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# Higher-dimensional linear rewriting and coherence in categorification and representation theory

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**Réécriture de dimension supérieure et cohérence  
appliquées à la catégorification et la théorie des  
représentations**

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# Résumé

Dans cette thèse, nous présentons des applications de la réécriture à l'étude de problèmes issus de la catégorification et de la théorie des représentations. En particulier, nous appliquons les méthodes de réécriture aux problèmes de cohérence dans les catégories linéaires et au calcul de décatégorifications. Des méthodes de réécriture ont été développées pour obtenir des résultats de cohérence dans les monoïdes et les catégories monoïdales présentés par des systèmes de réécriture nommés polygraphes. Ces constructions basées sur des résultats de Squier permettent en particulier de calculer des présentations cohérentes de catégories de dimension supérieure à partir des diagrammes de confluence de polygraphes convergents. Dans ce mémoire, nous étendons ces constructions pour obtenir des résultats de cohérence dans les catégories linéaires de dimension supérieure. Nous introduisons les polygraphes linéaires afin de présenter les catégories linéaires de dimension supérieure par des systèmes de réécriture. Nous étudions ensuite les propriétés de réécriture de ces systèmes. Nous donnons une description polygraphique du calcul de décatégorification de Grothendieck. Nous généralisons également la procédure de Knuth-Bendix appliquée aux polygraphes de dimension supérieure. Cette procédure permet de compléter des présentations de catégories de dimension supérieure n'admettant pas nécessairement d'ordre de terminaison induit par une orientation des règles. De plus, nous étudions des problèmes de cohérence dans les catégories de dimension supérieure. Étant donné un polygraphe confluent et quasi-terminant, nous introduisons une notion de complétion de Squier de ce polygraphe composée de diagrammes de décroissance. Nous prouvons que cette complétion rend asphérique la catégorie de dimension supérieure libre sur ce polygraphe. Ce résultat généralise un résultat de Squier au cas des présentations quasi-terminantes. Nous présentons enfin les applications des propriétés des polygraphes linéaire à l'étude de la catégorie  $\mathcal{AOB}$  définie par Brundan, Comes, Nash et Reynolds. Nous retrouvons par des méthodes de réécriture les bases des espaces de morphismes de  $\mathcal{AOB}$  exhibées par Brundan, Comes, Nash and Reynolds.



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# Abstract

In this thesis, we study applications of rewriting theory to categorification problems and representation theory. We apply rewriting methods to coherence problems in linear categories and computation of decategorifications.

Proofs of coherence results for monoids and monoidal categories by rewriting methods are well known. In particular, several constructions based on Squier's results lead to the computation of coherent presentations of higher-dimensional categories from the confluence diagrams of convergent rewriting systems. In this memoir, we extend those constructions to coherence results for higher-dimensional linear categories.

We introduce linear polygraphs to present higher-dimensional linear categories by rewriting systems. We then develop the main rewriting properties of these systems. We focus next on the applications of those properties to the study of categorification problems such that the computation of Grothendieck decategorification by rewriting methods. Another result we obtain on higher-dimensional polygraphs is a generalization of the Knuth-Bendix procedure to higher-dimensional polygraphs. This new procedure allows us to complete presentations of higher-dimensional categories which do not necessarily admit a termination order induced by any orientation of rules.

We also study general coherence problems. Given a confluent and quasi-terminating polygraph, we define a globular extension of this polygraph called decreasing Squier's completion. We prove that this extension makes aspherical the free higher-dimensional category over the given polygraph. This result generalizes a result of Squier to the case of non terminating presentations.

Finally, we focus on the applications of those properties to higher-dimensional linear categories such that the category  $\mathcal{AOB}$  defined by Brundan, Comes, Nash and Reynolds. We find by rewriting methods the bases of the morphisms spaces of  $\mathcal{AOB}$  that Brundan, Comes, Nash and Reynolds exhibited.





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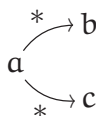
# Introduction

## 1.1. AN OVERVIEW OF REWRITING THEORY

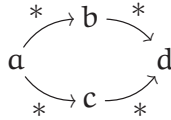
**1.1.1. Abstract rewriting.** In 1914, Thue [?] introduced rewriting to study the word problem in semi-groups. Thue's method consists in orienting the relations of a semi-group to study the properties of normal forms, or irreducible expressions. Explicitly, a rewriting system is a data made of objects and transformation rules on these objects. Given an equational structure and two expressions in this structure, the word problem is the question of whether or not those two expressions are equal modulo the relations of the structure. An instance of the word problem appears for monoids. In a monoid generated by an alphabet, the word problem is the question of whether or not two strings in the alphabet are equal in the monoid.

Beyond the word problem, rewriting theory finds applications in computer science. Those applications led to multiple variants of rewriting systems such that term rewriting, graph rewriting or tree rewriting. Other applications of rewriting theory appear in algebra with Gröbner bases. Abstract rewriting is a model of computation on a set of objects and a relation on this set called the rewrite relation encompassing all other variants of rewriting. This model is also Turing complete.

Two important properties of rewriting systems are termination, which ensures that all computations end, and confluence, which ensures that all computations from a same expression



lead to a same result.



A rewriting system which is both terminating and confluent is called *convergent*. Other properties like local confluence, a property weaker than confluence and easier to decide are also studied. The conjunction of termination and confluence is called convergence. In 1942, Newman [?] introduced a criterion linking local confluence to confluence under termination hypothesis. He proved that local confluence and confluence are equivalent in a terminating rewriting system. This result is called Newman's Lemma [?]. Newman's Lemma is also called diamond's Lemma due to its proof. Another important result of abstract rewriting is Church-Rosser's Theorem [?] stating that confluence is equivalent to a property called Church-Rosser property. All results of abstract rewriting are valid for the other variants of rewriting.

**1.1.2. Word rewriting and polygraphs.** In the context of higher-dimensional categories, rewriting is formalized by the notion of polygraph. Polygraphs were introduced in 1987 by Street [?] under the name of computads and independently by Burroni in 1993 [?] as systems of generators and oriented relations, for higher-dimensional categories. Those oriented relations are called *rewriting rules*. Polygraphs find applications in algebra to study structures like monoids or monoidal categories by rewriting [?]. Thus, a polygraph is a rewriting system on a higher-dimensional category.

Polygraphs are defined by induction on their dimension using the notion of globular extension. A 0-polygraph is a set. For  $n > 0$ , an  $n$ -polygraph is a data made of an  $(n - 1)$ -polygraph  $\Sigma$  and a globular extension of the free  $(n - 1)$ -category over  $\Sigma$ . This globular extension makes a set of rewriting rules over the  $(n - 1)$ -cells of a free  $(n - 1)$ -category. Quotienting an  $(n + 1)$ -polygraph by the congruence relation generated by its  $(n + 1)$ -cells, one obtains an  $n$ -category which is said to be presented by the  $(n + 1)$ -polygraph. In particular, a 2-polygraph is a string rewriting system and a 3-polygraph is a rewriting system on the 2-cells of a 2-category. Another particular case of 3-polygraphs are rewriting systems on monoidal categories. In computer science, polygraphs are used to study rewriting on Petri nets [?].

**1.1.3. Confluence and decreasingness.** In a rewriting system, a conflict between two rewriting rules is called a *local branching*. Local confluence of a polygraph  $\Sigma$  can be decided by examining some local branchings of  $\Sigma$  which are called the *critical branchings*. More explicitly, if all critical branchings of a polygraph are confluent, then all its local branchings are confluent. We thus get local confluence from the confluence of a family of local branchings. This result is called critical pairs Lemma [?]. The main confluence problem for rewriting systems is to decide confluence from local confluence and supplementary conditions. Since Newman's work, this is

a well-studied problem in rewriting. Newman's Lemma states that in a terminating rewriting system, local confluence and confluence are equivalent properties. A whole series of works [?] led to criteria dropping the termination condition of Newman's Lemma in favor of the properties of the local branchings to prove confluence from local confluence. For example, an abstract rewriting system in which all local branching are confluent by using only one rewriting step is confluent. In 1994, van Oostrom defined decreasingness, a criterion used to prove confluence from local confluence. Decreasingness is a refinement of an unpublished earlier criterion, weak diamond property, defined by de Bruijn. We refer the reader to [?] for the links between those properties. Then, van Oostrom proves the following result:

**Theorem** [?, Theorem 2.3.5.] *Any decreasing abstract rewriting system is confluent.*

This result is stronger than Newman's Lemma because any terminating rewriting system is decreasing. Furthermore, decreasingness is a universal criterion to decide confluence of countable rewriting systems. Any confluent countable rewriting system is indeed decreasing.

**1.1.4. Knuth-Bendix's procedure.** A completion procedure is a procedure adding generators and rewriting rules to a presentation to obtain a presentation of the same structure. The redundant generators and rules are expected to give a confluent rewriting system.

A general procedure to find a convergent, a fortiori confluent, term rewriting system given an equational presentation has been introduced by Knuth and Bendix in [?]. We will develop Knuth-Bendix's procedure for  $(n + 1)$ -polygraphs. Given  $\Sigma$  an  $(n + 1)$ -polygraph, Knuth-Bendix's procedure applied to  $\Sigma$  uses an order on the set of  $n$ -cells of  $\Sigma$  which is compatible with the rewrite relation. The procedure is based on the critical pairs Lemma and adds  $(n + 1)$ -cells to an  $(n + 1)$ -polygraph. For each non confluent critical pair leading to two chosen normal forms  $u$  and  $v$ , we add an  $(n + 1)$ -cell from  $u$  to  $v$  if  $v \prec u$  or from  $v$  to  $u$  if  $u \prec v$ . Knuth-Bendix's procedure either terminates in the case where all critical pairs are made confluent, creates an infinite increasing sequence of  $(n + 1)$ -polygraphs or fails. In the first two cases, we obtain a convergent  $(n + 1)$ -polygraph in a finite or infinite number of steps. The procedure fails when we encounter two  $n$ -cells which cannot be compared. If the order we use is total, we call it a total termination order.

**1.1.5. Coherence.** A coherent presentation of an  $n$ -category  $\mathcal{C}$  is a data made of an  $(n + 1)$ -polygraph  $\Sigma$  presenting  $\mathcal{C}$  and a family of  $(n + 2)$ -cells  $\Sigma_{n+2}$  such that the quotient of the free  $(n + 1, n)$ -category over  $\Sigma$ , or  $(n + 1)$ -category of congruences generated by  $\Sigma$ , by the congruence generated by  $\Sigma_{n+2}$  is aspherical, that is for any ordered pair  $(\alpha, \beta)$  of  $(n + 1)$ -cells, there exists an  $(n + 2)$ -cell from  $\alpha$  to  $\beta$ .

The syzygies problem was first introduced by Hilbert [?]. A syzygy in a module  $M$  over a ring  $\mathcal{R}$  generated by a family of  $n$  elements  $(g_1, \dots, g_n)$  is a  $n$ -uple  $(a_1, \dots, a_n)$  of elements

of  $\mathcal{R}$  such that

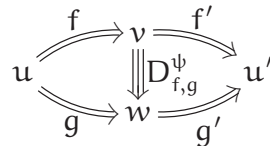
$$\alpha_1 g_1 + \cdots + \alpha_n g_n = 0$$

The set of all syzygies of  $M$  forms a module called the *module of syzygies*. Multiple methods have been developed to compute the module of syzygies by rewriting. Coherence problems in higher-dimensional categories have been studied in [?] and lead to the notion of *polygraphic resolution*, a generalization of the module of syzygies. A polygraphic resolution of an  $n$ -category is a cofibrant approximation of this  $n$ -category in the category of  $(\infty, n)$ -categories. The construction of polygraphic resolutions is a problem arising in the context of Squier's theory, which has been developed in a different context than the one of the study of polygraphs.

**1.1.6. Squier's theory.** Some properties of a higher-dimensional category can be derived from a presentation of this higher-dimensional category by a polygraph. In 1987, Squier proved [?] that all finitely presented monoids do not admit a presentation by a finite convergent rewriting system. In Squier's work, the notion of presentation of monoids is extended into the notion of coherent presentation [?]. A coherent presentation of a monoid  $\mathbf{M}$  is a data  $\Sigma$  made of a 2-polygraph  $(\Sigma_0, \Sigma_1, \Sigma_2)$  presenting  $\mathbf{M}$  and a globular extension  $\Sigma_3$  of the free  $(2, 1)$ -category over  $(\Sigma_0, \Sigma_1, \Sigma_2)$ . The 3-cells of  $\Sigma_3$  are relations between the relations of  $\mathbf{M}$ , or *coherence rules*. Given a coherent presentation  $\Sigma$  of a monoid  $\mathbf{M}$ , we can construct a resolution of the  $\mathbb{Z}\mathbf{M}$ -modules, where  $\mathbb{Z}\mathbf{M}$  denotes the ring generated by the elements of  $\mathbf{M}$  and whose multiplication is induced by the monoid law of  $\mathbf{M}$ :

$$\mathbb{Z}\mathbf{M}[\Sigma_3] \xrightarrow{d_3} \mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where for any  $k$ ,  $\mathbb{Z}\mathbf{M}[\Sigma_k]$  denotes the free  $\mathbb{Z}\mathbf{M}$ -module over  $\Sigma_k$ . In particular, a monoid with a finite coherent presentation is of homological type left-FP<sub>3</sub> and a finitely presented monoid is of homological type left-FP<sub>2</sub>. Furthermore, from a convergent presentation of a monoid, it is possible to construct a coherent presentation of this monoid. For each critical branching  $(f, g)$  of this monoid we choose a confluence diagram  $(f \cdot f', g \cdot g')$  for this branching and a 3-cell filling this diagram.



This process is called *Squier's completion*.

**Theorem [?, Theorem 5.2.]** *Let  $\mathcal{C}$  be a category and  $\Sigma$  a convergent presentation of  $\mathcal{C}$ . Squier's completion of  $\Sigma$  is a coherent presentation of  $\mathcal{C}$ .*

We say that a monoid has *finite derivation type* if it admits a finite coherent presentation. Moreover, if a monoid admits a finite convergent presentation, Squier's completion of this monoid is finite. This implies the following result linking a combinatorial property of monoids to a homological one.

**Theorem** [?, **Theorem 4.1.**] *If a monoid admits a finite convergent presentation, it is of finite derivation type.*

Finite derivation type can be generalized to higher dimensions. A monoid admitting a finite convergent presentation admits a finite coherent presentation. In the same way, we say that a higher-dimensional category is of finite derivation type if it admits a coherent presentation.

**1.1.7. Linear rewriting.** In parallel to Squier's theory, rewriting was developed in a linear context for the case of algebras presented by generators and relations. A first instance of linear rewriting was introduced in 1965 by Buchberger under the name of Gröbner bases. A Gröbner basis for an ideal  $I$  of a commutative algebra  $\mathbf{A}$  endowed with a monomial order  $<$  is a generating set  $G$  for the ideal  $I$  such that the rewrite relation on  $\mathbf{A}$  defined by  $G$  and  $<$  is confluent. The notion of Gröbner basis was then generalized by Mora [?] for two-sided ideal of non commutative algebras.

Gröbner bases find applications in homological algebra like the study of the Koszulity property. It is proved in [?] that an algebra with a quadratic Gröbner basis is Koszul. Another application of Gröbner bases to homological algebra is the construction of Anick's resolutions [?]. Indeed, the free modules appearing in this resolution are generated by the critical pairs arising from a Gröbner basis.

Gröbner bases make use of monomial orders. However multiple finitely presented algebras do not admit a finite Gröbner basis whichever monomial order is chosen. The algebra on three generators  $x, y$  and  $z$  and verifying the relation

$$xyz = x^3 + y^3 + z^3$$

is such an example [?, Example 3.4.7.]. Two-dimensional linear polygraphs were introduced in [?] as a generalization of non commutative Gröbner bases. In particular, linear polygraphs do not use monomial orders. In particular, the above algebra can be presented by a linear polygraph with only one 2-cell

$$xyz \Rightarrow x^3 + y^3 + z^3$$

whose orientation does not correspond to any monomial order. We will develop in this work the notion of linear polygraph for any dimension.

Extending a presentation of a monoid by coherence rules returns a polygraphic resolution of this monoid, see [?]. Extending a coherent presentation of a monoid also returns a cofibrant replacement of this monoid in the category of  $(\infty, 1)$ -categories [?, ?]. Finding a linear structure for polygraphs would enable us to find polygraphic resolutions for algebras.



## 1.2. CATEGORIFICATION IN REPRESENTATION THEORY

**1.2.1. Grothendieck decategorification.** The terminology "categorification" was introduced in 1995 by Crane. Categorification is a process giving from an algebra  $\mathbf{A}$  a linear monoidal category  $\mathcal{C}$  whose Grothendieck group is isomorphic to the algebra  $\mathbf{A}$  [?, ?]. We say that  $\mathcal{C}$  is a categorification of  $\mathbf{A}$  or, *categorifies*  $\mathbf{A}$ . The Grothendieck group of a linear category  $\mathcal{C}$  is the abelian group generated by the isomorphism classes of  $\mathcal{C}$  and subject to the relation  $[a] = [b] + [c]$  whenever an object  $a$  is direct sum of two objects  $b$  and  $c$ , that is there are 1-cells  $a \xrightarrow{p_b} b$ ,  $a \xrightarrow{p_c} c$ ,  $b \xrightarrow{i_b} a$  and  $c \xrightarrow{i_c} a$  such that:

- $p_b \star_0 i_b + p_c \star_0 i_c = 1_a$ ,
- $i_b \star_0 p_b = 1_b$ ,
- $i_c \star_0 p_c = 1_c$ .

The process of constructing the Grothendieck group of a linear category is a case of *decategorification*. This process can be generalized to any higher-dimensional linear category and is called *Grothendieck decategorification*.

Categorification finds many applications in algebra. An example of a categorification is the Khovanov homology [?], a categorification of the Jones polynomials. This categorification was used to give a new proof of Milnor's conjecture [?]. A classical categorification problem is to exhibit from a ring  $\mathbf{A}$  a categorification of  $\mathbf{A}$  with properties such that having all its idempotents split.

**1.2.2. The category of Bott-Samelson bimodules.** In 1979, Kazhdan and Lusztig constructed a basis for the Hecke algebra and conjectured that the base change polynomials from this basis to the classical basis had positive coefficients. This conjecture was proved in 1981 by Beilinson and Bernstein, and independently by Brylinski and Kashiwara, in the case of Hecke algebras of Weyl groups. The conjecture was proved in the general case in 2013 by Elias and Williamson [?] using categorification methods. Inspired by categorification, Elias and Williamson categorified the Hecke algebra by a category called the *category of Soergel bimodules*, which is the idempotent-complete grading shift closure of a category called *category of Bott-Samelson bimodules*. Then, from the properties of the category of Soergel bimodules, Elias and Williamson proved Kazhdan-Lusztig's conjecture.

Elias and Williamson category of Bott-Samelson bimodules is defined diagrammatically by generators and relations and can be studied from a combinatorial point of view. An open problem on this category is the study of its syzygies. This problem is a coherence problem for the category Bott-Samelson bimodules.

**1.2.3. Brauer algebras and the affine oriented Brauer category.** Schur-Weyl duality [?] is a result stating that there is an isomorphism between the  $\mathbb{C}$ -representation of the symmetric group  $\mathfrak{S}_k$  and the endomorphism algebra of the  $\mathfrak{gl}_n(\mathbb{C})$ -algebra  $(\mathbb{C}^n)^{\otimes k}$  whenever the integers  $k$  and  $n$  satisfy  $n \geq k$ . In 1989, Turaev [?] and Koike [?] introduced independently the walled Brauer algebra  $B_{r,s}(\delta)$ , an algebra having a Schur-Weyl duality with the tensor product  $V^{\otimes r} \otimes W^{\otimes s}$  where  $V$  is the natural representation of  $GL_n(\mathbb{C})$  and  $W$  is the dual of  $V$ .

Another variation of Schur-Weyl duality for superalgebras was studied by Rui and Su [?], and independently by Sartori [?], and led to the introduction of affine walled Brauer algebras. They proved that there is a Schur-Weyl duality between general Lie superalgebras and affine walled Brauer algebras. Those algebras are further studied in [?]. A linear monoidal category, the affine oriented Brauer category  $\mathcal{AOB}$  was introduced in [?] to encode each walled Brauer algebra as one of its morphism spaces. The category  $\mathcal{AOB}$  is presented by generators and relations using dotted oriented string diagrams. Brundan and al prove that a given family of those diagrams provides a basis for each morphism space of  $\mathcal{AOB}$ .

### 1.3. THESIS SUMMARY AND MAIN CONTRIBUTIONS

**1.3.1. Applications of rewriting to representation theory.** In this work, we use rewriting to study higher-dimensional linear categories appearing in representation theory.

We first define linear polygraphs. A first definition of two-dimensional linear polygraphs was given by Guiraud, Hoffbeck and Malbos [?]. We generalize their definition to all dimensions. A linear  $(2, 1)$ -polygraph is a data  $\Sigma$  made of a directed graph  $(\Sigma_0, \Sigma_1)$  and a globular extension  $\Sigma_2$  of the free linear category over  $(\Sigma_0, \Sigma_1)$ . In particular, if  $\Sigma_0$  is made of only one 0-cell, the directed graph  $(\Sigma_0, \Sigma_1)$  generates a free algebra over the 1-cells of  $\Sigma_1$  and the set of 2-cells  $\Sigma_2$  is a set of oriented relations for the quotient of this algebra by  $\Sigma_2$ . We say that the linear  $(2, 1)$ -polygraph  $\Sigma$  is a *presentation* of an algebra  $\mathbf{A}$  if  $\mathbf{A}$  is isomorphic to the quotient of the free algebra over  $(\Sigma_0, \Sigma_1)$  by  $\Sigma_2$ .

In this thesis, we generalize the notion of coherent presentation to higher-dimensional linear categories. Then, we generalize Squier's theory to linear polygraphs. This leads to the construction of coherent presentations of higher-dimensional linear categories and Squier's completion of higher-dimensional linear polygraphs. Coherent presentations in the linear case can be used to construct for an algebra  $\mathbf{A}$  a resolution

$$\mathbf{A}[\Sigma_3] \xrightarrow{d_3} \mathbf{A}[\Sigma_2] \xrightarrow{d_2} \mathbf{A}[\Sigma_1] \xrightarrow{d_1} \mathbf{A} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where  $\Sigma$  is a coherent presentation of  $\mathbf{A}$  and for any  $k$ ,  $\mathbf{A}[\Sigma_k]$  denotes the free  $\mathbf{A}$ -module over  $\Sigma_k$ . A first result in the context of higher-dimensional linear polygraph is that Squier's completion of a convergent linear polygraph yields a coherent presentation of a higher-dimensional linear

category. We call this result *linear Squier's Theorem*. Furthermore, it is proved in [?] that any convergent presentation of an algebra can be extended into a free infinite resolution of this algebra. Such a resolution is called a *polygraphic resolution*.

Another combinatorial result obtained from the construction of convergent presentations for higher-dimensional linear categories is the construction of bases for the morphism spaces of this higher-dimensional linear category. More precisely, a higher-dimensional linear category with a confluent and normalizing presentation has a basis for each of its morphism spaces indexed by the normal forms of this presentation. This result is motivated by the bases computation problems appearing in representation theory such that the study of the affine oriented Brauer category  $\mathcal{AOB}$  presented in [?]. We study this problem from a rewriting point of view. For this, we use the notion of decreasingness defined by van Oostrom [?]. More precisely, we give a decreasing and quasi-terminating presentation for  $\mathcal{AOB}$  and use van Oostrom's Theorem. This leads to a confluent and quasi-terminating presentation of  $\mathcal{AOB}$ . Then, we prove that the equivalence classes of the quasi-normal forms for this presentation index a basis for each morphism spaces of  $\mathcal{AOB}$ .

Another part of this memoir is the use of decreasingness techniques in Squier's theory. We prove in [?] that a decreasing and quasi-terminating presentation of a monoid yields a coherent presentation of this monoid when some other properties are verified. It is also shown by Yudin in [?] that the geometric proof of van Oostrom's Theorem presented in [?] leads to a version of Squier's Theorem with quasi-termination. Yudin's result uses stronger hypotheses than [?] on the Peiffer confluences and the compatibility between the product and the decreasing labelling. This result leads to the construction of a coherent presentation of the 0-Hecke monoid.

Finally, we use rewriting methods to study categorification problems. For this, we first define the notion of Karoubi envelope for polygraphs. This leads to the construction of Karoubi envelope for (coherent) presentations of higher-dimensional linear categories. We prove that for any higher-dimensional linear category  $\mathcal{C}$ , the Karoubi envelope of a (coherent) presentation of  $\mathcal{C}$  is a (coherent) presentation of the Karoubi envelope of  $\mathcal{C}$ . Another important remark in linear polygraph is that rewriting paths and their inverses generate *equality proofs* in the presented higher-dimensional linear category. This leads us to define *isomorphisms proofs* and *direct sum proofs* in the free higher-dimensional linear category over a linear polygraph. Isomorphism proofs and direct sum proofs are then used to construct Grothendieck decategorifications of higher-dimensional linear categories.

**1.3.2. Higher-dimensional linear categories and rewriting.** In the first chapter, before we introduce the rewriting tools used to study higher-dimensional linear categories, we present the classical theory of rewriting. In particular, we state and prove Church-Rosser's Theorem and Newman's Lemma.

Next, after having presented polygraphs, we generalize the families of local branchings appearing in word rewriting. Those families are aspherical, Peiffer and overlapping branchings.

Among the overlapping branchings, we can define critical branching as in the case of word rewriting. Then, we recall the critical pairs Lemma for polygraphs. This lemma links local confluence of a polygraph to the confluence of its critical pairs. This result allows us to use Newman's Lemma [?], stating that in a terminating abstract rewriting system, local confluence and confluence are equivalent, to decide the convergence of a terminating polygraph by examining all its critical pairs.

**1.3.3. Computing bases with linear polygraphs.** In the second chapter, we present linear polygraphs and their rewriting theory. A first definition of 2-polygraphs was given in [?]. We present in this memoir the notion of linear polygraph introduced in [?]. For  $n$  and  $p$  two integers such that  $n \geq p$ , we define the category of *linear  $(n, p)$ -polygraphs*. This category is constructed by induction by first defining the category of linear  $(n, n)$ -polygraphs, and then by defining the category of linear  $(n + 1, p)$ -polygraphs from the category of linear  $(n, p)$ -polygraphs.

Adding a linear structure to polygraphs creates multiple rewriting problems which do not appear in the set-theoretic setting. The first of those problems is the definition of rewriting steps. Indeed, in a linear  $(n + 1, n)$ -category, all  $(n + 1)$ -cells are invertible. As a consequence defining rewriting steps as  $(n + 1)$ -cells in this free linear  $(n + 1, n)$ -category requires to find some conditions to ensure that the inverse of a rewriting step is not always a rewriting step. Furthermore, if a linear  $(n + 1, n)$ -polygraph contains two  $(n + 1)$ -cells with respective sources of the form  $u + v$  and  $v + w$ , defining rewriting steps on this  $(n + 1, n)$ -polygraph would lead to overlapping branchings with source of the form  $u + v + w$ . We will thus define rewriting steps of linear polygraphs only when the sources of the rewriting rules are monomials.

The local branchings of linear polygraphs are also more complex. They belong to four families: aspherical branchings, Peiffer branchings, additive branchings and overlapping branchings. Contrarily to the set-theoretical case, local branchings other than overlapping branchings are not always confluent. Examples of this fact are given in [?]. As a consequence, the critical pairs Lemma for linear polygraphs needs more conditions than in the set-theoretical case. The new critical pairs Lemma becomes:

**Theorem ??** Let  $\Sigma$  be a quasi-terminating exponentiation free left-monomial linear polygraph. If all critical branchings of  $\Sigma$  are confluent, all local branchings of  $\Sigma$  are confluent.

See Chapter 3 for the context of this statement.

Defining the rewriting theory associated to linear polygraphs allows us to construct bases for the morphism spaces of higher-dimensional linear categories. This problem appears in [?] for the affine oriented Brauer category  $\mathcal{AOB}$ . We propose to compute bases of morphisms spaces by constructive methods. Linear rewriting links the elements of the bases to normal forms in a linear polygraph. More precisely, we have the following result:

**Proposition ??** *Let  $\Sigma$  be a left-monomial linear  $(n + 1, n)$ -polygraph presenting an  $(n, n)$ -linear category  $\mathcal{C}$ . Let us assume that  $\Sigma$  is confluent and normalizing. Then, for all parallel  $(n - 1)$ -cells  $u$  and  $v$  of  $\mathcal{C}$ , a basis of  $\mathcal{C}_n(u, v)$  is given by the set of equivalence classes of monomials in normal form of  $\Sigma_n^l(u, v)$ .*

The constructive part of this result is the explicit computation which can be used to find the elements of the bases. Indeed, given a confluent and normalizing rewriting system with an enumerable set of rewriting rules, it is possible to determine algorithmically all normal forms.

**1.3.4. Completion procedure in quasi-terminating polygraphs.** Many rewriting problems as the word problem can be solved by giving for a higher-dimensional category a polygraph presenting this higher-dimensional category. For the word problem, such a presentation would lead to apply the normal form procedure to decide the question of whether or not two  $n$ -cells of an  $n$ -category presented by generators and relations are equal. If  $\Sigma$  is a convergent  $(n + 1)$ -polygraph, two  $n$ -cells of  $\Sigma^*$  are equal in the  $n$ -category presented by  $\Sigma$  if and only if they have the same normal form. The normal form procedure consists in computing the normal forms of the two  $n$ -cells we wish to compare and checking if the two normal forms are equal or not.

Given an  $n$ -category presented by an  $(n + 1)$ -polygraph, we study in the third chapter of this memoir how to compute a confluent presentation of this  $n$ -category. Given a non confluent  $(n + 1)$ -polygraph, this problem can be solved by constructing a Tietze-equivalent confluent  $(n + 1)$ -polygraph, that is a polygraph presenting the same  $n$ -category. This approach is taken by Knuth and Bendix in [?] for term rewriting systems. They give a procedure to add generators and rewriting rules to a presentation of an equational structure to obtain a confluent presentation of the same structure. This procedure is thus called *Knuth-Bendix's completion procedure*, or Knuth-Bendix's procedure. If a well-founded order on the terms compatible with the rewrite relation is given, we can guarantee the existence of the constructed presentation, that is the procedure does not fail, see [?]. Such an order is called a *total termination order*.

In the case of  $n$ -categories, Knuth-Bendix's procedure is applied to  $(n + 1)$ -polygraphs. Given  $\Sigma$  an  $(n + 1)$ -polygraph, Knuth-Bendix's procedure applied to  $\Sigma$  uses an order on the set of  $n$ -cells of the free  $(n + 1)$ -category over  $\Sigma$  which is compatible with the rewrite relation. The procedure is based on the critical pairs Lemma.

**Lemma [?, 4.1.5.]** *An  $n$ -polygraph  $\Sigma$  is locally confluent if and only if all critical branchings of  $\Sigma$  are confluent.*

Knuth-Bendix's procedure adds  $(n + 1)$ -cells to an  $(n + 1)$ -polygraph. For each non confluent critical pair leading to two chosen normal forms  $u$  and  $v$ , we add an  $(n + 1)$ -cell from  $u$  to  $v$  if  $v \prec u$  or from  $v$  to  $u$  if  $u \prec v$ . Knuth-Bendix's procedure either terminates in the case where all critical pairs are made confluent, creates an infinite increasing sequence of  $(n + 1)$ -polygraphs or fails. In the first two cases, we obtain a convergent  $(n + 1)$ -polygraph in a finite or infinite

number of steps. The procedure fails when we encounter two  $n$ -cells which cannot be compared. If the order we use is total, we call it a total termination order and Knuth-Bendix's procedure does not fail. In this case, we construct a confluent polygraph in a finite number of steps if the procedure terminates or we obtain a confluent polygraph as an infinite increasing union of polygraph if the the procedure does not terminate.

Knuth-Bendix's procedure for  $(n + 1)$ -polygraphs gives a convergent  $(n + 1)$ -polygraph from an  $(n + 1)$ -polygraph with a total termination order. However, note that in general not all terminating  $(n + 1)$ -polygraphs have a total termination order, see ???. This raises the problem of finding another completion procedure for terminating  $(n + 1)$ -polygraphs. We introduce such a procedure, the generalized Knuth-Bendix's procedure on  $(n + 1)$ -polygraphs. This procedure takes for input an  $(n + 1)$ -polygraph  $\Sigma$  and an order on the  $n$ -cells of the free  $(n + 1)$ -category over  $\Sigma$ . We then define quasi-termination orders and maximal quasi-termination orders which are generalizations of termination orders. We then prove the following result:

**Proposition ??** *Let  $\Sigma$  be an  $(n + 1)$ -polygraph and let  $\prec$  be a maximal quasi-termination order on  $\Sigma$ . Then, the application of the generalized Knuth-Bendix's procedure on  $(\Sigma, \prec)$  does not fail.*

Finally, we prove that generalized Knuth-Bendix's procedure returns a confluent, not necessarily terminating, polygraphs.

**Proposition ??** *Let  $\Sigma$  be an  $(n + 1)$ -polygraph and  $\prec$  a maximal quasi-termination order on  $\Sigma$ . Let us assume that for each congruent  $n$ -cells  $f$  and  $g$  of  $\Sigma_n^*$  such that  $f \prec g$ , the  $n$ -cell  $g$  rewrites into  $f$ . Then, the  $(n + 1)$ -polygraph  $\Sigma$  is confluent.*

**1.3.5. Coherence by decreasingness.** In the fourth chapter, we present the applications of decreasingness to the construction of coherent presentations using quasi-termination. Decreasingness is criterion introduced by van Oostrom to prove confluence of abstract rewriting systems. Van Oostrom's Theorem can be applied to terminating rewriting systems by labelling all rewriting steps by the distance of its target to a normal form. On the other hand, Squier's Theorem is a coherence result using Newman's Lemma to prove that a family of confluence diagrams for each critical pair of a convergent  $(n + 1)$ -polygraph gives a coherent presentation of an  $n$ -category. The question we study is the generalization of Squier's Theorem to non terminating polygraphs using decreasingness.

Given an  $n$ -category  $\mathcal{C}$  and a convergent  $(n + 1)$ -polygraph  $\Sigma$  presenting  $\mathcal{C}$ , the homotopy basis given by Squier's Theorem is indexed by the critical branchings of  $\Sigma$ . For each critical branching of  $\Sigma$ , we chose a confluence diagram for this branching and fill this diagram by an  $(n + 2)$ -cell to obtain a coherent presentation. Such a family of confluence diagrams is called a *family of generating confluences*. In [?], we weaken the termination hypothesis to proves that for confluent quasi-terminating polygraphs verifying some hypotheses, a homotopy basis can be



constructed from a family of generating confluences and a family of elementary loops, which can be non empty since we work with quasi-terminating polygraphs.

Our result uses decreasingness as one of its conditions. We can define multiple notions of decreasingness for polygraphs. We say that a polygraph is (strictly) decreasing if it is (strictly) decreasing as an abstract rewriting system. Decreasingness being a property defined on a labelling of the rewriting steps of an abstract rewriting system, we can refine the notion of decreasingness for polygraph. We say that a labelling is *compatible with contexts* if applying any context to a decreasing diagram also gives a decreasing diagram. Another compatibility condition between decreasingness and the structure of polygraph is *Peiffer compatibility*. Indeed, in a polygraph, all Peiffer branchings are decreasing by using confluence diagrams called Peiffer confluences. But, in a decreasing polygraph, Peiffer confluences are not decreasing. We say that a polygraph is Peiffer decreasing if all its Peiffer confluences are homotopically equivalent to a decreasing diagram. Other compatibility conditions appear in Yudin's work, see [?]. The result we prove in [?] is:

**Theorem ??** *Let  $(\Sigma, \psi)$  be a strictly decreasing  $n$ -polygraph. Let  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)$  be a strictly decreasing Squier's completion of  $\Sigma$ . If  $\psi$  is compatible with contexts and  $(\Sigma, \psi)$  is Peiffer decreasing with respect to the extension  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)$ , then  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)$  is a coherent presentation of the category presented by  $\Sigma$ .*

**1.3.6. Rewriting applied to categorification.** In the fifth chapter of this thesis, we present applications of rewriting to categorification. We will present two main questions. First, we want to define polygraphs presenting Karoubi envelopes of higher-dimensional categories. Algebras like Hecke algebras [?] can be categorified by the Karoubi envelope of a monoidal category. Khovanov conjectured that the Karoubi envelope of some diagrammatic category [?] categorifies the Heisenberg algebra. Categorifications of algebras are generally defined as Karoubi envelopes of explicit categories.

The Karoubi envelope of a 1-category  $\mathcal{C}$  contains the objects of  $\mathcal{C}$  plus one object for each non identity idempotent of  $\mathcal{C}$  and is generated by the morphisms of  $\mathcal{C}$  plus a family of injections and projections morphisms verifying relations splitting all idempotents. In particular, if all idempotents of the category  $\mathcal{C}$  are split, the category  $\text{Kar}(\mathcal{C})$  is equivalent to  $\mathcal{C}$ . First, we extend the notion of Karoubi envelope to higher-dimensional categories. This higher-dimensional notion is consistent with the classical notion of Karoubi envelope and the definitions of Karoubi envelope of a monoidal category considered either as a 1-category or a 2-category with only one 0-cell are equivalent. This leads to define Karoubi envelopes for polygraphs. Given an  $n$ -category  $\mathcal{C}$  presented by an  $(n + 1)$ -polygraph  $\Sigma$ , we define an  $(n + 1)$ -polygraph  $\text{Kar}(\Sigma)$  presenting the Karoubi envelope of  $\mathcal{C}$  by universal construction.

**Theorem ??** *Let  $\mathcal{C}$  be an  $n$ -category presented by an  $(n + 1)$ -polygraph  $\Sigma$ . The Karoubi envelope of  $\mathcal{C}$  is presented by the  $(n + 1)$ -polygraph  $\text{Kar}(\Sigma)$ .*

Furthermore, we extend this result by defining Karoubi envelopes of globular extension to treat coherence problems in Karoubi envelopes. We prove the following result:

**Theorem ??** *Let  $\mathcal{C}$  be an  $n$ -category and let  $(\Sigma, \Sigma_{n+2})$  be a coherent presentation of  $\mathcal{C}$ . The  $(n+2, n)$ -polygraph  $(\text{Kar}(\Sigma), \text{Kar}(\Sigma_{n+2}))$  is a coherent presentation of the Karoubi envelope of  $\mathcal{C}$ .*

As a consequence of this result, we have a computation of coherent presentation of a higher-dimensional category which does not appeal to a confluent or terminating presentation.

The second question we study is the construction of decategorifications by rewriting methods. The decategorification process for higher-dimensional linear categories, is called *Grothendieck decategorification*. This process creates a linear  $(n-1, n-1)$ -category from a linear  $(n, n)$ -category. *Isomorphisms proofs* and *direct sum proofs* in a linear  $(n, n)$ -category  $\mathcal{C}$  presented by an  $(n+1, n)$ -polygraph  $\Sigma$  can be interpreted as  $(n+1)$ -cells in the free linear  $(n+1, n)$ -category over  $\Sigma$ . This allows us to define Grothendieck decategorification for linear polygraphs. We finally prove the following result:

**Theorem ??** *Let  $\mathcal{C}$  be a linear  $(n, n)$ -category presented by a linear  $(n+1, n)$ -polygraph  $\Sigma$ . The Grothendieck decategorification  $\mathbb{K}(\Sigma)$  of  $\Sigma$  presents the Grothendieck decategorification  $\mathbb{K}(\mathcal{C})$  of  $\mathcal{C}$ .*

**1.3.7. A confluent presentation of the affine oriented Brauer category.** In the last chapter, we give an application of decreasingness to the study of the affine oriented Brauer category  $\mathcal{AOB}$ .

The basis theorem of [?] uses an intermediate result on cyclotomic quotients of  $\mathcal{AOB}$ . For each of those quotients, a basis is given. With these multiple bases, each morphism space of  $\mathcal{AOB}$  is given a generating family which is proved to be linearly independent. This result is not constructive in  $\mathcal{AOB}$ . A constructive proof of this result by rewriting can however be given. For this, we use the results of Section ?? of and give a confluent presentation of  $\mathcal{AOB}$ . This presentation is a linear  $(3, 2)$ -polygraph that we call  $\overline{\mathcal{AOB}}$ .

**Theorem ??** *The linear  $(3, 2)$ -polygraph  $\overline{\mathcal{AOB}}$  is a confluent presentation of  $\mathcal{AOB}$ .*

The linear  $(3, 2)$ -polygraph  $\overline{\mathcal{AOB}}$  is not terminating. This will prevent us to prove that  $\overline{\mathcal{AOB}}$  is confluent by using Newman's Lemma. To prove that  $\overline{\mathcal{AOB}}$  is confluent, we will use the decreasingness property. We will prove that  $\overline{\mathcal{AOB}}$  is decreasing and use van Oostrom's Theorem to prove that  $\overline{\mathcal{AOB}}$  is confluent. We first prove that  $\overline{\mathcal{AOB}}$  is quasi-terminating by expliciting monomials we call quasi-reduced and we prove them to be quasi-normal forms. Then, we prove that  $\overline{\mathcal{AOB}}$  is a presentation of the linear monoidal category  $\mathcal{AOB}$ . Using decreasingness, we finally prove that all monomials of  $\overline{\mathcal{AOB}}$  can be rewritten into a linear combination of quasi-reduced monomials.





