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# Martingales sur les variétés de valeur terminale donnée

Jonathan Harter

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# THÈSE

PRÉSENTÉE À

**L'UNIVERSITÉ DE BORDEAUX**

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET  
D'INFORMATIQUE

par **Jonathan HARTER**

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POUR OBTENIR LE GRADE DE

**DOCTEUR**

SPÉCIALITÉ — MATHÉMATIQUES

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**Martingales sur les variétés de valeur terminale donnée**

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**Résumé** Définies il y a quelques décennies, les martingales dans les variétés sont maintenant des objets bien connus. Des questions très simples restent en suspens cependant. Par exemple, étant donnée une variable aléatoire à valeurs dans une variété complète, et une filtration continue (dont toute martingale réelle possède une version continue), existe-t-il une martingale continue dans cette variété qui a pour valeur terminale donnée cette variable aléatoire ? Que dire des semimartingales de valeur terminale et dérivées données ? Le but principal de cette thèse est d'apporter des réponses à ces questions. Sous des hypothèses de géométrie convexe, des réponses sont données dans les articles de Kendall (1990), Picard (1991), Picard (1994), Darling (1995) ou encore Arnaudon (1997). Le cas des semimartingales a plus largement été traité par Blache (2004). Les martingales dans les variétés permettent de définir les barycentres associés à une filtration, qui sont parfois plus simples à calculer que les barycentres usuels ou les moyennes, et qui possèdent une propriété d'associativité. Ils sont fortement reliés à la théorie du contrôle, à l'optimisation stochastique, ainsi qu'aux équations différentielles stochastiques rétrogrades (EDSRs). La résolution du problème avec des arguments géométriques donne par ailleurs des outils pour résoudre des EDSRs quadratiques multidimensionnelles.

Au cours de cette thèse deux principales méthodes ont été employées pour étudier le problème de l'existence de martingale de valeur terminale donnée. La première est basée sur un algorithme stochastique. La variable aléatoire que l'on cherchera à atteindre sera d'abord déformée en une famille  $\xi(a)$  de classe  $\mathcal{C}^1$ , et on se posera la question suivante : existe-t-il une martingale  $X(a)$  de valeur terminale  $\xi(a)$  ? Une méthode de tir, selon un principe similaire au tir géodésique déterministe, sera employée relativement au paramètre  $a$  en direction de la variable aléatoire  $\xi(a)$ .

La seconde est basée sur la résolution générale d'une EDSR multidimensionnelle à croissance quadratique. La principale problématique de cette partie sera d'adapter au cadre multidimensionnel une stratégie récente développée par Briand et Elie (2013) permettant de traiter les EDSRs quadratiques multidimensionnelles. Cette approche nouvelle permet de retrouver la plupart des résultats partiels obtenus par des méthodes différentes. Au-delà de l'intérêt unifiant, cette nouvelle approche ouvre la voie à de potentiels futurs travaux.

**Mots-clés** variétés, calcul stochastique, martingales, EDSRs, EDSRs quadratiques

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**Title** Martingales in manifolds with prescribed terminal value

**Abstract** Defined several decades ago, martingales in manifolds are very canonical objects. About these objects very simple questions are still unresolved. For instance, given a random variable with values in a complete manifold and a continuous filtration (one with respect to which all real-valued martingales admit a continuous version), does there exist a continuous martingale in the manifold with terminal value given by this random variable? What about semimartingales with prescribed drift and terminal value? The main aim of this thesis is to provide answers to these questions. Under convex geometry assumption, answers are given in the articles of Kendall (1990), Picard (1991), Picard (1994), Darling (1995) or Arnaudon (1997). The case of semimartingales was widely treated by Blache (2004). The martingales in the manifolds make it possible to define the barycenters associated to a filtration, which are sometimes simpler to compute than the usual barycenters or averages, and which have an associative property. They are strongly related to control theory, stochastic optimization, and backward stochastic differential equations (BSDEs). Solving the problem with geometric arguments also gives tools for solving multidimensional quadratic EDSRs.

During the thesis, two methods have been used for studying the problem of existence of a martingale with prescribed terminal value. The first one is based on a stochastic algorithm. The random variable that we try to reach will be deformed into a  $\mathcal{C}^1$ -family  $\xi(a)$ , and we deal with the following newer problem : does there exist a martingale  $X(a)$  with terminal value  $\xi(a)$ ? A shooting method, using the same kind of principle as the deterministic geodesic shooting, will be used with respect to a parameter  $a$  towards  $\xi(a)$ .

The second one is the resolution of a multidimensional quadratic BSDE. The aim of this part will be to adapt to the multidimensional framework a recent strategy developed by Briand and Elie (2013) to treat multidimensional quadratic BSDEs. This new approach makes it possible to rediscover the results obtained by different methods. Beyond the unification, this new approach paves the way for potential future works.

**Keywords** Manifolds, stochastic calculus, martingales, BSDEs, quadratic BSDEs

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## Introduction Générale

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L'objectif de ce chapitre introductif est de présenter la notion de martingale continue sur une variété, puis de poser le problème de recherche des martingales de valeur terminale fixée.

Nous rappellerons ensuite les différents résultats d'existence et d'unicité existants. L'unicité est bien comprise, et fortement reliée à l'existence d'une certaine fonction convexe. L'existence, quant à elle, peut être obtenue par des méthodes très variées.

En fin de partie nous présenterons les résultats principaux figurant dans cette thèse. Pour un panorama complet le lecteur pourra consulter les sous-sections [II.3](#) and [III.2](#) des deux principaux chapitres.

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## I.1 MARTINGALES SUR LES VARIÉTÉS

Les martingales vectorielles  $X$ , disons dans  $\mathbf{R}^d$  relativement à un espace probabilisé filtré  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$  avec  $T > 0$ , sont définies à partir d'une propriété de conditionnement. On appelle martingale continue sur  $\mathbf{R}^d$  tout processus continu adapté intégrable  $X$  tel que

$$X_t = \mathbf{E}(X_s | \mathcal{F}_t). \tag{I.1.1}$$

En n'exigeant cette propriété que localement, et en ajoutant un terme de dérive, on obtient la classe des semimartingales continues sur  $\mathbf{R}^d$  *i.e.* les sommes de martingales locales et de processus à variations finies. D'après la formule d'Itô, si  $f$  est de classe  $\mathcal{C}^2$  sur  $\mathbf{R}^d$ , on a aussi :

$$f(X_t) = f(X_0) + \int_0^t df(X_s)(dX_s) + \frac{1}{2} \int_0^t \text{Hess } f(X_s)(dX_s, dX_s), \tag{I.1.2}$$

où  $\text{Hess } f$  désigne ici le classique  $(0,2)$ -tenseur des dérivées partielles secondes. Une semimartingale se transforme donc en une semimartingale par une fonction  $\mathcal{C}^2$ . Comment définir à présent les semimartingales et martingales locales sur les espaces non plats ?

Pour les semimartingales, la définition vectorielle n'est pas utilisable en tant que telle, tout simplement parce qu'on ne peut pas ajouter entre eux les éléments d'une variété. P.A. Meyer répondit à ce problème dans par exemple [Mey81b, Mey82, Mey81a], Dunkan dans [Dun76], et Darling dans [Dar82], formalisme repris par Emery dans [Eme89], et écrit un peu plus tard dans [Eme99] avec le langage de la géométrie différentielle du second ordre, *i.e.* celle des fibrés osculateurs et opérateurs différentielles d'ordre deux.

Soit  $\mathcal{M}$  une variété différentiable réelle de classe  $\mathcal{C}^p$  avec  $p \geq 2$ , elle est dépourvue de structure linéaire, mais, possède des fonctions lisses  $f$  sur elle au moins de classe  $\mathcal{C}^2$ . Au moyen de la formule d'Itô on obtient sans peine la classe des semimartingales continues sur  $\mathcal{M}$  : ce sont les processus sur  $\mathcal{M}$  tels que pour toute fonction  $f \in \mathcal{C}^2(\mathcal{M}, \mathbf{R})$ ,  $f(X)$  soit une semimartingale continue réelle. C'est une définition totalement intrinsèque, et consistante avec le cas plat.

Le problème qui se pose à présent est : comment distinguer, dans la classe des semimartingales, les processus qui pourraient correspondre aux martingales ? En réalité, tout comme pour les géodésiques, il est nécessaire d'ajouter une structure de connexion linéaire sur la variété pour opérer cette distinction. On peut voir cet objet de plusieurs manières.

**Avec la géométrie du second ordre, d'après [Eme99].** La formule d'Itô (I.1.2) peut être réécrite de la manière ci-dessous en considérant une semimartingale  $X$  sur  $\mathcal{M}$  et  $f$  une fonction  $\mathcal{C}^2$  :

$$f(X_t) = f(X_0) + \int_0^t d^2 f(X_s)(\underline{d}X_s),$$

où  $\underline{d}X_s$  est défini de manière naturelle en coordonnées par  $\underline{d}X_s = dX_s^i D_i + \frac{1}{2} d\langle X^i, X^j \rangle_s D_{i,j}$ . Pour plus de détails sur ce type d'intégrale, voir [Eme99]. C'est selon ce principe, appelé aussi *principe de Schwartz*, que l'on identifie la partie droite de la formule (I.1.2) comme une intégrale du co-diffuseur  $d^2 f(X_s)$  (processus à valeurs dans le dual de l'espace tangent d'ordre deux noté  $\tau^* \mathcal{M}$ ), contre le « diffuseur »  $\underline{d}X_s$  contenant une partie martingale et une dérive.

Avec ce point de vue, nous voyons une connexion sur  $\mathcal{M}$  comme une application  $\Gamma_x : \tau_x \mathcal{M} \rightarrow T_x \mathcal{M}$ , qui préserve les éléments de  $T_x \mathcal{M}$  en tout point  $x \in \mathcal{M}$ . Comme détaillé dans [Eme99], cette notion est équivalente à la vision classique de dérivée covariante des géomètres. Emery arrive alors à une définition des  $\Gamma$ -martingales : *i.e.* un processus  $X$  à valeurs dans  $\mathcal{M}$  tel que pour tout processus  $\Sigma$  localement borné dans  $\tau^* \mathcal{M}$  au-dessus de  $X$ , le processus  $\int_0^\cdot \Gamma^*(\Sigma_s)(\underline{d}X_s)$  soit une martingale locale réelle.

De manière plus heuristique,  $X$  est une martingale si  $\underline{d}X$  peut se décomposer en une différentielle de martingale locale réelle (d'ordre un), et une partie purement d'ordre deux : c'est typiquement ce que l'on observe dans le cas Euclidien en regardant la formule d'Itô.

**En travaillant avec la formule d'Itô.** Dans le cas plat, on a une caractérisation intrinsèque des martingales locales du même type que celle des semimartingales. La semimartingale précédente  $X$  est une martingale si pour toute fonction  $f$ , le processus

$$f(X) - f(X_0) - \frac{1}{2} \int_0^\cdot \text{Hess } f(dX_s, dX_s)$$

est une martingale locale réelle. Emery dans [Eme89] a donné une définition de la variation quadratique d'une semimartingale contre n'importe quel tenseur de type  $(0, 2)$  et définie à l'aide d'un plongement. Il nous suffit donc de remplacer  $\text{Hess } f$  (pour le cas plat) par le vrai Hessian  $\nabla df$  sur une variété munie d'une connexion  $\nabla$ , comprise ici comme une dérivée covariante. On obtient alors la classe des  $\nabla$ -martingales sur  $\mathcal{M}$ . Dans tout le manuscrit les connexions seront vues généralement comme des dérivées covariantes. Enfin, cette relation écrite dans un système de coordonnées globales signifie que le terme

$$dX^i + \frac{1}{2} \Gamma_{jk}^i(X) d\langle X^j, X^k \rangle$$

est pour tout  $i$  la différentielle  $dM_t^i$  d'une martingale locale réelle. Cette propriété, comme le mentionne Blache dans [Bla04b] peut aussi s'écrire formellement

$$X_{t+dt} = \exp_{X_t}(dM_t).$$

**Autres notions de martingales ; transformation par les fonctions convexes et barycentres.**

Les  $\nabla$ -martingales sur une variété jouissent du même type de propriété que les martingales euclidiennes : elles sont transformées par les fonctions  $\nabla$ -convexes en sous-martingales locales réelles. On appellera  $\mathcal{C}$ -martingales les processus satisfaisant cette propriété.

En général la réciproque est fautive, les  $\mathcal{C}$ -martingales n'étant pas toujours continues, on ne peut donc se servir de cette propriété comme caractérisation des  $\nabla$ -martingales. Cependant les

$\mathcal{C}$ -martingales discrètes de valeur terminale donnée peuvent se construire facilement ; voir la section I.5.2 consacrée aux barycentres.

Enfin, on peut définir aussi la notion de martingales *au sens des barycentres*, qui contient la définition ci-dessus (voir là encore la section I.5.2 pour plus de détails).

**Martingales de valeur terminale donnée sur une variété.** Dans le cas plat de  $\mathbf{R}^d$ , on sait, par définition même des martingales, construire une martingale de valeur terminale donnée via l'espérance conditionnelle.

Sur une variété une telle notion d'espérance n'existe pas, et il se pose donc la question légitime suivante. Fixons-nous une connexion  $\nabla$  sur une variété différentiable  $\mathcal{M}$  de classe au moins deux, et  $\xi$  une variable aléatoire dans  $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$  où  $\mathcal{B}(\mathcal{M})$  désigne la tribu borélienne attachée à  $\mathcal{M}$ ,  $T > 0$  et  $(\mathcal{F}_t)_{0 \leq t \leq T}$  une filtration vérifiant

**(Filtration)** toute  $(\mathcal{F}_t)_{0 \leq t \leq T}$ -martingale admet une version continue.

Existe-t-il une  $\nabla$ -martingale  $X$ , associée à une filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$ , telle que  $X_T = \xi$  ? Est-elle unique ?

Les filtrations engendrées par un mouvement brownien satisfont **(Filtration)** par le théorème de représentation.

*Remarque – I.1.1* (À propos de l'horizon). Dans toute cette thèse nous travaillerons avec un horizon fini déterministe. Mais le problème est en bijection avec celui qui ferait intervenir des horizons infinis ou même aléatoires. En effet, soit un intervalle  $[0, \tau]$  fixé, et  $X$  une martingale telle que  $X_\tau = \xi$ . Alors on prolonge  $X$  sur  $[0, \infty[$  par la valeur  $X_\tau$ . On fait ensuite un changement de temps pour se ramener à un intervalle  $[0, T]$  avec  $T > 0$  déterministe. Les processus obtenus restent des martingales en changeant la filtration de départ.

Essentiellement, dans le [Chapitre II](#) l'horizon sera  $T = 1$  alors que dans le [Chapitre III](#) nous prendrons un horizon quelconque déterministe  $T$ , selon les notations propres à la littérature des équations différentielles stochastiques rétrogrades (ce type d'équation différentielle stochastique sera abrégé « EDSR » dans la suite).

Les résultats d'existence et d'unicité déjà existants font intervenir principalement deux types d'hypothèse. Le *premier type* porte sur la taille de la variété  $\mathcal{M}$  et sur la connexion  $\nabla$ . Nous rappellerons dans la section I.3 *infra* les résultats obtenus par Arnaudon dans [Arn97], Picard dans [Pic91, Pic06] et [Pic94], ainsi que ceux de Kendall dans [Ken90]. Le *second type* porte sur la filtration ; plus précisément si la filtration provient d'un mouvement brownien, on est capable de reformuler le problème en terme d'EDSR à croissance quadratique multidimensionnelle (dimension égale à la celle de la variété). Cette restriction du côté de la filtration possède cependant un énorme avantage, celui de pouvoir utiliser les résultats propres aux EDSRs déjà très nombreux. De plus, les EDSRs nous permettent de faire un peu mieux : trouver les semimartingales de valeur terminale fixée.

Nous verrons dans la [Section I.4](#) le détail de ce point de vue. D'une part, il est possible de regarder le problème dans  $\mathbf{R}^d$  grâce à un système de cartes globales, et aboutir à une EDSR quadratique multidimensionnelle (idée de Darling dans [Dar95]). D'autre part, nous pouvons résoudre directement une EDSR à valeurs dans  $\mathcal{M}$  (idée de Blache dans [Bla04a], [Bla06]). Nous travaillerons dans le [Chapitre III](#) de cette thèse avec la première vision.

## I.2 UNICITÉ DES MARTINGALES

Analysons tout d'abord le cas Euclidien  $\mathcal{M} = \mathbf{R}^d$ , où  $\nabla$  est la connexion plate associée, et fixons  $\xi \in L^1(\mathbf{R}^d, \mathcal{F}_T)$  une variable aléatoire que l'on cherche à atteindre. Les  $\nabla$ -martingales sont alors les vecteurs de martingales locales réelles. Comme le mentionne Picard dans [Pic06], il existe plusieurs martingales de valeur terminale  $\xi$  à moins de localiser et donc de réduire un peu la classe où l'on se pose le problème d'unicité. Mais il y a unicité au moins dans la classe des vraies martingales (non locales), ou même dans la classe  $\mathcal{C}$  des martingales locales pour lesquelles  $X^{\lfloor \tau}$  est uniformément intégrable pour un certain temps d'arrêt  $\tau$ . En effet, si  $X^{\lfloor \tau}$  et  $X'^{\lfloor \tau}$  vérifient  $X_\tau = \xi$  et  $X'_{\tau'} = \xi$ , la propriété de martingale donne  $X_t^{\lfloor \tau} = \mathbf{E}(\xi | \mathcal{F}_t) = X_t'^{\lfloor \tau}$ .

Revenons au cas général d'une variété quelconque  $\mathcal{M}$ . Le problème est là encore bien compris puisqu'il se reformule à l'aide de *fonctions séparantes*. Soit  $\mathcal{V}$  un ouvert relativement compact de  $\mathcal{M}$ , notons  $\text{Mart}_{\nabla}(\mathcal{V})$  l'ensemble des  $\nabla$ -martingales à valeurs dans  $\mathcal{V}$ .

**Définition – I.2.1.** Une application  $\Phi : \mathcal{V} \times \mathcal{V} \rightarrow \mathbf{R}^+$  est appelée *application séparante* si  $\Phi$  est  $\nabla \otimes \nabla$ -convexe et  $\Phi^{-1}(\{0\}) = \{(x, x), x \in \mathcal{M}\}$ .

Soient deux éléments  $X, X'$  de  $\text{Mart}_{\nabla}(\mathcal{V})$  telles que  $X_T = X'_T = \xi$ . On sait (voir la Proposition 4.1 de [Eme99] par exemple) qu'une fonction convexe transforme les martingales (au sens des variétés) en sous-martingale locale réelle. Il vient immédiatement :

$$\Phi(X_t, X'_t) \leq \mathbf{E}(\Phi(X_1, X'_1) | \mathcal{F}_t) = 0 \quad \text{p.s.} \quad (\text{I.2.1})$$

le processus  $X$  est alors une version de  $X'$  puisque  $\Phi$  est une séparante sur  $\mathcal{V}$  et que les processus considérés sont continus. L'existence d'une séparante sur  $\mathcal{V}$  garantit donc l'unicité dans  $\text{Mart}_{\nabla}(\mathcal{V})$  pour le problème de recherche de martingales de valeur terminale donnée.

Inversement, Kendall a montré dans [Ken90], que l'unicité permet de construire une séparante  $\Phi$  sur  $\mathcal{V}$  définie par :

$$\Phi(x, y) = \inf_{\substack{X, X' \text{ martingale dans } \mathcal{V} \\ X_0 = x, X'_0 = y}} \mathbf{P}(X_1 \neq X'_1). \quad (\text{I.2.2})$$

L'unicité est donc fortement reliée à la géométrie de la variété. Emery a montré que localement toute variété totalement géodésique possède une séparante, elle a donc toujours lieu si l'on restreint la taille de la variété. Un espace possédant une application séparante sur lui-même sera dit à *géométrie convexe*.

**Proposition – I.2.1** ([Eme89], Corollaire 4.61). *En tout point d'une sous-variété totalement géodésique de  $\mathcal{M}$ , il existe un voisinage ouvert  $\mathcal{V}$  inclus dans  $\mathcal{M}$  tel que pour toutes martingales  $X, X'$  à valeurs dans  $\mathcal{V}$  définies sur le même espace probabilisé et satisfaisant  $X_1 = X'_1$  p.s., le processus  $X$  est une version de  $X'$ .*

*Remarque – I.2.1.* Sans restriction sur la taille du domaine où vit la variable terminale  $\xi$ , on peut trouver très facilement des contre-exemples justifiant la non-unicité. On pourra trouver un

contre-exemple dans le cas du cercle  $\mathbb{S}^1$  dans la section I.3.2 *infra*.

Pour la sphère  $\mathbb{S}^2$  plongée dans  $\mathbf{R}^3$  munie de la métrique et connexion de Levi-Civita, on peut définir les deux processus suivants sur l'hémisphère nord en coordonnées sphériques  $(\theta, \lambda)$  :

$$\begin{cases} \theta_t^1 = B_t \\ \lambda_t^1 = \arctan \exp(-t/2) \end{cases}, \quad \begin{cases} \theta_t^2 = \pi - B_t \\ \lambda_t^2 = \arctan \exp(-t/2) \end{cases},$$

avec  $B$  un mouvement brownien réel issu de zéro. Les deux processus repérés par  $(\theta_1, \lambda_1)$  et  $(\theta_2, \lambda_2)$  sont deux martingales sur  $\mathbb{S}^2$ , et si on les arrête au temps d'atteinte de  $\{x = 0\}$  elles ont même valeur terminale.

## I.3 EXISTENCE DES MARTINGALES SANS EDSR

### § I.3.1. Existence sur un espace Euclidien

Reprenons les mêmes notations que précédemment. Notamment on suppose toujours que  $\xi \in L^1(\mathbf{R}^d, \mathcal{F}_T)$ . Pour tout espace espace probabilisé  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P})$  où  $(\mathcal{F}_t)_{0 \leq t \leq T}$  satisfait (**Filtration**), on est capable de former une version continue  $(X_t)_{0 \leq t \leq T}$  du processus  $M$  défini par :

$$M_t = \mathbf{E}(\xi | \mathcal{F}_t). \tag{I.3.1}$$

Alors  $X$  est une martingale Euclidienne continue de valeur terminale  $\xi$  donc *a fortiori* une martingale locale. Bien sûr le cas particulier où  $(\mathcal{F}_t)_{0 \leq t \leq T}$  provient d'un mouvement brownien est inclus dans celui-ci.

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Nous prendrons  $T = 1$  pour toute la suite de cette partie dédiée aux méthodes géométriques.

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### § I.3.2. Cas particulier du cercle sur l'espace de Wiener attaché à $\mathcal{C}^0(\Omega, \mathbf{R}^k)$

Le cercle unité  $\mathbb{S}^1$  possède une structure de variété différentiable de dimension un, la projection  $\pi^{-1}$  est une carte locale où  $\pi : \mathbf{R} \rightarrow \mathbb{S}^1$  peut être définie par  $\pi(\theta) = e^{i\theta}$ . L'application  $\pi$  n'est pas inversible, la notation  $\pi^{-1}$  est réservée pour une détermination particulière. Si  $X$  est une semimartingale continue sur  $\mathbb{S}^1$  on notera dans la suite  $\langle X, X \rangle$  la variation quadratique (réelle) de  $Y$  où  $Y$  est continu et vérifiant  $\pi(Y) = X$ . Un processus  $X$  sera appelé *martingale sur  $\mathbb{S}^1$*  si  $\pi^{-1}(X)$  en est une sur  $\mathbf{R}$  (comme les déterminations diffèrent d'un multiple de  $2\pi$  et que les processus sont supposés continus, ces définitions sont consistantes). Nous rappelons ici des résultats obtenus par Picard dans [Pic06].

Nous désignerons par  $\text{Mart}(\mathbb{S}^1)$  l'espace des martingales sur  $\mathbb{S}^1$  et  $\text{Mart}(\mathbb{S}^1, \xi)$  l'espace des martingales sur  $\mathbb{S}^1$  de valeur terminale  $\xi$ . Picard obtient une première classe d'unicité :

**Théorème – I.3.1** (Picard, [Pic06], Théorème 1). *Soit  $\xi \in L^1(\mathbb{S}^1)$ . Il y a unicité dans l'espace  $\text{Mart}(\mathbb{S}^1, \xi) \cap \mathcal{C}$  avec  $\mathcal{C}$  l'ensemble des martingales  $X$  de  $\mathbb{S}^1$  telles que :*

$$\mathbf{E}(\exp(\langle X, X \rangle_1)) < \infty.$$

L'autre classe d'unicité que Picard obtient fait intervenir une condition de régularité issue du calcul de Malliavin sur l'espace de Wiener  $\mathcal{C}^0(\Omega, \mathbf{R}^k)$ . On supposera connue ici la construction de l'espace  $\mathbb{D}^{1,2}(\mathbf{R})$  comme fermeture de l'espace des fonctions de Wiener pour la norme

$$\|F\|_{1,2} := \left[ \mathbf{E}(|F|^2) + \mathbf{E}(\|\mathbf{D}F\|_{L^2(dx)}^2) \right]^{1/2},$$

où  $\mathbf{D}F \in L^2([0,1] \times \Omega, \mathcal{B}([0,1]) \otimes \mathcal{F}, dx \otimes d\mathbf{P})$  est la dérivée de Malliavin de  $F$ . On pourra consulter [Nua06] ou l'item III.1.1 de ce manuscrit pour plus de détails. On peut facilement remplacer  $\mathbf{R}$  par  $\mathbb{S}^1$  dans les définitions (ou même par une variété générale  $\mathcal{M}$ , puisqu'elle possède des fonctions lisses sur elle) et ainsi définir l'espace des variables aléatoires sur  $\mathbb{S}^1$  dérivables au sens de Malliavin. La variable aléatoire  $(F, \mathbf{D}^l F)$  pour tout entier  $l$  compris entre 1 et  $k$  est, par construction, une variable aléatoire à valeurs dans  $T_F \mathbb{S}^1$ . Dans le cas du cercle les espaces tangents s'identifient à la droite réelle  $\mathbf{R}$ . On appellera donc  $\mathbb{D}^{1,2}(\mathbb{S}^1)$  la fermeture pour  $\|\cdot\|_{1,2}$ , définie après identification, de l'espace des fonctions de Wiener sur  $\mathbb{S}^1$ . Picard obtient le résultat d'unicité suivant :

**Théorème – I.3.2** (Picard, [Pic06], Théorème 3). *Soient  $r > 1$  et  $\xi \in L^1(\mathbb{S}^1)$ . Il y a unicité dans l'espace  $\text{Mart}(\mathbb{S}^1, \xi) \cap \mathcal{C}_r^l$  avec  $\mathcal{C}_r^l$  l'ensemble des martingales  $X$  de  $\mathbb{S}^1$  telles que :*

- (i) pour tout  $t$ ,  $X_t \in \mathbb{D}^{1,2}(\mathbb{S}^1)$ ,
- (ii) il existe  $\varepsilon > 0$  tel que  $\mathbf{E} \left( \int_0^1 \left( \frac{d\langle X, X \rangle_t}{dt} \right)^{1+2r+\varepsilon} dt \right) < \infty$ ,
- (iii)  $\int_0^1 \int_0^1 \|\mathbf{D}_s X_t\|_{L^{2r/r-1}(\Omega)}^2 dt ds$ .

Quant à l'existence, on peut déjà remarquer que  $\text{Mart}(\mathbb{S}^1, \xi)$  contient une infinité d'éléments grâce à la structure géométrique simple du cercle. En effet, soit  $\eta$  une variable aléatoire réelle telle que  $\pi(\eta) = \xi$ . Il existe une infinité d'éléments  $\eta$  satisfaisant cette condition. Définissant alors une martingale  $X$  par

$$X_t = \pi[\mathbf{E}(\eta | \mathcal{F}_t)], \quad 0 \leq t \leq T,$$

on construit une infinité d'éléments de  $\text{Mart}(\mathbb{S}^1, \xi)$  en choisissant des versions continues pour chaque choix de  $\eta$ . Dans les sous-classes données par les Théorème – I.3.1 et Théorème – I.3.2 l'unicité est cependant vérifiée. Pour finir, Picard donne une construction d'un autre élément de  $\text{Mart}(\mathbb{S}^1, \xi)$  qui est en tout temps dans  $\mathbb{D}^{1,2}(\mathbb{S}^1)$  en se basant sur la formule de Clark-Haussmann-Ocone.



**Théorème – I.3.3** (Picard, [Pic06], Théorème 4). *Si  $\xi \in \mathbb{D}^{1,2}(\mathbb{S}^1)$ , alors le processus*

$$X_t = \xi - \pi \left( \sum_{l=1}^k \int_t^1 \mathbf{E} \left( \mathbf{D}_s^l \xi \mid \mathcal{F}_s \right) dW_s^l \right)$$

*est dans  $\text{Mart}(\mathbb{S}^1, \xi) \cap \mathbb{D}^{1,2}(\mathbb{S}^1)$ . Sa dérivée de Malliavin est donnée par :*

$$\mathbf{D}_s X_t = \mathbf{E}(\mathbf{D}_s \eta \mid \mathcal{F}_t) \mathbb{1}_{[0,t]}(s).$$

**§ I.3.3. Cas général sur l'espace de Wiener attaché à  $\Omega = \mathcal{C}^0([0, 1], \mathbf{R}^m)$**

Les résultats de cette partie généralisent en quelque sorte les résultats obtenus dans le cas du cercle, certaines techniques employées sont similaires. Ils sont extraits de [Pic91].

Commençons à nouveau par regarder l'unicité. On conserve le cadre habituel, où  $\mathcal{M}$  est une variété différentiable munie d'une connexion  $\nabla$ , et on fixe un espace probabilisé où  $(\mathcal{F}_t)_{0 \leq t \leq 1}$  satisfait (**Filtration**). Ce triplet n'est pas encore supposé être l'espace de Wiener. Picard obtient un résultat d'unicité dans une classe de martingales admettant des moments exponentiels assez grands. Plus précisément :

**Théorème – I.3.4** (Théorème 2.2.1, Picard, [Pic91]). *Supposons que  $\mathcal{M}$  ait des rayons d'injectivité  $R$  strictement positifs, et que les courbures sectionnelles soient majorées uniformément par une quantité  $\kappa$  positive. Soit  $\lambda \in \mathbf{R}^+$  tel que  $\lambda \geq \max \left\{ \frac{\kappa}{2}, \frac{1}{2} \left( \frac{\pi}{R} \right)^2 \right\}$ . Il y a unicité dans l'espace des martingales  $X$  vérifiant*

$$\mathbf{E} \left( \exp \left( \int_0^1 \lambda \langle dX_s, dX_s \rangle_g \right) \right) < \infty. \quad (\text{I.3.2})$$

On conserve donc le **Théorème – I.3.1** *supra* dans une variété générale, pourvu que l'ordre d'intégrabilité des moments exponentiels de  $X$  soit suffisamment grand ; autrement dit que l'on réduise suffisamment la classe d'unicité. Le point crucial utilisé par Picard est que les martingales exponentiellement intégrables satisfont un *principe du maximum* au sens suivant : sous une hypothèse de convexité supplémentaire, toute martingale exponentiellement intégrable de valeur terminale disons dans un espace  $\mathcal{V}$ , est entièrement dans  $\mathcal{V}$  en tout temps (voir le Théorème 2.1.3 dans [Pic91]). Appliquant ce résultat pour un couple de martingales  $(X, X')$  et  $\mathcal{V}$  la diagonale de  $\mathcal{M}$ , si  $X$  et  $X'$  ont même valeur terminale alors elles sont nécessairement égales.

Des résultats intermédiaires de convexité seront repris dans le **Chapitre II**, notamment la construction de fonctions convexes sur des boules géodésiques régulières de rayon assez petit. Ces fonctions fournissent l'intégrabilité exponentielle des moments Riemanniens d'une martingale (voir la **Proposition – II.1.4**).

Si  $\Omega = \mathcal{C}^0([0, 1], \mathbf{R}^m)$ , Picard est capable de construire comme dans le cas du cercle une martingale particulière ayant l'avantage d'être dans la classe d'unicité donnée par le **Théorème – I.3.4**, propriété clef dans bon nombre d'articles sur le sujet.

**Théorème – I.3.5** (Picard, [Pic06], Théorème 4). *Supposons que  $\mathcal{M}$  a des rayons d'injectivité  $R$  strictement positifs, et que les courbures sectionnelles sont majorées uniformément par une quantité  $\kappa$  positive.*

*Si  $\xi \in \mathbb{D}^{1,2}(\mathcal{M})$  et s'il existe une fonction positive  $\rho : \mathbf{R}^+ \rightarrow \mathbf{R}$  telle que*

$$\int_0^1 \rho_t^2 dt < \frac{1}{\kappa}, \quad \left( \sum_{j=1}^m \mathbf{E} \left( \left| \mathbf{D}_t^j \xi \right| \middle| \mathcal{F}_t \right)^2 \right)^{1/2} \leq \rho_t,$$

*alors il existe une martingale  $X$  de valeur terminale  $\xi$ . Elle satisfait de plus*

$$\frac{d}{dt} \langle X, X \rangle_{g,t} \leq \frac{\rho_t^2}{1 - \kappa \int_t^1 \rho_s^2 ds}.$$

En particulier le crochet Riemannien de  $X$  est dans  $L^\infty(\Omega)$ , donc  $X$  est dans la classe d'unicité du Théorème – I.3.4. Nous n'avons plus d'expression explicite de  $X$  comme pour le cercle, la construction de  $X$  est faite par approximations successives.

#### § I.3.4. En approchant $\nabla$ par une famille de connexions analytiques

Nous exposons dans cette section les résultats obtenus par Arnaudon dans [Arn97]; certains d'entre eux, notamment des résultats de régularisation de martingale à partir de celle de la valeur terminale, seront repris et adaptés dans le Chapitre II.

Le contexte est le suivant : on fixe un espace probabilisé  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq 1}, \mathbf{P})$  où la filtration satisfait (**Filtration**), et une variable aléatoire  $\xi \in L^1(\Omega, \mathcal{F}_1)$  dans une variété différentiable  $\mathcal{M}$ . Un des résultats principaux obtenu est le suivant.

**Théorème – I.3.6** (Arnaudon, [Arn97], Théorème 7.3). *Supposons  $\mathcal{M}$  munie d'une connexion  $\nabla$  de classe  $\mathcal{C}^1$ , et soit  $\mathcal{V} \subset \mathcal{M}$  un espace compact convexe à géométrie convexe tel que  $\xi \in \mathcal{V}$  presque-sûrement.*

*Alors il existe une  $\nabla$ -martingale  $X$  à valeurs dans  $\mathcal{V}$  telle que  $X_1 = \xi$ .*

Notons également que l'unicité a lieu dans  $\text{Mart}(\mathcal{V}, \xi)$  puisque  $\mathcal{V}$  possède une fonction séparante. Le lemme 7.1. qui précède ce théorème précise qu'il suffit d'établir cette propriété localement en tout point de  $\mathcal{V}$ , cette localisation permet de supposer l'existence d'une séparante ou de supposer le domaine  $\mathcal{V}$  inclus dans des supports de cartes normales par exemple.

#### Si $\mathcal{M}$ et $\nabla$ sont analytiques–

**Corollaire – I.3.1** (Arnaudon, [Arn97], Corollary 7.1). *Soit  $\mathcal{M}^{An}$  une variété analytique réelle, munie d'une connexion  $\nabla^{An}$  analytique. Soit  $\mathcal{V}^{An} \subset \mathcal{M}^{An}$  un espace compact convexe à géométrie convexe tel que  $\xi \in \mathcal{V}^{An}$  presque-sûrement.*

*Alors il existe une  $\nabla^{An}$ -martingale  $X$  à valeurs dans  $\mathcal{V}^{An}$  telle que  $X_1 = \xi$ .*

Donnons l'idée générale du cas analytique. La première chose à faire est de déformer de manière analytique la valeur terminale – cette idée sera d'ailleurs fortement reprise dans l'algorithme sto-

chastique du **Chapitre II** – en une famille analytique en  $a$  notée  $(\xi(a))_{a \in [0,1]}$  telle que  $\xi(1) = \xi$ . Arnaudon dans [Arn97] construit alors une famille  $(X(a))_{a \in [0,1]}$  de  $\nabla^{\text{An}}$ -martingales de valeur terminale  $\xi(a)$ .

Supposons que la famille  $(X(a))_{a \in [0,1]}$  soit une solution. Puisque la variété est supposée analytique,  $X(a)$  est entièrement déterminé par les processus  $W^n = \frac{\partial^n X(a)}{\partial a^n} \Big|_{a=0}$  à valeurs dans les tangents itérés  $T_{X(0)}^n \mathcal{M}$ . Un autre résultat d'Arnaudon dans [Arn97], Théorème 3.3., ou rappelé dans la **Proposition – II.1.8** de ce manuscrit, montre que les processus  $W^n$  sont des  $\nabla^{c,(n)}$ -martingales de valeur terminale  $\frac{\partial^n \xi(a)}{\partial a^n} \Big|_{a=0}$ , où  $\nabla^{c,(n)}$  est définie en itérant le relèvement complet de  $\nabla$  :

$$\nabla^{c,(n)} = \left( \nabla^{c,(n-1)} \right)^c.$$

On peut alors montrer que les  $W^n$  satisfont une relation de récurrence ascendante, qui dépend uniquement de la valeur terminale  $\xi(a)$  et de constantes dépendant principalement des symboles de Christoffel et de leurs dérivées, mais pas de  $X(a)$ . Cette méthode permet donc de construire successivement les  $W^n$  puis  $X(a)$ .

Nous rappelons tout de même un résultat important établi dans la section 3 de [Arn97]. Sous des hypothèses de convexité, la régularité de la valeur terminale  $\xi(a)$  se transmet automatiquement à la famille de martingale  $(X(a))_{a \in [0,1]}$ .

**Proposition – I.3.1** (Proposition 3.7, [Arn97]). *Soit  $p \geq 1$ , et  $(X(a))_{0 \leq a \leq 1}$  une famille de  $\nabla$ -martingales telles  $a \mapsto X_1(a)$  soit une application différentiable.*

*Soit par ailleurs une  $\nabla^c$ -martingale  $Y$  dans  $T\mathcal{M}$  de valeur terminale  $Y_1 = \partial|_{a=0} \xi(a)$ . Sous les hypothèses*

- (i) *pour tout  $a$ ,  $X(a)$  est presque-sûrement à valeurs dans un ensemble compact de  $\mathcal{M}$  à géométrie  $p$ -convexe,*
- (ii) *pour tout  $a$ ,  $Y(a)$  et  $\partial_a \xi(a)$  sont presque-sûrement à valeurs dans un ensemble compact de  $T\mathcal{M}$ ,*
- (iii) *tout sous-espace compact de  $T\mathcal{M}$  possède un voisinage ouvert à géométrie strictement convexe,*

*on a que pour tout  $t \in [0, 1]$ , l'application  $a \mapsto X_t(a)$  est différentiable en  $a = 0$  de dérivée  $Y_t$ .*

**Si  $\mathcal{M}$  et  $\nabla$  sont différentiables–** L'idée principale réside dans l'approximation, en un sens à préciser, de la connexion  $\nabla$  par une suite de connexions analytiques  $\nabla^{(n)}$  pour lesquelles le **Corollaire – I.3.1** s'applique, puis de regarder la convergence vers une martingale limite.

Avec un argument de convolution, on peut construire une suite de symboles de Christoffel  $\left( \Gamma_{i,j}^k \right)^{(n)}$ , analytiques sur  $\mathcal{V}$ , convergeant uniformément sur  $\mathcal{V}$  vers les  $\nabla$ -symboles  $\Gamma_{i,j}^k$ , ainsi que leurs dérivées premières. Cette famille approchante de symboles définit des connexions analytiques  $\nabla^{(n)}$  pour tout entier  $n$ . Par ailleurs, quitte à diminuer le domaine  $\mathcal{V}$ , on peut supposer qu'une séparante sur  $\mathcal{V}$  est donnée par la séparante d'Emery (voir item (4.59) de [Eme89])

donnée en coordonnées par :

$$\Phi(x, y) = \frac{1}{2} \left( \varepsilon^2 + |x + y|^2 \right) |x - y|^2, \quad (\text{I.3.3})$$

et justifier qu'elle est  $\nabla^{(n)} \otimes \nabla^{(n)}$ -convexe pour  $n$  assez grand. Cette propriété est ensuite utilisée pour établir la convergence de  $X^n$  vers une  $\nabla$ -martingale  $X$  de valeur terminale  $\xi$ , où  $X^n$  est une  $\nabla^{(n)}$ -martingale de valeur terminale  $\xi$  donnée par le [Corollaire – I.3.1](#).

### § I.3.5. Nouveaux résultats

Dans le [Chapitre II](#) un algorithme stochastique permettant d'approcher une martingale de valeur terminale donnée sur une variété sera construit. L'hypothèse sur la filtration est générale ; on supposera à nouveau l'hypothèse (**Filtration**) déjà introduite plus haut, à savoir que toute  $(\mathcal{F}_t)_{0 \leq t \leq T}$ -martingale admet une version continue. En outre, la connexion  $\nabla$  sera la connexion de Levi-Civita. Pour certaines propriétés stochastiques sur les martingales, il est déjà nécessaire qu'elle soit sans torsion. Mais en plus, et de manière beaucoup plus centrale, nous aurons besoin d'estimées sur le transport géodésique déformé et du fait que sa norme au carré soit à variation finie. Pour toutes ces raisons, le fait que  $\nabla$  soit la connexion de Levi-Civita est indispensable.

La variable aléatoire  $\xi$  que l'on cherche à atteindre est d'abord déformée en une famille  $\xi(a)$  de classe  $\mathcal{C}^1$ , et on se pose la question suivante : existe-t-il une  $\nabla$ -martingale  $X(a)$  de valeur terminale  $\xi(a)$  ? On initialisera l'algorithme à une variable aléatoire constante presque-sûrement. Ceci implique notamment que la variable aléatoire  $\xi$  est presque-sûrement dans un domaine fixé de  $\mathcal{M}$ . Ensuite cette variable aléatoire initiale sera déformée en une nouvelle famille  $\overline{X(a)}$  de  $\nabla$ -martingales, où la dérivée  $\partial_a X(a)$  est une fonctionnelle bien choisie du vecteur  $\overline{X_1(a)} \xi$  (voir la [Section II.1](#) pour les notations). C'est ce vecteur  $\overline{X_1(a)} \xi$  qui va venir corriger, étape par étape, la valeur terminale  $X_1(a_1)$  pour  $a_1$  assez petit. Ensuite on recommence en prenant  $X(a_1)$  construit précédemment, on obtient ainsi une suite de martingales  $(X(a_n))_{n \in \mathbf{N}}$  dont on montrera qu'elle converge vers une solution du problème.

Un bon cadre pour tout ceci, comme dans [[Pic94](#), [Pic91](#)] ou encore dans [[Bla04a](#)] si la connexion de Levi-Civita est utilisée, est celui d'une boule géodésique régulière  $\mathcal{B}$ . Elle possède les bonnes propriétés suivantes : les courbures sectionnelles sont positivement majorées, ce qui nous donne des estimées sur le transport déformé (qui intervient dans l'étape de tir). D'autre part, les vecteurs du type  $\overrightarrow{xy}$  seront bien définis si  $x, y$  sont dans  $\mathcal{B}$ . Enfin,  $\mathcal{B}$  possède des fonctions numériques convexes intéressantes, qui seront très utiles pour établir la convergence de l'algorithme.

Présentons tout ceci de manière plus formelle. Soient  $\tilde{\xi}$  une variable aléatoire sur  $\mathcal{M}$ , et  $\mathcal{B}$  une boule géodésique régulière de rayon  $R(\mathcal{B})$  et centre  $O_{\mathcal{B}}$  telles que :

- (RegBall,Algo)** (i)  $\tilde{\xi} \in \mathcal{B}$  almost surely,  
(ii)  $R(\mathcal{B}) \leq \frac{\pi}{2\sqrt{\kappa^+(\mathcal{B})}q}$  with  $q > 1$  such that  $q^* \in 2\mathbf{N}$ , and  $\kappa^-(\mathcal{B}) < \infty$ .

Nous souhaitons résoudre le problème suivant plus général : trouver une famille  $(X(a))_{a \in [0,1]}$  de

martingales telle que

$$\forall t \in [0, T], \quad \partial_a X_t(a) = \mathbf{E} \left( \Theta_{t,1}^{-1}(X(a)) (\partial_a \xi(a)) \middle| \mathcal{F}_t \right), \quad (\text{I.3.4})$$

où, comme annoncé précédemment,  $a \in [0, 1] \mapsto \xi(a)$  est une trajectoire  $\mathcal{C}^1$  telle que  $\xi = \xi(1) \in L^2(\Omega, \mathcal{M})$ , et  $\Theta_{0,\cdot}(X(a))$  est le transport géodésique le long de  $X(a)$  (voir la [Section II.1](#) pour plus de détails sur les notations ci-dessus). Prenant  $a = 1$ , on voit que ce problème contient strictement celui qui consiste à chercher une martingale de valeur terminale  $\xi$ .

**Théorème – I.3.7** (Existence d'une solution à (I.3.4)). *Supposons que :*

- (i) l'hypothèse (**RegBall, Algo**) est satisfaite,
- (ii)  $\xi(a) \in \mathcal{B}$  pour tout  $a \in [0, 1]$  avec  $\partial \xi^* := \sup_{a \in [0,1]} \|\partial_a \xi(a)\|_{L^\infty} < \infty$ .

Alors il existe une unique famille  $\mathcal{C}^1$  de  $\nabla$ -martingales vérifiant (I.3.4).

En particulier, il existe une unique famille de  $\nabla$ -martingales à valeurs dans  $\mathcal{B}$  de valeur terminale  $\xi$ .

Pour terminer, donnons le principe de construction de l'algorithme solution, noté  $X$ , pendant la première étape de tir, *i.e.* la construction d'une famille  $X(a)$  indexée par  $a \in [0, a_1]$ . On le définit comme la solution de l'EDS

$$\begin{cases} d^\nabla X(a) = a \ll_{X(0), X(a)} (dJ(0)), & 0 \leq t \leq \tau(a), \\ X_0(a) = \exp_{X_0(0)}(aJ_0(0)), \\ X_t(a) = X_{\tau(a)}, & \tau(a) \leq t \leq 1, \end{cases}$$

où :

$$\tau(a) = \inf \{ t \in [0, 1], \quad X_t(a) \in \partial \mathcal{B} \}, \quad (\text{I.3.5})$$

$$J_t(a) = \mathbf{E} \left( \Theta_{t,1}^{-1}(X(a)) \overrightarrow{X_1(a)\xi} \middle| \mathcal{F}_t \right), \quad (\text{I.3.6})$$

$$Z_t(a) = \mathbf{E} \left( \Theta_{0,1}^{-1}(X(a)) \overrightarrow{X_1(a)\xi} \middle| \mathcal{F}_t \right). \quad (\text{I.3.7})$$

On établira ensuite que  $X(a)$  est solution de (I.3.4) en  $a = 0$ , puis en  $a_1$  et ensuite en chaque point de tir  $a_2, \dots, a_n$ . On prouve après que si l'on arrête assez tôt le paramètre  $a$ , la valeur terminale  $X_1(a_i)$  en  $a_i$  est plus proche de  $\xi$  que  $X_1(a_{i-1})$  à l'étape précédente.

*Remarque – I.3.1.* Cette EDS est en fait une version simplifiée du véritable algorithme présenté dans le [Chapitre II](#). En effet, la preuve de la convergence repose sur l'étude de :

$$\lim_{n \rightarrow \infty} \mathbf{E} (\rho^2(X_1(a_n); \xi)).$$

On constate qu'elle converge vers zéro si sa dérivée seconde

$$\partial_a^2 \mathbf{E} (\rho^2(X_1(a); \xi))$$

est uniformément bornée en  $a$ . Cette borne nécessitera d'avoir des estimées  $\mathcal{S}^\infty$ , uniformes en  $a$ , sur le transport géodésique  $\Theta_{0,\cdot}(X(a))$ . Et donc, comme nous le verrons, d'avoir une estimée  $\mathcal{H}^\infty$  sur  $X(a)$ . Défini en tant que tel, l'algorithme ne satisfait pas cette propriété, mais il est possible de localiser à chaque étape le processus solution pour que ce soit le cas et que la convergence ne soit pas altérée.

## I.4 EXISTENCE DES MARTINGALES EN RÉSOUVANT UNE EDSR

### § I.4.1. Reformulation du problème dans une carte

Nous présentons ici le point de vue développé par Darling dans [Dar95]. Rappelons qu'une semimartingale  $X$  sur  $\mathcal{M}$  est une  $\nabla$ -martingale si pour toute fonction  $f \in \mathcal{C}^2(\mathcal{M}, \mathbf{R}^d)$  comme précédemment, le processus

$$f(X) - f(X_0) - \frac{1}{2} \int_0^\cdot \nabla df(dX_s, dX_s) \quad (\text{I.4.1})$$

est une martingale locale réelle. Supposons à présent que la filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$  est issue d'un mouvement brownien standard  $W$  de  $\mathbf{R}^d$ ; en particulier, toutes les martingales locales continues réelles sont des processus d'Itô. Supposons également que la variété  $\mathcal{M}$  possède un système de coordonnées  $(x^1, \dots, x^d)$ , globales pour simplifier. Alors on peut écrire (I.4.1) avec  $f = x^i$ , on obtient alors que :

$X$  est une  $(\mathcal{F}, \nabla)$ -martingale sur  $\mathcal{M}$  de valeur terminale  $\xi$

$$\begin{aligned} & \iff \\ \left\{ \begin{array}{l} \text{pour tout } i, \text{ il existe } Z^i \text{ adapté} \end{array} \right. / & \begin{array}{l} dX^i + \frac{1}{2} \Gamma_{jk}^i(X) d\langle X^j, X^k \rangle = Z^i dW^i, \\ Y_T^i = \xi^i, \end{array} \\ & \iff \\ \left\{ \begin{array}{l} \text{pour tout } i, \text{ il existe } Z^i \text{ adapté} \end{array} \right. / & \begin{array}{l} dX^i + \frac{1}{2} \Gamma_{jk}^i(X) Z^j Z^k dt = Z^i dW^i, \\ Y_T^i = \xi^i. \end{array} \end{aligned}$$

Le dernier système est une équation différentielle stochastique rétrograde sur  $\mathbf{R}^d$ . En effet, on peut l'écrire sous la forme

$$X_t = \xi + \int_t^T f(s, X_s, Z_s) ds - \int_t^T Z_s dW_s, \quad (\text{I.4.2})$$

avec  $f : [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^d$  définie par  $f(s, y, z) = \frac{1}{2} \left( \Gamma_{ij}^k(y) z^{(i,:)} \cdot z^{(j,:)} \right)_{1 \leq k \leq d}$ . Pour plus de détail sur ces notations, le lecteur peut consulter la section III.1 et la Section I.4.2 qui suit. Depuis les années 90 (voir le papier fondateur [PP90]), de nombreux résultats sur les équations

différentielles stochastiques rétrogrades sont connus lorsque  $f$  est lipschitzienne en  $y$  et  $z$ . Cependant, il est généralement plus difficile d'en obtenir lorsque  $f$  varie de manière quadratique en  $z$ . Dans la section III.1, nous présentons la littérature propre aux EDSRs quadratiques multidimensionnelles, pour le moment dans cette introduction nous nous limitons au générateur  $f$  associé aux  $\nabla$ -martingales sur une variété  $\mathcal{M}$ , avec parfois un terme supplémentaire à variations finies (cf. résultats de Blache dans la section I.4.4).

### § I.4.2. À propos des EDSR multidimensionnelles à croissance quadratique

Résoudre une EDSR consiste à trouver un couple de processus  $(Y, Z)$ , où  $Y$  est continu et à valeurs dans  $\mathbf{R}^d$  et  $Z$  dans  $\mathbf{R}^{d \times k}$  progressivement mesurable, satisfaisant une équation du type :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad p.s. \quad (\text{I.4.3})$$

où  $W$  est un mouvement brownien  $k$ -dimensionnel relativement à une filtration  $(\mathcal{F}_t)_{t \in \mathbf{R}^+}$ , et  $\xi$  est une variable aléatoire  $\mathcal{F}_T$ -mesurable appelée condition terminale. L'application  $f$ , appelée générateur de l'EDSR, peut être aléatoire. Lorsque la fonction  $f(s, \cdot, \cdot)$  est lipschitzienne en ses deux coordonnées et  $\xi, \int_0^T f(s, 0, 0) ds$  sont de carré intégrable, on a le résultat d'existence et d'unicité de Pardoux-Peng (voir [PP90]). La preuve, comme dans le cas des EDS, utilise un argument de point fixe.

Considérons le jeu d'hypothèses plus général suivant :

- (H) (i) pour tout  $(y, y', z, z') \in (\mathbf{R}^d)^2 \times (\mathbf{R}^{d \times k})^2$ , on suppose qu'il existe  $(K_y, L_y, K_z, L_z) \in (\mathbf{R}^+)^4$  tel que  $\mathbf{P}$ -p.s. pour tout  $t \in [0, T]$  :

$$|f(t, y, z) - f(t, y', z)| \leq (K_y + L_y |z|^2) |y - y'|,$$

$$|f(t, y, z) - f(t, y, z')| \leq (K_z + L_z (|z| + |z'|)) |z - z'|,$$

- (ii)  $\mathbf{E} \left( |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds \right) < +\infty$ .

En dimension une, commençons par regarder le cas plus simple où (H) est vérifiée pour  $f$ , celui d'un carré en  $Z$  :

$$Y_t = \xi + \frac{1}{2} \int_t^T |Z_s|^2 ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T.$$

En faisant le changement de variable  $y = \exp(Y)$ ,  $z = yZ$ , on obtient cette nouvelle EDSR en  $(y, z)$  grâce à la formule d'Itô :

$$y_t = \exp(\xi) - \int_t^T z_s dW_s, \quad 0 \leq t \leq T.$$

Il semble nécessaire d'avoir mieux que  $\xi \in L^2$  pour l'existence, l'intégrabilité de  $\exp(\xi)$  (voir [BH06]). Dans le cadre multidimensionnel, considérons un autre exemple encore plus instructif.

Frei et dos Reis construisent dans [FDR11] une EDSR quadratique en dimension deux, avec une condition terminale bornée qui n'admet aucune solution. Plus précisément, il s'agit du système d'EDSRs dans  $\mathbf{R}^2$  suivant :

$$\begin{cases} Y_t^1 &= \xi^1 - \int_t^T Z_s^1 dW_s, \\ Y_t^2 &= \xi^2 + \int_t^T \left( |Z_s^1|^2 + \frac{1}{2} |Z_s^2|^2 \right) ds - \int_t^T Z_s^2 dW_s. \end{cases}$$

**Théorème – I.4.1** (Frei & Dos Reis, 2011, [FDR11]).

*Il existe une condition terminale  $\xi = (\xi^1, \xi^2) \in L^\infty$  telle que l'EDSR n'admette aucune solution.*

Les conditions de l'hypothèse **(H)**, même en restreignant la classe de la condition terminale, ne semblent donc pas suffisantes pour assurer l'existence et l'unicité de solutions aux EDSRs multidimensionnelles quadratiques. Cela provient du fait que les outils classiques utilisés en dimension une, tels que le théorème de Girsanov ou de convergence monotone, ne sont plus applicables en dimension quelconque. En conclusion, le résultat de Pardoux-Peng ne permet pas de traiter le problème de recherche de martingale de valeur terminale donnée sur une variété, même s'il sera tout-de-même utilisé comme outil intermédiaire.

### § I.4.3. Les résultats de Darling

Darling, dans [Dar95], obtient des résultats d'existence et d'unicité par une méthode d'EDSR selon le point de vue décrit plus haut. La variété  $\mathcal{M}$  est ici supposée générale tandis que la filtration sera brownienne. Aucune condition n'est supposée sur la connexion  $\nabla$ .

*Remarque – I.4.1.* Darling traite même le cas où  $\nabla$  n'est pas linéaire, *i.e.* la classe  $X$  des martingales telles que (I.4.2) soit satisfaite avec  $f(s, y, z) = \Gamma(y, z, z)$ , où  $\Gamma$  est une application de  $\mathbf{R}^d \times \mathbf{R}^{d \times k} \otimes \mathbf{R}^{d \times k}$  dans  $\mathbf{R}^d$ .

On retrouve le cas linéaire lorsque  $\Gamma$  dépend linéairement de  $y$ . Comme le mentionne Darling dans [Dar95], les deux cas se traitent de la même manière selon sa méthode, dans la suite de ce paragraphe nous supposerons donc sans restriction que la connexion est linéaire.

Comment souvent avec les EDSRs à croissance quadratique, le point de départ consiste à *localiser* l'équation *i.e.* à la transformer en une autre EDSR où  $f$  est légèrement modifié en un générateur à croissance lipschitzienne, donc pour lequel il y a existence et unicité d'après le résultat original de Pardoux-Peng (voir le [Chapitre III](#) pour plus de détails). C'est ce que commence par faire Darling, en définissant pour tout  $\varepsilon \in \mathbf{R}^{+*}$  une application  $h^\varepsilon : \mathbf{R}^{d \times k} \rightarrow \mathbf{R}$  telle que :

$$h^\varepsilon(z) = \begin{cases} 0 & \text{if } |z| \leq 1/\varepsilon, \\ \left( \frac{|z|}{1/\varepsilon + 1} \right)^2 - 1 & \text{if } |z| \geq 1/\varepsilon + 1, \end{cases}$$



et  $\rho^\varepsilon : \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^{d \times k}$  donnée par

$$\rho^\varepsilon(z) = \frac{z}{\sqrt{1+h^\varepsilon(z)}}.$$

On définit alors un nouveau générateur  $f^\varepsilon$  par  $f^\varepsilon(t, y, z) = f(t, y, \rho^\varepsilon(z))$ . Rappelons que pour  $1 \leq p \leq \infty$ ,  $\mathcal{S}^p$  désigne l'espace des processus de supremum dans  $L^p$ , et  $\mathcal{H}^p$  les processus dont la racine carrée de la variation quadratique en  $T$  est dans  $L^p$ . D'après le résultat classique d'existence et d'unicité des solutions d'EDSRs à générateur lipschitzien (voir le résultat de Pardoux et Peng, [PP90], rappelé dans le Chapitre III), il existe une unique solution  $(X^\varepsilon, Z^\varepsilon)$  dans  $\mathcal{S}^2 \times \mathcal{H}^2$  pour la version localisée :

$$X_t^\varepsilon = \xi + \int_t^T f^\varepsilon(s, X_s^\varepsilon, Z_s^\varepsilon) ds - \int_t^T Z_s^\varepsilon dW_s, \quad (\text{I.4.4})$$

ou de manière équivalente :

$$X_t^\varepsilon = \xi + \int_t^T f(s, X_s^\varepsilon, \bar{Z}_s^\varepsilon) ds - \int_t^T \bar{Z}_s^\varepsilon dW_s, \quad (\text{I.4.5})$$

notant  $\bar{Z}^\varepsilon = \rho^\varepsilon(Z^\varepsilon)$ . Tout l'enjeu est donc de regarder la convergence de  $(X^\varepsilon, Z^\varepsilon)$  lorsque  $\varepsilon \rightarrow 0$ .

