcal{P}(\Omega)$. We then have $\Phi_s(\{\omega(r \wedge s) \in A\}) = \mathcal{A}_s \cap \{\mathbb{P} : \mathbb{P}(\omega(r \wedge s) \in A) = 1\}$ which is Borel being the intersection of $\mathcal{A}_s$ which is Borel by Lemma 5.5.10, and of the preimage of $\{1\}$ by the Borel function $\mathbb{P} \mapsto \mathbb{P}(F)$ with $F = \{\omega(r \wedge s) \in A\}$. So Φ_s^{-1} is a Borel bijection which maps the Borel set $\mathcal{A}_s$ of the Polish space $\mathcal{P}(\Omega)$ into the Polish space Ω^s. By Kuratowsky theorem (see Corollary 3.3 in [75]), $\Phi_s : \begin{array}{ccc} \eta & \longmapsto & \mathbb{P}^{s,\eta} \\ \Omega^s & \longrightarrow & \mathcal{P}(\Omega) \end{array}$ is Borel.

Let us justify the second part of the statement. Since by Proposition 5.5.3, $\mathcal{B}(\Omega^s) = \Omega^s \cap \mathcal{F}_s^o$ for all s, it is clear that $\begin{pmatrix} \eta & \longmapsto & \eta^s \\ \Omega & \longrightarrow & \Omega^s \end{pmatrix}$ is $(\mathcal{F}_s^o, \mathcal{B}(\Omega^s))$-measurable and therefore that $\begin{pmatrix} \eta & \longmapsto & \mathbb{P}^{s,\eta} \\ \Omega & \longrightarrow & \mathcal{P}(\Omega) \end{pmatrix}$ is $\mathcal{F}_s^o$-measurable.

□

Proposition 5.5.12. *Let χ be a countable set of cadlag $\mathbb{F}^o$-adapted processes which are uniformly bounded on each interval $[0, T]$, and assume that the martingale problem associated to χ is well-posed, see Definition 5.5.7. Then $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ is a path-dependent canonical class verifying Hypothesis 5.3.5 .*

Proof. The first two items of Definition 5.3.4 are directly implied by Proposition 5.5.11 and the fact that $\mathbb{P}^{s,\eta} \in MP^{s,\eta}(\chi)$ hence $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$ for all (s, η). It remains to show the validity of Hypothesis 5.3.5.

We fix $(s, \eta) \in \mathbb{R}_+ \times \Omega$ and $t \geq s$. Since Ω is Polish and $\mathcal{F}_t$ is a sub σ-field of its Borel σ-field, there exists a regular conditional expectation of $\mathbb{P}^{s,\eta}$ by $\mathcal{F}_t$ (see Theorem 1.1.6 in [85]), meaning a set of probability measures $(\mathbb{Q}^{t,\zeta})_{\zeta\in\Omega}$ on $(\Omega, \mathcal{F})$ such that

1. for any $F \in \mathcal{F}$, $\zeta \mapsto \mathbb{Q}^{t,\zeta}(F)$ is $\mathcal{F}_t$-measurable;
2. for any $F \in \mathcal{F}$, $\mathbb{P}^{s,\eta}(F|\mathcal{F}_t)(\zeta) = \mathbb{Q}^{t,\zeta}(F)$ $\mathbb{P}^{s,\eta}$ a.s.

We will now show that for $\mathbb{P}^{s,\eta}$ almost all ζ, we have

$$\mathbb{Q}^{t,\zeta} = \mathbb{P}^{t,\zeta}, \tag{5.5.6}$$

so that item 2. above will imply Hypothesis 5.3.5. In order to show that equality, we will show that for $\mathbb{P}^{s,\eta}$ almost all ζ, $\mathbb{Q}^{t,\zeta}$ solves the Martingale problem associated to χ starting in (t, ζ) and conclude (5.5.6) since $MP^{t,\zeta}(\chi)$ is a singleton, taking into account the fact the corresponding martingale problem is well-posed.

For any $F \in \mathcal{F}_t^o$, by item 2. above we have $\mathbb{Q}^{t,\zeta}(F) = \mathbb{1}_F(\zeta)$ $\mathbb{P}^{s,\eta}$ a.s. Since Π_t is countable, there exists a $\mathbb{P}^{s,\eta}$-null set N_1 such that for all $\zeta \in N_1^c$ we have $\mathbb{Q}^{t,\zeta}(F) = \mathbb{1}_F(\zeta)$ for all $F \in \Pi_t$. Then since Π_t is a π-system generating $\mathcal{F}_t^o$ and since sets verifying the previous relation define a λ-system, we have by Dynkin's lemma (see 4.11 in [1]) that for all $\zeta \in N_1^c$, $\mathbb{Q}^{t,\zeta}(F) = \mathbb{1}_F(\zeta)$ for all $F \in \mathcal{F}_t^o$. Now for every fixed $\zeta \in N_1^c$, since $\{\omega : \omega^t = \zeta^t\} \in \mathcal{F}_t^o$, we have $\mathbb{Q}^{t,\zeta}(\omega^t = \zeta^t) = \mathbb{1}_{\{\omega:\omega^t=\zeta^t\}}(\zeta) = 1$, which is the first item of Definition 5.5.4 related to $MP^{t,\zeta}(\chi)$.

We then show that for $\mathbb{P}^{s,\eta}$-almost all ζ, the elements of χ are $(\mathbb{Q}^{t,\zeta}, \mathbb{F}^o)$-martingales, which constitutes the second item of Definition 5.5.4.

For any $t_1 \leq t_2$ in $[t, +\infty[$, $M \in \chi$ and $F \in \mathcal{F}^o_{t_1}$, we have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^{t,\zeta}}[(M_{t_2} - M_{t_1})\mathbb{1}_F] &= \mathbb{E}^{s,\eta}[(M_{t_2} - M_{t_1})\mathbb{1}_F|\mathcal{F}_t](\zeta) \\ &= \mathbb{E}^{s,\eta}[\mathbb{E}^{s,\eta}[(M_{t_2} - M_{t_1})\mathbb{1}_F|\mathcal{F}_{t_1}]|\mathcal{F}_t](\zeta) \\ &= \mathbb{E}^{s,\eta}[\mathbb{E}^{s,\eta}[(M_{t_2} - M_{t_1})|\mathcal{F}_{t_1}]\mathbb{1}_F|\mathcal{F}_t](\zeta) \\ &= 0, \end{aligned} \tag{5.5.7}$$

for $\mathbb{P}^{s,\eta}$ almost all ζ by Remark 5.5.5 since M is a $(\mathbb{P}^{s,\eta}, \mathbb{F})$-martingale on$[s, +\infty[$ and $F \in \mathcal{F}^o_{t_1} \subset \mathcal{F}_{t_1}$. Since χ and the set of rational numbers are countable and taking into account the fact that for any $r \geq 0$, $\mathcal{F}^o_r$ is countably generated, there exists a $\mathbb{P}^{s,\eta}$-null set N_2 such that for any $\zeta \in N_2^c$, we have for any $t_1 \leq t_2$ in $[t, +\infty[\cap\mathbb{Q}$, $M \in \chi$, $F \in \mathcal{F}^o_{t_1}$, that $\mathbb{E}^{\mathbb{Q}^{t,\zeta}}[(M_{t_2} - M_{t_1})\mathbb{1}_F] = 0$.

Let $\zeta \in N_2^c$. We will now show that this still holds for any $t_1 \leq t_2$ in $[t, +\infty[$, $M \in \chi$, $F \in \mathcal{F}^o_{t_1}$. We consider rational valued sequences $(t_1^n)_n$ (resp. $(t_2^n)_n$) which converge to t_1 (resp. to t_2) strictly from the right and such that $t_1^n \leq t_2^n$ for all n. For all n, $\mathbb{E}^{\mathbb{Q}^{t,\zeta}}[(M_{t_2^n} - M_{t_1^n})\mathbb{1}_F] = 0$; since M is right-continuous and bounded on finite intervals, by dominated convergence, we can pass to the limit in n and we obtain $\mathbb{E}^{\mathbb{Q}^{t,\zeta}}[(M_{t_2} - M_{t_1})\mathbb{1}_F] = 0$. Therefore if $\zeta \notin N_1 \bigcup N_2$ which is $\mathbb{P}^{s,\eta}$-negligible, then $\mathbb{Q}^{t,\zeta}(\omega^t = \zeta^t) = 1$ and all the elements of χ are $(\mathbb{Q}^{t,\zeta}, \mathbb{F}^o)$-martingales. This means that $\mathbb{Q}^{t,\zeta} = \mathbb{P}^{t,\zeta}$ by well-posedness and concludes the proof of Proposition 5.5.12. □

5.5.2 Martingale problem associated to an operator and weak generators

This section links the notion of martingale problem with respect to a natural notion of (weak) generator. In this section Notation 5.3.1 will be again in force. Let $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ be a path-dependent canonical class and let the corresponding path-dependent system of projectors be denoted $(P_s)_{s\in\mathbb{R}_+}$, see Definition 5.3.12. Let $V : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ be a non-decreasing cadlag function.

In the sequel of this section, we are given a couple $(\mathcal{D}(A), A)$ verifying the following.

Hypothesis 5.5.13.

1. *$\mathcal{D}(A)$ is a linear subspace of the space of $\mathbb{F}^o$-progressively measurable processes;*
2. *A is a linear mapping from $\mathcal{D}(A)$ into the space of $\mathbb{F}^o$-progressively measurable processes;*
3. *for all $\Phi \in \mathcal{D}(A)$, $\omega \in \Omega$, $t \geq 0$, $\int_0^t |A\Phi_r(\omega)|dV_r < +\infty$;*
4. *for all $\Phi \in \mathcal{D}(A)$, $(s,\eta) \in \mathbb{R}_+ \times \Omega$ and $t \in [s, +\infty[$, we have $\mathbb{E}^{s,\eta}\left[\int_s^t |A(\Phi)_r|dV_r\right] < +\infty$ and $\mathbb{E}^{s,\eta}[|\Phi_t|] < +\infty$.*

Inspired from the classical literature (see 13.28 in [61]) we introduce a notion of weak generator.

Definition 5.5.14. *We say that $(\mathcal{D}(A), A)$ is a **weak generator** of the path-dependent system of projectors $(P_s)_{s\in\mathbb{R}_+}$ if for all $\Phi \in \mathcal{D}(A)$, $(s,\eta) \in \mathbb{R}_+ \times \Omega$ and $t \in [s, +\infty[$, we have*

$$P_s[\Phi_t](\eta) = \Phi_s(\eta) + \int_s^t P_s[A(\Phi)_r](\eta)dV_r. \tag{5.5.8}$$

Definition 5.5.15. *We will call **martingale problem associated to** $(\mathcal{D}(A), A)$ the martingale problem (in the sense of Definition 5.5.4) associated to the set of processes χ constituted by the processes $\Phi - \int_0^\cdot A(\Phi)_r dV_r$, $\Phi \in \mathcal{D}(A)$. It will be said to be **well-posed** if it is well-posed in the sense of Definition 5.5.7.*

Proposition 5.5.16. *$(\mathcal{D}(A), A)$ is a weak generator of $(P_s)_{s\in\mathbb{R}_+}$ iff $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ solves the martingale problem associated to $(\mathcal{D}(A), A)$.*

Moreover, if $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ solves the well-posed martingale problem associated to $(\mathcal{D}(A), A)$ then $(P_s)_{s\in\mathbb{R}_+}$ is the unique path-dependent system of projectors for which $(\mathcal{D}(A), A)$ is a weak generator.

Proof. We start assuming that $(\mathcal{D}(A), A)$ is a weak generator of $(P_s)_{s\in\mathbb{R}_+}$. Let $\Phi \in \mathcal{D}(A)$, $s \le t \le u$. $\mathbb{P}^{s,\eta}$ a.s. we have

$$\begin{array}{rl} & \mathbb{E}^{s,\eta}[\Phi_u - \Phi_t - \int_t^u A(\Phi)_r dV_r | \mathcal{F}_t^o](\omega) \\ = & \mathbb{E}^{t,\omega}[\Phi_u - \Phi_t - \int_t^u A(\Phi)_r dV_r] \\ = & P_t[\Phi_u](\omega) - \Phi_t(\omega) - \int_t^u P_t[A(\Phi_r)](\omega) dV_r \\ = & 0, \end{array} \tag{5.5.9}$$

where the first equality holds by Remark 5.3.6, the second one by Fubini's theorem and the third one because $(\mathcal{D}(A), A)$ is assumed to be a weak generator of $(P_s)_{s\in\mathbb{R}_+}$. By definition of path-dependent canonical class, we have $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$. By (5.5.9), for all $\Phi \in \mathcal{D}(A)$, $\Phi - \int_s^\cdot A(\Phi)_r dV_r$ is a $(\mathbb{P}^{s,\eta}, \mathbb{F}^o)$-martingale, and therefore $\mathbb{P}^{s,\eta}$ solves the martingale problem associated to $(\mathcal{D}(A), A)$ starting in (s, η).

Conversely, let us assume that $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ solves the martingale problem associated to $(\mathcal{D}(A), A)$. Let $\Phi \in \mathcal{D}(A)$ and $(s, \eta) \in \mathbb{R}_+ \times \Omega$ be fixed. By Definitions 5.5.15 and 5.5.7, $M[\Phi] := \Phi - \int_0^\cdot A(\Phi)_r dV_r$, is a $(\mathbb{P}^{s,\eta}, \mathbb{F}^o)$-martingale on $[s, +\infty[$. Moreover, since $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$ and Φ_s being $\mathcal{F}_s^o$-measurable, we obtain $\Phi_s = \Phi_s(\eta)$ $\mathbb{P}^{s,\eta}$ a.s. Therefore, for any $t \ge s$, $\Phi_t - \Phi_s(\eta) - \int_s^t A(\Phi)_r dV_r = M[\Phi]_t - M[\Phi]_s$ a.s.; so taking the $\mathbb{P}^{s,\eta}$ expectation, by Fubini's Theorem and Definition 5.3.12 it yields

$$\begin{array}{rl} & P_s[\Phi_t](\eta) - \Phi_s(\eta) - \int_s^t P_s[A(\Phi)_r](\eta) dV_r \\ = & \mathbb{E}^{s,\eta}\left[\Phi_t - \Phi_s(\eta) - \int_s^t A(\Phi)_r dV_r\right] \\ = & \mathbb{E}^{s,\eta}\left[M[\Phi]_t - M[\Phi]_s\right] \\ = & 0, \end{array} \tag{5.5.10}$$

hence that $(\mathcal{D}(A), A)$ is a weak generator of $(P_s)_{s\in\mathbb{R}_+}$.

Finally assume moreover that the martingale problem is well-posed and that $(\mathcal{D}(A), A)$ is a weak generator of another path-dependent system of projectors $(Q_s)_{s\in\mathbb{R}_+}$ with associated path-dependent canonical class $(\mathbb{Q}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$. Then by the first statement of the present proposition, $(\mathbb{Q}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ solves the martingale problem associated to $(\mathcal{D}(A), A)$. Since that martingale problem is well-posed we have $(\mathbb{Q}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega} = (\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ and by Proposition 5.3.11, $(Q_s)_{s\in\mathbb{R}_+} = (P_s)_{s\in\mathbb{R}_+}$. □

Remark 5.5.17. *When the conditions of previous proposition are verified, one can therefore associate analytically to $(\mathcal{D}(A), A)$ a unique path-dependent system of projectors $(P_s)_{s\in\mathbb{R}_+}$ through Definition 5.5.14.*

Combining Proposition 5.5.16 and Lemma 5.4.4 yields the following.

Corollary 5.5.18. *Assume that $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ is progressive and fulfills Hypothesis 5.3.5. Suppose that $(\mathcal{D}(A), A)$ is a weak generator of $(P_s)_{s\in\mathbb{R}_+}$. Let $\Phi \in \mathcal{D}(A)$, and fix (s, η). Then $\Phi - \int_0^\cdot A(\Phi)_r dV_r$ admits on $[s, +\infty[$ a $\mathbb{P}^{s,\eta}$ version $M[\Phi]^{s,\eta}$ which is a $(\mathbb{P}^{s,\eta}, \mathbb{F}^{s,\eta})$-cadlag martingale. In particular, the random field defined by $M[\Phi]_{t,u}(\omega) := \Phi_u(\omega) - \Phi_t(\omega) - \int_t^u A\Phi_r(\omega) dV_r$ defines a MAF with cadlag version $M[\Phi]^{s,\eta}$ under $\mathbb{P}^{s,\eta}$.*

We insist on the fact that in previous corollary, Φ is not necessarily cadlag. That result will be crucial in the following Chapter 6.

5.6 Weak solutions of path-dependent SDEs

We will now focus on a more specific type of martingale problem which will be associated to a path-dependent Stochastic Differential Equation with jumps. In this section we will refer to notions of [61] Chapters II, III, VI and [60] Chapter XIV.5.

We fix $m \in \mathbb{N}^*$, $E = \mathbb{R}^m$, the associated canonical space, see Definition 5.3.1, and a finite positive measure F on $\mathcal{B}(\mathbb{R}^m)$ not charging 0.

Definition 5.6.1. *$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, W, p)$ will be called a **space of driving processes** if $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ is a stochastic basis fulfilling the usual conditions, W is an m-dimensional Brownian motion and p is a Poisson measure of intensity $q(dt, dx) := dt \otimes F(dx)$, and W, p are optional for the underlying filtration.*

We now fix the following objects defined on the canonical space.

- β, an $\mathbb{R}^m$-valued $\mathbb{F}^o$-predictable process;
- σ, a $\mathbb{M}_m(\mathbb{R})$-valued $\mathbb{F}^o$-predictable process;
- γ, an $\mathbb{R}^m$-valued $\mathcal{P}re^o \otimes \mathcal{B}(\mathbb{R}^m)$-measurable function on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^m$,

where $\mathbb{M}_m(\mathbb{R})$ denotes the set of real-valued square matrices of size m.

Definition 5.6.2. *Let $(s, \eta) \in \mathbb{R}_+ \times \Omega$. We call a **weak solution of the SDE with coefficients β, σ, γ and starting in** (s, η) any probability measure $\mathbb{P}^{s,\eta}$ on $(\Omega, \mathcal{F})$ such that there exists a space of driving processes $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, W, p)$, on it an m-dimensional $\tilde{\mathbb{F}}$-adapted cadlag process $\tilde{X}$ such that $\mathbb{P}^{s,\eta} = \tilde{\mathbb{P}} \circ \tilde{X}^{-1}$ and such that the following holds.*

Let $\tilde{\beta} := \beta_\cdot(\tilde{X}(\cdot))$, $\tilde{\sigma} := \sigma_\cdot(\tilde{X}(\cdot))$ and $\tilde{\gamma} := \gamma(\cdot, \tilde{X}(\cdot), \cdot)$. We have the following.

- *for all $t \in [0, s]$, $\tilde{X}_t = \eta(t)$ $\tilde{\mathbb{P}}$ a.s.;*
- *$\int_s^\cdot \left(\|\tilde{\beta}_r\| + \|\tilde{\sigma}_r\|^2 + \int_{\mathbb{R}^m} (\|\tilde{\gamma}(r, \cdot, y)\| + \|\tilde{\gamma}(r, \cdot, y)\|^2) F(dy) \right) dr$ takes finite values $\tilde{\mathbb{P}}$ a.s.;*
- *$\tilde{X}_t^i = \eta_i(s) + \int_s^t \tilde{\beta}_r^i dr + \sum_{j \leq m} \int_s^t \tilde{\sigma}_r^{i,j} dW_r^j + \tilde{\gamma}^i \star (p - q)_t$ $\tilde{\mathbb{P}}$ a.s. for all $t \geq s$, $i \leq m$,*

where $\star$ is the integration against random measures, see [61] Chapter II.2.d for instance.

Remark 5.6.3. *Previous Definition 5.6.2 corresponds to Definition 14.73 in [60]. However, in the second item we have required that*

$$\int_s^\cdot \int_{\mathbb{R}^m} (\|\tilde{\gamma}(r, \cdot, y)\| + \|\tilde{\gamma}(r, \cdot, y)\|^2) F(dy) dr$$

takes finite values a.s. so that $\tilde{\gamma} \star (p - q)$ is a well-defined purely discontinuous locally square integrable martingale with angle bracket the $\mathbb{M}_m(\mathbb{R})$-valued process $\int_s^{\cdot \vee s} \int_{\mathbb{R}^m} \tilde{\gamma}\tilde{\gamma}^\intercal(r, \cdot, y) F(dy) dr$, (see Definition 1.27, Proposition 1.28 and Theorem 1.33 in [60] chapter II) and we will not need to use any truncation function.

With this definition, if $\mathbb{P}^{s,\eta}$ is a weak solution of the SDE starting at some (s, η), then under $\mathbb{P}^{s,\eta}$, $(X_t)_{t \geq s}$ is a special semimartingale.

Definition 5.6.4. *Let $s \in \mathbb{R}_+$ and $(Y_t)_{t \geq s}$ be a cadlag special semimartingale defined on the canonical space with (unique) decomposition $Y = Y_s + B + M^c + M^d$ where B is predictable with bounded variation, M^c a continuous local martingale, M^d a purely discontinuous local martingale, all three vanishing at the initial time $t = s$. We will call **characteristics** of Y the triplet (B, C, ν) where $C = \langle M^c \rangle$ and ν is the predictable compensator of the measure of the jumps of Y.*

There are several known equivalent characterizations of weak solutions of path-dependent SDEs with jumps which we will now state in our setup.

Notation 5.6.5. *For every $f \in \mathcal{C}_b^2(\mathbb{R}^m)$ and $t \geq 0$, we denote by $A_t f$ the r.v.*

$$\beta_t \cdot \nabla f(X_t) + \frac{1}{2} Tr(\sigma_t \sigma_t^\intercal \nabla^2 f(X_t)) + \int_{\mathbb{R}^m} (f(X_t + \gamma(t, \cdot, y)) - f(X_t) - \nabla f(X_t) \cdot \gamma(t, \cdot, y)) F(dy). \quad (5.6.1)$$

Proposition 5.6.6. *Let $(s, \eta) \in \mathbb{R}_+ \times \Omega$ be fixed and let $\mathbb{P} \in \mathcal{P}(\Omega)$. There is equivalence between the following properties.*

1. *$\mathbb{P}$ is a weak solution of the SDE with coefficients β, σ, γ starting in (s, η);*
2. *$\mathbb{P}(\omega^s = \eta^s) = 1$ and $(X_t)_{t \geq s}$ is under $\mathbb{P}$ a special semimartingale with characteristics*
   - *$B = \int_s^\cdot \beta_r dr$;*
   - *$C = \int_s^\cdot (\sigma\sigma^\intercal)_r dr$;*
   - *$\nu : (\omega, G) \mapsto \int_s^{+\infty} \int_E \mathbb{1}_G(r, \gamma(\omega, r, y)) \mathbb{1}_{\{\gamma(\omega, r, y) \neq 0\}} F(dy) dr$;*
3. *$\mathbb{P}$ solves $MP^{s,\eta}(\chi)$ where χ is constituted of processes $f(X_\cdot) - \int_0^\cdot A_r f dr$ for all $f \in \mathcal{C}_b^2(\mathbb{R}^m)$.*
4. *$\mathbb{P}$ solves $MP^{s,\eta}(\chi')$ where χ' is constituted of processes $f(X_\cdot) - \int_0^\cdot A_r f dr$ for all functions $f : x \mapsto cos(\theta \cdot x)$ and $f : x \mapsto sin(\theta \cdot x)$ with $\theta \in \mathbb{Q}^m$.*

Proof. Equivalence between items 1. and 2. is a consequence of Theorem 14.80 in [60]. The equivalence between items 2., 3. and 4. if θ was ranging in $\mathbb{R}^m$ is shown in Theorem 2.42 of [61] chapter II. Observe that 4. is stated for $\theta \in \mathbb{R}^m$; however the proof of the implication $(4. \implies 2.)$ in Theorem 2.42 of [61] chapter II only uses the values of θ in $\mathbb{Q}^m$. □

Theorem 5.6.7. *Assume that for any $(s, \eta) \in \mathbb{R}_+ \times \Omega$, the SDE with coefficients β, σ, γ and starting in (s, η) admits a unique weak solution $\mathbb{P}^{s,\eta}$. Then $(\mathbb{P}^{s,\eta})_{(s,\eta) \in \mathbb{R}_+ \times \Omega}$ is a path-dependent canonical class verifying Hypothesis 5.3.5.*

Proof. By Proposition 5.6.6, $\mathbb{P}^{s,\eta}$ is for each (s, η) the unique solution of $MP^{s,\eta}(\chi)$ where χ is constituted of the processes $f(X_\cdot) - \int_s^\cdot A_r f dr$ for all functions $f : x \mapsto cos(\theta \cdot x)$ or $f : x \mapsto sin(\theta \cdot x)$ with $\theta \in \mathbb{Q}^m$. Since χ is a countable set of cadlag $\mathbb{F}^o$-adapted processes which are bounded on bounded intervals, we can conclude by Proposition 5.5.12. □

We recall two classical examples of conditions on the coefficients for which it is known that there is existence and uniqueness of a weak solution for the path-dependent SDE, hence for which the above theorem applies, see Theorem 14.95 and Corollary 14.82 in [60].

Example 5.6.8. *We suppose β, σ, γ to be bounded. Moreover we suppose that for all $n \in \mathbb{N}^*$ there exist $K_2^n \in L^1_{loc}(\mathbb{R}_+)$ and a Borel function $K_3^n : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}$ such that $\int_{\mathbb{R}^m} K_3^n(\cdot, y) F(dy) \in L^1_{loc}(\mathbb{R}_+)$ verifying the following.*

For all $x \in \mathbb{R}^m$, $t \geq 0$ and $\omega, \omega' \in \Omega$ such that $\sup_{r \leq t} \|\omega(r)\| \leq n$ and $\sup_{r \leq t} \|\omega'(r)\| \leq n$, we have

- *$\|\sigma_t(\omega) - \sigma_t(\omega')\| \leq K_2^n(t) \sup_{r \leq t} \|\omega(r) - \omega'(r)\|^2$;*
- *$\|\gamma(t, \omega, x) - \gamma(t, \omega', x)\| \leq K_3^n(t, x) \sup_{r \leq t} \|\omega(r) - \omega'(r)\|^2$.*

Finally we suppose that one of the two following hypotheses is fulfilled.

1. *For all $n \in \mathbb{N}^*$, there exists $K_1^n \in L^1_{loc}(\mathbb{R}_+)$ such that for all $t \geq 0$ and $\omega, \omega' \in \Omega$ verifying $\sup_{r\leq t} \|\omega(r)\| \leq n$ and $\sup_{r\leq t} \|\omega'(r)\| \leq n$, we have $\|\beta_t(\omega) - \beta_t(\omega')\| \leq K_1^n(t) \sup_{r\leq t} \|\omega(r) - \omega'(r)\|$;*
2. *there exists $c > 0$ such that for all $x \in \mathbb{R}^m$, $t \geq 0$ and $\omega \in \Omega$, $x^\intercal \sigma_t(\omega)\sigma_t(\omega)^\intercal x \geq c\|x\|^2$.*

If the assumptions of Theorem 5.6.7 are fulfilled and β, σ (resp. γ) are bounded and continuous in ω for fixed t (resp. fixed t, y), then $(s, \eta) \longmapsto \mathbb{P}^{s,\eta}$ is continuous for the topology of weak convergence, and in particular, the path-dependent canonical class is progressive hence all results of Section 5.4 can be applied with respect to $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$.

Proposition 5.6.9. *Assume that that β, σ, γ are bounded. Let $(s_n, \eta_n)_n$ be a sequence in Λ which converges to some (s, η). For every $n \in \mathbb{N}$, let $\mathbb{P}^n$ be a weak solution starting in (s_n, η_n) of the SDE with coefficients β, σ, γ. Then $(\mathbb{P}^n)_{n\geq 0}$ is tight.*

We recall some notations from [61] Chapter VI which we will use in the proof of Proposition 5.6.9.

Notation 5.6.10. *For any $\omega \in \Omega$ and interval $\mathcal{I}$ of $\mathbb{R}_+$, we denote $W(\omega, \mathcal{I}) := \sup_{s,t\in\mathcal{I}} \|\omega(t) - \omega(s)\|$. For any $\omega \in \Omega$, $N \in \mathbb{N}^\star$ and $\theta > 0$, we write*

$$W_N(\omega, \theta) := \sup_{0\leq t\leq t+\theta\leq N} W(\omega, [t, t+\theta]) = \sup_{s,t\in[0,N]:\, |t-s|\leq\theta} \|\omega(t) - \omega(s)\|.$$

For any $\omega \in \Omega$, $N \in \mathbb{N}^\star$ and $\theta > 0$, we denote

$$W'_N(\omega, \theta) := \inf \left\{ \max_{i\leq r} W(\omega, [t_{i-1}, t_i[) : \quad 0 = t_0 < \cdots < t_r = N; \quad \forall 1 \leq i \leq r : t_i - t_{i-1} \geq \theta \right\}.$$

We will also recall the classical general tightness criterion in $\mathcal{P}(\Omega)$ which one can find for example in Theorem 3.21 of [61] Chapter VI.

Theorem 5.6.11. *Let $(\mathbb{P}^n)_{n\geq 0}$ be a sequence of elements of $\mathcal{P}(\Omega)$, then it is tight iff it verifies the two following conditions.*

$$\begin{cases} \forall N \in \mathbb{N}^* \quad \forall \epsilon > 0 \quad \exists K > 0 \quad \forall n \in \mathbb{N} : \quad \mathbb{P}^n \left(\sup_{t\leq N} \|\omega(t)\| > K \right) \leq \epsilon \\ \forall N \in \mathbb{N}^* \quad \forall \epsilon > 0 \quad \forall \alpha > 0 \quad \exists \theta \quad \forall n \in \mathbb{N} : \quad \mathbb{P}^n(W'_N(\omega, \theta) < \alpha) \geq 1 - \epsilon. \end{cases} \tag{5.6.2}$$

Finally we will also need to introduce a definition.

Definition 5.6.12. *A sequence of probability measures on $(\Omega, \mathcal{F})$ is called $\mathcal{C}$-tight if it is tight and if each of its limiting points has all its support in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^m)$.*

Proof of Proposition 5.6.9.
We fix a converging sequence $(s_n, \eta_n) \underset{n}{\longrightarrow} (s, \eta)$ in Λ, and for every n, a weak solution $\mathbb{P}^n$ of the SDE with coefficients β, σ, γ starting in (s_n, η_n). In order to show that $(\mathbb{P}^n)_{n\geq 0}$ is tight, we will use Theorem 5.6.11. The main idea consists in combining the fact that the canonical process X under $\mathbb{P}^n$ is deterministic on $[0, s_n]$, where it coincides with η_n with the fact that on $[s_n, +\infty[$ it is a semimartingale with known characteristics. So we will split the study of the modulus of continuity of path ω on these two intervals $[0, s_n]$ and $[s_n, +\infty[$.

Since η_n tends to η, the set $\{\eta_n : n \geq 0\}$ is relatively compact in Ω so by Theorem 1.14.b in [61] Chapter VI we have

$$\begin{cases} \forall N \in \mathbb{N}^* \quad \exists K_1 > 0 \quad \forall n \in \mathbb{N} : \quad \sup_{t\in[0,N]} \|\eta_n(t)\| \leq K_1 \\ \forall N \in \mathbb{N}^* \quad \forall \alpha > 0 \quad \exists \theta_1 > 0 \quad \forall n \in \mathbb{N} : \quad W'_N(\eta_n, \theta_1) < \alpha. \end{cases} \tag{5.6.3}$$

For fixed $n \in \mathbb{N}$, we now introduce the process
$X^n : \omega \longmapsto \eta_n(s_n)\mathbb{1}_{[0,s_n[} + \omega\mathbb{1}_{[s_n,+\infty[}$, we denote by $\mathbb{Q}^n := \mathbb{P}^n \circ (X^n)^{-1} \in \mathcal{P}(\Omega)$ its law under $\mathbb{P}^n$ and we now show that $(\mathbb{Q}^n)_{n\geq 0}$ is tight.

