



Controllability and observability of non autonomous evolution equations

Duc-Trung Hoang

► To cite this version:

Duc-Trung Hoang. Controllability and observability of non autonomous evolution equations. Optimization and Control [math.OC]. Université de Bordeaux, 2018. English. NNT : 2018BORD0083 . tel-01898943

HAL Id: tel-01898943

<https://theses.hal.science/tel-01898943>

Submitted on 19 Oct 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THÈSE

PRÉSENTÉE À

L'UNIVERSITÉ DE BORDEAUX

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET
D'INFORMATIQUE

par **Duc-Trung HOANG**

POUR OBTENIR LE GRADE DE

DOCTEUR

SPÉCIALITÉ : MATHÉMATIQUE

**Controllability and observability of non-autonomous
evolution equations**

Date de soutenance : Juin, 2018.

Devant la commission d'examen composée de :

Mme Assia Ben Addallah	Professeur, Université de Marseille	Rapporteur
Mme Birgit Jacob	Professeur, Université de Wuppertal	Rapporteur
Mme Fatiha Alabau-Boussouira		Professeur, Université de Lorraine	Examineur
M. Farid Ammar-Khodja	MCF-HDR, Université de Franche-Comté		Examineur
M. Marius Tucsnak	Professeur, Université de Bordeaux	Examineur
M. Bernhard Haak	MCF-HDR, Université de Bordeaux	Directeur de these
M. El Maati Ouhabaz	Professeur, Université de Bordeaux	Directeur de these

Résumé Cette thèse est consacrée à la contrôlabilité et à l'observabilité de l'équation d'évolution non autonome. Dans la première partie, nous donnons un aperçu de la théorie du contrôle ainsi que quelques résultats classiques sur le contrôle des systèmes autonomes et non autonomes. En fait, nous rappellerons les techniques de la théorie des semi-groupes, théorie de l'évolution familiale, théorie de la dualité et de l'opérateur. Dans la deuxième partie, nous sommes intéressés à étudier le problème de contrôle pour les systèmes EDP définis sur des domaines dépendant du temps. Nous développons de nouvelles techniques pour obtenir les résultats sur l'observabilité exacte des équations de l'onde et de Schrödinger 1D, puis par dualité nous établissons la contrôlabilité exacte du système adjoint. Le dernier résultat est une généralisation des tests de Hautus pour l'observabilité du système d'évolution non autonome. Notre méthode peut s'appliquer aux équations de Schrödinger et à l'équation d'onde avec des potentiels dépendant du temps.

Title

Abstract This thesis is devoted to the controllability and observability of non-autonomous evolution equation. In the first part, we give an overview on control theory as well as some classical results on control of both autonomous and non-autonomous systems. In fact, we will recall the technique in semigroup theory, evolution families theory, duality theory and operator theory. In the second part, we are interested to investigate the control problem for PDEs systems defined on time-dependent domains. We develop some new techniques to obtain the results on exact observability for one dimensional wave and Schrödinger equations, then by duality we establish exact controllability of adjoint system. The last result is a generalization of Hautus tests for observability of non- autonomous evolution system. Our method can be applied for Schrödinger equations with time dependent potentials and to a damped wave-equation with time-dependent damping.

Keywords Observability, controllability, non-autonomous evolution equation, Hautus test, moving domain problem

Mots-clés Observabilité, contrôlabilité, équation d'évolution non autonome, test de hautus, problème de domaineburgere

Laboratoire d'accueil Bordeaux University, Institut de Mathématiques (IMB), CNRS UMR 5251. Address: 351, Cours de la Libération 33405 Talence, France.

Acknowledgement

Firstly, I would like to express my sincere gratitude to my advisors Prof. Bernhard Haak and Prof. El Maati Ouhabaz for the continuous support of my PhD study and related researches. Their patiences, motivations, and immense knowledges inspired me to take long journey into the research. Their suggestions also allowed me to improve the manuscript of this thesis. I could not have imagined having better advisors and mentors for my PhD study.

Besides my advisors, I would like to thank the rest of my thesis committee: Prof. Fatiha Alabau, Prof. Farid Ammar-Khodja, Prof. Assia Benabdallahand, Prof. Birgit Jacob, and Prof. Marius Tucsnak for their insightful comments and encouragement, but also for the hard questions which incented me to widen my research from various perspectives.

My sincere thanks also goes to Prof. Duc-Manh Nguyen (Bordeaux) who gave access to the laboratory and research facilities. I also want to my greet friends PhD students, Achache Mahdi (Bordeaux), Huu-Nghi Nguyen (Lyon), Van-Thang Pham (Lausanne), Nguyen Hoang Thanh (Bordeaux), Marouane Assan (Nice), as well as the other PhD students, give me a lot of valuable advices into the researches. Without they precious support it would not be possible to conduct this research.

I thank my fellow labmates in for the stimulating discussions, for the sleepless nights we were working together before deadlines, and for all the fun we have had in the last three years. Also I thank to my friends in all the conferences I attended. In particular, I am grateful to Prof. Lorenzo Brandolese (Lyon), Prof. Moshe Rosenfeld (Seattle) and Prof. Vu Hoang Linh (Hanoi) for enlightening me the first glance of research.

Last but not the least, I would like to thank my family: my parents and my younger sister for supporting me spiritually throughout writing this thesis and my my life in general. I also thanks to Le Phuong Tram Anh for her encouragement helped me to overcome the difficulties encountered during the three years of the PhD program.

Contents

Contents	vii
Résumé	1
Introduction	5
1 Definitions and tools	13
1.1 Preliminaries	13
1.2 Autonomous evolution equations	14
1.2.1 Semigroup theory	14
1.2.2 Existence and uniqueness	18
1.3 Non-autonomous evolution equation	18
1.3.1 Evolution family theory	18
1.3.2 Existence and uniqueness	25
1.4 General concepts	25
2 Duality results	27
2.1 Definitions	27
2.2 Duality of admissibility for control and observation	27
2.3 Duality of controllability and observability	29
2.3.1 The autonomous case	30
2.3.2 The non autonomous case	35
3 Exact observability of a 1D wave on a non-cylindrical domain	47
3.1 Introduction and main results	47
3.1.1 Existence of solutions	47
3.1.2 Energy estimates	50
3.2 Point Observations	52
3.2.1 Boundary Observation	52
3.2.2 Internal Point observation	54
3.2.3 Simultaneous exact observability	58
3.2.4 Duality results	60
3.3 Differentiable solutions for general boundary functions	61

4	Observability of a 1D Schrödinger equation with time-varying boundaries	65
4.1	Introduction	65
4.2	Main Results	66
4.3	Proof of the main results	70
4.3.1	The multiplier Lemma	70
4.3.2	Energy estimates	72
4.3.3	Admissibility of Neumann observations at the boundary . . .	75
4.3.4	Results for linear moving walls	79
4.4	Boundary controllability of dual problem	83
5	The averaged Hautus test for non-autonomous linear evolution equation	87
5.1	Introduction	87
5.2	Preliminary results	89
5.3	The averaged Hautus test: skew-adjoint operators	94
5.4	The averaged Hautus test: a more general class of operators	98
5.5	Applications to the wave and Schrödinger equations with time dependent potentials	100
	Conclusion	105
	Bibliography	107

Résumé

Cette thèse est consacrée à l'étude de la contrôlabilité et de l'observabilité du système d'évolution non autonome suivant:

$$\begin{cases} x'(t) &= A(t)x(t) + B(t)u(t) \\ x(0) &= x_0 \\ y(t) &= C(t)x(t) \end{cases} \quad (0.0.1)$$

Plus précisément, nous considérons les systèmes abstraits avec les opérateurs A, B et C en fonction du temps t . Des exemples intéressants d'opérateurs $A(t)$ qui changent avec le temps incluent, entre autres, des opérateurs de diffusion avec des coefficients de conductivité dépendant du temps, ou des opérateurs sur des domaines variants dans le temps ou l'équation de chaleur avec des conditions aux limites dynamiques. Dans le cas dépendant du temps, de nouvelles difficultés apparaissent et la théorie du contrôle en dimension infini est moins développée que dans le cas autonome. Nous considérons la famille des opérateurs dépendant du temps $\{A(t)\}$ génère la famille d'évolution $\{U(t, s)\}_{s \leq t}$. Les familles d'évolution apparaissent comme une solution aux équations d'évolution non autonomes, bien que toutes les familles d'évolution ne résolvent pas un tel problème. Il a été noté que les systèmes d'équation aux dérivées partielles (EDP) définis sur des domaines spatiaux dépendant du temps sont intrinsèquement non autonomes, même si les EDP ne contiennent pas de coefficients dépendant du temps. Quelques travaux récents ont été consacrés à l'étude des EDP paraboliques sur des domaines spatiaux dépendant du temps [22, 5, 8]. Dans ces études, diverses approches ont été utilisées pour établir les propriétés d'existence et de régularité des solutions, y compris l'utilisation des transformations qui cartographient le système sur un nouveau domaine spatial fixe, tandis que d'autres décrivent l'évolution temporelle du domaine spatial par difféomorphismes différentiables. Pour les systèmes non autonomes sous forme variationnelle, il existe une approche bien connue due à Lions [79]. Pour le système non autonome de type hyperbolique, il y a des résultats par T.Kato, Pazy et autres.

Nous décrivons maintenant nos principales contributions. Ceux-ci sont donnés aux chapitres 3, 4 et 5. Le premier chapitre de la thèse rappelle quelques techniques de base et les théories classiques. En fait, nous présentons les approches générales du système d'évolution autonome et non autonome par les semi-groupes et les familles d'évolutions. Nous passons également en revue les résultats sur l'existence et l'unicité pour le système (0.0.1).

Dans le chapitre 2, nous étudierons l'admissibilité, la contrôlabilité et l'observabilité des systèmes de la forme (0.0.1) en dimension fini et infini. Nous nous concentrons principalement sur la dualité entre contrôlabilité et observabilité et la dualité sur l'admissibilité entre opérateur de contrôle et opérateur d'observation. Le reste de la thèse est divisé en trois chapitres qui présentent nos contributions. Nous considérons deux types de problèmes: les domaines dépendant du temps et l'observabilité pour les systèmes d'évolution non autonomes.

Le chapitre 3 est consacré à l'étude de l'équation des ondes en 1D sur un certain domaine $s(t)$ dépendant du temps. Nous discutons l'admissibilité et l'observabilité exactes à l'intérieur et sur le bord avec les courbes $s(t)$ suffisamment régulières. Nous discutons également des observateurs mobiles dans le domaine non cylindrique et des résultats d'observabilité simultanés. En fait, nous développons de nouvelles techniques pour trouver la solution de série générale exacte de l'équation d'onde et ensuite établir la limite et l'observabilité exacte interne. Le temps d'observabilité minimum τ , en fonction de la courbe $s(t)$, il est prouvé qu'il est assez grand est due à la vitesse de propagation finie des solutions à l'équation des ondes. Fait intéressant, nous prouvons que l'inégalité d'observabilité tenir pour un sous-ensemble arbitraire $\omega \subset (0, 1)$, même à chaque point interne où cette affirmation n'est pas vraie pour l'équation d'onde autonome. Par dualité, nous pouvons obtenir le résultat de la contrôlabilité en zéro pour le système dual dans le sens de "transposition". Dans le cas autonome (borné), il existe plusieurs façons de prouver l'observabilité de l'équation des ondes, comme la méthode du multiplicateur, l'analyse micro-locale ou l'estimation de Carleman. Pour l'équation d'onde classique, il existe une bonne façon de prouver l'inégalité d'observabilité des ondes en dimension un en utilisant les séries de Fourier des solutions et l'inégalité d'Ingham. Dans des dimensions supérieures, C. Bardos, G. Lebeau et J. Rauch [7] ont prouvé en utilisant l'analyse micro-locale que dans la classe des domaines lisses $\Omega \subset \mathbb{R}^n$ et pour chaque sous-ensemble $\omega \subset \Omega$, l'inégalité d'observabilité est vérifiée si et seulement si (ω, τ) satisfait certaines conditions de contrôle géométrique (CCG) dans Ω : chaque rayon d'optique géométrique qui se propage en Ω et se reflète sur sa limite Γ entre ω dans le temps moins que τ . Cette approche ne semble pas s'appliquer à l'équation d'onde sur le domaine mobile. Il est bien connu que les équations des ondes servent à modéliser de nombreux phénomènes physiques tels que la petite vibration des corps élastiques et la propagation du son. De plus, l'équation d'onde est aussi un prototype pour la classe des équations hyperboliques possédant des propriétés majeures comme l'absence d'effets de régularisation, la vitesse de propagation finie qui a des conséquences très importantes dans la théorie du contrôle. Ce chapitre présente le résultat de [42] qui est soumis pour publication.

Le chapitre 4 étudie l'équation de Schrödinger en dimension un dans des domaines non-fixe avec une condition aux limites de Dirichlet. En mécanique quantique, cette équation décrit les changements dans le temps d'un système physique dans lequel les effets quantiques sont significatifs. Comme au chapitre 3, nous discutons l'admissibilité et l'observabilité exacte de l'équation de Schrödinger sur un certain domaine $\ell(t)$ dépendant du temps. En transformant en domaine borné, nous

traitons le système avec des coefficients variant dans le temps et donnons l'existence et l'unicité des solutions de Schrödinger dues à la stabilité de Kato. Dans le cas d'un déplacement linéaire où $\ell(t) = 1 + \varepsilon t$ ($\varepsilon > 0$), L_p -admissibilité et l'observabilité des observations ponctuelles sont établis. Dans le cas général où $\ell(t)$ est une fonction C^2 strictement positive avec $\frac{\ell'}{\ell} \in L_\infty$, la méthode pour trouver une solution analytique exacte ne fonctionne pas bien comme dans le cas linéaire mobile. Sur la base de l'idée de [84], nous développons une nouvelle version de la méthode du multiplicateur avec un multiplicateur dépendant du temps pour gérer l'observabilité correspondante du problème de transformation, puis nous revenons à l'observabilité de l'équation de Schrödinger. L'opérateur $C(t)$ est considéré comme une observation de Neumann dans toutes les parties de la frontière. Par la théorie de la dualité, nous établissons la contrôlabilité du système adjoint. Dans le cas autonome (borné), l'observabilité exacte et sa propriété dual, la contrôlabilité exacte, des systèmes régis par les équations de Schrödinger ont été largement étudiées [11, 76, 84]. Un des résultats les plus importants est probablement celui dû à G. Lebeau [76] qui garantit que la condition de contrôle géométrique (CCG) pour la contrôlabilité exacte de l'équation des ondes est suffisante pour la contrôlabilité exacte de l'équation de Schrödinger à tout moment τ . Ce résultat nous implique que l'équation de Schrödinger peut, dans une certaine mesure, être considérée comme une équation d'onde à vitesse de propagation infinie. En effet, le fait que la CCG soit satisfait pour un temps fini τ^* suffit pour que la contrôlabilité exacte de l'équation de Schrödinger soit valide pour tout $\tau > 0$. Du point de vue de la contrôlabilité, l'équation de Schrödinger est légèrement meilleure qu'une équation des ondes puisqu'elle a une vitesse de propagation infinie. Ce chapitre présente le résultat [56] qui est soumis pour publication.

Le chapitre 5 est consacré à l'étude du test d'observabilité des systèmes d'évolution non autonomes. Pour l'équation d'évolution autonome, ce test est bien connu en tant que condition de rang de Kalman pour la dimension finie et le test de Hautus pour la dimension infini. Pour la condition suffisante, Russell et Weiss [111] ont conjecturé la généralisation suivante du test de Hautus à la situation de dimension infini: il existe un $\delta > 0$ tel que:

$$\|(\lambda I - A)x\|^2 + |\operatorname{Re}(\lambda)|^2 \|Cx\|^2 \geq \delta |\operatorname{Re}(\lambda)|^2 \|x\|^2 \quad (0.0.2)$$

pour tout $\lambda \in \mathbb{C}$ avec une partie réelle négative et pour tout $x \in \mathcal{D}(A)$. Sous l'hypothèse que le semigroupe $\mathbb{T}(\cdot)$ est exponentiellement stable, ils ont montré que (0.0.2) est nécessaire pour l'observabilité exacte à l'infini ($t = \infty$) des systèmes d'évolution autonomes et qu'il est suffisant pour l'observabilité approximative à l'infini ($t = \infty$). La conjecture de Russell et Weiss n'est pas vraie en général. Jacob et Zwart [51] ont construit un contre-exemple avec un semigroupe analytique. Un autre contre-exemple de Jacob et Zwart [52] montre que (0.0.2) n'implique même pas une observabilité approximative, si nous affaiblissons l'hypothèse de stabilité exponentielle à une forte stabilité. Au système non-autonome (0.0.1) pour la dimension finie où l'opérateur $A(t)$ est une matrice dépendant du temps, nous rappelons le résultat bien connu dans [115]. Pour le cas de dimension infinie, nous introduisons une condition de Hautus moyennée pour les familles d'opérateurs anti-adjoints $A(t)$. En fait, nous supposons que pour $t \in [0, \tau]$, l'opérateur $A(t)$ génère un semi-groupe

de contraction fortement continu $(e^{-sA(t)})_{s \geq 0}$ et le domaine de $A(t)$ est indépendant de t . En utilisant une idée de [52], nous étendons ceci à une classe plus générale d'opérateurs dans une condition de croissance de la famille d'évolution associée. Nos résultats sont limités aux familles d'évolution inversibles sous certaines contraintes de croissance. Dans les applications, nous prêtons attention au problème de perturbation où le test de Hautus passe de l'opérateur indépendant du temps à celui qui dépend du temps. Si le test de Hautus est satisfait par le problème non perturbé, nous obtenons immédiatement l'observabilité du système perturbé. Dans l'exemple concret des EDP, nous appliquons le test de Hautus généralisé aux équations des ondes et de Schrödinger avec des potentiels dépendants du temps. Ce chapitre présente le résultat [43] qui est soumis pour publication.

Introduction

Mathematical control theory is an important field that was widely investigated by scientists, researchers or engineers. Some of the most important developments in control theory for PDEs occurred in the periods 1960s-1970s. The purpose of this theory is to determine the targets, so that one can drive the state of some dynamical system, by means of a control parameter present in the given equation. In terms of applications, the theory can be applied to various fields such as fluid dynamics [80], quantum systems [10], networks of structures [72], wave propagation, etc. These models are represented by an infinite number of degrees of freedom led to an evolution equation followed by some partial differential equation. Moreover, the problem on finding active controls in order to properly influence the dynamics of these systems are highly involved. The controllability problem, among the control theory for PDEs, is a mathematical description of such situations. As a result, any dynamical system which is represented by a PDE can be the object of a study from this point of view. We consider the general system:

$$\begin{cases} \dot{x} = F(x, t, u) \\ y = G(x, t) \end{cases} \quad (0.0.3)$$

where x be the description of the state system, \dot{x} be the derivative with respect to the time t , y be some output function, and u denotes the control parameter, which we can choose in a suitable range. For the finite dimensional problem, the state x of the system can be described by a finite number of degrees of freedom if it belongs to an Euclidean space or to a manifold. On the contrary, for the infinite dimensional problem considered on Banach or Hilbert spaces, the systems would have an infinite number of degree of freedom. In the infinite dimensional problem, the equation (0.0.3) typically has the form of some partial differential equations in which F acts as a differential operator on the function x . Here, u can take multiple different roles. For examples, u can be an additional force term in the right-hand side of the equation, localized in a part of the domain, or it can also appear in the boundary conditions.

The general problem of controllability is to study whether or not in a given system, it is possible to bring any initial state to a given target in an fixed initially time. In term of mathematical sense, the problem of controllability can be stated in the following way:

"Given a time $\tau > 0$, an initial state x_0 and a target x_1 . Is it possible to find a control function u (depending on the time), such that the solution of the system, starting from x_0 provided with this function u reaches the state x_1 at time τ ?"

The means used to act on a system to direct it to a target are called controls. Moreover, the observability is to determine the initial data if we know the knowledge of the output function y . Controllability and observability are basic concepts in system theory and control theory. They are important structural properties which have close relationships with the stability of state feedback controllers and state observers. The controllability and observability theory for finite dimensional linear systems was introduced by Kalman [57], which is now the basis of the control theory. In practice, the finite dimensional system is usually, to some extent, only an approximation of the infinite dimensional system. For infinite dimension, the following abstract formulation appears frequently:

$$\begin{cases} x'(t) &= Ax(t) + Bu(t) \\ x(0) &= x_0 \\ y(t) &= Cx(t) \end{cases} \quad (0.0.4)$$

where A, B and C are unbounded operators in general and A is the generator of a semigroup e^{tA} . The control operator B is called admissible if the solution $x(\cdot)$ depends continuously on the control function $u(\cdot)$. Many natural questions raise on the admissibility, controllability and observability for autonomous evolution equation. Systems of the form (0.0.4) have been largely studied in the literature and controllability and observability have been characterised in many ways. For example, the contribution of J. Lions [79] and D. Russell [110] are primary works on controllability/observability of PDEs. For nonlinear controllability and observability problems for PDEs, we refer to Coron [21], Fursikov [36] and Li [81]. It is well-known that the exact boundary controllability problems are studied by the Hilbert Uniqueness Method (HUM). This approach, introduced by Lions [80] in 1986, is based on uniqueness theorems leading to the construction of suitable Hilbert spaces of the controllable spaces. The control given by HUM method is the best control in the sense that it is the minimal L_2 control. In the infinite dimensional problem, it is more convenient to use an alternative method which consists in converting the control problem into a problem of observability for the dual system. There are many various approaches to obtain observability estimates for evolution equations such as micro-local analysis, multiplies methods, Fourier methods or Carleman estimates. Systems governed by partial differential equations (PDEs for short) are typically infinite dimensional. It is well-known that the controllability/observability theory of PDEs depends very strongly on its nature and, in particular, on its time-reversibility properties, and therefore, the related problems are much more difficult than that for the finite dimensional setting [77].

Concerning existence and uniqueness for evolution equations, there is a big difference between autonomous and non-autonomous linear evolution equations. For autonomous equation, the theory is well understood and it is based on strongly continuous operator semigroups and their generalizations. By the Hille–Yosida theorem,

one can expand the question of well-posedness to a great extension, so that many perturbation and approximation results were established. Moreover, the asymptotic behaviour problem can be studied by using spectral theory and transform methods. For non-autonomous problems, however, there is a little coherent and general theory. We can rely on several sophisticated existence theorems due to T. Kato [61], J. Lions [79], and Schnaubelt [113], but these facts cannot be combined into a unified approach. Accordingly, some other interesting subjects such as perturbation or duality problem can be treated only in some special cases and leaving many questions open. The asymptotic properties could not work well for non-autonomous system since spectral and transform theory cannot be applied directly. The available results are usually restricted to some problems that close to equations with known behaviour (in particular an autonomous one).

This thesis is devoted to study the controllability and observability of non-autonomous evolution system. More precisely, we consider abstract systems with operators A, B and C depending on time t . Interesting examples of operators $A(t)$ that change with time include, among others, diffusion operators with time-dependent conductivity coefficients, or operators on time-varying domains or the heat equation with dynamic boundary conditions. In the time-dependent case, new difficulties appear and infinite dimensional control theory is less developed than in the autonomous case. We consider the system:

$$\begin{cases} x'(t) &= A(t)x(t) + B(t)u(t) \\ x(0) &= x_0 \\ y(t) &= C(t)x(t) \end{cases} \quad (0.0.5)$$

where the family of time dependent operators $\{A(t)\}$ generates evolution family $\{U(t, s)\}_{s \leq t}$. Evolution families arise as solution to non-autonomous evolution equations, although not every evolution family solves such a problem. It has been noted that PDE systems defined on time-dependent spatial domains are inherently non-autonomous even if the PDEs do not contain time-dependent coefficients. There are a few number of recent works which have been dedicated to the study of parabolic PDEs on time-dependent spatial domains [22, 5, 8]. In these studies, a variety of approaches have been taken to establish existence and regularity properties of solutions including the utilization of transformations which map the system onto a new fixed spatial domain, while others have described the time evolution of the spatial domain via continuously differentiable diffeomorphisms. For non-autonomous systems in variational form, there is the well-known approach due to Lions [79]. For non-autonomous system of hyperbolic type, there are results by T.Kato, Pazy and others.

Now we describe our main contributions. These are given in Chapter 3,4 and 5. The first chapter of the thesis recalls some basic techniques and classical theories. In fact, we present the general approaches to autonomous and non-autonomous evolution system by semigroup and evolution families. We also review the results on the existence and uniqueness for both system (0.0.4) and (0.0.5).

In Chapter 2, we will investigate admissibility, controllability and observability

of systems of the form (0.0.5) on finite and infinite dimensional system. We focus mainly on the duality between controllability and observability and the duality on admissibility between control operator and observation operator.

The rest of the thesis is divided into three chapters that present our contributions. We consider two types of problems: time-dependent domains and observability for non-autonomous evolution systems. The idea is whether controllability/observability of non-autonomous system:

$$\begin{cases} x'(t) &= A(t)x(t) \\ x(0) &= x_0 \\ y(t) &= C(t)x(t) \end{cases} \quad (0.0.6)$$

is related to controllability/observability of autonomous system:

$$\begin{cases} x'(t) &= A(s)x(t) \\ x(0) &= x_0 \\ y(t) &= C(t)x(t) \end{cases} \quad (0.0.7)$$

for each (or some) fixed $s \geq 0$. We shall see that the answer is no in general.

Chapter 3 is devoted to study one-dimensional wave equation in non-cylindrical domains. We discuss admissibility and exact observability estimates of boundary observation and interior point observation of a one-dimensional wave equation on a time dependent domain for sufficiently regular boundary functions. We also discuss moving observers inside the noncylindrical domain and simultaneous observability results. The system is given as:

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x, t) \in \Omega \\ u(0, t) = u(s(t), t) = 0 & t \geq 0 \\ u(x, 0) = g(x) & x \in [0, 1] \\ u_t(x, 0) = f(x) & x \in [0, 1] \end{cases} \quad (0.0.8)$$

where $\Omega = [0, s(t)] \times [0, \tau]$ and $s : \mathbb{R}_+ \rightarrow (0, \infty)$ with $s(0) = 1$ satisfying certain conditions. Let $X = H_0^1(\mathbb{R}_+) \times L_2(\mathbb{R}_+)$. It is useful to formulate the wave equation (0.0.8) as an abstract non-autonomous Cauchy problem $z'(t) = \mathbf{a}(t)z(t)$ where $z = \begin{pmatrix} u \\ u_t \end{pmatrix}$ and

$$\mathbf{a}(t) = \begin{pmatrix} 0 & I \\ A(t) & 0 \end{pmatrix} \quad A(t)u = u_{xx} \quad (0.0.9)$$

$$\mathcal{D}(A(t)) = H^1([0, s(t)]) \cap H_0^1([0, s(t)]).$$

The initial condition is $x(0) = x_0 = (g, f)$. It is not important here that $\mathcal{D}(\mathbf{a}(t)) = H_0^1(\mathbb{R}_+) \times D(A(t))$ is not dense in X . The observation operator $C(t)$ is taken as:

- $C(t) \begin{pmatrix} u \\ u_t \end{pmatrix} = u_x(0, t)$ or $u_x(s(t), t)$ for boundary observation

- $C(t) \begin{pmatrix} u \\ u_t \end{pmatrix} = u_x(a, t) \text{ or } u_t(a, t)$ for internal point observation where $a \in (0, 1)$

The admissibility and exact observability for the wave equation (0.0.8) in the case of homogeneous Dirichlet boundary conditions with Neumann boundary and internal observation is equivalent to:

$$m(s, \tau) \|(g, f)\|_{H_0^1 \times L_2} \leq \left\| C(t) \begin{pmatrix} u \\ u_t \end{pmatrix} \right\|_{L_2(0, \tau)} \leq M(s, \tau) \|(g, f)\|_{H_0^1 \times L_2} \quad (0.0.10)$$

where the constants m and M depend on the curve s and τ . In fact, we develop new techniques to find the exact general series solution of (0.0.8) and then establish the boundary and internal exact observability. The minimum observability time τ , depending on the curve $s(t)$, is proved to be large enough since it is due to the finite propagation speed of solutions to the wave equation. Interestingly, we prove that the observability inequality holds for arbitrary subset $\omega \subset (0, 1)$, even at every internal point where this statement is not true for autonomous wave equation. By duality, we can get the null-controllability result for the dual system in the sense of 'transposition'. In the autonomous (bounded) case, there are many ways to prove observability for the wave equation such as the multiplier method, the micro-local analysis, or the Carleman estimate. For the classical wave equation, there is a nice way of proving observability inequality in one dimension using Fourier expansion of solutions and Ingham's inequality. In higher dimensions, C. Bardos, G. Lebeau and J. Rauch [7] has proved using micro-local analysis that in the class of smooth domains $\Omega \subset \mathbb{R}^n$ and for every subset $\omega \subset \Omega$, the observability inequality holds if and only if (ω, τ) satisfies certain geometric control condition (GCC) in Ω : every ray of geometric optics that propagates in Ω and is reflected on its boundary Γ enters ω in time less than τ . This approach does not seem to apply to the wave equation on moving domain. It is well known that wave equations used to model many physical phenomena such as small vibration of elastic bodies and propagation of sound. Moreover, the wave equation is also a prototype for the class of hyperbolic equations possessing major properties as the lack of regularizing effects, finite speed of propagation which have very important consequences in control theory. This chapter presents the result of [42] which is submitted for publication.

Chapter 4 studies the one-dimension Schrödinger equation in non-cylindrical domains with Dirichlet boundary condition. In quantum mechanics, this equation describes the changes over time of a physical system in which quantum effects are significant. As in Chapter 3, we discuss the admissibility and exact observability of Schrödinger equation on certain time dependent domain. The system is given as:

$$\begin{cases} i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0 & x \in [0, \ell(t)] \\ u(0, t) = u(\ell(t), t) = 0 & \tau \geq t \geq 0 \\ u(x, 0) = u_0 & x \in [0, 1] \end{cases} \quad (0.0.11)$$

where $\ell(0) = 1$ and the boundary curve $\ell(t)$ satisfies some certain conditions. Let $X = L_2(0, \mathbb{R}_+)$, we reformulate the Schrödinger equation (0.0.11) as an abstract non-autonomous Cauchy problem $u'(t) = A(t)u(t)$ where $A(t)u = iu_{xx}$ with $\mathcal{D}(A(t)) =$

$H^1([0, \ell(t)])$ is not dense in X . By transforming to the bounded domain, we treat the system with time-variant coefficients and give the existence and uniqueness for solutions of (0.0.11) due to the Kato's stability. In linear moving case where $\ell(t) = 1 + \varepsilon t$ ($\varepsilon > 0$), we take the observed operator $C(t)$ as:

- $C(t)u = u_x(0, t)$ or $u_x(\ell(t), t)$ (Neumann boundary observation).
- $C(t)u = u(a, t)$ for internal point observation where $a \in (0, 1)$.

Using the same idea for the wave equation, we find exact series solution of (0.0.11). Then we prove that admissibility and exact L_2 boundary and pointwise internal observability hold for arbitrary time τ . That is:

$$k_1(\tau)\|u_0\|_{H_0^1} \leq \|C(t)u\|_{L_2(0,\tau)} \leq K_1(\tau)\|u_0\|_{H_0^1} \quad \text{for boundary observation.} \quad (0.0.12)$$

$$k_2(\tau)\|u_0\|_{L_2} \leq \|C(t)u\|_{L_2(0,\tau)} \leq K_2(\tau)\|u_0\|_{L_2} \quad \text{for internal observation.} \quad (0.0.13)$$

where k_1, K_1, k_2 and K_2 be the constants depending on τ and decay exponentially. More generally, L_p -admissibility and observability of point observations are followed. In general case where $\ell(t)$ a strictly positive C^2 -function with $\frac{\ell'}{\ell} \in L_\infty$, the method to find exact analytical solution does not work well as in the linear moving case. Based on the idea on [84] we develop a new version of the multiplier method with time-dependent multiplier to handle the corresponding observability of transforming problem, then reverse back to observability of (0.0.11). The operator $C(t)$ is taken as Neumann observation in all parts of boundary. i.e: $C(t)u = (u_x(0, t), u_x(\ell(t), t))$. By duality theory, we establish the controllability of adjoint system. In the autonomous (bounded) case, the exact observability and its dual property, the exact controllability, of systems governed by Schrödinger equations have been extensively studied [11, 76, 84]. One of the most important result is probably that due to G. Lebeau [76] which guarantees that the geometric control condition (GCC) for the exact controllability of the wave equation is sufficient for the exact controllability of the Schrödinger equation (0.0.11) in any time τ . This result implies us that the Schrödinger equation, to some extent, can be viewed as a wave equation with infinite speed of propagation. Indeed, the fact that the GCC is satisfied for some finite time τ^* suffices for the exact controllability of the Schrödinger equation (0.0.11) to hold for all $\tau > 0$. From the point of view of controllability, the Schrödinger equation is slightly better than a wave equation since it has infinite velocity of propagation. This chapter presents the result [56] which is submitted for publication.

Chapter 5 is devoted to study the observability test for non-autonomous evolution systems. For autonomous evolution equation, this test is well-known as Kalman rank condition for the finite dimension and the Hautus test for the infinite dimension. That is:

$$\|Cx\|^2 + \|(\lambda I - A)x\|^2 \geq \kappa\|x\|^2. \quad (0.0.14)$$

for all $\lambda \in \mathbb{C}$. For the sufficient condition, Russell and Weiss [111] conjectured the following generalization of the Hautus test to the infinite-dimensional situation: there exists a $\delta > 0$ such that:

$$\|(\lambda I - A)x\|^2 + |\operatorname{Re}(\lambda)|^2\|Cx\|^2 \geq \delta|\operatorname{Re}(\lambda)|^2\|x\|^2 \quad (0.0.15)$$

for all $\lambda \in \mathbb{C}$ with negative real part and for all $x \in \mathcal{D}(A)$. Under the assumption that the semigroup $\mathbb{T}(\cdot)$ is exponentially stable, they showed that (0.0.15) is necessary for the exact infinite-time observability of (0.0.4) and that it is sufficient for the approximate infinite-time observability of (0.0.4). The conjecture of Russell and Weiss is not true in general. Jacob and Zwart [51] constructed a counter-example with an analytic semigroup. Another counter-examle in Jacob and Zwart [52] shows that (0.0.15) does not even imply approximate observability, if we weaken the exponential stability assumption to strong stability. To the non-autonomous system (0.0.5) for finite dimension where the operator $A(t)$ be time-dependent matrices, we recall the well-known result in [115]. For infinite dimension case, we introduce an averaged Hautus condition for families of skew-adjoint operators $A(t)$. In fact, we assume that for $t \in [0, \tau]$, the operator $A(t)$ generates a strongly continuous contraction semigroup $(e^{-sA(t)})_{s \geq 0}$ and the domain of $A(t)$ is independent of t . The test is given as:

$$\|x\|^2 \leq m^2 \left(\frac{1}{\tau} \int_0^\tau \|C(s)e^{\lambda s}x\|^2 ds \right) + M^2 \left(\frac{1}{\tau} \int_0^\tau e^{\operatorname{Re} \lambda \cdot s} \|(\lambda + A(s))x\| ds \right)^2$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{D}(A(t))$. Using an idea from [52], we extend this to more general class of operators under a growth condition on the associated evolution family. Our results are restricted to invertible evolution families under certain growth constraints. In applications, we pay attention to the perturbation problem where the Hautus test carries over from the time independent operator to time dependent ones. If the Hautus test is satisfied by the unperturbed problem we immediately obtain observability of the perturbed system. In concrete PDEs example, we apply the generalized Hautus test to Schrödinger equations with time dependent potentials:

$$\begin{cases} z'(t, x) &= i\Delta z(t, x) + iV(t)xz(t, x) & (t, x) \in [0, \tau] \times \Omega \\ z(0, \cdot) &= z_0 \in H^2(\Omega) \cap H_0^1(\Omega) \\ z(t, x) &= 0 & (t, x) \in [0, \tau] \times \Gamma. \end{cases} \quad (0.0.16)$$

and to a damped wave-equation with time dependent damped term:

$$\begin{cases} z''(t, x) &= \Delta z(t, x) + b(t, x)z'(t, x) + V(t, x)z(t, x) \in [0, \tau] \times \Omega \\ z(0, \cdot) &= z_0 \in H_0^1(\Omega), z'(0, \cdot) = z_1 \in L^2(\Omega) \\ z(t, x) &= 0 & (t, x) \in [0, \tau] \times \Gamma. \end{cases} \quad (0.0.17)$$

Here Ω be a bounded smooth domain of \mathbb{R}^n and Γ be the boundary of Ω . This chapter present the result of [43] which is submitted for publication.