**L'hypothèse de convexité.** Considérons l'hypothèse suivante.

**(Conv, Darling)** Il existe  $\Phi \in \mathcal{C}^2(\mathcal{M}, \mathbf{R})$  à dérivées premières et secondes bornées, telle que :

- (i) l'ensemble  $\mathcal{V} = \Phi^{-1}(\mathbf{R}^-)$  est compact, à géométrie convexe de séparante  $\Psi$  doublement convexe sur  $B \times B$ , avec  $B$  un ouvert tel que  $\mathcal{V} \subset B$ ,
- (ii)  $\Phi$  est doublement convexe sur  $\mathcal{V}^c$ ,
- (iii) pour tous  $(y, z) \in \mathbf{R}^d \times \mathbf{R}^{d \times k}$ ,  $\nabla d\Phi(y)(z, z) \leq c|z|$ ,
- (iv) il existe de plus une fonction  $f \in \mathcal{C}^2(\mathcal{M}, \mathbf{R})$  qui est strictement doublement convexe sur  $\mathcal{V}$ .

Comme le précise Darling, une fonction lisse définie sur  $\mathcal{M}$  est doublement convexe si son  $\nabla$ -hessien et son hessien euclidien sont positifs. Une conséquence immédiate de **(Conv, Darling)** est que pour toute  $\nabla$ -géodésique  $\gamma : [a, b] \rightarrow \mathcal{M}$  telle que  $\Psi(\gamma(a)) = 0 = \Psi(\gamma(b))$ , alors  $\Psi(\gamma(t)) = 0$  pour tout  $t \in [a, b]$ . Autrement dit si les points extrémaux d'une géodésique sont dans  $\mathcal{V}$  alors toute la géodésique l'est également. Ceci est étendu dans [Dar95], Lemme 3.3, à une version stochastique *i.e.* pour les  $\nabla$ -martingales de valeur terminale  $\xi$  vivant dans l'espace  $\mathcal{V}$  défini *supra*. Ce résultat s'apparente au principe du maximum énoncé par Picard dans [Pic91] et rappelé précédemment. Plus précisément, la formule d'Itô appliquée à  $\Psi(Y^\varepsilon)$  permet d'obtenir :

**Lemme – I.4.1** (Lemme 3.3, Darling, [Dar95]). *Si  $\xi$  est presque-sûrement dans  $\mathcal{V}$ , alors pour tout  $\varepsilon > 0$ ,  $X_t^\varepsilon \in \mathcal{V}$  pour tout  $t$  presque-sûrement.*

**L'hypothèse sur la connexion.** Comme précisé plus haut nous énonçons ici les résultats de Darling associés aux connexions linéaires. Les hypothèses de régularité sur les connexions non linéaires intervenant dans les conditions (3.2), (5.1), (6.1) de [Dar95] se réécrivent ainsi dans le cas linéaire :

**(Gamma,Darling)** pour tout  $i, j, k$ , la fonction  $y \in \mathcal{M} \mapsto \Gamma_{j,k}^i(y)$  est globalement lipschitzienne, et bornée sur  $\mathcal{V} \subset \mathcal{M}$ .

Darling obtient alors le résultat d'existence et d'unicité ci-dessous.

**Theorem – I.4.1** (Théorème 7.1, Darling, [Dar95]). *Sous (Gamma,Darling) et (Conv,Darling), il existe une unique martingale  $X$  de valeur terminale  $\xi$  telle que  $Z \in \mathcal{H}^2$ .*

De plus Darling montre que  $X$  est à valeurs dans l'ensemble  $\mathcal{V}$ , ce qui n'est surprenant puisque  $X$  est construite à l'aide des  $X^\varepsilon$ . Comme d'habitude, l'unicité vient alors directement en remarquant que  $\Psi(X, X')$  est une sous-martingale réelle, donc  $\Psi(X, X')$  est presque-sûrement égale à zéro puisque  $X$  et  $X'$  ont même valeur terminale.

Concernant la construction de  $X$ , rappelons l'existence de la solution approchée  $(X^\varepsilon, \bar{Z}^\varepsilon)$  définie dans (I.4.5). Darling s'y prend de la manière suivante. Introduisons tout d'abord un processus  $M^\varepsilon$  tel que  $M_0^\varepsilon = Y_0^\varepsilon$  et

$$dM_t^\varepsilon = \bar{Z}_t^\varepsilon dW_t - \frac{1}{2} f(M_t^\varepsilon, \bar{Z}_t^\varepsilon) dt. \quad (\text{I.4.6})$$

L'existence de la diffusion  $M^\varepsilon$  peut être vérifiée facilement sous l'hypothèse **(Gamma,Darling)**, elle a même valeur initiale mais pas la même valeur terminale que  $Y^\varepsilon$ . On ne peut pas appliquer en tant que tel le **Lemme – I.4.1** au processus  $M^\varepsilon$ , mais en arrêtant  $M^\varepsilon$  une fois sorti de  $\mathcal{V}$ , Darling montre que la valeur terminale de ce processus arrêté converge en probabilité vers la bonne valeur terminale  $\xi$  lorsque  $\varepsilon \rightarrow 0$ . Le résultat de stabilité énoncé dans le Corollaire 5.6 de Darling permet ensuite de conclure sur la convergence.

Dans le **Chapitre III**, une approche par approximation sera également utilisée, partant d'une forme localisée d'EDSR comme dans (I.4.4).

#### § I.4.4. Les résultats de Blache

Les résultats de Blache dans [Bla04a] et [Bla06] généralisent en quelque sorte ceux de Darling aux semimartingales de dérive markovienne fixée (*i.e.* avec un terme à variations finies supplémentaire dépendant d'une diffusion), toujours sous l'hypothèse de filtration brownienne engendrée par un brownien  $W$ .

La valeur terminale  $\xi$  est toujours supposée être dans un domaine particulier, noté  $\bar{\mathcal{V}}$ , qui devra s'écrire aussi comme  $\bar{\mathcal{V}} = \{\Phi \leq c\}$  avec  $c \in \mathbf{R}$  et  $\Phi : \mathcal{M} \rightarrow \mathbf{R}$  une fonction convexe. Ces conditions de convexité sont toujours vérifiées sur une boule géodésique régulière (cf. **Proposition – II.1.4** de ce manuscrit).

**L'hypothèse de convexité.** Plus précisément, il est nécessaire de supposer :

- (Conv,Blache)** (i) L'ensemble  $\mathcal{V}$  est un ouvert non vide d'un ouvert  $O$ , lui-même relativement compact dans une variété  $\mathcal{M}$ , et  $\overline{\mathcal{V}} = \{\Phi \leq c\}$  avec  $c \in \mathbf{R}$  et  $\Phi : \mathcal{M} \rightarrow \mathbf{R}$  une fonction strictement convexe,
- (ii) deux points de  $O$  peuvent être joints par une unique géodésique à valeurs dans  $O$ ,
- (iii) l'ensemble  $\overline{\mathcal{V}}$  est à géométrie  $p$ -convexe avec  $p$  entier pair.

La condition (ii) implique notamment que  $O$  est inclus dans une carte locale. La notion d'EDSR sera donc comprise relativement à cette carte locale dans la suite.

Blache travaille avec une dérive déterministe  $f : \mathbf{R}^l \times \mathbf{R}^d \times \mathbf{R}^k \rightarrow \mathbf{R}^d$ , l'aléa dépendra d'une diffusion  $B^y$  sur  $\mathbf{R}^l$  définie par :

$$\begin{cases} dB_t^y &= b(B_t^y)dt + \sigma(B_t^y)dW_t, \\ B_0^y &= \quad \quad y \in \mathbf{R}^l, \end{cases} \quad (\text{I.4.7})$$

l'entier  $k$  est la dimension du brownien  $W$  engendrant la filtration, et  $b, \sigma$  sont des fonctions régulières satisfaisant de bonnes propriétés pour assurer l'existence de  $B^y$ .

**L'hypothèse sur la dérive  $f$ .** Elle est supposée satisfaire en plus :

- (Drift,Blache)** (i) l'application  $f$  est sortante sur la frontière de  $\overline{\mathcal{V}}$ ,
- (ii)  $f|_{\mathbf{R}^l \times O \times \mathbf{R}^{d \times k}}$  est lipschitzienne en la première et la deuxième variable, localement lipschitzienne en la troisième,
- (iii) il existe  $x_0 \in O$  tel que  $\sup_{b \in \mathbf{R}^l} |f(b, x_0, 0)| < \infty$ .

Les points (ii) et (iii) sont les conditions (1.4) et (1.5) dans [Bla04a], et sont habituelles pour les EDSRs quadratiques multidimensionnelles. La condition (i) intervient naturellement lorsque l'on souhaite des solutions à l'EDSR ci-dessous confinées dans  $\overline{\mathcal{V}}$ .

Considérons l'EDSR principale suivante sur  $\mathbf{R}^d$  :

$$X_t = \xi + \int_t^T \left( f(B_s^y, X_s, Z_s) - \frac{1}{2} \Gamma_{ij}^k(Y_s) Z_s^{(i,:)} \cdot Z_s^{(j,:)} \right) ds - \int_t^T Z_s dW_s. \quad (\text{I.4.8})$$

**Théorème – I.4.2** (Blache, [Bla04a], cas de la connexion de Levi-Civita). *On suppose que  $\Gamma$  est la connexion de Levi-Civita.*

Si  $\xi \in \overline{\mathcal{V}}$ , si l'ensemble  $\overline{\mathcal{V}}$  vérifie les hypothèses **(Conv,Blache)**, et  $f$  vérifie **(Drift,Blache)**, alors :

- (i) si  $f$  ne dépend pas de  $z$ , l'EDSR (I.4.8) admet une unique solution  $(X, Z)$  telle que  $X \in \overline{\mathcal{V}}$  presque-sûrement,
- (ii) si  $f$  dépend de  $z$  et  $\mathcal{M}$  est une variété de Cartan-Hadamard, alors l'EDSR (I.4.8) admet une unique solution  $(X, Z)$  telle que  $X \in \overline{\mathcal{V}}$  presque-sûrement.

La dépendance en  $z$  est un point central vis-à-vis de la propriété d'unicité. En effet, la démarche habituelle (et déjà présentée plus haut) afin d'établir l'unicité dans le problème de recherche de martingale de valeur terminale donnée, est d'étudier la sous-martingale  $\Psi(X, X')$  avec  $\Psi$

une séparante donnée, et  $X, X'$  deux martingales solutions. Ici, les processus étant seulement des semimartingales,  $\Psi(X, X')$  n'est plus une sous-martingale. Cependant, il est possible de justifier dans le cas où  $f$  ne dépend pas de  $z$  que  $\left(e^{\lambda t} \Psi(X_t, X'_t)\right)_{0 \leq t \leq T}$  est une sous-martingale pour un certain  $\lambda$  en utilisant les hypothèses de régularité sur  $f$  et la formule d'Itô. L'unicité suit alors dans ce cas. Le second cadre, où  $\mathcal{M}$  est une variété de Cartan-Hadamard se traite en utilisant comme séparante la distance géodésique au carré  $\rho^2$ . La sous-martingale à étudier devient alors  $\left(e^{A_t} \Psi(X_t, X'_t)\right)_{0 \leq t \leq T}$ , avec  $A$  un processus à variations finies, dont Blache justifie son exponentielle-intégrabilité dans le Lemme 3.4.5. (voir [Bla04a]). Dans les deux cas l'unicité est ainsi établie.

Quant à l'existence, une approche par stabilité est utilisée pour la valeur terminale, comme dans la section III.4 de ce manuscrit. Le résultat de stabilité correspondant est la réunion des Propositions 4.1.3 et 4.1.4 de [Bla04a]. Une fois le problème réduit à des variables aléatoires terminales fonctions de mouvements brownien, Blache localise le générateur  $f$ , comme Darling précédemment (voir l'EDSR (I.4.5)). Il s'agit ensuite de regarder la convergence lorsque  $\varepsilon \rightarrow 0$  de  $(X^\varepsilon, Z^\varepsilon)$ , en cherchant une estimée sur  $Z^\varepsilon$ . La formule de Feynman–Kac est alors largement utilisée. Une borne sur un intervalle de temps  $[T^\varepsilon, T]$  est établie dans un premier temps. On peut étendre les solutions sur  $[0, T]$  sous une hypothèse de sortie stricte de  $\overline{\mathcal{V}}$  pour  $f$ , hypothèse qui peut être ensuite supprimée par approximations de  $f$  en convoluant. Le théorème ci-dessus sera généralisé dans [Bla06] au cas d'une connexion plus générale.

### § I.4.5. Nouveaux résultats

Nous avons vu précédemment comment le problème d'existence de martingales de valeur terminale donnée pouvait se reformuler en terme d'une EDSR quadratique multidimensionnelle. Plus précisément, il s'agissait de l'EDSR suivante :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

avec  $f : [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^d$  définie par  $f(s, y, z) = \frac{1}{2} \left( \Gamma_{ij}^k(y) z^{(i,:)} \cdot z^{(j,:)} \right)_{1 \leq k \leq d}$ . Plus généralement, rappelons qu'une EDSR quadratique multidimensionnelle est une EDSR vérifiant l'hypothèse de structure **(H)** mentionnée plus haut.

Les résultats obtenus s'appliquent aux EDSRs quadratiques multidimensionnelles, et sont donc plus généraux que ceux permettant de traiter le problème de recherche de martingales de valeurs terminale donnée sur une variété. Ils sont détaillés entièrement dans la Section III.2 mais ici, nous rappellerons uniquement leur version propre au générateur des martingales sur une variété.

La technique employée ici est une approche par stabilité. Nous commençons par modifier l'EDSR de départ en une version approchée pour laquelle le résultat historique de Pardoux et Peng (voir [PP90]) s'applique. Ce point de départ est donc identique à celui de Darling précisé plus haut et à beaucoup d'articles propres aux EDSR quadratiques, citons par exemple [BE13].

Plus précisément, on considère l'EDSR dite localisée suivante :

$$Y_t^M = \xi + \int_t^T f^M(s, Y_s^M, Z_s^M) ds - \int_t^T Z_s^M dW_s, \quad 0 \leq t \leq T,$$

où  $f^M(t, y, z) = f(t, y, \rho^M(z))$  avec  $\rho^M : \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^{d \times k}$  qui satisfait les conditions suivantes :

- $\rho^M$  est l'identité sur  $\mathcal{B}_{\mathbf{R}^{d \times k}}(0, M)$ ,
- $\rho^M$  est la projection sur  $\mathcal{B}_{\mathbf{R}^{d \times k}}(0, M+1)$  en dehors de  $\mathcal{B}_{\mathbf{R}^{d \times k}}(0, M+1)$ ,
- $\rho^M$  une fonction  $\mathcal{C}^\infty$  avec  $|\nabla \rho^M(z)| \leq 1$  pour tout  $z \in \mathbf{R}^{d \times k}$ .

Ainsi  $f^M$  est une fonction globalement Lipschitzienne, de constante de Lipschitz dépendant de  $M$ , il y a donc existence et unicité d'une solution  $(Y^M, Z^M)$  dans  $\mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$ . La seconde propriété sur la fonction  $\rho^M$  nous apprend la chose suivante : si on a l'estimée

$$\sup_{M \in \mathbf{R}^+} \|Z^M\|_{\mathcal{S}^\infty} < \infty,$$

alors pour  $M$  suffisamment grand,  $(Y^M, Z^M)$  est une solution de l'EDSR non localisée. Le point central de la méthode est donc cette estimation uniforme en  $M$ .

Afin d'obtenir une telle estimée on commence par dériver au sens de Malliavin (voir [EKPQ97]) l'EDSR localisée. On obtient pour tous  $0 \leq u \leq t \leq T$  :

$$\begin{aligned} \mathbf{D}_u Y_t^M &= \mathbf{D}_u \xi + \int_t^T \left( \nabla_y f^M(s, Y_s^M, Z_s^M) \mathbf{D}_u Y_s^M + \nabla_z f^M(s, Y_s^M, Z_s^M) \mathbf{D}_u Z_s^M \right. \\ &\quad \left. + (\mathbf{D}_u f^M)(s, Y_s^M, Z_s^M) \right) ds - \int_t^T \mathbf{D}_u Z_s^M dW_s. \end{aligned} \quad (\text{I.4.9})$$

Le processus  $(\mathbf{D}_u Y^M, \mathbf{D}_u Z^M) = (\mathbf{D}_u Y_t^M, \mathbf{D}_u Z_t^M)_{0 \leq t \leq T}$  est donc solution d'une EDSR linéaire, et  $(\mathbf{D}_t Y_t)_{0 \leq t \leq T}$  est une version de  $(Z_t)_{0 \leq t \leq T}$ . Le problème se reformule donc encore : comment obtenir une estimée uniforme en  $M$  sur la première composante d'une solution d'EDSR linéaire dont les coefficients dépendent de  $M$  ?

On connaît une formulation explicite de telles solutions, qui fait intervenir notamment une solution d'EDS *ad hoc* (notée  $S$ ) dépendant des coefficients de l'EDSR linéaire. Si cette solution satisfait une inégalité de type Hölder inverse et que  $\xi, f$  sont à dérivée de Malliavin bornée, alors on peut obtenir l'estimée désirée (voir [DT08] pour les détails, et la Section III.3 de ce manuscrit). Toujours d'après [DT08], une telle inégalité existe si  $Z^M \star W$  est dans l'espace des martingales BMO, avec une norme BMO assez petite (voir la Section III.1 pour une définition de l'espace BMO). Ces estimations propres au cas linéaire nous donnent en plus des résultats de stabilité *via* une technique de linéarisation, fournissant d'une part une classe d'unicité pour l'EDSR, et d'autre part justifiant les passages à la limite lorsque l'on cherche à lever l'hypothèse de bornitude sur la dérivée de la Malliavin de  $\xi$  et  $f$ .

C'est ici que nous aurons besoin de fonctions convexes particulières sur la variété, comme Darling ou Blache *supra*, afin d'obtenir des estimées BMO sur le processus  $Z^M \star W$ . Nous supposerons aussi l'hypothèse de régularité ci-dessous sur les symboles de Christoffel, vérifiée sur n'importe quel espace relativement compact d'une variété :

**(HGam)** il existe deux constantes  $L_y$  et  $L_z$  telles que pour tous  $i, j, k \in \{1, \dots, d\}$  :

$$|\Gamma_{ij}^k(y) - \Gamma_{ij}^k(y')| \leq 2L_y |y - y'|, \quad |\Gamma_{ij}^k(y)| \leq 2L_z.$$

**Définition – I.4.1.** Nous dirons qu'une fonction  $F \in \mathcal{C}^2(\mathcal{M}, \mathbf{R})$  (vue comme une fonction sur  $\mathbf{R}^d$ ) est doublement convexe sur un ensemble  $G \subset \mathbf{R}^d$  si pour tout  $y \in G$  et  $z \in \mathbf{R}^d$ ,

$$\min \{\text{Hess} F(y)(z, z), \nabla dF(y)(z, z)\} \geq 0,$$

et, pour  $\alpha > 0$ ,  $F$  est  $\alpha$ -strictement doublement convexe sur  $G$  si pour tout  $y \in G$  et  $z \in \mathbf{R}^d$ ,

$$\min \{\text{Hess} F(y)(z, z), \nabla dF(y)(z, z)\} \geq \alpha |z|^2.$$

**Théorème – I.4.3** (Existence et Unicité). *Supposons que :*

- (i) il existe une fonction  $F^{dc} \in \mathcal{C}^2(\mathcal{M}, \mathbf{R})$ , telle que  $G = (F^{dc})^{-1}([-\infty, 0])$  est compact et  $\xi \in G$ ,
- (ii)  $F^{dc}$  is doubly convex on  $\mathcal{M}$ , et il existe  $\alpha > 0$  et  $m \geq 1$  tels que  $F^{dc}$  soit  $\alpha$ -strictly doublement convexe sur  $G$  et satisfait

$$\left( \sup_{(x,y) \in G^2} \{F^{dc}(x) - F^{dc}(y)\} \right)^{1/2} \leq \sqrt{\frac{\alpha}{2}} \times \mathbb{B}^m(L_y, L_z),$$

où  $\mathbb{B}^m(L_y, L_z)$  est une constante dépendant de  $m, L_y$  et  $L_z$  (voir (III.1.7) pour une formulation explicite).

- (iii) **(HGam)** est vérifiée.

Alors il existe une unique  $\nabla$ -martingale  $Y$  dans  $\mathcal{S}^\infty(\mathbf{R}^d)$  de valeur terminale  $\xi$  telle que

$$\left\| \sqrt{|\langle Y, Y \rangle|} \star W \right\|_{BMO} < \mathbb{B}^m(L_y, L_z).$$

De plus, si  $\xi$  est à dérivée de Malliavin bornée, on a de plus :

$$\text{esssup}_{\Omega \times [0, T]} |\langle Y, Y \rangle| < +\infty.$$

*Remarque – I.4.2.* La principale limitation de ce théorème vient de l'hypothèse (ii), qui elle-même donne l'estimée BMO uniforme en  $M$  sur le processus  $Z^M \star W$  dont nous avons parlé précédemment. Comme déjà précisé ci-dessus, cette hypothèse BMO peut être obtenue si une inégalité Hölder inverse est vérifiée pour une certaine solution d'EDS notée  $S$ .

Nous savons que si  $d = 1$ , il existe un certain exposant  $m$  tel que  $S$  vérifie une telle inégalité (voir le théorème 3.1 de [Kaz94]). Si cette propriété peut se généraliser, on est donc capable avec cette méthode de prouver le même type de résultat sans l'hypothèse (ii).

Ce sont les hypothèses (i) et (ii) qui nous garantissent l'estimée BMO uniforme en  $M$  sur  $Z^M \star W$  mentionnée plus haut. Dans le cas où  $\xi$  est à dérivée de Malliavin bornée, la solution est construite sans passage à la limite (ni en  $f$ , ni en  $\xi$ ), voir le théorème d'existence III.2.1 du

chapitre consacré aux EDSRs. Une solution de l'EDSR localisée dans  $\mathcal{S}^\infty(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$  restera dans  $\mathcal{S}^\infty(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$  pour  $M$  assez grand. Ce cas n'est contenu ni dans [Dar95] ni dans [Bla04a, Bla06], mais impose une condition plus restrictive sur le supremum de  $F^{dc}$  (voir (ii)). Ce résultat est donc un cas très particulier des théorèmes d'existence et d'unicité III.2.1 et III.2.3.

D'autre part, le générateur associé aux martingales sur une variété et l'existence de certaines fonctions convexes n'est pas le seul jeu d'hypothèses permettant d'aboutir à des estimées uniformes en  $M$  sur la norme BMO de  $Z^M \star W$ . Nous traiterons dans le Chapitre III d'autres cas : celui d'un générateur à croissance quadratique diagonale, markovien ou encore monotone.

## I.5 APPLICATIONS

Dans cette partie nous détaillons les applications directement reliées au problème initial posé dans cette thèse. Mais le lien riche entre ce dernier et les EDSRs quadratiques multidimensionnelles en induit de nombreuses autres propres aux EDSRs, comme par exemple la théorie des jeux ou les Mathématiques financières. Elles seront détaillées plus largement dans la section III.1 du Chapitre III.

### § I.5.1. Construction d'applications harmoniques

Les martingales trouvent d'importantes applications en analyse, par exemple pour les équations aux dérivées partielles en fournissant une expression probabiliste des solutions de certaines équations.

Un exemple classique est l'utilisation du mouvement brownien pour résoudre le problème de Dirichlet sur un ouvert  $D$  de  $\mathbf{R}^d$  : *i.e.* chercher fonctions  $u : \bar{D} \rightarrow \mathbf{R}$  vérifiant

$$\begin{cases} \Delta u = 0 & \text{sur } D \\ u|_{\partial D} = f \end{cases}, \quad (\text{I.5.1})$$

où  $f : \partial D \rightarrow \mathbf{R}$  est une fonction continue définie sur le bord de  $D$ . Soit  $B$  un mouvement brownien issu de  $x \in \bar{D}$ . Si  $\mathbf{E}(|f(B_{\tau_D})|) < \infty$ , alors la fonction

$$u(x) = \mathbf{E}(f(B_{\tau_D}))$$

est une solution de (I.5.1), où  $\tau_D$  désigne le temps de sortie de  $B$  de l'ouvert  $D$ . La condition d'intégrabilité est bien sûre satisfaite par exemple dans le cas où  $D$  est borné. Un point important pour justifier la formule (I.5.1) est que toute fonction harmonique transforme un brownien en une martingale locale, ceci peut servir de définition pour les fonctions harmoniques sur les variétés.

**Fonctions harmoniques à valeurs réelles sur une variété.** Plus généralement, si  $\mathcal{L}$  est un opérateur différentiel sans terme constant sur l'espace des fonctions lisses d'une variété  $\mathcal{M}$ , une application  $u : \mathcal{M} \rightarrow \mathbf{R}$  est dite  $\mathcal{L}$ -harmonique si elle vérifie  $\mathcal{L}u = 0$ . De manière similaire

au cas du laplacien, si  $x \in \mathcal{M}$  et  $Z^x$  est la diffusion issue de  $x$  associée au générateur  $\mathcal{L}$ , on a la propriété caractéristique suivante :  $u$  est  $\mathcal{L}$ -harmonique si et seulement si  $u(Z^x)$  est une martingale locale réelle continue pour tout  $x$ .

Une conséquence de ce qui précède est que les solutions de certaines équations différentielles peuvent se représenter à l'aide de martingales. Ainsi, si  $D$  est un ouvert de  $\mathcal{M}$  et  $f : \bar{D} \rightarrow \mathbf{R}$  est une application, alors une fonction  $u$  est solution de

$$\begin{cases} \mathcal{L}u = 0 & \text{sur } D, \\ u|_{\partial D} = f, \end{cases} \quad (\text{I.5.2})$$

si et seulement si  $u(Z^{\uparrow \tau_D})$  est une martingale prenant la valeur  $f(Z_{\tau_D})$  en  $t = \tau$ , et si  $\tau_D < \infty$  presque-sûrement. La notation  $Z^{\uparrow \tau_D}$  désigne le processus  $Z$  arrêté au temps  $\tau_D$ .

Si on autorise les fonctions à dépendre du temps, on a le critère suivant :

$$\begin{cases} \left( \frac{\partial}{\partial t} + \mathcal{L} \right) u(t, \cdot) = 0 & \text{sur } D, \\ u(1, \cdot)|_{\partial D} = f, \end{cases} \quad (\text{I.5.3})$$

si et seulement si  $u(\cdot, Z)$  est une martingale locale prenant la valeur  $f(Z_1)$  en  $t = 1$ . Dans cet exemple simple, on voit donc que les fonctions harmoniques  $u$  avec condition de bord admettent une formulation probabiliste en formant une diffusion associée à  $\mathcal{L}$ . L'image de cette diffusion est alors une martingale réelle de valeur terminale fixée.

**Fonctions harmoniques entre variétés.** Ces faits précédents peuvent s'étendre aux fonctions  $u$  à valeurs sur les variétés à l'aide des  $\nabla$ -martingales relatives à une certaine connexion. Plus précisément, soit  $u : \mathcal{M} \rightarrow \mathcal{N}$  où  $\mathcal{N}$  est également une variété différentiable munie d'une connexion  $\nabla^{\mathcal{N}}$ . Si  $\mathcal{L}$  désigne toujours un opérateur différentiel sur les fonctions lisses de  $\mathcal{M}$  dans  $\mathbf{R}$ , la quantité  $\mathcal{L}(u)$  n'est pas bien définie, cependant elle l'est pour toute fonction  $g \circ u$  avec  $g : \mathcal{N} \rightarrow \mathbf{R}$ .

L'application  $L_{\mathcal{N}}(u)(z) : g \in \mathcal{C}^2(\mathcal{N}, \mathbf{R}) \mapsto \mathcal{L}(g \circ u)(z)$  est un élément de  $\tau_{h(z)}\mathcal{N}$ , l'espace des diffuseurs en  $h(z)$  de la variété  $\mathcal{N}$ . D'après Emery ([Eme89], item ), la donnée d'une connexion sur  $\mathcal{N}$  permet de définir un opérateur de projection  $\Gamma$  de l'espace tangent du second ordre sur le premier, on dit alors qu'une application  $u : \mathcal{M} \rightarrow \mathcal{N}$  est  $\mathcal{L}$ -harmonique si  $\Gamma[L_{\mathcal{N}}(h)] := \mathcal{L}_{\mathcal{N}}(h) = 0$ . Après cette étape de projection qui dépend de la connexion  $\nabla^{\mathcal{N}}$ , on a la caractérisation suivante :  $u$  est  $\mathcal{L}$ -harmonique si et seulement si  $u(X)$  est une  $\nabla^{\mathcal{N}}$ -martingale pour toute condition initiale. Peu importe le cadre on est donc amené à rechercher les martingales de valeur terminale donnée sur une variété.

### § I.5.2. Calculs de barycentres associés à une filtration

Cette partie sera dédiée aux résultats d'Emery et Mokobodzki dans [EM91], ceux de Picard dans [Pic94] et d'Arnaudon dans [Arn95].

Il peut être utile de chercher un *barycentre* associé à une certaine mesure  $\mu$ , i.e. intuitivement



un point (ou plusieurs) contenant  $\mu$ -presque sûrement la majorité du support de  $\mu$ . Un premier exemple crucial est le suivant : si  $x, y$  sont deux points d'une variété pour laquelle chaque point est lié de manière unique par une seule géodésique, alors la mesure  $(1-t)\delta_x + t\delta_y$  devra avoir  $\gamma(t)$  comme barycentre, où  $\gamma$  est l'unique géodésique reliant  $x$  à  $y$  et  $\delta_x, \delta_y$  sont les masses de Dirac en  $x$  et  $y$  respectivement. Il existe plusieurs définitions du barycentre, nous en présentons une ici qui est intéressante vis-à-vis des martingales continues. Le lecteur pourra considérer par exemple [EM91] ou [Arn95] pour la notion de barycentre exponentiel, et [Lob89] pour une toute autre définition.

Cette notion de moyenne permet notamment de palier l'absence d'espérance et d'espérance conditionnelle sur une variété. Si l'on sait définir le barycentre d'une mesure, l'espérance d'une variable aléatoire sera simplement le barycentre de sa loi. Admettons qu'une telle notion existe,  $\mathbf{E}(\xi|\mathcal{F}_t)$  est alors l'unique  $\mathcal{F}$ -martingale de valeur terminale  $\xi$ . Le barycentre conditionnelle semble donc fortement reliée à la notion de martingale de valeur terminale donnée sur une variété.

Avant de l'expliquer, rappelons la définition du barycentre proposée par Arnaudon dans [Arn95] ou Emery et Mokobodzki dans [EM91], inspirée de l'inégalité de Jensen satisfaite par l'espérance conditionnelle dans  $\mathbf{R}^d$ .

**Définition – I.5.1.** Soit  $\mu$  une probabilité sur  $(\mathcal{M}, \nabla)$ . Le  $\nabla$ -barycentre de  $\mu$  est l'ensemble des points  $x \in \mathcal{M}$ , noté  $\mathbf{Bar}(\mu) \subset \mathcal{M}$ , tels que pour toute fonction  $\nabla$ -convexe bornée  $f$  sur  $\mathcal{M}$ , on ait :

$$f(x) \leq \int_{\mathcal{M}} f(y) d\mu(y).$$

L'espérance d'une variable aléatoire  $\zeta$  sur  $\mathcal{M}$  est alors l'ensemble des barycentres de sa loi. L'espérance conditionnelle d'une variable aléatoire  $\zeta$  sachant une tribu  $\mathcal{F}$  est l'ensemble de toutes les variables aléatoires  $\mathcal{F}$ -mesurables  $E$ , noté  $\mathbf{E}(\zeta|\mathcal{F})$ , telles que  $f \circ E \leq \mathbf{E}(f \circ E|\mathcal{F})$  pour toute fonction  $f$ ,  $\nabla$ -convexe bornée.

Dans la suite on parlera seulement de barycentre en omettant la connexion. L'inconvénient principal de cette définition réside dans l'absence d'unicité. Par exemple dans le cas euclidien tout point se trouve être barycentre de n'importe quelle mesure puisque les fonctions convexes bornées sont alors constantes. En revanche, l'élément  $\gamma(t)$  défini plus haut est bien dans  $\mathbf{Bar}((1-t)\delta_x + t\delta_y)$ . Avec la Définition – I.5.1, on peut justifier une propriété dite d'associativité. En particulier, il devient licite de définir les  $\nabla$ -martingales au sens des barycentres de la manière suivante.

**Définition – I.5.1.** Un processus  $X$  continu à valeur dans  $\mathcal{M}$  est appelé  $\nabla$ -martingale continue au sens des barycentres s'il vérifie :

$$X_s \in \mathbf{E}(X_t|\mathcal{F}_s), \tag{I.5.4}$$

pour tous  $s < t$ .

En utilisant le fait que toute martingale continue est transformée en sous-martingale locale réelle par une fonction convexe, on peut vérifier que si  $X$  est une  $\nabla$ -martingale continue telle que  $X_1 = \xi$  au sens habituel, alors  $X_t \in \mathbf{E}(\xi|\mathcal{F}_t)$  pour tout  $0 \leq t \leq 1$ .

Inversement étant donnée une variable aléatoire  $\xi$ , il paraît cependant difficile de construire  $X$  à rebours puisque les ensembles  $\mathbf{E}(\xi | \mathcal{F}_t)$  ne sont en général pas réduits à un singleton. Cependant cela est possible pour les  $\mathcal{C}$ -martingales à temps discret modulo une condition *ad hoc* garantissant l'unicité dans le choix  $X_{t_k} \in \mathbf{E}(X_{t_{k+1}} | \mathcal{F}_{t_k})$  pour toute subdivision  $(t_k)_k$  de  $[0, 1]$  (voir [Arm95] pour plus de détails).

Soit  $X$  une  $\nabla$ -martingale telle que  $X_0 = x \in \mathcal{M}$ . Comme  $X$  est transformée par toute fonction convexe en une sous-martingale locale,  $x \in \mathbf{E}(X_T)$  pour tout temps d'arrêt fini  $T$ . Le théorème suivant précise la réciproque, et fournit ainsi une autre preuve d'existence d'une martingale de valeur terminale donnée.

**Théorème – I.5.1** (Théorème 2, [EM91]). *Soit  $\mathcal{V}$  un espace affinement difféomorphe à un ouvert relativement compact de  $\mathcal{M}$ , et supposons qu'il existe une fonction  $\Psi$  de classe  $\mathcal{C}^2$  strictement convexe.*

*Pour toute mesure de probabilité  $\mu$  sur  $\mathcal{V}$ , et  $x \in \mathbf{Bar}(\mu)$ , il existe un espace probabilisé et une martingale continue  $X$  à valeurs dans  $\mathcal{V}$  relativement à cet espace probabilisé, telle que  $X_0 = x$  et  $X_1$  ait pour loi  $\mu$ .*

En particulier, prenant  $\mu = \mathbf{P}_\xi$ , la loi de  $\xi$ , avec ce théorème il est possible de construire une martingale de loi terminale  $\mu$  sous l'hypothèse principale  $f(x) \leq \mathbf{E}(f(\xi))$  pour toute fonctionnelle convexe bornée  $f$ .



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## Méthode d'algorithmique stochastique

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Dans cette partie, un algorithme stochastique est proposé afin d'approcher les martingales de valeur terminale donnée dans une boule géodésique. Il est construit *via* une méthode de tir, en tenant compte de propriétés relatives aux variations de martingales sur une variété.

In this chapter, we define a stochastic algorithm approaching the martingales with prescribed terminal value on a regular geodesic ball by using a shooting method. The construction is based on results about the variations of martingales in a manifold.

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## II.1 INTRODUCTION

### § II.1.1. Martingales with prescribed terminal values on manifolds

In an Euclidean space, we know how to construct a martingale with prescribed terminal value in terms of the conditional expectation. This object has no intrinsic representation on a general manifold and the question don't have an obvious answer. Given a connection  $\nabla$  on a manifold  $\mathcal{M}$ , a random variable  $\xi$  in  $\mathcal{M}$ , and  $(\mathcal{F}_t)_{0 \leq t \leq 1}$  a filtration, the problem is the following: are we able to find a  $(\mathcal{F}, \nabla)$ -martingale  $X$  such that  $X_1 = \xi$ ? Does uniqueness hold?

The main assumption on the manifold will be convex geometry. This property is routine in order to get stability results by using interesting convex maps. The problem has been solved with this kind of setting in [Ken90, Pic91] assuming a Brownian filtration. Later, in [Pic94], Picard has dealt with more general filtrations like filtrations generated by independent Brownian motions and Poisson processes. The method used is a discretization of the filtration, and the construction of a family of discrete martingales converging to a solution. Other methods are possible, as Darling in [Dar95] and Blache in [Bla04a, Bla06] using backward stochastic differential equations, or Arnaudon in [Arn97] using analytic connections together with an analytic deformation of the terminal value.

The method used in this work is quite hybrid in the sense that a discrete algorithm is employed to reach a smooth deformation of the terminal value. It is quite directly related to the deterministic geodesic shooting, devoted to approach a geodesic linking two points on a manifold, knowing the exponential map (see Remark – II.3.6).

### § II.1.2. Our approach

We set a random variable  $\xi$  in a manifold. The smooth families of martingales satisfy a kind of ordinary differential equation which will be used to construct an algorithm; that is a family of martingales, admitting a subsequence along which the terminal values tend to  $\xi$  with respect to the squared geodesic distance. The terminal condition will be supposed to be in a small regular geodesic ball  $\mathcal{B}$ .

We begin to consider a constant martingale in  $\mathcal{B}$ . Next, we distort it into a smooth family of martingales, indexed by an interval  $[0, a_1)$ ,  $a_1 \in \mathbf{R}^+$ , such that the terminal value at  $a_1$  is closer to  $\xi$  than the latest at zero. We repeat this procedure to reach  $\xi$  asymptotically. Some results on regular geodesic balls will be widely used to prove the convergence. For example, we know that all martingales in  $\mathcal{B}$  have exponential moments and  $\mathcal{B}$  has convex geometry. Convex geometry property will be used to obtain stability results; that is convergence with respect to the topology of semimartingales of the algorithm when the terminal values converge.

Finally, this method will allow us to solve a more general problem. If we consider a smooth target included in a small regular geodesic ball, we are able to construct a smooth family of

martingales reaching the image of the target.

### § II.1.3. Structure of the paper

In the remaining of the introduction, we introduce notations, the setting and general properties relatively to the stochastic geometry notions for a sake of completeness. In section two, we gather the preliminary results to make easier the comprehension of the algorithm ; mainly a result of localization in  $\mathcal{H}^\infty$ . We have collected in section three all our main results in order to improve the readability of the paper, and introduce the main stochastic algorithm. The last sections are devoted to all the technical proofs: convergences and existence of the algorithms.

### § II.1.4. Notations

◇ We will deal with a manifold  $\mathcal{M}$  of dimension  $d$ . The first tangent bundle over  $\mathcal{M}$  is assumed to be equipped by a compatible Riemannian metric  $\langle \cdot, \cdot \rangle_g \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M})$ , that is a smooth field of bilinear maps compatible with the connection  $\nabla$ . The corresponding Euclidean norm is  $|\cdot|$ .

In the whole paper we endow the manifold  $\mathcal{M}$  with the **Levi-Civita connection**  $\nabla$  associated to  $g$ , and understood as a covariant derivative on the space of smooth vector fields  $\Gamma(T\mathcal{M})$  on  $\mathcal{M}$ .

The  $\nabla$ -geodesic distance on  $\mathcal{M}$  is noted  $\rho$ , and for all  $x \in \mathcal{M}$  the cut locus at  $x$  will be  $\text{Cut}_x\mathcal{M}$ . If  $(x, y) \in \mathcal{M}^2$  such that  $y \notin \text{Cut}_x\mathcal{M}$ , we denote  $\vec{xy} = \exp_x^{-1}(y) \in T_x\mathcal{M}$ , the map  $\pi : T\mathcal{M} \rightarrow \mathcal{M}$  is the canonical projection of the fiber bundle  $T\mathcal{M}$  characterized by: for all  $X \in T\mathcal{M}$ ,  $\pi(X) = x \in \mathcal{M}$  if and only if  $X \in T_x\mathcal{M}$ , for all  $k \geq 2$ ,  $\pi^k$  the canonical projection of  $T^k\mathcal{M}$ ,  $\pi^E$  for a general fiber bundle  $E$  above  $\mathcal{M}$ . The bundle where each fiber is a space of second order operators is denoted by  $\tau\mathcal{M}$  and the space given by all the smooth sections of the bundle  $E$  is noted  $\Gamma(E)$ .

We define an other connection  $\nabla^c$  on  $T\mathcal{M}$ , called the complete lift of  $\nabla$  (see [YI73] for more details), by

$$\nabla_{A^c}^c B^c := (\nabla_A B)^c, \quad (A, B) \in \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}),$$

where  $A^c$  is the vector field  $A^c(df) = dA(f)$  for all smooth function  $f$  on  $\mathcal{M}$ . We know explicitly the geodesics of  $\nabla^c$ : they are precisely the Jacobi fields, *i.e.* the vector fields  $\mathcal{J}$  along a curve  $\gamma$  satisfying the following second order equation (see below for a definition of the  $\nabla$ -curvature tensor  $R$ ):

$$\nabla_t \nabla_t \mathcal{J} + R(\dot{\gamma}, \mathcal{J}) \dot{\gamma} = 0. \quad (\text{II.1.1})$$

The  $\nabla$ -parallel transport along a curve  $\gamma$  is denoted by  $\parallel_{0, \cdot}(\gamma)$ , and is a linear map from  $T_{\gamma(0)}\mathcal{M}$  to  $T\mathcal{M}$ , so an element of  $\mathcal{L}(T_{\gamma(0)}\mathcal{M}, T\mathcal{M})$ , and more generally  $\parallel_{s_1, s_2}(\gamma)$ , with  $s_1, s_2$  two times in the support of  $\gamma$ , will state for  $\parallel_{0, s_2}(\gamma) \circ \parallel_{0, s_1}(\gamma)^{-1} \in \mathcal{L}(T_{\gamma(s_1)}\mathcal{M}, T_{\gamma(s_2)}\mathcal{M})$ .

Sometimes, if the extremal points are most central, we will use the notation  $\parallel_{x, y}$  where  $(x, y) \in \mathcal{M}^2$  such that  $y \notin \text{Cut}_x\mathcal{M}$  for the parallel transport following the unique minimal  $\nabla$ -geodesic between  $x$  and  $y$ , if the previous uniqueness holds true.

**Proposition – II.1.1.** *There exists a constant  $C$  such that: for all  $(x, y) \in \mathcal{M}^2$ ,  $(v, w) \in (T_x \mathcal{M})^2$ ,*

$$|\nabla_v \llbracket y, \cdot \rrbracket| \leq C \times \rho(x, y) |v|, \quad |\nabla_w \nabla_v \llbracket y, \cdot \rrbracket| \leq C |v| |w|.$$

We recall also a geometrical result of Blache in [Bla06], valid on a relatively compact domain  $\mathbf{B}$  included in  $\mathcal{M}$ , and having a global chart  $(x^1, \dots, x^d)$ .

**Proposition – II.1.2** (Proposition 2.2.1 of [Bla06]). *There is  $C > 0$  such that for all  $(x, x') \in \mathbf{B} \times \mathbf{B}$  and  $(z, z') \in T_x \mathcal{M} \times T_{x'} \mathcal{M}$ , we have:*

$$\|\llbracket_{x, x'} z - z' \rrbracket\| \leq C (|z - z'|_2 + \rho(x, x') (|z|_2 + |z'|_2)).$$

*Remark – II.1.1.* Note that the term  $|z - z'|_2$  is computed relatively to the local chart  $(x^1, \dots, x^d)$  and for the Euclidean norm.

◇ *Decomposition of the second fiber bundle and lifting* — If  $\mathcal{M}$  carries a connection  $\nabla$ , we have a decomposition of  $TT\mathcal{M}$  into a vertical subbundle  $V\mathcal{M}$  and horizontal  $H\mathcal{M}$ .

More precisely: we set  $V_A \mathcal{M} = \text{Ker } d\pi(A)$ , where  $d\pi : TT\mathcal{M} \rightarrow T\mathcal{M}$ , for the vertical space at the point  $A \in T\mathcal{M}$ . To define  $H_A \mathcal{M}$ , we consider the space spanned by all the vectors

$$\left. \frac{\partial}{\partial s} \right|_{s=0} \llbracket_{0, s}(\gamma)(A) \in T_A T\mathcal{M},$$

where  $\gamma$  is a curve in  $\mathcal{M}$  such that  $\dot{\gamma}(0) = A$ . We have for all  $A \in T\mathcal{M}$ :

$$T_A(T\mathcal{M}) = V_A \mathcal{M} \oplus H_A \mathcal{M}.$$

If  $X \in TT\mathcal{M}$ ,  $X^{\text{vert}}$  and  $X^{\text{hor}}$  will state for the vertical and horizontal parts of  $X$ . We can easily lift all the vectors from the first tangent bundle to the second one. Firstly, given  $A \in T\mathcal{M}$ , the vertical lift at  $A$  is the map

$$v_A = (d\pi(A)|_{V_A \mathcal{M}})^{-1} : T_{\pi(A)} \mathcal{M} \rightarrow V_A \mathcal{M} \subset T_A T\mathcal{M}.$$

Secondly the horizontal lift at  $A$  is the map

$$h_A : T_{\pi(A)} \mathcal{M} \rightarrow H_A \mathcal{M} \subset T_A T\mathcal{M}, \quad u \mapsto \left. \frac{\partial}{\partial s} \right|_{s=0} \llbracket_{0, s}(\gamma)(u),$$

where  $\gamma$  is a curve such that  $\gamma(0) = \pi(u)$  and  $\dot{\gamma}(0) = A$ .

If  $(x^i)_{i=1}^d$  is a local chart around  $\pi(A) \in \mathcal{M}$  and  $A = A^i \partial_{x^i}$ , we have in local coordinates

$$v_A : \begin{cases} T_{\pi(A)} \mathcal{M} & \longrightarrow & V_A \mathcal{M} \\ u = u^i \partial_{x^i} & \longmapsto & (x^i, u^i, 0, A^i) \end{cases}, \quad h_A : \begin{cases} T_{\pi(A)} \mathcal{M} & \longrightarrow & H_A \mathcal{M} \\ u = u^i \partial_{x^i} & \longmapsto & (x^i, u^i, A^i, -\Gamma_{jk}^i(\pi(A)) A^j u^k) \end{cases}.$$

A link between the time and the covariant derivative of a curve  $a \mapsto u(a)$  on  $T\mathcal{M}$  above  $\gamma$  in

$\mathcal{M}$  is given by:

$$\nabla_a u(a) = v_{u(a)}^{-1} (\partial_a u(a)^{\text{vert}}).$$

◇ *Curvature, sectional curvatures and regular geodesic balls* — For all  $(X, Y, Z) \in (\Gamma(T\mathcal{M}))^3$ , the (1, 3)-curvature tensor  $R$  is

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

If a local chart  $(x^i)_{i=1}^d$  is set, the coefficients are given by

$$R_{lij}^k = \frac{\partial \Gamma_{jl}^k}{\partial x^i} - \frac{\partial \Gamma_{il}^k}{\partial x^j} + \Gamma_{im}^k \Gamma_{jl}^m - \Gamma_{jm}^k \Gamma_{il}^m.$$

If  $V \in \Gamma(T^* \mathcal{M} \otimes T^* \mathcal{M} \otimes T\mathcal{M})$ , we define a modified curvature tensor  $R_V$  by: for all  $(X, Y, Z) \in (\Gamma(T\mathcal{M}))^3$ ,

$$R_V(X, Y)Z = R(X, Y)Z - 2 \times \nabla_X V(Y, Z).$$

The subordinate norms of  $R_V$  and  $\nabla R_V$  at  $m \in \mathcal{M}$  are defined by

$$|R_V(m)| = \sup_{\substack{(X, Y, Z) \in \Gamma(T\mathcal{M})^3 \\ X(m) \neq 0, Y(m) \neq 0, Z(m) \neq 0}} \frac{|(R_V(X, Y)Z)(m)|}{|X(m)| |Y(m)| |Z(m)|}, \quad (\text{II.1.2})$$

$$|\nabla R_V(m)| = \sup_{\substack{(X, Y, Z, T) \in \Gamma(T\mathcal{M})^4 \\ X(m) \neq 0, Y(m) \neq 0, Z(m) \neq 0, T(m) \neq 0}} \frac{|(\nabla_T R_V(X, Y)Z)(m)|}{|X(m)| |Y(m)| |Z(m)| |T(m)|}. \quad (\text{II.1.3})$$

Let  $\sigma_x$  a plane in  $T\mathcal{M}$ , and let us consider  $(X, Y)$  an orthonormal basis of  $\sigma_x$ . The sectional curvature  $\kappa_V(\sigma_x)$  of  $\sigma_x$  is defined by:

$$\kappa_V(X, Y) = \langle R_V(X, Y)Y, X \rangle_g. \quad (\text{II.1.4})$$

Besides, we have the formula

$$\kappa_V(X, Y) = \kappa_0(X, Y) - 2 \langle \nabla_X V(Y, Y), X \rangle_g.$$

For  $y \in \mathcal{M}$  and  $\mathcal{N}$  a submanifold of  $\mathcal{M}$ , we define:

$$(\kappa_V^\uparrow)_y = \sup_{\substack{\sigma_y \\ 2\text{-plane at } y}} \kappa(\sigma_y), \quad (\kappa_V^\downarrow)_y = \inf_{\substack{\sigma_y \\ 2\text{-plane at } y}} \kappa(\sigma_y), \quad (\text{II.1.5})$$

$$(\kappa_V^-)_y = -\min \left\{ \kappa_y^\downarrow; 0 \right\}, \quad (\kappa_V^+)_y = \max \left\{ \kappa_y^\uparrow; 0 \right\}, \quad (\text{II.1.6})$$

$$\kappa_V^-(\mathcal{N}) = \inf_{y \in \mathcal{N}} \kappa_y^-, \quad \kappa_V^+(\mathcal{N}) = \sup_{y \in \mathcal{N}} \kappa_y^+. \quad (\text{II.1.7})$$



If  $V = 0$  we will omit the subscripts  $V$  in the sequel.

*Remark – II.1.2.* Let us emphasize that by definition of  $\Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes T\mathcal{M})$ , the tensor field  $V$  is differentiable, which make sense to the definition of  $R_V$ . The motivation for the last covariant differential equation is purely probabilistic (see [Proposition – II.1.9](#) below for more details).

To achieve this subsection, let us introduce a class of geodesic balls having positively bounded sectional curvatures, and very interesting stochastic properties for our purpose.

**Definition – II.1.1** (Regular Geodesic Ball, Definition 1.6 of [Ken90]). Let  $\mathcal{B}$  a closed geodesic ball in  $\mathcal{M}$  of radius  $R$  and center  $O \in \mathcal{M}$ .

The set  $\mathcal{B}$  is called a regular geodesic ball of radius  $R$  and center  $O$  if the following conditions are fulfilled:

$$\kappa^+(\mathcal{B}) < \sqrt{\frac{\pi}{2R}} \quad \text{and} \quad \text{Cut}_O \mathcal{M} \cap \mathcal{B} = \emptyset.$$

Let  $\mathcal{B}$  a regular geodesic ball.

**Proposition – II.1.3.** Let  $x, x', y \in \mathcal{B}$ . There exists a constant  $C(d, \kappa^+(\mathcal{B}), R(\mathcal{B}))$  depending only on  $d$  and  $\mathcal{B}$  such that:

$$\left| \left\|_{x,x'} \overrightarrow{xy} - \overrightarrow{x'y} \right\| \leq C(d, \kappa^+(\mathcal{B}), R(\mathcal{B})) \times \rho(x, x').$$

Proof. Let  $\gamma$  be the geodesic of minimal length linking  $x$  to  $x'$ . The map  $v : t \mapsto \overrightarrow{\gamma(t)y}$  is a vector field along  $\gamma$  corresponding to  $V : z \mapsto \overrightarrow{zy}$ . We note  $f(t) = \left\|_{0,t}(\gamma)v(0) - v(t) \right\|$  and looking for an estimation of  $f(t)$  for all  $t$ . Taking particularly  $t = 1$ , the claim will follow.

Let us see firstly that  $f(0) = 0$ , and since the connection is metric the following equality holds:

$$\frac{\partial}{\partial t} f(t) = -\frac{1}{f(t)} \langle \nabla_t v(t), \left\|_{0,t}(\gamma)v(0) - v(t) \right\rangle.$$

With the Cauchy-Schwarz inequality one has:

$$\left| \frac{\partial}{\partial t} f(t) \right| \leq |\nabla_t v(t)|,$$

and

$$\left\|_{x,x'} \overrightarrow{xy} - \overrightarrow{x'y} \right\| \leq \sup_{t \in [0,1]} |\nabla_t v(t)| \leq \rho(x, x') \times \sup_{z \in \text{Im } \gamma} |\nabla V(z)|.$$

We know a formula for the gradient of  $r(z) = \rho(z, y)$  (according to [Proposition – II.1.5](#) below):

$$V = -r \text{ grad } r. \tag{II.1.8}$$

Consequently,

$$\nabla V = -dr \otimes \text{ grad } r - r \nabla \text{ grad } r.$$

Let  $z$  in the image of  $\gamma$ . Using again the formula (II.1.8) and the fact that  $x, x', y$  are in  $\mathcal{B}$ , we obtain the existence of a constant  $C$  depending only on the radius of  $\mathcal{B}$  and satisfying:

$$|\nabla V(z)| \leq C(1 + |\nabla \text{grad} r(z)|).$$

Since  $\kappa^+(\mathcal{M}) < \infty$ , the Hessian comparison theorem shows that  $\nabla \text{grad} r(z)$  is bounded by a constant depending only on the dimension of  $\mathcal{M}$  and  $\mathcal{B}$  (through the radius  $R(\mathcal{B})$  and the upper bound  $\kappa^+(\mathcal{B})$  on the sectional curvatures). The claim follows.  $\square$

◇ *Martingales and stochastic transports* — Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbf{P})$  a complete filtered probability space such that every martingale has a continuous version.

A semimartingale on  $\mathcal{M}$  is a continuous process  $Y$  with values in  $\mathcal{M}$  such that for all  $f \in \mathcal{C}^2(\mathcal{M}, \mathbf{R})$ ,  $f \circ Y$  is a real semimartingale. A  $\nabla$ -martingale  $X$  on  $\mathcal{M}$  is a continuous semimartingale with values in  $\mathcal{M}$ , such that for every function  $f \in \mathcal{C}^2(\mathcal{M}, \mathbf{R})$ ,

$$f(X) - f(X_0) - \frac{1}{2} \int_0^{\cdot} \nabla df(dX_s, dX_s)$$

is a real continuous local martingale. Let  $X$  and  $Y$  two real processes. The notations  $X \stackrel{m}{=} Y$  (*resp.*  $X \leq^m Y$ ) mean that  $X - Y$  is a real local martingale (*resp.*  $X - Y$  is less or equal to a real local martingale).

The spaces  $\text{Mart}_{\nabla}(\mathcal{M})$  and  $\text{SemMart}(\mathcal{M})$  are respectively the sets of  $\nabla$ -martingales and semimartingales on  $\mathcal{M}$ .

Let  $Y$  a semimartingale. The curved Itô differential against  $Y$  with respect to  $\nabla$  will be denoted by  $d^{\nabla}Y$  and the Stratonovich version with  $\circ dY$ .

In local coordinates, if  $Y$  takes their values in a domain having a global chart  $(y^i)_{i=1}^d$ , for all

$\alpha.(\omega) = \sum_{i=1}^d \alpha_i(\omega) \partial_{y^i}(Y)$  above  $Y$  in  $T^*\mathcal{M}$  we have in local coordinates:

$$\alpha(\circ dY) = \alpha_i(Y) dY^i + \frac{1}{2} \frac{\partial \alpha_i}{\partial y^j}(X) d\langle Y^i, Y^j \rangle, \quad \alpha(d^{\nabla}Y) = \alpha_i(Y) \left( dY^i + \frac{1}{2} \Gamma_{jk}^i(Y) d\langle Y^j, Y^k \rangle \right). \quad (\text{II.1.9})$$

The formal differential of order two coming from the Schwartz principle is denoted  $\underline{\mathbf{d}}Y$ . In a local chart we have for all  $\Theta.(\omega) = \theta_i(\omega) d^2y^i + \theta_{ij}(\omega) dy^i \cdot dy^j \in \tau^*\mathcal{M}$  above  $Y$ ,

$$\Theta(\underline{\mathbf{d}}Y) = \theta_i dY^i + \frac{1}{2} \theta_{ij}(Y) d\langle Y^i, Y^j \rangle.$$

If  $J$  is a semimartingale with values in  $T\mathcal{M}$ , the stochastic covariant derivative  $\mathbf{D}$  and deformed stochastic covariant derivative  $\mathbf{D}^{\text{def}}$  are defined by the following identities (see below

for the definition of  $\parallel_{0,\cdot}(\pi J)$  and  $\Theta_{0,\cdot}(\pi J)$ ):

$$\mathbf{D}J = \parallel_{0,\cdot}(\pi J) d\left(\parallel_{0,\cdot}^{-1}(\pi J)(J)\right) = v_J^{-1}\left(\left(d^{\nabla^h} J\right)^{\text{vert}}\right), \quad \mathbf{D}^{\text{def}}J = \Theta_{0,\cdot}(\pi J) d\left(\Theta_{0,\cdot}^{-1}(\pi J)(J)\right),$$

$\nabla^h$  being the horizontal lift of  $\nabla$  (see [AT03] for more details).

**Definition – II.1.2** (Parallel transport along  $Y$ ). The parallel transport along  $Y$  (with respect to  $\nabla$ ) is the  $\mathcal{L}(T_{Y_0}\mathcal{M}, T\mathcal{M})$ -valued semimartingale  $\parallel_{0,\cdot}(Y)$  defined by

$$\parallel_{0,0}(Y) = \text{Id}_{T_{Y_0}\mathcal{M}}, \quad \circ d(\parallel_{0,\cdot}(Y)) = h_{\parallel_{0,\cdot}(Y)}(\circ dY).$$

The stochastic parallel transport satisfies the same isometry property as for the deterministic version, when the connection is Levi-Civita. Particularly, in the whole paper, all the stochastic parallel transports according to  $\nabla$  along a continuous semimartingale  $X$  in  $\mathcal{M}$  will be an isometry. This statement is not true for the deformed stochastic transport defined below as we will see later.

Let  $X$  a continuous  $\nabla$ -martingale.

**Definition – II.1.3** (Deformed stochastic transport along  $X$ ). The geodesic transport (also called Dorhn-Guerra transport or deformed transport) along  $X$  with respect to  $\nabla$  is the  $\mathcal{L}(T_{X_0}\mathcal{M}, T\mathcal{M})$ -valued semimartingale  $\Theta_{0,\cdot}(X)$  defined by

$$\Theta_{0,0}(X) = \text{Id}_{T_{X_0}\mathcal{M}}, \quad d^{\nabla^c}\Theta_{0,\cdot}(X) = h_{\Theta_{0,\cdot}(X)}\left(d^{\nabla}X\right),$$

where  $\nabla^c$  is seen as a linear connection on the bundle  $\mathcal{L}(T_{X_0}\mathcal{M}, T\mathcal{M})$ .