# Chapter 2

## Rewriting, polygraphs and coherence

In this chapter, we introduce the rewriting and categorical notions used in the rest of this thesis. The notion of rewriting system was introduced in 1914 by Thue [?] for the study of the word problem in combinatorial structures. Rewriting is a computation model in which calculus is expressed as a sequence of elementary transformations called rewriting steps. Each rewriting step comes from a rewriting rule. The set of rewriting rules defines a rewriting system. Two important properties of rewriting systems are termination, which ensures that all computations end, and confluence, which ensures that all computations from a same expression lead to a same result. The conjunction of termination and confluence is called convergence.

In the case of a monoid presented by generators and relations, the word problem amounts to deciding if two words correspond to the same element of the monoid. Rewriting can be used to decide this problem by presenting the monoid by a finite convergent rewriting system. The decision algorithm using such a rewriting system is called the normal form algorithm. Rewriting also has applications in computer science. In particular, rewriting appear in operational semantic, formal calculus or automatic theorem proving.

In the context of higher-dimensional categories, rewriting is formalized by the notion of polygraph. Polygraphs were introduced in 1987 by Street [?] under the name of computads and independently by Burroni in 1993 [?] as systems of generators and oriented relations, or rewriting rules, for higher-dimensional categories. For  $n \geq 1$ , an  $(n + 1)$ -polygraph is a presentation of an  $n$ -category by generators and relations. Polygraphs find applications in algebra to study structures like monoids or monoidal categories by rewriting.

In 1987, Squier proved [?] that all finitely presented monoids do not admit a presentation by a finite convergent rewriting system. Thus, the normal form algorithm is not a universal decision procedure for the word problem. He linked the existence of a finite convergent presentation for a finitely presented monoid to a homological property by showing that the critical branchings of a convergent string rewriting system generate the module of the 2-homological syzygies of

the presentation. A combinatorial approach is then presented in [?] to the question of whether or not a finitely presented monoid admits a finite convergent presentation. The existence of such a presentation is linked to a finiteness condition of finitely presented monoids, called *finite derivation type*, that extends the properties of being finitely generated and finitely presented. He then gives an example of finitely presented monoid which does not have finite derivation type. In term of higher-dimensional rewriting, a monoid admitting a finite convergent presentation admits a finite coherent presentation. A coherent presentation of an  $n$ -category  $\mathcal{C}$  is a data made of an  $(n + 1)$ -polygraph  $\Sigma$  presenting  $\mathcal{C}$  and a family of  $(n + 2)$ -cells  $\Sigma_{n+2}$  such that the quotient of the free  $(n + 1, n)$ -category over  $\Sigma$ , of  $(n + 1)$ -category of congruences generated by  $\Sigma$ , by the congruence generated by  $\Sigma_{n+2}$  is aspherical, that is for any ordered pair  $(\alpha, \beta)$  of  $(n + 1)$ -cells, there exists an  $(n + 1)$ -cell from  $\alpha$  to  $\beta$ .

**Plan of Chapter.** In Section ??, we begin by introducing the notions of abstract rewriting used and the main properties of abstract rewriting systems. Those notions can also be found in [?, ?]. In Section ??, we present  $n$ -polygraphs and their associated rewriting systems. Rewriting properties of  $n$ -polygraphs are given by exhibiting the families of local branchings in an  $n$ -polygraph. This chapter ends by an introduction of coherence for  $n$ -categories and by the statement of Squier’s Theorem ??.

## 2.1. ABSTRACT REWRITING

We recall in this section the classical notions of abstract rewriting. Abstract rewriting is treated in many references such as [?].

**2.1.1. Abstract rewriting systems.** An *abstract rewriting system* is a data made of a set  $S$  and a relation  $\rightarrow$  on  $S$ . For all  $x$  and  $y$  in  $S$ , we denote  $x \rightarrow y$  if  $(x, y)$  is in  $\rightarrow$ .

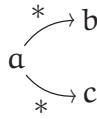
In the rest of this section, we fix  $(S, \rightarrow)$  an abstract rewriting system.

**2.1.2. Rewriting sequences.** The transitive closure of  $\rightarrow$  will be denoted by  $\xrightarrow{+}$ . The transitive reflexive closure of  $\rightarrow$  will be denoted by  $\xrightarrow{*}$ . The symmetric transitive closure of  $\rightarrow$  will be denoted by  $\leftrightarrow^*$ . We recall that for all  $x$  and  $y$  in  $S$ , we have

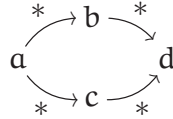
- $x \xrightarrow{+} y$  if and only if there exists  $n \geq 1$  and a family  $(x_k)_{k \in [0, n]}$  of elements of  $S$  such that  $x = x_1, y = x_n$  and  $x_k \rightarrow x_{k+1}$  for any  $0 \leq k \leq n$ ,
- $x \xrightarrow{*} y$  if and only if  $x = y$  or  $x \xrightarrow{+} y$ ,
- $x \leftrightarrow^* y$  if and only if  $x \xrightarrow{*} y$  or  $y \xrightarrow{*} x$ .

We say that  $x$  *rewrites* into  $y$  if  $x \xrightarrow{+} y$ . We say that there is a *rewriting sequence* from  $x$  to  $y$  if  $x \xrightarrow{*} y$ .

**2.1.3. Confluence.** We say that the relation  $\rightarrow$  is *confluent*, or that  $(S, \rightarrow)$  is confluent, if for any triple  $(a, b, c)$  of elements of  $S$  such that  $a \xrightarrow{*} b$  and  $a \xrightarrow{*} c$



there exists  $d$  in  $S$  such that  $b \xrightarrow{*} d$  and  $c \xrightarrow{*} d$ .



**2.1.4. Church-Rosser's property.** We say that  $(S, \rightarrow)$  verifies *Church-Rosser's property*, or is Church-Rosser, if for any pair  $(a, b)$  of elements of  $S$  such that  $a \leftrightarrow^* b$ , there exists  $c$  in  $S$  such that  $a \xrightarrow{*} c$  and  $b \xrightarrow{*} c$ .

**2.1.5. Theorem (Church-Rosser's Theorem [?]).** *An abstract rewriting system is confluent if and only if it is Church-Rosser.*

*Proof.* Let us assume that  $(S, \rightarrow)$  is Church-Rosser. Let  $a, b$  and  $c$  in  $S$  such that  $a \xrightarrow{*} b$  and  $a \xrightarrow{*} c$ . So, we have  $b \leftrightarrow^* c$ , which implies the existence of an element  $d$  in  $S$  such that  $b \xrightarrow{*} d$  and  $c \xrightarrow{*} d$ . Thus, the abstract rewriting system  $(S, \rightarrow)$  is confluent.

Let us assume that  $(S, \rightarrow)$  is confluent. Let  $a$  and  $b$  be in  $S$ . We prove by induction on the smallest  $n$  which makes exist a family  $(x_k)_{k \in \llbracket 0, n \rrbracket}$  of elements of  $S$  such that  $a = x_1$ ,  $b = x_n$  and  $x_k \rightarrow x_{k+1}$  for tout  $0 \leq k \leq n$  that there exists  $c \in S$  verifying  $a \xrightarrow{*} c$  and  $b \xrightarrow{*} c$ . If  $n = 0$ , we have  $a = b$  and we can choose  $c = a = b$ . If  $n > 0$ , there exists by induction hypothesis  $c'$  in  $S$  such that  $a \xrightarrow{*} c'$  and  $x_{n-1} \xrightarrow{*} c'$ . If  $b$  rewrites into  $x_{n-1}$ , we can chose  $c = c'$  to prove that  $(S, \rightarrow)$  is Church-Rosser. Else, we have  $x_{n-1} \xrightarrow{*} c'$  and  $x_{n-1} \xrightarrow{*} b$ , which allows us to construct  $c$  by confluence.  $\square$

**2.1.6. Normal forms and quasi-normal forms.** We say that  $a \in S$  is in *normal form* (or is a normal form) if there does not exist  $b$  in  $S$  such that  $a \rightarrow b$ . A *normal form of*  $a$  is a normal form  $a' \in S$  such that  $a$  rewrites into  $a'$ . We say that  $(S, \rightarrow)$  is *normalizing* if all elements of  $S$  have a normal form.

We say that  $a \in S$  is in *quasi-normal form* (or is a quasi-normal form) if for all  $a$  and  $b$  in  $S$  such that  $a \rightarrow b$ , we have  $b \xrightarrow{*} a$ . A *quasi-normal form of  $a$*  is a normal form  $a' \in S$  such that  $a$  rewrites into  $a'$ . We say that  $(S, \rightarrow)$  is *quasi-normalizing*, see [?], if all elements of  $S$  have a quasi-normal form.

**2.1.7. Termination and quasi-termination.** We say that  $(S, \rightarrow)$  is *terminating* if there does not exist any sequence  $(u_k)_{k \in \mathbb{N}}$  of elements of  $S$  such that  $u_k \rightarrow u_{k+1}$  for all  $k$ .

$$u_0 \rightarrow u_1 \rightarrow \cdots u_n \rightarrow \cdots$$

The terminologies *strongly normalizing* and *noetherian* are also used in literature.

We say that  $(S, \rightarrow)$  is *quasi-terminating*, see [?], if any sequence  $(u_k)_{k \in \mathbb{N}}$  of elements of  $S$  such that  $u_k \rightarrow u_{k+1}$  for all  $k$  contains infinitely many occurrences of the same element of  $S$ .

**2.1.8. Noetherian induction.** Let us assume that  $(S, \rightarrow)$  is terminating. The relation  $\xrightarrow{+}$  is thus well-founded, that is there does not exist any infinite decreasing sequence for the relation  $\xrightarrow{+}$ . An induction on this relation is called a *noetherian induction*.

**2.1.9. Lemma.** *Any terminating abstract rewriting system is normalizing.*

*Proof.* We prove this lemma by noetherian induction. Let us assume that  $(S, \rightarrow)$  is terminating and let  $a \in S$ . If  $a$  is a normal form,  $a$  is normal form of  $a$ . If there exists  $a'$  such that  $a \xrightarrow{+} a'$  and  $a'$  has a normal form, then any normal form of  $a'$  is a normal form of  $a$ . This concludes the proof.  $\square$

**2.1.10. Lemma [?, Lemma 2.6].** *Let  $(S, \rightarrow)$  be an normalizing abstract rewriting system. Then,  $(S, \rightarrow)$  is confluent if and only if all elements of  $S$  have a unique normal form.*

*Proof.* Let us assume that  $(S, \rightarrow)$  is normalizing. Let  $a \in S$ . If  $a$  has two distinct normal forms  $a'$  and  $a''$ , we have  $a \xrightarrow{*} a'$  and  $a \xrightarrow{*} a''$ . By definition of a normal form, there does not exist any  $b \in S$  such that  $a' \xrightarrow{*} b$  and  $a'' \xrightarrow{*} b$ . Thus,  $(S, \rightarrow)$  is not confluent.

Let us now assume that any element of  $S$  has a unique normal form. Let  $a, b$  and  $c$  in  $S$  such that  $a \xrightarrow{*} b$  and  $a \xrightarrow{*} c$ . Let  $d$  be the unique normal form of  $a$ . The unique normal form of  $b$  and the unique normal form of  $c$  are then necessarily equal to  $d$ . Then, we have  $b \xrightarrow{*} d$  and  $c \xrightarrow{*} d$ , which proves the confluence of  $(S, \rightarrow)$ .  $\square$

**2.1.11. Convergence.** We say that  $(S, \rightarrow)$  is *convergent*, or *converges*, if  $(S, \rightarrow)$  is both confluent and terminating.

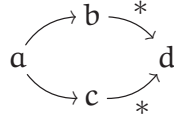
**2.1.12. Lemma.** *Let  $(S, \rightarrow)$  be a convergent abstract rewriting system. Any element of  $S$  has a unique normal form. Moreover, any pair  $(a, b)$  of elements of  $S$  verifies  $a \leftrightarrow^* b$  if and only if  $a$  and  $b$  have the same normal form.*

*Proof.* The system  $(S, \rightarrow)$  being convergent, it is in particular terminating and normalizing by Lemma ???. Moreover,  $(S, \rightarrow)$  is confluent. By Lemma ???, any element of  $S$  has a unique normal form. Let now  $a$  and  $b$  be in  $S$ .

Let us assume that the unique normal form of  $a$  is equal to the unique normal form of  $b$ , and let  $c$  be this normal form. Then, we have  $a \xrightarrow{*} c$  and  $b \xrightarrow{*} c$ , which implies  $a \leftrightarrow^* b$ .

If  $a \leftrightarrow^* b$ , by Theorem ???, there exists  $c \in S$  such that  $a \xrightarrow{*} c$  and  $b \xrightarrow{*} c$  because  $(S, \rightarrow)$  is confluent. The unique normal form of  $a$  is then equal to the unique normal form of  $c$ , which is equal to the unique normal form of  $b$ .  $\square$

**2.1.13. Local confluence.** We say that the relation  $\rightarrow$  is *locally confluent*, or that  $(S, \rightarrow)$  is locally confluent, if for all triple  $(a, b, c)$  of elements of  $S$  such that  $a \rightarrow b$  and  $a \rightarrow c$ , there exists  $d \in S$  such that  $b \xrightarrow{*} d$  and  $c \xrightarrow{*} d$ .



**2.1.14. Lemma (Newman's Lemma [?, Theorem 3]).** *A terminating abstract rewriting system is confluent if and only if it is locally confluent.*

*Proof.* Let  $(S, \rightarrow)$  be a terminating abstract rewriting system. If  $(S, \rightarrow)$  is confluent, then  $(S, \rightarrow)$  is locally confluent.

Let us now assume that  $(S, \rightarrow)$  is locally confluent. We prove by noetherian induction that  $(S, \rightarrow)$  is confluent. Let  $a \in S$ . If  $a$  is a normal form, then the only pair  $(b, c)$  such that  $a \xrightarrow{*} b$  and  $a \xrightarrow{*} c$  is  $(a, a)$ . If  $a$  is not a normal form, we can consider a pair  $(b, c)$  of elements of  $S$  verifying  $a \xrightarrow{+} b$  and  $a \xrightarrow{+} c$ . Then, there exists  $b'$  and  $c'$  such that  $a \rightarrow b' \xrightarrow{*} b$  and  $a \rightarrow c' \xrightarrow{*} c$ . By local confluence of  $(S, \rightarrow)$ , there exists  $d \in S$  such that  $b' \xrightarrow{*} d$  and  $c' \xrightarrow{*} d$ . By induction hypothesis, there exists  $d' \in S$  such that  $b \xrightarrow{*} d'$  and  $d \xrightarrow{*} d'$ . By applying the induction hypothesis a second time, there exists  $d''$  such that  $c \xrightarrow{*} d''$  and  $d' \xrightarrow{*} d''$ , which allows to conclude that  $c \xrightarrow{*} d''$  and  $b \xrightarrow{*} d''$ .  $\square$

**2.1.15. Example.** Let us give an example of locally confluent abstract rewriting system which is not confluent. By Newman's Lemma, such an abstract rewriting system is not terminating. The following abstract rewriting system

$$a \leftarrow b \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} c \longrightarrow d$$

has two local branchings  $a \leftarrow b \rightarrow c$  and  $b \leftarrow c \rightarrow d$ . Both are confluent. However, the branching

$$a \leftarrow b \rightarrow c \rightarrow d$$

is not confluent.

## 2.2. POLYGRAPHS

In this section, we recall the definition of polygraphs. We also recall for any  $n \geq 1$  the presentation of an  $n$ -category by generators and relation by an  $(n + 1)$ -polygraph.

**2.2.1. Categorical notations.** We will denote by **Set** the category of sets. For  $n \in \mathbb{N}$ , we will denote by  $\mathbf{Cat}_n$  the category of small  $n$ -categories. All  $n$ -categories will be considered small in this document. An  $(n + 1)$ -category will then be a small category enriched in  $\mathbf{Cat}_n$ , that is a small category whose morphism spaces have a structure of  $n$ -category and whose composition is  $n$ -bifunctorial.

In an  $n$ -category, for any integer  $0 \leq k < n - 1$ , we will denote the  $k$ -composition in diagrammatic order by  $\star_k$ . We will also denote by  $1_u$  the  $(k + 1)$ -cell identity of  $u$  for any  $0 \leq k < n - 1$  and  $k$ -cell  $u$ . If  $u$  is a  $k$ -cell and  $\alpha$  is a  $(k + 1)$ -cell such that the composition  $1_u \star_j \alpha$  (respectively  $\alpha \star_j 1_u$ ) is defined for an integer  $0 \leq j \leq k$ , we use the notation  $u \star_j \alpha$  (respectively  $\alpha \star_j u$ ) for  $1_u \star_j \alpha$  (respectively  $\alpha \star_j 1_u$ ).

**2.2.2. Spheres.** A  $0$ -sphere in an  $n$ -category is a pair of  $0$ -cells. For any integer  $k > 0$ , a  $k$ -sphere in an  $n$ -category is a pair of parallel  $k$ -cells.

**2.2.3.  $n$ -graphs.** An  $n$ -graph in a category  $\mathbf{C}$  is a diagram in  $\mathbf{C}$

$$G_0 \begin{array}{c} \xleftarrow{s_0} \\ \xleftarrow{t_0} \end{array} G_1 \begin{array}{c} \xleftarrow{s_1} \\ \xleftarrow{t_1} \end{array} \cdots \begin{array}{c} \xleftarrow{s_{n-2}} \\ \xleftarrow{t_{n-2}} \end{array} G_{n-1} \begin{array}{c} \xleftarrow{s_{n-1}} \\ \xleftarrow{t_{n-1}} \end{array} G_n$$

such that for all  $1 \leq k \leq n - 1$ , we have  $s_{k-1} \circ s_k = s_{k-1} \circ t_k$  and  $t_{k-1} \circ s_k = t_{k-1} \circ t_k$ . These relations are called the *globular relations*. We just call  $n$ -graph an  $n$ -graph in **Set**. The maps  $s_k$  and  $t_k$  are respectively called  $k$ -source and  $k$ -target maps.

A *morphism of  $n$ -graphs*  $F$  from  $G$  to  $G'$  is a collection  $(F_k : G_k \rightarrow G'_k)$  of maps such that

for all  $0 < k \leq n$ , the following diagrams commute:

$$\begin{array}{ccc}
 & s_{k-1} & \\
 G_{k-1} & \longleftarrow & G_k \\
 \downarrow F_{k-1} & & \downarrow F_k \\
 G'_{k-1} & \xleftarrow{s'_{k-1}} & G'_k
 \end{array}
 \qquad
 \begin{array}{ccc}
 & t_{k-1} & \\
 G_{k-1} & \longleftarrow & G_k \\
 \downarrow F_{k-1} & & \downarrow F_k \\
 G'_{k-1} & \xleftarrow{t'_{k-1}} & G'_k
 \end{array}$$

We denote by  $\mathbf{Grph}_n$  the category of  $n$ -graphs. There exists forgetful functor  $\mathcal{U}_n : \mathbf{Cat}_n \rightarrow \mathbf{Grph}_n$ .

**2.2.4. Globular extensions.** The category  $\mathbf{Cat}_n^+$  the  $n$ -categories with a *globular extension* is defined by the following pullback diagram:

$$\begin{array}{ccc}
 \mathbf{Cat}_n^+ & \longrightarrow & \mathbf{Grph}_{n+1} \\
 \downarrow & \lrcorner & \downarrow \mathcal{U}_n^G \\
 \mathbf{Cat}_n & \xrightarrow{\mathcal{U}_n} & \mathbf{Grph}_n
 \end{array}$$

A globular extension of an  $n$ -category  $\mathcal{C}$  is thus a data made of a set  $\Gamma$  and of two applications  $s_n$  and  $t_n$  from  $\Gamma$  to  $\mathcal{C}_n$  making the diagram

$$\begin{array}{ccccccc}
 \mathcal{C}_0 & \xleftarrow{s_0} & \mathcal{C}_1 & \xleftarrow{s_1} & \cdots & \xleftarrow{s_{n-2}} & \mathcal{C}_{n-1} & \xleftarrow{s_{n-1}} & \mathcal{C}_n & \xleftarrow{s_n} & \Gamma \\
 & \xleftarrow{t_0} & & \xleftarrow{t_1} & & \xleftarrow{t_{n-2}} & & \xleftarrow{t_{n-1}} & & \xleftarrow{t_n} & 
 \end{array}$$

an  $n$ -graph. Consequently, there exists a forgetful functor from  $\mathbf{Cat}_{n+1}^+$  to  $\mathbf{Cat}_n^+$ . This functor has a left adjoint. This left adjoint, constructed in [?], is the free functor from  $\mathbf{Cat}_n^+$  to  $\mathbf{Cat}_{n+1}^+$  and we denote this functor by  $\mathcal{F}_{n+1}^W$ .

**2.2.5. The category of polygraphs.** We define the category  $\mathbf{Pol}_n$  of  $n$ -polygraphs and the *free functor*  $\mathcal{F}_n$  from  $\mathbf{Pol}_n$  to  $\mathbf{Cat}_n$  by induction on  $n$ . We define first  $\mathbf{Pol}_0 = \mathbf{Set}$  and define  $\mathcal{F}_0$  as the identity functor. Let us assume then that  $\mathbf{Pol}_n$  and  $\mathcal{F}_n$  are defined for some  $n$ . We define



then  $\mathbf{Pol}_{n+1}$  by the following pullback diagram:

$$\begin{array}{ccc}
 \mathbf{Pol}_{n+1} & \xrightarrow{\mathcal{U}_{n+1}^{\text{GP}}} & \mathbf{Grph}_{n+1} \\
 \mathcal{U}_n^{\text{P}} \downarrow \lrcorner & & \downarrow \mathcal{U}_n^{\text{G}} \\
 \mathbf{Pol}_n & \xrightarrow{\mathcal{F}_n} \mathbf{Cat}_n \xrightarrow{\mathcal{U}_n} & \mathbf{Grph}_n
 \end{array}$$

To define  $\mathcal{F}_{n+1}$ , we set first  $\mathcal{F}_{n+1}^{\text{P}}$  the unique functor making the following diagram commutative:

$$\begin{array}{ccccc}
 \mathbf{Pol}_{n+1} & & & & \\
 \mathcal{U}_n^{\text{P}} \downarrow & \searrow \mathcal{F}_{n+1}^{\text{P}} & & \searrow \mathcal{U}_{n+1}^{\text{GP}} & \\
 & & \mathbf{Cat}_n^+ & \xrightarrow{\quad} & \mathbf{Grph}_{n+1} \\
 & & \downarrow \lrcorner & & \downarrow \mathcal{U}_n^{\text{G}} \\
 \mathbf{Pol}_n & \xrightarrow{\mathcal{F}_n} & \mathbf{Cat}_n & \xrightarrow{\mathcal{U}_n} & \mathbf{Grph}_n
 \end{array}$$

The functor  $\mathcal{F}_{n+1}$  is then the composition

$$\mathbf{Pol}_{n+1} \xrightarrow{\mathcal{F}_{n+1}^{\text{P}}} \mathbf{Cat}_n^+ \xrightarrow{\mathcal{F}_{n+1}^{\text{W}}} \mathbf{Cat}_{n+1}$$

Given an  $n$ -polygraph  $\Sigma$ , we call *free  $n$ -category on  $\Sigma$*  the  $n$ -category  $\mathcal{F}_n(\Sigma)$ . We denote by  $\Sigma^*$  this  $n$ -category.

For  $n > 0$ , an  $n$ -polygraph can be defined as a data of an  $(n - 1)$ -polygraph  $\Pi$  and a globular extension of  $\Pi$ . For all  $n$ -polygraph  $\Sigma$  and all  $0 \leq k \leq n$ , we will denote by  $\Sigma_k$  the set of  $k$ -cells of  $\Sigma$ .

We now consider  $\Sigma$  an  $n$ -polygraph with  $n > 0$  for the rest of this section.

**2.2.6. Example.** A 3-polygraph is a data made of a generating set of 0-cells, 1-cells and 2-cells of a free 2-category and a globular extension of this 2-category. A set made of only one 0-cell, one 1-cell and two 2-cells represented by:

$$\cap, \quad \cup$$

generates a free 2-category. Let  $\mathcal{C}$  be this free 2-category. We denote the 2-cells by planar diagrams with an upper boundary and a lower boundary. The upper boundary corresponds to the 1-source of the 2-cell and the lower boundary corresponds to its 1-target. The 0-composition is represented by the horizontal concatenation and the 1-composition is represented by the vertical concatenation. An example of globular extension of  $\mathcal{C}$  is given by the 3-cell

$$\left| \begin{array}{c} \bigcirc \\ \Rightarrow \\ \bigcirc \end{array} \right|$$

**2.2.7. Rewriting steps.** A *rewriting step* of  $\Sigma$  is an  $n$ -cell of the  $n$ -category  $\Sigma^*$  of the form

$$u_1 \star_{n-2} \cdots (u_{n-1} \star_0 \alpha \star_0 u_n) \cdots \star_{n-2} u_{2n-2}$$

where  $\alpha$  is an  $n$ -cell of  $\Sigma$  and  $u_i$  is an  $(n-1)$ -cell of  $\Sigma^*$  for  $0 \leq i \leq 2n-2$ .

**2.2.8. Interpretation of  $n$ -polygraphs as rewriting systems.** The *rewriting system defined by  $\Sigma$*  is the abstract rewriting system  $(\Sigma_{n-1}^*, \rightarrow)$  where we have  $u \rightarrow v$  if and only if there exists a rewriting step from  $u$  to  $v$ . This abstract rewriting system will be confused with  $\Sigma$ .

**2.2.9. Remark.** In the  $n$ -category  $\Sigma_n^*$ , any  $n$ -cell can be written as an  $(n-1)$ -composition of rewriting steps. So, the relation  $\overset{*}{\rightarrow}$  of the system  $(\Sigma_{n-1}^*, \rightarrow)$  is defined by  $u \overset{*}{\rightarrow} v$  if and only if there exists an  $n$ -cell from  $u$  to  $v$  in  $\Sigma_n^*$ .

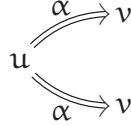
**2.2.10. Branchings.** A *branching* of  $\Sigma$  is a pair of  $n$ -cells of  $\Sigma_n^*$  with the same  $(n-1)$ -source. A *local branching* of  $\Sigma$  is a pair of rewriting steps of  $\Sigma_n$  with the same  $(n-1)$ -source. A branching  $(\alpha, \beta)$  is said to be *confluent* if there exists two  $n$ -cells  $\alpha'$  and  $\beta'$  of  $\Sigma_n^*$  such that  $\alpha \star_{n-1} \alpha'$  and  $\beta \star_{n-1} \beta'$  have the same  $(n-1)$ -target. We then say that the  $n$ -sphere  $(\alpha \star_{n-1} \alpha', \beta \star_{n-1} \beta')$  is a *confluence* of  $(\alpha, \beta)$ .

**2.2.11. Branchings of an abstract rewriting system.** An abstract rewriting system  $(S, \rightarrow)$  can be seen as a 1-polygraph whose 0-cells are the elements of  $S$  and whose 1-cells have source  $x$  and target  $y$  for each  $(x, y)$  in  $S \times S$  such that  $x \rightarrow y$ . The branchings of an abstract rewriting system are then defined as its branchings when seen as a 1-polygraph.

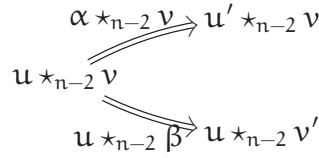
**2.2.12. Remark.** As an abstract rewriting system, the  $n$ -polygraph  $\Sigma$  is (locally) confluent if and only if all its (local) branchings are confluent.

**2.2.13. Classification of local branchings.** The local branchings of  $\Sigma$  can be classified into three families.

- i) An *aspherical* branching of  $\Sigma$  is a local branching of  $\Sigma$  made of two identical rewriting steps.



- ii) A *Peiffer* branching is a local branching of  $\Sigma$  of the form  $(\alpha \star_{n-2} v, u \star_{n-2} \beta)$  where  $u$  and  $v$  are  $(n-1)$ -cells and  $\alpha$  and  $\beta$  are rewriting steps.



- iii) An *overlapping* branching is a local branching of  $\Sigma$  which is neither aspherical nor Peiffer.

Let  $\sqsubseteq$  be the order relation on  $\Sigma_{n-1}^*$  defined by  $u \sqsubseteq v$  if there exists an integer  $0 \leq k \leq n-2$  and an  $(n-1)$ -cell  $w$  of  $\Sigma_{n-1}^*$  such that  $u \star_k w \sqsubseteq v$  or  $w \star_k u \sqsubseteq v$ . A *critical branching*, or *critical pair*, of  $\Sigma$  is an overlapping branching of  $\Sigma$  of source  $u$  such that no  $(n-1)$ -cell  $v$  of  $\Sigma_{n-1}^*$  verifying  $v \sqsubseteq u$  is the source of an overlapping branching.

**2.2.14. Lemma [?, 4.1.5.] (Critical pairs Lemma).** *An  $n$ -polygraph  $\Sigma$  is locally confluent if and only if all critical branchings of  $\Sigma$  are confluent.*

*Proof.* If  $\Sigma$  is locally confluent, then the critical branchings of  $\Sigma$  are confluent. Let us now assume that all critical branchings of  $\Sigma$  are confluent and let  $(\alpha, \beta)$  be a local branching of  $\Sigma$ .

If  $(\alpha, \beta)$  is an aspherical branching, we have  $s_{n-1}(\alpha \star_{n-1} 1_{s_{n-1}(\alpha)}) = s_{n-1}(\beta \star_{n-1} 1_{s_{n-1}(\beta)})$ . The branching  $(\alpha, \beta)$  is then confluent.

If  $(\alpha, \beta)$  is a Peiffer branching, we can assume without loss of generality that there exists two rewriting steps  $\alpha'$  and  $\beta'$  and two  $(n-1)$ -cells  $u$  and  $v$  such that  $\alpha = \alpha' \star_{n-2} v$  and  $\beta = u \star_{n-2} \beta'$ . We have then  $(\alpha' \star_{n-2} v) \star_{n-1} (u \star_{n-2} \beta') = \alpha' \star_{n-2} \beta' = (u \star_{n-2} \beta') \star_{n-1} (\alpha' \star_{n-2} v)$ . The branching  $(\alpha, \beta)$  is then confluent.

If  $(\alpha, \beta)$  is an overlapping branching, there exists a critical branching  $(\alpha', \beta')$  and two families  $(u_k)_{k \in \llbracket 1, 2n \rrbracket}$  and  $(v_k)_{k \in \llbracket 1, 2n \rrbracket}$  of  $(n-1)$ -cells of  $\Sigma_n^*$  such that

$$\alpha = u_1 \star_{n-1} (u_2 \star_{n-2} (\cdots u_n \star_0 \alpha' \star_0 u_n) \cdots \star_{n-2} u_{2n-1}) \star_{n-1} u_{2n},$$

$$\beta = u_1 \star_{n-1} (u_2 \star_{n-2} (\cdots u_n \star_0 \beta' \star_0 u_n) \cdots \star_{n-2} u_{2n-1}) \star_{n-1} u_{2n}.$$

Let  $\alpha''$  and  $\beta''$  be two  $n$ -cells of  $\Sigma_n^*$  such that  $t_{n-1}(\alpha' \star_{n-1} \alpha'') = t_{n-1}(\beta' \star_{n-1} \beta'')$ . We have then

$$\begin{aligned} & t_{n-1}(u_1 \star_{n-1} (u_2 \star_{n-2} (\cdots u_n \star_0 \alpha'' \star_0 u_n) \cdots \star_{n-2} u_{2n-1}) \star_{n-1} u_{2n}) \\ &= t_{n-1}(u_1 \star_{n-1} (u_2 \star_{n-2} (\cdots u_n \star_0 \beta'' \star_0 u_n) \cdots \star_{n-2} u_{2n-1}) \star_{n-1} u_{2n}). \end{aligned}$$

Thus, the branching  $(\alpha, \beta)$  is confluent. This proves that  $\Sigma$  is locally confluent.  $\square$

## 2.3. COHERENCE

In this section, we define the notion of coherence which will be used in Chapter 3. This notion is based on  $(n, p)$ -polygraphs.

**2.3.1.  $(n, p)$ -categories.** Let  $p \leq n$ . An  $(n, p)$ -category is an  $n$ -category having all its  $k$ -cells invertible for any  $k > p$ . The category of  $(n, p)$ -categories will be denoted by  $\mathbf{Cat}_{n,p}$ . The forgetful functor from  $\mathbf{Cat}_{n,p}$  to  $\mathbf{Grph}_n$  will be denoted by  $\mathcal{U}_{n,p}$ . The category  $\mathbf{Cat}_{n,p}^+$  of  $(n, p)$ -categories with a globular extension will be defined by the following pullback diagram:

$$\begin{array}{ccc} \mathbf{Cat}_{n,p}^+ & \longrightarrow & \mathbf{Grph}_{n+1} \\ \downarrow & \lrcorner & \downarrow \mathcal{U}_n^G \\ \mathbf{Cat}_{n,p} & \xrightarrow{\mathcal{U}_n} & \mathbf{Grph}_n \end{array}$$

The free functor from  $\mathbf{Cat}_{n,p}^+$  to  $\mathbf{Cat}_{n,p}$ , restriction of the functor  $\mathcal{F}_{n+1}^W$ , will finally be denoted by  $\mathcal{F}_{n+1,p}^W$ .

**2.3.2.  $(n, p)$ -polygraphs.** We define the category  $\mathbf{Pol}_{n,p}$  of  $(n, p)$ -polygraphs and the free functor  $\mathcal{F}_{n,p}$  from  $\mathbf{Pol}_n$  to  $\mathbf{Cat}_n$  by induction on  $n \geq p$ . We define first  $\mathbf{Pol}_{n,n} = \mathbf{Pol}_n$  and  $\mathcal{F}_{n,n} = \mathcal{F}_n$ . Let us now assume that  $\mathbf{Pol}_{n,p}$  and  $\mathcal{F}_{n,p}$  are defined for a certain pair of oriented

integers  $(n, p)$  verifying  $n \geq p$ . We define then  $\mathbf{Pol}_{n+1,p}$  by the following pullback diagram:

$$\begin{array}{ccc}
 \mathbf{Pol}_{n+1,p} & \xrightarrow{\mathcal{U}_{n+1,p}^{\text{GP}}} & \mathbf{Grph}_{n+1} \\
 \mathcal{U}_{n,p}^{\text{P}} \downarrow \lrcorner & & \downarrow \mathcal{U}_n^{\text{G}} \\
 \mathbf{Pol}_{n,p} & \xrightarrow{\mathcal{F}_{n,p}} \mathbf{Cat}_{n,p} \xrightarrow{\mathcal{U}_{n,p}} & \mathbf{Grph}_n
 \end{array}$$

To define  $\mathcal{F}_{n+1,p}$ , we set first  $\mathcal{F}_{n+1,p}^{\text{P}}$  the unique functor making the following diagram commutative:

$$\begin{array}{ccccc}
 \mathbf{Pol}_{n+1,p} & & & \xrightarrow{\mathcal{U}_{n+1,p}^{\text{GP}}} & \mathbf{Grph}_{n+1} \\
 & \searrow \mathcal{F}_{n+1,p}^{\text{P}} & & & \downarrow \mathcal{U}_n^{\text{G}} \\
 \mathbf{Pol}_{n+1,p} & & \mathbf{Cat}_{n,p}^+ & \xrightarrow{\quad} & \mathbf{Grph}_{n+1} \\
 \mathcal{U}_{n,p}^{\text{P}} \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \mathcal{U}_n^{\text{G}} \\
 \mathbf{Pol}_{n,p} & \xrightarrow{\mathcal{F}_{n,p}} \mathbf{Cat}_{n,p} \xrightarrow{\mathcal{U}_{n,p}} & \mathbf{Grph}_n & & 
 \end{array}$$

The functor  $\mathcal{F}_{n+1,p}$  is then the composition

$$\mathbf{Pol}_{n+1,p} \xrightarrow{\mathcal{F}_{n+1,p}^{\text{P}}} \mathbf{Cat}_{n,p}^+ \xrightarrow{\mathcal{F}_{n+1,p}^{\text{W}}} \mathbf{Cat}_{n+1,p}$$

Given an  $(n, p)$ -polygraph  $\Sigma$ , we call *free  $(n, p)$ -category over  $\Sigma$*  the  $(n, p)$ -category  $\mathcal{F}_{n,p}(\Sigma)$ . The functor  $\mathcal{F}_{n,p}$  is called the *free functor* from  $\mathbf{Pol}_{n,p}$  to  $\mathbf{Cat}_{n,p}$ . The fact that this functor is free is proven in [?]. We denote by  $\Sigma^\top$  this  $(n, p)$ -category.

For  $n > p$ , an  $(n, p)$ -polygraph can be defined as a data made of an  $(n - 1, p)$ -polygraph  $\Pi$  and a globular extension of  $\Pi$ . Any  $(n, p)$ -polygraph is in particulier an  $n$ -polygraph.

We now fix  $\mathcal{C}$  an  $n$ -category for the rest of this section.

**2.3.3. Homotopy bases.** A *homotopy basis* of  $\mathcal{C}$  is a globular extension  $\Gamma$  of  $\mathcal{C}$  such that for any ordered pair  $(\alpha, \beta)$  of parallel  $n$ -cells of  $\mathcal{C}$ , there exists an  $(n + 1)$ -cell from  $\alpha$  to  $\beta$  in the free  $(n + 1)$ -category over  $(\mathcal{C}, \Gamma)$ .

**2.3.4. Presentation of an  $n$ -category.** Let  $\Sigma$  be an  $(n + 1)$ -polygraph. We say that  $\Sigma$  is a *presentation* of  $\mathcal{C}$ , or that  $\Sigma$  *presents*  $\mathcal{C}$ , if the quotient of the  $n$ -category  $\Sigma_n^*$  by the globular extension  $\Sigma_{n+1}$  is isomorphic to  $\mathcal{C}$ , by this, we mean the quotient of the set  $\Sigma_n^*$  by the equivalence relation spanned by  $\Sigma_{n+1}$ .

**2.3.5. Coherent presentation of an  $n$ -category.** Let  $\Sigma$  be an  $(n + 2, n)$ -polygraph. We say that  $\Sigma$  is a *coherent presentation* of  $\mathcal{C}$ , if the underlying  $(n + 1)$ -polygraph of  $\Sigma$  presents  $\mathcal{C}$  and the globular extension  $\Sigma_{n+2}$  is a homotopy basis of the  $(n + 1, n)$ -category  $\Sigma_{n+1}^\top$ .

**2.3.6. Squier's completion.** Let  $\Sigma$  be an  $(n + 1)$ -polygraph. A *Squier's completion* of  $\Sigma$  is an  $(n + 2, n)$ -polygraph  $\mathcal{S}(\Sigma)$  defined by

- $\mathcal{S}(\Sigma)$  has the same  $k$ -cells than  $\Sigma$  for  $0 \leq k \leq n + 1$ ,
- for each critical branching  $(\alpha, \beta)$  of  $\Sigma$ , there exists a unique confluence  $(\alpha \star_n \alpha', \beta \star_n \beta')$  of  $(\alpha, \beta)$  and a unique  $(n + 2)$ -cell  $A$  from  $\alpha \star_n \alpha'$  to  $\beta \star_n \beta'$  such that  $A \in \Sigma_{n+2}$ .

We recall Squier's Theorem for  $(n + 1)$ -polygraphs.

**2.3.7. Theorem [?, Theorem 5.2].** *Let  $\Sigma$  be an  $(n + 1)$ -convergent polygraph presenting an  $n$ -category  $\mathcal{C}$ . Let  $\mathcal{S}(\Sigma)$  be a Squier's completion of  $\mathcal{C}$ . The  $(n + 2, n)$ -polygraph  $\mathcal{S}(\Sigma)$  is then a coherent presentation of  $\mathcal{C}$ .*

*Proof.* We prove this theorem in three steps.

**Step 1.** Let  $(\alpha, \beta)$  be a local branching of  $\Sigma$ . We prove that there exists a confluence  $(\alpha \star_n \alpha', \beta \star_n \beta')$  of  $(\alpha, \beta)$  such that  $\mathcal{S}(\Sigma)^\top$  contains an  $(n + 2)$ -cell from  $\alpha \star_n \alpha'$  to  $\beta \star_n \beta'$ .

If  $(\alpha, \beta)$  is an aspherical branching, the branching  $(\alpha, \beta)$  is an  $n$ -sphere and the wanted  $(n + 2)$ -cell is an identity.

If  $(\alpha, \beta)$  is a Peiffer branching, we can write  $(\alpha, \beta) = (\alpha' \star_{n-1} v, u \star_{n-1} \beta')$  where  $u$  and  $v$  are  $n$ -cells and  $\alpha$  and  $\beta$  are rewriting steps. The  $n$ -sphere  $(\alpha' \star_{n-1} v, \alpha' \star_{n-1} \beta')$  is then a confluence of  $(\alpha, \beta)$  and the wanted  $(n + 2)$ -cell is an identity.