Chapter 1

Definitions and tools

1.1 Preliminaries

Let us represent by Ω a bounded open set of \mathbb{R}^n with boundary Γ . By \mathbb{Q} we represent the cylinder $\Omega \times [0, \tau]$, $\tau > 0$ be real number. For $1 \leq p < +\infty$, we denote by $L_p(\Omega)$ the classical Lebesgue space. This is a Banach space with the norm:

$$\|v\|_{L_p(\Omega)} = \left(\int_{\Omega} |v(x)|^p dx \right)^{1/p}.$$

When $p = \infty$, $L_{\infty}(\Omega)$ means the space of all essentially bounded real functions in Ω , with the norm:

$$\|v\|_{\infty} = \text{ess sup}_{x \in \Omega} |v(x)|.$$

$L_{\infty}(\Omega)$ is also a Banach space. When $p = 2$ we have a Hilbert space $L_2(\Omega)$ with the inner product:

$$\langle u, v \rangle = \int_{\Omega} u(x)v(x) dx$$

and induced norm

$$\|v\|_2^2 = \int_{\Omega} |v(x)|^2 dx.$$

By C_0^{∞} or $\mathcal{D}(\Omega)$, we represent the space of real functions defined in Ω , infinitely differentiable and with compact support in Ω . By $W^{m,p}(\Omega)$ we denote the Sobolev spaces of order m , that is, the space of all real functions $v \in L_p(\Omega)$ such that the distribution $D^{\alpha}v \in L_p(\Omega)$ for all $|\alpha| \leq m$. On $W^{m,p}(\Omega)$ we define the norm:

$$\|v\|_{m,p}^p = \sum_{|\alpha| \leq m} \int_{\Omega} |D^{\alpha}v(x)|^p dx.$$

It is known that $W^{m,p}(\Omega)$ with this norm is a Banach space. By $W_0^{m,p}(\Omega)$ we represent the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$. When $p = 2$, the space $W^{m,2}(\Omega)$ is represented by $H^m(\Omega)$, which is a Hilbert space with the inner product:

$$\langle u, v \rangle = \sum_{|\alpha| \leq m} \int_{\Omega} D^{\alpha}u(x).D^{\alpha}v(x) dx.$$

and norm:

$$\|v\|_{m,2}^2 = \sum_{|\alpha| \leq m} \int_{\Omega} |D^{\alpha}v(x)|^2 dx.$$

In particular, we use, frequently, the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$. We have:

$$H^1(\Omega) = \{v \in L_2(\Omega); \frac{\partial v}{\partial x_i} \in L_2(\Omega) \quad i = 1, 2, \dots, n\}$$

with the inner product

$$\langle u, v \rangle = \int_{\Omega} u(x) \cdot v(x) dx + \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx$$

and norm:

$$\|v\|^2 = \int_{\Omega} |v(x)|^2 dx + \int_{\Omega} |\nabla v(x)|^2 dx.$$

Here ∇ denotes the gradient operator. When Ω is bounded or has finite measure, we obtain an equivalent norm in $H_0^1(\Omega)$ given by

$$\|v\|_{H_0^1}^2 = \int_{\Omega} |\nabla v(x)|^2 dx.$$

Let us consider the Dirichlet Laplacian operator Δ defined by the triplet $\{H_0^1(\Omega), L_2(\Omega); \langle \cdot, \cdot \rangle\}$. Its domain is, for smooth Γ :

$$\mathcal{D}(\Delta) = H_0^1(\Omega) \cap H^2(\Omega).$$

If Ω is bounded with C^2 boundary, then by Poincaré inequality the norm of $H_0^1(\Omega) \cap H^2(\Omega)$ is equivalent to $\|v\|_{\Delta}^2 = \int_{\Omega} |\Delta v|^2 dx$.

Given a Banach space X and a real number $\tau > 0$, we denote by $L_p(0, \tau; X)$, for $1 \leq p < \infty$, the space of vector functions $v :]0, \tau[\rightarrow X$, measurable and such that $\|v(t)\|_X^p$ is integrable in $]0, \tau[$. On $L_p(0, \tau; X)$ we define the norm:

$$\|v\|_{L_p(0, \tau; X)} = \left(\int_0^{\tau} \|v\|_X^p dt \right)^{1/p}.$$

For $p = +\infty$, the norm is:

$$\|v\|_{L_{\infty}(0, \tau; X)} = \operatorname{ess\,sup}_{0 \leq t \leq \tau} \|v(t)\|_X.$$

Note that $L_p(0, \tau; X)$, $1 \leq p \leq \infty$ are Banach spaces.

1.2 Autonomous evolution equations

1.2.1 Semigroup theory

Let us take a quick view on the strongly continuous semigroups theory that play a very important role in many areas of modern mathematical analysis. For detail and sophisticated arguments, we refer to [29].

Strongly continuous semigroup

Denote H be the Hilbert space.

Definition 1.2.1. The operator value function $\mathbb{T}(t) : \mathbb{R}_+ \rightarrow \mathcal{L}(H)$ is called a strongly continuous semigroup (or C_0 -semigroup) if it satisfies:

- i) $\mathbb{T}(t+s) = \mathbb{T}(t)\mathbb{T}(s)$ for $t, s \geq 0$
- ii) $\mathbb{T}(0) = I$
- iii) $\|\mathbb{T}(t)x - x\| \rightarrow 0$ as $t \rightarrow 0$ for all $x \in H$.

Example 1.2.2. Let $A \in \mathcal{L}(H)$ and let

$$e^{tA} = \sum_{k=0}^{+\infty} \frac{(tA)^k}{k!}$$

then e^{tA} defines a C_0 -semigroup on H .

Example 1.2.3. Let $\{\phi_n\}$ be an orthonormal basis in separable Hilbert space H , and $\{\mu_n\}_{n \geq 1}$ be the sequence of real number. For $x \in H$, we define:

$$\mathbb{T}(t)x = \sum_{k=1}^{+\infty} e^{\mu_k t} \langle x, \phi_k \rangle \phi_k \quad (1.2.1)$$

Then $\mathbb{T}(t)$ defines a C_0 -semigroup on H if $\sup_{k \geq 1} \mu_k < +\infty$.

Theorem 1.2.4. Every C_0 -semigroup $\mathbb{T}(t)$ has the following properties:

- i) $\mathbb{T}(t)$ is bounded on every finite sub-interval of $[0, +\infty)$
- ii) $\mathbb{T}(t)$ is strongly continuous for all $t \in [0, +\infty)$
- iii) $\frac{1}{t} \int_0^t \mathbb{T}(s)x ds \rightarrow x$ as $t \rightarrow 0$.
- iv) If $\omega_0 = \inf_{t>0} \frac{\log \|\mathbb{T}(t)\|}{t}$ then $\omega_0 = \lim_{t \rightarrow +\infty} \frac{\log \|\mathbb{T}(t)\|}{t} < +\infty$.
- v) For all $\omega > \omega_0$, there exist the constant ν_ω such that: $\|\mathbb{T}(t)\| \leq \nu_\omega e^{\omega t}$.

The constant ω_0 is call the growth bound of the semigroup.

Definition 1.2.5. Let A be the operator defined by:

$$\mathcal{D}(A) = \{x \in H \mid \lim_{t \rightarrow 0^+} \frac{\mathbb{T}(t)x - x}{t} \text{ exists in } H\}$$

$$Ax = \lim_{t \rightarrow 0^+} \frac{\mathbb{T}(t)x - x}{t} \quad \forall x \in \mathcal{D}(A)$$

then A is called the infinitesimal generator of C_0 -semigroup on H .

Theorem 1.2.6. Considering the strongly continuous semigroup $\mathbb{T}(t)$ on the Hilbert space H with infinitesimal generator A . Then we have:

- i) $\mathbb{T}(t)x \in \mathcal{D}(A)$ for all $x \in \mathcal{D}(A)$, $t \geq 0$
- ii) $\frac{d^n}{dt}(\mathbb{T}(t)x) = A^n \mathbb{T}(t)x = \mathbb{T}(t)A^n x$ for all $x \in \mathcal{D}(A^n)$, $t \geq 0$ and $n \geq 1$
- iii) $\bigcap_{n=1}^{+\infty} \mathcal{D}(A^n)$ is dense in H .

We now introduce some basic spectral properties for generators of strongly continuous semigroups. We denote $\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible}\}$ be the spectrum, $\rho(A) = \mathbb{C}/\sigma(A)$, and $R(\lambda, A) = (\lambda I - A)^{-1}$ be the resolvent operator of A .

Theorem 1.2.7. (*Hille-Yosida's theorem*)

A closed and densely defined operator A on the Hilbert space H is the infinitesimal generator of a C_0 -semigroup $\mathbb{T}(t)$ if and only if there exist real numbers M and ω such that for all real number $\gamma > \omega$, $\gamma \in \rho(A)$ and

$$\|R(\gamma, A)^n\| \leq \frac{M}{(\gamma - \omega)^n} \quad \forall n \geq 1. \quad (1.2.2)$$

Moreover, we have the boundedness for the semigroup $\|\mathbb{T}(t)\| \leq Me^{\omega t}$

Contraction semigroup

Definition 1.2.8. A C_0 -semigroup $\mathbb{T}(t)$ is called a contraction semigroup if $\|\mathbb{T}(t)\|_{\mathcal{L}(H)} \leq 1$ for all $t \geq 0$.

The following result derived by Hille-Yosida:

Theorem 1.2.9. For a closed, densely defined operator A with domain $\mathcal{D}(A)$ in the Hilbert space H , the operator $A - \omega I$ is the infinitesimal generator of the contraction semigroup $\mathbb{T}(t)$ on H if and only if

$$\|(\gamma - A)x\| \geq (\gamma - \omega)\|x\| \quad \forall x \in \mathcal{D}(A)$$

$$\|(\gamma - A^*)x\| \geq (\gamma - \omega)\|x\| \quad \forall x \in \mathcal{D}(A^*)$$

holds for all $\gamma > \omega$ where A^* is the adjoint operator of A .

Corollary 1.2.10. If a closed and densely defined operator A satisfies:

$$\operatorname{Re}(\langle Ax, x \rangle) \leq \omega \|x\|^2 \quad \forall x \in \mathcal{D}(A)$$

$$\operatorname{Re}(\langle A^*x, x \rangle) \leq \omega \|x\|^2 \quad \forall x \in \mathcal{D}(A^*)$$

then A is the infinitesimal generator of a C_0 -semigroup satisfying $\|\mathbb{T}(t)\| \leq e^{\omega t}$.

Theorem 1.2.11. If A is the infinitesimal generator of a C_0 -semigroup $\mathbb{T}(t)$ on the Hilbert space H , then A^* is the infinitesimal generator of the C_0 -semigroup $\mathbb{T}(t)^*$ on H .

Definition 1.2.12. A linear operator A is called dissipative if for all $\lambda > 0$ and all $x \in \mathcal{D}(A)$

$$\|(\lambda I - A)x\| \geq \lambda \|x\|.$$

Moreover, A is called maximally dissipative if it is dissipative and for all $\lambda > 0$, the operator $\lambda I - A$ is surjective, meaning that the range when applied to the domain $\mathcal{D}(A)$ is the whole of the space H .

Theorem 1.2.13. (Lumer–Phillips theorem)[29, Chapter 2]

Let A be a linear operator and densely defined. Then the following assertions hold:

- i) If A is dissipative and $\lambda_0 I - A$ is surjective for some $\lambda_0 > 0$, then A generates a contraction semigroup.
- ii) If A generates a contraction semigroup, then A is dissipative, $\mathbb{C}_+ \subset \rho(A)$ and $\|R(\lambda, A)\| \leq \frac{1}{\operatorname{Re} \lambda}$ for all $\lambda \in \mathbb{C}_+$.

Diagonalizable operators

We introduce diagonalizable operators, which can be described entirely in terms of their eigenvalues and eigenvectors. If a semigroup generator is diagonalizable then so is the semigroup. Many examples of semigroups discussed in the PDEs literature are diagonalizable.

Definition 1.2.14. A is called diagonalizable if $\rho(A) \neq \emptyset$ and there exists a Riesz basis $\{\phi_k\}$ in H consisting of eigenvectors of A .

Proposition 1.2.15. [123]

Let $\{\phi_k\}$ be a Riesz basis in H and let $\{\tilde{\phi}_k\}$ be the biorthogonal sequence to $\{\phi_k\}$. Let λ_k be a sequence in \mathbb{C} which is not dense in \mathbb{C} . Define an operator $\tilde{A} : \mathcal{D}(\tilde{A}) \rightarrow H$ by:

$$\mathcal{D}(\tilde{A}) = \left\{ x \in H \quad \text{s.t.} \quad \sum_k (1 + |\lambda_k|^2) |\langle x, \tilde{\phi}_k \rangle|^2 < +\infty \right\}$$

$$\tilde{A}x = \sum_k \lambda_k \langle x, \tilde{\phi}_k \rangle \phi_k, \quad \forall x \in \mathcal{D}(\tilde{A}).$$

Then \tilde{A} is diagonalizable, and for every $s \in \rho(\tilde{A})$ we have:

$$(sI - \tilde{A})^{-1}z = \sum_{k \in \mathbb{N}} \frac{1}{s - \lambda_k} \langle z, \tilde{\phi}_k \rangle \phi_k \quad \forall z \in H.$$

Proposition 1.2.16. [123]

Let $A : \mathcal{D}(A) \rightarrow H$ be a diagonalizable operator and $\{\phi_k\}$, the set of eigenvectors of A which form a Riesz base of H . Let $\{\tilde{\phi}_k\}$ be the biorthogonal sequence to $\{\phi_k\}$ and denote the eigenvalue corresponding to the eigenvector ϕ_k by λ_k . Then

$$\mathcal{D}(A) = \left\{ x \in H \quad \text{s.t.} \quad \sum_k (1 + |\lambda_k|^2) |\langle x, \tilde{\phi}_k \rangle|^2 < +\infty \right\}$$

$$Ax = \sum_k \lambda_k \langle x, \tilde{\phi}_k \rangle \phi_k, \quad \forall x \in \mathcal{D}(A).$$

Proposition 1.2.17. [123]

A is the generator of a strongly continuous semigroup $\mathbb{T}(t)$ on H if and only if:

$$\sup_{k \in \mathbb{N}} \operatorname{Re} \lambda_k < +\infty$$

If this is the case, then for every $t \geq 0$:

$$\mathbb{T}(t)x = \sum_{k \in \mathbb{N}} e^{\lambda_k t} \langle x, \tilde{\phi}_k \rangle \phi_k \quad \forall x \in H.$$

Here, $\{\lambda_k\}$ and $\{\phi_k\}$ are defined as in Proposition 1.2.16.

1.2.2 Existence and uniqueness

We consider the abstract non-homogeneous Cauchy problem:

$$\begin{cases} x'(t) &= Ax(t) + f(t) \\ x(0) &= x_0 \end{cases} \quad (1.2.3)$$

where $x_0 \in H$, $f : [0, \tau] \rightarrow H$ is integrable and $f \in C(0, \tau; H)$.

Definition 1.2.18. $x(t)$ is called the classical solution of (1.2.3) on $[0, \tau]$ if for all $t \in [0, \tau]$, $x(t) \in \mathcal{D}(A) \cap C^1(0, \tau; H)$ and $x(t)$ satisfies (1.2.3).

Lemma 1.2.19. *If the operator $A : \mathcal{D}(A) \rightarrow H$ is maximal dissipative and $f \in C(0, \tau; H)$, then for all $x_0 \in H$, the function $x : [0, \tau] \rightarrow H$ defined by:*

$$x(t) = \mathbb{T}(t)x_0 + \int_0^t \mathbb{T}(t-s)f(s) ds \quad t \in [0, \tau] \quad (1.2.4)$$

is continuous from $[0, \tau]$ to H . Here, $\mathbb{T}(t)$ is the semigroup generated by A .

Definition 1.2.20. If $f \in L_1(0, \tau; H)$, $x(t)$ is a weak solution of (1.2.3) if $x \in C([0, \tau]; H)$ and given by (1.2.4).

Theorem 1.2.21. *For all function $f \in L_1(0, \tau; H)$ and initial data $x_0 \in \mathcal{D}(A)$, the system (1.2.3) admits an unique solution $x \in C([0, \tau]; \mathcal{D}(A)) \cap C^1([0, \tau]; H)$.*

1.3 Non-autonomous evolution equation

1.3.1 Evolution family theory

Let us denote $\diamond = \{(t, s) \in \mathbb{R}_+^2 : s \leq t\}$. We recall that the Banach space

$$C_0 = C_0(\mathbb{R}, H) = \{f : \mathbb{R} \rightarrow H : f \text{ is continuous and } \lim_{|t| \rightarrow +\infty} f(t) = 0\}$$

equipped with the norm $\|f\| := \sup_{t \in \mathbb{R}} \|f(t)\|$ for $f \in C_0$.

Let f be a locally integrable H -valued function on \mathbb{R} and consider the following homogeneous and non-homogeneous systems :

$$\begin{cases} x'(t) &= A(t)x(t) \\ x(0) &= x_0 \end{cases} \quad (1.3.1)$$

and

$$\begin{cases} x'(t) &= A(t)x(t) + f(t) \\ x(0) &= x_0 \end{cases} \quad (1.3.2)$$

on H . As in the autonomous case when the operators A does not depend on t and generate a semigroup $\mathbb{T}(t) = e^{-tA}$, we are able to derive a representation formula for the solution (for example, see [103], [73]) :

$$x(t) = \mathbb{T}(t)x_0 + \int_0^t \mathbb{T}(t-s)f(s) ds.$$

From the point of view of semigroup theory, it is reasonable to develop a suitable approach to non homogeneous initial boundary value problems based on the theory of evolution families.

Definition 1.3.1. (Evolution family)

A family $\{U(t, s)_{(t,s) \in \diamond}\}$ of operators is called an evolution family on H if it satisfies the following conditions :

1. $U(t, t)x = x$ for all $t \geq 0$ and $x \in H$;
2. $U(t, s) = U(t, r)U(r, s)$ for all $t \geq r \geq s \geq 0$
3. $U(t, s)x$ is jointly continuous with respect to t, s for every $x \in H$.

Assume, for a moment, that (1.3.1) is well-posed. It means that there exists an evolution family of operators $\{U(t, s)_{(t,s) \in \diamond}\}$ so that $x(t) = U(t, 0)x_0$ is a solution to (1.3.1).

If A is the infinitesimal generator of a strongly continuous semigroup on H , then $U(t, s) = e^{(t-s)A}$ for $t \geq s$ is a strongly continuous, exponentially bounded evolution family. Futhurmore, we have:

$$\frac{\partial}{\partial t}U(t, s)x = A(t)U(t, s)x \quad \frac{\partial}{\partial s}U(t, s)x = U(t, s)A(s)x \quad (1.3.3)$$

Definition 1.3.2. Suppose that the linear equation (1.3.1) is well-posed. Then $x(t)$ defined by:

$$x(t) = U(t, 0)x_0 + \int_0^t U(t, \sigma)f(\sigma) d\sigma \quad (1.3.4)$$

is called a mild solution of non autonomous system (1.3.2). Futhurmore, the equation (1.3.2) is said to generate an evolution family $(U(t, s))_{(t,s) \in \diamond}$ if for every $x \in H$ and $s \in \mathbb{R}_+$, the map $t \mapsto U(t, 0)x$ is the unique solution of equation (1.3.1).

We denote the growth bound:

$$\omega(U) = \inf\{\omega : \text{there exist } M = M(\omega) \text{ such that } \|U(t, s)\|_{\mathcal{L}(H)} \leq Me^{(t-s)\omega}\} \quad (1.3.5)$$

Definition 1.3.3. The evolution family $\{U(t, s)\}_{0 \leq s \leq t}$ is called uniformly exponentially stable if the growth bound $\omega(U)$ is negative.

Example 1.3.4. We consider:

$$A_0 = \begin{bmatrix} -1 & -5 \\ 0 & -1 \end{bmatrix}, \quad W(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}, \quad A(t) = W(t)^{-1}A_0W(t)$$

We note that $\sigma(A(t)) = \sigma(A_0) = \{-1\}$ for $t \in \mathbb{R}$. The time dependent change of variables given by $z(t) = W(t)x(t)$ transforms the non-autonomous differential equation $\dot{x}(t) = A(t)x(t)$ to the autonomous differential equation $\dot{z} = Bz$ where

$$B = A_0 + \dot{W}(t)W(t)^{-1} = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix}$$

since $\sigma(B) = \{-2, 2\}$, the semigroup generated by B for the differential equation $\dot{z} = Bz$ is unstable. Clearly, this also implies that the evolution family generated by $\{A(t)\}$ for the equation $\dot{x} = A(t)x$ is unstable.

Example 1.3.5. Let us denote $H = H_0^1[0, \pi] \times L_2[0, \pi]$ which is a Hilbert space with the scalar product

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle = \int_0^\pi \frac{df_1}{dx}(x) \overline{\frac{df_2}{dx}(x)} dx + \int_0^\pi g_1(x) \overline{g_2(x)} dx.$$

Let $a : [0, \tau] \rightarrow \mathbb{R}^-$ be a Hölder continuous function with parameter $0 < \delta < 1$. We consider the following non-autonomous wave equation with Dirichlet boundary condition:

$$\begin{cases} \frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2} + a(t)z & x \in [0, \pi], \quad t \in [0, \tau] \\ z(0, t) = z(\pi, t) = 0 & \tau \geq t \geq 0 \\ z(x, 0) = z_0 & x \in [0, \pi] \end{cases} \quad (\text{P2})$$

The above system can be written in the abstract form:

$$\begin{bmatrix} \dot{z} \\ z_t \end{bmatrix} = \begin{bmatrix} 0 & I \\ \Delta + a(t) & 0 \end{bmatrix} \begin{bmatrix} z \\ z_t \end{bmatrix} = A(t) \begin{bmatrix} z \\ z_t \end{bmatrix}.$$

Where the time dependent operator $A(t) : \mathcal{D}(A(t)) \rightarrow H$ is defined by

$$\mathcal{D}(A(t)) = [H^2(0, \pi) \cap H_0^1(0, \pi)] \times H_0^1(0, \pi)$$

$$A(t) \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{d^2 f}{dx^2} + a(t)f \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A(t)).$$

It is well known that $\varphi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$ is an orthonormal basis in $L_2[0, \pi]$, Hence, the family $(\phi_n)_{n \in \mathbb{Z}}$ defined by

$$\phi_n = \begin{bmatrix} \frac{1}{i\sqrt{2n^2 - a(t)}} \varphi_n \\ \frac{\sqrt{n^2 - a(t)}}{\sqrt{2n^2 - a(t)}} \varphi_n \end{bmatrix} \quad \text{for} \quad n \in \mathbb{Z}$$

is an orthonormal basis in H . These ϕ_n , which is eigenvectors of $A(t)$, corresponds to the eigenvalues $\lambda_n(t) = i\sqrt{n^2 - a(t)}$ with $n \in \mathbb{R}$. Thus for all $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A(t))$, it holds that

$$A(t) \begin{bmatrix} f \\ g \end{bmatrix} = \sum_{n=1}^{+\infty} \lambda_n(t) \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \phi_n \right\rangle \phi_n.$$

$\{A(t)\}$ generates an evolution semigroup $U(t, s)$ of the form

$$U(t, s) = T(t - s) \exp\left(\int_s^t a(r) dr\right)$$

where $T(t)$ is a compact analytic semigroup generated by the operator A satisfying

$$A \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} g \\ \frac{d^2 f}{dx^2} \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{D}(A) = \mathcal{D}(A(t)).$$

Now for all $\begin{bmatrix} f \\ g \end{bmatrix} \in H$, we have

$$U(t, s) \begin{bmatrix} f \\ g \end{bmatrix} = \sum_{n=1}^{+\infty} e^{\mu_n(t) - \mu_n(s)} \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \phi_n \right\rangle \phi_n$$

where $\mu_n(t) \in C^2([0, \tau])$ satisfying $\frac{d}{dt}\mu_n(t) = \lambda_n(t)$. That implies

$$\mu_n(t) = \mu_n(0) + \int_0^t i\sqrt{n^2 - a(s)} ds$$

It is easy to verify that the family of two parameter $\{U(t, s)\}$ satisfy all the criterion of evolution family. Now from the definititon we have

$$\begin{aligned} U(t, s) \begin{bmatrix} f \\ g \end{bmatrix} &= \sum_{n=1}^{+\infty} e^{\mu_n(t) - \mu_n(s)} \left(\frac{i}{\sqrt{2n^2 - a(t)}} \left\langle \frac{df}{dx}, \frac{d\varphi_n}{dx} \right\rangle_{L_2[0, \pi]} \right. \\ &\quad \left. + \frac{\sqrt{n^2 - a(t)}}{\sqrt{2n^2 - a(t)}} \langle g, \varphi_n \rangle_{L_2[0, \pi]} \right) \phi_n. \end{aligned}$$

Example 1.3.6. Let $H = L_2(0, \pi)$. We consider the one dimensional heat equation on $\Omega = (0, \pi)$ satisfying Neuman boundary condition with distributed control

$$\begin{cases} \frac{\partial \Psi}{\partial t} = \alpha(t) \frac{\partial^2 \Psi}{\partial x^2} + Bu(t) & x \in [0, \pi] \\ \frac{\partial \Psi}{\partial x}(0, t) = \frac{\partial \Psi}{\partial x}(\pi, t) = 0 & t \geq 0 \\ \Psi(x, 0) = \Psi_0(x) & x \in [0, \pi]. \end{cases} \quad (1.3.6)$$

Here $\alpha(t)$ describes processes in which the conductivity or diffusivity changes over time. We rewrite (1.3.6) as the non-autonomous form $\dot{\Psi} = A(t)\Psi(t) + Bu(t)$. The homogeneous equation $\dot{\Psi} = A(t)\Psi(t)$ has the solution $\Psi(t) = U(t, 0)\Psi_0$ where the evolution family $\{U(t, s)\}$ is generated by $A(t)$ and given by:

$$U(t, s)x = \sum_{k=0}^{+\infty} e^{\int_s^t \lambda_k(r) dr} \langle x, \phi_k \rangle \phi_k$$

where the eigenvalues $\lambda_k(t) = -\alpha(t)(k\pi)^2$ and the eigenfunctions $\phi_k = \sqrt{2} \cos(k\pi)$. The system (1.3.6) has the solution of following form:

$$\Psi(x, t) = U(t, 0)\Psi_0 + \int_0^t U(t, s)Bu(s) ds. \quad (1.3.7)$$

Existence of evolution families

The representation the initial and boundary value problem (1.3.1) was widely investigated in many works (e.g see [103], [78], [74]). Also, the existence problem for solutions has been studied by many authors (e.g see [89], [102], [103]). We now survey the existences properties of evolution family for system (1.3.1) in some cases: **The parabolic case** We assumes that each $A(t)$ generates analytic C_0 -semigroup

of the same type and the map $t \mapsto A(t)$ is regular. Then there exists an evolution family $U(t, s)_{(s,t) \in \diamond}$ on H solving (1.3.1) on $\mathcal{D}(A(t))$ such that:

$$U(t, s)H \subset \mathcal{D}(A(t)), \quad \frac{\partial}{\partial t} U(t, s) = A(t)U(t, s) \quad \text{in } \mathcal{L}(H)$$

$$\|A(t)U(t, s)\| \leq \frac{C}{t-s} \quad 0 \leq s < t \leq \tau.$$

The operators $U(t, s)_{(s,t) \in \diamond}$ can be constructed as solutions to certain integral equations:

$$U(t, s) = e^{(t-s)A(s)} + \int_s^t U(t, r)(A(r) - A(s))e^{(r-s)A(s)} dr.$$

In the particular case where the domain of $A(t)$ is independent of t , we use the assumptions on the family of operators $\{A(t)\}$ that was proposed by Pazy [103]:

(A1) The domain $\mathcal{D}(A)$ of the family $\{A(t) : 0 \leq t \leq \tau\}$ is dense in H and does not depend on t . Moreover, $A(t)$ is a closed linear operator.

(A2) For every $t \in [0, \tau]$, the resolvent $R(\lambda, A(t)) = (\lambda - A(t))^{-1}$ of $A(t)$ exists for all λ such that $\operatorname{Re}(\lambda) \leq 0$, and there also exists $K > 0$ such that $\|R(\lambda, A(t))\| \leq \frac{K}{|\lambda|+1}$

(A3) There exists $0 < \delta \leq 1$ and $K > 0$ such that $\|(A(t) - A(s))A^{-1}(y)\| \leq K|t-s|^\delta$ for all $t, s, y \in [0, \tau]$

(A4) For each $t \in [0, \tau]$ and some $\lambda \in \rho(A(t))$, the resolvent set $R(\lambda, A(t))$ of linear operator $A(t)$ is a compact operator.

Under these assumptions, the family $\{A(t)\}$ generates a unique linear evolution family which satisfy $U(t, s)_{(s,t) \in \Delta}$ satisfy the following conditions:

- a) For every $x \in H$, the mapping $(t, s) \rightarrow U(t, s)x$ is continuous
- b) $U(t, s)U(s, y) = U(t, y)$ for all $0 \leq y \leq s \leq t \leq \tau$
- c) $U(t, t) = I$
- d) $U(t, s)$ is a compact operator for $s < t$, and there exist the constant $M \geq 1$ such that $\|U(t, s)\| \leq M$

Due to [98], the condition for invertibility of the evolution family $U(t, s)$ is following:

Lemma 1.3.7. [98, Lemma 4.3]

An evolution family $(U(t, s))_{s \leq t}$ consists of invertible operators if and only if there exists a family of invertible bounded operators $Q(t)_{t \in \mathbb{R}}$ such that

$$U(t, s) = Q(t)Q(s)^{-1}, \quad t, s \in \mathbb{R}$$

Moreover, by setting $U(s, t) := U(t, s)^{-1}$ for $s < t$, the evolution family can be extended to an evolution family $(U(t, s))_{(t,s) \in \mathbb{R}^2}$. Precisely, we obtain $U(t, 0) = Q(t)$ and $U(0, t) = Q^{-1}(t)$

Proposition 1.3.8. [98, Proposition 4.5]

The strongly continuous evolution family $(U(t, s))_{t > s}$ consists of invertible operators on H satisfying:

1. $\lim_{s \rightarrow t} U(t, s)^{-1}x = x$ and

2. $\lim_{t \rightarrow s} U(t, s)^{-1}x = x$ for every $x \in H$ and $t \leq s$ if and only if there exists a strongly continuous family of invertible bounded operators $(Q(t))_{t \in \mathbb{R}}$ with strongly continuous inverse such that

$$U(t, s) = Q(t)Q(s)^{-1}, \quad t, s \in \mathbb{R}.$$

Moreover, the evolution family can be extended to a strongly continuous evolution family $(U(t, s))_{(t, s) \in \mathbb{R}^2}$ by setting $U(s, t) := U(t, s)^{-1}$ for $s < t$. The family Q can then be given by $Q(t) := U(t, 0)$.

The hyperbolic case We can extend the Hille-Yosida theorem for semigroups to non autonomous situation. Following [113], we need to assume the family of operators $\{A(t)\}$ are densely defined and stable in the sense:

$$\|R(\lambda, A(t_n))R(\lambda, A(t_{n-1})) \dots R(\lambda, A(t_1))\| \leq M(\lambda - \omega)^{-n} \quad (1.3.8)$$

for all $0 \leq t_1 \leq \dots \leq t_n \leq \tau$, $n \in \mathbb{N}$ and $\lambda > \omega$ and some constants $M \geq 1$ and $\omega \in \mathbb{R}$. By Hille-Yosida theorem, the operator $A(t)$ then generates a C_0 -semigroup. Suppose in addition that there exists a Banach space Y such that $Y \subseteq \mathcal{D}(A(t))$ for all t , and Y is densely and continuously embedded into H . Kato [61] constructed an evolution family $U(t, s)$ satisfying $\frac{\partial}{\partial t}U(t, s)|_{t=s} = A(s)x$ for $t \geq s$ and $x \in Y$ due to the assumption that $A(\cdot) \in C([0, \tau], \mathcal{L}(Y, H))$.

The general case also established by Kato in 1970 [60] where he introduced a time discretization $A_n(\cdot)$ of $A(\cdot)$ solved the Cauchy problem by a finite product of discretization evolution family $U_n(t, s)$ of the given operator $e^{\tau A(\tau)}$. He proved that $U_n(t, s)$ strongly converges to an operator $U(t, s)$ that have all properties of evolution family.

Also in [60], Kato imposed a regularity hypothesis on $t \mapsto A(t)$ and proved that $U(t, s)Y \subseteq Y$, $U(t, s)$ is strongly continuous on Y and $\frac{\partial}{\partial t}U(t, s)x = A(t)U(t, s)x$ for $t \geq s$ and $x \in Y$. In particular, if $\mathcal{D}(A(t)) = Y$ and $A(\cdot)Y \in C^1(J, H)$ for $y \in Y$ and $J \subset \mathbb{R}$ be a closed interval, the above extra assumption is also true.

Evolution semigroup

Next, we introduce the notion of the evolution semigroup, which is a very important definition and widely studied in the past 40 years (e.g see [99], [31], [46]). For every exponentially bounded evolution family, we can associate a C_0 semigroup on H valued functional spaces. Furthermore, it is possible to characterize some asymptotic behavior properties of evolution families by spectral theory due to the corresponding evolution semigroup and generators.