The formulas  $\parallel_{s,t}(Y) := \parallel_{0,t}(Y)\parallel_{0,s}^{-1}(Y)$  and  $\Theta_{s,t}(X) := \Theta_{0,t}(X)\Theta_{0,s}^{-1}(X)$  for  $0 \leq s \leq t$  extend the last definitions for general couple of times. Equivalently, they satisfy the following stochastic differential equations:

$$\mathbf{D}_s \parallel_{t,s}(Y) = 0, \quad \mathbf{D}_s \Theta_{t,s}(X) = -\frac{1}{2}R(\Theta_{t,s}(X), dX_s) dX_s, \quad s \in [t, 1].$$

With the Itô formula we deduce easily a link between  $\mathbf{D}J$  and  $\mathbf{D}^{\text{def}}J$  defined above:

$$\mathbf{D}J = \mathbf{D}^{\text{def}}J - \frac{1}{2}R(J, d\pi J) d\pi J,$$

mainly using that  $\mathbf{D}\Theta_{0,\cdot}(X)$  has finite variations.

The processes  $\mathcal{A}(Y)_t = \int_0^t \parallel_{0,s}^{-1}(Y) \left(d^{\nabla}Y_s\right)$  (*resp.*  $\mathcal{A}^{\text{def}}(X)_t = \int_0^t \Theta_{0,s}^{-1}(X) \left(d^{\nabla}X_s\right)$ ) with values in  $T_{Y_0}\mathcal{M}$  are called the antidevelopment of  $Y$  (*resp.* the deformed antidevelopment of  $X$ ). The notations  $\mathcal{A}^c$  and  $\mathcal{A}^{\text{def},c}$  are borrowed for the antidevelopments with respect to  $\nabla^c$ . Conversely, let  $A$  a process in  $T_x\mathcal{M}$  for  $x \in \mathcal{M}$ . There exists a predictable stopping time  $\zeta_A > 0$

and a semimartingale  $\mathcal{D}(A)$  in  $\mathcal{M}$  with lifetime  $\zeta_A$  and exploding at the time  $\zeta_A$ , such that

$$\mathcal{A}(\mathcal{D}(A)) = A \quad \text{on} \quad [0, \zeta_A], \quad \text{and} \quad \mathcal{D}(A)_0 = x.$$

**Definition – II.1.4.** Let  $Y$  a continuous semimartingale in  $\mathcal{M}$ . We shall say that  $Y$  is a  $\nabla$ -martingale with drift  $V_Y(dY, dY)$ ,  $V \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes T\mathcal{M})$ , if  $\|_{0,\cdot}^{-1}(Y)V_Y(dY, dY)$  is the finite variation part of  $\mathcal{A}(X)$ :

$$d\mathcal{A}(Y) - \|_{0,\cdot}^{-1}(Y)V_Y(dY, dY) = d\mathcal{A}(Y)^{\text{mart}}.$$

Alternatively, if  $d_{\text{mart}}Y = \|_{0,\cdot}(Y) d\mathcal{A}(Y)^{\text{mart}}$  states for the martingale part of  $d^\nabla Y$ , which does not depend on  $\nabla$ , we have

$$d^\nabla Y = d_{\text{mart}}Y + V_Y(dY, dY).$$

To simplify the notations in the sequel, a drift will be denoted by  $V(dY, dY)$  instead of  $V_Y(dY, dY)$ . Sometimes if it is necessary we will use again the first notation. The process  $V$  does not exist for general semimartingales. The set of all  $\nabla$ -martingales with drift  $V$  as in Definition – II.1.4 is strictly included in the set of all continuous semimartingales. For this reason, we will use two different terminologies in the sequel. For now, the deformed stochastic transport is only defined against the  $\nabla$ -martingales, we are going to give an other definition for the processes having a drift.

Let  $Y$  a  $\nabla$ -martingale with prescribed drift  $V_Y(dY, dY)$ .

**Definition – II.1.5** (Deformed stochastic transport along  $Y$ ). The deformed transport (also called Dorhn-Guerra transport or geodesic transport) along  $Y$  with respect to  $\nabla$  is the  $\mathcal{L}(T_{X_0}\mathcal{M}, T\mathcal{M})$ -valued semimartingale  $\Theta_{0,\cdot}(X)$  defined by

$$\mathbf{D}\Theta_{0,\cdot}(Y) = -\frac{1}{2}R_V(\Theta_{0,\cdot}(Y), dY) dY, \quad (\text{II.1.10})$$

where  $\nabla^c$  is seen as a linear connection on the bundle  $\mathcal{L}(T_{X_0}\mathcal{M}, T\mathcal{M})$ .

- ◇ *Convexity* – The manifold  $\mathcal{M}$  is said to have *convex geometry* if there exists a  $\nabla \otimes \nabla$ -convex function  $\Psi : \mathcal{M} \times \mathcal{M} \rightarrow \mathbf{R}^+$ , also called a *separant*, such that  $\Psi^{-1}(0) = \{(x, x), x \in \mathcal{M}\}$ . We shall say that a subset  $C$  of  $\mathcal{M}$  has convex geometry if there exists an open neighborhood of  $C$  having convex geometry.

Let  $p \geq 1$ , we shall say that  $\mathcal{M}$  has *p-convex geometry* if  $\mathcal{M}$  has convex geometry with a separant function  $\Psi$  satisfying moreover:

$$c\rho^p \leq \Psi \leq C\rho^p, \quad c, C \in \mathbf{R}^+.$$

We can see briefly that *p-convex geometry* implies *q-convex geometry* for all  $q \geq p$  mainly because  $\Psi^{q/p}$  is convex as soon as  $q \geq p$ .

If  $\mathcal{M}$  has a  $p$ -convex geometry, we define the map  $T^{\otimes p}\Psi : T\mathcal{M} \rightarrow \mathbf{R}^+$  by

$$T^{\otimes p}\Psi(v) = \lim_{\substack{t \rightarrow 0 \\ >}} \frac{1}{t^p} \Psi(\pi v, \exp_{\pi v}(tv)), \quad v \in T\mathcal{M}. \quad (\text{II.1.11})$$

In [Arn97], it is shown that as soon as  $p \in 2\mathbf{N}^*$  the map  $T^{\otimes p}\Psi$  is  $\nabla^c$ -convex and satisfies for all  $v \in T\mathcal{M}$ ,

$$c|v|^p \leq T^{\otimes p}\Psi(v) \leq C|v|^p.$$

Besides, the map

$$(v, v') \in \mathcal{D}(\mathcal{M}) := \{(v, v') \in T\mathcal{M}^2, \pi(v) = \pi(v')\} \mapsto T^{\otimes p}\Psi(v - v') \quad (\text{II.1.12})$$

is  $\nabla^c \otimes \nabla^c$ -convex.

We shall say that a semimartingale  $Y$  has *exponential moments of order*  $\lambda \geq 0$  if

$$\mathbf{E} \left( \exp \left( \lambda \int_0^1 \langle dY, dY \rangle_g \right) \right) < \infty,$$

where  $\int_0^\cdot \langle dY, dY \rangle_g$  states for the Riemannian quadratic variation of  $Y$ . Emery has shown that locally a manifold has always  $p$ -convex geometry for some  $p > 1$  (see [Eme89] item (4.59)). Kendall in [Ken91] proves that every regular geodesic ball with small radius has convex geometry, it is the statement (i) recalled above. Picard, in [Pic91] Proposition 2.1.2., has proved that in the same kind of small regular geodesic ball, the martingales have always exponential Riemannian moments. The proof is widely based on the convexity of the map  $G$  defined below and satisfying (ii), (iii). Finally, Blache in [Bla04a] Lemma 1.12 extends this property to the solutions of some backward stochastic differential equations.

Let us consider a field  $V \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes T\mathcal{M})$ , a regular geodesic ball  $\mathbf{B}$  with radius  $\mathbf{R} > 0$  and center  $\mathbf{O}_{\mathbf{B}}$  such that  $\kappa^+(\mathbf{B}) < \frac{\pi^2}{4\mathbf{R}^2}$ , and the following map:

$$G(x) = \cos \left( \sqrt{\kappa^+(\mathbf{B})q(\mathbf{B})} \times \rho(\mathbf{O}_{\mathbf{B}}, x) \right)$$

where  $q(\mathbf{B}) > 1$  is set such that  $G > 0$  on  $\mathbf{B}$ , i.e.  $\kappa^+(\mathbf{B})q(\mathbf{B}) \leq \frac{\pi^2}{4\mathbf{R}^2}$ . In the sequel we will consider the following assumptions.

(V1) There exists a constant  $\mathbf{V}(\mathbf{B})$  such that for all  $u \in T\mathcal{M}$ :

$$|V_{\pi u}(u, u)| \leq \mathbf{V}(\mathbf{B}) \langle u, u \rangle_g, \quad 0 \leq \mathbf{V}(\mathbf{B}) \leq \frac{c\sqrt{\kappa^+(\mathbf{B})q(\mathbf{B})}}{2},$$

where  $c$  is defined in Proposition – II.1.4 below.

(V2) For all  $u \in T\mathcal{B}$ :  $\left\langle \overrightarrow{\pi(u)\mathbf{O}_{\mathbf{B}}}, V_{\pi u}(u, u) \right\rangle_g \leq 0$ .

We are able to prove now the following proposition.

**Proposition – II.1.4.** *Firstly, the following statements hold:*

- (i) *the set  $\mathbf{B}$  has  $q(\mathbf{B})^*$ -convex geometry,*
- (ii) *for all  $x \in \mathbf{B}$  and  $u \in T_x\mathbf{B}$ :  $\nabla dG(x)(u, u) + \kappa^+(\mathbf{B})q(\mathbf{B})G(x)|u|^2 \leq 0$ ,*
- (iii) *there exists  $c > 0$  and  $C \geq c$  such that:  $c \leq G \leq C$ .*

*Secondly, if  $X$  is a  $\nabla$ -martingale with drift  $V_X(dX, dX)$  and taking values in  $\mathbf{B}$ ,*

- (iv) *if (VI) is in force,  $X$  has exponential moments of order  $\frac{\kappa^+(\mathbf{B})q(\mathbf{B})}{2} - \frac{V(\mathbf{B})\sqrt{\kappa^+(\mathbf{B})q(\mathbf{B})}}{c}$ , depending only on  $c, C$ ,*
- (v) *if (V2) is in force,  $X$  has exponential moments of order  $\frac{\kappa^+(\mathbf{B})q(\mathbf{B})}{2}$ , depending only on  $c, C$ .*

*Remark – II.1.3.* The point (i) is established by Picard with a probabilistic approach. Besides, a separant  $\Psi$  on  $\mathcal{B}$  is also given by Kendall in [Ken91] with the following formula:

$$\Psi(x, y) = \left( \frac{1 - \cos\left(\sqrt{\kappa^+(\mathcal{B})}\rho(x, y)\right)}{\cos\left(\sqrt{\kappa^+(\mathcal{B})}\rho(x, \mathbf{O}_\mathbf{B})\right)\cos\left(\sqrt{\kappa^+(\mathcal{B})}\rho(y, \mathbf{O}_\mathbf{B})\right) - h^2} \right)^{r(\mathbf{B})},$$

with  $h > 0$  small enough. As mentioned in [Ken91], we can take  $r(\mathbf{B}) \geq 2$  large enough such that  $\Psi$  is strictly convex outside the diagonal, therefore globally strictly convex on  $\mathcal{B} \times \mathcal{B} \setminus \Psi^{-1}([0, \varepsilon))$  for all  $\varepsilon > 0$ .

Proof. To show the last statements (iv) and (v) about semimartingales, let us consider  $X$  a  $\nabla$ -martingale with drift  $V_X(dX, dX)$ , and taking values in  $\mathbf{B}$ . We can compute easily the differential of  $G$ . For all  $u \in T_x\mathcal{M}$ , one has  $d\rho^2(\mathbf{O}_\mathbf{B}, \cdot)(x)(u) = -2\left\langle \overrightarrow{x\mathbf{O}_\mathbf{B}}, u \right\rangle_g$  and consequently,

$$dG(x)V_X(u, u) = \sqrt{\kappa^+(\mathbf{B})q(\mathbf{B})} \times \sin\left(\sqrt{\kappa^+(\mathbf{B})q(\mathbf{B})} \times \rho(\mathbf{O}_\mathbf{B}, x)\right) \frac{\left\langle \overrightarrow{x\mathbf{O}_\mathbf{B}}, V_X(u, u) \right\rangle_g}{\rho(\mathbf{O}_\mathbf{B}, x)}. \quad (\text{II.1.13})$$

*Assuming (VI)*, we consider the same kind of process as Blache in [Bla04a]. Let us denote by  $\mathbf{E}$  the process:

$$\mathbf{E}_t = \exp\left(\frac{\kappa^+(\mathbf{B})q(\mathbf{B})}{2} \int_0^t \langle dX_s, dX_s \rangle_g - \frac{1}{c} \int_0^t |dG(X_s) \cdot V_{X_s}(dX_s, dX_s)|\right).$$

The Itô formula and the statement (ii) give us:

$$\begin{aligned}
 & G(X_t)\mathbf{E}_t \\
 & \stackrel{m}{=} G(X_0) + \frac{1}{2} \left( \int_0^t \mathbf{E}_s \nabla dG(X_s)(dX_s, dX_s) + \kappa^+(\mathbf{B})q(\mathbf{B}) \int_0^t \mathbf{E}_s G(X_s) \langle dX_s, dX_s \rangle_g \right) \\
 & + \int_0^t \mathbf{E}_s \left( dG(X_s)V_{X_s}(dX_s, dX_s) - \frac{G(X_s)}{c} |dG(X_s) \cdot V_{X_s}(dX_s, dX_s)| \right) \\
 & \stackrel{m}{\leq} G(X_0).
 \end{aligned}$$

We obtain, using the upper and lower bounds on  $G$ :

$$\mathbf{E}(\mathbf{E}_t) = \mathbf{E} \left( \exp \left( \frac{\kappa^+(\mathbf{B})q(\mathbf{B})}{2} \int_0^t \langle dX_s, dX_s \rangle_g - \frac{1}{c} \int_0^t |dG(X_s) \cdot V_{X_s}(dX_s, dX_s)| \right) \right) \leq \frac{C}{c}.$$

Using the formula (II.1.13) and (V1), we obtain the following estimate:

$$|dG(X_s) \cdot V_{X_s}(dX_s, dX_s)| \leq \sqrt{\kappa^+(\mathbf{B})q(\mathbf{B})} |V_{X_s}(dX_s, dX_s)| \leq \mathbf{V}(\mathbf{B}) \sqrt{\kappa^+(\mathbf{B})q(\mathbf{B})} \langle dX_s, dX_s \rangle_g.$$

Finally, the following integrability property hold:

$$\mathbf{E} \left( \exp \left( \left[ \frac{\kappa^+(\mathbf{B})q(\mathbf{B})}{2} - \frac{\mathbf{V}(\mathbf{B})\sqrt{\kappa^+(\mathbf{B})q(\mathbf{B})}}{c} \right] \int_0^t \langle dX_s, dX_s \rangle_g \right) \right) \leq \frac{C}{c}.$$

Assuming (V2), let us consider  $\mathbf{F}$  the following process:

$$\mathbf{F}_t = \exp \left( \frac{\kappa^+(\mathbf{B})q(\mathbf{B})}{2} \int_0^t \langle dX_s, dX_s \rangle_g \right).$$

The Itô formula and the statement (ii) give us again:

$$G(X_t)\mathbf{F}_t \stackrel{m}{\leq} G(X_0) + \int_0^t \mathbf{F}_s dG(X_s)V_{X_s}(dX_s, dX_s). \quad (\text{II.1.14})$$

Using (V2) and (II.1.13) again, this inequality yields the following:

$$c\mathbf{E}(\mathbf{F}_t) \leq \mathbf{E}(G(X_t)\mathbf{F}_t) \leq C,$$

and thus the statement (v). □

- ◇ *Families of geodesics and variation formula* – Let  $(\gamma^a)_{a \in (-1,1)}$  a smooth family of  $\nabla$ -geodesics time indexed by  $[0, 1]$ , the map  $a \in (-1, 1) \mapsto \mathcal{E}(\gamma^a) = \int_0^1 |\partial_s \gamma^a(s)| ds$  is two times differentiable and we recall here the first order term.

We set  $\mathcal{J} := \partial_{|a=0} \gamma^a$  the associated Jacobi field, and assume that  $\mathcal{J}$  vanishes at  $s = 1$  (the end point of each geodesic  $\gamma^a$  is equal to  $\xi \in \mathcal{M}$ ).

**Proposition – II.1.5** (First variation of the energy). *The energy functional  $\mathcal{E}(\gamma^a)$  is differentiable with respect to  $a$  at zero, and:*

$$\partial|_{a=0}\mathcal{E}(\gamma^a) = -\langle \mathcal{J}(0), \partial|_{s=0}\gamma^0(s) \rangle_g.$$

Consequently, if  $x_0 \in \mathcal{M}$ , for all  $x$  joined with  $x_0$  by a unique geodesic, the map  $F : y \mapsto \rho^2(y, x_0)$  is differentiable at  $x$  and  $\text{grad}F(x) = -2\overrightarrow{xx_0}$ .

- ◇ *Estimations on the deformed stochastic transport* – Arnaudon, Li and Thalmaier establish in [ALT99], Proposition 2.8, an upper bound on  $|\Theta_{s,t}(Y)|$  for all  $(s, t) \in [0, 1]^2$  in terms of the exponential moments of  $X$ , under assumptions about the sectional curvatures. We are able to extend easily the proposition to the  $\nabla$ -martingales with prescribed drift  $V$ .

**Proposition – II.1.6.** *Let  $X$  be a  $\nabla$ -martingale in  $\mathcal{M}$  with prescribed drift  $V_X(dX, dX)$ . The geodesic transport along  $X$  is estimated in terms of the Riemannian quadratic variation of  $X$ :*

- (i) if  $\kappa_V^-(\mathcal{M}) < \infty$ , then  $|\Theta_{t,s}(X)| \leq \exp\left(\int_t^s \frac{(\kappa_V^-)_{X_u}}{2} \langle dX_u, dX_u \rangle_g\right)$ ,
- (ii) if  $\kappa_V^+(\mathcal{M}) < \infty$ , then  $|\Theta_{t,s}^{-1}(X)| \leq \exp\left(\int_t^s \frac{(\kappa_V^+)_{X_u}}{2} \langle dX_u, dX_u \rangle_g\right)$ .

In particular, if  $\int_0^1 \langle dX_u, dX_u \rangle_g \in L^\infty$ , then  $\Theta_{0,\cdot}(X) \in \mathcal{S}^\infty$  and:

$$\|\Theta_{0,\cdot}(X)\|_{\mathcal{S}^\infty} \leq \exp\left(\frac{\kappa_V^-(\mathcal{M})}{2} \left\| \int_0^1 \langle dX_u, dX_u \rangle_g \right\|_{L^\infty}\right).$$

Proof. Setting  $t \in [0, 1]$ , the Itô formula gives us:

$$d|\Theta_{t,\cdot}(X)|^2 = 2 \langle \Theta_{t,\cdot}(X), \mathbf{D}\Theta_{t,\cdot}(X) \rangle_g = - \langle \Theta_{t,\cdot}(X), R_V(\Theta_{t,\cdot}(X), dX) dX \rangle_g.$$

Using the  $V$ -sectional curvatures, we obtain:

$$-(\kappa_V^+)_{X|\Theta_{t,\cdot}(X)|^2} \langle dX, dX \rangle_g \leq d(|\Theta_{t,\cdot}(X)|^2) \leq (\kappa_V^-)_{X|\Theta_{t,\cdot}(X)|^2} \langle dX, dX \rangle_g,$$

and one has for all  $s \geq t$ :

$$\exp\left(\int_t^s -(\kappa_V^+)_{X_u} \langle dX_u, dX_u \rangle_g\right) \leq |\Theta_{t,s}(X)|^2 \leq \exp\left(\int_t^s (\kappa_V^-)_{X_u} \langle dX_u, dX_u \rangle_g\right).$$

□

- ◇ *Functionnal spaces* – Throughout this paper, Euclidean, Riemannian and subordinate norms will be denoted by  $|\cdot|$  while relatively to  $\omega$  and  $t$  by  $\|\cdot\|$ . For an  $\mathcal{F}$ -adapted continuous process



$J$  with values in  $T\mathcal{M}$ , and  $1 \leq p \leq \infty$ , let us define

$$\|J\|_{\mathcal{S}^p} = \mathbf{E} \left( \sup_{0 \leq s \leq 1} |J_s|^p \right)^{1/p}, \quad \|Y\|_{\mathcal{S}^\infty} = \text{esssup}_{0 \leq s \leq 1} |J_s|.$$

The notation  $p^*$  states for the real satisfying  $\frac{1}{p} + \frac{1}{p^*} = 1$ , with convention  $1^* = \infty$  and  $\infty^* = 1$ . For each field of bilinear forms  $b$  we define the  $b$ -norm  $\mathcal{H}^p$  of a  $\nabla$ -martingale  $Y$  by

$$\|Y\|_{b, \mathcal{H}^p} = \left\| \sqrt{\int_0^1 b(dY, dY)} \right\|_{L^p}.$$

- ◇ *Families of martingales* – Let  $I$  an open set of  $\mathbf{R}$ , we denote by  $(\text{Mart}_\nabla(\mathcal{M}))^I$  (resp.  $(\text{SemMart}(\mathcal{M}))^I$ ) the set of all continuous families indexed by  $I$  of  $\nabla$ -martingales on  $\mathcal{M}$  (resp. semimartingales in  $\mathcal{M}$ ), and  $(\text{Mart}_\nabla(\mathcal{M}))^{I, \mathcal{C}^1}$  (resp.  $(\text{SemMart}(\mathcal{M}))^{I, \mathcal{C}^1}$ ) the set of all  $\mathcal{C}^1$ -families of  $\nabla$ -martingales (resp. semimartingales)  $(X(a))_{a \in I}$  with respect to  $a$  and the topology of semimartingales (see [AT98b] for more details about these topologies on manifolds). For the reader convenience, we recall some results about families of differentiable martingales. In [AT98a], Arnaudon and Thalmaier show an explicit characterisation of the  $\nabla^c$ -martingales in  $T\mathcal{M}$  amongst the semimartingales.

**Proposition – II.1.7** ( $\nabla^c$ -martingales on  $T\mathcal{M}$ , Theorem 4.12. of [AT98a]). *Let  $J$  a continuous semimartingale with values in  $T\mathcal{M}$ . Then  $J$  is  $\nabla^c$ -martingale if and only if*

- (i)  $X = \pi(J)$  is a  $\nabla$ -martingale,
- (ii)  $\Theta_{0,\cdot}^{-1}(X)J$  is vectorial local martingale in  $T_{X_0}\mathcal{M}$ .

The torsion-free property of  $\nabla$  is essential for this proposition. Besides, the Corollary 4.3. in [AT98b] claims a stochastic version of the fact that each Jacobi field can be seen as a variation of geodesics.

**Proposition – II.1.8.** *Let  $J$  a continuous semimartingale with values in  $T\mathcal{M}$  and lifetime  $\zeta^J$ . Then  $J$  is a  $\nabla^c$ -martingale if and only if there exists a  $\mathcal{C}^1$ -family  $(X(a))_{a \in (-1,1)}$  of  $\nabla$ -martingales with family of lifetimes  $(\zeta(a))_{a \in (-1,1)}$ , such that  $J = \partial|_{a=0}X(a)$  is fulfilled.*

*In addition,  $\zeta(a) \wedge \zeta^J$  converges in probability to  $\zeta(0)$  when  $a$  goes to zero.*

Combining the both propositions, we obtain that  $\Theta_{0,\cdot}^{-1}(X(a))\partial_a X(a)$  is a vectorial local martingale if  $X(\cdot) \in (\text{Mart}_\nabla(\mathcal{M}))^{I, \mathcal{C}^1}$ . This statement can be easily extended to the families of  $\nabla$ -martingales with prescribed drift  $V$ , the aim of the following proposition.

**Proposition – II.1.9.** *Let  $(X(a))_{a \in [0,1]} \in (\text{SemMart}(\mathcal{M}))^{I, \mathcal{C}^1}$ , a family with prescribed family of drifts  $(V(a))_{a \in [0,1]}$ .*

Then  $(\Theta_{0,\cdot})(X(a))^{-1}\partial_a X(a)$  is a vectorial local martingale in  $T_{X_0(a)}\mathcal{M}$  for all  $a$ , and:

$$\mathbf{D}^{def}\partial_a X(a) = \Theta_{0,\cdot}(X(a)) \mathbf{d} \left( \Theta_{0,\cdot}^{-1}(X(a))\partial_a X(a) \right) = \nabla_a \mathbf{d}_{mart}X(a).$$

Proof. First, we can compute easily the covariant differential equation of  $\Theta_{0,\cdot}^{-1}(X(a))$ :

$$\mathbf{D}\Theta_{0,\cdot}^{-1}(X(a)) = \frac{1}{2}R_V(a) \left( \Theta_{0,\cdot}^{-1}(X(a)), dX(a) \right) dX(a). \quad (\text{II.1.15})$$

Since  $\Theta_{0,\cdot}^{-1}(X(a)) \parallel_{0,\cdot}(X(a))$  has finite variations, the Ito formula gives us

$$\begin{aligned} & \mathbf{d} \left( \Theta_{0,\cdot}^{-1}(X(a)) \parallel_{0,\cdot}(X(a)) \parallel_{0,\cdot}^{-1}(X(a))\partial_a X(a) \right) \\ &= -\Theta_{0,\cdot}^{-1}(X(a)) (\mathbf{D}\Theta_{0,\cdot}(X(a)) \Theta_{0,\cdot}^{-1}(X(a))\partial_a X(a) + \Theta_{0,\cdot}^{-1}(X(a))\mathbf{D}\partial_a X(a)), \end{aligned}$$

according to [Proposition – II.1.10](#) (see below), we have

$$\mathbf{D}\partial_a X(a) = \nabla_a dX(a) - \frac{1}{2}R(\partial_a X(a), dX(a)) dX(a),$$

and one has

$$\begin{aligned} & \mathbf{d} \left( \Theta_{0,\cdot}^{-1}(X(a)) \parallel_{0,\cdot}(X(a)) \parallel_{0,\cdot}^{-1}(X(a))\partial_a X(a) \right) \\ &= -\Theta_{0,\cdot}^{-1}(X(a)) \left( -\frac{1}{2}R_V(a) (\Theta_{0,\cdot}(X(a)), dX(a)) dX(a) \right) \Theta_{0,\cdot}^{-1}(X(a))\partial_a X(a) \\ &+ \Theta_{0,\cdot}^{-1}(X(a))\nabla_a dX(a) - \frac{1}{2}\Theta_{0,\cdot}^{-1}(X(a))R(\partial_a X(a), dX(a)) dX(a). \end{aligned}$$

Using the link between  $R_V$  and  $R$ , we obtain that the finite variation part is equal to zero. The first claim is thus established, together with the formula:

$$\mathbf{D}^{def}\partial_a X(a) = \nabla_a \mathbf{d}_{mart}X(a).$$

□

If  $J$  is a  $\nabla^c$ -martingale above a martingale  $X$ , then  $J$  has the form  $J_t = \Theta_{0,t}(X)E_t$  where  $E_t$  is a  $T_{X_0}\mathcal{M}$ -valued local martingale. If moreover

$$\mathbf{E} \left( \sup_{t \in [0,1]} |E_t|^2 \right) = \mathbf{E} \left( \sup_{t \in [0,1]} \left| \Theta_{0,t}^{-1}(X)J_t \right|^2 \right) < \infty, \quad (\text{II.1.16})$$

we deduce from the martingale property and  $\mathcal{F}$ -adaptability of  $\Theta_{0,\cdot}(X)$  that

$$\Theta_{0,t}^{-1}(X)J_t = \mathbf{E}\left(\Theta_{0,1}^{-1}(X)J_1 \middle| \mathcal{F}_t\right) \quad \text{a.s.} \implies J_t = \mathbf{E}\left(\Theta_{t,1}^{-1}(X)J_1 \middle| \mathcal{F}_t\right) \quad \text{a.s.} \quad (\text{II.1.17})$$

The same expression can be obtained using  $\partial_a X(a)$  instead of  $J(a)$  where  $X(a)$  is a family of  $\nabla$ -martingales with prescribed drift  $V(a) = V_{X(a)}(dX(a), dX(a))$ .

To achieve this subpart we recall two other results given stochastic differential equations satisfied by  $\partial_a X(a)$  and  $J(a)$ , where  $X(a)$  and  $J(a)$  will state for two  $\mathcal{C}^1$ -families of respectively  $\nabla$ -martingales in  $\mathcal{M}$ , and semimartingales in  $T\mathcal{M}$ .

The first one comes from equation (4.7) of [AT03]. If  $\nabla$  is a torsion-free connection, one has for all semimartingale  $J$  in  $T\mathcal{M}$ :

$$d^{\nabla^c} J = d^{\nabla^h} J + \frac{1}{2} v_J R(J, dX) dX. \quad (\text{II.1.18})$$

Taking the vertical part of (II.1.18) and using the definition  $\mathbf{D}J = v_J^{-1} \left[ (d^{\nabla^h} J)^{\text{vert}} \right]$ , we obtain

$$\mathbf{D}J = v_J^{-1} \left( (d^{\nabla^c} J)^{\text{vert}} \right) - \frac{1}{2} R(J, dX) dX. \quad (\text{II.1.19})$$

Taking particularly a family  $J(a) = \partial_a X(a)$ , the first term of the right member of (II.1.19) is equal to  $\nabla_a d^{\nabla} X(a)$ . The second result claims a commutation formula between the stochastic covariant derivative  $\mathbf{D}$  and the deterministic covariant derivative  $\nabla_a$  ([AT03], Theorem 4.5.). We gather the both propositions in the following result.

**Proposition – II.1.10.** *Let  $(J(a))_{a \in [0,1]} \in (\text{SemMart}(\mathcal{M}))^{I, \mathcal{C}^1}$ ,  $X(a) = \pi(J(a))$ . Then*

$$\begin{aligned} \mathbf{D}\partial_a X(a) &= \nabla_a d^{\nabla} X(a) - \frac{1}{2} R(\partial_a X(a), dX(a)) dX(a) \\ \mathbf{D}\nabla_a J(a) &= \nabla_a \mathbf{D}J(a) + R\left(d^{\nabla} X(a), \partial_a X(a)\right) J(a) - \frac{1}{2} R(\mathbf{D}\partial_a X(a), dX(a)) J(a) \\ &\quad + R(dX(a), \partial_a X(a)) \mathbf{D}J(a) - \frac{1}{2} \nabla_{dX(a)} R(\partial_a X(a), dX(a)) J(a). \end{aligned}$$

Let us emphasize that  $\nabla_a$  is a shortcut for the notation  $\nabla_{\partial_a X(a)}$ . Hence, for the sake of clarity, let us note that  $\nabla_a \parallel_{0,a}(X(\cdot)) = 0$ , but generally  $\nabla_a \parallel_{X(0), X(a)} \neq 0$  (but it is true if  $a \mapsto X_t(a)$  is for all  $t$  a  $\nabla$ -geodesic in  $\mathcal{M}$  for example).

## II.2 PRELIMINARY RESULTS

### § II.2.1. About a few stochastic inequalities

Let  $p > 1$ . Generally, a Burkholder-Davis-Gundy inequality claims that the norms  $\mathcal{S}^p$  and  $\mathcal{H}^p$  are equivalent on the space of local vectorial martingales vanishing at time zero, where the spaces  $\mathcal{H}^p$  and  $\mathcal{S}^p$  have been defined in the introduction.

**Burkholder-Davis-Gundy inequalities on  $\mathbf{R}^d$ .** Let  $p > 1$ . There exists two universal constants  $C_p$  and  $C'_p$  such that for all real continuous local martingale  $X$  vanishing at zero,

$$\|X\|_{\mathcal{H}^p} \leq C_p \|M\|_{\mathcal{S}^p}, \quad (\text{II.2.1})$$

and

$$\|X\|_{\mathcal{S}^p} \leq C'_p \|M\|_{\mathcal{H}^p}. \quad (\text{II.2.2})$$

We can choose constants valid for every dimension. In [MR16], Marinelli and Röckner consider the general framework of martingales with values in a separable Hilbert space. Particularly the upper constant  $C'$  defined below is valid for all the dimensions:

$$C'_p = \begin{cases} \left(\frac{p}{p-1}\right)^{\frac{p}{2}} \left(\frac{p(p-1)}{2}\right)^2 & \text{if } p > 2, \\ 4\sqrt{\frac{2}{p}} & \text{if } p < 2, \\ 4 & \text{if } p = 2. \end{cases} \quad (\text{II.2.3})$$

The first two cases are shown in [MR16] (Proposition 2.1 and Proposition 3.1), and the third one is the constant obtained in the scalar case which still valid in the general case.

**Burkholder-Davis-Gundy inequality on manifolds.** As mentioned in the introduction we deal with a manifold  $\mathcal{M}$  of dimension  $d$  equipped by a linear connection  $\nabla$ , and let  $X$  a  $\nabla$ -martingale in  $\mathcal{M}$ . When the stochastic integral is a vector in an Euclidean space  $(E, \mathbf{g})$ , that is to say  $\int_0^\cdot \alpha(d^\nabla X)$  where  $\alpha : T\mathcal{M} \rightarrow (E, \mathbf{g})$  such that  $\alpha|_{T_x\mathcal{M}}$  is linear for all  $x$ , we have the following statement.

**Proposition – II.2.1.** *There exists two constants  $D_{d,p}$  and  $D'_{d,p}$  such that for all  $\nabla$ -martingale  $X$  and  $\alpha : T\mathcal{M} \rightarrow (E, \mathbf{g})$  above  $X$  satisfying*

$$-\infty < \inf_{s \in [0,1]} |\alpha_s|_{\mathbf{g}} = \gamma \leq \beta = \sup_{s \in [0,1]} |\alpha_s|_{\mathbf{g}} < \infty \quad \text{a.s.},$$

the process  $\int_0^\cdot \alpha(d^\nabla X) = \left( \int_0^\cdot \alpha^1(d^\nabla X), \dots, \int_0^\cdot \alpha^d(d^\nabla X) \right)$  having values in  $E$  satisfies

$$D_{d,p} \left\| \sqrt{\int_0^1 \gamma^2 \langle dX, dX \rangle_{\mathbf{g}}} \right\|_{L^p} \leq \left\| \int_0^\cdot \alpha(d^\nabla X) \right\|_{\mathcal{S}^p} \leq D'_{d,p} \left\| \sqrt{\int_0^1 \beta^2 \langle dX, dX \rangle_{\mathbf{g}}} \right\|_{L^p}.$$

**Norm  $\mathcal{S}^p$  of a  $\nabla$ -stochastic integral.** Let us recall an inequality due to Emery in [Eme78], Proposition 1, giving an estimate on the  $\mathcal{S}^p$  norms of the stochastic integrals, it is a consequence of the BDG inequality. We extend this statement plainly to the stochastic integrals on a manifold.

**Proposition – II.2.2.** Let  $(p, q, r) \in [1, \infty]^2 \times [1, \infty)$  satisfying  $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$  and  $M \in \mathcal{S}^p(\mathbf{R}^{d \times d})$ ,  $X \in \mathcal{H}^q(\mathbf{R}^d)$  two semimartingales. The stochastic integral  $\int_0^\cdot M dX$  is in  $\mathcal{S}^r(\mathbf{R}^d)$  and:

$$\left\| \int_0^\cdot M dX \right\|_{\mathcal{S}^r} \leq C'_r \|M\|_{\mathcal{S}^p} \|X\|_{\mathcal{H}^q}.$$

When the stochastic integral is a vector in an Euclidean space  $(E, \mathbf{g})$ , that is to say  $\alpha : T\mathcal{M} \rightarrow (E, \mathbf{g})$  such that  $\alpha|_{T_x\mathcal{M}}$  is linear for all  $x$ , we have the following statement.

**Corollary – II.2.1.** Let  $(p, q, r) \in [1, \infty]^2 \times [1, \infty)$  satisfying  $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ , there exists a constant  $D''_{d,p}$  such that for all  $\nabla$ -martingale  $X$ , and  $\alpha : T\mathcal{M} \rightarrow (E, \mathbf{g})$  above  $X$  locally bounded,  $\int_0^\cdot \alpha(d^\nabla X) = \left( \int_0^\cdot \alpha^1(d^\nabla Y), \dots, \int_0^\cdot \alpha^d(d^\nabla Y) \right)$  having values in  $E$  satisfies

$$\left\| \int_0^\cdot \alpha(d^\nabla X) \right\|_{\mathcal{S}^r} \leq D''_{d,r} \|\alpha\|_{\mathcal{S}^p} \left\| \sqrt{\int_0^1 \langle dX, dX \rangle_{\mathbf{g}}} \right\|_{L^q}.$$

**Kunita-Watanabe inequality.** The manifold  $\mathcal{M}$  is assumed to be equipped by a tensor  $\mathbf{b} : T^*\mathcal{M} \otimes T^*\mathcal{M} \rightarrow (E, \mathbf{g})$  where  $(E, \mathbf{g})$  is again an Euclidean space. As for the usual vectorial quadratic variation, the following Kunita-Watanabe inequality holds true.

**Proposition – II.2.3.** Let  $X$  a semimartingale and  $J$  a semimartingale above  $X$ . Then for all  $p, q, r \geq 1$  satisfying  $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$  we have:

$$\left\| \int_0^\cdot |\langle dX, \mathbf{D}J \rangle_{\mathbf{b}}| \right\|_{L^r} \leq \left\| \sqrt{\int_0^\cdot |\langle dX, dX \rangle_{\mathbf{b}}|} \right\|_{L^p} \left\| \sqrt{\int_0^\cdot |\langle \mathbf{D}J, \mathbf{D}J \rangle_{\mathbf{b}}|} \right\|_{L^q}.$$

Proof. The Euclidean case can be used by considering the antidevelopment of  $X$ . First, we write

$$\langle dX, \mathbf{D}J \rangle_{\mathbf{b}} = \left\langle \int_{0,\cdot}^{-1}(X) dX, \int_{0,\cdot}^{-1}(X) \mathbf{D}J \right\rangle_{\mathbf{b}} = \left\langle d\mathcal{A}(X), d \left( \int_{0,\cdot}^{-1}(X) J \right) \right\rangle_{\mathbf{b}}.$$

Writing the last term in local coordinates in  $T_{X_0}\mathcal{M}$ , and using the classical scalaire version of the Kunita-Watanabe inequality, we obtain:

$$\left\| \int_0^\cdot |\langle dX, \mathbf{D}J \rangle_{\mathbf{b}}| \right\|_{L^r} \leq \left\| \sqrt{\int_0^\cdot |\langle d\mathcal{A}(X), d\mathcal{A}(X) \rangle|} \right\|_{L^p} \left\| \sqrt{\int_0^\cdot |\langle d \left( \int_{0,\cdot}^{-1}(X) J \right), d \left( \int_{0,\cdot}^{-1}(X) J \right) \rangle|} \right\|_{L^q},$$

and the claim follows by using again the isometry property of  $\int_{0,\cdot}^{-1}(X)$ .  $\square$

### § II.2.2. Localization in $\mathcal{H}^\infty$

In this subpart we use the notations of [Proposition – II.1.4](#). We give a result about approximations of  $\nabla$ -martingales with prescribed drift in  $\mathcal{H}^\infty$  taking values in a small set. Let us consider  $V \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes T\mathcal{M})$ .

**Proposition – II.2.4** (Approximation in  $\mathcal{H}^\infty$ ). *Let  $\mathbf{B}$  a regular geodesic ball with radius  $\mathbf{R} > 0$ , center  $\mathbf{O}_\mathbf{B}$ , such that  $\kappa_V^+(\mathbf{B}) < \infty$ .*

*Let a point  $\xi_\mathbf{B} \in \mathbf{B}$  and  $X$  a  $\nabla$ -martingale with drift  $V$ , satisfying [\(V1\)](#) or [\(V2\)](#), taking values in  $\mathbf{B}$ . Let  $J$  and  $Z$  the following processes:*

$$J_t = \mathbf{E} \left( \Theta_{t,1}^{-1}(X) \overrightarrow{X_1 \xi_\mathbf{B}} \middle| \mathcal{F}_t \right), \quad Z_t = \Theta_{0,t}^{-1}(X) J_t.$$

Let us set  $\varepsilon > 0$  and  $r > 1$ . Then there exists:

- a semimartingale  $X^\varepsilon$  with values in  $\mathbf{B}$ , a constant  $\mathbf{R}(\kappa^+(\mathbf{B}), q(\mathbf{B}), r, c, C, \varepsilon) > 0$ ,
- a semimartingale  $J^\varepsilon$  above  $X^\varepsilon$  with values in  $T\mathbf{B}$ ,
- a local martingale  $Z^\varepsilon = \Theta_{0,\cdot}^{-1}(X^\varepsilon) J^\varepsilon$  with values in  $T_{X_0}\mathbf{B}$ , two constants  $\mathbf{Q}(r, \varepsilon, c, C, \mathbf{B}, d) > 0$ ,  $\mathbf{S}(\mathbf{B}, r) > 0$ ,

such that the following statements hold:

- (i)  $\left\| \int_0^1 \langle dX^\varepsilon, dX^\varepsilon \rangle_g \right\|_{L^\infty} \leq \mathbf{R}(\kappa^+(\mathbf{B}), q(\mathbf{B}), r, c, C, \varepsilon), \quad \mathbf{E}(\rho^r(X_1, X_1^\varepsilon)) \leq \varepsilon,$
- (ii)  $\left\| \int_0^1 \langle dZ^\varepsilon, dZ^\varepsilon \rangle_g \right\|_{L^\infty} \leq \mathbf{Q}(r, \varepsilon, c, C, \mathbf{B}, d),$
- (iii)  $\left\| J_1 - \int_{X_1^\varepsilon, X_1} J_1^\varepsilon \right\|_{L^r(\Omega)} \leq \varepsilon \times \mathbf{S}(\mathbf{B}, r).$

Moreover, if  $V = 0$ , the process  $X^\varepsilon$  is a  $\nabla$ -martingale, and  $J^\varepsilon$  is a  $\nabla^c$ -martingale.

Proof. Taking into account [Proposition – II.1.4](#) together with [\(V1\)](#) or [\(V2\)](#), there exists a non-negative constant  $\lambda$  such that:

$$\mathcal{E}_\lambda := \mathbf{E} \left( \exp \left( \lambda \int_0^1 \langle dX, dX \rangle_g \right) \right) < \infty.$$

Moreover the constant  $\lambda$  depends only on  $\mathbf{B}$ , and eventually on  $\mathbf{V}(\mathbf{B})$ .

**Step 1 – Localization of  $X$ .** Let us set  $\tau^D = \inf \left\{ t \geq 0 : \int_0^t \langle dX, dX \rangle_g \geq D \right\}$  for all  $D > 0$  and set  $X' = X^{\lfloor \tau^D \rfloor}$ . The process  $X'$  has a quadratic variation less than or equal to  $D$ . Let us show the statement [\(i\)](#). One has

$$\mathbf{E} (\rho^r(X_1, X'_1)) = \mathbf{E} \left( \mathbb{1}_{\left\{ \int_0^1 \langle dX, dX \rangle_g \geq D \right\}} \rho^r(X_1, X'_1) \right) + \mathbf{E} \left( \mathbb{1}_{\left\{ \int_0^1 \langle dX, dX \rangle_g < D \right\}} \rho^r(X_1, X'_1) \right),$$

and on the set  $\left\{ \int_0^1 \langle dX, dX \rangle_g < D \right\}$  we have  $X_1 = X'_1$ . Consequently, the following estimation holds true by using the Markov inequality:

$$\mathbf{E} (\rho^r(X_1, X'_1)) \leq (2\mathbf{R})^r e^{-D\lambda} \mathcal{E}_\lambda. \quad (\text{II.2.4})$$

According to [Proposition – II.1.4](#), we have  $\mathcal{E}_\lambda \leq \frac{C}{c}$  where  $c$  and  $C$  are defined in this proposition.

We obtain the statement by choosing  $D$  as the smallest real satisfying:  $\frac{2^r \mathbf{R}^r C}{c} e^{-D\lambda} \leq \varepsilon$ . Letting  $D = \mathbf{R}(\kappa^+(\mathbf{B}), q(\mathbf{B}), r, c, C, \varepsilon)$  and  $X^\varepsilon = X'$  with the previous choice of  $D$ , we get the first claim.

**Step 2 – Localization of  $J$  and  $Z$ .** Let us define  $\Theta = \Theta(X)$ ,  $\Theta^\varepsilon = \Theta(X^\varepsilon)$  the geodesic transports along  $X$ ,  $X^\varepsilon$  and  $M_t^\varepsilon = \mathbf{E} \left( \left( \Theta_{t,1}^\varepsilon \right)^{-1} \overrightarrow{X_1^\varepsilon \xi_{\mathbf{B}}} \middle| \mathcal{F}_t \right)$ . We have:

$$\mathbf{E} \left( \left| J_1 - \left\|_{X_1^\varepsilon, X_1} M_1^\varepsilon \right\|^r \right) \right) = \mathbf{E} \left( \left| \overrightarrow{X_1 \xi_{\mathbf{B}}} - \left\|_{X_1^\varepsilon, X_1} \overrightarrow{X_1^\varepsilon \xi_{\mathbf{B}}} \right\|^r \right) \right).$$

Using [Proposition – II.1.3](#), there exists a constant  $L_{\mathbf{B}}$  depending only on  $\mathbf{B}$  such that:

$$\left\| \left\| J_1 - \left\|_{X_1^\varepsilon, X_1} M_1^\varepsilon \right\|^r \right\|_{L^r} \leq L_{\mathbf{B}}^r \times \mathbf{E} (\rho^r(X_1, X_1^\varepsilon)) \leq L_{\mathbf{B}}^r \times \varepsilon. \quad (\text{II.2.5})$$

Let  $D > 0$ , we define also  $Z'_t = \left( \Theta_{0,t}^{\varepsilon, -1} M_t^\varepsilon \right)^{\lfloor \sigma^D \rfloor} \in T_{X_0^\varepsilon} \mathcal{M}$ , and  $J'_t = \Theta_{0,t}^\varepsilon Z'_t \in T_{X_t^\varepsilon} \mathcal{M}$  for all  $t$ , with

$$\sigma^D = \inf \left\{ t \geq 0 : \int_0^t \left\langle d \left( \Theta_{0,\cdot}^{\varepsilon, -1} M^\varepsilon \right), d \left( \Theta_{0,\cdot}^{\varepsilon, -1} M^\varepsilon \right) \right\rangle_g \geq D \right\}.$$

One has, using the same method as above,

$$\begin{aligned}
 & \|J'_1 - M_1^\varepsilon\|_{L^r(\Omega)}^r \\
 &= \mathbf{E} \left( \mathbf{1}_{\left\{ \int_0^1 \langle d(\Theta_{0,\cdot}^{\varepsilon,-1} M^\varepsilon), d(\Theta_{0,\cdot}^{\varepsilon,-1} M^\varepsilon) \rangle_g \geq D \right\}} |J'_1 - M_1^\varepsilon|^r \right) + \mathbf{E} \left( \mathbf{1}_{\left\{ \int_0^1 \langle d(\Theta_{0,\cdot}^{\varepsilon,-1} M^\varepsilon), d(\Theta_{0,\cdot}^{\varepsilon,-1} M^\varepsilon) \rangle_g < D \right\}} |J'_1 - M_1^\varepsilon|^r \right) \\
 &= \mathbf{E} \left( \mathbf{1}_{\left\{ \int_0^1 \langle d(\Theta_{0,\cdot}^{\varepsilon,-1} M^\varepsilon), d(\Theta_{0,\cdot}^{\varepsilon,-1} M^\varepsilon) \rangle_g \geq D \right\}} |J'_1 - M_1^\varepsilon|^r \right). \tag{II.2.6}
 \end{aligned}$$

Since  $M_1^\varepsilon = \overline{X_1^\varepsilon \xi_{\mathbf{B}}}$ , the random variable  $J'_1$  is essentially bounded, and there exists a constant  $K$  depending on  $r, \kappa_V^+(\mathbf{B}), q(\mathbf{B}), r, c, C, \mathbf{R}$  and  $\varepsilon$  such that:

$$\begin{aligned}
 \|J'_1 - M_1^\varepsilon\|_{L^r(\Omega)}^r &\leq K \times \mathbf{P} \left( \int_0^1 \langle d(\Theta_{0,\cdot}^{\varepsilon,-1} M^\varepsilon), d(\Theta_{0,\cdot}^{\varepsilon,-1} M^\varepsilon) \rangle_g \geq D \right) \\
 &\leq \frac{K}{D^r} \mathbf{E} \left( \left| \int_0^1 \langle d(\Theta_{0,\cdot}^{\varepsilon,-1} M^\varepsilon), d(\Theta_{0,\cdot}^{\varepsilon,-1} M^\varepsilon) \rangle_g \right|^r \right). \tag{II.2.7}
 \end{aligned}$$

Let us show that the last expectation is finite. We have

$$\left| \Theta_{0,t}^{\varepsilon,-1} M_t^\varepsilon \right| \leq \mathbf{E} \left( \left| \Theta_{0,1}^{\varepsilon,-1} \overline{X_1^\varepsilon \xi_{\mathbf{B}}} \right| \middle| \mathcal{F}_t \right),$$

since  $\Theta_{0,\cdot}^{\varepsilon,-1} M^\varepsilon = \mathbf{E} \left( \left( \Theta_{0,1}^{\varepsilon,-1} \overline{X_1^\varepsilon \xi_{\mathbf{B}}} \right) \middle| \mathcal{F}_\cdot \right)$  is a vectorial martingale. In addition, we have the following estimation of  $\Theta_{0,1}^{\varepsilon,-1}$  according to [Proposition – II.1.6](#):

$$\left| \Theta_{0,1}^{\varepsilon,-1} \right| \leq \exp \left( \frac{\kappa_V^+(\mathbf{B})}{2} \int_0^1 \langle dX^\varepsilon, dX^\varepsilon \rangle_g \right) \leq \exp \left( \frac{\kappa_V^+(\mathbf{B})}{2} \times \mathbf{R}(\mathbf{B}, r, c, C, \varepsilon) \right).$$

Finally,  $\Theta_{0,\cdot}^{\varepsilon,-1} M^\varepsilon$  is a vectorial martingale in  $\mathcal{S}^\infty$  and is consequently in  $\mathcal{S}^p$  for all  $p > 1$ . According to the Burkholder-Davis-Gundy inequality ([II.2.2](#)), we get:

$$\mathbf{E} \left( \left| \int_0^1 \langle d(\Theta_{0,\cdot}^{\varepsilon,-1} M^\varepsilon), d(\Theta_{0,\cdot}^{\varepsilon,-1} M^\varepsilon) \rangle_g \right|^r \right) < \infty,$$

with an expectation depending only on the dimension  $d, \mathbf{R}(\mathbf{B}, r, c, C, \varepsilon)$  and  $\mathbf{R}$ . Combining the inequalities ([II.2.5](#)) and ([II.2.7](#)) with  $D$  large enough (depending on  $\varepsilon$ ), we obtain finally a constant  $\mathbf{S}(\mathbf{B}, r)$  such that:

$$\left\| J_1 - \overline{X_1^\varepsilon \xi_{\mathbf{B}}} \right\|_{L^r}^r \leq \varepsilon \times \mathbf{S}(\mathbf{B}, r).$$

The smallest choice of  $D$  satisfying [\(iii\)](#) will be denoted by  $\mathbf{Q}(r, \varepsilon, c, C, \mathbf{B}, d)$ , and  $Z^\varepsilon = Z', J^\varepsilon = J'$  are the processes  $Z'$  and  $J'$  corresponding to  $D = \mathbf{Q}(r, \varepsilon, c, C, \mathbf{B}, d)$ .  $\square$



§ II.2.3. A result of Emery about the linear stochastic differential equations

Let the following stochastic differential equation driven by a general semimartingale  $Y$  on  $\mathbf{R}^d$ :

$$dQ_t = F(Q)_t dY_t + dH_t, \quad (\text{II.2.8})$$

where  $Q, H$  are two vectorial c.à.d.l.à.g. and adapted processes in  $\mathbf{R}^{d \times d}$ , and  $Y$  is a semimartingale with values in  $\mathbf{R}^d$ . The map  $F$  is assumed to satisfy:

- (Lip(F))** (i) for all optional stopping time  $T$ , we have  
 $X^{\lrcorner T} = Y^{\lrcorner T} \implies F(X)^{\lrcorner T} = F(Y)^{\lrcorner T}$ ,  
(ii) there exists  $L \in \mathbf{R}$  such that for all  $t$ ,  $\sup_{s \leq t} |F(Y)_s - F(X)_s| \leq L \times \sup_{s \leq t} |Y_s - X_s|$ .

**Proposition – II.2.5** (Emery, [Eme78] Lemma 3). *Let us assume that  $H \in \mathcal{S}^p$ ,  $F$  satisfies (Lip(F)) with  $F(0) = 0$  and  $Y$  is in  $\mathcal{H}^\infty$  with*

$$\|Y\|_{\mathcal{H}^\infty} \leq \frac{L}{2} C'_p, \quad (\text{II.2.9})$$

where  $C'_p$  is a constant defined in (II.2.3). Then the equation (II.2.8) admits a unique solution in  $\mathcal{S}^p$  satisfying

$$\|Q\|_{\mathcal{S}^p} \leq b(L, \|Y\|_{\mathcal{H}^\infty}) \|H\|_{\mathcal{S}^p},$$

where  $b(L, \|Y\|_{\mathcal{H}^\infty})$  depends only on  $L$  and  $\|Y\|_{\mathcal{H}^\infty}$ .

### II.3 MAIN RESULTS

In the subpart II.1.4, we have seen that there are two ways to understand the  $\nabla^c$ -martingales. On the one hand they are the derivatives of the  $\mathcal{C}^1$ -families of  $\nabla$ -martingales. On the other hand the geodesic transports of vectorial martingales above the  $\nabla$ -martingales. Let  $\xi$  a random variable in  $\mathcal{M}$ , and  $(\xi(a))_{a \in [0,1]}$  a smooth random curve in  $\mathcal{M}$  such that  $\xi(1) = \xi$ . Taking into account the equation (II.1.17), our motivation is to consider the following differential equation with respect to  $a$ , and unknown  $(X(a))_{a \in [0,1]} \in (\text{Mart}_{\nabla}(\mathcal{M}))^{[0,1], \mathcal{C}^1}$ :

$$\forall t \in [0, 1], \quad \partial_a X_t(a) = \mathbf{E} \left( \Theta_{t,1}^{-1}(X(a)) \partial_a \xi(a) \middle| \mathcal{F}_t \right), \quad (\text{II.3.1})$$

and particularly:

$$\begin{cases} X(a) & \text{is a } \nabla\text{-martingale,} \\ X_1(a) & = \xi(a). \end{cases}$$

*Remark – II.3.1.* Particularly if we find a family of processes satisfying the equality (II.3.1), by taking  $a = 1$  we obtain a  $\nabla$ -martingale with prescribed terminal value  $\xi$ .

Because of the lack of vector or manifold structure on the set of martingales, we do not have general theorems as the so-called Cauchy-Lipschitz one, and the existence of a solution becomes non-trivial.

#### Sketch of the procedure.

- (1) Construction of a sequence  $(Y^n)_n$  of  $\nabla$ -martingales with terminal values converging to  $\tilde{\xi}$  lying in a small compact subset of  $\mathcal{M}$  ( $\mathcal{B}$  in the sequel).
- (2) Show the existence of  $Y = \lim_{n \rightarrow \infty} Y^n$  for the topology of semimartingales with the help of a convexity property, and prove that  $Y_1 = \tilde{\xi}$  almost surely.
- (3) Start the algorithm (step (1) and (2)) with  $\tilde{\xi} \leftarrow \xi(a)$  for all  $a \in [0, 1]$ , giving a family  $(Y(a))_{a \in [0,1]}$  of  $\nabla^c$ -martingales with terminal value  $\xi(a)$ .
- (4) Show that for all  $t$ , the map  $a \mapsto Y_t(a)$  is  $\mathcal{C}^1$ . By recalling that  $\Theta_{0,t}^{-1}(Y(a)) \partial_a Y(a)$  is an Euclidean local martingale, under integrability conditions, the family  $(Y(a))_{a \in [0,1]}$  will be a solution of (II.3.1) as already noted in equation (II.1.17).

The sequence of step (2) will be based on a shooting procedure towards the expected terminal value  $\tilde{\xi}$ , inside a small regular geodesic ball satisfying the following statements.

**Assumption on the terminal value.** Let  $\mathcal{B}$  a regular geodesic ball of radius  $R(\mathcal{B})$  and center  $O_{\mathcal{B}}$  satisfying

- (RegBall, Algo) (i)  $\tilde{\xi} \in \mathcal{B}$  almost surely,
- (ii)  $R(\mathcal{B}) \leq \frac{\pi}{2\sqrt{\kappa^+(\mathcal{B})q}}$  with  $q > 1$  such that  $q^* \in 2\mathbf{N}$ , and  $\kappa^-(\mathcal{B}) < \infty$ .

Particularly, the assumption on the cut locus of  $\mathcal{B}$  mentioned in Definition – II.1.1 make sense to the parallel transports between two points of  $\mathcal{B}$ , and the velocity vectors  $\vec{xy}$ . In addition, since  $\mathcal{B}$  is a relatively compact domain of  $\mathcal{M}$ , one has:

$$R^*(\mathcal{B}) := \sup_{y \in \mathcal{B}} |R(y)| < \infty, \quad (\nabla R)^*(\mathcal{B}) := \sup_{y \in \mathcal{B}} |\nabla R(y)| < \infty,$$

where  $|R(y)|$  and  $|\nabla R(y)|$  are defined in (II.1.2) and (II.1.3).

In the sequel we are going to give a procedure to build the sequence  $(Y^n)_{n \in \mathbf{N}}$ . We will introduce a family of  $\nabla$ -martingales  $X(a)$ ,  $a \geq 0$ , defined piece by piece in  $a$  with a size of pieces depending on two additional parameters  $h$  and  $\varepsilon$ . After, we will be able to find a sequence  $(a_n^h)_n$  of non-negative reals such that  $Y = X(a_n^h)$  be a solution of Step (2).