By Proposition 5.6.6, under $\mathbb{P}^n$, $(X_t)_{t\in[s_n,+\infty[}$ is a special semimartingale with initial value $\eta_n(s_n)$ and characteristics (see Definition 5.6.4) $\int_{s_n}^{\cdot} \beta_r dr$, $\int_{s_n}^{\cdot} (\sigma\sigma^\intercal)_r dr$ and
$(\omega, A) \mapsto \int_{s_n}^{+\infty} \int_{\mathbb{R}^m} \mathbb{1}_A(r, \gamma(r, \omega, y))\mathbb{1}_{\{\gamma(r,\omega,y)\neq 0\}} F(dy)dr$. Therefore, since X^n is constant on $[0, s_n[$ and since on $[s_n, +\infty[$ its law under $\mathbb{P}^n$ coincides with the one of X, we can say that $\mathbb{Q}^n$ is the law of a special semimartingale (starting at time $t = 0$) with initial value $\eta_n(s_n)$, and characteristics $\int_0^{\cdot} \mathbb{1}_{[s_n,+\infty[}(r)\beta_r dr$, $\int_0^{\cdot} \mathbb{1}_{[s_n,+\infty[}(r)(\sigma\sigma^\intercal)_r dr$ and
$(\omega, G) \mapsto \int_0^{+\infty} \mathbb{1}_{[s_n,+\infty[}(r) \int_{\mathbb{R}^m} \mathbb{1}_G(r, \gamma(r, \omega, y))\mathbb{1}_{\{\gamma(r,\omega,y)\neq 0\}} F(dy)dr$.
Theorem 4.18 in [61] chapter VI implies that $(\mathbb{Q}^n)_{n\geq 0}$ is tight if and only if the properties below hold true.

1. $(\mathbb{Q}^n \circ X_0^{-1})_{n\geq 0}$ is tight;
2. the following sequences are $\mathcal{C}$-tight (under $(\mathbb{Q}^n)_{n\geq 0}$):
   (a) $(B^n := \int_0^{\cdot} \mathbb{1}_{[s_n,+\infty[}(r)\beta_r dr)_{n\geq 0}$;
   (b) $\left(\tilde{C}^n := \int_0^{\cdot} \mathbb{1}_{[s_n,+\infty[}(r) \left((\sigma\sigma^\intercal)_r + \int_{\mathbb{R}^m} (\gamma\gamma^\intercal)(r, \cdot, y)F(dy)\right) dr\right)_{n\geq 0}$;
   (c) $\left(G_p^n := \int_0^{\cdot} \mathbb{1}_{[s_n,+\infty[}(r) \int_{\mathbb{R}^m} \mathbb{1}_{\{\gamma(r,\omega,y)\neq 0\}} ((p\|\gamma(r, \cdot, y)\| - 1)^+) \wedge 1F(dy)dr\right)_{n\geq 0}$ for all $p \in \mathbb{N}$;
3. for all $N > 0$, $\epsilon > 0$,

$$\lim_{a\to\infty} \sup_n \mathbb{Q}^n \left(\int_{s_n}^N \int_{\mathbb{R}^m} \mathbb{1}_{\{\|\gamma(r,\cdot,y)\|>a\}} F(dy)dr > \epsilon \right) = 0. \tag{5.6.4}$$

Item 3. trivially holds since γ is bounded. At this point $\eta_n(s_n)$ is a bounded sequence according to the first line of (5.6.3) and the fact that the sequence $(s_n)_{n\geq 0}$ is bounded, so $(\mathbb{Q}^n \circ X_0^{-1})_{n\geq 0} = (\delta_{\eta_n(s_n)})_{n\geq 0}$ is obviously tight. We are left to show item 2. By Proposition 3.36 in [61] chapter VI, items 2. (a) and 2. (b) hold if $(Var(B^n))_{n\geq 0} = (\int_0^{\cdot} \mathbb{1}_{[s_n,+\infty[}(r)\|\beta_r\|dr)_{n\geq 0}$ and
$(Tr(\tilde{C}^n))_{n\geq 0} = \left(\int_0^{\cdot} \mathbb{1}_{[s_n,+\infty[}(r) \left(Tr(\sigma\sigma_r^\intercal) + \int_{\mathbb{R}^m} Tr(\gamma\gamma^\intercal(r, \cdot, y))F(dy)\right) dr\right)_{n\geq 0}$ are $\mathcal{C}$-tight. Finally, β, σ, γ, F being bounded, there exists some strictly positive constant K such that all the processes given below are increasing:

- $t \mapsto Kt - Var(B^n)_t, \quad n \geq 0$;
- $t \mapsto Kt - Tr(\tilde{C}_t^n), \quad n \geq 0$;
- $t \mapsto Kt - (G_p^n)_t, \quad n, p \geq 0$.

In the terminology of [61] chapter VI, this means that the increasing processes $Var(B^n), \quad n \geq 0$, $Tr(\tilde{C}^n), \quad n \geq 0$, $G_p^n \quad n, p \geq 0$ are strongly dominated by the increasing function $t \mapsto Kt$. The singleton $t \mapsto Kt$ being trivially $\mathcal{C}$-tight, Proposition 3.35 in [61] chapter VI implies that the dominated sequences of processes $(Var(B^n))_{n\geq 0}$, $(Tr(\tilde{C}^n))_{n\geq 0}$ and $(G_p^n)_{n\geq 0}$ for all p are $\mathcal{C}$-tight. Finally $(\mathbb{Q}^n)_{n\geq 0}$ is tight.

Now by Theorem 5.6.11 this implies that

$$\begin{cases} \forall N \in \mathbb{N}^* \quad \forall \epsilon > 0 \quad \exists K_2 > 0 \quad \forall n \in \mathbb{N} : \quad \mathbb{Q}^n \left(\sup_{t\leq N} \|\omega(t)\| > K_2 \right) \leq \epsilon \\ \forall N \in \mathbb{N}^* \quad \forall \epsilon > 0 \quad \forall \alpha > 0 \quad \exists \theta_2 \quad \forall n \in \mathbb{N} : \quad \mathbb{Q}^n(W'_N(\omega, \theta_2) < \alpha) \geq 1 - \epsilon. \end{cases} \tag{5.6.5}$$

Combining the first line of (5.6.3) and the first line of (5.6.5) and by construction of $\mathbb{Q}^n$, taking $K = K_1 + K_2$ for instance, we have

$$\forall N \in \mathbb{N}^* \quad \forall \epsilon > 0 \quad \exists K > 0 \quad \forall n \in \mathbb{N} : \quad \mathbb{P}^n \left(\sup_{t \leq N} \|\omega(t)\| > K \right) \leq \epsilon. \tag{5.6.6}$$

Our aim is now to show that

$$\forall N \in \mathbb{N}^* \quad \forall \epsilon > 0 \quad \forall \alpha > 0 \quad \exists \theta \quad \forall n \in \mathbb{N} : \quad \mathbb{P}^n(W'_N(\omega, \theta) < \alpha) \geq 1 - \epsilon; \tag{5.6.7}$$

this combined with (5.6.6) will imply by Theorem 5.6.11 that $(\mathbb{P}^n)_{n \geq 0}$ is tight.

In what follows, if $\eta, \omega \in \Omega$ and $s \in \mathbb{R}_+$, $\eta \otimes_s \omega$ will denote the path $\eta \mathbb{1}_{[0,s[} + \omega \mathbb{1}_{[s,+\infty[}$, which still belongs to Ω.

By construction of $\mathbb{Q}^n$, for every n, $\mathbb{P}^n$ is the law of $\eta_n \otimes_{s_n} \omega$ under $\mathbb{Q}^n$. Therefore, (5.6.7) is equivalent to

$$\forall N \in \mathbb{N}^* \quad \forall \epsilon > 0 \quad \forall \alpha > 0 \quad \exists \theta \quad \forall n \in \mathbb{N} : \quad \mathbb{Q}^n(W'_N(\eta_n \otimes_{s_n} \omega, \theta) < \alpha) \geq 1 - \epsilon, \tag{5.6.8}$$

and this is what we will now show to conclude the proof of Proposition 5.6.9. So we prove (5.6.8).

We fix some $N \in \mathbb{N}^*$, $\alpha > 0$ and $\epsilon > 0$. Combining the second lines of (5.6.3) and of (5.6.5), there exists $\theta > 0$ such that for all $n \geq 0$,

$$\begin{cases} W'_N(\eta_n, \theta) < \frac{\alpha}{4} \\ \mathbb{Q}^n(W'_N(\omega, \theta) < \frac{\alpha}{4}) \geq 1 - \epsilon. \end{cases} \tag{5.6.9}$$

We show below that, for every n

$$\{\omega | W'_N(\omega, \theta) < \frac{\alpha}{4}\} \subset \{\omega | W'_N(\eta_n \otimes_{s_n} \omega, \theta) < \alpha\}. \tag{5.6.10}$$

This together with (5.6.9) will imply that for all n,

$$\mathbb{Q}^n(W'_N(\eta_n \otimes_{s_n} \omega, \theta) < \alpha) \geq \mathbb{Q}^n(W'_N(\omega, \theta) < \frac{\alpha}{4}) \geq 1 - \epsilon,$$

hence that (5.6.8) is verified.

We fix n. To establish (5.6.10) let ω be such that $W'_N(\omega, \theta) < \frac{\alpha}{4}$; we need to show that

$$W'_N(\eta_n \otimes_{s_n} \omega, \theta) < \alpha. \tag{5.6.11}$$

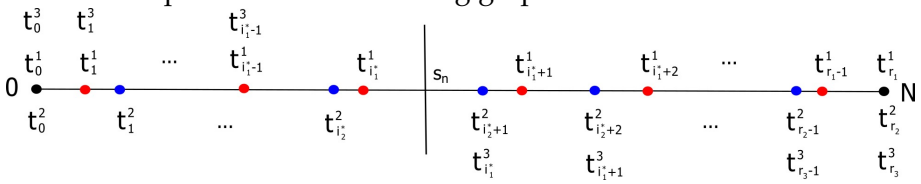
By the first line of (5.6.9) and the definition of W'_N (see Notation 5.6.10), there exist two subdivisions of $[0, N]$ $0 = t_0^1 < \cdots < t_{r_1}^1 = N$, $0 = t_0^2 < \cdots < t_{r_2}^2 = N$ with increments $t_i^j - t_{i-1}^j \geq \theta$ for all $1 \leq i \leq r_j$ and $j = 1, 2$, such that

$$\begin{cases} W(\eta_n, [t_{i-1}^1, t_i^1[) & \leq & \frac{\alpha}{4} \text{ for all } 1 \leq i \leq r_1 \\ W(\omega, [t_{i-1}^2, t_i^2[) & \leq & \frac{\alpha}{4} \text{ for all } 1 \leq i \leq r_2. \end{cases} \tag{5.6.12}$$

We set $i_j^* := \max \{i : t_i^j \leq s_n\}$ for $j = 1, 2$ and introduce the third subdivision

$$(t_0^3, \cdots, t_{r_3}^3) := (t_0^1, \cdots, t_{i_1^*-1}^1, t_{i_2^*+1}^2, \cdots, t_{r_2}^2), \tag{5.6.13}$$

which we represent in the following graphic.



As for the other two, the subdivision of $[0,N]$ above verifies $t^3_i - t^3_{i-1} \geq \theta$ for all i. Indeed, $t^3_i - t^3_{i-1}$ is either equal to $t^1_i - t^1_{i-1} \geq \theta$ for some i, or to $t^2_j - t^2_{j-1} \geq \theta$ for some j, or to $t^2_{i^*_2+1} - t^1_{i^*_1-1} \geq t^1_{i^*_1} - t^1_{i^*_1-1} \geq \theta$ where the first inequality follows by the fact that $t^1_{i^*_1-1} \leq t^1_{i^*_1} \leq s_n < t^2_{i^*_2+1}$.

Now by definition of $W'_N(\eta_n \otimes_{s_n} \omega, \theta)$, in order to show (5.6.11) and conclude this proof, it is enough to show that

$$W(\eta_n \otimes_{s_n} \omega, [t^3_{i-1}, t^3_i[) < \alpha, \tag{5.6.14}$$

for all $1 \leq i \leq r_3$.

If $i \leq i^*_1 - 1$, then $[t^3_{i-1}, t^3_i[= [t^1_{i-1}, t^1_i[\subset [0, s_n[$ where $\eta_n \otimes_{s_n} \omega$ coincides with η_n so $W(\eta_n \otimes_{s_n} \omega, [t^3_{i-1}, t^3_i[) = W(\eta_n, [t^1_{i-1}, t^1_i[) \leq \frac{\alpha}{4} < \alpha$ by the first line of (5.6.12). Similarly, if $i \geq i^*_1 + 1$, then $[t^3_{i-1}, t^3_i[= [t^2_{i-i^*_1+i^*_2}, t^2_{i-i^*_1+i^*_2+1}[\subset [s_n, +\infty[$ where $\eta_n \otimes_{s_n} \omega$ coincides with ω so $W(\eta_n \otimes_{s_n} \omega, [t^3_{i-1}, t^3_i[) = W(\omega, [t^2_{i-i^*_1+i^*_2}, t^2_{i-i^*_1+i^*_2+1}[) \leq \frac{\alpha}{4} < \alpha$ by the second line of (5.6.12). Finally, we consider the specific case $i = i^*_1$ meaning that $[t^3_{i-1}, t^3_i[= [t^1_{i^*_1-1}, t^2_{i^*_2+1}[$ contains s_n. We have

$$\begin{array}{rcl} W(\eta_n \otimes_{s_n} \omega, [t^1_{i^*_1-1}, t^2_{i^*_2+1}[) & \leq & W(\eta_n \otimes_{s_n} \omega, [t^1_{i^*_1-1}, t^1_{i^*_1}[) \\ & + & W(\eta_n \otimes_{s_n} \omega, [t^1_{i^*_1}, s_n[) + W(\eta_n \otimes_{s_n} \omega, [s_n, t^2_{i^*_2+1}[) \\ & \leq & W(\eta_n, [t^1_{i^*_1-1}, t^1_{i^*_1}[) + W(\eta_n, [t^1_{i^*_1}, s_n[) + W(\omega, [s_n, t^2_{i^*_2+1}[) \\ & \leq & W(\eta_n, [t^1_{i^*_1-1}, t^1_{i^*_1}[) + W(\eta_n, [t^1_{i^*_1}, t^1_{i^*_1+1}[) \\ & + & W(\omega, [t^2_{i^*_2}, t^2_{i^*_2+1}[) \\ & \leq & \frac{\alpha}{4} + \frac{\alpha}{4} + \frac{\alpha}{4} \\ & < & \alpha, \end{array} \tag{5.6.15}$$

by (5.6.12). So (5.6.14) is verified for all i and the proof is complete.

□

Proposition 5.6.13. *Assume that β, σ (resp. γ) are bounded and that for Lebesgue almost all t (resp. $dt \otimes dF$ almost all (t,y)), $\beta(t,\cdot), \sigma(t,\cdot)$ (resp. $\gamma(t,\cdot,y)$) are continuous. Assume that for any $(s,\eta) \in \mathbb{R}_+ \times \Omega$ there exists a unique weak solution $\mathbb{P}^{s,\eta}$ of the SDE of coefficients β, σ, γ starting in (s,η). Then* $\begin{array}{ccc} (s,\eta) & \longmapsto & \mathbb{P}^{s,\eta} \\ \Lambda & \longrightarrow & \mathcal{P}(\Omega) \end{array}$ *is continuous. Moreover the path-dependent canonical class $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ is progressive.*

Remark 5.6.14. *Taking Theorem 5.6.7 into account, the family of probabilities $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ of Proposition 5.6.13 constitutes a progressive path-dependent canonical class verifying Hypothesis 5.3.5. It therefore verifies Hypothesis 5.4.1 and all results of Section 5.4 apply.*

Proof. of Proposition 5.6.13.
We consider a convergent sequence $(s_n, \eta_n) \underset{n}{\longrightarrow} (s,\eta)$ in Λ. Since β, σ, γ are bounded, by Proposition 5.6.9 $(\mathbb{P}^{s_n,\eta_n})_{n\in\mathbb{N}}$ is tight, hence relatively compact by Prokhorov's theorem. We consider a subsequence $\mathbb{P}^{s_{n_k},\eta_{n_k}} \underset{k}{\longrightarrow} \mathbb{Q}$ and we show below that $\mathbb{Q}$ is a weak solution of the SDE with coefficients β, σ, γ, starting at (s,η). Since that problem has a unique solution, we will have $\mathbb{Q} = \mathbb{P}^{s,\eta}$. This will imply that $\mathbb{P}^{s_n,\eta_n} \underset{n}{\longrightarrow} \mathbb{P}^{s,\eta}$, hence the announced continuity.

We will indeed verify item 3. of Proposition 5.6.6. For the convenience of the reader, we will omit the extraction of the subsequence in the notations.

We start by showing

$$\mathbb{Q}(\omega^s = \eta^s) = 1. \tag{5.6.16}$$

The set

$$D := \{t \in \mathbb{R}_+ : \mathbb{Q}(X_t \neq X_{t^-}) > 0\} \cup \left\{t \in [0,s] : \eta(t) \neq \eta(t^-)\right\}, \tag{5.6.17}$$

is countable first because η is a cadlag function and second because of Proposition 3.12 in [61] Chapter VI which states that, for every probability $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, the set $D_0 := \{t \in \mathbb{R}_+ : \mathbb{Q}(X_t \neq X_{t^-}) > 0\}$ is countable. If $t \notin D_0$ then

$$\mathbb{P}^{s_n,\eta_n} \circ X_t^{-1} \underset{n}{\Longrightarrow} \mathbb{Q} \circ X_t^{-1}, \tag{5.6.18}$$

by Proposition 3.14 ibidem. Since η_n converges to η in the Skorokhod topology, if $t \notin D$ (t is a continuity point of η), then it follows that $\eta_n(t) \underset{n}{\longrightarrow} \eta(t)$, see Proposition 2.3 of [61] Chapter VI.

Let $\epsilon > 0$, $t \in [0, s-\epsilon] \cap D^c$ be fixed. Since s_n tends to s, we can suppose without loss of generality that $s_n \geq s - \epsilon$ for all n, so that $\mathbb{P}^{s_n,\eta_n} \circ X_t^{-1} = \delta_{\eta_n(t)}$. By (5.6.18) this sequence converges to $\mathbb{Q} \circ X_t^{-1}$ which is therefore necessarily equal to $\delta_{\eta(t)}$ since $\eta_n(t)$ tends to $\eta(t)$ being $t \notin D$. This means that

$$\mathbb{Q}(\omega(t) = \eta(t)) = 1, \tag{5.6.19}$$

for all $t \in [0, s-\epsilon] \cap D^c$. Since $\epsilon > 0$ is arbitrary, (5.6.19) holds for all $t \in [0, s[\cap D^c$; and since ω is right-continuous and D is countable, (5.6.19) holds for all $t \in [0, s[$. We will now show that (5.6.19) also holds for $t = s$. We first note that

$$\eta_n(s_n) \underset{n}{\longrightarrow} \eta(s). \tag{5.6.20}$$

Indeed, without restriction of generality we can consider that $s_n \leq s + 1$, so since $(s_n, \eta_n(s_n)) \in \Lambda$, η_n is constantly equal to $\eta_n(s_n)$ on $[s_n, +\infty[$ which contains $[s+1, +\infty[$. On the other hand η is constantly equal to $\eta(s)$ on $[s, +\infty[$ which also contains $[s+1, +\infty[$, and η_n tends to η almost everywhere on that interval, because it converges in the Skorokhod sense. So necessarily (5.6.20) holds.

We fix now some $f \in \mathcal{C}_c^\infty(\mathbb{R}^m)$. For all n, since $\mathbb{P}^{s_n,\eta_n}$ is a weak solution of the SDE starting at (s_n, η_n) and by Proposition 5.6.6, it follows that $f(\omega(\cdot)) - f(\eta_n(s_n)) - \int_{s_n}^{\cdot} A_r f(\omega) dr$ (see Notation 5.6.5) is a martingale on $[s_n, +\infty[$ under $\mathbb{P}^{s_n,\eta_n}$ vanishing in s_n. We consider a sequence $(t_p)_{p\in\mathbb{N}}$ in D^c converging to t strictly from the right. For all n, p we have

$$\begin{aligned} \mathbb{E}^{s_n,\eta_n}[f(\omega(t_p))] &= f(\eta_n(s_n)) + \mathbb{E}^{s_n,\eta_n}\left[\int_{s_n}^{t_p} A_r f(\omega) dr\right] \\ &= f(\eta_n(s_n)) + \mathbb{E}^{s_n,\eta_n}\left[\int_{s}^{t_p} A_r f(\omega) dr\right] + \int_{s_n}^{s} \mathbb{E}^{s_n,\eta_n}[A_r f(\omega)] dr, \end{aligned} \tag{5.6.21}$$

where the second equality holds by Fubini's theorem since $A_r f(\omega)$ is uniformly bounded for r varying on bounded intervals. We now pass to the limit in n. Since $t_p \notin D$, taking into account (5.6.18), we have $\mathbb{P}^{s_n,\eta_n} \circ X_{t_p}^{-1} \underset{n}{\Longrightarrow} \mathbb{Q} \circ X_{t_p}^{-1}$; moreover f is bounded and continuous, so

$$\mathbb{E}^{s_n,\eta_n}[f(\omega(t_p))] \underset{n}{\longrightarrow} \mathbb{E}^{\mathbb{Q}}[f(\omega(t_p))]. \tag{5.6.22}$$

Since β, σ, γ are bounded and $\beta(r,\cdot), \sigma(r,\cdot)$ (resp. $\gamma(r,\cdot,y)$) are continuous for Lebesgue almost all r (resp. $dt \otimes dF$ almost all (r,y)) and since $f \in \mathcal{C}_c^\infty$, then $\Phi : \omega \longmapsto \int_s^{t_p} A_r f(\omega) dr$ is a bounded continuous functional for the Skorokhod topology, so

$$\mathbb{E}^{s_n,\eta_n}\left[\int_s^{t_p} A_r f(\omega) dr\right] \underset{n}{\longrightarrow} \mathbb{E}^{\mathbb{Q}}\left[\int_s^{t_p} A_r f(\omega) dr\right]. \tag{5.6.23}$$

Finally since s_n tends to s and $A_r f$ is uniformly bounded for r varying on bounded intervals, we have

$$\int_{s_n}^{s} \mathbb{E}^{s_n,\eta_n}[A_r f(\omega)] dr \underset{n}{\longrightarrow} 0. \tag{5.6.24}$$

Combining relations (5.6.21), (5.6.20), (5.6.22), (5.6.23), (5.6.24), for all p, we get

$$\mathbb{E}^{\mathbb{Q}}[f(\omega(t_p))] = f(\eta(s)) + \mathbb{E}^{\mathbb{Q}}\left[\int_s^{t_p} A_r f(\omega) dr\right]. \tag{5.6.25}$$

We now pass to the limit in p. Since t_p tends to s from the right and ω is right-continuous, the left-hand side of (5.6.25) tends to $\mathbb{E}^{\mathbb{Q}}[f(\omega(s))]$. By dominated convergence, the second term in the right-hand side of (5.6.25) tends to 0. This yields $\mathbb{E}^{\mathbb{Q}}[f(\omega(s))] = f(\eta(s))$ and this holds for all $f \in \mathcal{C}_c^\infty(\mathbb{R}^m)$, which implies that $\mathbb{Q} \circ X_s^{-1} = \delta_{\eta(s)}$. So we have shown (5.6.16).

We will proceed showing that $\mathbb{Q}$ solves weakly the SDE with respect to β, σ, γ starting in (s, η). By Proposition 5.6.6 this holds iff for any $f \in \mathcal{C}_b^2(\mathbb{R}^m)$, $f(X_\cdot) - \int_s^\cdot A_r f dr$ is a $(\mathbb{Q}, \mathbb{F}^o)$-martingale. We fix such an f, some $t \leq u$ in $]s, +\infty[\cap D^c$, $N \in \mathbb{N}^*$, $t_1 \leq \cdots \leq t_N \in [s,t] \cap D^c$ and $\phi_1, \cdots, \phi_N \in \mathcal{C}_b(\mathbb{R}^m, \mathbb{R})$. Taking into account Proposition 5.6.6, since $s < t$, for n large enough, we can suppose that $f(X_\cdot) - \int_t^\cdot A_r f dr$ is under every $\mathbb{P}^{s_n,\eta_n}$ a martingale on the interval $[t, +\infty[$. Therefore, for all n, we have

$$\mathbb{E}^{s_n,\eta_n}\left[\left(f(\omega(u)) - f(\omega(t)) - \int_t^u A_r f(\omega) dr\right) \prod_{1\leq i\leq N} \phi_i(\omega(t_i))\right] = 0. \tag{5.6.26}$$

We wish to pass to the limit in n. By Theorem 12.5 in [16], for any $r \in \mathbb{R}_+$, the mapping X_r is continuous on the set $C_r := \{\omega \in \Omega : \omega(r) = \omega(r^-)\}$. By construction of D and since $t, u, t_1, \cdots, t_N \notin D$, then $C_t, C_u, C_{t_1}, \cdots, C_{t_N}$ are of full $\mathbb{Q}$-measure hence $\Phi := (X, X_u, X_t, X_{t_1}, \cdots, X_{t_N})$ is continuous on a set of full $\mathbb{Q}$-measure. By the mapping theorem (see Theorem 2.7 in [16] for instance), since $\mathbb{P}^{s_n,\eta_n} \underset{n}{\Longrightarrow} \mathbb{Q}$ and Φ is continuous on a set of full $\mathbb{Q}$-measure, then $\mathbb{P}^{s_n,\eta_n} \circ \Phi^{-1} \underset{n}{\Longrightarrow} \mathbb{Q} \circ \Phi^{-1}$, meaning $\mathbb{P}^{s_n,\eta_n} \circ (X, X_u, X_t, X_{t_1}, \cdots, X_{t_N})^{-1} \underset{n}{\Longrightarrow} \mathbb{Q} \circ (X, X_u, X_t, X_{t_1}, \cdots, X_{t_N})^{-1}$. Since $\omega \mapsto \int_t^u A_r f(\omega) dr, f, \phi_1, \cdots, \phi_N$ are bounded continuous functions, the previous convergence in law allows to pass to the limit in n in (5.6.26) so that for any $t \leq u \in]s, +\infty[\cap D^c$ and $t_1, \cdots, t_N \in [s,t] \cap D^c$

$$\mathbb{E}^{\mathbb{Q}}\left[\left(f(\omega(u)) - f(\omega(t)) - \int_t^u A_r f(\omega) dr\right) \prod_{1\leq i\leq N} \phi_i(\omega(t_i))\right] = 0. \tag{5.6.27}$$

Equality (5.6.27) still holds if $t = s$ and if some of the values $t, u, t_1, \cdots, t_N$ belong to D. Indeed to show this statement we approximate from the right such values by sequences of times not belonging to D and strictly greater than s and we then use the right-continuity of ω and the dominated convergence theorem.

By use of the functional monotone class theorem (see Theorem 21 in [32] Chapter I), we have

$$\mathbb{E}^{\mathbb{Q}}\left[\left(f(\omega(u)) - f(\omega(t)) - \int_t^u A_r f(\omega) dr\right) \mathbb{1}_G\right] = 0, \tag{5.6.28}$$

for any $s \leq t \leq u$ and $G \in \sigma(X_r | r \in [s,t])$. Since $\mathbb{Q}(\omega^s = \eta^s) = 1$ then $\mathcal{F}_s^o$ is $\mathbb{Q}$-trivial, so equality (5.6.28) holds for all $G = G_s \cap G_t^s$ where $G_s \in \mathcal{F}_s^o$ and $G_t^s \in \sigma(X_r | r \in [s,t])$. Events of such type form a π-system generating $\mathcal{F}_t^o$ so by Dynkin's Lemma, (5.6.28) holds for all $G \in \mathcal{F}_t^o$. For all $s \leq t \leq u$, then we have

$$\mathbb{E}^{\mathbb{Q}}\left[\left(f(\omega(u)) - f(\omega(t)) - \int_t^u A_r f(\omega)_r dr\right) \middle| \mathcal{F}_t^o\right] = 0. \tag{5.6.29}$$

So $f(X) - \int_s^\cdot A_r f_r dr$ is a $(\mathbb{Q}, \mathbb{F}^o)$-martingale. This implies that $\mathbb{Q}$ is a weak solution of the SDE with coefficients β, σ, γ starting in (s, η). As anticipated, since the SDE is well-posed for every (s, η), we have $\mathbb{Q} = \mathbb{P}^{s,\eta}$ and the proof of the first statement is complete.

The second statement follows from the fact that a continuous function is Borel and that $\mathcal{B}(\Lambda) = \Lambda \cap \mathcal{P}ro^o$, see Proposition 5.5.3. □

Appendix

5.A Proofs of Section 5.4

Proof of Proposition 5.4.6.
In the whole proof $t < u$ will be fixed. We consider a sequence of subdivisions of $[t,u]$: $t = t_1^k < t_2^k < \cdots < t_k^k = u$ such that $\min_{i<k}(t_{i+1}^k - t_i^k) \underset{k\to\infty}{\longrightarrow} 0$. Let $(s,\eta) \in [0,t]\times\Omega$ with corresponding probability $\mathbb{P}^{s,\eta}$. For any k, we have $\sum_{i<k}\left(M_{t_i^k,t_{i+1}^k}\right)^2 = \sum_{i<k}(M^{s,\eta}_{t_{i+1}^k} - M^{s,\eta}_{t_i^k})^2$ $\mathbb{P}^{s,\eta}$ a.s., so by definition of quadratic variation we know that $\sum_{i<k}\left(M_{t_i^k,t_{i+1}^k}\right)^2 \underset{k\to\infty}{\overset{\mathbb{P}^{s,\eta}}{\longrightarrow}} [M^{s,\eta}]_u - [M^{s,\eta}]_t$. In the sequel we will construct an $\mathcal{F}_u^o$-measurable random variable $[M]_{t,u}$ such that for any $(s,\eta)\in[0,t]\times\Omega$, $\sum_{i<k}\left(M_{t_i^k,t_{i+1}^k}\right)^2 \underset{k\to\infty}{\overset{\mathbb{P}^{s,\eta}}{\longrightarrow}} [M]_{t,u}$. In that case $[M]_{t,u}$ will then be $\mathbb{P}^{s,\eta}$ a.s. equal to $[M^{s,\eta}]_u - [M^{s,\eta}]_t$.

Let $\eta\in\Omega$. $[M^{t,\eta}]$ is $\mathbb{F}^{t,\eta}$-adapted, so $[M^{t,\eta}]_u - [M^{t,\eta}]_t$ is $\mathcal{F}_u^{t,\eta}$-measurable and by Corollary 5.3.21, there is an $\mathcal{F}_u^o$-measurable variable which depends on (t,u,η), that we denote $\omega\mapsto a_{t,u}(\eta,\omega)$ such that $a_{t,u}(\eta,\omega) = [M^{t,\eta}]_u - [M^{t,\eta}]_t$, $\mathbb{P}^{t,\eta}$ a.s.

We will show below that there is a jointly $\mathcal{F}_t^o\otimes\mathcal{F}_u^o$ -measurable version of $(\eta,\omega)\mapsto a_{t,u}(\eta,\omega)$.