Definition 1.3.9. An evolution family $\{U(t, s)\}_{0 \leq s \leq t}$ is said to have an exponential dichotomy (with constants $M > 0$ and $\beta > 0$) if there exists a projection-valued function $P : \mathbb{R} \rightarrow \mathcal{L}(H)$ such that, for each $x \in H$, the function $y \mapsto P(y)x$ is continuous and bounded, and, for all $y \geq r$, the following conditions hold:

- i) $P(y)U(y, r) = U(y, r)P(r)$
- ii) $U_Q(y, r)$ is an invertible operator from $\text{Im}(Q(r))$ to $\text{Im}(Q(y))$
- iii) $\|U_P(y, r)\| \leq Me^{-\beta(y-r)}$

$$\text{iv)} \quad \|U_Q^{-1}(y, r)\| \leq M e^{-\beta(y-r)}$$

The existence of an exponential dichotomy gives an important insight into the long term behaviour of an evolution family. It is also used to study the asymptotic properties of mild solutions and the existence criterion for the solution of inhomogeneous problem (2.1.1).

Definition 1.3.10. (Evolution semigroup)

For every evolution family $(U(t, s))_{(t,s) \in \diamond}$, we define the corresponding evolution semigroup $(\Upsilon(s))_{s \geq 0}$ on the space $C_0(\mathbb{R}, H)$ by

$$(\Upsilon(s)f)(r) := U(r, r-s)f(r-s), \quad (1.3.9)$$

for $f \in C_0$, $r \in \mathbb{R}$ and $s \geq 0$. This semigroup is also called the Howland evolution semigroup on the real line. We denote its generator by $(\Gamma, \mathcal{D}(\Gamma))$.

Noting that each operator of the above evolution semigroup is a product of a multiplication operator and a translation, we consider the following

$$C_1 := \{f \in C^1(\mathbb{R}, H) : f, f' \in C_0\}$$

the domain of the generator $\left(-\frac{d}{dx}, \mathcal{D}\left(-\frac{d}{dx}\right)\right)$ of the right translation semigroup $(\Upsilon_1(s))_{s \geq 0}$ on the space C_0 defined by

$$(\Upsilon_1(s)f)(r) := f(r-s), \quad f \in C_0, \quad r \in \mathbb{R}, s \geq 0$$

Therefore, we can rewrite the evolution semigroup as

$$\Upsilon(s)f = U(., .-s)\Upsilon_1(s)f.$$

Proposition 1.3.11. *If $\{U(t, s)\}_{0 \leq s \leq t}$ is an exponentially bounded evolution family on a H , then the semigroup $(\Upsilon(t))_{t \geq 0}$, as defined in (1.3.9) is a strongly continuous semigroup on $L_p(\mathbb{R}, H)$ for $p \in (1, +\infty)$ and on $C_0(\mathbb{R}, H)$.*

Proposition 1.3.12. [99, Proposition 1.3]

Let $(\Upsilon(s))_{s \geq 0}$ be a C_0 semigroup with the generator $(\Gamma, \mathcal{D}(\Gamma))$ on the Banach space C_0 . Then the following assertions are equivalent:

- i) The semigroup $(\Upsilon(s))_{s \geq 0}$ is an evolution semigroup, i.e: there exists an evolution family $(U(t, s))_{s \leq t}$ on the H such that $\Upsilon(s) = U(., .-s)\Upsilon_1(s)$
- ii) For all $\eta \in C_0(\mathbb{R})$, $f \in C_0$ we have

$$(\Upsilon(t)\eta f)(r) = \eta(r-s)(\Upsilon(s)f)(r), \quad r \in \mathbb{R}, s \geq 0$$

- iii) There exists a core D for G such that for all $\eta \in C_c^1(\mathbb{R})$ and $f \in D$ we have $\eta f \in D(G)$ and $G\eta f = -\eta' f + \eta Gf$.

Theorem 1.3.13. (Dichotomy Theorem)[17, Theorem 3.17]

Let $\{U(t, s)\}_{0 \leq s \leq t}$ be a strongly continuous, exponentially bounded evolution family on H , let $\Upsilon(s)$ be the corresponding evolution semigroup given by $(\Upsilon(s)f)(r) = U(r, r-s)f(r-s)$ on $L_p(\mathbb{R}, H)$ for $1 < p < +\infty$ and let Γ denote its infinitesimal generator. The following statements are equivalent:

- i) $\{U(t, s)\}_{0 \leq s \leq t}$ has an exponential dichotomy on H .
- ii) $0 \in \rho(\Gamma)$.

The Dichotomy Theorem also holds if the space $L_p(\mathbb{R}, H)$ is replaced by $C_0(\mathbb{R}, H)$. We see that a strongly continuous evolution family $\{U(t, s)\}_{0 \leq s \leq t}$ has exponential dichotomy on H if and only if the spectrum $\sigma(\Gamma), t > 0$, does not intersect the unit circle or, equivalently, the operator Γ^{-1} is bounded on $C_0(\mathbb{R}, H)$ or $L_p(\mathbb{R}, H)$.

1.3.2 Existence and uniqueness

Theorem 1.3.14. [75, Theorem 1.1]

Assume the evolution family $\{U(t, s)\}_{(s,t) \in \diamond}$ is an exponentially bounded evolution family. Then $\{U(t, s)\}_{(s,t) \in \diamond}$ has exponential dichotomy if and only if for every $f \in C_b(\mathbb{R}, H)$ there exists a unique solution $u \in C_b(\mathbb{R}, H)$ to (1.3.2).

The sufficiency for above theorem will use the spectral mapping theorem for evolution semigroups. The proof is a series of three lemmas that can be found in [75]. The necessity of existence for solution is proven in the following proposition which involves the Green's function for $\{U(t, s)\}_{t \geq s}$.

Proposition 1.3.15. [75, Proposition 1.2]

Let $\{U(t, s)\}_{t \geq s}$ be an evolution family. If $\{U(t, s)\}_{t \geq s}$ has exponential dichotomy, then there exists a unique solution $u \in C_b(\mathbb{R}, H)$ to (1.3.2).

Noting that in this proposition, we do not assume exponential boundedness of $\{U(t, s)\}_{t \geq s}$.

1.4 General concepts

There are several possibilities to introduce a control problem for partial differential equation. In fact, we are interested in learning some basic techniques connected to the problem of infinite-dimensional controllability. As in the finite dimensional setting, one can distinguish between the linear systems, where the partial differential equation under view is linear (as well as the action of the control), and the non-linear one. Given an evolution system which one can exert a certain influence. The question is that whether or not it is possible to use this influence to make the system reach a certain state? Now we give some general definitions of the typical controllability problems, associated to a control system.

Definition 1.4.1. A control system is said to be exactly controllable in time $\tau > 0$ if and only if for all y_0 and y_1 in Y , there exists control function $u : [0, \tau] \rightarrow U$ such that the unique solution of the system satisfies

$$\begin{cases} \dot{y} &= F(y, t, u) & t \in [0, \tau] \\ y|_{t=0} &= y_0 \end{cases} \quad (1.4.1)$$

satisfies

$$y|_{t=\tau} = y_1 \quad (1.4.2)$$

Definition 1.4.2. We suppose that the space Y is endowed with a metric d . The control system is said to be approximately controllable in time $\tau > 0$ if and only if, for all y_0, y_1 in Y and for any $\varepsilon > 0$, there exists a control function $u : [0, \tau] \rightarrow U$ such that the unique solution of the system (1.4.1) satisfies

$$d(y|_{t=\tau}, y_1) < \varepsilon \quad (1.4.3)$$

Definition 1.4.3. We consider a particular element $0 \in Y$. A control system is said to be null-controllable (or zero-controllable) in time $\tau > 0$ if and only if, for all y_0 in Y , there exists a control function $u : [0, \tau] \rightarrow U$ such that the unique solution of the system (1.4.1) satisfies:

$$y|_{t=\tau} = 0. \quad (1.4.4)$$

Definition 1.4.4. A control system is said to be controllable to trajectories in time $\tau > 0$ if and only if, for all y_0 in Y and any trajectory y of the system (typically but not necessarily satisfying (1.4.1) with $u = 0$), there exists a control function $u : [0, \tau] \rightarrow U$ such that the unique solution of the system (1.4.1) satisfies

$$y|_{t=\tau} = y(\tau) \quad (1.4.5)$$

Remark 1.4.5. The problems of controllability should be distinguished from the problems of optimal control, which give another viewpoint on control theory. In general, problems of optimal control look for a control u minimizing some functional $J(u, y(u))$, where $y(u)$ be the trajectory associated to the control u .

Remark 1.4.6. For the linear systems, there is no difference between controllability to zero and controllability to trajectories. In that case, it is indeed equivalent to bring y_0 to $y(\tau)$ or to bring $y_0 - y(0)$ to zero. On the contrary, noting that even for linear systems, approximate controllability and exact controllability are different.

Example 1.4.7. (The heat equation with distributed control)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $\omega \subseteq \Omega$ be the open subset. We consider the following initial boundary problem:

$$\begin{cases} \partial_t v - \Delta v &= \chi_\omega u & t \in [0, \tau] \\ v|_{\partial\Omega} &= 0 \end{cases} \quad (1.4.6)$$

The state y of the system is given by the function $v(., t)$ that belong to the space $L_2(\Omega)$. Here, we choose the control function u acting on the set ω and belong to $L_2([0, \tau]; L_2(\omega))$.

Example 1.4.8. (The wave equation with boundary control)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let the open non-empty subset $\Sigma \subseteq \partial\Omega$. We consider the following initial boundary problem:

$$\begin{cases} \partial_{tt} v - \Delta v &= 0 & t \in [0, \tau] \\ v|_{\partial\Omega} &= \chi_\Sigma u \end{cases} \quad (1.4.7)$$

The state y of the system is given by the function $v(., t)$ belong to the space $L_2(\Omega)$ as in the previous example, but here the control u is imposed on a part of the boundary. One can for instance consider the set of controls as $L_2([0, \tau]; L_2(\Sigma))$.

Chapter 2

Duality results

In this chapter, we will study the concepts of admissibility, controllability and the observability for the non autonomous linear system. We also summarize the duality of admissibility for control and observation and the duality between controllability and observability. These properties were studied well for the autonomous system.

2.1 Definitions

Let H and H_{-1} be the Hilbert space where H is dense in H_{-1} . Considering the family of unbounded operators $A(t) : \mathcal{D}(A(t)) \rightarrow H$. Let U be another Hilbert space and suppose $B(t) : \mathcal{D}(B(t)) \subset U \rightarrow H_{-1}$ be also family of unbounded operators and $C : \mathcal{D}(C(t)) \subset H \rightarrow U$ be a linear operator. For simplicity, we denote the above system as $(A(t), B(t), C(t))$. We consider the system :

$$\begin{cases} x'(t) &= A(t)x(t) + B(t)u(t) \\ x(0) &= x_0 \\ y(t) &= C(t)x(t). \end{cases} \quad (2.1.1)$$

The admissibility for control and observation operators as well as observability and controllability about a certain "duality" to each other that will be detailed on the following sections.

2.2 Duality of admissibility for control and observation

In the autonomous case, the duality of admissible control and observation operators has already been discussed earlier in many texts books (for example see Dolecki and Russell [27] or [123, chapter 4]). This duality allows us to translate most of the results given in the autonomous linear system for observation operators into result concerning control operators. We call B is an infinite-time admissible control operator for semigroup $(\mathbb{T}(t))_{t \geq 0}$, that is, $B \in \mathcal{L}(U, \mathcal{D}(A^*)')$ and there exists a constant $\kappa > 0$ such that

$$\left\| \int_0^t \mathbb{T}(t) B u(t) dt \right\|_H \leq \kappa \|u\|_{L_2(0, \infty; U)}, \quad u \in L_2(0, \infty; U).$$

Definition 2.2.1. An operator $C \in \mathcal{L}(\mathcal{D}(A), U)$ is called an (infinite-time) admissible observation operator for $(\mathbb{T}(t))_{t \geq 0}$ if there exists a constant $M > 0$ such that:

$$\|C\mathbb{T}(\cdot)x_0\|_{L_2(0,\infty;U)} \leq M\|x_0\|_H, \quad x_0 \in \mathcal{D}(A).$$

The following well-known result comes from [123].

Theorem 2.2.2. [123, Theorem 4.4.3] Suppose that $B \in \mathcal{L}(U, H_{-1})$. The operator B is an infinite-time admissible control operator for $(\mathbb{T}(t))_{t \geq 0}$ if and only if B^* is an infinite-time admissible observation operator for the dual semigroup $(\mathbb{T}(t)^*)_{t \geq 0}$.

For the non-autonomous case, let $A(t)$ be such that the initial value problem

$$\begin{cases} x'(t) &= A(t)x(t) + B(t)u(t) \\ x(0) &= x_0 \end{cases} \quad (2.2.1)$$

admits a evolution (solution) family $U(t, s)$. We observe that:

$$\frac{d}{dt}U(t, s)x = A(t)U(t, s)x, \quad x \in \mathcal{D}(A(t)).$$

Moreover, we have:

$$U(t, s+h)x - U(t, s)x = U(t, s+h)[I - U(s+h, s)]x$$

and so, dividing by $h > 0$ and letting $h \rightarrow 0+$,

$$\frac{d}{ds+}U(t, s)x = -U(t, s)A(s)x \quad x \in \mathcal{D}(A(s)).$$

Our classical solution satisfies the following mild integral equation:

$$x(t) = U(t, 0)x_0 + \int_0^t U(t, s)B(s)u(s) ds. \quad (2.2.2)$$

Now we take adjoints:

$$\frac{d}{ds+}\langle U(\tau, s)^*z_\tau, x \rangle = \frac{d}{ds+}\langle z_\tau, U(\tau, s)x \rangle = \langle z_\tau, -U(\tau, s)A(s)x \rangle = \langle -A(s)^*U(\tau, s)^*z_\tau, x \rangle.$$

This holds for all x , so we may drop duality pairing and obtain that $z(t) := U(\tau, t)^*z_\tau$ will solve the dual final time problem

$$\begin{cases} z'(t) + A(t)^*z(t) &= 0 \\ z(\tau) &= z_\tau \\ y(t) &= B(t)^*z(t). \end{cases} \quad (2.2.3)$$

We use the following definitions:

Definition 2.2.3 (Averaged admissible controls). Let $(B(t))_{t \in [0, \tau]}$ be a family of bounded operators in $\mathcal{L}(U; H_{-1})$. We say that $(B(t))_t$ are averaged admissible controls for $(A(t))_{t \in [0, \tau]}$ if there exists a constant $M_\tau > 0$ such that the solution x to (A,B) satisfies $x(t) \in H$ and for all $s \in [0, \tau]$

$$\left\| \int_s^\tau U(\tau, r)B(r)u(r) dr \right\|^2 \leq M_\tau^2 \|u\|_{L_2(s, \tau; U)}^2$$

for all $u \in \mathcal{D}(0, \tau; U)$ (one can also consider a weaker admissibility notions by requiring the above inequality for $s = 0$, only).

Definition 2.2.4 (Averaged admissible observations). Let $(C(t))_{t \in [0, \tau]}$ be a family of bounded operators in $\mathcal{L}(\mathcal{D}, Y)$, where Y is some Hilbert space. We say that $(C(t))_t$ are averaged admissible observations for $(A(t))_{t \in [0, \tau]}$ if there exists a constant $M_\tau > 0$ such that

$$\int_s^\tau \|C(t)U(t, s)x\|^2 dt \leq M_\tau^2 \|x\|^2 \quad \forall x \in \mathcal{D}, s \in [0, \tau]. \quad (2.2.4)$$

(one can also consider a weaker admissibility notion by requiring the above inequality for $s = 0$, only). For a single operator $C(t_0)$ is admissible if $C(t) := C(t_0)$ for all t satisfies (2.2.4).

Lemma 2.2.5. *The family $(B(t))_{t \in [0, \tau]}$ are averaged admissible controls for the system (2.2.1) if and only if the family $(C(t))_{t \in [0, \tau]} = (B(t)^*)_{t \in [0, \tau]}$ are averaged admissible observations for the retrograde equation (2.2.3).*

Proof. The following calculation is standard.

$$\begin{aligned} \sup_{\|u\|_2 \leq 1} \left\| \int_s^\tau U(\tau, r)B(r)u(r) dr \right\| &= \sup_{\|u\|_2 \leq 1} \sup_{\|x^*\| \leq 1} \left| \int_s^\tau \langle U(\tau, r)B(r)u(r), x^* \rangle dr \right| \\ &= \sup_{\|x^*\| \leq 1} \sup_{\|u\|_2 \leq 1} \left| \int_s^\tau \langle u(r), B(r)^*U(\tau, r)^*x^* \rangle dr \right| \\ &= \sup_{\|x^*\| \leq 1} \left(\int_s^\tau \|B(r)^*U(\tau, r)^*x^*\|^2 dr \right)^{1/2}. \quad \square \end{aligned}$$

With the observation operators $C(t) = B(t)^*$, we get the duality of admissibility between the system (2.2.1) and the retrograde final-value system (2.2.3).

2.3 Duality of controllability and observability

The idea of controllability is the following: given two states $x_0 \in H$ and $x_1 \in H$ of the system (2.1.1), does there exist a function u (called control function) allowing to "pass" the state x_0 in a fixed time $\tau > 0$? There are many ways to interpret the term "pass". For example, it may mean that the value at time $t = \tau$ of the solution starting from the state x_0 at time $t = 0$ is exactly equal to x_1 , in which case we have the definition of exact controllability. On the contrary, it can also mean that the value of solution at time τ is sufficiently close to x_1 , without necessarily being equal to, in which case we have the notion of approximate controllability. The notions of controllability may also differ according to the form of the targets that one seeks to attain. For example, one can seek to reach only the zero state $x_1 = 0$, in which case we have the notion of null controllability (controllability at zero).

Definition 2.3.1. The system (2.1.1) is said to be exactly controllable at time $\tau > 0$ if for every (x_0, x_1) in H , there exist $u \in L_2(0, \tau; U)$ such that the solution satisfy $x_1 = x(\tau)$.

Definition 2.3.2. The system (2.1.1) is said to be approximately controllable at time $\tau > 0$ if for every (x_0, x_1) in H and for all $\varepsilon > 0$, there exist $u \in L_2(0, \tau; U)$ such that the solution satisfy $\|x_\tau - x_1\| < \varepsilon$.

Definition 2.3.3. The system (2.1.1) is said to be exactly null controllable at time τ if for every x_0 in H , there exist $u \in L_2(0, \tau; U)$ such that the solution satisfy $x(\tau) = 0$.

We associate with (2.1.1) the operator:

$$(\Phi_\tau x)(t) = \begin{cases} C(t)U(t, 0)x & t \in [0, \tau] \\ 0 & t > \tau. \end{cases}$$

Definition 2.3.4. The system (2.1.1) is said to be exactly observable in $[0, \tau]$ if the map $O : H \rightarrow L_2(0, \tau; U): x_0 \rightarrow y(\cdot)$ is injective, i.e: the following inequality is true for all solutions x :

$$C\|x_0\|_H \leq \|y\|_{L_2(0, \tau; U)}$$

where $C > 0$ be a constant.

Definition 2.3.5. The system (2.1.1) is said to be exactly observable at time τ if and only if there exist the constant $C > 0$ such that the following inequality would be true for all solutions x :

$$C\|x(\tau)\|_H \leq \|y\|_{L_2(0, \tau; U)}.$$

Definition 2.3.6. The system (2.1.1) is said to be approximate observable at time τ if $\ker \Phi_\tau = \{0\}$.

2.3.1 The autonomous case

Finite dimensional systems

In the finite dimensional autonomous setting where $A(t) = A \in \mathcal{L}(\mathbb{C}^n)$ and $B(t) = B \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$ are constant matrices, we consider the variable spaces $H = H_{-1} = \mathbb{C}^n$ and the control space $U = \mathbb{R}^m$ where $m, n \in \mathbb{N}$ and $m, n \geq 1$. We consider the system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0 \in \mathbb{C}^n \end{cases} \quad (2.3.1)$$

where $u \in L^2(0, \tau; \mathbb{C}^m)$ is the control. We also consider the dual system:

$$\begin{cases} \dot{z}(t) = -A^T z(t) \\ z(\tau) = z_0 \in \mathbb{C}^n \\ y(t) = B^T z(t). \end{cases} \quad (2.3.2)$$

Definition 2.3.7. [103, Chapter 4]

Let $x_0 \in \mathbb{C}^n$ and $Bu \in L_1(0, \tau; \mathbb{C}^n)$. The function $x \in C(0, \tau; \mathbb{C}^n)$ given by

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)ds, \quad \forall t \in [0, \tau],$$

is the mild solution of the initial value problem (2.3.1) on $(0, \tau)$.

We define the set of reachable states:

$$R(\tau, x_0) = \{x(\tau) \in \mathbb{R}^n : x \text{ solution of (2.3.1) with } u \in (L_2(0, \tau))^m\}.$$

Since e^{tA} is a group, then the exact controllability property is equivalent to the fact that $R(\tau, 0) = \mathbb{R}^n$ for any $x_0 \in \mathbb{R}^n$. The controllability Gramian is a $n \times n$ matrix and defined as:

$$W_c(\tau) = \int_0^\tau e^{(t-s)A} B B^T e^{(t-s)A^T} ds. \quad (2.3.3)$$

Theorem 2.3.8. *The linear time varying control system $x'(t) = Ax + Bu$ is controllable if and only if its controllability Gramian is invertible.*

The simplest and best known criterion for controllability is called "Kalman rank condition" [59] that was proposed by Kalman, Ho and Narendra in 1963. In this paper, they defined the controllability matrix as

$$[A|B] = (B, AB, A^2B, \dots, A^{n-1}B) \in \mathcal{L}(\mathbb{C}^{nm}; \mathbb{C}^n).$$

Theorem 2.3.9. [21, Chapter 1, pages 10-11] *The time invariant linear control system $x' = Ax + Bu$ is exactly controllable at time $\tau > 0$ if and only if $\text{rank } [A|B] = n$.*

Remark 2.3.10. Noting that the Kalman rank condition is indepent of τ , so it implies that the system (2.3.1) is exact controllable in time τ if and only if it is exact controllable at any time τ .

Theorem 2.3.11. *The following statement are equivalent:*

- i) *The system (2.3.1) is controllable*
- ii) *The controllability Gramian $W_c(t)$ is non-singular for all $t \geq 0$*
- iii) *The controllability matrix $[A|B] = (B, AB, A^2B, \dots, A^{n-1}B)$ has rank n*
- iv) *The $n \times (n + m)$ matrix $[A - \lambda I, B]$ has the full row rank at every eigenvalue λ of A*
- v) *If all eigenvalues of A have negative real parts, then the unique solution of*

$$AW_c + W_c A^T = -BB^T$$

is definite positive. The solution can be expressed as

$$W_c = \int_0^{+\infty} e^{\tau A} B B^T e^{\tau A^T} ds.$$

- vi) *The system (2.3.2) is observable*

From the above theorem, we concule that the pair (A, C) is observable if and only if

$$\text{Rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \quad \forall \lambda \in \sigma(A).$$

It follows also that (A, C) is observable implies $Cz \neq 0$ for every eigenvector z of A , and there exists $k > 0$ such that for every $s \in \mathbb{C}$:

$$\|(sI - A)z\|^2 + \|Cz\|^2 > k^2 \|z\|^2.$$

Infinite dimensional systems

In the infinite dimensional linear system, the Kalman's rank condition, in general, does not a useful tool to characterize the controllability. As a result, it is reasonable to find an alternative method that convert the control problem in to the problem of observability for the dual system. The duality between the controllability of a system and the observability of its adjoint system can be described in an abstract form due to Dolecki and Russell [27]. Considering the system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0 \in H \end{cases} \quad (2.3.4)$$

where the operator A is closed in H , densely defined (i.e. $\overline{\mathcal{D}(A)} = H$), and A generate a strongly continuous semigroup $\mathbb{T}(t) = e^{tA}$. Then the solution of (2.3.4) can be represented by the variation of constant formula:

$$x(\tau) = e^{\tau A}x_0 + \int_0^\tau e^{(\tau-s)A}Bu(s)ds. \quad (2.3.5)$$

Put $z(t) = e^{A^*(\tau-t)}z_0$, so z is the solution of the dual system

$$\begin{cases} z'(t) &= -A^*z, \quad t \in (0, \tau) \\ z(\tau) &= z_0. \end{cases} \quad (2.3.6)$$

Here the operator $-A^*$ generates $\mathbb{T}(t)^*$ on reflexive space (see [103, chapter I.10]), and so $e^{(\tau-t)A^*}$ is well defined on $[0, \tau]$ and satisfies (2.3.6). Let take $y \in L_2(0, \tau; U)$, $y(t) = B^*e^{(\tau-t)A^*}z_0$ as an observation output. We recall the following important lemma:

Lemma 2.3.12. (see [123, Proposition 12.1.2])

Suppose that Z_1, Z_2, Z_3 are Hilbert spaces, the operators $F \in \mathcal{L}(Z_1, Z_3)$ and $G \in \mathcal{L}(Z_2, Z_3)$. Then the following statements are equivalent:

- a) $\text{Ran}(F) \subset \text{Ran}(G)$
- b) There exists a constant $c > 0$ such that $\|F^*z\|_{Z_1} \leq c\|G^*z\|_{Z_2}$ for all $z \in Z_3$
- c) There exist an operator $U \in \mathcal{L}(Z_1, Z_2)$ such that $F = GU$.

By mean of classical range comparison, we conclude that the system (2.3.4) is exact controllable at some time τ if and only if there exist constant $\delta > 0$ such that:

$$\int_0^\tau \|B^*e^{A^*(\tau-t)}z_0\|^2 dt \geq \delta\|z_0\|^2 \quad \forall z_0 \in H. \quad (2.3.7)$$

Theorem 2.3.13. (Duality)[123, chapter 11]

1. The exact controllability of system (2.3.4) is equivalent to the observability of system (2.3.6).

2. Duality results

2. The null controllability of system (2.3.4) is equivalent to the observability at time 0 of system (2.3.6).
3. The approximate controllability of system (2.3.4) in H is equivalent to unique continuation property for system (2.3.6).

Remember that the controllability (or observability) time for the finite dimensional case occur at any time $\tau > 0$. On the contrary, there are some new phenomena in the infinite dimensional case. For example, in the case of hyperbolic equation such as the wave equation, the controllability (or observability) time when acting on the boundary or internal open sets need to be large enough because of finite velocity of propagation.

Examples

The observability inequality for the wave equation [123, chapter 7]. We consider the following uncontrolled problem

$$\begin{cases} \frac{\partial^2}{\partial t^2} \eta(x, t) = \Delta \eta(x, t) & (x, t) \in \Omega \times (0, \tau) \\ \eta(x, t) = 0 & (x, t) \in \Gamma \times (0, \tau) \\ \eta(x, 0) = f(x), \eta_t(x, 0) = g(x) & x \in \Omega \end{cases} \quad (2.3.8)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2 boundary $\partial\Omega$, and Γ be a subset of $\partial\Omega$. Now we consider the space $H = L_2(\Omega)$. The operator $A : H \rightarrow H$ defined by $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $A = \Delta$ is strictly positive. Let us set $H_1 = H^2(\Omega) \cap H_0^1(\Omega)$, $X = H_0^1(\Omega) \times L_2(\Omega)$, the operator $\mathbf{a} : \mathcal{D}(\mathbf{a}) \rightarrow X$ with $\mathcal{D}(\mathbf{a}) = H_1 \times H_0^1(\Omega)$

$$\mathbf{a} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \quad (2.3.9)$$

Since \mathbf{a} be a skew adjoint operator, so by Stone's theorem \mathbf{a} generates an unitary group \mathbb{T} . This semigroup can be restricted to an operator semigroup on $X_1 = H_0^1(\Omega) \times L_2(\Omega)$. The generator of this restriction is $\mathbf{a}|_{\mathcal{D}(\mathbf{a}^2)}$ where $\mathcal{D}(\mathbf{a}^2) = H_1 \times H_0^1(\Omega)$. Let denote the output space $Y = L_2(\Gamma)$, we consider the observation operator $C \in \mathcal{L}(H_1 \times H_0^1(\Omega), Y)$ defined by

$$C \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \frac{\partial \varphi}{\partial \nu} \Big|_{\Gamma} \quad \forall \varphi \in H_1 \times H_0^1(\Omega)$$

where ν is the unit outward normal vector field on $\partial\Omega$. As shown in [123, Theorem 7.1.3], we have that C is an admissible operator, i.e: there exist the constant $C_\tau > 0$ such that for all $f \in H^2(\Omega) \cap H_0^1(\Omega)$ and $g \in H_0^1(\Omega)$ we have:

$$\int_0^\tau \int_{\partial\Omega} \left| \frac{\partial \eta}{\partial \nu} \right|^2 d\sigma dt \leq C_\tau (\|\nabla f\|^2 + \|g\|^2) \quad (2.3.10)$$

where $d\sigma$ be the surface measure on $\partial\Omega$. Let us assume that there exist $x_0 \in \mathbb{R}^n$ such that

$$\{x \in \partial\Omega | (x - x_0) \cdot \nu(x) > 0\} \subset \Gamma$$

and denote $r(x_0) = \sup_{x \in \Omega} |x - x_0|$. We have

Theorem 2.3.14. (see [123, Theorem pp 233]) For every $f \in H^2(\Omega) \cap H_0^1(\Omega)$ and $g \in H_0^1(\Omega)$, the solution of (2.3.8) satisfies:

$$\int_0^\tau \int_\Gamma \left| \frac{\partial \eta}{\partial \nu} \right|^2 d\sigma dt \geq \frac{\tau - 2r(x_0)}{r(x_0)} (\|\nabla f\|^2 + \|g\|^2) \quad (2.3.11)$$

The system (2.3.8) is exactly observable in any time $\tau > 2r(x_0)$.

By duality theory, the following initial and boudary value problem

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(x, t) = \Delta u(x, t) & (x, t) \in \Omega \times (0, \tau) \\ u(x, t) = v(x, t) & (x, t) \in \Gamma \times (0, \tau) \\ u(x, t) = 0 & (x, t) \in (\partial\Omega/\Gamma) \times (0, \tau) \\ u(x, 0) = f(x), u_t(x, 0) = g(x) & x \in \Omega \end{cases} \quad (2.3.12)$$

has exactly controllable for any time $\tau > 2r(x_0)$ with the initial data $f \in L_2(\Omega)$ and $g \in H^{-1}(\Omega)$.

The Hautus test

The Hautus test is a powerful tool in control theory for checking the observability of various linear systems, for example: approximate observability of exponentially stable systems [49], polynomially stable system [49], exact observability of strongly stable Riesz-spectral systems with finite dimensional output spaces [50], exponentially stable C_0 -groups [52], etc.. Russell and Weiss [111] showed that a necessary condition for exact observability is the following Hautus test :

Theorem 2.3.15. If the system (A, C) is exactly observable and C is admissible observation operator in $(0, \tau)$, then there exists the positive constants m and M such that for every $\lambda \in \mathbb{C}_+$ and every $x \in \mathcal{D}(A)$:

$$M \|(\lambda I + A)x\|^2 + |\operatorname{Re} \lambda| \|Cx\|^2 \geq m |\operatorname{Re} \lambda|^2 \|x\|^2$$

where \mathbb{C}_+ denotes the open right half plane.

Proof. Observe that

$$d/ds \left(e^{-\lambda s} \mathbb{T}(t-s)x \right) = -\lambda e^{-\lambda s} \mathbb{T}(t-s)x - e^{-\lambda s} A \mathbb{T}(t-s)x$$

and so, integrating on $[0, t]$,

$$e^{-\lambda t} x - \mathbb{T}(t)x = - \int_0^t \mathbb{T}(t-s)(\lambda + A)x e^{-\lambda s} ds$$

We apply C on both sides and get

$$C \mathbb{T}(t)x = e^{-\lambda t} Cx + \int_0^t C \mathbb{T}(t-s)(\lambda + A)x e^{-\lambda s} ds$$

2. Duality results

Suppose that C is exactly observable, i.e. $m(\tau)\|x\| \leq \|C\mathbb{T}(t)x\|_{L_2(0,\tau)}$. We distinguish two cases: for $\operatorname{Re}(\lambda) \neq 0$,

$$m(\tau)\|x\|^2 \leq \|Cx\|^2 \frac{1-e^{-2\tau\operatorname{Re}(\lambda)}}{\operatorname{Re}(\lambda)} + 2\left\|t \mapsto \int_0^t C\mathbb{T}(t-s)(\lambda+A)xe^{-\lambda s} ds\right\|_{L_2(0,\tau;H)}^2$$

whereas for $\lambda = i\xi$ and $\xi \in \mathbb{R}$ we have

$$m(\tau)\|x\|^2 \leq \frac{\|Cx\|^2}{\operatorname{Re}(\lambda)} + 2\left\|t \mapsto \int_0^t C\mathbb{T}(t-s)(\lambda+A)xe^{-\lambda s} ds\right\|_{L_2(0,\tau;H)}^2$$

The second term can be treated in both cases simultaneously: let $g \in L_2$ of norm one. Then using admissibility of C^* for $\mathbb{T}(t-s)^*$,

$$\begin{aligned} & \left\|t \mapsto \int_0^t C\mathbb{T}(t-s)(\lambda+A)xe^{-\lambda s} ds\right\|_{L_2(0,\tau;H)} \\ &= \sup_{\|g\| \leq 1} \left| \left\langle \int_0^t C\mathbb{T}(t-s)(\lambda+A)xe^{-\lambda s} ds, g \right\rangle_{L_2(0,\tau;H)} \right| \\ &= \sup_{\|g\| \leq 1} \left| \int_0^\tau \int_0^t \langle C\mathbb{T}(t-s)(\lambda+A)xe^{-\lambda s}, g(t) \rangle_H ds dt \right| \\ &= \sup_{\|g\| \leq 1} \left| \int_0^\tau \langle (\lambda+A)x e^{-\lambda s}, \int_s^\tau \mathbb{T}(t-s)^* C^* g(t) dt \rangle_H ds \right| \\ &\leq M(\tau) \sup_{\|g\| \leq 1} \int_0^\tau \|(\lambda+A)x e^{-\lambda s}\|_H \|g\|_{L_2(s,\tau)} ds \\ &\leq M(\tau) \int_0^\tau \|(\lambda+A)x\|_H e^{-\operatorname{Re}(\lambda)s} ds \\ &= M(\tau) \frac{1-e^{-\operatorname{Re}(\lambda)\tau}}{\operatorname{Re}(\lambda)} \|(\lambda+A)x\|_H \end{aligned}$$

we obtain the **Hautus condition** as a necessary condition for exact observability : for all $x \in \mathcal{D}(A)$, and all $\operatorname{Re}(\lambda) \neq 0$,

$$m(\tau)^2\|x\|^2 \leq \frac{1-e^{-2\tau\operatorname{Re}(\lambda)}}{\operatorname{Re}(\lambda)} \|Cx\|^2 + 2M^2 \frac{(1-e^{-\operatorname{Re}(\lambda)\tau})^2}{\operatorname{Re}(\lambda)^2} \|(\lambda+A)x\|_H^2 \quad (\text{Hautus})$$

□

Remark 2.3.16. In finite dimensional space, the Kalman condition is equivalent to the Hautus test.