### § II.3.1. A stochastic algorithm obtained with parallel coupling

#### STARTUP AT $a = 0$ .

Let us consider  $\tilde{\xi}^0 \in \mathcal{B}$  deterministic. We define  $X(0)$  as the almost surely constant  $\nabla$ -martingale equaled to  $\tilde{\xi}^0$ ,  $J(0)$  the  $\nabla^c$ -martingale in  $T\mathcal{M}$  with terminal value  $\overrightarrow{\tilde{\xi}^0 \tilde{\xi}}$  over  $X(0)$ :

$$J_t(0) = \mathbf{E} \left( \Theta_{t,1}^{-1}(X(0)) \overrightarrow{\tilde{\xi}^0 \tilde{\xi}} \middle| \mathcal{F}_t \right) = \mathbf{E} \left( \overrightarrow{\tilde{\xi}^0 \tilde{\xi}} \middle| \mathcal{F}_t \right).$$

#### FIRST STEP.

Let  $\delta > 0$ ,  $\lambda \geq 0$ ,  $h > 0$ ,  $\varepsilon > 0$ , and let us expose to start the algorithm on the first small interval

$$\left[ 0, a_1^h \right] = \left[ 0, \frac{2h \times \mathbf{E} \left( \rho^2 \left( \tilde{\xi}^0, \tilde{\xi} \right) \right)}{\delta} - 2\lambda \varepsilon \right].$$

**The shooting.** Let  $a \in \left[ 0, a_1^h \right)$ . We define  $X^{h,\varepsilon}(a)$  as the unique solution of the following stochastic differential equation:

$$\begin{cases} d^\nabla X^{h,\varepsilon}(a) = a \parallel_{X(0), X^{h,\varepsilon}(a)} \left( dJ^{h,\varepsilon}(0) \right), & 0 \leq t \leq \tau^{h,\varepsilon}(a), \\ X_0^{h,\varepsilon}(a) = \exp_{X_0(0)}(aJ_0(0)), \\ X_t^{h,\varepsilon}(a) = X_{\tau^{h,\varepsilon}(a)}^{h,\varepsilon}, & \tau^{h,\varepsilon}(a) \leq t \leq 1, \end{cases}$$

where:

$$\tau^{h,\varepsilon}(a) = \inf \left\{ t \in [0, 1], \quad X_t^{h,\varepsilon}(a) \in \partial \mathcal{B} \right\}. \quad (\text{II.3.2})$$

We define in addition the following processes:

$$J_t^{h,\varepsilon} = \mathbf{E} \left( \Theta_{t,1}^{-1} \left( X^{h,\varepsilon}(a) \overline{X_1^{h,\varepsilon}(a) \tilde{\xi}} \right) \middle| \mathcal{F}_t \right), \quad Z_t^{h,\varepsilon} = \mathbf{E} \left( \Theta_{0,1}^{-1} \left( X^{h,\varepsilon}(a) \overline{X_1^{h,\varepsilon}(a) \tilde{\xi}} \right) \middle| \mathcal{F}_t \right). \quad (\text{II.3.3})$$

**Localization in  $\mathcal{H}^\infty$ .** According to Proposition – II.2.4 with

$$r=2, \quad V=0, \quad X = \widetilde{X_1^{h,\varepsilon}} := \lim_{a \rightarrow a_1^h} X^{h,\varepsilon}(a), \quad Z = \widetilde{Z_1^{h,\varepsilon}} := \lim_{a \rightarrow a_1^h} Z^{h,\varepsilon}(a), \quad J = \widetilde{J_1^{h,\varepsilon}} := \lim_{a \rightarrow a_1^h} J^{h,\varepsilon}(a),$$

there exists: a  $\nabla$ -martingale  $X^{h,\varepsilon}(a_1^h)$ , a  $\nabla^c$ -martingale  $J^{h,\varepsilon}(a_1^h)$  above  $X^{h,\varepsilon}(a_1^h)$  and a vectorial martingale  $Z^{h,\varepsilon}(a_1^h)$  in  $T_{X_0^{h,\varepsilon}(a_1^h)} \mathcal{M}$  satisfying:

- (i)  $\left\| \int_0^1 \left\langle dX^{h,\varepsilon}(a_1^h), dX^{h,\varepsilon}(a_1^h) \right\rangle_g \right\|_{L^\infty} \leq R(\varepsilon), \quad \mathbf{E} \left( \rho^2 \left( X_1^{h,\varepsilon}(a_1^h), \left( \widetilde{X_1^{h,\varepsilon}} \right)_{t=1} \right) \right) \leq \varepsilon,$
- (ii)  $\left\| \int_0^1 \left\langle dZ^{h,\varepsilon}(a_1^h), dZ^{h,\varepsilon}(a_1^h) \right\rangle_g \right\|_{L^\infty} \leq Q(\varepsilon),$
- (iii)  $\left\| J_1^{h,\varepsilon}(a_1^h) - \left\| \widetilde{X_n^{h,\varepsilon}, X_1^{h,\varepsilon}}(a_1^h) \left( \widetilde{J_1^{h,\varepsilon}} \right)_{t=1} \right\|_{L^2(\Omega)} \right\| \leq \varepsilon S,$

where  $R(\varepsilon), Q(\varepsilon)$  are two constants depending mainly on the regular ball  $\mathcal{B}$  and  $\varepsilon$ , but in order to simplify the notations we will mention only the dependence with respect to  $\varepsilon$ , and  $S$  depends on  $\mathcal{B}$ .

*Remark – II.3.2.* Let us emphasize that we can use Proposition – II.2.4: since  $J^h(a_1^h)$  is a  $\nabla^c$ -martingale essentially bounded (this claim will be shown in Proposition – II.4.1 below), and since (ii) in (RegBall, Algo) is in force.

### DISCRETIZATION.

Let us define the following sequence  $(a_n^h)_{n \in \mathbf{N}}$  driving the algorithm:

$$a_0^h = 0, \quad a_{n+1}^h = a_n^h + \frac{2h \times \mathbf{E} \left( \rho^2 \left( X_1^{h,\varepsilon}(a_n^h), \tilde{\xi} \right) \right)}{\delta} - 2\lambda \varepsilon.$$

### OTHER STEPS BY INDUCTION.

**The shooting.** If  $(X^{h,\varepsilon}, J^{h,\varepsilon}, Z^{h,\varepsilon})$  is defined on  $[0, a_n^h]$ , we continue to shoot in the direction of  $\tilde{\xi}$  by setting for all  $a \in [a_n^h, a_{n+1}^h)$ :

$$\begin{cases} d^\nabla X^{h,\varepsilon}(a) = \mathbb{I}_{X^{h,\varepsilon}(a_n^h), X^{h,\varepsilon}(a)} \left( d^\nabla X^{h,\varepsilon}(a_n^h) + (a - a_n^h) \times \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) dZ^{h,\varepsilon}(a_n^h) \right), & 0 \leq t \leq \tau^{h,\varepsilon}(a), \\ X_0^{h,\varepsilon}(a) = \exp_{X_0^{h,\varepsilon}(a_n^h)} \left( (a - a_n^h) J_0^{h,\varepsilon}(a_n^h) \right), \\ X_t^{h,\varepsilon}(a) = X_{\tau^{h,\varepsilon}(a)}^{h,\varepsilon}, & \tau^{h,\varepsilon}(a) \leq t \leq 1, \end{cases} \quad (\text{II.3.4})$$

where:

$$\tau^{h,\varepsilon}(a) = \inf \left\{ t \in [0, 1], \quad X_t^{h,\varepsilon}(a) \in \partial \mathcal{B} \right\}. \quad (\text{II.3.5})$$

On  $(a_n^h, a_{n+1}^h)$ , we define  $J^{h,\varepsilon}(a)$  and  $Z^{h,\varepsilon}(a)$  with the same formulas as in (II.3.3).

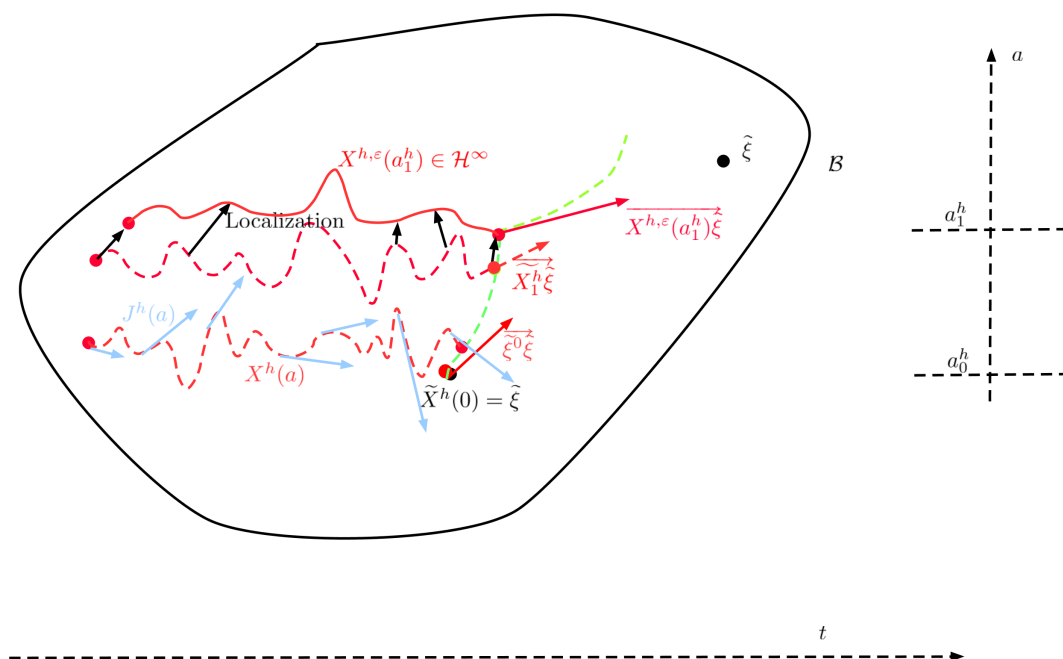
Let us note  $\left( \widetilde{X}_{n+1}^{h,\varepsilon}, \widetilde{J}_{n+1}^{h,\varepsilon}, \widetilde{Z}_{n+1}^{h,\varepsilon} \right) = \lim_{a \rightarrow a_{n+1}^h} \left( X^{h,\varepsilon}(a), J^{h,\varepsilon}(a), Z^{h,\varepsilon}(a) \right)$ . They are the terminal processes obtained at the end of the shooting on  $[a_n^h, a_{n+1}^h)$ , hence before the localization step. By definition, one has:

$$\left( \widetilde{J}_{n+1}^{h,\varepsilon} \right)_{t=1} = \overrightarrow{\left( \widetilde{X}_{n+1}^{h,\varepsilon} \right)_{t=1}} \widetilde{\xi}.$$

**Localization in  $\mathcal{H}^\infty$ .** According to Proposition – II.2.4 with  $r = 2$ ,  $X = \widetilde{X}_{n+1}^{h,\varepsilon}$ ,  $Z = \widetilde{Z}_{n+1}^{h,\varepsilon}$  and  $J = \widetilde{J}_{n+1}^{h,\varepsilon}$ , there exists a  $\nabla$ -martingale  $X^{h,\varepsilon}(a_{n+1}^h)$ , a  $\nabla^c$ -martingale  $J^{h,\varepsilon}(a_{n+1}^h)$  and a vectorial martingale  $Z^{h,\varepsilon}(a_{n+1}^h)$  in  $T_{X_0^{h,\varepsilon}(a_{n+1}^h)} \mathcal{M}$  satisfying:

- (i)  $\left\| \int_0^1 \left\langle dX^{h,\varepsilon}(a_{n+1}^h), dX^{h,\varepsilon}(a_{n+1}^h) \right\rangle_g \right\|_{L^\infty} \leq R(\varepsilon), \quad \mathbf{E} \left( \rho^2 \left( X_1^{h,\varepsilon}(a_{n+1}^h), \left( \widetilde{X}_{n+1}^{h,\varepsilon} \right)_{t=1} \right) \right) \leq \varepsilon,$
- (ii)  $\left\| \int_0^1 \left\langle dZ^{h,\varepsilon}(a_{n+1}^h), dZ^{h,\varepsilon}(a_{n+1}^h) \right\rangle_g \right\|_{L^\infty} \leq Q(\varepsilon),$
- (iii)  $\left\| J_1^{h,\varepsilon}(a_{n+1}^h) - \mathbb{I}_{\widetilde{X}_n^{h,\varepsilon}, X_1^{h,\varepsilon}(a_{n+1}^h)} \left( \widetilde{J}_{n+1}^{h,\varepsilon} \right)_{t=1} \right\|_{L^2(\Omega)} \leq \varepsilon S.$

*Remark – II.3.3.* With this procedure the family  $J^{h,\varepsilon}(\cdot)$  is not smooth with respect to  $a$  at the shooting points  $a_n^h$ . This smoothness will be not required for the convergence, but it will for the process  $X^{h,\varepsilon}(\cdot)$ .


 Figure II.1 – First shooting step, on  $[0, a_1^h]$ .

*Remark – II.3.4* (About the approximations in  $\mathcal{H}^\infty$ ). For all  $y \in \mathcal{B}$ , if one has  $\kappa_y^- \leq \kappa_y^+$ , the approximation procedure is not needed.

The approximation is useful in order to obtain estimations in  $\mathcal{S}^p$  with  $1 < p \leq \infty$  on  $\Theta_{0,\cdot}(X)$  if  $X$  is a  $\nabla$ -martingale in  $\mathcal{B}$ , see equations (II.4.20) below. If the inequality  $\kappa^- \leq \kappa^+$  is in force, we can write, according to [Proposition – II.1.6](#):

$$|\Theta_{0,\cdot}(X)| \leq \exp\left(\frac{1}{2} \int_0^\cdot \kappa_{X_u}^- \langle dX_u, dX_u \rangle_g\right) \leq \exp\left(\frac{1}{2} \int_0^\cdot \kappa_{X_u}^+ \langle dX_u, dX_u \rangle_g\right),$$

and the right member is in  $L^{\tilde{q}}$  ( $\tilde{q}$  given by [Proposition – II.1.4](#)), giving the  $\mathcal{S}^p$ -estimation of  $\Theta_{0,\cdot}(X)$  claimed above.

From now, we are able to understand the interest of the localization steps. The terminal values of the non-approximated process will be very close to the approximated process. Thus, on the one hand, by taking  $\varepsilon$  small enough we will see that the converge is always in force. On the other hand, we win estimates in  $\mathcal{S}^\infty$  for the stochastic deformed transports along the algorithm.

*Remark – II.3.5* (Why the stochastic differential equation (II.3.4) is used?). An answer will be given in the [Section II.4.2](#). The keystone is the following formula:  $\partial_{|a \geq a_n^h} X^{h,\varepsilon}(a) = J^{h,\varepsilon}(a_n^h)$ , implying that  $\partial_{|a=a_n^h} \mathbf{E}\left(\rho^2\left(X_1^{h,\varepsilon}(a); \tilde{\xi}\right)\right) \leq 0$ . The reader can find all the details in the [Section II.4](#) below.

*Remark – II.3.6* (Comparison with the deterministic geodesic shooting). Let  $p_0, p_1$  in  $\mathcal{M}$  and let us find a geodesic linking  $p_0$  and  $p_1$ .

As mentioned in [LB17], if we know how to compute the exponential map on a manifold, the problem is equivalent to find a vector  $u^{p_0, p_1} \in T_{p_0} \mathcal{M}$  such that  $\exp_{p_0}(u^{p_0, p_1}) = p_1$ . We start with an arbitrary vector  $u_0 \in T \mathcal{M}$ , analyze the position of  $\exp_{p_0}(u_0)$  by evaluating the gap between the point  $\exp_{p_0}(u_0)$  and  $p_1$ . If it is necessary, Le Brigant corrects the vector  $u_0$  in an other one  $u_1$ . By induction, if  $u_k$  is defined, we construct iteratively  $u_{k+1}$  by evaluating the gap between the point  $\exp_{p_0}(u_k)$  and  $p_1$ . At the end we hope a vector-limit corresponding to  $u^{p_0, p_1}$ .

For the algorithm  $X^{h, \varepsilon}(a)$ , the initial point is  $\tilde{\xi}^0$  and corresponds to  $a = 0$ . We define  $X_1^{h, \varepsilon}(a)$ , and evaluate the gap with  $\tilde{\xi}$  by computing the corresponding square geodesic distances. We minimize it by stopping the parameter  $a$  at  $a_1$ . The step  $u_0 \rightarrow u_1$  is replaced by a new stochastic differential equation depending on the positions at  $a = a_1$ . Afterward, we repeat this procedure.

### § II.3.2. Convergence of $X^{h, \varepsilon}$

In the sequel we set  $h, \varepsilon > 0$  and investigate the values of the terminal positions  $X_1^{h, \varepsilon}(a_n^h)$  through the expectation of the squared geodesic distance:

$$\Delta^{h, \varepsilon} : a \in \mathbf{R}^+ \mapsto \mathbf{E} \left( \rho^2 \left( X_1^{h, \varepsilon}(a); \tilde{\xi} \right) \right).$$

It is well-know that  $\Delta^{h, \varepsilon}$  are differentiable (see Proposition – II.1.5) and we know its first derivative:

$$\partial_a \Delta^{h, \varepsilon}(a) = -2 \times \mathbf{E} \left( \left\langle \overrightarrow{X_1^{h, \varepsilon}(a) \tilde{\xi}}, \partial_a X_1^{h, \varepsilon}(a) \right\rangle \right), \quad (\text{II.3.6})$$

To analyze the sequence  $\left( \Delta^{h, \varepsilon}(a_n^h) \right)_{n \in \mathbf{N}}$ , we are going to compute a Taylor expansion of order two at  $a = a_n^h$ . The second order part has to be bounded with a constant which does not depend on  $a$  and  $h$ . Consequently we have to establish a few estimations on the second derivative of  $\Delta^{h, \varepsilon}$  under assumption (**RegBall, Algo**), and it is the purpose of the following and main technical proposition.

**Proposition – II.3.1.** *Let  $h > 0$  and  $\varepsilon > 0$ . If (**RegBall, Algo**) is fulfilled, then  $\partial_a \Delta^{h, \varepsilon}$  is differentiable on  $\bigcup_{n \in \mathbf{N}} [a_n^h, a_{n+1}^h)$ , and the second derivative is given by:*

$$\partial_a^2 \Delta^{h, \varepsilon}(a) = -2 \times \mathbf{E} \left( \left\langle \overrightarrow{\nabla_a X_1^{h, \varepsilon}(a) \tilde{\xi}}, \partial_a X_1^{h, \varepsilon}(a) \right\rangle \right) - 2 \times \mathbf{E} \left( \left\langle \overrightarrow{X_1^{h, \varepsilon}(a) \tilde{\xi}}, \nabla_a \partial_a X_1^{h, \varepsilon}(a) \right\rangle \right).$$

Moreover there exists  $\delta_\varepsilon^*$  such that for all  $a \in \bigcup_{n \in \mathbf{N}} [a_n^h, a_{n+1}^h)$  and  $h \leq \delta$ :

$$|\partial_a^2 \Delta^{h, \varepsilon}(a)| \leq \delta_\varepsilon^*.$$

With the help of this upper bound, we will able to show the following convergence.

**Proposition – II.3.2** (Convergence of the scheme). *Let  $h > 0, \varepsilon > 0, \lambda > 0$ , and assume that the assumption **(RegBall, Algo)** is fulfilled together with:*

(i)  $0 < h < 2$  and  $\mathbf{E} \left( \rho^2 \left( \tilde{\xi}^0, \tilde{\xi} \right) \right) < \lambda \varepsilon + \frac{\delta_\varepsilon^*}{4h(2-h)},$

(ii) the sequence  $\left( a_n^h \right)_{n \in \mathbf{N}}$  is constructed with  $(\delta, \lambda, \varepsilon) = (\delta_\varepsilon^*, \lambda, \varepsilon).$

Then there exists  $\lambda = \lambda^* > 0, h_{\tilde{\xi}^0}^\varepsilon > 0$ , such that:

$$\lim_{n \rightarrow \infty} \Delta_{h_{\tilde{\xi}^0}^\varepsilon, \varepsilon} \left( a_n^h \right) = \lim_{n \rightarrow \infty} \mathbf{E} \left( \rho^2 \left( X_1^{h_{\tilde{\xi}^0}^\varepsilon, \varepsilon} \left( a_n^h \right), \tilde{\xi} \right) \right) = \lambda^* \varepsilon.$$

Consequently, there exists a  $\nabla$ -martingale lying in  $\mathcal{B}$  with terminal value  $\tilde{\xi}$ .

*Remark – II.3.7.* Let us emphasize that the assumption (i) is automatically fulfilled by taking  $h$  small enough.

### § II.3.3. Solving the main equation on the space of martingales

We come back to the main ordinary equation

$$\forall t \in [0, T], \quad \partial_a X_t(a) = \mathbf{E} \left( \Theta_{t,1}^{-1}(X(a)) (\partial_a \xi(a)) \middle| \mathcal{F}_t \right), \quad (\text{II.3.7})$$

where  $a \in [0, 1] \mapsto \xi(a)$  is a  $\mathcal{C}^1$ -random geodesic with end point  $\xi := \xi(1) \in L^2(\Omega, \mathcal{M}).$

**Theorem – II.3.1** (Existence of a solution to (II.3.7)). *Let us assume that:*

(i) the assumption **(RegBall, Algo)** is fulfilled,

(ii)  $\xi(a) \in \mathcal{B}$  for all  $a \in [0, 1]$  with  $\partial \xi^* := \sup_{a \in [0,1]} \|\partial_a \xi(a)\|_{L^\infty} < \infty.$

Then there exists a unique  $\mathcal{C}^1$ -family of  $\nabla$ -martingales satisfying (II.3.7).

Consequently, there exists a unique  $\nabla$ -martingale with prescribed terminal value  $\xi$  lying in  $\mathcal{B}.$



## II.4 PROOFS OF THE CONVERGENCE

For all  $a \geq 0$ ,  $h > 0$  and  $\varepsilon > 0$ , the stochastic interval  $\left\{ (\omega, t) \in \Omega \times [0, 1] \mid 0 \leq t \leq \tau^{h, \varepsilon}(a)(\omega) \right\}$  will be denoted by  $\mathcal{I}^{h, \varepsilon}(a)$ .

## § II.4.1. First estimates and properties.

**Proposition – II.4.1.** *Under (RegBall, Algo), the following statements hold true.*

(i) For all  $\varepsilon > 0$ , and  $\delta > 0$ , there exists a constant  $\mathcal{R}(\varepsilon)$  such that

$$\sup_{a \geq 0, h \leq \delta} \left\| \int_0^1 \left\langle dX^{h, \varepsilon}(a), dX^{h, \varepsilon}(a) \right\rangle_g \right\|_{L^\infty} \leq \mathcal{R}(\varepsilon).$$

(ii) Consequently,

$$\sup_{a \geq 0, h \leq \delta} \left\{ \sup_{0 \leq t \leq s \leq 1} \left\| \Theta_{t, s}(X^{h, \varepsilon}(a)) \right\|_{L^\infty} \right\} < \infty, \quad \sup_{a \geq 0, h \leq \delta} \left\{ \sup_{0 \leq t \leq s \leq 1} \left\| \Theta_{t, s}^{-1}(X^{h, \varepsilon}(a)) \right\|_{L^\infty} \right\} < \infty.$$

(iii) The set  $\mathcal{B}$  has  $q^*$ -convex geometry.

(iv) We have:  $\sup_{a \geq 0, h > 0} \left\| J^{h, \varepsilon}(a) \right\|_{\mathcal{I}^{h, \varepsilon}(a)} < \infty$ .

Proof. According to Proposition – II.1.6, and taking into account the localization step:

$$\left\| \Theta_{0, \cdot}(X^{h, \varepsilon}(a_n^h)) \right\|_{\mathcal{I}^{h, \varepsilon}(a_n^h)} \leq \exp \left( \frac{\kappa^-(\mathcal{M})}{2} \left\| \int_0^1 \left\langle dX^{h, \varepsilon}(a_n^h), dX^{h, \varepsilon}(a_n^h) \right\rangle_g \right\|_{L^\infty} \right) \leq \exp \left( \frac{\kappa^-(\mathcal{M})}{2} R(\varepsilon) \right).$$

Just note that, for all  $a \in [a_n^h, a_{n+1}^h)$ , since the  $\nabla$ -parallel transport is an isometry:

$$\begin{aligned} \int_0^1 \left\langle dX^{h, \varepsilon}(a), dX^{h, \varepsilon}(a) \right\rangle_g &= \int_0^1 \left\langle dX^{h, \varepsilon}(a_n^h), dX^{h, \varepsilon}(a_n^h) \right\rangle_g \\ &\quad + 2(a - a_n^h) \int_0^1 \left\langle dX^{h, \varepsilon}(a_n^h), \Theta_{0, \cdot}(X^{h, \varepsilon}(a_n^h)) dZ^{h, \varepsilon}(a_n^h) \right\rangle_g \\ &\quad + (a - a_n^h)^2 \int_0^1 \left\langle \Theta_{0, \cdot}(X^{h, \varepsilon}(a_n^h)) dZ^{h, \varepsilon}(a_n^h), \Theta_{0, \cdot}(X^{h, \varepsilon}(a_n^h)) dZ^{h, \varepsilon}(a_n^h) \right\rangle_g. \end{aligned}$$

The first term is essentially bounded by  $R(\varepsilon)$ . Taking into account the Kunita-Watanabe inequality and the fact that the term  $a - a_n^h$  is bounded by nothing else than  $h/\delta$  (which is by assumption less or equal to one) and the radius of  $\mathcal{B}$ , it is sufficient to show the existence of a constant  $\mathcal{R}(\varepsilon)$

satisfying

$$\left\| \left\langle \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) dZ^{h,\varepsilon}(a_n^h), \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) dZ^{h,\varepsilon}(a_n^h) \right\rangle_g \right\|_{L^\infty} \leq \mathcal{R}(\varepsilon).$$

But,

$$\begin{aligned} & \left| \int_0^1 \left\langle \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) dZ^{h,\varepsilon}(a_n^h), \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) dZ^{h,\varepsilon}(a_n^h) \right\rangle_g \right| \\ & \leq \left\| \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) \right\|_{\mathcal{S}^\infty}^2 \left\| \int_0^1 \left\langle dZ^{h,\varepsilon}(a_n^h), dZ^{h,\varepsilon}(a_n^h) \right\rangle_g \right\|_{L^\infty} \\ & \leq \exp(\kappa^-(\mathcal{M})R(\varepsilon)) Q(\varepsilon). \end{aligned}$$

We have established (i). According to Proposition – II.1.6 again, we have for all  $0 \leq t \leq s \leq 1$ :

$$\begin{aligned} \left\| \Theta_{t,s} \left( X^{h,\varepsilon}(a) \right) \right\|_{\mathcal{S}^\infty} & \leq \exp \left( \frac{\kappa^-(\mathcal{M})}{2} \left\| \int_0^1 \left\langle dX_s^{h,\varepsilon}(a), dX_s^{h,\varepsilon}(a) \right\rangle_g \right\|_{L^\infty} \right) \leq \exp \left( \frac{\kappa^-(\mathcal{M})}{2} \mathcal{R}(\varepsilon) \right), \\ \left\| \Theta_{t,s}^{-1} \left( X^{h,\varepsilon}(a) \right) \right\|_{\mathcal{S}^\infty} & \leq \exp \left( \frac{\kappa^+(\mathcal{M})}{2} \left\| \int_0^1 \left\langle dX_s^{h,\varepsilon}(a), dX_s^{h,\varepsilon}(a) \right\rangle_g \right\|_{L^\infty} \right) \leq \exp \left( \frac{\kappa^+(\mathcal{M})}{2} \mathcal{R}(\varepsilon) \right). \end{aligned}$$

The statement (ii) is shown and (iii) follows directly from Proposition – II.1.4. The last claim (iv) derives from (ii) taking into account that  $X^{h,\varepsilon}(a)$  lies in  $\mathcal{B}$ .  $\square$

#### § II.4.2. Main property satisfied by $X^{h,\varepsilon}(a)$ .

Plainly we will see that the family  $(X^{h,\varepsilon}(a))_{a \geq 0}$  solve the main equation (II.3.1) at the shooting points  $a = a_n^h$  for all  $n \in \mathbf{N}$  and  $h > 0$ .

*Remark – II.4.1* (Differentiability of  $X^{h,\varepsilon}(\cdot)$ ). The map  $a \mapsto X_t^{h,\varepsilon}(a)$  is not differentiable for all  $a$  if  $t = \tau^{h,\varepsilon}(a)$ . We can not apply Proposition – II.1.10 directly on the whole stochastic interval  $\mathcal{I}^{h,\varepsilon}(a)$ , but in the sequel we will suppose that we can. Let us explain why.

It is possible to consider a bigger geodesic ball  $\mathcal{B}'$ , containing strictly  $\mathcal{B}$ , with a corresponding exit time denoted by  $(\tau')^{h,\varepsilon}(a)$  and satisfying  $\tau^{h,\varepsilon}(a) \leq (\tau')^{h,\varepsilon}(a)$  almost surely. We will use Proposition – II.1.10 in order to obtain estimates in  $\mathcal{S}^p$ ,  $1 \leq p \leq \infty$  of the processes  $\partial_a X^{h,\varepsilon}(a)$  and  $\nabla_a \partial_a X^{h,\varepsilon}(a)$ . And if this kind of estimate is true on

$$\left\{ (\omega, t) \in \Omega \times [0, 1] / 0 \leq t \leq (\tau')^{h,\varepsilon}(a)(\omega) \right\},$$

it is also on  $\mathcal{I}^{h,\varepsilon}(a)$ .

**Proposition – II.4.2.** *The family of processes  $(X^{h,\varepsilon}(a))_{a \geq 0}$  is right-differentiable at each  $a_n^h$  for all  $n \in \mathbf{N}$ ,  $h > 0$ ,  $\varepsilon > 0$  and one has:*

$$\partial_{|a \geq a_n^h} X^{h,\varepsilon}(a) = J^{h,\varepsilon}(a_n^h) \quad \text{on} \quad \mathcal{I}^{h,\varepsilon}(a_n^h).$$

In addition,  $X^{h,\varepsilon}(a)$  is a  $\nabla$ -martingale for all  $a, h > 0$  and  $\varepsilon > 0$ .

**Proof.** We are able to write both processes  $\partial|_{a \geq a_n^h} X^{h,\varepsilon}(a)$  and  $J^{h,\varepsilon}(a_n^h)$  as a solution of the same stochastic differential equation. Firstly, with **Proposition – II.1.10** on  $[a_n^h, a_{n+1}^h)$ , and taking into account the **Remark – II.4.1**, one has:

$$\begin{aligned} \mathbf{D}\partial_a X^{h,\varepsilon}(a) &= \left( \nabla_a \parallel_{X^{h,\varepsilon}(a_n^h), X^{h,\varepsilon}(a)} \right) \left( d^\nabla X^{h,\varepsilon}(a_n^h) + (a - a_n^h) \times \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) dZ^{h,\varepsilon}(a_n^h) \right) \\ &\quad + \parallel_{X^{h,\varepsilon}(a_n^h), X^{h,\varepsilon}(a)} \left( \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) dZ^{h,\varepsilon}(a_n^h) \right) - \frac{1}{2} R \left( \partial_a X^{h,\varepsilon}(a), dX^{h,\varepsilon}(a) \right) dX^{h,\varepsilon}(a), \end{aligned}$$

and with  $a = a_n^h$ , one has  $\mathcal{J}^{h,\varepsilon}(a_n^h)$ :

$$\mathbf{D}\partial|_{a \geq a_n^h} X^{h,\varepsilon}(a) = \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) dZ^{h,\varepsilon}(a_n^h) - \frac{1}{2} R \left( \partial|_{a \geq a_n^h} X^{h,\varepsilon}(a), dX^{h,\varepsilon}(a_n^h) \right) dX^{h,\varepsilon}(a_n^h).$$

Combining this equation with

$$\mathbf{D}J^{h,\varepsilon}(a) = \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a) \right) dZ^{h,\varepsilon}(a) - \frac{1}{2} R \left( J^{h,\varepsilon}(a), dX^{h,\varepsilon}(a) \right) dX^{h,\varepsilon}(a) \quad (\text{II.4.1})$$

at  $a = a_n^h$ , we obtain the first claim. For the second one, looking at the algorithm (II.3.4), and since  $X^{h,\varepsilon}(a_n^h)$  is a  $\nabla$ -martingale, it is sufficient to show that for all  $\alpha$  a form locally bounded above  $X^{h,\varepsilon}(a)$  the process  $\int_0^\cdot \alpha \left( \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) dZ^{h,\varepsilon}(a_n^h) \right)$  is real local martingale. It is equal to

$$\int_0^\cdot \alpha \left( \mathbf{D}^{\text{def}} J^{h,\varepsilon}(a_n^h) \right),$$

and according to **Proposition – II.1.7** we obtain the second claim.  $\square$

### § II.4.3. Convergence of the algorithm $X^{h,\varepsilon}(a)$ .

**Proof.** [of **Proposition – II.3.1**] Let us recall  $\partial_a \Delta^{h,\varepsilon}(a) = -2 \times \mathbf{E} \left( \left\langle \overrightarrow{X_1^{h,\varepsilon}(a)} \xi, \partial_a X_1^{h,\varepsilon}(a) \right\rangle \right)$  for all  $a \geq 0$ . Since the connection  $\nabla$  is assumed to be metric, we have

$$\partial_a \left( \left\langle \overrightarrow{X_1^{h,\varepsilon}(a)} \xi, \partial_a X_1^{h,\varepsilon}(a) \right\rangle \right) = \left\langle \nabla_a \overrightarrow{X_1^{h,\varepsilon}(a)} \xi, \partial_a X_1^{h,\varepsilon}(a) \right\rangle + \left\langle \overrightarrow{X_1^{h,\varepsilon}(a)} \xi, \nabla_a \partial_a X_1^{h,\varepsilon}(a) \right\rangle. \quad (\text{II.4.2})$$

**Step 1 — The first term with deterministic tools.** Let us consider  $\gamma^a(t) = \exp_{X_1^{h,\varepsilon}(a)} \left( t \times \overrightarrow{X_1^{h,\varepsilon}(a)\tilde{\xi}} \right)$  the geodesic linking  $X_1^{h,\varepsilon}(a)$  and  $\tilde{\xi}$ , and compute

$$\overrightarrow{\nabla_a X_1^{h,\varepsilon}(a)\tilde{\xi}} = \nabla_a \partial_{|s=0} \gamma^a(s) = \nabla_{|s=0} \partial_a \gamma^a(s) = \nabla_{|s=0} \mathcal{J}^a(s),$$

where  $\mathcal{J}^a$  states for the Jacobi field  $\partial_a \gamma^a$ . It is well-known that each Jacobi field  $\mathcal{J}^a$  along a geodesic  $\gamma^a$  has the form

$$\begin{aligned} \mathcal{J}^a(t) &= (\langle \nabla_{|t=0} \mathcal{J}^a(t), \dot{\gamma}^a(0) \rangle + t \langle \mathcal{J}^a(0), \dot{\gamma}^a(0) \rangle) \dot{\gamma}^a(t) + (\mathcal{J}^a(t))^\perp, \quad (\mathcal{J}^a)^\perp \perp_g \dot{\gamma}^a \\ &= \left( \left\langle \nabla_{|t=0} \mathcal{J}^a(t), \overrightarrow{X_1^{h,\varepsilon}(a)\tilde{\xi}} \right\rangle + t \left\langle \partial_a X_1^{h,\varepsilon}(a), \overrightarrow{X_1^{h,\varepsilon}(a)\tilde{\xi}} \right\rangle \right) \dot{\gamma}^a(t) + (\mathcal{J}^a(t))^\perp. \end{aligned}$$

And consequently, since  $\gamma^a$  is a geodesic, we obtain:

$$\nabla_{|t=0} \mathcal{J}^a(t) = \left\langle \partial_a X_1^{h,\varepsilon}(a), \overrightarrow{X_1^{h,\varepsilon}(a)\tilde{\xi}} \right\rangle \overrightarrow{X_1^{h,\varepsilon}(a)\tilde{\xi}} + \nabla_{|t=0} (\mathcal{J}^a(t))^\perp,$$

and we get

$$\left\langle \overrightarrow{\nabla_a X_1^{h,\varepsilon}(a)\tilde{\xi}}, \partial_a X_1^{h,\varepsilon}(a) \right\rangle = \left\langle \partial_a X_1^{h,\varepsilon}(a), \overrightarrow{X_1^{h,\varepsilon}(a)\tilde{\xi}} \right\rangle^2 + \left\langle \partial_a X_1^{h,\varepsilon}(a), \nabla_{|t=0} (\mathcal{J}^a(t))^\perp \right\rangle. \quad (\text{II.4.3})$$

The aim is now to find an upper bound on the covariant derivative of the orthogonal component  $(\mathcal{J}^a)^\perp$ . Let us recall an estimate due to Picard in [Pic91] (estimate (1.1.16)), since **(RegBall, Algo) (ii)** is in force:

$$\partial_{|t=0} \left[ \left| (\mathcal{J}^a(t))^\perp \right| \right] \leq \alpha \sqrt{\kappa^+(\mathcal{B})} \times \rho \left( X_1^{h,\varepsilon}(a); \tilde{\xi} \right) \times \cos(\beta), \quad (\text{II.4.4})$$

where  $\alpha$  and  $\beta$  satisfy:

$$\alpha \leq \frac{\left| \partial_a X_1^{h,\varepsilon}(a) \right|}{\sin \beta}, \quad \beta = \pi - \sqrt{\kappa^+(\mathcal{B})} \times \rho \left( X_1^{h,\varepsilon}(a); \tilde{\xi} \right).$$

We deduce easily the following inequality computing the derivative of  $\left| (\mathcal{J}^a)^\perp \right|^2$ , and using:

$$\left\langle \nabla_{|t=0} (\mathcal{J}^a(t))^\perp, (\mathcal{J}^a(0))^\perp \right\rangle \leq \left| (\mathcal{J}^a(0))^\perp \right| \sqrt{\kappa^+(\mathcal{B})} \times \rho \left( X_1^{h,\varepsilon}(a), \tilde{\xi} \right) \left| \partial_a X_1^{h,\varepsilon}(a) \right| \cotan(\beta),$$

and since  $(\mathcal{J}^a(0))^\perp = \mathcal{J}^a(0) - V_a = \partial_a X_1^h(a) - V_a$  where  $V_a$  is collinear to  $\dot{\gamma}^a(0)$ , we obtain:

$$\left\langle \nabla_{|t=0} (\mathcal{J}^a(t))^\perp, \partial_a X_1^{h,\varepsilon}(a) \right\rangle \leq \left| \partial_a X_1^{h,\varepsilon}(a) \right| \sqrt{\kappa^+(\mathcal{B})} \times \rho \left( \partial_a X_1^{h,\varepsilon}(a), \tilde{\xi} \right) \left| X_1^{h,\varepsilon}(a) \right| \cotan(\beta).$$

Combining with the equality (II.4.3), the first term can be estimated with:

$$\left| \left\langle \overrightarrow{\nabla_a X_1^{h,\varepsilon}(a) \xi}, \partial_a X_1^{h,\varepsilon}(a) \right\rangle \right| \leq (2R(\mathcal{B}))^2 \left| \partial_a X_1^{h,\varepsilon}(a) \right|^2 + 2R(\mathcal{B}) \left| \partial_a X_1^{h,\varepsilon}(a) \right|^2 \sqrt{\kappa^+(\mathcal{B})} \times \cotan(\beta).$$

It is an inequality in terms of  $\mathcal{B}$  and  $\partial_a X_1^{h,\varepsilon}(a)$ . We just have to find an upper bound on  $\partial_a X_1^{h,\varepsilon}(a)$  in the sequel to achieve the first term.

We have to take the expectation on  $[a_n^h, a_{n+1}^h]$  in (II.4.2) and to swap derivative and expectation.

Our aim is thus to find  $p > 2$  such that the families  $\left( \partial_a X^{h,\varepsilon}(a) \right)_a$  and  $\left( \nabla_a \partial_a X^{h,\varepsilon}(a) \right)_a$  are uniformly bounded in  $\mathcal{S}^p$  with respect to  $a$ . This upper bound, if there exists, will give also an upper bound on the second derivative of  $\Delta^{h,\varepsilon}$ , as clammed in the Proposition. In fact we are able to show that the previous families are uniformly bounded in  $\mathcal{S}^p$  for all  $p > 1$ .

**Step 2 — Stochastic differential equations satisfied by  $\partial_a X^{h,\varepsilon}(a)$  and  $\nabla_a \partial_a X^{h,\varepsilon}(a)$ .** Let  $a \geq 0$ ,  $n \in \mathbf{N}$ ,  $h > 0$  and  $\varepsilon > 0$  such that  $a \in [a_n^h, a_{n+1}^h]$ . With the Cauchy-Schwarz inequality we write

$$\left| \left\langle \overrightarrow{X_1^{h,\varepsilon}(a) \xi}, \nabla_a \partial_a X_1^{h,\varepsilon}(a) \right\rangle \right| \leq 2R(\mathcal{B}) \left| \nabla_a \partial_a X_1^{h,\varepsilon}(a) \right|.$$

With Proposition – II.1.10, one has

$$\mathbf{D} \partial_a X^{h,\varepsilon}(a) = \nabla_a d^\nabla X^{h,\varepsilon}(a) - \frac{1}{2} R \left( \partial_a X^{h,\varepsilon}(a), dX^{h,\varepsilon}(a) \right) dX^{h,\varepsilon}(a).$$

Observing the algorithm, we deduce

$$\nabla_a \partial_a X_0^{h,\varepsilon}(a) = 0,$$

and on  $\mathcal{S}^{h,\varepsilon}(a)$ :

$$\begin{aligned} \nabla_a d^\nabla X^{h,\varepsilon}(a) &= \left( \nabla_a \parallel_{X^{h,\varepsilon}(a_n^h), X^{h,\varepsilon}(a)} \right) \left( d^\nabla X^{h,\varepsilon}(a_n^h) + (a - a_n^h) \times \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) dZ^{h,\varepsilon}(a_n^h) \right) \\ &\quad + \parallel_{X^{h,\varepsilon}(a_n^h), X^{h,\varepsilon}(a)} \left( \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) dZ^{h,\varepsilon}(a_n^h) \right). \end{aligned} \quad (\text{II.4.5})$$

We obtain, on  $\mathcal{S}^{h,\varepsilon}(a)$  again, the following stochastic differential equation:

$$\begin{aligned} \mathbf{D}^{\text{def}} \partial_a X^{h,\varepsilon}(a) &= \left( \nabla_a \parallel_{X^{h,\varepsilon}(a_n^h), X^{h,\varepsilon}(a)} \right) \left( d^\nabla X^{h,\varepsilon}(a_n^h) + (a - a_n^h) \times \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) dZ^{h,\varepsilon}(a_n^h) \right) \\ &\quad + \parallel_{X^{h,\varepsilon}(a_n^h), X^{h,\varepsilon}(a)} \left( \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) dZ^{h,\varepsilon}(a_n^h) \right). \end{aligned} \quad (\text{II.4.6})$$

Consequently, from (II.4.5), we deduce also:

$$\begin{aligned} \nabla_a^2 \mathbf{d}^\nabla X^{h,\varepsilon}(a) &= \left( \nabla_a^2 \mathbb{I}_{X^{h,\varepsilon}(a_n^h), X^{h,\varepsilon}(a)} \right) \left( \mathbf{d}^\nabla X^{h,\varepsilon}(a_n^h) + (a - a_n^h) \times \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) \mathbf{d}Z^{h,\varepsilon}(a_n^h) \right) \\ &\quad + 2 \left( \nabla_a \mathbb{I}_{X^{h,\varepsilon}(a_n^h), X^{h,\varepsilon}(a)} \right) \left( \Theta_{0,\cdot} \left( X^{h,\varepsilon}(a_n^h) \right) \mathbf{d}Z^{h,\varepsilon}(a_n^h) \right). \end{aligned} \quad (\text{II.4.7})$$

The process  $\nabla_a \partial_a X^{h,\varepsilon}(a)$  satisfies a stochastic differential equation given by Proposition – II.1.10:

$$\begin{aligned} \mathbf{D}\nabla_a \partial_a X^{h,\varepsilon}(a) &= \nabla_a \mathbf{D}\partial_a X^{h,\varepsilon}(a) + R \left( \mathbf{d}^\nabla X^{h,\varepsilon}(a), \partial_a X^{h,\varepsilon}(a) \right) \partial_a X^{h,\varepsilon}(a) \\ &\quad - \frac{1}{2} R \left( \mathbf{D}\partial_a X^{h,\varepsilon}(a), \mathbf{d}X^{h,\varepsilon}(a) \right) \partial_a X^{h,\varepsilon}(a) + R \left( \mathbf{d}X^{h,\varepsilon}(a), \partial_a X^{h,\varepsilon}(a) \right) \mathbf{D}\partial_a X^{h,\varepsilon}(a) \\ &\quad - \frac{1}{2} \nabla_{\mathbf{d}X^{h,\varepsilon}(a)} R \left( \partial_a X^{h,\varepsilon}(a), \mathbf{d}X^{h,\varepsilon}(a) \right) \partial_a X^{h,\varepsilon}(a), \end{aligned}$$

or, equivalently:

$$\begin{aligned} \mathbf{D}\nabla_a \partial_a X^{h,\varepsilon}(a) &= \nabla_a \nabla_a \mathbf{d}^\nabla X^{h,\varepsilon}(a) - \frac{1}{2} (\nabla_a R) \left( \partial_a X^{h,\varepsilon}(a), \mathbf{d}X^{h,\varepsilon}(a) \right) \mathbf{d}X^{h,\varepsilon}(a) \\ &\quad - \frac{1}{2} R \left( \nabla_a \partial_a X^{h,\varepsilon}(a), \mathbf{d}X^{h,\varepsilon}(a) \right) \mathbf{d}X^{h,\varepsilon}(a) \\ &\quad - \frac{1}{2} R \left( \partial_a X^{h,\varepsilon}(a), \nabla_a \mathbf{d}X^{h,\varepsilon}(a) \right) \mathbf{d}X^{h,\varepsilon}(a) - \frac{1}{2} R \left( \partial_a X^{h,\varepsilon}(a), \mathbf{d}X^{h,\varepsilon}(a) \right) \nabla_a \mathbf{d}X^{h,\varepsilon}(a) \\ &\quad + R \left( \mathbf{d}^\nabla X^{h,\varepsilon}(a), \partial_a X^{h,\varepsilon}(a) \right) \partial_a X^{h,\varepsilon}(a) - \frac{1}{2} R \left( \nabla_a \mathbf{d}X^{h,\varepsilon}(a), \mathbf{d}X^{h,\varepsilon}(a) \right) \partial_a X^{h,\varepsilon}(a) \\ &\quad + R \left( \mathbf{d}X^{h,\varepsilon}(a), \partial_a X^{h,\varepsilon}(a) \right) \nabla_a \mathbf{d}X^{h,\varepsilon}(a) - \frac{1}{2} \nabla_{\mathbf{d}X^{h,\varepsilon}(a)} R \left( \partial_a X^{h,\varepsilon}(a), \mathbf{d}X^{h,\varepsilon}(a) \right) \partial_a X^{h,\varepsilon}(a). \end{aligned} \quad (\text{II.4.8})$$

By using the Jacobi identity

$$\begin{aligned} &-\frac{1}{2} R \left( \partial_a X^{h,\varepsilon}(a), \nabla_a \mathbf{d}X^{h,\varepsilon}(a) \right) \mathbf{d}X^{h,\varepsilon}(a) - \frac{1}{2} R \left( \mathbf{d}X^{h,\varepsilon}(a), \partial_a X^{h,\varepsilon}(a) \right) \nabla_a \mathbf{d}X^{h,\varepsilon}(a) \\ &-\frac{1}{2} R \left( \nabla_a \mathbf{d}X^{h,\varepsilon}(a), \mathbf{d}X^{h,\varepsilon}(a) \right) \partial_a X^{h,\varepsilon}(a) = 0, \end{aligned}$$



The assumption **(Lip(F))** is thus satisfied. Moreover:

- the process  $Y$  is in  $\mathcal{H}^\infty$  according to Proposition – II.4.1, with a norm depending only on  $\varepsilon$ ,
- the process  $H$  is in  $\mathcal{S}^p$  for all  $1 \leq p < \infty$  using Proposition – II.4.1 and Corollary – II.2.1, the isometry property of the parallel transport and the inequality

$$\left\| \int_0^1 \left\langle dZ^{h,\varepsilon}(a_n^{h,\varepsilon}), dZ^{h,\varepsilon}(a_n^{h,\varepsilon}) \right\rangle_g \right\|_{L^\infty} \leq Q(\varepsilon).$$

To achieve this step we can note that  $\lim_{p \rightarrow \infty} C'_p = \infty$  which authorize to pick  $p$  large enough (larger than  $p_0$ ) such that condition (II.2.9) be fulfilled:

$$\left\| \int_0^1 \langle dY, dY \rangle_g \right\|_{L^\infty} \leq CR(\mathcal{B}) \exp\left(\frac{2\kappa^-(\mathcal{M}) + \kappa^+(\mathcal{M})}{2} \mathcal{R}(\varepsilon)\right) C'_p.$$

Finally the process  $Q = \Theta_{0,\cdot}^{-1}(X^{h,\varepsilon}(a)) \partial_a X^{h,\varepsilon}(a)$  is in  $\mathcal{S}^p$  for all  $p \geq p_0$ , and the  $\mathcal{S}^p$  norm depends only on:  $R(\mathcal{B}), \kappa^+(\mathcal{M}), \kappa^+(\mathcal{M}), \varepsilon, p$ . Writing

$$\partial_a X^{h,\varepsilon}(a) = \Theta_{0,\cdot}(X^{h,\varepsilon}(a)) \Theta_{0,\cdot}^{-1}(X^{h,\varepsilon}(a)) \partial_a X^{h,\varepsilon}(a)$$

we can conclude that:

$$\sup_{a \geq 0, h \leq \delta} \left\| \partial_a X^{h,\varepsilon}(a) \right\|_{\mathcal{S}^p} \leq \exp\left(\frac{\kappa^-(\mathcal{M})}{2} \mathcal{R}(\varepsilon)\right) \left( \sup_{a \geq 0, h \leq \delta} \left\| \Theta_{0,\cdot}^{-1}(X^{h,\varepsilon}(a)) \partial_a X^{h,\varepsilon}(a) \right\|_{\mathcal{S}^p} \right) < \infty. \quad (\text{II.4.10})$$

**Step 4 — estimation of  $\nabla_a \partial_a X^{h,\varepsilon}(a)$ .** Integrating between zero and  $t \in [0, 1]$  equation (II.4.9), we obtain:

$$\begin{aligned} & \Theta_{0,t}^{-1}(X^{h,\varepsilon}(a)) \nabla_a \partial_a X_t^{h,\varepsilon}(a) \\ &= \int_0^t \Theta_{0,s}^{-1}(X^{h,\varepsilon}(a)) \left( (a - a_n^h) \left( \nabla_a^2 \parallel_{X_s^{h,\varepsilon}(a_n^h), X_s^{h,\varepsilon}(a)} \right) + 2 \left( \nabla_a \parallel_{X_s^{h,\varepsilon}(a_n^h), X_s^{h,\varepsilon}(a)} \right) \right) \left( \Theta_{0,s} \left( X^{h,\varepsilon}(a_n^h) \right) dZ_s^{h,\varepsilon}(a_n^h) \right) \end{aligned} \quad (\text{II.4.11})$$

$$+ \int_0^t \Theta_{0,s}^{-1}(X^{h,\varepsilon}(a)) \left( \left( \nabla_a^2 \parallel_{X_s^{h,\varepsilon}(a_n^h), X_s^{h,\varepsilon}(a)} \right) d^\nabla X_s^{h,\varepsilon}(a_n^h) + R \left( d^\nabla X_s^{h,\varepsilon}(a), \partial_a X_s^{h,\varepsilon}(a) \right) \partial_a X_s^{h,\varepsilon}(a) \right) \quad (\text{II.4.12})$$

$$- \frac{1}{2} \int_0^t \Theta_{0,s}^{-1}(X^{h,\varepsilon}(a)) \left( (\nabla_a R) \left( \partial_a X_s^{h,\varepsilon}(a), dX_s^{h,\varepsilon}(a) \right) dX_s^{h,\varepsilon}(a) + \nabla_{dX_s^{h,\varepsilon}(a)} R \left( \partial_a X_s^{h,\varepsilon}(a), dX_s^{h,\varepsilon}(a) \right) \partial_a X_s^{h,\varepsilon}(a) \right) \quad (\text{II.4.13})$$

$$+ 2 \times \int_0^t \Theta_{0,s}^{-1}(X^{h,\varepsilon}(a)) R \left( dX_s^{h,\varepsilon}(a), \partial_a X_s^{h,\varepsilon}(a) \right) \left( \nabla_a d^\nabla X_s^{h,\varepsilon}(a) \right). \quad (\text{II.4.14})$$



Let  $p > 1$ . By applying [Proposition – II.1.1](#), we have the following estimations:

$$\left\| \nabla_a \llbracket X^{h,\varepsilon}(a_n^h), X^{h,\varepsilon}(a) \rrbracket \right\|_{\mathcal{F}^p} \leq 2CR(\mathcal{B}) \left\| \partial_a X^{h,\varepsilon}(a) \right\|_{\mathcal{F}^p}, \quad (\text{II.4.15})$$

$$\left\| \nabla_a^2 \llbracket X^{h,\varepsilon}(a_n^h), X^{h,\varepsilon}(a) \rrbracket \right\|_{\mathcal{F}^p} \leq C \left\| \partial_a X^{h,\varepsilon}(a) \right\|_{\mathcal{F}^{2p}}^2. \quad (\text{II.4.16})$$

Let us begin to deal with the second part of term [\(II.4.12\)](#). In the sequel, the constants  $K$  which do not depend on  $a$  can change from line to line. It is a real martingale for which we can use [Proposition – II.2.1](#), and [Proposition – II.4.1](#):

$$\left\| \int_0^{\cdot} \Theta_{0,s}^{-1}(X^{h,\varepsilon}(a)) R \left( d^\nabla X_s^{h,\varepsilon}(a), \partial_a X_s^{h,\varepsilon}(a) \right) \partial_a X_s^{h,\varepsilon}(a) \right\|_{\mathcal{F}^p} \quad (\text{II.4.17})$$

$$\begin{aligned} &\leq K \left\| \sqrt{\int_0^1 \left| \partial_a X_s^{h,\varepsilon}(a) \right|^2 \left\langle dX_s^{h,\varepsilon}(a), dX_s^{h,\varepsilon}(a) \right\rangle_g} \right\|_{L^p} \\ &\leq K \times \left\| \partial_a X^{h,\varepsilon}(a) \right\|_{\mathcal{F}^p} \mathcal{R}(\varepsilon)^{1/2}, \end{aligned} \quad (\text{II.4.18})$$

where  $K$  is a non-negative constant which does not depend on  $a$ . Let us continue to analyze the martingale local parts: that is [\(II.4.11\)](#) and the first part of [\(II.4.12\)](#). Since the stochastic deformed transports are essentially bounded, we obtain by using the BDG inequality and the estimates [\(II.4.15\)](#) and [\(II.4.16\)](#):

$$\begin{aligned} &\left\| \int_0^t \Theta_{0,s}^{-1}(X^{h,\varepsilon}(a)) \left( (a - a_n^h) \left( \nabla_a^2 \llbracket X_s^{h,\varepsilon}(a_n^h), X_s^{h,\varepsilon}(a) \rrbracket \right) + 2 \left( \nabla_a \llbracket X_s^{h,\varepsilon}(a_n^h), X_s^{h,\varepsilon}(a) \rrbracket \right) \right) \left( \Theta_{0,s} \left( X^{h,\varepsilon}(a_n^h) \right) dZ_s^{h,\varepsilon}(a_n^h) \right) \right\|_{\mathcal{F}^p} \\ &\leq K \left\| \sqrt{\int_0^1 \left( \left| \nabla_a^2 \llbracket X_s^{h,\varepsilon}(a_n^h), X_s^{h,\varepsilon}(a) \rrbracket \right|^2 + \left| \nabla_a \llbracket X_s^{h,\varepsilon}(a_n^h), X_s^{h,\varepsilon}(a) \rrbracket \right|^2 \right) \left\langle dZ_s^{h,\varepsilon}(a), dZ_s^{h,\varepsilon}(a) \right\rangle_g} \right\|_{L^p} \\ &\leq K \left( \left\| \nabla_a \llbracket X^{h,\varepsilon}(a_n^h), X^{h,\varepsilon}(a) \rrbracket \right\|_{\mathcal{F}^p} + \left\| \nabla_a^2 \llbracket X^{h,\varepsilon}(a_n^h), X^{h,\varepsilon}(a) \rrbracket \right\|_{\mathcal{F}^p} \right) \mathcal{Q}(\varepsilon)^{1/2} \\ &\leq K \left( 2CR(\mathcal{B}) \left\| \partial_a X^{h,\varepsilon}(a) \right\|_{\mathcal{F}^p} + C \left\| \partial_a X^{h,\varepsilon}(a) \right\|_{\mathcal{F}^{2p}}^2 \right) \mathcal{Q}(\varepsilon)^{1/2}. \end{aligned}$$

The first part of [\(II.4.12\)](#) can be treated with the same tools. Let us investigate now the finite variation part [\(II.4.13\)](#) and [\(II.4.14\)](#). Using [Proposition – II.4.1](#), we obtain the following estimates:

$$\begin{aligned} &\left\| \int_0^{\cdot} \Theta_{0,s}^{-1}(X^{h,\varepsilon}(a)) \left( \nabla_a R \left( \partial_a X_s^{h,\varepsilon}(a), dX_s^{h,\varepsilon}(a) \right) dX_s^{h,\varepsilon}(a) + \nabla_{dX_s^{h,\varepsilon}(a)} R \left( \partial_a X_s^{h,\varepsilon}(a), dX_s^{h,\varepsilon}(a) \right) \partial_a X_s^{h,\varepsilon}(a) \right) \right\|_{\mathcal{F}^p} \\ &\leq K \left\| \int_0^1 \left| \partial_a X_s^{h,\varepsilon}(a) \right|^2 \left\langle dX_s^{h,\varepsilon}(a), dX_s^{h,\varepsilon}(a) \right\rangle_g \right\|_{L^p} \\ &\leq K \left\| \partial_a X^{h,\varepsilon}(a) \right\|_{\mathcal{F}^{2p}}^2 \left\| \int_0^1 \left\langle dX_s^{h,\varepsilon}(a), dX_s^{h,\varepsilon}(a) \right\rangle_g \right\|_{L^\infty} \\ &\leq K \mathcal{R}(\varepsilon) \times \left\| \partial_a X^{h,\varepsilon}(a) \right\|_{\mathcal{F}^{2p}}^2. \end{aligned} \quad (\text{II.4.19})$$

To achieve the proof, let us analyse the term (II.4.14). The  $\mathcal{S}^p$ -norm satisfies:

$$\begin{aligned} & \left\| 2 \times \int_0^\cdot \Theta_{0,s}^{-1}(X^{h,\varepsilon}(a)) R \left( dX_s^{h,\varepsilon}(a), \partial_a X_s^{h,\varepsilon}(a) \right) \left( \nabla_a d^\nabla X_s^{h,\varepsilon}(a) \right) \right\|_{\mathcal{S}^p} \\ & \leq 2 \left\| \int_0^\cdot \Theta_{0,s}^{-1}(X^{h,\varepsilon}(a)) R \left( dX_s^{h,\varepsilon}(a), \partial_a X_s^{h,\varepsilon}(a) \right) \left( \nabla_a \parallel_{X_s^{h,\varepsilon}(a_n^h), X_s^{h,\varepsilon}(a)} \right) d^\nabla X_s^{h,\varepsilon}(a_n^h) \right\|_{\mathcal{S}^p} \\ & + 2(a - a_n^h) \left\| \int_0^\cdot \Theta_{0,s}^{-1}(X^{h,\varepsilon}(a)) R \left( dX_s^{h,\varepsilon}(a), \partial_a X_s^{h,\varepsilon}(a) \right) \left( \nabla_a \parallel_{X_s^{h,\varepsilon}(a_n^h), X_s^{h,\varepsilon}(a)} \right) \left( \Theta_{0,s} \left( X_s^{h,\varepsilon}(a_n^h) \right) dZ_s^{h,\varepsilon}(a_n^h) \right) \right\|_{\mathcal{S}^p} \\ & + 2 \left\| \int_0^\cdot \Theta_{0,s}^{-1}(X^{h,\varepsilon}(a)) R \left( dX_s^{h,\varepsilon}(a), \partial_a X_s^{h,\varepsilon}(a) \right) \left( \parallel_{X_s^{h,\varepsilon}(a_n^h), X_s^{h,\varepsilon}(a)} \left( \Theta_{0,s} \left( X_s^{h,\varepsilon}(a_n^h) \right) dZ_s^{h,\varepsilon}(a_n^h) \right) \right) \right\|_{\mathcal{S}^p}. \end{aligned}$$