For every integer $n\geq 0$, we set $a^n_{t,u}(\eta,\omega) := n\wedge a_{t,u}(\eta,\omega)$ which is in particular limit in probability of $n\wedge\sum_{i\leq k}\left(M_{t_i^k,t_{i+1}^k}\right)^2$ under $\mathbb{P}^{t,\eta}$. For any integers k,n and any $\eta\in\Omega$, we define the finite positive measures $\mathbb{Q}^{k,n,\eta}$, $\mathbb{Q}^{n,\eta}$ and $\mathbb{Q}^{\eta}$ on $(\Omega,\mathcal{F}_u^o)$ by

1. $\mathbb{Q}^{k,n,\eta}(F) := \mathbb{E}^{t,\eta}\left[\mathbb{1}_F\left(n\wedge\sum_{i<k}\left(M_{t_i^k,t_{i+1}^k}\right)^2\right)\right]$;
2. $\mathbb{Q}^{n,\eta}(F) := \mathbb{E}^{t,\eta}[\mathbb{1}_F\left(a^n_{t,u}(\eta,\omega)\right)]$;
3. $\mathbb{Q}^{\eta}(F) := \mathbb{E}^{t,\eta}[\mathbb{1}_F\left(a_{t,u}(\eta,\omega)\right)]$.

When k and n are fixed integers and F is a fixed event, by Remark 5.3.6,
$\eta\longmapsto\mathbb{E}^{t,\eta}\left[F\left(n\wedge\sum_{i<k}\left(M_{t_i^k,t_{i+1}^k}\right)^2\right)\right]$, is $\mathcal{F}_t^o$ -measurable.

Then $n\wedge\sum_{i<k}\left(M_{t_i^k,t_{i+1}^k}\right)^2 \underset{k\to\infty}{\overset{\mathbb{P}^{t,\eta}}{\longrightarrow}} a^n_{t,u}(\eta,\omega)$, and this sequence is uniformly bounded by the constant n, so the convergence takes place in L^1, therefore $\eta\longmapsto\mathbb{Q}^{n,\eta}(F)$ is also $\mathcal{F}_t^o$-measurable as the pointwise limit in k of the functions $\eta\longmapsto\mathbb{Q}^{k,n,\eta}(F)$. Similarly, $a^n_{t,u}(\eta,\omega) \underset{n\to\infty}{\overset{\mathbb{P}^{t,\eta}-a.s.}{\longrightarrow}} a_t(\eta,\omega)$ and is non-decreasing, so by monotone convergence theorem, the function $\eta\longmapsto\mathbb{Q}^{\eta}(F)$ is $\mathcal{F}_t^o$-measurable being a pointwise limit in n of the functions $\eta\longmapsto\mathbb{Q}^{n,\eta}(F)$.

We make then use of Theorem 58 Chapter V in [33]: the property above, the separability of $\mathcal{F}$ and the fact that for any η, $\mathbb{Q}^{\eta}\ll\mathbb{P}^{t,\eta}$ by item 3. above, imply the existence of a jointly measurable (for

$\mathcal{F}_t^o \otimes \mathcal{F}_u^o$) version of $(\eta, \omega) \mapsto a_{t,u}(\eta, \omega)$. That version will still be denoted by the same symbol. We recall that for any η, $a_{t,u}(\eta, \cdot)$ is the Radon-Nikodym density of $\mathbb{Q}^\eta$ with respect to $\mathbb{P}^{t,\eta}$.

We can now set $[M]_{t,u}(\omega) := a_{t,u}(\omega, \omega)$, which is a well-defined $\mathcal{F}_u^o$-measurable random variable. Since $a_{t,u}$ is $\mathcal{F}_t^o$-measurable in the first variable and for any η $\mathbb{P}^{t,\eta}(\omega^t = \eta^t) = 1$ we have the equalities

$$[M]_{t,u}(\omega) = a_{t,u}(\omega, \omega) = a_{t,u}(\eta, \omega) = [M^{t,\eta}]_u(\omega) - [M^{t,\eta}]_t(\omega) \ \mathbb{P}^{t,\eta}\text{a.s.} \tag{5.A.1}$$

We can then show that

$$[M]_{t,u} = [M^{s,\eta}]_u - [M^{s,\eta}]_t \ \ \mathbb{P}^{s,\eta} \text{ a.s.}, \tag{5.A.2}$$

holds for every $(s, \eta) \in [0, t] \times \Omega$, and not just in the case $s = t$ that we have just established in (5.A.1). This can be done reasoning as in the proof of Proposition 1.4.4 in Chapter 1, replacing the use of the Markov property with item 3. of Definition 5.3.4.

So we have built an $\mathcal{F}_u^o$-measurable variable $[M]_{t,u}$ such that under any $\mathbb{P}^{s,\eta}$ with $s \leq t$, $[M^{s,\eta}]_u - [M^{s,\eta}]_t = [M]_{t,u}$ a.s. and this concludes the proof. □

Proof of Proposition 5.4.7.
We start defining $A_{t,t} = 0$ for every $t \geq 0$. We then recall a property of the predictable dual projection which we will have to extend slightly. Let us fix (s, η) and the corresponding stochastic basis $(\Omega, \mathcal{F}^{s,\eta}, \mathbb{F}^{s,\eta}, \mathbb{P}^{s,\eta})$. For any $F \in \mathcal{F}^{s,\eta}$, let $N^{s,\eta,F}$ be the cadlag version of the martingale $r \longmapsto \mathbb{E}^{s,\eta}[\mathbb{1}_F|\mathcal{F}_r]$. Then for any $0 \leq t \leq u$, the predictable projection of the process $r \mapsto \mathbb{1}_F \mathbb{1}_{[t,u[}(r)$ is $r \mapsto N^{s,\eta,F}_{r^-} \mathbb{1}_{[t,u[}(r)$, see the proof of Theorem 43 Chapter VI in [33]. Therefore by definition of the dual predictable projection (see Definition 73 Chapter VI in [33]), for any $0 \leq t \leq u$ and $F \in \mathcal{F}^{s,\eta}$ we have $\mathbb{E}^{s,\eta}\left[\mathbb{1}_F(A^{s,\eta}_{u^-} - A^{s,\eta}_t)\right] = \mathbb{E}^{s,\eta}\left[\int_t^{u^-} N^{s,\eta,F}_{r^-} dB^{s,\eta}_r\right]$. Then, at fixed t, u, F, since for every $\epsilon > 0$ we have $\mathbb{E}^{s,\eta}\left[\mathbb{1}_F(A^{s,\eta}_{(u+\epsilon)^-} - A^{s,\eta}_t)\right] = \mathbb{E}^{s,\eta}\left[\int_t^{(u+\epsilon)^-} N^{s,\eta,F}_{r^-} dB^{s,\eta}_r\right]$, letting ϵ tend to zero we obtain by dominated convergence theorem that

$$\mathbb{E}^{s,\eta}\left[\mathbb{1}_F(A^{s,\eta}_u - A^{s,\eta}_t)\right] = \mathbb{E}^{s,\eta}\left[\int_t^u N^{s,\eta,F}_{r^-} dB^{s,\eta}_r\right], \tag{5.A.3}$$

taking into account the right-continuity of $A^{s,\eta}, B^{s,\eta}$ and the fact that they are both non-decreasing processes with $\mathcal{L}^1$ -terminal value.

For any $F \in \mathcal{F}$, we introduce the process $N^F : (t, \omega) \longmapsto \mathbb{P}^{t,\omega}(F)$. N^F takes values in $[0, 1]$ for every (t, ω), and by Definition 5.3.4, it is an $\mathbb{F}^o$-progressively measurable process such that for any $(s, \eta) \in \mathbb{R}_+ \times \Omega$, $N^{s,\eta,F}$ is a $\mathbb{P}^{s,\eta}$ cadlag version of N^F on $[s, +\infty[$.

For the rest of the proof, $0 \leq t < u$ are fixed. Following the same proof than that of Lemma 1.4.7 in Chapter 1 but with parameter (s, x) replaced with (s, η), we obtain the following.

Lemma 5.A.1. *Let $F \in \mathcal{F}$. There exists an $\mathcal{F}_u$-measurable random variable which we will call $\int_t^u N^F_{r^-} dB_r$ such that for any $(s, \eta) \in [0, t] \times \Omega$, $\int_t^u N^F_{r^-} dB_r = \int_t^u N^{s,\eta,F}_{r^-} dB^{s,\eta}_r$ $\mathbb{P}^{s,\eta}$ a.s.*

Remark 5.A.2. *By definition, the r.v. $\int_t^u N^F_{r^-} dB_r$ will not depend on (s, η).*

We continue now the proof of Proposition 5.4.7 by showing that for given $0 \leq t < u$ there is an $\mathcal{F}_u^o$-measurable r.v. $A_{t,u}$ such that for every $(s, \eta) \in [0, t] \times \Omega$, $(A^{s,\eta}_u - A^{s,\eta}_t) = A_{t,u}$ $\mathbb{P}^{s,\eta}$ a.s.

Similarly to what we did with the quadratic variation in Proposition 5.4.6, we start noticing that for any $\eta \in \Omega$, $(A^{t,\eta}_u - A^{t,\eta}_t)$ being $\mathcal{F}^{t,\eta}_u$-measurable, there exists by Corollary 5.3.21 an $\mathcal{F}_u^o$-measurable r.v. $\omega \mapsto a_{t,u}(\eta, \omega)$ such that

$$a_{t,u}(\eta, \omega) = A^{t,\eta}_u(\omega) - A^{t,\eta}_t(\omega) \quad \mathbb{P}^{t,\eta} \text{ a.s.} \tag{5.A.4}$$

As in the proof of Proposition 5.4.6, we show below the existence of a jointly-measurable version of $(\eta, \omega) \mapsto a_{t,u}(\eta, \omega)$.

For every $\eta \in \Omega$ we define on $\mathcal{F}_u^o$ the positive measure

$$\mathbb{Q}^\eta : F \longmapsto \mathbb{E}^{t,\eta}\left[\mathbb{1}_F a_{t,u}(\eta,\omega)\right] = \mathbb{E}^{t,\eta}\left[\mathbb{1}_F(A_u^{t,\eta} - A_t^{t,\eta})\right]. \tag{5.A.5}$$

By Lemma 5.A.1 and (5.A.3), for every $F \in \mathcal{F}_u^o$ we have

$$\mathbb{Q}^\eta(F) = \mathbb{E}^{t,\eta}\left[\int_t^u N_{r-}^F dB_r\right], \tag{5.A.6}$$

where we recall that $\int_t^u N_{r-}^F dB_r$ does not depend on η. So by Remark 5.3.6, $\eta \longmapsto \mathbb{Q}^\eta(F)$ is $\mathcal{F}_t^o$-measurable for any F. Moreover, by (5.A.5) for any η, $\mathbb{Q}^\eta \ll \mathbb{P}^{t,\eta}$. Again by Theorem 58 Chapter V in [33], there exists a version $(\eta, \omega) \mapsto a_{t,u}(\eta, \omega)$ -measurable for $\mathcal{F}_t^o \otimes \mathcal{F}_u^o$ of the related Radon-Nikodym densities.

We can now set $A_{t,u}(\omega) := a_{t,u}(\omega, \omega)$ which is then an $\mathcal{F}_u^o$-measurable r.v.
It yields for any $\eta \in \Omega$

$$A_{t,u}(\omega) = a_{t,u}(\omega,\omega) = a_{t,u}(\eta,\omega) = A_u^{t,\eta}(\omega) - A_t^{t,\eta}(\omega) \quad \mathbb{P}^{t,\eta} \text{ a.s.} \tag{5.A.7}$$

Indeed the second equality holds given that $a_{t,u}$ is $\mathcal{F}_t^o$-measurable with respect to the first variable, taking into account that $\mathbb{P}^{t,\eta}(\omega^t = \eta^t) = 1$; the third equality follows by (5.A.4).

We now set $s < t$ and $\eta \in \Omega$. We want to show that we still have
$A_{t,u} = A_u^{s,\eta} - A_t^{s,\eta}$ $\mathbb{P}^{s,\eta}$ a.s. So we consider $F \in \mathcal{F}_u^o$; we compute

$$\begin{array}{rlcl} & \mathbb{E}^{s,\eta}\left[\mathbb{1}_F(A_u^{s,\eta} - A_t^{s,\eta})\right] & = & \mathbb{E}^{s,\eta}\left[\int_t^u N_{r-}^F dB_r\right] \\ = & \mathbb{E}^{s,\eta}\left[\mathbb{E}^{s,\eta}\left[\int_t^u N_{r-}^F dB_r | \mathcal{F}_t\right]\right] & = & \mathbb{E}^{s,\eta}\left[\mathbb{E}^{t,\omega}\left[\int_t^u N_{r-}^F dB_r\right]\right] \\ = & \mathbb{E}^{s,\eta}\left[\mathbb{E}^{t,\omega}\left[\mathbb{1}_F A_{t,u}\right]\right] & = & \mathbb{E}^{s,\eta}\left[\mathbb{E}^{s,\eta}\left[\mathbb{1}_F A_{t,u} | \mathcal{F}_t\right]\right] \\ = & \mathbb{E}^{s,\eta}\left[\mathbb{1}_F A_{t,u}\right]. & & \end{array} \tag{5.A.8}$$

Indeed, the first equality comes from (5.A.3) and Lemma 5.A.1; concerning the fourth equality we recall that, by (5.A.5), (5.A.6) and (5.A.7), we have $\mathbb{E}^{t,\omega}\left[\int_t^u N_{r-}^F dB_r\right] = \mathbb{E}^{t,\omega}\left[\mathbb{1}_F A_{t,u}\right]$ for all ω. The third and fifth equalities come from Remark 5.3.6.

Since adding $\mathbb{P}^{s,\eta}$-null sets does not change the validity of (5.A.8), by Proposition 5.3.20 for any $F \in \mathcal{F}_u^{s,\eta}$ we have $\mathbb{E}^{s,\eta}\left[\mathbb{1}_F(A_u^{s,\eta} - A_t^{s,\eta})\right] = \mathbb{E}^{s,\eta}\left[\mathbb{1}_F A_{t,u}\right]$.

Finally, since both $A_u^{s,\eta} - A_t^{s,\eta}$ and $A_{t,u}$ are $\mathcal{F}_u^{s,\eta}$-measurable, we can conclude that $A_u^{s,\eta} - A_t^{s,\eta} = A_{t,u}$ $\mathbb{P}^{s,\eta}$ a.s. We emphasize that this holds for any $t \le u$ and $(s, \eta) \in [0, t] \times \Omega$, so $(A_{t,u})_{(t,u)\in\Delta}$ is the desired path-dependent AF, which ends the proof of Proposition 5.4.7. □

Proof of Proposition 5.4.10.
We set

$$C_{t,u} = A_{t,u} + (V_u - V_t) + (u - t), \tag{5.A.9}$$

which is a path-dependent AF with cadlag versions $C_t^{s,\eta} = A_t^{s,\eta} + V_t + t$ and we start by showing the statement for A and C instead of A and V.

The reason of the introduction of the intermediary function C is that for any $u > t$ we have $\frac{A_u^{s,\eta} - A_t^{s,\eta}}{C_u^{s,\eta} - C_t^{s,\eta}} \in [0, 1]$; that property will be used extensively in connection with the application of dominated convergence theorem.

Since $A^{s,\eta}$ is non-decreasing for any $(s,\eta) \in \mathbb{R}_+ \times \Omega$, A can be taken positive (in the sense that $A_{t,u}(\omega) \geq 0$ for any $(t,u) \in \Delta$ and $\omega \in \Omega$) by considering A^+ (defined by $(A^+)_{t,u}(\omega) := A_{t,u}(\omega)^+$) instead of A. On $\mathbb{R}_+$ we set

$$\begin{aligned} K_t &= \liminf_{n\to\infty} \frac{A_{t,t+\frac{1}{n}}}{A_{t,t+\frac{1}{n}} + \frac{1}{n} + (V_{t+\frac{1}{n}} - V_t)} \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\min_{n\leq p\leq m} \frac{A_{t,t+\frac{1}{p}}}{A_{t,t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}. \end{aligned} \tag{5.A.10}$$

This liminf always exists and belongs to $[0,1]$ since the sequence belongs to $[0,1]$. For any $(s,\eta) \in \mathbb{R}_+ \times \Omega$, since for all $t \geq s$ and $n \geq 0$,

$A_{t,t+\frac{1}{n}} = A^{s,\eta}_{t+\frac{1}{n}} - A^{s,\eta}_t$ $\mathbb{P}^{s,\eta}$ a.s., then $K^{s,\eta}$ defined by $K^{s,\eta}_t := \liminf_{n\to\infty} \frac{A^{s,\eta}_{t+\frac{1}{n}} - A^{s,\eta}_t}{C^{s,\eta}_{t+\frac{1}{n}} - C^{s,\eta}_t}$ is a $\mathbb{P}^{s,\eta}$-version of K, for $t \in [s,+\infty[$.

By Lebesgue Differentiation theorem (see Theorem 12 Chapter XV in [34] for a version of the theorem with a general atomless measure), for any (s,η), for $\mathbb{P}^{s,\eta}$-almost all ω, since $dC^{s,\eta}(\omega)$ is absolutely continuous with respect to $dA^{s,\eta}(\omega)$, $K^{s,\eta}(\omega)$ is a density of $dA^{s,\eta}(\omega)$ with respect to $dC^{s,\eta}(\omega)$.
For any $t \geq 0$, K_t is measurable with respect to $\bigcap_{n\geq 0} \mathcal{F}^o_{t+\frac{1}{n}} = \mathcal{F}_t$, by definition of the canonical filtration.
For any $(t,\omega) \in \mathbb{R}_+ \times \Omega$, we now set

$$k_t(\omega) := \mathbb{E}^{t,\omega}[K_t]. \tag{5.A.11}$$

Remark 5.3.6 implies that k is an $\mathbb{F}^o$-adapted process. The path-dependent canonical class verifies Hypothesis 5.3.5, and K_t is $\mathcal{F}_t$-measurable then for any $(s,\eta) \in [t,+\infty[\times\Omega$, $K_t(\omega) = \mathbb{E}^{s,\eta}[K_t|\mathcal{F}_t](\omega) = \mathbb{E}^{t,\omega}[K_t] = k_t(\omega)$ $\mathbb{P}^{s,\eta}$-a.s. hence k is on $[s,+\infty[$ a $\mathbb{P}^{s,\eta}$-version of K, and therefore of $K^{s,\eta}$.

The next main object of this proof is to show that k is an $\mathbb{F}^o$-progressively measurable process. For any integers (n,m), we define

$$k^{n,m} : (t,\eta) \mapsto \mathbb{E}^{t,\eta}\left[\min_{n\leq p\leq m} \frac{A_{t,t+\frac{1}{p}}}{A_{t,t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right],$$

and for all n,

$$k^{n} : (t,\eta) \mapsto \mathbb{E}^{t,\eta}\left[\inf_{p\geq n} \frac{A_{t,t+\frac{1}{p}}}{A_{t,t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right]. \tag{5.A.12}$$

We start showing that

$$\begin{aligned} \tilde{k}^{n,m} : \quad ((s,\eta),t) &\longmapsto \mathbb{E}^{s,\eta}\left[\min_{n\leq p\leq m} \frac{A_{t,t+\frac{1}{p}}}{A_{t,t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right] \mathbb{1}_{s\leq t}, \\ (\mathbb{R}_+ \times \Omega) \times \mathbb{R}_+ &\longrightarrow [0,1], \end{aligned} \tag{5.A.13}$$

is measurable with respect to $\mathcal{P}ro^o \otimes \mathcal{B}(\mathbb{R}_+)$. In order to do so, we will show that it is measurable in the first variable (s,η), and right-continuous in the second variable t, and conclude with Lemma 1.4.12 in Chapter 1.

We fix $t \in \mathbb{R}_+$. Since the path-dependent canonical class is progressive, by Remark 5.3.6, the map

$$\begin{aligned} (s,\eta) &\longmapsto \mathbb{E}^{s,\eta}\left[\min_{n\leq p\leq m} \frac{A_{t,t+\frac{1}{p}}}{A_{t,t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right] \\ \mathbb{R}_+ \times \Omega &\longrightarrow [0,1], \end{aligned} \tag{5.A.14}$$

is measurable with respect to $\mathcal{P}ro^o$. The map $(s,\eta) \longmapsto \mathbb{1}_{[t,+\infty[}(s)$ is also trivially measurable with respect to $\mathcal{P}ro^o$; therefore the product of the latter map and (5.A.14), that we denote by $\tilde{k}(\cdot,\cdot,t)$ is also measurable with respect to $\mathcal{P}ro^o$. Moreover, if we fix $(s,\eta) \in \mathbb{R}_+ \times \Omega$, reasoning exactly as in the proof of Proposition 1.4.13 in Chapter 1 we see that $t \mapsto \tilde{k}^{n,m}(s,\eta,t)$ is right-continuous, which by Lemma 1.4.12 in Chapter 1 implies the joint measurability of $\tilde{k}^{n,m}$.

Since $k^{n,m}(t,\eta) = \tilde{k}^{n,m}(t,t,\eta)$, and since $(t,\eta) \mapsto (t,\eta,t)$ is obviously $(\mathcal{P}ro^o, \mathcal{P}ro^o \otimes \mathcal{B}(\mathbb{R}_+))$-measurable, then by composition we can deduce that for any n,m, $k^{n,m}$ is an $\mathbb{F}^o$-progressively measurable process. By the dominated convergence theorem, $k^{n,m}$ tends pointwise to k^n when m goes to infinity, so k^n also is an $\mathbb{F}^o$-progressively measurable process for every n. Finally, since $K_t = \liminf_{n\to\infty} \liminf_{p\geq n} \frac{A_{t,t+\frac{1}{p}}}{A_{t,t+\frac{1}{p}}+\frac{1}{p}+(V_{t+\frac{1}{p}}-V_t)}$, taking the expectation and again by the dominated convergence theorem, k^n (defined in (5.A.12)) tends pointwise to k (defined in (5.A.11)), when n goes to infinity, so k is an $\mathbb{F}^o$-progressively measurable process.

Considering that $(t,u,\omega) \mapsto V_u - V_t$ also trivially defines a non-negative non-decreasing path-dependent AF absolutely continuous with respect to C, defined in (5.A.9), we proceed similarly as at the beginning of the proof, replacing the path-dependent AF A with V.

Let the process K' be defined by $K'_t = \liminf_{n\to\infty} \frac{V_{t+\frac{1}{n}}-V_t}{A_{t,t+\frac{1}{n}}+\frac{1}{n}+(V_{t+\frac{1}{n}}-V_t)}$, and for any (s,η), let $K'^{s,\eta}$ be defined on $[s,+\infty[$ by $K'^{s,\eta}_t = \liminf_{n\to\infty} \frac{V_{t+\frac{1}{n}}-V_t}{A^{s,\eta}_{t+\frac{1}{n}}-A^{s,\eta}_t+\frac{1}{n}+(V_{t+\frac{1}{n}}-V_t)}$. Then, for any (s,η), $K'^{s,\eta}$ on $[s,+\infty[$ is a $\mathbb{P}^{s,\eta}$-version of K', and it constitutes a density of dV with respect to $dC^{s,\eta}(\omega)$ on $[s,+\infty[$, for almost all ω. One shows then the existence of an $\mathbb{F}^o$-progressively measurable process k' such that for any (s,η), k' is a $\mathbb{P}^{s,\eta}$-version of K' and of $K'^{s,\eta}$ on $[s,+\infty[$.

By the considerations after (5.A.10), for any (s,η), under $\mathbb{P}^{s,\eta}$, we can write

$$\begin{cases} A^{s,\eta} &= \int_s^{\cdot\vee s} K^{s,\eta}_r dC^{s,\eta}_r \\ V_{\cdot\vee s} - V_s &= \int_s^{\cdot\vee s} K'^{s,\eta}_r dC^{s,\eta}_r. \end{cases}$$

Now since $dA^{s,\eta} \ll dV$, we have for $\mathbb{P}^{s,\eta}$ almost all ω that the set $\{r \in [s,+\infty[\colon |K'^{s,\eta}_r(\omega) = 0\}$ is negligible with respect to dV so also for $dA^{s,\eta}(\omega)$ and therefore we can write

$$\begin{aligned} A^{s,\eta} &= \int_s^{\cdot\vee s} K^{s,\eta}_r dC^{s,\eta}_r \\ &= \int_s^{\cdot\vee s} \tfrac{K^{s,\eta}_r}{K'^{s,\eta}_r} \mathbb{1}_{\{K'^{s,\eta}_r \neq 0\}} K'^{s,\eta}_r dC^{s,\eta}_r + \int_s^{\cdot\vee s} \mathbb{1}_{\{K'^{s,\eta}_r = 0\}} dA^{s,\eta}_r \\ &= \int_s^{\cdot\vee s} \tfrac{K^{s,\eta}_r}{K'^{s,\eta}_r} \mathbb{1}_{\{K'^{s,\eta}_r \neq 0\}} dV_r, \end{aligned}$$

where we use the convention that for any two functions ϕ,ψ then $\frac{\phi}{\psi}\mathbb{1}_{\psi\neq 0}$ is defined by

$$\frac{\phi}{\psi}\mathbb{1}_{\{\psi\neq 0\}}(x) = \begin{cases} \frac{\phi(x)}{\psi(x)} \text{ if } \psi(x) \neq 0 \\ 0 \text{ if } \psi(x) = 0. \end{cases}$$

We now set $h := \frac{k}{k'}\mathbb{1}_{\{k'_r\neq 0\}}$ which is an $\mathbb{F}^o$-progressively measurable process, and clearly for any (s,η), h is a $\mathbb{P}^{s,\eta}$-version of $H^{s,\eta} := \frac{K^{s,\eta}}{K'^{s,\eta}}\mathbb{1}_{\{K'^{s,\eta}\neq 0\}}$ on $[s,+\infty[$. So by Lemma 2.5.13 in Chapter 2, $H^{s,\eta} = h\ dV \otimes d\mathbb{P}^{s,\eta}$ a.e. on $[s,+\infty[$ and finally we have shown that under any $\mathbb{P}^{s,\eta}$, $A^{s,\eta} = \int_s^{\cdot\vee s} h_r dV_r$. □

Chapter 6

Decoupled mild solutions of path-dependent PDEs and IPDEs represented by BSDEs driven by cadlag martingales

This chapter is the object of paper [13].

Abstract

We focus on a class of path-dependent problems which include path-dependent (possibly Integro) PDEs, and their representation via BSDEs driven by a cadlag martingale. For those equations we introduce the notion of *decoupled mild solution* for which, under general assumptions, we study existence and uniqueness and its representation via the afore mentioned BSDEs. This concept generalizes a similar notion introduced by the authors in previous papers in the framework of classical PDEs and IPDEs. For every initial condition (s, η), where s is an initial time and η an initial path, the solution of such BSDE produces a couple of processes $(Y^{s,\eta}, Z^{s,\eta})$. In the classical (Markovian or not) literature the function $u(s, \eta) := Y_s^{s,\eta}$ constitutes a viscosity type solution of an associated PDE (resp. IPDE); our approach allows not only to identify u as (in our language) the unique decoupled mild solution, but also to solve quite generally the so called *identification problem*, i.e. to also characterize the $(Z^{s,\eta})_{s,\eta}$ processes in term of a deterministic function v associated to the (above decoupled mild) solution u.

6.1 Introduction

We focus on a family of path-dependent problems of the type

$$\begin{cases} AY + f(\cdot, \cdot, Y, \Gamma(\Psi, Y)) = 0 \text{ on } [0, T] \times \Omega \\ Y_T = \xi \text{ on } \Omega, \end{cases} \tag{6.1.1}$$

where A is a linear map from some linear subspace $\mathcal{D}(A)$ of the space of progressively measurable processes into the space of progressively measurable processes, $\Psi := (\Psi^1, \cdots, \Psi^d)$ is a given vector of elements of $\mathcal{D}(A)$ and Γ is a *carré du champs* type operator defined by $\Gamma(\Phi, \Phi') := A(\Phi\Phi') - \Phi A(\Phi') - \Phi' A(\Phi)$. Associated with this map, there is a path-dependent system of projectors $(P_s)_{s\in\mathbb{R}_+}$, which extends the notion of semigroups from the Markovian case, for which A is a weak generator, see

Definition 6.3.16. A typical example is to consider $\Psi := X$ the canonical process, and a map A given by

$$\begin{array}{rl}(A\Phi)_t(\omega) := & (D\Phi)_t(\omega) + \frac{1}{2}Tr(\sigma_t\sigma_t^\intercal(\nabla^2\Phi)_t(\omega)) + \beta_t(\omega)\cdot(\nabla\Phi)_t(\omega) \\ & + \int(\Phi_t(\omega+\gamma_t(\omega,y)\mathbb{1}_{[t,+\infty[}) - \Phi_t(\omega) - \gamma_t(\omega,y)\cdot(\nabla\Phi)_t(\omega))F(dy),\end{array} \tag{6.1.2}$$

where β, σ, γ are bounded path-dependent predictable coefficients and F is a bounded positive measure not charging 0. In (6.1.2), D is the horizontal derivative and ∇ is the vertical gradient intended in the sense of [37, 27]. In that case one has

$$\Gamma(X,\Phi)_t = (\sigma\sigma^\intercal\nabla\Phi)_t + \int_{\mathbb{R}^d}\gamma_t(\cdot,y)(\Phi_t(\cdot+\gamma_t(\cdot,y)\mathbb{1}_{[t,+\infty[}) - \Phi_t)F(dy). \tag{6.1.3}$$

If $\gamma \equiv 0$ then (6.1.1) becomes the path-dependent PDE

$$\left\{\begin{array}{l} DY + \frac{1}{2}Tr(\sigma\sigma^\intercal\nabla^2 Y) + \beta\cdot\nabla Y + f(\cdot,\cdot,Y,\sigma\sigma^\intercal\nabla Y) = 0 \text{ on } [0,T]\times\Omega \\ Y_T = \xi \text{ on } \Omega. \end{array}\right. \tag{6.1.4}$$

We introduce a notion of *decoupled mild solution* which is inspired by the one for classical (I)PDEs introduced in Chapters 3, 4, which can be represented by solutions of Markovian BSDEs. Concerning the corresponding notion for (6.1.1) the intuition behind is the following. We decouple the first line of equation (6.1.1) into

$$\left\{\begin{array}{rcl} AY & = & -f(\cdot,\cdot,Y,Z) \\ Z^i & = & \Gamma(\Psi^i,Y), \quad 1\le i\le d, \end{array}\right. \tag{6.1.5}$$

which we can also write

$$\left\{\begin{array}{rcl} AY & = & -f(\cdot,\cdot,Y,Z) \\ A(Y\Psi^i) & = & Z^i + YA\Psi^i + \Psi^i AY, \quad 1\le i\le d, \end{array}\right. \tag{6.1.6}$$

and finally

$$\left\{\begin{array}{rcl} AY & = & -f(\cdot,\cdot,Y,Z) \\ A(Y\Psi^i) & = & Z^i + YA\Psi^i - \Psi^i f(\cdot,\cdot,Y,Z), \quad 1\le i\le d. \end{array}\right. \tag{6.1.7}$$

Taking (6.1.7) into account and inspired by the classical notion of mild solution of an evolution problem, we define a decoupled mild solution of equation (6.1.1) as a functional Y for which there exists an auxiliary $\mathbb{R}^d$-valued functional $Z := (Z^1,\cdots,Z^d)$ such that for all $(s,\eta)\in[0,T]\times\Omega$ we have

$$\left\{\begin{array}{rl} Y_s(\eta) & = P_s[\xi](\eta) + \int_s^T P_s\left[f\left(r,\cdot,Y_r,Z_r\right)\right](\eta)dr \\ (Y\Psi^1)_s(\eta) & = P_s[\xi\Psi^1_T](\eta) - \int_s^T P_s\left[\left(Z^1_r + Y_rA\Psi^1_r - \Psi^1_r f\left(r,\cdot,Y_r,Z_r\right)\right)\right](\eta)dr \\ & \cdots \\ (Y\Psi^d)_s(\eta) & = P_s[\xi\Psi^d_T](\eta) - \int_s^T P_s\left[\left(Z^d_r + Y_rA\Psi^d_r - \Psi^d_r f\left(r,\cdot,Y_r,Z_r\right)\right)\right](\eta)dr. \end{array}\right. \tag{6.1.8}$$

The couple (Y,Z) will be called *solution of the identification problem related to* (f,ξ) because it can be strictly related to BSDEs driven by cadlag martingales which are one natural generalization of classical Brownian BSDEs. We consider for any (s,η) the BSDE

$$Y^{s,\eta} = \xi + \int_\cdot^T f\left(r,\cdot,Y^{s,\eta}_r,\frac{d\langle M^{s,\eta},M[\Psi]^{s,\eta}\rangle_r}{dr}\right)dr - (M^{s,\eta}_T - M^{s,\eta}_\cdot), \tag{6.1.9}$$

in the (completed) stochastic basis $(\Omega,\mathcal{F}^{s,\eta},\mathbb{F}^{s,\eta},\mathbb{P}^{s,\eta})$, where $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ solves a martingale problem associated to $(\mathcal{D}(A),A)$. In (6.1.9), $M[\Psi]^{s,\eta}$ is the driving martingale of the BSDE, and is the

martingale part of the process Ψ under $\mathbb{P}^{s,\eta}$. These BSDEs were considered in a more general framework by the authors in Chapter 4. A significant contribution about BSDEs driven by cadlag martingales and beyond was provided by [22] and [67]. Those BSDEs have however a *forward* component which is modeled in law by the fixed family $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$. An important application for path-dependent (I)PDEs is Theorem 6.5.32 that states the following. Suppose that the path-dependent SDE with coefficients β, σ, γ admits existence and uniqueness in law for every initial condition (s, η); we suppose moreover that β_t, σ_t (resp. $\gamma_t(\cdot, x)$) are continuous for the Skorokhod topology in ω for almost all t (resp. $dt \otimes dF$ a.e.), that $f(\cdot, \cdot, 0, 0), \xi$ have polynomial growth and that f is Lipschitz in (y, z) uniformly in (t, ω). Then there is a unique decoupled mild solution Y for (6.1.1) with $\Psi := X$ and A given in (6.1.2). Moreover, both processes Y, Z appearing in (6.1.8) can be represented through the associated BSDEs (6.1.9). In particular, (6.1.8) gives an analytical meaning to the second process Z obtained through those BSDEs. In general the way of linking the first component Y of the solution (Y, Z) of a BSDE with the solution of a PDE is made by means of viscosity solutions. However, even when the BSDE is Markovian, this does not allow to identify Z. In particular, when $\gamma \equiv 0$, our technique allows to characterize Z as a *generalized gradient* even if the solution does not have the vertical derivative, contrarily to the case in [50].