If  $(\alpha, \beta)$  is an overlapping branching, we can write

$$\alpha = u_1 \star_{n-1} (u_2 \star_{n-2} (\cdots u_n \star_0 \alpha' \star_0 u_{n+1}) \cdots \star_1 u_{2n+1}) \star_{n-1} u_{2n},$$

$$\beta = u_1 \star_{n-1} (u_2 \star_{n-2} (\cdots u_n \star_0 \beta' \star_0 u_{n+1}) \cdots \star_1 u_{2n+1}) \star_{n-1} u_{2n}$$

where  $(\alpha', \beta')$  is a critical branching. Let  $(\alpha' \star_n \alpha'', \beta' \star_n \beta'')$  be the confluence of  $(\alpha', \beta')$  such that the  $(n + 2, n)$ -polygraph  $\mathcal{S}(\Sigma)$  contains an  $(n + 2)$ -cell  $A$  from  $\alpha' \star_n \alpha''$  to  $\beta' \star_n \beta''$  or from  $\beta' \star_n \beta''$  to  $\alpha' \star_n \alpha''$ . The  $(n + 2)$ -cell

$$u_1 \star_{n-1} (u_2 \star_{n-2} (\cdots u_n \star_0 A \star_0 u_{n+1}) \cdots \star_1 u_{2n+1}) \star_{n-1} u_{2n}$$

or

$$u_1 \star_{n-1} (u_2 \star_{n-2} (\cdots u_n \star_0 A^{-1} \star_0 u_{n+1}) \cdots \star_1 u_{2n+1}) \star_{n-1} u_{2n}$$

is thus an  $(n+2)$ -cell from  $\alpha$  to  $\beta$  in  $\mathcal{S}(\Sigma)^\top$ .

**Step 2.** Let  $(\alpha, \beta)$  be an  $(n+1)$ -sphere of  $\Sigma^\top$ . We prove by noetherian induction that there exists an  $(n+2)$ -cell from  $\alpha$  to  $\beta$ . If the source of  $(\alpha, \beta)$  is a normal form, we have  $\alpha = \beta$  and the wanted  $(n+2)$  cell is an identity. If the source of  $(\alpha, \beta)$  is not a normal form and that  $\alpha$  and  $\beta$  are not identities, we can consider two decompositions  $\alpha = \alpha_1 \star_n \alpha'$  and  $\beta = \beta_1 \star_n \beta'$  where  $\alpha_1$  and  $\beta_1$  are rewriting steps. By step 1, there exists then a confluence  $(\alpha_1 \star_n \alpha_2, \beta_1 \star_n \beta_2)$  of  $(\alpha_1, \beta_1)$  such that  $\mathcal{S}(\Sigma)^\top$  contains an  $(n+2)$ -cell  $A$  from  $\alpha_1 \star_n \alpha_2$  to  $\beta_1 \star_n \beta_2$ . Let  $u$  be the target of  $(\alpha_1 \star_n \alpha_2, \beta_1 \star_n \beta_2)$  and  $u'$  be the target of  $(\alpha, \beta)$ . Let  $\bar{u}$  be the common normal form of  $u$  and  $u'$ . Let  $\gamma$  be an  $(n+1)$ -cell from  $u$  to  $\bar{u}$  and  $\gamma'$  be an  $(n+1)$ -cell of  $u'$  to  $\bar{u}$ . By induction hypothesis,  $\mathcal{S}(\Sigma)^\top$  contains an  $(n+2)$ -cell  $B'$  from  $\alpha' \star_n \gamma'$  to  $\alpha_2 \star_n \gamma$  and an  $(n+2)$ -cell  $C'$  from  $\beta_2 \star_n \gamma$  to  $\beta' \star_n \gamma'$ . Let us pose  $B = B' \star_n \gamma'^{-1}$  and  $C = C' \star_n \gamma'^{-1}$ . The  $(n+2)$ -cell  $(\alpha_1 \star_n B) \star_{n+1} (A \star_n \gamma'^{-1}) \star_{n+1} (\beta_1 \star_n C)$  of  $\mathcal{S}(\Sigma)^\top$  is thus an  $(n+2)$ -cell of source  $\alpha$  and of target  $\beta$ .

**Step 3.** Let  $(\alpha, \beta)$  be an  $(n+1)$ -sphere of  $\mathcal{S}(\Sigma)^\top$ . We prove that there exists an  $(n+2)$ -cell from  $\alpha$  to  $\beta$ . One can write for a certain odd integer  $k$

$$\begin{aligned} \alpha &= \alpha_1 \star_n \alpha_2^{-1} \star_n \cdots \star_n \alpha_{k-1}^{-1} \star_n \alpha_k, \\ \beta &= \beta_1 \star_n \beta_2^{-1} \star_n \cdots \star_n \beta_{k-1}^{-1} \star_n \beta_k \end{aligned}$$

where  $\alpha_i$  and  $\beta_i$  are  $(n+1)$ -cells, possibly identities, of  $\Sigma^\top$  for tout  $0 \leq i \leq k$ . By convergence of  $\Sigma$ , there exists a normal form  $u$  of  $\Sigma$  and a family  $(\sigma_i)_{i \in \llbracket 1, k+1 \rrbracket}$  of  $(n+1)$ -cells of  $\Sigma^*$  such that

- $s_n(\sigma_i) = s_n(\alpha_i)$  for for all odd  $i$  of  $\llbracket 1, k+1 \rrbracket$ ,
- $s_n(\sigma_i) = t_n(\alpha_i)$  for for all even  $i$  of  $\llbracket 1, k+1 \rrbracket$ ,
- $t_n(\sigma_{k+1}) = u$  for tout  $i \in \llbracket 1, k+1 \rrbracket$ .

By step 2,  $\mathcal{S}(\Sigma)^\top$  contains a family  $(A_i)_{i \in \llbracket 1, k \rrbracket}$  of  $(n+2)$ -cells of  $\mathcal{S}(\Sigma)^\top$  such that

- $s_{n+1}(A_i) = \alpha_i$  for for all odd  $i$  of  $\llbracket 1, k+1 \rrbracket$ ,
- $s_{n+1}(A_i) = \alpha_i^{-1}$  for for all even  $i$  of  $\llbracket 1, k+1 \rrbracket$ ,
- $t_{n+1}(A_i) = \sigma_i \star_n \sigma_{i+1}^{-1}$  for tout  $i \in \llbracket 1, k+1 \rrbracket$ .

The composition

$$A = (A_1 \star_n \alpha_2^{-1} \star_n \cdots \star_n \alpha_k) \star_{n+1} \cdots (\sigma_1 \star_n \sigma_k^{-1} \star_n A_k)$$

is then an  $(n+2)$ -cell from  $\alpha$  to  $\sigma_1 \star_n \sigma_{k+1}^{-1}$ . Moreover, the convergence of  $\Sigma$  implies the existence of a family  $(\gamma_i)_{i \in \llbracket 1, k+1 \rrbracket}$  of  $(n+1)$ -cells of  $\Sigma^*$  such that

- $\gamma_1 = \sigma_1$  and  $\gamma_{k+1} = \sigma_{k+1}$ ,
- $s_n(\gamma_i) = s_n(\beta_i)$  for for all odd  $i$  of  $\llbracket 1, k+1 \rrbracket$ ,
- $s_n(\gamma_i) = t_n(\beta_i)$  for for all even  $i$  of  $\llbracket 1, k+1 \rrbracket$ ,
- $t_n(\gamma_{k+1}) = u$  for tout  $i \in \llbracket 1, k+1 \rrbracket$ .

By step 2,  $\mathcal{S}(\Sigma)^\top$  contains a family  $(B_i)_{i \in \llbracket 1, k \rrbracket}$  of  $(n+2)$ -cells of  $\mathcal{S}(\Sigma)^\top$  such that

- $s_{n+1}(B_i) = \gamma_i \star_n \gamma_{i+1}^{-1}$  for tout  $i \in \llbracket 1, k+1 \rrbracket$ ,
- $t_{n+1}(B_i) = \beta_i$  for for all odd  $i$  of  $\llbracket 1, k+1 \rrbracket$ ,
- $t_{n+1}(B_i) = \beta_i^{-1}$  for for all even  $i$  of  $\llbracket 1, k+1 \rrbracket$ .

The composition

$$B = (B_1 \star_n \gamma_2 \star_n \gamma_{k+1}^{-1}) \star_{n+1} \cdots (\beta_1 \star_n \beta_2^{-1} \star_n \cdots \star_n \beta_{k-1}^{-1} \star_n B_k)$$

is then an  $(n+2)$ -cell of  $\sigma_1 \star_n \sigma_{k+1}^{-1}$  to  $\beta$ . The  $(n+2, n)$ -category  $\mathcal{S}(\Sigma)^\top$  contains then the  $(n+2)$ -cell  $A \star_{n+1} B$  from  $\alpha$  to  $\beta$ . This concludes the proof.  $\square$





## Higher-dimensional linear rewriting

As seen in Chapter 2, the structure of polygraph is a model for higher-dimensional rewriting. This model can be used to study the relations in a higher-dimensional category. The rewriting paths arising from an  $(n + 1)$ -polygraph are the  $(n + 1)$ -cells of a free  $(n + 1)$ -category. The congruences arising from an  $(n + 1)$ -polygraph are the  $(n + 1)$ -cells of a free  $(n + 1, n - 1)$ -category. Those higher-dimensional cells give us relations between relations, also called syzygies, and are used to compute coherent presentations of higher-dimensional categories. A coherent presentation of a monoid is the first step to construct a resolution of this monoid. Finding a linear structure for polygraphs would enable us to find polygraphic resolutions for algebras.

A linear version of 2-polygraphs was defined in [?]. In this chapter, we present the notion of linear polygraph we introduced in [?], which is slightly more general than the one in [?]. Adding a linear structure to polygraphs creates multiple rewriting problems which do not appear in the set-theoretic setting. First, rewriting steps of linear polygraphs are only defined when the sources of the rewriting rules are monomial. The local branchings of linear polygraphs are also more complex. They belong to four families: aspherical branchings, Peiffer branchings, additive branchings and overlapping branchings. Contrarily to the set-theoretical case, local branchings other than overlapping branchings are not always confluent. As a consequence, as pointed out in [?] the critical pairs Lemma for linear polygraphs needs more conditions than in the set-theoretical case.

**Plan of Chapter.** We begin in Section ?? by defining linear  $(n, p)$ -categories. Then, we introduce linear  $(n, p)$ -polygraphs. The main rewriting properties of linear  $(n, p)$ -polygraphs are also given. Finally, in Section ??, we present a criterion allowing to give bases for the  $n$ -cells spaces of an  $(n, n)$ -linear category using some linear  $(n + 1, n)$ -polygraphs.

### 3.1. LINEAR POLYGRAPHS

**3.1.1. Internal  $n$ -categories.** Let  $\mathbf{C}$  be a category with pullbacks. An  $n$ -category internal to  $\mathbf{C}$  is a data made of an  $n$ -graph in  $\mathbf{C}$

$$G_0 \begin{array}{c} \xleftarrow{s_0} \\ \xleftarrow{t_0} \end{array} G_1 \begin{array}{c} \xleftarrow{s_1} \\ \xleftarrow{t_1} \end{array} \cdots \begin{array}{c} \xleftarrow{s_{n-2}} \\ \xleftarrow{t_{n-2}} \end{array} G_{n-1} \begin{array}{c} \xleftarrow{s_{n-1}} \\ \xleftarrow{t_{n-1}} \end{array} G_n$$

and an identity morphism  $x \mapsto 1_x$  from  $G_i$  to  $G_{i+1}$  for all  $i \in \llbracket 0, n-1 \rrbracket$ , and a composition morphism  $\star_k$  of  $G_k \times_{G_l} G_k$  for all  $k \in \llbracket 0, n \rrbracket$  and  $l \in \llbracket 0, n-1 \rrbracket$  such that the identity and composition morphisms verify the axioms of  $n$ -category.

**3.1.2. Linear  $(n, p)$ -categories.** Let us fix  $k$  a commutative ring and let  $\mathbf{kMod}$  be the category of  $k$ -modules. We define the category  $\mathbf{LinCat}_{n,p}$  of *linear  $(n, p)$ -categories* by induction on  $n \geq p$ .

The category  $\mathbf{LinCat}_{n,0}$  is the category of  $n$ -categories internal to  $\mathbf{kMod}$ .

Let us assume that the category  $\mathbf{LinCat}_{n,p}$  is defined for two integers  $n$  and  $p$  verifying  $n \geq p$ . Then, an object of  $\mathbf{LinCat}_{n,p}$  is defined by a data made of a set  $\mathcal{C}_0$  and

- for all  $a$  and  $b$  of  $\mathcal{C}_0$ , a linear  $(n, p)$ -category  $\mathcal{C}(a, b)$ ,
- for all  $a$  of  $\mathcal{C}_0$ , an identity morphism  $i_a$  from the terminal  $n$ -category  $I_n$  to  $\mathcal{C}(a, a)$ ,
- for all  $a, b$  and  $c$  in  $\mathcal{C}_0$ , a composition morphism  $\star^{a,b,c}$  from  $\mathcal{C}(a, b) \times \mathcal{C}(b, c)$  to  $\mathcal{C}(a, c)$  which is bilinear in each morphisms space of  $\mathcal{C}(a, b) \times \mathcal{C}(b, c)$  internal to  $\mathbf{kMod}$ .

such that

- $\star^{a,c,d} \circ (\star^{a,b,c} \times \text{id}_{\mathcal{C}(c,d)}) = \star^{a,b,d} \circ (\text{id}_{\mathcal{C}(a,b)} \times \star^{b,c,d})$ ,
- $\star^{a,a,b} \circ (i_a \times \text{id}_{\mathcal{C}(a,b)}) \circ \text{is}_l = \text{id}_{\mathcal{C}(a,b)} = \star^{a,b,b} \circ (\text{id}_{\mathcal{C}(a,b)} \times i_q) \circ \text{is}_r$  where  $\text{is}_l$  and  $\text{is}_r$  respectively correspond to the canonical isomorphism from  $\mathcal{C}(a, b)$  to  $I_n \times \mathcal{C}(a, b)$  and from  $\mathcal{C}(a, b)$  to  $\mathcal{C}(a, b) \times I_n$ .

In particular, a linear  $(n+1, p)$ -category has a structure of  $(n+1)$ -category. A morphism in  $\mathbf{LinCat}_{n+1,p}$  is then an  $(n+1)$ -functor which is linear in each internal part to  $\mathbf{kMod}$ .

The category  $\mathbf{LinCat}_{n,p}$  of linear  $(n, p)$ -categories with a globular extension is defined by the following pullback diagram:

$$\begin{array}{ccc}
 \mathbf{LinCat}_{n,p}^+ & \longrightarrow & \mathbf{Grph}_{n+1} \\
 \downarrow & \lrcorner & \downarrow \mathcal{U}_n^G \\
 \mathbf{LinCat}_{n,p} & \xrightarrow{\mathcal{U}_{n,p}^\ell} & \mathbf{Grph}_n
 \end{array}$$

where  $\mathcal{U}_{n,p}^\ell$  is the forgetful functor from  $\mathbf{LinCat}_{n,p}$  to  $\mathbf{Grph}_n$ .

**3.1.3. Notation.** For  $p \leq k < n$ , if  $u$  is a  $k$ -cell of a linear  $(n, p)$ -category  $\mathcal{C}$  and  $\alpha$  is a  $(k+1)$ -cell of  $\mathcal{C}$ , we denote by  $\alpha + u$  the  $(n+1)$ -cell  $\alpha + 1_u$ .

**3.1.4. Free construction of linear  $(n, p)$ -categories.** Let us define the free functor  $\mathcal{F}_{n,p}^c$  from  $\mathbf{Cat}_n$  to  $\mathbf{LinCat}_{n,p}$ . Let  $\mathcal{C}$  be an  $n$ -category. We define first  $\mathcal{F}_{n,0}^c(\mathcal{C})$  to be the linear  $(n, 0)$ -category such that for any  $0 \leq k \leq n$ , the module  $\mathcal{F}_{n,0}^c(\mathcal{C})_k$  is the free module over  $\mathcal{C}_k$ . Let us now assume that  $p \neq 0$ . We define  $\mathcal{F}_{n,p}^c(\mathcal{C})$  to be the linear  $(n, p)$ -category such that:

- for any  $0 \leq k < p$ , the linear  $(n, p)$ -category  $\mathcal{F}_{n,p}^c(\mathcal{C})$  has the same  $k$ -cells than  $\mathcal{C}$ ,
- for any parallel  $(p-1)$ -cells  $a$  and  $b$  of  $\mathcal{C}$  and any  $p \leq k < n$ , the module  $\mathcal{F}_{n,p}^c(\mathcal{C})_k(a, b)$  is the free module over  $\mathcal{C}_k(a, b)$ .

The compositions of  $\mathcal{F}_{n,p}^c(\mathcal{C})$  are defined by:

- for any  $0 \leq k < n$ , the compositions of  $k$ -cells of  $\mathcal{C}$  remain unchanged,
- for any  $0 \leq k < p$ , the composition  $\star_k$  is bilinear.
- for any parallel  $(p-1)$ -cells  $a$  and  $b$  of  $\mathcal{C}$ , for any  $p \leq i < n$ , any  $i < j \leq n$ , any scalars  $\lambda$  and  $\mu$ , any  $i$ -composable  $j$ -cells  $f$  and  $f'$  of  $\mathcal{C}_j(a, b)$  and any  $i$ -composable  $j$ -cells  $g$  and  $g'$  of  $\mathcal{C}_j(a, b)$ , we have

$$(\lambda f + \mu g) \star_i (\lambda f' + \mu g') = \lambda(f \star_i f') + \mu(g \star_i g').$$

The composition  $\star_i$  is linear on  $\mathcal{C}_j(a, b) \times_{\mathcal{C}_i} \mathcal{C}_j(a, b)$ .

Note that the compositions still verify the exchange relations. Indeed, for any  $0 \leq i < p - 1$  and any  $i < j \leq p - 1$ , we have

$$\begin{aligned}
 & \left( \left( \sum_{x \in X} \lambda_x f_x \right) \star_i \left( \sum_{y \in Y} \mu_y g_y \right) \right) \star_j \left( \left( \sum_{x' \in X'} \lambda'_{x'} f'_{x'} \right) \star_i \left( \sum_{y' \in Y'} \mu'_{y'} g'_{y'} \right) \right) \\
 &= \left( \sum_{x \in X, y \in Y} \lambda_x \mu_y f_x \star_i g_y \right) \star_j \left( \sum_{x' \in X', y' \in Y'} \lambda'_{x'} \mu'_{y'} f'_{x'} \star_i g'_{y'} \right) \\
 &= \sum_{x \in X, y \in Y, x' \in X', y' \in Y'} \lambda_x \mu_y \lambda'_{x'} \mu'_{y'} (f_x \star_i g_y) \star_j (f'_{x'} \star_i g'_{y'}) \\
 &= \sum_{x \in X, y \in Y, x' \in X', y' \in Y'} \lambda_x \mu_y \lambda'_{x'} \mu'_{y'} (f_x \star_j f'_{x'}) \star_i (g_y \star_j g'_{y'}) \\
 &= \left( \left( \sum_{x \in X} \lambda_x f_x \right) \star_j \left( \sum_{x' \in X'} \lambda'_{x'} f'_{x'} \right) \right) \star_i \left( \left( \sum_{y \in Y} \mu_y g_y \right) \star_j \left( \sum_{y' \in Y'} \mu'_{y'} g'_{y'} \right) \right).
 \end{aligned}$$

For any  $0 \leq i < p - 1$  and any  $p - 1 < j \leq n - 1$ , we have

$$\begin{aligned}
 & \left( \left( \sum_{x \in X} \lambda_x f_x \right) \star_i \left( \sum_{y \in Y} \mu_y g_y \right) \right) \star_j \left( \left( \sum_{x \in X} \lambda_x f'_x \right) \star_i \left( \sum_{y \in Y} \mu_y g'_y \right) \right) \\
 &= \left( \sum_{x \in X, y \in Y} \lambda_x \mu_y f_x \star_i g_y \right) \star_j \left( \sum_{x \in X, y \in Y} \lambda_x \mu_y f'_x \star_i g'_y \right) \\
 &= \sum_{x \in X, y \in Y} \lambda_x \mu_y (f_x \star_i g_y) \star_j (f'_x \star_i g'_y) \\
 &= \sum_{x \in X, y \in Y} \lambda_x \mu_y (f_x \star_j f'_x) \star_i (g_y \star_j g'_y) \\
 &= \left( \left( \sum_{x \in X} \lambda_x f_x \right) \star_j \left( \sum_{x \in X} \lambda_x f'_x \right) \right) \star_i \left( \left( \sum_{y \in Y} \mu_y g_y \right) \star_j \left( \sum_{y \in Y} \mu_y g'_y \right) \right).
 \end{aligned}$$

For any  $p - 1 < i < n - 1$  and any  $i < j \leq n - 1$ , we have

$$\begin{aligned}
 & \left( \left( \sum_{x \in X} \lambda_x f_x \right) \star_i \left( \sum_{x \in X} \lambda_x g_x \right) \right) \star_j \left( \left( \sum_{x \in X} \lambda_x f'_x \right) \star_i \left( \sum_{x \in X} \lambda_x g'_x \right) \right) \\
 &= \left( \sum_{x \in X} \lambda_x (f_x \star_i g_x) \right) \star_j \left( \sum_{x \in X} \lambda_x (f'_x \star_i g'_x) \right) \\
 &= \sum_{x \in X} \lambda_x (f_x \star_i g_x) \star_j (f'_x \star_i g'_x) \\
 &= \left( \left( \sum_{x \in X} \lambda_x f_x \right) \star_j \left( \sum_{x \in X} \lambda_x f'_x \right) \right) \star_i \left( \left( \sum_{x \in X} \lambda_x g_x \right) \star_j \left( \sum_{x \in X} \lambda_x g'_x \right) \right).
 \end{aligned}$$

The functor  $\mathcal{F}_{n,p}^c$  extends all  $n$ -functors between  $n$ -categories by linearity into morphisms of linear  $(n, p)$ -categories.

**3.1.5. Remark.** Each linear  $(n, p)$ -category has a structure of  $(n, p)$ -category. Let  $p < k \leq n$  and let  $\alpha$  be a  $k$ -cell of a linear  $(n, p)$ -category  $\mathcal{C}$ . The  $k$ -cell  $\alpha$  is then invertible for the  $(k-1)$ -composition and its inverse is  $s_{k-1}(\alpha) + t_{k-1}(\alpha) - \alpha$ .

**3.1.6. Linear  $(n, p)$ -polygraphs.** We define the category  $\mathbf{LinPol}_{n,p}$  of *linear  $(n, p)$ -polygraphs* and the free functor  $\mathcal{F}_{n,p}^\ell$  from  $\mathbf{LinPol}_{n,p}$  to  $\mathbf{LinCat}_{n,p}$  by induction on  $n \geq p$ . We define first  $\mathbf{LinPol}_{n,n} = \mathbf{Pol}_n$  and  $\mathcal{F}_{n,n}^\ell$  as the composition of the functor  $\mathcal{F}_n$  defined in ?? with the free functor  $\mathcal{F}_{n,n}^c$  from  $\mathbf{Cat}_n$  to  $\mathbf{LinCat}_{n,n}$ .

$$\mathbf{LinPol}_{n,n} \xrightarrow{\mathcal{F}_{n,n}^\ell} \mathbf{Cat}_n \xrightarrow{\mathcal{F}_{n,n}^c} \mathbf{LinCat}_{n,n}$$

Let us assume then that  $\mathbf{LinPol}_{n,p}$  and  $\mathcal{F}_{n,p}^\ell$  are defined for an ordered pair of integers  $(n, p)$  verifying  $n \geq p$ . We define then  $\mathbf{LinPol}_{n+1,p}$  by the following pullback diagram:

$$\begin{array}{ccc} \mathbf{LinPol}_{n+1,p} & \xrightarrow{\mathcal{U}_{n+1,p}^{\text{GP}^\ell}} & \mathbf{Grph}_{n+1} \\ \mathcal{U}_{n,p}^{\text{P}^\ell} \downarrow & \lrcorner & \downarrow \mathcal{U}_n^{\text{G}} \\ \mathbf{LinPol}_{n,p} & \xrightarrow{\mathcal{F}_{n,p}^\ell} \mathbf{LinCat}_{n,p} \xrightarrow{\mathcal{U}_{n,p}^\ell} & \mathbf{Grph}_n \end{array}$$

To define  $\mathcal{F}_{n+1,p}^\ell$ , we set first  $\mathcal{F}_{n+1,p}^{\text{P}^\ell}$  the unique functor making the following diagram commutative:

$$\begin{array}{ccccc} \mathbf{LinPol}_{n+1,p} & & & & \\ \mathcal{U}_{n,p}^{\text{P}^\ell} \downarrow & \searrow \mathcal{F}_{n+1,p}^{\text{P}^\ell} & & \searrow \mathcal{U}_{n+1,p}^{\text{GP}^\ell} & \\ & \mathbf{LinCat}_{n,p}^+ & \xrightarrow{\quad} & \mathbf{Grph}_{n+1} & \\ & \downarrow & \lrcorner & \downarrow \mathcal{U}_n^{\text{G}} & \\ \mathbf{LinPol}_{n,p} & \xrightarrow{\mathcal{F}_{n,p}^\ell} \mathbf{LinCat}_{n,p} \xrightarrow{\mathcal{U}_{n,p}^\ell} & & \mathbf{Grph}_n & \end{array}$$

The functor  $\mathcal{F}_{n+1,p}^\ell$  is then the composition

$$\mathbf{LinPol}_{n+1,p} \xrightarrow{\mathcal{F}_{n+1,p}^P} \mathbf{LinCat}_{n,p}^+ \xrightarrow{\mathcal{F}_{n+1,p}^W} \mathbf{LinCat}_{n+1,p}$$

Given a linear  $(n, p)$ -polygraph  $\Sigma$ , we call *free linear  $(n, p)$ -category over  $\Sigma$*  the linear  $(n, p)$ -category  $\mathcal{F}_{n,p}^\ell(\Sigma)$ . We denote by  $\Sigma^\ell$  this linear  $(n, p)$ -category.

For  $n > p$ , a linear  $(n, p)$ -polygraph can be defined as a data made of an  $(n-1, p)$ -linear polygraph  $\Pi$  and a globular extension of  $\Pi^\ell$ .

**3.1.7. Remark.** By Remark ??, any linear  $(n, p)$ -polygraph is also an  $(n, p)$ -polygraph.

**3.1.8. Presentation of a linear  $(n, p)$ -category.** Let  $n \geq p$  and let  $\Sigma$  be a linear  $(n+1, p)$ -polygraph. We say that  $\Sigma$  is a *presentation* of a linear  $(n, p)$ -category  $\mathcal{C}$ , or that  $\Sigma$  *presents*  $\mathcal{C}$ , if the quotient of the linear  $(n, p)$ -category  $\Sigma_n^\ell$  by the globular extension  $\Sigma_{n+1}$  is isomorphic to  $\mathcal{C}$ .

**3.1.9. Coherent presentation of a linear  $(n, p)$ -category.** Let  $n \geq p$  and let  $\Sigma$  be an  $(n+2, p)$ -linear polygraph. We say that  $\Sigma$  is a *coherent presentation* of  $\mathcal{C}$  if the underlying linear  $(n+1, p)$ -polygraph to  $\Sigma$  presents  $\mathcal{C}$  and the globular extension  $\Sigma_{n+2}$  is a homotopy basis of the linear  $(n+1, p)$ -category  $\Sigma_{n+1}^\ell$ .

## 3.2. REWRITING PROPERTIES OF LINEAR POLYGRAPHS

For the rest of this section,  $n$  will be a non negative integer and  $\Sigma$  a linear  $(n+1, n)$ -polygraph.

**3.2.1. Monomials.** A *monomial* of  $\Sigma$  is an  $n$ -cell of the  $(n+1)$ -category  $\Sigma^*$ . We say that  $\Sigma$  is *left-monomial* if all  $(n+1)$ -cells of the linear  $(n+1, n)$ -polygraph  $\Sigma$  have a monomial source.

**3.2.2. Remark.** Any  $n$ -cell of  $\Sigma^\ell$  has a unique decomposition as a linear combination of monomials.

For the rest of this section, we assume that  $\Sigma$  is left-monomial.

**3.2.3. Rewriting steps of a linear  $(n+1, n)$ -polygraph.** A *rewriting step* of  $\Sigma$  is an  $(n+1)$ -cell of  $\Sigma^\ell$  of the form

$$\lambda u_1 \star_{n-1} \cdots (u_n \star_0 \alpha \star_0 u_{n+1}) \cdots \star_{n-1} u_{2n} + h$$

where  $\lambda$  is a non zero scalar,  $\alpha$  is an  $(n+1)$ -cell of  $\Sigma$ ,  $u_i$  is a monomial of  $\Sigma$  for  $1 \leq i \leq 2n$  and  $h$  is an  $n$ -cell of  $\Sigma^\ell$  such that  $u_1 \star_{n-1} \cdots (u_n \star_0 \alpha \star_0 u_{n+1}) \cdots \star_{n-1} u_{2n}$  does not appear in the monomial decomposition of  $h$ .

**3.2.4. Local branchings of a linear  $(n + 1, n)$ -polygraph.** The local branchings of  $\Sigma$  can be classified into five families.

- i) An *aspherical* branching of  $\Sigma$  is a local branching of  $\Sigma$  made of two identical rewriting steps.

$$\begin{array}{c} \alpha \rightarrow v \\ u \\ \alpha \rightarrow v \end{array}$$

- ii) A *Peiffer* branching is a local branching of  $\Sigma$  of the form  $(\alpha \star_{n-1} v + h, u \star_{n-1} \beta + h)$  where  $u$  and  $v$  are monomials,  $\alpha$  and  $\beta$  are rewriting steps and  $h$  is an  $n$ -cell of  $\Sigma^\ell$ .

$$\begin{array}{c} \alpha \star_{n-1} v + h \rightarrow u' \star_{n-1} v + h \\ u \star_{n-1} v + h \\ u \star_{n-1} \beta + h \rightarrow u \star_{n-1} v' + h \end{array}$$

- iii) An *additive* branching of  $\Sigma$  is a local branching of  $\Sigma$  of the form  $(\alpha + v, u + \beta)$  where  $\alpha$  and  $\beta$  are rewriting steps and  $u$  and  $v$  are  $n$ -cells of  $\Sigma^\ell$ .

$$\begin{array}{c} \alpha + v \rightarrow u' + v \\ u + v + h \\ u + \beta \rightarrow u + v' \end{array}$$

- iv) An *overlapping* branching is a local branching of  $\Sigma$  which is not aspherical, Peiffer or additive.

Let  $\sqsubseteq$  be the order relation on the monomials of  $\Sigma$  defined in ???. A *critical branching*, or *critical pair*, of  $\Sigma$  is an overlapping branching of  $\Sigma$  with a monomial source  $u$  such that no monomial  $v$  of  $\Sigma$  verifying  $v \sqsubseteq u$  is source of an overlapping branching.

**3.2.5. Elementary  $(n + 1)$ -cells.** An  $(n + 1)$ -cell of  $\Sigma^\ell$  is said to be *elementary* if it is of the form

$$\lambda u_1 \star_{n-1} \cdots (u_n \star_0 \alpha \star_0 u_{n+1}) \cdots \star_{n-1} u_{2n} + h$$

where  $\lambda$  is a non zero scalar,  $\alpha$  is an  $(n + 1)$ -cell of  $\Sigma$ ,  $u_i$  is a monomial of  $\Sigma$  for  $1 \leq i \leq 2n$  and  $h$  is an  $n$ -cell of  $\Sigma^\ell$ .



**3.2.6. Lemma.** Let  $\alpha$  an elementary  $(n + 1)$ -cell of  $\Sigma^\ell$ . Then, there exist two  $(n + 1)$ -cells  $\beta$  and  $\gamma$  of  $\Sigma^\ell$  such that  $\alpha = \beta \star_2 \gamma^{-1}$  and  $\beta$  and  $\gamma$  are either identities or rewriting steps.

*Proof.* Let us write  $\alpha = \alpha' + w$  where  $\alpha'$  is a rewriting step from an  $n$ -cell  $u$  to an  $n$ -cell  $v$  and  $w$  is an  $n$ -cell. Let us write  $w = \lambda u + h$  where  $v$  does not appear in the monomial decomposition of  $h$ . Then,  $(\lambda + 1)u + h$  rewrites into  $(\lambda + 1)v + h$  by a rewriting step if  $(\lambda + 1)u + h \neq (\lambda + 1)v + h$ . Moreover,  $u + v$  rewrites into  $(\lambda + 1)u + h$  by a rewriting step if  $u + v \neq (\lambda + 1)u + h$ .  $\square$

**3.2.7. Rewrite orders.** The *rewrite order* of the linear  $(n + 1, n)$ -polygraph  $\Sigma$  is the relation  $\preceq_\Sigma$  on  $\Sigma_n^\ell$  defined by

- if  $u$  and  $v$  are monomials, then  $v \preceq_\Sigma u$  is  $u = v$  or  $u$  rewrites into  $v$ ,
- if for any monomial  $y$  of  $v$  not appearing in  $u$ , there is a monomial  $x$  of  $u$  not appearing in  $v$  such that  $y \preceq_\Sigma x$ , then  $v \preceq_\Sigma u$ .

The *strict rewrite order* of  $\Sigma$  is the strict order relation  $\prec_\Sigma$  on  $\Sigma_n^\ell$  defined by  $v \prec_\Sigma u$  if we have  $v \preceq_\Sigma u$  but not  $u \preceq_\Sigma v$ .

**3.2.8. Remark.** The relation  $\preceq_\Sigma$  is not necessarily an order relation. In particular if two distinct  $n$ -cells of  $\Sigma^\ell$  rewrite into each other, the antisymmetry is not satisfied.

**3.2.9. Exponentiation freedom.** We say that  $\Sigma$  is (*scalar*) *exponentiation free* if no monomial  $m$  of  $\Sigma^\ell$  can be rewritten into  $\lambda m + f$  for some scalar  $\lambda$  other than 0 or 1 and some non zero  $n$ -cell  $f$  which does not contain  $m$  in its monomial decomposition. Note that if  $\Sigma$  is quasi-terminating, exponentiation freedom is equivalent to the fact that for every monomial  $m$  rewriting into a  $n$ -cell  $f$  containing  $m$  in its monomial decomposition, we have  $f = m$ . With the terminology of Dershowitz, see [?, Definition 2., Definition 3.], the conjunction of exponentiation freedom and quasi-termination means that in the associated rewriting system, all loops are cycles.

The following lemma is needed to prove the critical pairs Lemma for linear polygraph. Note that there is a gap in [?] which does not make use of this lemma. Indeed, exponentiation freedom hypothesis is not mentioned although it is necessary.

**3.2.10. Lemma.** Let  $\Sigma$  be a quasi-terminating exponentiation free left-monomial linear polygraph. All additive branchings of  $\Sigma$  are confluent.

*Proof.* Let  $\alpha$  be a rewriting step of monomial source  $u$ ,  $\beta$  be a rewriting step of monomial source  $v$ ,  $\lambda$  and  $\mu$  non zero scalars and  $h$  a  $n$ -cell of  $\Sigma^\ell$  which does not contain  $u$  or  $v$  in its monomial decomposition. Let us denote  $a = t_1(\alpha)$  and  $b = t_1(\beta)$ . We prove that the additive branching  $(\lambda\alpha + \mu\nu + h, \lambda u + \mu\beta + h)$  is confluent by considering four cases.

### 3.2. REWRITING PROPERTIES OF LINEAR POLYGRAPHS

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**Case 1.** If  $u$  does not appear in the monomial decomposition of  $b$  and  $v$  does not appear in the monomial decomposition of  $a$ , the 2-cells  $\lambda\alpha + \mu\beta + h$  and  $\lambda\alpha + \mu b + h$  are rewriting steps and make the branching  $(\lambda\alpha + \mu\nu + h, \lambda u + \mu\beta + h)$  confluent.

$$\begin{array}{ccccc}
 & & \lambda\alpha + \mu\nu + h & \xrightarrow{\quad} & \lambda\alpha + \mu\nu + h & \xrightarrow{\quad} & \lambda\alpha + \mu\beta + h & & \\
 & & \nearrow & & \searrow & & \nearrow & & \\
 \lambda u + \mu\nu + h & & & & & & & & \lambda\alpha + \mu b + h \\
 & & \searrow & & \nearrow & & \searrow & & \\
 & & \lambda u + \mu\beta + h & \xrightarrow{\quad} & \lambda u + \mu b + h & \xrightarrow{\quad} & \lambda\alpha + \mu b + h & & 
 \end{array}$$

**Case 2.** If  $u$  appears in the monomial decomposition of  $b$  and  $v$  does not appear in the monomial decomposition of  $a$ , we consider the  $n$ -cell  $f$  and the scalar  $\gamma$  such that  $\mu b + \lambda u = \gamma u + f$  and  $u$  does not appear in the monomial decomposition of  $f$ . If  $\gamma = 0$ , the  $n$ -cell  $\lambda\alpha + \mu\nu + h$  rewrites into  $f + h$ , using the rewriting step  $\lambda\alpha + \mu\beta + h$ .

$$\begin{array}{ccccc}
 & & \lambda\alpha + \mu\nu + h & \xrightarrow{\quad} & \lambda\alpha + \mu\nu + h & \xrightarrow{\quad} & \lambda\alpha + \mu\beta + h & & \\
 & & \nearrow & & \searrow & & \nearrow & & \\
 \lambda u + \mu\nu + h & & & & & & & & \lambda(a - u) + f + h \\
 & & \searrow & & \nearrow & & \searrow & & \\
 & & \lambda u + \mu\beta + h & \xrightarrow{\quad} & f + h & \xleftarrow{\quad} & \lambda(a - \alpha) + f + h & & 
 \end{array}$$

Else,  $\lambda\alpha + \mu\nu + h$  rewrites into  $\lambda\alpha + (\gamma - \lambda)u + f + h$ , which rewrites into  $\gamma\alpha + f + h$  and  $\gamma u + f + h$  rewrites into  $\gamma\alpha + f + h$ . This makes the branching  $(\lambda\alpha + \mu\nu + h, \lambda u + \mu\beta + h)$  confluent.

$$\begin{array}{ccccc}
 & & \lambda\alpha + \mu\nu + h & \xrightarrow{\quad} & \lambda\alpha + \mu\nu + h & \xrightarrow{\quad} & \lambda\alpha + (\gamma - \lambda)u + f + h & & \\
 & & \nearrow & & \searrow & & \nearrow & & \\
 \lambda u + \mu\nu + h & & & & & & & & \lambda\alpha + (\gamma - \lambda)\alpha + f + h \\
 & & \searrow & & \nearrow & & \nearrow & & \\
 & & \lambda u + \mu\beta + h & \xrightarrow{\quad} & \gamma u + f + h & \xrightarrow{\quad} & \lambda\alpha + \mu\beta + h & & \\
 & & & & & & & & \gamma\alpha + f + h
 \end{array}$$

**Case 3.** If  $u$  does not appear in the monomial decomposition of  $b$  and  $v$  appears in the monomial decomposition of  $a$ , we are in a case symmetric to Case 2.

**Case 4.** If  $u$  appears in the monomial decomposition of  $b$  and  $v$  appears in the monomial decomposition of  $a$ , we can write

$$a = \gamma_u u + \gamma_v v + f,$$

$$b = \sigma_u u + \sigma_v v + g$$

where  $f$  and  $g$  are  $n$ -cells which do not contain  $u$  or  $v$  in their monomial decompositions,  $\gamma_v$  and  $\sigma_u$  are non zero scalars, and  $\gamma_u$  and  $\sigma_v$  are scalars in  $\{0, 1\}$  because  $\Sigma$  is exponentiation free.

Because  $\Sigma$  is quasi-terminating, we also have  $\gamma_u = \sigma_v = 0$ . By quasi-termination hypothesis, we also have  $f = g = 0$ . This implies  $\gamma_v = \sigma_u = 1$ . Thus, we have

$$a = v,$$

$$b = u.$$

The targets of the branching  $(\lambda\alpha + \mu\nu + h, \lambda u + \mu\beta + h)$  are thus  $(\lambda + \mu)u + h$  and  $(\lambda + \mu)v + h$ . Then, the branching  $(\lambda\alpha + \mu\nu + h, \lambda u + \mu\beta + h)$  is confluent because  $u$  rewrites into  $v$ .

$$\begin{array}{ccc}
 & \lambda\alpha + \mu\nu + h & \xrightarrow{\quad} & (\lambda + \mu)u + h \\
 \lambda u + \mu\nu + h & \begin{array}{c} \nearrow \\ \searrow \end{array} & & \uparrow \Big) (\lambda + \mu)\alpha + h \\
 & \lambda u + \mu\beta + h & \xrightarrow{\quad} & (\lambda + \mu)v + h
 \end{array}$$

This concludes the proof. □

In this way, we prove the following theorem by well-founded induction on the order  $\prec_\Sigma$ .

**3.2.11. Example.** Let us give an example of quasi-terminating left-monomial linear  $(2, 2)$ -polygraph which is not locally confluent. By Lemma ??, such a linear  $(2, 2)$ -polygraph is not exponentiation free. Let us consider the left-monomial linear  $(2, 2)$ -polygraph  $\Sigma$  with

- only 0-cell and two 1-cells  $x$  and  $y$ ,
- two 2-cells  $x \Rightarrow y$  and  $y \Rightarrow -x$ .

It has a non confluent local branching of source  $x + y$ . Indeed, the 2-cell  $x + y$  rewrites both into 0 and  $2y$  and  $2y$  does not rewrite into 0. Because of the rewriting sequence

$$x \Rightarrow y \Rightarrow -x$$

the linear  $(2, 2)$ -polygraph  $\Sigma$  does not satisfy the conditions of Lemma ??.

**3.2.12. Theorem.** Let  $\Sigma$  be a quasi-terminating exponentiation free left-monomial linear polygraph. If all critical branchings of  $\Sigma$  are confluent, all local branchings of  $\Sigma$  are confluent.