We are going to use the following statements: the deformed stochastic transports are in  $\mathcal{S}^\infty$  (see Proposition – II.4.1), inequality (II.4.15), isometry property of parallel transports and:

$$\left\| \int_0^1 \left\langle dZ^{h,\varepsilon}(a_n^h), dZ^{h,\varepsilon}(a_n^h) \right\rangle_g \right\|_{L^\infty} \leq Q(\varepsilon).$$

We obtain

$$\begin{aligned} & \left\| 2 \times \int_0^\cdot \Theta_{0,s}^{-1}(X^{h,\varepsilon}(a)) R \left( dX_s^{h,\varepsilon}(a), \partial_a X_s^{h,\varepsilon}(a) \right) \left( \nabla_a d^\nabla X_s^{h,\varepsilon}(a) \right) \right\|_{\mathcal{S}^p} \\ & \leq K \left( \|\partial_a X^{h,\varepsilon}(a)\|_{\mathcal{S}^{2p}}^2 \mathcal{R}(\varepsilon) + \|\partial_a X^{h,\varepsilon}(a)\|_{\mathcal{S}^{2p}}^2 \mathcal{R}(\varepsilon)^{1/2} Q(\varepsilon)^{1/2} + \|\partial_a X^{h,\varepsilon}(a)\|_{\mathcal{S}^p} \mathcal{R}(\varepsilon)^{1/2} Q(\varepsilon)^{1/2} \right), \end{aligned}$$

where  $K$  do not depend on  $a$ . By writing finally

$$\nabla_a \partial_a X^{h,\varepsilon}(a) = \Theta_{0,\cdot}(X^{h,\varepsilon}(a)) \Theta_{0,\cdot}^{-1}(X^{h,\varepsilon}(a)) \nabla_a \partial_a X^{h,\varepsilon}(a),$$

and using in addition the step three, we get:

$$\sup_{a \geq 0, h \leq \delta} \|\nabla_a \partial_a X^{h,\varepsilon}(a)\|_{\mathcal{S}^p} \leq \exp\left(\frac{\kappa^-(\mathcal{M})}{2} \mathcal{R}(\varepsilon)\right) \left( \sup_{a \geq 0, h \leq \delta} \left\| \Theta_{0,\cdot}^{-1}(X^{h,\varepsilon}(a)) \nabla_a \partial_a X^{h,\varepsilon}(a) \right\|_{\mathcal{S}^p} \right) < \infty. \quad (\text{II.4.20})$$

**Step 5 — Conclusion.** According to the last steps, the following formula holds true:

$$\partial_a^2 \Delta^{h,\varepsilon}(a) = -2 \times \mathbf{E} \left( \left\langle \overline{\nabla_a X_1^{h,\varepsilon}(a)} \overset{\rightharpoonup}{\xi}, \partial_a X_1^{h,\varepsilon}(a) \right\rangle_g \right) - 2 \times \mathbf{E} \left( \left\langle \overline{X_1^{h,\varepsilon}(a)} \overset{\rightharpoonup}{\xi}, \nabla_a \partial_a X_1^{h,\varepsilon}(a) \right\rangle_g \right).$$

Moreover, for all  $p > 1$ , there exists a constant  $\delta_\varepsilon^*$  satisfying : for all  $h \leq \delta$ :

$$|\partial_a^2 \Delta^{h,\varepsilon}(a)| \leq 2 \times \left\| \left\langle \overline{\nabla_a X_1^{h,\varepsilon}(a)} \overset{\rightharpoonup}{\xi}, \partial_a X_1^{h,\varepsilon}(a) \right\rangle_g \right\|_{L^p} + 2 \times \left\| \left\langle \overline{X_1^{h,\varepsilon}(a)} \overset{\rightharpoonup}{\xi}, \nabla_a \partial_a X_1^{h,\varepsilon}(a) \right\rangle_g \right\|_{L^p} \leq \delta_\varepsilon^*.$$

□

Proof. [of Proposition – II.3.2.] Taking into account Proposition – II.4.2, we get

$$\partial|_{a=a_n^h} \Delta^{h,\varepsilon}(a) = -2 \times \mathbf{E} \left( \left\langle \left( \widetilde{J}_n^{h,\varepsilon} \right)_{t=1}, J_1^{h,\varepsilon}(a_n^h) \right\rangle \right).$$

Recalling equation (II.3.6), a Taylor's expansion of  $\Delta^{h,\varepsilon}$  with respect to  $a$  at  $a = a_n^h$  gives us for all  $a \in [a_n^h, a_{n+1}^h]$ :

$$\Delta^{h,\varepsilon}(a_{n+1}^h) \leq \mathbf{E} \left( \rho^2 \left( X_1^{h,\varepsilon}(a_n^h), \widetilde{\xi} \right) \right) - 2(a_{n+1}^h - a_n^h) \times \mathbf{E} \left( \left\langle \left( \widetilde{J}_n^{h,\varepsilon} \right)_{t=1}, J_1^{h,\varepsilon}(a_n^h) \right\rangle \right) + \frac{(a_{n+1}^h - a_n^h)^2}{2} \delta_\varepsilon^*,$$

where  $\delta_\varepsilon^*$  is the supremum on  $[a_n^h, a_{n+1}^h]$  of the second order derivative of  $\Delta^{h,\varepsilon}$  given by Proposition – II.3.1. Let us analyze the first order term. We have by definition  $\left( \widetilde{J}_n^{h,\varepsilon} \right)_{t=1} = \overrightarrow{X_1^{h,\varepsilon}(a_n^h) \widetilde{\xi}}$ , and the  $\nabla^c$ -martingale  $\widetilde{J}_n^{h,\varepsilon}$  satisfies:

$$\left\| J_1^{h,\varepsilon}(a_n^h) - \overrightarrow{X_n^{h,\varepsilon}, X_1^{h,\varepsilon}(a_n^h)} \left( \widetilde{J}_n^{h,\varepsilon} \right)_{t=1} \right\|_{L^2(\Omega)} \leq \varepsilon S.$$

We can thus split the first order term:

$$\mathbf{E} \left( \left\langle \overrightarrow{X_1^{h,\varepsilon}(a_n^h) \widetilde{\xi}}, J_1^{h,\varepsilon}(a_n^h) \right\rangle_g \right) = \mathbf{E} \left( \left\langle \overrightarrow{X_1^{h,\varepsilon}(a_n^h) \widetilde{\xi}}, J_1^{h,\varepsilon}(a_n^h) - \overrightarrow{X_n^{h,\varepsilon}, X_1^{h,\varepsilon}(a_n^h)} \left( \widetilde{J}_n^{h,\varepsilon} \right)_{t=1} \right\rangle_g \right) \quad (\text{II.4.21})$$

$$+ \mathbf{E} \left( \left\langle \overrightarrow{X_1^{h,\varepsilon}(a_n^h) \widetilde{\xi}}, \overrightarrow{X_n^{h,\varepsilon}, X_1^{h,\varepsilon}(a_n^h)} \left( \left( \widetilde{J}_n^{h,\varepsilon} \right)_{t=1} - \overrightarrow{X_n^{h,\varepsilon} \widetilde{\xi}} \right) \right\rangle_g \right) \quad (\text{II.4.22})$$

$$+ \mathbf{E} \left( \left\langle \overrightarrow{X_1^{h,\varepsilon}(a_n^h) \widetilde{\xi}}, \overrightarrow{X_n^{h,\varepsilon} \widetilde{\xi}} - \overrightarrow{X_1^{h,\varepsilon}(a_n^h) \widetilde{\xi}} \right\rangle_g \right) \quad (\text{II.4.23})$$

$$+ \mathbf{E} \left( \rho^2 \left( X_1^{h,\varepsilon}(a_n^h), \widetilde{\xi} \right) \right).$$

The term (II.4.22) is equal to zero, (II.4.21) and (II.4.23) are small by taking  $\varepsilon$  small enough. More precisely, using the inequality

$$\left| \overrightarrow{X_1^{h,\varepsilon}(a_n^h) \widetilde{\xi}} \right| \leq 2R(\mathcal{B}),$$

the Cauchy-Schwarz inequality, Proposition – II.1.2 and Proposition – II.1.3 on the relatively

compact set  $\mathcal{B}$ : there exists a constant  $L$  (depending on  $d$  and  $\mathcal{B}$ ) such that for all  $n, h, \varepsilon$ :

$$\begin{aligned} & \left\| J_1^{h,\varepsilon}(a_n^h) - \left\|_{\widetilde{X}_n^{h,\varepsilon}, X_1^{h,\varepsilon}(a_n^h)} \left( \widetilde{J}_n^{h,\varepsilon} \right)_{t=1} \right\|_{L^2(\Omega)} \right\| \leq \varepsilon S, \\ & \left| \left\|_{\widetilde{X}_n^{h,\varepsilon}, X_1^{h,\varepsilon}(a_n^h)} \left( \widetilde{X}_n^{h,\varepsilon} \right)_{t=1} \right\|_{\widetilde{\xi}} - \left\|_{X_1^{h,\varepsilon}(a_n^h)} \left( \widetilde{X}_n^{h,\varepsilon} \right)_{t=1} \right\|_{\widetilde{\xi}} \right| \leq L \times \rho \left( \left( \widetilde{X}_n^{h,\varepsilon} \right)_{t=1}, X_1^{h,\varepsilon}(a_n^h) \right) \leq L\varepsilon, \end{aligned}$$

there exists thus a non-negative constant  $\lambda^*$  (depending only on  $d, \mathcal{B}$  and  $S, L$ ) such that

$$\begin{aligned} & \mathbf{E} \left( \left\langle \left\|_{X_1^{h,\varepsilon}(a_n^h)} \left( \widetilde{X}_n^{h,\varepsilon} \right)_{t=1} \right\|_{\widetilde{\xi}}, J_1^{h,\varepsilon}(a_n^h) - \left\|_{\widetilde{X}_n^{h,\varepsilon}, X_1^{h,\varepsilon}(a_n^h)} \left( \widetilde{J}_n^{h,\varepsilon} \right)_{t=1} \right\|_g \right\rangle \right) \\ & + \mathbf{E} \left( \left\langle \left\|_{X_1^{h,\varepsilon}(a_n^h)} \left( \widetilde{X}_n^{h,\varepsilon} \right)_{t=1} \right\|_{\widetilde{\xi}}, \left\|_{\widetilde{X}_n^{h,\varepsilon}, X_1^{h,\varepsilon}(a_n^h)} \left( \widetilde{X}_n^{h,\varepsilon} \right)_{t=1} \right\|_{\widetilde{\xi}} - \left\|_{X_1^{h,\varepsilon}(a_n^h)} \left( \widetilde{X}_n^{h,\varepsilon} \right)_{t=1} \right\|_{\widetilde{\xi}} \right\rangle_g \right) \leq \lambda^* \varepsilon. \end{aligned} \quad (\text{II.4.24})$$

Finally:

$$\Delta^{h,\varepsilon}(a_{n+1}^h) \leq \left( 1 - 2(a_{n+1}^h - a_n^h) \right) \Delta^{h,\varepsilon}(a_n^h) + 2\lambda^*(a_{n+1}^h - a_n^h)\varepsilon + \frac{(a_{n+1}^h - a_n^h)^2}{2} \delta_\varepsilon^*. \quad (\text{II.4.25})$$

The sequence defined by  $a_{n+1}^h = a_n^h + \frac{2h}{\delta_\varepsilon^*} \left( \Delta^{h,\varepsilon}(a_n^h) - \lambda^* \varepsilon \right)$  is a rescaled (in  $h$ ) minimizer with respect to  $a_{n+1}^h - a_n^h$  of the right member (II.4.25). We obtain with a straightforward calculation:

$$\begin{aligned} \Delta^{h,\varepsilon}(a_{n+1}^h) & \leq \Delta^{h,\varepsilon}(a_n^h) \left( 1 + \frac{2h(h-2)}{\delta_\varepsilon^*} \left( \Delta^{h,\varepsilon}(a_n^h) - 2\lambda^* \varepsilon \right) \right) + \frac{2(\lambda^*)^2 \varepsilon^2 h(h-2)}{\delta_\varepsilon^*} \\ & \leq \Delta^{h,\varepsilon}(a_n^h) \left( 1 + \frac{2h(h-2)}{\delta_\varepsilon^*} \Delta^{h,\varepsilon}(a_n^h) \right) + \frac{2\lambda^* \varepsilon h(h-2)}{\delta_\varepsilon^*} \left( \lambda^* \varepsilon - 2\Delta^{h,\varepsilon}(a_n^h) \right). \end{aligned}$$

Let us consider the mapping  $f^{h,\varepsilon}$  defined below:

$$f^{h,\varepsilon} : x \mapsto x \left( 1 + \frac{2h(h-2)}{\delta_\varepsilon^*} x \right) + \frac{2\lambda^* \varepsilon h(h-2)}{\delta_\varepsilon^*} (\lambda^* \varepsilon - 2x),$$

whose unique fixed point is  $\lambda^* \varepsilon$  and maximum at  $x = \lambda^* \varepsilon + \frac{\delta_\varepsilon^*}{4h(2-h)}$ . Hence, the sequence  $\left( \Delta^{h,\varepsilon}(a_n^h) \right)_{n \in \mathbf{N}}$  is non-increasing and tends to  $\lambda^* \varepsilon$  as soon as

$$h \in (0, 2), \quad \Delta^{h,\varepsilon}(a_0^h) = \mathbf{E} \left( \rho^2 \left( \widetilde{\xi}^0; \widetilde{\xi} \right) \right) \in \left[ 0, \lambda^* \varepsilon + \frac{\delta_\varepsilon^*}{4h(2-h)} \right). \quad (\text{II.4.26})$$

In the sequel  $h_{\xi^0}^\varepsilon$  will state for  $h \in (0, 2)$  small enough such that (II.4.26) is fulfilled, implying

$$\lim_{n \rightarrow \infty} \mathbf{E} \left( \rho^2 \left( X_1^{h_{\xi^0}^\varepsilon, \varepsilon} \left( a_n^{h_{\xi^0}^\varepsilon} \right), \tilde{\xi} \right) \right) = \lambda^* \varepsilon.$$

Let us denote  $X^{\varepsilon, n} = X^{h_{\xi^0}^\varepsilon, \varepsilon} \left( a_n^{h_{\xi^0}^\varepsilon} \right)$ , lying in  $\mathcal{B}$  for all  $n$ . Considering a subsequence of  $(X^{\varepsilon, n})_{n \in \mathbf{N}}$ , we can assume that there exists  $\tilde{\xi}^\varepsilon \in \mathcal{B}$  satisfying  $\mathbf{E} \left( \rho^2 \left( \tilde{\xi}^\varepsilon, \tilde{\xi} \right) \right) = \lambda^* \varepsilon$  and

$$\lim_{n \rightarrow \infty} \mathbf{E} \left( \rho^2 \left( X_1^{\varepsilon, n}, \tilde{\xi}^\varepsilon \right) \right) = 0.$$

Using the separant  $\Psi$  defined on  $\mathcal{B}$ , the submartingale property of  $\Psi \left( X^{\varepsilon, n}, X^{\varepsilon, n'} \right)$ , and since  $q^* > 2$  and  $X^{\varepsilon, n}$  is in  $\mathcal{B}$ , we deduce that for all  $t \in [0, 1]$ :

$$\Psi \left( X_t^{\varepsilon, n}, X_t^{\varepsilon, n'} \right) \leq K \left[ \mathbf{E} \left( \rho^2 \left( X_1^{\varepsilon, n}, \tilde{\xi}^\varepsilon \right) \middle| \mathcal{F}_t \right) + \mathbf{E} \left( \rho^2 \left( X_1^{\varepsilon, n'}, \tilde{\xi}^\varepsilon \right) \middle| \mathcal{F}_t \right) \right].$$

Using the Doob inequality, once again the convex geometry property and passing to the limit, we obtain:

$$\limsup_{n, n' \rightarrow \infty} \mathbf{E} \left( \sup_{t \in [0, 1]} \rho^{q^*} \left( X_t^{\varepsilon, n}, X_t^{\varepsilon, n'} \right) \right) = 0. \quad (\text{II.4.27})$$

Consequently, the family of  $\nabla$ -martingales  $(X^{\varepsilon, n})_{n \in \mathbf{N}}$  converge with respect to the semimartingale topology to a  $\nabla$ -martingale  $X^\varepsilon$  satisfying  $X_1^\varepsilon = \tilde{\xi}^\varepsilon$ . With the same tools, mainly the convex geometry of  $\mathcal{B}$ , we establish that  $X^\varepsilon$  is convergent when  $\varepsilon$  goes to zero, and the limit process is a  $\nabla$ -martingale lying in  $\mathcal{B}$  with terminal value  $\tilde{\xi}$ .  $\square$

#### § II.4.4. Construction of a solution for the main ODE (II.3.1).

Proof. [of Theorem – II.3.1]

**Step 1 — Construction of  $X(a)$ .** For all  $a \in [0, 1]$ , with Proposition – II.3.2, there exists a  $\nabla$ -martingale  $X(a)$  such that  $X_1(a) = \xi(a)$ .

**Step 2 — Differentiability of the family  $X(a)$ .** Let us show firstly that for all  $t \in [0, 1]$ ,  $X_t(\cdot)$  is differentiable by using the same kind of procedure as Arnaudon in [Arn97], Proposition 3.7.

Let  $\tilde{J}(a) = \mathbf{E} \left( \Theta_{\cdot, 1}^{-1}(X(a)) \partial_a \xi(a) \middle| \mathcal{F}_\cdot \right)$  a  $\nabla^c$ -martingale with terminal value  $\xi(a)$ . We establish that  $a \mapsto X(a)$  is differentiable with derivative  $\tilde{J}(a)$ .

Using (RegBall, Algo), there exists a separant function  $\Psi$  and  $p = q^* > 1$  such that for all  $b \in$

$[0, 1 - a]$ ,

$$c \times \rho^p(X(a), X(a+b)) \leq \Psi(X(a), X(a+b)) \leq C \times \rho^p(X(a), X(a+b)).$$

But  $\Psi(X(a), X(a+b))$  is a real non-negative submartingale, implying for all  $t \in [0, 1]$ :

$$\rho^p(X_t(a), X_t(a+b)) \leq \frac{C}{c} \times \mathbf{E}(\rho^p(\xi(a), \xi(a+b)) | \mathcal{F}_t).$$

The curve  $\xi$  takes values in  $\mathcal{B}$ . With the mean-value theorem, there exists a constant  $K$  such that

$$\rho^p(\xi(a), \xi(a+b)) \leq K^p b^p,$$

and consequently,

$$\rho(X_t(a), X_t(a+b)) \leq \left(\frac{C}{c}\right)^{1/p} \times Kb.$$

When  $b$  tends to zero,  $\rho^p(X_t(a), X_t(a+b))$  tends to zero also, and we can consider a relatively compact open domain  $\mathcal{B}'_a$  of  $T\mathcal{M}$ ,  $b_0(a) > 0$ , such that for all  $b \in (0, b_0(a))$ ,

$$\frac{1}{b} \overrightarrow{X(a)X(a+b)} \in \mathcal{B}'_a, \quad \partial_a X(a) \in \mathcal{B}'_a, \quad \mathcal{B}'_a \subset \mathcal{B}_a \subset \mathcal{B},$$

where  $\mathcal{B}_a$  is an open set such that  $l_a : \begin{cases} \mathcal{B}_a & \longrightarrow & l_a(\mathcal{B}_a) \subset \mathcal{B} \times \mathcal{B} \\ u & \longmapsto & (\pi(u), \exp_{\pi(u)}(au)) \end{cases}$  is a diffeomorphism.

Let us emphasize that  $\overrightarrow{X(a)X(a+b)}$  is well-defined since  $X$  takes values in the regular geodesic ball  $\mathcal{B}$ . Let  $\nabla^{(a)}$  the connection on  $T\mathcal{M}$  induced by  $l_a^{-1}$  and  $\nabla \otimes \nabla$ : more precisely, for all  $(X, Y) \in \Gamma(TT\mathcal{M})^2$ , we define

$$\nabla^{(a)}_Y X = (\nabla \otimes \nabla) X^{l_a},$$

where  $X^{l_a} = d(l_a)(X) \in \Gamma(T\mathcal{M} \times T\mathcal{M})$  is the push-forward of  $X$  by  $l_a$ . For all  $\varepsilon > 0$ , the mapping

$$\chi : (u, v) \in TM \times TM \longmapsto T^{\otimes q^*} \Psi(u - v)$$

is strictly  $\nabla^c \otimes \nabla^c$ -convex on the set  $\delta_\varepsilon = \{(u, v) \in TM^2, \pi(u) = \pi(v), |u - v| < \varepsilon\}$ . According to Proposition 3.1 of Arnaudon in [Arn97], the following convergence hold:

$$\lim_{b \rightarrow 0} (\nabla^{(b)} \otimes \nabla^c) d\chi(x)(u, u) = (\nabla^c \otimes \nabla^c) d\chi(x)(u, u) > 0, \quad (\text{II.4.28})$$

for all  $x$  uniformly in  $u$ , yielding the  $\nabla^{(b)} \otimes \nabla^c$ -convexity with  $b$  small enough of  $\chi$ , on the set  $\delta_\varepsilon$ . Let us consider now  $\chi^\varepsilon = \sup\{\chi, \varepsilon\}$ . The map  $\chi^\varepsilon$  is plainly  $\nabla^{(b)} \otimes \nabla^c$ -convex on  $\delta = \{(u, v) \in TM^2, \pi(u) = \pi(v)\}$ . Besides, the process

$$\left( \frac{1}{b} \overrightarrow{X(a)X(a+b)}, \widetilde{J(a)} \right)$$

is a  $\nabla^{(b)} \otimes \nabla^c$ -martingale. The process  $\chi^\varepsilon \left( \frac{1}{b} \overrightarrow{X(a)X(a+b)}, \widetilde{J(a)} \right)$  is thus a real nonnegative submartingale and we have:

$$\begin{aligned} & \chi^\varepsilon \left( \frac{1}{b} \overrightarrow{X_t(a)X_t(a+b)}, \widetilde{J_t(a)} \right) \\ & \leq \mathbf{E} \left( \chi^\varepsilon \left( \frac{1}{b} \overrightarrow{\xi(a)\xi(a+b)}, \widetilde{J_1(a)} \right) \middle| \mathcal{F}_t \right) = \mathbf{E} \left( \sup \left\{ \chi \left( \frac{1}{b} \overrightarrow{\xi(a)\xi(a+b)}, \partial_a \xi(a) \right), \varepsilon \right\} \middle| \mathcal{F}_t \right). \end{aligned} \quad (\text{II.4.29})$$

Since

$$\chi \left( \frac{1}{b} \overrightarrow{\xi(a)\xi(a+b)}, \partial_a \xi(a) \right) \leq C \left| \frac{1}{b} \overrightarrow{\xi(a)\xi(a+b)} - \partial_a \xi(a) \right|^p,$$

and  $\xi(\cdot)$  is  $\mathcal{C}^1$ , the right-member of (II.4.29) converges to  $\varepsilon$  when  $b$  goes to zero. Besides,

$$c \left| \frac{1}{b} \overrightarrow{X_t(a)X_t(a+b)} - \widetilde{J(a)} \right|^p \leq \chi^\varepsilon \left( \frac{1}{b} \overrightarrow{X_t(a)X_t(a+b)}, \widetilde{J_t(a)} \right),$$

hence, one has finally:

$$\limsup_{b \rightarrow 0} \left| \frac{1}{b} \overrightarrow{X_t(a)X_t(a+b)} - \widetilde{J(a)} \right|^p \leq \frac{C\varepsilon}{c}. \quad (\text{II.4.30})$$

When  $\varepsilon$  goes to zero in (II.4.30) (see [Ken90], Lemma 4.5 for more details), we obtain the claimed differentiability and the formula  $\partial_a X(a) = \widetilde{J(a)}$ .

**Step 3 — The family  $a \mapsto \partial_a X(a)$  is continuous and conclusion.** Since  $\partial_a X(a)$  is a  $\nabla^c$ -martingale, the process  $\Theta_{0,\cdot}^{-1}(X(a))\partial_a X(a)$  is a local martingale. Besides,  $T^{\otimes q^*}(\partial_a X(a))$  is a submartingale, yielding for all  $t \in [0, 1]$ :

$$c |\partial_a X_t(a)|^{q^*} \leq \mathbf{E} \left( T^{\otimes q^*}(\partial_a X_1(a)) \middle| \mathcal{F}_t \right) \leq C \|\partial_a \xi(a)\|_{L^\infty}^{q^*} \leq C(\partial \xi^*)^{q^*}. \quad (\text{II.4.31})$$

The process  $\partial_a X(a)$  is thus in  $\mathcal{S}^\infty$ , and taking into account that  $X(a)$  has exponential moments of order  $\frac{\kappa^+(\mathcal{B})q}{2}$  (see Proposition – II.1.4), we obtain with Proposition – II.1.6 the following estimate:

$$\left\| \Theta_{0,\cdot}^{-1}(X(a))\partial_a X(a) \right\|_{\mathcal{S}^2} \leq \left\| \exp \left( \frac{\kappa^+(\mathcal{B})}{2} \int_0^1 \langle dX(a), dX(a) \rangle_g \right) \right\|_{L^q} \|\partial_a X(a)\|_{\mathcal{S}^\infty}.$$

From here we deduce that  $\Theta_{0,\cdot}^{-1}(X(a))\partial_a X(a)$  is a martingale in  $T_{X_0(a)}\mathcal{M}$  and the following formula hold:

$$\partial_a X_t(a) = \mathbf{E} \left( \Theta_{t,1}^{-1}(X(a))\partial_a \xi(a) \middle| \mathcal{F}_t \right),$$

the family  $X(\cdot)$  satisfies the ordinary differential equation (II.3.1). The set

$$\widehat{K^X} = \left\{ u \in T\mathcal{M} : \pi u \in \mathcal{B}, \quad |u| \leq \frac{C}{c} (\partial \xi^*)^{q^*} \right\}$$

is a compact set of  $T\mathcal{M}$ , and  $\partial_a X(a)$  lies in  $\widehat{K^X}$ . Let us recall briefly how to construct a separant  $\tilde{\Psi}$  on  $\widehat{K^X}$  relatively to  $\nabla^c$  by checking the conditions of Corollary 3.4 of Kendall in [Ken92a]. Let us consider a random variable  $u$  with values in  $\widehat{K^X}$ , and show the uniqueness of a  $\nabla^c$ -martingale  $J$  in  $\widehat{K^X}$  such that  $J_1 = u$ . If  $J, J'$  are two solutions, then  $\pi J$  and  $\pi J'$  lie in  $\mathcal{B}$ , a set having convex geometry, and have the same terminal value  $\pi u$ . Hence  $\pi J = \pi J'$ . The process  $T^{\otimes q^*}(J - J')$  is thus well defined, and is a local submartingale. Since  $J$  and  $J'$  are bounded,  $T^{\otimes q^*}(J - J')$  is a submartingale and we obtain  $J = J'$  taking into account  $J_1 = J'_1$ .

Finally, for all  $a \in [0, 1]$  and  $h \in [0, 1 - a]$ , the process  $\tilde{\Psi}(\partial_{|a=a+h} X(a), \partial_a X(a))$  is a submartingale. Using (II.4.31), one has for all  $t \in [0, 1]$ :

$$\tilde{\Psi}(\partial_{|a=a+h} X_t(a), \partial_a X_t(a)) \leq \mathbf{E} \left( \tilde{\Psi}(\partial_{|a=a+h} \xi(a), \partial_a \xi(a)) \middle| \mathcal{F}_t \right). \quad (\text{II.4.32})$$

Since  $\tilde{\Psi}$  is continuous,  $\partial \xi^* < \infty$  and  $a \mapsto \partial_a \xi(a)$  is continuous, if  $h$  goes to zero in (II.4.32) we obtain:

$$\lim_{h \rightarrow 0} \tilde{\Psi}(\partial_{|a=a+h} X_t(a), \partial_a X_t(a)) = 0,$$

yielding the continuity of  $\partial X_t(a)$  since  $\tilde{\Psi}$  is continuous and vanishes exactly on the diagonal of  $\widehat{K^X}$ .  $\square$





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## Méthode basée sur les équations différentielles stochastiques rétrogrades.

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Nous établissons un résultat d’existence et d’unicité pour une classe d’équations différentielles stochastiques rétrogrades quadratiques (EDSR). Cette classe est caractérisée par des conditions *a priori* sur les solutions d’une suite d’EDSR approchées, dite *localisée*. Nous présentons des exemples d’applications. Notre approche est reliée à celle développée par Briand et Elie dans [BE13] [Stochastic Process. Appl. **123** 2921–2939] concernant les EDSR quadratiques scalaires.

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We establish an existence and uniqueness result for a class of multidimensional quadratic backward stochastic differential equations (BSDE). This class is characterized by constraints on some uniform *a priori* estimate on solutions of a sequence of approximated BSDEs. We also present effective examples of applications. Our approach relies on the strategy developed by Briand and Elie in [BE13] [Stochastic Process. Appl. **123** 2921–2939] concerning scalar quadratic BSDEs.

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## III.1 INTRODUCTION

**Backward Stochastic Differential Equations** Backward stochastic differential equations (BSDEs) have been first introduced in a linear version by Bismut [Bis73], but since the early nineties and the seminal work of Pardoux and Peng [PP90], there has been an increasing interest for these equations due to their wide range of applications in stochastic control, in finance or in the theory of partial differential equations. Let us recall that, solving a BSDE consists in finding an adapted pair of processes  $(Y, Z)$ , where  $Y$  is a  $\mathbf{R}^d$ -valued continuous process and  $Z$  is a  $\mathbf{R}^{d \times k}$ -valued progressively measurable process, satisfying the equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad a.s. \quad (\text{III.1.1})$$

where  $W$  is a  $k$ -dimensional Brownian motion with filtration  $(\mathcal{F}_t)_{t \in \mathbf{R}^+}$ ,  $\xi$  is a  $\mathcal{F}_T$ -measurable random variable called the terminal condition, and  $f$  is a (possibly random) function called the generator. Since the seminal paper of Pardoux and Peng [PP90] that gives an existence and uniqueness result for BSDEs with a Lipschitz generator, a huge amount of paper deal with extensions and applications. In particular, the class of BSDEs with generators of quadratic growth with respect to the variable  $z$ , has received a lot of attention in recent years. Concerning the scalar case, *i.e.*  $d = 1$ , existence and uniqueness of solutions for quadratic BSDEs has been first proved by Kobylanski in [Kob00]. Since then, many authors have worked on this question and the theory is now well understood: we refer to [Kob00, Tev08, BE13] when the terminal condition is bounded and to [BH06, BEK13, DHR11] for the unbounded case. We refer also to [GY14] for a study of BMO properties of  $Z$ .

In this paper we will focus on existence and uniqueness results for quadratic BSDEs in the multidimensional setting, *i.e.*  $d > 1$ . Let us remark that, in addition to its intrinsic mathematical interest, this question is important due to many applications of such equations. We can mention for example following applications: nonzero-sum risk-sensitive stochastic differential games in [EKH03, HT16], financial market equilibrium problems for several interacting agents in [ET15, FDR11, Fre14, BLDR15], financial price-impact models in [KP16b, KP16a], principal agent contracting problems with competitive interacting agents in [EP16], stochastic equilibria problems in incomplete financial markets [KXŽ15, XŽ16] or existence of martingales on curved spaces with a prescribed terminal condition [Dar95].

Let us note that moving from the scalar framework to the multidimensional one is quite challenging since tools usually used when  $d = 1$ , like monotone convergence or Girsanov transform, can no longer be used when  $d > 1$ . Moreover, Frei and dos Reis provide in [FDR11] an example of multidimensional quadratic BSDE with a bounded terminal condition and a very simple generator such that there is no solution to the equation. This informative counterexample show that it is hopeless to try to obtain a direct generalization of the Kobylanski existence and uniqueness theorem in the multidimensional framework or a direct extension of the Pardoux and Peng existence and uniqueness theorem for locally-Lipschitz generators. Nevertheless, we can find in the literature several papers that deal with special cases of multidimensional quadratic BSDEs and we give now a really brief summary of them.

First of all, a quite general result was obtain by Tevzadze in [Tev08], when the bounded terminal condition is small enough, by using a fixed-point argument and the theory of BMO martingales. Some generalizations with somewhat more general terminal conditions are considered in [Fre14, KP16a]. In [CN15], Cheridito and Nam treat some quadratic BSDEs with very specific generators. Before these papers, Darling was already able to construct a martingale on a manifold with a prescribed terminal condition by solving a multidimensional quadratic BSDE (see [Dar95]). Its proof relies on a stability result obtained by coupling arguments. Recently, the so-called quadratic diagonal case has been considered by Hu and Tang in [HT16]. To be more precise, they assume that the  $n$ th line of the generator has only a quadratic growth with respect to the  $n$ th line of  $Z$ . This type of assumption allows authors to use Girsanov transforms in their *a priori* estimates calculations. Some little bit more general assumptions are treated by Jamneshan, Kupper and Luo in [JKL14] (see also [LT15]). Finally, in the very recent paper [XŽ16], Xing and Žitković obtained a general result in a Markovian setting with weak regularity assumptions on the generator and the terminal condition. Instead of assuming some specific hypotheses on the generator, they suppose the existence of a so called Liapounov function which allows to obtain a uniform *a priori* estimate on some sequence  $(Y^n, Z^n)$  of approximations of  $(Y, Z)$ . Their approach relies on analytic methods. We refer to this paper for references on analytic and PDE methods for solving systems of quadratic semilinear parabolic PDEs.

**Our approach** Our approach for solving multidimensional quadratic BSDEs relies on the theory of BMO martingales and stability results as in [BE13]. To get more into the details about our strategy, let us recall the sketch of the proof used by Briand and Elie in [BE13]. The generator  $f$  is assumed to be locally Lipschitz and, to simplify, we assume that it depends only

on  $z$ . First of all, they consider the following approximated BSDE

$$Y_t^M = \xi + \int_t^T f(\rho^M(Z_s^M)) ds - \int_t^T Z_s^M dW_s, \quad 0 \leq t \leq T, \quad a.s.$$

where  $\rho^M$  is a projection on the centered Euclidean ball of radius  $M$ . Then existence and uniqueness of  $(Y^M, Z^M)$  is obvious since this new BSDE has a Lipschitz generator. Now, if we assume that  $\xi$  is Malliavin differentiable with a bounded Malliavin derivative, they show that  $Z^M$  is bounded uniformly with respect to  $M$ . Thus,  $(Y^M, Z^M) = (Y, Z)$  for  $M$  large enough. Importantly, the uniform bound on  $Z^M$  is obtained thanks to a uniform (with respect to  $M$ ) *a priori* estimate on the BMO norm of the martingale  $\int_0^\cdot Z_s^M dW_s$ . Subsequently, they extend their existence and uniqueness result for a general bounded terminal condition:  $\xi$  is approximated by a sequence  $(\xi^n)_{n \in \mathbb{N}}$  of bounded terminal conditions with bounded Malliavin derivatives and they consider  $(Y^n, Z^n)$  the solution of the following BSDE

$$Y_t^n = \xi^n + \int_t^T f(Z_s^n) ds - \int_t^T Z_s^n dW_s, \quad 0 \leq t \leq T, \quad a.s.$$

By using a stability result for quadratic BSDEs, they show that  $(Y^n, Z^n)$  is a Cauchy sequence that converges to the solution of the initial BSDE (III.1.1). Once again, the stability result used by Briand and Elie relies on a uniform (with respect to  $n$ ) *a priori* estimate on the BMO norm of the martingale  $\int_0^\cdot Z^n dW_s$ .

The aim of this paper is to adapt this approach in our multidimensional setting. In the first approximation step, we are able to show that  $Z^M$  is bounded uniformly with respect to  $M$  if we have a small enough uniform (with respect to  $M$ ) *a priori* estimate on the BMO norm of the martingale  $\int_0^\cdot Z_s^M dW_s$ . But, contrarily to the scalar case, it is not possible to show that we have an *a priori* estimate on the BMO norm of the martingale  $\int_0^\cdot Z_s^M dW_s$  under general quadratic assumptions on the generator (let us recall the counterexample provided by Frei and dos Reis in [FDR11]). So, this *a priori* estimate on the BMO norm of the martingale  $\int_0^\cdot Z_s^M dW_s$  becomes in our paper an *a priori* assumption and this assumption has to be verified on a case-by-case basis according to the BSDE structure. In the second approximation step, we are facing the same issue: we are able to show the existence and uniqueness of a solution to (III.1.1) by using a stability result if we have a small enough uniform (with respect to  $n$ ) *a priori* estimate on the BMO norm of the martingale  $\int_0^\cdot Z_s^n dW_s$ , and this *a priori* estimate becomes, once again, an assumption that has to be verified on a case-by-case basis according to the BSDE structure. Let us emphasize that the estimate on the boundedness of  $Z^M$  and the stability result used in the second step come from an adaptation of results obtained by Delbaen and Tang in [DT08]. The fact that our results are true only when we have a small enough uniform estimate on the BMO norm of the martingale  $\int_0^\cdot Z_s^M dW_s$  or  $\int_0^\cdot Z_s^n dW_s$  is the main limitation of our results. Nevertheless, we emphasize that this limitation is related to a crucial open question that could be independently investigated.

To be precise, we would like to know if the classical reverse Hölder inequality for exponential of BMO martingales (see Theorem 3.1 in [Kaz94]) stays true in a multidimensional setting, i.e. when we have a matrix valued BMO martingale. For further details we refer the reader to Remark III.3.2.

To show the interest of these theoretical results, we have to find now some frameworks for which we are able to obtain estimates on the BMO norm of martingales  $\int_0^\cdot Z_s^M dW_s$  and  $\int_0^\cdot Z_s^n dW_s$ . This is the purpose of Section III.5 where results of [Tev08, Dar95, HT16] are revisited. Let us note that one interest of our strategy comes from the fact that we obtain these estimates by very simple calculations that allow to easily get new results: for example, we are able to extend the result of Tevzadze when the generator satisfies a kind of monotone assumption with respect to  $y$  (see subsection III.2.2). Moreover, we can remark that obtaining such estimates is strongly related to finding a so-called Liapounov function in [XŽ16]. Result on the boundedness of  $Z$  is also interesting in itself since it allows to consider the initial quadratic BSDE (III.1.1) as a simple Lipschitz one which gives access to numerous results on Lipschitz BSDEs: numerical approximation schemes, differentiability, stability, and so on.

**Structure of the paper** In the remaining of the introduction, we introduce notations, the framework and general assumptions. We have collected in Section 2 all our main results in order to improve the readability of the paper. Section 3 contains some general results about SDEs and linear BSDEs adapted from [DT08]. Section 4 is devoted to the proof of stability properties, existence and uniqueness theorems for multidimensional quadratic BSDEs. Finally, proofs of the applications of previous theoretical results are given in Section 5.

### § III.1.1. Notations

◇ Let  $T > 0$ . We consider  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbf{P})$  a complete probability space where  $(\mathcal{F}_t)_{t \in [0, T]}$  is a Brownian filtration satisfying the usual conditions. In particular every càdlàg process has a continuous version. Every Brownian motion will be considered relatively to this filtered probability space. A  $k$ -dimensional Brownian motion  $W = (W^i)_{1 \leq i \leq k}$  is a process with values in  $\mathbf{R}^k$  and with independent Brownian components. Almost every process will be defined on a finite horizon  $[0, T]$ , either we will precise it explicitly. The stochastic integral of an adapted process  $H$  will be denoted by  $H \star W$ , and the Euclidean quadratic variation by  $\langle \cdot, \cdot \rangle$ . The Dolean-Dade exponential of a continuous real local martingale  $M$  is denoted by

$$\mathcal{E}(M) := \exp\left(M - \frac{1}{2}\langle M, M \rangle\right).$$

◇ *Linear notions* – On each  $\mathbf{R}^p$ , the scalar product will be simply denoted by a dot, including the canonical scalar product on  $\mathcal{M}_{dk}(\mathbf{R})$ :

$$M.N = \sum_{1 \leq i \leq d, 1 \leq j \leq k} M_{i,j} N_{i,j}.$$

For  $A \in \mathcal{M}_{dk}(\mathbf{R})$ ,  $A^{(\cdot,p)}$  will be the column  $p \in \{1, \dots, k\}$  of  $A$ , and  $A^{(l,\cdot)}$  the line  $l \in \{1, \dots, d\}$ . If  $B \in \mathcal{L}(\mathbf{R}^{d \times k}, \mathbf{R}^d)$ , we write for  $i \in \{1, \dots, k\}$ ,  $B^{(\cdot,i,\cdot)} \in \mathcal{L}(\mathbf{R}^d, \mathbf{R}^d)$  the linear map such that  $Bx = \sum_{i=1}^k B^{(\cdot,i,\cdot)} x^{(\cdot,i)}$  for all  $x \in \mathbf{R}^{d \times k}$ . If  $A$  and  $B$  are two processes with values in  $\mathcal{M}_{dk}(\mathbf{R})$  and  $\mathbf{R}^k$ , the quadratic variation  $\langle A, B \rangle$  is the  $\mathbf{R}^d$  vector process

$$\left( \sum_{l=1}^d \langle A^{il}, B^l \rangle \right)_{i=1}^d$$

and we have the integration by part formula  $d(AB) = dA \cdot B + A \cdot dB + d\langle A, B \rangle$ . We can also define the covariation of  $(A, B) \in \mathcal{M}_{dk}(\mathbf{R}) \times \mathcal{M}_{kd'}(\mathbf{R})$  by

$$\left( \sum_{l=1}^d \langle A^{il}, B^{lj} \rangle \right)_{i,j=1}^{d,d'}.$$

◇ *Functional spaces* – In a general way, Euclidean norms will be denoted by  $|\cdot|$  while norms relatively to  $\omega$  and  $t$  will be denoted by  $\|\cdot\|$ .

For a  $\mathcal{F}$ -adapted continuous process  $Y$  with values in  $\mathbf{R}^d$  and  $1 \leq p \leq \infty$ , let us define

$$\|Y\|_{\mathcal{S}^p} = \mathbf{E} \left( \sup_{0 \leq s \leq T} |Y_s|^p \right)^{1/p}, \quad \text{and} \quad \|Y\|_{\mathcal{S}^\infty} = \text{esssup}_{0 \leq s \leq T} |Y_s|.$$

If  $Z$  is a random variable with values in  $\mathbf{R}^d$ , we define

$$\|Z\|_{L^p} = \mathbf{E}(|Z|^p)^{1/p}.$$

A continuous martingale  $M$  with values in  $\mathbf{R}$  is in  $\mathcal{H}^p(\mathbf{R})$ , or only  $\mathcal{H}^p$  when it is not necessary to specify the state space, if  $\sqrt{\langle M, M \rangle_T} \in L^p$ . And we define the  $\mathcal{H}^p$  norm by

$$\|M\|_{\mathcal{H}^p} := \mathbf{E} \left( \langle M, M \rangle_T^{p/2} \right)^{1/p} < \infty.$$

If  $M$  is a martingale with values in  $\mathbf{R}^d$ ,  $M$  is in  $\mathcal{H}^p(\mathbf{R}^d)$  if  $|M|$  is in  $\mathcal{H}^p(\mathbf{R})$ . A real martingale  $M = (M_t)_{0 \leq t \leq T}$  is said to be BMO (bounded in mean oscillation) if there exists a constant  $C \geq 0$  such that for every stopping time  $0 \leq \tau \leq T$ :

$$\mathbf{E}((M_T - M_\tau)^2 | \mathcal{F}_\tau) \leq C^2 \text{ a.s.}$$

The best constant  $C$  is called the BMO norm of  $M$ , denoted by  $\|M\|_{BMO(\mathbf{P})}$  or sometimes only  $\|M\|_{BMO}$ . In particular, the one dimensional local martingale  $Z \star W^1 = \int_0^\cdot Z_s dW_s^1$  is BMO if

### III. Méthode basée sur les équations différentielles stochastiques rétrogrades.

there exists a constant  $C \geq 0$  such that, for all stopping time  $\tau$  with values in  $[0, T]$ , we have

$$\mathbf{E} \left( \int_{\tau}^T |Z_s|^2 ds \middle| \mathcal{F}_{\tau} \right) \leq C^2 \text{ a.s.}$$

In the sequel, to simplify notations we will skip the superscript  $\cdot^1$  on the Brownian motion after a star. For more details about BMO martingales, we can refer to [Kaz94].

For  $k \geq 1$ ,  $\mathcal{C}_b^{\infty}(\mathbf{R}^k)$  is the set of all  $\mathcal{C}^{\infty}$  functions with values in  $\mathbf{R}$  defined on  $\mathbf{R}^k$ , which have bounded derivatives.

Given  $b_0 \in \mathbf{R}^d$  and a sequence  $(\alpha_n) \in (0, 1]^{\mathbf{N}}$ , a function  $g : [0, T] \times \mathbf{R}^d \rightarrow \mathbf{R}^d$  is said to be in  $\mathcal{C}_{b_0}^{(\alpha_n), loc}$  if there exists a sequence  $(c_n)$  of positive constants, such that, for all  $n \in \mathbf{N}$ ,

$$\sup_{(t,x) \in [0,T] \times B_n(b_0)} |g(t,x)| + \sup_{(t,x) \neq (t',x') \in [0,T] \times B_n(b_0)} \frac{|g(t,x) - g(t',x')|}{|t - t'|^{\alpha_n/2} + |x - x'|^{\alpha_n}} \leq c_n,$$

where  $B_n(b_0)$  states for the Euclidean ball on  $\mathbf{R}^d$  of center  $b_0$  and radius  $n$ . If the last term does not depend on  $b_0$ , we shall say that  $g$  is in  $\mathcal{C}^{(\alpha_n), loc}$ . Finally, for a given  $\alpha \in (0, 1]$ , a function  $g : [0, T] \times \mathbf{R}^d \rightarrow \mathbf{R}^d$  is said to be in  $\mathcal{C}^{\alpha}$  if

$$\sup_{(t,x) \neq (t',x') \in [0,T] \times \mathbf{R}^d} \frac{|g(t,x) - g(t',x')|}{|t - t'|^{\alpha/2} + |x - x'|^{\alpha}} < +\infty.$$

*Remark – III.1.1.* We can plainly show that if there exists  $(\alpha_n) \in (0, 1]^{\mathbf{N}}$  such that a bounded solution  $v$  is in  $\mathcal{C}^{(\alpha_n), loc}$ , then  $v \in \mathcal{C}^{\alpha_1}$ .

- ◇ *Inequalities –* BDG inequalities claim that  $\|\cdot\|_{\mathcal{S}^p}$  and  $\|\cdot\|_{\mathcal{H}^p}$  are equivalent on martingale spaces with two universal constants denoted  $C'_p, C_p$ . It means that for all continuous local martingales  $M$  vanishing at 0,

$$\|M\|_{\mathcal{H}^p} \leq C_p \|M\|_{\mathcal{S}^p}$$

and

$$\|M\|_{\mathcal{S}^p} \leq C'_p \|M\|_{\mathcal{H}^p}.$$

In [MR16], Marinelli and Röckner deal with martingales taking values in a separable Hilbert space. In particular, the upper constant  $C'$  (Proposition 2.1 and Proposition 3.1) defined below is valid for all dimensions:

$$C'_p = \begin{cases} \left( \frac{p}{p-1} \right)^{\frac{p}{2}} \left( \frac{p(p-1)}{2} \right)^2 & \text{if } p > 2, \\ 4\sqrt{\frac{2}{p}} & \text{if } p < 2, \\ 4 & \text{if } p = 2. \end{cases} \quad (\text{III.1.2})$$

We remark that in the case  $p = 2$ , the scalar BDG constant is valid. In the following every BDG inequality should be understood with this choice of  $C'$ . The Doob maximal inequality



claims that for every  $\mathbf{R}^d$ -valued martingale  $M$  and  $p > 1$ ,

$$\|M\|_{\mathcal{S}^p} \leq \frac{p}{p-1} \|M_T\|_{L^p},$$

and for  $p = \infty$ ,

$$\|M\|_{\mathcal{S}^\infty} \leq \|M_T\|_{L^\infty}.$$

If  $p \in ]1, \infty[$ , we will denote by  $p^*$  the conjugated exponent of  $p$  such that  $\frac{1}{p} + \frac{1}{p^*} = 1$ . Finally, we say that a process  $L = (L_t)_{0 \leq t \leq T}$  with values in  $\mathbf{R}^d$  satisfies a reverse Hölder inequality for some integer  $1 \leq p < \infty$  if there exists some constant  $K_p$  such that for every stopping time  $0 \leq \tau \leq T$  a.s.,

$$\mathbf{E}(|L_T|^p | \mathcal{F}_\tau) \leq K_p |L_\tau|^p \text{ a.s.}$$

◇ *BMO martingales properties* – We recall here several results on BMO martingales that will be useful in the sequel. The energy inequality (see [Kaz94]) tells us that for every BMO martingale  $M$  and every integer  $n \geq 1$ , we have

$$\mathbf{E}(\langle M \rangle_T^n) \leq n! \|M\|_{BMO}^{2n}, \quad (\text{III.1.3})$$

and a conditional version of this inequality is also true: for all  $t \in [0, T]$ ,

$$\mathbf{E}(\langle M \rangle_T - \langle M \rangle_t)^n | \mathcal{F}_t) \leq n! \|M\|_{BMO}^{2n}. \quad (\text{III.1.4})$$

Consequently the space of BMO martingales is a subset of  $\bigcap_{p \geq 1} \mathcal{H}^p$ . We recall also the so-called Fefferman inequality: for  $X \in \mathcal{H}^1$  and  $Y \in BMO$ ,

$$\mathbf{E} \left( \int_0^T |d\langle X, Y \rangle_s| \right) \leq \|X\|_{\mathcal{H}^1} \|Y\|_{BMO}.$$

This inequality yields the following technical lemma (see [BB88] and [DT08] for more details).

**Lemma – III.1.1.** *Let  $m \geq 1$ . We consider  $X$  an adapted process and  $M$  a local martingale.*

(i) *If  $X \in \mathcal{S}^m$  and  $M \in BMO$ , then  $X \star M \in \mathcal{H}^m$  and*

$$\|X \star M\|_{\mathcal{H}^m} \leq \sqrt{2} \|X\|_{\mathcal{S}^m} \|M\|_{BMO}.$$

(ii) *If  $X \in \mathcal{H}^m$  and  $M \in BMO$ , then  $\langle X, M \rangle_T \in L^m$  and*

$$\|\langle X, M \rangle_T\|_{L^m} \leq \sqrt{2} m \|X\|_{\mathcal{H}^m} \|M\|_{BMO}.$$

The John-Nirenberg inequality gives a useful estimation on exponential moments of  $\langle Z \star W \rangle_T$ :

if  $\|Z \star W\|_{BMO} < 1$ , for any stopping time  $\tau \in [0, T]$  we have

$$\mathbf{E} \left( e^{\int_{\tau}^T |Z_s|^2 ds} \middle| \mathcal{F}_{\tau} \right) \leq \frac{1}{1 - \|Z \star W\|_{BMO}^2}. \quad (\text{III.1.5})$$

We have also a result about changes of probability law and equivalence of BMO norms on a BMO ball (see Lemma A.4 in [HT16] and Theorem 3.6 in [Kaz94]).

**Proposition – III.1.1.** *Let  $B > 0$ . There are two constants  $c_1 > 0$  and  $c_2 > 0$  depending only on  $B$ , such that for any BMO martingale  $M$ , we have for any BMO martingale  $N$  such that  $\|N\|_{BMO(\mathbf{P})} \leq B$ ,  $c_1 \|M\|_{BMO(\mathbf{P})} \leq \|M - \langle M, N \rangle\|_{BMO(\mathbf{Q})} \leq c_2 \|M\|_{BMO(\mathbf{P})}$ , where  $d\mathbf{Q} = \mathcal{E}(N)_T d\mathbf{P}$ .*

To conclude this paragraph, let us show a technical proposition that will be useful in this paper.

**Proposition – III.1.2.** *Let  $m \geq 1$  and a sequence of BMO-uniformly bounded local martingales  $(Z^n \star W)_{n \in \mathbf{N}}$ . We denote  $K = \sup_{n \in \mathbf{N}} \|Z^n \star W\|_{BMO} < \infty$  and assume that  $Z^n \star W$  converge in  $\mathcal{H}^m$  to a martingale  $Z \star W$ . Then  $Z \star W$  is BMO too and satisfies the same inequality  $\|Z \star W\|_{BMO} \leq K$ .*

Proof. Let us define by  $\mathcal{M}$  the measure  $d\mathcal{M} = d\mathbf{P} \otimes dx$ . Firstly we show that convergence in  $\mathcal{H}^m$  implies the convergence for the measure  $\mathcal{M}$ . Indeed, if  $m \geq 2$ , the Jensen inequality gives us

$$\mathbf{E} \left( \int_0^T |Z_s^n - Z_s|^2 ds \right) \leq \|Z^n \star W - Z \star W\|_{\mathcal{H}^m}^2,$$

and thus we get the convergence in measure, since for all  $\varepsilon > 0$ ,

$$\mathcal{M}(|Z^n - Z| > \varepsilon) \leq \frac{1}{\varepsilon^2} \|Z^n \star W - Z \star W\|_{\mathcal{H}^2}^2 \leq \frac{1}{\varepsilon^2} \|Z^n \star W - Z \star W\|_{\mathcal{H}^m}^2.$$

Moreover, if  $m < 2$  we also have

$$\mathcal{M}(|Z^n - Z| > \varepsilon) \leq \frac{1}{\varepsilon^m} \mathbf{E} \left( \int_0^T |Z_s^n - Z_s|^m ds \right) \leq \frac{T^{1-m/2}}{\varepsilon^m} \mathbf{E} \left( \left( \int_0^T |Z_s^n - Z_s|^2 ds \right)^{m/2} \right).$$

For the both cases, we get convergence in measure. Hence there exists a subsequence  $(n_k)_{k \in \mathbf{N}}$  such that

$$|Z^{n_k}|^2 \xrightarrow[k \rightarrow \infty]{} |Z|^2 \quad \mathcal{M} - \text{a.s.}$$

The Fatou lemma gives us for all stopping time  $\tau \in [0, T]$ ,

$$\int_{\tau}^T |Z_s|^2 ds \leq \liminf_{k \rightarrow \infty} \int_{\tau}^T |Z_s^{n_k}|^2 ds \quad \text{a.s.}$$

and taking the conditional expectation

$$\mathbf{E}\left(\int_{\tau}^T |Z_s|^2 ds \middle| \mathcal{F}_{\tau}\right) \leq \mathbf{E}\left(\liminf_{k \rightarrow \infty} \int_{\tau}^T |Z_s^{n_k}|^2 ds \middle| \mathcal{F}_{\tau}\right) \leq \liminf_{k \rightarrow \infty} \mathbf{E}\left(\int_{\tau}^T |Z_s^{n_k}|^2 ds \middle| \mathcal{F}_{\tau}\right) \leq K \quad \text{a.s.}$$

Finally  $Z \star W$  is BMO and  $\|Z \star W\|_{BMO} \leq K$ .  $\square$

- ◇ *Sliceability* – For a process  $X$  and a stopping time  $\tau$  we denote by  ${}^{\tau}X$  the process started at time  $\tau$ , that is  ${}^{\tau}X = X - X^{\downarrow\tau}$  where  $X^{\downarrow\tau}$  is the process stopped at  $\tau$ . For two stopping times  $\tau \leq \sigma$  a.s., we denote by  ${}^{\tau}X^{\downarrow\sigma}$  the process started at  $\tau$  and stopped at  $\sigma$ :

$${}^{\tau}X^{\downarrow\sigma} = ({}^{\tau}X)^{\downarrow\sigma}.$$

Associativity property of the stochastic integral can be rewritten with this notation:

$${}^{\tau}(H \star W)^{\downarrow\sigma} = H \star {}^{\tau}W^{\downarrow\sigma}.$$

Between  $\tau$  and  $\sigma$ , the started and stopped process is simply a translation of the stopped process: for all  $u$  such that  $\tau \leq u \leq \sigma$  a.s.,

$${}^{\tau}X_u^{\downarrow\sigma} = X_u - X_{\tau}.$$

This process is constant after  $\sigma$  and vanishes before  $\tau$ . Let us suppose that  $X$  is a BMO martingale. We say that  $X$  is  $\varepsilon$ -sliceable if there exists a subsequence of stopping times  $0 = T_0 \leq T_1 \leq \dots \leq T_N = T$ , where  $N \in \mathbf{N}$  is deterministic, such that

$$\left\| {}^{T_n}(X)^{\downarrow T_{n+1}} \right\|_{BMO} \leq \varepsilon.$$

The set of all  $\varepsilon$ -sliceable processes will be denoted by  $BMO_{\varepsilon}$ . Schachermayer proved in [Sch96] that

$$\bigcap_{\varepsilon > 0} BMO_{\varepsilon} = \overline{\mathcal{H}^{\infty}}^{BMO}.$$

Moreover the BMO norm of a started and stopped stochastic integral process  ${}^{\tau}Z \star W^{\downarrow\sigma}$  has a simple expression:

**Proposition – III.1.3.**

$$\left\| {}^{\tau}Z \star W^{\downarrow\sigma} \right\|_{BMO} = \text{esssup} \sup_{\tau' \in \mathcal{T}_{\tau, \sigma}} \mathbf{E}\left(\int_{\tau'}^{\sigma} |Z_s|^2 ds \middle| \mathcal{F}_{\tau'}\right),$$

where  $\mathcal{T}_{\tau, \sigma} = \{\tau' \text{ stopping time} : \tau \leq \tau' \leq \sigma \text{ a.s.}\}$ .

A proof of this proposition is given in the appendix part.

- ◇ *Malliavin calculus* – We denote by

$$\mathcal{P} = \{f((g_1 \star W)_T, \dots, (g_n \star W)_T) : f \in \mathcal{C}_b^{\infty}(\mathbf{R}^n), g_i \text{ adapted}, n \geq 1\},$$

the set of all Wiener functions. For  $F \in \mathcal{P}$ , the Malliavin derivative of  $F$  is a progressively measurable process  $\mathbf{D}F \in L^2([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F}, dx \otimes d\mathbf{P})$  defined by

$$\mathbf{D}_t F = \sum_{i=1}^n \partial_i f((g_1 \star W)_T, \dots, (g_n \star W)_T) g_i(t).$$

In particular  $\mathbf{D}((h \star W)_T) = h$  for all adapted process  $h$ . We define a kind of Sobolev norm on  $\mathcal{P}$  with the following definition

$$\|F\|_{1,2} := \left[ \mathbf{E}(|F|^2) + \mathbf{E}(\|\mathbf{D}F\|_{L^2(dx)}^2) \right]^{1/2}.$$

We can show that  $\mathbf{D}$  is closable, consequently it is possible to extend the definition of  $\mathbf{D}$  to  $\mathbb{D}^{1,2} = \overline{\mathcal{P}}^{1,2}$ . Besides,  $\mathbb{D}^{1,2}$  is dense in  $L^2(\Omega)$ . For further considerations on Malliavin calculus we can refer to [Nua06]. We finish this paragraph by the following useful result proved in [Nua06] (Proposition 1.2.4).