Brownian Backward stochastic differential equations (BSDEs) were introduced in [71], after a pioneering work of [18]. When those involve a forward dynamic described by the solution X of a Brownian Markovian SDE, they are said to be Markovian, and are naturally linked to a parabolic PDE, see [73]. In particular, under reasonable conditions, which among others ensure well-posedness, the solutions of BSDEs produce *viscosity* type solutions for the mentioned PDE. Recently Brownian BSDEs of the type

$$Y^{s,\eta} = \xi\left((B_t^{s,\eta})_{t\in[0,T]}\right) + \int_\cdot^T f\left(r, (B_t^{s,\eta})_{t\in[0,r]}, Y_r^{s,\eta}, Z_r^{s,\eta}\right) dr - \int_\cdot^T Z_r^{s,\eta} dB_r, \tag{6.1.10}$$

where B is a Brownian motion and for any $s \in [0, T]$, $\eta \in \mathbb{D}([0, T], \mathbb{R}^d)$, $B^{s,\eta} = \eta(\cdot \wedge s) + (B_{\cdot\vee s} - B_s)$ were associated to the path-dependent semi-linear PDE

$$\begin{cases} D\Phi + \frac{1}{2}Tr(\nabla^2\Phi) + f(\cdot, \cdot, \Phi, \nabla\Phi) = 0 \quad \text{ on } [0, T[\times\Omega \\ \Phi_T = \xi. \end{cases} \tag{6.1.11}$$

Path-dependent PDEs of previous type have been investigated by several methods. For instance strict (classical, regular) solutions have been studied in [35, 49, 29] under the point of view of Banach space valued stochastic processes. It was shown for instance in [29, 77] that under some assumptions the mapping $(s, \eta) \longmapsto Y_s^{s,\eta}$ is the unique smooth solution of (6.1.11). Another popular approach is the one of *viscosity solutions*, which was considered by several authors. For instance it was shown in [42] that if f is bounded, continuous in t , uniformly continuous in the second variable, and uniformly Lipschitz continuous in (y, z) and if ξ is bounded uniformly continuous, $(s, \eta) \longmapsto Y_s^{s,\eta}$ is a viscosity solution of (6.1.11) in some specific sense, where the sense of solutions involved the underlying probability. On another level, [28] considered the so called *strong-viscosity* solutions (based on approximation techniques), which are an analytic concept, the first under non-smoothness conditions. Another interesting approach (probabilistic) but still based on approximation (discretizations) was given by [65]. More recently, [17] produced a viscosity solution to a more general path-dependent (possibly integro)-PDE through Dynamic risk measures. In all those cases the solution Φ of (6.1.11) was associated to the process $Y^{s,\eta}$ of the solution couple $(Y^{s,\eta}, Z^{s,\eta})$ of (6.1.10) with initial time s and initial condition η. As mentioned earlier a challenging link to be explored was the link between $Z^{s,\eta}$ and the solution of the path-dependent PDE Φ. For instance in the case of Fréchet $C^{0,1}$ solutions Φ defined on $C([-T, 0])$, then $Z^{s,\eta}$ is equal to the *vertical* derivative $\nabla\Phi$, see for instance [50].

The paper is organized as follows. After Section 6.2 devoted to fix some notations and basic vocabulary, Section 6.3 recalls some fundamental tools from the previous Chapter 5. In Section 6.4, we are given a general path dependent canonical class, its associated path-dependent system of projectors $(P_s)_{s\in\mathbb{R}_+}$ and we treat BSDEs driven by a general path-dependent MAF, see Definition 6.3.19. In Subsection 6.4.2 we are given a weak generator A of $(P_s)_{s\in\mathbb{R}_+}$, and a corresponding abstract equation. We define the notion of decoupled mild solution of that equation and prove under some conditions, existence and uniqueness of such a solution in Theorem 6.4.19. In Section 6.5, we focus on the framework of (I)PDEs. In Subsection 6.5.1 (resp. 6.5.2) we recall some results concerning path-dependent SDEs (resp. path-dependent differential operators). In Subsection 6.5.3, we consider an IPDE of coefficients β, σ, γ (which when $\gamma \equiv 0$ is given by (6.1.4)) and Theorem 6.5.32 states the existence and uniqueness of a decoupled mild solution. Proposition 6.5.34 compares classical and decoupled mild solutions for that IPDE.

6.2 Basic vocabulary and Notations

For fixed $d, k \in \mathbb{N}^*$, $\mathcal{C}_b^k(\mathbb{R}^d)$ will denote the set of functions k times differentiable with bounded continuous derivatives. A topological space E will always be considered as a measurable space equipped with its Borel σ-field which shall be denoted $\mathcal{B}(E)$.

Let $(\Omega, \mathcal{F})$, $(E, \mathcal{E})$ be two measurable spaces. A measurable mapping from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$ shall often be called a **random variable** (with values in E), or in short r.v. If $\mathbb{T}$ is some index set, a family $(X_t)_{t\in\mathbb{T}}$ of r.v. with values in E, will be called **random field** (indexed by $\mathbb{T}$ with values in E). In particular, if $\mathbb{T}$ is an interval included in $\mathbb{R}_+$, $(X_t)_{t\in\mathbb{T}}$ will be called a **stochastic process** (indexed by $\mathbb{T}$ with values in E).

Given a measurable space $(\Omega, \mathcal{F})$, for any $p \geq 1$, the set of real valued random variables with finite p-th moment under probability $\mathbb{P}$ will be denoted $\mathcal{L}^p(\mathbb{P})$ or $\mathcal{L}^p$ if there can be no ambiguity concerning the underlying probability. Two random fields (or stochastic processes) $(X_t)_{t\in\mathbb{T}}$, $(Y_t)_{t\in\mathbb{T}}$ indexed by the same set and with values in the same space will be said to be **modifications (or versions) of each other** if for every $t \in \mathbb{T}$, $\mathbb{P}(X_t = Y_t) = 1$.

A filtered probability space $\big(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P}\big)$ will be called called **stochastic basis** and will be said to **fulfill the usual conditions** if the filtration is right-continuous, if the probability space is complete and if $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible sets. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis. Let Y be a process and τ a stopping time, we denote Y^τ the process $t \mapsto Y_{t\wedge\tau}$. If $\mathcal{C}$ is a set of processes, we will say that Y is **locally in** $\mathcal{C}$ (resp. locally verifies some property) if there exists an a.s. increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ tending a.s. to infinity such that for every n, Y^{τ_n} belongs to $\mathcal{C}$ (resp. verifies the mentioned property). In this paper we will consider martingales (with respect to a given filtration and probability), which are not necessarily cadlag. For any cadlag local martingales M, N, we denote $[M]$ (resp. $[M, N]$) the **quadratic variation** of M (resp. **quadratic covariation** of M, N). If moreover M, N are locally square integrable, $\langle M, N\rangle$ (or simply $\langle M\rangle$ if $M = N$) will denote their (predictable) **angular bracket**.

6.3 Fundamental tools

6.3.1 Path-dependent canonical classes and systems of projectors

We start by recalling some notions and results of Section 5.3 of Chapter 5. that will be used all along the paper. The first definition refers to the canonical space that one can find in [60], see paragraph 12.63.

Notation 6.3.1. *In the whole section E will be a fixed Polish space (a separable completely metrizable topological space), which will be called the **state space**.*

*$\Omega := \mathbb{D}(\mathbb{R}_+, E)$ will denote the Skorokhod space of functions from $\mathbb{R}_+$ to E right-continuous with left limits (e.g. cadlag). For every $t \in \mathbb{R}_+$ we denote the coordinate mapping $X_t : \omega \mapsto \omega(t)$ and we define on Ω the σ-field $\mathcal{F} := \sigma(X_r | r \in \mathbb{R}_+)$. On the measurable space $(\Omega, \mathcal{F})$, we introduce **initial filtration** $\mathbb{F}^o := (\mathcal{F}^o_t)_{t\in\mathbb{R}_+}$, where $\mathcal{F}^o_t := \sigma(X_r | r \in [0,t])$, and the (right-continuous) **canonical filtration** $\mathbb{F} := (\mathcal{F}_t)_{t\in\mathbb{R}_+}$, where $\mathcal{F}_t := \bigcap_{s>t} \mathcal{F}^o_s$. $(\Omega, \mathcal{F}, \mathbb{F})$ will be called the **canonical space** (associated to E), and X the **canonical process**. On $\mathbb{R}_+ \times \Omega$, we will denote by $\mathcal{P}ro^o$ (resp. $\mathcal{P}re^o$) the $\mathbb{F}^o$-progressive (resp. $\mathbb{F}^o$-predictable) σ-field. Ω will be equipped with the Skorokhod topology which makes it a Polish space since E is itself Polish (see Theorem 5.6 in chapter 3 of [44]), and for which the Borel σ-field is $\mathcal{F}$, see Proposition 7.1 in chapter 3 of [44]. This in particular implies that $\mathcal{F}$ is separable, as the Borel σ-field of a separable metric space.*

$\mathcal{P}(\Omega)$ will denote the set of probability measures on Ω and will be equipped with the topology of weak convergence of measures which also makes it a Polish space since Ω is Polish, see Theorems 1.7 and 3.1 in [44] chapter 3. It will also be equipped with the associated Borel σ-field.

Notation 6.3.2. *For any $\omega \in \Omega$ and $t \in \mathbb{R}_+$, the path ω stopped at time t $r \mapsto \omega(r \wedge t)$ will be denoted ω^t.*

Definition 6.3.3. *A **path-dependent canonical class** will be a set of probability measures $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ defined on the canonical space $(\Omega, \mathcal{F})$. It will verify the three following items.*

1. *For every $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$;*
2. *for every $s \in \mathbb{R}_+$ and $F \in \mathcal{F}$, the mapping*
$$\begin{array}{ccc} \eta & \longmapsto & \mathbb{P}^{s,\eta}(F) \\ \Omega & \longrightarrow & [0,1] \end{array}$$
is $\mathcal{F}^o_s$-measurable;
3. *for every $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $t \geq s$ and $F \in \mathcal{F}$,*
$$\mathbb{P}^{s,\eta}(F|\mathcal{F}^o_t)(\omega) = \mathbb{P}^{t,\omega}(F) \text{ for } \mathbb{P}^{s,\eta} \text{ almost all } \omega. \tag{6.3.1}$$

This implies in particular that for every $(s,\eta) \in \mathbb{R}_+ \times \Omega$ and $t \geq s$, then $(\mathbb{P}^{t,\omega})_{\omega\in\Omega}$ is a regular conditional expectation of $\mathbb{P}^{s,\eta}$ by $\mathcal{F}^o_t$, see the Definition above Theorem 1.1.6 in [85] for instance.

*A path-dependent canonical class $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ will be said to be **progressive** if for every $F \in \mathcal{F}$, the mapping $(t,\omega) \longmapsto \mathbb{P}^{t,\omega}(F)$ is $\mathbb{F}^o$-progressively measurable.*

Very often path-dependent canonical classes will verify the following important hypothesis which is a reinforcement of (6.3.1).

Hypothesis 6.3.4. *For every $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $t \geq s$ and $F \in \mathcal{F}$,*
$$\mathbb{P}^{s,\eta}(F|\mathcal{F}_t)(\omega) = \mathbb{P}^{t,\omega}(F) \text{ for } \mathbb{P}^{s,\eta} \text{ almost all } \omega. \tag{6.3.2}$$

Remark 6.3.5. *By approximation through simple functions, one can easily show the following.*

- *For $s \geq 0$ and random variable Z we have that $\eta \longmapsto \mathbb{E}^{s,\eta}[Z]$ is $\mathcal{F}^o_s$-measurable and for every $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $t \geq s$, $\mathbb{E}^{s,\eta}[Z|\mathcal{F}^o_t](\omega) = \mathbb{E}^{t,\omega}[Z]$ for $\mathbb{P}^{s,\eta}$ almost all ω, provided previous expectations are finite;*
- *if the path-dependent canonical class is progressive, $(t,\omega) \longmapsto \mathbb{E}^{t,\omega}[Z]$ is $\mathbb{F}^o$-progressively measurable, provided previous expectations are finite.*

Notation 6.3.6. *$\mathcal{B}_b(\Omega)$ stands for the set of real bounded measurable functions on Ω. Let $s \in \mathbb{R}_+$, $\mathcal{B}_b^s(\Omega)$ will denote the set of real bounded $\mathcal{F}_s^o$-measurable functions on Ω. We also denote by $\mathcal{B}_b^+(\Omega)$ the subset of r.v. $\phi \in \mathcal{B}_b(\Omega)$ such that $\phi(\omega) \geq 0$ for all $\omega \in \Omega$.*

Definition 6.3.7.

1. *A linear map $Q : \mathcal{B}_b(\Omega) \to \mathcal{B}_b(\Omega)$ is said* **positivity preserving monotonic** *if for every $\phi \in \mathcal{B}_b^+(\Omega)$ then $Q[\phi] \in \mathcal{B}_b^+(\Omega)$ and for every increasing converging (in the pointwise sense) sequence $f_n \underset{n}{\longrightarrow} f$ we have that $Q[f_n] \underset{n}{\longrightarrow} P_s[f]$ in the pointwise sense.*

2. *A family $(P_s)_{s\in\mathbb{R}_+}$ of positivity preserving monotonic linear operators on $\mathcal{B}_b(\Omega)$ will be called a* ***path-dependent system of projectors*** *if it verifies the three following items.*

   - *For all $s \in \mathbb{R}_+$, the restriction of P_s on $\mathcal{B}_b^s(\Omega)$ coincides with the identity;*
   - *for all $s \in \mathbb{R}_+$, P_s maps $\mathcal{B}_b(\Omega)$ into $\mathcal{B}_b^s(\Omega)$;*
   - *for all $s, t \in \mathbb{R}_+$ with $t \geq s$, $P_s \circ P_t = P_s$.*

The proposition below states a correspondence between path-dependent canonical classes and path-dependent systems of projectors.

Proposition 6.3.8. *The mapping*

$$(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega} \longmapsto \left(\begin{array}{ccc} Z & \longmapsto & (\eta \mapsto \mathbb{E}^{s,\eta}[Z]) \\ \mathcal{B}_b(\Omega) & \longrightarrow & \mathcal{B}_b(\Omega) \end{array} \right)_{s\in\mathbb{R}_+}, \tag{6.3.3}$$

is a bijection between the set of path-dependent system of probability measures and the set of path-dependent system of projectors.

Definition 6.3.9. *From now on, two elements in correspondence through the previous bijection will be said to be* ***associated****.*

Notation 6.3.10. *Let $(P_s)_{s\in\mathbb{R}_+}$ be a path-dependent system of projectors, and $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ the associated path-dependent system of probability measures. Then for any r.v. $\phi \in \mathcal{L}^1(\mathbb{P}^{s,\eta})$, $P_s[\phi](\eta)$ will still denote the expectation of ϕ under $\mathbb{P}^{s,\eta}$. In other words we extend the linear form $\phi \longmapsto P_s[\phi](\eta)$ from $\mathcal{B}_b(\Omega)$ to $\mathcal{L}^1(\mathbb{P}^{s,\eta})$.*

For the results of the whole section, we are given a progressive path-dependent canonical class $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ satisfying Hypothesis 6.3.4 and the corresponding path-dependent system of projectors $(P_s)_{s\in\mathbb{R}_+}$.

Notation 6.3.11. *For any $(s, \eta) \in \mathbb{R}_+ \times \Omega$ we will consider the stochastic basis $\big(\Omega, \mathcal{F}^{s,\eta}, \mathbb{F}^{s,\eta} := (\mathcal{F}_t^{s,\eta})_{t\in\mathbb{R}_+}, \mathbb{P}^{s,\eta}\big)$ where $\mathcal{F}^{s,\eta}$ (resp. $\mathcal{F}_t^{s,\eta}$ for all t) is $\mathcal{F}$ (resp. $\mathcal{F}_t$) augmented with the $\mathbb{P}^{s,\eta}$ negligible sets. $\mathbb{P}^{s,\eta}$ is extended to $\mathcal{F}^{s,\eta}$.*

We remark that, for any $(s, \eta) \in \mathbb{R}_+ \times \Omega$, $(\Omega, \mathcal{F}^{s,\eta}, \mathbb{F}^{s,\eta}, \mathbb{P}^{s,\eta})$ is a stochastic basis fulfilling the usual conditions, see 1.4 in [61] Chapter I.

Proposition 6.3.12. *Let $(s, \eta) \in \mathbb{R}_+ \times \Omega$ be fixed, Z be a positive r.v. or in $\mathcal{L}^1(\mathbb{P}^{s,\eta})$ and $t \geq s$. Then $\mathbb{E}^{s,\eta}[Z|\mathcal{F}_t] = \mathbb{E}^{s,\eta}[Z|\mathcal{F}_t^{s,\eta}]$ $\mathbb{P}^{s,\eta}$ a.s.*

So when considering conditional expectations, we will always drop the (s, η) superscript on the filtration.

Definition 6.3.13. *Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$ and $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$, we say that $\mathcal{G}$ is $\mathbb{P}$-**trivial** if for any element G of $\mathcal{G}$, then $\mathbb{P}(G) \in \{0, 1\}$.*

Proposition 6.3.14. *For every $(s, \eta) \in \mathbb{R}_+ \times \Omega$, $\mathcal{F}_s$ is $\mathbb{P}^{s,\eta}$-trivial. In particular, an $\mathcal{F}_s^{s,\eta}$-measurable r.v. will be $\mathbb{P}^{s,\eta}$-a.s. equal to a constant.*

The last notions and results of this subsection are taken from Subsection 5.5.2 of Chapter 5.

From now on we are given a non-decreasing continuous function V and a couple $(\mathcal{D}(A), A)$ verifying the following.

Hypothesis 6.3.15.

1. *$\mathcal{D}(A)$ is a linear subspace of the space of $\mathbb{F}^o$-progressively measurable processes;*
2. *A is a linear mapping from $\mathcal{D}(A)$ into the space of $\mathbb{F}^o$-progressively measurable processes;*
3. *for all $\Phi \in \mathcal{D}(A)$, $\omega \in \Omega$, $t \geq 0$, $\int_0^t |A\Phi_r(\omega)| dV_r < +\infty$;*
4. *for all $\Phi \in \mathcal{D}(A)$, $(s, \eta) \in \mathbb{R}_+ \times \Omega$ and $t \in [s, +\infty[$, we have $\mathbb{E}^{s,\eta}\left[\int_s^t |A(\Phi)_r| dV_r\right] < +\infty$ and $\mathbb{E}^{s,\eta}[|\Phi_t|] < +\infty$.*

Inspired from the classical literature (see 13.28 in [61]) we introduce the following notion of a weak generator.

Definition 6.3.16. *We say that $(\mathcal{D}(A), A)$ is a **weak generator** of a path-dependent system of projectors $(P_s)_{s \in \mathbb{R}_+}$ if for all $\Phi \in \mathcal{D}(A)$, $(s, \eta) \in \mathbb{R}_+ \times \Omega$ and $t \in [s, +\infty[$, we have*

$$P_s[\Phi_t](\eta) = \Phi_s(\eta) + \int_s^t P_s[A(\Phi)_r](\eta) dV_r. \tag{6.3.4}$$

Definition 6.3.17.

1. *$(\mathbb{P}^{s,\eta})_{(s,\eta) \in \mathbb{R}_+ \times \Omega}$ will be said to solve the **martingale problem associated to** $(\mathcal{D}(A), A)$ if for every $(s, \eta) \in \mathbb{R}_+ \times \Omega$,*
   - *$\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$;*
   - *$\Phi - \int_0^\cdot A(\Phi)_r dr$, is on $[s, +\infty[$ a $(\mathbb{P}^{s,\eta}, \mathbb{F}^o)$-martingale for all $\Phi \in \mathcal{D}(A)$.*
2. *The martingale problem associated to $(\mathcal{D}(A), A)$ will be said to be **well-posed** if for every $(s, \eta) \in \mathbb{R}_+ \times \Omega$ there exists a unique $\mathbb{P}^{s,\eta}$ verifying both items above.*

Proposition 6.3.18. *$(\mathcal{D}(A), A)$ is a weak generator of $(P_s)_{s \in \mathbb{R}_+}$ if and only if $(\mathbb{P}^{s,\eta})_{(s,\eta) \in \mathbb{R}_+ \times \Omega}$ solves the martingale problem associated to $(\mathcal{D}(A), A)$.*

In particular, if $(\mathbb{P}^{s,\eta})_{(s,\eta) \in \mathbb{R}_+ \times \Omega}$ solves the well-posed martingale problem associated to $(\mathcal{D}(A), A)$ then $(P_s)_{s \in \mathbb{R}_+}$ is the unique path-dependent system of projectors for which $(\mathcal{D}(A), A)$ is a weak generator.

Indeed, the last statement allows to associate analytically to $(\mathcal{D}(A), A)$ a unique path-dependent system of projectors $(P_s)_{s \in \mathbb{R}_+}$ through Definition 6.3.16.

6.3.2 Path-dependent martingale additive functionals

We now recall the notion of Path-dependent Martingale Additive Functionals that we use in the chapter. This was introduced in Chapter 5 and can be conceived as a path-dependent extension of the notion of non-homogeneous Martingale Additive Functionals of a Markov processes developed in Chapter 1. In this subsection, all results come from Section 5.4 in Chapter 5. In this subsection we are again given a progressive path-dependent canonical class $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ satisfying Hypothesis 6.3.4 and the corresponding path-dependent system of projectors $(P_s)_{s\in\mathbb{R}_+}$.

Definition 6.3.19. *On $(\Omega,\mathcal{F})$, a **path-dependent Martingale Additive Functional**, in short path-dependent MAF will be a real-valued random-field $M := (M_{t,u})_{0\le t\le u}$ verifying the two following conditions.*

1. *For any $0\le t\le u$, $M_{t,u}$ is $\mathcal{F}^o_u$-measurable;*
2. *for any $(s,\eta)\in\mathbb{R}_+\times\Omega$, there exists a real cadlag $(\mathbb{P}^{s,\eta},\mathbb{F}^{s,\eta})$- martingale $M^{s,\eta}$ (taken equal to zero on $[0,s]$ by convention) such that for any $\eta\in\Omega$ and $s\le t\le u$,*

$$M_{t,u} = M^{s,\eta}_u - M^{s,\eta}_t \quad \mathbb{P}^{s,\eta} \text{ a.s.}$$

*$M^{s,\eta}$ will be called the **cadlag version of M under $\mathbb{P}^{s,\eta}$**.*

A path-dependent MAF will be said to verify a certain property (being square integrable, having angular bracket absolutely continuous with respect to some non-decreasing function) if under any $\mathbb{P}^{s,\eta}$ its cadlag version verifies it.

Proposition 6.3.20. *Let $(\mathcal{D}(A),A)$ be a weak generator of $(P_s)_{s\in\mathbb{R}_+}$ and $(s,\eta)\in\mathbb{R}_+\times\Omega$. Then for every $\Phi\in\mathcal{D}(A)$, $\Phi - \int_0^\cdot A(\Phi)_r dV_r$ admits for all (s,η) on $[s,+\infty[$ a $\mathbb{P}^{s,\eta}$ version $M[\Phi]^{s,\eta}$ which is a $(\mathbb{P}^{s,\eta},\mathbb{F}^{s,\eta})$-cadlag martingale. In particular, the random field defined by $M[\Phi]_{t,u}(\omega) := \Phi_u(\omega) - \Phi_t(\omega) - \int_t^u A\Phi_r(\omega)dV_r$ defines a MAF with cadlag version $M[\Phi]^{s,\eta}$ under $\mathbb{P}^{s,\eta}$.*

Proposition 6.3.21. *Let M and N be two square integrable path-dependent MAFs and let $M^{s,\eta}$ (respectively $N^{s,\eta}$) be the cadlag version of M (respectively N) under a fixed $\mathbb{P}^{s,\eta}$. Assume that N has an angular bracket absolutely continuous with respect to V (introduced above Hypothesis 6.3.15). Then there exists an $\mathbb{F}^o$-progressively measurable process k such that for any $(s,\eta)\in\mathbb{R}_+\times\Omega$,*

$$\langle M^{s,\eta}, N^{s,\eta}\rangle = \int_s^{\cdot\vee s} k_r dV_r.$$

Notation 6.3.22. *The process k whose existence is stated in Proposition 6.3.21 will be denoted $\frac{d\langle M,N\rangle}{dV}$.*

6.4 BSDEs and abstract analytical problem

6.4.1 BSDEs driven by a path-dependent MAF

We keep using Notation 6.3.1. We fix a progressive path-dependent canonical class $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ verifying Hypothesis 6.3.4, and $(P_s)_{s\in\mathbb{R}_+}$ the associated path-dependent system of projectors. $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ will model the forward process evolution in the BSDEs.

In this section, we fix $T>0$ and a non-decreasing continuous function $V:[0,T]\longmapsto\mathbb{R}_+$. By convention, any process (resp. function) Y defined on $[0,T]\times\Omega$ (resp. $[0,T]$) will be extended taking value Y_T after time T.

Notation 6.4.1. *For a fixed $(s,\eta)\in[0,T]\times\Omega$, we denote by $dV\otimes d\mathbb{P}^{s,\eta}$ the measure on $\mathcal{B}([s,T])\otimes\mathcal{F}$ defined by $dV\otimes d\mathbb{P}^{s,\eta}(C)=\mathbb{E}^{s,\eta}\left[\int_s^T \mathbb{1}_C(r,\omega)dV_r\right]$. For any $p\in\mathbb{N}^*$ we denote by $\mathcal{L}^p(dV\otimes d\mathbb{P}^{s,\eta})$ the space of $(\mathcal{F}_t^{s,\eta})_{t\in[s,T]}$-progressively measurable processes Y such that $\|Y\|_{p,s,\eta}:=\left(\mathbb{E}^{s,\eta}\left[\int_s^T|Y_r|^p dV_r\right]\right)^{\frac{1}{p}}<\infty$. By a slight abuse of notation we will also say that a process indexed by $[0,T]$ belongs to $\mathcal{L}^p(dV\otimes d\mathbb{P}^{s,\eta})$ if its restriction to $[s,T]\times\Omega$ does.*

$\mathcal{H}_0^2(\mathbb{P}^{s,\eta})$ will denote the space of $(\mathbb{P}^{s,\eta},(\mathcal{F}_t^{s,\eta})_{t\in[0,T]})$-square integrable martingales vanishing at time s, hence on the interval $[0,s]$ since $\mathcal{F}_s$ is $\mathbb{P}^{s,\eta}$-trivial, see Proposition 6.3.14; they will be defined up to indistinguishability.

For any $p\geq 1$, we define $\mathcal{L}^p_{uni}$ as the linear space of $\mathbb{F}^o$-progressively measurable processes such that for all $(s,\eta)\in[0,T]\times\Omega$, $(Y_t)_{t\in[s,T]}$ belongs to $\mathcal{L}^p(dV\otimes d\mathbb{P}^{s,\eta})$. Let $\mathcal{N}$ be the linear subspace of $\mathcal{L}^p_{uni}$ constituted of elements which are equal to 0 $dV\otimes d\mathbb{P}^{s,\eta}$ a.e. for all $(s,\eta)\in[0,T]\times\Omega$. We denote $L^p_{uni}:=\mathcal{L}^p_{uni}\backslash\mathcal{N}$. L^p_{uni} can be equipped with the topology generated by the family of semi-norms $(\|\cdot\|_{p,s,\eta})_{(s,\eta)\in[0,T]\times\Omega}$ which makes it a separate locally convex topological vector space, see Theorem 5.76 in [1].

Definition 6.4.2. *A set $C\in\mathcal{P}ro^o$ will be said to be* **of zero potential** *if $\mathbb{1}_C\in\mathcal{N}$, meaning that $\mathbb{E}^{s,\eta}\left[\int_s^T\mathbb{1}_C(t,\omega)dV_t\right]=0$ for all (s,η) and equivalently that $\mathbb{1}_C$ is equal to 0 in L^2_{uni}.*

A property holding everywhere in $[0,T]\times\Omega$ except on a set of zero potential will said to hold ***quasi surely*** *abbreviated by q.s.*

Remark 6.4.3. *The terminology zero potential is inspired from classical potential theory in the Markovian setup, whereas the terminology quasi surely comes from the theory of capacities and is justified by the fact A is of zero potential iff $\sup_{(s,\eta)\in[0,T]\times\Omega} dV\otimes d\mathbb{P}^{s,\eta}(A\cap([s,T]\times\Omega))=0$.*

We now fix some some $d\in\mathbb{N}^*$ and d square integrable path-dependent MAFs (see Definition 6.3.19) $(N^1_{t,u})_{0\leq t\leq u},\cdots,(N^d_{t,u})_{0\leq t\leq u}$ with cadlag versions $N^{s,\eta}:=(N^{1,s,\eta},\cdots,N^{d,s,\eta})$ under $\mathbb{P}^{s,\eta}$ for fixed (s,η).

Definition 6.4.4. *$N:=(N^1,\cdots,N^d)$ will be called the* ***driving MAF****.*

In relation to this driving MAF we introduce the following hypothesis, which will be in force for the rest of the section.