2.3.2 The non autonomous case

Finite dimensional system

We consider the system (2.1.1) where A , B and C are the time dependent matrices with $A \in \mathbb{C}^{n-2}([0, \tau], \mathcal{L}(\mathbb{R}^n))$, $B \in \mathbb{C}^{n-1}([0, \tau], \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$, $C \in \mathbb{C}^{n-1}([0, \tau], \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$

and $u \in L_2(0, \tau; \mathbb{R}^n)$ be the control. From [115], we know that the solution of (2.1.1) can be represented as

$$x(t) = X(t)X^{-1}(0)x_0 + \int_0^t X(t)X^{-1}(s)B(s)u(s)ds, \quad (2.3.13)$$

where $X(t)$ is the fundamental matrix of the homegeneous problem

$$\dot{X}(t) - A(t)X(t) = 0 \quad (2.3.14)$$

The output of system is then given by

$$y(t) = C(t)X(t)(X^{-1}(0)x_0 + \int_0^t X^{-1}(s)B(s)u(s)ds)$$

We define the sequence of matrices $\{B_i(t)\}_{1 \leq i \leq n}$ and $\{C_j(t)\}_{1 \leq j \leq n}$ as follow:

$$B_0(t) = B(t), \quad B_i(t) = -A(t)B_{i-1}(t) + \frac{d}{dt}B_{i-1}(t) \quad \forall i \in [1, n-1]$$

$$C_0(t) = C(t), \quad C_j(t) = A(t)C_{j-1}(t) + \frac{d}{dt}C_{j-1}(t) \quad \forall j \in [1, n-1]$$

Then we can define the controllability and observability matrices :

$$H(t) = (B_0(t), B_1(t), \dots, B_{n-1}(t))$$

$$Q(t) = (C_0(t), C_1(t), \dots, C_{n-1}(t))$$

Noting that $H(t)$ and $Q(t) \in C^0([0, \tau]; \mathcal{L}(\mathbb{R}^{mn}; \mathbb{R}^n))$ and are types of Wronskian matrix.

Definition 2.3.17. The Wronskian matrix: Considering the set of n -dimensional row continuous vector functions $f_1(t), f_2(t), \dots, f_n(t)$ together with their first $n-1$ derivatives. We define the Wronskian matrix as

$$\text{Wronskian}(f_1(t), f_2(t), \dots, f_n(t)) = [F(t), F^{(1)}(t), \dots, F^{(n-1)}(t)]$$

where

$$F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

Due to [115], we take $\theta(t) = X^{-1}B(t)$ and $\Psi(t) = C(t)X(t)$ the Wronskian matrix is defined as

$$[\theta(t), \theta^{(1)}(t), \dots, \theta^{(n-1)}(t)] = X^{-1}(t)H(t)$$

$$[\Psi(t), \Psi^{(1)}(t), \dots, \Psi^{(n-1)}(t)] = Q(t)X(t)$$

Since $X(t)$ is non-singular for all t , the ranks of $H(t)$ and $Q(t)$ are equal to the rank of Wronskian matrix.

Due to Silverman and Meadows, we have the following theorem:

Theorem 2.3.18. [115]

- i) If there exist $t_0 \in [0, \tau]$ such that $\text{Rank } H(t_0) = n$, then the system (2.1.1) is exactly controllable on $(0, \tau)$, i.e: for every $x_0, x_d \in \mathbb{R}^n$, there exists a control $u \in L^2(0, \tau)^m$ such that the solution $x(t)$ of (2.1.1) corresponding to the initial condition $x(0) = x_0$ satisfies $x(\tau) = x_d$
- ii) The autonomous system (2.1.1) is total exactly controllable on $(0, \tau)$ if and only if there exist a dense subset E of $(0, \tau)$ such that: $\text{Rank } H(t) = n$ for all $t \in E$.

Theorem 2.3.19. [115]

- i) If there exist $t_0 \in [0, \tau]$ such that $\text{Rank } Q(t_0) = n$, then the system (2.1.1) is exactly observable on $(0, \tau)$.
- ii) The autonomous system (2.1.1) is totally exactly observable on $(0, \tau)$ if and only if there exist a dense subset E of $(0, \tau)$ such that: $\text{Rank } Q(t) = n$ for all $t \in E$.

Remark 2.3.20. When A and B are the constant matrices, we have the Kalman matrix $H = [A|B] = (B, AB, \dots, AB^{n-1})$ with $\text{Rank } H = n$. The definition of complete and total controllability are the same, so the exact controllability of system (2.1.1) is equivalent to Kalman's rank condition.

Example 2.3.21. We consider the case $n = 3$ and $m = 1$ and:

$$A(t) = \begin{bmatrix} t & 1 & 0 \\ 1 & t^3 & 0 \\ 0 & 0 & t^2 \end{bmatrix} \quad B(t) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

We can compute that:

$$B_0(t) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad B_1(t) = \begin{bmatrix} t+1 \\ t^3+1 \\ 2t^2 \end{bmatrix}, \quad B_2(t) = \begin{bmatrix} t^3+t^2+t \\ t^6+t^3-3t^2+t+1 \\ 4t^4-4t \end{bmatrix}$$

The controllability matrix:

$$H(0) = (B_0(0), B_1(0), B_2(0)) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix}$$

has rank 3. So, the system is controllable on every interval $[0, \tau]$ due to the theorem 2.3.18.

Example 2.3.22. Considering the case $n = 2$ and $m = 1$ and

$$A(t) = A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad B(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

By calculation, we see that $B_0(t) = B(t)$ and

$$B_1(t) = AB_0(t) - \frac{d}{dt}B_0(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} - \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, the controllability matrix

$$H(t) = (B_0(t), B_1(t)) = \begin{bmatrix} \cos t & 0 \\ -\sin t & 0 \end{bmatrix}$$

has the rank less than 2. Then by theorem 2.3.18, the system (2.1.1) is not controllable for any interval $(0, \tau)$. However, for fix $\alpha \in \mathbb{R}$, the matrix:

$$(B(\alpha), AB(\alpha)) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ -\sin \alpha & -\cos \alpha \end{bmatrix}$$

has the rank 2. Hence, the frozen system $(A, B(\alpha))$ is controllable.

Example 2.3.23. For $t \in (0, 1)$, we consider the matrices:

$$A(t) = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} \quad B(t) = \begin{bmatrix} e^t \\ e^{t^2/2} \end{bmatrix}$$

By calculation, we see that $B_0(t) = B(t)$ and

$$B_1(t) = A(t)B_0(t) - \frac{d}{dt}B_0(t) = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} e^t \\ e^{t^2/2} \end{bmatrix} - \begin{bmatrix} e^t \\ te^{t^2/2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, the controllability matrix

$$H(t) = (B_0(t), B_1(t)) = \begin{bmatrix} e^t & 0 \\ e^{t^2/2} & 0 \end{bmatrix}$$

has the rank less than 2. Then by theorem 2.3.18, the system (2.1.1) is not controllable on interval $(0, 1)$. However, for fix $\beta \in (0, 1)$, the matrix:

$$(B(\beta), A(\beta)B(\beta)) = \begin{bmatrix} e^\beta & e^\beta \\ e^{\beta^2/2} & \beta e^{\beta^2/2} \end{bmatrix}$$

has the rank 2 since its determinant is different from 0. Hence, the frozen system $(A(\beta), B(\beta))$ is controllable.

Remark 2.3.24. In the autonomous case when A and B are constant matrices, Rank $H = n$ if and only if Rank $Q = n$. It implies the duality between controllability and observability in the autonomous case. In the non-autonomous case, we do not sure that $H(t)$ and $Q(t)$ have the same ranks.

Example 2.3.25. (Final time observability for a system of couple integro-differential equations). We consider a system as following

$$\begin{cases} \dot{x}(t) = a(t)x(t) \\ \dot{y}(t) = b(t)y(t) \end{cases} \quad (2.3.15)$$

where a and $b : [0, \tau] \rightarrow \mathbb{R}$ and the initial condition provide $(x(t), y(t))|_{t=0} = (x(0), y(0))$. We rewrite the system as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} a(t) & 0 \\ 0 & b(t) \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

2. Duality results

We define the observability map $C(t) = C = (a_1 a_2)$ where $a_1, a_2 \in \mathbb{R}$ then $C \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = a_1 x(t) + a_2 y(t)$.

We will check the condition for exact observability. In fact, the Wroskian matrix is:

$$Q(t) = (C_0(t), C_1(t)) = \begin{bmatrix} a_1 & a_2 \\ a_1 a(t) & a_2 b(t) \end{bmatrix}$$

Since $a(0) \neq b(0)$, it is easy to see that $\text{Rank}(Q(0)) = 2$ then by theorem 2.3.19, the system (2.3.15) is exactly observable in $[0, \tau]$. The equation $\dot{x}(t) = a(t)x(t)$ gives the solution $x(t) = e^{-\int_0^t a(s)ds} x_0$, and the equation $\dot{y}(t) = b(t)y(t)$ gives the solution $y(t) = e^{-\int_0^t b(s)ds} y_0$. For the condition for the final observability in the time τ of the system (2.3.15), we will prove the following inequality:

$$\int_0^\tau \left\| C \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \right\|^2 dt \geq M_\tau \left\| \begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} \right\|^2$$

In fact, it is equivalent to

$$\begin{aligned} & \int_0^\tau a_1^2 e^{-2A(t)} x_0^2 dt + \int_0^\tau a_2^2 e^{-2B(t)} y_0^2 dt + \int_0^\tau 2a_1 a_2 e^{-A(t)-B(t)} dt \\ & \geq M_\tau (e^{-2A(\tau)} |x_0|^2 + e^{-2B(\tau)} |y_0|^2) \end{aligned}$$

where $A(t) = \int_0^t a(s) ds$ and $B(t) = \int_0^t b(s) ds$. Now we take the functions Ψ_1 and Ψ_2 as:

$$\Psi_1(x_0, y_0) = \int_0^\tau a_1^2 e^{-2A(t)} x_0^2 dt + \int_0^\tau a_2^2 e^{-2B(t)} y_0^2 dt + \int_0^\tau 2a_1 a_2 e^{-A(t)-B(t)} dt$$

and

$$\Psi_2(x_0, y_0) = e^{-2A(\tau)} |x_0|^2 + e^{-2B(\tau)} |y_0|^2$$

The condition for $\Psi_1(x_0, y_0) \geq M_\tau \Psi_2(x_0, y_0)$ holds is that there exist $\lambda > 0$ satisfying the following system of equations

$$\begin{cases} \frac{\partial}{\partial x_0} \Psi_1 = \lambda \frac{\partial}{\partial x_0} \Psi_2 \\ \frac{\partial}{\partial y_0} \Psi_1 = \lambda \frac{\partial}{\partial y_0} \Psi_2 \end{cases} \quad (2.3.16)$$

It is equivalent to

$$\begin{cases} a_1^2 x_0 \int_0^\tau e^{-2A(t)} dt + a_1 a_2 y_0 \int_0^\tau e^{-A(t)-B(t)} dt = \lambda e^{-2A(\tau)} x_0 \\ a_2^2 y_0 \int_0^\tau e^{-2B(t)} dt + a_1 a_2 x_0 \int_0^\tau e^{-A(t)-B(t)} dt = \lambda e^{-2B(\tau)} y_0 \end{cases} \quad (2.3.17)$$

That means λ be the positive eigenvalue respected to the eigenvale $(x_0, y_0)^T$ of the matrix

$$G = \begin{bmatrix} a_1^2 \int_0^\tau e^{2A(\tau)-2A(t)} dt & a_1 a_2 \int_0^\tau e^{2A(\tau)-A(t)-B(t)} dt \\ a_1 a_2 \int_0^\tau e^{2B(\tau)-A(t)-B(t)} dt & a_2^2 \int_0^\tau e^{2B(\tau)-2B(t)} dt \end{bmatrix}$$

Noting that $\text{Trace}(G) > 0$, then it should has $\det(G) > 0$. Consequently, we need to have:

$$\left(\int_0^\tau e^{-2A(t)} dt \right) \left(\int_0^\tau e^{-2B(t)} dt \right) \geq \left(\int_0^\tau e^{-A(t)-B(t)} dt \right)^2$$

This is the form of Cauchy-Schwarz inequality. Hence, we always get the final time observability for all τ .

Remark 2.3.26. A generalization of the Kalman rank condition was proposed by F. Khodija, A. Benabdallah, C. Dupaix and M. Burgos (see [64]). In this paper, the authors consider a system of $n \geq 2$ coupled parabolic equations (posed in time interval $(0, \tau)$ with $\tau > 0$) where the coupling matrices A and B depend on t .

$$\begin{cases} \dot{x}(t) + L(t)x = Ax(t) + Bu(t)\chi_\omega & \text{in } Q = \Omega \times (0, \tau) \\ x = 0 & \text{on } \Sigma = \partial\Omega \times (0, \tau) \end{cases} \quad (2.3.18)$$

They proved a sufficient condition (not necessary) that the consider parabolic system is exactly controllable if Kalman rank condition provides $[A|B](t_0) = n$ with $t_0 \in (0, \tau)$.

Infinite dimensional system

Exact controllability and exact observability

We assume that $\{B(t)\}_{0 \leq t \leq \tau}$ is an admissible family of control operators for the evolution equation. The operator $\Psi_\tau : L_2(0, \tau; U) \rightarrow H$ that continuously map $u \mapsto x(\tau)$ and defined by

$$\Psi_\tau u = \int_0^\tau U(\tau, s)B(s)u(s) ds$$

Now for $x \in H$ and $y \in L_2(0, \tau; U)$, by computation:

$$\begin{aligned} \langle x, \Psi_\tau y \rangle_H &= \left\langle x, \int_0^\tau U(\tau, s)B(s)y(s) ds \right\rangle_H = \int_0^\tau \langle x, U(\tau, s)B(s)y(s) \rangle_H ds \\ &= \int_0^\tau \langle B^*(s)U(\tau, s)^*x, y \rangle_U ds = \left\langle \int_0^\tau B^*(s)U(\tau, s)^*x ds, y \right\rangle_U \\ &= \langle \Psi_\tau^*x, y \rangle_{L_2(0, \tau; U)} \end{aligned}$$

This shows that $\Psi_\tau^*x = B^*(s)U(\tau, s)^*x$.

Theorem 2.3.27. *The system (2.2.1) is exactly controllable if and only if the system (2.2.3) is exactly observable.*

Proof. (i) We assume the exact controllability of the system (2.2.1), i.e. that for any $x_\tau \in X$, we can find some $u \in L_2(0, \tau; U)$ such that the solution of the initial value problem (2.2.1) satisfies $x(\tau) = x_\tau$. Then Ψ_τ is bounded and surjective. According to the lemma 2.3.12, Ψ_τ is surjective if and only if Ψ_τ^* allows lower estimates, i.e. there exist $\delta > 0$ such that

$$\delta \|x^*\| \leq \|B(t)^*U(\tau, t)^*x^*\|_{L_2(0, \tau)} \quad \forall x^* \in H.$$

We can therefore simply let $x^* = z_\tau$ for the dual system (2.2.3) and obtain the observability inequality:

$$\delta \|z_\tau\|_H^2 \leq \int_0^\tau \|B(t)^*U(\tau, t)^*z_\tau\|_H^2 dt. \quad (2.3.19)$$

(ii) For the converse direction we assume (2.3.19), i.e. "exact observability" of the dual system (2.2.3). We aim to obtain surjectivity of Ψ_τ . We define the controllability Gramian as

$$W_\tau = \Psi_\tau \Psi_\tau^* = \int_0^\tau U(\tau, s) B(s) B(s)^* U(\tau, s)^* ds.$$

We have

$$\|z_\tau\|_H \leq C \left(\int_0^\tau \|B(s)^* U(\tau, s)^* z_\tau\|_H^2 ds \right).$$

Or it is equivalent to

$$\|z_\tau\|_H^2 \leq C^2 \|\Psi_\tau^* z_\tau\|^2 = C^2 \langle \Psi_\tau^* z_\tau, \Psi_\tau^* z_\tau \rangle = C^2 \langle \Psi_\tau \Psi_\tau^* z_\tau, z_\tau \rangle = C^2 \langle W_\tau z_\tau, z_\tau \rangle.$$

Hence, we conclude that W_τ is self-adjoint, injective and coercive operator. Then W_τ is boundedly invertible. Hence, $\text{Im}(W_\tau) = \mathcal{D}(W_\tau^{-1}) = H$. This implies $\text{Im}(\Psi_\tau) = H$ since $H = \text{Im}(W_\tau) \subset \text{Im}(\Psi_\tau)$. This indicates the controllability of the initial system. \square

The generlization of Riccati's equation:

If we take $G(t, s) = U(t, s) B(s) B(s)^* U(t, s)^*$ be the function of two variables s, t with $0 \leq s \leq t$, then

$$\begin{aligned} \frac{d}{dt} G(t, s) &= \left(\frac{d}{dt} U(t, s) \right) B(s) B(s)^* U(t, s)^* + U(t, s) B(s) B(s)^* \left(\frac{d}{dt} U(t, s)^* \right) \\ &= A(t) G(t, s) + G(t, s) A(t)^*. \end{aligned}$$

Intergrating from 0 to t with respected to the variable s , we have:

$$\int_0^t \left(\frac{d}{dt} G(t, s) \right) ds = \int_0^t (A(t) G(t, s) + G(t, s) A(t)^*) ds = A(t) W(t) + W(t) A(t)^*.$$

The left hand side can be written as: $\frac{d}{dt} \left(\int_0^t G(t, s) ds \right) - G(t, t)$. As a result, the controllability $W(t)$ is the solution of the generlized Riccati's equation

$$\frac{d}{dt} W(t) = A(t) W(t) + W(t) A(t)^* + B(t) B(t)^*. \quad (2.3.20)$$

The minimum cost controls:

We take the specific control \tilde{u} as:

$$\tilde{u} = B(s)^* U(\tau, s)^* W_\tau^{-1} (x_\tau - U(\tau, 0) x_0) \quad (2.3.21)$$

where W_τ be the controllability Gramian. It is easy to check that this control satisfies the equation:

$$x_\tau = U(\tau, 0) x_0 + \int_0^\tau U(\tau, s) B(s) \tilde{u}(s) ds$$

so that the initial value problem (2.2.1) is exact controllable by \tilde{u} .

Proposition 2.3.28. *The function \tilde{u} minimize the $L_2(0, \tau; U)$ -norm among all the controls that steer (2.2.1) to the final value x_τ .*

Proof. Supposing that we have both controls u and \tilde{u} and satisfy the equations:

$$\begin{cases} x_\tau = U(\tau, 0)x_0 + \int_0^\tau U(\tau, s)B(s)\tilde{u}(s) ds \\ x_\tau = U(\tau, 0)x_0 + \int_0^\tau U(\tau, s)B(s)u(s) ds \end{cases} \quad (2.3.22)$$

Subtracting both sides of (2.3.2), we get:

$$\int_0^\tau U(\tau, s)B(s)(u(s) - \tilde{u}(s)) ds = 0.$$

Therefore, for all elements $\eta \in H$:

$$\left\langle \int_0^\tau U(\tau, s)B(s)(u(s) - \tilde{u}(s)) ds, \eta \right\rangle = 0$$

By choosing $\eta = W_\tau^{-1}(x_\tau - U(\tau, 0)x_0)$, and using (2.3.21), we have:

$$\begin{aligned} 0 &= \left\langle \int_0^\tau U(\tau, s)B(s)(u(s) - \tilde{u}(s)) ds, W_\tau^{-1}(x_\tau - U(\tau, 0)x_0) \right\rangle \\ &= \int_0^\tau \left\langle U(\tau, s)B(s)(u(s) - \tilde{u}(s)), W_\tau^{-1}(x_\tau - U(\tau, 0)x_0) \right\rangle ds \\ &= \int_0^\tau \left\langle (u(s) - \tilde{u}(s)), B(s)^*U(\tau, s)^*W_\tau^{-1}(x_\tau - U(\tau, 0)x_0) \right\rangle ds \\ &= \int_0^\tau \left\langle u(s) - \tilde{u}(s), \tilde{u}(s) \right\rangle ds \end{aligned}$$

By using orthogonality, we have

$$\|u\|^2 = \|u - \tilde{u} + \tilde{u}\|^2 = \|u - \tilde{u}\|^2 + \|\tilde{u}\|^2 \geq \|\tilde{u}\|^2.$$

Futhurmore, the minimum norm can be computed explicitly:

$$\begin{aligned} \|\tilde{u}\|_{L^2}^2 &= \int_0^\tau \langle \tilde{u}, \tilde{u} \rangle ds \\ &= \int_0^\tau \left\langle B(s)^*U(\tau, s)^*W_\tau^{-1}(x_\tau - U(\tau, 0)x_0), B(s)^*U(\tau, s)^*W_\tau^{-1}(x_\tau - U(\tau, 0)x_0) \right\rangle ds \\ &= \int_0^\tau \left\langle U(\tau, s)B(s)B(s)^*U(\tau, s)^*W_\tau^{-1}(x_\tau - U(\tau, 0)x_0), W_\tau^{-1}(x_\tau - U(\tau, 0)x_0) \right\rangle ds \\ &= \left\langle \int_0^\tau U(\tau, s)B(s)B(s)^*U(\tau, s)^* ds W_\tau^{-1}(x_\tau - U(\tau, 0)x_0), W_\tau^{-1}(x_\tau - U(\tau, 0)x_0) \right\rangle \\ &= \left\langle (x_\tau - U(\tau, 0)x_0), W_\tau^{-1}(x_\tau - U(\tau, 0)x_0) \right\rangle = W_\tau^{-1}\|(x_\tau - U(\tau, 0)x_0)\|^2. \end{aligned}$$

□

Null controllability and final time observability

We assume that the evolution family $\{U(t, s)\}_{s \leq t}$ is exponentially bounded, i.e: there exists constant $M \geq 0$ and $v > 0$ such that:

$$\|U(t, s)\| \leq M e^{-v(t-s)}. \quad (2.3.23)$$

We assume further that $\{B(t)\}$ is a family of admissible control operator. Supposing that the system (2.2.1) is exactly null controllable on $[0, \tau]$. By definition, for all $x_0 \in H$, we can find the control $u \in L_2(0, \tau; U)$ such that

$$0 = U(\tau, 0)x_0 + \int_0^\tau U(\tau, s)B(s)u(s) ds.$$

That implies the following:

$$\text{Ran}(U(\tau, s)) \subset \text{Ran}(u \rightarrow \int_0^\tau U(\tau, s)B(s)u(s) ds) = \text{Ran}(\Psi_\tau).$$

We define the operator $\Lambda : x_0 \rightarrow U(\tau, s)x_0$. Hence, Λ is a bounded operator from $H \rightarrow H$. Moreover, we have:

$$\text{Ran}(\Lambda) \subset \text{Ran}(\Psi_\tau). \quad (2.3.24)$$

Lemma 2.3.29. *Supposing that $B \in \mathcal{L}(U, H)$. The operator*

$$\Psi_\tau : u \rightarrow \int_0^\tau U(\tau, s)B(s)u(s) ds$$

is bounded linear map from $L^2([0, \tau]; U) \rightarrow H$.

Proof. We know that the two parameter evolution semigroup $U(t, s)$ is strongly continuous on $0 \leq s \leq t \leq \tau$, and for the input function $u \in L_2([0, T]; U)$ the function $\langle x, U(\tau, s)B(s)u(s) \rangle$ is measurable in s for every $x \in H$. We have:

$$\Psi_\tau u = \int_0^\tau U(\tau, s)B(s)u(s) ds \leq \int_0^\tau \|U(\tau, s)B(s)u(s)\|_H ds.$$

Using (2.3.23) and noting that the operator $B \in \mathcal{L}(U, H)$, we have:

$$\begin{aligned} \Psi_\tau u &\leq \int_0^\tau M e^{-v(\tau-s)} \|B(s)\|_{\mathcal{L}(U, H)} \|u(s)\|_H ds \\ &\leq M e^{-v\tau} \|B(t)\|_{\mathcal{L}(U, H)} \int_0^\tau e^{vs} \|u(s)\|_H ds \\ &\leq M e^{-v\tau} \|B(t)\|_{\mathcal{L}(U, H)} \left(\int_0^\tau e^{2vs} ds \int_0^\tau \|u(s)\|_H^2 ds \right)^{1/2} \\ &\leq M e^{-v\tau} \|B(t)\|_{\mathcal{L}(U, H)} \frac{1}{2} \left(\frac{e^{2v\tau} - 1}{v} \right)^{1/2} \|u(s)\|_{L^2([0, \tau]; U)} \end{aligned}$$

Therefore, $\Psi_\tau \in \mathcal{L}(L^2([0, \tau]; U), H)$. □

Using the property (2.3.24) and the lemma (2.3.12), there exists a constant $c > 0$ such that

$$\|\Lambda^*x\| \leq c\|\Psi_\tau^*x\|.$$

It is also easy to see $\Lambda^*x = U(\tau, s)^*x$. As a result, we obtain the inequality:

$$c\|B^*(s)U(\tau, s)^*x\| \geq \|U(\tau, s)^*x\|.$$

The key seems to be valid if and only if the control u yields a solution $x(\cdot)$ satisfying $x(\tau) = x_\tau$. Therefore, we can steer the system (2.2.1) via the control u to the desired endpoint if and only if

$$\int_0^\tau \langle u(s), B(s)^*z(s) \rangle ds \leq C^2 \int_0^\tau \|B(s)^*z(s)\|_H^2 ds$$

This, of course admits a trivial solution: just put $u(s) = B(s)^*z(s)$.

Compare the choice of the control to the homogeneous situation: If $A(t) = A$, $z(s) = T(\tau-s)^*z_\tau$ satisfies (2.2.3), and so $u(s) = B(s)^*T(\tau-s)^*z_\tau$, whence

$$x(\tau) = \left(\int_0^\tau T(\tau-s)B(s)B(s)^*T(\tau-s)^* ds \right) z_\tau.$$

Lemma 2.3.30. *Suppose that the family of bounded operator $\{B(t)\}$ is admissible for (2.2.1), and $\{U(t, s)\}_{s \leq t}$ is exponentially bounded evolution family. Then the null controllability for (2.2.1) intime τ is equivalent to final observability of $z(s)$, $0 \leq s < \tau$ where z is the solution of the retrograde final-value problem (2.2.3) with the observation operators $C(t) = B(t)^*$.*

Approximate controllability and approximate observability

It has been shown in [35] that under a suitable condition on the resolvent operator and the Schauder fixed point theorem, we have the approximate controllability of (2.1.1). We propose the following assumption:

$$\lambda R(\lambda, -W_\tau) \rightarrow 0 \quad \text{for } \lambda \rightarrow 0^+ \quad \text{in the strong operator topology.} \quad (2.3.25)$$

Theorem 2.3.31. [35, Theorem 1] *The following statements are equivalent:*

- i) *The system (2.1.1) is approximate controllable on $[0, \tau]$*
- ii) *If $B^*U(t, 0)y = 0$ for all $t \in [0, \tau]$ then $y = 0$*
- iii) *The condition (2.3.25) holds*

Example 2.3.32. [35] We consider the following non-autonomous heat equation

$$\begin{cases} \frac{\partial}{\partial t} z(x, t) = \frac{\partial^2}{\partial x^2} z(x, t) + a(t)z(x, t) + u(t) & (x, t) \in [0, \pi] \times [0, \tau] \\ z(0, t) = u(\pi, t) = 0 & \tau \geq t \geq 0 \\ z(x, 0) = z_0 & x \in [0, \pi]. \end{cases} \quad (2.3.26)$$

The family of operator $\{A(t)\}$ is defined as $A(t)f = -f'' - a(t)f$ with the domain $\mathcal{D}(A) = \{f(\cdot) \in X \mid f, f' \text{ are absolutely continuous, } f'' \in X, f(0) = f(\pi) = 0\}$. Then

we can check that $A(t)$ satisfies the assumptions from [103] and generates evolution family $\{U(t, s)\}$ of the form:

$$U(t, s) = e^{-(t-s)A} \exp\left(\int_s^t a(r) dr\right) \quad (2.3.27)$$

where e^{-tA} is a compact analytic semigroup generated by $(-A)$ where $Af = f''$ for any $f \in \mathcal{D}(A)$. Noting that A is diagonal operator with the set of eigenvalues $\{n^2\}_{n \in \mathbb{N}}$ and the normal eigenvector $z_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$. Then we have the representation:

$$A(t)f = \sum_{n=1}^{+\infty} (n^2 - a(t)) \langle f, z_n \rangle z_n \quad (2.3.28)$$

Moreover, for any $f \in X$ we have:

$$U(t, s)f = \sum_{n=1}^{+\infty} e^{-n^2(t-s) + \int_s^t a(r) dr} \langle f, z_n \rangle z_n. \quad (2.3.29)$$

Since $B = I$, we have:

$$W_\tau = \int_0^\tau U(t, s) B B^* U(t, s)^* ds = \int_0^\tau W(\tau, s) W^*(\tau, s) ds.$$

Indeed, if $U^*(\tau, 0)y = 0$ then

$$\sum_{n=1}^{+\infty} \int_0^\tau e^{-2n^2\tau - 2 \int_0^\tau a(r) dr} ds \langle y, z_n \rangle^2 = 0.$$

It implies immediately that $\langle y, z_n \rangle^2 = 0$ for every $n \geq 0$, so $y = 0$. From theorem 2.3.31, we conclude that $\lambda(\lambda I + W_\tau)^{-1} \rightarrow 0$ in the strong topology for $\lambda \rightarrow 0$. The system (2.3.26) is approximately controllable on the interval $[0, \tau]$.

Lemma 2.3.33. *Approximate controllability for (2.2.1) in time τ is equivalent to approximate observability of the retrograde final-value problem (2.2.3) with the observation operators $C(t) = B(t)^*$.*

Chapter 3

Exact observability of a 1D wave on a non-cylindrical domain

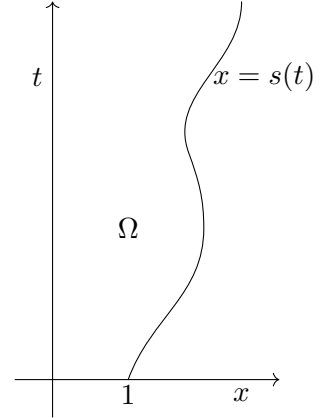
3.1 Introduction and main results

In this chapter we are concerned with exact observability of the 1D wave equation on a domain with time-dependent boundary. To be precise, let $s : \mathbb{R}_+ \rightarrow (0, \infty)$ and let

$$\Omega = \{(x, t) \in \mathbb{R}^2 : t \geq 0 \text{ and } 0 \leq x \leq s(t)\},$$

Where $s(0) = 1$ and $\|s'(t)\|_{L^\infty(\mathbb{R})} < 1$. The last condition ensures amongst other things that the characteristic emerging from the origin hits the boundary in finite time. Let $f \in L_2([0, 1])$ and $g \in H_0^1([0, 1])$ be initial values. We consider a wave equation on Ω with Dirichlet boundary conditions

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x, t) \in \Omega \\ u(0, t) = u(s(t), t) = 0 & t \geq 0 \\ u(x, 0) = g(x) & x \in [0, 1] \\ u_t(x, 0) = f(x) & x \in [0, 1] \end{cases} \quad (\text{W.Eq})$$



3.1.1 Existence of solutions

There are several natural approaches to (W.Eq). One may for example transform the domain Ω to a cylindrical domain. Instead, seeking a natural and more simple approach, we try to develop the solution u into a series of the form

$$u(x, t) := \sum_{n \in \mathbb{Z}} A_n \left(e^{2\pi i n \varphi(t+x)} - e^{2\pi i n \varphi(t-x)} \right) \quad (3.1.1)$$

where the coefficients A_n are given by the initial data (g, f) . This approach has almost a century of history, dating back to Nicolai [100] in the case of a linear

moving boundary $s(t) = 1 + \varepsilon t$ and Moore [92] for general boundary curves (however only asymptotic developments for φ are given). We refer to Donodov [25] for a large number of references. In order to satisfy the Dirichlet boundary condition, we need a solution φ to the functional equation

$$\varphi(t + s(t)) - \varphi(t - s(t)) = 1. \quad (3.1.2)$$

Because of the importance of this functional equation we fix the notation $\alpha(t) := t + s(t)$ and $\beta(t) := t - s(t)$ and mention that both are strictly increasing bijections from \mathbb{R}_+ to $[\pm s(0), \infty)$, respectively. We will also consider $\gamma = \alpha \circ \beta^{-1} : [-s(0), \infty) \rightarrow [s(0), \infty)$. Most solutions to (3.1.2) are useless for our purposes*. On the other hand side, under reasonable assumptions on the boundary function, differentiable solutions to (3.1.2) are unique, at least up to an additive constant. This is of course what we look for. In some easy cases a differentiable solution φ can be found by calculus, see the following table for some examples. We refer to a detailed discussion on the general situation in the last section 3.3.

Name	Boundary function	
linear moving boundary	$s(t) = 1 + \varepsilon t$	$\varepsilon \in (0, 1)$
parabolic boundary	$s(t) = \sqrt{1 + \varepsilon t}$	$\varepsilon \in (0, 2)$
hyperbolic boundary	$s(t) = \frac{1}{\varepsilon}(-1 + \sqrt{1 + (1 + \varepsilon t)^2})$	$\varepsilon > 0$
shrinking domain	$s(t) = \frac{1}{1 + \varepsilon t}$	$\varepsilon \in (0, 1)$.
Solution to (3.1.2)		

$$\begin{aligned} \varphi(t) &= \ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{-1} \ln(1+\varepsilon t) \\ \varphi(t) &= \frac{1}{2\varepsilon} \sqrt{\varepsilon^2 + 4\varepsilon t + 4} \\ \varphi(t) &= \frac{\varepsilon t}{1+\varepsilon t} \\ \varphi(t) &= \frac{\varepsilon}{4} \left(t + \frac{1}{\varepsilon}\right)^2. \end{aligned}$$

For simplicity of notation, we shall always assume $s(0) = 1$; in case of hyperbolic boundaries some straight-forward modifications have to be made. The common denominator of these examples is the following: $\varphi \in C^2([-1, \infty))$ and $\varphi'(t) > 0$ for all $t \geq -1$. We call s an **admissible boundary function** if (3.1.2) admits such a solution φ .

Proposition 3.1.1. *Let s be an admissible boundary function and assume the initial data $f, g \in \mathcal{D}((0, 1))$. Then (g, f) determine uniquely a sequence $(A_n)_{n \in \mathbb{Z}} \in \ell_2$ such that for $t \geq 0$ and $0 \leq x \leq s(t)$, the function (3.1.1) is the solution of the moving boundary wave equation (W.Eq).*

We start the proof with the following trivial observation.

Lemma 3.1.2. *For fixed $t_0 \geq 0$, the family $\{e^{2\pi i n \varphi(x)} : n \in \mathbb{Z}\}$, is a complete orthonormal system in $H := L_2([t_0 - s(t_0), t_0 + s(t_0)], \varphi'(x) dx)$.*

*It is indeed easy to construct solutions depending on an arbitrary function by using the axiom of choice

3. Exact observability of a 1D wave on a non-cylindrical domain

For $t_0=0$, we obtain as a particular case that the family (b_n) with $b_n(x) = e^{2\pi i n \varphi(x)}$ is an orthonormal basis in $H := L_2([-1, 1], \varphi'(x) dx)$. Since there is $C > 0$ such that $\frac{1}{C} \leq \varphi'(x) \leq C$ on $[0, 1]$, we have $L_2([-1, 1], \varphi'(x) dx) = L_2([-1, 1], dx)$ as sets with equivalent respective norms[†].