*Proof.* By Lemma ??, all additive branchings of  $\Sigma$  are confluent. Because aspherical branchings are always confluent, all that remains to prove is that Peiffer and overlapping branchings of  $\Sigma$  are confluent. Let  $(\alpha, \beta)$  be a branching of  $\Sigma$  that is aspherical or Peiffer. We proceed by well-founded induction on the order  $\prec_\Sigma$ .

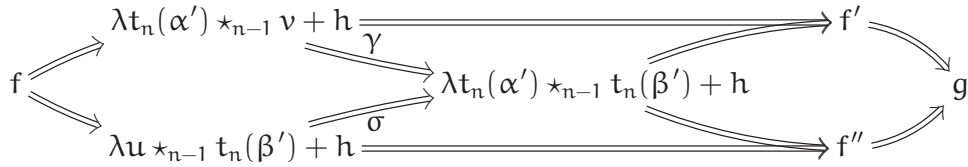
### 3.3. BASES IN HIGHER-DIMENSIONAL LINEAR CATEGORIES

If  $(\alpha, \beta)$  is an overlapping branching, we can write  $f = \lambda u + h$  where  $u$  is a monomial which is the source of an overlapping branching  $(\alpha', \beta')$  and  $\lambda$  is a nonzero scalar. The overlapping branching can be  $(\alpha, \beta)$  written  $(\lambda\alpha' + h, \lambda\beta' + h)$  where  $(\lambda\alpha', \lambda\beta')$  is confluent by hypothesis to a common target  $g$ . If  $t_n(\lambda\alpha') = t_n(\lambda\beta')$  or if we do not have  $f \prec_\Sigma g + h$ , we have either  $t_n(\lambda\alpha') \preceq_\Sigma t_n(\lambda\beta') + h$  or  $t_n(\lambda\beta') + h \preceq_\Sigma t_n(\lambda\alpha') + h$ , and the overlapping branching is confluent. Else, because  $\prec_\Sigma$  is well-founded, there is a monomial  $m$  of  $t_n(\lambda\alpha') - t_n(\lambda\beta')$  such that:

- $m$  can be rewritten into a linear combination of the others monomials of  $t_n(\lambda\alpha') - t_n(\lambda\beta')$ ,
- $m$  only appears in  $t_n(\lambda\alpha')$  or  $t_2(\beta)$ .

This makes all overlapping branchings confluent.

If  $(\alpha, \beta)$  is a Peiffer branching, we can write  $f = \lambda u \star_{n-1} v + h$  with  $u$  and  $v$  two monomials. Let  $\alpha'$  and  $\beta'$  be the rewriting steps such that  $\alpha = \lambda\alpha' \star_{n-1} v + h$  and  $\beta = \lambda u \star_{n-1} \beta' + h$ . If we do not have  $\lambda t_n(\alpha') \star_{n-1} v + h \prec_\Sigma f$  or  $\lambda u \star_{n-1} t_n(\beta') + h \prec_\Sigma f$ , then  $\lambda t_n(\alpha') \star_{n-1} v + h$  rewrites into  $f$  or  $\lambda u \star_{n-1} t_n(\beta') + h$  rewrites into  $f$  because  $\Sigma$  is quasi-terminating and exponentiation free. Let us now assume without that  $\lambda t_n(\alpha') \star_{n-1} v + h \prec_\Sigma f$  and  $\lambda u \star_{n-1} t_n(\beta') + h \prec_\Sigma f$ . Then, we have a confluence diagram of the form



where the  $(n+1)$ -cells  $\gamma$  and  $\sigma$  are elementary, the two  $(n+1)$ -cells of sources  $\lambda t_n(\alpha') \star_{n-1} v + h$  and  $\lambda u \star_{n-1} t_n(\beta') + h$  are rewriting steps and the others  $(n+1)$ -cells are compositions of rewriting steps. The branching of source  $\lambda t_n(\alpha') \star_{n-1} t_n(\beta') + h$  is confluent by induction hypothesis.

This makes the branching  $(\alpha, \beta)$  confluent and concludes the last case of local branching.  $\square$

### 3.3. BASES IN HIGHER-DIMENSIONAL LINEAR CATEGORIES

In this section, we set  $n > 0$ . Given an  $(n, n)$ -linear category  $\mathcal{C}$ , we study the question of how to explicit a basis for each  $n$ -cells module of  $\mathcal{C}$ . In particular, from a presentation  $\mathcal{C}$ , we want to compute a basis for the module  $\mathcal{C}_n(u, v)$  for any parallel  $(n-1)$ -cells  $u$  and  $v$ . We give a result, Theorem ??, using rewriting to exhibit such bases. This result is proved using the following lemma.

**3.3.1. Lemma [?, Lemma 4.2.14].** Let  $\Sigma$  be a left-monomial linear  $(n + 1, n)$ -polygraph presenting an  $(n, n)$ -linear category  $\mathcal{C}$ . Let us assume that  $\Sigma$  is confluent. Then, for all parallel  $(n - 1)$ -cells  $u$  and  $v$  of  $\mathcal{C}$ , the linear map sending each element of  $\Sigma_n^\ell(u, v)$  on its equivalence class in  $\mathcal{C}_n(u, v)$  has for kernel the submodule of  $\Sigma_n^\ell(u, v)$  made of the  $n$ -cells having  $0$  as a normal form.

*Proof.* Let  $\tau$  be the canonical surjection of  $\Sigma_n^\ell(u, v)$  into  $\mathcal{C}_n(u, v)$ . If an element of  $\Sigma_n^\ell(u, v)$  rewrites into  $0$ , then this element has the same equivalence class as  $0$  and is in  $\text{Ker}(\tau)$ . Let now  $A$  be an element of  $\text{Ker}(\tau)$ . Because  $0$  is a normal form and Church-Rosser's property is equivalent to confluence,  $A$  rewrites into  $0$ . This concludes the proof.  $\square$

**3.3.2. Theorem [?, Proposition 4.2.15].** Let  $\Sigma$  a be left-monomial linear  $(n + 1, n)$ -polygraph presenting an  $(n, n)$ -linear category  $\mathcal{C}$ . Let us assume that  $\Sigma$  is confluent and normalizing. Then, for all parallel  $(n - 1)$ -cells  $u$  and  $v$  of  $\mathcal{C}$ , a basis of  $\mathcal{C}_n(u, v)$  is given by the set of equivalence classes of monomials in normal form of  $\Sigma_n^\ell(u, v)$ .

*Proof.* Let  $\tau$  be the canonical surjection of  $\Sigma_n^\ell(u, v)$  into  $\mathcal{C}_n(u, v)$ . All elements of  $\Sigma_n^\ell(u, v)$  have a normal form because  $\Sigma$  is normalizing. Then, the image of the set of monomials in normal form of  $\Sigma_n^\ell(u, v)$  by  $\tau$  is a generating family. The confluence of  $\Sigma$  allows us to use Lemma ?? to conclude that this family is free.  $\square$

## Completion of higher-dimensional polygraphs

As we have seen in chapter 2, higher-dimensional categories can be presented by generators and relations using the structure of polygraphs, and rewriting is a tool to study those presentations. When a polygraph presenting a higher-dimensional category satisfies properties such that termination and confluence, we can use results like Squier's Theorem to compute a coherent presentation or Theorem ?? to compute bases. Given a higher-dimensional category presented by generators and relations, one problem is the computation of a confluent presentation of this higher-dimensional category. Given a non confluent polygraph, this problem can be solved by constructing a Tietze-equivalent confluent polygraph. This can be done by adding cells to a polygraph.

In this chapter, we present a completion method for higher-dimensional categories. A completion procedure is a procedure adding generators and rewriting rules to a presentation to obtain a presentation of the same structure. The redundant generators and rules we add are expected to give us a confluent rewriting system. A general procedure to find a convergent, a fortiori confluent, rewriting system given an equational presentation has been introduced by Knuth and Bendix in [?]. In the case of  $n$ -categories, Knuth-Bendix's procedure is applied to  $(n + 1)$ -polygraphs. Given  $\Sigma$  an  $(n + 1)$ -polygraph, Knuth-Bendix's procedure applied to  $\Sigma$  uses an order on the set of  $n$ -cells of  $\Sigma^*$  which is compatible with the rewrite relation. The procedure is based on the critical pairs Lemma and adds  $(n+1)$ -cells to an  $(n+1)$ -polygraph. For each non confluent critical pair leading to two chosen normal forms  $u$  and  $v$ , we add an  $(n + 1)$ -cell from  $u$  to  $v$  if  $v \prec u$  or from  $v$  to  $u$  if  $u \prec v$ . Knuth-Bendix's procedure and either terminates in the case where all critical pairs are made confluent, creates an infinite increasing sequence of  $(n + 1)$ -polygraphs or fails. In the first two cases, we obtain a convergent  $(n + 1)$ -polygraph in a finite or infinite number of steps. The procedure fails when we encounter two  $n$ -cells which

cannot be compared. If the order we use is total, we call it a total termination order.

Note that to effectively apply Knuth-Bendix's procedure, one must be able to compute the set of critical pairs of the  $(n + 1)$ -polygraph  $\Sigma$ . If  $\Sigma$  is a finite 2-polygraphs, this can be done algorithmically by enumerating all the 2-cells of  $\Sigma$ . The set of critical pairs is always finite in this case. However, there is no known algorithm to compute this set in the case of  $(n + 1)$ -polygraphs if  $n \geq 2$ . Even in the case of finite 3-polygraphs, the set of critical pairs can be infinite, see [?].

Knuth-Bendix completion procedure for  $(n + 1)$ -polygraphs gives a convergent  $(n + 1)$ -polygraph from an  $(n + 1)$ -polygraph with a total termination order, meaning the procedure does not fail if a total termination order is used, see [?]. Not all terminating  $(n + 1)$ -polygraphs have a total termination order. This raises the problem of finding another completion procedure. For finite terminating  $(n + 1)$ -polygraphs, we introduce a completion procedure, the generalized Knuth-Bendix's procedure which gives us local confluence.

**Plan of Chapter.** We will recall first in Section ?? the notion of total termination order and give an example of 3-polygraph which does not admit such an order. We also recall Knuth-Bendix's procedure for polygraphs. Then, in Section ?? we introduce quasi-termination orders, they are generalizations of total termination orders to all terminating  $(n + 1)$ -polygraphs. Next, we introduce the generalized Knuth-Bendix's procedure. We prove this procedure produces a locally confluent  $(n + 1)$ -polygraph when it does not fail. Moreover, we give a criterion on the used quasi-termination order guaranteeing the procedure does not fail. We discuss after when the produced locally confluent  $(n + 1)$ -polygraph is confluent. We finally give a criterion to guarantee confluence.

## 4.1. KNUTH-BENDIX'S PROCEDURE FOR POLYGRAPHS

Suppose  $n > 1$ . We set  $\Sigma$  a finite  $(n + 1)$ -polygraph. In this section, we recall Knuth-Bendix's procedure for polygraphs. This procedure takes the set of critical pairs of the  $(n + 1)$ -polygraph as an input. Computing this set is not part of the procedure. In the case of 2-polygraphs, algorithms are known to compute the critical pairs. If  $n > 2$ , we do not know general algorithms to compute the set of critical pairs of  $\Sigma$ , even if this set can be given in the cases we will study.

**4.1.1. Total termination orders.** A *total termination order* on the  $(n + 1)$ -polygraph  $\Sigma$  is a strict order relation  $\prec$  on  $\Sigma_n^*$  such that:

- i) for each parallel  $(n - 1)$ -cells  $u$  and  $v$  of  $\Sigma_{n-1}^*$ , the restriction of  $\prec$  to  $\Sigma_n^*(u, v)$  is a well-founded total order,
- ii) for any  $n$ -cells  $f$  and  $g$  of  $\Sigma_n^*$  such that  $g$  rewrites into  $f$ , we have  $f \prec g$ ,

#### 4.1. KNUTH-BENDIX'S PROCEDURE FOR POLYGRAPHS

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- iii) for any parallel  $n$ -cells  $f$  and  $g$  such that  $f \prec g$ , for every integer  $0 \leq i < n$  and every  $n$ -cell  $h$  such that  $f \star_i h$  is defined, we have  $f \star_i h \prec g \star_i h$ ,
- iv) for any parallel  $n$ -cells  $f$  and  $g$  such that  $f \prec g$ , for every integer  $0 \leq i < n$  and every  $n$ -cell  $h$  such that  $h \star_i f$  is defined, we have  $h \star_i f \prec h \star_i g$ .

Even if the  $(n + 1)$ -polygraph  $\Sigma$  is terminating, it does not always have a total termination order.

**4.1.2. Example.** Let  $\Sigma_{ex}$  be the 3-polygraph with only one 0-cell, one 1-cell and two 2-cells represented by:

$$\cap, \quad \cup$$

If there is a total termination order  $\prec$  on  $\Sigma_{ex}$ , we have one of the following inequalities:

$$\left| \begin{array}{c} \circ \\ \prec \\ \circ \end{array} \right| \quad \text{or} \quad \circ \left| \begin{array}{c} \prec \\ \circ \end{array} \right|$$

Let us assume that the first inequality is true, the other case being symmetric. Then, we have:

$$\left| \begin{array}{c} \cap \\ \circ \\ \prec \\ \circ \end{array} \right| \prec \circ \cap = \cap \circ \prec \left| \begin{array}{c} \cap \\ \circ \end{array} \right|$$

which contradicts the existence of a total termination order  $\Sigma_{ex}$  because a total termination order is strict.

Knuth-Bendix's procedure [?] is a completion procedure used in term rewriting to complete non confluent terminating term rewriting systems into convergent ones. We present the application of this procedure for  $(n + 1)$ -polygraphs.

**4.1.3. Knuth-Bendix's procedure for  $(n + 1)$ -polygraphs.** We give on the set  $\Sigma_n^*$  a well-founded order  $\prec$  compatible with the rewrite relation, that is for all  $n$ -cells  $f$  and  $g$  such that  $g$  rewrites into  $f$ , we have  $f \prec g$ . Knuth-Bendix's procedure can then complete the  $(n + 1)$ -polygraph  $\Sigma$  into a convergent  $(n + 1)$ -polygraph  $KB(\Sigma)$  or fail. Knuth-Bendix's procedure is defined as follows

- Input:  $\Sigma$  a terminating  $(n + 1)$ -polygraph
- Attribute to CP the set of non confluent critical branchings of  $\Sigma$
- Attribute to  $KB(\Sigma)$  the  $(n + 1)$ -polygraph  $\Sigma$



```

while  $CP \neq \emptyset$  do
    Choose  $(\alpha, \beta)$  in  $CP$  where  $\alpha$  is a rewriting step with target  $a$  and  $\beta$  is a rewriting step with
    target  $b$ 
    Attribute to  $A$  the set  $\{a' \in \Sigma_n^* \mid a \text{ rewrites into } a'\}$ 
    Choose in  $A$  a minimal element  $u_\alpha$  with respect to  $\prec$ 
    Attribute to  $B$  the set  $\{b' \in \Sigma_n^* \mid b \text{ rewrites into } b'\}$ 
    Choose in  $B$  a minimal element  $u_\beta$  with respect to  $\prec$ 
    if  $u_\alpha \prec u_\beta$  then
        Add to  $KB(\Sigma)$  an  $(n + 1)$ -cell from  $u_\beta$  to  $u_\alpha$ 
    end if
    if  $u_\beta \prec u_\alpha$  then
        Add to  $KB(\Sigma)$  an  $(n + 1)$ -cell from  $u_\alpha$  to  $u_\beta$ 
    end if
    if we do not have  $u_\alpha \prec u_\beta$  or  $u_\beta \prec u_\alpha$  then
        Return "Fail"
    end if
    Attribute to  $CP$  the set of non confluent critical branchings of  $KB(\Sigma)$ 
end while
Return  $KB(\Sigma)$ 
    
```

The minimal elements for  $A$  and  $B$  exist because  $\prec$  is well-founded. More precisely, by  $u_\alpha$  minimal, we mean no element  $a'$  of  $A$  verifies  $a' \prec u_\alpha$ . This procedure is indeed a completion procedure of the critical branchings because  $a$  rewrites into  $u_\alpha$  and  $b$  rewrites into  $u_\beta$ . We also remark  $u_\alpha$  and  $u_\beta$  are normal forms because  $\Sigma$  is terminating. Comparing  $u_\alpha$  and  $u_\beta$  and adding an  $(n + 1)$ -cell from the greatest of those  $n$ -cells to the lowest has for goal to keep termination and ensure convergence of the obtained  $(n + 1)$ -polygraph.

**4.1.4. Correctness of the Knuth-Bendix's procedure.** To guarantee that Knuth-Bendix's procedure does not fail,  $\prec$  must be a total termination order, see [?]. What remains to prove is that  $KB(\Sigma)$  is confluent if  $\prec$  is a total termination order.

If Knuth-Bendix's procedure ends in a finite number of steps, then all critical branchings of  $KB(\Sigma)$  are confluent, which implies that  $KB(\Sigma)$  is locally confluent by the critical pairs Lemma, and confluent by Newman's Lemma.

## 4.2. GENERALIZED KNUTH-BENDIX'S PROCEDURE

**4.2.1. Quasi-termination orders.** A *quasi-termination order* on an  $(n + 1)$ -polygraph  $\Sigma$  is a strict order relation  $\prec$  on  $\Sigma_n^*$  such that:

- i) for each parallel  $(n - 1)$ -cells  $u$  and  $v$  of  $\Sigma_{n-1}^*$ , the restriction of  $\prec$  to  $\Sigma_n^*(u, v)$  is a well-founded order,
- ii) for any  $n$ -cells  $f$  and  $g$  of  $\Sigma_n^*$  such that  $g$  rewrites into  $f$  and  $f$  does not rewrite into  $g$ , we have  $f \prec g$ ,
- iii) for any parallel  $n$ -cells  $f$  and  $g$  such that  $f \prec g$ , for every integer  $0 \leq i < n$  and every  $n$ -cell  $h$  such that  $f \star_i h$  is defined, we do not have  $g \star_i h \prec f \star_i h$ ,
- iv) for any parallel  $n$ -cells  $f$  and  $g$  such that  $f \prec g$ , for every integer  $0 \leq i < n$  and every  $n$ -cell  $h$  such that  $h \star_i f$  is defined, we do not have  $h \star_i g \prec h \star_i f$ .

A quasi-termination order can also be defined as a strict order contained into a well-founded quasi-order compatible with the rewrite relation, see [?]. We say the order  $\prec$  is *maximal* if for every congruent  $n$ -cells  $f$  and  $g$  of  $\Sigma_n^*$  relatively to  $\Sigma_{n+1}$ , one of the following properties holds:

- $f \prec g$ ,
- $g \prec f$ ,
- there is no quasi-termination order  $\prec'$  on  $\Sigma$  such that  $f \prec' g$  or  $g \prec' f$ .

**4.2.2. Generalized Knuth-Bendix's procedure for  $(n + 1)$ -polygraphs.** We give on the set  $\Sigma_n^*$  a quasi-termination order  $\prec$ . Generalized Knuth-Bendix's procedure is defined by the same procedure as Knuth-Bendix's procedure where the termination order is replaced by the quasi-termination order  $\prec$ . Like in the case of Knuth-Bendix's procedure, we note that the minimal elements for  $A$  and  $B$  exist because  $\prec$  is well-founded.

**4.2.3. Proposition.** Let  $\Sigma$  be an  $(n + 1)$ -polygraph and let  $\prec$  be a maximal quasi-termination order on  $\Sigma$ . Then, the application of the generalized Knuth-Bendix's procedure on  $(\Sigma, \prec)$  does not fail.

*Proof.* Let us assume that the generalized Knuth-Bendix's procedure fails on  $(\Sigma, \prec)$ . This implies the existence of a branching of  $\Sigma$  leading to  $n$ -cells  $f$  and  $g$  of  $\Sigma_n^*$  such that:

- there is no  $n$ -cell  $f'$  such that  $f' \prec f$  and  $f$  rewrites into  $f'$ ,
- there is no  $n$ -cell  $g'$  such that  $g' \prec g$  and  $f$  rewrites into  $g'$ ,
- $f$  does not rewrite into  $g$ ,
- $g$  does not rewrite into  $f$ ,

– we have neither  $f \prec g$  nor  $g \prec f$ .

Because  $\prec$  is maximal, this implies no quasi-termination order  $\prec'$  on  $\Sigma$  verifies  $f \prec' g$  or  $g \prec' f$ . Let  $\Sigma'$  be the union of the  $(n + 1)$ -polygraph  $\Sigma$  and an  $(n + 1)$ -cell from  $f$  to  $g$ . So,  $f$  rewrites into  $g$  relatively to  $\Sigma'$ . Besides,  $\prec$  is also a quasi-termination order on  $\Sigma'$ . And because we do not have  $g \prec f$ , this implies the  $n$ -cells  $f$  and  $g$  rewrite into each other relatively to the  $(n + 1)$ -polygraph  $\Sigma'$ . Thus, we can rewrite  $g$  into  $f$  only by using  $(n + 1)$ -cells of  $\Sigma_{n+1}^*$ . This contradicts the failure of the generalized Knuth-Bendix's procedure.  $\square$

**4.2.4. Proposition.** Let  $\Sigma$  be an  $(n + 1)$ -polygraph and  $\prec$  a maximal quasi-termination order on  $\Sigma$ . Let us assume that for each congruent  $n$ -cells  $f$  and  $g$  of  $\Sigma_n^*$  such that  $f \prec g$ , the  $n$ -cell  $g$  rewrites into  $f$ . Then, the  $(n + 1)$ -polygraph  $\Sigma$  is confluent.

*Proof.* Let  $(\alpha, \beta)$  be a branching of  $\Sigma$  such that  $t_n(\alpha) = f$  and  $t_n(\beta) = g$ . If  $f \prec g$  or  $g \prec f$ , the branching  $(\alpha, \beta)$  is confluent by hypothesis. If we do not have  $f \prec g$  or  $g \prec f$ , the  $n$ -cells  $f$  and  $g$  rewrite into each other because  $\prec$  is maximal. We conclude that the branching  $(\alpha, \beta)$  is confluent.  $\square$

## Coherence by decreasingness

Proving confluence of rewriting systems from their local confluence and some supplementary conditions is a problem treated in [?]. We recall from chapter 2 that Newman's Lemma states that in a terminating rewriting system, local confluence and confluence are equivalent properties. Some criteria drop the termination condition of Newman's Lemma in favor of the properties of the local branchings to prove confluence from local confluence. For example, an abstract rewriting system in which all local branchings are confluent by using only one rewriting step is confluent. In 1994, inspired by an unpublished work of de Bruijn, see [?], van Oostrom defined decreasingness [?], a criterion used to prove confluence from local confluence. Any decreasing rewriting system is confluent. This result is stronger than Newman's Lemma because any terminating rewriting system is decreasing. Furthermore, decreasingness is a universal criterion to decide confluence of countable rewriting systems. Indeed, any confluent countable rewriting system is decreasing.

We recall from chapter 2 that Squier showed that there are finitely presented monoids with a decidable word problem that cannot be presented by a finite convergent string rewriting system, [?]. Beyond the questions of decidability of the word problem and of the existence of finite convergent presentations, the graph-theoretical tools associated to convergent presentations of monoids developed in [?] were applied to question of coherence problems for monoids such that Artin monoids [?] or plactic monoids [?] and monoidal categories [?]. In particular, one of the problems is to compute a *coherent presentation* of a monoid presented by a string rewriting system. A method is given in [?] to solve this problem from a convergent string rewriting system. However, in some situations it is difficult to get both confluence and termination on a finite set of generators and a finite set of rules. This leads to the goal of weakening the Squier's termination condition to construct coherent presentations.

**Plan of Chapter.** In this chapter, we present the applications of decreasingness, a property defined by van Oostrom [?], to the construction of coherent presentations introduced in [?] using quasi-termination. We begin in Section ?? by recalling the notion of decreasingness in abstract rewriting. We also recall van Oostrom’s main result stating decreasingness implies confluence. In Section ??, we introduce labelled polygraphs. Multiple notions of decreasing polygraphs follow. We will present those notions. In Section ??, we generalize Squier’s Theorem to some confluent, non terminating, polygraphs. Finally, in Section ??, we give a coherence result confluent, non terminating, linear polygraphs.

## 5.1. DECREASINGNESS IN ABSTRACT REWRITING

**5.1.1. Labellings.** Let  $(S, \rightarrow)$  be an abstract rewriting system. A *labelling* of  $(S, \rightarrow)$  is a data made of

- a partition of  $\rightarrow$  indexed by a set  $W$ ,
- a well-founded order  $\prec$  on  $W$ .

We call the data  $(S, \rightarrow, W, \prec)$  a *labelled rewriting system*. We denote for each rewriting step  $f$

$$a \rightarrow b$$

the label of  $f$  by  $\psi(f)$ .

We denote for each rewriting sequence  $f$

$$u_0 \rightarrow u_1 \rightarrow \cdots \rightarrow u_n$$

the set of labels of  $f$  by  $L^W(f)$ .

**5.1.2. Distance.** Let  $(S, \rightarrow)$  be an abstract rewriting system. Let  $a$  and  $b$  be elements of  $S$  such that  $a$  rewrites into  $b$ . The *distance* from  $a$  to  $b$ , denoted by  $d(a, b)$  is the length of the shortest rewriting sequence from  $a$  to  $b$ .

**5.1.3. Decreasingness.** A local branching of  $(S, \rightarrow, W, \prec)$  is said to be *decreasing* (resp. *strictly decreasing*) if there is a confluence diagram of the following form

$$\begin{array}{ccc}
 & \xrightarrow{f} & \\
 g \downarrow & & \downarrow f' \\
 & & g'' \\
 & \xrightarrow{g'} \xrightarrow{f''} \xrightarrow{h_2} & \downarrow h_1 \\
 & & \\
 & & \xrightarrow{g'} \\
 & & \downarrow f' \\
 & & g'
 \end{array}
 \quad (\text{resp.} \quad
 \begin{array}{ccc}
 & \xrightarrow{f} & \\
 g \downarrow & & \downarrow f' \\
 & & \\
 & \xrightarrow{g'} & \\
 & & \downarrow f' \\
 & & g'
 \end{array}
 ).$$

such that the following properties hold

- $k \prec \psi(f)$ , for all  $k$  in  $L^W(f')$ ,
- $k \prec \psi(g)$ , for all  $k$  in  $L^W(g')$ ,
- $f''$  is an identity or a rewriting step labelled by  $\psi(f)$ ,
- $g''$  is an identity or a rewriting step labelled by  $\psi(g)$ ,
- $k \prec \psi(f)$  or  $k \prec \psi(g)$ , for all  $k$  in  $L^W(h_1) \cup L^W(h_2)$ .

Such a diagram is then called a *decreasing confluence diagram* (resp. *strictly decreasing confluence diagram*). A labelled rewriting system  $(S, \rightarrow, W, \prec)$  is decreasing if all its local branchings decreasing. An abstract rewriting system  $(S, \rightarrow)$  is decreasing if it admits a labelling  $(W, \prec)$  making all its local branchings decreasing.

**5.1.4. Measure of a branching [?, Definition 3.1].** Let  $(S, \rightarrow, W, \prec)$  be a well-founded labelled rewriting system. Let  $w = w_1 \dots w_n$  and  $w' = w'_1 \dots w'_m$  be 1-cells in the free monoid  $W^*$  with  $w_i$  and  $w'_j$  in  $W$ . We denote by  $w^{(w')}$  the 1-cell  $\bar{w}_1 \dots \bar{w}_n$  such that for every  $0 \leq k \leq n$ , the 1-cell  $\bar{w}_k$  is defined by

$$\bar{w}_k = \begin{cases} 1 & \text{if } w_k \prec w'_j \text{ for some } 1 \leq j \leq m, \\ w_k & \text{otherwise.} \end{cases}$$

Following [?, Definition 3.1], we consider the measure  $|\cdot|$  from the free monoid  $W^*$  to the set of multisets over  $W$  and defined as follows:

- i) for every  $i$  in  $W$ , the multiset  $|i|$  is the singleton  $\{i\}$ ,

ii) for every  $i$  in  $W$  and every 1-cell  $w$  in  $W^*$ , we have  $|i w| = |i| \cup |w^{(i)}|$ .

The measure  $|\cdot|$  is extended to the set of finite rewriting sequences of  $S$  by setting, for every rewriting sequence  $f_1 \cdot \dots \cdot f_n$ , with  $f_i$  labelled by  $k_i$  for all  $i$ ,

$$|f_1 \cdot \dots \cdot f_n| = |k_1 \dots k_n|,$$

where  $k_1 \dots k_n$  is a product in the monoid  $W^*$ . Finally, the measure  $|\cdot|$  is extended to the set of finite branchings  $(f, g)$  of  $\Sigma$ , by setting

$$|(f, g)| = |f| \cup |g|.$$

Recall from [?, Lemma 3.2], that for every 1-cells  $w_1, w_2$  in  $W^*$ , we have

$$|w_1 w_2| = |w_1| \cup |w_2^{(w_1)}|.$$

As a consequence, for any rewriting sequences  $f$  and  $g$  of  $\Sigma$  the following relation holds

$$|f \cdot g| = |f| \cup |g^{(f)}|,$$

where  $|g^{(f)}|$  is defined by

$$|g^{(f)}| = |k_1 \dots k_m^{(l_1 \dots l_n)}|,$$

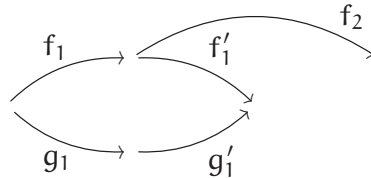
with  $f = f_1 \cdot \dots \cdot f_n$  and  $g = g_1 \cdot \dots \cdot g_m$  and  $f_i$  labelled by  $l_i$  and  $g_j$  labelled by  $k_j$ .

**5.1.5. Multiset order [?, ?].** Given a well-founded set of labels  $(W, \prec)$ , we consider the partial order  $\prec_{\text{mult}}$  on the multisets over  $W$  defined as follows. For any multisets  $M$  and  $N$  over  $W$ , we set  $M \prec_{\text{mult}} N$  if there exist multisets  $X, Y$  and  $Z$  such that:

- i)  $M = Z \cup X, N = Z \cup Y$  and  $Y$  is not empty,
- ii) for every  $i$  in  $W$  such that  $X(i) \neq 0$ , there exists  $j$  in  $W$  such that  $Y(j) \neq 0$  and  $i \prec j$ .

The order  $\prec_{\text{mult}}$  is well-founded because  $\prec$  is. We call  $\preceq_{\text{mult}}$  the symmetric closure of  $\prec_{\text{mult}}$ .

**5.1.6. Lemma.** Let  $(S, \rightarrow, W, \prec)$  be a decreasing labelled rewriting system. For every diagram in  $(S, \rightarrow, W, \prec)$  of the following form



where  $f_1$  is a non empty rewriting sequences,  $f_2$  and  $g_1$  are rewriting sequence and the confluence diagram  $(f_1 \cdot f'_1, g_1 \cdot g'_1)$  is decreasing, the inequality  $|(f'_1, f_2)| \preceq_{\text{mult}} |(g_1, f_1 \cdot f_2)|$  holds.

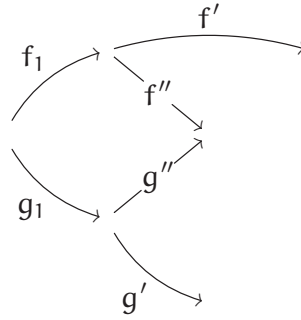
*Proof.* We use the inequalities

$$|(f'_1, f_2)| \preceq_{\text{mult}} |f_1| + |(f_1^{(f_1)}, f_2^{(f_1)})| = |f_1 \star_0 f'_1| + |f_2^{(f_1)}| \preceq_{\text{mult}} |(g_1, f_1)| + |f_2^{(f_1)}| = |(g_1, f_1 \star_0 f_2)|.$$

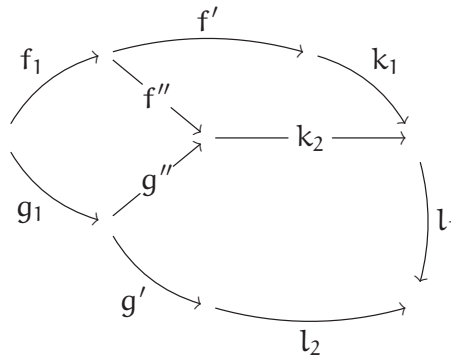
□

**5.1.7. Proposition.** Let  $(S, \rightarrow, W, \prec)$  a labelled rewriting system. Then  $(S, \rightarrow, W, \prec)$  is strictly decreasing if and only if any branching of  $(S, \rightarrow, W, \prec)$  is strictly decreasing.

*Proof.* One implication is trivial. Let us assume that  $(S, \rightarrow, W, \prec)$  is strictly decreasing and let  $(f, g)$  be a branching of  $(S, \rightarrow, W, \prec)$ . We prove by induction on  $|(f, g)|$  that  $(f, g)$  is strictly decreasing. If  $f$  or  $g$  is an empty rewriting sequence, the strict decreasingness of  $(f, g)$  is trivial. Else, we can write



such that the confluence diagram  $(f_1 \cdot f'', g_1 \cdot g'')$  is strictly decreasing. By Lemma ??, we have  $|(f', f'')| \preceq_{\text{mult}} |(f_1, g)|$ . Thus, we have  $|(f', f'')| \preceq_{\text{mult}} |(f, g)|$  and we can use the induction hypothesis to construct a strictly decreasing confluence diagram  $(f' \cdot k_1, f'' \cdot k_2)$ . By using again Lemma ??, we have  $|(g'' \cdot k_2, g')| \preceq_{\text{mult}} |(f, g)|$ . Thus, by applying again the induction hypothesis, we have a diagram



where the diagram  $(f \cdot k_1 \cdot l_1, g \cdot l_2)$  is strictly decreasing.

□



**5.1.8. Lemma (Pasting property [?, Lemma 2.3.17].)** *Let  $(S, \rightarrow, W, \prec)$  be a decreasing labelled rewriting system. For every diagram of the form:*

$$\begin{array}{ccccc}
 & & \xrightarrow{\gamma_1} & \xrightarrow{\gamma_2} & \\
 \delta_0 \downarrow & & & \downarrow \delta_1 & \downarrow \delta_2 \\
 & & \xrightarrow{\tau_1} & \xrightarrow{\tau_2} & 
 \end{array}$$

such that:

- $|\delta_0 \star \tau_1| \preceq_{\text{mult}} |(\delta_0, \gamma_1)|$  and  $|\gamma_1 \star \delta_1| \preceq_{\text{mult}} |(\delta_0, \gamma_1)|$ ,
- $|\delta_1 \star \tau_2| \preceq_{\text{mult}} |(\delta_1, \gamma_2)|$  and  $|\gamma_2 \star \delta_2| \preceq_{\text{mult}} |(\delta_1, \gamma_2)|$ .

We have  $|\delta_0 \star \tau_1 \star \tau_2| \preceq_{\text{mult}} |(\delta_0, \gamma_1 \star \gamma_2)|$  and  $|\gamma_1 \star \gamma_2 \star \delta_2| \preceq_{\text{mult}} |(\delta_0, \gamma_1 \star \gamma_2)|$ .

*Proof.* We use the inequalities

$$|\delta_0 \star \tau_1 \star \tau_2| = |\delta_0 \star \tau_1| + |\tau_2^{(\delta_0 \star \tau_1)}| \preceq_{\text{mult}} |(\delta_0, \gamma_1)| + |\tau_2^{(\delta_0 \star \tau_1)(\gamma_1)}| \preceq_{\text{mult}} |(\delta_0, \gamma_1 \star \gamma_2)|,$$

$$|\gamma_1 \star \gamma_2 \star \delta_2| = |\gamma_1 \star \gamma_2| + |\delta_2^{(\gamma_1)(\gamma_2)}| \preceq_{\text{mult}} |\gamma_1 \star \gamma_2| + |\delta_2^{(\gamma_2)}| \preceq_{\text{mult}} |(\delta_0, \gamma_1 \star \gamma_2)|.$$

□

**5.1.9. van Oostrom's Theorem [?, Theorem 2.3.5].** Any decreasing abstract rewriting system is confluent.

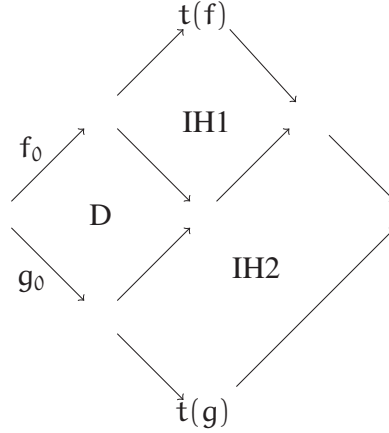
*Proof.* Let  $(S, \rightarrow, W, \prec)$  be a decreasing labelled rewriting system. Let  $(f, g)$  be a branching of  $(S, \rightarrow, W, \prec)$  such that  $f$  and  $g$  are not empty. We prove by well-founded induction that  $(f, g)$  can be completed into a confluence diagram  $(f \cdot f', g \cdot g')$  such that

$$|f \cdot f'| \preceq_{\text{mult}} |(f, g)|, \tag{5.1}$$

$$|g \cdot g'| \preceq_{\text{mult}} |(f, g)|. \tag{5.2}$$

Let us consider  $f_0$  the first rewriting step of  $f$  and  $g_0$  the first rewriting step of  $g$ . We have a

confluence diagram



where  $D$  verifies (??) and (??) by decreasingness of  $(S, \rightarrow, W, \prec)$ . The diagram  $IH1$  exists because of the induction hypothesis and verifies (??) and (??) by the pasting property. Finally, the induction hypothesis allows us to construct  $IH2$ . The pasting property proves that all the diagram verifies (??) and (??). This proves in particular by well-founded induction that  $\Sigma$  is confluent.  $\square$

## 5.2. DECREASING POLYGRAPHS

In this subsection, we introduce decreasing polygraphs. The notions of decreasingness and strict decreasingness for polygraphs will be the same as in the case of abstract rewriting systems by viewing polygraphs as abstract rewriting systems, see ??.

**5.2.1. Labelled polygraphs.** A *labelled  $n$ -polygraph* is a data  $(\Sigma, W, \prec, \psi)$  made of an  $n$ -polygraph  $\Sigma$ , a set  $W$ , a well-founded order  $\prec$  on  $W$  and a map  $\psi : \Sigma_{stp} \rightarrow W$  where  $\Sigma_{stp}$  is the set of rewriting steps of  $\Sigma$ . The map  $\psi$  is called a *well-founded labelling* of  $\Sigma$  and associates to a rewriting step  $f$  a label  $\psi(f)$ . In particular, labelled  $n$ -polygraphs are also labelled rewriting systems.

**5.2.2. Labelling to the quasi-normal form.** Let  $n > 0$ . Let  $\Sigma$  be an  $n$ -polygraph (respectively a linear  $(n, n - 1)$ -polygraph). If  $\Sigma$  is quasi-terminating, any  $(n - 1)$ -cell  $u$  of  $\Sigma_{n-1}^*$  (respectively  $\Sigma_{n-1}^\ell$ ) admits a quasi-normal form. Let us fix a family of quasi-normal forms  $Q$  such that any  $(n - 1)$ -cell  $u$  in  $\Sigma_{n-1}^*$  (respectively  $\Sigma_{n-1}^\ell$ ) rewrites into an  $(n - 1)$ -cell in  $Q$ . For each  $(n - 1)$ -cell  $u$  in  $\Sigma_{n-1}^*$  (respectively  $\Sigma_{n-1}^\ell$ ), let us choose  $\tilde{u}$  a quasi-normal form of  $u$  in  $Q$

such that  $d(u, \tilde{u})$  is minimal. The *labelling to the quasi-normal form*, labelling QNF for short, is the map  $\psi : \Sigma_{\text{stp}} \longrightarrow \mathbb{N}$  defined by

$$\psi(f) = d(t_{n-1}(f), \widetilde{t_{n-1}(f)}),$$

for any rewriting step  $f$  of  $\Sigma$ . Note that if  $\Sigma$  is confluent, then any labelling QNF of  $\Sigma$  is decreasing.

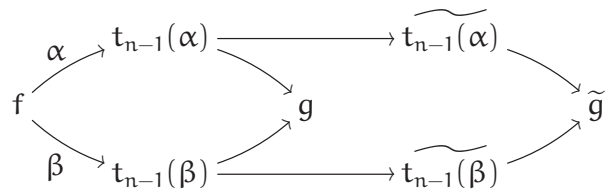
**5.2.3. Remark.** If  $\Sigma$  is confluent, any two congruent  $(n-1)$ -cells of  $\Sigma_{n-1}^*$  (respectively  $\Sigma_{n-1}^\ell$ ) can be rewritten into a same quasi-normal form.

**5.2.4. Theorem.** Let  $n > 0$ . Let  $\Sigma$  a decreasing left-monomial linear  $(n, n-1)$ -polygraph. Then,  $\Sigma$  is confluent.

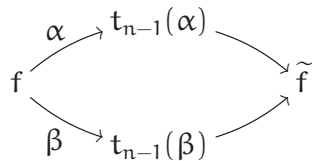
*Proof.* By van Oostrom's Theorem, any decreasing abstract is confluent. Thus, any decreasing left-monomial linear  $(n, n-1)$ -polygraph is confluent.  $\square$

**5.2.5. Theorem.** Let  $n$  be a non negative integer. Let  $\Sigma$  be an exponentiation free linear  $(n, n-1)$ -polygraph and  $\psi$  a labelling to the quasi-normal form on  $\Sigma$ . If all critical branchings of  $\Sigma$  are decreasing with respect to  $\psi$ , then  $\Sigma$  is decreasing.