**Proposition – III.1.4.** *Let  $\varphi : \mathbf{R}^d \rightarrow \mathbf{R}$ . We assume that there exists a constant  $K$  such that for all  $x, y \in \mathbf{R}^d$ ,*

$$|\varphi(x) - \varphi(y)| \leq K|x - y|.$$

*Let  $(F^1, \dots, F^d)$  a vector in  $\mathbb{D}^{1,2}(\mathbf{R}^d) \cap L^\infty(\Omega)$ . Then  $\varphi(F) \in \mathbb{D}^{1,2}(\mathbf{R}^d)$  and there exists a random vector  $(G^1, \dots, G^d)$  such that*

$$\mathbf{D}\varphi(F) = \sum_{i=1}^d G^i \mathbf{D}F^i, \quad \text{and} \quad |G| \leq K.$$

### § III.1.2. Framework and first assumptions

In this paper we consider the following quadratic BSDE on  $\mathbf{R}^d$ :

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad a.s. \quad (\text{III.1.6})$$

where  $f$  is a random function  $\Omega \times [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^d$  called the *generator* of the BSDE such that for all  $(y, z) \in \mathbf{R}^d \times \mathbf{R}^{d \times k}$  and  $t \in [0, T]$ ,  $(f(t, y, z))_{0 \leq t \leq T}$  is progressively measurable,  $(Y, Z)$  is a process with values in  $\mathbf{R}^d \times \mathbf{R}^{d \times k}$  and  $\xi \in L^2(\mathcal{F}_T, \mathbf{R}^d)$ .

**Definition – III.1.1.** A solution of BSDE (III.1.6) is a process  $(Y, Z) \in \mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$  satisfying usual integrability conditions and solving initial BSDE:

$$(i) \int_0^T (|f(s, Y_s, Z_s)|^2 + |Z_s|^2) ds < \infty \quad a.s.,$$

$$(ii) Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, a.s.$$

Some locally Lipschitz assumptions on  $f$  and integrability assumptions on  $\xi$  and  $f$  will be assumed all along this paper.

(H) (i) For all  $(y, y', z, z') \in (\mathbf{R}^d)^2 \times (\mathbf{R}^{d \times k})^2$ , we assume that there exists  $(K_y, L_y, K_z, L_z) \in (\mathbf{R}^+)^4$  such that  $\mathbf{P}$  – a.s for all  $t \in [0, T]$ :

$$|f(t, y, z) - f(t, y', z)| \leq (K_y + L_y |z|^2) |y - y'|,$$

$$|f(t, y, z) - f(t, y, z')| \leq (K_z + L_z (|z| + |z'|)) |z - z'|,$$

(ii)  $\mathbf{E} \left( |\xi|^2 + \int_0^T |f(s, 0, 0)|^2 ds \right) < +\infty$ .

We denote by  $\mathbb{B}^m(L_y, L_z)$  the following quantity depending on  $L_y$  and  $L_z$ :

$$\mathbb{B}^m(L_y, L_z) := \begin{cases} \frac{-L_z C'_m + \sqrt{m L_y + (L_z C'_m)^2}}{\sqrt{2m L_y}} & \text{if } L_y \neq 0, \\ \frac{1}{2\sqrt{2} L_z C'_m} & \text{if } L_y = 0. \end{cases} \quad (\text{III.1.7})$$

For all  $m > 1$ , let us denote by  $\mathcal{L}_{BMO}^m$  the set

$$\mathcal{L}_{BMO}^m = \left\{ Z, \mathbf{R}^{d \times k} \text{-valued process} \left/ \begin{array}{l} 2m L_y \| |Z| \star W \|_{BMO}^2 + 2\sqrt{2} L_z C'_m \| |Z| \star W \|_{BMO} < 1 \end{array} \right. \right\},$$

which can be rewritten as

$$\mathcal{L}_{BMO}^m = \left\{ Z, \mathbf{R}^{d \times k} \text{-valued process} \left/ \begin{array}{l} \| |Z| \star W \|_{BMO} < \mathbb{B}^m(L_y, L_z) \end{array} \right. \right\},$$

where  $\mathbb{B}^m(L_y, L_z)$  is defined in (III.1.7). We also denote by  $\mathcal{L}_{BMO}^{slic, m}$  the set of all  $\mathbf{R}^{d \times k}$ -valued processes  $Z$  for which there exists a sequence of stopping times  $0 = T_0 \leq T_1 \leq \dots \leq T_N = T$  such that  ${}^{T_i}Z|^{T_{i+1}} \in \mathcal{L}_{BMO}^m$  for all  $i \in \{0, \dots, N\}$ .

To conclude this introduction, we finally consider an approximation of the BSDE (III.1.6). To this purpose let us introduce a localisation of  $f$  defined by  $f^M(t, y, z) = f(t, y, \rho^M(z))$  where  $\rho^M : \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^{d \times k}$  satisfies the following properties :

- $\rho^M$  is the identity on  $\mathcal{B}_{\mathbf{R}^{d \times k}}(0, M)$ ,
- $\rho^M$  is the projection on  $\mathcal{B}_{\mathbf{R}^{d \times k}}(0, M+1)$  outside  $\mathcal{B}_{\mathbf{R}^{d \times k}}(0, M+2)$ ,
- $\rho^M$  is a  $\mathcal{C}^\infty$  function with  $|\nabla \rho^M(z)| \leq 1$  for all  $z \in \mathbf{R}^{d \times k}$ .

Thus  $f^M$  is a globally Lipschitz function with constants depending on  $M$ . Indeed we have for all

$$(t, y, y', z, z') \in [0, T] \times (\mathbf{R}^d)^2 \times (\mathbf{R}^{d \times k})^2,$$

$$\begin{aligned} |f^M(t, y, z) - f^M(t, y', z')| &\leq |f(t, y, \rho^M(z)) - f(t, y', \rho^M(z))| + |f(t, y', \rho^M(z)) - f(t, y', \rho^M(z'))| \\ &\leq (K_y + L_y |\rho^M(z)|^2) |y - y'| + (K_z + L_z (|\rho^M(z)| + |\rho^M(z')|)) |z - z'| \\ &\leq (K_y + L_y (M + 1)^2) |y - y'| + (K_z + 2L_z (M + 1)) |z - z'|. \end{aligned}$$

Then, according to the classical result of Pardoux and Peng in [PP90], there exists a unique solution  $(Y^M, Z^M) \in \mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$  of the localized BSDE

$$Y_t^M = \xi + \int_t^T f^M(s, Y_s^M, Z_s^M) ds - \int_t^T Z_s^M dW_s, \quad 0 \leq t \leq T. \quad (\text{III.1.8})$$

## III.2 MAIN RESULTS

We have collect in this section principal results proved in our article. All proofs are postponed to sections III.4 and III.5. The following subsection gives some existence and uniqueness results while subsection III.2.2 is dedicated to particular frameworks where these existence and uniqueness results apply.

### § III.2.1. Some general existence and uniqueness results

#### EXISTENCE AND UNIQUENESS RESULTS WHEN THE TERMINAL CONDITION AND THE GENERATOR HAVE BOUNDED MALLIAVIN DERIVATIVES

We consider here a particular framework where the terminal condition and the random part of the generator have bounded Malliavin derivatives. More precisely, let us consider the following assumptions.

**(Dxi,b)** The Malliavin derivative of  $\xi$  is bounded:

$$\|\mathbf{D}\xi\|_{\mathcal{S}^\infty} = \sup_{0 \leq t \leq T} \|\mathbf{D}_t \xi\|_{L^\infty} < \infty.$$

**(Df,b)** (i) For all  $(t, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$ , we have

$$f(t, y, z) \in \mathbb{D}^{1,2}(\mathbf{R}^d), \quad \text{and} \quad \mathbf{E} \left( \int_0^T \int_0^T |\mathbf{D}_u f(s, y, z)| du ds \right) < \infty.$$

(ii) There exists  $C \geq 0$  such that for all  $(u, t, y, z) \in [0, T]^2 \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$ ,

$$|\mathbf{D}_u f(t, y, z)| \leq C \left( 1 + |z|^2 \right) \text{ a.s.}$$

(iii) For all  $(u, t) \in [0, T]^2$ , there exists a random variable  $C_u(t)$  such that for all

$$\begin{aligned}
 (y^1, z^1, y^2, z^2) &\in (\mathbf{R}^d \times \mathbf{R}^{d \times k})^2, \\
 |\mathbf{D}_u f(t, y^1, z^1) - \mathbf{D}_u f(t, y^2, z^2)| \\
 &\leq C_u(t) \left( (1 + |z^1|^2 + |z^2|^2) |y^1 - y^2| + (1 + |z^1| + |z^2|) |z^1 - z^2| \right) \text{ a.s.}
 \end{aligned}$$

By recalling that  $(Y^M, Z^M)$  is the unique solution of (III.1.8), we will also assume that we have an *a priori* estimate on  $|Z^M| \star W$  uniform in  $M$  and small enough. For a given  $m > 1$  we consider the following assumption:

**(BMO,m)** there exists a constant  $\mathbb{K} < \mathbb{B}^m(L_y, L_z)$  such that

$$\sup_{M \in \mathbf{R}^+} \| |Z^M| \star W \|_{BMO} \leq \mathbb{K}.$$

**Theorem – III.2.1** (Existence and uniqueness (1)). *Let  $m > 1$ . Under the main assumption (H), the BMO a priori estimate (BMO,m), and the boundedness of the Malliavin derivatives of  $\xi$  and  $f$ , (Dxi,b)–(Df,b), the quadratic BSDE (III.1.6) has a unique solution  $(Y, Z) \in \mathcal{S}^2(\mathbf{R}^d) \times \mathcal{L}_{BMO}^m$  such that*

$$\text{esssup}_{\Omega \times [0, T]} |Z| < +\infty.$$

A result similar to Theorem – III.2.1 can be obtained when the quadratic growth of  $z$  has essentially a diagonal structure. Thus, we replace assumption (H) by the following one:

**(Hdiag)**

— There exist  $f_{\text{diag}} : \Omega \times [0, T] \times \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^d$  and  $g : \Omega \times [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^d$  such that for all  $i \in \{1, \dots, d\}$  we have

$$f^i(t, y, z) = f_{\text{diag}}^i(t, z) + g^i(t, y, z).$$

— There exist five nonnegative constants  $L_d, K_{d,y}, L_{d,y}, K_{d,z}, L_{d,z}$  such that for all  $(t, y, y', z, z') \in [0, T] \times (\mathbf{R}^d)^2 \times (\mathbf{R}^{d \times k})^2$  and  $i \in \{1, \dots, d\}$ :

$$|f_{\text{diag}}^i(t, z) - f_{\text{diag}}^i(t, z')| \leq L_d \left( |z^{(i, \cdot)}| + |(z')^{(i, \cdot)}| \right) |(z - z')^{(i, \cdot)}|,$$

$$|g(t, y, z) - g(t, y', z)| \leq (K_{d,y} + L_{d,y} |z|^2) |y - y'|,$$

$$|g(t, y, z) - g(t, y, z')| \leq (K_{d,z} + L_{d,z} (|z| + |z'|)) |z - z'|.$$

This kind of framework has been introduced by Hu and Tang in [HT16] (see also [JKL14]). The following result of existence and uniqueness is specific, and do not follows directly from Theorem – III.2.1. Indeed, if an uniform upper bound is assumed (assumption (i) below), we can use specific tools in the diagonal case to obtain an upper bound small enough.

**Theorem – III.2.2** (Existence and uniqueness (1) - Diagonal Case). *We assume that (Hdiag), (Dxi,b), (Df,b) hold true and that there exists a constant  $\mathbb{B}$  such that*

$$(i) \sup_{M \in \mathbf{R}^+} \| |Z^M| \star W \|_{BMO} \leq \mathbb{B},$$

$$(ii) c_2^2 d L_{d,y} \mathbb{B}^2 < 1, \quad \left( \frac{c_2}{c_1} \sqrt{L_{d,y}} + \frac{2c_2^2 \sqrt{d}}{c_1^2} L_{d,z} \right) \frac{4\sqrt{d} c_2^2 L_{d,z} \mathbb{B}^2}{1 - c_2^2 d L_{d,y} \mathbb{B}^2} < 1, \text{ where } c_1 \text{ and } c_2 \text{ are given}$$

by Proposition – III.1.1 with  $B = 2L_d \mathbb{B}$ .

We also assume that  $\xi \in L^\infty(\Omega, \mathcal{F}_T)$  and  $f(\cdot, 0, 0) \in \mathcal{S}^\infty(\mathbf{R}^d)$ . Then, the quadratic BSDE (III.1.6) has a unique solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbf{R}^d) \times BMO_{\mathbb{B}}$  such that

$$\text{esssup}_{\Omega \times [0, T]} |Z| < +\infty.$$

The main difference between assumptions in Theorem III.2.1 and Theorem III.2.2 comes from the form of constants used in the bound of the BMO norm. In particular, for any  $L_d > 0$ , there exists  $\varepsilon > 0$  such that (ii) in Theorem III.2.2 is fulfilled as soon as  $L_{d,y} < \varepsilon$  and  $L_{d,z} < \varepsilon$  while we cannot take  $L_z$  as large as we want in Theorem III.2.1.

### EXTENSION TO GENERAL TERMINAL VALUES AND GENERATORS

Now we are able to relax assumptions (Dxi,b) and (Df,b) with some density arguments. To do so, we assume that we can write  $f$  as a deterministic function  $\mathbf{f}$  of a progressively measurable continuous process: the randomness of the generator will be contained into this process.

- (H') (i) There exists a progressively measurable continuous process  $\alpha \in \bigcap_{p \in \mathbf{N}^*} \mathcal{S}^p$  with values in  $\mathbf{R}^{d'}$ ,  $d' \geq 1$ , and a function  $\mathbf{f} : \mathbf{R}^{d'} \times \mathbf{R}^d \times \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^d$  such that for all  $(t, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$ :

$$f(t, y, z) = \mathbf{f}(\alpha_t, y, z).$$

Besides, we assume that (H) holds true for  $\mathbf{f}$ .

- (ii) There exists  $D \in \mathbf{R}^+$  and  $\delta \in (0, 1]$  such that for all  $(y, z) \in \mathbf{R}^d \times \mathbf{R}^{d \times k}$ ,  $(\beta, \beta') \in (\mathbf{R}^{d'})^2$ :

$$|\mathbf{f}(\beta, y, z) - \mathbf{f}(\beta', y, z)| \leq D \left( 1 + |z|^2 \right) |\beta - \beta'|^\delta. \quad (\text{III.2.1})$$

For  $\eta \in L^2(\Omega, \mathcal{F}_T)$ ,  $\beta \in \mathcal{S}^\infty$  and  $M \in \mathbf{R}^+$ , we denote by  $(Y^{(M, \eta, \beta)}, Z^{(M, \eta, \beta)})$  the unique solution of the BSDE

$$Y_t^{(M, \eta, \beta)} = \eta + \int_t^T \mathbf{f}^M(\beta_s, Y_s^{(M, \eta, \beta)}, Z_s^{(M, \eta, \beta)}) ds - \int_t^T Z_s^{(M, \eta, \beta)} dW_s, \quad 0 \leq t \leq T, \quad (\text{III.2.2})$$

where for all  $(t, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$  and  $\mathbf{R}^{d'}$ -valued processes  $\alpha$ , we have  $\mathbf{f}^M(\alpha_t, y, z) = f^M(t, y, z)$ . Finally, assumption (BMO,m) will be replaced by the following one.

(BMO2,m) We assume that  $\xi \in L^{2m^*}(\Omega, \mathcal{F}_T)$  and that there exists a constant  $\mathbb{K} < \mathbb{B}^m(L_y, L_z)$



such that

$$\sup_{M \in \mathbf{R}^+} \sup_{\substack{\|\eta\|_{L^{2m^*}(\Omega, \mathcal{F}_T)} \leq \|\xi\|_{L^{2m^*}(\Omega, \mathcal{F}_T)} \\ \|\beta\|_{L^2(\Omega \times [0, T])} \leq \|\alpha\|_{L^2(\Omega \times [0, T])}} \left\| |Z^{(M, \eta, \beta)}| \star W \right\|_{BMO} \leq \mathbb{K}.$$

**Theorem – III.2.3** (Existence and uniqueness (2)). *Let  $m > 1$ . Under the main assumption  $(H')$  and the BMO estimation  $(BMO2, m)$ , the quadratic BSDE (III.1.6) has a unique solution in  $\mathcal{S}^{2m^*}(\mathbf{R}^d) \times (\mathcal{H}^{m^*}(\mathbf{R}^{d \times k}) \cap \mathcal{L}_{BMO}^m)$ .*

*Remark – III.2.1.*

- Let us emphasize that the uniqueness result in Theorem III.2.3 lies in a different space than the space used in Theorem III.2.1.
- It is also possible to extend the result of Theorem – III.2.2 (diagonal case) to more general terminal conditions and generators. Nevertheless, the result obtained would be less general than Theorem – III.2.3. See Remark – III.4.8 for more details.

### § III.2.2. Applications to multidimensional quadratic BSDEs with special structures

In this subsection we give some explicit frameworks where assumptions  $(BMO, m)$  and  $(BMO2, m)$  or assumptions (i) and (ii) of Theorem – III.2.2 are fulfilled. The aim is to show that numerous results on multidimensional quadratic BSDEs already proved in the literature can be obtained with similar assumptions by our approach. We want to underline the simplicity of this approach since we just have to obtain some *a priori* estimates on the BMO norm of  $|Z| \star W$  by using classical tools as explained in section III.5. Moreover, it is quite easy to construct some « new » frameworks where  $(BMO, m)$  and  $(BMO2, m)$  or assumptions (i) and (ii) of Theorem – III.2.2 are also fulfilled.

#### AN EXISTENCE AND UNIQUENESS RESULT FOR BSDEs WITH A SMALL TERMINAL CONDITION

In [Tev08], Tevzadze obtains a result of existence and uniqueness for multidimensional quadratic BSDEs when the terminal condition is small enough by using a contraction argument in  $\mathcal{S}^\infty \times BMO$ . We are able to deal with this kind of assumption with our approach. We consider the following hypothesis.

- (HQ)** (i) There exists  $\gamma \in \mathbf{R}^+$  such that for all  $(t, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$ , we have  $|f(t, y, z)| \leq \gamma |z|^2$ ,  
 (ii)  $32\gamma^2 \|\xi\|_{L^\infty}^2 \leq 1$ .

**Proposition – III.2.1.** *Let  $m > 1$ . Under  $(H')$ — $(HQ)$ , and the following condition on  $\gamma$ :*

$$\frac{1}{2\sqrt{2}\gamma} \left( 1 - \sqrt{1 - 32\gamma^2 \|\xi\|_{L^\infty}^2} \right)^{1/2} < \mathbb{B}^m(L_y, L_z),$$

the BSDE (III.1.6) has a unique solution in  $\mathcal{S}^{2m^*}(\mathbf{R}^d) \times (\mathcal{H}^{m^*}(\mathbf{R}^{d \times k}) \cap \mathcal{L}_{BMO}^m)$ . If in addition **(Dxi,b)** and **(Df,b)** hold true, there exists a unique solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbf{R}^d) \times \mathcal{L}_{BMO}^m$  such that

$$\text{esssup}_{\Omega \times [0, T]} |Z| < +\infty.$$

### AN EXISTENCE AND UNIQUENESS RESULT FOR BSDEs WITH A MONOTONE GENERATOR

In this part we investigate the case where we have for  $f$  a kind of monotonicity assumption with respect to  $y$ .

**(HMon)** (i) There exists  $\mu > 0$  and  $\alpha, \gamma \geq 0$  such that for all  $(s, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$

$$y \cdot f(s, y, z) \leq \alpha |y| - \mu |y|^2 + \gamma |y| |z|^2,$$

(ii)  $32\gamma^2 A^2 \leq 1$ , where  $A = \max\left(\|\xi\|_{L^\infty}, \frac{\alpha}{\mu}\right)$ .

**Proposition – III.2.2.** Let  $m > 1$ . Under **(H')**–**(HMon)** and the following estimate on  $\gamma$ :

$$\frac{1}{2\sqrt{2}\gamma} \left(1 - \sqrt{1 - 32\gamma^2 A^2}\right)^{1/2} < \mathbb{B}^m(L_y, L_z),$$

the quadratic BSDE (III.1.6) has a solution in  $\mathcal{S}^{2m^*}(\mathbf{R}^d) \times (\mathcal{H}^{m^*}(\mathbf{R}^{d \times k}) \cap \mathcal{L}_{BMO}^m)$ . If in addition **(Dxi,b)** and **(Df,b)** hold true, there exists a unique solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbf{R}^d) \times \mathcal{L}_{BMO}^m$  such that

$$\text{esssup}_{\Omega \times [0, T]} |Z| < +\infty.$$

### AN EXISTENCE AND UNIQUENESS RESULT FOR DIAGONAL QUADRATIC BSDEs

Now we consider the diagonal framework introduced in section III.2.1. We assume that the generator satisfies **(Hdiag)**, i.e. the generator  $f$  can be written as  $f(t, y, z) = f_{\text{diag}}(t, z) + g(t, y, z)$  where  $f_{\text{diag}}$  has a diagonal structure with respect to  $z$ .

**Proposition – III.2.3.** We assume that

(i) **(Hdiag)**, **(Dxi,b)** and **(Df,b)** hold true,

(ii) there exist nonnegative constants  $G_d$  and  $G$  such that, for all  $(t, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$ , we have

$$|f_{\text{diag}}(t, z)| \leq G_d |z|^2, \quad |g(t, y, z)| \leq G |z|^2, \quad (\text{III.2.3})$$

(iii)

$$\frac{4 \sum_{i=1}^d e^{2G_d \|\xi^i\|_{L^\infty}}}{G_d} G \leq 1,$$

(iv)  $c_2^2 d L_{d,y} (4G_d G)^{-1} < 1$ ,  $\left(\frac{c_2}{c_1} \sqrt{L_{d,y}} + \frac{2c_2^2 \sqrt{d}}{c_1} L_{d,z}\right) \frac{4\sqrt{d} c_2^2 L_{d,z} (4G_d G)^{-1}}{1 - c_2^2 d L_{d,y} (4G_d G)^{-1}} < 1$ , where  $c_1$

and  $c_2$  are given by Proposition – III.1.1 with  $B = 2L_d (4G_d G)^{-1/2}$ .

Then, the quadratic BSDE (III.1.6) has a unique solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbf{R}^d) \times BMO_{(4G_d G)^{-1/2}}$  such that

$$\text{esssup}_{\Omega \times [0, T]} |Z| < +\infty.$$

*Remark – III.2.2.* The growing assumption (III.2.3) is only one example of hypothesis that can be tackled by our approach. It is also possible to obtain the same kind of result by replacing (III.2.3) by one of the following assumption:

- We assume that for all  $(t, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$ ,

$$|g(t, y, z)| \leq C(1 + |y|) + \varepsilon |z|^2$$

and  $T, \varepsilon$  are supposed to be small enough. This framework is studied in [HT16, JKL14].

- We assume that for all  $(t, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$ ,

$$|g(t, y, z)| \leq C(1 + |y|).$$

This situation is already studied in [HT16].

### EXISTENCE AND UNIQUENESS OF MARTINGALES IN MANIFOLDS WITH PRESCRIBED TERMINAL CONDITION

The problem of finding martingales on a manifold with prescribed terminal value has generated a huge amount of literature. On the one hand with geometrical methods, Kendall in [Ken90] treats the case where the terminal value lies in a geodesic ball and is expressed as a functional of the Brownian motion. Kendall gives also a characterisation of the uniqueness in terms of existence of a convex separative function, i.e. a convex function on the product space which vanishes exactly on the diagonal. Besides, in [Ken92b], Kendall proved that the property *every couple of points are connected by a unique geodesic* is not sufficient to ensure existence of a separative convex function, which was conjectured by Émery. An approach by barycenters, of the martingale notion on a manifold, is used by Picard in [Pic94] for Brownian filtrations. Arnaudon in [Arn97] solved the problem in a complex analytic manifold having a convex geometry property for continuous filtrations: the main idea is to consider a differentiable family of martingales. For all these results, a convex geometry property is assumed. The first approach using the tool of BSDEs is proposed by Darling in [Dar95].

Let us now define more precisely the problem. A so-called linear connection structure is required to define martingales on a manifold  $\mathcal{M}$  in an intrinsic way. *A contrario*, for semimartingales, a differential structure is enough. The definition of a martingale can be rewritten with a system of coupled BSDEs having a quadratic growth, so we begin to recall it. We can refer to [Eme89] for more details about stochastic calculus on manifolds.

Let us consider  $(\mathcal{M}, \nabla)$  a differential manifold equipped with a linear connection  $\nabla$ . This is equivalent to give ourselves a Hessian notion or a covariant derivative. We say that a continuous process  $X$  is a semimartingale on  $M$  if for all  $F \in \mathcal{C}^2(\mathcal{M})$ ,  $F \circ X$  is a real semimartingale. Consistence of the definition is simply due to the Itô formula. We say that a continuous process

$Y$  is a (local)  $\nabla$ -martingale if for all  $F \in \mathcal{C}^2(\mathcal{M})$ ,

$$F(Y)_t - \frac{1}{2} \int_0^t \nabla dF(dY, dY)_s$$

is a real local martingale on  $[0, T]$ . Again it is not very hard to see with the Itô formula that this definition is equivalent to the Euclidean one in the flat case. Let us remember that  $\int_0^\cdot \nabla dF(dY, dY)_s$  is a notation for the quadratic variation of  $Y$  with respect to the  $(0, 2)$ -tensor field  $\nabla dF$ . This notion is defined by considering a proper embedding  $(x_i)_{1 \leq i \leq d}$  into  $\mathbf{R}^d$  such that every bilinear form  $b$  can be written as  $b = b_{ij} dx^i \otimes dx^j$  (implicit summation). On the other hand it can be proved that the quantity

$$\int_0^\cdot b(dY, dY)_s := \int_0^\cdot b_{ij}(Y_s) d\langle Y^i, Y^j \rangle_s$$

does not depend on  $(x^i)_{1 \leq i \leq d}$  and so the quantity  $\int_0^\cdot \nabla dF(dY, dY)_s$  is intrinsic. It is well-known that for all  $m \in \mathcal{M}$ ,

$$(\nabla dF)_{ij}(m) = D_{ij}F(m) - \Gamma_{ij}^k(m) D_k F(m),$$

where  $\Gamma_{ij}^k(m)$  denotes a  $\nabla$ -Christoffel symbol at the point  $m$ . The coefficients are symmetric with respect to  $i, j$ . Hence martingale property in the domain of a local chart is equivalent to the existence of a process  $Z$  such that  $(Y, Z)$  solves the following BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

with  $f : [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^d$  defined by  $f(s, y, z) = \frac{1}{2} \left( \Gamma_{ij}^k(y) z^{(i,:)} \cdot z^{(j,:)} \right)_{1 \leq k \leq d}$ . It is an easy consequence of the representation theorem for Brownian martingales and the definition applied to  $F = x^i$ . We consider in addition the following assumption

**(HGam)** there exists two constants  $L_y$  and  $L_z$  such that for all  $i, j, k \in \{1, \dots, d\}$

$$|\Gamma_{ij}^k(y) - \Gamma_{ij}^k(y')| \leq 2L_y |y - y'|, \quad |\Gamma_{ij}^k(y)| \leq 2L_z.$$

For example **(HGam)** is in force if the domain of the chart is a compact set. It is also true if we choose an exponential chart. Without loss of generality we can suppose that  $\mathcal{M}$  has a global system of coordinates: all the Christoffel symbols will be computed in this system.

Under **(HGam)**, assumption **(H)** is in force: for all  $(y, y', z, z') \in (\mathbf{R}^d)^2 \times (\mathbf{R}^{d \times k})^2$ ,

$$|f(t, y, z) - f(t, y', z)| \leq L_y |z|^2 |y - y'|,$$

and with the symmetric property of the Christoffel symbols, we have

$$\begin{aligned} f(t, y, z') - f(t, y, z) &= -\frac{1}{2} \sum_{i,j} \Gamma_{ij}^\bullet(y) \left( z^{(i,:)} \cdot z^{(j,:)} - (z')^{(i,:)} \cdot (z')^{(j,:)} \right) \\ &= -\frac{1}{2} \sum_{i,j} \Gamma_{ij}^\bullet(y) \left( z^{(i,:)} - (z')^{(i,:)} \right) \left( z^{(j,:)} + (z')^{(j,:)} \right), \end{aligned}$$

which implies that

$$|f(t, y, z) - f(t, y, z')| \leq L_z (|z| + |z'|) |z - z'|.$$

To obtain some important *a priori* estimate for the BMO norm of  $Z \star W$ , Darling introduce in [Dar95] a convex geometry assumption.

**Definition – III.2.1.** We say that a function  $F \in \mathcal{C}^2(\mathcal{M}, \mathbf{R})$  (seen as a function on  $\mathbf{R}^d$ ) is doubly convex on a set  $G \subset \mathbf{R}^d$  if for all  $y \in G$  and  $z \in \mathbf{R}^d$ ,

$$\min \{ \text{Hess } F(y)(z, z), \nabla dF(y)(z, z) \} \geq 0,$$

and, for  $\alpha > 0$ ,  $F$  is  $\alpha$ -strictly doubly convex on  $G$  if for all  $y \in G$  and  $z \in \mathbf{R}^d$ ,

$$\min \{ \text{Hess } F(y)(z, z), \nabla dF(y)(z, z) \} \geq \alpha |z|^2.$$

This property means that  $F$  is convex with respect to the flat connection, and, with respect to the connection  $\nabla$ .

**Theorem – III.2.4.** Let us assume that:

- (i) there exists a function  $F^{dc} \in \mathcal{C}^2(\mathcal{M}, \mathbf{R})$ , such that  $G = (F^{dc})^{-1}([-\infty, 0])$  is compact and  $\xi \in G$ ,
- (ii)  $F^{dc}$  is doubly convex on  $\mathcal{M}$ , and there exists  $\alpha > 0$  and  $m \geq 1$  such that  $F^{dc}$  is  $\alpha$ -strictly doubly convex on  $G$  and satisfies

$$\left( \sup_{(x,y) \in G^2} \{ F^{dc}(x) - F^{dc}(y) \} \right)^{1/2} \leq \sqrt{\frac{\alpha}{2}} \times \mathbb{B}^m(L_y, L_z),$$

- (iii) **(HGam)** holds true.

Then there exists a unique  $\nabla$ -martingale  $Y$  in  $\mathcal{S}^\infty(\mathbf{R}^d)$  with terminal value  $\xi$  such that  $\sqrt{|\langle Y, Y \rangle|} \star W \in \mathcal{L}_{BMO}^m$ . Moreover, if **(Dxi, b)** holds true we have

$$\text{esssup}_{\Omega \times [0, T]} |\langle Y, Y \rangle| < +\infty.$$

**Remark – III.2.3.** By using the same approach, it should be possible to extend the previous result to  $\nabla$ -Christoffel symbols that depend on time or even that are progressively measurable random processes.

### THE MARKOVIAN SETTING

The aim of this subsection is to refine some results of Xing and Žitković obtained in [XŽ16]: in this paper, authors establish existence and uniqueness results for a general class of Markovian multidimensional quadratic BSDEs. Let us start by introducing the Markovian framework. For all  $t \in [0, T]$  and  $x \in \mathbf{R}^k$  we denote  $X^{t,x}$  a diffusion process satisfying the following SDE

$$\begin{cases} dX_s^{t,x} = b(s, X_s^{t,x}) dt + \sigma(s, X_s^{t,x}) dW_s, & s \in [t, T], \\ X_s^{t,x} = x, & s \in [0, t]. \end{cases} \quad (\text{III.2.4})$$

In all this part, we assume following assumptions that ensure, in particular, that for all  $(t, x) \in [0, T] \times \mathbf{R}^k$ , there exists a unique strong solution of (III.2.4).

- (HX) — The drift vector  $b : [0, T] \times \mathbf{R}^k \rightarrow \mathbf{R}^k$  is measurable and uniformly bounded,  
 — The dispersion matrix  $\sigma : [0, T] \times \mathbf{R}^k \rightarrow \mathbf{R}^{k \times k}$  is symmetric, measurable and there exists a constant  $\Lambda > 0$  such that  $\Lambda |u|^2 \geq |\sigma(t, x)u|^2 \geq \frac{1}{\Lambda} |u|^2$  for all  $(t, x) \in [0, T] \times \mathbf{R}^k$  and  $u \in \mathbf{R}^k$ ,  
 —  $b$  and  $\sigma$  are Lipschitz functions with respect to  $x$ .

The aim of this subsection is to study the following Markovian BSDE

$$Y_u^{t,x} = \mathcal{G}(X_T^{t,x}) + \int_u^T \mathbf{f}(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds - \int_u^T Z_s^{t,x} dW_s, \quad t \leq u \leq T, \quad (\text{III.2.5})$$

for which we assume following assumptions:

- (HMark) —  $(s, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k} \mapsto \mathbf{f}(s, X_s^{t,x}, y, z)$  satisfies (H),  
 — There exists  $D \in \mathbf{R}^+$  and  $\kappa \in (0, 1]$  such that for all  $(x, x') \in (\mathbf{R}^k)^2$ ,

$$|\mathcal{G}(x)| \leq D, \quad |\mathcal{G}(x) - \mathcal{G}(x')| \leq D|x - x'|^\kappa.$$

As in [XŽ16] we say that a pair  $(v, w)$  of functions is a continuous Markovian solution of (III.2.5) if

- $v : [0, T] \times \mathbf{R}^k \rightarrow \mathbf{R}^d$  is a continuous function and  $w : [0, T] \times \mathbf{R}^k \rightarrow \mathbf{R}^{d \times k}$  is a measurable function,
- for all  $(t, x) \in [0, T] \times \mathbf{R}^k$ ,  $(Y^{t,x}, Z^{t,x}) := (v(\cdot, X^{t,x}), w(\cdot, X^{t,x}))$  is a solution of (III.2.5).

Two Markovian solutions,  $(v, w)$  and  $(v', w')$ , are considered equal if  $v(t, x) = v'(t, x)$  for all  $(t, x) \in [0, T] \times \mathbf{R}^k$  and  $w = w'$  a.s. with respect to the Lebesgue measure on  $[0, T] \times \mathbf{R}^k$ .

Some existence and uniqueness results about continuous Markovian solutions of (III.2.5) are obtained in [XŽ16] by assuming the existence of a so-called *Lyapunov function*. We recall here the definition of these functions given in [XŽ16].

**Definition – III.2.2** (Lyapunov function associated to  $\mathbf{g}$ ). Let  $\mathbf{g} : [0, T] \times \mathbf{R}^k \times \mathbf{R}^d \times \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^d$  a Borel function. A nonnegative function  $F \in \mathcal{C}^2(\mathbf{R}^d, \mathbf{R})$  is said to be a Lyapunov function

associated to  $\mathbf{g}$  if for all  $(t, x, y, z) \in [0, T] \times \mathbf{R}^k \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$ :

$$\frac{1}{2} \sum_{l=1}^d \text{Hess} F(y) \left( z^{(:,l)}, z^{(:,l)} \right) - \mathbf{d}F(y) \mathbf{g}(t, x, y, z) \geq |z|^2.$$

We are now able to give a uniqueness result that partially refine the result given by [XŽ16].

**Theorem – III.2.5** (Uniqueness for the Markovian case). *We assume that*

- (i) **(HX)** and **(HMark)** are in force.
- (ii) there exists a Lyapunov function  $F$  associated to  $\mathbf{f}$ .

Then (III.2.5) admits at most one continuous Markovian solution  $(v, w)$  such that  $v$  is bounded.

Moreover, we are also able to precise the regularity of the solution when it exists.

**Theorem – III.2.6** (Regularity of the Markovian solution). *We assume that:*

- (i) **(HX)** and **(HMark)** are in force,
- (ii) there exists  $D \in \mathbf{R}^+$  and  $\kappa \in (0, 1]$  (same constant  $\kappa$  as in **(HMark)**) such that for all  $(s, x, x', y, z) \in [0, T] \times (\mathbf{R}^k)^2 \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$ ,

$$|\mathbf{f}(s, x, y, z)| \leq D(1 + |y| + |z|^2), \quad |\mathbf{f}(s, x, y, z) - \mathbf{f}(s, x', y, z)| \leq D(1 + |z|^2) |x - x'|^\kappa,$$

- (iii) there exists a Lyapunov function  $F$  associated to  $\mathbf{f}$ .

If  $(v, w)$  is a continuous Markovian solution of (III.2.5) such that  $v$  is bounded, then  $v \in \mathcal{C}^\kappa$ . Particularly, if  $\kappa = 1$  then  $w$  is essentially bounded: the multidimensional quadratic BSDE (III.2.5) becomes a standard multidimensional Lipschitz BSDE by a localisation argument.

*Remark – III.2.4.* An existence result is given by Theorem 2.7 in [XŽ16]. A less general existence result can be obtained thanks to our approach by combining estimates obtained by Xing and Žitković in Theorem 2.5 of [XŽ16], small BMO estimates obtained in the proof of Theorem – III.2.5 and Remark – III.4.7 but the approach is less direct than in [XŽ16]. Concerning the uniqueness, Xing and Žitković have proved a uniqueness result for generators that do not depend on  $y$ : our result allows to fill this small gap. Finally, Xing and Žitković prove that there exists a Markovian solution that satisfies  $v \in \mathcal{C}^{\kappa', loc}$  with  $\kappa' \in (0, \kappa]$ . Thus, our regularity result gives a better estimation of the solution regularity since the regularity of the terminal condition and the generator is retained. In particular, we obtain that  $Z$  is bounded when  $\kappa = 1$  which can have important applications, as pointed out in the introduction.

*Remark – III.2.5.* The existence of a Lyapunov function seems to be an *ad hoc* theoretical assumption at first sight but Xing and Žitković provide in [XŽ16] a lot of examples and concrete criteria to obtain such kind of functions. Moreover we can note that the Lyapunov function can be used to obtain *a priori* estimates on  $\| |Z| \star W \|_{BMO}$  (see the proof of Theorem – III.2.5 and Theorem – III.2.6).

### III.3 GENERALITIES ABOUT SDES AND LINEAR BSDEs

We collect in this section some technical results that will be useful for section III.4 and section III.5.

#### § III.3.1. The linear case: representation of the solutions

We investigate here the following linear BSDE

$$U_t = \zeta + \int_t^T (A_s U_s + B_s V_s + f_s) ds - \int_t^T V_s dW_s, \quad 0 \leq t \leq T, \quad (\text{III.3.1})$$

where  $\zeta \in L^2(\mathcal{F}_T, \mathbf{R}^d)$ ,  $f \in L^2(\Omega \times [0, T])$  and  $A, B, f$  are three processes with values in  $\mathcal{L}(\mathbf{R}^d, \mathbf{R}^d)$ ,  $\mathcal{L}(\mathbf{R}^{d \times k}, \mathbf{R}^d)$  and  $\mathbf{R}^d$ . For the linear case we have an explicit formulation of the solution. Let us begin to recall the classical scalar formula which can be obtained using the Girsanov transform.

*Remark – III.3.1* (One-dimensional case ( $d = 1$ )). It is well-known that the solution of (III.3.1) is given by the formula

$$U_t = \mathbf{E} \left( S_t^{-1} S_T \zeta + \int_t^T S_t^{-1} S_s f_s ds \middle| \mathcal{F}_t \right), \quad 0 \leq t \leq T,$$

where

$$S_t = \exp \left( \int_0^t B_s dW_s - \frac{1}{2} \int_0^t |B_s|^2 ds + \int_0^t A_s ds \right) = \mathcal{E}(B \star W)_t \exp \left( \int_0^t A_s ds \right).$$

To extend this last formula in the general case we define, as in [DT08], a process  $S$  as the unique strong solution of

$$dS_t = \sum_{p=1}^k S_t B_t^{(:,p,:)} dW_t^p + S_t A_t dt, \quad S_0 = I_{d \times d}.$$

**Proposition – III.3.1** (Formula for  $U$ ).

(i) The process  $S$  is almost surely invertible for all  $t \in [0, T]$  and  $S^{-1}$  is the solution of

$$dS_t^{-1} = \left[ \left( \sum_{p=1}^k \left( B_t^{(:,p,:)} \right)^2 - A_t \right) dt - \sum_{p=1}^k B_t^{(:,p,:)} dW_t^p \right] S_t^{-1}, \quad S_0^{-1} = I_{d \times d}.$$

(ii) The BSDE (III.3.1) has a unique solution  $(U, V)$  in  $\mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$ , and  $U$  is given by:

$$U_t = \mathbf{E} \left( S_t^{-1} S_T \zeta + \int_t^T S_t^{-1} S_s f_s ds \middle| \mathcal{F}_t \right). \quad (\text{III.3.2})$$



Proof. Existence and uniqueness of a solution  $(U, V)$  in  $\mathcal{L}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$  is guaranteed by the Pardoux and Peng result in [PP90]. The solution  $(U, V)$  satisfies

$$U_t = \zeta + \int_t^T \left( A_s U_s + \sum_{p=1}^k B_s^{(:,p,:)} V_s^{(:,p)} + f_s \right) ds - \sum_{p=1}^k \int_t^T V_s^{(:,p)} dW_s^p.$$

The Itô formula gives the invertibility of  $S$  and the formula for  $S^{-1}$  on the one hand. On the other hand:

$$d(S_t U_t) = -S_t f_t dt + \sum_{p=1}^k \left( S_t B_t^{(:,p,:)} U_t + S_t V_t^{(:,p)} \right) dW_t^p,$$

and thus we get, for all  $t \in [0, T]$ ,

$$S_t U_t = S_T \zeta + \int_t^T S_s f_s ds - \int_t^T \sum_{p=1}^k \left( S_s B_s^{(:,p,:)} U_s + S_s V_s^{(:,p)} \right) dW_s^p.$$

By taking the conditional expectation  $S_t U_t = \mathbf{E} \left( S_T \zeta + \int_t^T S_s f_s ds \middle| \mathcal{F}_t \right)$ . Adaptability and invertibility of  $S$  give the result.  $\square$

### § III.3.2. A result about SDEs

We consider a SDE on  $\mathbf{R}^{d \times d}$  of the form

$$X_t = X_0 + \int_0^t F(s, X_s) ds + \sum_{p=1}^k \int_0^t G^p(s, X_s) dW_s^p, \quad (\text{III.3.3})$$

where  $F : \Omega \times [0, T] \times \mathbf{R}^{d \times d} \rightarrow \mathbf{R}^{d \times d}$  and for all  $p \in \{1, \dots, k\}$ ,  $G^p : \Omega \times [0, T] \times \mathbf{R}^{d \times d} \rightarrow \mathbf{R}^{d \times d}$  are progressively measurable functions. We start by recalling a result of Delbaen and Tang (see [DT08], Theorem 2.1) about existence and uniqueness of a solution to the equation (III.3.3), under BMO assumptions.

**Proposition – III.3.2.** *Let  $m \geq 1$ . We suppose that there are two non-negative adapted processes  $\alpha$  and  $\beta$  such that*

(i) (Regularity)  $F(t, 0) = 0$ ,  $G(t, 0) = 0$  and for all  $(x_1, x_2, t) \in (\mathbf{R}^{d \times d})^2 \times [0, T]$ ,

$$|F(t, x_1) - F(t, x_2)| \leq \alpha_t |x_1 - x_2| \quad \text{a.s.},$$

$$\sum_{p=1}^k |G^p(t, x_1) - G^p(t, x_2)|^2 \leq \beta_t^2 |x_1 - x_2|^2 \quad \text{a.s.}$$

(ii) (Sliceability)  $(\sqrt{\alpha} \star W, \beta \star W) \in BMO_{\varepsilon_1} \times BMO_{\varepsilon_2}$  with the condition

$$2m\varepsilon_1^2 + \sqrt{2}\varepsilon_2 C'_m < 1,$$

where  $C'_m$  is a constant defined in (III.1.2).

Then there exists a solution  $X \in \mathcal{S}^m(\mathbf{R}^d)$  to the equation (III.3.3) and a constant  $K_{m,\varepsilon_1,\varepsilon_2}$  such that

$$\|X\|_{\mathcal{S}^m} \leq K_{m,\varepsilon_1,\varepsilon_2} \|X_0\|_{L^m}.$$

For the reader convenience a proof of this result can be found in the appendix. From this last proposition we can deduce the following corollary (see [DT08], Corollary 2.1)

**Corollary – III.3.1.** *Let  $m \geq 1$ . We suppose that there are two non-negative adapted processes  $\alpha$  and  $\beta$  such that*

(i) (Regularity)  $F(t,0) = 0$ ,  $G(t,0) = 0$  and for all  $(x_1, x_2, t) \in (\mathbf{R}^{d \times d})^2 \times [0, T]$ ,

$$|F(t, x_1) - F(t, x_2)| \leq \alpha_t |x_1 - x_2| \text{ a.s.},$$

$$\sum_{p=1}^k |G^p(t, x_1) - G^p(t, x_2)|^2 \leq \beta_t^2 |x_1 - x_2|^2 \text{ a.s.}$$

(ii) (Sliceability)  $(\sqrt{\alpha} \star W, \beta \star W) \in BMO_{\varepsilon_1} \times BMO_{\varepsilon_2}$  with the condition

$$2m\varepsilon_1^2 + \sqrt{2}\varepsilon_2 C'_m < 1.$$

For  $t \in [0, T]$ , let  $X^{t, I_d}$  the unique solution defined on  $[t, T]$  of the SDE (III.3.3) such that  $X_t^{t, I_d} = I_d$ . Then  $X^{t, I_d}$  is in  $\mathcal{S}^m(\mathbf{R}^d)$  and satisfies for a constant  $K_m$  depending only on  $C'_m$ ,  $m$ ,  $k$  and  $\varepsilon_1, \varepsilon_2$ :

$$\mathbf{E} \left( \sup_{t \leq s \leq T} |X_s^{t, I_d}|^m \middle| \mathcal{F}_t \right) \leq K_{m,\varepsilon_1,\varepsilon_2}^m. \quad (\text{III.3.4})$$

In particular, if  $X$  is an invertible solution to the equation (III.3.3) and if  $F$  and  $G$  are linear with respect to  $x$ , we get the reverse Hölder inequality

$$\mathbf{E} \left( \sup_{t \leq s \leq T} |X_t^{-1} X_s|^m \middle| \mathcal{F}_t \right) \leq K_{m,\varepsilon_1,\varepsilon_2}^m.$$

Proof. We can use Proposition – III.3.2. For all  $t \in [0, T]$  and all event  $\mathbb{A} \in \mathcal{F}_t$ ,

$$\|X^{t, I_d} \times \mathbf{1}_{\mathbb{A}}\|_{\mathcal{S}^m([t, T])} \leq K_{m,\varepsilon_1,\varepsilon_2} \|I_d \times \mathbf{1}_{\mathbb{A}}\|_{L^m}.$$

Then we get, for all  $t \in [0, T]$ ,

$$\mathbf{E} \left( \sup_{t \leq s \leq T} |X_s^{t, I_d} \times \mathbf{1}_{\mathbb{A}}|^m \right) \leq K_{m,\varepsilon_1,\varepsilon_2}^m \mathbf{E}(|\mathbf{1}_{\mathbb{A}}|^m),$$

and we have

$$\mathbf{E} \left( \sup_{t \leq s \leq T} |X_s^{t, I_d}|^m \times \mathbf{1}_{\mathbb{A}} \right) \leq K_{m, \varepsilon_1, \varepsilon_2}^m \mathbf{E}(\mathbf{1}_{\mathbb{A}}).$$

Finally, the definition of conditional expectation gives us the result. If  $X$  is invertible and if  $F$  and  $G$  are linear with respect to  $x$ , the process  $X_t^{-1}X$  is for all  $t$  a solution taking the value  $I_d$  at  $s = t$ . The particular case is shown using (III.3.4).  $\square$

*Remark – III.3.2.* The main limitation of Corollary – III.3.1 comes from assumption (ii): we need to have a small BMO norm estimate on processes  $(\sqrt{\alpha} \star W, \beta \star W)$  to get a reverse Hölder inequality. It is well known that we have a more general result when  $d = 1$ : if  $\alpha = 0$  and  $\beta \star W \in BMO$  then there exists  $m > 1$  (that depends on the BMO norm of  $\beta \star W \in BMO$ ) such that  $X$  satisfies a reverse Hölder inequality with the exponent  $m$  (see Theorem 3.1 in [Kaz94] and references inside, or [CM13] for a new recent proof). We do not know if this result stays true in the multidimensional framework but we emphasize that this is a crucial open question. Indeed, if such a result is true, then we would be able to prove that Theorem – III.2.1 and Theorem – III.2.3 stay true without assuming  $\mathbb{K} < \mathbb{B}^m(L_y, L_z)$  in hypothesis (BMO, m) (at least when the generator is Lipschitz with respect to  $y$ , i.e.  $L_y = 0$ ).

### § III.3.3. Estimates for the solution to BSDE (III.3.1)

We come back to the linear BSDE (III.3.1), and we want to obtain some  $\mathcal{S}^q$ -estimations for  $U$  with  $q$  large enough, including  $q = \infty$ , under BMO assumptions.

**Proposition – III.3.3.** *Let  $m \geq 1$ . We assume that  $B$  and  $A$  are adapted, bounded respectively by two non negative processes  $\beta$  and  $\alpha$  such that:  $(\sqrt{\alpha} \star W, \beta \star W) \in BMO_{\varepsilon_1} \times BMO_{\varepsilon_2}$  with the condition*

$$2m\varepsilon_1^2 + \sqrt{2}\varepsilon_2 C'_m < 1.$$

Then

(i) *If  $\zeta \in L^\infty(\Omega, \mathcal{F}_T)$  and  $f \in \mathcal{S}^\infty$ , then  $U \in \mathcal{S}^\infty(\mathbf{R}^d)$  and*

$$\|U\|_{\mathcal{S}^\infty} \leq K_{m, \varepsilon_1, \varepsilon_2} (\|\zeta\|_{L^\infty} + T \|f\|_{\mathcal{S}^\infty}),$$

(ii) *Let us assume that  $m > 1$ . If  $\zeta \in L^\infty(\Omega, \mathcal{F}_T)$ , and  $\sqrt{|f|} \star W \in BMO$ , then  $U \in \mathcal{S}^\infty$  and*

$$\|U\|_{\mathcal{S}^\infty} \leq (m^*)! K_{m, \varepsilon_1, \varepsilon_2} \left( \|\zeta\|_{L^\infty} + \left\| \sqrt{|f|} \star W \right\|_{BMO}^2 \right).$$

(iii) *Let us assume that  $m > 1$ . For all  $q > m^* = \frac{m}{m-1}$ , if  $(\zeta, \int_0^T |f_s| ds) \in L^q \times L^q$ , then  $U \in \mathcal{S}^q(\mathbf{R}^d)$  and*

$$\|U\|_{\mathcal{S}^q}^q \leq 2^{q-1} K_{m, \varepsilon_1, \varepsilon_2}^q \left( \frac{q}{q-m^*} \right)^{q/m^*} \left( \|\zeta\|_{L^q}^q + \left\| \int_0^T |f_s| ds \right\|_{L^q}^q \right).$$

In the following we will denote simply  $\mathcal{K}_{q,m,\varepsilon_1,\varepsilon_2} = 2^{q-1} K_{m,\varepsilon_1,\varepsilon_2}^q \left( \frac{q}{q-m^*} \right)^{q/m^*}$ .

Proof. The formula (III.3.2) gives us, for all  $t \in [0, T]$ :

$$|U_t| \leq \mathbf{E}(|S_t^{-1} S_T| |\zeta| | \mathcal{F}_t) + \mathbf{E} \left( \int_t^T |S_t^{-1} S_s| |f_s| ds \middle| \mathcal{F}_t \right),$$

with

$$dS_t = \sum_{p=1}^k S_t B_t^{(:,p,:)} dW_t^p + S_t A_t dt, \quad S_0 = I_{d \times d}.$$

$S$  is the solution of an SDE on  $\mathbf{R}^{d \times d}$  for which we can use Corollary – III.3.1 by taking, for all  $1 \leq p \leq k$  and  $(x, y) \in (\mathbf{R}^{d \times d})^2$ ,  $G^p(s, x) = x B_s^{(:,p,:)}$  and  $F(s, y) = y A_s$ . Let us note that  $|B^{(:,p,:)}| \leq |B|$  for all  $p \in \{1, \dots, k\}$ . Thus there exists a constant  $K_{m,\varepsilon_1,\varepsilon_2}$  such that:

$$\mathbf{E} \left( \sup_{t \leq s \leq T} |S_t^{-1} S_s|^m \right) \leq K_{m,\varepsilon_1,\varepsilon_2}^m.$$

◊ If  $\zeta \in L^\infty$  and  $f \in \mathcal{S}^\infty$ , by using the Hölder inequality we have

$$\begin{aligned} |U_t| &\leq \|\zeta\|_{L^\infty} K_{m,\varepsilon_1,\varepsilon_2} + \|f\|_{\mathcal{S}^\infty} \mathbf{E} \left( (T-t) \sup_{t \leq s \leq T} |S_t^{-1} S_s|^m \middle| \mathcal{F}_t \right)^{1/m} \\ &\leq K_{m,\varepsilon_1,\varepsilon_2} (\|\zeta\|_{L^\infty} + T \|f\|_{\mathcal{S}^\infty}). \end{aligned}$$

◊ Let us consider  $m > 1$  and assume that  $\zeta \in L^\infty$ ,  $\sqrt{|f|} \star W$  is BMO. Then, by using Hölder and energy inequalities

$$\begin{aligned} |U_t| &\leq K_{m,\varepsilon_1,\varepsilon_2} \|\zeta\|_{L^\infty} + K_{m,\varepsilon_1,\varepsilon_2} \mathbf{E} \left( \left( \int_t^T |f_s| ds \right)^{m^*} \middle| \mathcal{F}_t \right)^{1/m^*} \\ &\leq (m^*)! K_{m,\varepsilon_1,\varepsilon_2} \left( \|\zeta\|_{L^\infty} + \left\| \sqrt{|f|} \star W \right\|_{BMO}^2 \right). \end{aligned}$$

◇ Let us consider  $m > 1$  and  $q > m^*$ . We get, for all  $t \in [0, T]$ ,

$$\begin{aligned} |U_t|^q &\leq 2^{q-1} \left( \mathbf{E}(|S_t^{-1} S_T| |\zeta| | \mathcal{F}_t)^q + \mathbf{E} \left( \int_t^T |S_t^{-1} S_s| |f_s| ds \middle| \mathcal{F}_t \right)^q \right) \\ &\leq 2^{q-1} \mathbf{E}(|S_t^{-1} S_T|^m | \mathcal{F}_t)^{q/m} \mathbf{E}(|\zeta|^{m^*} | \mathcal{F}_t)^{q/m^*} \\ &\quad + 2^{q-1} \mathbf{E} \left( \sup_{t \leq s \leq T} |S_t^{-1} S_s|^m \middle| \mathcal{F}_t \right)^{q/m} \mathbf{E} \left( \left( \int_t^T |f_s| ds \right)^{m^*} \middle| \mathcal{F}_t \right)^{q/m^*} \\ &\leq 2^{q-1} K_{m, \varepsilon_1, \varepsilon_2}^q \left( \mathbf{E}(|\zeta|^{m^*} | \mathcal{F}_t)^{q/m^*} + \mathbf{E} \left( \left( \int_0^T |f_s| ds \right)^{m^*} \middle| \mathcal{F}_t \right)^{q/m^*} \right). \end{aligned}$$

The processes  $M_t = \mathbf{E}(|\zeta|^{m^*} | \mathcal{F}_t)$  and  $N_t = \mathbf{E} \left( \left( \int_0^T |f_s| ds \right)^{m^*} \middle| \mathcal{F}_t \right)$  are two martingales with terminal values, respectively given by  $|\zeta|^{m^*}$  and  $\left( \int_0^T |f_s| ds \right)^{m^*}$ . Hence the Doob maximal inequality gives us, if  $q > m^*$ ,

$$\mathbf{E} \left( \sup_{0 \leq t \leq T} |M_t|^{q/m^*} \right) = \|M\|_{\mathcal{S}^{q/m^*}}^{q/m^*} \leq \left( \frac{q}{q-m^*} \right)^{q/m^*} \|M_T\|_{L^{q/m^*}}^{q/m^*} = \left( \frac{q}{q-m^*} \right)^{q/m^*} \|\zeta\|_{L^q}^q,$$

and

$$\mathbf{E} \left( \sup_{0 \leq t \leq T} |N_t|^{q/m^*} \right) \leq \left( \frac{q}{q-m^*} \right)^{q/m^*} \left\| \int_0^T |f_s| ds \right\|_{L^q}^q.$$

So we obtain the announced result:

$$\|U\|_{\mathcal{S}^q}^q \leq 2^{q-1} K_{m, \varepsilon_1, \varepsilon_2}^q \left( \frac{q}{q-m^*} \right)^{q/m^*} \left( \|\zeta\|_{L^q}^q + \left\| \int_0^T |f_s| ds \right\|_{L^q}^q \right).$$

□

**Corollary – III.3.2** (Affine upper bound). *Let  $m \geq 1$ . Let us consider  $A$  and  $B$  adapted, bounded respectively by two real processes  $\alpha$  and  $\beta$  of the form*

$$\alpha_s = K + L\mathcal{A}_s, \quad \beta_s = K' + L'\mathcal{B}_s,$$

with  $(K, L, K', L') \in (\mathbf{R}^+)^4$ ,  $\mathcal{A}$ ,  $\mathcal{B}$  two non negative real processes such that  $\sqrt{\mathcal{A}} \star W$  and  $\mathcal{B} \star W$  are BMO with the condition

$$2mL \left\| \sqrt{\mathcal{A}} \star W \right\|_{BMO}^2 + \sqrt{2}L' \|\mathcal{B} \star W\|_{BMO} C'_m < 1.$$

We have the following estimates, with constants  $K_m$ ,  $\mathcal{H}_{q,m}$  depending only on  $m, q, K_y, K_z, L_y, L_z$

### III. Méthode basée sur les équations différentielles stochastiques rétrogrades.

and the BMO norms  $\left\| \sqrt{\mathcal{A}} \star W \right\|_{BMO}$ ,  $\|\mathcal{B} \star W\|_{BMO}$ :

(i) If  $\zeta \in L^\infty(\Omega, \mathcal{F}_T)$  and  $f \in \mathcal{S}^\infty$ , then  $U \in \mathcal{S}^\infty(\mathbf{R}^d)$  and

$$\|U\|_{\mathcal{S}^\infty} \leq K_m (\|\zeta\|_{L^\infty} + T \|f\|_{\mathcal{S}^\infty}).$$

(ii) Let us assume that  $m > 1$ . If  $\zeta \in L^\infty(\Omega, \mathcal{F}_T)$ ,  $\sqrt{|f|} \star W \in BMO$ , then  $U \in \mathcal{S}^\infty$  and

$$\|U\|_{\mathcal{S}^\infty} \leq (m^*)! K_m \left( \|\zeta\|_{L^\infty} + \left\| \sqrt{|f|} \star W \right\|_{BMO}^2 \right).$$

(iii) Let us assume that  $m > 1$ . For all  $q > m^* = \frac{m}{m-1}$ , if  $\left( \zeta, \int_0^T |f_s| \, ds \right) \in L^q \times L^q$ , then  $U \in \mathcal{S}^q(\mathbf{R}^d)$  and

$$\|U\|_{\mathcal{S}^q}^q \leq 2^{q-1} K_m^q \left( \frac{q}{q-m^*} \right)^{q/m^*} \left( \|\zeta\|_{L^q}^q + \left\| \int_0^T |f_s| \, ds \right\|_{L^q}^q \right).$$

In the following we will denote simply  $\mathcal{K}_{q,m} = 2^{q-1} K_m^q \left( \frac{q}{q-m^*} \right)^{q/m^*}$ .