Proof of Proposition 3.1.1 . We let $F(x) = - \int_x^1 f(s) ds$ and

$$h(x) := \begin{cases} \frac{1}{2}g(x) + \frac{1}{2\varphi'(0)}F(x) & \text{for } 0 \leq x \leq 1 \\ -\frac{1}{2}g(-x) + \frac{1}{2\varphi'(0)}F(-x) & \text{for } -1 \leq x < 0 \end{cases}$$

By assumption, $h \in H$ that we develop into the orthonormal basis: $h = \sum_{\mathbb{Z}} \langle h, b_n \rangle b_n$. We shall always note

$$A_n = \langle h, b_n \rangle = \int_{-1}^1 h(x) e^{2\pi i n \varphi(x)} \varphi'(x) dx \quad (3.1.3)$$

Since $g(0)=g(1)=0$, we have $h(1)=h(-1)=0$. Hence the sequences (A_n) and $(n A_n)$ are square-summable. Taking sum and difference, we find $F(x) = \varphi'(0)(h(x) + h(-x))$ and $g(x) = h(x) - h(-x)$, so

$$F(x) = \varphi'(0) \sum_{n \in \mathbb{Z}} A_n \left(e^{2\pi i n \varphi(x)} + e^{2\pi i n \varphi(-x)} \right), \quad x \in [0, 1]$$

and

$$g(x) = \sum_{n \in \mathbb{Z}} A_n \left(e^{2\pi i n \varphi(x)} - e^{2\pi i n \varphi(-x)} \right), \quad x \in [0, 1].$$

Since we suppose $f, g \in \mathcal{D}((0, 1))$, h satisfies the periodicity condition $h^{(\alpha)}(-1)=h^{(\alpha)}(1)$ for all derivative orders $\alpha \geq 0$. As a consequence, the series of F , g and h above may be differentiated term by term. We let

$$u(x, t) := \sum_{n \in \mathbb{Z}} A_n \left(e^{2\pi i n \varphi(t+x)} - e^{2\pi i n \varphi(t-x)} \right)$$

Since $\varphi \in C^2([-1, \infty))$, u is twice differentiable and with respect to x and t . Moreover, partial derivatives can be calculated term by term. As an immediate consequence, $u_{xx} - u_{tt} = 0$ in the interior domain Ω° . Moreover, u satisfies the Dirichlet condition since for $x = 0$

$$u(0, t) = \sum_{n \in \mathbb{Z}} A_n \left(e^{2\pi i n \varphi(t)} - e^{2\pi i n \varphi(t)} \right) = 0$$

whereas for $x = s(t)$, thanks to the functional equation (3.1.2),

$$\begin{aligned} u(s(t), t) &= \sum_{n \in \mathbb{Z}} A_n \left(e^{2\pi i n \varphi(t+s(t))} - e^{2\pi i n \varphi(t-s(t))} \right) \\ &= \sum_{n \in \mathbb{Z}} A_n e^{2\pi i n \varphi(t+s(t))} (1 - e^{-2\pi i n}) = 0. \end{aligned}$$

Finally, $u(x, 0) = g(t)$ and $u_t(x, 0) = f(t)$ by direct calculation. □

[†]In particular, (b_n) is a Riesz basis in $L_2([-1, 1])$.

The series representation of the solution is the key to obtain explicit and precise constants for admissibility and exact observability in different situations, since they can be played back to classical Fourier analysis.

Let us fix some often appearing constants:

$$\begin{aligned} m(t) &= \min\{\varphi'(x) : x \in [t - s(t), t + s(t)]\} \quad \text{and} \\ M(t) &= \max\{\varphi'(x) : x \in [t - s(t), t + s(t)]\}. \end{aligned} \quad (3.1.4)$$

Since on $[0, 1]$, $m(0) \leq \varphi'(x) \leq M(0)$, we may use the unweighted Poincaré inequality on $[0, 1]$ to show that

$$\|(g, f)\|_{H_0^1([0,1]; \frac{dx}{\varphi'(x)}) \times L_2([0,1]; \frac{dx}{\varphi'(x)})}^2 := \|\nabla g\|_{L_2([0,1]; \frac{dx}{\varphi'(x)})}^2 + \|f\|_{L_2([0,1]; \frac{dx}{\varphi'(x)})}^2. \quad (3.1.5)$$

is an equivalent to $\|g\|_{L_2([0,1]; \frac{dx}{\varphi'(x)})}^2 + \|g'\|_{L_2([0,1]; \frac{dx}{\varphi'(x)})}^2 + \|f\|_{L_2([0,1]; \frac{dx}{\varphi'(x)})}^2$. The notation

$$\|(g, f)\|_{H_0^1 \times L_2}^2 := \|g'\|_{L_2(0,1)}^2 + \|f\|_{L_2(0,1)}^2$$

(without specifying intervals or weights) always refers to the unweighted norms on $[0, s(0)] = [0, 1]$.

Proposition 3.1.3. *We have the following estimate*

$$8\pi^2 m(0) \sum_{n \in \mathbb{Z}} n^2 |A_n|^2 \leq \|(g, f)\|_{H_0^1 \times L_2}^2 \leq 8\pi^2 M(0) \sum_{n \in \mathbb{Z}} n^2 |A_n|^2,$$

where the constants are given by (3.1.4).

Proof. Recall that $g(x) = h(x) - h(-x)$ and $F(x) = h(x) + h(-x)$ on $[0, 1]$. Therefore

$$\begin{aligned} \|(g, f)\|_{H_0^1 \times L_2}^2 &= \|g'\|_{L_2([0,1])}^2 + \|F'\|_{L_2([0,1])}^2 \\ &= \|h'(\cdot) + h'(-\cdot)\|_{L_2([0,1])}^2 + \|h'(\cdot) - h'(-\cdot)\|_{L_2([0,1])}^2 \\ &= 2\|h'\|_{L_2([0,1])}^2 + 2\|h'(-\cdot)\|_{L_2([0,1])}^2 = 2\|h'\|_{L_2([-1,1])}^2 \end{aligned}$$

by parallelogram identity. Estimating the maximum of φ' and $\frac{1}{\varphi'}$ on $[-1, 1]$ allows to relate $\|h'\|_{L_2([-1,1], \varphi'(x) dx)}^2$ and $\|h'\|_{L_2([-1,1])}^2$, and the result follows by Parseval's identity. \square

Observe that for the concrete examples we discuss later, the minimum respectively maximum is easy to calculate; we obtain therefore explicit constants in Proposition 3.1.3.

3.1.2 Energy estimates

Define the energy of the problem (W.Eq) as

$$E_u(t) = 1/2 \int_0^{s(t)} |u_x(x, t)|^2 + |u_t(x, t)|^2 dx.$$

for all $t \geq 0$. When $t = 0$, we see that $E_u(0) = 1/2 \|(g, f)\|_{H_0^1 \times L_2(0,1)}^2$. In the case of a 1D-wave equation with time-invariant boundary (i.e. $s \equiv 1$) the energy is constant. In time-dependent domains it decays when $s'(t) > 0$ and increases when $s'(t) < 0$.

Lemma 3.1.4. *The function $t \mapsto E_u(t)$ is decreasing for $t \geq 0$ if $s'(t) > 0$ and increasing when $s'(t) < 0$. More precisely,*

$$\frac{d}{dt} E_u(t) = \frac{s'(t)}{2} (s'(t)^2 - 1) |u_x(s(t), t)|^2. \quad (3.1.6)$$

Proof. Differentiating the constant zero function $u(s(t), t)$ with respect to t yields $u_t(s(t), t) = -s'(t) u_x(s(t), t)$. We use this twice in the following calculation.

$$\begin{aligned} \frac{d}{dt} E_u(t) &= \frac{1}{2} s'(t) (u_t^2 + u_x^2)|_{x=s(t)} + \frac{1}{2} \int_0^{s(t)} \frac{\partial}{\partial t} (u_t^2 + u_x^2) dx \\ &= \frac{s'(t)}{2} (1 + s'(t)^2) (u_x^2)|_{x=s(t)} + \int_0^{s(t)} (u_t u_{tt} + u_x u_{tx}) dx \\ &= \frac{s'(t)}{2} (1 + s'(t)^2) (u_x^2)|_{x=s(t)} + \int_0^{s(t)} (u_t u_{xx} + u_x u_{tx}) dx \\ (\text{integration by parts}) \quad &= \frac{s'(t)}{2} (1 + s'(t)^2) (u_x^2)|_{x=s(t)} + \left[u_t u_x \right]_{x=0}^{x=s(t)} \\ &= \frac{s'(t)}{2} (1 + s'(t)^2) (u_x^2)|_{x=s(t)} + u_t u_x|_{x=s(t)} \\ &= \frac{s'(t)}{2} (s'(t)^2 - 1) |u_x(s(t), t)|^2. \end{aligned}$$

Recall that $\|s'\|_\infty < 1$ to conclude that $\text{sign}(\frac{d}{dt} E_u(t)) = -\text{sign}(s'(t))$. \square

Proposition 3.1.5. *For (W.Eq) the following energy estimate holds*

$$\frac{m(t)}{2M(0)} \|(g, f)\|_{H_0^1 \times L_2}^2 \leq E_u(t) \leq \frac{M(t)}{2m(0)} \|(g, f)\|_{H_0^1 \times L_2}^2 \quad (3.1.7)$$

where the constants are given by (3.1.4).

Proof. Taking term by term derivatives in (3.1.1) gives

$$\begin{aligned} u_x(x, t) &= 2\pi i \sum_{n \in \mathbb{Z}} n A_n (\varphi'(t+x) e^{2\pi i n \varphi(t+x)} + \varphi'(t-x) e^{2\pi i n \varphi(t-x)}) \\ u_t(x, t) &= 2\pi i \sum_{n \in \mathbb{Z}} n A_n (\varphi'(t+x) e^{2\pi i n \varphi(t+x)} - \varphi'(t-x) e^{2\pi i n \varphi(t-x)}) \end{aligned}$$

Therefore, using parallelogram identity as in the proof of Proposition 3.1.3,

$$\begin{aligned} 2E_u(t) &= \int_0^{s(t)} |u_x(x, t)|^2 + |u_t(x, t)|^2 dx \\ &= 8\pi^2 \left(\int_0^{s(t)} \left| \sum_{n \in \mathbb{Z}} n A_n \varphi'(t+x) e^{2\pi i n \varphi(t+x)} \right|^2 dx + \int_0^{s(t)} \left| \sum_{n \in \mathbb{Z}} n A_n \varphi'(t-x) e^{2\pi i n \varphi(t-x)} \right|^2 dx \right) \\ &= 8\pi^2 \int_{t-s(t)}^{t+s(t)} \left| \sum_{n \in \mathbb{Z}} n A_n (\varphi'(y) e^{2\pi i n \varphi(y)}) \right|^2 dy. \end{aligned}$$

This yields the double inequality

$$4\pi^2 m(t) a(t) \leq E_u(t) \leq 4\pi^2 M(t) a(t)$$

where

$$a(t) = \int_{t-s(t)}^{t+s(t)} \left| \sum_{n \in \mathbb{Z}} n A_n e^{2\pi i n \varphi(y)} \right|^2 \varphi'(y) dy.$$

By Lemma 3.1.2 and Proposition 3.1.3 we conclude. \square

3.2 Point Observations

3.2.1 Boundary Observation

Recall the notation $\alpha(t) = t + s(t)$, $\beta(t) = t - s(t)$ and $\gamma = \alpha \circ \beta^{-1}$.

Theorem 3.2.1. *For any admissible boundary curve $s(t)$ and solution u to the moving boundary wave equation (W.Eq) given by (3.1.1) the following double inequality holds:*

$$2 \frac{m(\beta^{-1}(0))}{M(0)} \|(g, f)\|_{H_0^1 \times L_2}^2 \leq \int_0^{\gamma(0)} |u_x(0, t)|^2 dt \leq 2 \frac{M(\beta^{-1}(0))}{m(0)} \|(g, f)\|_{H_0^1 \times L_2}^2 \quad (3.2.1)$$

In particular, with the observations $C\psi = \psi_x(0)$ the problem (W.Eq) is exactly observable in time τ if and only if $\tau \geq \gamma(0)$.

Proof. Differentiating u term by term, and evaluating at $x = 0$ we have for all $\tau > 0$

$$\|u_x(0, t)\|_{L_2(0, \tau, \frac{1}{\varphi'(t)})} = \int_0^\tau \left| 4\pi i \sum_{n \in \mathbb{Z}} n A_n \varphi'(t) e^{2\pi i n \varphi(t)} \right|^2 \frac{dt}{\varphi'(t)}.$$

Consider $\beta(t) = t - s(t)$ with domain $t \in [0, +\infty)$. Clearly, $\beta(t)$ is strictly increasing and since $\beta(0) = -1 < 0$, there exist a unique t_0 such that $\beta(t_0) = 0$. Let $\tau_0 := t_0 + s(t_0) = \gamma(0)$. Then, by Lemma 3.1.2,

$$\|u_x(0, t)\|_{L_2(0, \tau_0, \frac{1}{\varphi'(t)})}^2 = 16\pi^2 \sum_{n \in \mathbb{Z}} n^2 |A_n|^2$$

Clearly,

$$\frac{1}{M(t_0)} \|u_x(0, t)\|_{L_2(0, \tau_0)}^2 \leq \|u_x(0, t)\|_{L_2(0, \tau_0, \frac{1}{\varphi'(t)})}^2 \leq \frac{1}{m(t_0)} \|u_x(0, t)\|_{L_2(0, \tau_0)}^2.$$

Combining this with Proposition 3.1.3, we find our double inequality. From this is obvious that observation times $\tau \geq \tau_0$ suffice. On the other hand, if $\tau < \tau_0$, $\|u_x(0, t)\|_{L_2(0, \tau, \frac{1}{\varphi'(t)})}^2$ and $\sum n^2 |A_n|^2$ cannot be comparable, which is easy to see by a change of variables bringing it back to the standard trigonometric orthonormal basis of $L_2(0, 1)$. This shows, again by Proposition 3.1.3, that exact observation is impossible. \square

Theorem 3.2.2. *For the solution u given by (3.1.1) to the moving boundary wave equation (W.Eq) the following double inequality holds:*

$$C_1 \|(g, f)\|_{H_0^1 \times L_2}^2 \leq \int_0^{\gamma^{-1}(0)} |u_x(s(t), t)|^2 dt \leq C_2 \|(g, f)\|_{H_0^1 \times L_2}^2 \quad (3.2.2)$$

3. Exact observability of a 1D wave on a non-cylindrical domain

where $C_1 = \frac{m(0)}{2M(0)(1+\|s'\|_\infty)}(1+\frac{m(t_0)}{M(t_0)})^2$ and $C_2 = \frac{M(0)}{2m(0)(1-\|s'\|_\infty)}(1+\frac{M(t_0)}{m(t_0)})^2$.

In particular, with the observations $M(t)\psi = \psi_x(s(t))$ the problem (W.Eq) is exactly observable in time τ if and only if $\tau \geq \gamma^{-1}(0)$.

Proof. Next we consider observation on the right boundary $x = s(t)$. As in the proof of Theorem 3.2.1, let t_0 be such that $\beta(t_0) = t_0 - s(t_0) = 0$ and define $\tau_0 := \gamma^{-1}(0)$. Taking the derivative of $u(x, t)$ with respect to x term by term, substituting $x = s(t)$ and exploiting (3.1.2) yields

$$\begin{aligned} u_x(s(t), t) &= 2\pi i \sum_{n \in \mathbb{Z}} n A_n \left(e^{2\pi i n \varphi(t+s(t))} \varphi'(t+s(t)) + e^{2\pi i n \varphi(t-s(t))} \varphi'(t-s(t)) \right) \\ &= 2\pi i \sum_{n \in \mathbb{Z}} \varphi'(t-s(t)) e^{2\pi i n \varphi(t-s(t))} n A_n \left(1 + \frac{\varphi'(t+s(t))}{\varphi'(t-s(t))} \right) \end{aligned} \quad (3.2.3)$$

Then

$$\left(1 + \frac{m(t_0)}{M(t_0)} \right) \leq \left(1 + \frac{\varphi'(t+s(t))}{\varphi'(t-s(t))} \right) \leq \left(1 + \frac{M(t_0)}{m(t_0)} \right) \quad (3.2.4)$$

Let $\omega(t) = \frac{1-s'(t)}{\varphi'(t-s(t))}$. Then

$$\|u_x(s(t), t)\|_{L_2(0, \tau_0, \omega(t) dt)}^2 \sim 4\pi^2 \int_0^{\tau_0} \left| \sum_{n \in \mathbb{Z}} e^{2\pi i n \varphi(t-s(t))} n A_n \right|^2 \varphi'(t-s(t)) (1-s'(t)) dt$$

where the equivalence comes from (3.2.4). We make the change of variables $\xi = \varphi(t-s(t))$ and observe that (3.1.2) gives an upper bound of the integral to be $\varphi(\beta(\tau_0)) = 1 + \varphi(\beta(0))$. So

$$\|u_x(s(t), t)\|_{L_2(0, \tau_0, \omega(t) dt)}^2 \sim 4\pi^2 \int_{\varphi(\beta(0))}^{\varphi(\beta(0))+1} \left| \sum_{n \in \mathbb{Z}} e^{2\pi i n \xi} n A_n \right|^2 d\xi = 4\pi^2 \sum_{n \in \mathbb{Z}} n^2 |A_n|^2$$

We summarise:

$$4\pi^2 \left(1 + \frac{m(t_0)}{M(t_0)} \right)^2 \sum_{n \in \mathbb{Z}} n^2 |A_n|^2 \leq \|u_x(s(t), t)\|_{L_2(0, \tau_0, \omega(t) dt)}^2 \leq 4\pi^2 \left(1 + \frac{M(t_0)}{m(t_0)} \right)^2 \sum_{n \in \mathbb{Z}} n^2 |A_n|^2$$

We conclude the proof observing that $\frac{1-\|s'\|_\infty}{M(0)} \leq \omega(t) \leq \frac{1+\|s'\|_\infty}{m(0)}$ which allows to remove the weight function:

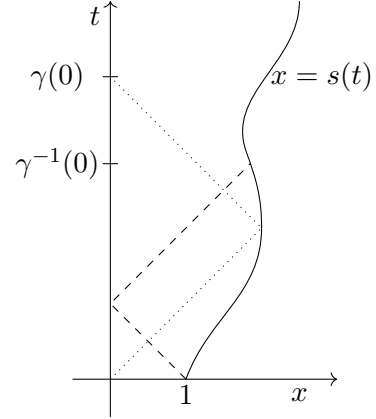
$$\frac{4\pi^2 m(0)}{1+\|s'\|_\infty} \left(1 + \frac{m(t_0)}{M(t_0)} \right)^2 \sum_{n \in \mathbb{Z}} n^2 |A_n|^2 \leq \|u_x(s(t), t)\|_{L_2(0, \tau_0)}^2 \leq \frac{4\pi^2 M(0)}{1-\|s'\|_\infty} \left(1 + \frac{M(t_0)}{m(t_0)} \right)^2 \sum_{n \in \mathbb{Z}} n^2 |A_n|^2$$

We conclude using Proposition 3.1.3. □

Let us finish this paragraph with a little observation. The optimal times for boundary observations given in Theorems 3.2.1 and 3.2.2 are precisely the times where a characteristic emerging from the left (resp. right) boundary point $x = 0$, resp. $x = 1$ hit again the boundary curve, see the picture on the right.

A second remark is that since $u(s(t), t) = 0$, taking derivative with respect to t gives $s'(t)u_x(s(t), t) = -u_t(s(t), t)$. We may hence replace u_x by u_t in the inequality (3.2.2), at the only price to modify the constants by a factor $\|s'\|_\infty$.

Somehow a similar result to Theorem 3.2.2 in a dual setting in terms of controllability have been shown in [23] for the special case of a linear moving wall $s(t) = 1 + \varepsilon t$ by a transformation to a cylindrical domain proposed by Miranda [90]. The minimal control time estimate was however far from optimal. Their result (again only for the linear moving wall case) was subsequently improved in [117] who found the same minimal control time as ourselves by a different method[‡].



3.2.2 Internal Point observation

Next, we turn our attention to observation on an internal point. In the situation where $s(t) = 1$ and hence $\varphi(x) = x$, the solution u to (W.Eq) is given by a sine-series (due to Dirichlet boundary conditions),

$$u(x, t) = \sum_{n \in \mathbb{Z}} a_n e^{i\pi n t} \sin(n\pi x).$$

Consequently, internal point observation at $x=a$ is not possible when $a \in \mathbb{Q}$ since then infinitely many terms in the sum vanish, independently of the leading coefficient. One way to counter this problem is to obtain observability results for the average of $|u|^2$ in a small neighbourhood of a fixed internal point a , see [32]. It is also well known that another way to counter this problem is to consider a moving interior point, see for example [14, 62, 63]. We follow in this chapter the idea that fixed domain with moving observers should somehow behave similar to moving domains with fixed observers. The following result confirms this intuition: for any fixed point $a \in (0, 1)$, consider a Neumann observer defined by $Cu = u_x(a, t)$ to the solution u of the moving boundary wave equation (W.Eq).

Theorem 3.2.3. *Let s be an monotonic admissible boundary curve and φ be a C^2 -solution to (3.1.2). Assume additionally that φ' is strictly decreasing if $s(\cdot)$ is increasing or that φ' is strictly increasing if $s(\cdot)$ is decreasing, respectively.*

Then solution u to the wave equation (W.Eq) satisfies the following double in-

[‡]Caution: when writing out the parametrisation of the boundary integral in [117, formula (2.2)], the authors forget a factor $(1+\varepsilon)^{1/2}$. This wrong factor then appears in many subsequent estimates in their paper.

3. Exact observability of a 1D wave on a non-cylindrical domain

equality:

$$C_1(a) \|(g, f)\|_{H_0^1 \times L_2}^2 \leq \int_0^{a+\gamma(-a)} |u_x(a, t)|^2 dt \leq C_2(a) \|(g, f)\|_{H_0^1 \times L_2}^2,$$

where the constants C_1 and C_2 depend only on s and a . We provide them explicitly in the proof.

Proof. Let $t_1 = \beta^{-1}(-a)$ and $\tau_a = a + \gamma(-a)$. Term by term differentiation of (3.1.1) with respect to x gives

$$u_x(a, t) = 2\pi i \sum_{n \in \mathbb{Z}} n A_n \left(e^{2\pi i n \varphi(t+a)} \varphi'(t+a) + e^{2\pi i n \varphi(t-a)} \varphi'(t-a) \right)$$

First we suppose that φ' is strictly decreasing. We first calculate a weighted L_2 -norm with $\omega_a(t) = \frac{1}{\varphi'(t-a)}$:

$$A - B \leq \|u_x(a, t)\|_{L_2(0, \tau_a, \omega_a(t) dt)} \leq A + B$$

with

$$\begin{aligned} A &:= 2\pi \left\| \sum_{n \in \mathbb{Z}} n A_n e^{2\pi i n \varphi(t-a)} \varphi'(t-a) \right\|_{L_2(0, \tau_a, \omega_a(t) dt)} \\ B &:= 2\pi \left\| \sum_{n \in \mathbb{Z}} n A_n e^{2\pi i n \varphi(t+a)} \varphi'(t+a) \right\|_{L_2(0, \tau_a, \omega_a(t) dt)}. \end{aligned}$$

To estimate A , the change of variables $s = t - a$ together with Lemma 3.1.2 therefore gives

$$A^2 = 4\pi^2 \sum_{n \in \mathbb{Z}} n^2 |A_n|^2.$$

For B , we have

$$B^2 = 4\pi^2 \int_0^{\tau_a} \left| \sum_{n \in \mathbb{Z}} n A_n (e^{2\pi i n \varphi(t+a)} \varphi'(t+a)) \right|^2 \omega_a(t) dt$$

Since φ' is strictly decreasing, $0 < \frac{\varphi'(t+a)}{\varphi'(t-a)} < 1$ for all $t \in [0, \tau_a]$ and so $q_a := \max_{[0, \tau_a]} \frac{\varphi'(t+a)}{\varphi'(t-a)} < 1$. We then have

$$\begin{aligned} B^2 &\leq 4\pi^2 q_a \int_0^{\tau_a} \left| \sum_{n \in \mathbb{Z}} n A_n e^{2\pi i n \varphi(t+a)} \varphi'(t+a) \right|^2 \frac{1}{\varphi'(t+a)} dt \\ &= 4\pi^2 q_a \int_a^{a+\tau_a} \left| \sum_{n \in \mathbb{Z}} n A_n e^{2\pi i n \varphi(s)} \right|^2 \varphi'(s) ds \end{aligned}$$

Recall that $a + \tau_a = 2a + \gamma(-a)$. Since $s' \geq 0$, we have $\gamma' \geq 1$ and so $2a + \gamma(-a) \leq \gamma(a)$. By Lemma 3.1.2 we infer

$$B^2 \leq 4\pi^2 q_a \int_a^{\gamma(a)} \left| \sum_{n \in \mathbb{Z}} n A_n e^{2\pi i n \varphi(s)} \right|^2 \varphi'(s) ds = 4\pi^2 q_a \sum_{n \in \mathbb{Z}} n^2 |A_n|^2.$$

Putting both on A and B estimates together, and using Proposition 3.1.3, we get the lower estimate

$$\begin{aligned} \|u_x(a, t)\|_{L_2(0, \tau_a)}^2 &\geq m(t_1) \|u_x(a, t)\|_{L_2(0, \tau_a, \omega_a(t) dt)}^2 \\ &\geq 4\pi^2 m(t_1) (1 - \sqrt{q_a})^2 \sum_{n \in \mathbb{Z}} n^2 |A_n|^2 \\ &\geq C_1(a) \|(g, f)\|_{H_0^1 \times L_2}^2 \end{aligned}$$

with $C_1(a) = \frac{m(t_1)}{2M(0)}(1 - \sqrt{q_a})^2$. The upper estimate is similar; we find $C_2(a) = \frac{M(t_1)}{2m(0)}(1 + \sqrt{q_a})^2$.

In the case where φ' is strictly increasing we use $\widetilde{\omega}_a(t) = \frac{1}{\varphi'(t+a)}$ as a weight function and change the rôles of A and B . The result follows the same lines then. \square

We observe that the same proof also gives the double inequality

$$C_1(a) \|(g, f)\|_{H_0^1 \times L_2}^2 \leq \int_0^{a+\gamma(-a)} |u_t(a, t)|^2 dt \leq C_2(a) \|(g, f)\|_{H_0^1 \times L_2}^2.$$

Discussion

One may formulate (W.Eq) as an abstract non-autonomous Cauchy problem, for example as follows: let $H_t = L_2([0, s(t)])$ and define

$$\mathcal{D}(A(t)) = H_0^1([0, s(t)] \cap H^2([0, s(t)]) \quad \text{and} \quad A(t)f = f''$$

Then $A(t)$ is the generator of an analytic semigroup on H_t . For $t \geq 0$, we let $\mathcal{H}_t = H_0^1([0, s(t)]) \times L_2([0, s(t)])$ and

$$\mathcal{D}(\mathbf{a}(t)) = \mathcal{D}(A(t)) \times H_0^1([0, s(t)]) \quad \text{and} \quad \mathbf{a}(t) = \begin{pmatrix} 0 & I \\ A(t) & 0 \end{pmatrix}.$$

With this notation (W.Eq) rewrites as

$$\begin{cases} x'(t) = \mathbf{a}(t)x(t) \\ x(0) = x_0 = (g, f) \in \mathcal{H}_0. \end{cases} \quad (3.2.5)$$

The observation of $t \mapsto u_x(a, t)$ discussed in the theorem is then realised with observation operators $C(t) : \mathcal{D}(\mathbf{a}(t)) \rightarrow \mathbb{C}$ defined by $C(t)(v, w)^t = v_x(a)$. Theorem 3.2.3 states in particular exact observability on $[0, \tau]$ if and only if $\tau \geq a + \gamma(-a)$. It is remarkable that this holds true, although, for a dense subset of values of t_0 (precisely if $a/s(t_0) \in \mathbb{Q}$) the “frozen” evolution equations

$$x'(t) + \mathbf{a}(t_0)x(t) = 0 \quad y(t) = C(t)x(t)$$

are *not* exactly observable by the sine-series argument given above for the case $s(t) = 1$. This could now lead to the intuition that the non-observability on for all $t > 0$ such that $a/s(t) \in \mathbb{Q}$ is an “almost everywhere phenomenon”, and may be ignored. This idea is partially contradicted by the following result, where the observation position depends on time and may be such that the ratio $a(t)/s(t) \in \mathbb{Q}$ for all $t > 0$.

3. Exact observability of a 1D wave on a non-cylindrical domain

Theorem 3.2.4. *Let $s(t) = 1 + \varepsilon t$ and $a(t) = as(t)$ for some $a \in (0, 1)$. Then the solution u to the wave equation (W.Eq) satisfies the following admissibility and observation inequality:*

$$C_1(a, \varepsilon) \|(g, f)\|_{H_0^1 \times L_2}^2 \leq \int_0^{\frac{2}{1-\varepsilon}} |u_t(a(t), t)|^2 dt \leq C_2(a, \varepsilon) \|(g, f)\|_{H_0^1 \times L_2}^2$$

The constants C_1 and C_2 depend only on a and ε . We provide them explicitly in the proof.

Proof. Recall that the solution u of the equation (W.Eq) can be written in the form (3.1.1) with $\varphi(t) = C_\varepsilon \ln(1+\varepsilon t)$, see the table on page 48. Taking the derivative respected to t gives

$$u_t(x, t) = 2\pi i \sum_{n \in \mathbb{Z}} n A_n \left(e^{2\pi i n \varphi(t+x)} \varphi'(t+x) - e^{2\pi i n \varphi(t-x)} \varphi'(t-x) \right)$$

Substituting $x = a(t)$, we get

$$u_t(a(t), t) = 2\pi i \sum_{n \in \mathbb{Z}} n A_n \left(e^{2\pi i n \varphi(t+a(1+\varepsilon t))} \varphi'(t+a(1+\varepsilon t)) - e^{2\pi i n \varphi(t-a(1+\varepsilon t))} \varphi'(t-a(1+\varepsilon t)) \right)$$

By calculation, we have the followings identities

$$\begin{aligned} \varphi(t \pm a(1+\varepsilon t)) &= \varphi(t) + \varphi(\pm a) \\ \varphi_t(t \pm a(1+\varepsilon t)) &= \frac{1}{\varepsilon} \varphi'(t) \varphi'(\pm a) \end{aligned}$$

Plugging them into the preceding equation we get

$$\begin{aligned} u_t(a(t), t) &= \frac{2\pi i}{\varepsilon} \sum_{n \in \mathbb{Z}} A_n \left(e^{2\pi i n (\varphi(t) + \varphi(a))} \varphi'(t) \varphi'(a) - e^{2\pi i n (\varphi(t) + \varphi(-a))} \varphi'(t) \varphi'(-a) \right) \\ &= \frac{2\pi i}{\varepsilon} \sum_{n \in \mathbb{Z}} A_n e^{2\pi i n \varphi(t)} \varphi'(t) \left(e^{2\pi i n \varphi(a)} \varphi'(a) - e^{2\pi i n \varphi(-a)} \varphi'(-a) \right) \end{aligned}$$

Let $t_0 = \frac{1}{1-\varepsilon}$. Then $[t_0 - s(t_0), t_0 + s(t_0)] = [0, \frac{2}{1-\varepsilon}]$ and so, using Lemma 3.1.2,

$$\begin{aligned} &\|u_t(a(t), t)\|_{L_2(0, \frac{2}{1-\varepsilon}, \frac{1}{\varphi'(t)})}^2 \\ &= \frac{4\pi^2}{\varepsilon^2} \int_0^{\frac{2}{1-\varepsilon}} \left| \sum_{n \in \mathbb{Z}} e^{2\pi i n \varphi(t)} \varphi'(t) n A_n \left(e^{2\pi i n \varphi(a)} \varphi'(a) - e^{2\pi i n \varphi(-a)} \varphi'(-a) \right) \right|^2 \frac{1}{\varphi'(t)} dt \\ &= \frac{4\pi^2}{\varepsilon^2} \sum_{n \in \mathbb{Z}} n^2 |A_n|^2 \left| e^{2\pi i n \varphi(a)} \varphi'(a) - e^{2\pi i n \varphi(-a)} \varphi'(-a) \right|^2 \end{aligned}$$

Now we need to estimate the multiplicative term

$$\begin{aligned} M_n^2 &= \left| e^{2\pi i n \varphi(a)} \varphi'(a) - e^{2\pi i n \varphi(-a)} \varphi'(-a) \right|^2 \\ &= \varphi'(a)^2 + \varphi'(-a)^2 - 2\varphi'(a)\varphi'(-a) \cos\left(2\pi n(\varphi(a) - \varphi(-a))\right). \end{aligned}$$

Clearly, $(\varphi'(a) - \varphi'(-a))^2 \leq M_n^2 \leq (\varphi'(a) + \varphi'(-a))^2$; by direct calculation,

$$(\varphi'(a) - \varphi'(-a))^2 = C_\varepsilon^2 \frac{4\varepsilon^4 a^2}{(1-\varepsilon^2 a^2)^2} \quad \text{and} \quad (\varphi'(a) + \varphi'(-a))^2 = C_\varepsilon^2 \frac{4\varepsilon^2}{(1-\varepsilon^2 a^2)^2}$$

Therefore, by Proposition 3.1.3,

$$\frac{16\pi^2 \varepsilon^2 a^2}{(1-\varepsilon^2 a^2)^2 \eta_\varepsilon^2} \sum_{n \in \mathbb{Z}} n^2 |A_n|^2 \leq \|u_t(a(t), t)\|_{L_2(0, \frac{2}{1-\varepsilon}, \frac{1}{\varphi'(t)})}^2 \leq \frac{16\pi^2}{(1-\varepsilon^2 a^2)^2 \eta_\varepsilon^2} \sum_{n \in \mathbb{Z}} n^2 |A_n|^2$$

Now we apply Proposition 3.1.3 to conclude. We find

$$C_1(a, \varepsilon) = \frac{1-\varepsilon}{1+\varepsilon} \frac{2\varepsilon^2 a^2}{(1-\varepsilon^2 a^2)^2 \eta_\varepsilon^2} \quad \text{and} \quad C_2(a, \varepsilon) = \frac{1+\varepsilon}{1-\varepsilon} \frac{2}{(1-\varepsilon^2 a^2)^2 \eta_\varepsilon^2}. \quad \square$$

3.2.3 Simultaneous exact observability

A last result in this section concerns simultaneous exact observability : consider a system of two coupled 1D wave equations, one of which has a fixed boundary, and the second has the moving domain $0 \leq x \leq s(t)$ as above. Assume that we can observe only the combined force exerted by the strings at the common endpoint $\varphi(t) = u_x^{(1)}(0, t) + u_x^{(2)}(0, t)$, for $t \in [0, T]$. The question is whether we can still exactly observe all initial data. Our system is defined as

$$\begin{cases} u_{tt} - u_{xx} = 0 & (x, t) \in \Omega \\ v_{tt} - v_{xx} = 0 & -1 \leq x \leq 0 \\ u(0, t) = u(s(t), t) = v(-1, t) = v(0, t) = 0 & t \geq 0 \\ u(x, 0) = g(x), u_t(x, 0) = f(x) & x \in [0, 1] \\ v(x, 0) = \tilde{g}(x), v_t(x, 0) = \tilde{f}(x) & x \in [-1, 0] \end{cases} \quad (W_2)$$

Theorem 3.2.5. *Let $s(\cdot)$ be an admissible boundary curve and assume additionally that either*

$$\liminf_{t \rightarrow \infty} \gamma'(t) > 1 \quad \text{or} \quad \gamma'(t) = 1 + ax^{-\delta} + o(t^{-\delta}), \quad 0 < \delta < 1, a > 0.$$

Moreover assume that φ' is bounded on \mathbb{R}_+ . Let (u, v) be the solution to (W_2) . Then, for all $\lambda > 0$ there exists $\tau_0 > 2$ such that for all $\tau \geq \tau_0$

$$\lambda \left(\|(g, f)\|_{H_1^0 \times L_1}^2 + \|(\tilde{g}, \tilde{f})\|_{H_1^0 \times L_2}^2 \right) \leq \int_0^\tau |u_x(0, t) + v_x(0, t)|^2 dt \quad (3.2.6)$$

Our assumptions include the cases of linear moving boundaries, parabolic boundaries and hyperbolic boundaries. However, for the shrinking domain they are not satisfied.