*Proof.*  $\Sigma$ , having a labelling QNF, is quasi-terminating. By Theorem ??, this implies that  $\Sigma$  is locally confluent because  $\Sigma$  is also exponentiation free. Let  $(\alpha, \beta)$  be a local branching of  $\Sigma$ . Let us prove that the branching  $(\alpha, \beta)$  is decreasing with respect to  $\psi$ . Let us set  $f = s_{n-1}(\alpha) = s_{n-1}(\beta)$ . By local confluence of  $\Sigma$ , we have a diagram



where we have either  $\tilde{f} = \widetilde{t_{n-1}(\alpha)}$  or  $\tilde{f} = \widetilde{t_{n-1}(\beta)}$  and the confluence diagrams of sources  $\widetilde{t_{n-1}(\alpha)}$  and  $\widetilde{t_{n-1}(\beta)}$  exist because all overlapping branchings of  $\Sigma$  are confluent. Because  $\widetilde{t_{n-1}(\alpha)}$  and  $\widetilde{t_{n-1}(\beta)}$  are quasi-normal forms, they rewrite into each other. This implies the existence of a decreasing diagram



and conclude the proof.  $\square$

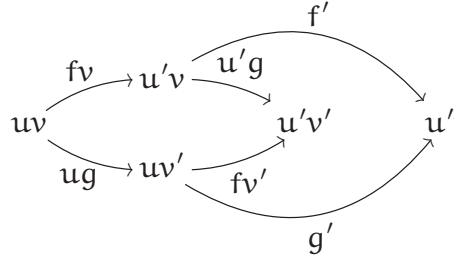
**5.2.6. Stability by contexts.** Let  $n > 0$ . A *context* in an  $n$ -category  $\mathcal{C}$  is an application  $C$  from  $\mathcal{C}_n(a, b)$  for some  $(n-1)$ -cells  $a$  and  $b$  to  $\mathcal{C}_n$  defined by

$$C(x) = u_1 \star_{n-1} (u_2 \star_{n-2} (\cdots u_n \star_0 x \star_0 u_n) \cdots \star_{n-2} u_{2n-1}) \star_{n-1} u_{2n}$$

for some family  $(u_i)_{i \in \llbracket 1, 2n \rrbracket}$  of  $n$ -cells and for any  $x$  in  $\mathcal{C}_n(a, b)$ .

Let  $(\Sigma, \psi)$  be a labelled  $n$ -polygraph. The labelling  $\psi$  is *compatible with contexts* if for any decreasing (resp. strictly decreasing) confluence diagram  $(f \cdot f', g \cdot g')$ , where  $(f, g)$  is a local branching, and any context  $C$  of the free  $n$ -category  $\Sigma^*$  such that  $(C(f) \cdot C(f'), C(g) \cdot C(g'))$  is defined, the diagram  $(C(f) \cdot C(f'), C(g) \cdot C(g'))$  is decreasing (resp. strictly decreasing).

**5.2.7. Peiffer decreasingness.** A decreasing (resp. strictly decreasing)  $n$ -polygraph  $(\Sigma, \psi)$  is *Peiffer decreasing* with respect to a globular extension  $\Gamma$  of the free  $(n, n-1)$ -category  $\Sigma_n^\top$  over  $\Sigma$  if, for any Peiffer branching  $(fv, ug) : uv \Rightarrow (u'v, uv')$ , there exists a decreasing (resp. strictly decreasing) confluence diagram  $(fv \cdot f', ug \cdot g')$ :



such that  $u'g \star_{n-1} (fv')^- \equiv_\Gamma f' \star_{n-1} (g')^-$ . Here we denote for any  $n$ -cell  $x$  of  $\Sigma_n^\top$ , the inverse of  $x$  by  $x^-$ .

## 5.3. SQUIER'S DECREASING COMPLETION

**5.3.1. Loops in an  $n$ -polygraph.** An  $n$ -loop in the  $n$ -category  $\Sigma_n^*$  is an  $n$ -cell  $f$  of  $\Sigma_n^*$  such that  $s_1(f) = t_1(f)$ . Two  $n$ -loops  $f$  and  $g$  in  $\Sigma_n^*$  are *equivalent* if there exist a decomposition

$$f = f_1 \star_{n-1} \cdots \star_{n-1} f_p$$

where  $f_i$  is a rewriting step of  $\Sigma$  for any  $1 \leq i \leq p$ , and a circular permutation  $\sigma$  such that

$$g = f_{\sigma(1)} \star_{n-1} \cdots \star_{n-1} f_{\sigma(p)}.$$

This defines an equivalence relation on  $n$ -cells of  $\Sigma_n^*$ . We will denote by  $\mathcal{L}(f)$  the equivalence class of an  $n$ -loop  $f$  in  $\Sigma_n^*$  for this relation.

**5.3.2. Support of an  $n$ -cell.** Any  $n$ -cell  $f$  in  $\Sigma_n^*$  can be written as a composite of finitely many rewriting steps  $C_1(\varphi_1), \dots, C_k(\varphi_k)$ , where the  $C_i$  are contexts and the  $\varphi_i$  are  $n$ -cell of  $\Sigma_n$ . We define the *support* of the  $n$ -cell  $f$  as the multiset, denoted by  $\text{Supp}(f)$ , consisting of the  $n$ -cells  $\varphi_i$  occurring in this decomposition. The support is well-defined because any decomposition of  $f$  in  $\Sigma_n^*$  into a composite of rewriting steps involves the same elements of  $\Sigma_n$ . Note also that any such a decomposition is finite and thus the support of an  $n$ -cell is a finite multiset. As a consequence, the multiset inclusion is a well-founded order on supports, allowing us to prove some properties by induction on the support of  $n$ -cells.

**5.3.3. Minimal and elementary loops.** We say that an  $n$ -loop  $f$  in  $\Sigma_n^*$  is

- i) *minimal with respect to  $(n-1)$ -composition*, if any decomposition  $f = g \star_{n-1} h \star_{n-1} k$  in  $\Sigma_n^*$  with  $h$  an  $n$ -loop implies that  $h$  is either an identity or equal to  $f$ ,
- ii) *minimal by context*, if there is no decomposition  $f = C(g)$ , where  $C$  is a context and  $g$  is a loop in  $\Sigma_n^*$ .

An  $n$ -loop  $f$  in  $\Sigma_n^*$  is *elementary* if it is minimal both with respect to  $(n-1)$ -composition and by context. As an immediate consequence of these definitions, any  $n$ -loop  $f$  minimal for  $(n-1)$ -composition can be written  $f = C(g)$ , where  $g$  is an elementary loop and  $C$  is a context.

**5.3.4. Lemma.** For any equivalent  $n$ -loops  $f$  and  $g$  in  $\Sigma_n^*$ , there exist  $n$ -cells  $h$  and  $k$  of  $\Sigma_n^\top$  such that  $f = h \star_{n-1} g \star_{n-1} k$ .

*Proof.* Let us decompose  $f$  into a sequence  $f = f_1 \star_{n-1} \dots \star_{n-1} f_p$  of rewriting steps and let  $\sigma$  be a circular permutation such that  $g = f_{\sigma(1)} \star_{n-1} \dots \star_{n-1} f_{\sigma(p)}$ . Let  $i$  be the integer such that  $\sigma(i) = 1$ . Let  $k$  be the  $n$ -cell  $f_{\sigma(1)} \star_{n-1} \dots \star_{n-1} f_{\sigma(i-1)}$ . Let  $h = k^-$  be the inverse of  $k$  for the  $(n-1)$ -composition. Then, we have  $f = h \star_{n-1} g \star_{n-1} k$ .  $\square$

**5.3.5. Lemma.** Let  $f$  be a nonidentity  $n$ -loop in  $\Sigma_n^*$ . Then, there exists a decomposition

$$f = f_1 \star_{n-1} f' \star_{n-1} f_2$$

in  $\Sigma_n^*$ , where  $f'$  is an  $n$ -loop minimal with respect to  $(n-1)$ -composition and  $f_1$  and  $f_2$  are  $n$ -cells such that  $f_1 \star_{n-1} f_2$  is an  $n$ -loop.

*Proof.* Let  $f$  be a nonidentity  $n$ -loop in  $\Sigma_n^*$ . The proof is by induction on the support  $\text{Supp}(f)$ . If the  $n$ -loop  $f$  is minimal for  $(n-1)$ -composition, we can write  $f = 1_{s_1(f)} \star_{n-1} f \star_{n-1} 1_{s_1(f)}$ . If  $f$  is not minimal for  $(n-1)$ -composition, there exists a decomposition  $f = g \star_{n-1} h \star_{n-1} k$ , where  $h$  is an  $n$ -loop that is neither an identity nor equal to  $f$ . Hence,  $\text{Supp}(h)$  is strictly included in  $\text{Supp}(f)$ . This proves the decomposition.  $\square$

**5.3.6. Lemma.** Let  $\mathcal{L}(\Sigma)$  be a loop extension of  $\Sigma$ . For any  $n$ -loop  $f$  in  $\Sigma_n^*$ , there exists an  $(n+1)$ -cell from  $f$  to  $1_{s_1(f)}$  in the free  $(n+1, n-1)$ -category  $\mathcal{L}(\Sigma)^\top$  generated by the  $(n+1, n-1)$ -polygraph  $(\Sigma, \mathcal{L}(\Sigma))$ .

*Proof.* Let us fix a loop extension  $\mathcal{L}(\Sigma)$ . Let  $f$  be  $n$ -loop in  $\Sigma_n^*$ . We proceed by induction on the support  $\text{Supp}(f)$ .

**Step 1.** Suppose that  $f$  is elementary. By definition of  $\mathcal{L}(\Sigma)$ , the equivalence class  $\mathcal{L}(f)$  contains an elementary  $n$ -loop  $e$  such that  $\mathcal{L}(\Sigma)$  contains an  $(n+1)$ -cell  $A_e$  from  $e$  to  $1_{s_1(e)}$ . The  $n$ -loop  $e$  being equivalent to  $f$ , by Lemma ?? there exist two  $n$ -cells  $h$  and  $k$  of  $\Sigma_n^\top$  such that  $f = h \star_{n-1} e \star_{n-1} k$ . Thus, the  $(n+1)$ -cell  $h \star_{n-1} A_e \star_{n-1} k$  in  $\mathcal{L}(\Sigma)^\top$  goes from  $f$  to  $h \star_{n-1} k$ . By construction the  $n$ -cell  $h \star_{n-1} k$  is equal  $1_{s_1(f)}$ . In this way we construct an  $(n+1)$ -cell in  $\mathcal{L}(\Sigma)^\top$  from  $f$  to  $1_{s_1(f)}$ .

**Step 2.** Suppose that  $f$  is minimal with respect to  $(n-1)$ -composition. Then, there is a decomposition  $f = ugv$ , where  $u$  and  $v$  are  $(n-1)$ -cells in  $\Sigma_{n-1}^*$  and  $g$  is an elementary  $n$ -loop in  $\Sigma_n^*$ . By Step 1, there exists an  $(n+1)$ -cell  $A_g$  from  $g$  to  $1_{s_1(g)}$  in  $\mathcal{L}(\Sigma)^\top$ . Thus,  $uA_gv$  is an  $(n+1)$ -cell in  $\mathcal{L}(\Sigma)^\top$  from  $f$  to  $1_{s_1(f)}$ .

**Step 3.** Suppose that  $f$  is a nonidentity  $n$ -loop. By Lemma ??, the  $n$ -loop  $f$  can be written as  $f_1 \star_{n-1} f' \star_{n-1} f_2$  where  $f'$  is an  $n$ -loop minimal for  $(n-1)$ -composition and  $f_1$  and  $f_2$  are  $n$ -cells such that  $f_1 \star_{n-1} f_2$  is an  $n$ -loop. By Step 2, there exists an  $(n+1)$ -cell  $A_{f'}$  in  $\mathcal{L}(\Sigma)^\top$  from  $f'$  to  $1_{s_1(f')}$ . Hence, the  $(n-1)$ -composite  $f_1 \star_{n-1} A_{f'} \star_{n-1} f_2$  is an  $(n+1)$ -cell from  $f$  to  $f_1 \star_{n-1} f_2$  in  $\mathcal{L}(\Sigma)^\top$ . The support of  $f_1 \star_{n-1} f_2$  being strictly included in the support of  $f$ , this proves the lemma by induction on the support of  $f$ .  $\square$

**5.3.7. Loop extension.** We will denote by  $\mathcal{E}(\Sigma)$  the set of equivalence classes of elementary  $n$ -loops of  $\Sigma_n^*$ . A *loop extension* of  $\Sigma$  is a globular extension of the  $(n, n-1)$ -category  $\Sigma_n^\top$  made of a family of  $(n+1)$ -cells  $A_\alpha : \alpha \Rightarrow 1_{s_1(\alpha)}$  indexed by exactly one  $\alpha$  for each equivalence class in  $\mathcal{E}(\Sigma)$ .

**5.3.8. Generating decreasing confluences.** A *family of generating decreasing confluences of  $\Sigma$  with respect to  $\psi$*  is a globular extension of the  $(n, n-1)$ -category  $\Sigma_n^\top$  that contains, for every critical branching  $(f, g) : u \Rightarrow (v, w)$  of  $\Sigma$ , exactly one  $(n+1)$ -cell  $D_{f,g}^\psi$  of the following form

$$\begin{array}{ccccc}
 & & f & \rightarrow & v & \xrightarrow{f'} & & & \\
 & & \searrow & & & \searrow & & & \\
 u & & & & & & & & u' \\
 & & \swarrow & & & \swarrow & & & \\
 & & g & \rightarrow & w & \xrightarrow{g'} & & & 
 \end{array}
 \quad \Downarrow D_{f,g}^\psi$$

and where the confluence diagram  $(f \cdot f', g \cdot g')$  is decreasing with respect to  $\psi$ . We denote here the  $n$ -cells by 1-arrows and the  $(n+k)$ -cells by  $(n+k)$ -arrows for any positive integer  $k$ . Any decreasing  $n$ -polygraph admits such a family of generating decreasing confluences. Indeed, any

critical branching is local and thus confluent by decreasingness hypothesis. However, such a family is not unique in general.

**5.3.9. Squier's decreasing completion.** Let  $(\Sigma, \psi)$  be a decreasing  $n$ -polygraph. A *Squier's decreasing completion of  $\Sigma$  with respect to  $\psi$*  is an  $(n + 1, n - 1)$ -polygraph that extends the  $n$ -polygraph  $\Sigma$  by a globular extension

$$\mathcal{O}(\Sigma, \psi) \cup \mathcal{L}(\Sigma)$$

where  $\mathcal{O}(\Sigma, \psi)$  is a chosen family of generating decreasing confluences with respect to  $\psi$  and  $\mathcal{L}(\Sigma)$  is a loop extension of  $\Sigma$ . If  $(\Sigma, \psi)$  is a strictly decreasing  $n$ -polygraph, a *strictly decreasing Squier's completion* is a Squier's decreasing completion, whose generating decreasing confluences are required to be strict.

**5.3.10. Lemma.** Let  $(\Sigma, \psi)$  be a strictly decreasing  $n$ -polygraph. Let  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)$  be a strictly decreasing Squier's completion of  $\Sigma$ . Suppose that  $\psi$  is compatible with contexts and that  $(\Sigma, \psi)$  is Peiffer decreasing with respect to the extension  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)$ . Then, for any  $n$ -sphere  $(f, g)$  in  $\Sigma_n^*$ , there exists an  $(n + 1)$ -cell from  $f$  to  $g$  in the  $(n + 1, n - 1)$ -category  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)^\top$ .

*Proof.* We proceed in two steps.

**Step 1.** We prove that, for every local branching  $(f, g) : u \Rightarrow (v, w)$  of  $\Sigma$ , there exists a confluence  $(f', g') : (v, w) \Rightarrow u'$  of  $\Sigma$  and an  $(n + 1)$ -cell  $A : f \star_{n-1} f' \Rightarrow g \star_{n-1} g'$  in  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)^\top$  such that the confluence diagram  $(f \cdot f', g \cdot g')$  is strictly decreasing.

In the case of an aspherical branching, we can choose  $f'$  and  $g'$  to be identity  $n$ -cells,  $A$  to be an identity  $(n + 1)$ -cell and the confluence diagram  $(f, f)$  is trivially strictly decreasing.

Suppose that  $(f, g)$  is a Peiffer branching  $(f_1 v_1, u_1 g_1) : u_1 v_1 \Rightarrow (u'_1 v_1, u_1 v'_1)$ . By hypothesis, the Peiffer confluence  $(f_1 v_1 \cdot u'_1 g_1, u_1 g_1 \cdot f_1 v'_1)$  is equivalent to a strictly decreasing confluence diagram  $(f_1 v_1 \cdot f'_1, u_1 g_1 \cdot g'_1)$ . Hence, there exists an  $(n + 1)$ -cell  $A : f_1 v_1 \star_{n-1} f'_1 \Rightarrow u_1 g_1 \star_{n-1} g'_1$  in the  $(n + 1, n - 1)$ -category  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)^\top$ .

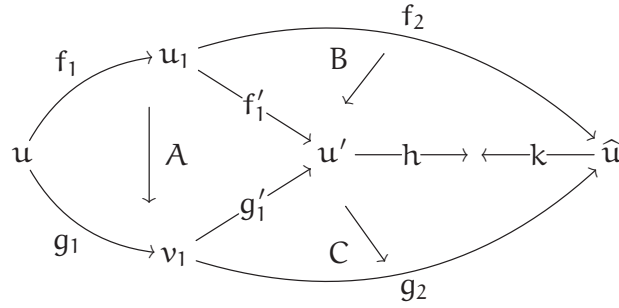
If  $(f, g)$  is an overlapping branching, we have  $(f, g) = (C(h), C(k))$  with  $(h, k)$  a critical branching and  $C$  a context. We consider the  $(n + 1)$ -cell  $D_{h,k}^\psi : h \star_{n-1} h' \Rightarrow k \star_{n-1} k'$  of  $\mathcal{O}(\Sigma, \psi)$  corresponding to the strict generating decreasing confluence of the critical branching  $(h, k)$  with respect to the labelling  $\psi$ , or its inverse. Let us define the  $n$ -cells  $f' = C(h')$  and  $g' = C(k')$  and the  $(n + 1)$ -cell  $A = C(D_{h,k}^\psi)$ . The labelling  $\psi$  being compatible with contexts, the confluence diagram corresponding to the  $(n + 1)$ -cell  $A$  is strictly decreasing.

**Step 2.** Let  $(f, g)$  be an  $n$ -sphere in  $\Sigma_n^*$ . This  $n$ -sphere defines a branching with source  $s_{n-1}(f) = s_{n-1}(g)$ . The  $n$ -polygraph  $\Sigma$  being strictly decreasing, we prove the lemma by well-founded induction on the measure  $|(f, g)|$  of the branching  $(f, g)$ . If  $f$  or  $g$  is an identity  $n$ -cell, say  $g = 1$ , the  $n$ -cell  $f$  is an  $n$ -loop. By Lemma ??, there exists an  $(n + 1)$ -cell  $E :$

### 5.3. SQUIER'S DECREASING COMPLETION

$f \Rightarrow 1_{s_1(f)}$  in the  $(n+1, n-1)$ -category  $\mathcal{L}(\Sigma)^\top$ . Else, we have decompositions  $f = f_1 \star_{n-1} f_2$  and  $g = g_1 \star_{n-1} g_2$  in  $\Sigma_n^*$  where  $(f_1, g_1)$  is a local branching. Note that  $f_2$  or  $g_2$  can be equal to an identity  $n$ -cell. The local branching  $(f_1, g_1)$  is confluent by decreasingness. Moreover, by Step 1, there exists an  $(n+1)$ -cell  $A : f_1 \star_{n-1} f'_1 \Rightarrow g_1 \star_{n-1} g'_1$  in the  $(n+1, n-1)$ -category  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)^\top$ , where the confluence diagram  $(f_1 \cdot f'_1, g_1 \cdot g'_1)$  is strictly decreasing.

The branching  $(f'_1, f_2)$  is confluent by decreasingness. Moreover, the  $n$ -polygraph  $\Sigma$  being strictly decreasing, by Proposition ??, there exist rewriting sequences  $h$  and  $k$  as indicated in the following diagram:



such that the confluence diagrams  $(f'_1 \cdot h, f_2 \cdot k)$  is strictly decreasing.

Consider the multiset order  $\preccurlyeq_{\text{mult}}$  associated to the order  $\prec$ . The confluence diagram  $(f_1 \cdot f'_1, g_1 \cdot g'_1)$  being strictly decreasing, for any  $k$  in  $L^W(f'_1)$  and any  $l$  in  $L^W(g'_1)$ , we have  $k \prec \psi(f_1)$  and  $l \prec \psi(g_1)$ . Thus  $|f_1 \cdot f'_1| = |f_1|$  and  $|g_1 \cdot g'_1| = |g_1|$ . This implies the following equality

$$|(f_1, g_1)| = |(f, g)|.$$

The confluence diagram  $(f'_1 \cdot h, f_2 \cdot k)$  being strictly decreasing, by the same argument, we have

$$|(f'_1 \cdot h, f_2 \cdot k)| = |(f'_1, f_2)|.$$

Moreover, by Lemma ??, we have  $|(f'_1, f_2)| \preccurlyeq_{\text{mult}} |(f, g_1)|$ . It follows that

$$|(f'_1 \cdot h, f_2 \cdot k)| \preccurlyeq_{\text{mult}} |(f, g)|.$$

By induction hypothesis, we deduce that there exists an  $(n+1)$ -cell  $B : f_2 \star_{n-1} k \Rightarrow f'_1 \star_{n-1} h$  in  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)^\top$ .

Finally, let us prove that there exists an  $(n+1)$ -cell  $C : g'_1 \star_{n-1} h \Rightarrow g_2 \star_{n-1} k$  in  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)^\top$ . We have

$$|(g'_1 \cdot h, g_2 \cdot k)| = |g'_1| \cup |h^{(g'_1)}| \cup |g_2| \cup |k^{(g_2)}|.$$

On the other hand, we have

$$|(f, g)| = |f| \cup |g| = |f| \cup |g_1| \cup |g_2^{(g_1)}|.$$



Furthermore, there exists a multiset  $R$ , possibly empty, such that  $|g_2| = |g_2^{(g_1)}| \cup R$ . Hence

$$|(g'_1 \cdot h, g_2 \cdot k)| = |g_2^{(g_1)}| \cup X \quad \text{and} \quad |(f, g)| = |g_2^{(g_1)}| \cup Y.$$

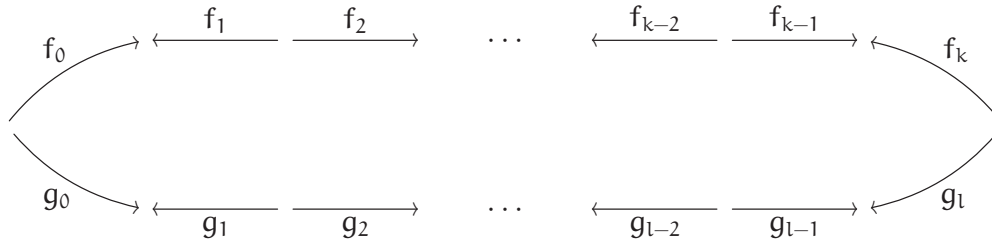
where  $X = |g'_1| \cup |h^{(g'_1)}| \cup R \cup |k^{(g_2)}|$  and  $Y = |f| \cup |g_1|$ . Moreover, we check that for every  $i$  in  $W$  such that  $X(i) \neq 0$ , there exists  $j$  in  $W$  such that  $Y(j) \neq 0$  and  $i \prec j$ . Hence, we have

$$|(g'_1 \cdot h, g_2 \cdot k)| \preceq_{\text{mult}} |(f, g)|.$$

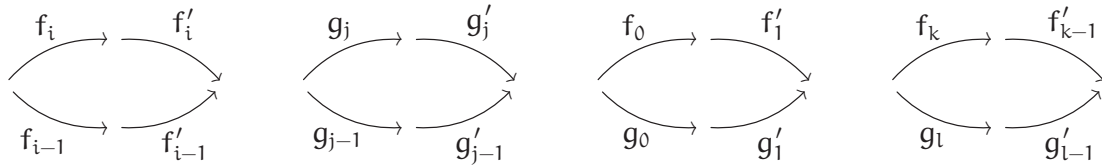
The existence of the  $(n+1)$ -cell  $C$  follows by induction hypothesis. In this way, we have constructed an  $(n+1)$ -cell in  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)^\top$  from  $f$  to  $g$  obtained by composition of the  $(n+1)$ -cells  $A$ ,  $B$  and  $C$ .  $\square$

**5.3.11. Theorem.** *Let  $(\Sigma, \psi)$  be a strictly decreasing  $n$ -polygraph. Let  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)$  be a strictly decreasing Squier's completion of  $\Sigma$ . If  $\psi$  is compatible with contexts and  $(\Sigma, \psi)$  is Peiffer decreasing with respect to the extension  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)$ , then  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)$  is a coherent presentation of the  $(n-1)$ -category presented by  $\Sigma$ .*

*Proof.* Let  $(f, g)$  be an  $n$ -sphere of the  $(n, n-1)$ -category  $\Sigma_n^\top$ . By definition of  $\Sigma_n^\top$ , the  $n$ -cell  $f \star_{n-1} g^-$  can be decomposed into a zigzag

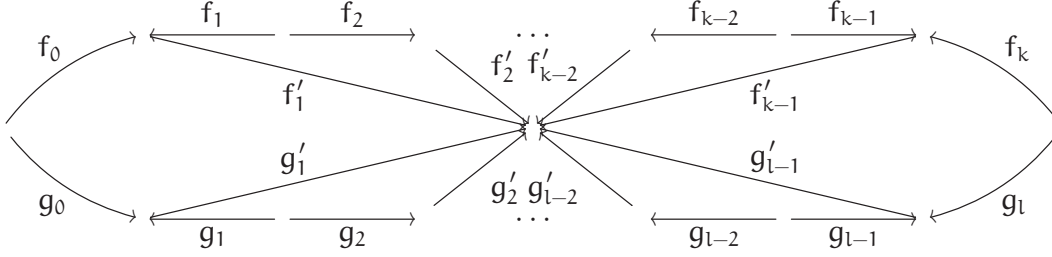


where the  $n$ -cells  $f_0, \dots, f_k$  and  $g_0, \dots, g_l$  are  $n$ -cells of the  $n$ -category  $\Sigma_n^*$ . Note that some of those  $n$ -cells can be identities. By confluence of the  $n$ -polygraph  $\Sigma$ , there exist families of  $n$ -spheres of  $\Sigma_n^*$



with same  $(n-1)$ -target, for all  $2 \leq i \leq k-1$  and  $2 \leq j \leq l-1$ . Note that some of these  $n$ -spheres can be trivial. Then the  $n$ -sphere  $(f, g)$  can be filled up by these  $n$ -spheres as

follows:



By Lemma ??, these  $n$ -spheres can be filled up by  $(n + 1)$ -cells of the  $(n + 1, n - 1)$ -category  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)^\top$ . Finally, the composition of these  $(n + 1)$ -cells gives an  $(n + 1)$ -cell of  $\mathcal{S}^{\text{sd}}(\Sigma, \psi)^\top$  from  $f$  to  $g$ .  $\square$

## 5.4. COHERENCE BY DECREASINGNESS

We fix in this section  $\Sigma$  a left-monomial linear  $(n, n - 1)$ -polygraph. We will also denote  $\psi : \Sigma_{\text{stp}} \rightarrow W$  a well-founded labelling of  $\Sigma$  where  $W$  is a set endowed with a well-founded order  $\prec$ . We first state the following result:

**5.4.1. Linear contexts.** A *context* in a linear  $(n, n)$ -category  $\mathcal{C}$  is an application  $C$  from  $\mathcal{C}_n(a, b)$  for some  $(n - 1)$ -cells  $a$  and  $b$  to  $\mathcal{C}_n$  defined by

$$C(x) = \lambda u_1 \star_{n-1} (u_2 \star_{n-2} (\cdots u_n \star_0 x \star_0 u_n) \cdots \star_{n-2} u_{2n-1}) \star_{n-1} u_{2n} + h$$

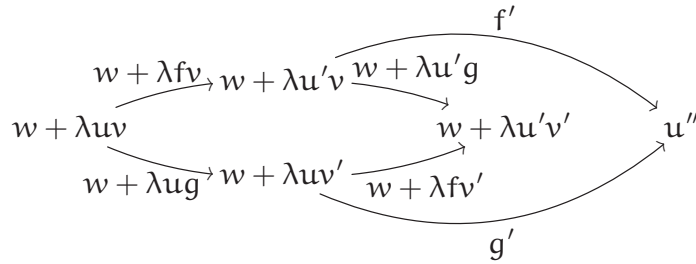
for some family  $(u_i)_{i \in [1, 2n]}$  of  $n$ -cells,  $n$ -cell  $h$ , scalar  $\lambda$  and for any  $x$  in  $\mathcal{C}_n(a, b)$ .

**5.4.2. Minimal and elementary loops in linear polygraphs.** A loop  $f$  of  $\Sigma^\ell$  is minimal by context if there is no decomposition  $f = C(g)$ , where  $C$  is a context of  $\Sigma^\ell$  and  $g$  is a loop in  $\Sigma_n^*$ . An elementary loop in  $\Sigma^\ell$  is a loop which is both minimal with respect to  $(n - 1)$ -composition and by context.

**5.4.3. Stability by contexts in labelled linear polygraph.** The well-founded labelling  $\psi$  is compatible with contexts if for any decreasing (resp. strictly decreasing) confluence diagram  $(f \cdot f', g \cdot g')$ , where  $(f, g)$  is a local branching, and any context  $C$  of the free  $n$ -category  $\Sigma^\ell$  such that  $(C(f) \cdot C(f'), C(g) \cdot C(g'))$  is defined, the diagram  $(C(f) \cdot C(f'), C(g) \cdot C(g'))$  is decreasing (resp. strictly decreasing).

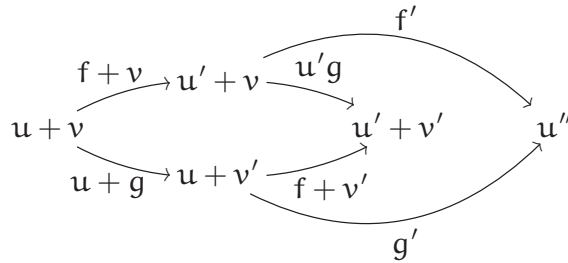
**5.4.4. Peiffer decreasingness in linear polygraphs.** Let  $\Gamma$  be a globular extension of the linear  $(n, n)$ -category  $\Sigma^\ell$ . The labelled  $(n, n)$ -linear polygraph  $(\Sigma, \psi)$  is (strictly) Peiffer decreasing with respect to  $\Gamma$  if the following conditions hold

- i)  $\Sigma$  is decreasing with respect to the labelling  $(W, \prec, \psi)$ ,
- ii) for any Peiffer branching  $(w + \lambda fv, w + \lambda ug)$ , there exists a (strictly) decreasing confluence diagram:



such that  $(w + \lambda fv) \star_{n-1} f' \equiv_{\Gamma} (w + \lambda ug) \star_{n-1} g'$ .

- iii) for any additive branching  $(f + v, u + g)$ , there exists a (strictly) decreasing confluence diagram:



such that  $(f + v) \star_{n-1} f' \equiv_{\Gamma} (u + g) \star_{n-1} g'$ .

**5.4.5. Proposition.** Let  $\Sigma$  be a strictly decreasing left-monoidal linear  $(n, n - 1)$ -polygraph with respect to a strict well-founded labelling  $(W, \psi, \prec)$ . Let  $\mathcal{D}(\Sigma, \psi)$  be a strict decreasing Squier's completion of  $\Sigma$ . If the following conditions hold

- i)  $\psi$  is whisker compatible,
- ii)  $\psi$  is strictly Peiffer decreasing with respect to  $\mathcal{D}(\Sigma, \psi)$ ,

then the strict decreasing Squier's completion  $\mathcal{D}(\Sigma, \psi)$  is a coherent presentation of the linear  $(n - 1, n - 1)$ -category presented by  $\Sigma$ .

*Proof.* To prove this Proposition, we need to prove that any local branching of  $\Sigma$  can be completed into a strict decreasing diagram. Step 2 of the proof of Lemma [?, Lemma 3.1.4.] is the same. The cases of aspherical, Peiffer and overlapping branchings are treated in Step 1 of the proof of Lemma [?, Lemma 3.1.4.]. By strict Peiffer decreasingness of the linear polygraph  $\Sigma$ , we conclude the proof.  $\square$



## Categorification of linear categories

**Categorification.** The terminology "categorification" was introduced in 1995 by Crane. Categorification is a process giving from an (associative) algebra  $\mathbf{A}$  a linear monoidal category  $\mathcal{C}$  whose Grothendieck group is isomorphic to  $\mathbf{A}$  [?, ?]. We say that  $\mathcal{C}$  is a categorification of  $\mathbf{A}$  or, *categorifies*  $\mathbf{A}$ . An example of a categorification is the Khovanov homology [?], a categorification of the Jones polynomials. More precisely, Khovanov defines a family of chain complexes whose homologies have the Jones polynomials as Euler's characteristics. This categorification was used to give a new proof of Milnor's conjecture [?]. A classical categorification problem is to exhibit from a ring  $\mathbf{A}$  a categorification of  $\mathbf{A}$  with properties such that having all its idempotents split. Algebras like Hecke algebras [?] can be categorified by the Karoubi envelope of a diagrammatic category, that is a monoidal category in which the morphism spaces are depicted by string diagrams, the 0-composition by horizontal concatenation and the 1-composition by vertical concatenation. Khovanov conjectured that the Karoubi envelope of some diagrammatic category [?] categorifies the Heisenberg algebra [?]. More applications of categorification can be found in [?].

**Karoubi envelopes.** Categorifications of algebras are generally defined as Karoubi envelopes of explicit categories. Karoubi envelopes of categories were introduced as a way to classify the idempotents of a category [?]. The Karoubi envelope of a category  $\mathcal{C}$  is a completion  $\text{Kar}(\mathcal{C})$  of  $\mathcal{C}$  splitting all idempotents. More precisely, the Karoubi envelope of  $\mathcal{C}$  contains the objects of  $\mathcal{C}$  plus one object for each non identity idempotent of  $\mathcal{C}$  and is generated by the morphisms of  $\mathcal{C}$  plus a family of injections and projections morphisms verifying relations splitting all idempotents. In particular, if all idempotents of the category  $\mathcal{C}$  are split, the category  $\text{Kar}(\mathcal{C})$  is equivalent to  $\mathcal{C}$ . Karoubi envelopes also find applications in topology when considering some enriched categories [?, ?]. The construction of Karoubi envelopes can also be defined for higher-dimensional categories. We wish to present Karoubi envelopes of higher-dimensional categories

by generators and relations. Karoubi envelopes being used to construct categorifications of algebras, computing a presentation of  $\text{Kar}(\mathcal{C})$  from a presentation of  $\mathcal{C}$  is the first step to present categorification of algebras presented by generators and relations.

**Higher-dimensional categorification.** The categorification process can be defined for higher-dimensional linear categories, a case that generalizes the one of algebras. This definition uses a generalization of the Grothendieck group we present in this chapter: *Grothendieck decategorification*. The process we present in this chapter to compute Karoubi envelopes and Grothendieck decategorification can be generalized to any dimension.

**Plan of chapter.** In this chapter, we present the applications of rewriting to categorification we introduced in [?]. We begin in Section ?? by recalling the notion of Karoubi envelope for 1-categories from [?]. We also introduce notion of Karoubi envelope for  $n$ -categories and linear  $(n, n)$ -categories given in [?]. In Section ??, we recall the definitions of Grothendieck group and Grothendieck decategorification from [?]. In Section ??, we introduce the notions of Karoubi envelope and decategorification for polygraphs. We also prove that those notions respectively present the Karoubi envelope ?? and the Grothendieck decategorification ?? of higher-dimensional categories. This chapter ends in ?? by a presentation of a categorification of the Hecke algebra given in [?], the category of Soergel bimodules.

## 6.1. KAROUBI ENVELOPES

**6.1.1. Idempotents.** Let  $n \geq 1$  be an integer and  $\mathcal{C}$  be an  $n$ -category. An *idempotent* of  $\mathcal{C}$  is an  $n$ -cell  $e$  of  $\mathcal{C}$  such that  $e \star_{n-1} e = e$ . Note that the  $(n-1)$ -source and the  $(n-1)$ -target of an idempotent are necessarily equal. If there are no integer  $k < n-1$  and non identities idempotents  $e'$  and  $e''$  such that,  $e = e' \star_k e'' \neq e' \star_{n-1} e''$ , we say that the idempotent  $e$  is *minimal*. We say that the idempotent  $e$  is *split* if there exists an  $(n-1)$ -cell  $A$  of  $\mathcal{C}$ , an  $n$ -cell  $p$  from  $s_{n-1}(e)$  to  $A$  and an  $n$ -cell  $i$  from  $A$  to  $s_{n-1}(e)$  such that:

- $p \star_{n-1} i = e$ ,
- $i \star_{n-1} p = 1_A$ .

**6.1.2. Karoubi envelope of an  $n$ -category.** The *Karoubi envelope* of the  $n$ -category  $\mathcal{C}$  is the  $n$ -category  $\text{Kar}(\mathcal{C})$  such that:

- $\text{Kar}(\mathcal{C})$  has the same  $k$ -cells as  $\mathcal{C}$  for  $k < n-1$ ,
- $\text{Kar}(\mathcal{C})$  has an  $(n-1)$ -cell  $A_e$  from  $s_{n-2}(e)$  to  $t_{n-2}(e)$  for each idempotent  $e$  of  $\mathcal{C}$ ,

- for  $k < n - 1$ , for each  $k$ -composable idempotents  $e$  and  $e'$  of  $\mathcal{C}$ , we have

$$A_e \star_k A_{e'} = A_{e \star_k e'},$$

- $\text{Kar}(\mathcal{C})$  has an  $n$ -cell  $\alpha(e, f, e')$  from  $A_e$  to  $A_{e'}$  for each triple  $(e, f, e')$  of  $n$ -cells of  $\mathcal{C}$  such that  $e$  and  $e'$  are idempotents verifying  $f = e \star_{n-1} f \star_{n-1} e'$ ,
- for  $k < n - 1$ , for each pairs of  $k$ -composable idempotents  $(e_1, e_2)$  and  $(e'_1, e'_2)$  of  $\mathcal{C}$  and each  $k$ -composable  $n$ -cells  $f_1$  and  $f_2$  of  $\mathcal{C}$  such that  $\alpha(e_1, f_1, e'_1)$  and  $\alpha(e_2, f_2, e'_2)$  are defined, we have

$$\alpha(e_1, f_1, e'_1) \star_k \alpha(e_2, f_2, e'_2) = \alpha(e_1 \star_k e_2, f_1 \star_k f_2, e'_1 \star_k e'_2),$$

- for each  $(n - 1)$ -composable  $n$ -cells  $f$  and  $g$  of  $\mathcal{C}$  and each triple  $(e, e', e'')$  of idempotents of  $\mathcal{C}$  such that  $f = e \star_{n-1} f \star_{n-1} e'$  and  $g = e' \star_{n-1} g \star_{n-1} e''$ , we have

$$\alpha(e, f, e') \star_{n-1} \alpha(e', g, e'') = \alpha(e, f \star_{n-1} g, e'').$$

**6.1.3. The surjective  $n$ -functor CS.** Let  $\mathcal{C}$  be an  $n$ -category. There is a surjective  $n$ -functor CS from  $\text{Kar}(\mathcal{C})$  to  $\mathcal{C}$  defined by:

- the restriction of CS to  $\mathcal{C}$  is an identity,
- $\text{CS}_{n-1}(A_e) = s_{n-1}(e)$  for any idempotent  $e$  of  $\mathcal{C}$ ,
- $\text{CS}_n(\alpha(e, f, e')) = f$  for each triple  $(e, f, e')$  of  $n$ -cells of  $\mathcal{C}$  such that  $e$  and  $e'$  are idempotents verifying  $f = e \star_{n-1} f \star_{n-1} e'$ .

We call this  $n$ -functor the *canonical surjection  $n$ -functor* from  $\text{Kar}(\mathcal{C})$  to  $\mathcal{C}$ .

**6.1.4. Karoubi envelope of a linear  $(n, n)$ -category.** Let  $\mathcal{C}$  be a linear  $(n, n)$ -category. In particular,  $\mathcal{C}$  is an  $n$ -category. Let us denote by  $\text{Kar}(\mathcal{C})$  its Karoubi envelope. There is a structure of linear  $(n, n)$ -category on  $\text{Kar}(\mathcal{C})$  defined by  $\alpha(e, \lambda f + g, e') = \lambda \alpha(e, f, e') + \alpha(e, g, e')$  for each scalar  $\lambda$ , each parallel  $n$ -cells  $f$  and  $g$  of  $\mathcal{C}$  and each idempotents  $e$  and  $e'$  of  $\mathcal{C}$  such that  $f = e \star_{n-1} f \star_{n-1} e'$  and  $g = e \star_{n-1} g \star_{n-1} e'$ .

## 6.2. DECATEGORIFICATION

**6.2.1. Direct sums.** Let  $n > 1$  be an integer and  $\mathcal{C}$  be a linear  $(n, n)$ -category. Let  $a$ ,  $b$  and  $c$  be  $(n - 1)$ -cells of  $\mathcal{C}$ . We say that  $a$  is *direct sum* of  $b$  and  $c$  if there exist  $n$ -cells  $a \xrightarrow{p_b} b$ ,  $a \xrightarrow{p_c} c$ ,  $b \xrightarrow{i_b} a$  and  $c \xrightarrow{i_c} a$  such that:



- $p_b \star_{n-1} i_b + p_c \star_{n-1} i_c = 1_a$ ,
- $i_b \star_{n-1} p_b = 1_b$ ,
- $i_c \star_{n-1} p_c = 1_c$ .