Proof. We obtain easily estimates about BMO-norms of  $\sqrt{\alpha} \star W$  and  $\beta \star W$  by using the triangle inequality,

$$\|\sqrt{\alpha} \star W\|_{BMO} \leq \sqrt{KT} + \sqrt{L} \|\sqrt{\mathcal{A}} \star W\|_{BMO}, \quad \|\beta \star W\|_{BMO} \leq K' \sqrt{T} + L' \|\mathcal{B} \star W\|_{BMO},$$

and it follows that  $\sqrt{\alpha} \star W, \beta \star W$  are BMO. To use [Proposition – III.3.3](#) we just have to show that  $\sqrt{\alpha} \star W$  and  $\beta \star W$  are respectively  $\varepsilon_1$  and  $\varepsilon_2$  sliceable with  $2m\varepsilon_1^2 + \sqrt{2}\varepsilon_2 C'_m < 1$ . To this end, we consider the following uniform sequence of deterministic stopping times

$$T_j = j \frac{T}{N}, \quad j \in \{0, \dots, N\},$$

and a parameter  $\eta > 0$ . With [Proposition – III.1.3](#) and defining  $\eta = \frac{T}{N}$ , previous inequalities become on  $[T_i, T_{i+1}]$

$$\left\| \int_{T_i}^{T_{i+1}} \sqrt{\alpha} \star W \right\|_{BMO} \leq \sqrt{K\eta} + \sqrt{L} \|\sqrt{\mathcal{A}} \star W\|_{BMO}, \quad (\text{III.3.5})$$

$$\left\| \int_{T_i}^{T_{i+1}} \beta \star W \right\|_{BMO} \leq K' \sqrt{\eta} + L' \|\mathcal{B} \star W\|_{BMO}. \quad (\text{III.3.6})$$

By taking  $\eta$  small enough, we get  $2m\varepsilon_1^2 + \sqrt{2}\varepsilon_2 C'_m < 1$  since the following upper bound holds true

$$2mL \left\| \sqrt{\mathcal{A}} \star W \right\|_{BMO}^2 + \sqrt{2}L' \|\mathcal{B} \star W\|_{BMO} C'_m < 1.$$

□

*Remark – III.3.3.* In inequalities (III.3.5) and (III.3.6), we have used that  $\left\| \int_{T_i}^{T_{i+1}} \mathcal{B} \star W \right\|_{BMO} \leq \|\mathcal{B} \star W\|_{BMO}$  and  $\left\| \int_{T_i}^{T_{i+1}} \sqrt{\mathcal{A}} \star W \right\|_{BMO} \leq \|\sqrt{\mathcal{A}} \star W\|_{BMO}$ . We can easily obtain a more general result by replacing the following assumption:  $\mathcal{A}, \mathcal{B}$  are two positive real processes such that  $\sqrt{\mathcal{A}} \star W, \mathcal{B} \star W$  are BMO with the condition

$$2mL \left\| \sqrt{\mathcal{A}} \star W \right\|_{BMO}^2 + \sqrt{2L'} \|\mathcal{B} \star W\|_{BMO} C'_m < 1,$$

by the new one:  $\mathcal{A}, \mathcal{B}$  are two positive real processes such that  $\sqrt{\mathcal{A}} \star W, \mathcal{B} \star W$  are in  $BMO_{\varepsilon_1}$  and  $BMO_{\varepsilon_2}$  with the condition

$$2mL\varepsilon_1^2 + \sqrt{2L'}\varepsilon_2 C'_m < 1.$$

*Remark – III.3.4.* We have not mentioned the dependence of the constants with respect to  $\left\| \sqrt{\mathcal{A}} \star W \right\|_{BMO}$  and  $\|\mathcal{B} \star W\|_{BMO}$  in notations but we will precise it explicitly when it is important.

## III.4 STABILITY, EXISTENCE AND UNIQUENESS RESULTS FOR GENERAL MULTIDIMENSIONAL QUADRATIC BSDEs

### § III.4.1. Proofs of Theorem – III.2.1 and Theorem – III.2.2

We recall first that  $(Y^M, Z^M)$  is the unique solution of the localized BSDE (III.1.8). To show Theorem – III.2.1 we begin to prove the following proposition which gives a uniform estimates for  $Z^M$ . This is the keystone of our procedure.

**Proposition – III.4.1.** *Let  $m > 1$ . If assumptions (H)—(BMO,m)—(Dxi,b)—(Df,b) hold true then*

$$\sup_M \operatorname{esssup}_{\Omega \times [0, T]} |Z^M| < +\infty.$$

Proof.

**Step 1 — Malliavin differentiation.** We assume that  $f$  is continuously differentiable with respect to  $(y, z)$ . This assumption is not restrictive by considering a smooth regularization of  $f$ .

Recalling assumptions (Dxi,b) and (Df,b), Proposition 5.3 in [EKPQ97] gives us that for all  $0 \leq u \leq t \leq T$ ,  $Y_t^M$  and  $Z_t^M$  are respectively in  $\mathbb{D}^{1,2}(\mathbf{R}^d)$  and  $\mathbb{D}^{1,2}(\mathbf{R}^{d \times k})$ . Moreover the process

$(\mathbf{D}_u Y^M, \mathbf{D}_u Z^M) = (\mathbf{D}_u Y_t^M, \mathbf{D}_u Z_t^M)_{0 \leq t \leq T}$  solves for all  $u$  the following linear BSDE in  $\mathbf{R}^{d \times k}$ :

$$\begin{aligned} \mathbf{D}_u Y_t^M &= \mathbf{D}_u \xi + \int_t^T \left( \nabla_y f^M(s, Y_s^M, Z_s^M) \mathbf{D}_u Y_s^M + \nabla_z f^M(s, Y_s^M, Z_s^M) \mathbf{D}_u Z_s^M \right. \\ &\quad \left. + (\mathbf{D}_u f^M)(s, Y_s^M, Z_s^M) \right) ds - \int_t^T \mathbf{D}_u Z_s^M dW_s, \end{aligned} \quad (\text{III.4.1})$$

and  $(\mathbf{D}_t Y_t)_{0 \leq t \leq T}$  is a version of  $(Z_t)_{0 \leq t \leq T}$ . In particular, there exists a continuous version of  $Z$ . Let us emphasize that BSDE (III.4.1) means that for each  $p \in \{1, \dots, k\}$ ,

$$\begin{aligned} \mathbf{D}_u^p Y_t^M &= \mathbf{D}_u^p \xi + \int_t^T \left( \nabla_y f^M(s, Y_s^M, Z_s^M) \mathbf{D}_u^p Y_s^M + \nabla_z f^M(s, Y_s^M, Z_s^M) \mathbf{D}_u^p Z_s^M \right. \\ &\quad \left. + (\mathbf{D}_u^p f^M)(s, Y_s^M, Z_s^M) \right) ds - \int_t^T \mathbf{D}_u^p Z_s^M dW_s, \end{aligned} \quad (\text{III.4.2})$$

besides  $\mathbf{D}^p Y^M$  is a process with values in  $\mathbf{R}^d$  for each  $p \in \{1, \dots, k\}$ .

**Step 2 —  $\mathcal{S}^\infty$ -Estimation.** We are looking for an  $\mathcal{S}^\infty$ -estimate of  $\mathbf{D}_u Y^M$  for all  $u \in [0, T]$  applying results of section III.3. Since  $|\nabla_z \rho^M(z)| \leq 1$ , we obtain the following inequalities by recalling the main assumption (H),

$$\begin{aligned} |\nabla_y f^M(s, Y_s^M, Z_s^M)| &= |\nabla_y f(s, Y_s^M, \rho^M(Z_s^M))| \leq K_y + L_y |Z_s^M|^2, \\ |\nabla_z f^M(s, Y_s^M, Z_s^M)| &= |\nabla_z f(s, Y_s^M, \rho^M(Z_s^M))| \leq K_z + 2L_z |Z_s^M|. \end{aligned}$$

Let us consider the two positive processes  $\alpha^M$  and  $\beta^M$  defined below,

$$\alpha^M = K_y + L_y |Z^M|^2, \quad \beta^M = K_z + 2L_z |Z^M|.$$

For all  $p \in \{1, \dots, k\}$ , by recalling (BMO,m), we can apply Corollary – III.3.2 (iii), to the BSDE (III.4.2) with the following constants and processes:

$$L = L_y, \quad K = K_y, \quad K' = K_z, \quad L' = 2L_z, \quad \mathcal{A} = |Z^M|^2, \quad \mathcal{B} = |Z^M|.$$

Thus, we obtain, for all  $u \in [0, T]$ ,

$$\begin{aligned} \|\mathbf{D}_u Y^M\|_{\mathcal{S}^\infty} &\leq \sum_{p=1}^k \|\mathbf{D}_u^p Y^M\|_{\mathcal{S}^\infty} \leq (m^*)! K_m \sum_{p=1}^k \left( \|\mathbf{D}_u^p \xi\|_{L^\infty} + \left\| \sqrt{|\mathbf{D}_u^p f^M(\cdot, Y^M, Z^M)|} \star W \right\|_{BMO}^2 \right) \\ &\leq C_m \left( \|\mathbf{D}_u \xi\|_{L^\infty} + \left\| \sqrt{|\mathbf{D}_u f^M(\cdot, Y^M, Z^M)|} \star W \right\|_{BMO}^2 \right), \end{aligned}$$

where  $C_m$  does not depend on  $M$ . Indeed, it is important to remark that the constant  $K_m$  given by Corollary – III.3.2 depends on  $\left\| |Z^M| \star W \right\|_{BMO}$  and so, could depend on  $M$ . But, by checking the proof of Proposition – III.3.2 in the Appendix it is easy to see that the constant  $K_m$  given by



Corollary – III.3.2 is equal to

$$\sum_{i=0}^{N-1} \left( \frac{1}{1 - 2mL_y \|Z^M \star W\|_{BMO}^2 - 2L_z \|Z^M \star W\|_{BMO} C'_m} \right)^i \leq \sum_{i=0}^{N-1} \left( \frac{1}{1 - 2mL_y \mathbb{K}^2 - 2L_z \mathbb{K} C'_m} \right)^i,$$

where  $N$  is an integer large enough and the uniform bound with respect to  $M$  follows.

Under the assumption **(Df,b)** together with **(BMO,m)**, the last term has a  $\mathcal{S}^\infty$ -upper bound uniform with respect to  $M$ . Indeed we have, for all  $(u, t) \in [0, T]^2$ ,

$$\mathbf{E} \left( \int_t^T |\mathbf{D}_u f^M(s, Y_s^M, Z_s^M)| ds \middle| \mathcal{F}_t \right) \leq C \left( T + \| |Z^M| \star W \|_{BMO}^2 \right),$$

hence we deduce

$$\sup_M \left\| \sqrt{|\mathbf{D}_u f^M(\cdot, Y^M, Z^M)| \star W} \right\|_{BMO} \leq \sqrt{C} \left( \sqrt{T} + \sup_M \| |Z^M| \star W \|_{BMO} \right).$$

The last supremum is finite under assumption **(BMO,m)** and we obtain the announced result since

$$\sup_M \|Z^M\|_{\mathcal{S}^\infty} = \sup_M \|(\mathbf{D}_t Y_t^M)_{t \in [0, T]}\|_{\mathcal{S}^\infty} \leq \sup_M \sup_u \|\mathbf{D}_u Y^M\|_{\mathcal{S}^\infty} < +\infty.$$

When  $f$  is not continuously differentiable with respect to  $(y, z)$  we consider a smooth regularization of  $f$  and we obtain by this classical approximation that

$$\sup_M \operatorname{esssup}_{\Omega \times [0, T]} |Z^M| < +\infty.$$

□

We are now able to prove **Theorem – III.2.1**.

**Proof.** [of **Theorem – III.2.1**] For the existence result, we can fix  $M^* > \sup_M \operatorname{esssup}_{\Omega \times [0, T]} |Z^M|$  according to **Proposition – III.4.1**. Thanks to assumptions on  $f$  and  $f^M$ , we get

$$f^{M^*} \left( s, Y_s^{M^*}, Z_s^{M^*} \right) = f \left( s, Y_s^{M^*}, \rho^{M^*} \left( Z_s^{M^*} \right) \right) = f \left( s, Y_s^{M^*}, Z_s^{M^*} \right) \quad \mathbb{P} \otimes [0, T] \quad \text{a.e.}$$

Then,  $(Y^{M^*}, Z^{M^*})$  becomes a solution of the quadratic BSDE **(III.1.6)** in  $\mathcal{S}^2(\mathbf{R}^d) \times (\mathcal{S}^\infty(\mathbf{R}^{d \times k}) \cap \mathcal{L}_{BMO}^m)$ . The uniqueness comes from the classical uniqueness result of Pardoux and Peng **[PP90]**: indeed, if we have two solutions  $(Y^1, Z^1)$  and  $(Y^2, Z^2)$  with  $\operatorname{esssup}_{\Omega \times [0, T]} |Z^1| + |Z^2| < +\infty$  then they are solution of the Lipschitz localized BSDE **(III.1.8)** where  $M = \operatorname{esssup}_{\Omega \times [0, T]} |Z^1| + |Z^2|$ . □

**Remark – III.4.1.** By using **Remark – III.3.3**, **Theorem – III.2.1** can be adapted if we replace

**(BMO,m)** by the following one: there exist a constant  $\mathbb{K}$  and a sequence  $0 = T_0 \leq T_1 \leq \dots \leq T_N = T$  of stopping times (that does not depend on  $M$ ) such that

(i)  $2mL_y\mathbb{K}^2 + \sqrt{2}L_z\mathbb{K}C'_m < 1$ , or equivalently  $\mathbb{K} < \mathbb{B}^m(L_y, L_z)$ ,

(ii) for all  $i \in \{0, \dots, N-1\}$ ,  $\sup_{M \in \mathbf{R}^+} \left\| \int_{T_i}^{T_{i+1}} |Z^M| \star W \right\|_{BMO} \leq \mathbb{K}$ .

In this case, if all the other assumptions of **Theorem – III.2.1** are fulfilled, then the quadratic BSDE (III.1.6) has a unique solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbf{R}^d) \times \mathcal{L}_{BMO}^{slc,m}$  such that

$$\text{esssup}_{\Omega \times [0, T]} |Z| < +\infty.$$

We do not give the proof of **Theorem – III.2.2** since it is quite similar to the proof of **Theorem – III.2.1**. Indeed, the main point is to show that **Proposition – III.4.1** stays true. To do that we just have to mimic its proof and replace the application of **Corollary – III.3.2 (iii)** by a new tailored one adapted to the diagonal framework and proved by using the same strategy as in the proof of **Theorem – III.4.2**.

### § III.4.2. Stability result

With the classical linearisation tool we can prove a stability theorem for the BSDE (III.1.6) by using results of section III.3. Let us consider two solutions of (III.1.6) in  $\mathbf{R}^d \times \mathbf{R}^{d \times k}$ , denoted  $(Y^1, Z^1)$  and  $(Y^2, Z^2)$ , with terminal conditions  $\xi^1$  and  $\xi^2$  and generators respectively  $f_1$  and  $f_2$ :

$$Y_t^1 = \xi^1 + \int_t^T f_1(s, Y_s^1, Z_s^1) ds - \int_t^T Z_s^1 dW_s, \quad 0 \leq t \leq T,$$

$$Y_t^2 = \xi^2 + \int_t^T f_2(s, Y_s^2, Z_s^2) ds - \int_t^T Z_s^2 dW_s, \quad 0 \leq t \leq T.$$

We assume that  $f_1, f_2$  satisfies the usual conditions **(H)**. Let us denote

$$\delta Y_s = Y_s^1 - Y_s^2, \quad \delta Z_s = Z_s^1 - Z_s^2, \quad \delta F_s = f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2),$$

$$\delta f_s = f_1(s, Y_s^2, Z_s^2) - f_2(s, Y_s^2, Z_s^2) \quad \text{and} \quad \delta \xi = \xi^1 - \xi^2.$$

The process  $(\delta Y, \delta Z)$  solves the BSDE

$$\delta Y_t = \delta \xi + \int_t^T \delta F_s ds - \int_t^T \delta Z_s dW_s, \quad 0 \leq t \leq T. \tag{III.4.3}$$

**Theorem – III.4.1** (Stability result). *Let  $m > 1$ ,  $p > \frac{m^*}{2}$  and let us suppose that*

(i)  $2mL_y \left\| \int |Z^1| \star W \right\|_{BMO}^2 + \sqrt{2}L_z \left( \left\| \int |Z^1| \star W \right\|_{BMO} + \left\| \int |Z^2| \star W \right\|_{BMO} \right) C'_m < 1$  or  $(Z^1, Z^2) \in \mathcal{L}_{BMO}^m \times \mathcal{L}_{BMO}^m$ ,

(ii)  $(\xi_1, \xi_2) \in (L^{2p})^2$ ,

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$$(iii) \int_0^T |\delta f_s| ds \in L^{2p}.$$

Then, there exists a constant  $\widetilde{K}_p$  ( $\| |Z^1| \star W \|_{BMO}, \| |Z^2| \star W \|_{BMO}$ ) (depending only on  $p, K_y, L_y, K_z, L_z, T$  and the BMO norms of  $|Z^1| \star W$  and  $|Z^2| \star W$ ) such that

$$\| \delta Y \|_{\mathcal{L}^{2p}}^p + \| | \delta Z | \star W \|_{\mathcal{L}^{2p}}^p \leq \widetilde{K}_p (\| |Z^1| \star W \|_{BMO}, \| |Z^2| \star W \|_{BMO}) \left( \| \delta \xi \|_{L^{2p}}^p + \left\| \int_0^T |\delta f_s| ds \right\|_{L^{2p}}^p \right).$$

Proof. We firstly assume that

$$2mL_y \| |Z^1| \star W \|_{BMO}^2 + \sqrt{2}L_z (\| |Z^1| \star W \|_{BMO} + \| |Z^2| \star W \|_{BMO}) C'_m < 1.$$

By using the classical linearisation tool, we can rewrite (III.4.3) as

$$\delta Y = \delta \xi + \int_t^T (A_s \delta Y_s + B_s (\delta Z_s) + \delta f_s) ds - \int_t^T \delta Z_s dW_s,$$

where

◇  $B$  is a  $\mathcal{L}(\mathbf{R}^{d \times k}, \mathbf{R}^d)$  process defined by blocks by, for all  $i \in \{1, \dots, k\}$ ,

$$B_s^i = \begin{cases} \frac{f_1(s, Y_s^2, Z_s^1) - f_1(s, Y_s^2, Z_s^2)}{|\delta Z_s|^2} (\mathbf{T} \delta Z_s^{(:,i)}) & \text{if } \delta Z_s \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{and } B_s \delta Z_s = \sum_{i=1}^k B_s^i \delta Z_s^{(:,i)},$$

◇  $A$  is a  $\mathcal{L}(\mathbf{R}^d, \mathbf{R}^d)$ -process defined by

$$A_s = \begin{cases} \frac{f_1(s, Y_s^1, Z_s^1) - f_1(s, Y_s^2, Z_s^1)}{|\delta Y_s|^2} (\mathbf{T} \delta Y_s) & \text{if } \delta Y_s \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

Assumption (H) on  $f_1$  and  $f_2$  gives the following inequalities:

$$|B_s| \leq K_z + L_z (|Z_s^1| + |Z_s^2|),$$

$$|A_s| \leq K_y + L_y |Z_s^1|^2.$$

**Step 1 – Control of  $\delta Y$ .**  $A$  and  $B$  are bounded respectively by two real processes  $\alpha$  and  $\beta$  defined by

$$\alpha = K_y + L_y |Z^1|^2, \quad \beta = K_z + L_z (|Z^1| + |Z^2|),$$

and  $(\delta Y, \delta Z)$  solves a linear BSDE of the form (III.3.1) with  $\delta f$  instead of  $f$ . We can apply Corollary – III.3.2, (iii) with

$$\mathcal{B} = |Z^1| + |Z^2|, \quad \mathcal{A} = |Z^1|^2, \quad L' = L_z, \quad K = K_y, \quad K' = K_z, \quad \text{and} \quad L = L_y,$$

which gives, for all  $q > 1$  such that  $q > m^*$ ,

$$\|\delta Y\|_{\mathcal{H}^q}^q \leq \mathcal{K}_{q,m} \left( \|\delta \xi\|_{L^q}^q + \left\| \int_0^T |\delta f_s| ds \right\|_{L^q}^q \right). \quad (\text{III.4.4})$$

**Step 2 – Control of  $\delta Z$ .** The Itô formula applied to  $|\delta Y|^2$  gives us

$$\begin{aligned} \int_0^T |\delta Z_s|^2 ds &= |\delta \xi|^2 - |\delta Y_0|^2 - 2 \int_0^T \delta Y_s \cdot (\delta Z_s dW_s) + 2 \int_0^T (\delta Y \cdot \delta F)_s ds \\ &\leq |\delta \xi|^2 - 2 \int_0^T \delta Y_s \cdot (\delta Z_s dW_s) + 2 \int_0^T (\delta Y \cdot \delta F)_s ds. \end{aligned} \quad (\text{III.4.5})$$

Recalling assumption (H) we have

$$|\delta F_s| = |f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)| \leq (K_y + L_y |Z_s^1|^2) |\delta Y_s| + (K_z + L_z (|Z_s^1| + |Z_s^2|)) |\delta Z_s| + |\delta f_s|.$$

With the Cauchy-Schwarz and Young inequalities, we get

$$\begin{aligned} 2 \int_0^T (\delta Y \cdot \delta F)_s ds &\leq 2 \int_0^T |\delta Y_s| |\delta F_s| ds \\ &\leq 2 \int_0^T \left[ (K_y + L_y |Z_s^1|^2) |\delta Y_s|^2 + (K_z + L_z (|Z_s^1| + |Z_s^2|)) |\delta Y_s| |\delta Z_s| + |\delta f_s| |\delta Y_s| \right] ds \\ &\leq 2 \left( \sup_{0 \leq s \leq T} |\delta Y_s|^2 \right) \int_0^T \left[ K_y + L_y |Z_s^1|^2 + (K_z + L_z (|Z_s^1| + |Z_s^2|))^2 + \frac{1}{2} \right] ds + \frac{1}{2} \int_0^T |\delta Z_s|^2 ds + \left( \int_0^T |\delta f_s| ds \right)^2. \end{aligned}$$

By using this last inequality in (III.4.5) we obtain

$$\begin{aligned} \frac{1}{2} \int_0^T |\delta Z_s|^2 ds &\leq |\delta \xi|^2 - 2 \int_0^T \delta Y_s \cdot (\delta Z_s dW_s) \\ &\quad + 2 \left( \sup_{0 \leq s \leq T} |\delta Y_s|^2 \right) \int_0^T \left[ K_y + L_y |Z_s^1|^2 + (K_z + L_z (|Z_s^1| + |Z_s^2|))^2 + \frac{1}{2} \right] ds + \left( \int_0^T |\delta f_s| ds \right)^2. \end{aligned}$$

Thus, for all  $p \geq 1$ , there exists a constant  $K$  depending only on  $p$  such that

$$\begin{aligned} \|\delta Z\|_{\star W}^p &\leq K \left[ \|\delta \xi\|_{L^p}^p + \mathbf{E} \left( \left( \sup_{0 \leq t \leq T} \left| \int_0^t \delta Y_s \cdot (\delta Z_s dW_s) \right| \right)^{p/2} \right) \right. \\ &\quad \left. + \mathbf{E} \left( \left( \sup_{0 \leq s \leq T} |\delta Y_s|^2 \int_0^T \left[ K_y + L_y |Z_s^1|^2 + \frac{1}{2} (K_z + L_z (|Z_s^1| + |Z_s^2|))^2 + 1 \right] ds \right)^{p/2} \right) + \mathbf{E} \left( \left( \int_0^T |\delta f_s| ds \right)^p \right) \right]. \end{aligned}$$

### III.4. Stability, existence and uniqueness results for general multidimensional quadratic BSDEs

In the following we keep the notation  $K$  for all constants appearing in the upper bounds. Then, according to the *BDG* inequalities, we get for all  $p \geq 1$ :

$$\mathbf{E} \left( \left( \sup_{0 \leq t \leq T} \left| \int_0^t \delta Y_s \cdot (\delta Z_s dW_s) \right| \right)^{p/2} \right) = \left\| \int_0^\cdot \delta Y_s \cdot (\delta Z_s dW_s) \right\|_{\mathcal{H}^{p/2}}^{p/2} \leq (C'_{p/2})^{p/2} \left\| \int_0^\cdot \delta Y_s \cdot (\delta Z_s dW_s) \right\|_{\mathcal{H}^{p/2}}^{p/2}.$$

Since we have

$$\left\| \int_0^\cdot \delta Y_s \cdot (\delta Z_s dW_s) \right\|_{\mathcal{H}^{p/2}}^{p/2} = \mathbf{E} \left( \left( \sum_{i=1}^k \int_0^T (\delta Y_s \cdot \delta Z_s^{(:,i)})^2 ds \right)^{p/4} \right) \leq \mathbf{E} \left( \left( \sup_{0 \leq s \leq T} |\delta Y_s|^2 \times \int_0^T |\delta Z_s|^2 ds \right)^{p/4} \right),$$

then the Cauchy-Schwartz inequality gives us

$$\mathbf{E} \left( \left( \sup_{0 \leq t \leq T} \left| \int_0^t \delta Y_s \cdot (\delta Z_s dW_s) \right| \right)^{p/2} \right) \leq (C'_{p/2})^{p/2} \|\delta Z\|_{\star W}^{p/2} \|\delta Y\|_{\mathcal{S}^p}^{p/2}.$$

Moreover we obtain with Cauchy-Schwarz and Young inequalities:

$$\begin{aligned} & \|\delta Z\|_{\star W}^p_{\mathcal{H}^p} \\ & \leq K \left[ \|\delta \xi\|_{L^p}^p + \|\delta Z\|_{\star W}^{p/2}_{\mathcal{H}^p} \|\delta Y\|_{\mathcal{S}^p}^{p/2} + \right. \\ & \quad \left. \|\delta Y\|_{\mathcal{S}^{2p}}^p \mathbf{E} \left( \left( \int_0^T [1 + |Z_s^1|^2 + |Z_s^2|^2] ds \right)^p \right)^{1/2} + \mathbf{E} \left( \left( \int_0^T |\delta f_s| ds \right)^p \right) \right] \\ & \leq K \left[ \|\delta \xi\|_{L^p}^p + \|\delta Y\|_{\mathcal{S}^p}^p + \|\delta Y\|_{\mathcal{S}^{2p}}^p \mathbf{E} \left( \left( \int_0^T [1 + |Z_s^1|^2 + |Z_s^2|^2] ds \right)^p \right)^{1/2} + \mathbf{E} \left( \left( \int_0^T |\delta f_s| ds \right)^p \right) \right] \\ & \quad + \frac{1}{2} \|\delta Z\|_{\star W}^p_{\mathcal{H}^p}. \end{aligned}$$

The energy inequality allows us to bound  $\mathbf{E} \left( \left( \int_0^T [1 + |Z_s^1|^2 + |Z_s^2|^2] ds \right)^p \right)$  by

$$K \left( 1 + \|\delta Z^1\|_{\star W}^{2p}_{BMO} + \|\delta Z^2\|_{\star W}^{2p}_{BMO} \right),$$

which is finite recalling assumption **(i)**. Finally, for all  $p \geq 1$ , there exists a constant  $K$  (which depends only on  $p, K_y, L_y, K_z, L_z, T$  and the BMO norms of  $\delta Z^1, \delta Z^2$ ) such that

$$\|\delta Z\|_{\star W}^p_{\mathcal{H}^p} \leq K \left( \|\delta \xi\|_{L^p}^p + \|\delta Y\|_{\mathcal{S}^{2p}}^p + \left\| \int_0^T |\delta f_s| ds \right\|_{L^p}^p \right). \quad (\text{III.4.6})$$

**Step 3 – Stability.** Considering  $p > \frac{m^*}{2}$  and combining (III.4.4) where  $q = 2p$  with (III.4.6), we obtain existence of a constant  $\widetilde{K}_p$  ( $\|\delta Z^1\|_{\star W}_{BMO}, \|\delta Z^2\|_{\star W}_{BMO}$ ) which depends only on

$p, K_y, L_y, K_z, L_z, T, K$  and the BMO norms of  $|Z^1| \star W, |Z^2| \star W$  such that

$$\|\delta Y\|_{\mathcal{G}^{2p}}^p + \|\delta Z \star W\|_{\mathcal{H}^p}^p \leq \widetilde{K}_p \left( \left\| |Z^1| \star W \right\|_{BMO}, \left\| |Z^2| \star W \right\|_{BMO} \right) \left( \|\delta \xi\|_{L^{2p}}^p + \left\| \int_0^T |\delta f_s| ds \right\|_{L^{2p}}^p \right).$$

**Step 4 – The alternative assumption (i)** We can deal with assumption  $(Z^1, Z^2) \in \mathcal{Z}_{BMO}^m \times \mathcal{Z}_{BMO}^m$  by changing the linearization step in the proof. We can remark that  $\delta F_s = \widetilde{A}_s \delta Y_s + \widetilde{B}_s \delta Z_s + \delta f_s$ , where

$$\widetilde{A}_s \delta Y_s = \frac{1}{2} \left( f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^1) + f^1(s, Y_s^1, Z_s^2) - f^1(s, Y_s^2, Z_s^2) \right),$$

$$\widetilde{B}_s \delta Z_s = \frac{1}{2} \left( f^1(s, Y_s^2, Z_s^1) - f^1(s, Y_s^2, Z_s^2) + f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^1, Z_s^2) \right),$$

and we get symmetric bounds for  $\widetilde{A}$  and  $\widetilde{B}$ :

$$\left| \widetilde{A}_s \right| \leq K_y + \frac{L_y}{2} \left( |Z_s^1|^2 + |Z_s^2|^2 \right), \quad \left| \widetilde{B}_s \right| \leq K_z + L_z \left( |Z_s^1| + |Z_s^2| \right).$$

Then (i) becomes

$$mL_y \left( \left\| |Z^1| \star W \right\|_{BMO}^2 + \left\| |Z^2| \star W \right\|_{BMO}^2 \right) + \sqrt{2} L_z C'_m \left( \left\| |Z^1| \star W \right\|_{BMO} + \left\| |Z^2| \star W \right\|_{BMO} \right) < 1,$$

wich is fulfilled as soon as we have  $(Z^1, Z^2) \in \mathcal{Z}_{BMO}^m \times \mathcal{Z}_{BMO}^m$ .  $\square$

*Remark – III.4.2.* A more restrictive stability result is already obtained in [KP16b] (see Theorem 2.1).

*Remark – III.4.3.* By using Remark – III.3.3, it is clear that Theorem – III.4.1 stays true when  $Z^1$  and  $Z^2$  are only in  $\mathcal{Z}_{BMO}^{slc,m}$ . Indeed, if we denote  $0 = T_0^j \leq T_1^j \leq \dots \leq T_{N^j}^j$  the sequence of stopping times associated to  $Z^j \star W$  for  $j \in \{1, 2\}$ , we can define a new common sequence of stopping times:

$$T_{kN^1+i} = (T_i^1 \vee T_k^2) \wedge T_{k+1}^2, \quad i \in \{0, \dots, N^1 - 1\}, k \in \{0, \dots, N^2 - 1\}.$$

Then, by applying the stability result on each interval  $[T_i, T_{i+1}]$  for  $i \in \{0, N^1 N^2 - 1\}$  we obtain

$$\begin{aligned} & \|\delta Y\|_{\mathcal{G}^{2p}}^p + \|\delta Z \star W\|_{\mathcal{H}^p}^p \\ & \leq N^1 N^2 \prod_{k=0}^{N^1 N^2 - 1} \widetilde{K}_p \left( \left\| |Z^1| \star W \right\|_{BMO}^{T_{k+1}}, \left\| |Z^2| \star W \right\|_{BMO}^{T_{k+1}} \right) \left( \|\delta \xi\|_{L^{2p}}^p + \left\| \int_0^T |\delta f_s| ds \right\|_{L^{2p}}^p \right). \end{aligned}$$

Obviously, when sequences of stopping times are the same for  $|Z^1| \star W$  and  $|Z^2| \star W$ , we can use it directly as the common sequence of stopping time.

§ III.4.3. Stability result for the diagonal quadratic case

We give here a specific result when the quadratic growth of  $z$  has essentially a diagonal structure: we assume that assumption **(Hdiag)** is in force. As explained in section III.2.1, this kind of framework has been introduced by Hu and Tang in [HT16] (see also [JKL14]).

To simplify notations in this paragraph, the line  $i$  of  $z$  will be denoted in a simple way by  $(z)^i$ , or  $z^i$  if there is no ambiguity, instead of  $z^{(i,:)}$ . Let us consider two solutions  $(Y^1, Z^1)$  and  $(Y^2, Z^2)$  which correspond to terminal conditions  $\xi^1, \xi^2$  and generators  $f_1 = f^{\text{diag},1} + g_1, f_2 = f^{\text{diag},2} + g_2$ . We have for all  $i \in \{1, \dots, d\}$ ,

$$\delta Y_t^i = \delta \xi^i + \int_t^T \delta F_s^i ds - \int_t^T \delta Z_s^i \cdot dW_s,$$

with

$$\begin{aligned} \delta Y_s &:= Y_s^1 - Y_s^2, & \delta Z_s &:= Z_s^1 - Z_s^2, \\ \delta F_s &:= (f^{\text{diag},1}(s, Z_s^1) - f^{\text{diag},2}(s, Z_s^2)) + (g_1(s, Y_s^1, Z_s^1) - g_2(s, Y_s^2, Z_s^2)) & \text{and} & \quad \delta \xi := \xi^1 - \xi^2. \end{aligned}$$

We also define

$$\delta f_s = f_1(s, Y_s^2, Z_s^2) - f_2(s, Y_s^2, Z_s^2).$$

**Theorem – III.4.2** (Stability result for the diagonal quadratic case). *Let us assume that*

- (i)  $f_1$  and  $f_2$  satisfy **(Hdiag)**,
- (ii) there exists  $\mathbb{B} > 0$  such that  $(Y^1, Z^1)$  and  $(Y^2, Z^2)$  are in  $\mathcal{S}^\infty(\mathbf{R}^d) \times BMO_{\mathbb{B}}$  and

$$c_2^2 d L_{d,y} \mathbb{B}^2 < 1, \quad \left( \frac{c_2}{c_1} \sqrt{L_{d,y}} + \frac{2\sqrt{d}c_2^2}{c_1^2} L_{d,z} \right) \frac{4\sqrt{d}c_2^2 L_{d,z} \mathbb{B}^2}{1 - c_2^2 d L_{d,y} \mathbb{B}^2} < 1, \quad (\text{III.4.7})$$

where  $c_1$  and  $c_2$  are given by Proposition – III.1.1 with  $B = 2L_d \mathbb{B}$ .

Then there exists a constant  $\widetilde{K}^{\text{diag}} (\| |Z^1| \star W \|_{BMO}, \| |Z^2| \star W \|_{BMO})$  depending only on  $\mathbb{B}$  and constants in **(Hdiag)** such that

$$\|\delta Y\|_{\mathcal{S}^\infty} + \| |\delta Z| \star W \|_{BMO} \leq \widetilde{K}^{\text{diag}} (\| |Z^1| \star W \|_{BMO}, \| |Z^2| \star W \|_{BMO}) \left( \|\delta \xi\|_{L^\infty} + \left\| \sqrt{|\delta f|} \star W \right\|_{BMO}^2 \right).$$

*Remark – III.4.4.* For a given  $L_d$  and a given  $\mathbb{B}$ , condition (III.4.7) is fulfilled as soon as  $L_{d,y}$  and  $L_{d,z}$  are small enough.

Proof.

**Step 1 – Control of  $\delta Y$ .** We write  $\delta F^i$  as

$$\begin{aligned} \delta F_s^i &= (f^{\text{diag},1}(s, Z_s^1) - f^{\text{diag},1}(s, Z_s^2))^i + (g_1(s, Y_s^1, Z_s^1) - g_1(s, Y_s^2, Z_s^2))^i + \delta f_s^i \\ &= \beta_s^i \delta Z_s^i + \alpha_s^i \delta Y_s + \text{Tr}(\gamma_s^i \delta Z_s) + \delta f_s^i, \end{aligned}$$

where  $\beta^i, \alpha^i$  and  $\gamma^i$  are defined by:

$$\beta_s^i = \begin{cases} \frac{f_{\text{diag},1}^i(s, Z_s^1) - f_{\text{diag},1}^i(s, Z_s^2)}{|\delta Z_s^i|^2} (\mathbf{T} \delta Z_s^i) & \text{if } \delta Z_s^i \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\alpha_s^i = \begin{cases} \frac{g_1^i(s, Y_s^1, Z_s^1) - g_1^i(s, Y_s^2, Z_s^1)}{|\delta Y_s|^2} (\mathbf{T} \delta Y_s) & \text{if } \delta Y_s \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma_s^i = \begin{cases} \frac{g_1^i(s, Y_s^2, Z_s^1) - g_1^i(s, Y_s^2, Z_s^2)}{|\delta Z_s|^2} (\mathbf{T} \delta Z_s) & \text{if } \delta Z_s \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since we have the following estimate on  $\beta^i$ , for all  $i \in \{1, \dots, d\}$ ,

$$|\beta^i| \leq L_d (|(Z^1)^i| + |(Z^2)^i|),$$

and that  $(|(Z^1)^i| \star W, |(Z^2)^i| \star W) \in BMO \times BMO$ , we deduce that  $|\beta^i| \star W$  is BMO too and  $\mathcal{E}(\beta^i \star W)$  is an uniform integrable martingale. Consequently we can apply the Girsanov theorem:

$$\begin{aligned} \delta Y_t^i &= \delta \xi^i + \int_t^T (\alpha_s^i \delta Y_s + \text{Tr}(\gamma_s^i \delta Z_s) + \delta f_s^i) ds - \int_t^T \delta Z_s^i \cdot (dW_s - \beta_s^i ds) \\ &= \delta \xi^i + \int_t^T (\alpha_s^i \delta Y_s + \text{Tr}(\gamma_s^i \delta Z_s) + \delta f_s^i) ds - \int_t^T \delta Z_s^i \cdot d\bar{W}_s^i, \end{aligned} \quad (\text{III.4.8})$$

where  $\bar{W}^i$  is a Brownian motion with respect to the probability  $\mathbf{Q}^i$  defined by  $d\mathbf{Q}^i = \mathcal{E}(\beta^i \star W)_T d\mathbf{P}$ . Taking the  $\mathbf{Q}^i$ -conditional expectation we get

$$\delta Y_t^i = \mathbf{E}^{\mathbf{Q}^i} \left( \delta \xi^i + \int_t^T (\alpha_s^i \delta Y_s + \text{Tr}(\gamma_s^i \delta Z_s) + \delta f_s^i) ds \middle| \mathcal{F}_t \right).$$

Following estimates hold true:

$$|\alpha| \leq K_{d,y} + L_{d,y} |Z^1|^2, \quad |\gamma| \leq K_{d,z} + L_{d,z} (|Z^1| + |Z^2|),$$

and consequently, we obtain

$$\begin{aligned} \left\| \sqrt{|\alpha|} \star W \right\|_{BMO}^2 &\leq K_{d,y} T + L_{d,y} \left\| |Z^1| \star W \right\|_{BMO}^2, \\ \left\| |\gamma| \star W \right\|_{BMO} &\leq K_{d,z} \sqrt{T} + L_{d,z} (\left\| |Z^1| \star W \right\|_{BMO} + \left\| |Z^2| \star W \right\|_{BMO}). \end{aligned}$$



### III.4. Stability, existence and uniqueness results for general multidimensional quadratic BSDEs

Since  $\sqrt{(|\alpha^i| |\delta Y| + |\gamma^i| |\delta Z| + |\delta f^i|)} \star W$  is a BMO martingale, we can apply [Proposition – III.1.1](#): there exists a constant  $c_2$  that depend only on  $L_d$  and  $\mathbb{B}$  such that

$$|\delta Y_t^i| \leq \|\delta \xi^i\|_{L^\infty} + c_2^2 \left( \|\delta Y\|_{\mathcal{S}^\infty} \left\| \sqrt{|\alpha^i|} \star W \right\|_{BMO}^2 + \|\delta Z \star W\|_{BMO} \|\gamma^i \star W\|_{BMO} + \left\| \sqrt{|\delta f^i|} \star W \right\|_{BMO}^2 \right),$$

and consequently we get

$$\|\delta Y\|_{\mathcal{S}^\infty} \tag{III.4.9}$$

$$\begin{aligned} &\leq \sum_{i=1}^d \|\delta Y^i\|_{\mathcal{S}^\infty} \\ &\leq \sqrt{d} \|\delta \xi\|_{L^\infty} + c_2^2 \left( d \|\delta Y\|_{\mathcal{S}^\infty} \left\| \sqrt{|\alpha|} \star W \right\|_{BMO}^2 + d \|\delta Z \star W\|_{BMO} \|\gamma \star W\|_{BMO} + d \left\| \sqrt{|\delta f|} \star W \right\|_{BMO}^2 \right) \\ &\leq \sqrt{d} \|\delta \xi\|_{L^\infty} + c_2^2 d \|\delta Y\|_{\mathcal{S}^\infty} \left( K_{d,y} T + L_{d,y} \|\mathbb{B}\|_{BMO}^2 \right) \\ &+ c_2^2 \|\delta Z \star W\|_{BMO} \left( d K_{d,z} \sqrt{T} + d L_{d,z} (\|\mathbb{B}\|_{BMO} + \|\mathbb{B}\|_{BMO}^2) \right) + c_2^2 d \left\| \sqrt{|\delta f|} \star W \right\|_{BMO}^2. \end{aligned} \tag{III.4.10}$$

As in the proof of [Proposition – III.3.2](#), now we slice  $[0, T]$  in small pieces. We consider  $\eta = \frac{T}{N}$  with  $N \in \mathbf{N}^*$  and we set  $T_i = i\eta$  for  $i \in \{0, \dots, N\}$ . The process  $\delta Y$  is equal to

$$\delta Y_t = \delta Y_T \mathbb{1}_{\{T\}}(t) + \sum_{k=1}^{N-1} \widetilde{\delta Y}_t^k \mathbb{1}_{[T_k, T_{k+1}]}(t), \quad \text{with} \quad \widetilde{\delta Y}_t^k = \delta Y_t \mathbb{1}_{[T_k, T_{k+1}]}(t), \quad k \in \{0, \dots, N-1\}. \tag{III.4.11}$$

On the interval  $[T_k, T_{k+1}]$  the inequality [\(III.4.10\)](#) becomes:

$$\begin{aligned} \left\| \widetilde{\delta Y}^k \right\|_{\mathcal{S}^\infty} &\leq \sqrt{d} \left\| \widetilde{\delta Y}_{T_{k+1}}^k \right\|_{L^\infty} + d c_2^2 \left\| \sqrt{|\delta f|} \star W \right\|_{BMO}^2 \\ &+ c_2^2 \left( d \left\| \widetilde{\delta Y}^k \right\|_{\mathcal{S}^\infty} (K_{d,y} \eta + L_{d,y} \mathbb{B}^2) + \left\| \mathbb{B} \right\|_{BMO}^{T_k} \|\delta Z \star W\|_{BMO}^{T_{k+1}} (d K_{d,z} \sqrt{\eta} + 2d L_{d,z} \mathbb{B}) \right). \end{aligned}$$

Then, we can choose  $N$  large enough to get  $1 - c_2^2 d (K_{d,y} \eta + L_{d,y} \mathbb{B}^2) > 0$ . Finally we obtain

$$\begin{aligned} \left\| \widetilde{\delta Y}^k \right\|_{\mathcal{S}^\infty} &\leq \frac{\sqrt{d}}{1 - c_2^2 d (K_{d,y} \eta + L_{d,y} \mathbb{B}^2)} \left\| \widetilde{\delta Y}_{T_{k+1}}^k \right\|_{L^\infty} \\ &+ \left\| \mathbb{B} \right\|_{BMO}^{T_k} \|\delta Z \star W\|_{BMO}^{T_{k+1}} \frac{c_2^2 d (K_{d,z} \sqrt{\eta} + 2L_{d,z} \mathbb{B})}{1 - c_2^2 d (K_{d,y} \eta + L_{d,y} \mathbb{B}^2)} + \left\| \sqrt{|\delta f|} \star W \right\|_{BMO}^2 \frac{c_2^2 d}{1 - c_2^2 d (K_{d,y} \eta + L_{d,y} \mathbb{B}^2)}. \end{aligned} \tag{III.4.12}$$

**Step 2 – Control of  $\delta Z$ .** Applying the Itô formula for the process  $|\delta Y^i|^2$  and taking the  $\mathbf{Q}^i$ -conditional expectation, we get for all  $t \in [0, T]$ ,

$$\begin{aligned} |\delta Y_t^i|^2 + \mathbf{E}^{\mathbf{Q}^i} \left( \int_t^T |\delta Z_s^i|^2 ds \middle| \mathcal{F}_t \right) &= |\delta \xi^i|^2 + 2\mathbf{E}^{\mathbf{Q}^i} \left( \int_t^T (\delta Y_s^i \cdot (g_1(s, Y_s^1, Z_s^1) - g_1(s, Y_s^2, Z_s^2)))^i ds \middle| \mathcal{F}_t \right) \\ &\quad + 2\mathbf{E}^{\mathbf{Q}^i} \left( \int_t^T \delta Y_s^i \cdot (\delta f^i)_s ds \middle| \mathcal{F}_t \right). \end{aligned}$$

Martingales  $\sqrt{|\delta Y^i| |(g_1(\cdot, Y^1, Z^1) - g_1(\cdot, Y^2, Z^2))^i|} \star W$  and  $\sqrt{|\delta Y^i| |\delta f^i|} \star W$  are BMO, since under **(Hdiag)** we have, for all  $s \in [0, T]$ ,

$$|g_1(s, Y_s^1, Z_s^1) - g_1(s, Y_s^2, Z_s^2)| \leq (K_{d,y} + L_{d,y} |Z_s^1|^2) |\delta Y_s| + (K_{d,z} + L_{d,z} (|Z_s^1| + |Z_s^2|)) |\delta Z_s|.$$

By using Proposition – III.1.1, there exist two constants  $c_1 > 0$  and  $c_2 > 0$  that depend only on  $L_d$  and  $\mathbb{B}$  such that

$$\begin{aligned} c_1^2 \|\delta Z^i \star W\|_{BMO}^2 &\leq \|\delta \xi^i\|_{L^\infty}^2 + 2c_2^2 \left\| \sqrt{|\delta Y^i| |(g_1(\cdot, Y^1, Z^1) - g_1(\cdot, Y^2, Z^2))^i|} \star W \right\|_{BMO}^2 \\ &\quad + 2c_2^2 \left\| \sqrt{|\delta Y^i| |\delta f^i|} \star W \right\|_{BMO}^2. \end{aligned}$$

By summing with respect to  $i$  and by using assumption **(Hdiag)** we obtain

$$\begin{aligned} c_1^2 \|\delta Z \star W\|_{BMO}^2 &\leq \|\delta \xi\|_{L^\infty}^2 \\ &\quad + 2c_2^2 d \|\delta Y\|_{\mathcal{S}^\infty} \left\| \sqrt{|g_1(\cdot, Y^1, Z^1) - g_1(\cdot, Y^2, Z^2)|} \star W \right\|_{BMO}^2 + 2c_2^2 d \|\delta Y\|_{\mathcal{S}^\infty} \left\| \sqrt{|\delta f|} \star W \right\|_{BMO}^2 \\ &\leq \|\delta \xi\|_{L^\infty}^2 + 2c_2^2 v d \left( TK_{d,y} + L_{d,y} \|\delta Z\|_{BMO}^2 \right) \|\delta Y\|_{\mathcal{S}^\infty}^2 \\ &\quad + 2c_2^2 d (K_{d,z} T + L_{d,z} (\|\delta Z^1\|_{BMO} + \|\delta Z^2\|_{BMO})) \|\delta Z \star W\|_{BMO} \|\delta Y\|_{\mathcal{S}^\infty} \\ &\quad + 2c_2^2 d \|\delta Y\|_{\mathcal{S}^\infty} \left\| \sqrt{|\delta f|} \star W \right\|_{BMO}^2. \end{aligned}$$

Once again, for each  $k \in \{0, \dots, N-1\}$  we can write this inequality on  $[T_k, T_{k+1}]$ , and with the same notations as in (III.4.11) we obtain

$$\begin{aligned}
c_1^2 \left\| \left\| T_k |\delta Z| \star W \right\|^{T_{k+1}} \right\|_{BMO}^2 &\leq \left\| \widetilde{\delta Y}_{T_{k+1}}^k \right\|_{L^\infty}^2 + 2c_2^2 d (\eta K_{d,y} + L_{d,y} \mathbb{B}^2) \left\| \widetilde{\delta Y}^k \right\|_{\mathcal{H}^\infty}^2 \\
&\quad + 2c_2^2 d (\eta K_{d,z} + 2\mathbb{B}L_{d,z}) \left\| \widetilde{\delta Y}^k \right\|_{\mathcal{H}^\infty} \left\| \left\| T_k |\delta Z| \star W \right\|^{T_{k+1}} \right\|_{BMO} \\
&\quad + 2c_2^2 d \left\| \widetilde{\delta Y}^k \right\|_{\mathcal{H}^\infty} \left\| \sqrt{|\delta f|} \star W \right\|_{BMO}^2.
\end{aligned}$$

We apply Young inequality to the terms  $\left\| \left\| T_k |\delta Z| \star W \right\|^{T_{k+1}} \right\|_{BMO}$  and  $\left\| \sqrt{|\delta f|} \star W \right\|_{BMO}^2$  and we obtain

$$\begin{aligned}
2c_2^2 d (\eta K_{d,z} + 2\mathbb{B}L_{d,z}) \left\| \widetilde{\delta Y}^k \right\|_{\mathcal{H}^\infty} \left\| \left\| T_k |\delta Z| \star W \right\|^{T_{k+1}} \right\|_{BMO} &\leq 2 \frac{c_2^4 d^2}{c_1^2} (\eta K_{d,z} + 2\mathbb{B}L_{d,z})^2 \left\| \widetilde{\delta Y}^k \right\|_{\mathcal{H}^\infty}^2 \\
+ \frac{c_1^2}{2} \left\| \left\| T_k |\delta Z| \star W \right\|^{T_{k+1}} \right\|_{BMO}^2,
\end{aligned}$$

and for all  $\varepsilon > 0$

$$2c_2^2 d \left\| \widetilde{\delta Y}^k \right\|_{\mathcal{H}^\infty} \left\| \sqrt{|\delta f|} \star W \right\|_{BMO}^2 \leq \varepsilon^2 \left\| \widetilde{\delta Y}^k \right\|_{\mathcal{H}^\infty}^2 + \frac{c_2^4 d^2}{\varepsilon^2} \left\| \sqrt{|\delta f|} \star W \right\|_{BMO}^4.$$

Consequently we get

$$\frac{c_1}{\sqrt{2}} \left\| \left\| T_k |\delta Z| \star W \right\|^{T_{k+1}} \right\|_{BMO} \tag{III.4.13}$$

$$\begin{aligned}
&\leq \left\| \widetilde{\delta Y}_{T_{k+1}}^k \right\|_{L^\infty} + \left( \sqrt{2} c_2 \sqrt{d} (\sqrt{\eta K_{d,y}} + \sqrt{L_{d,y} \mathbb{B}}) + \sqrt{2} \frac{c_2^2 d}{c_1} (\eta K_{d,z} + 2\mathbb{B}L_{d,z}) + \varepsilon \right) \left\| \widetilde{\delta Y}^k \right\|_{\mathcal{H}^\infty} \\
&\quad + \frac{c_2^2 d}{\varepsilon} \left\| \sqrt{|\delta f|} \star W \right\|_{BMO}^2. \tag{III.4.14}
\end{aligned}$$

**Step 3 – Stability.** Combining (III.4.12) and (III.4.13), we can obtain a stability result on  $[T_k, T_{k+1}]$  as soon as  $\eta$  and  $\varepsilon$  are sufficiently small to get  $c_2^2 d (K_{d,y} \eta + L_{d,y} \mathbb{B}^2) < 1$  and

$$\frac{\sqrt{2}}{c_1} \left( \sqrt{2} c_2 \sqrt{d} (\sqrt{\eta K_{d,y}} + \sqrt{L_{d,y} \mathbb{B}}) + \sqrt{2} \frac{c_2^2 d}{c_1} (\eta K_{d,z} + 2\mathbb{B}L_{d,z}) + \varepsilon \right) \frac{c_2^2 d (K_{d,z} \sqrt{\eta} + 2L_{d,z} \mathbb{B})}{1 - c_2^2 d (K_{d,y} \eta + L_{d,y} \mathbb{B}^2)} < 1.$$

We obtain the existence of a constant  $K$  which does not depend on  $k$  such that

$$\left\| \widetilde{\delta Y}^k \right\|_{\mathcal{H}^\infty} + \left\| \left\| T_k |\delta Z| \star W \right\|^{T_{k+1}} \right\|_{BMO} \leq K \left( \left\| \widetilde{\delta Y}_{T_{k+1}}^k \right\|_{L^\infty} + \left\| \sqrt{|\delta f|} \star W \right\|_{BMO}^2 \right).$$

Since  $\|\delta Y\|_{\mathcal{S}^\infty} \leq \sum_{k=0}^{N-1} \|\widetilde{\delta Y}^k\|_{\mathcal{S}^\infty}$  and  $\|\delta Z \star W\|_{BMO} \leq \sum_{k=0}^{N-1} \left\| \int_{T_k}^{T_{k+1}} \delta Z \star W \right\|_{BMO}$ , by a direct iteration we finally obtain a constant  $K$  such that

$$\|\delta Y\|_{\mathcal{S}^\infty} + \|\delta Z \star W\|_{BMO} \leq K \left( \|\delta \xi\|_{L^\infty} + \left\| \sqrt{|\delta f|} \star W \right\|_{BMO}^2 \right),$$

and  $K$  depends only on  $\mathbb{B}$  and constants in **(Hdiag)**.  $\square$

#### § III.4.4. Proof of Theorem – III.2.3

Theorem – III.2.3 is proved by relaxing assumptions **(Dxi,b)** and **(Df,b)** of Theorem – III.2.1 thanks to some density arguments. To ensure the convergence, the keystone result will be the stability Theorem – III.4.1.

Proof. [of Theorem – III.2.3]

**Step 1– Approximations.** We can approach  $\xi$  with a sequence of random variables  $(\xi^n)_{n \in \mathbb{N}}$  such that for every  $n$ ,  $\xi^n$  has a bounded Malliavin derivative:

$$\|\mathbf{D}\xi^n\|_{\mathcal{S}^\infty} < \infty.$$

More precisely  $\xi^n$  can be chosen of the form  $\Phi^n(W_{t_1}, \dots, W_{t_n})$  where  $\Phi^n \in \mathcal{C}_b^\infty(\mathbf{R}^n)$ ,  $(t_1, \dots, t_n) \in [0, T]^n$  and  $\xi^n$  tends to  $\xi$  in every  $L^p$  for  $p \geq 1$  (see [Nua06], Exercise 1.1.7).

Since  $\alpha$  is adapted, we can approach this process with a sequence of sample processes  $\alpha^n$  of the form

$$\alpha_t^n = \sum_{i=0}^{p_n-1} \alpha_t^{i,n} \mathbb{1}_{[t_i^n, t_{i+1}^n[}(t),$$

where  $(t_i^n)_{i=0}^{p_n}$  is a sequence of subdivisions of  $[0, T]$ , with  $\sup_{0 \leq i \leq p_n-1} |t_{i+1}^n - t_i^n| \xrightarrow{n \rightarrow \infty} 0$ , and, for

all  $0 \leq i \leq p_n - 1$ ,  $n \in \mathbb{N}$ ,  $\alpha^{i,n}$  is a  $\mathcal{F}_{t_i^n}$ -measurable random variable. We have a convergence of this sequence to  $\alpha$  in  $L^2(\Omega \times [0, T])$ :

$$\mathbf{E} \left( \int_0^T |\alpha_s^n - \alpha_s|^2 ds \right) \xrightarrow{n \rightarrow \infty} 0.$$

We can assume in addition that for all  $n$  and for all  $0 \leq i \leq p_n$ ,  $\alpha^{i,n}$  has a bounded Malliavin derivative since this set is dense in  $L^2(\Omega)$ . It is obvious that for all  $0 \leq u \leq T$  and  $0 \leq t \leq T$ ,

$$\mathbf{D}_u \alpha_t^n = \sum_{i=0}^{p_n-1} \mathbf{D}_u \alpha_t^{i,n} \mathbb{1}_{[t_i^n, t_{i+1}^n[}(u).$$

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According to [Proposition – III.1.4](#) applied to  $\varphi = \mathbf{f}(\cdot, y, z)$ , there exists for all  $n \in \mathbf{N}$  and  $t \in [0, T]$  a bounded random variable  $\mathbf{G}$  such that

$$\mathbf{D}_t \mathbf{f}(\alpha_t^n, y, z) = \mathbf{G} \cdot \mathbf{D}_t \alpha_t^n, \quad \text{and} \quad |\mathbf{G}| \leq D(1 + |z|^2).$$

For each  $n \in \mathbf{N}$ :  $\xi^n$  satisfies **(Dxi,b)**,  $\mathbf{f}(\alpha_t^n, \cdot, \cdot)$  satisfies **(Df,b)** and **(BMO,m)** is fulfilled. So, we can apply [Theorem – III.2.1](#): there exists an unique solution  $(Y^n, Z^n) \in \mathcal{S}^2(\mathbf{R}^d) \times \mathcal{L}_{BMO}^m$  of the equation

$$Y_t^n = \xi^n + \int_t^T \mathbf{f}(\alpha_s^n, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n dW_s, \quad 0 \leq t \leq T.$$

**Step 2– Application of the stability result.** We can assume that for all  $n$ ,  $\|\xi^n\|_{L^{2m^*}} \leq \|\xi\|_{L^{2m^*}}$ . If it is not true, we consider the sequence  $\tilde{\xi}^n = \frac{\|\xi\|_{L^{2m^*}}}{\|\xi\|_{L^{2m^*}} + \|\xi^n - \xi\|_{L^{2m^*}}} \xi^n$  instead of  $\xi^n$ . The same argument allows us to assume that

$$\|\alpha^n\|_{L^{2m^*}(\Omega \times [0, T])} \leq \|\alpha\|_{L^{2m^*}(\Omega \times [0, T])}.$$

Under **(BMO2,m)**, we have the estimate

$$mL_y \| |Z^n| \star W \|_{BMO}^2 + \sqrt{2}L_z \| |Z^n| \star W \|_{BMO} C'_m \leq mL_y \mathbb{K}^2 + \sqrt{2}L_z \mathbb{K} C'_m < \frac{1}{2}.$$

Hence, for all  $n_1, n_2 \in \mathbf{N}$ , we can use [Theorem – III.4.1](#) for  $p = m^*$  which gives us:

$$\begin{aligned} & \|Y^{n_1} - Y^{n_2}\|_{\mathcal{S}^{2m^*}}^{m^*} + \| |Z^{n_1} - Z^{n_2}| \star W \|_{\mathcal{H}^{m^*}}^{m^*} \\ & \leq \tilde{K}_{m^*} \left( \|\xi^{n_1} - \xi^{n_2}\|_{L^{2m^*}}^{m^*} + \mathbf{E} \left( \left( \int_0^T |\mathbf{f}(\alpha_t^{n_1}, Y_t^{n_2}, Z_t^{n_2}) - \mathbf{f}(\alpha_t^{n_2}, Y_t^{n_2}, Z_t^{n_2})| dt \right)^{2m^*} \right)^{1/2} \right). \end{aligned}$$

where the constant  $\tilde{K}_{m^*}$  appearing does not depend on  $n$  under **(BMO2,m)**. This fact was already highlighted in the proof of [Proposition – III.4.1](#) where an explicit formula for  $\tilde{K}_{m^*}$  was given. We recall that  $(\xi^n)_{n \in \mathbf{N}}$  is a Cauchy sequence in  $L^{2m^*}$ , so

$$\|\xi^{n_1} - \xi^{n_2}\|_{L^{2m^*}} \xrightarrow{n_1, n_2 \rightarrow \infty} 0.$$

For the second term, we use the Hölder inequality:

$$\begin{aligned}
 & \mathbf{E} \left( \left( \int_0^T |\mathbf{f}(\alpha_t^{n_1}, Y_t^{n_2}, Z_t^{n_2}) - \mathbf{f}(\alpha_t^{n_2}, Y_t^{n_2}, Z_t^{n_2})| dt \right)^{2m^*} \right) \\
 & \leq D^{2m^*} \mathbf{E} \left( \left( \int_0^T (1 + |Z_t^{n_2}|^2) |\alpha_t^{n_1} - \alpha_t^{n_2}|^\delta dt \right)^{2m^*} \right) \\
 & \leq D^{2m^*} \|\alpha^{n_1} - \alpha^{n_2}\|_{\mathcal{S}^{2m^*} \delta}^{2m^*} \mathbf{E} \left( \left( \int_0^T (1 + |Z_t^{n_2}|^2) dt \right)^{4m^*} \right)^{1/2}. \tag{III.4.15}
 \end{aligned}$$

Since  $|Z^n| \star W \in BMO$ , and **(BMO2,m)** holds true, we have

$$\sup_{n_2 \in \mathbf{N}} \mathbf{E} \left( \left( \int_0^T (1 + |Z_t^{n_2}|^2) dt \right)^{4m^*} \right)^{1/2} < +\infty.$$

Moreover, by using the uniform continuity of trajectories of  $\alpha$  on  $[0, T]$ , we get:

$$\sup_{t \in [0, T]} |\alpha_t^{n_1} - \alpha_t^{n_2}| \xrightarrow{n_1, n_2 \rightarrow \infty} 0 \quad \text{a.s.}$$

Since we have

$$\mathbf{E} \left( \sup_{t \in [0, T]} |\alpha_t^{n_1} - \alpha_t^{n_2}|^{4m^*+1} \right) \leq 2C \mathbf{E} \left( \sup_{t \in [0, T]} |\alpha_t|^{4m^*+1} \right) < +\infty,$$

then, a uniform integrability argument gives us

$$\mathbf{E} \left( \sup_{t \in [0, T]} |\alpha_t^{n_1} - \alpha_t^{n_2}|^{4m^*} \right) \xrightarrow{n_1, n_2 \rightarrow \infty} 0.$$

Finally we get

$$\mathbf{E} \left( \left( \int_0^T |\mathbf{f}(\alpha_t^{n_1}, Y_t^{n_2}, Z_t^{n_2}) - \mathbf{f}(\alpha_t^{n_2}, Y_t^{n_2}, Z_t^{n_2})| dt \right)^{2m^*} \right) \xrightarrow{n_1, n_2 \rightarrow \infty} 0.$$

Consequently  $(Y^n, Z^n \star W)_{n \in \mathbf{N}}$  is a Cauchy sequence in  $\mathcal{S}^{2m^*}(\mathbf{R}^d) \times \mathcal{H}^{m^*}(\mathbf{R}^{d \times k})$ , thus it converges in  $\mathcal{S}^{2m^*}(\mathbf{R}^d) \times \mathcal{H}^{m^*}(\mathbf{R}^{d \times k})$  to a process  $(Y, Z \star W)$ , and  $(Y, Z)$  solves the BSDE **(III.1.6)**. Finally, according to **Proposition – III.1.2** the upper bound for  $\| |Z^n| \star W \|_{BMO}$  holds true for  $\| |Z| \star W \|_{BMO}$  and so the uniqueness follows from the stability theorem.  $\square$

*Remark – III.4.5.* If  $f$  is a deterministic function, then the assumption **(Df,b)** is not required.