Hypothesis 6.4.5. *For every integer i such that $1\leq i\leq d$, N^i has an angular bracket which is absolutely continuous with respect to V (see Definition 6.3.19) and $\frac{d\langle N^i\rangle}{dV}$ (see Notation 6.3.22) is q.s. bounded.*

We consider some ξ,f verifying the following hypothesis.

Hypothesis 6.4.6.

1. *ξ is an $\mathcal{F}_T$-measurable r.v. which belongs to $\mathcal{L}^2(\mathbb{P}^{s,\eta})$ for every (s,η);*
2. *$f:([0,T]\times\Omega)\times\mathbb{R}\times\mathbb{R}^d\longmapsto\mathbb{R}$ is measurable with respect to $\mathcal{P}ro^o\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}^d)$ and such that*
   - *(a) $f(\cdot,\cdot,0,0)\in\mathcal{L}^2_{uni}$;*
   - *(b) there exists $K>0$ such that for all $(t,\omega,y,y',z,z')\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$*

$$|f(t,\omega,y',z')-f(t,\omega,y,z)|\leq K(|y'-y|+\|z'-z\|). \tag{6.4.1}$$

An immediate application of Theorem 4.3.3 and Remark 4.3.4 in Chapter 4 is the following existence and uniqueness theorem.

Theorem 6.4.7. *Assume the validity of Hypotheses 6.4.5, 6.4.6. For every $(s,\eta)\in[0,T]\times\Omega$, there exists a unique couple of processes $(Y^{s,\eta},M^{s,\eta})\in\mathcal{L}^2(dV\otimes d\mathbb{P}^{s,\eta})\times\mathcal{H}_0^2(\mathbb{P}^{s,\eta})$ such that on $[s,T]$*

$$Y_\cdot^{s,\eta}=\xi+\int_\cdot^T f\left(r,\cdot,Y_r^{s,\eta},\frac{d\langle M^{s,\eta},N^{s,\eta}\rangle_r}{dV_r}\right)dV_r-(M_T^{s,\eta}-M_\cdot^{s,\eta}),\tag{6.4.2}$$

in the sense of indistinguishability, under probability $\mathbb{P}^{s,\eta}$.

Notation 6.4.8. *For the rest of this section, at fixed $(s,\eta)\in[0,T]\times\Omega$, the previous equation will be denoted $BSDE^{s,\eta}(f,\xi)$. Its unique solution will be denoted $(Y^{s,\eta},M^{s,\eta})$ and we will use the notation $Z^{s,\eta}:=(Z^{1,s,\eta},\cdots,Z^{d,s,\eta}):=\left(\frac{d\langle M^{s,\eta},N^{s,\eta}\rangle_t}{dV_t}\right)_{t\in[s,T]}$.*

Remark 6.4.9. *We emphasize that equation $BSDE^{s,\eta}(f,\xi)$ of the present paper corresponds to equation $BSDE(\xi,f,V,N^{s,\eta})$ in Chapter 4.*

The following proposition can be seen as a path-dependent extension of Theorem 4.5.18 in Chapter 4. Its proof is similar to the one in the Markovian setup and is therefore postponed to the Appendix.

Proposition 6.4.10. *Assume the validity of Hypotheses 6.4.5, 6.4.6. For any (s,η) let $(Y^{s,\eta},M^{s,\eta})$ be as introduced at Theorem 6.4.7. There exists a unique process $Y\in\mathcal{L}^2_{uni}$, a square integrable path-dependent MAF $(M_{t,u})_{0\leq t\leq u}$ and $Z^1,\cdots,Z^d\in\mathcal{L}^2_{uni}$ unique up to zero potential sets (see Definition 6.4.2) such that for all $(s,\eta)\in[0,T]\times\Omega$ the following holds.*

1. *$Y^{s,\eta}$ is on $[s,T]$ a $\mathbb{P}^{s,\eta}$-modification of Y;*
2. *$M^{s,\eta}$ is the cadlag version of M under $\mathbb{P}^{s,\eta}$;*
3. *For all integers $i\in\{1,\dots,d\}$, $Z^i=\frac{d\langle M^{s,\eta},N^{i,s,\eta}\rangle}{dV}\,dV\otimes d\mathbb{P}^{s,\eta}$ a.e.*

Moreover, Y is given by $Y:(s,\eta)\longmapsto Y_s^{s,\eta}$.

6.4.2 Decoupled mild solutions for abstract operators

In this subsection, we assume that we are given some $(\mathcal{D}(A),A)$ satisfying Hypothesis 6.3.15 and being a weak generator of $(P_s)_{s\in\mathbb{R}_+}$. We recall that by Proposition 6.3.18, this implies that $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ solves the martingale problem associated to $(\mathcal{D}(A),A)$. By convention we will assume that every $\Phi\in\mathcal{D}(A)$ is constant after time T (meaning $\Phi=\Phi_{\cdot\wedge T}$) and $A\Phi=0$ on $]T,+\infty[$.

Notation 6.4.11. *For every $\Phi\in\mathcal{D}(A)$ we introduce the MAF (see Proposition 6.3.20)*

$$M[\Phi]_{t,u}(\omega):=\Phi_u(\omega)-\Phi_t(\omega)-\int_t^u A\Phi_r(\omega)dV_r.\tag{6.4.3}$$

For all (s,η) we denote by $M[\Phi]^{s,\eta}$ its cadlag version.
We also denote $\Phi^{s,\eta}:=\Phi_s(\eta)+\int_s^{s\vee\cdot}(A\Phi)_r dV_r+M[\Phi]^{s,\eta}$.

Remark 6.4.12. *By Proposition 6.3.20, $M[\Phi]^{s,\eta}$ is a cadlag $(\mathbb{P}^{s,\eta},\mathbb{F}^{s,\eta})$-martingale which is a $\mathbb{P}^{s,\eta}$-version of $\Phi_{s\vee\cdot\wedge T}-\Phi_s(\eta)-\int_s^{s\vee\cdot}(A\Phi)_r dV_r$ and therefore $\Phi^{s,\eta}$ is a cadlag special semimartingale which is a $\mathbb{P}^{s,\eta}$-version of Φ on $[s,T]$.*

Notation 6.4.13. *Let $\Phi,\Psi\in\mathcal{D}(A)$ be such that $\Phi\Psi\in\mathcal{D}(A)$. We denote by $\Gamma(\Phi,\Psi)$ the process $A(\Phi\Psi)-\Phi A(\Psi)-\Psi A(\Phi)$. If Φ or Ψ is multidimensional, then we define $\Gamma(\Phi,\Psi)$ as a vector or matrix, coordinate by coordinate.*

When it exists, $\Gamma(\Phi,\Phi)$ will be denoted $\Gamma(\Phi)$.

Γ can be interpreted as a path-dependent extension of the concept of carré du champ operator in the theory of Markov processes.

Lemma 6.4.14. *Let $\Phi, \Psi \in \mathcal{D}(A)$ be such that $\Phi\Psi \in \mathcal{D}(A)$ and assume that both $M[\Psi], M[\Phi]$ are square integrable MAFs. Then for any $(s,\eta) \in [0,T] \times \Omega$ we have $\langle M^{s,\eta}[\Phi], M[\Psi]^{s,\eta}\rangle = \int_s^{s\vee\cdot\wedge T} \Gamma(\Phi,\Psi)dV_r$ in the sense of $\mathbb{P}^{s,\eta}$-indistinguishability. In particular, $\Gamma(\Phi,\Psi) = \frac{d\langle M[\Phi],M[\Psi]\rangle}{dV}$ q.s., see Notation 6.3.22.*

Proof. We fix $\Phi, \Psi \in \mathcal{D}(A)$, $(s,\eta) \in [0,T] \times \Omega$ and the associated probability $\mathbb{P}^{s,\eta}$. We recall that on $[s,T]$, $\Phi^{s,\eta} = \Phi_s(\eta) + \int_s^\cdot A(\Phi)_r dV_r + M^{s,\eta}[\Phi]$ and $\Psi^{s,\eta} = \Psi_s(\eta) + \int_s^\cdot A(\Psi)_r dV_r + M^{s,\eta}[\Psi]$ are both cadlag special semimartingales; since $M^{s,\eta}[\Phi], M^{s,\eta}[\Psi]$ are assumed to be square integrable martingales, they have a well-defined quadratic covariation and angular bracket. Therefore, by integration by parts on $[s,T]$ and by Lemma 6.A.2 we have

$$\begin{aligned}\Phi^{s,\eta}\Psi^{s,\eta} = \;& \Phi_s(\eta)\Psi_s(\eta) + \textstyle\int_s^\cdot (\Psi_r A(\Phi)_r + \Phi_r A(\Psi)_r)dV_r + \langle M^{s,\eta}[\Phi], M^{s,\eta}[\Psi]\rangle \\ & + \textstyle\int_s^\cdot \Phi^{s,\eta}_{r^-} dM^{s,\eta}[\Psi]_r + \int_s^\cdot \Psi^{s,\eta}_{r^-} dM^{s,\eta}[\Phi]_r \\ & + \left([M^{s,\eta}[\Phi], M^{s,\eta}[\Psi]] - \langle M^{s,\eta}[\Phi], M^{s,\eta}[\Psi]\rangle\right).\end{aligned}$$

On the other hand, since $\Phi\Psi$ belongs to $\mathcal{D}(A)$, we also have that

$$\Phi^{s,\eta}\Psi^{s,\eta} = (\Phi\Psi)^{s,\eta} = \Phi_s(\eta)\Psi_s(\eta) + \int_s^\cdot A(\Phi\Psi)_r dV_r + M^{s,\eta}[\Phi\Psi]. \tag{6.4.4}$$

By uniqueness of the decomposition of a special semimartingale, we can identify the predictable bounded variation part in the two previous decompositions, and we get

$$\int_s^\cdot (\Psi_r A(\Phi)_r + \Phi_r A(\Psi)_r)dV_r + \langle M^{s,\eta}[\Phi], M^{s,\eta}[\Psi]\rangle = \int_s^\cdot A(\Phi\Psi)_r dV_r, \tag{6.4.5}$$

hence that

$$\langle M^{s,\eta}[\Phi], M^{s,\eta}[\Psi]\rangle = \int_s^\cdot (A(\Phi\Psi)_r - \Psi_r A(\Phi)_r - \Phi_r A(\Psi)_r)dV_r, \tag{6.4.6}$$

and the proof is complete. □

Lemma 6.4.15. *If $\Phi \in \mathcal{D}(A)$, $\Phi^2 \in \mathcal{D}(A)$ and $A\Phi \in \mathcal{L}^2_{uni}$, then the MAF $M[\Phi]$ is square integrable. Moreover, $\sup_{t\in[s,T]} |\Phi^{s,\eta}_t| \in \mathcal{L}^2(\mathbb{P}^{s,\eta})$ for all (s,η).*

Proof. Let (s,η) be fixed. We have $(\Phi^{s,\eta})^2 = \Phi_s^2(\eta) + \int_s^\cdot A(\Phi^2)_r dV_r + M^{s,\eta}[\Phi^2]$, where $M^{s,\eta}[\Phi^2]$ is a martingale, hence takes $\mathcal{L}^1$-values; moreover $A(\Phi^2) \in \mathcal{L}^1(dV \otimes d\mathbb{P}^{s,\eta})$, see Hypothesis 6.3.15. It is therefore clear that $\Phi^{s,\eta}$ has $\mathcal{L}^2$-values. Since

$$\Phi^{s,\eta} = \Phi_s(\eta) + \int_s^\cdot A(\Phi)_r dV_r + M^{s,\eta}[\Phi], \tag{6.4.7}$$

by Cauchy-Schwarz inequality we have

$$(M^{s,\eta}[\Phi]_T)^2 \le 4(\Phi_s^2(\eta) + (V_T - V_s)\int_s^T A(\Phi)_r^2 dV_r + (\Phi_T^{s,\eta})^2).$$

The right-hand side belongs to $\mathcal{L}^1$ because $A\Phi \in \mathcal{L}^2_{uni}$, $\Phi_s^2(\eta)$ is deterministic and $\Phi^{s,\eta}$ takes $\mathcal{L}^2$ values, therefore $M^{s,\eta}[\Phi] \in \mathcal{H}^2(\mathbb{P}^{s,\eta})$.

Concerning the second statement, (6.4.7) yields

$$\sup_{t\in[s,T]} (\Phi_t^{s,\eta})^2 \le 4(\Phi_s^2(\eta) + (V_T - V_s)\int_s^T A(\Phi)_r^2 dV_r + \sup_{t\in[s,T]} (M^{s,\eta}[\Phi]_t)^2) \in \mathcal{L}^1,$$

by Doob's inequality because $M^{s,\eta}[\Phi] \in \mathcal{H}^2(\mathbb{P}^{s,\eta})$. □

We will now be interested in a specific type of driving MAF N, see Definition 6.4.4.
We fix $\Psi^1, \cdots, \Psi^d \in \mathcal{D}(A)$ verifying the following.

Hypothesis 6.4.16. *For every $1 \leq i \leq d$,*

1. $\Psi^i, A(\Psi^i) \in \mathcal{L}^2_{uni}$;
2. $(\Psi^i)^2 \in \mathcal{D}(A)$;
3. $\Gamma(\Psi^i)$ *is bounded.*

Remark 6.4.17.

1. $(M[\Psi^1], \cdots, M[\Psi^d])$ *is a vector of square integrable MAFs verifying Hypothesis 6.4.5. This follows because of Lemmas 6.4.14 and 6.4.15.*
2. *In the sequel of Section 6.4, we therefore work with the driving MAF $N = M[\Psi] := (M[\Psi^1], \cdots, M[\Psi^d])$, see Definition 6.4.4. With this choice we fit the framework of Section 6.4.1.*
3. *In particular, Theorem 6.4.7 and Proposition 6.4.10 apply: for all (s, η) there is a unique solution $(Y^{s,\eta}, M^{s,\eta})$ of the $BSDE^{s,\eta}(f, \xi)$ (6.4.2), where the driving MAF is $N = M[\Psi]$.*

We now consider the following abstract path-dependent non linear equation.

$$\begin{cases} A\Phi + f(\cdot, \cdot, \Phi, \Gamma(\Phi, \Psi)) = 0 \text{ on } [0, T] \times \Omega \\ \Phi_T = \xi \text{ on } \Omega. \end{cases} \tag{6.4.8}$$

Inspired by an analogous notion in the Markovian framework (see Chapter 4) and by the classical notion of mild solution, we introduce the corresponding notion of *decoupled mild solution* for a path-dependent evolution equation.

Definition 6.4.18. *Let $Y, Z^1, \cdots, Z^d \in \mathcal{L}^2_{uni}$ (see Notation 6.4.1) and denote $Z := (Z_1, \cdots, Z_d)$.*

1. *The couple (Y, Z) will be called* **solution of the identification problem related to** (f, ξ) *or simply* **solution of** $IP(f, \xi)$ *if for every $(s, \eta) \in [0, T] \times \Omega$,*

$$\begin{cases} Y_s(\eta) &= P_s[\xi](\eta) + \int_s^T P_s\left[f\left(r, \cdot, Y_r, Z_r\right)\right](\eta) dV_r \\ (Y\Psi^1)_s(\eta) &= P_s[\xi\Psi^1_T](\eta) - \int_s^T P_s\left[\left(Z^1_r + Y_r A(\Psi^1)_r - \Psi^1_r f\left(r, \cdot, Y_r, Z_r\right)\right)\right](\eta) dV_r \\ &\cdots \\ (Y\Psi^d)_s(\eta) &= P_s[\xi\Psi^d_T](\eta) - \int_s^T P_s\left[\left(Z^d_r + Y_r A(\Psi^d)_r - \Psi^d_r f\left(r, \cdot, Y_r, Z_r\right)\right)\right](\eta) dV_r. \end{cases} \tag{6.4.9}$$

2. *A process Y will be called* **decoupled mild solution** *of* (6.4.8) *if there exist some Z such that the couple (Y, Z) is a solution of $IP(f, \xi)$.*

Theorem 6.4.19. *Assume the validity of Hypothesis 6.4.16 for $\Psi^1, \ldots, \Psi^d$ and of Hypothesis 6.4.6 for (f, ξ).*

1. *$IP(f, \xi)$ admits a unique solution $(Y, Z) \in \mathcal{L}^2_{uni} \times (L^2_{uni})^d$. By uniqueness we mean more precisely the following: if (Y, Z) and $(\bar{Y}, \bar{Z})$ are two solutions then Y and $\bar{Y}$ are identical and $Z = \bar{Z}$ q.s. In particular, there is a unique decoupled mild solution Y of* (6.4.8).
2. *For every (s, η), let $(Y^{s,\eta}, M^{s,\eta})$ be the solution of $BSDE^{s,\eta}(f, \xi)$ (6.4.2) with $N^{s,\eta} = M[\Psi]^{s,\eta}$. Then, for every (s, η), $Y_s(\eta) = Y^{s,\eta}_s$, and $(Z^1, \ldots, Z^d)$ are identified as in item 3. of Proposition 6.4.10 with $N = M[\Psi]$.*

Proof. We start establishing existence in item 1. Let Y, Z be the processes introduced in Proposition 6.4.10. A direct consequence of that proposition, of Lemma 6.A.2 and of equations $BSDE^{s,\eta}(f,g)$ is that for every $(s,\eta) \in [0,T] \times \Omega$, we have

$$Y_s = \xi + \int_s^T f\left(r, \cdot, Y_r, Z_r\right) dV_r - (M_T^{s,\eta} - M_s^{s,\eta}) \quad \mathbb{P}^{s,\eta} \text{ a.s.} \tag{6.4.10}$$

Taking the expectation, applying Fubini's theorem and using the fact that $M^{s,\eta}$ is a martingale, in agreement with Remark 6.3.5 we get

$$Y_s(\eta) = P_s[\xi](\eta) + \int_s^T P_s[f(r,\cdot,Y_r,Z_r)](\eta) dV_r. \tag{6.4.11}$$

We now fix an integer $1 \leq i \leq d$ and $(s,\eta) \in [0,T] \times \Omega$. By Remark 6.4.12. we recall that the process Ψ^i admits on $[s,T]$ a $\mathbb{P}^{s,\eta}$-modification which is a cadlag special semimartingale with decomposition

$$\Psi^{i,s,\eta} = \Psi_s^i(\eta) + \int_s^{\cdot} A(\Psi^i)_r dV_r + M[\Psi^i]^{s,\eta}. \tag{6.4.12}$$

Applying the integration by parts formula to $Y^{s,\eta}\Psi^{i,s,\eta}$, we get

$$\begin{aligned} d(Y^{s,\eta}\Psi^{i,s,\eta})_t &= -\Psi_t^{i,s,\eta} f(t,\cdot,Y_t^{s,\eta},Z_t^{s,\eta}) dV_t + \Psi_{t^-}^{i,s,\eta} dM_t^{s,\eta} + Y_t^{s,\eta} A(\Psi^i)_t dV_t \\ &+ Y_{t^-}^{s,\eta} dM[\Psi^i]_t^{s,\eta} + d[M^{s,\eta}, M[\Psi^i]^{s,\eta}]_t, \end{aligned}$$

hence integrating between s and T, by Proposition 6.4.10 and by Lemma 6.A.2,

$$\begin{aligned} Y_s(\eta)\Psi_s^i(\eta) &= \xi\Psi_T^i - \int_s^T (Y_r A(\Psi^i)_r - \Psi_r^i f(r,\cdot,Y_r,Z_r)) dV_r - \int_s^T \Psi_{r^-}^{i,s,\eta} dM_r^{s,\eta} \\ &- \int_s^T Y_{r^-}^{s,\eta} dM[\Psi^i]_r^{s,\eta} - [M^{s,\eta}, M[\Psi^i]^{s,\eta}]_T. \end{aligned} \tag{6.4.13}$$

We wish once again to take the expectation and to use Fubini's theorem. Since $Y, A(\Psi^i), \Psi^i, f(\cdot,\cdot,Y,Z) \in \mathcal{L}^2_{uni}$ then $(Y_r A(\Psi^i)_r - \Psi_r^i f(r,\cdot,Y_r,Z_r)) \in \mathcal{L}^1_{uni}$.

By Lemma 6.4.15 we have $\sup_{t\in[s,T]} |\Psi_t^{i,s,\eta}| \in \mathcal{L}^2(\mathbb{P}^{s,\eta})$ and $M[\Psi^i]^{s,\eta} \in \mathcal{H}^2(\mathbb{P}^{s,\eta})$. We also have $M^{s,\eta} \in \mathcal{H}^2(\mathbb{P}^{s,\eta})$, and by Remark 4.A.11 in Chapter 4, $\sup_{t\in[s,T]} |Y_t^{s,\eta}| \in \mathcal{L}^2(\mathbb{P}^{s,\eta})$. We recall that this implies that both $\int_s^{\cdot} \Psi_{r^-}^{i,s,\eta} dM_r^{s,\eta}$ and $\int_s^{\cdot} Y_{r^-}^{s,\eta} dM[\Psi^i]_r^{s,\eta}$ are martingales, see for example Lemma 2.3.17 in Chapter 2. Finally, since by Lemma 6.4.15, $\Psi_T^i \in \mathcal{L}^2(\mathbb{P}^{s,\eta})$ and since $\xi \in \mathcal{L}^2(\mathbb{P}^{s,\eta})$, then $\xi\Psi_T^i \in \mathcal{L}^1(\mathbb{P}^{s,\eta})$. So we can take the expectation in (6.4.13) to get

$$\begin{aligned} Y_s(\eta)\Psi_s^i(\eta) &= \mathbb{E}^{s,\eta}\left[\xi\Psi_T^i - \textstyle\int_s^T (Y_r A(\Psi^i)_r - \Psi_r^i f(r,\cdot,Y_r,Z_r)) dV_r - [M^{s,\eta}, M[\Psi^i]^{s,\eta}]_T\right] \\ &= \mathbb{E}^{s,\eta}\left[\xi\Psi_T^i - \textstyle\int_s^T (Y_r A(\Psi^i)_r - \Psi_r^i f(r,\cdot,Y_r,Z_r)) dV_r - \langle M^{s,\eta}, M[\Psi^i]^{s,\eta}\rangle_T\right] \\ &= \mathbb{E}^{s,\eta}\left[\xi\Psi_T^i - \textstyle\int_s^T (Z_r^i + Y_r A(\Psi^i)_r - \Psi_r^i f(r,\cdot,Y_r,Z_r)) dV_r\right], \end{aligned} \tag{6.4.14}$$

where the latter equality yields from Proposition 6.4.10. Since $Z^i \in \mathcal{L}^2_{uni}$ and $(Y_r A(\Psi^i)_r - \Psi_r^i f(r,\cdot,Y_r,Z_r)) \in \mathcal{L}^1_{uni}$, by Fubini's Theorem we have

$$Y_s(\eta)\Psi_s^i(\eta) = P_s[\xi\Psi_T^i](\eta) - \int_s^T P_s[Z_r^i + Y_r A(\Psi^i)_r - \Psi_r^i f(r,\cdot,Y_r,Z_r)](\eta) dV_r. \tag{6.4.15}$$

This shows existence in item 1. The validity of item 2. comes from the choice of (Y, Z) and by Proposition 6.4.10.

We will now proceed showing uniqueness in item 1. We assume the existence of $U, W^1, \cdots, W^d \in \mathcal{L}^2_{uni}$ such that for all $(s,\eta) \in [0,T] \times \Omega$,

$$\left\{ \begin{array}{rl} U_s(\eta) & = P_s[\xi](\eta) + \int_s^T P_s[f(r,\cdot,U_r,W_r)](\eta) dV_r \\ U_s(\eta)\Psi^i_s(\eta) & = P_s[\xi\Psi^i_T](\eta) - \int_s^T P_s[W^i_r + U_r A(\Psi^i)_r - \Psi^i_r f(r,\cdot,U_r,W_r)](\eta) dV_r \\ & 1 \leq i \leq d. \end{array} \right. \tag{6.4.16}$$

We will show that $U = Y$ and that for all (s,η), $1 \leq i \leq d$,
$W^i = \frac{d\langle M^{s,\eta}, M[\Psi^i]^{s,\eta}\rangle_r}{dV_r}$, $\quad dV \otimes d\mathbb{P}^{s,\eta}$a.e. hence that $W = Z$ q.s.

We fix $(s,\eta) \in [0,T] \times \Omega$. We define the process $\bar{M}$ as being equal to 0 on $[0,s]$ and to $U_t - U_s(\eta) + \int_s^t f(r,\cdot,U_r,W_r)dV_r$ for $t \in [s,T]$. Let us fix $t \in [s,T]$. We emphasize that U and $\bar{M}$ are a priori not cadlag. Applying the first line of (6.4.16) to $(s,\eta) := (t,\omega)$, we get, $\mathbb{P}^{s,\eta}$ a.s.

$$\begin{array}{rcl} U_t(\omega) & = & P_t[\xi](\omega) + \int_t^T P_t[f(r,\cdot,U_r,W_r)](\omega) dV_r \\ & = & \mathbb{E}^{t,\omega}\left[\xi + \int_t^T f(r,\cdot,U_r,W_r)dV_r\right] \\ & = & \mathbb{E}^{s,\eta}\left[\xi + \int_t^T f(r,\cdot,U_r,W_r)dV_r | \mathcal{F}_t\right], \end{array} \tag{6.4.17}$$

thanks to Remark 6.3.5. From this we deduce that for all $t \in [s,T]$,

$$\bar{M}_t = \mathbb{E}^{s,\eta}\left[\xi + \int_s^T f(r,\cdot,U_r,W_r)dV_r \middle| \mathcal{F}_t\right] - U_s(\eta) \quad \mathbb{P}^{s,\eta}\text{a.s.} \tag{6.4.18}$$

In particular $\bar{M}$ is a $\mathbb{P}^{s,\eta}$-martingale on $[s,T]$. Since $(\Omega, \mathcal{F}^{s,\eta}, \mathbb{F}^{s,\eta}, \mathbb{P}^{s,\eta})$ fulfills the usual conditions, $\bar{M}$ admits a cadlag version which we will denote $\bar{M}^{s,\eta}$. Then U admits on $[s,T]$ a cadlag version $U^{s,\eta}$ which is a special semimartingale with decomposition

$$\begin{array}{rcl} U^{s,\eta} & := & U_s(\eta) - \int_s^\cdot f(r,\cdot,U_r,W_r)dV_r + \bar{M}^{s,\eta} \\ & = & U_s(\eta) - \int_s^\cdot f(r,\cdot,U^{s,\eta}_r,W_r)dV_r + \bar{M}^{s,\eta}, \end{array} \tag{6.4.19}$$

where the second equality follows by the fact that $U^{s,\eta}$ is a version of U and by Lemma 6.A.2. By (6.4.17), we have $U^{s,\eta}_T = U_T = \xi$, $\mathbb{P}^{s,\eta}$-a.s. so,

$$U^{s,\eta} = \xi + \int_\cdot^T f(r,\cdot,U^{s,\eta}_r,W_r)dV_r - (\bar{M}^{s,\eta}_T - \bar{M}^{s,\eta}). \tag{6.4.20}$$

We show below that $(U^{s,\eta}, \bar{M}^{s,\eta})$ solves $BSDE^{s,\eta}(f,\xi)$ on $[s,T]$. For this, we are left to show that $W = \frac{d\langle \bar{M}^{s,\eta}, M[\Psi]^{s,\eta}\rangle}{dV}$, $dV \otimes d\mathbb{P}^{s,\eta}$ a.e., and that $(U^{s,\eta}, \bar{M}^{s,\eta}) \in \mathcal{L}^2(dV \otimes d\mathbb{P}^{s,\eta}) \times \mathcal{H}^2_0(\mathbb{P}^{s,\eta})$. By (6.4.18) together with Jensen's and Cauchy-Schwarz inequalities we have

$$\begin{array}{rcl} & & \mathbb{E}^{s,\eta}[\bar{M}^2_T] \\ & = & \mathbb{E}^{s,\eta}\left[(\xi - U_s(\eta) + \int_s^T f(r,\cdot,U_r,W_r)dV_r)^2\right] \\ & \leq & 4\left(U_s(\eta)^2 + \mathbb{E}^{s,\eta}[\xi^2] + (V_T - V_s)\mathbb{E}^{s,\eta}\left[\int_s^T f^2(r,\cdot,U_r,W_r)dV_r)\right]\right), \end{array} \tag{6.4.21}$$

where the latter term is finite because $f(\cdot,\cdot,0,0), U, W^1, \cdots, W^d \in \mathcal{L}^2_{uni}$ and because of the Lipschitz condition on f. So the cadlag version $\bar{M}^{s,\eta}$ of $\bar{M}$, belongs to $\mathcal{H}^2_0(\mathbb{P}^{s,\eta})$. We fix again some $1 \leq i \leq d$.

Combining the second line of (6.4.16) with this fixed integer i, applied with (t,ω) instead of (s,η), Fubini's lemma and Remark 6.3.5, we get the $\mathbb{P}^{s,\eta}$-a.s. equalities

$$\begin{array}{rl} & U_t(\omega)\Psi^i_t(\omega) \\ = & P_t[\xi\Psi^i_T](\omega) - \int_t^T P_t\left[\left(W^i_r + U_r A(\Psi^i)_r - \Psi^i_r f\left(r,\cdot,U_r,W_r\right)\right)\right](\omega) dV_r \\ = & \mathbb{E}^{t,\omega}\left[\xi\Psi^i_T - \int_t^T \left(W^i_r + U_r A(\Psi^i)_r - \Psi^i_r f\left(r,\cdot,U_r,W_r\right)\right) dV_r\right] \\ = & \mathbb{E}^{s,\eta}\left[\xi\Psi^i_T - \int_t^T \left(W^i_r + U_r A(\Psi^i)_r - \Psi^i_r f\left(r,\cdot,U_r,W_r\right)\right) dV_r | \mathcal{F}_t\right](\omega). \end{array} \tag{6.4.22}$$

We introduce the process $\tilde{M}^i$ equal to 0 on $[0,s[$ and defined on $[s,T]$ by

$$\tilde{M}^i_t := U_t\Psi^i_t - U_s(\eta)\Psi^i_s(\eta) - \int_s^t \left(W^i_r + U_r A(\Psi^i)_r - \Psi^i_r f\left(r,\cdot,U_r,W_r\right)\right) dV_r. \tag{6.4.23}$$

Similarly as for (6.4.18), we deduce from (6.4.22) that for all $t \in [s,T]$, $\mathbb{P}^{s,\eta}$ a.s., $\tilde{M}^i_t = \mathbb{E}^{s,\eta}\left[\xi\Psi^i_T - \int_s^T \left(W^i_r + U_r A(\Psi^i)_r - \Psi^i_r f\left(r,\cdot,U_r,W_r\right)\right) dV_r | \mathcal{F}_t\right] - U_s(\eta)\Psi^i_s(\eta)$ So $\tilde{M}^i$ is a $\mathbb{P}^{s,\eta}$-martingale. Under $\mathbb{P}^{s,\eta}$, we consider the cadlag version $\tilde{M}^{i,s,x}$ of $\tilde{M}^i$.