In this case, we denote  $a \simeq b \oplus c$ .

**6.2.2. Grothendieck decategorification.** The *Grothendieck decategorification*, or *Grothendieck group*, of a linear category  $\mathcal{C}$  is the group generated by the isomorphism classes of  $\mathcal{C}$  and subject to the relation  $[a] = [b] + [c]$  whenever an object  $a$  is direct sum of two objects  $b$  and  $c$ . If the category  $\mathcal{C}$  is monoidal, this case corresponding to a linear  $(2, 2)$ -category the Grothendieck decategorification of  $\mathcal{C}$  is also a ring with a product defined by  $[a \otimes b] = [a][b]$  for any objects  $a$  and  $b$  of  $\mathcal{C}$ . In this section, we extend the definition of Grothendieck decategorification to arbitrary linear  $(n, n)$ -categories and give a construction for presenting such Grothendieck decategorifications.

**6.2.3. Grothendieck decategorification of a linear  $(n, n)$ -category.** Let  $n > 1$  be an integer and  $\mathcal{C}$  be a linear  $(n, n)$ -category. Two  $(n-1)$ -cells  $u$  and  $v$  of  $\mathcal{C}$  are *isomorphic* if there is an  $n$ -cell from  $u$  to  $v$  which is invertible for the  $n$ -composition. We will call  $[u]$  the isomorphism class of the  $(n-1)$ -cell  $u$ . The *Grothendieck decategorification* of  $\mathcal{C}$  is the linear  $\mathbb{Z}$ -linear  $(n-1, n-1)$ -category  $K(\mathcal{C})$  defined by:

- for  $k < n-1$ , the linear  $\mathbb{Z}$ -linear  $(n-1, n-1)$ -category  $K(\mathcal{C})$  has the same  $k$ -cells than  $\mathcal{C}$ ,
- for any parallel  $(n-2)$ -cells  $x$  and  $y$  of  $K(\mathcal{C})$ , the  $\mathbb{Z}$ -module  $K(\mathcal{C})_{n-1}[x, y]$  is the abelian group generated by the isomorphisms classes of  $\mathcal{C}_{n-1}[x, y]$  and subject to the relation  $[a] = [b] + [c]$  for each  $(n-1)$ -cells  $a, b$  and  $c$  such that  $a \simeq b \oplus c$ ,
- for any  $0 \leq k \leq n-2$  and any  $k$ -composable  $(n-1)$ -cells  $u$  and  $v$  of  $\mathcal{C}_{n-1}$ , we have  $[u] \star_k [v] = [u \star_k v]$ .

If the spaces of  $(n-1)$ -cells of  $\mathcal{C}$  are  $\mathbb{Z}$ -graded,  $K(\mathcal{C})$  has a structure of  $\mathbb{Z}[q, q^{-1}]$ -linear  $(n-1, n-1)$ -category where for any  $(n-1)$ -cell  $u$  of  $\mathcal{C}$ , we have  $[u[1]] = q[u]$ .

We say that  $\mathcal{C}$  *categorifies*  $K(\mathcal{C})$ , or is a *categorification* of  $K(\mathcal{C})$ .

## 6.3. CATEGORIFICATION AND REWRITING

**6.3.1. Karoubi envelope of an  $(n+1)$ -polygraph.** Let  $\Sigma$  be an  $(n+1)$ -polygraph. The Karoubi envelope of  $\Sigma$  is the  $(n+1)$ -polygraph  $\text{Kar}(\Sigma)$  defined by:

- $\text{Kar}(\Sigma)_k = \Sigma_k$  for  $k < n - 1$ ,
- $\text{Kar}(\Sigma)_{n-1} = \Sigma_{n-1} \cup \{A_e \mid e \text{ is a minimal idempotent of } \mathcal{C}\}$ ,
- for each minimal idempotent  $e$  of  $\mathcal{C}$ , we have  $s_{n-2}(A_e) = s_{n-2}(e)$  and  $t_{n-2}(A_e) = t_{n-2}(e)$ ,
- $\text{Kar}(\Sigma)_n = \Sigma_n \cup \{p_e, i_e \mid e \text{ is a minimal idempotent of } \mathcal{C}\}$ ,
- for each minimal idempotent  $e$  of  $\mathcal{C}$ , we have  $s_{n-1}(p_e) = s_{n-1}(e)$  and  $t_{n-1}(p_e) = A_e$ ,
- for each minimal idempotent  $e$  of  $\mathcal{C}$ , we have  $s_{n-1}(i_e) = A_e$  and  $t_{n-1}(i_e) = t_{n-1}(e)$ ,
- $\text{Kar}(\Sigma)_{n+1} = \Sigma_{n+1} \cup \{\pi_e, \iota_e \mid e \text{ is a minimal idempotent of } \mathcal{C}\}$ ,
- for each minimal idempotent  $e$  of  $\mathcal{C}$ , we have  $s_n(\pi_e) = e$  and  $t_n(\pi_e) = p_e \star_n i_e$ ,
- for each minimal idempotent  $e$  of  $\mathcal{C}$ , we have  $s_n(\iota_e) = i_e \star_n p_e$  and  $t_n(\iota_e) = 1_{A_e}$ .

**6.3.2. Proposition [?, Proposition 3.1.2.].** Let  $\mathcal{C}$  be an  $n$ -category presented by an  $(n + 1)$ -polygraph  $\Sigma$ . The Karoubi envelope of  $\mathcal{C}$  is presented by the  $(n + 1)$ -polygraph  $\text{Kar}(\Sigma)$ .

*Proof.* Let  $\text{Kar}(\mathcal{C})$  be the Karoubi envelope of the  $n$ -category  $\mathcal{C}$ . For  $k < n - 1$ , the  $n$ -category  $\text{Kar}(\mathcal{C})$  has the same  $k$ -cells than  $\mathcal{C}$  and  $\text{Kar}(\Sigma)_k = \Sigma_k$ . Then, the  $(n+1)$ -polygraph  $\text{Kar}(\Sigma)$  presents an  $n$ -category with the same  $k$ -cells than  $\text{Kar}(\mathcal{C})$ . Let us now prove that  $\text{Kar}(\Sigma)$  presents an  $n$ -category with the same  $(n - 1)$ -cells than  $\text{Kar}(\mathcal{C})$ . Let  $e$  be an idempotent of  $\mathcal{C}$  and let us write for some implied bracketing:

$$e = e_0 \star_{k_1} e_1 \star_{k_2} \cdots \star_{k_m} e_m$$

where all  $e_i$  are minimal idempotents or identities and all  $k_i$  are integer smaller than  $n - 1$ . By denoting  $A_{e_i} = 1_{e_i}$  if  $e_i$  is an identity, we can write for some implied bracketing:

$$A_e = A_{e_0} \star_{k_1} A_{e_1} \star_{k_2} \cdots \star_{k_m} A_{e_m}$$

which corresponds to an  $(n - 1)$ -cell in the  $n$ -category presented by  $\text{Kar}(\Sigma)$ . What remains to prove is that the  $n$ -category  $\text{Kar}(\Sigma)^*/\text{Kar}(\Sigma)_{n+1}$  has the same  $n$ -cells and relations on  $n$ -cells than  $\text{Kar}(\mathcal{C})$ . There is an injective  $n$ -functor  $F$  from  $\text{Kar}(\Sigma)^*/\text{Kar}(\Sigma)_{n+1}$  to  $\text{Kar}(\mathcal{C})$  defined by:

- for  $k \leq n - 1$ ,  $F$  sends each  $k$ -cell of  $\text{Kar}(\Sigma)_k$  onto its representative in  $\text{Kar}(\mathcal{C})$ ,
- for each minimal idempotent  $e$  of  $\mathcal{C}$ , the  $n$ -functor  $F$  sends the  $n$ -cell  $p_e$  onto  $\alpha(1_{s_{n-1}(e)}, e, e)$ ,
- for each minimal idempotent  $e$  of  $\mathcal{C}$ , the  $n$ -functor  $F$  sends the  $n$ -cell  $i_e$  onto  $\alpha(e, e, 1_{s_{n-1}(e)})$ .

Let us prove that the  $n$ -functor  $F$  is surjective. Let  $e$  be an idempotent of  $\mathcal{C}$  and let us write again the decomposition into minimal idempotents:

$$e = e_0 \star_{k_1} e_1 \star_{k_2} \cdots \star_{k_m} e_m.$$

We then have the decompositions:

$$\alpha(1_{s_{n-1}(e)}, e, e) = \alpha(1_{s_{n-1}(e_0)}, e_0, e_0) \star_{k_1} \alpha(1_{s_{n-1}(e_1)}, e_1, e_1) \star_{k_2} \cdots \star_{k_m} \alpha(1_{s_{n-1}(e_m)}, e_m, e_m),$$

$$\alpha(e, e, 1_{s_{n-1}(e)}) = \alpha(e_0, e_0, 1_{s_{n-1}(e_0)}) \star_{k_1} \alpha(e_1, e_1, 1_{s_{n-1}(e_1)}) \star_{k_2} \cdots \star_{k_m} \alpha(e_m, e_m, 1_{s_{n-1}(e_m)}).$$

Thus, the  $n$ -cells  $\alpha(1_{s_{n-1}(e)}, e, e)$  and  $\alpha(e, e, 1_{s_{n-1}(e)})$  are images by  $F$  of  $n$ -cells of  $\text{Kar}(\Sigma)^*/\text{Kar}(\Sigma)_{n+1}$ . Let now  $\alpha(e, f, e')$  be an  $n$ -cell of the  $\text{Kar}(\mathcal{C})$  such that  $e$  and  $e'$  are idempotents of  $\mathcal{C}$ . We have:

$$\alpha(e, f, e') = \alpha(e, e, 1_{s_{n-1}(e)}) \star_{n-1} \alpha(1_{s_{n-1}(e)}, f, 1_{s_{n-1}(e)}) \star_{n-1} \alpha(1_{s_{n-1}(e)}, e', e').$$

Thus, the  $n$ -cell  $\alpha(e, f, e')$  is the image by  $F$  of an  $n$ -cell of  $\text{Kar}(\Sigma)^*/\text{Kar}(\Sigma)_{n+1}$ . This concludes the proof.  $\square$

Let us remark that, because of this proposition, the construction of Karoubi envelope of a polygraph verifies a universal property. If two polygraphs are Tietze-equivalent, their Karoubi envelopes are also Tietze-equivalent.

**6.3.3. Karoubi envelope of a globular extension.** Let  $\mathcal{C}$  be an  $n$ -category presented by an  $(n+1)$ -polygraph  $\Sigma$ . Let  $\Gamma$  be a globular extension of  $\Sigma^\top$ . We define a globular extension  $\text{Kar}(\Gamma)$  of  $\text{Kar}(\Sigma)^\top$  as follows. For each  $(n+2)$ -cell  $A$  of  $\Gamma$  with  $n$ -source  $f$  and  $n$ -target  $g$ , let  $P(f, g)$  be the set of parallel  $(n+1)$ -cells  $f'$  and  $g'$  of  $\text{Kar}(\mathcal{C})$  such that  $\text{CS}(f') = f$  and  $\text{CS}(g') = g$  with  $\text{CS}$  being the canonical surjection  $(n+1)$ -functor from  $\text{Kar}(\Sigma)^\top$  to  $\Sigma^\top$ . We fix for each  $(f', g')$  in  $P(f, g)$  an  $(n+2)$ -cell  $A_{f', g'}$  from  $f'$  to  $g'$ . Then, we define

$$\text{CS}^{-1}(A) = \bigcup_{(f', g') \in P(f, g)} A_{f', g'}.$$

The Karoubi envelope of the globular extension  $\Gamma$  is the globular extension of  $\text{Kar}(\Sigma)^\top$  defined by

$$\text{Kar}(\Gamma) = \bigcup_{A \in \Gamma} \text{CS}^{-1}(A).$$

**6.3.4. Theorem [?, Theorem 3.1.5].** *Let  $\mathcal{C}$  be an  $n$ -category and let  $(\Sigma, \Sigma_{n+2})$  be a coherent presentation of  $\mathcal{C}$ . The  $(n+2, n)$ -polygraph  $(\text{Kar}(\Sigma), \text{Kar}(\Sigma_{n+2}))$  is a coherent presentation of the Karoubi envelope of  $\mathcal{C}$ .*

*Proof.* We proceed in four steps.

**Step 1.** Let  $f$  and  $g$  be parallel  $(n+1)$ -cells of  $\text{Kar}(\Sigma)^\top$  such that there is an  $(n+2)$ -cell  $A$  from  $\text{CS}(f)$  to  $\text{CS}(g)$  in  $\Sigma_{n+2}$ . We prove that there is an  $(n+2)$ -cell from  $g$  to  $f$  in  $\text{Kar}(\Sigma_{n+2})^\top$ . There is an  $(n+2)$ -cell of  $\text{CS}^{-1}(A)$  from  $f$  to  $g$ . The inverse of this  $(n+2)$ -cell is in  $\text{Kar}(\Sigma)^\top$ .

**Step 2.** Let  $f$  and  $g$  be parallel  $(n+1)$ -cells of  $\text{Kar}(\Sigma)^\top$  such that there is an  $(n+2)$ -cell  $A$  from  $\text{CS}(f)$  to  $\text{CS}(g)$  in  $\Sigma_{n+2}$ . Let  $f'$  and  $g'$  be parallel  $(n+1)$ -cells of  $\text{Kar}(\Sigma)^\top$  such that there is an  $(n+2)$ -cell  $A'$  from  $\text{CS}(f')$  to  $\text{CS}(g')$  in  $\Sigma_{n+2}$ . Let us assume that the  $(n+1)$ -cells  $f \star_k f'$  and  $g \star_k g'$  are defined for an integer  $k < n$ . We prove that there is an  $(n+2)$ -cell from  $f \star_k f'$  to  $g \star_k g'$  in  $\text{Kar}(\Sigma_{n+2})^\top$ . There is an  $(n+2)$ -cell of  $\text{CS}^{-1}(A)$  from  $f$  to  $g$  and an  $(n+2)$ -cell of  $\text{CS}^{-1}(A')$  from  $f'$  to  $g'$ . Their  $k$ -composition is in  $\text{Kar}(\Sigma)^\top$ .

**Step 3.** Let  $f, g$  and  $h$  be parallel  $(n+1)$ -cells of  $\text{Kar}(\Sigma)^\top$  such that there is an  $(n+2)$ -cell  $A$  from  $\text{CS}(f)$  to  $\text{CS}(g)$  in  $\Sigma_{n+2}$  and an  $(n+2)$ -cell  $B$  from  $\text{CS}(g)$  to  $\text{CS}(h)$  in  $\Sigma_{n+2}$ . We prove that there is an  $(n+2)$ -cell from  $f$  to  $h$  in  $\text{Kar}(\Sigma_{n+2})^\top$ . There is an  $(n+2)$ -cell of  $\text{CS}^{-1}(A)$  from  $f$  to  $g$  and an  $(n+2)$ -cell of  $\text{CS}^{-1}(B)$  from  $g$  to  $h$ . Their  $n$ -composition is in  $\text{Kar}(\Sigma)^\top$ .

**Step 4.** Let  $f$  and  $g$  be parallel  $(n+1)$ -cells of  $\text{Kar}(\Sigma)^\top$ . We prove that there is an  $(n+2)$ -cell from  $f$  to  $g$  in  $\text{Kar}(\Sigma_{n+2})^\top$ . Because  $\Sigma_{n+2}$  is a homotopy basis of  $\Sigma^\top$ , there is an  $(n+2)$ -cell from  $\text{CS}(f)$  to  $\text{CS}(g)$  obtained by compositions and inversions of  $(n+2)$ -cells of  $\Sigma_{n+2}$  and identities  $(n+2)$ -cells. By steps 1, 2 and 3, this allows us to construct an  $(n+2)$ -cell from  $f$  to  $g$  in  $\text{Kar}(\Sigma_{n+2})^\top$ .  $\square$

**6.3.5. Remark.** The free  $(n+1)$ -category over the Karoubi envelope of an  $(n+1)$ -polygraph  $\Sigma$  is not the Karoubi envelope of the free  $(n+1)$ -category  $\Sigma^*$ . Indeed, the only idempotents of  $\Sigma^*$  are the identities  $(n+1)$ -cells. Thus,  $\text{Kar}(\Sigma^*)$  is isomorphic to  $\Sigma^*$  and not to  $\text{Kar}(\Sigma)^*$ . This implies a homotopy basis of  $\text{Kar}(\Sigma^*)$  is not a homotopy basis of  $\text{Kar}(\Sigma)^*$  in general.

**6.3.6. Coherent presentation of the Karoubi envelope of a linear  $(n, n)$ -category.** Let  $\mathcal{C}$  be a linear  $(n, n)$ -category and let  $(\Sigma, \Sigma_{n+2})$  be a coherent presentation of  $\mathcal{C}$ . Let  $\text{Kar}(\Sigma_{n+2})$  be the globular extension of  $\text{Kar}(\Sigma)$  defined as in ???. The  $(n+2, n)$ -polygraph  $(\text{Kar}(\Sigma), \text{Kar}(\Sigma_{n+2}))$  is a coherent presentation of the Karoubi envelope of  $\mathcal{C}$ .

*Proof.* To prove this proposition, we just have to prove that for each parallel  $(n+1)$ -cells  $f$  and  $g$  and each parallel  $(n+1)$ -cells  $f'$  and  $g'$  of  $\text{Kar}(\Sigma)^\ell$  and each scalar  $\lambda$  such that  $\lambda f + f'$  and  $\lambda g + g'$  are defined, we can construct an  $(n+2)$ -cell from  $\lambda f + f'$  to  $\lambda g + g'$  in  $\text{Kar}(\Sigma_{n+2})^\ell$  if there is an  $(n+2)$ -cell  $A$  from  $\text{CS}(f)$  to  $\text{CS}(g)$  and an  $(n+2)$ -cell  $A'$  from  $\text{CS}(f')$  to  $\text{CS}(g')$

in  $\Sigma_{n+2}$ . There is an  $(n+2)$ -cells  $B$  from  $f$  to  $g$  in  $CS^{-1}(A)$  and is an  $(n+2)$ -cells  $B'$  from  $f'$  to  $g'$  in  $CS^{-1}(A')$ . Then,  $\text{Kar}(\Sigma_{n+2})^\ell$  contains the  $(n+2)$ -cell  $\lambda B + B'$  from  $\lambda f + f'$  to  $\lambda g + g'$ . This concludes the proof.  $\square$

**6.3.7. Isomorphism proofs.** Let  $\Sigma$  be a linear  $(n+1, n)$ -polygraph. Let  $u$  and  $v$  be distinct  $(n-1)$ -cells of the free linear  $(n+1, n)$ -category  $\Sigma^\ell$ . An *isomorphism proof* between  $u$  and  $v$  is a data  $(\alpha_u, \alpha_v)$  made of two  $(n+1)$ -cells in  $\Sigma^\ell$  such that there exist  $n$ -cells  $u \xrightarrow{a_u} v$  and  $v \xrightarrow{a_v} u$  verifying:

- $\alpha_u$  is an  $(n+1)$ -cell from  $a_u \star_{n-1} a_v$  to  $1_u$ ,
- $\alpha_v$  is an  $(n+1)$ -cell from  $a_v \star_{n-1} a_u$  to  $1_v$ .

An isomorphism proof  $(\alpha_u, \alpha_v)$  is *minimal* if there is no  $(n-1)$ -cell  $w$ , no integer  $k < n-1$  and no isomorphism proof  $(\alpha_{u'}, \alpha_{v'})$  other than  $(\alpha_u, \alpha_v)$  such that

$$(\alpha_u, \alpha_v) = (w \star_{n-1} \alpha_{u'}, w \star_{n-1} \alpha_{v'})$$

or

$$(\alpha_u, \alpha_v) = (\alpha_{u'} \star_{n-1} w, \alpha_{v'} \star_{n-1} w).$$

**6.3.8. Direct sum proofs.** Let  $\Sigma$  be a linear  $(n+1, n)$ -polygraph. Let  $a, b$  and  $c$  be  $(n-1)$ -cells of the free linear  $(n+1, n)$ -category  $\Sigma^\ell$ . A *direct sum proof* of  $a \simeq b \oplus c$  is a data  $(\alpha_a, \alpha_b, \alpha_c)$  made of three  $(n+1)$ -cells in  $\Sigma^\ell$  such that there exist  $n$ -cells  $a \xrightarrow{p_b} b$ ,  $a \xrightarrow{p_c} c$ ,  $b \xrightarrow{i_b} a$  and  $c \xrightarrow{i_c} a$  in  $\Sigma^\ell$  verifying:

- $\alpha_a$  is an  $(n+1)$ -cell from  $p_b \star_{n-1} i_b + p_c \star_{n-1} i_c$  to  $1_a$ ,
- $\alpha_b$  is an  $(n+1)$ -cell from  $i_b \star_{n-1} p_b$  to  $1_b$ ,
- $\alpha_c$  is an  $(n+1)$ -cell from  $i_c \star_{n-1} p_c$  to  $1_c$ .

A direct sum proof  $(\alpha_a, \alpha_b, \alpha_c)$  is said to be *minimal* if there are no  $(n-1)$ -cell  $u$ , no integer  $k < n-1$  and no direct sum proof  $(\alpha_{a'}, \alpha_{b'}, \alpha_{c'})$  other than  $(\alpha_a, \alpha_b, \alpha_c)$  such that

$$(\alpha_a, \alpha_b, \alpha_c) = (u \star_{n-1} \alpha_{a'}, u \star_{n-1} \alpha_{b'}, u \star_{n-1} \alpha_{c'})$$

or

$$(\alpha_a, \alpha_b, \alpha_c) = (\alpha_{a'} \star_{n-1} u, \alpha_{b'} \star_{n-1} u, \alpha_{c'} \star_{n-1} u).$$

**6.3.9. Grothendieck decategorification of a linear  $(n + 1, n)$ -polygraph.** Let  $\mathcal{C}$  be a linear  $(n, n)$ -category presented by a linear  $(n + 1, n)$ -polygraph  $\Sigma$ . The *Grothendieck decategorification* of  $\Sigma$  is the linear  $\mathbb{Z}$ -linear  $(n, n - 1)$ -polygraph  $K(\Sigma)$  defined by:

- for  $k \leq n - 1$ , the linear  $\mathbb{Z}$ -linear  $(n, n - 1)$ -polygraph  $K(\Sigma)$  has the same  $k$ -cells than  $\Sigma$ ,
- for each  $(n - 1)$ -cells  $u$  and  $v$  of  $\Sigma_{n-1}^\ell$  such that  $u \neq v$  and there is a minimal isomorphism proof between  $u$  and  $v$ , there is an  $n$ -cell in  $K(\Sigma)$  from  $u$  to  $v$ .
- for each  $(n - 1)$ -cells  $a, b$  and  $c$  of  $\Sigma_{n-1}^\ell$  such that there is a minimal direct sum proof of  $a \simeq b \oplus c$ , there is an  $n$ -cell in  $K(\Sigma)$  from  $a$  to  $b + c$ .

**6.3.10. Theorem [?, Theorem 4.1.8].** *Let  $\mathcal{C}$  be a linear  $(n, n)$ -category presented by a linear  $(n + 1, n)$ -polygraph  $\Sigma$ . The Grothendieck decategorification  $K(\Sigma)$  of  $\Sigma$  presents the Grothendieck decategorification  $K(\mathcal{C})$  of  $\mathcal{C}$ .*

*Proof.* By definition, the  $(n - 1, n - 1)$ -category presented by  $K(\Sigma)$  has the same  $k$ -cells as  $K(\mathcal{C})$  for  $k < n - 1$ . This linear  $(n - 1, n - 1)$ -category is also generated by the same  $(n - 1)$ -cells as  $K(\mathcal{C})$ . Each relation verified by the  $(n - 1)$ -cells of the  $(n - 1, n - 1)$ -category presented by  $K(\Sigma)$  is also verified by the  $(n - 1)$ -cells of  $K(\mathcal{C})$ . Let us now prove that each relation verified by the  $(n - 1)$ -cells of  $K(\mathcal{C})$  is verified by the  $(n - 1)$ -cells of the  $(n - 1, n - 1)$ -category presented by  $K(\Sigma)$ .

Let  $a \simeq b \oplus c$  be a direct sum in  $\mathcal{C}$ . If there is a minimal proof of this direct sum, then there is an  $n$ -cell in  $K(\Sigma)_n^\ell$  from  $[a]$  to  $[b] + [c]$ . Else, there are decompositions:

- $a = u_1 \star_{n-2} (u_2 \star_{n-3} (\cdots (u_{n-1} \star_0 a' \star_0 u_n) \cdots) \star_{n-3} u_{2n-3}) \star_{n-2} u_{2n-2}$ ,
- $b = u_1 \star_{n-2} (u_2 \star_{n-3} (\cdots (u_{n-1} \star_0 b' \star_0 u_n) \cdots) \star_{n-3} u_{2n-3}) \star_{n-2} u_{2n-2}$ ,
- $c = u_1 \star_{n-2} (u_2 \star_{n-3} (\cdots (u_{n-1} \star_0 c' \star_0 u_n) \cdots) \star_{n-3} u_{2n-3}) \star_{n-2} u_{2n-2}$

where all  $u_i$  are  $(n - 1)$ -cells of  $\mathcal{C}$  and the direct sum  $a' \simeq b' \oplus c'$  has a minimal proof. Hence, by distributivity of the compositions there is an  $n$ -cell in  $K(\Sigma)_n^\ell$  from  $[a]$  to  $[b] + [c]$ . This concludes the proof.  $\square$

## 6.4. CATEGORY OF SOERTEL BIMODULES

**6.4.1. Coxeter groups.** A *Coxeter group* is a group  $W$  generated by a finite set  $S$  and, for any  $s$  and  $t$  in  $S$ , subject to the relations

$$(st)^{m_{st}} = 1$$

where each  $m_{st}$  is an element of  $\mathbb{N}_{\geq 2} \cup \{\infty\}$  such that

- for each  $s$  in  $S$ , we have  $m_{ss} = 2$ ,
- for each  $s$  in  $S$  and each  $t$  in  $S$ , we have  $m_{st} = m_{ts}$ .

If  $m_{st} = \infty$ , there is no relation between  $s$  and  $t$ . The data made of such a set  $S$  and the group  $W$  it generates is called a *Coxeter system*.

In the rest of this section, we fix  $(W, S)$  a Coxeter system.

**6.4.2. Hecke algebras.** The *Hecke algebra* associated to  $(W, S)$  is the  $\mathbb{Z}[q, q^{-1}]$ -algebra generated by the set  $\{T_s | s \in S\}$  and, for any  $s$  in  $S$  subject to the relations

$$T_s^2 = (q-1)T_s + q,$$

and, for all  $s \neq t$ , the relations

$$\underbrace{T_s T_t \cdots}_{m_{st}} = \underbrace{T_t T_s \cdots}_{m_{st}}.$$

This algebra is denoted by  $\mathcal{H}(W, S)$ .

**6.4.3. Category of Bott-Samelson bimodules.** Let  $\mathfrak{h}$  be a vector space with basis  $\{\alpha_s^\vee | s \in S\}$ . Let us consider a subset  $\{\alpha_s | s \in S\}$  of  $\mathfrak{h}^*$  verifying the following properties:

- for each  $s$  in  $S$ , we have  $\alpha_s(\alpha_s^\vee) = 2$ ,
- for each  $s$  in  $S$  and each  $t$  in  $S$ , we have  $\alpha_s(\alpha_t^\vee)\alpha_t(\alpha_s^\vee) = 4(\cos(\frac{\pi}{m_{st}}))^2$ .

Let  $\mathcal{B}$  be a basis of  $\mathfrak{h}^*$ . For each  $s$  in  $S$ , the *Demazure operator* associated to  $s$  is the linear form  $\partial_s$  of  $\mathfrak{h}^*$  sending each element  $f$  of  $\mathfrak{h}^*$  to  $f(\alpha_s^\vee)$ . The *category of Bott-Samelson bimodules*, denoted by  $\mathcal{D}_{W,S}$  or  $\mathcal{D}$  if not ambiguous, is the linear  $(2, 2)$  category defined by:

- $\mathcal{D}$  has only one 0-cell,
- $\mathcal{D}$  has  $S$  as a set of generating 1-cells,
- the elements of  $\mathcal{B}$  are generating 2-cells of  $\mathcal{D}$
- for each  $s \in S$ ,  $\mathcal{D}$  has the generating 2-cells:

$$\circlearrowleft, \circlearrowright, \nabla, \triangleleft.$$

- for each  $s \in S$  and each  $f \in \mathfrak{h}^*$ ,  $\mathcal{D}$  is subject to the relations

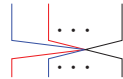
$$\begin{aligned} \diamond &= 0 | & \triangleleft \nabla &= \nabla \triangleleft & \nabla \triangleleft &= \triangleleft \nabla \\ \triangleleft \circlearrowleft &= | & \circlearrowleft \triangleleft &= | & \nabla \circlearrowleft &= | & \circlearrowleft \nabla &= | \\ \circlearrowleft &= \alpha_s & |f = s.f| + \partial_s f \circlearrowleft & & & & & \end{aligned}$$

6.4. CATEGORY OF SOERGEL BIMODULES

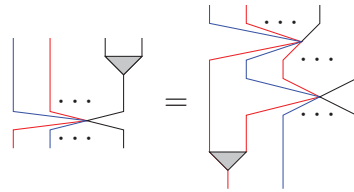
- for each  $(s, t)$  of  $S \times S$  such that  $s \neq t$  and  $m_{s,t} \neq \infty$ , we have a 2-cell

$$\underbrace{st \cdots}_{m_{st}} \Rightarrow \underbrace{ts \cdots}_{m_{st}}$$

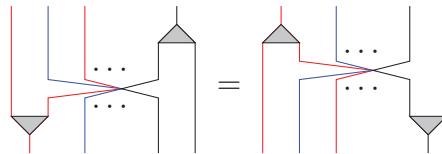
represented by



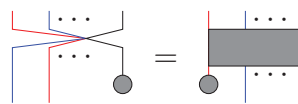
- for each  $(s, t)$  of  $S \times S$  such that  $s \neq t$  and  $m_{s,t} \neq \infty$ , we have the relation



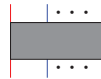
- for each  $(s, t)$  of  $S \times S$  such that  $s \neq t$  and  $m_{s,t} \neq \infty$ , we have the relation



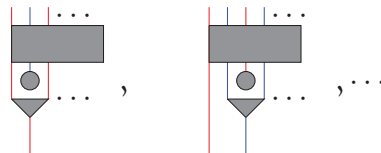
- for each  $(s, t)$  of  $S \times S$  such that  $s \neq t$  and  $m_{s,t} \neq \infty$ , we have the relation



where



is the only idempotent of  $\mathcal{D}$  such that the 2-cells



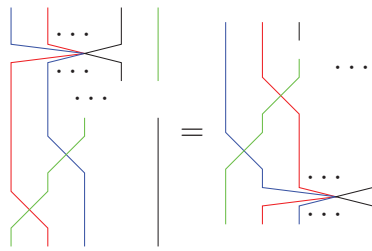
are zero,



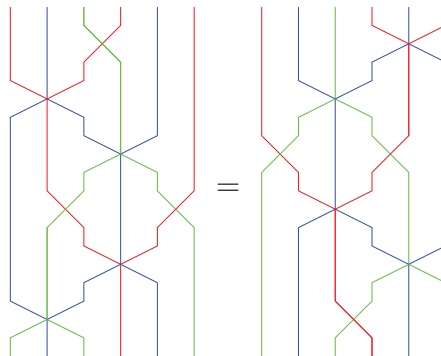
CHAPTER 6. CATEGORIFICATION OF LINEAR CATEGORIES

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- for each triple  $(s, t, u)$  of distinct elements of  $S$  such that  $m_{s,u} = m_{t,u} = 2$ , we have the relation

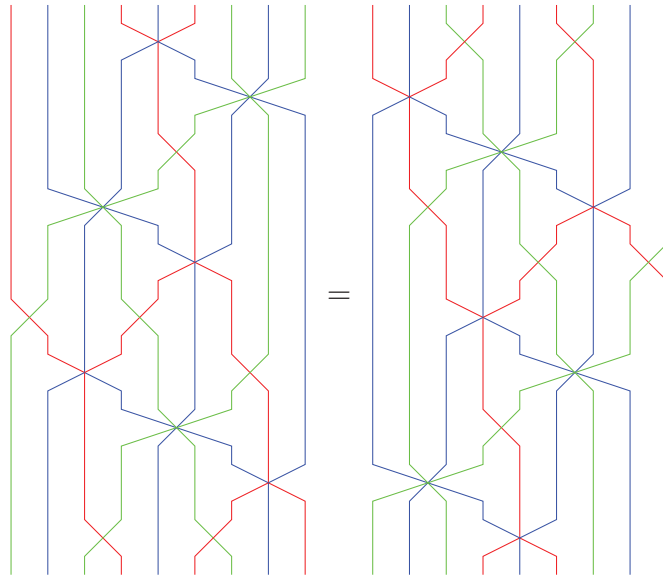


- for each triple  $(s, t, u)$  of distinct elements of  $S$  such that  $m_{s,t} = m_{t,u} = 3$  and  $m_{s,u} = 2$ , we have the relation



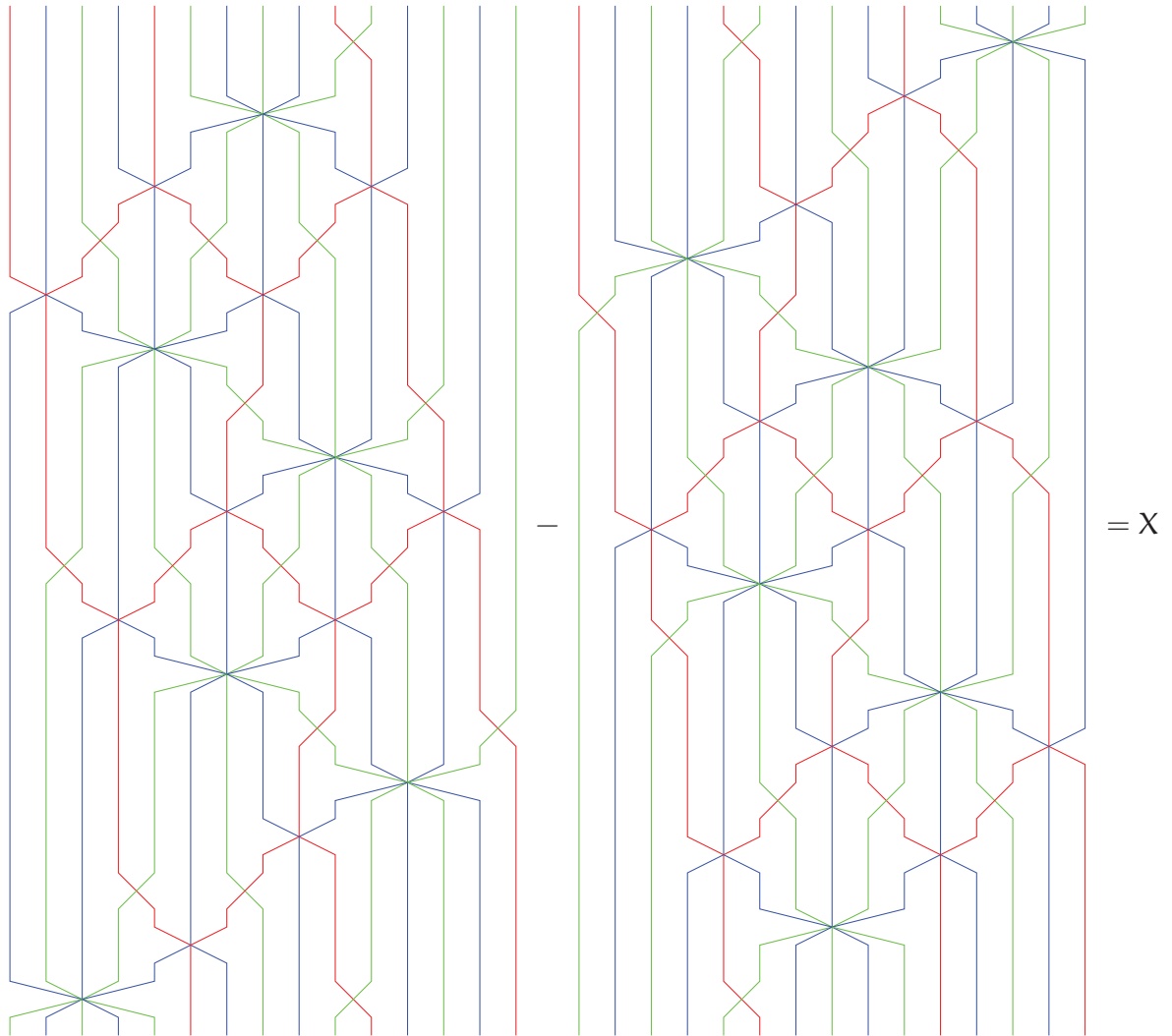
- for each triple  $(s, t, u)$  of distinct elements of  $S$  such that  $m_{s,t} = 3$ ,  $m_{s,u} = 2$  and  $m_{t,u} = 4$ ,

we have the relation



- for each triple  $(s, t, u)$  of distinct elements of  $S$  such that  $m_{s,t} = 3$ ,  $m_{s,u} = 2$  and  $m_{t,u} = 5$ ,

we have the relation



where  $X$  is a 2-cell which factors through some 1-cell  $w$  for  $w$  a subword of  $stutsututsututu$ , see [?].

**6.4.4. Category of Soergel bimodules.** The *category of Soergel bimodules* associated to  $(W, S)$ , denoted by  $\mathcal{SBim}_{W,S}$  or  $\mathcal{SBim}$  if not ambiguous, is the closure of the linear  $(2, 2)$ -category  $\text{Kar}(\mathcal{D})$  by grading shift. This linear  $(2, 2)$ -category categorifies the Hecke algebra  $\mathcal{H}(W, S)$ .