*Remark – III.4.6.* If we replace the inequality **(III.2.1)** by the new one: there exist  $\eta > 0, D > 0$

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and  $\delta \in (0, 1]$  such that for all  $(\beta, \beta', y, z) \in (\mathbf{R}^d)^2 \times \mathbf{R}^d \times \mathbf{R}^{d \times k}$  we have

$$|\mathbf{f}(\beta, y, z) - \mathbf{f}(\beta', y, z)| \leq D \left(1 + |z|^{2-\eta}\right) |\beta - \beta'|^\delta,$$

then we do not have to assume that  $\alpha$  is a continuous process. Indeed, we can change the inequality (III.4.15) by the following one: by using the Hölder and Cauchy-Schwartz inequalities we have, for all  $p > 1$ ,

$$\begin{aligned} & \mathbf{E} \left( \left( \int_0^T |\mathbf{f}(\alpha_t^{n_1}, Y_t^{n_2}, Z_t^{n_2}) - \mathbf{f}(\alpha_t^{n_2}, Y_t^{n_2}, Z_t^{n_2})| dt \right)^{2m^*} \right) \\ & \leq D^{2m^*} \mathbf{E} \left( \left( \int_0^T (1 + |Z_t^{n_2}|^{2-\eta}) |\alpha_t^{n_1} - \alpha_t^{n_2}|^\delta dt \right)^{2m^*} \right) \\ & \leq D^{2m^*} \mathbf{E} \left( \left( \int_0^T (1 + |Z_t^{n_2}|^{2-\eta})^p dt \right)^{2m^*/p} \times \left( \int_0^T |\alpha_t^{n_1} - \alpha_t^{n_2}|^{\delta p^*} dt \right)^{2m^*/p^*} \right) \\ & \leq D^{2m^*} \mathbf{E} \left( \left( \int_0^T (1 + |Z_t^{n_2}|^{2-\eta})^p dt \right)^{2m^*} \right)^{1/p} \mathbf{E} \left( \left( \int_0^T |\alpha_t^{n_1} - \alpha_t^{n_2}|^{\delta p^*} dt \right)^{2m^*} \right)^{1/p^*} \\ & \leq D^{2m^*} T^{\frac{4m^*}{p^*} - 2m^*} \mathbf{E} \left( \left( \int_0^T (1 + |Z_t^{n_2}|^{2-\eta})^p dt \right)^{2m^*} \right)^{1/p} \left\| |\alpha^{n_1} - \alpha^{n_2}|^{\delta p^*} \star W \right\|_{\mathcal{H}^{4m^*}}^{4m^*/p^*}. \end{aligned}$$

With the energy inequality, the first term is uniformly bounded with respect to  $n_2$  under the assumption (BMO2,m) by choosing  $1 < p \leq \frac{2}{2-\eta}$ . The second one tends to zero when  $n_1, n_2$  go to infinity since the convergence in every  $\mathcal{H}^r$  for  $r > 1$  holds true.

*Remark – III.4.7.* By using Remark – III.3.3 once again, Theorem – III.2.3 can be adapted if we replace the assumption (BMO2,m) by the following one:  $\xi \in L^{2m^*}$  and there exist a constant  $\mathbb{K}$  and a sequence  $0 = T_0 \leq T_1 \leq \dots \leq T_N = T$  of stopping times (that does not depend on  $M$ ) such that

(i)  $2mL_y \mathbb{K}^2 + 2\sqrt{2}L_z \mathbb{K} C'_m < 1,$

(ii) for all  $i \in \{0, \dots, N-1\}$ ,  $\sup_{M \in \mathbf{R}^+} \sup_{\substack{\|\eta\|_{L^{2m^*}} \leq \|\xi\|_{L^{2m^*}} \\ \|\beta\|_{L^2(\Omega \times [0, T])} \leq \|\alpha\|_{L^2(\Omega \times [0, T])}}} \left\| \left| Z^{(M, \eta, \beta)} \right| \star W \right|_{T_i+1} \right\|_{BMO} \leq \mathbb{K}.$

In this case, if all the other assumptions of Theorem – III.2.3 are fulfilled, then the quadratic BSDE (III.1.6) has a unique solution  $(Y, Z) \in \mathcal{S}^\infty(\mathbf{R}^d) \times \mathcal{L}_{BMO}^{stic, m}$  such that

$$\text{esssup}_{\Omega \times [0, T]} |Z| < +\infty.$$

*Remark – III.4.8.* It is possible to extend the existence and uniqueness result for the diagonal case given by Theorem – III.2.2 to more general terminal conditions and generators. More precisely, it is possible to apply the same strategy as for the proof of Theorem – III.2.3 by applying the stability result given by Theorem – III.4.2 instead of Theorem – III.4.1. Nevertheless we can

only obtain an existence and uniqueness result for terminal conditions (resp. generators) that can be approximated in  $L^\infty$  (resp.  $BMO$ ) by terminal conditions satisfying assumption **(Dxi,b)** (resp. generators satisfying assumption **(Df,b)**).

### III.5 PROOFS OF SECTION III.2.2 RESULTS

#### § III.5.1. Proof of Proposition – III.2.1

We start by proving some uniform (with respect to  $M$ ) *a priori* estimates on  $(Y^M, Z^M)$ .

**Proposition – III.5.1.** *Let us assume that **(H)** and **(HQ)** are in force. Then  $|Z^M| \star W \in BMO$ ,  $Y^M \in \mathcal{S}^\infty$  and we have the following estimates:*

$$(i) \quad \||Z^M| \star W\|_{BMO}^2 \leq \frac{1}{8\gamma^2} \left( 1 - \sqrt{1 - 32\gamma^2 \|\xi\|_{L^\infty}^2} \right),$$

$$(ii) \quad \|Y^M\|_{\mathcal{S}^\infty} \leq \|\xi\|_{L^\infty} + \gamma \||Z^M| \star W\|_{BMO}^2.$$

We can note that upper bound do not depend on  $M$ .

Proof. To simplify notations in the proof, we skip the superscript  $M$  on  $(Y^M, Z^M)$  and  $f^M$ . The unique solution  $(Y, Z) \in \mathcal{S}^2 \times \mathcal{H}^2$  of **(III.1.8)** can be constructed with a Picard principle as in the seminal paper of Pardoux and Peng (see **[PP90]**). We consider a sequence  $(Y^n, Z^n)_{n \in \mathbf{N}}$  such that  $(Y^n, Z^n)_{n \in \mathbf{N}}$  tends to  $(Y, Z)$  in  $\mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$ . This sequence is given by

$$Y_t^{n+1} = \xi + \int_t^T f(s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^{n+1} dW_s, \quad 0 \leq t \leq T, \quad (Y^0, Z^0) = (0, 0).$$

We will prove with an induction that: for all  $n \in \mathbf{N}$ ,  $Y^n \in \mathcal{S}^\infty$ ,  $|Z^n| \star W \in BMO$  and

$$\||Z^n| \star W\|_{BMO}^2 \leq \frac{1}{8\gamma^2} \left( 1 - \sqrt{1 - 32\gamma^2 \|\xi\|_{L^\infty}^2} \right).$$

The case  $n = 0$  is obviously satisfied. Let us suppose that  $Y^n \in \mathcal{S}^\infty$  and  $|Z^n| \star W \in BMO$ . Then for all  $t \in [0, T]$ , under **(HQ)**,

$$|Y_t^{n+1}| \leq \mathbf{E}(\|\xi\| | \mathcal{F}_t) + \gamma \times \mathbf{E} \left( \int_t^T |Z_s^n|^2 ds \middle| \mathcal{F}_t \right). \quad (\text{III.5.1})$$

We get  $Y^{n+1} \in \mathcal{S}^\infty$  since  $\|Y^{n+1}\|_{\mathcal{S}^\infty} \leq \|\xi\|_{L^\infty} + \gamma \||Z^n| \star W\|_{BMO}^2$ . Itô formula gives the following equality

$$|Y_t^{n+1}|^2 = \|\xi\|^2 + 2 \int_t^T Y_s^{n+1} \cdot f(s, Y_s^n, Z_s^n) ds - 2 \int_t^T Y_s^{n+1} \cdot (Z_s^{n+1} dW_s) - \int_t^T |Z_s^{n+1}|^2 ds.$$



By taking conditional expectation we get for every stopping time  $\tau$ :

$$\begin{aligned} |Y_\tau^{n+1}|^2 + \mathbf{E} \left( \int_\tau^T |Z_s^{n+1}|^2 ds \middle| \mathcal{F}_\tau \right) &\leq \mathbf{E} \left( |\xi|^2 \middle| \mathcal{F}_t \right) + 2\mathbf{E} \left( \int_\tau^T Y_s^{n+1} \cdot f(s, Y_s^n, Z_s^n) ds \middle| \mathcal{F}_\tau \right) \\ &\leq \|\xi\|_{L^\infty}^2 + 2\|Y^{n+1}\|_{\mathcal{S}^\infty} \mathbf{E} \left( \int_\tau^T \gamma |Z_s^n|^2 ds \middle| \mathcal{F}_\tau \right) \\ &\leq \|\xi\|_{L^\infty}^2 + 2\gamma \|Y^{n+1}\|_{\mathcal{S}^\infty} \|Z^n \star W\|_{BMO}^2. \end{aligned}$$

Taking the essential supremum with respect to  $\tau$  in following inequalities, we obtain

$$\begin{aligned} |Y_\tau^{n+1}|^2 &\leq \|\xi\|_{L^\infty}^2 + 2\gamma \|Y^{n+1}\|_{\mathcal{S}^\infty} \|Z^n \star W\|_{BMO}^2 \\ \mathbf{E} \left( \int_\tau^T |Z_s^{n+1}|^2 ds \middle| \mathcal{F}_\tau \right) &\leq \|\xi\|_{L^\infty}^2 + 2\gamma \|Y^{n+1}\|_{\mathcal{S}^\infty} \|Z^n \star W\|_{BMO}^2. \end{aligned}$$

Thus  $|Z^{n+1}| \star W \in BMO$  and we have:

$$\begin{aligned} \|Y^{n+1}\|_{\mathcal{S}^\infty}^2 + \| |Z^{n+1}| \star W \|_{BMO}^2 &\leq 2\|\xi\|_{L^\infty}^2 + 4\gamma \|Y^{n+1}\|_{\mathcal{S}^\infty} \|Z^n \star W\|_{BMO}^2 \\ &\leq 2\|\xi\|_{L^\infty}^2 + \|Y^{n+1}\|_{\mathcal{S}^\infty}^2 + 4\gamma^2 \|Z^n \star W\|_{BMO}^4, \end{aligned}$$

which leads to

$$\| |Z^{n+1}| \star W \|_{BMO}^2 \leq 2\|\xi\|_{L^\infty}^2 + 4\gamma^2 \|Z^n \star W\|_{BMO}^4.$$

Using the induction assumption we obtain

$$\| |Z^{n+1}| \star W \|_{BMO}^2 \leq \frac{1}{8\gamma^2} \left( 1 - \sqrt{1 - 32\gamma^2 \|\xi\|_{L^\infty}^2} \right).$$

The induction is achieved. Now we can use [Proposition – III.1.2](#) with  $K = \frac{1}{8\gamma^2} \left( 1 - \sqrt{1 - 32\gamma^2 \|\xi\|_{L^\infty}^2} \right)$ :

since  $Z^n \star W$  tends to  $Z \star W$  in  $\mathcal{H}^2$ , we conclude that  $\| |Z| \star W \|_{BMO}^2 \leq \frac{1}{8\gamma^2} \left( 1 - \sqrt{1 - 32\gamma^2 \|\xi\|_{L^\infty}^2} \right)$ .

Finally, we use that  $Y^n$  tends to  $Y$  in  $\mathcal{S}^2$  to pass to the limit into [\(III.5.1\)](#) and to obtain the final upper bound on  $\|Y\|_{\mathcal{S}^\infty}$ .  $\square$

**Proof.** [of [Proposition – III.2.1](#)] The proof of the proposition is a direct consequence of [Theorem – III.2.3](#) together with [Proposition – III.5.1](#): since the map

$$x \in \mathbf{R}^+ \mapsto \frac{1}{2\sqrt{2}\gamma} \left( 1 - \sqrt{1 - 32\gamma^2 x^2} \right)^{\frac{1}{2}}$$

is nondecreasing, the assumption **(BMO2,m)** is satisfied.  $\square$

§ III.5.2. Proof of Proposition – III.2.2

Once again, we start by proving some uniform (with respect to  $M$ ) *a priori* estimates on  $(Y^M, Z^M)$ .

**Proposition – III.5.2.** *Let us assume that (H) and (HMon) are in force. Then  $|Z^M| \star W \in BMO$ ,  $Y^M \in \mathcal{S}^\infty$  and we have*

$$(i) \operatorname{esssup}_{t \in [0, T]} \sup \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^M|^2 ds \middle| \mathcal{F}_t \right) \leq \frac{1}{8\gamma^2} \left( 1 - \sqrt{1 - 32\gamma^2 A^2} \right),$$

$$(ii) \|Y^M\|_{\mathcal{S}^\infty} \leq \frac{1}{4\gamma} \left( 1 - \sqrt{1 - 32\gamma^2 A^2} \right) + \sqrt{2A^2 + \frac{1}{16\gamma^2} \left( 1 - \sqrt{1 - 32\gamma^2 A^2} \right)^2}.$$

$$\text{with } A = \max \left( \|\xi\|_{L^\infty}, \frac{\alpha}{\mu} \right).$$

Proof. To simplify notations in the proof, we skip the superscript  $M$  on  $(Y^M, Z^M)$  and  $f^M$ . The unique solution  $(Y, Z)$  of (III.1.8) can be constructed with a Picard principle. We consider a sequence  $(Y^n, Z^n)_{n \in \mathbf{N}}$  such that  $(Y^n, Z^n \star W)_{n \in \mathbf{N}}$  tends to  $(Y, Z \star W)$  in  $\mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$ , with

$$Y_t^{n+1} = \xi + \int_t^T f(s, Y_s^{n+1}, Z_s^n) ds - \int_t^T Z_s^{n+1} dW_s, \quad 0 \leq t \leq T, \quad (Y^0, Z^0) = (0, 0), \dots$$

We can easily show that replacing  $Y^{n+1}$  by  $Y^n$  in the generator does not affect the convergence of the scheme since  $f$  is a Lipschitz function. Moreover, applying Itô formula to  $e^{Kt} |Y_t^{n+1}|^2$  with  $K$  large enough, we justify with classical inequalities that for all  $n \in \mathbf{N}$ ,  $Y^{n+1} \in \mathcal{S}^\infty$ , with a bound that depend on  $M$  for the moment. Applying Itô formula to the process  $e^{-\mu t} |Y_t^{n+1}|^2$ , we obtain

$$\begin{aligned} e^{-\mu t} |Y_t^{n+1}|^2 &= e^{-\mu T} |\xi|^2 - \int_t^T \left( -\mu e^{-\mu s} |Y_s^{n+1}|^2 - 2e^{-\mu s} Y_s^{n+1} \cdot f(s, Y_s^{n+1}, Z_s^n) + e^{-\mu s} |Z_s^{n+1}|^2 \right) ds \\ &\quad - 2 \int_t^T e^{-\mu s} Y_s^{n+1} \cdot (Z_s^{n+1} dW_s). \end{aligned}$$

Taking conditional expectation, and using assumption (HMon), we get:

$$\begin{aligned} |Y_t^{n+1}|^2 &\leq e^{-\mu(T-t)} \|\xi\|_{L^\infty}^2 + \mathbf{E} \left( \int_t^T 2e^{-\mu(s-t)} \left( \alpha |Y_s^{n+1}| - \frac{\mu}{2} |Y_s^{n+1}|^2 + \gamma |Y_s^{n+1}| |Z_s^n|^2 \right) ds \middle| \mathcal{F}_t \right) \\ &\quad - \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^{n+1}|^2 ds \middle| \mathcal{F}_t \right). \end{aligned}$$

With the Young inequality we have the following estimate for all  $n$  and  $s \in [0, T]$ :

$$\alpha |Y_s^{n+1}| \leq \frac{\mu}{2} |Y_s^{n+1}|^2 + \frac{\alpha^2}{2\mu},$$

and thus

$$\begin{aligned} |Y_t^{n+1}|^2 &\leq e^{-\mu(T-t)} \|\xi\|_{L^\infty}^2 + \mathbf{E} \left( \int_t^T 2e^{-\mu(s-T)} \left( \frac{\alpha^2}{2\mu} + \gamma |Y_s^{n+1}| |Z_s^n|^2 \right) ds \middle| \mathcal{F}_t \right) \\ &\quad - \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^{n+1}|^2 ds \middle| \mathcal{F}_t \right) \\ &\leq \underbrace{e^{-\mu(T-t)} \|\xi\|_{L^\infty}^2 + \frac{\alpha^2}{\mu^2} (1 - e^{-\mu(T-t)})}_{\leq A^2} + 2\gamma \times \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Y_s^{n+1}| |Z_s^n|^2 ds \middle| \mathcal{F}_t \right) \\ &\quad - \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^{n+1}|^2 ds \middle| \mathcal{F}_t \right). \end{aligned}$$

Finally we obtain

$$|Y_t^{n+1}|^2 + \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^{n+1}|^2 ds \middle| \mathcal{F}_t \right) \leq A^2 + 2\gamma \times \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Y_s^{n+1}| |Z_s^n|^2 ds \middle| \mathcal{F}_t \right).$$

Then

$$\begin{aligned} \|Y^{n+1}\|_{\mathcal{S}^\infty}^2 + \text{esssup} \sup_{t \in [0, T]} \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^{n+1}|^2 ds \middle| \mathcal{F}_t \right) \\ \leq 2A^2 + 4\gamma \|Y^{n+1}\|_{\mathcal{S}^\infty} \text{esssup} \sup_{t \in [0, T]} \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^n|^2 ds \middle| \mathcal{F}_t \right) \quad (\text{III.5.2}) \\ \leq 2A^2 + \|Y^{n+1}\|_{\mathcal{S}^\infty}^2 + (2\gamma)^2 \text{esssup} \sup_{t \in [0, T]} \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^n|^2 ds \middle| \mathcal{F}_t \right)^2. \end{aligned}$$

Once again with an induction we show easily that for all  $n \in \mathbf{N}$ ,  $|Z^n| \star W \in BMO$ ,  $Y^n \in \mathcal{S}^\infty$  and

$$\text{esssup} \sup_{t \in [0, T]} \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^n|^2 ds \middle| \mathcal{F}_t \right) \leq \frac{1}{8\gamma^2} (1 - \sqrt{1 - 32\gamma^2 A^2}).$$

Moreover the inequality (III.5.2) gives us

$$\begin{aligned} \|Y^{n+1}\|_{\mathcal{S}^\infty} &\leq 2\gamma \times \text{esssup} \sup_{t \in [0, T]} \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^n|^2 ds \middle| \mathcal{F}_t \right) \\ &\quad + \sqrt{2A^2 + 4\gamma^2 \left( \text{esssup} \sup_{t \in [0, T]} \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^n|^2 ds \middle| \mathcal{F}_t \right) \right)^2}. \end{aligned}$$

Letting  $n$  to infinity, and with the Proposition – III.1.2, we finally get

$$\text{esssup}_{t \in [0, T]} \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s|^2 ds \middle| \mathcal{F}_t \right) \leq \frac{1}{8\gamma^2} \left( 1 - \sqrt{1 - 32\gamma^2 A^2} \right).$$

and so we deduce that

$$\|Y\|_{\mathcal{S}^\infty} \leq \frac{1}{4\gamma} \left( 1 - \sqrt{1 - 32\gamma^2 A^2} \right) + \sqrt{2A^2 + \frac{1}{16\gamma^2} \left( 1 - \sqrt{1 - 32\gamma^2 A^2} \right)^2}.$$

□

*Remark – III.5.1.* What about the *BMO* norm of  $Z^M$ ? — we can slice  $[0, T]$  with a uniform sequence  $(T_i)_{i=1}^N$  such that  $0 = T_0 \leq T_1 \leq \dots \leq T_N = T$  and for all  $i$ ,  $h = |T_{i+1} - T_i| = \frac{T}{N}$ . The last inequality can be used for each started and stopped process  $T_i |Z^M| \star W^{T_{i+1}}$ :

$$\begin{aligned} \text{esssup}_{T_i \leq t \leq T_{i+1}} \mathbf{E} \left( \int_t^{T_{i+1}} |Z_s^M|^2 ds \middle| \mathcal{F}_t \right) &\leq e^{\mu h} \left( \text{esssup}_{T_i \leq t \leq T_{i+1}} \mathbf{E} \left( \int_t^{T_{i+1}} e^{-\mu(s-t)} |Z_s^M|^2 ds \middle| \mathcal{F}_t \right) \right) \\ &\leq e^{\mu h} \left( \text{esssup}_{T_i \leq t \leq T_{i+1}} \mathbf{E} \left( \int_t^T e^{-\mu(s-t)} |Z_s^M|^2 ds \middle| \mathcal{F}_t \right) \right) \\ &\leq \frac{e^{\mu h}}{8\gamma^2} \left( 1 - \sqrt{1 - 32\gamma^2 A^2} \right). \end{aligned}$$

We are now in position to prove Proposition – III.2.2.

Proof. [of Proposition – III.2.2] The previous Remark – III.5.1 shows that for all  $M \in \mathbf{R}^+$  and all  $h > 0$ , the process  $|Z^M| \star W$  is

$$\left( \frac{e^{\frac{1}{2}\mu h}}{2\sqrt{2}\gamma} \left( 1 - \sqrt{1 - 32\gamma^2 A^2} \right)^{1/2} \right) - \text{sliceable}.$$

We just have to apply an adaptation of Theorem – III.2.3 given by Remark – III.4.7. □

### § III.5.3. Proof of Proposition – III.2.3

We consider the diagonal framework introduced in section III.2 and subsection III.4.3. We assume that the generator satisfies **(Hdiag)**, so the generator  $f$  can be written as  $f = f_{\text{diag}}(t, z) + g(t, y, z)$  where  $f_{\text{diag}}$  is diagonal with respect to  $z$ . If we want to apply Theorem – III.2.2, we have to obtain a uniform estimate on  $\| |Z^M| \star W \|_{BMO}$  where  $(Y^M, Z^M)$  is the unique solution of the Lipschitz localized BSDE (III.1.8). This is the purpose of the following lemma.

**Proposition – III.5.3.** *Let us assume that there exist nonnegative constants  $G_d$  and  $G$  such that*

$$(i) \text{ for all } (t, y, z) \in [0, T] \times \mathbf{R}^d \times \mathbf{R}^{d \times k}, \quad |f_{\text{diag}}(t, z)| \leq G_d |z|^2, \quad |g(t, y, z)| \leq G |z|^2.$$

(ii)

$$\frac{4 \sum_{i=1}^d e^{2G_d} \|\xi^i\|_{L^\infty}}{G_d} G \leq 1. \quad (\text{III.5.3})$$

Then,  $|Z^M| \star W \in BMO$ ,  $Y^M \in \mathcal{S}^\infty$  and we have following estimates:

$$\| |Z^M| \star W \|_{BMO} \leq (4G_d G)^{-1/2}, \quad \|Y^M\|_{\mathcal{S}^\infty} \leq \|\xi\|_{L^\infty} + \frac{\sqrt{d} \log 2}{2G_d}.$$

Proof. To simplify notations in the proof, we skip once again the superscript  $M$  on  $(Y^M, Z^M)$  and  $f^M$ . The unique solution  $(Y, Z) \in \mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$  of (III.1.8) can be constructed with a Picard principle slightly different than the one used in the seminal paper of Pardoux and Peng (see [PP90]). We consider a sequence  $(Y^n, Z^n)_{n \in \mathbf{N}}$  defined by

$$Y_t^{n+1} = \xi + \int_t^T f_{\text{diag}}(s, Z_s^{n+1}) + g(s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^{n+1} dW_s, \quad 0 \leq t \leq T, \quad (Y^0, Z^0) = (0, 0).$$

Obviously, we can easily show that  $(Y^n, Z^n)_{n \in \mathbf{N}}$  tends to  $(Y, Z \star W)$  in  $\mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$  since we are in the Lipschitz framework. We will prove by induction that: for all  $n \in \mathbf{N}$ ,  $Y^n \in \mathcal{S}^\infty$ ,  $|Z^n| \star W \in BMO$  and

$$\| |Z^n| \star W \|_{BMO} \leq (4G_d G)^{-1/2}, \quad \|Y^n\|_{\mathcal{S}^\infty} \leq \|\xi\|_{L^\infty} + \frac{\sqrt{d} \log 2}{2G_d}. \quad (\text{III.5.4})$$

The result is obvious for  $n = 0$ . Let us assume that for a given  $n \in \mathbf{N}$  we have  $Y^n \in \mathcal{S}^\infty(\mathbf{R}^d)$ ,  $|Z^n| \star W \in BMO$  and (III.5.4) is true. The Lipschitz regularity of  $f$  gives us that  $(Y^{n+1}, |Z^{n+1}| \star W) \in \mathcal{S}^\infty(\mathbf{R}^d) \times BMO$ . By following the idea of [BH06], we introduce the  $C^2$  function  $\varphi : \mathbf{R} \mapsto (0, +\infty)$  defined by

$$\varphi : x \mapsto \frac{e^{2G_d|x|} - 1 - 2G_d|x|}{|2G_d|^2}, \quad \text{so that} \quad \varphi''(\cdot) - 2G_d|\varphi'(\cdot)| = 1.$$

We pick a stopping time  $\tau$  and applying Itô's formula to the regular function  $\phi$ , we compute, for

all components  $i \in \{1, \dots, d\}$ ,

$$\begin{aligned} \varphi(Y_\tau^{n+1,i}) &= \varphi(\xi^i) + \int_\tau^T \left( \varphi'(Y_s^{n+1,i}) (f_{\text{diag}}(s, Z_s^{n+1,i}) + g(s, Y_s^{n,i}, Z_s^{n,i})) - \frac{\varphi''(Y_s^{n+1,i}) |Z_s^{n+1,i}|^2}{2} \right) ds \\ &\quad - \int_\tau^T \varphi'(Y_s^{n+1,i}) Z_s^{n+1,i} dW_s \\ &\leq \varphi(\|\xi^i\|_{L^\infty}) + \int_\tau^T \left( \frac{2G_d |\varphi'(Y_s^{n+1,i})| - \varphi''(Y_s^{n+1,i})}{2} |Z_s^{n+1,i}|^2 + G |\varphi'(Y_s^{n+1,i})| |Z_s^n|^2 \right) ds \\ &\quad - \int_\tau^T \varphi'(Y_s^{n+1,i}) Z_s^{n+1,i} dW_s. \end{aligned}$$

Since  $\varphi''(\cdot) - 2G_d |\varphi'(\cdot)| = 1$ ,  $\varphi \geq 0$  and  $|\varphi'(x)| \leq (2G_d)^{-1} e^{2G_d \|Y^{n+1,i}\|_{\mathcal{F}_\infty}}$  whenever  $|x| \leq \|Y^{n+1,i}\|_{\mathcal{F}_\infty}$ , taking the conditional expectation with respect to  $\mathcal{F}_\tau$ , we compute

$$\frac{1}{2} \mathbf{E} \left( \int_\tau^T |Z_s^{n+1,i}|^2 ds \middle| \mathcal{F}_\tau \right) \leq \varphi(\|\xi^i\|_{L^\infty}) + G(2G_d)^{-1} e^{2G_d \|Y^{n+1,i}\|_{\mathcal{F}_\infty}} \| |Z^n| \star W \|_{BMO}^2.$$

Thus, we get the estimate

$$\frac{1}{2} \| |Z^{n+1,i}| \star W \|_{BMO}^2 \leq \varphi(\|\xi^i\|_{L^\infty}) + G(2G_d)^{-1} e^{2G_d \|Y^{n+1,i}\|_{\mathcal{F}_\infty}} \| |Z^n| \star W \|_{BMO}^2. \quad (\text{III.5.5})$$

By using the *a priori* estimate given by Proposition 1 in [BH06] we also have, for all stopping time  $\tau$  and  $i \in \{1, \dots, d\}$ ,

$$e^{2G_d |Y_\tau^{n+1,i}|} \leq e^{2G_d \|\xi^i\|_{L^\infty}} \mathbf{E} \left( e^{2G_d G \int_\tau^T |Z_s^n|^2 ds} \middle| \mathcal{F}_\tau \right).$$

Then, the John-Nirenberg inequality (III.1.5) coupled with the induction assumption on  $Z^n$  gives us

$$e^{2G_d \|Y^{n+1,i}\|_{\mathcal{F}_\infty}} \leq \frac{e^{2G_d \|\xi^i\|_{L^\infty}}}{1 - 2G_d G \| |Z^n| \star W \|_{BMO}^2}.$$

We put this last inequality into (III.5.5) to obtain

$$\| |Z^{n+1}| \star W \|_{BMO}^2 \leq \frac{\sum_{i=1}^d e^{2G_d \|\xi^i\|_{L^\infty}}}{2G_d^2} \frac{1}{1 - 2G_d G \| |Z^n| \star W \|_{BMO}^2}.$$

Since we have assumed that  $\| |Z^n| \star W \|_{BMO}^2 \leq (4G_d G)^{-1}$  and (III.5.3) is fulfilled, we get

$$\| |Z^{n+1}| \star W \|_{BMO}^2 \leq (4G_d G)^{-1}$$

and, by using previous calculations,

$$\|Y^{n+1,i}\|_{\mathcal{S}^\infty} \leq \|\xi^i\|_{L^\infty} + \frac{\log 2}{2G_d}, \quad \text{and so} \quad \|Y^{n+1}\|_{\mathcal{S}^\infty} \leq \|\xi\|_{L^\infty} + \frac{\sqrt{d} \log 2}{2G_d},$$

which concludes the induction. Finally, we just have to use the fact that  $(Y^n, Z^n \star W)_{n \in \mathbf{N}}$  tends to  $(Y, Z \star W)$  in  $\mathcal{S}^2(\mathbf{R}^d) \times \mathcal{H}^2(\mathbf{R}^{d \times k})$  and Proposition – III.1.2 to obtain the desired result.  $\square$

Then the proof of Proposition – III.2.3 is direct: we just have to apply Theorem – III.2.2 by using Proposition – III.5.3.

*Remark – III.5.2.* As explained in Remark – III.4.8, it is possible to extend Proposition – III.2.3 to more general terminal conditions and generators.

#### § III.5.4. Proof of Theorem – III.2.4

Proof. [of Theorem – III.2.4] Let us consider for all  $M \in \mathbf{R}^+$  a smooth map  $h^M : \mathbf{R}^{d \times k} \rightarrow \mathbf{R}$  satisfying:

$$h^M(z) = \begin{cases} 0 & \text{if } |z| \leq M, \\ \left(\frac{|z|}{M+1}\right)^2 - 1 & \text{if } |z| \geq M+1, \end{cases}$$

and let us define a localisation  $\rho^M : \mathbf{R}^{d \times k} \rightarrow \mathbf{R}^{d \times k}$  given by

$$\rho^M(z) = \frac{z}{\sqrt{1+h^M(z)}}.$$

As usual we denote by  $(Y^M, Z^M)$  the solution obtained by replacing  $f$  by  $f^M$ . This choice of  $\rho^M$  will be useful in the following computations. For  $F \in \mathcal{C}^2(\mathcal{M}, \mathbf{R})$ , the Itô formula with  $F$  seen as a function on  $\mathbf{R}^d$  gives for all stopping time  $\tau$ :

$$\mathbf{E}(F(\xi) - F(Y_\tau^M) | \mathcal{F}_\tau) = \mathbf{E} \left( \int_\tau^T \left( \frac{1}{2} \sum_{l=1}^k \text{Hess } F(Y_s^M) \left( Z_s^{M,(:,l)}, Z_s^{M,(:,l)} \right) - dF(Y_s^M) f^M(Y_s^M, Z_s^M) \right) ds \middle| \mathcal{F}_\tau \right),$$

since the local martingale part is a martingale because  $F$  has bounded first derivative. By using the definition of  $f$ , its formulation in the local chart and the link between  $Z^M$  and  $\rho^M(Z^M)$ , we get

$$\begin{aligned} & \mathbf{E}(F(\xi) - F(Y_\tau^M) | \mathcal{F}_\tau) \\ &= \frac{1}{2} \mathbf{E} \left( \int_\tau^T \sum_{l=1}^k \left( h^M(Z_s^M) \text{Hess} F(Y_s^M) \left( \rho^M(Z_s^M)^{(:,l)}, \rho^M(Z_s^M)^{(:,l)} \right) \right. \right. \end{aligned} \quad (\text{III.5.6})$$

$$\left. \left. + \nabla dF(Y_s^M) \left( \rho^M(Z_s^M)^{(:,l)}, \rho^M(Z_s^M)^{(:,l)} \right) \right) ds \middle| \mathcal{F}_\tau \right). \quad (\text{III.5.7})$$

By using same arguments as Darling in [Dar95] we can show that  $Y^M$  is in  $G$  almost surely. Indeed, we know with (III.5.6) applied to  $F = F^{dc}$ , integrating between  $\tau$  and  $\sigma$  together with (ii), that for all  $\sigma \geq \tau$  a.s.,

$$\mathbf{E} \left( F^{dc}(Y_\sigma^M) | \mathcal{F}_\tau \right) \geq F^{dc}(Y_\tau^M) \text{ a.s.}$$

Let us consider for all  $n \in \mathbf{N}$  the following sequence of stopping times:  $\sigma^n = \inf \left\{ u \geq \tau \mid F^{dc}(Y_u^M) \leq \frac{1}{n} \text{ a.s.} \right\}$ .

Each  $\sigma^n$  is finite almost surely since  $\xi \in G$ . Continuity of  $Y^M$  gives for all  $n \in \mathbf{N}$ ,  $F^{dc}(Y_{\sigma^n}^M) \leq \frac{1}{n}$  a.s. So we get for all stopping time  $\tau$ :

$$F^{dc}(Y_\tau^M) \leq \mathbf{E} \left( F^{dc}(Y_{\sigma^n}^M) | \mathcal{F}_\tau \right) \leq \frac{1}{n} \quad \text{a.s.}$$

and, consequently,  $\mathbf{P}(Y_t \in G) = 1$  for all  $t \in [0, T]$ . Moreover the  $\alpha$ -strictly doubly convexity on  $G$  gives us

$$\begin{aligned} & \mathbf{E} \left( F^{dc}(\xi) - F^{dc}(Y_\tau^M) | \mathcal{F}_\tau \right) \\ & \geq \frac{\alpha}{2} \mathbf{E} \left( \int_\tau^T \sum_{l=1}^k \left( h^M(Z_s^M) \left| \rho^M(Z_s^M)^{(:,l)} \right|^2 + \left| \rho^M(Z_s^M)^{(:,l)} \right|^2 \right) ds \middle| \mathcal{F}_\tau \right) = \frac{\alpha}{2} \mathbf{E} \left( \int_\tau^T |Z_s^M|^2 ds \middle| \mathcal{F}_\tau \right). \end{aligned} \quad (\text{III.5.8})$$

And finally, continuity of  $F^{dc}$  on  $G$  yields

$$\| |Z^M| \star W \|_{BMO} \leq \sqrt{\frac{2}{\alpha} \times \left( \sup_{(x,y) \in G \times G} \{F^{dc}(x) - F^{dc}(y)\} \right)}.$$

Thus, assumption (ii) ensures assumption **(BMO,m)**. Since the terminal value is bounded (in  $G$ ), Theorem – III.2.3 together with Remark – III.4.5 gives the result.  $\square$



## § III.5.5. Proofs of Theorem – III.2.5 and Theorem – III.2.6

Proof. [of Theorem – III.2.5 and Theorem – III.2.6]

**Uniqueness** We start by proving Theorem – III.2.5. Let us consider two continuous Markovian solutions  $(v, w)$  and  $(\tilde{v}, \tilde{w})$  such that  $v$  and  $\tilde{v}$  are bounded. We set  $(t, x) \in [0, T] \times \mathbf{R}^k$  and we denote  $(Y^{t,x}, Z^{t,x})$  (resp.  $(\tilde{Y}^{t,x}, \tilde{Z}^{t,x})$ ) the solution of the BSDE (III.2.5) associated to  $(v, w)$  (resp.  $(\tilde{v}, \tilde{w})$ ). The idea of the proof is to compare the two solutions by using the stability result given by Remark III.4.3. In order to do that, we must show that  $|Z^{t,x}| \star W$  and  $|\tilde{Z}^{t,x}| \star W$  are  $\varepsilon$ -sliceable BMO martingales. By Remark 2.6 part (2) in [XŽ16], we know that there exist  $b_0 \in \mathbf{R}^d$  and  $(\alpha_n) \in (0, 1]^{\mathbf{N}}$  such that  $v, \tilde{v} \in \mathcal{C}_{b_0}^{(\alpha_n), loc}$ . Moreover, by following same arguments than in the proof of Theorem 2.9 in [XŽ16] we can show that  $v, \tilde{v} \in \mathcal{C}^{(\alpha_n), loc}$ . Finally, we just have to apply Remark – III.1.1 to conclude that there exists  $\kappa' \in (0, 1]$  such that  $v, \tilde{v} \in \mathcal{C}^{\kappa'}$ .

Now, let us apply the Itô formula to  $F(Y^{t,x})$ : we consider two stopping times  $\tau$  and  $\sigma$  such that  $\tau \leq \sigma$  a.s and we take the conditional expectation

$$\begin{aligned} & \mathbf{E}(F(v(\sigma, X_\sigma^{t,x})) - F(v(\tau, X_\tau^{t,x})) | \mathcal{F}_\tau) \\ &= \mathbf{E}\left(-\int_\tau^\sigma dF(v(u, X_u^{t,x})) \mathbf{f}(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x}) du + \frac{1}{2} \sum_{l=1}^k \int_\tau^\sigma \text{Hess } F(v(u, X_u^{t,x})) \left( (Z_u^{t,x})^{(:,l)}, (Z_u^{t,x})^{(:,l)} \right) du \middle| \mathcal{F}_\tau\right) \\ &\geq \mathbf{E}\left(\int_\tau^\sigma |Z_u^{t,x}|^2 du \middle| \mathcal{F}_\tau\right). \end{aligned}$$

The map  $F$  is a Lipschitz function on the centred Euclidean ball of radius  $\|v\|_{L^\infty([0,T] \times \mathbf{R}^k)} \vee \|\tilde{v}\|_{L^\infty([0,T] \times \mathbf{R}^k)}$ . Denoting by  $L$  its Lipschitz constant, we obtain for all  $n \in \mathbf{N}$ ,

$$\begin{aligned} \mathbf{E}\left(\int_\tau^\sigma |Z_u^{t,x}|^2 du \middle| \mathcal{F}_\tau\right) &\leq L \times \mathbf{E}(|v(\sigma, X_\sigma^{t,x}) - v(\tau, X_\tau^{t,x})| | \mathcal{F}_\tau) \\ &\leq L \times \mathbf{E}\left(|\sigma - \tau|^{\kappa'/2} + |X_\sigma^{t,x} - X_\tau^{t,x}|^{\kappa'} \middle| \mathcal{F}_\tau\right) \\ &\leq C |\sigma - \tau|^{\kappa'/2}, \end{aligned}$$

where we have used in the last inequality a classical estimate for SDEs when  $b$  and  $\sigma$  are bounded. For  $N \in \mathbf{N}^*$  we set  $T_i = \frac{iT}{N}$ . Then, for all  $i \in \{0, \dots, N-1\}$  and stopping time  $T_i \leq \tau \leq T_{i+1}$  we get

$$\mathbf{E}\left(\int_\tau^{T_{i+1}} |Z_u^{t,x}|^2 du \middle| \mathcal{F}_\tau\right) \leq C \left(\frac{T}{N}\right)^{\kappa'/2},$$

and finally, for  $N$  large enough, we have that

$$\left\| \int_{T_i}^{T_{i+1}} |Z^{(t,x)}| \star W \middle|_{T_{i+1}} \right\|_{BMO} \leq \mathbb{K}, \quad \forall i \in \{0, \dots, N-1\}$$

with  $2L_y\mathbb{K}^2 + 2\sqrt{2}L_z\mathbb{K}C_2' < 1$ . Obviously, this estimate is also true for  $\tilde{Z}^{t,x}$  which means that  $Z^{t,x} \star W$  and  $\tilde{Z}^{t,x} \star W$  are in  $\mathcal{L}_{BMO}^{slc,2}$ . By using [Remark – III.4.3](#) we get that  $v(t,x) = \tilde{v}(t,x)$  and  $\mathbf{E} \left( \int_t^T |w(s, X_s^{t,x}) - w'(s, X_s^{t,x})|^2 ds \right) = 0$ . Since this is true for all  $(t,x) \in [0,T] \times \mathbf{R}^k$  and, due to [\(HX\)](#),  $X_s^{0,x}$  has positive density on  $\mathbf{R}^k$  for  $s \in (0,T]$ ,  $w = \tilde{w}$  a.s. with respect to the Lebesgue measure on  $[0,T] \times \mathbf{R}^k$ . Then,  $(v,w)$  and  $(v',w')$  are equal.

**Regularity** Now we prove [Theorem – III.2.6](#). We consider  $(v,w)$  a continuous Markovian solution of [\(III.2.5\)](#) such that  $v$  is bounded. We set  $t, t' \in [0,T]$  and  $x, x' \in \mathbf{R}^k$ . Without restriction, we can assume that  $t \leq t'$ . Then, we can write

$$|v(t,x) - v(t',x')| = \left| Y_t^{t,x} - Y_{t'}^{t',x'} \right| \leq \left| Y_t^{t,x} - Y_t^{t',x'} \right| + \left| Y_t^{t',x'} - Y_{t'}^{t',x'} \right|,$$

where we have extend the definition of  $(Y_u^{t',x'}, Z_u^{t',x'})_{t' \leq u \leq T}$  to  $[0,t']$  by setting

$$Y_s^{t',x'} = Y_{t'}^{t',x'} + \int_s^{t'} \mathbf{f}(u, x', Y_u^{t',x'}, 0) du, \quad Z_s^{t',x'} = 0, \quad 0 \leq s \leq t',$$

which is equivalent to take  $\sigma(s, \cdot) = 0$  and  $b(s, \cdot) = 0$  for  $s \in [0,t']$ . By the same token we can extend  $(Y_u^{t,x}, Z_u^{t,x})_{t \leq u \leq T}$  to  $[0,t]$ . Then, a standard estimate gives us

$$\left| Y_t^{t',x'} - Y_{t'}^{t',x'} \right| \leq C |t - t'|,$$

where  $C$  does not depend on  $(t,x)$  and  $(t',x')$ . To conclude we just have to study the remaining term  $\left| Y_t^{t,x} - Y_{t'}^{t',x'} \right|$ . Thanks to calculations done in the uniqueness part of the proof, we know that there exist some deterministic times  $0 = T_0 \leq T_1 \leq \dots \leq T_N = T$  such that  ${}^{T_i}Z^{t,x} \star W \upharpoonright^{T_{i+1}} \in \mathcal{L}_{BMO}^2$  and  ${}^{T_i}Z^{t',x'} \star W \upharpoonright^{T_{i+1}} \in \mathcal{L}_{BMO}^2$  for all  $i \in \{0, \dots, N-1\}$ . Let us emphasize that times  $0 = T_0 \leq T_1 \leq \dots \leq T_N = T$  can be chosen independently from  $(t,x)$  and  $(t',x')$ . Then, by using once again [Remark – III.4.3](#) and assumptions on  $\mathcal{G}$  and  $\mathbf{f}$ , we obtain

$$\begin{aligned} & \left| Y_t^{t,x} - Y_{t'}^{t',x'} \right| \\ & \leq \left\| Y^{t,x} - Y^{t',x'} \right\|_{\mathcal{G}^4} \\ & \leq C \left( \mathbf{E} \left( \left| \mathcal{G}(X_T^{t,x}) - \mathcal{G}(X_T^{t',x'}) \right|^4 \right)^{1/4} + \mathbf{E} \left( \left| \int_0^T \left| \mathbf{f}(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) - \mathbf{f}(s, X_s^{t',x'}, Y_s^{t',x'}, Z_s^{t',x'}) \right| ds \right|^4 \right)^{1/4} \right) \\ & \leq C \left( \mathbf{E} \left( \left| X_T^{t,x} - X_T^{t',x'} \right|^{4\kappa} \right)^{1/4} + \mathbf{E} \left( \sup_{s \in [0,T]} \left| X_s^{t,x} - X_s^{t',x'} \right|^{4\kappa} \left( 1 + \int_0^T |Z_s^{t,x}|^2 ds \right)^4 \right)^{1/4} \right) \\ & \leq C \mathbf{E} \left( \left| X_T^{t,x} - X_T^{t',x'} \right|^{4\kappa} \right)^{1/4} + C \mathbf{E} \left( \sup_{s \in [0,T]} \left| X_s^{t,x} - X_s^{t',x'} \right|^{8\kappa} \right)^{1/8} \left( 1 + \mathbf{E} \left( \left( \int_0^T |Z_s^{t,x}|^2 ds \right)^8 \right)^{1/8} \right). \end{aligned}$$

where  $C$  depends on  $N$  and the BMO norms of  ${}^{T_i}Z^{t,x} \star W^{\downarrow T_{i+1}}$  (see an explicit value in Remark – III.4.3). However  $N$  stays finite in the sequel and, as already mentioned in the Proof of Theorem – III.2.1 and Theorem – III.2.2, an uniform estimation on  $\left\| {}^{T_i}Z^{t,x} \star W^{\downarrow T_{i+1}} \right\|_{BMO}$  and  $\left\| {}^{T_i}Z^{t',x'} \star W^{\downarrow T_{i+1}} \right\|_{BMO}$  gives an uniform upper bound on  $C$  (particularly  $C$  does not depend on  $(t,x)$  and  $(t',x')$ ). Then an energy inequality gives us

$$|Y_t^{t,x} - Y_t^{t',x'}| \leqslant CE \left( \sup_{s \in [0,T]} |X_s^{t,x} - X_s^{t',x'}|^{8\kappa} \right)^{1/8}.$$

Thus, we just have to use the classical estimate on SDEs given by

$$\left\| X^{t',x'} - X^{t,x} \right\|_{\mathcal{S}^8} \leqslant C \left( |x - x'| + |t - t'|^{1/2} \right),$$

to obtain that

$$|Y_t^{t,x} - Y_t^{t',x'}| \leqslant C \left( |x - x'|^\kappa + |t - t'|^{\kappa/2} \right).$$

□

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### III.6 APPENDIX – TECHNICAL PROOFS

**Proof.** [of Proposition – III.1.3] Let us show the BMO property for a started and stopped process. Let us consider a stopping time  $\tau'$  such that  $0 \leqslant \tau' \leqslant T$  a.s. we have

$$\begin{aligned} \mathbf{E} \left( \left\langle {}^\tau |Z| \star W^{\downarrow \sigma} \right\rangle_T - \left\langle {}^\tau |Z| \star W^{\downarrow \sigma} \right\rangle_{\tau'} \middle| \mathcal{F}_{\tau'} \right) &= \mathbf{E} \left( \left( \left\langle {}^\tau |Z| \star W \right\rangle_\sigma - \left\langle {}^\tau |Z| \star W \right\rangle_{\min(\tau', \sigma)} \right) \middle| \mathcal{F}_{\tau'} \right) \\ &= \mathbf{E} \left( \left( \left\langle {}^\tau |Z| \star W \right\rangle_\sigma - \left\langle {}^\tau |Z| \star W \right\rangle_{\tau'} \right) \mathbf{1}_{(0 \leqslant \tau' \leqslant \sigma)} \middle| \mathcal{F}_{\tau'} \right). \end{aligned}$$

Since  ${}^\tau|Z| \star W$  vanishes before  $\tau$  and  $\mathcal{F}_{\tau'} \subset \mathcal{F}_{\max(\tau', \tau)}$ , we get:

$$\begin{aligned} & \mathbf{E}(\langle \langle {}^\tau|Z| \star W \rangle_\sigma - \langle {}^\tau|Z| \star W \rangle_{\tau'} \mathbb{1}_{(0 \leq \tau' \leq \sigma)} \mid \mathcal{F}_{\tau'} \rangle) \\ &= \mathbf{E}(\langle \langle |Z| \star W \rangle_\sigma - \langle |Z| \star W \rangle_{\max(\tau', \tau)} \mathbb{1}_{(0 \leq \tau' \leq \sigma)} \mid \mathcal{F}_{\tau'} \rangle) \\ &= \mathbf{E}(\mathbf{E}(\langle \langle |Z| \star W \rangle_\sigma - \langle |Z| \star W \rangle_{\max(\tau', \tau)} \mathbb{1}_{(0 \leq \tau' \leq \sigma)} \mid \mathcal{F}_{\max(\tau, \tau')} \rangle) \mid \mathcal{F}_{\tau'}) \\ &\leq \text{esssup} \sup_{\tilde{\tau} \in \mathcal{F}^{\tau, \sigma}} \mathbf{E}(\langle |Z| \star W \rangle_\sigma - \langle |Z| \star W \rangle_{\tilde{\tau}} \mid \mathcal{F}_{\tilde{\tau}}). \end{aligned}$$

Finally we have shown that

$$\text{esssup} \sup_{0 \leq \tau' \leq T} \mathbf{E}(\langle \langle {}^\tau|Z| \star W \rangle_T^\sigma - \langle {}^\tau|Z| \star W \rangle_{\tau'}^\sigma \mid \mathcal{F}_{\tau'}) \leq \text{esssup} \sup_{\tilde{\tau} \in \mathcal{F}^{\tau, \sigma}} \mathbf{E}(\langle |Z| \star W \rangle_\sigma - \langle |Z| \star W \rangle_{\tilde{\tau}} \mid \mathcal{F}_{\tilde{\tau}}),$$

and the inequality is obviously an equality.  $\square$

Proof. [of Proposition – III.3.2] We are going to use inequalities given by Lemma – III.1.1. Let us suppose the existence of a solution  $X$  for the equation (III.3.3). We have for all  $m \geq 1$ ,

$$\begin{aligned} \|X\|_{\mathcal{S}^m} &\leq \|X_0\|_{L^m} + \left\| \int_0^\cdot F(s, X_s) ds \right\|_{\mathcal{S}^m} + \left\| \sum_{p=1}^k \int_0^\cdot G^p(s, X_s) dW_s^p \right\|_{\mathcal{S}^m} \\ &\leq \|X_0\|_{L^m} + \mathbf{E} \left( \sup_{0 \leq u \leq T} \left( \int_0^u \alpha_s |X_s| ds \right)^m \right)^{1/m} + C'_m \mathbf{E} \left( \left( \sum_{p=1}^k \int_0^T |G^p(s, X_s)|^2 ds \right)^{m/2} \right)^{1/m} \\ &\leq \|X_0\|_{L^m} + \mathbf{E} \left( \left( \int_0^T \alpha_s |X_s| ds \right)^m \right)^{1/m} + C'_m \mathbf{E} \left( \left( \int_0^T \beta_s^2 |X_s|^2 ds \right)^{m/2} \right)^{1/m}. \end{aligned}$$

On the one hand, according to Lemma – III.1.1 we have

$$\begin{aligned} \mathbf{E} \left( \left( \int_0^T \alpha_s |X_s| ds \right)^m \right)^{1/m} &= \|\langle \sqrt{\alpha} \star W, (\sqrt{\alpha} |X|) \star W \rangle_T\|_{L^m} \\ &\leq \sqrt{2} m \|\sqrt{\alpha} \star W\|_{BMO} \|(\sqrt{\alpha} |X|) \star W\|_{\mathcal{H}^m} \\ &\leq 2m \|X\|_{\mathcal{S}^m} \|\sqrt{\alpha} \star W\|_{BMO}^2. \end{aligned}$$

On the other hand, we get for the last term

$$\mathbf{E} \left( \left( \int_0^T \beta_s^2 |X_s|^2 ds \right)^{m/2} \right)^{1/m} = \|(\beta |X|) \star W\|_{\mathcal{H}^m} = \| |X| \star (\beta \star W) \|_{\mathcal{H}^m} \leq \sqrt{2} \|X\|_{\mathcal{S}^m} \|\beta \star W\|_{BMO}.$$

Hence we obtain the following inequality

$$\|X\|_{\mathcal{S}^m} \left( 1 - 2m \|\sqrt{\alpha} \star W\|_{BMO}^2 - \sqrt{2}C'_m \|\beta \star W\|_{BMO} \right) \leq \|X_0\|_{L^m}. \quad (\text{III.6.1})$$

The constant behind  $\|X\|_{\mathcal{S}^m}$  is not always positive, but we can use the sliceability assumption in order to construct piece by piece the process  $X$ , and on each piece the constant will be positive.

More precisely there exists a sequence of stopping times  $0 = T_0 \leq T_1 \leq \dots \leq T_N = T$  a.s such that for all  $i \in \{0, \dots, N-1\}$ :

$$\left\| \int_{T_i}^{T_{i+1}} \sqrt{\alpha} \star W \right\|_{BMO} \leq \varepsilon_1, \quad \left\| \int_{T_i}^{T_{i+1}} \beta \star W \right\|_{BMO} \leq \varepsilon_2.$$

The process  $X$  is equal to

$$X_t = \sum_{i=1}^{N-1} \tilde{X}_t^i \mathbb{1}_{[T_i, T_{i+1}[}(t)$$

where each  $\tilde{X}^i$  is the restriction of  $X$  to the stochastic interval  $[T_i, T_{i+1}]$ . By convention we extend  $\tilde{X}^i$  to  $[0, T]$  by zero outside  $[T_i, T_{i+1}]$ .  $\tilde{X}_i$  satisfies the following SDE:

$$\tilde{X}_t^i = \tilde{X}_{T_i}^{i-1} + \int_{T_i}^t F(s, \tilde{X}_s^i) ds + \sum_{p=1}^k \int_{T_i}^t G^p(s, \tilde{X}_s^i) d^T W_s^{T_{i+1}}, \quad t \in [T_i, T_{i+1}[ , \quad \text{and} \quad \tilde{X}^{-1} = X_0.$$

For all  $i \in \{0, \dots, N-1\}$ , by considering above computations on each  $[T_i, T_{i+1}[$ , (III.6.1) becomes

$$\left\| \tilde{X}^i \right\|_{\mathcal{S}^m} \left( 1 - 2m\varepsilon_1^2 - \varepsilon_2 \sqrt{2}C'_m \right) \leq \left\| \tilde{X}_{T_i}^{i-1} \right\|_{L^m}.$$

Denoting by  $\mathbf{K}_{\varepsilon_1, \varepsilon_2}$  the constant

$$\mathbf{K}_{\varepsilon_1, \varepsilon_2} := \frac{1}{1 - 2m\varepsilon_1^2 - \varepsilon_2 \sqrt{2}C'_m} > 0,$$

we have

$$\left\| \tilde{X}^i \right\|_{\mathcal{S}^m} \leq \mathbf{K}_{\varepsilon_1, \varepsilon_2}^i \|X^0\|_{L^m},$$

and finally we obtain

$$\|X\|_{\mathcal{S}^m} \leq \sum_{i=0}^{N-1} \left\| \tilde{X}^i \right\|_{\mathcal{S}^m} \leq \left( \sum_{i=0}^{N-1} \mathbf{K}_{\varepsilon_1, \varepsilon_2}^i \right) \|X^0\|_{L^m}.$$

The result follows by setting  $K_{m, \varepsilon_1, \varepsilon_2} = \sum_{i=0}^{N-1} \mathbf{K}_{\varepsilon_1, \varepsilon_2}^i$ . □

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