It follows by (6.4.23) that the process

$$U_s(\eta)\Psi^i_s(\eta) + \int_s^\cdot \left(W^i_r + U_r A(\Psi^i)_r - \Psi^i_r f\left(r,\cdot,U_r,W_r\right)\right) dV_r + \tilde{M}^{i,s,\eta}, \tag{6.4.24}$$

is a cadlag special semimartingale which is a $\mathbb{P}^{s,\eta}$-version of $U\Psi^i$ on $[s,T]$ hence indistinguishable from $U^{s,\eta}\Psi^{i,s,\eta}$ on $[s,T]$ since $U^{s,\eta}$ (resp. $\Psi^{i,s,\eta}$) is a cadlag version of U (resp. Ψ^i). Using (6.4.19) and integrating by parts, on $[s,T]$ we also get for $U^{s,\eta}\Psi^{i,s,\eta}$ the decomposition

$$\begin{array}{rl} & U^{s,\eta}\Psi^{i,s,\eta} \\ = & U_s(\eta)\Psi^i_s(\eta) + \int_s^\cdot U^{s,\eta}_r A(\Psi^i)_r dV_r + \int_s^\cdot U^{s,\eta}_{r^-} dM[\Psi^i]^{s,\eta}_r \\ & - \int_s^\cdot \Psi^{i,s,\eta}_r f(r,\cdot,U_r,W_r) dV_r + \int_s^\cdot \Psi^{i,s,\eta}_{r^-} d\bar{M}^{s,\eta}_r + [\bar{M}^{s,\eta}, M[\Psi^i]^{s,\eta}] \\ = & U_s(\eta)\Psi^i_s(\eta) + \int_s^\cdot (U^{s,\eta}_r A(\Psi^i)_r - \Psi^{i,s,\eta}_r f(r,\cdot,U_r,W_r)) dV_r + \langle \bar{M}^{s,\eta}, M[\Psi^i]^{s,\eta}\rangle \\ & + \int_s^\cdot U^{s,\eta}_{r^-} dM[\Psi^i]^{s,\eta}_r + \int_s^\cdot \Psi^{i,s,\eta}_{r^-} d\bar{M}^{s,\eta}_r + \left([\bar{M}^{s,\eta}, M[\Psi^i]^{s,\eta}] - \langle \bar{M}^{s,\eta}, M[\Psi^i]^{s,\eta}\rangle\right). \end{array} \tag{6.4.25}$$

(6.4.24) and (6.4.25) provide now two ways of decomposing the special semimartingale $U^{s,\eta}\Psi^{i,s,\eta}$ into the sum of an initial value, a bounded variation predictable process vanishing at time s and of a local martingale vanishing at time s.

By uniqueness of the decomposition of a special semimartingale, identifying the bounded variation predictable components and using Lemma 6.A.2 we get

$$\begin{array}{rl} & \int_s^\cdot (W^i_r + U_r A(\Psi^i)_r - \Psi^i_r f(r,\cdot,U_r,W_r)) dV_r \\ = & \langle \bar{M}^{s,\eta}, M[\Psi^i]^{s,\eta}\rangle + \int_s^\cdot (U_r A(\Psi^i)_r - \Psi^i_r f(r,\cdot,U_r,W_r)) dV_r. \end{array} \tag{6.4.26}$$

This yields that $\langle \bar{M}^{s,\eta}, M[\Psi^i]^{s,\eta}\rangle$ and $\int_s^\cdot W^i_r dV_r$ are indistinguishable on $[s,T]$. Since this holds for all i, thanks to (6.4.20) we have

$$U^{s,\eta} = \xi + \int_\cdot^T f\left(r,\cdot,U^{s,\eta}_r, \frac{d\langle \bar{M}^{s,\eta}, M[\Psi]^{s,\eta}\rangle_r}{dV_r}\right) dV_r - (\bar{M}^{s,\eta}_T - \bar{M}^{s,\eta}), \tag{6.4.27}$$

with $\bar{M}^{s,\eta} \in \mathcal{H}_0^2(\mathbb{P}^{s,\eta})$. This implies of course that $(U^{s,\eta}, \bar{M}^{s,\eta})$ is a solution of $BSDE^{s,\eta}(f,\xi)$. Thanks to the uniqueness statement for BSDEs, see Theorem 6.4.7, this shows $U^{s,\eta} = Y^{s,\eta}$ and $\bar{M}^{s,\eta} = M^{s,\eta}$ in the sense of indistinguishability. In particular, the first equality gives

$$U_s(\eta) = U_s^{s,\eta} = Y_s^{s,\eta} = Y_s(\eta) \text{ a.s.} \tag{6.4.28}$$

Since this holds for all (s,η), we have $U = Y$. On the other hand, for all i we have $W_t^i = \frac{d\langle \bar{M}^{s,\eta}, M[\Psi^i]^{s,\eta}\rangle_t}{dV_t} = \frac{d\langle M^{s,\eta}, M[\Psi^i]^{s,\eta}\rangle_t}{dV_t} = Z_t^i \; dV \otimes dP^{s,\eta}$ a.e. for all (s,η) hence $W^i = Z^i$ q.s. This concludes the proof of uniqueness. □

6.5 Decoupled mild solutions of path-dependent PDEs and IPDEs

In this section we keep using Notation 6.3.1, but $E = \mathbb{R}^d$ for some $d \in \mathbb{N}^*$ and $(X_t^1, \cdots, X_t^d)_{t\in\mathbb{R}_+}$ will denote the coordinates of the canonical process,see Notation 6.3.1. $T > 0$ will be a fixed horizon.

For the convenience of the reader, the stopped canonical process $(X^1_{\cdot\wedge T}, \cdots, X^d_{\cdot\wedge T})$ will still be by denoted $(X^1, \cdots, X^d)$.

6.5.1 Path-dependent SDEs

We now recall some notions and results concerning a family of path-dependent SDEs with jumps whose solution provide examples of path-dependent canonical classes. In this subsection, all results come from Subsection 5.6 in Chapter 5. We will also refer to notions of [61] Chapters II, III, VI and [60] Chapter XIV.5.

Notation 6.5.1. *For any $t \in \mathbb{R}_+$, we denote $\Omega^t := \{\omega \in \Omega : \omega = \omega^t\}$ the set of paths which are constant after time t. We also denote $\Lambda := \{(s,\eta) \in \mathbb{R}_+ \times \Omega : \eta \in \Omega^s\}$.*

Proposition 6.5.2. *Λ is a closed subspace of $\mathbb{R}_+ \times \Omega$, hence a Polish space when equipped with the induced topology. The Borel σ-field $\mathcal{B}(\Lambda)$ is equal to the trace σ-field $\Lambda \cap \mathcal{P}ro^o$.*

From now on, Λ, introduced in Notation 6.5.1, is equipped with the induced topology and the trace σ-field.

We fix a bounded positive measure F on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ not charging 0, and some coefficients:

- β, a bounded $\mathbb{R}^d$-valued $\mathbb{F}^o$-predictable process;
- σ, a bounded $\mathbb{M}_d(\mathbb{R})$-valued $\mathbb{F}^o$-predictable process;
- γ, a bounded $\mathbb{R}^d$-valued $\mathcal{P}re^o \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d$,

defined on the canonical space.

Definition 6.5.3. *Let $(s,\eta) \in \mathbb{R}_+ \times \Omega$. We call **weak solution of the SDE with coefficients** β, σ, γ **and starting in** (s,η) any probability measure $\mathbb{P}^{s,\eta}$ on $(\Omega, \mathcal{F})$ such that there exists a stochastic basis fulfilling the usual conditions $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ on which is defined a d-dimensional Brownian motion W and a Poisson measure p of intensity $q(dt,dx) := dt \otimes F(dx)$, W, p being optional for the filtration $\tilde{\mathbb{F}}$, a d-dimensional $\tilde{\mathbb{F}}$-adapted cadlag process $\tilde{X}$ such that $\mathbb{P}^{s,\eta} = \tilde{\mathbb{P}} \circ \tilde{X}^{-1}$ and such that the following holds.*

Let $\tilde{\beta}$, $\tilde{\sigma}$, $\tilde{\gamma}$ be defined by $\tilde{\beta}_t := \beta_t \circ \tilde{X}$, $\tilde{\sigma}_t := \sigma_t \circ \tilde{X}$ for all $t \in \mathbb{R}_+$ and $\tilde{\gamma}_t(\cdot, x) := \gamma_t(\tilde{X}, x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Then

- *for all $t \in [0,s]$, $\tilde{X}_t = \eta(t)$ $\tilde{\mathbb{P}}$ a.s.;*
- *$\tilde{X}_t = \eta(s) + \int_s^t \tilde{\beta}_r dr + \int_s^t \tilde{\sigma}_r dW_r + \tilde{\gamma} \star (p-q)_t$ $\tilde{\mathbb{P}}$ a.s. for all $t \geq s$,*

where $\star$ is the integration against random measures, see [61] Chapter II.2.d for instance.

Definition 6.5.4. *Let $s \in \mathbb{R}_+$ and $(Y_t)_{t \geq s}$ be a cadlag special semimartingale defined on the canonical space with (unique) decomposition $Y = Y_s + B + M^c + M^d$, where B is predictable with bounded variation, M^c a continuous local martingale, M^d a purely discontinuous martingale, all three vanishing at time s. We will call **characteristics** of Y the triplet (B, C, ν) where $C = \langle M^c \rangle$ and ν is the predictable compensator of the measure of the jumps of Y.*

There are several equivalent characterizations of weak solutions of path-dependent SDEs with jumps which we will now state in our setup.

Notation 6.5.5. *For any $f \in \mathcal{C}_b^2(\mathbb{R}^d)$ and $t \geq 0$, we denote by $A_t f$ the r.v.*

$$\begin{array}{rl} & \beta_t \cdot \nabla f(X_t) + \frac{1}{2} Tr(\sigma\sigma^\intercal \nabla^2 f(X_t)) \\ + & \int_{\mathbb{R}^d} (f(X_t + \gamma_t(\cdot, y)) - f(X_t) - \nabla f(X_t) \cdot \gamma_t(\cdot, y)) F(dy). \end{array} \tag{6.5.1}$$

Proposition 6.5.6. *Let $(s, \eta) \in \mathbb{R}_+ \times \Omega$ be fixed and let $\mathbb{P} \in \mathcal{P}(\Omega)$. There is equivalence between the following items.*

1. *$\mathbb{P}$ is a weak solution of the SDE with coefficients β, σ, γ starting in (s, η).*
2. *$\mathbb{P}(\omega^s = \eta^s) = 1$ and, under $\mathbb{P}$, $(X_t)_{t \geq s}$ is a special semimartingale with characteristics*

$$\begin{array}{rcl} B & = & \int_s^\cdot \beta_r dr; \\ C & = & \int_s^\cdot (\sigma\sigma^\intercal)_r dr; \\ \nu & : & (\omega, A) \longmapsto \int_s^{+\infty} \int_{\mathbb{R}^d} \mathbb{1}_A(r, \gamma_r(\omega, y)) \mathbb{1}_{\{\gamma_r(\omega,y) \neq 0\}} F(dy) dr. \end{array} \tag{6.5.2}$$

3. *$\mathbb{P}(\omega^s = \eta^s) = 1$ and $f(X_\cdot) - \int_s^\cdot A_r f dr$ is on $[s, +\infty[$ a $(\mathbb{P}, \mathbb{F}^o)$-martingale for all $f \in \mathcal{C}_b^2(\mathbb{R}^d)$.*

Theorem 6.5.7. *Assume that for every $(s, \eta) \in \mathbb{R}_+ \times \Omega$, the SDE with coefficients β, σ, γ and starting in (s, η) admits a unique weak solution $\mathbb{P}^{s,\eta}$. Then $(\mathbb{P}^{s,\eta})_{(s,\eta) \in \mathbb{R}_+ \times \Omega}$ is a path-dependent canonical class verifying Hypothesis 6.3.4.*

We introduce the following hypothesis on the coefficients β, σ, γ.

Hypothesis 6.5.8.

1. *β, σ (resp. γ) are bounded and for Lebesgue almost all t (resp. $dt \otimes dF$ almost all (t, y)), β_t, σ_t (resp. $\gamma_t(\cdot, y)$) are continuous.*
2. *For every $(s, \eta) \in \mathbb{R}_+ \times \Omega$ there exists a unique weak solution $\mathbb{P}^{s,\eta}$ of the SDE of coefficients β, σ, γ starting in (s, η), see Definition 6.5.3.*

We recall two classical examples of conditions on the coefficients for which it is known that there is existence and uniqueness of a weak solution for the path-dependent SDE, see Theorem 14.95 and Corollary 14.82 in [60].

Example 6.5.9. *Assume that β, σ, γ are bounded. Moreover we suppose that for all $n \in \mathbb{N}^*$ there exist $K_2^n \in L^1_{loc}(\mathbb{R}_+)$ and a Borel function $K_3^n : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ such that $\int_{\mathbb{R}^d} K_3^n(\cdot, y) F(dy) \in L^1_{loc}(\mathbb{R}_+)$ verifying the following. For all $x \in \mathbb{R}^d$, $t \geq 0$ and $\omega, \omega' \in \Omega$ such that $\sup_{r \leq t} \|\omega(r)\| \leq n$ and $\sup_{r \leq t} \|\omega'(r)\| \leq n$, we have*

- $\|\sigma_t(\omega) - \sigma_t(\omega')\| \leq K_2^n(t) \sup_{r \leq t} \|\omega(r) - \omega'(r)\|^2$;
- $\|\gamma(t, \omega, x) - \gamma(t, \omega', x)\| \leq K_3^n(t, x) \sup_{r \leq t} \|\omega(r) - \omega'(r)\|^2$.

Finally assume that one of the two following hypotheses are fulfilled.

1. *For all $n \in \mathbb{N}^*$, there exists $K_1^n \in L^1_{loc}(\mathbb{R}_+)$ such that for all $t \geq 0$ and $\omega, \omega' \in \Omega$ verifying $\sup_{r \leq t} \|\omega(r)\| \leq n$ and $\sup_{r \leq t} \|\omega'(r)\| \leq n$, we have $\|\beta_t(\omega) - \beta_t(\omega')\| \leq K_1^n(t) \sup_{r \leq t} \|\omega(r) - \omega'(r)\|$;*
2. *there exists $c > 0$ such that for all $x \in \mathbb{R}^d$, $t \geq 0$ and $\omega \in \Omega$, $x^\intercal \sigma_t(\omega) \sigma_t(\omega)^\intercal x \geq c\|x\|^2$.*

Then item 2. of Hypothesis 6.5.8 is satisfied.

Proposition 6.5.10. *Assume that Hypothesis 6.5.8 holds. Then $(\mathbb{P}^{s,\eta})_{(s,\eta) \in \mathbb{R}_+ \times \Omega}$ is a progressive path-dependent canonical class verifying Hypothesis 6.3.4.*

6.5.2 Dupire's derivatives and path-dependent stochastic calculus

We will recall here some notions and results introduced in [37] and later developed in [27].

Definition 6.5.11. *From now on, an $\mathbb{F}^o$-progressively measurable process with values in $\mathbb{R}^n$ for some $n \in \mathbb{N}^*$ will also be called an $\mathbb{R}^n$-**valued functional**. If $n = 1$, Φ will be said real valued functional.*

We recall that such an $\mathbb{R}^n$-**valued functional** can also be seen (by considering its restriction on Λ) as a Borel function from Λ to $\mathbb{R}^n$, see Definition 6.5.1 and Proposition 6.5.2. In the sequel we will not distinguish between an $\mathbb{F}^o$-progressively measurable process and its restriction to Λ.

Notation 6.5.12. *For all $t \geq 0$, we denote $\Lambda_t := \{(s, \eta) \in [0, t] \times \Omega : \eta \in \Omega^s\}$ which is clearly a closed subspace of Λ and of $\mathbb{R}_+ \times \Omega$. On Λ_T we denote by d_∞ the distance defined by $d_\infty((s_1, \eta_1), (s_2, \eta_2)) := \sup_{t \in [0,T]} |\eta_2(t) - \eta_1(t)| + |s_2 - s_1|$.*

This distance induces a topology on Λ_T which is stronger than its natural induced topology inherited from $\mathbb{R}_+ \times \Omega$.

Definition 6.5.13. *Let Φ be some $\mathbb{R}^n$-valued functional. Φ will be said to be* **continuous** *if it is continuous with respect to d_∞.*

The following definitions and notations are adapted from [25].

Definition 6.5.14. *In the whole definition, we fix Φ a real valued functional, constant after time T, i.e. such that $\Phi_t(\omega) = \Phi_{t \wedge T}(\omega)$ for all (t, ω).*

Let $(s, \eta) \in \Lambda_T$. We say that Φ is ***vertically differentiable at*** *(s, η) if*

$$\begin{array}{rcl} x & \longmapsto & \Phi_s(\eta + x \mathbb{1}_{[s,T]}) \\ \mathbb{R}^d & \longrightarrow & \mathbb{R}, \end{array} \tag{6.5.3}$$

is differentiable in 0. The corresponding gradient at 0 is denoted $\nabla \Phi_s(\eta)$.

We say that Φ is ***vertically differentiable*** *if it is vertically differentiable in (s, η) for all $(s, \eta) \in \Lambda_T$. In this case, $\nabla \Phi : (s, \eta) \longmapsto \nabla \Phi(s, \eta)$ defined on Λ_T, will be called the* ***gradient*** *of Φ. We remark that, whenever that derivation gradient is Borel, it defines an $\mathbb{R}^d$-valued functional. Its coordinates will be denoted $(\nabla_i \Phi)_{i \leq d}$.*

Similarly, we can define the **Hessian** $\nabla^2\Phi_s(\eta)$ *of* Φ *at some point* (t,η)*. It belongs to the space of symmetric matrices of size* d *and its coordinates will be denoted* $(\nabla^2_{i,j}\Phi_s(\eta))_{i,j\leq d}$*.*

Let $(s,\eta)\in\Lambda_T$ *(implying that* η *is constant after time* s*). We say that* Φ *is* ***horizontally differentiable at*** $(s,\eta)\in\Lambda_T$*,* $s<T$*; if*

$$\begin{array}{rcl} t & \longmapsto & \Phi_t(\eta) \\ [s,T] & \longrightarrow & \mathbb{R}, \end{array} \tag{6.5.4}$$

admits a right-derivative at s*. The corresponding derivative will be denoted* $D\Phi_s(\eta)$*.*

We say that Φ *is* ***horizontally differentiable*** *if it is horizontally differentiable in* (s,η) *for all* $(s,\eta)\in\Lambda_T$ *such that* $s<T$ *and the limit* $D\Phi_T(\eta):=\lim_{s\uparrow T}D\Phi_s(\eta^s)$ *exists for every* $\eta\in\Omega^T$*. In this case,* $D\Phi:(s,\eta)\longmapsto D\Phi_s(\eta)$ *will be called the* ***horizontal derivative*** *of* Φ*.*

If it is Borel, it defines a real valued functional. Φ *will be said* ***continuous at fixed times*** *if for all* $t\in[0,T]$*,* $\Phi_t(\cdot):\Omega^t\longmapsto\mathbb{R}$ *is continuous with respect to the sup norm on* Ω^t*.*

By convention, if $\Psi=D\Phi,\nabla\Phi,\nabla^2\Phi$ *is well-defined on* Λ_T*, it will be extended on* $[0,T]\times\Omega$ *by setting* $\Psi_t(\omega):=\Psi_t(\omega^t)$ *and on* $]T,+\infty[\times\Omega$ *by the value* 0*.*

Φ *will be said* ***left-continuous*** *if for all* $t\in[0,T]$*,* $\epsilon>0$*,* $\omega\in\Omega^t$*, there exists* $\zeta>0$ *such that for any* $(t',\omega')\in\Lambda_t$ *verifying* $d_\infty((t,\omega),(t',\omega'))<\zeta$ *we have* $|\Phi_t(\omega)-\Phi_{t'}(\omega')|\leq\epsilon$*.*

Φ *will be said* ***boundedness preserving*** *if for any compact set* K *of* $\mathbb{R}^d$ *there exists a constant* $C_K>0$ *such that for all* $t\in[0,T]$ *and* $\omega\in\Omega^t$ *taking values in* K*, we have* $|\Phi_t(\omega)|\leq C_K$*.*

Φ *will be said to* ***have the horizontal local Lipschitz property*** *if for all* $(t,\omega)\in\Lambda_T$*, there exists* $C>0$ *and* $\zeta>0$ *such that for all* $(s,\eta)\in\Lambda_T$ *verifying* $d_\infty((t,\omega),(s,\eta))\leq\zeta$ *we have for all* $0\leq t_1\leq t_2\leq s$ *that* $|\Phi_{t_2}(\eta^{t_1})-\Phi_{t_1}(\eta^{t_1})|\leq C|t_2-t_1|$*.*

Notation 6.5.15. *We denote by* $\mathbb{C}_b^{1,2}(\Lambda_T)$ *the space of real valued functionals* Φ *constant after time* T*, which admit a horizontal derivative and vertical derivatives up to order two such that* $\Phi,D\Phi,\nabla\Phi,\nabla^2\Phi$ *are boundedness preserving, left-continuous and are continuous at fixed time.*

This space is stable by pointwise sum and product.

Notation 6.5.16. *For every* $\omega\in\Omega$*, and* $t\geq0$*, we denote by* ω^{t^-} *the element of* Ω^t *defined by* $\omega^{t^-}(r)=\omega(r)$ *if* $r\in[0,t[$ *and* $\omega^{t^-}(r)=\omega(t^-)$ *otherwise.*

For any process Y *and time* $t\geq0$*, we denote* $\Delta Y_t:=Y_t-Y_{t^-}$*.*

The following path-dependent Itô formula comes from [25] Proposition 6.1. We formulate it in our setup.

Theorem 6.5.17. *Let* $\mathbb{P}$ *be a probability measure on the canonical space* $(\Omega,\mathcal{F})$*. Let* $s\in[0,T]$ *and assume that under probability* $\mathbb{P}$*, the canonical process* X *is such that* $(X_t)_{t\in[s,T]}$ *is an* $(\mathcal{F}_t)_{t\in[s,T]}$*-semimartingale. Let* $\Phi\in\mathbb{C}_b^{1,2}(\Lambda_T)$ *and assume that* $\nabla\Phi$ *has the horizontal local Lipschitz property. Then* $(\Phi_t)_{t\in[s,T]}$ *is an* $(\mathcal{F}_t)_{t\in[s,T]}$*-semimartingale and we have*

$$\begin{array}{rcl} \Phi & = & \Phi_s+\displaystyle\int_s^{\cdot}D\Phi_r dr+\int_s^{\cdot}\nabla\Phi_r dX_r+\frac{1}{2}\int_s^{\cdot}Tr((\nabla^2\Phi_r)^\intercal d\langle X^c\rangle_r) \\ & + & \displaystyle\sum_{r\in]s,\cdot]:\Delta X_r\neq0}(\Phi_r(\omega)-\Phi_r(\omega^{r-})-\nabla\Phi_r(\omega^{r-})\cdot\Delta X_r), \end{array}$$

in the sense of $\mathbb{P}$*-indistinguishability.*

We recall the following elementary example.

Lemma 6.5.18. *Let* $h\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^d)$ *then* $H:(t,\omega)\longmapsto h(t\wedge T,\omega(t\wedge T))$ *belongs to* $\mathbb{C}_b^{1,2}(\Lambda_T)$ *and* $DH:(t,\omega)\mapsto\partial_t h(t,\omega(t))\mathbb{1}_{[0,T]}(t)$*,* $\nabla H:(t,\omega)\mapsto\nabla_x h(t,\omega(t))\mathbb{1}_{[0,T]}(t)$*,* $\nabla^2H:(t,\omega)\mapsto\nabla_x^2h(t,\omega(t))\mathbb{1}_{[0,T]}(t)$*.*

Remark 6.5.19.

1. *In Lemma 6.5.18, it is moreover clear that if h does not depend on t then ∇H has the horizontal local Lipschitz property, hence that Theorem 6.5.17 above applies for $\Phi = H$.*
2. *In particular, for any $i \in [\![1,d]\!]$, $X^i \in \mathbb{C}_b^{1,2}(\Lambda_T)$, $DX^i \equiv 0$, $\nabla X^i \equiv e_i \mathbb{1}_{[0,T]}$ and $\nabla^2 X^i \equiv 0$, where $(e_1, \cdots, e_d)$ denotes the Euclidean basis of $\mathbb{R}^d$.*

6.5.3 Decoupled mild solutions of Path-dependent (I)PDEs

From now on, we suppose $V(t) \equiv t$. We fix some coefficients β, σ, γ verifying Hypothesis 6.5.8 but vanishing after time T. We denote by $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ the weak solution of the corresponding SDE. By Proposition 6.5.10 it defines a progressive path-dependent canonical class verifying Hypothesis 6.3.4. We denote again by $(P_s)_{s\in\mathbb{R}_+}$ the associated path-dependent system of projectors, see Definition 6.3.9.

Definition 6.5.20. *Let Φ be an $\mathbb{F}^o$-progressively measurable process constant after time T. Φ will be said to have **polynomial growth** if there exists $C > 0$, $p \in \mathbb{N}^*$ such that for all $(t,\omega) \in [0,T]\times\Omega$, $|\Phi_t(\omega)| \leq C(1 + \sup_{r\leq t}\|\omega(r)\|^p)$.*

*A r.v. ξ will be said to have **polynomial growth** if there exists $C > 0$, $p \in \mathbb{N}^*$ such that $|\xi(\omega)| \leq C(1 + \sup_{r\leq T}\|\omega(r)\|^p)$.*

Lemma 6.5.21. *For any finite $p \geq 1$, $(s,\eta) \in [0,T]\times\Omega$, $\sup_{t\in[s,T]} |X_t^i| \in \mathcal{L}^p(\mathbb{P}^{s,\eta})$.*

Proof. We fix some $(s,\eta) \in [0,T]\times\Omega$, $1 \leq i \leq d$ and some finite $p \geq 1$. A direct consequence of Proposition 6.5.6 item 2. and of Definition 6.5.4 a is that under $\mathbb{P}^{s,\eta}$, X^i may be decomposed on $[s,T]$ as $\eta_i(s) + \int_s^\cdot \beta_r^i dr + M^c + M^d$ where M^c (resp. M^d) is a continuous (resp. purely discontinuous) local martingale. $\int_s^\cdot \beta_r^i dr$ is bounded and M^c is a continuous local martingale with bounded bracket $\langle M^c\rangle = [M^c] = \int_s^\cdot (\sigma\sigma^\intercal)_r^i dr$ hence by BDG inequality, $\sup_{t\in[s,T]} |M_t^c| \in \mathcal{L}^p(\mathbb{P}^{s,\eta})$. In order to conclude, we are therefore left to show that the same holds for M^d. We have $M_t^d = \sum_{r\leq t}\Delta X_r^i - \int_s^t \int_{\mathbb{R}^d} \gamma_r^i(\cdot,y)F(dy)dr$, where the integral in previous formula is bounded, because γ is bounded. So we need to show that $\left(\sum_{r\leq T}|\Delta X_r^i|\right) \in \mathcal{L}^p(\mathbb{P}^{s,\eta})$. Observing the definition of $\mathbb{P}^{s,\eta}$ and $\tilde{X}$ in Definition 6.5.3 it is clear that since γ is bounded then the jumps of X under $\mathbb{P}^{s,\eta}$ are bounded. So finally, we are left to show that the number of jumps of X^i, meaning $\sum_{r\leq T}\mathbb{1}_{\Delta X_r^i\neq 0}$, belongs to $\mathcal{L}^p(\mathbb{P}^{s,\eta})$. This holds since X can jump only if the underlying Poisson measure p jumps (see Definition 6.5.3) and the number of jumps of p on $[s,T]$ is a Poisson r.v. of parameter $(T-s)F(\mathbb{R}^d)$ which admits a finite p-th moment.

□

Notation 6.5.22. *We set $\mathcal{D}(A)$ to be the space of real valued functionals $\Phi \in \mathbb{C}_b^{1,2}(\Lambda_T)$ such that $\Phi, D\Phi, \nabla\Phi, \nabla^2\Phi$ have polynomial growth and such that $\nabla\Phi$ has the horizontal local Lipschitz property. We define the map A on $\mathcal{D}(A)$ by setting for every $\Phi \in \mathcal{D}(A)$*

$$\begin{aligned}(A\Phi)_t(\omega) := \quad & D\Phi_t(\omega) + \tfrac{1}{2}Tr((\sigma\sigma^\intercal)_t(\omega)\nabla^2\Phi_t(\omega)) + \beta_t(\omega)\cdot\nabla\Phi_t(\omega) \\ & + \textstyle\int_{\mathbb{R}^d}(\Phi_t(\omega + \gamma_t(\omega,y)\mathbb{1}_{[t,+\infty[}) - \Phi_t(\omega) - \gamma_t(\omega,y)\cdot\nabla\Phi_t(\omega))F(dy).\end{aligned} \tag{6.5.5}$$

We also set $M[\Phi]$ as in (6.4.3) *in Notation 6.4.11. It defines an AF.*

Remark 6.5.23. *By Lemma 6.5.18, and expression* (6.5.5)*, the coordinates $X^i, 1 \leq i \leq d$ of the canonical process belong to $\mathcal{D}(A)$ and $A(X^i) = \beta^i$.*

Definition 6.5.20, Lemma 6.5.21, Notation 6.5.22 and the fact that β, σ, γ are bounded and F is finite yield the following.

Corollary 6.5.24.

1. *for every $\Phi \in \mathcal{D}(A)$, $A\Phi$ and $\Phi - \int_0^{\cdot}(A\Phi)_r dr$ have polynomial growth;*
2. *every Φ with polynomial growth verifies that $\sup_{r\in[s,T]} |\Phi_r| \in \mathcal{L}^p(\mathbb{P}^{s,\eta})$ for all finite $p \geq 1$, $(s,\eta) \in [0,T] \times \Omega$;*
3. *for all $\Phi \in \mathcal{D}(A)$, $(s,\eta) \in [0,T] \times \Omega$ and finite $p \geq 1$, we have $\sup_{t\in[s,T]} \left|\Phi_t - \Phi_s(\eta) - \int_s^t A(\Phi)_r dr\right| \in \mathcal{L}^p(\mathbb{P}^{s,\eta})$;*
4. *$(\mathcal{D}(A), A)$ verifies Hypothesis 6.3.15.*

Proposition 6.5.25. *Let $(s,\eta) \in [0,T] \times \Omega$. A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a weak solution of the SDE with coefficients β, σ, γ starting in (s,η) iff it solves the martingale problem associated to $(\mathcal{D}(A), A)$, defined in Notation 6.5.22, in the sense of Definition 6.3.17.*

Proof. We fix (s,η). Let $\mathbb{P}$ be a weak solution of the SDE with coefficients β, σ, γ starting in (s,η). We show that it fulfills the martingale problem in the sense of Definition 6.3.17. By Proposition 6.5.6, we immediately see that $\mathbb{P}(\omega^s = \eta^s) = 1$ which constitutes item 1. of Definition 6.3.17. By Proposition 6.5.6 it follows that, under $\mathbb{P}$, $(X_t)_{t\in[s,+\infty[}$ is a semimartingale with characteristics $\int_s^{\cdot} \beta_r dr$, $\int_s^{\cdot} (\sigma\sigma^\intercal)_r dr$ and $\nu : (\omega, C) \mapsto \int_s^{+\infty} \int_{\mathbb{R}^d} \mathbb{1}_C(r, \gamma_r(\omega, y)) \mathbb{1}_{\{\gamma_r(\omega,y)\neq 0\}} F(dy)dr$.

Now let $\Phi \in \mathcal{D}(A)$. Since for every ω, the set of jump times $\{t : \Delta\omega(t) \neq 0\}$ is countable hence Lebesgue negligible, then $\Phi_r(\omega^r) = \Phi_r(\omega^{r-}),\ dr$ a.e., and so

$$\int_s^{\cdot} \int_{\mathbb{R}^d} (\Phi_r(\omega + \gamma_r(\omega, y)\mathbb{1}_{[r,+\infty[}) - \Phi_r(\omega) - \gamma_r(\omega, y) \cdot \nabla\Phi_r(\omega)) F(dy)dr,$$

is indistinguishable from

$$\int_s^{\cdot} \int_{\mathbb{R}^d} (\Phi_r(\omega^{r-} + \gamma_r(\omega, y)\mathbb{1}_{[r,+\infty[}) - \Phi_r(\omega^{r-}) - \gamma_r(\omega, y) \cdot \nabla\Phi_r(\omega^{r-})) F(dy)dr,$$

which is the compensator of

$$\sum_{r\in]s,\cdot]:\Delta X_r\neq 0} \Phi_r(\omega) - \Phi_r(\omega^{r-}) - \Delta X_r \cdot \nabla\Phi_r(\omega^{r-}),$$

i.e. their difference is a local martingale. By Theorem 6.5.17, we therefore have that, $\Phi - \int_s^{\cdot} A\Phi_r dr$ is a local martingale, and by item 3. of Corollary 6.5.24 it is a martingale. Since this holds for any $\Phi \in \mathcal{D}(A)$, $\mathbb{P}$ also verifies item 2. of Definition 6.3.17. This concludes the proof of the direct implication.