**6.4.5. A confluent presentation of  $\mathcal{D}$ .** In the linear  $(2, 2)$ -category  $\mathcal{D}$ , all identity 2-cells can be written as a sum of orthogonal non trivial idempotents, that is two of such distinct idempotents  $e_1$

6.4. CATEGORY OF SOERGEL BIMODULES

and  $e_2$  verify  $e_1 \star_1 e_2 = 0$ . All non identity idempotents of  $\mathcal{D}$  appear in such a decomposition. Let us consider the linear  $(3, 2)$ -polygraph  $\Sigma$  defined by:

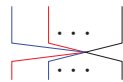
- $\Sigma$  has only one 0-cell,
- $\Sigma_1 = S$ ,
- for each  $s \in S$ ,  $\Sigma$  contains the 2-cells:

$$\circ, \phi, \nabla, \triangleleft.$$

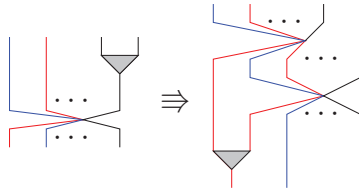
- for each  $(s, t)$  of  $S \times S$  such that  $s \neq t$  and  $m_{s,t} \neq \infty$ ,  $\Sigma$  contains the 2-cells

$$\underbrace{st \cdots}_{m_{st}} \Rightarrow \underbrace{ts \cdots}_{m_{st}}$$

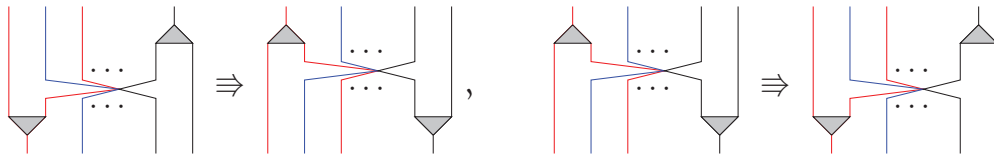
represented by



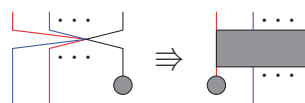
- for each  $(s, t)$  of  $S \times S$  such that  $s \neq t$  and  $m_{s,t} \neq \infty$ ,  $\Sigma$  contains the 3-cell



- for each  $(s, t)$  of  $S \times S$  such that  $s \neq t$  and  $m_{s,t} \neq \infty$ ,  $\Sigma$  contains the 3-cells



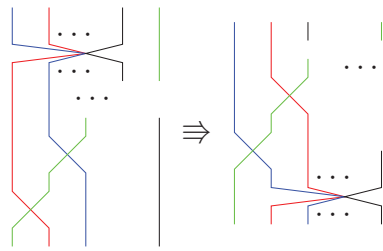
- for each  $(s, t)$  of  $S \times S$  such that  $s \neq t$  and  $m_{s,t} \neq \infty$ ,  $\Sigma$  contains the 3-cell



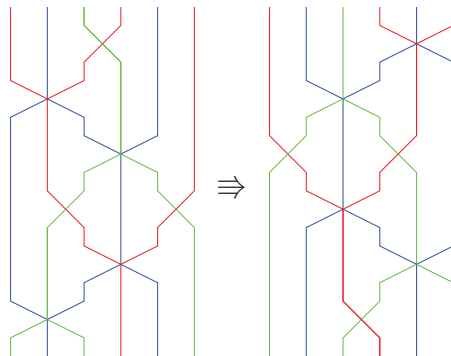
CHAPTER 6. CATEGORIFICATION OF LINEAR CATEGORIES

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- for each triple  $(s, t, u)$  of distinct elements of  $S$  such that  $m_{s,u} = m_{t,u} = 2$ ,  $\Sigma$  contains the 3-cell

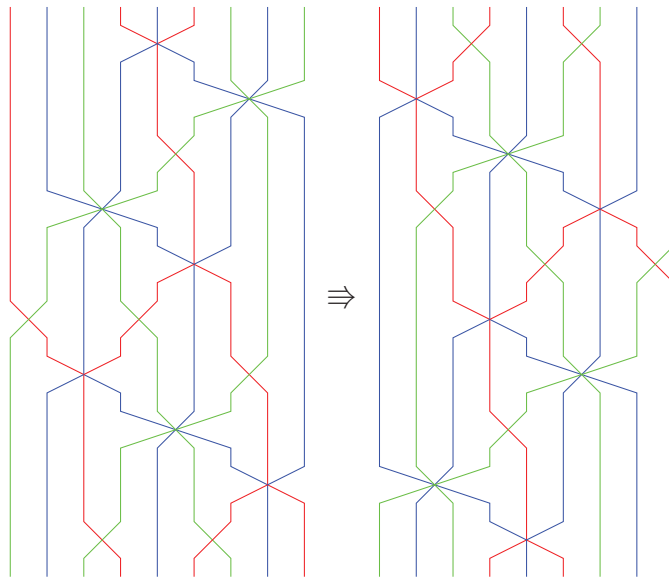


- for each triple  $(s, t, u)$  of distinct elements of  $S$  such that  $m_{s,t} = m_{t,u} = 3$  and  $m_{s,u} = 2$ ,  $\Sigma$  contains the 3-cell



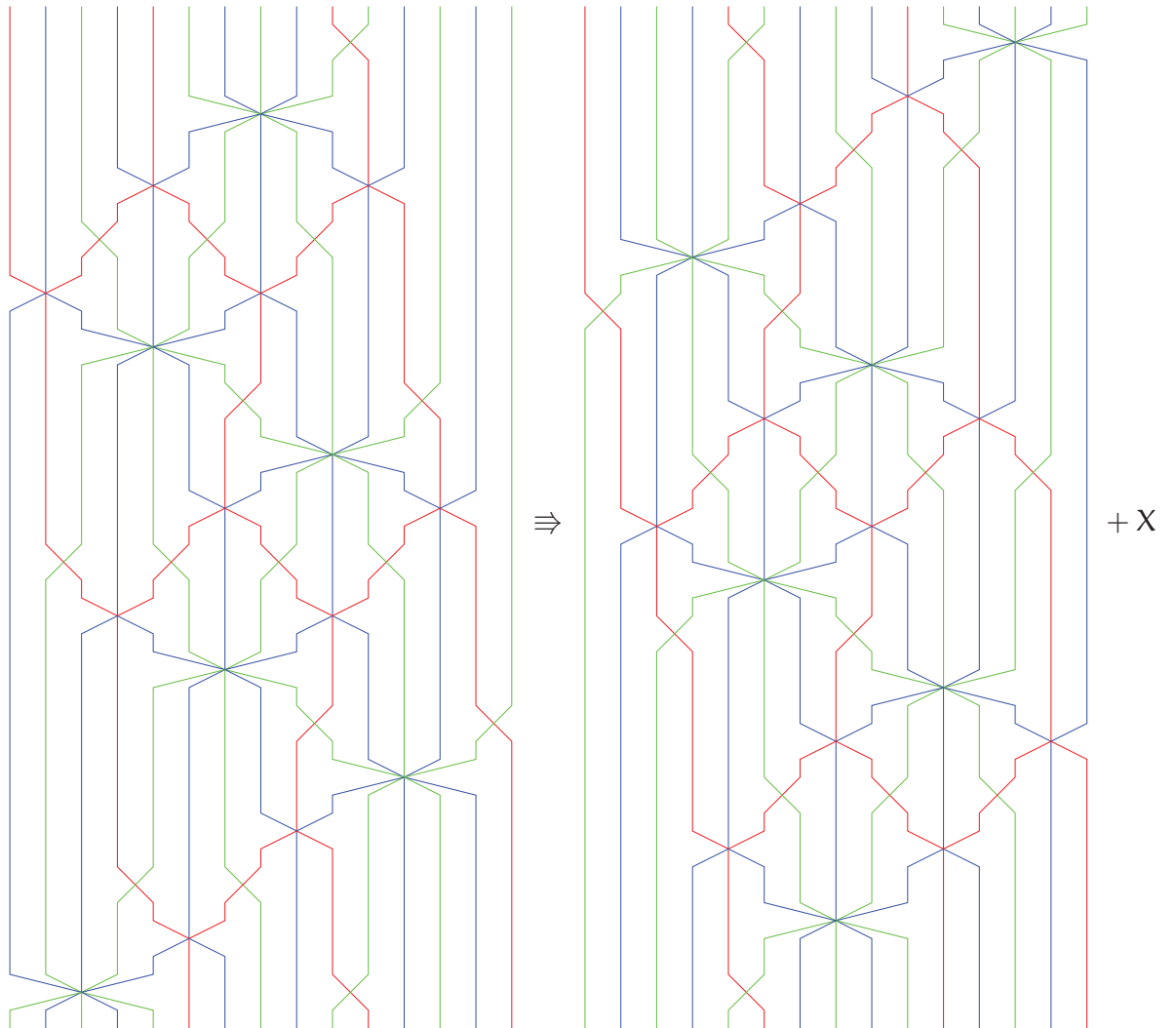
- for each triple  $(s, t, u)$  of distinct elements of  $S$  such that  $m_{s,t} = 3$ ,  $m_{s,u} = 2$  and  $m_{t,u} = 4$ ,

$\Sigma$  contains the 3-cell



- for each triple  $(s, t, u)$  of distinct elements of  $S$  such that  $m_{s,t} = 3$ ,  $m_{s,u} = 2$  and  $m_{t,u} = 5$ ,

$\Sigma$  contains the 3-cell



– for each 1-cell  $w$  of  $\Sigma^\ell$ ,  $\Sigma$  contains the 3-cell

$$1_w \Rightarrow \sum_{k=1}^n e_k$$

for a maximal integer  $n$  such that each  $e_k$  is a non trivial idempotent.

The linear  $(3, 2)$ -polygraph  $\Sigma$  is a presentation of  $\mathcal{D}$ . Because of the last family of 3-cells, we have rewriting sequences of the form

$$1_w \Rightarrow \sum_{k=1}^n e_k \Rightarrow \left( \sum_{k=1}^n e_k \right) \star_1 \left( \sum_{k=1}^n e_k \right) \Rightarrow \dots$$

and  $\Sigma$  is not terminating. However,  $\Sigma$  is confluent. Indeed, each 2-cell of  $\Sigma^\ell$  can be rewritten into a unique linear combination of 2-cells whose class in  $\mathcal{D}$  form a basis of each space of 2-cells. Those bases are called light-leaves bases, see [?].





## Affine oriented Brauer category and rewriting

Affine walled Brauer algebras were introduced by Rui and Su [?], and independently by Sartori [?], in the study of super Schur-Weyl duality. They show the Schur-Weyl duality between general Lie superalgebras and affine walled Brauer algebras. The affine oriented Brauer category  $\mathcal{AOB}$  is a linear monoidal category introduced in [?] to encode walled Brauer algebras. Each walled Brauer algebra is expressed as a morphism space of  $\mathcal{AOB}$ . In particular, the category  $\mathcal{AOB}$  is used to prove basis theorems for the affine walled Brauer algebras given in [?]. More precisely, Brundan, Comes, Nash and Reynolds provide an explicit basis for each affine walled Brauer algebra. The proof of this theorem uses an intermediate result on the cyclotomic quotients of  $\mathcal{AOB}$ . A basis is then given for each morphism space of those quotients. Brundan, Comes, Nash and Reynolds finally use those bases to construct a basis for each morphism space of  $\mathcal{AOB}$ .

In this memoir, we give a constructive proof of the mentioned basis result. We study the linear monoidal category  $\mathcal{AOB}$  by rewriting methods. For this, we use the result from Section ?? of Chapter 4 and give a confluent presentation of  $\mathcal{AOB}$ . This presentation is a linear  $(3, 2)$ -polygraph denoted by  $\overline{\mathcal{AOB}}$ . The confluence property of this linear  $(3, 2)$ -polygraph gives us the bases of the morphisms spaces of  $\mathcal{AOB}$  explicated in [?]. The linear  $(3, 2)$ -polygraph  $\overline{\mathcal{AOB}}$  will not be terminating, which will prevent us to prove that  $\overline{\mathcal{AOB}}$  is confluent by using Newman's Lemma, see Lemma ?. To prove that  $\overline{\mathcal{AOB}}$  is confluent, we will use the decreasingness property introduced by van Oostrom in [?], see also [?]. We will prove that  $\overline{\mathcal{AOB}}$  is decreasing and then use the theorem from [?] stating that decreasingness implies confluence.

**Plan of Chapter.** In Section ??, we recall from [?] the definition of the linear monoidal category  $\mathcal{AOB}$  using dotted oriented Brauer diagrams. Then, in Section ??, we define the linear  $(3, 2)$ -polygraph  $\overline{\mathcal{AOB}}$ . We first prove that  $\overline{\mathcal{AOB}}$  is quasi-terminating by expliciting monomials we call *quasi-reduced*. Then, we prove that  $\overline{\mathcal{AOB}}$  is a presentation of the linear monoidal category  $\mathcal{AOB}$ . We finally prove in ?? that  $\overline{\mathcal{AOB}}$  is confluent.

## 7.1. THE CATEGORY $\mathcal{AOB}$

**7.1.1. Dotted oriented Brauer diagrams with bubbles.** A dotted oriented Brauer diagram with bubbles is a planar diagram such that:

- edges are oriented,
- edges are either bubbles or have a boundary as source and target,
- each edge is decorated with an arbitrary number of dots not allowed to pass through the crossings.

A dotted oriented Brauer diagram is *normally ordered* if:

- all bubbles are clockwise,
- all bubbles are in the leftmost side region,
- all dots are either on a bubble or a segment pointing toward a boundary.

**7.1.2. Equivalence of dotted oriented Brauer diagrams with bubbles.** Two dotted oriented Brauer diagrams are *equivalent* if one can be transformed into the other with isotopies and Reidemeister moves. A description of those moves can be found in [?, Chapter I, Section 4]. An isotopy can move a dot along an edge but cannot make a dot go through a crossing.

**7.1.3. Definition of  $\mathcal{AOB}$ .** The *affine oriented Brauer category*  $\mathcal{AOB}$  is the linear  $(2, 2)$ -category with one 0-cell, two generating 1-cells and with 2-cells from  $a$  to  $b$  given by linear combinations of dotted oriented Brauer diagrams with bubbles with source  $a$  and target  $b$  subject to the following relations:

- invariance by equivalence given in ??,

–

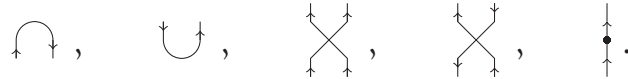
$$\begin{array}{c} \nearrow \\ \times \\ \searrow \end{array} = \begin{array}{c} \nearrow \\ \times \\ \searrow \\ \bullet \end{array} + \begin{array}{c} \uparrow \\ | \\ \uparrow \end{array}.$$

**7.1.4. Presentation of  $\mathcal{AOB}$ .** Let us recall the presentation of  $\mathcal{AOB}$  given by Brundan, Comes, Nash and Reynolds in [?]. The generators of this presentation are:

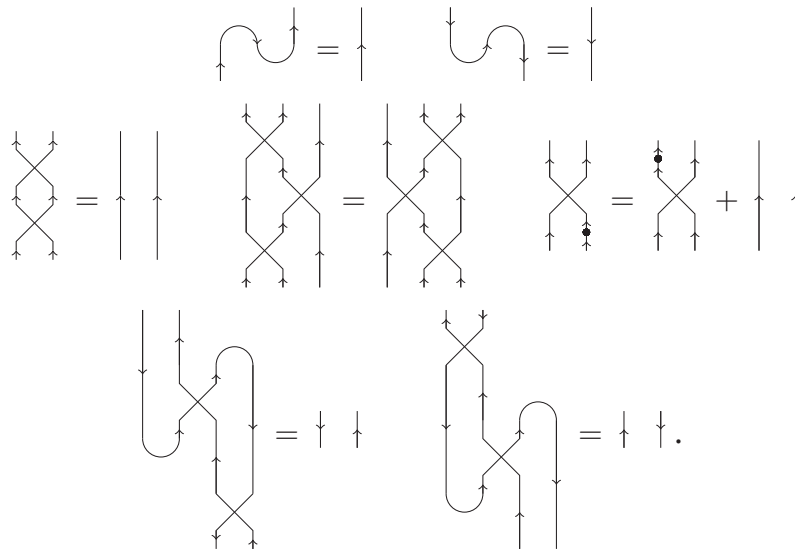
- i) only one 0-cell,
- ii) two generating 1-cells denoted by  $\wedge$  and  $\vee$ ,
- iii) five generating 2-cells

$$1 \xrightarrow{c} \wedge \vee, \quad \vee \wedge \xrightarrow{d} 1, \quad \wedge \wedge \xrightarrow{s} \wedge \wedge, \quad \wedge \vee \xrightarrow{t} \vee \wedge, \quad \wedge \xrightarrow{x} \wedge$$

respectively represented by:



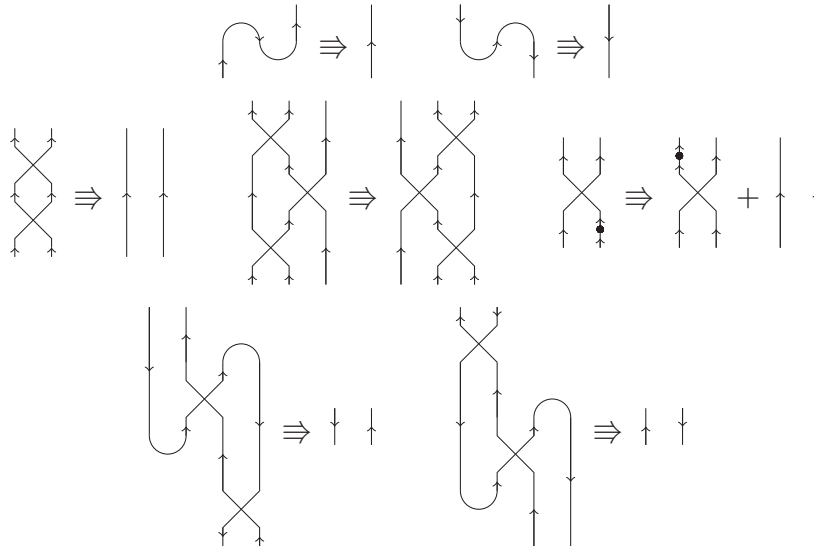
The relations are:



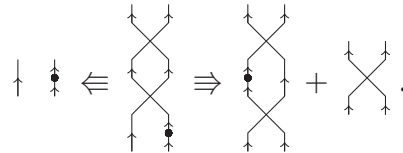
**7.1.5. Polygraphic presentation of  $\mathcal{AOB}$ .** Let us define the linear  $(3, 2)$ -polygraph  $\mathcal{AOB}$  with only one 0-cell and:

- $\mathcal{AOB}_1 = \{\wedge, \vee\}$ ,
- $\mathcal{AOB}_2 = \{ \text{cup}, \text{cap}, \text{crossing}, \text{braid}, \text{dot} \}$ ,

–  $\text{AOB}_3$  is made of the following 3-cells:



**7.1.6. Remark.** The linear  $(3, 2)$ -polygraph  $\text{AOB}$  presents the linear  $(2, 2)$ -category  $\mathcal{AOB}$  but is not confluent. For example, the following critical branching is not confluent:



Note that  $\mathcal{AOB}$  verifies the relations

$$\left| \begin{array}{c} \circlearrowleft \\ \uparrow \end{array} \right. = \left. \begin{array}{c} \circlearrowleft \\ \uparrow \end{array} \right|, \quad \left| \begin{array}{c} \circlearrowright \\ \downarrow \end{array} \right. = \left. \begin{array}{c} \circlearrowright \\ \downarrow \end{array} \right|.$$

Because of those relations, finding a terminating, a fortiori convergent presentation of  $\mathcal{AOB}$  with a finite number of 2-cells is not possible. We have indeed seen in ?? that such relations cannot be oriented to make a terminating presentation. We will thus search for a confluent and quasi-terminating presentation of  $\mathcal{AOB}$ .

## 7.2. A QUASI-CONVERGENT PRESENTATION OF $\mathcal{AOB}$

In this section, we define another presentation of  $\mathcal{AOB}$ . We will call this presentation  $\overline{\text{AOB}}$ . Then, we will prove that  $\overline{\text{AOB}}$  is a confluent and quasi-terminating presentation of the linear  $(2, 2)$ -category  $\mathcal{AOB}$ .

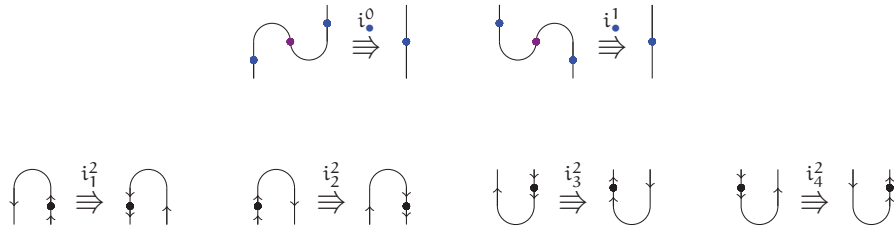
**7.2.1. Definition of  $\overline{\mathcal{AOB}}$ .** We define the linear  $(3, 2)$ -polygraph  $\overline{\mathcal{AOB}}$  by:

–  $\overline{\mathcal{AOB}}$  has the same 0-cells and 1-cells than the linear  $(3, 2)$ -polygraph  $\mathcal{AOB}$ ,

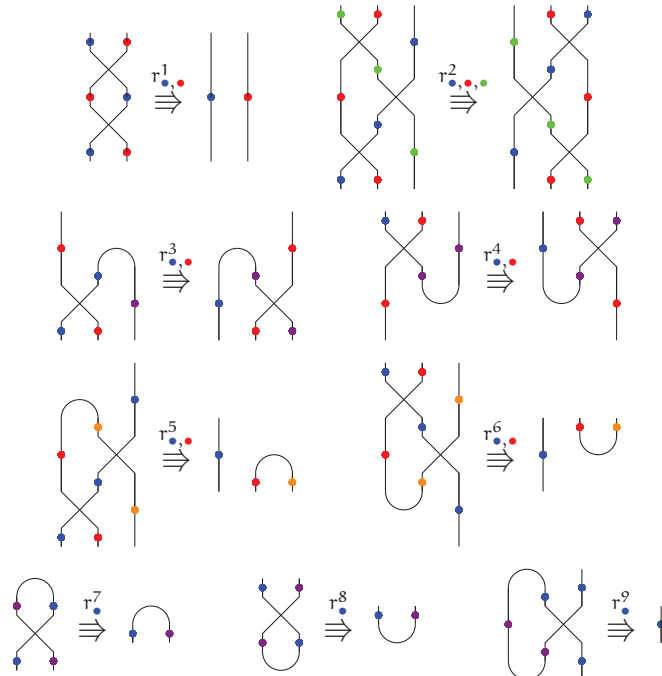
–  $\overline{\mathcal{AOB}}_2 = \mathcal{AOB}_2 \cup \{ \downarrow \curvearrowright, \curvearrowleft, \downarrow \cdot, \begin{array}{c} \downarrow \quad \downarrow \\ \diagup \quad \diagdown \\ \downarrow \quad \downarrow \end{array}, \begin{array}{c} \downarrow \quad \downarrow \\ \diagdown \quad \diagup \\ \downarrow \quad \downarrow \end{array} \},$

–  $\overline{\mathcal{AOB}}_3$  is made of the following four families of 3-cells:

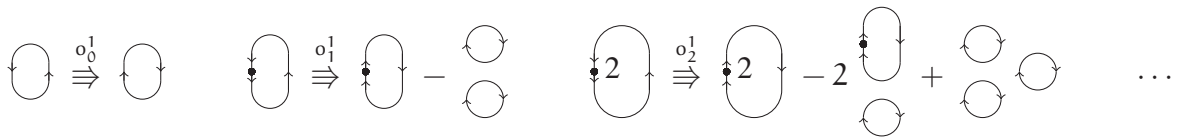
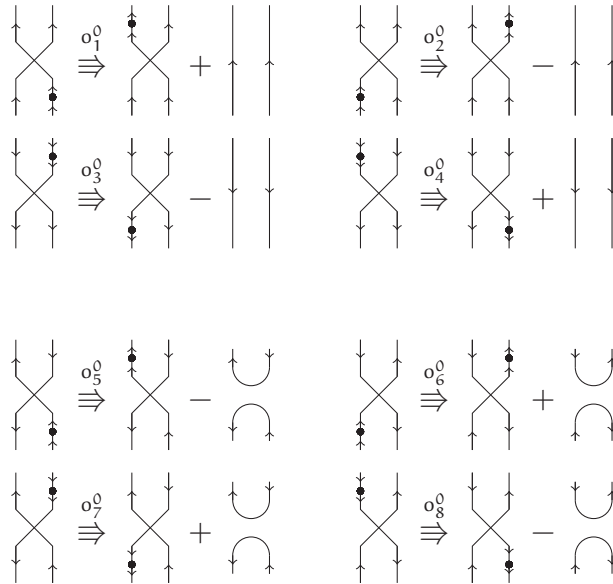
Isotopy 3-cells,  $\forall (\bullet, \bullet) \in \{(\wedge, \vee), (\vee, \wedge)\}$ , we have the following 3-cells



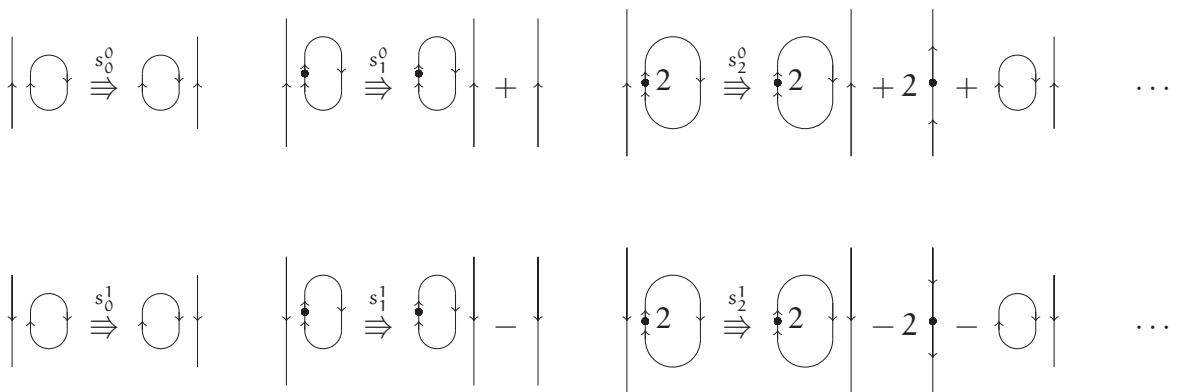
Reidemeister 3-cells,  $\forall (\bullet, \bullet) \in \{(\wedge, \vee), (\vee, \wedge)\}, \forall (\bullet, \bullet) \in \{(\wedge, \vee), (\vee, \wedge)\}, \forall \bullet \in \{\wedge, \vee\}$ , we have the following 3-cells



Ordering 3-cells



Sliding 3-cells



where the last three families of 3-cells correspond to infinite families of relations which can be calculated by induction. The first induction formula is given in [?]. If we denote the

counterclockwise bubble with  $n$  dots



as  $u_{n,0}$  and by expressing the clockwise bubbles with  $m$  dots



as  $u_{0,m}$ , we have for any  $n$  and  $m$  of  $\mathbb{N}$ :

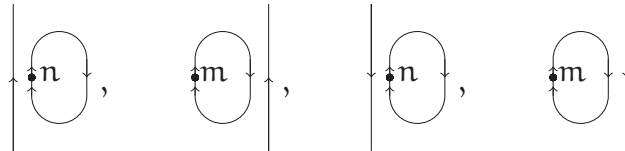
$$u_{n+1,m} = u_{n,m+1} - u_{n,0} \star_1 u_{0,m}.$$

Which can be used to rewrite  $u_{n,0}$  as a linear combination of the family  $(u_{0,i}^j)_{0 \leq i \leq n, j \in \mathbb{N}}$ . The others induction formulas are:

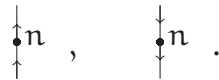
$$v_{n+1,m} = v_{n,m+1} + w_{n+m} + \sum_{k=0}^{n-1} w_{m+n-k} \star_1 v_{k,0},$$

$$v'_{n+1,m} = v'_{n,m+1} + w'_{n+m} - \sum_{k=0}^{m-1} v'_{0,k} \star_1 w'_{m+n-k}$$

where  $v_{n,0}$ ,  $v_{0,m}$ ,  $v'_{n,0}$  and  $v'_{0,m}$  respectively denote:



and where  $w_n$  and  $w'_n$  respectively denote:



Let us first expand on the first chapter of [?]. This chapter defines first ribbon categories as braided monoidal categories with duals and twist. The twist is a natural transformation  $\theta$  from the identity functor to itself satisfying:

$$\theta_{V \otimes W} = b_{W,V} \circ (\theta_V \otimes \theta_W) \circ b_{V,W}$$

for each objects  $V$  and  $W$ , where  $b$  denotes the braiding. A ribbon category satisfies the axiom  $\theta^* = \theta$ .

An example of ribbon category is the category of ribbon tangles on a set  $S$  of colors. This strict monoidal category  $\mathcal{RTB}_S$  is defined by:



- the objects of  $\mathcal{RIB}_S$  are the words of the free monoid on  $S \times \{\vee, \wedge\}$ ,
- the morphisms of  $\mathcal{RIB}_S$  from a word  $u$  to a word  $v$  are the oriented tangles of colored ribbons such that the word  $u$  is on the upper boundary and the word  $v$  is on the lower boundary,
- two isotopic tangles are equal.

In this category, each object the twist corresponds to transversally twisting a ribbon by 360 degrees. Turaev then gives a presentation by generators and relations of  $\mathcal{RIB}_S$ . The generators are the cups, caps, crossings and twistings of each ribbon colors and directions. The relations are invariance by multiple moves Turaev describes as elementary isotopies and Reidemeister moves.

From this presentation of  $\mathcal{RIB}_S$  by generators and relations, we can present the category  $\mathcal{RIB}$  of ribbon tangles with only one color by generators and relations. Because  $\mathcal{RIB}$  is a monoidal category, we can describe it as a 2-category with only one 0-cell. The linearization of  $\mathcal{RIB}$  is thus a linear  $(2, 2)$ -category with the same 0-cell and 1-cells than  $\mathcal{RIB}$ . We will call  $\mathcal{RIB}_{\mathbb{K}}$  this linear  $(2, 2)$ -category, where  $\mathbb{K}$  is our fixed field. Let  $\mathcal{OB}$  be the subcategory of  $\mathcal{AOB}$  defined by:

- $\mathcal{OB}_0 = \mathcal{AOB}_0$  and  $\mathcal{OB}_1 = \mathcal{AOB}_1$ ,
- $\mathcal{OB}_2$  is made of the oriented Brauer diagrams with bubbles (without dots).

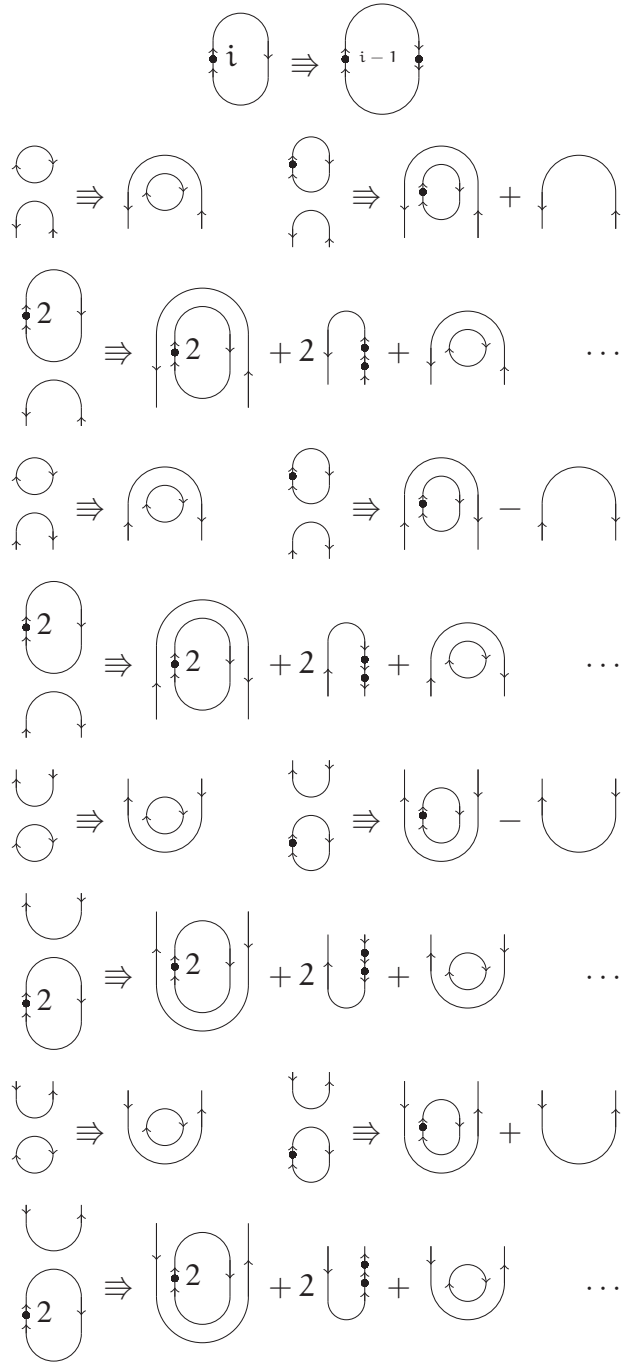
The linear 2-functor from  $\mathcal{RIB}_{\mathbb{K}}$  to  $\mathcal{OB}$  sending each ribbon to a string with the same direction and ignoring the twists is full as noted in [?]. We derive from this fact a description of elementary isotopies and Reidemeister moves first described in [?] for the linear  $(2, 2)$ -category  $\mathcal{AOB}$ .

**7.2.2. Proposition.** The linear  $(3, 2)$ -polygraph  $\overline{\mathcal{AOB}}$  is a presentation of the linear  $(2, 2)$ -category  $\mathcal{AOB}$ .

*Proof.* All 3-cells of  $\overline{\mathcal{AOB}}$  correspond to a relation verified in the linear  $(2, 2)$ -category  $\mathcal{AOB}$  by Definition ???. Moreover, the set of 3-cells of type  $i_*^0, r_*^0, r_*^1, r_*^2, r_*^4,$  and  $r_*^5$  contains all elementary isotopies and Reidemeister moves given in [?] (Chapter 1, Section 4), and thus generates the equivalence of dotted oriented Brauer diagram with bubbles. As a consequence, the 3-cells of  $\mathcal{AOB}$  are sufficient to find any relation verified in  $\overline{\mathcal{AOB}}$ .  $\square$

Then, we define some particular 2-cells of  $\overline{\mathcal{AOB}}_2^\ell$ .

**7.2.3. Quasi-reduced monomials.** We call a monomial of  $\overline{\mathcal{AOB}}$  *quasi-reduced* if the only 3-cells we can apply to it are of the form:



We call a 2-cell of the linear  $(2, 2)$ -category  $\overline{\text{AOB}}^\ell$  quasi-reduced if all monomials in its monomial decomposition are quasi-reduced.

**7.2.4. Lemma.** *The linear  $(3, 2)$ -polygraph  $\overline{\text{AOB}}$  is locally confluent.*

*Proof.* The linear  $(3, 2)$ -polygraph  $\overline{\text{AOB}}$  is quasi-terminating by the fact that all quasi-reduced monomials are quasi-normal forms. The linear  $(3, 2)$ -polygraph  $\overline{\text{AOB}}$  is exponentiation free because no quasi-reduced monomial  $m$  can be rewritten into  $\lambda m$  for  $\lambda$  a scalar other than 1. We will deduce local confluence of  $\overline{\text{AOB}}$  by Lemma ???. We have to check that all critical branchings of  $\overline{\text{AOB}}$  are confluent.

We enumerate first the sources of the critical branchings which do not involve dotted bubbles, starting with the overlapping of the isotopy 3-cells and continuing in the order in which the 3-cells are given. The final enumeration is given in the first section of the appendix ??. By doing this, we remark we can eliminate multiple critical branchings similar to others. First, we can ignore the directions of the ups and downs for each of the following interactions:

- all critical branchings involving two isotopy 3-cells other than  $i_k^2$  for some  $1 \leq k \leq 4$ ,
- all critical branchings involving an isotopy 3-cell other than  $i_k^2$  for some  $1 \leq k \leq 4$  and a Reidemeister 3-cell,
- all critical branchings involving two Reidemeister 3-cells.

Indeed, the above 3-cells correspond to the relations verified by the category  $\mathcal{OB}$  defined in [?]. Those relations do not depend on the up or down directions. Those relations are also invariant by axial symmetry. Those facts allow us to withdraw multiple critical branchings from our enumeration. In the same way, any critical branchings involving two isotopy 3-cells can be studied up to axial symmetry.

The interactions between an ordering 3-cell and a Reidemeister 3-cell are not invariant by symmetry. But, when a critical branching involving an ordering 3-cell and a Reidemeister 3-cell  $r_l^k$  is confluent, all critical branchings involving an ordering 3-cell and a 3-cell of the form  $r_*^k$  or a 3-cell whose source has an axial symmetry with the source of  $r_*^k$  are confluent. This fact allows us to treat multiple cases of critical branching simultaneously.

We now verify each critical branching is confluent. In the second section of the appendix ??, we give the source of each critical branching and a target attained by a confluence diagram for this critical branching.

The sliding 3-cells can be seen as making a bubble going through an identity and creating some additive residues. For each  $n$  of  $\mathbb{N}$ , the only bubbles created by the 3-cells  $s_n^0$  and  $s_n^1$  have less than  $n$  dots. Thus, treating the confluence of the critical branchings involving no sliding 3-cell other than  $s_0^0$  and  $s_0^1$  shows by induction on  $n$  all critical branchings involving sliding 3-cells are confluent. This deals with the totality of the critical branchings.  $\square$

**7.2.5. Weight functions.** Let  $\mathcal{C}$  be a linear  $(2, 2)$ -category. A *weight function* on  $\mathcal{C}$  is a function  $\tau$  from  $\mathcal{C}_2$  to  $\mathbb{N}$  such that:

- for each 0-composable monomials  $a$  and  $b$  we have  $\tau(a \star_0 b) = \tau(a) + \tau(b)$ ,
- for each 1-composable monomials  $a$  and  $b$  we have  $\tau(a \star_1 b) = \tau(a) + \tau(b)$ ,
- for each 2-cell  $x$ , we have  $\tau(x) = \max\{\tau(a) \mid a \text{ appears in the monomial decomposition of } x\}$ .

**7.2.6. Lemma.** Each 2-cell of the linear  $(2, 2)$ -category  $\overline{\mathcal{AOB}}^\ell$  can be rewritten into a quasi-reduced 2-cell.

*Proof.* To prove that each 2-cell of  $\overline{\mathcal{AOB}}_2^\ell$  rewrites into a normally dotted oriented Brauer diagram, we give first a weight function  $\tau$  on  $\overline{\mathcal{AOB}}_2^\ell$  defined by:

$$\begin{aligned} \tau \begin{array}{c} \curvearrowright \\ \downarrow \downarrow \end{array} &= \tau \begin{array}{c} \curvearrowleft \\ \downarrow \downarrow \end{array} = \tau \begin{array}{c} \cup \\ \downarrow \downarrow \end{array} = \tau \begin{array}{c} \cup \\ \downarrow \downarrow \end{array} = 1 \\ \tau \begin{array}{c} \downarrow \downarrow \\ \times \\ \downarrow \downarrow \end{array} &= \tau \begin{array}{c} \downarrow \downarrow \\ \times \\ \downarrow \downarrow \end{array} = \tau \begin{array}{c} \downarrow \downarrow \\ \times \\ \downarrow \downarrow \end{array} = \tau \begin{array}{c} \downarrow \downarrow \\ \times \\ \downarrow \downarrow \end{array} = 3 \\ \tau \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} &= \tau \begin{array}{c} \downarrow \\ \bullet \\ \downarrow \end{array} = 0 \end{aligned}$$

Then, we give the following procedure taking a monomial  $m$  as an entry:

- While  $m$  can be rewritten into a 2-cell  $m'$  such that  $\tau(m') < \tau(m)$  do:
  - assign  $m$  to  $m'$



- assign  $m$  to  $m'$

This procedure terminates because  $\Sigma$  is terminating without the sliding 3-cells and the isotopy 3-cells that do not decrease the weight. This results in rewriting any 2-cell into a linear combination of normally ordered Brauer diagrams. For each equivalence class of dotted oriented Brauer diagram with bubbles, exactly one representative of this class can appear in a 2-cell attained by the given procedure because there is no isotopy or Reidemeister 3-cell to apply anymore. Those are the quasi-reduced monomials.  $\square$

**7.2.7. Lemma.** The linear  $(3, 2)$ -polygraph  $\overline{\mathcal{AOB}}$  is decreasing.

*Proof.* We define the labelling to the quasi-normal form  $(\mathbb{N}, <, \psi)$  on  $\overline{\mathcal{AOB}}$  where for each rewriting step  $\alpha$  of  $\overline{\mathcal{AOB}}$ , the integer  $\psi(\alpha)$  is the distance from  $t_2(\alpha)$  to a quasi-reduced 2-cell. By Lemma ??, every 2-cell of  $\overline{\mathcal{AOB}}_2^\ell$  can be rewritten into a quasi-reduced one. This makes the map  $\psi$  well-defined because rewriting sequences leading to a quasi-reduced 2-cell always exist. The branchings of  $\overline{\mathcal{AOB}}$  are decreasing for this well-founded labelling. The decreasing diagrams for the critical branchings of  $\overline{\mathcal{AOB}}$  are given in the third section of the appendix ??.

All critical branchings of  $\overline{\mathcal{AOB}}$ , by Theorem ??, all local branchings of  $\overline{\mathcal{AOB}}$  are decreasing because  $\overline{\mathcal{AOB}}$  is exponentiation free.  $\square$

As an immediate consequence of Theorem ??, Lemma ?? implies the following result:

**7.2.8. Theorem.** The linear  $(3, 2)$ -polygraph  $\overline{\mathcal{AOB}}$  is a confluent presentation of  $\mathcal{AOB}$ .

## 7.3. BASES OF THE 2-CELLS SPACES OF $\mathcal{AOB}$

Let  $\Sigma$  be a left-monomial linear  $(3, 2)$ -polygraph presenting the linear  $(2, 2)$ -category  $\mathcal{AOB}$  verifying the two following properties:

- i) each monomial of the free linear  $(2, 2)$ -category  $\Sigma_2^\ell$  corresponds to a dotted oriented Brauer diagram with bubbles,
- ii) each 2-cell of  $\Sigma_2^\ell$  is congruent to a linear combination of normally ordered dotted oriented Brauer diagrams.

Then, for any  $a$  and  $b$  in  $\mathcal{AOB}_1$ , any element of  $\mathcal{AOB}_2(a, b)$  is equal to a linear combination of normally ordered dotted oriented Brauer diagrams. The vector space  $\mathcal{AOB}_2(a, b)$  has then a generating family given by the equivalence classes of normally ordered dotted oriented Brauer diagrams with bubbles with source  $a$  and target  $b$

Let  $\Sigma$  be a left-monomial linear  $(3, 2)$ -polygraph presenting the linear  $(2, 2)$ -category  $\mathcal{AOB}$  verifying the two following properties:

- i) each monomial of the free linear  $(2, 2)$ -category  $\Sigma_2^\ell$  corresponds to a dotted oriented Brauer diagram with bubbles,
- ii) a normally ordered dotted oriented Brauer is not congruent to a linear combination of non-equivalent others.

The vector space  $\mathcal{AOB}_2(\alpha, \beta)$  has then a free family given by the equivalence classes of normally ordered dotted oriented Brauer diagrams with bubbles with source  $\alpha$  and target  $\beta$ . Note that, by Church-Rosser Theorem ??, the second property is implied by the following two properties:

- i)  $\Sigma$  is confluent,
- ii) a normally ordered dotted oriented Brauer does not rewrite into a linear combination of non-equivalent others.

The linear  $(3, 2)$ -polygraph  $\overline{\text{AOB}}$  is a presentation of  $\mathcal{AOB}$  verifying:

- i) each monomial of the free linear  $(2, 2)$ -category  $\overline{\text{AOB}}_2^\ell$  corresponds to a dotted oriented Brauer diagram with bubbles,
- ii) each 2-cell of  $\overline{\text{AOB}}_2^\ell$  is congruent to a linear combination of normally ordered dotted oriented Brauer diagrams.
- iii)  $\overline{\text{AOB}}$  is confluent,
- iv) a normally ordered dotted oriented Brauer does not rewrite into a linear combination of non-equivalent others.