Proof. By the triangle inequality we have

$$\left(\int_0^\tau |u_x(0, t) + v_x(0, t)|^2 dt \right)^{1/2} \geq A(\tau) - B(\tau)$$

where

$$A(\tau) = \left(\int_0^\tau |v_x(0, t)|^2 dt \right)^{1/2} \quad \text{and} \quad B(\tau) = \left(\int_0^\tau |u_x(0, t)|^2 dt \right)^{1/2}$$

It is well known that the solution v of the wave equation with the fixed boundary can be expressed as a pure sine series

$$v(x, t) = \sum_{n \in \mathbb{Z}} a_n e^{\pi i n t} \sin(n\pi x), \quad (3.2.7)$$

where $(na_n)_{n \in \mathbb{Z}} \in \ell_2$ and hence $(a_n)_{n \in \mathbb{Z}} \in \ell_2$. Consequently, for all $t \geq 0$, the energy of v is constant: indeed, by direct computation,

$$E_v(t) = \frac{1}{2} \int_0^1 \left| \frac{\partial v(x, t)}{\partial t} \right|^2 + \left| \frac{\partial v(x, t)}{\partial x} \right|^2 dx = \frac{\pi^2}{2} \sum_{n \in \mathbb{Z}} n^2 a_n^2$$

We also have

$$\int_0^2 |v_x(0, t)|^2 dt = \int_0^2 \left| \sum_{n \in \mathbb{Z}} \pi n a_n e^{i\pi n t} \right|^2 dt = 4E_v(0).$$

Hence, using periodicity of v , we obtain (recall $\tau \geq 2$)

$$A(\tau)^2 = \int_0^\tau |v_x(0, t)|^2 dt \geq 4 \lfloor \frac{\tau}{2} \rfloor E_v(0)$$

Next we turn to an estimate for $B(\tau)$. Recall that

$$u_x(0, t) = 4\pi i \sum_{n \in \mathbb{Z}} n A_n \varphi'(t) e^{2\pi i n \varphi(t)}$$

Let $t_0 = 0$ and $t_n = \gamma^{(n)}(t_0)$. By construction of t_n and (3.1.2),

$$\varphi(t_{n+1}) - \varphi(t_n) = \varphi(\gamma(t_n)) - \varphi(t_n) = 1.$$

Hence, by Lemma 3.1.2, $e^{2\pi i n \varphi(x)}$ is an orthonormal system on $L_2([t_n, t_{n+1}], \varphi'(t) dt)$.

An inspection of the proof of Theorems 3.3.1 and 3.3.2 shows that if $\liminf_{t \rightarrow \infty} \gamma' > 1$, $t_n \rightarrow +\infty$ exponentially, whereas the asymptotics $\gamma'(t) = 1 + at^{-\delta} + o(t^{-\delta})$ ensures $t_n \sim cn^{1/\delta}$. Let $N(\tau)$ be the unique integer satisfying $t_n \leq \tau < t_{n+1}$. Let $C = \sup\{\varphi'(t) : t \geq 0\}$. Then

$$\begin{aligned} B(\tau) &= \int_0^\tau |u_x(0, t)|^2 dt \leq \int_0^\tau |u_x(0, t)|^2 \frac{1}{\varphi'(t)} dt \\ &\leq C \sum_{j=0}^{N(\tau)} \int_{t_j}^{t_{j+1}} |u_x(0, t)|^2 \frac{1}{\varphi'(t)} dt \\ &\leq 16\pi^2 C(N(\tau)+1) \sum_{n \in \mathbb{Z}} n^2 |A_n|^2 \\ &\leq \frac{2C}{m(0)} (N(\tau)+1) \left(\|g^{(1)}(x)\|_{H_1^0(0,1)}^2 + \|f^{(1)}(x)\|_{L_2(0,1)}^2 \right). \end{aligned}$$

We obtained so far that

$$\begin{aligned} \int_0^\tau |u_x(0, t) + v_x(0, t)|^2 dt &\geq A(\tau)^2 - B(\tau)^2 \\ &\geq 4\lfloor \frac{\tau}{2} \rfloor E_v(0) - \frac{2C}{m(0)}(N(\tau)+1) \left(\|g^{(1)}(x)\|_{H_1^0(0,1)}^2 + \|f^{(1)}(x)\|_{L_2(0,1)}^2 \right) \end{aligned}$$

The first term grows linearly in τ . The second term is $o(\tau)$ since in case of exponential growth of the sequence t_n , $N(\tau)$ behaves logarithmically and in case that $t_n \sim cn^{1/\delta}$, $N(\tau) \sim \tau^\delta$ with $\delta < 1$. Hence, the difference tends to infinity with $\tau \rightarrow +\infty$, which means that for all $\lambda > 0$ there exists $\tau_0 > 0$ such that for $\tau \geq \tau_0$,

$$\begin{aligned} \int_0^\tau |u_x(0, t) + v_x(0, t)|^2 dt &\geq 2\lambda(E(u)(0) + E_v(0)) \\ &= \lambda \left(\|(g, f)\|_{H_0^1 \times L_2}^2 + \|(\tilde{g}, \tilde{f})\|_{H_0^1 \times L_2}^2 \right). \quad \square \end{aligned}$$

3.2.4 Duality results

Without detailed proofs we state dual results to our results formulated as null-controllability in the sense of 'transposition'.

Dirichlet control on boundary

Let s be an admissible boundary curve, v the solution to the wave equation on Ω . Let $(Gv)(t) = (v(0, t), v(s(t), t))$ be the trace of v on the two boundary points. Then for either choice, $\zeta(t) = (y(t), 0)$ or $\zeta(t) = (0, y(t))$ the boundary controlled wave equation

$$\begin{cases} v_{tt} - v_{xx} = 0 & (x, t) \in \Omega \\ (Gv)(t) = \zeta(t) & t \geq 0 \\ v(x, 0) = g \in L_2([0, 1]) & x \in [0, 1] \\ v_t(x, 0) = f \in H^{-1}([0, 1]) & x \in [0, 1] \end{cases} \quad (3.2.8)$$

is null-controllable in times $\tau = \gamma(0)$ in case $\zeta(t) = (y(t), 0)$ and in time $\tau = \gamma^{-1}(0)$ in case $\zeta(t) = (0, y(t))$. The null control can be achieved by the control function $y(t) = -u_x(0, t)$, or $y(t) = -u_x(s(t), t)$, respectively where $u(\cdot)$ is the solution to (W.Eq).

Simultaneous Null Control

Next we focus on the dual statement to Theorem 3.2.3 in terms of null-controllability. Instead of one wave equation on Ω , we consider two wave equations with mixed boundary conditions, one on the cylindrical domain $[0, a] \times \mathbb{R}_+$ and one on the non-cylindrical domain $\{(x, t) : a \leq x \leq s(t)\}$. Both equations are coupled via the

control function ζ in the following way:

$$\begin{cases} v_{tt} - v_{xx} = 0 & 0 \leq x \leq a \\ w_{tt} - w_{xx} = 0 & a \leq x \leq s(t) \\ v(0, t) = w(s(t), t) = 0 & t \geq 0 \\ v(a-, t) = w(a+, t) & t \geq 0 \\ v_x(a-, t) - w_x(a+, t) = \zeta(t) & t \geq 0 \\ v(x, 0) = g(x), \quad v_t(x, 0) = f(x) & x \in [0, a] \\ w(x, 0) = g(x), \quad w_t(x, 0) = f(x) & x \in [a, 1] \end{cases} \quad (3.2.9)$$

Then Theorem 3.2.3 implies that (3.2.9) is null-controllable in time $\tau \geq a + \gamma(-a)$. The control can be achieved by letting $\zeta(t) = u_x(a, t)$ where $u(\cdot)$ is the solution to (W.Eq).

3.3 Differentiable solutions for general boundary functions

In this section we discuss the solvability of (3.1.2) by a differentiable function φ . Our hypotheses are that the boundary function s be of class C^1 at least and that $\lim_{t \rightarrow \infty} s'(t) = s$ exists. This last condition is of course only of interest if we seek for solutions φ satisfying (3.1.2) for $t \in \mathbb{R}_+$, since it can easily be arranged if we consider only $t \in [0, \tau]$.

Let $s(\cdot)$ be of class C^1 and $\|s'\|_\infty < 1$. Let $\alpha(t) = t + s(t)$ and $\beta(t) = t - s(t)$. Both functions, α and β are strictly increasing and continuous. Moreover, $\alpha(t) = \alpha(0) + t\alpha'(\xi_t) > \alpha(0) + t(1 - \|s'\|_\infty)$ yields $\lim_{t \rightarrow +\infty} \alpha(t) = +\infty$. Hence α is a bijection from $[0, \infty)$ to $[1, \infty)$; similarly β is a bijection from $[0, \infty)$ to $[-1, \infty)$. We then consider the bijection

$$\gamma := \alpha \circ \beta^{-1} : [-1, \infty) \rightarrow [+1, \infty).$$

Observe that

$$\gamma'(t) = \frac{\alpha' \circ \beta^{-1}}{\beta' \circ \beta^{-1}} = \frac{1 + s'(\beta^{-1}(t))}{1 - s'(\beta^{-1}(t))},$$

so that γ is strictly increasing by $\|s'\|_\infty < 1$. The sign of $s'(\beta^{-1}(t))$ determines whether γ is strictly contractive or strictly expansive. We also note for further reference that if $s \in C^2$,

$$\gamma''(t) = \frac{2s''(\beta^{-1}(t))}{(1 - s'(\beta^{-1}(t)))^3}.$$

The functional equation (3.1.2) can now be rephrased as

$$\varphi \circ \gamma = \varphi + 1. \quad (A)$$

This equation is known as 'Abel's equation' and intensively studied, see for example [70, 71] and references therein.

We will consider only the case where $\lim s(t) = s$ exists. Since $s(t) > 0$ for all t , $\lim s'(t) = s < 0$ is impossible. We may therefore either have $s = 0$ or $s \in (0, 1)$. We first discuss the situation of a non-zero limit, which means that $\gamma'(t) \rightarrow \ell = \frac{1+s}{1-s} > 1$.

Theorem 3.3.1. *Let $\ell > 1$ and assume that $\gamma'(x) = \ell + \mathcal{O}(x^{-\delta})$ for $\delta > 0$. Then Abel's equation (A) admits a strictly increasing solution $\varphi \in C^1([-1, \infty))$. If additionally $\gamma \in C^2[0, \infty)$, $\gamma'' = \mathcal{O}(x^{-1-\delta})$ and γ' is decreasing, then φ is of class $C^2([-1, \infty))$.*

Proof of Theorem 3.3.1. Put $\psi = \ell\varphi$. Then ψ satisfies the Schröder equation $\psi \circ \gamma = \ell\psi$. Since $\gamma(-1) = +1$ and γ has no fixed points (otherwise $s(t) = 0$), $\gamma(x) > x$ for all $x \geq -1$. Observe that by assumption, there exists some $\xi > 0$ such that $\gamma'(x) \geq \frac{1+\ell}{2} > 1$ for all $x \geq \xi$. Let $a_0 = -1$ and $a_n = \gamma^{(n)}(a_0)$. If (a_n) were bounded, we could extract a subsequence that converges to a fixed point of γ . So $a_n \rightarrow \infty$. Let k be such that $a_k > \xi$. Hence

$$a_{n+k+1} - \xi \geq \gamma(a_{n+k}) - \gamma(\xi) > \frac{1+\ell}{2}(a_{n+k} - \xi)$$

shows that $a_n \rightarrow +\infty$ exponentially. By monotonicity of γ we infer the same for $\gamma^{(n)}(x) \geq a_n$ for all $x \geq -1$. This, together with $\gamma'(x) = \ell + \mathcal{O}(x^{-\delta})$ shows that

$$P(x) = \prod_{n=0}^{\infty} \frac{\gamma'(\gamma^{(n)}(x))}{\ell}$$

converges absolutely and uniformly on $[-1, \infty)$. P vanishes nowhere and satisfies $P \circ \gamma = \frac{\ell}{\gamma}P$. We define

$$\psi(x) := \int_1^x P(t) dt + C$$

where the constant C is to be determined. By construction, ψ is strictly increasing and satisfies

$$\psi \circ \gamma(x) = \int_{\gamma(-1)}^{\gamma(x)} P(t) dt + C = \ell \int_{-1}^x P(t) dt + C = \ell \int_{-1}^1 P(t) dt + \ell\psi + C(1 - \ell)$$

So that, letting $C = \frac{\ell}{\ell-1} \int_{-1}^1 P(t) dt > 0$ ensures $\psi \circ \gamma = \ell\psi$ as required. Then $\varphi := \frac{\ln \psi}{\ln(\ell)}$ is of class C^1 , strictly increasing.

If additionally γ' decreases towards ℓ at infinity, a new lecture of the above growth rate of (x_n) shows that $\limsup \frac{\ell^n}{x_n} \leq 1$ for any $x_0 \geq -1$. Therefore, the (termwise differentiated product P) yields a series

$$\sum_n \gamma''(x_n) \left(\prod_{j=0}^{n-1} \gamma'(x_j) \right) \left(\prod_{k \neq n} \frac{\gamma'(x_k)}{\ell} \right)$$

that normally on $[-1, \infty)$. We infer that P is of class C^1 , hence ψ and φ of class C^2 . \square

In the situation that $\lim s'(t) = s = 0$ and hence $\lim \gamma'(t) = 1$ things are more delicate. If γ is such that $\gamma'(x) = 1 + o(x^{-\delta})$ at infinity, for all x, y ,

$$\lim_{n \rightarrow \infty} \frac{\gamma^{(n+1)}(x) - \gamma^{(n)}(x)}{\gamma^{(n+1)}(y) - \gamma^{(n)}(y)} = 1.$$

3. Exact observability of a 1D wave on a non-cylindrical domain

We leave the proof as exercise, as it is a modification of [70, Lemma 7.3]. Consequently, whenever

$$\varphi(x) := \lim_{n \rightarrow \infty} \frac{\gamma^{(n)}(x) - \gamma^{(n)}(x_0)}{\gamma^{(n+1)}(x_0) - \gamma^{(n)}(x_0)}$$

exists, φ is a solution to Abel's equation (A). This is the P. Lévy's algorithm, see e.g. [70, Chapter VII]. In order to ensure existence of a solution we will in general have to get a finer control of the asymptotics. The next result in this direction is based on ideas of Szekeres [118, Theorem 1c], see also [70, Theorem 7.2]). The principal idea is similar to Theorem 3.3.1, but we have to transform differently and to be more careful how to construct an infinite product.

Theorem 3.3.2. *If $\gamma'(x) = 1 + a(1 - \delta)x^{-\delta} + o(x^{-\delta})$ at infinity, where $a > 0$ and $\delta > 0$, $\delta \neq 1$, then Abel's equation (A) has a strictly positive and strictly increasing C^1 -solution φ .*

Proof. First observe that $\frac{\gamma(x)}{x} = 1 + ax^{-\delta} + o(x^{-\delta})$, by integrating γ' on $[0, x]$ or $[x, \infty)$ according to $\delta < 1$ or $\delta > 1$. First we transform our problem into a multiplicative version. To this end, let $g : [-1, \infty) \rightarrow (0, \infty)$ be a C^1 -function. Then, whenever φ solves Abel's equation (A), $\psi(x) = g(x)\varphi'(x)$ satisfies

$$(\psi \circ \gamma)(x) = g(\gamma(x))\varphi'(\gamma(x)) = g(\gamma(x))\frac{\varphi'(x)}{\gamma'(x)} = \frac{g(\gamma(x))}{g(x)\gamma'(x)}\psi(x) =: m(x)\psi(x)$$

Let $x_n = \gamma^{(n)}(x)$. If (x_n) were bounded, it would converge to a fixed point of γ — but there is none. So $x_n \rightarrow +\infty$. Assume that we chose the function g such that

$$\sum_n \left| \frac{g(x_n)\gamma'(x_n)}{g(x_{n+1})} - 1 \right| \quad (3.3.1)$$

converges uniformly on compact intervals. Then the infinite product

$$P(x) = \prod_{n=0}^{\infty} \frac{1}{m(\gamma^{(n)}(x))} = \prod_{n=0}^{\infty} \frac{g(x_n)\gamma'(x_n)}{g(x_{n+1})}, \quad (3.3.2)$$

defines a continuous function P that solves $\psi \circ \gamma = m \cdot \psi$. From P we then easily regain φ . We chose $g(x) = \gamma(x)^{1-\delta}$. Then $P(x) > 0$ for all x . Moreover we have the following asymptotics for $x \rightarrow \infty$:

$$\begin{aligned} 1 - \gamma'(x) \left(\frac{x}{\gamma(x)} \right)^{1-\delta} &= 1 - \frac{1}{(1+ax^{-\delta}+r_1(x))^{1-\delta}} \left(1 + a(1-\delta)x^{-\delta} + \tilde{r}_1(x) \right) \\ &= 1 - \left(1 - a(1-\delta)x^{-\delta} + r_2(x) \right) \left(1 + a(1-\delta)x^{-\delta} + \tilde{r}_2(x) \right) \\ &= a^2(1-\delta)^2 x^{-2\delta} + r(x). \end{aligned}$$

where $r_1, r_2, \tilde{r}_1, \tilde{r}_2 = o(x^{-\delta})$ and $r = o(x^{-2\delta})$ for $x \rightarrow \infty$. Next, we need a growth rate for the orbits $x_n = \gamma^{(n)}(x_0)$: Observe that $a = \lim_{n \rightarrow \infty} \frac{\gamma(x_n) - x_n}{x_n^{1-\delta}} = \lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{x_n^{1-\delta}}$. Rewriting the right hand side we obtain

$$a = \lim_{n \rightarrow \infty} (x_n^\delta - x_{n+1}^\delta) \left(\frac{x_{n+1}}{x_n} \right)^{-\delta} \frac{\frac{x_{n+1}}{x_n} - 1}{\left(\frac{x_{n+1}}{x_n} \right)^{-\delta} - 1}.$$

Using $\frac{x_{n+1}}{x_n} = \frac{\gamma(x_n)}{x_n} \rightarrow 1$ as $n \rightarrow \infty$ the last fraction has limit $-1/\delta$ and we obtain

$$\delta a = \lim_{n \rightarrow \infty} (x_{n+1}^\delta - x_n^\delta).$$

Taking Cesaro sums,

$$\delta a = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} (x_{j+1}^\delta - x_j^\delta) = \lim_{n \rightarrow \infty} \frac{1}{n} x_n^\delta.$$

We infer finally $x_n \sim c n^{1/\delta}$ when $n \rightarrow \infty$. Putting both parts together,

$$\left| \frac{g(x_n)\gamma'(x_n)}{g(x_{n+1})} - 1 \right| = a^2(1-\delta)^2 x_n^{-2\delta} + r(x_n) = a^2(1-\delta)^2 n^{-2} + r(x_n)$$

where $r(x_n) = o(n^{-2})$. Therefore (3.3.1) converges absolutely and uniformly on compact intervals so that (3.3.2) converges to a strictly positive function P . For $C > 0$ to be determined in a moment, we let

$$\varphi(x) := C \int_1^x \frac{P(t)}{\gamma(t)^{1-\delta}} dt.$$

P and γ being strictly positive, φ is positive, strictly increasing and of class C^1 . Moreover,

$$\begin{aligned} \varphi(\gamma(x)) &= C \int_{\gamma(-1)}^{\gamma(x)} \frac{P(t)}{\gamma(t)^{1-\delta}} dt = C \int_{-1}^x \frac{P(\gamma(s))}{\gamma(\gamma(s))^{1-\delta}} \gamma'(s) ds \\ &= C \int_{-1}^x \frac{P(s)m(s)}{\gamma(\gamma(s))^{1-\delta}} \gamma'(s) ds = C \int_{-1}^x \frac{P(t)}{\gamma(t)^{1-\delta}} dt \\ &= \varphi(x) + C \int_{-1}^1 \frac{P(t)}{\gamma(t)^{1-\delta}} dt, \end{aligned}$$

so that adjusting C (the integral being strictly positive) we obtain a solution of Abel's equation (A). \square

Chapter 4

Observability of a 1D Schrödinger equation with time-varying boundaries

We discuss the observability of a one-dimensional Schrödinger equation on certain time dependent domain. In linear moving case, we give the exact boundary and pointwise internal observability for arbitrary time. For the general moving, we provide exact boundary observability when the curve satisfies some certain conditions. By duality theory, we establish the controllability of adjoint system.

4.1 Introduction

Let $\tau > 0$, and $\ell(t) : [0, \tau] \rightarrow \mathbb{R}_+$ a strictly positive C^2 -function satisfying $\ell(0) = 1$ and $\frac{\ell'}{\ell} \in L_\infty$. We consider the following system as a initial boundary value problem in a time dependent domain.

$$\begin{cases} i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0 & x \in [0, \ell(t)] \\ u(0, t) = u(\ell(t), t) = 0 & t \geq 0 \\ u(x, 0) = u_0 & x \in [0, 1] \end{cases} \quad (S_{\text{moving}})$$

For Neumann boundary observations we obtain estimates like

$$c(\tau) \|u_0\|_{H_0^1(0,1)}^2 \leq \int_0^\tau |u_x(0, t)|^2 + |u_x(\ell(t), t)|^2 dt \leq C(\tau) \|u_0\|_{H_0^1(0,1)}^2,$$

see Theorems 4.2.1, 4.2.2 and 4.2.3. We refer to the first estimate as observability estimate and to the second as admissibility estimate. The two first mentioned results rely on a transformation of (S_{moving}) to a non-autonomous equation on the fixed domain $[0, 1]$: the change of variables $y = \frac{x}{\ell(t)}$ and new function $w(y, t) := u(x, t)$ gives an equivalent differential equation for w , namely

$$\begin{cases} i \frac{\partial w}{\partial t} &= \frac{-1}{\ell(t)^2} \frac{\partial^2 w}{\partial y^2} + i \frac{\ell'(t)}{\ell(t)} y \frac{\partial w}{\partial y}, \\ w(0, t) &= w(1, t) = 0 \\ w_y(0, t) &= \ell(t) u_x(0, t) \quad \text{and} \\ w_y(1, t) &= \ell(t) u_x(\ell(t), t) \end{cases} \quad (S_{\text{fixed}})$$

which can easily be obtained by the chain rule.

To obtain Theorems 4.2.1 and 4.2.2 we apply the ‘multiplier technique’: This powerful method has been developed by Morawetz [93] and was later extended by Ho [44] and Lions [80]. We extend a version of Machtyngier [84] to time-dependent multipliers. The observability estimate relies then on the “uniqueness-compactness” lemma 4.3.5. The pitfall of this proof strategy is that it only proves existence of some positive constant, without explicit estimates. This is in contrast with Theorem 4.2.3 which is a specific result for the boundary curve $\ell(t) = 1 + \varepsilon t$. In this linear moving wall case, we mimic a successful approach for a one-dimensional wave-equation obtained by Haak and the author in [42] and develop the solution of (S_{moving}) into a series of eigenfunctions. This allows to use results from Fourier analysis; the obtained admissibility estimates are sharper than those obtained in the previous results, and the observation estimate is provided with explicit constants. Moreover, we obtain in this case admissibility and exact observability of internal point observations:

$$k(\tau) \|u_0\|_{L_2(0,1)}^2 \leq \int_0^\tau |u(a, t)|^2 dt \leq K(\tau) \|u_0\|_{L_2(0,1)}^2,$$

see Theorem 4.2.5. It is remarkable that the lower estimate cannot be true when $\varepsilon = 0$ on any rational point a ; the fact that the considered domains extend however, seem to ‘middle out’ this obstacle. Closely related to this observation are works of Castro and Khapalov [14, 62, 63] where on a fixed domain Ω a moving point observer is considered, with similar conclusions. We also mention results from Moyano [95, 96] where in a two-dimensional circle the radius $\ell(t)$ is used as a control parameter.

An additional result on L_p -admissibility and observability of point observations are presented as well, see Theorem 4.2.7.

It is well-known that exact observability for an (autonomous) wave equation implies observability for the associated Schrödinger equation, see e.g. [123, Chapter 6.7 ff.]. An inspection of the proof gives several obstacles when one passes to non-autonomous problems, and we were not able to use this approach to directly infer our results from those for the wave equation in [42]. We mention that some results on the so-called Hautus-test will be subject of an independent publication [43].

4.2 Main Results

Before giving precise formulations of the aforementioned results, let us start by proving that the Schrödinger equation (S_{fixed}) admits a solution: to this end, we reformulate it as an abstract non-autonomous Cauchy problem in the following way: let $X = L_2(0, 1)$ and the family of operators $\{A(t)\}$ be defined as

$$A(t)w = \frac{i}{\ell(t)^2} w_{yy} + \frac{\ell'(t)}{\ell(t)} y w_y \quad (4.2.1)$$

with natural domain $D(A(t)) = H^2(0, 1) \cap H_0^1(0, 1) =: D$. Moreover, by assumption, the map $t \mapsto A(t)u$ is continuously differentiable for all $u \in D$. Let $\omega > 0$. Then

integration by parts gives

$$\begin{aligned}
 \langle (A(t) + \omega I)w, w \rangle &= \int_0^1 \left(\frac{i}{\ell(t)^2} w_{yy} \bar{w} + \frac{\ell'(t)}{\ell(t)} y w_y \bar{w} + \omega |w|^2 \right) dy \\
 &= \frac{-i}{\ell(t)^2} \int_0^1 |w_y|^2 dy + \frac{\ell'(t)}{\ell(t)} \int_0^1 y w_y \bar{w} dy + \omega \int_0^1 |w|^2 dy \\
 &= \frac{-i}{\ell(t)^2} \int_0^1 |w_y|^2 dy - \frac{\ell'(t)}{\ell(t)} \int_0^1 (|w|^2 + y w \bar{w}_y) dy + \omega \int_0^1 |w|^2 dy
 \end{aligned} \tag{4.2.2}$$

Taking real parts and observing that

$$\operatorname{Re} \left(\int_0^1 y w \bar{w}_y dy \right) = \operatorname{Re} \left(\frac{\ell'(t)}{\ell(t)} \int_0^1 y w_y \bar{w} dy \right) = -\operatorname{Re} \left(\frac{\ell'(t)}{2\ell(t)} \int_0^1 |w|^2 dy \right)$$

we obtain

$$\operatorname{Re} \left(\langle (A(t) + \omega I)w, w \rangle \right) = \left(\omega - \frac{\ell'(t)}{2\ell(t)} \right) \int_0^1 |w|^2 dy \tag{4.2.3}$$

For $\omega > \left\| \frac{\ell'}{2\ell} \right\|_{L^\infty}$, the left hand side of (4.2.3) becomes positive, and the Lumer-Philips theorem asserts that $\omega + A(t)$ generates a contraction semigroup, i.e.

$$\forall t \geq 0 \quad \|e^{-sA(t)}\| \leq e^{\omega s}$$

This ensures in particular that the family $(A(t))_{t \in [0, \tau]}$ satisfies the Kato stability condition. We apply [103, Theorem V.4.8 pp.145] to conclude that $(A(t))$ generates a unique evolution family $\{U(t, s)\}_{0 \leq s \leq t \leq \tau}$ on X satisfying $w(t) = U(t, 0)w_0$. From this we infer a solution to ($\mathcal{S}_{\text{moving}}$) as well, by transforming the fixed domain back to the time-dependent domain.

Suppose that we are given observation operators $C(t) : D \rightarrow Y$ where Y is another Hilbert space. Define the output function $y(t) = C(t)w(t)$. The operator $C(t)$ is called (Y, Z) -admissible if there exist $\gamma > 0$ such that:

$$\int_0^\tau \|C(t)w(t)\|_Y^2 dt \leq \gamma \|w_0\|_Z^2.$$

We say that the system ($\mathcal{S}_{\text{fixed}}$) is exactly (Y, Z) -observable in time $\tau > 0$ if there exist $\delta > 0$ such that:

$$\int_0^\tau \|C(t)w(t)\|_Y^2 dt \geq \delta \|w_0\|_Z^2.$$

If the spaces Y, Z are fixed, we simply speak of admissibility and exact observability. Exact observation in time $\tau > 0$ means that the knowledge of $y_{[0, \tau]}$ allows to recover the initial value w_0 . It is well known that exact observability is equivalent to exact controllability of the retrograde adjoint system:

$$z'(t) = -A(t)^* z(t) - C(t)^* w(t) \quad \text{with} \quad z(\tau) = 0$$

Moreover, it is easy to see that admissibility or observability of ($\mathcal{S}_{\text{fixed}}$) is equivalent to those of ($\mathcal{S}_{\text{moving}}$).

Results on Neumann observations

Theorem 4.2.1. *Let $\tau > 0$ and $\ell : [0, \tau] \rightarrow \mathbb{R}_+^*$ be a strictly positive, twice continuously differentiable function satisfying $\frac{\ell'}{\ell} \in L_\infty$ and $\ell(0) = 1$. Then there exists a constants $C(\tau)$ such that the following admissibility inequalities hold:*

$$\int_0^\tau |u_x(0, t)|^2 + |u_x(\ell(t), t)|^2 dt \leq C(\tau) \|u_0\|_{H_0^1(0,1)}^2$$

An explicit estimate of constant $C(\tau)$ is given in the proof, see (4.3.8).

Concerning observability, we will have the following result. Let $\tau > 0$ and $\ell : [0, \tau] \rightarrow \mathbb{R}_+^*$ be a strictly positive, twice continuously differentiable function satisfying:

$$\ell'(t) > 0, \quad \ell(0) = 1 \quad \text{and} \quad \ell'(t)\ell(t) < \frac{1}{\pi} \quad \forall t \in (0, \tau) \quad (4.2.4)$$

Integrating for 0 to τ of the second condition, we have $2\tau + \pi(1 - \ell(\tau)^2) > 0$. From the condition (4.2.4), $\ell(t)$ is an increasing function, and then $\ell'(t) < \frac{1}{\pi}$. It follows that $\frac{\ell'(t)}{\ell(t)} < \frac{1}{\pi}$, and so the condition $\frac{\ell'}{\ell} \in L_\infty$ guaranteeing admissibility is satisfied.

Theorem 4.2.2. *For all τ satisfying (4.2.4), the following observability inequality holds:*

$$c(\tau) \|u_0\|_{H_0^1(0,1)}^2 \leq \int_0^\tau (|u_x(0, t)|^2 + |u_x(\ell(t), t)|^2) dt.$$

Here $c(\tau)$ is some positive constant depending on τ .

A direct application of theorem 4.2.2 can be used for periodic moving boundary $\ell(t) = 1 + \varepsilon \sin(\omega t)$ where $\varepsilon \in (0, 1)$ and $\omega \in (0, \frac{1}{\pi\varepsilon(1+\varepsilon)})$. For all $\tau \in (0, \frac{\pi}{2\omega})$, we have

$$\ell'(t) = \varepsilon\omega \cos(\omega t) > 0 \quad \text{since} \quad \omega t \in \left(0, \frac{\pi}{2}\right) \quad \forall 0 \leq t \leq \tau$$

$$\ell(0) = 1 \quad \text{and} \quad \ell'(t)\ell(t) = \varepsilon\omega \cos(\omega t)(1 + \varepsilon \sin(\omega t)) < \varepsilon\omega(1 + \varepsilon) < \frac{1}{\pi}$$

Hence, $\ell(t)$ satisfies the condition (4.2.4), so the curve is admissible. The problem of particles moving inside one dimensional square-well of oscillating width was proposed by Fermi and Ulam [34] in order to explain the mechanism of particles containing high energies. This model that plays an important role on theory of quantum chaos and it seems difficult to give an exact solution formula. Glasser [39] investigated the behavior of wave functions and energy in a given instantaneous eigenstate by assumptions on the smoothness of boundary. As far as we know, there are no results in the literature concerning observability and controllability with periodic boundary functions.

In the case that $\ell(t) = 1 + \varepsilon t$, the condition (4.2.4) is ensured when $\varepsilon \in (0, \frac{2}{\pi})$ and $0 < t < \frac{1}{\varepsilon}(\frac{2}{\varepsilon\pi} - 1)$. We have the following exact analytic solution for S_{moving} , due to Doescher and Rice [26]

$$u(x, t) = \sum_{n=1}^{+\infty} a_n \sqrt{\frac{2}{\ell(t)}} \sin\left(\frac{n\pi x}{\ell(t)}\right) e^{i\left(\frac{\varepsilon x^2}{4\ell(t)} - n^2 \pi^2 \frac{t}{\ell(t)}\right)} \quad (4.2.5)$$

where the coefficients (a_n) are defined by the sine-series development of the initial value u_0 . A similar exact solution in the case of two-variable moving wall can be found in [126] where the author uses the fundamental transformation to change the moving boundary problem into a solvable one side fixed boundary problem.

Based on formula (4.2.5) we obtain a first result on Neumann observability at the boundary $\{(x, t) : x \in \{0, \ell(t)\}\}$. Compared to Theorem 4.2.2 the admissibility constant is sharper. In contrast with Theorem 4.2.2, where we can only prove existence of some positive constant $c(\tau)$, we obtain now an explicit estimate for the observability constant. The proof is presented in section 4.3.

Theorem 4.2.3. *For every $\tau > 0$ there exist explicit constants $c(\tau, \varepsilon), C(\tau, \varepsilon)$ such that:*

$$c(\tau, \varepsilon) \|u_0\|_{H_0^1(0,1)}^2 \leq \int_0^\tau |u_x(0, t)|^2 + |u_x(\ell(t), t)|^2 dt \leq C(\tau, \varepsilon) \|u_0\|_{H_0^1(0,1)}^2 \quad (4.2.6)$$

In particular, the Neumann observation at the boundary of the system ($\mathcal{S}_{\text{moving}}$) is exact observable in any time $\tau > 0$. Moreover, the observability coefficient $c(\tau, \varepsilon)$ decays $\sim \exp\left(\frac{-2k\pi^2}{\varepsilon\tau}\right)$ where $k > \frac{3}{2}$.