As far as the converse implication is concerned, let us assume that $\mathbb{P}$ satisfies both items of Definition 6.3.17. By Lemma 6.5.18 and Remark 6.5.19, we have the following. For any $h \in \mathcal{C}_b^2(\mathbb{R}^d)$, the functional $H : (t,\omega) \mapsto h(\omega(t \wedge T))$ belongs to $\mathcal{D}(A)$ and, for any $(t,\omega) \in \Lambda$, $AH(t,\omega) = A_t h(\omega)$, see Notation 6.5.5. Definition 6.3.17 therefore implies that $\mathbb{P}$ verifies item 3. in Proposition 6.5.6, hence that it is a weak solution of the SDE. □

Corollary 6.5.26. *Let $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ be the family introduced at the beginning of Section 6.5.3. We suppose the validity Hypothesis 6.5.8.*

1. *$(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ solves the well-posed martingale problem associated to $(\mathcal{D}(A), A)$, see Definition 6.3.17.*
2. *$(\mathcal{D}(A), A)$ is a weak generator of $(P_s)_{s\in\mathbb{R}_+}$, which is the unique path-dependent system of projectors for which this holds.*
3. *For all $\Phi \in \mathcal{D}(A)$, the AF $M[\Phi]$ is a square integrable MAF.*

Proof. The first statement follows by Proposition 6.5.25, the second one by the second statement of Proposition 6.3.18. The third statement holds because of Proposition 6.3.20. We are indeed in the framework of Section 6.4.2, see Notation 6.4.11. □

Proposition 6.5.27. *Let $\Phi \in \mathcal{D}(A)$ be such that for all $i \le d$, $\Phi X^i \in \mathcal{D}(A)$. Then for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$,*

$$\Gamma(X,\Phi)_t(\omega) = (\sigma\sigma^\intercal \nabla\Phi)_t(\omega) + \int_{\mathbb{R}^d} \gamma_t(\omega,y)(\Phi_t(\omega + \gamma_t(\omega,y)\mathbb{1}_{[t,+\infty[}) - \Phi_t(\omega))F(dy). \tag{6.5.6}$$

Proof. We fix Φ and $1 \le i \le d$. We recall that the usual product rules apply to both the horizontal and the vertical derivative so that

$$\begin{aligned}
& A(\Phi X^i)_t - \Phi_t A X^i_t - X^i_t A\Phi_t \\
= \;& D(\Phi X^i)_t - \Phi_t DX^i_t - X^i_t D\Phi_t \\
& + \tfrac{1}{2}Tr((\sigma\sigma^\intercal\nabla^2(\Phi X^i))_t) - \tfrac{1}{2}\Phi_t Tr((\sigma\sigma^\intercal\nabla^2 X^i)_t) - \tfrac{1}{2}X^i_t Tr((\sigma\sigma^\intercal\nabla^2\Phi)_t) \\
& + \beta_t\cdot\nabla(\Phi X^i)_t - \Phi_t\beta_t\cdot\nabla X^i_t - X^i_t\beta_t\cdot\nabla\Phi_t \\
& + \textstyle\int_{\mathbb{R}^d}(\Phi_t(\cdot+\gamma_t(\cdot,y)\mathbb{1}_{[t,+\infty[})(X^i_t+\gamma^i_t(\cdot,y)) - \Phi_t X^i_t - \gamma_t(\cdot,y)\cdot\nabla(X^i\Phi)_t)F(dy) \\
& - \Phi_t\textstyle\int_{\mathbb{R}^d}(\gamma^i_t(\cdot,y) - \gamma_t(\cdot,y)\cdot\nabla\Phi_t)F(dy) \\
& - X^i_t\textstyle\int_{\mathbb{R}^d}(\Phi_t(\cdot+\gamma_t(\cdot,y)\mathbb{1}_{[t,+\infty[}) - \Phi_t - \gamma_t(\cdot,y)\cdot\nabla\Phi_t)F(dy) \\
= \;& \tfrac{1}{2}Tr\left(\sigma\sigma^\intercal_t(\nabla^2(\Phi X^i)_t - \Phi_t\nabla^2X^i_t - X^i_t\nabla^2\Phi_t)\right) \\
& + \textstyle\int_{\mathbb{R}^d}(\Phi_t(\cdot+\gamma_t(\cdot,y)\mathbb{1}_{[t,+\infty[})(X^i_t+\gamma^i_t(\cdot,y)) - \Phi_t X^i_t \\
& - X^i_t(\Phi_t(\cdot+\gamma_t(\cdot,y)\mathbb{1}_{[t,+\infty[}) - \Phi_t) - \Phi_t\gamma^i_t(\cdot,y))F(dy) \\
= \;& \tfrac{1}{2}\textstyle\sum_{j,k}(\sigma\sigma^\intercal)^{j,k}_t(\nabla^2_{j,k}(\Phi X^i)_t - \Phi_t\nabla^2_{j,k}X^i_t - X^i_t\nabla^2_{j,k}\Phi_t) \\
& + \textstyle\int_{\mathbb{R}^d}\gamma^i_t(\cdot,y)(\Phi_t(\cdot+\gamma_t(\cdot,y)\mathbb{1}_{[t,+\infty[}) - \Phi_t)F(dy) \\
= \;& \tfrac{1}{2}\textstyle\sum_{j,k}(\sigma\sigma^\intercal)^{j,k}_t(\nabla_j\Phi_t\nabla_k X^i_t + \nabla^k\Phi_t\nabla^j X^i_t) \\
& + \textstyle\int_{\mathbb{R}^d}\gamma^i_t(\cdot,y)(\Phi_t(\cdot+\gamma_t(\cdot,y)\mathbb{1}_{[t,+\infty[}) - \Phi_t)F(dy) \\
= \;& \tfrac{1}{2}\left(\textstyle\sum_j(\sigma\sigma^\intercal)^{j,i}_t\nabla_j\Phi_t + \sum_k(\sigma\sigma^\intercal)^{i,k}_t\nabla_k\Phi_t\right) \\
& + \textstyle\int_{\mathbb{R}^d}\gamma^i_t(\cdot,y)(\Phi_t(\cdot+\gamma_t(\cdot,y)\mathbb{1}_{[t,+\infty[}) - \Phi_t)F(dy) \\
= \;& (\sigma\sigma^\intercal\nabla\Phi)^i_t + \textstyle\int_{\mathbb{R}^d}\gamma^i_t(\cdot,y)(\Phi_t(\cdot+\gamma_t(\cdot,y)\mathbb{1}_{[t,+\infty[}) - \Phi_t)F(dy),
\end{aligned} \tag{6.5.7}$$

where by Lemma 6.5.18, the fifth equality holds since $\nabla_j X^i$ is constantly equal to 1 if $j = i$ and 0 otherwise. □

Proposition 6.5.28. *X verifies Hypothesis 6.4.16.*

Proof. We fix $i \le d$. By Remark 6.5.23, $X^i \in \mathcal{D}(A)$ with $A(X^i) = \beta^i$. A consequence of Lemma 6.5.21 and of the fact that β is bounded is that X^i verifies item 1. of Hypothesis 6.4.16. By Remark 6.5.19 and since $(X^i)^2$ clearly has polynomial growth we have $(X^i)^2 \in \mathcal{D}(A)$ which is item 2. of Hypothesis 6.4.16. Finally by Proposition 6.5.27, we have $\Gamma(X^i)_t(\omega) = (\sigma\sigma^\intercal)^{i,i}_t(\omega) + \int_{\mathbb{R}^d}(\gamma^i)^2_t(\omega,y)F(dy)$ which is bounded being the coefficients σ, γ, F bounded; so item 3. of Hypothesis 6.4.16 is verified. □

From now on we fix $\Psi^i := X^i$ for all i and the corresponding driving MAF $M[X]$ and we will apply the results of Subsection 6.4.2 to this specific setup.

We now fix ξ, f verifying Hypothesis 6.4.6. We will be interested in the following path-dependent non linear IPDE with terminal condition, denoted by $IPDE(f,\xi)$:

$$\begin{cases} A(\Phi) + f(\cdot,\cdot,\Phi,\Gamma(\Phi,X)) = 0 \text{ on } [0,T]\times\Omega \\ \Phi_T = \xi \text{ on } \Omega, \end{cases} \tag{6.5.8}$$

where the explicit expression of $\Gamma(\Phi,X)$ is given by Proposition 6.5.27.

Remark 6.5.29. *When $\gamma \equiv 0$, the equation* (6.5.8) *is given by*

$$\begin{cases} D\Phi + \frac{1}{2}Tr(\sigma\sigma^\intercal\nabla^2\Phi) + \beta\nabla\Phi + f(\cdot,\cdot,\Phi,\sigma\sigma^\intercal\nabla\Phi) = 0 \text{ on } [0,T]\times\Omega \\ \Phi_T = \xi \text{ on } \Omega, \end{cases} \tag{6.5.9}$$

and it is a path-dependent PDE.

To the path-dependent IPDE (6.5.8), we will associate a family of BSDEs driven by a cadlag martingale, indexed by $(s,\eta) \in [0,T]\times\Omega$.

Notation 6.5.30. *$BSDE^{s,\eta}(f,\xi)$ will denote equation*

$$Y^{s,\eta} = \xi + \int_\cdot^T f\left(r,\cdot,Y_r^{s,\eta},\frac{d\langle M^{s,\eta},M^{s,\eta}[X]\rangle_r}{dr}\right)dr - (M_T^{s,\eta} - M_\cdot^{s,\eta}), \tag{6.5.10}$$

in the stochastic basis $(\Omega,\mathcal{F}^{s,\eta},\mathbb{F}^{s,\eta},\mathbb{P}^{s,\eta})$.

By Proposition 6.5.28, $\Psi := X$, where X is the canonical process, verifies Hypothesis 6.4.16; by item 1. of Remark 6.4.17, $M[X]$ satisfies Hypothesis 6.4.5. Now ξ, f verify Hypothesis 6.4.6; so by Theorem 6.4.7, for every $(s,\eta) \in [0,T]\times\Omega$, $BSDE^{s,\eta}(f,\xi)$ admits a unique solution $(Y^{s,\eta},M^{s,\eta})$ in $\mathcal{L}^2(dt\otimes d\mathbb{P}^{s,\eta})\times\mathcal{H}_0^2(\mathbb{P}^{s,\eta})$.

Definition 6.5.31. *Let $\Phi \in \mathcal{D}(A)$ such that $\Phi X^i \in \mathcal{D}(A)$ for all $i \le d$. We will say that Φ is a **classical solution** of $IPDE(f,\xi)$ if it verifies* (6.5.8).

We can now formulate the main result of this paper.

Theorem 6.5.32. *Assume that Hypothesis 6.5.8 holds, that $\xi, f(\cdot,\cdot,0,0)$ are Borel with polynomial growth and that f is Lipschitz in y,z uniformly in t,ω.*

1. *The identification problem $IP(f,\xi)$ (see Definition 6.4.18) admits a unique solution $(Y,Z) \in \mathcal{L}^2_{uni}\times(L^2_{uni})^d$.*
2. *$IPDE(f,\xi)$ admits a unique decoupled mild solution Y in the sense that whenever Y and $\bar{Y}$ are two decoupled mild solutions then Y and $\bar{Y}$ are identical.*
3. *If for every (s,η), $(Y^{s,\eta},M^{s,\eta})$ is the solution of $BSDE^{s,\eta}(f,\xi)$, i.e.* (6.5.10), *then the decoupled mild solution Y is given by $(s,\eta)\mapsto Y_s^{s,\eta}$. Moreover, for every (s,η), on $[s,T]$, $Y^{s,\eta}$ is a $\mathbb{P}^{s,\eta}$ version of Y and $Z_t = \frac{d\langle M^{s,\eta},M[X]^{s,\eta}\rangle_t}{dt}$, $dt\otimes\mathbb{P}^{s,\eta}$ a.e.*

Proof. It is a consequence of Theorem 6.4.19. Indeed firstly Hypothesis 6.4.16 holds by Proposition 6.5.28 and $\Psi := X$ satisfies Hypothesis 6.4.16; secondly ξ and $f(\cdot,\cdot,0,0)$, being of polynomial growth then (f,ξ) fulfill Hypothesis 6.4.6 because of item 2. of Corollary 6.5.24 and the Lipschitz property of f in (y,z).

□

Remark 6.5.33. *As anticipated in the introduction, given the family of solutions $(Y^{s,\eta}, M^{s,\eta})$ of $BSDE^{s,\eta}(f,\xi)$, we have obtained an analytical characterization of the process $\frac{d\langle M^{s,\eta}, M[X]^{s,\eta}\rangle}{dt}$. This constitutes indeed the "identification" of the "second component" of a solution to a BSDE via solving an analytical problem.*

1. *Indeed, by item 3. of Theorem 6.5.32, that process is $dt \otimes d\mathbb{P}^{s,\eta}$ a.e. equal to the q.s. unique functional Z such that (Y,Z) fulfills* (6.4.9).
2. *If $\Gamma(Y,X)$ (hence $\sigma\sigma^\intercal \nabla Y$ if $\gamma \equiv 0$) is well-defined then $(Y, \Gamma(Y,X))$ fulfills equation* (6.4.9)*, see item 3. of Proposition 6.5.34.*
3. *Previous analytical characterization of $\frac{d\langle M^{s,\eta}, M[X]^{s,\eta}\rangle}{dt}$ is not possible with the theory of viscosity solutions, even in the case of classical Pardoux-Peng Markovian Brownian BSDEs.*

The link between decoupled mild solutions and classical solutions is the following.

Proposition 6.5.34.

1. *Let Φ be a classical solution of $IPDE(f,\xi)$, see Definition 6.5.31. Then $(\Phi, \Gamma(\Phi,X))$ is a solution of the identification problem $IP(f,\xi)$ (see Definition 6.4.18) and in particular, Φ is a decoupled mild solution of $IPDE(f,\xi)$;*
2. *there is at most one classical solution of $IPDE(f,\xi)$;*
3. *assume that the unique decoupled mild solution Y of $IPDE(f,\xi)$ verifies $Y \in \mathcal{D}(A)$ and $YX^i \in \mathcal{D}(A)$, $0 \le i \le d$, then Y is a classical solution q.s., in the sense that $Y_T = \xi$ (for all ω) and that $A(Y) = -f(\cdot,\cdot,Y,\Gamma(Y,X))$ q.s., see Definition 6.4.2.*

Proof. 1. Let Φ be a classical solution. First, since Φ and ΦX^i belong to $\mathcal{D}(A)$ for all $i \le d$; taking into account Notation 6.4.13, by items 1. 2. of Corollary 6.5.24 we can show that $\Phi, \Gamma(\Phi,X^1), \cdots, \Gamma(\Phi,X^d)$ belong to $\mathcal{L}^2_{uni}$.

On the other hand, let $(s,\eta) \in [0,T]\times\Omega$. By Corollary 6.5.26 3. $M[\Phi]_{s,\cdot} := \Phi - \Phi_s(\eta) - \int_s^\cdot A\Phi_r dr$, and $M[\Phi X^i]_{s,\cdot} := \Phi X^i - \Phi_s(\eta)\eta^i(s) - \int_s^\cdot A(\Phi X^i)_r dr$, $1 \le i \le d$, are $\mathbb{P}^{s,\eta}$-martingales on $[s,T]$ vanishing at time s. By Definition 6.5.31 we have $A\Phi = -f(\cdot,\cdot,\Phi,\Gamma(\Phi,X))$ and by Remark 6.5.23 and Notation 6.4.13 we have

$$A(\Phi X^i) = \Gamma(\Phi X^i) + \Phi A X^i + X^i A\Phi = \Gamma(\Phi X^i) + \Phi\beta^i - X^i f(\cdot,\cdot,\Phi,\Gamma(\Phi,X)),$$

so the previously mentioned martingales indexed by $[s,T]$, can be rewritten

$$\left\{\begin{array}{rcl} M[\Phi]_{s,\cdot} &=& \Phi - \Phi_s(\eta) + \int_s^\cdot f(r,\cdot,\Phi_r,\Gamma(\Phi,X)_r)dr \\ M[\Phi X^i]_{s,\cdot} &=& \Phi X^i - \Phi_s(\eta)\eta^i(s) - \int_s^\cdot (\Gamma(\Phi X^i)_r + \Phi_r\beta^i_r - X^i_r f(r,\cdot,\Phi,\Gamma(\Phi,X)_r))dr, \\ && 1 \le i \le d. \end{array}\right. \tag{6.5.11}$$

Finally, again by Definition 6.5.31 we have $\Phi_T = \xi$, so, for any (s,η), taking the expectations in (6.5.11) at $s = T$, we get

$$\left\{\begin{array}{l} \mathbb{E}^{s,\eta}\left[\xi - \Phi_s(\eta) + \int_s^T f(r,\cdot,\Phi_r,\Gamma(\Phi,X)_r)dr\right] = 0; \\ \mathbb{E}^{s,\eta}\left[\xi X^i_T - \Phi_s(\eta)\eta^i(s) - \int_s^T (\Gamma(\Phi X^i)_r + \Phi_r\beta^i_r - X^i_r f(r,\cdot,\Phi,\Gamma(\Phi,X)_r))dr\right] = 0, \\ 1 \le i \le d, \end{array}\right. \tag{6.5.12}$$

which by Fubini's Lemma and Definition 6.3.9 yields

$$\left\{\begin{array}{rcl} \Phi_s(\eta) = & P_s[\xi](\eta) + \int_s^T P_s\left[f\left(r,\cdot,\Phi_r,\Gamma(\Phi,X)_r\right)\right](\eta)dr \\ (\Phi X^1)_s(\eta) = & P_s[\xi X^1_T](\eta) - \int_s^T P_s\left[\left(\Gamma(\Phi X^1)_r + \Phi_r\beta^1_r - X^1_r f\left(r,\cdot,\Phi_r,\Gamma(\Phi,X)_r\right)\right)\right](\eta)dr \\ \cdots & \\ (\Phi X^d)_s(\eta) = & P_s[\xi X^d_T](\eta) - \int_s^T P_s\left[\left(\Gamma(\Phi,X^d)_r + \Phi_r\beta^d_r - X^d_r f\left(r,\cdot,\Phi_r,\Gamma(\Phi,X)_r\right)\right)\right](\eta)dr, \end{array}\right. \tag{6.5.13}$$

and the first item is proven.

2. The second item follows from item 1. and by the uniqueness of a decoupled mild solution of $IPDE(f,\xi)$, see Theorem 6.5.32.

3. Concerning item 3. let (Y,Z) be the unique decoupled mild solution of $IP(f,\xi)$. We first note that the first line of (6.4.9) taken with $s=T$ yields $Y_T=\xi$.

   Let us now fix some $(s,\eta)\in[0,T]\times\Omega$. The fact that $Y\in\mathcal{D}(A)$ implies by Proposition 6.5.25 that $Y-Y_s-\int_s^{\cdot}AY_r dr$ is on $[s,T]$ a $\mathbb{P}^{s,\eta}$-martingale and by Theorem 6.5.17 that this martingale, which we shall denote $M^{s,\eta}[Y]$, is $\mathbb{P}^{s,\eta}$ a.s. cadlag. Hence Y is under $\mathbb{P}^{s,\eta}$ a cadlag special semi-martingale. Let us keep in mind the solution $(Y^{s,\eta},M^{s,\eta})$ of (6.5.10). A consequence of item 3. of Theorem 6.5.32 is that Y admits on $[s,T]$, $Y^{s,\eta}$ as $\mathbb{P}^{s,\eta}$ cadlag version which is a special semi-martingale verifying $Y_t^{s,\eta}=Y_s^{s,\eta}-\int_s^t f(r,Y_r,Z_r)dt+M_t^{s,\eta}, t\in[s,T]$. Since Y is $\mathbb{P}^{s,\eta}$ a.s. cadlag, then Y and $Y^{s,\eta}$ are actually $\mathbb{P}^{s,\eta}$-indistinguishable on $[s,T]$ and by uniqueness of the decomposition of the semi-martingale Y, we have that $(\int_s^{\cdot}AY_r dr, M^{s,\eta}[Y])$ and $(-\int_s^{\cdot}f(r,Y_r,Z_r)dr, M^{s,\eta})$ are $\mathbb{P}^{s,\eta}$-indistinguishable on $[s,T]$. Since this holds for all (s,η), by Definition 6.4.2 we have $AY=-f(\cdot,\cdot,Y,Z)$ q.s. so we are left to show that $Z=\Gamma(Y,X)$ q.s.

   We fix again (s,η). By item 3. of Theorem 6.5.32, $\langle M^{s,\eta},M^{s,\eta}[X]\rangle=\int_s^{\cdot}Z_r dr$. By item 3. of Corollary 6.5.26 and Lemma 6.4.14, $\langle M^{s,\eta}[Y],M^{s,\eta}[X]\rangle=\int_s^{\cdot}\Gamma(Y,X)_r dr$. As we have remarked above, $M^{s,\eta}=M^{s,\eta}[Y]$ so $\int_s^{\cdot}Z_r dr$ and $\int_s^{\cdot}\Gamma(Y,X)_r dr$ are $\mathbb{P}^{s,\eta}$-indistinguishable on $[s,T]$. Since this holds for all (s,η), we indeed have by Definition 6.4.2 that $Z=\Gamma(Y,X)$ q.s., and the proof is complete.

□

Appendix

6.A Some technicalities

In all the appendix, we are in the framework of Section 6.4.

Lemma 6.A.1. *Let $\tilde{f} \in \mathcal{L}^1_{uni}$. Then* $\begin{array}{ccc}(s,\eta) & \longmapsto & \mathbb{E}^{s,\eta}[\int_s^T \tilde{f}_r dV_r] \\ [0,T]\times\Omega & \longrightarrow & \mathbb{R}\end{array}$ *is $\mathbb{F}^o$-progressively measurable.*

Proof. We fix $T_0 \in]0,T]$ and we will show that on $[0,T_0]\times\Omega$, $(s,\eta) \longmapsto \mathbb{E}^{s,\eta}[\int_s^T \tilde{f}_r dV_r]$ is $\mathcal{B}([0,T_0])\otimes\mathcal{F}^o_{T_0}$-measurable. We will start by showing that on $[0,T_0]\times\Omega\times[0,T_0]$, the function
$k^n : (s,\eta,t) \mapsto \mathbb{E}^{s,\eta}[\int_t^T((-n)\vee\tilde{f}_r\wedge n)dV_r]$ is $\mathcal{B}([0,T_0])\otimes\mathcal{F}^o_{T_0}\otimes\mathcal{B}([0,T_0])$-measurable, where $n\in\mathbb{N}$.

Let $t\in[0,T_0]$ be fixed. Then by Remark 6.3.5
$(s,\eta)\mapsto\mathbb{E}^{s,\eta}[\int_t^T((-n)\vee\tilde{f}_r\wedge n)dV_r]$ is $\mathcal{B}([0,T_0])\otimes\mathcal{F}^o_{T_0}$-measurable.

Let $(s,\eta)\in[0,T_0]\times\Omega$ be fixed and $t_m \underset{m\to\infty}{\longrightarrow} t$ be a converging sequence in $[0,T_0]$. Since V is continuous,

$$\int_{t_m}^T((-n)\vee\tilde{f}_r\wedge n)dV_r \underset{m\to\infty}{\longrightarrow} \int_t^T((-n)\vee\tilde{f}_r\wedge n)dV_r \text{ a.s.} \tag{6.A.1}$$

This sequence is uniformly bounded by nV_T, so by dominated convergence theorem, the convergence in (6.A.1) also holds under the expectation, so that $t\mapsto\mathbb{E}^{s,\eta}[\int_t^T((-n)\vee\tilde{f}_r\wedge n)dV_r]$ is continuous. By Lemma 4.51 in [1], k^n is therefore $\mathcal{B}([0,T_0])\otimes\mathcal{F}^o_{T_0}\otimes\mathcal{B}([0,T_0])$-measurable.

The composition of $(s,\eta)\mapsto(s,\eta,s)$ with the maps k_n yields that, for any $n\geq 0$, $\tilde{k}^n : (s,\eta)\longmapsto\mathbb{E}^{s,\eta}[\int_s^T((-n)\vee\tilde{f}_r\wedge n)dV_r]$ is (on $[0,T_0]\times\Omega$) $\mathcal{B}([0,T_0])\otimes\mathcal{F}^o_{T_0}$-measurable. $\tilde{k}^n$ therefore defines an $\mathbb{F}^o$-progressively measurable process. Then by letting n tend to infinity, $((-n)\vee\tilde{f}\wedge n)$ tends $dV\otimes d\mathbb{P}^{s,\eta}$ a.e. to $\tilde{f}$ and since we assumed $\mathbb{E}^{s,\eta}[\int_s^T|\tilde{f}_r|dV_r]<\infty$, by dominated convergence, $\mathbb{E}^{s,\eta}[\int_s^T((-n)\vee\tilde{f}_r\wedge n)dV_r]$ tends to $\mathbb{E}^{s,\eta}[\int_s^T\tilde{f}_r dV_r]$. $(s,\eta)\longmapsto\mathbb{E}^{s,\eta}[\int_s^T\tilde{f}(r,X_r)dV_r]$ is therefore an $\mathbb{F}^o$-progressively measurable process as the pointwise limit of the $\tilde{k}^n$ which are $\mathbb{F}^o$-progressively measurable processes. □

We recall the following immediate consequence of Fubini's Theorem which corresponds to Lemma 2.5.13 in Chapter 1.

Lemma 6.A.2. *Let $\mathbb{P}$ be a probability measure on $(\Omega,\mathcal{F})$ and ϕ,ψ be two measurable processes. If ϕ and ψ are $\mathbb{P}$-modifications of each other, then they are equal $dV\otimes d\mathbb{P}$ a.e.*

The proof of Proposition 6.4.10 goes through a linearization lemma.

Lemma 6.A.3. *Let $\tilde{f}\in\mathcal{L}^2_{uni}$. Let, for every $(s,\eta)\in[0,T]\times\Omega$, $(\tilde{Y}^{s,\eta},\tilde{M}^{s,\eta})$ be the unique solution of*

$$\tilde{Y}^{s,\eta}_t = \xi + \int_t^T \tilde{f}_r dV_r - (\tilde{M}^{s,\eta}_T - \tilde{M}^{s,\eta}_t),\quad t\in[s,T], \tag{6.A.2}$$

in $(\Omega, \mathcal{F}^{s,\eta}, \mathbb{F}^{s,\eta}, \mathbb{P}^{s,\eta})$. *Then there exists a process* $\tilde{Y} \in \mathcal{L}^2_{uni}$, *a square integrable path-dependent MAF* $(\tilde{M}_{t,u})_{0\le t\le u}$ *and* $\tilde{Z}^1, \cdots, \tilde{Z}^d \in \mathcal{L}^2_{uni}$, *such that for all* $(s,\eta) \in [0,T]\times\Omega$ *the following holds.*

1. $\tilde{Y}^{s,\eta}$ *is on* $[s,T]$ *a* $\mathbb{P}^{s,\eta}$*-modification of* $\tilde{Y}$*;*
2. $\tilde{M}^{s,\eta}$ *is the cadlag version of* $\tilde{M}$ *under* $\mathbb{P}^{s,\eta}$.
3. *For each integer* $1 \le i \le d$, $\tilde{Z}^i = \frac{d\langle \tilde{M}^{s,\eta}, N^{i,s,\eta}\rangle}{dV}\, dV \otimes d\mathbb{P}^{s,\eta}$ *a.e.*

Remark 6.A.4. *The existence, for any* (s,η), *of a unique solution* $(\tilde{Y}^{s,\eta}, \tilde{M}^{s,\eta})$ *of* (6.A.2) *holds because* ξ *and* $(t,\omega,y,z) \mapsto \tilde{f}_t(\omega)$ *trivially verify the hypothesis of Theorem 6.4.7.*

Proof. We set $\tilde{Y} : (s,\eta) \mapsto \mathbb{E}^{s,\eta}\left[\xi + \int_s^T \tilde{f}_r dV_r\right]$ which is $\mathbb{F}^o$-progressively measurable by Remark 6.3.5 and Lemma 6.A.1. Therefore, for a fixed $t \in [s,T]$ we have $\mathbb{P}^{s,\eta}$-a.s.

$$\begin{aligned}
\tilde{Y}_t(\omega) &= \mathbb{E}^{t,\omega}\left[\xi + \int_t^T \tilde{f}_r dV_r\right] \\
&= \mathbb{E}^{s,\eta}\left[\xi + \int_t^T \tilde{f}_r dV_r \middle| \mathcal{F}_t\right](\omega) \\
&= \mathbb{E}^{s,\eta}\left[\tilde{Y}^{s,\eta}_t + (\tilde{M}^{s,\eta}_T - \tilde{M}^{s,\eta}_t) | \mathcal{F}_t\right](\omega) \\
&= \tilde{Y}^{s,\eta}_t(\omega).
\end{aligned}$$

The second equality follows by Remark 6.3.5 and the third one uses (6.A.2). For every $0 \le t \le u$ and $\omega \in \Omega$ we set

$$\tilde{M}_{t,u}(\omega) := \begin{cases} \tilde{Y}_{u\wedge T}(\omega) - \tilde{Y}_{t\wedge T}(\omega) - \int_{t\wedge T}^{u\wedge T} \tilde{f}_r(\omega) dV_r \text{ if } \int_{t\wedge T}^{u\wedge T} |\tilde{f}(\omega)|_r dV_r < +\infty, \\ 0 \text{ otherwise.} \end{cases} \tag{6.A.3}$$

For fixed (s,η), (6.A.2) implies $d\tilde{Y}^{s,\eta}_r = -\tilde{f}_r dV_r + d\tilde{M}^{s,\eta}_r$. On the other hand $\int_s^T |\tilde{f}|_r dV_r < +\infty$ $\mathbb{P}^{s,\eta}$ a.s.; so for any $s \le t \le u$ we have $\tilde{M}^{s,\eta}_u - \tilde{M}^{s,\eta}_t = \tilde{M}_{t,u}$ $\mathbb{P}^{s,\eta}$- a.s. Taking into account that $\tilde{M}^{s,\eta}$ is square integrable and the fact that previous equality holds for any (s,η) and $t \le u$, then $(\tilde{M}_{t,u})_{0\le t\le u}$ indeed defines a square integrable path-dependent MAF. Y belongs to $\mathcal{L}^2_{uni}$ because the validity of the two following arguments hold for all (s,η). First Y is a $\mathbb{P}^{s,\eta}$-modification of $Y^{s,\eta}$ on $[s,T]$, so by Lemma 6.A.2 $Y = Y^{s,\eta}\, dV \otimes d\mathbb{P}^{s,\eta}$ a.e.; second $Y^{s,\eta} \in \mathcal{L}^2(dV \otimes d\mathbb{P}^{s,\eta})$. The existence of Z follows setting for all i, $Z^i = \frac{d\langle \tilde{M}, N^i\rangle}{dV}$, see Notation 6.3.22 and Proposition 6.3.21. □

Notation 6.A.5. *For every fixed* $(s,\eta) \in [0,T]\times\Omega$, *we will denote by* $(Y^{k,s,\eta}, M^{k,s,\eta})_{k\in\mathbb{N}}$ *the Picard iterations associated to* $BSDE^{s,\eta}(f,\xi)$ *as defined in Notation 4.A.13 in Chapter 4 and* $Z^{k,s,\eta} := (Z^{1,k,s,\eta}, \cdots Z^{d,k,s,\eta})$ *will denote* $\frac{\langle M^{k,s,\eta}, N^{s,\eta}\rangle}{dV}$.