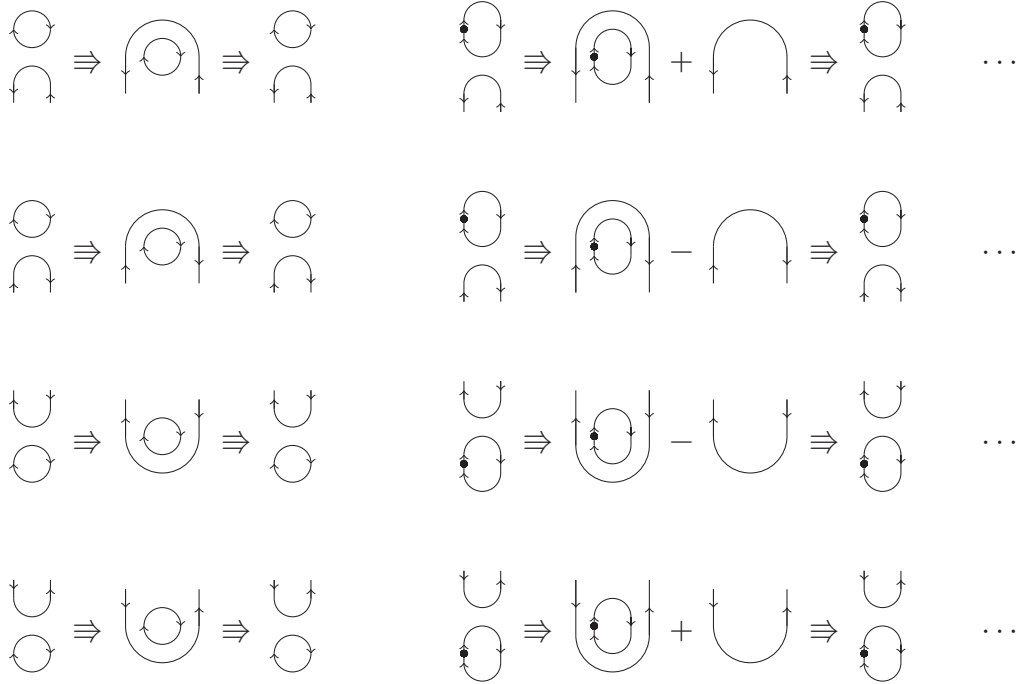
Property i) is a direct consequence of the definition of  $\mathcal{AOB}$ . Property ii) is a consequence of Lemma ?.?. Property iii) is a consequence of ?.?. Let us now prove Property iv).

**7.3.1. Lemma.** In the free linear  $(2, 2)$ -category  $\overline{\text{AOB}}_2^\ell$ , no normally ordered dotted oriented Brauer can be rewritten into a linear combination of non-equivalent others.

*Proof.* To prove this Lemma, it is sufficient to show that a quasi-reduced 2-cell does not rewrite into another quasi-reduced 2-cell. The only way to rewrite a quasi-reduced monomial is to apply sliding 3-cells or isotopy 3-cells that do not decrease the weight. So, the only way to rewrite a quasi-reduced monomial into a linear combination of others is to use rewriting paths of the forms:



7.3. BASES OF THE 2-CELLS SPACES OF  $\mathcal{AOB}$



Then, a quasi-reduced 2-cell does not rewrite into a linear combination of others. □

Thus, we get the following result as a consequence of Theorem ??:

**7.3.2. Corollary [?, Theorem 1.2.].** *Let  $a$  and  $b$  be two 1-cells of the linear  $(2, 2)$ -category  $\mathcal{AOB}$ . Then, the vector space  $\mathcal{AOB}_2(a, b)$  has a basis given by the equivalence classes of normally ordered dotted oriented Brauer diagrams with bubbles with source  $a$  and target  $b$ .*





## Appendix

### 8.1. CRITICAL BRANCHINGS OF $\overline{\text{AOB}}$

We give the enumeration of the critical branchings of the linear  $(3, 2)$ -polygraph  $\overline{\text{AOB}}$ . This enumeration is used in the proof of Lemma ??.

i) Critical branchings involving an isotopy 3-cell:

$$(i_{\bullet}^0, i_{\bullet}^1), (i_{\wedge}^0, i_{\vee}^2), (i_{\vee}^0, i_{\wedge}^2), (i_{\wedge}^1, i_{\vee}^2), (i_{\vee}^1, i_{\wedge}^2), (i_{\bullet}^0, r_{\bullet, \bullet}^3), (i_{\bullet}^1, r_{\bullet, \bullet}^4)$$

$$(i_{\wedge}^1, i_{\vee}^2), (i_{\wedge}^1, r_{\bullet, \bullet}^4), (i_{\wedge}^1, i_{\vee}^2), (i_{\vee}^1, r_{\vee, \wedge}^3), (i_{\vee}^1, r_{\vee, \vee}^3), (i_{\vee}^2, r_{\wedge, \vee}^4), (i_{\vee}^2, r_{\wedge, \wedge}^4)$$

ii) Other critical branchings involving a Reidemeister 3-cell:

$$(r_{\bullet, \bullet}^1, r_{\bullet, \bullet}^1), (r_{\bullet, \bullet}^1, r_{\bullet, \bullet}^2), (r_{\bullet, \bullet}^1, r_{\bullet, \bullet}^2), (r_{\bullet, \bullet}^1, r_{\bullet, \bullet}^4),$$

$$(r_{\bullet, \bullet}^1, r_{\bullet, \bullet}^5), (r_{\bullet, \bullet}^1, r_{\bullet, \bullet}^6), (r_{\bullet, \bullet}^1, r_{\bullet, \bullet}^7), (r_{\bullet, \bullet}^1, r_{\bullet, \bullet}^8), (r_{\vee, \wedge}^1, o_6^0), (r_{\vee, \wedge}^1, o_7^0), (r_{\wedge, \vee}^1, o_5^0), (r_{\wedge, \vee}^1, o_8^0),$$

$$(r_{\vee, \vee}^1, o_4^0), (r_{\vee, \vee}^1, o_7^0), (r_{\vee, \vee}^2, r_{\vee, \vee}^2), (r_{\wedge, \wedge}^2, o_1^0), (r_{\wedge, \wedge}^2, o_2^0), (r_{\vee, \wedge}^2, o_5^0), (r_{\vee, \wedge}^2, o_7^0),$$

$$(r_{\wedge, \vee}^2, o_5^0), (r_{\wedge, \vee}^2, o_6^0), (r_{\wedge, \vee}^2, o_7^0), (r_{\wedge, \vee}^2, o_8^0), (r_{\wedge, \vee}^2, o_5^0), (r_{\wedge, \vee}^2, o_6^0), (r_{\wedge, \vee}^2, o_7^0), (r_{\wedge, \vee}^2, o_8^0),$$

$$(r_{\vee, \vee}^2, o_3^0), (r_{\vee, \vee}^2, o_4^0), (r_{\vee, \vee}^2, o_5^0), (r_{\vee, \vee}^2, o_7^0), (r_{\wedge, \vee}^2, o_4^0), (r_{\wedge, \vee}^2, o_5^0), (r_{\wedge, \vee}^2, o_6^0),$$

$$(r_{\vee, \wedge}^2, o_3^0), (r_{\vee, \wedge}^2, o_5^0), (r_{\vee, \wedge}^2, o_8^0), (r_{\bullet, \bullet}^3, r_{\bullet, \bullet}^4), (r_{\wedge, \wedge}^3, o_1^0), (r_{\wedge, \wedge}^3, o_2^0),$$

$$(r_{\vee, \wedge}^3, o_5^0), (r_{\wedge, \wedge}^4, o_6^0), (r_{\vee, \vee}^3, o_4^0), (r_{\wedge, \wedge}^4, o_3^0), (r_{\wedge, \wedge}^4, o_4^0),$$

$$(r_{\vee, \wedge}^4, o_7^0), (r_{\wedge, \vee}^4, o_6^0), (r_{\wedge, \vee}^4, o_8^0), (r_{\vee, \vee}^4, o_2^0), (r_{\wedge, \wedge}^5, o_1^0), (r_{\wedge, \wedge}^5, o_2^0), (r_{\wedge, \vee}^5, o_5^0), (r_{\wedge, \vee}^5, o_6^0),$$

$$(r_{\vee, \wedge}^5, o_3^0), (r_{\vee, \wedge}^5, o_5^0), (r_{\vee, \vee}^5, r_{\vee, \vee}^3), (r_{\vee, \vee}^5, o_5^0), (r_{\vee, \vee}^5, o_7^0), (r_{\vee, \vee}^6, o_3^0), (r_{\vee, \vee}^6, o_4^0),$$

$$(r_{\vee, \wedge}^6, o_7^0), (r_{\vee, \wedge}^6, o_8^0), (r_{\wedge, \vee}^6, o_1^0), (r_{\wedge, \vee}^6, o_7^0), (r_{\wedge, \wedge}^6, o_5^0), (r_{\wedge, \wedge}^6, o_7^0),$$

$$(r_{\vee}^7, o_5^0), (r_{\vee}^8, o_8^0), (r_{\wedge}^8, o_7^0), (r_{\wedge}^9, o_1^0), (r_{\vee}^9, o_3^0),$$

$$(i_{\bullet}^0, s_*), (i_{\bullet}^1, s_*), (r_{\bullet, \bullet}^1, s_*), (r_{\bullet, \bullet}^2, s_*), (r_{\bullet, \bullet}^3, s_*), (r_{\bullet, \bullet}^4, s_*),$$

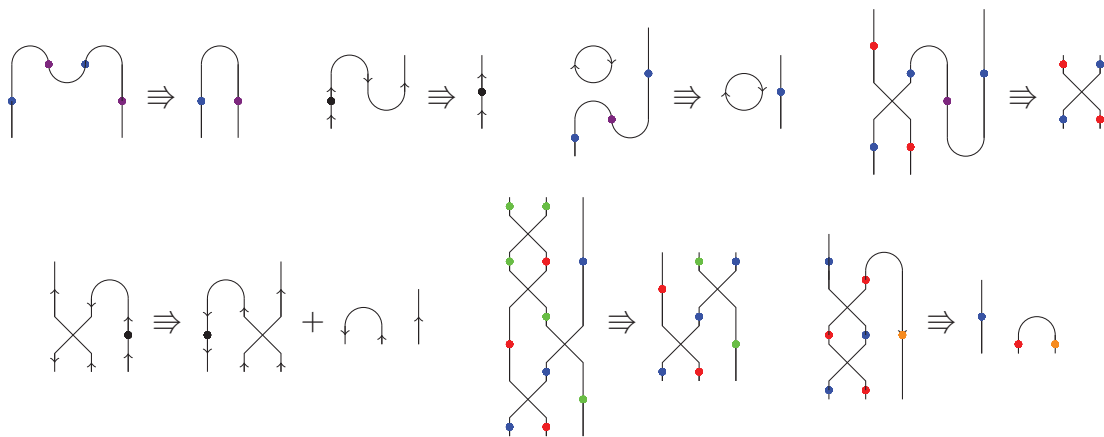
$$(r_{\bullet, \bullet}^5, s_*), (r_{\bullet, \bullet}^6, s_*), (r_{\bullet}^7, s_*), (r_{\bullet}^8, s_*), (r_{\bullet}^9, s_*),$$

iii) Other critical branchings involving a sliding 3-cell:

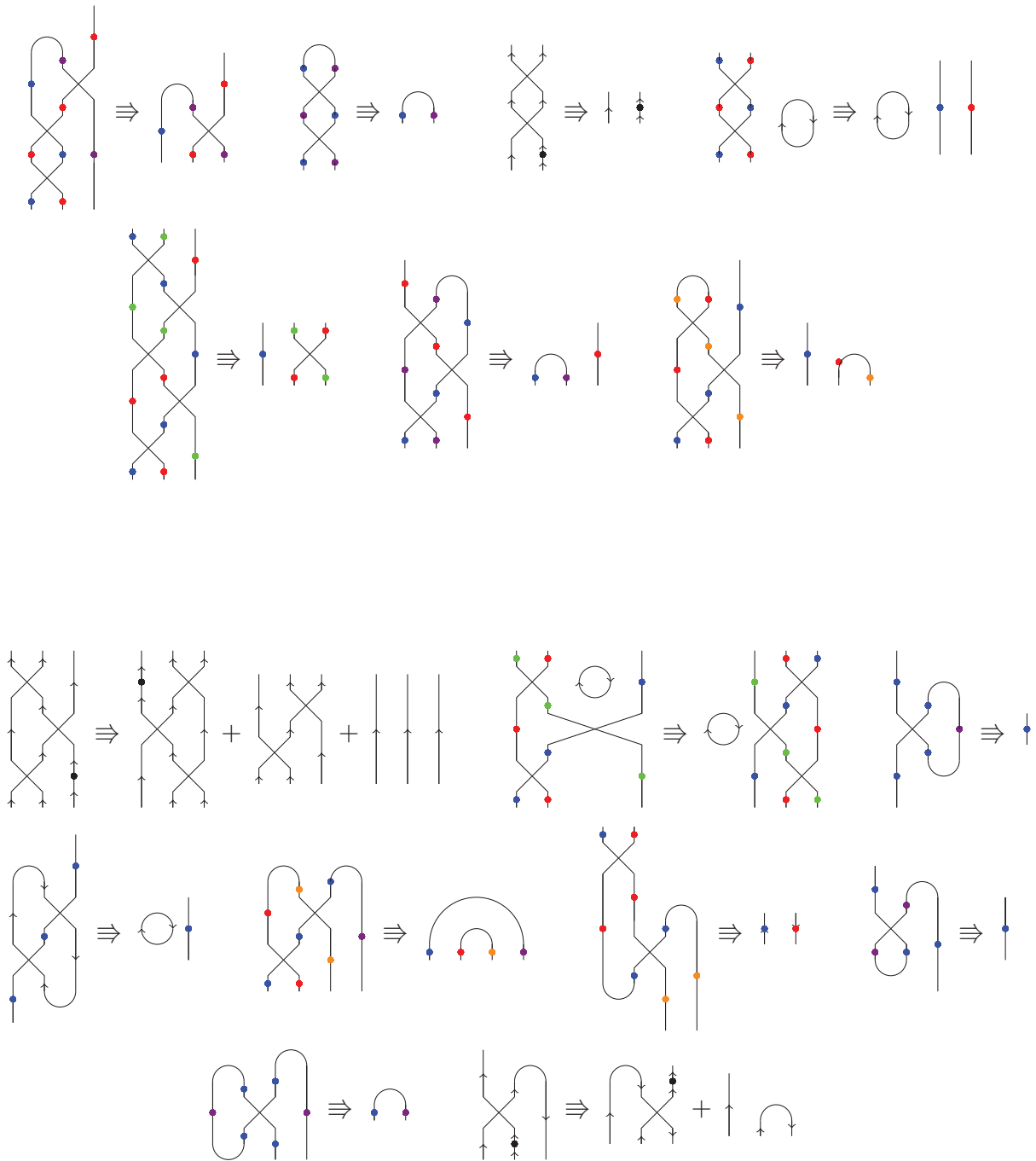
$$(o_*^1, s_*), (s_*, s_*).$$

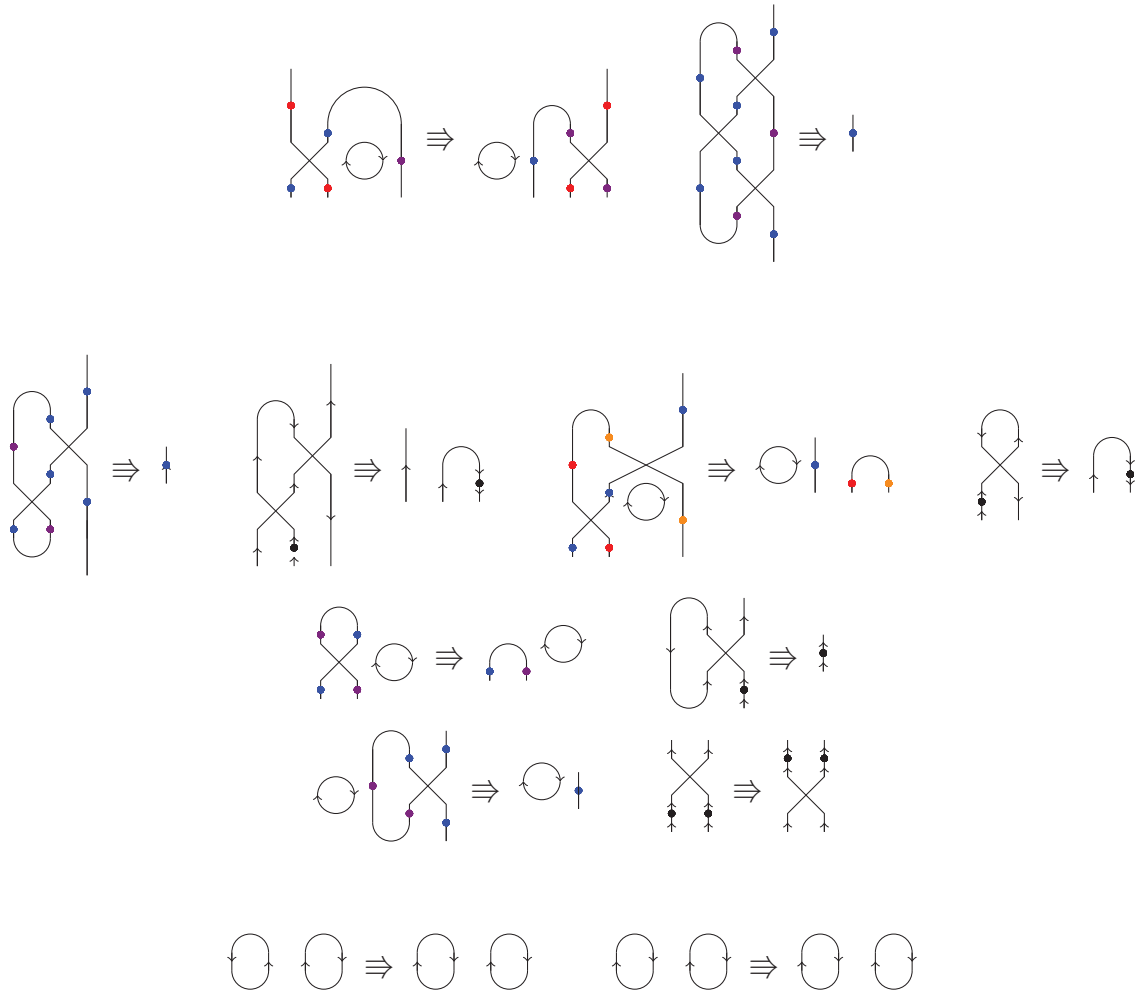
## 8.2. CONFLUENCE DIAGRAMS OF $\overline{\text{AOB}}$

We give a confluence diagram for each critical branching of the linear  $(3, 2)$ -polygraph  $\overline{\text{AOB}}$  we need to check in the proof of Lemma ??.



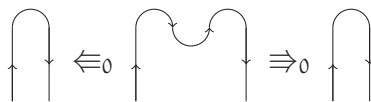
## 8.2. CONFLUENCE DIAGRAMS OF $\overline{\text{AOB}}$



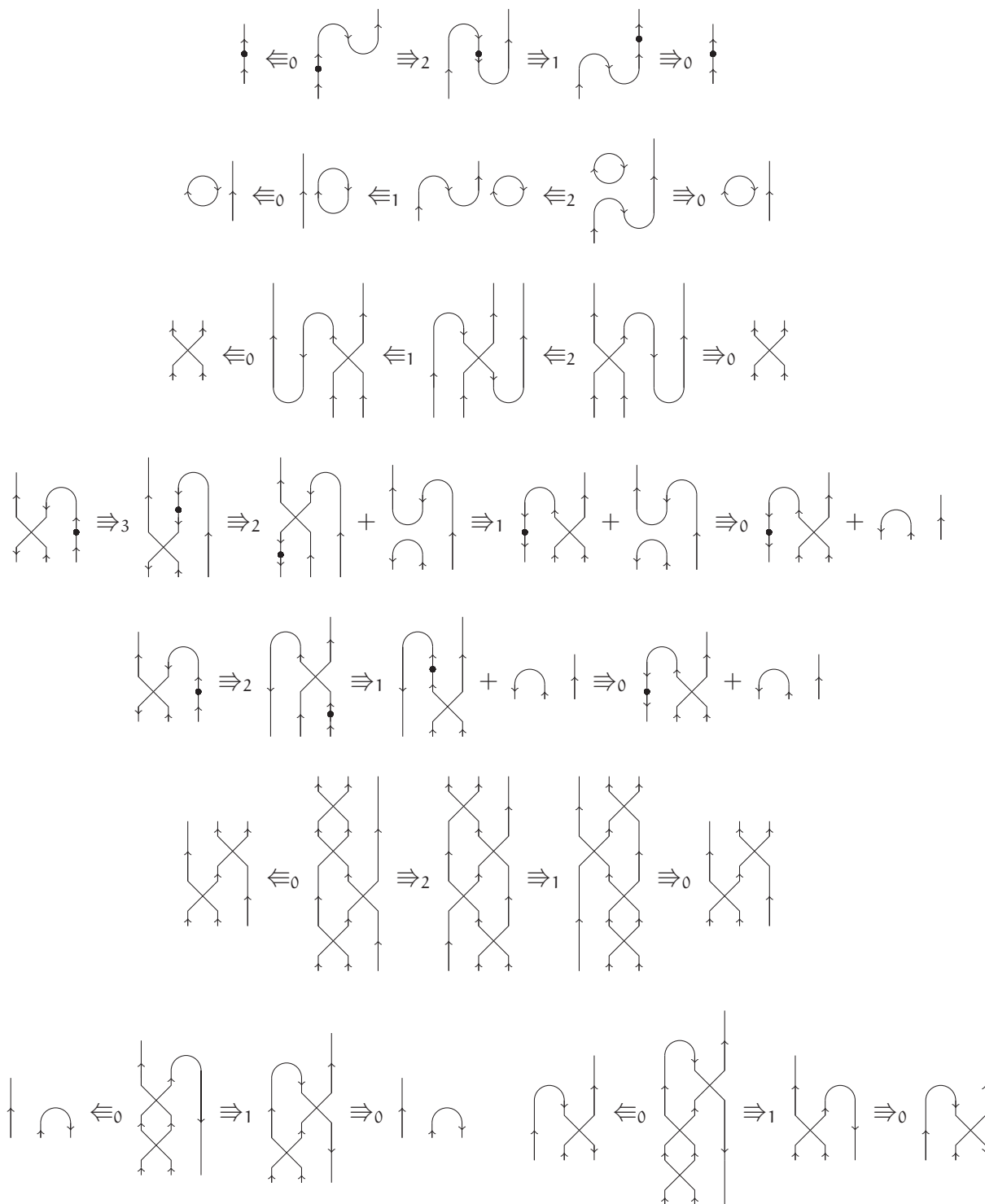


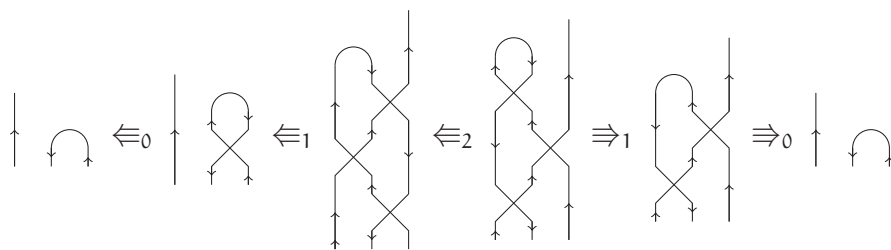
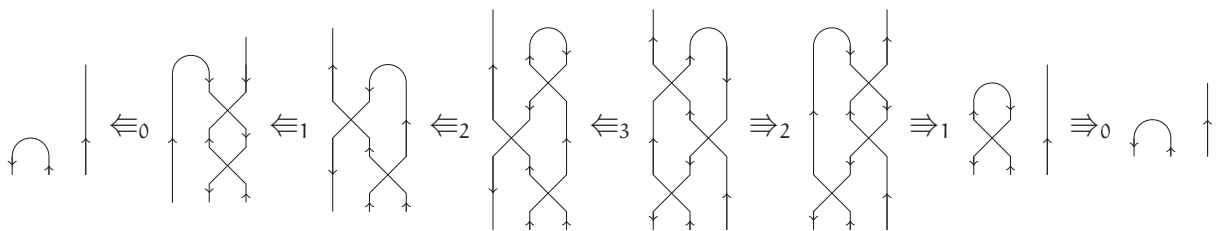
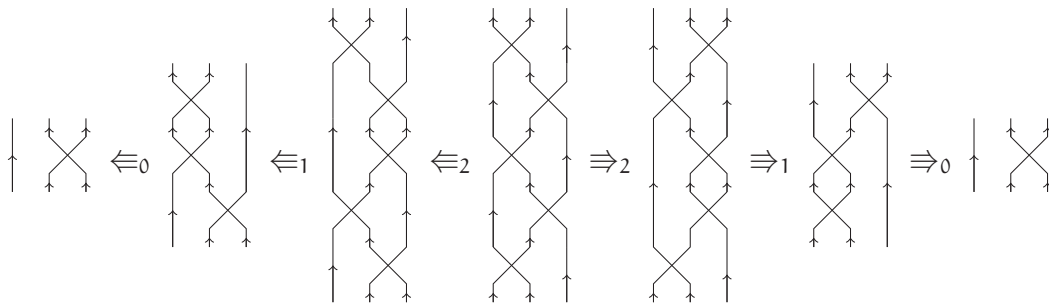
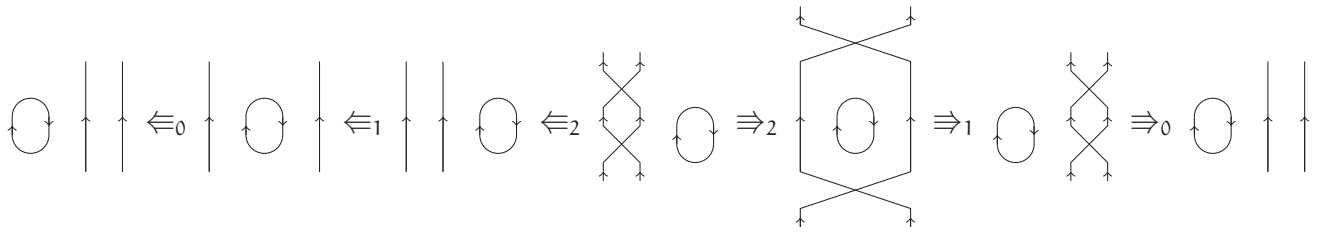
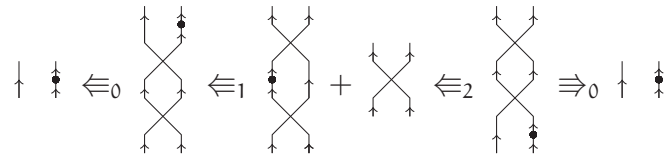
### 8.3. DECREASINGNESS OF $\overline{\text{AOB}}$

We give a decreasing diagram for each critical branching of the linear  $(3, 2)$ -polygraph  $\overline{\text{AOB}}$  we need to check in the proof of Lemma ??.

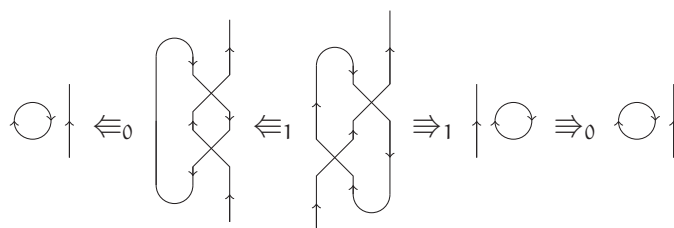
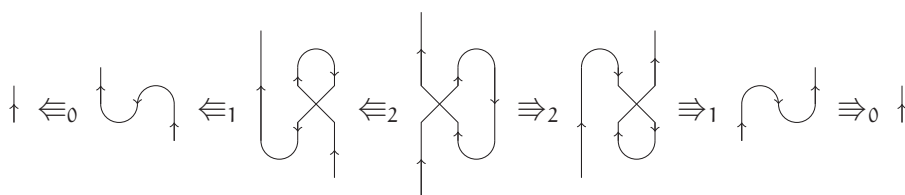
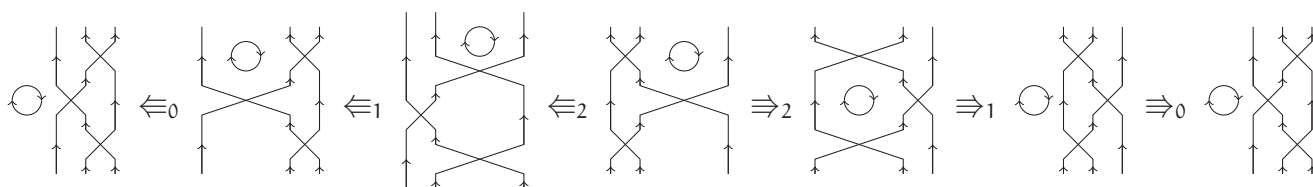
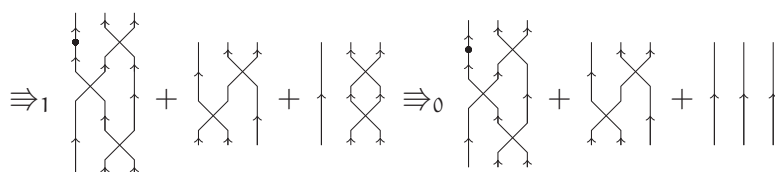
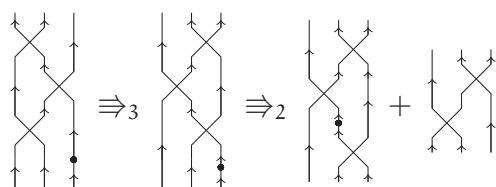
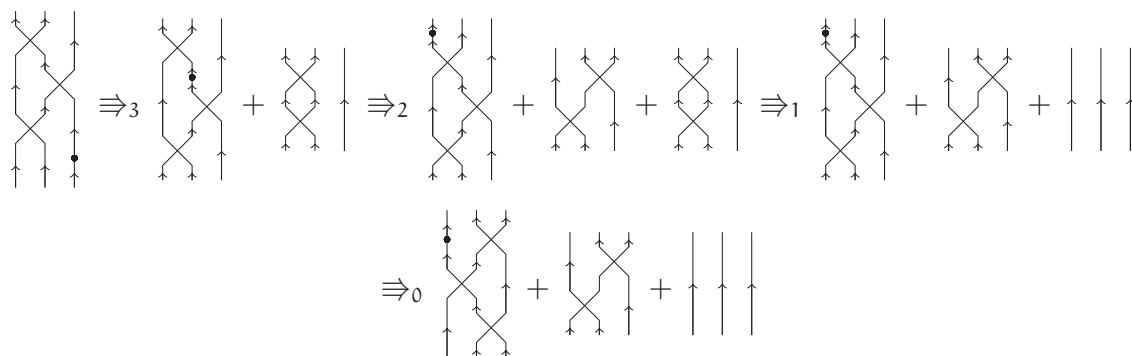


8.3. DECREASINGNESS OF  $\overline{\text{AOB}}$

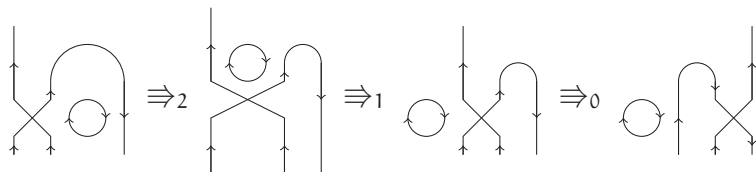
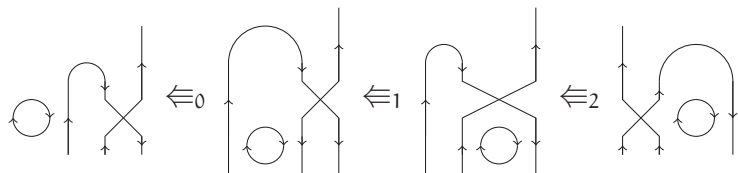
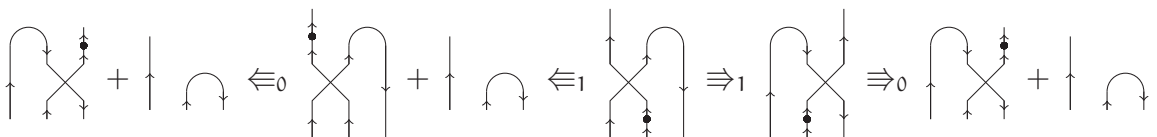
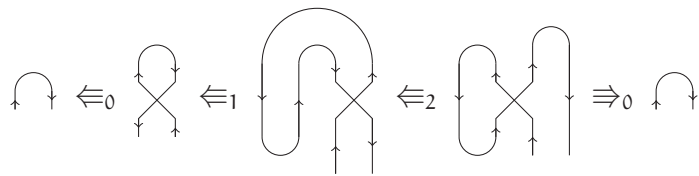
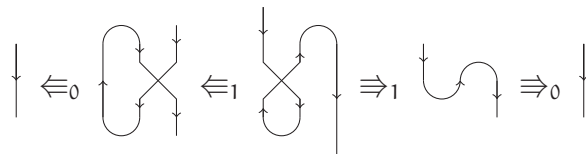
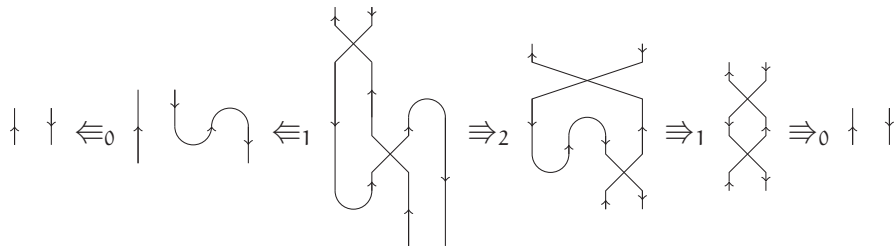
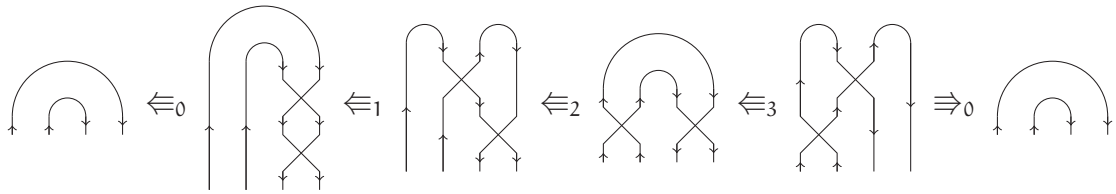




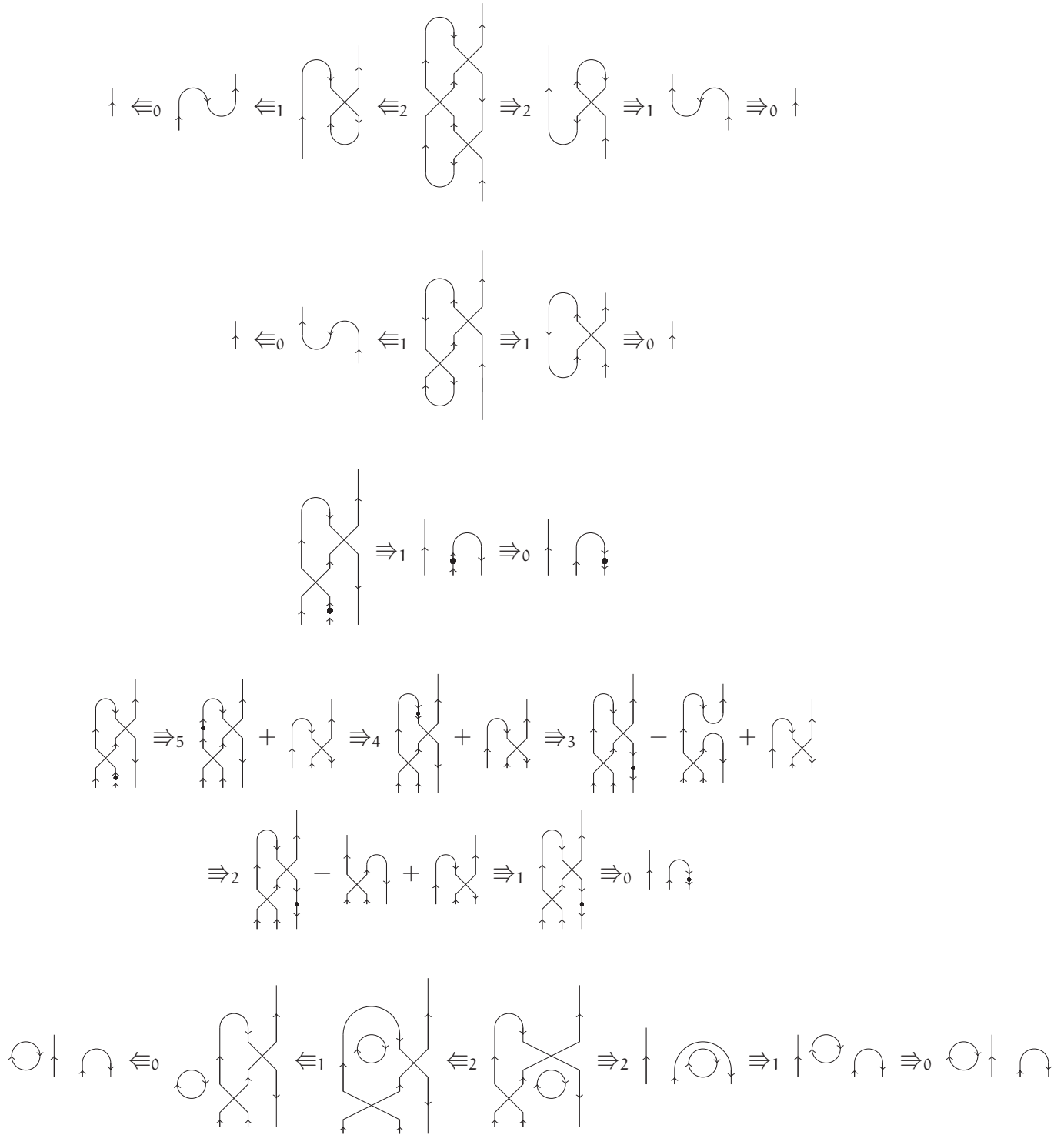
8.3. DECREASINGNESS OF  $\overline{\text{AOB}}$

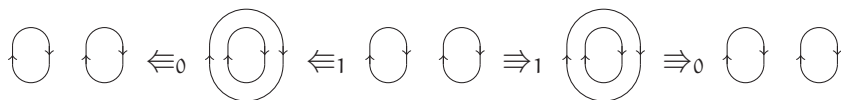
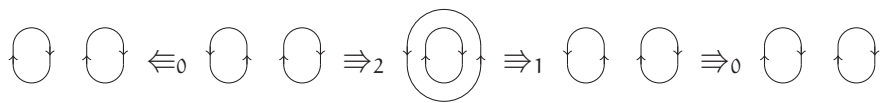
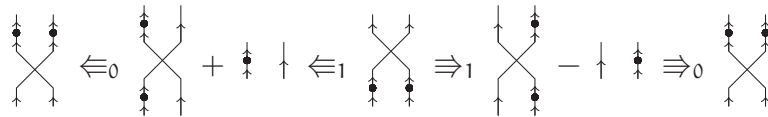
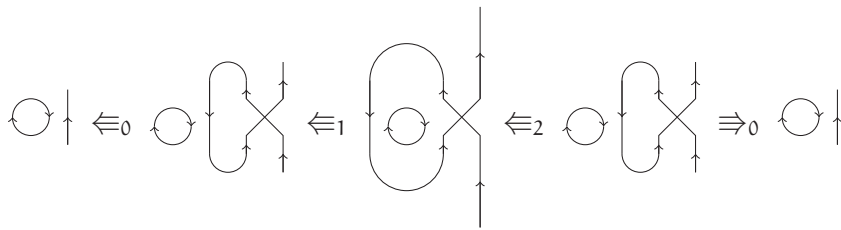
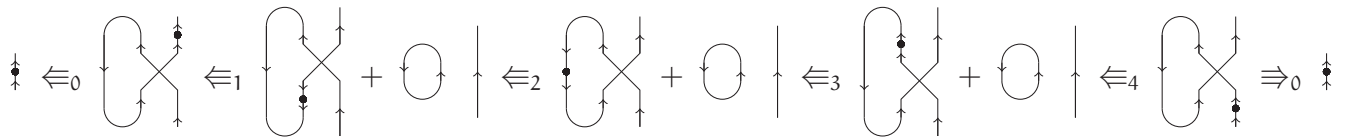
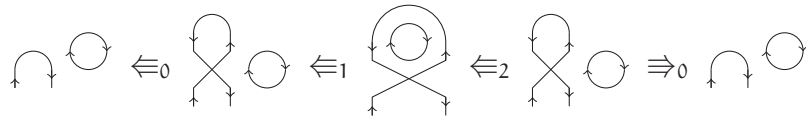
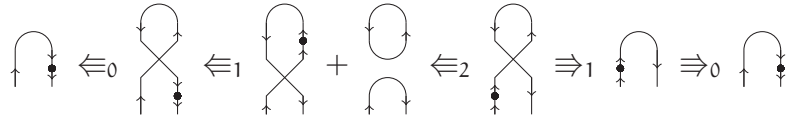






8.3. DECREASINGNESS OF  $\overline{\text{AOB}}$





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