Remark 4.2.4. By Dirichlet condition $u(\ell(t), t) = 0$ for all t . Differentiating yields $\ell'(t)u_x(\ell(t), t) + u_t(\ell(t), t) = 0$, and so $u_x(\ell(t), t) = \frac{-1}{\varepsilon}u_t(\ell(t), t)$. As a result, observing $u_t(\ell(t), t)$ or $u_x(\ell(t), t)$ is, up to a constant, the same.

Point observations

We now focus on point observations $u \mapsto u(a, t)$ in the case of a linearly moving wall $\ell(t) = 1 + \varepsilon t$. Observe that in the “degenerate” case that is, $\varepsilon = 0$, the (then) autonomous Schrödinger equation has the well-known solution

$$u(x, t) = \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 t} \sin(n\pi x).$$

Clearly, there is no reasonable observability possible at rational points x since infinitely many terms in the sum vanish, independently of the leading coefficient a_n . This changes when $\varepsilon > 0$: from (4.2.5) we obtain

$$u(a, t) = \sum_{n=1}^{+\infty} a_n \left(\frac{2}{\ell(t)}\right)^{\frac{1}{2}} \exp\left(\frac{i\varepsilon a^2}{4\ell(t)} - in^2\pi^2 \frac{t}{\ell(t)}\right) \sin\left(\frac{n\pi a}{\ell(t)}\right)$$

and so

$$\int_0^\tau |u(a, t)|^2 dt = \int_0^\tau \frac{2}{\ell(t)} \left| \sum_{n=1}^{+\infty} a_n e^{-in^2\pi^2 \frac{t}{\ell(t)}} \sin\left(\frac{n\pi a}{\ell(t)}\right) \right|^2 dt. \quad (4.2.7)$$

Based on a remarkable result of Tenenbaum and Tucsnaak we obtain the following result in section 4.3.

Theorem 4.2.5. Assume $\ell(t) = 1 + \varepsilon t$. Then, for every $\tau > 0$, we have:

$$K(\tau) \|u_0\|_{L_2(0,1)}^2 \gtrsim \int_0^\tau |u(a, t)|^2 dt \gtrsim k(\tau) \|u_0\|_{L_2(0,1)}^2 \quad (4.2.8)$$

More precisely, $k(\tau) \approx M e^{-\frac{c}{\tau}}$ where $T = \frac{1}{\ell(0)} - \frac{1}{\ell(\tau)}$ and M, c are some positive constants that appear in the proof.

Corollary 4.2.6. For all $a \in (0, 1)$ the point observation $C = \delta_a$ for the system (S_{moving}) is exactly observable in arbitrary short time.

L_p -estimates of point observations

Finally we have the following L_p admissibility and observability estimates.

Theorem 4.2.7. Let $\ell(t) = 1 + \varepsilon t$. We assume that $u_0 \in H_0^1(0, 1)$. For $0 < p < 2$ and $a \in (0, 1)$, we have

$$k_p(\tau) \|u_0\|_{L_2(0,1)}^{2/p} \|u_0\|_{H_0^1(0,1)}^{1-2/p} \leq \left(\int_0^\tau |u(a, t)|^p dt \right)^{1/p} \leq K_p(\tau) \|u_0\|_{L_2(0,1)}^{2/p} \|u_0\|_{H_0^1(0,1)}^{1-2/p}$$

where $k_p(\tau)$, are constants depending on τ and p .

The upper estimate is a direct consequence of (4.2.8). Indeed, by the continuity of the embeddings $H_0^1 \hookrightarrow L_2 \hookrightarrow L_p$ and the boundedness of $\ell(t)$ to obtain:

$$\|u(a, t)\|_{L_p} \lesssim \|u(a, t)\|_{L_2} \stackrel{\text{from (4.2.8)}}{\lesssim} \|u_0\|_{L_2} \lesssim \|u_0\|_{L_2(0,1)}^{2/p} \|u_0\|_{H_0^1(0,1)}^{1-2/p}$$

Hence, it serves only to show that the lower estimate is of the right order.

4.3 Proof of the main results

4.3.1 The multiplier Lemma

We follow E. Machtyngier [84, Lemma 2.2] by using multiplier method for (S_{fixed}) : Let w be a solution to (S_{fixed}) and $q \in C^2([0, 1] \times [0, \tau])$ be a real valued function. Then, due to the differential equation (S_{fixed}) ,

$$\operatorname{Re} \left(\int_0^\tau \int_0^1 (q \bar{w}_y + \frac{1}{2} \bar{w} q_y) \left(i w_t + \frac{1}{\ell(t)^2} w_{yy} - i \frac{\ell'(t)}{\ell(t)} y w_y \right) dy dt \right) = 0 \quad (4.3.1)$$

We separate the left hand side of (4.3.1) into three parts and simplify each of them.

Lemma 4.3.1. *The following identities hold.*

$$\left\{ \begin{array}{l} \operatorname{Re} \left(\int_0^\tau \int_0^1 (q \bar{w}_y + \frac{1}{2} \bar{w} q_y) i w_t \, dy \, dt \right) \\ = \operatorname{Re} \left(\int_0^1 \left[\frac{1}{2} i q \bar{w}_y w \right]_{t=0}^{t=\tau} dy \right) - \frac{1}{2} \operatorname{Re} \left(\int_0^\tau \int_0^1 i w q_t \bar{w}_y \, dy \, dt \right) \end{array} \right. \quad (4.3.2)$$

$$\left\{ \begin{array}{l} \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{w_{yy}}{\ell(t)^2} (q \bar{w}_y + \frac{1}{2} \bar{w} q_y) \, dy \, dt \right) \\ = \operatorname{Re} \left(\int_0^\tau \frac{1}{2 \ell(t)^2} (q(1, t) |w_y(1, t)|^2 - q(0, t) |w_y(0, t)|^2) \, dt \right) \\ - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{\ell(t)^2} |w_y|^2 q_y \, dy \, dt \right) - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{w_y \bar{w}}{2 \ell(t)^2} q_{yy} \, dy \, dt \right) \end{array} \right. \quad (4.3.3)$$

$$\left\{ \begin{array}{l} - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{i y \ell'(t)}{\ell(t)} w_y (q \bar{w}_y + \frac{1}{2} \bar{w} q_y) \, dy \, dt \right) \\ = - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{i y \ell'(t)}{\ell(t)} q |w_y|^2 \, dy \, dt \right) - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} \frac{i y \ell'(t)}{\ell(t)} w_y \bar{w} q_y \, dy \, dt \right) \end{array} \right. \quad (4.3.4)$$

Proof. To prove (4.3.2), we use integration by parts. Using $\bar{w}(0, t) = \bar{w}(1, t) = 0$, we have:

$$\begin{aligned} \frac{1}{2} \operatorname{Re} \left(i \int_0^\tau \int_0^1 q_y \cdot \bar{w} w_t \, dy \, dt \right) &= \frac{1}{2} \operatorname{Re} \left(i \int_0^\tau \left(\left[\bar{w} w_t q \right]_{y=0}^{y=1} - \int_0^1 q \cdot (\bar{w}_y w_t + \bar{w} w_{ty}) \, dy \right) dt \right) \\ &= - \frac{1}{2} \operatorname{Re} \left(i \int_0^\tau \int_0^1 q (\bar{w}_y w_t + \bar{w} w_{ty}) \, dy \, dt \right) \end{aligned}$$

Therefore, the left hand side of (4.3.2) equals

$$\begin{aligned} &\operatorname{Re} \left(\int_0^\tau \int_0^1 (q \bar{w}_y + \frac{1}{2} \bar{w} q_y) i w_t \, dy \, dt \right) = \frac{1}{2} \operatorname{Re} \left(i \int_0^\tau \int_0^1 (w_t \cdot q \bar{w}_y - q \bar{w} w_{ty}) \, dy \, dt \right) \\ &= \frac{1}{2} \operatorname{Re} \left(i \int_0^1 \left(\left[q \bar{w}_y w \right]_{t=0}^{t=\tau} - \int_0^\tau w (q_t \bar{w}_y + q \bar{w}_{yt}) \, dt \right) dy \right) - \frac{1}{2} \operatorname{Re} \left(\int_0^\tau \int_0^1 q i \bar{w} w_{ty} \, dy \, dt \right) \\ &= \frac{1}{2} \operatorname{Re} \left(i \int_0^1 \left(\left[q \bar{w}_y w \right]_{t=0}^{t=\tau} \right) dy \right) - \frac{1}{2} \operatorname{Re} \left(\int_0^1 \int_0^\tau i w q_t \bar{w}_y \, dy \, dt \right) \end{aligned}$$

Here, we already use the fact that

$$- \operatorname{Re} \left(\int_0^1 \int_0^\tau i q w \bar{w}_{ty} \, dt \, dy \right) = \operatorname{Re} \left(\int_0^1 \int_0^\tau i q \bar{w} w_{ty} \, dt \, dy \right).$$

To prove (4.3.3) we have

$$\begin{aligned} \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{w_{yy}}{\ell(t)^2} q \overline{w_y} dy dt\right) &= \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{d}{dy}(|w_y|^2) \cdot \frac{1}{2\ell(t)^2} q dy dt\right) \\ &= \operatorname{Re}\left(\int_0^\tau \frac{1}{2\ell(t)^2} (q(1,t)|w_y(1,t)|^2 - q(0,t)|w_y(0,t)|^2) dt\right) \\ &\quad - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} q_y |w_y|^2 dy dt\right) \end{aligned}$$

since we use $\operatorname{Re}(w_{yy}\overline{w_y}) = \operatorname{Re}(\overline{w_{yy}}w_y)$. Again, integration by parts shows

$$\begin{aligned} \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{w_{yy}}{2\ell(t)^2} \overline{w} q_y dy dt\right) &= \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} \overline{w} q_y d(w_y) dt\right) \\ &= \operatorname{Re}\left(\int_0^\tau \left(\left[\frac{1}{2\ell(t)^2} \overline{w} q_y w_y\right]_{y=0}^{y=1}\right) dt\right) - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} (\overline{w_y} q_y + \overline{w} q_{yy}) w_y dy dt\right) \\ &= -\operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{1}{2\ell(t)^2} (\overline{w_y} q_y + \overline{w} q_{yy}) w_y dy dt\right) \end{aligned}$$

Therefore we have:

$$\begin{aligned} &\operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{w_{yy}}{\ell(t)^2} (q \overline{w_y} + \frac{1}{2} \overline{w} q_y) dy dt\right) \\ &= \operatorname{Re}\left(\int_0^\tau \frac{1}{2\ell(t)^2} (q(1,t)w_y^2(1,t) - q(0,t)w_y^2(0,t)) dt\right) - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{1}{\ell(t)^2} |w_y|^2 q_y dy dt\right) \\ &\quad - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{w_y \overline{w}}{2\ell(t)^2} q_{yy} dy dt\right) \end{aligned}$$

Hence, part (4.3.3) is proved. The last part is obvious. \square

Now summing up the three parts and using (4.3.1) yields

Proposition 4.3.2. *For any real valued function $q \in C^2([0, 1] \times [0, \tau])$ and a solution w to ($\mathcal{S}_{\text{fixed}}$) we have*

$$\begin{aligned} 0 &= \operatorname{Re}\left(\int_0^1 \frac{i}{2} \left[q \overline{w_y} w\right]_{t=0}^{t=\tau} dy\right) - \frac{1}{2} \operatorname{Re}\left(\int_0^\tau \int_0^1 i w q_t \overline{w_y} dy dt\right) \\ &\quad + \operatorname{Re}\left(\int_0^\tau \frac{1}{2\ell(t)^2} (q(1,t)|w_y(1,t)|^2 - q(0,t)|w_y(0,t)|^2) dt\right) \\ &\quad - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{1}{\ell(t)^2} |w_y|^2 q_y dy dt\right) - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{w_y \overline{w}}{2\ell(t)^2} q_{yy} dy dt\right) \\ &\quad - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{i y \ell'(t)}{\ell(t)} q |w_y|^2 dy dt\right) - \operatorname{Re}\left(\int_0^\tau \int_0^1 \frac{1}{2} \frac{i y \ell'(t)}{\ell(t)} w_y \overline{w} q_y dy dt\right) \end{aligned}$$

4.3.2 Energy estimates

For a solution w to ($\mathcal{S}_{\text{fixed}}$) we define the first and second energy as

$$E(t) = \frac{1}{2} \int_0^1 |w(y, t)|^2 dy \quad \text{and} \quad F(t) = \frac{1}{2} \int_0^1 |w_y(y, t)|^2 dy$$

respectively.

Lemma 4.3.3. *We have $\ell(\tau)E(\tau) = E(0)$.*

Proof. Taking the derivative respected to t and using $\mathcal{S}_{\text{fixed}}$, we have

$$\begin{aligned} \frac{dE(t)}{dt} &= \frac{d}{dt} \frac{1}{2} \int_0^1 |w(y, t)|^2 dy = \frac{1}{2} \int_0^1 (w_t \bar{w} + w \bar{w}_t) dy \\ &= \frac{1}{2} \int_0^1 \left(\frac{i}{\ell(t)^2} w_{yy} + \frac{\ell'(t)}{\ell(t)} y w_y \right) \bar{w} + \overline{w \left(\frac{i}{\ell(t)^2} w_{yy} + \frac{\ell'(t)}{\ell(t)} y w_y \right)} dy \\ &= \frac{1}{2} \int_0^1 \left(\frac{i}{\ell(t)^2} (w_{yy} \bar{w} - \overline{w_{yy} w}) + \frac{\ell'(t)}{\ell(t)} y (w_y \bar{w} + \overline{w_y w}) \right) dy \end{aligned}$$

Now integration by parts gives

$$\begin{aligned} \int_0^1 \frac{i}{\ell(t)^2} (w_{yy} \bar{w} - \overline{w_{yy} w}) dy &= \int_0^1 \frac{i}{\ell(t)^2} \bar{w} d(w_y) - \int_0^1 \frac{i}{\ell(t)^2} w d(\bar{w}_y) \\ &= \frac{i}{\ell(t)^2} \left([\bar{w} w_y]_{y=0}^{y=1} - \int_0^1 |w_y|^2 \right) - \frac{i}{\ell(t)^2} \left([w \bar{w}_y]_{y=0}^{y=1} - \int_0^1 |w_y|^2 \right) \\ &= 0 \end{aligned}$$

whereas

$$\begin{aligned} \int_0^1 \frac{\ell'(t)}{\ell(t)} y (w_y \bar{w} + \overline{w_y w}) dy &= \int_0^1 \frac{\ell'(t)}{\ell(t)} y \bar{w} d(w) - \int_0^1 \frac{\ell'(t)}{\ell(t)} y w d(\bar{w}) \\ &= \left[\frac{\ell'(t)}{\ell(t)} y \bar{w} w \right]_{y=0}^{y=1} - \frac{\ell'(t)}{\ell(t)} \int_0^1 (\bar{w} + y \bar{w}_y) w dy + \left[\frac{\ell'(t)}{\ell(t)} y \bar{w} w \right]_{y=0}^{y=1} - \frac{\ell'(t)}{\ell(t)} \int_0^1 (w + y w_y) \bar{w} dy \\ &= -\frac{2\ell'(t)}{\ell(t)} \int_0^1 |w(y, t)|^2 dy - \int_0^1 \frac{\ell'(t)}{\ell(t)} y (w_y \bar{w} + \overline{w_y w}) dy. \end{aligned}$$

Therefore,

$$\int_0^1 \frac{\ell'(t)}{\ell(t)} y (w_y \bar{w} + \overline{w_y w}) dy = -\frac{\ell'(t)}{\ell(t)} \int_0^1 |w(y, t)|^2 dy,$$

so that

$$\frac{dE(t)}{dt} = -\frac{1}{2} \int_0^1 \frac{\ell'(t)}{\ell(t)} |w(y, t)|^2 dy = -\frac{\ell'(t)}{\ell(t)} E(t).$$

Using $\ell(0) = 1$, this implies easily $E(\tau) = \frac{E(0)}{\ell(\tau)}$. □

Lemma 4.3.4. *For all $\tau > 0$ and $\tau \in \left(0, \frac{\pi}{2\omega}\right)$, we have:*

$$\frac{\pi^2}{\ell(\tau)} E(0) \leq F(\tau) \leq \ell(\tau) F(0)$$

Proof. Concerning F we have

$$\begin{aligned} \frac{dF(t)}{dt} &= \frac{d}{dt} \frac{1}{2} \int_0^1 |w_y(y, t)|^2 dy = \frac{1}{2} \int_0^1 (w_{yt} \bar{w}_y + w_y \bar{w}_{yt}) dt \\ &= \frac{1}{2} \int_0^1 \left(\frac{i}{\ell(t)^2} w_{yy} + \frac{\ell'(t)}{\ell(t)} y w_y \right)_y \bar{w}_y + \overline{w_y \left(\frac{i}{\ell(t)^2} w_{yy} + \frac{\ell'(t)}{\ell(t)} y w_y \right)_y} dy \\ &= \frac{i}{2\ell(t)^2} \int_0^1 (w_{yyy} \bar{w}_y - \overline{w_{yyy} w_y}) dy + \frac{\ell'(t)}{2\ell(t)} \int_0^1 ((y w_y)_y \bar{w}_y + w_y \overline{(y w_y)_y}) dy. \end{aligned}$$

The first term on the right hand side simplifies as

$$\begin{aligned}
 & \frac{i}{2\ell(t)^2} \int_0^1 (w_{yy} \overline{w_y} - \overline{w_{yy}} w_y) dy \\
 &= \frac{i}{2\ell(t)^2} \int_0^1 \overline{w_y} d(w_{yy}) - \frac{i}{2\ell(t)^2} \int_0^1 w_y d(\overline{w_{yy}}) \\
 &= \frac{i}{2\ell(t)^2} \left[\overline{w_y} w_{yy} \right]_{y=0}^{y=1} - \frac{i}{2\ell(t)^2} \int_0^1 |w_{yy}|^2 dy - \frac{i}{2\ell(t)^2} \left[\overline{w_{yy}} w_y \right]_{y=0}^{y=1} + \frac{i}{2\ell(t)^2} \int_0^1 |w_{yy}|^2 dy \\
 &= \left[\frac{1}{2} \overline{w_y} (w_t - \frac{\ell'(t)}{\ell(t)} y w_y) \right]_{y=0}^{y=1} + \left[\frac{1}{2} w_y (\overline{w_t} - \frac{\ell'(t)}{\ell(t)} y \overline{w_y}) \right]_{y=0}^{y=1} = -\frac{\ell'(t)}{\ell(t)} |w_y(1, t)|^2
 \end{aligned}$$

whereas the second term simplifies as follows.

$$\begin{aligned}
 \frac{\ell'(t)}{2\ell(t)} \int_0^1 ((y w_y)_y \overline{w_y} + w_y \overline{(y w_y)_y}) dy &= \frac{\ell'(t)}{2\ell(t)} \int_0^1 (w_y + y w_{yy}) \overline{w_y} + w_y \overline{(w_y + y w_{yy})} dy \\
 &= \frac{\ell'(t)}{2\ell(t)} \int_0^1 2|w_y|^2 + y(w_{yy} \overline{w_y} + w_y \overline{w_{yy}}) dy \\
 &= \frac{\ell'(t)}{\ell(t)} \int_0^1 |w_y|^2 dy + \frac{\ell'(t)}{2\ell(t)} \int_0^1 y d(|w_y|^2) \\
 &= \frac{\ell'(t)}{2\ell(t)} \int_0^1 |w_y|^2 dy + \frac{\ell'(t)}{2\ell(t)} |w_y(1, t)|^2.
 \end{aligned}$$

We add both parts to obtain

$$\begin{aligned}
 \frac{dF(t)}{dt} &= \frac{\ell'(t)}{2\ell(t)} \int_0^1 |w_y(y, t)|^2 dt - \frac{1}{2} |w_y(1, t)|^2 \frac{\ell'(t)}{\ell(t)} \\
 &= \frac{\ell'(t)}{\ell(t)} \left(F(t) - \frac{1}{2} |w_y(1, t)|^2 \right),
 \end{aligned}$$

By Variation of constants, we get an explicit solution:

$$F(t) = \ell(t) F(0) - \ell(t) \int_0^t \frac{\ell'(s)}{2\ell(s)^2} |w_y(1, s)|^2 ds \quad (4.3.5)$$

One easily obtains an upper bound, namely $F(t) \leq F(0)\ell(t)$. For the lower bound, we use the Poincaré (or Wirtinger) inequality on $[0, 1]$ to obtain,

$$F(t) = \frac{1}{2} \int_0^1 |w_y(y, t)|^2 dy \geq \frac{\pi^2}{2} \int_0^1 |w(y, t)|^2 dy = \frac{\pi^2}{\ell(t)} E(0) \quad (4.3.6)$$

□

4.3.3 Admissibility of Neumann observations at the boundary

Proof of Theorem 4.2.2. We take the function $q(y, t) = q(y)\ell(t)$ on $(0, 1)$ satisfying $q(1) = 0$ and $q(0) = 1$. By Proposition 4.3.2, we have

$$\begin{aligned} & \operatorname{Re} \left(\int_0^\tau \frac{1}{2\ell(t)^2} q(0, t) |w_y(0, t)|^2 dt \right) = \operatorname{Re} \left(\int_0^1 \left[\frac{1}{2} i q \ell(t) \overline{w_y} w \right]_{t=0}^{t=\tau} dy \right) \\ & - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{\ell(t)} |w_y|^2 q_y dy dt \right) - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{w_y w}{2\ell(t)} q_{yy} dy dt \right) \\ & - \operatorname{Re} \left(\int_0^\tau \int_0^1 i y \ell'(t) q |w_y|^2 dy dt \right) - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i y \ell'(t) w_y \overline{w} q_y dy dt \right) \\ & - \frac{1}{2} \operatorname{Re} \left(\int_0^\tau \int_0^1 i w q \ell'(t) \overline{w_y} dy dt \right) \end{aligned}$$

Therefore, we have

$$\int_0^\tau \frac{1}{2\ell(t)} |w_y(0, t)|^2 dt \leq A + B + C + D + E + F,$$

where we estimate all five terms separately. Concerning A , we separate the products in the real part by $ab \leq \frac{1}{2}(a^2 + b^2)$, then use Lemmata 4.3.3 and 4.3.4 to obtain

$$\begin{aligned} A &= \left| \operatorname{Re} \left(\int_0^1 \left[\frac{1}{2} i q \ell(t) \overline{w_y} w \right]_{t=0}^{t=\tau} dy \right) \right| \\ &\leq \frac{1}{4} \|q\|_{L_\infty(0,1)} \left(\int_0^1 \ell(\tau) |w(y, \tau)|^2 + |w(y, 0)|^2 + \ell(\tau) |w_y(y, \tau)|^2 + |w_y(y, 0)|^2 dy \right) \\ &= \frac{1}{4} \|q\|_{L_\infty(0,1)} \left(\int_0^1 2|w(y, 0)|^2 + (1 + \ell(\tau)^2) |w_y(y, 0)|^2 dy \right) \\ &\leq \frac{1}{4} \|q\|_{L_\infty(0,1)} \left(\int_0^1 (2\pi^2 + 1 + \ell(\tau)^2) |w_y(y, 0)|^2 dy \right). \end{aligned}$$

The second term is easily estimated by Lemma 4.3.3:

$$\begin{aligned} B &= \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{\ell(t)} |w_y(y, t)|^2 q_y dy dt \right) \right| \leq \|q_y\|_{L_\infty(0,1)} \int_0^\tau \int_0^1 \frac{1}{\ell(t)} |w_y(y, t)|^2 dy dt \\ &\leq \|q_y\|_{L_\infty(0,1)} \int_0^\tau \int_0^1 |w_y(y, 0)|^2 dy dt \\ &= \|q_y\|_{L_\infty(0,1)} \tau \int_0^1 |w_y(y, 0)|^2 dy \end{aligned}$$

Part C is decoupled by Cauchy-Schwarz and then estimated using Lemma 4.3.4 as follows:

$$\begin{aligned}
 C &= \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{w_y w}{2\ell(t)} q_{yy} dy dt \right) \right| \leq \|q_{yy}\|_{L_\infty(0,1)} \left(\int_0^\tau \int_0^1 \frac{|w_y w|}{2\ell(t)} dy dt \right) \\
 &\leq \|q_{yy}\|_{L_\infty(0,1)} \left(\int_0^\tau \frac{1}{2\ell(t)} \left(\int_0^1 |w(y,t)|^2 dy \right)^{1/2} \left(\int_0^1 |w_y(y,t)|^2 dy \right)^{1/2} dt \right) \\
 &\leq \|q_{yy}\|_{L_\infty(0,1)} \left(\int_0^\tau \frac{\pi}{2\ell(t)} \left(\int_0^1 |w_y(y,t)|^2 dy \right) dt \right) \\
 &\leq \|q_{yy}\|_{L_\infty(0,1)} \left(\int_0^\tau \frac{\pi}{2} dt \right) \left(\int_0^1 |w_y(y,0)|^2 dy \right) \\
 &= \|q_{yy}\|_{L_\infty(0,1)} \frac{\pi\tau}{2} \left(\int_0^1 |w_y(y,0)|^2 dy \right).
 \end{aligned}$$

For the forth part, we use Lemma 4.3.4 to obtain

$$\begin{aligned}
 D &= \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 i y \ell'(t) q |w_y(y,t)|^2 dy dt \right) \right| \leq \|q\|_{L_\infty(0,1)} \int_0^\tau \int_0^1 \ell'(t) |w_y(y,t)|^2 dy dt \\
 &\leq \|q\|_{L_\infty(0,1)} \left(\int_0^\tau \ell'(t) \ell(t) dt \right) \left(\int_0^1 |w_y(y,0)|^2 dy \right) \\
 &= \|q\|_{L_\infty(0,1)} \frac{\ell(\tau)^2 - 1}{2} \left(\int_0^1 |w_y(y,0)|^2 dy \right).
 \end{aligned}$$

The estimate for fifth part E is similar to part C :

$$\begin{aligned}
 E &= \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i y \ell'(t) w_y \bar{w} q_y dy dt \right) \right| \leq \|q_y\|_{L_\infty(0,1)} \left(\int_0^\tau \int_0^1 \frac{1}{2} \ell'(t) |w_y| |\bar{w}| dy dt \right) \\
 &\leq \|q_y\|_{L_\infty(0,1)} \left(\int_0^\tau \frac{\ell'(t)}{2} \left(\int_0^1 |w(y,t)|^2 dy \right)^{1/2} \left(\int_0^1 |w_y(y,t)|^2 dy \right)^{1/2} dt \right) \\
 &\leq \|q_y\|_{L_\infty(0,1)} \left(\int_0^\tau \frac{\pi \ell'(t)}{2} \left(\int_0^1 |w_y(y,t)|^2 dy \right) dt \right) \\
 &\leq \|q_y\|_{L_\infty(0,1)} \left(\int_0^\tau \frac{\pi \ell'(t) \ell(t)}{2} dt \right) \left(\int_0^1 |w_y(y,0)|^2 dy \right) \\
 &= \|q_y\|_{L_\infty(0,1)} \frac{\pi}{4} (\ell(\tau)^2 - 1) \left(\int_0^1 |w_y(y,0)|^2 dy \right).
 \end{aligned}$$

Finally, the last part F is treated like part C and E :

$$\begin{aligned}
 F &= \frac{1}{2} \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 i w q \ell'(t) \bar{w}_y dy dt \right) \right| \leq \frac{1}{2} \|q\|_{L_\infty(0,1)} \left(\int_0^\tau \int_0^1 \ell'(t) |\bar{w}_y| |w| dy dt \right) \\
 &\leq \|q\|_{L_\infty(0,1)} \frac{\pi}{4} (\ell(\tau)^2 - 1) \left(\int_0^1 |w_y(y,0)|^2 dy \right).
 \end{aligned}$$

Summing up all five estimates, we obtain

$$\int_0^\tau \frac{1}{2\ell(t)} |w_y(0,t)|^2 dt \leq C_1(\tau) \|w_0\|_{H_0^1(0,1)}^2 \quad (4.3.7)$$

where the constant $C_1(\tau)$ is given by

$$\begin{aligned} C_1(\tau) = & \frac{(3 + \pi^2)\ell(\tau)^2 + \pi^2 - 1}{4} \|q\|_{L_\infty(0,1)} + \left(\tau + \frac{\pi}{4}(\ell(\tau)^2 - 1)\right) \|q_y\|_{L_\infty(0,1)} \\ & + \frac{\pi\tau}{2} \|q_{yy}\|_{L_\infty(0,1)} \end{aligned} \quad (4.3.8)$$

Replacing $w_y(0, t) = \ell(t)u_x(0, t)$ in (4.3.7) yields the admissibility inequality:

$$\int_0^\tau |u_x(0, t)|^2 dt \leq \int_0^\tau \ell(t) |u_x(0, t)|^2 dt \leq 2C_1(\tau) \|u_0\|_{H_0^1(0,1)}^2$$

The second admissibility estimate follows the same lines, using $q(y, t) = q(y)\ell(t)$ on $(0, 1)$ with $q(0) = 0$ and $q(1) = 1$. \square

Neumann Observability at the Boundary

Recall the following lemma

Lemma 4.3.5. *Let E_1, E_2 and E_3 be the Hilbert spaces. We consider the continuous linear operators $T : E_1 \rightarrow E_2$, $K : E_1 \rightarrow E_3$ and $L : E_1 \rightarrow E_1$ such that K is compact, L is bounded below and:*

$$\|Lu\|_{E_1} \approx \|Tu\|_{E_2} + \|Ku\|_{E_3} \quad (4.3.9)$$

Then the kernel of A has finite dimension and $\|Lu\|_{E_1} \approx \|Tu\|_{E_3}$

Proof. A similar proof can be found in [120, Lemma 1 pp.1] where we just replace u by Lu . \square

Proof of Theorem 4.2.2. For all τ satisfying $2\tau + \pi(1 - \ell(\tau)^2) > 0$, we choose two positive constants $\eta(\tau)$ and $\delta(\tau)$ such that:

$$\eta(\tau) + \delta(\tau) < \frac{4}{1 + \ell(\tau)^3} \left(\tau - \frac{\pi}{2}(\ell(\tau)^2 - 1) \right) \quad (4.3.10)$$

We choose $q(y) = (1 - y)\ell(t)$ where $y \in (0, 1)$. Proposition 4.3.2 is then equivalent to:

$$\begin{aligned} \int_0^\tau \frac{1}{2\ell(t)} |w_y(0, t)|^2 dt = & \int_0^\tau \int_0^1 \frac{1}{\ell(t)} |w_y|^2 dy dt - \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i(1 - y) \ell'(t) \overline{w_y} w dy dt \right) \\ & + \operatorname{Re} \left(\int_0^1 \left[\frac{1}{2} i(1 - y) \ell(t) \overline{w_y} w \right]_{t=0}^{t=\tau} dy \right) + \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i y \ell'(t) w_y \overline{w} dy dt \right) \end{aligned} \quad (4.3.11)$$

Taking the three last formula of the right hand side to the left, then taking the absolute to get:

$$\begin{aligned} \int_0^\tau \int_0^1 \frac{1}{\ell(t)} |w_y|^2 dy dt \leq & \int_0^\tau \frac{1}{2\ell(t)} |w_y(0, t)|^2 dt + \left| \operatorname{Re} \left(\int_0^1 \left[\frac{1}{2} i(1 - y) \ell(t) \overline{w_y} w \right]_{t=0}^{t=\tau} dy \right) \right| \\ & + \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i(1 - y) \ell'(t) \overline{w_y} w dy dt \right) \right| \\ & + \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i y \ell'(t) w_y \overline{w} dy dt \right) \right| \end{aligned}$$

The sum of third and fourth terms in the right hand side of above formula can be estimated as:

$$\begin{aligned}
 & \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i (1-y) \ell'(t) \overline{w_y} w \, dy \, dt \right) \right| + \left| \operatorname{Re} \left(\int_0^\tau \int_0^1 \frac{1}{2} i y \ell'(t) w_y \overline{w} \, dy \, dt \right) \right| \\
 & \leq \frac{1}{2} \int_0^\tau \int_0^1 \ell'(t) |w_y \overline{w}| \, dy \, dt + \frac{1}{2} \int_0^\tau \int_0^1 \ell'(t) |\overline{w_y} w| \, dy \, dt \\
 & \leq \int_0^\tau \ell'(t) \left(\int_0^1 |w|^2 \, dy \right)^{1/2} \left(\int_0^1 |w_y|^2 \, dy \right)^{1/2} \, dt \\
 & \leq \int_0^\tau \pi \ell'(t) \left(\int_0^1 |w_y|^2 \, dy \right) \, dt
 \end{aligned}$$

Due to the energy estimate in lemma 4.3.3 and 4.3.4, we have the upper bound for the second term:

$$\begin{aligned}
 & \left| \operatorname{Re} \left(\int_0^1 \left[\frac{1}{2} i (1-y) \ell(t) \overline{w_y} w \right]_{t=0}^{t=\tau} \, dy \right) \right| \\
 & \leq \frac{1}{4} \int_0^1 \left(\frac{|w(y, 0)|^2}{\eta(\tau)} + \frac{|w(y, \tau)|^2}{\eta(\tau)} + \eta(\tau) |w_y(y, 0)|^2 + \eta(\tau) \ell(\tau)^2 |w_y(y, \tau)|^2 \right) \, dy \\
 & \leq \frac{1}{4\eta(\tau)} \left(\frac{1}{\ell(\tau)} + 1 \right) \int_0^1 |w(y, 0)|^2 \, dy + \frac{(1+\ell(\tau)^3)\eta(\tau)}{4} \int_0^1 |w_y(y, 0)|^2 \, dy
 \end{aligned}$$

As a result, we combine these estimation and use (4.3.5) to obtain:

$$\begin{aligned}
 & \int_0^\tau \frac{1}{2\ell(t)} |w_y(0, t)|^2 \, dt + \frac{1}{4\eta(\tau)} \left(\frac{1}{\ell(\tau)} + 1 \right) \int_0^1 |w(y, 0)|^2 \, dy \\
 & \geq \int_0^\tau \int_0^1 \left(\frac{1}{\ell(t)} - \pi \ell'(t) \right) |w_y(y, t)|^2 \, dy \, dt - \frac{(1+\ell(\tau)^3)\eta(\tau)}{4} \int_0^1 |w_y(y, 0)|^2 \, dy \\
 & = \left(\int_0^\tau (1 - \pi \ell'(t) \ell(t)) \, dt - \frac{(1+\ell(\tau)^3)\eta(\tau)}{4} \right) \left(\int_0^1 |w_y(y, 0)|^2 \, dy \right) \\
 & \quad - \int_0^\tau (1 - \pi \ell'(t) \ell(t)) \int_0^t \frac{\ell'(s)}{\ell(s)^2} |w_y(1, s)|^2 \, ds \, dt \\
 & = \left(\tau + \frac{\pi}{2} (1 - \ell(\tau)^2) - \frac{(1+\ell(\tau)^3)\eta(\tau)}{4} \right) \left(\int_0^1 |w_y(y, 0)|^2 \, dy \right) \\
 & \quad - \int_0^\tau (1 - \pi \ell'(t) \ell(t)) \int_0^t \frac{\ell'(s)}{\ell(s)^2} |w_y(1, s)|^2 \, ds \, dt \\
 & \geq \frac{(1+\ell(\tau)^3)\delta(\tau)}{4} \left(\int_0^1 |w_y(y, 0)|^2 \, dy \right) - \int_0^\tau (1 - \pi \ell'(t) \ell(t)) \int_0^t \frac{\ell'(s)}{\ell(s)^2} |w_y(1, s)|^2 \, ds \, dt
 \end{aligned}$$

where the last inequality come from (4.3.10). Therefore, there exist the constants A_τ and B_τ such that:

$$\int_0^1 |w_y(y, 0)|^2 \, dy \leq A_\tau \int_0^\tau (|w_y(0, t)|^2 + |w_y(1, t)|^2) \, dt + B_\tau \int_0^1 |w(y, 0)|^2 \, dy \quad (4.3.12)$$

It is sufficient to prove that there exist a constant $K > 0$ such that

$$\int_0^1 |w(y, 0)|^2 dy \leq K \left(\int_0^\tau |w_y(0, t)|^2 dt + \int_0^\tau |w_y(1, t)|^2 dt \right) \quad (4.3.13)$$

Let us denote the operator T from $H_0^1(0, \tau)$ to $L_2(0, \tau) \times L_2(0, \tau)$ and the operator K from $H_0^1(0, 1)$ to $L_2(0, 1)$ that maps:

$$(Tw)(t) = (w_y(0, t), w_y(1, t)) \quad (4.3.14)$$

$$(Kw)(y) = w(y, 0) \quad (4.3.15)$$

From admissibility and (4.3.12), we have:

$$a_\tau \|Tw\|_{L_2}^2 + b_\tau \|Kw\|_{L_2}^2 \leq \|w_0\|_{H_0^1}^2 \leq A_\tau \|Tw\|_{L_2}^2 + B_\tau \|Kw\|_{L_2}^2 \quad (4.3.16)$$

It is easy to see that K is compact operator due to Rellich's embedding lemma. In order to use the unique-compactness lemma 4.3.5 for $L = K$, we need to check that T is injective. Observe that $Tw = 0$ means that w satisfies (S_{fixed}) with Dirichlet conditions and zero Neumann derivative. It is well known that w vanishes in this case, see for example [121, Theorem 3] or [47, Corollary 6.1]. As a consequence,

$$c_\tau \|Tw\|_{L_2}^2 \leq \|w_0\|_{H_0^1}^2 \leq C_\tau \|Tw\|_{L_2}^2$$

for some constants $c(\tau), C(\tau) > 0$.