This means that for all $(s,\eta) \in [0,T]\times\Omega$, $(Y^{0,s,\eta}, M^{0,s,\eta}) \equiv (0,0)$ *and for all* $k \ge 1$, *we have on* $[s,T]$

$$Y^{k,s,\eta} = \xi + \int_\cdot^T f(r,\cdot, Y^{k-1,s,\eta}_r, Z^{k-1,s,\eta}_r) dV_r - (M^{k,s,\eta}_T - M^{k,s,\eta}_\cdot), \tag{6.A.4}$$

in the sense of $\mathbb{P}^{s,\eta}$*-indistinguishability, and that for all* $(s,\eta) \in [0,T]\times\Omega$, $k \ge 0$, $Y^{k,s,\eta}, Z^{1,k,s,\eta}, \cdots Z^{d,k,s,\eta}$ *belong to* $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,\eta})$, *see Notation 4.A.13 and Lemma 4.A.2 in Chapter 4.*

A direct consequence of Proposition 4.A.15 in Chapter 4 and the lines above it, is the following.

Proposition 6.A.6. *For every* $(s,\eta) \in [0,T]\times\Omega$, *each component of* $(Y^{k,s,\eta}, Z^{1,k,s,\eta}, \cdots, Z^{d,k,s,\eta})$ *tends to each component of* $(Y^{s,\eta}, Z^{1,s,\eta}, \cdots, Z^{d,s,\eta})$ *in* $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,\eta})$ *and* $dV \otimes d\mathbb{P}^{s,\eta}$*-a.e. when* k *tends to infinity.*

Proposition 6.A.7. *For each $k \in \mathbb{N}$, there exist processes $Y^k \in \mathcal{L}^2_{uni}, Z^{k,1}, \cdots, Z^{k,d} \in \mathcal{L}^2_{uni}$, a square integrable path-dependent MAF $(M^k_{t,u})_{0\leq t\leq u}$ such that for all $(s,\eta) \in [0,T]\times\Omega$, we have the following.*

1. *$Y^{k,s,\eta}$ is on $[s,T]$ a $\mathbb{P}^{s,\eta}$-modification of Y^k;*
2. *$M^{k,s,\eta}$ is the cadlag version of M^k under $\mathbb{P}^{s,\eta}$.*
3. *For all $(s,\eta) \in [0,T]\times\Omega$ and $i \in [\![1;d]\!]$, $Z^{k,i} = \frac{d\langle M^{k,s,\eta}, N^{i,s,\eta}\rangle}{dV}\, dV\otimes d\mathbb{P}^{s,\eta}$ a.e.*

Proof. We prove the statement by induction on $k \geq 0$. It is clear that $Y^0 \equiv 0$ and $M^0 \equiv 0$ verify the assertion for $k = 0$.

Suppose the existence, for $k \geq 1$, of a square integrable path-dependent MAF M^{k-1} and processes $Y^{k-1}\ Z^{k-1,1}, \cdots, Z^{k-1,d} \in \mathcal{L}^2_{uni}$ such that the statements 1. 2. 3. above hold replacing k with $k-1$.

We fix $(s,\eta) \in [0,T]\times\Omega$. By Lemma 6.A.2, $(Y^{k-1,s,\eta}, Z^{k-1,s,\eta}) = (Y^{k-1}, Z^{k-1})\ dV\otimes d\mathbb{P}^{s,\eta}$ a.e. Therefore by (6.A.4)

$$Y^{k,s,\eta}_t = \xi + \int_t^T f\left(r,\cdot, Y^{k-1}_r, Z^{k-1}_r\right) dV_r - (M^{k,s,\eta}_T - M^{k,s,\eta}_t), t \in [s,T].$$

According to Notation 6.4.8, the equation (6.A.4) can be seen as a BSDE of the type $BSDE^{s,\eta}(\tilde{f},\xi)$ where $\tilde{f} : (t,\omega) \longmapsto f(t,\omega,Y^{k-1}_t(\omega), Z^{k-1}_t(\omega))$. We now verify that $\tilde{f}$ verifies the conditions under which Lemma 6.A.3 applies.

$\tilde{f}$ is $\mathbb{F}^o$-progressively measurable since Y^{k-1}, Z^{k-1} are $\mathbb{F}^o$-progressively measurable and f is $\mathcal{P}ro^o\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}^d)$-measurable. Since

$$|\tilde{f}(t,\omega)| = |f(t,\omega,Y^{k-1}_t(\omega), Z^{k-1}_t(\omega))| \leq |f(t,\omega,0,0)| + K(|Y^{k-1}_t(\omega)| + \|Z^{k-1}_t(\omega)\|),$$

for all t,ω, with $f(\cdot,\cdot,0,0), Y^{k-1}, Z^{k-1,1}, \cdots, Z^{k-1,d} \in \mathcal{L}^2_{uni}$ by recurrence hypothesis, it is clear that $\tilde{f} \in \mathcal{L}^2_{uni}$. Since $(Y^{k,s,\eta}, M^{k,s,\eta})$ is a solution of $BSDE^{s,\eta}(\tilde{f},\xi)$, Lemma 6.A.3 shows the existence of suitable $Y^k, M^k, Z^{k,1}, \cdots, Z^{k,d}$ verifying the statement for the integer k. □

Proof of Proposition 6.4.10. We define $\bar{Y}$ and $\bar{Z}^i, 1 \leq i \leq d$ by $\bar{Y}_s(\eta) := \limsup_{k\in\mathbb{N}} Y^k_s(\eta)$ and $\bar{Z}^i_s(\eta) := \limsup_{k\in\mathbb{N}} Z^{k,i}_s(\eta)$, for every $(s,\eta) \in [0,T]\times\Omega$.
$\bar{Y}$ and $\bar{Z} := (\bar{Z}^1, \cdots, \bar{Z}^d)$ are $\mathbb{F}^o$-progressively measurable. Combining Propositions 6.A.7, 6.A.6 and Lemma 6.A.2 it follows that for every $(s,\eta) \in [0,T]\times\Omega$,

$$\begin{cases} Y^k \underset{k\to\infty}{\longrightarrow} Y^{s,\eta} \quad dV\otimes d\mathbb{P}^{s,\eta} \\ Z^{k,i} \underset{k\to\infty}{\longrightarrow} Z^{i,s,\eta} \quad dV\otimes d\mathbb{P}^{s,\eta}, \text{ for all } 1 \leq i \leq d. \end{cases} \tag{6.A.5}$$

Let us fix $1 \leq i \leq d$ and $(s,\eta) \in [0,T]\times\Omega$. There is a set $A^{s,\eta}$ of full $dV\otimes d\mathbb{P}^{s,\eta}$ measure such that for all $(t,\omega) \in A^{s,\eta}$ we have

$$\begin{cases} \bar{Y}_t(\omega) = \limsup_{k\in\mathbb{N}} Y^k_t(\omega) = \lim_{k\in\mathbb{N}} Y^k_t(\omega) = Y^{s,\eta}_t(\omega) \\ \bar{Z}_t(\omega) = \left(\limsup_{k\in\mathbb{N}} Z^{k,i}_t(\omega)\right)_{i\leq d} = \left(\lim_{k\in\mathbb{N}} Z^{k,i}_t(\omega)\right)_{i\leq d} = Z^{s,\eta}_t(\omega). \end{cases} \tag{6.A.6}$$

This implies

$$\begin{aligned} \bar{Y}_t(\omega) &= Y^{s,\eta}\, dV \otimes d\mathbb{P}^{s,\eta}\text{a.e.} \\ \bar{Z}_t(\omega) &= Z^{s,\eta}\, dV \otimes d\mathbb{P}^{s,\eta}\text{a.e.} \end{aligned} \tag{6.A.7}$$

By (6.A.7) and (6.4.2), under every $\mathbb{P}^{s,\eta}$, we actually have

$$Y^{s,\eta} = \xi + \int_\cdot^T f\left(r, \cdot, \bar{Y}_r, \bar{Z}_r\right) dV_r - (M_T^{s,\eta} - M_\cdot^{s,\eta}), \tag{6.A.8}$$

in the sense of $\mathbb{P}^{s,\eta}$-indistinguishability, on the interval $[s,T]$. At this stage, in spite of (6.A.7), $\bar{Y}$ is not necessarily a modification of $Y^{s,\eta}$. We will construct processes Y, Z fulfilling indeed the statement of Proposition 6.4.10. In particular Y fulfills item 1. that is a bit stronger than (6.A.7).

We set now $\tilde{f} : (t,\omega) \mapsto f(t,\omega,\bar{Y}_t(\omega),\bar{Z}_t(\omega))$; equation (6.A.8) is now of the form (6.A.2) and we show that $\tilde{f}$ so defined verifies the conditions under which Lemma 6.A.3 applies. $\tilde{f}$ is $\mathbb{F}^o$-progressively measurable since f is $\mathcal{P}ro^o \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and $\bar{Y}, \bar{Z}$ are $\mathbb{F}^o$-progressively measurable.

Moreover, for any $(s,\eta) \in [0,T] \times \Omega$, $Y^{s,\eta}$ and $Z^{1,s,\eta}, \cdots, Z^{d,s,\eta}$ belong to $\mathcal{L}^2(dV \otimes d\mathbb{P}^{s,\eta})$; therefore by (6.A.6), so do $\bar{Y}$ and $\bar{Z}^1, \cdots, \bar{Z}^d$.

Since this holds for all (s,η), then $\bar{Y}$ and $\bar{Z}^1, \cdots, \bar{Z}^d$ belong to $\mathcal{L}^2_{uni}$.
Finally, since $|\tilde{f}(t,\omega)| = |f(t,\omega,\bar{Y}_t(\omega),\bar{Z}_t(\omega))| \leq |f(t,\omega,0,0)| + K(|\bar{Y}_t(\omega)| + \|\bar{Z}_t(\omega)\|)$ for all t,ω, with $f(\cdot,\cdot,0,0), \bar{Y}, \bar{Z}^1, \cdots, \bar{Z}^d \in \mathcal{L}^2_{uni}$, it is clear that $\tilde{f} \in \mathcal{L}^2_{uni}$. Now (6.A.8) can be considered as a BSDE where the driver does not depend on y and z of the form (6.A.2). We can therefore apply Lemma 6.A.3 to $\tilde{f}$ and conclude on the existence of $(Y, M, Z^1, \cdots, Z^d)$ verifying the three items of the proposition.

It remains to prove now the last assertion of Proposition 6.4.10. We fix some (s,η). The first item implies that $Y_s = Y_s^{s,\eta}$ $\mathbb{P}^{s,\eta}$ a.s. But since Y_s is $\mathcal{F}_s^o$-measurable and $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$, it also yields that Y_s is $\mathbb{P}^{s,\eta}$ a.s. equal to the deterministic value $Y_s(\eta)$ hence $Y_s^{s,\eta}$ is $\mathbb{P}^{s,\eta}$ a.s. equal to the deterministic value $Y_s(\eta)$. This also proves that Y is unique because it is given by $Y : (s,\eta) \longmapsto Y_s^{s,\eta}$. The uniqueness of Z up to zero potential sets is immediate by the third item of the proposition and Definition 6.4.2. □

Bibliography

[1] C. D. Aliprantis and K. C. Border. *Infinite-dimensional analysis*. Springer-Verlag, Berlin, second edition, 1999. A hitchhiker's guide.

[2] D. G. Aronson. Bounds for the fundamental solution of a parabolic equation. *Bull. Amer. Math. Soc.*, 73:890–896, 1967.

[3] D. Bakry, I. Gentil, and M. Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348. Springer Science & Business Media, 2013.

[4] V. Bally, E. Pardoux, and L. Stoica. Backward stochastic differential equations associated to a symmetric Markov process. *Potential Anal.*, 22(1):17–60, 2005.

[5] E. Bandini. Existence and uniqueness for backward stochastic differential equations driven by a random measure. *Electronic Communications in Probability*, 20(71):1–13, 2015.

[6] E. Bandini and F. Russo. Special weak Dirichlet processes and BSDEs driven by a random measure. *Bernoulli*, 24(4A):2569–2609, 2018.

[7] G. Barles, R. Buckdahn, and E. Pardoux. Backward stochastic differential equations and integral-partial differential equations. *Stochastics: An International Journal of Probability and Stochastic Processes*, 60(1-2):57–83, 1997.

[8] G. Barles and E. Lesigne. SDE, BSDE and PDE. In *Backward stochastic differential equations (Paris, 1995–1996)*, volume 364 of *Pitman Res. Notes Math. Ser.*, pages 47–80. Longman, Harlow, 1997.

[9] A. Barrasso and F. Russo. Backward Stochastic Differential Equations with no driving martingale, Markov processes and associated P seudo Partial Differential Equations. 2017. Preprint, hal-01431559, v1.

[10] A. Barrasso and F. Russo. Backward Stochastic Differential Equations with no driving martingale, Markov processes and associated Pseudo Partial Differential Equations. part II: Decoupled mild solutions and examples. 2017. Preprint, hal-01505974.

[11] A. Barrasso and F. Russo. Martingale driven BSDEs, PDEs and other related deterministic problems. 2017. Preprint, hal-01566883.

[12] A. Barrasso and F. Russo. A note on time-dependent additive functionals. *Communications on Stochastic Analysis*, 11 no 3:313–334, 9 2017.

[13] A. Barrasso and F. Russo. BSDEs and decoupled mild solutions of path-dependent PDEs and IPDEs. 2018. Preprint hal-01774823.

[14] A. Barrasso and F. Russo. Path-dependent Martingale Problems and Additive Functionals. 2018. Preprint, hal-01775200.

[15] A. Ben-Israel and Th. N. E. Greville. *Generalized inverses*, volume 15 of *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC*. Springer-Verlag, New York, second edition, 2003. Theory and applications.

[16] P. Billingsley. *Probability and measure*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, second edition, 1986.

[17] J. Bion-Nadal. Dynamic risk reasures and path-dependent second order PDEs. *Stochastics of Environmental and Financial Economics*, 138:147–178, 2016.

[18] J.M. Bismut. Conjugate convex functions in optimal stochastic control. *J. Math. Anal. Appl.*, 44:384–404, 1973.

[19] R. M. Blumenthal, R. K. Getoor, and H. P. McKean, Jr. Markov processes with identical hitting distributions. *Bull. Amer. Math. Soc.*, 68:372–373, 1962.

[20] R. Buckdahn. Backward stochastic differential equations driven by a martingale. *Unpublished*, 1993.

[21] G. Cannizzaro and K. Chouk. Multidimensional sdes with singular drift and universal construction of the polymer measure with white noise potential. *arXiv preprint arXiv:1501.04751*, 2015.

[22] R. Carbone, B. Ferrario, and M. Santacroce. Backward stochastic differential equations driven by càdlàg martingales. *Teor. Veroyatn. Primen.*, 52(2):375–385, 2007.

[23] C. Ceci, A. Cretarola, and F. Russo. BSDEs under partial information and financial applications. *Stochastic Process. Appl.*, 124(8):2628–2653, 2014.

[24] F. Confortola, M. Fuhrman, and J. Jacod. Backward stochastic differential equation driven by a marked point process: an elementary approach with an application to optimal control. *Ann. Appl. Probab.*, 26(3):1743–1773, 2016.

[25] R. Cont and D.-A. Fournié. Change of variable formulas for non-anticipative functionals on path space. *J. Funct. Anal.*, 259(4):1043–1072, 2010.

[26] R. Cont and D.-A. Fournié. A functional extension of the Itô formula. *C. R. Math. Acad. Sci. Paris*, 348(1-2):57–61, 2010.

[27] R. Cont and D.-A. Fournié. Functional Itô calculus and stochastic integral representation of martingales. *Ann. Probab.*, 41(1):109–133, 2013.

[28] A. Cosso and F. Russo. *Strong-viscosity solutions: semilinear parabolic PDEs and path-dependent PDEs*. 2015.

[29] A. Cosso and F. Russo. Functional Itô versus Banach space stochastic calculus and strict solutions of semilinear path-dependent equations. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 19(4):1650024, 44, 2016.

[30] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 152 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2014.

[31] F. Delarue and R. Diel. Rough paths and 1d SDE with a time dependent distributional drift: application to polymers. *Probab. Theory Related Fields*, 165(1-2):1–63, 2016.

[32] C. Dellacherie and P.-A. Meyer. *Probabilités et potentiel*, volume A. Hermann, Paris, 1975. Chapitres I à IV.

[33] C. Dellacherie and P.-A. Meyer. *Probabilités et potentiel. Chapitres V à VIII*, volume 1385 of *Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics]*. Hermann, Paris, revised edition, 1980. Théorie des martingales. [Martingale theory].

[34] C. Dellacherie and P.-A. Meyer. *Probabilités et potentiel. Chapitres XII–XVI*. Publications de l'Institut de Mathématiques de l'Université de Strasbourg [Publications of the Mathematical Institute of the University of Strasbourg], XIX. Hermann, Paris, second edition, 1987. Théorie des processus de Markov. [Theory of Markov processes].

[35] C. Di Girolami and F. Russo. Infinite dimensional stochastic calculus via regularization and applications. *Preprint* HAL-INRIA, inria-00473947 version 1, (Unpublished), 2010.

[36] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[37] B. Dupire. *Functional Itô calculus*. Portfolio Research Paper, Bloomberg, 2009.

[38] E. B. Dynkin. *Osnovaniya teorii markovskikh protsessov*. Teorija Verojatnosteĭ i Matematičeskaja Statistika. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959.

[39] E. B. Dynkin. Additive functionals of markov processes and stochastic systems. In *Annales de l'institut Fourier*, volume 25, pages 177–200, 1975.

[40] E. B. Dynkin. *Markov processes and related problems of analysis*, volume 54 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York, 1982.

[41] A. Einstein. *Investigations on the theory of the Brownian movement*. Dover Publications, Inc., New York, 1956. Edited with notes by R. Fürth, Translated by A. D. Cowper.

[42] I. Ekren, C. Keller, N. Touzi, and J. Zhang. On viscosity solutions of path dependent PDEs. *Ann. Probab.*, 42(1):204–236, 2014.

[43] N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Mathematical finance*, 7(1):1–71, 1997.

[44] S. N. Ethier and T. G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986. Characterization and convergence.

[45] G. Fabbri, F. Gozzi, and A. Świ¸ech. *Stochastic optimal control in infinite dimension*, volume 82 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2017. Dynamic programming and HJB equations, With a contribution by Marco Fuhrman and Gianmario Tessitore.

[46] F. Flandoli, E. Issoglio, and F. Russo. Multidimensional stochastic differential equations with distributional drift. *Trans. Amer. Math. Soc.*, 369(3):1665–1688, 2017.

[47] F. Flandoli, F. Russo, and J. Wolf. Some SDEs with distributional drift. I. General calculus. *Osaka J. Math.*, 40(2):493–542, 2003.

[48] F. Flandoli, F. Russo, and J. Wolf. Some SDEs with distributional drift. II. Lyons-Zheng structure, Itô's formula and semimartingale characterization. *Random Oper. Stochastic Equations*, 12(2):145–184, 2004.

[49] F. Flandoli and G. Zanco. An infinite-dimensional approach to path-dependent Kolmogorov equations. *Ann. Probab.*, 44(4):2643–2693, 2016.

[50] M. Fuhrman, F. Masiero, and G. Tessitore. Stochastic equations with delay: optimal control via BSDEs and regular solutions of Hamilton-Jacobi-Bellman equations. *SIAM J. Control Optim.*, 48(7):4624–4651, 2010.

[51] M. Fuhrman and G. Tessitore. Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control. *Ann. Probab.*, 30(3):1397–1465, 2002.

[52] M. Fuhrman and G. Tessitore. Generalized directional gradients, backward stochastic differential equations and mild solutions of semilinear parabolic equations. *Appl. Math. Optim.*, 51(3):279–332, 2005.

[53] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[54] W. Hoh. Pseudo differential operators generating markov processes. *Habilitations-schrift, Universität Bielefeld*, 1998.

[55] E. P. Hsu. *Stochastic analysis on manifolds*, volume 38. American Mathematical Soc., 2002.

[56] E. Issoglio and S. Jing. Forward-Backward SDEs with distributional coefficients. preprint - ArXiv 2016 (arXiv:1605.01558).

[57] N. Jacob. *Pseudo differential operators and Markov processes. Vol. I*. Imperial College Press, London, 2001. Fourier analysis and semigroups.

[58] N. Jacob. *Pseudo differential operators & Markov processes. Vol. II*. Imperial College Press, London, 2002. Generators and their potential theory.

[59] N. Jacob. *Pseudo Differential Operators & Markov Processes: Markov Processes And Applications*, volume 3. Imperial College Press, 2005.

[60] J. Jacod. *Calcul stochastique et problèmes de martingales*, volume 714 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[61] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.

[62] J. Jost. *Riemannian Geometry and Geometric Analysis*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011.

[63] T. Klimsiak. Semi-Dirichlet forms, Feynman-Kac functionals and the Cauchy problem for semilinear parabolic equations. *J. Funct. Anal.*, 268(5):1205–1240, 2015.

[64] I. Laachir and F. Russo. BSDEs, càdlàg martingale problems, and orthogonalization under basis risk. *SIAM J. Financial Math.*, 7(1):308–356, 2016.

[65] D. Leão, A. Ohashi, and A. B. Simas. Weak functional Itô calculus and applications. *Preprint* arXiv:1408.1423v2, 2014.

[66] G. Liang, T. Lyons, and Z. Qian. Backward stochastic dynamics on a filtered probability space. *Ann. Probab.*, 39(4):1422–1448, 2011.

[67] G. Liang, T. Lyons, and Zh. Qian. Backward stochastic dynamics on a filtered probability space. *Ann. Probab.*, 39(4):1422–1448, 2011.

[68] P.-A. Meyer. Fonctionelles multiplicatives et additives de Markov. *Ann. Inst. Fourier (Grenoble)*, 12:125–230, 1962.

[69] P.A. Meyer. *Séminaire de Probabilités, X*. Lecture Notes in Mathematics, Vol. 511. Springer-Verlag, Berlin-New York, 1976. Tenu à l'Université de Strasbourg, Strasbourg. Première partie (année universitaire 1974/1975). Seconde partie: Théorie des intégrales stochastiques (année universitaire 1974/1975). Exposés supplémentaires, Edité par P. A. Meyer.

[70] É. Pardoux. Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic PDEs of second order. In *Stochastic analysis and related topics, VI (Geilo, 1996)*, volume 42 of *Progr. Probab.*, pages 79–127. Birkhäuser Boston, Boston, MA, 1998.

[71] É. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, 14(1):55–61, 1990.

[72] É. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic partial differential equations and their applications (Charlotte, NC, 1991)*, volume 176 of *Lecture Notes in Control and Inform. Sci.*, pages 200–217. Springer, Berlin, 1992.

[73] É. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic partial differential equations and their applications (Charlotte, NC, 1991)*, volume 176 of *Lecture Notes in Control and Inform. Sci.*, pages 200–217. Springer, Berlin, 1992.

[74] E. Pardoux and A. Răşcanu. *Stochastic differential equations, backward SDEs, partial differential equations*, volume 69 of *Stochastic Modelling and Applied Probability*. Springer, Cham, 2014.

[75] K. R. Parthasarathy. *Probability measures on metric spaces*. Probability and Mathematical Statistics, No. 3. Academic Press, Inc., New York-London, 1967.

[76] S. Peng. Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. *Stochastics Stochastics Rep.*, 37(1-2):61–74, 1991.

[77] S. Peng and F. Wang. BSDE, path-dependent PDE and nonlinear Feynman-Kac formula. *Science China Mathematics*, 59(1):19–36, 2016.

[78] P. E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. Stochastic Modelling and Applied Probability.

[79] J.P. Roth. Opérateurs dissipatifs et semi-groupes dans les espaces de fonctions continues. *Ann. Inst. Fourier (Grenoble)*, 26(4):ix, 1–97, 1976.

[80] A. Rozkosz. Weak convergence of diffusions corresponding to divergence form operators. *Stochastics Stochastics Rep.*, 57(1-2):129–157, 1996.

[81] F. Russo and G. Trutnau. Some parabolic PDEs whose drift is an irregular random noise in space. *Ann. Probab.*, 35(6):2213–2262, 2007.

[82] F. Russo and L. Wurzer. Elliptic PDEs with distributional drift and backward SDEs driven by a càdlàg martingale with random terminal time. *To appear in Stochastics and Dynamics.*, 2015. arXiv:1407.3218v2.

[83] M. Schweizer. Approximating random variables by stochastic integrals. *Ann. Probab.*, 22(3):1536–1575, 1994.

[84] D. W. Stroock. Diffusion processes associated with Lévy generators. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 32(3):209–244, 1975.

[85] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes*. Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.

[86] F. Viens and J. Zhang. A Martingale Approach for Fractional Brownian Motions and Related Path Dependent PDEs. *ArXiv e-prints*, December 2017.

[87] R. Zhu. BSDE and generalized Dirichlet forms: the finite-dimensional case. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 15(4):1250022, 40, 2012.

Remerciements

Avant tout je voudrais remercier Francesco. Il est certain que tout.e élève remercie ici son directeur ou sa directrice de thèse avec plus ou moins de sincérité. Dans le cas présent, on ne pourrait être plus sincère. Je te remercie pour avoir été à la fois si disponible (qu'on se voit ou se parle trois fois par semaine a toujours épaté mes ami.e.s thésard.e.s), et si flexible. Il est vrai qu'à certaines périodes je mettais assez peu les pieds au labo, mais le froid glacial du bureau y était peut être aussi pour quelque chose. Au delà de me laisser travailler à mon rythme et d'où je voulais, tu m'as petit à petit laissé faire mes propres choix de sujets, tout en restant d'excellent conseil. Je pense que dans mon cas précis, on aurait pas pu imaginer meilleur encadrement. Merci à mes rapporteur.e.s pour leur relecture attentive et à mon Jury de soutenance de m'avoir accordé sa confiance et d'avoir posé des questions qui me permettront de pousser mes travaux plus loin. Je remercie aussi David et Olivier, avec qui ça a été (et ça continue d'être puisqu'on ne change pas une équipe qui gagne) un plaisir d'enseigner, et de découvrir que j'aimais enseigner. Une pensée à Christophe dont le sourire permanent est une valeur sûre et qui m'a aidé à me mettre à Linux, ainsi qu'à Corinne pour ses excellents conseils de lecture, et qui a sû me supporter malgré mon incapacité totale à conserver mes reçus de paiement en déplacement, ou à noter mes vacances sur l'intranet. Je remercie bien sûr toute l'UMA qui m'a très bien accueilli, bien qu'il faille reconnaître que j'ai été très peu présent dans la vie du labo. Le fait d'avoir des amis de longue date à l'X avec qui déjeuner y était pour quelque chose.

Merci à ma famille. À ma mère qui en a vu de toutes les couleurs pendant ces trois ans et qui a eu le toupet de réussir ses exams avant moi. Ma marraine, mon parrain et mes sœurs de cœur qui ont été la dans les moments durs, mais que j'ai aussi eu la chance d'accompagner quand leur famille s'agrandissait. Mon beau-père et ses fils, dont un excellent élève en maths qui mérite tous les trains du monde. À mes tantes qui ont suivi tout ça, et à nos débats parfois houleux autour d'un scrabble (avouez vous me laissiez gagner).

Merci à la colloc de l'amour, pour ces années qu'on ne peut résumer en quelques lignes, vous même vous savez. De NDDL à Bure, en passant par Ispahan, Pontpoint ou Bisca, on s'est pas laissé abattre. À tous ces dragons cachés dans le placard et toutes ces découvertes faites ensemble. À ma doctorante russe préférée qui a accompli l'exploit de faire d'Anthony, Palaiseau ou Orsay des villes dont je garderai de bons souvenirs; à mes ami.e.s de longue date; de teufs interminables qui nous laissaient le champ libre mais sur les quelles on ne fera pas de zoom; de manifs plan plan ou sauvages toujours fidèlement live-tweetées (bravo Moray).

Je termine par une pensée à mes camarades. Aux anarchistes, aux libertaires, aux syndicalistes révolutionnaires, aux autonaumes, aux zadistes, aux féministes, aux écologistes, aux antifascites, aux antiracistes, aux victimes des violences policières et leurs familles, aux prisonnier.e.s politiques ou non, aux situationistes, aux squateur.euse.s, aux bloqueur.euse.s, aux hiboux, aux communistes libertaires, aux mutuelistes libertaires, aux collectivistes libertaires, aux anarchosyndicalistes, aux collectifs de sans-papiers, à ceux de mal logé.e.s, aux antipubs, aux libristes, aux vegans, au cortège de tête, aux casseur.euse.s, aux grévistes, à toutes celles et ceux qui s'organisent, et qui s'assignent un bonheur difficile mais immédiat.



université PARIS-SACLAY | École doctorale de mathématiques Hadamard (EDMH)

Titre: Solutions mild découplées de problèmes d'évolution déterministes à coefficients singuliers ou dépendants de la trajectorie, et leur représentation par des EDS rétrogrades.

Mots-clés: EDS rétrogrades dirigées par des martingales cadlag; équations aux dérivées partielles semilinéaires; équations intégrales aux dérivées partielles; EDS et EDP à drift distributionnel; calcul stochastique et EDP dépendant de la trajectoire.

Résumé: Cette thèse introduit une nouvelle notion de solution pour des équations d'évolution non-linéaires déterministes, appellées solutions *mild* découplées. Nous revisitons les liens entre équations différentielles rétrogrades (EDSRs) markoviennes browniennes et EDPs paraboliques semilinéaires en montrant que, sous de très faibles hypothèses, les EDSRs produisent une unique solution *mild* découplée d'une EDP. Nous étendons ce résultat à de nombreuses autres équations déterministes telles que des Pseudo-EDPs, des Équations Intégrales aux Dérivées Partielles (EIDPs), des EDPs à drift distributionnel, ou des E(I)DPs à dépendance trajectorielle. Les solutions de ces équations sont représentées via des EDSRs qui peuvent être sans martingale de référence, ou dirigées par des martingales cadlag. En particulier, cette thèse résout le *problème d'identification*, qui consiste, dans le cas classique d'une EDSR markovienne brownienne, à donner un sens analytique au processus Z, second membre de la solution (Y, Z) de l'EDSR. Dans la littérature, Y détermine en général une solution de viscosité de l'équation déterministe et ce problème d'identification n'est résolu que quand cette solution de viscosité a un minimum de régularité. Notre méthode permet de résoudre ce problème même dans le cas général d'EDSRs à sauts (non nécéssairement markoviennes).

Title: Decoupled mild solutions of deterministic evolution problems with singular or path-dependent coefficients, represented by backward SDEs.

Keywords: BSDEs driven by cadlag martingales; semilinear partial differential equations; integro-partial differential equations; SDEs and PDEs with distributional drift; path-dependent stochastic calculus and PDEs.

Abstract: This thesis introduces a new notion of solution for deterministic non-linear evolution equations, called decoupled mild solution. We revisit the links between Markovian Brownian Backward stochastic differential equations (BSDEs) and parabolic semilinear PDEs showing that under very mild assumptions, the BSDEs produce a unique decoupled mild solution of some PDE. We extend this result to many other deterministic equations such as Pseudo-PDEs, Integro-PDEs, PDEs with distributional drift or path-dependent (I)PDEs. The solutions of those equations are represented through BSDEs which may either be without driving martingale, or driven by cadlag martingales. In particular this thesis solves the so called *identification problem*, which consists, in the case of classical Markovian Brownian BSDEs, to give an analytical meaning to the second component Z of the solution (Y, Z) of the BSDE. In the literature, Y generally determines a so called viscosity solution and the identification problem is only solved when this viscosity solution has a minimal regularity. Our method allows to treat this problem even in the case of general (even non-Markovian) BSDEs with jumps.