□

4.3.4 Results for linear moving walls

Recall the Doescher-Rice representation formula (4.2.5) that yields for $t = 0$

$$u(x, 0) = \sqrt{2} \sum_{n=1}^N a_n e^{\frac{i\epsilon x^2}{4}} \sin(n\pi x), \quad (4.3.17)$$

and denote by

$$u_n(x, t) := \sqrt{\frac{2}{\ell(t)}} \sin\left(\frac{n\pi x}{\ell(t)}\right).$$

For all fixed $t > 0$, the functions $(u_n(\cdot, t))_{n \geq 1}$ form an orthonormal basis in $L_2(0, \ell(t))$, since the change of variable $y = \frac{x}{\ell(t)}$ reduces $u_n(\cdot, t)$ to the standard trigonometric system on $L_2([0, 1])$.

Lemma 4.3.6. *For all finitely supported sequences (a_n) we have the following relation between (a_n) and the norms of the initial data u_0 .*

$$\|u(x, 0)\|_{L_2(0, 1)}^2 = \sum_{n=1}^{+\infty} |a_n|^2, \quad \|u(x, 0)\|_{H_0^1(0, 1)}^2 \sim \sum_{n=1}^{+\infty} |a_n|^2 n^2$$

Proof. Observe that

$$\begin{aligned} \|e^{-\frac{i\varepsilon x^2}{4}} u_N(x)\|_{L_2(0,1)}^2 &= \|u_N(x)\|_{L_2(0,1)}^2 = 2 \int_0^1 \left| \sum_{n=1}^N a_n \sin(n\pi x) \right|^2 dx \\ &= 2 \int_0^1 \left| \sum_{n=1}^N a_n \sin(n\pi x) \right|^2 dx = \sum_{n=1}^{\infty} |a_n|^2. \end{aligned}$$

Since (a_n) is a finite sequence we may interchange differentiation and summation and obtain

$$\frac{d}{dx} u(x) = \sqrt{2} \sum_{n=1}^N a_n e^{\frac{i\varepsilon x^2}{4}} (ix \frac{\varepsilon}{2} \sin(n\pi x) + n\pi \cos(n\pi x))$$

so that, squaring real and imaginary parts, we find

$$\begin{aligned} \|u(x)\|_{H_0^1(0,1)}^2 &= 2 \int_0^1 \left| \sum_{n=1}^N a_n n\pi \cos(n\pi x) \right|^2 dx + 2 \int_0^1 \left| \sum_{n=1}^N a_n x \frac{\varepsilon}{2} \sin(n\pi x) \right|^2 dx \\ &= \pi^2 \sum_{n=1}^N |a_n|^2 n^2 + 2 \int_0^1 \left| \sum_{n=1}^N a_n x \frac{\varepsilon}{2} \sin(n\pi x) \right|^2 dx \\ &\leq \pi^2 \sum_{n=1}^N |a_n|^2 n^2 + \frac{\varepsilon^2}{2} \int_0^1 \left| \sum_{n=1}^N a_n \sin(n\pi x) \right|^2 dx \\ &= \pi^2 \sum_{n=1}^N |a_n|^2 n^2 + \frac{\varepsilon^2}{2} \sum_{n=1}^N |a_n|^2 \leq C(\varepsilon) \sum_{n=1}^N |a_n|^2 n^2 \quad \square \end{aligned}$$

Lemma 4.3.7. *Let $\varepsilon \in (0, \frac{\pi}{2})$ and $\tau = \frac{2}{\pi-2\varepsilon}$, then the functions $b_n(t) = \frac{\sqrt{\pi}}{\sqrt{2\ell(t)}} e^{-i\pi^2 n^2 \frac{t}{\ell(t)}}$ for $n \geq 1$ form an orthonormal system in $L_2(0, \tau)$.*

Proof. Note that $(\frac{t}{\ell(t)})' = \frac{\ell(t) - t\ell'(t)}{\ell(t)^2} = \frac{1}{\ell(t)^2}$. Therefore, the obvious change of variable $x = \frac{t}{\ell(t)}$ reduces f_n to a standard trigonometric function on $[0, \frac{\tau}{\ell(\tau)}]$. Observe that $\frac{\tau}{\ell(\tau)} = \frac{2}{\pi-2\varepsilon}(1 + \frac{2\varepsilon}{\pi-2\varepsilon})^{-1} = \frac{2}{\pi}$. Now orthonormality easily follows. \square

Observe that the above sequence $\{b_n(t)\}_{n \geq 1}$ is not an orthonormal basis. Indeed, with $f(t) = \frac{\sqrt{\pi}}{\sqrt{2\ell(t)}} e^{3i\pi^2 \frac{t}{\ell(t)}}$, we have $\langle f(t), b_n(t) \rangle = 0$ for all $n \in \mathbb{N}$.

Neumann observation at the Boundary

Proof of Theorem 4.2.3. We start considering only the first term at $x = 0$. As in the proof of Lemma 4.3.6 we consider for a moment only initial data associated with finitely supported sequences (a_n) . Differentiating the representation formula (4.2.5) u term by term yields

$$u_x(0, t) = \sum_{n=1}^{+\infty} a_n \left(\frac{2}{\ell(t)} \right)^{1/2} e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \frac{n\pi}{\ell(t)},$$

and therefore

$$\|u_x(0, t)\|_{L_2(0, \tau)}^2 = \int_0^\tau \frac{2\pi^2}{\ell(t)^3} \left| \sum_{n=1}^{+\infty} na_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \right|^2 dt.$$

Using the monotonicity of $\ell(t)$ in $[0, \tau]$, we have $\frac{2\pi^2}{\ell(\tau)} J \leq \|u_x(0, \cdot)\|_{L_2(0, \tau)}^2 \leq 2\pi^2 J$ where

$$J = \int_0^\tau \left| \sum_{n=1}^{+\infty} na_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \right|^2 \frac{dt}{\ell(t)^2}.$$

This allows to focus only on the integral J , where we abbreviate $b_n = na_n e^{-i\pi^2 n^2 / \varepsilon}$ and make a change of variable $\xi = \frac{1}{\ell(t)} + \frac{1}{2}(\frac{1}{\ell(0)} + \frac{1}{\ell(\tau)})$. Letting $T = \frac{1}{\ell(0)} - \frac{1}{\ell(\tau)}$, the above double inequality rewrites as

$$\int_{-T/2}^{+T/2} \left| \sum_{n=1}^{+\infty} b_n e^{-i\frac{\pi^2 n^2}{\varepsilon} \xi} \right|^2 d\xi \approx \|u_x(0, t)\|_{L_2(0, \tau)}^2$$

The sequence $\lambda_n = \frac{\pi^2 n^2}{\varepsilon}$ satisfies the hypotheses of [122, Theorem 3.1 and Corollary 3.3] so that, for all $k > \frac{3}{2}\pi^2$ and $r = \varepsilon/\pi^2$

$$\int_{-T/2}^{+T/2} \left| \sum_{n=1}^{+\infty} b_n e^{-i\frac{\pi^2 n^2}{\varepsilon} \xi} \right|^2 d\xi \gg e^{-\frac{2k}{r\tau}} \sum_{n=1}^{+\infty} |b_n|^2 = e^{-\frac{2k}{r\tau}} \sum_{n=1}^{+\infty} |na_n|^2.$$

On the other hand side, if $T \in [m\frac{\varepsilon}{\pi}, (m+1)\frac{\varepsilon}{\pi})$, we have by periodicity and Parseval's identity

$$\int_{-T/2}^{+T/2} \left| \sum_{n=1}^{+\infty} b_n e^{-in^2 \frac{\pi^2}{\varepsilon} \xi} \right|^2 d\xi \leq \int_{-(m+1)\frac{\varepsilon}{\pi}}^{(m+1)\frac{\varepsilon}{\pi}} \left| \sum_{n=1}^{+\infty} b_n e^{-in^2 \frac{\pi^2}{\varepsilon} \xi} \right|^2 d\xi = (m+1) \sum_{n=1}^{+\infty} |b_n|^2.$$

We conclude by Lemma 4.3.6 that

$$c(\varepsilon) \|u_0\|_{H_0^1(0,1)}^2 \leq \|u_x(0, t)\|_{L_2(0, \tau)}^2 \leq C(\varepsilon) \|u_0\|_{H_0^1(0,1)}^2.$$

This inequality being true for all u_0 leading to finitely supported sequences (a_n) , it is true for any $u_0 \in H_0^1(0, 1)$ by density.

For second term at $x = \ell(t)$, we see for finitely supported sequences (a_n) that

$$u_x(\ell(t), t) = \sum_{n=1}^{+\infty} (-1)^n a_n \left(\frac{2}{\ell(t)}\right)^{1/2} e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \frac{n\pi}{\ell(t)} e^{i\frac{\varepsilon}{4} \ell(t)}$$

Taking the L_2 -norm, one get the equivalent between $\|u_x(\ell(t), t)\|_{L_2}$ and $\|u_x(0, t)\|_{L_2}$

$$\begin{aligned} \|u_x(\ell(t), t)\|_{L_2(0, \tau)}^2 &= \int_0^\tau \left| \sum_{n=1}^{+\infty} (-1)^n a_n \left(\frac{2}{\ell(t)}\right)^{1/2} e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \frac{n\pi}{\ell(t)} e^{i\frac{\varepsilon}{4} \ell(t)} \right|^2 dt \\ &= \int_0^\tau \frac{2\pi^2}{\ell(t)} \left| \sum_{n=1}^{+\infty} ((-1)^n na_n) e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \right|^2 \frac{dt}{\ell(t)^2}. \end{aligned}$$

Clearly, the rest proof follows the lines above. \square

Internal Point Observability

Proof of Theorem 4.2.5. Since $\ell(t) \geq 1$ for all t ,

$$\int_0^\tau \frac{2}{\ell(t)} \left| \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \sin\left(\frac{n\pi a}{\ell(t)}\right) \right|^2 dt \geq \int_0^\tau \frac{2}{\ell(t)^2} \left| \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \sin\left(\frac{n\pi a}{\ell(t)}\right) \right|^2 dt.$$

By definition, $\sin\left(\frac{n\pi a}{\ell(t)}\right) = \frac{1}{2i} (\exp(i\frac{n\pi a}{\ell(t)}) - \exp(-i\frac{n\pi a}{\ell(t)}))$. Therefore,

$$\begin{aligned} \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \sin\left(\frac{n\pi a}{\ell(t)}\right) &= \frac{1}{2i} \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \left(e^{i\frac{n\pi a}{\ell(t)}} - e^{-i\frac{n\pi a}{\ell(t)}} \right) \\ &= \frac{1}{2i} \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{1}{\varepsilon}} \left(e^{-\frac{i\pi^2 n^2}{\varepsilon \ell(t)} + \frac{i n \pi a}{\ell(t)}} - e^{-\frac{i\pi^2 n^2}{\varepsilon \ell(t)} - \frac{i n \pi a}{\ell(t)}} \right) \end{aligned}$$

For $n \in \mathbb{Z}$, we extend the series by $a_n = a_{-n}$, and $\lambda_n = \frac{\pi^2 n^2}{\varepsilon} + \text{sign}(n)n\pi a$. The sequence $\lambda_n = \frac{\pi^2 n^2}{\varepsilon}$ is regular and satisfies the hypotheses of [122, Theorem 3.1] with $r = \frac{\varepsilon}{\pi^2}$ and $C = a\pi$. We follow the lines of the proof of Theorem 4.2.3: changing the variable $\xi = \frac{-1}{\ell(t)}$ gives with the notation $T = \frac{1}{\ell(0)} - \frac{1}{\ell(\tau)}$,

$$\int_0^\tau \frac{1}{\ell(t)^2} \left| \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \sin\left(\frac{n\pi a}{\ell(t)}\right) \right|^2 dt = \frac{1}{\varepsilon} \int_{-T/2}^{+T/2} \left| \sum_{n \in \mathbb{Z}} e^{-\frac{i\pi^2 n^2}{\varepsilon}} a_n e^{i\lambda_n \xi} \right|^2 d\xi$$

we write $b_n = e^{-\frac{i\pi^2 n^2}{\varepsilon}} a_n$ and use [122, Corollary 3.3] with $k > \frac{3\pi^2}{2}$:

$$\frac{1}{\varepsilon} \int_{-T}^T \left| \sum_{n \in \mathbb{Z}} a_n e^{-\frac{i\pi^2 n^2}{\varepsilon}} e^{-i\lambda_n \xi} \right|^2 d\xi \gg e^{-\frac{2k}{rT}} \sum_{n \in \mathbb{Z}} \left| a_n e^{-\frac{i\pi^2 n^2}{\varepsilon}} \right|^2 \geq e^{-\frac{2k}{rT}} \sum_{n=1}^{+\infty} |a_n|^2. \quad \square$$

For the upper estimate, we use similar method as in theorem (4.2.3). More precisely,

$$\|u(a, t)\|_{L_2} \leq \int_0^\tau \frac{2}{\ell(t)} \left| \sum_{n=1}^{+\infty} a_n e^{-i\pi^2 n^2 \frac{t}{\ell(t)}} \right|^2 dt \lesssim (m+1) \sum_{n=1}^{+\infty} |a_n|^2$$

where m be the integer number such that $\frac{\pi\varepsilon}{T} \in [m, m+1]$ with $T = \frac{1}{\ell(0)} - \frac{1}{\ell(\tau)}$.

L_p -admissibility and observability

Proof of Theorem 4.2.7. The upper estimate yielding $K_p(\tau)$ is obtained by interpolation of the two upper estimates in Theorem 4.2.3. We are left with the lower estimate. Since $u \in H_0^1$, $(na_n) \in \ell_2$, and so $(a_n) \in \ell_1$ by the Cauchy-Schwarz inequality. Let $p \in (0, 2)$ and let $\theta = \frac{2}{4-p} \in (0, 1)$ which is chosen to satisfy $p\theta + 4(1-\theta) = 2$. By Hölder's inequality we then have

$$\begin{aligned} \int_0^\tau |u(a, t)|^2 dt &= \int_0^\tau |u(a, t)|^{p\theta} \cdot |u(a, t)|^{4(1-\theta)} dt \\ &\leq \left(\int_0^\tau |u(a, t)|^p dt \right)^\theta \cdot \left(\int_0^\tau |u(a, t)|^4 dt \right)^{1-\theta} \end{aligned} \tag{4.3.18}$$

From trivial argument on boundedness of $\sin(\frac{n\pi a}{\ell(t)})$ and $e^{\frac{i\varepsilon a^2}{4\ell(t)} - i\pi^2 n^2 \frac{t}{\ell(t)}}$:

$$|u(a, t)|^2 = \left| \sum_{n=1}^{+\infty} a_n e^{\frac{i\varepsilon a^2}{4\ell(t)} - i\pi^2 n^2 \frac{t}{\ell(t)}} \sin\left(\frac{n\pi a}{\ell(t)}\right) \right|^2 \leq \left(\sum_{n=1}^{+\infty} |a_n| \right)^2$$

Combining with the estimate (4.3.18), one get:

$$\int_0^\tau |u(a, t)|^4 dt \leq \left(\sum_{n=1}^{+\infty} |a_n| \right)^2 \left(\int_0^\tau |u(a, t)|^2 dt \right) \quad (4.3.19)$$

From inequalities (4.3.18) and (4.3.19) and Theorem (4.2.5) we deduce now

$$\begin{aligned} \int_0^\tau |u(a, t)|^p dt &\geq \left(\int_0^\tau |u(a, t)|^2 dt \right)^{1/\theta} \left(\int_0^\tau |u(a, t)|^4 dt \right)^{\frac{\theta-1}{\theta}} \\ &\geq \left(\int_0^\tau |u(a, t)|^2 dt \right)^{1/\theta} \left(\sum_{n=1}^{+\infty} |a_n| \right)^{\frac{2(\theta-1)}{\theta}} \left(\int_0^\tau |u(a, t)|^2 dt \right)^{\frac{\theta-1}{\theta}} \\ &\geq k \left(\sum_{n=1}^{+\infty} |a_n|^2 \right) \left(\sum_{n=1}^{+\infty} |na_n|^2 \right)^{\frac{2\theta-1}{\theta}} \geq k \|u_0\|_{L_2(0,1)}^2 \|u_0\|_{H_0^1}^{\frac{2\theta-1}{\theta}}. \end{aligned}$$

Since $\frac{\theta-1}{\theta} = \frac{p-2}{2}$, the result follows. \square

4.4 Boundary controllability of dual problem

Since we have already stated several theorems that can be interpreted as exact observation we will briefly sketch the duality theory that allows to rephrase these assertions in terms of exact control, then the solution z to adjoint problem

$$z'(t) = -A(t)^* z(t) - C(t)^* C(t) w(t) \quad z(\tau) = 0 \quad (4.4.1)$$

satisfies $\langle w_0, z(0) \rangle = -\int_0^\tau \frac{d}{dt} \langle w(t), z(t) \rangle dt = \int_0^\tau \|C(t)w(t)\|^2 dt$ by injection of the respective differential equations of w and z . Hence exact observability implies that the Gramian $Q : w_0 \mapsto z(0)$ satisfies $\|Qw_0\| \|w_0\| \geq \langle w_0, Qw_0 \rangle \geq \delta \|w_0\|$ to the effect that Q has closed image. Moreover, if $Q^* w_0 = 0$, taking scalar product with w_0 reveals $w_0 = 0$, so Q^* is injective and hence Q has dense range. By the open mapping theorem, Q is therefore an isomorphism on X . This means that the adjoint problem (4.4.1) can be steered to any state $z(0) \in X$ by an appropriate choice of the initial value w_0 . Indeed, for $u, v \in D(A(t))$ we have

$$\begin{aligned} \langle A(t)u, v \rangle_X &= \left\langle \frac{i}{\ell(t)^2} u_{yy} + \frac{\ell'(t)}{\ell(t)} y u_y, v \right\rangle_X = \int_0^1 \frac{i}{\ell(t)^2} u_{yy} \bar{v} dy + \int_0^1 \frac{\ell'(t)}{\ell(t)} y u_y \bar{v} dy \\ (\text{int. by parts}) &= -\frac{i}{\ell(t)^2} \int_0^1 u_y \bar{v}_y dy - \frac{\ell'(t)}{\ell(t)} \int_0^1 (y u \bar{v}_y + u \bar{v}) dy \\ (\text{int. by parts}) &= \frac{i}{\ell(t)^2} \int_0^1 u \bar{v}_{yy} dy - \frac{\ell'(t)}{\ell(t)} \int_0^1 (y u \bar{v}_y + u \bar{v}) dy \\ &= -\left\langle u, \frac{i}{\ell(t)^2} v_{yy} + \frac{\ell'(t)}{\ell(t)} y v_y \right\rangle - \left\langle u, \frac{\ell'(t)}{\ell(t)} v \right\rangle = \left\langle u, -(A(t) + \frac{\ell'(t)}{\ell(t)}) v \right\rangle \end{aligned}$$

It turns out that in our case $A(t)^* = -A(t) - \frac{\ell'(t)}{\ell(t)}$. So exact observation of the Schrödinger equation (S_{moving}) can be reformulated as exact control for the Schrödinger equation with zero final time. We turn back to these ideas after stating our first theorem. In the case of linear moving $\ell(t) = 1 + \varepsilon t$, let $C(t) : D(A(t)) \rightarrow \mathbb{C}$ be given by $C(t)(\varphi) := \varphi_y(b)$ where $b \in \{0, 1\}$. The (lower) estimate in theorems 4.2.3 and 4.2.2 then reformulates as exact observability of $C(t)$ for the non-autonomous Cauchy problem (4.2.1). Some care has to be taken since $C(t)$ is unbounded on X . Indeed, $C(t)^* : \mathbb{C} \rightarrow D(A(t))'$ is given by $C(t)^*\alpha = -\alpha \frac{d}{dy}\delta_{y=b}$, then we obtain exact controllability of (4.4.1) in a distributional sense:

$$z_t = \frac{i}{\ell(t)^2} z_{yy} + \frac{\ell'(t)}{\ell(t)} y z_y + \frac{\ell'(t)}{\ell(t)} z + w_y(b, t) \frac{d}{dy} \delta_{y=b} \quad \text{and} \quad z(y, \tau) = 0$$

Multiplying with a test function $\eta \in D((0, 1))$, and integrating on $[0, 1]$ we obtain by partial integration

$$\begin{aligned} \int_0^1 z_t \eta(y) dy &= \int_0^1 \left(\frac{i}{\ell(t)^2} z_{yy} + \frac{\ell'(t)}{\ell(t)} (yz)_y \right) \eta(y) dy - w_y(b, t) \eta'(b) \\ &= \int_0^1 \left(\frac{i}{\ell(t)^2} z \eta''(y) - \frac{\ell'(t)}{\ell(t)} y z \eta'(y) \right) dy + \left(\frac{i}{\ell(t)^2} z(b, t) - w_y(b, t) \right) \eta'(b) \end{aligned}$$

This is possible for any test function η only if the point evaluation vanishes. The dual statement of the lower estimate in theorems 4.2.3 and 4.2.2 is thus exact controllability of a Schrödinger equation with Dirichlet control on the right boundary,

$$\begin{cases} z_t = \frac{i}{\ell(t)^2} z_{yy} + \frac{\ell'(t)}{\ell(t)} y z_y + \frac{\ell'(t)}{\ell(t)} z & (y, t) \in (0, 1) \times (0, \tau) \\ z(\bar{b}, t) = 0 & \{\bar{b}\} \cup \{b\} = \{0, 1\}, t \geq 0 \\ z(b, t) = -i\ell(t)^2 w_y(b, t) & t \geq 0 \\ z(y, \tau) = 0 & y \in [0, 1] \end{cases} \quad (4.4.2)$$

We reverse back to the moving boundary problem by taking $x = \ell(t)y$ and $h(x, t) = z(y, t)$. Then the problem can be written as:

$$\begin{cases} i h_t + h_{xx} - i \frac{\ell'(t)}{\ell(t)} h = 0 & (x, t) \in (0, \ell(t)) \times (0, \tau) \\ h(\ell(t), t) = 0 & t \geq 0 \\ h(0, t) = -i\ell(t)^3 u_x(0, t) & t \geq 0 \\ h(x, \tau) = 0 & x \in [0, \ell(t)] \end{cases} \quad (4.4.3)$$

or

$$\begin{cases} i h_t + h_{xx} - i \frac{\ell'(t)}{\ell(t)} h = 0 & (x, t) \in (0, \ell(t)) \times (0, \tau) \\ h(0, t) = 0 & t \geq 0 \\ h(\ell(t), t) = -i\ell(t)^3 u_x(\ell(t), t) & t \geq 0 \\ h(x, \tau) = 0 & x \in [0, \ell(t)] \end{cases} \quad (4.4.4)$$

In general situation of $\ell(t)$ satisfying condition (4.2.4), one take $C(t) : D(A(t)) \rightarrow \mathbb{C} \times \mathbb{C}$ be given by $C(t)(\varphi) := (\varphi_y(0), \varphi_y(1))$. Therefore, the dual operator $C(t)^* : \mathbb{C} \times \mathbb{C} \rightarrow D(A(t))'$ is given by $C(t)^*(\alpha, \beta) = -\alpha \frac{d}{dy} \delta_{y=0} - \beta \frac{d}{dy} \delta_{y=1}$. Using similar

argument, we obtain exact controllability of a Schrödinger equation with Dirichlet control applied on both of boundaries

$$\left\{ \begin{array}{ll} ih_t + h_{xx} - i\frac{\ell'(t)}{\ell(t)}h &= 0 & (x, t) \in (0, \ell(t)) \times (0, \tau) \\ h(0, t) &= -i\ell(t)^3 u_x(0, t) & t \geq 0 \\ h(\ell(t), t) &= -i\ell(t)^3 u_x(\ell(t), t) & t \geq 0 \\ h(x, \tau) &= 0 & x \in [0, \ell(t)] \end{array} \right. \quad (4.4.5)$$

Chapter 5

The averaged Hautus test for non-autonomous linear evolution equation

We consider the observability problem for non-autonomous evolution systems (i.e., the operators governing the system depend on time). We introduce an averaged Hautus condition and prove that for skew-adjoint operators it characterizes exact observability. Next, we extend this to more general class of operators under a growth condition on the associated evolution family. We give an application to the Schrödinger equation with time dependent potential and the damped wave equation with a time dependent damping coefficient.

5.1 Introduction

Observability is an important concept in system and control theory. It treats the question to which extent an observation, i.e., partial knowledge of the solution of an evolution equation, determines its initial or final state. The theory has been studied for several decades for systems of the form:

$$\begin{cases} x'(t) + Ax(t) &= 0 & t \in [0, T] \\ x(0) &= x_0 \\ y(t) &= Cx(t) \end{cases} \quad (5.1.1)$$

in which the two operators A and C are independent of time t and satisfy appropriate conditions such as $-A$, with domain $\mathcal{D}(A)$, generates a strongly continuous semigroup on a Hilbert space H and C is bounded from $\mathcal{D}(A)$ into another Hilbert space Y .

Observability consists of unique determination or recovery of the initial (or final) time state under the knowledge of the observed solution $y(\cdot)$. Recall that in the case of matrices A and C (finite dimensional setting), all observation concepts coincide and can be characterized in various manners. The Kalman rank condition is certainly the most known version; it states that C is observable if and only if the matrix

$$[C \mid CA \mid CA^2 \mid \dots \mid CA^{n-1}]$$

has full rank. An equivalent statement is the Hautus lemma: it characterizes observability by the condition

$$\forall \lambda \in \mathbb{C} : \quad \text{rank}[\lambda I - A, C] = n$$

that clearly is equivalent to the condition

$$\|Cx\|^2 + \|(\lambda I - A)x\|^2 \geq \kappa \|x\|^2. \quad (5.1.2)$$

In an infinite-dimensional setting with operators A, C , instead of matrices, rank conditions are not appropriate. However, the Hautus test in the form (5.1.2) can be generalized, and has actually been proposed in [111] as a criterion for observability. Russell and Weiss conjectured in [111] that this inequality characterizes exact observability. They proved in [111] that the conjecture is valid for bounded and invertible operators A . Later, Jacob and Zwart [50] showed equivalence for diagonal semigroup generator on a Riesz basis if the output space Y is finite dimensional. The general conjecture was later proved to be wrong, see [51]. Note however, that if C is admissible and A has a bounded H^∞ -calculus on a suitable sector (which is, in turn a consequence of admissibility and exact observation, see Proposition 5.1 in [41]), then it does not seem to be known whether the Hautus condition implies observability. There exist other formulations of the Hautus condition (or spectral condition) and there are several cases where it implies exact observability. This holds for example if A generates a unitary group. We refer to [82, 127] for early results with bounded observations, and [12, 88] for successive extensions. These have subsequently been generalized (see [52]) to groups with certain growth bounds. See also [123] for more information and references on this subject.

In this chapter we consider first order non-autonomous evolution equations of the following form:

$$\begin{cases} x'(t) + A(t)x(t) &= 0 & t \in [0, \tau] \\ x(0) &= x_0 \\ y(t) &= C(t)x(t). \end{cases} \quad (\text{A,C})$$

The difference with (5.1.1) is that we allow operators A and C to depend on time t . To be precise, let $\tau > 0$ and assume that for $t \in [0, \tau]$, the operator $A(t)$ generates a strongly continuous contraction semigroup $(e^{-sA(t)})_{s \geq 0}$ on the Hilbert space H . We suppose further that there exists a densely and continuously embedded subspace $\mathcal{D} \hookrightarrow H$ such that for all $t \in [0, T]$, $\mathcal{D}(A(t)) = \mathcal{D}$ and that $t \mapsto A(t)v$ is continuously differentiable in H for every $v \in \mathcal{D}$. These assumptions are sufficient to guarantee that the Cauchy problem $x'(t) = A(t)x(t)$, $x(0) = x_0$ admits a solution, see e.g. [103, Sections 5.3 and 5.4]. For each t , $C(t) : \mathcal{D} \rightarrow Y$ is a bounded operator. Then, for initial data $x_0 \in \mathcal{D}$, the solution x to (A,C) satisfies $x(t) \in \mathcal{D}$ for each $t \geq 0$ and hence $y(t)$ is well defined. We define observability concepts (and controllability concepts for the adjoint system) as in the autonomous case (5.1.1).

In the case of time-dependent matrices, a famous result of Silverman and Meadows [115] characterizes exact observability and controllability. Their arguments have been adapted to certain infinite dimensional settings, see for example [1, 64, 4]. Our main objective is different. We seek to prove observability from a certain Hautus

type condition. In order to do this, we introduce the following averaged Hautus conditions:

$$\|x\|^2 \leq m^2 \left(\frac{1}{\tau} \int_0^\tau \|C(s)e^{\lambda s}x\|^2 ds \right) + M^2 \left(\frac{1}{\tau} \int_0^\tau e^{\operatorname{Re}\lambda \cdot s} \|(\lambda + A(s))x\| ds \right)^2$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{D}$, or

$$\|x\|^2 \leq m^2 \left(\frac{1}{\tau} \int_0^\tau \|C(s)x\|^2 ds \right) + M^2 \left(\frac{1}{\tau} \int_0^\tau \|(i\xi + A(s))x\|^2 ds \right)$$

for all $\xi \in \mathbb{R}$ and $x \in \mathcal{D}$. These inequalities do coincide with the usual Hautus conditions if the operators A and C are independent of t . We prove that these averaged Hautus conditions imply exact observability when the operators $A(t)$ are skew-adjoint. This result is refined to the case of invertible evolution families (not necessarily unitary) under certain growth constraints. We apply these results to Schrödinger equations with time dependent potentials and to a damped wave-equation with time-dependent damping.

Finally, we mention the papers [28], [64] and the references therein on observability (or controllability) of parabolic equations (with time dependent coefficients). The approach in these papers is based on Carleman estimates and it differs from ours.

5.2 Preliminary results

Recall that we suppose $A(t) : \mathcal{D} \rightarrow H$ to have a fixed domain, that $t \mapsto A(t)v$ is continuously differentiable in H for every $v \in \mathcal{D}$ and each semigroup $e^{-sA(t)}$ is a contraction on H . By [103, Sections 5.3 and 5.4] there exists a unique evolution family $(U(t, s))_{0 \leq s \leq t \leq \tau}$ on H generated by $A(t)_{0 \leq t \leq \tau}$. This evolution family satisfies the following properties.

1. $\|U(t, s)\| \leq Me^{-\omega(t-s)}$ for some $\omega \in \mathbb{R}$
2. For all $v \in \mathcal{D}$, $\frac{\partial^+}{\partial t} U(t, s)v|_{t=s} = -A(s)v$, $\frac{\partial^+}{\partial t} U(t, s)v = -A(t)U(t, s)v$.
3. For all $v \in \mathcal{D}$, $\frac{\partial}{\partial s} U(t, s)v = U(t, s)A(s)v$.
4. $U(t, s)\mathcal{D} \subseteq \mathcal{D}$
5. For all $v \in \mathcal{D}$, $(s, t) \mapsto U(t, s)v$ is continuous in \mathcal{D} for $0 \leq s \leq t \leq T$.

For every $v \in \mathcal{D}$, the evolution equation

$$\begin{cases} \frac{d}{dt}\eta(t) + A(t)\eta(t) &= 0 & 0 \leq s \leq t \leq \tau \\ \eta(s) &= v \end{cases} \quad (\text{CP})$$

has a unique solution. This solution is given by $\eta(t) = U(t, s)v$. For $f \in L^1(0, \tau; H)$, the non homogeneous problem

$$\begin{cases} \frac{d}{dt}\eta(t) + A(t)\eta(t) &= f(t) & 0 \leq s \leq t \leq \tau \\ \eta(s) &= v \in H. \end{cases} \quad (\text{NHCP})$$

has then a mild solution given by

$$\eta(t) = U(t, s)v + \int_s^t U(t, r)f(r) \, dr, \quad (5.2.1)$$

see e.g. [103, p.146]. If, in addition to the standing assumptions, $f \in C^1([s, \tau], H)$ then (NHCP) has a unique classical solution which coincides with the mild solution, see for example [103, Theorem 5.2, p.146].

We associate with (A,C) the operator

$$(\Psi_{s,\tau}x)(t) = \begin{cases} C(t)U(t, s)x & t \in [s, \tau] \\ 0 & t > \tau \end{cases}$$

and define the following notions:

Definition 5.2.1 (Averaged admissible observations). Let $(C(t))_{t \in [0, \tau]}$ be a family of bounded operators in $\mathcal{L}(\mathcal{D}, Y)$, where Y is some Hilbert space. We say that $(C(t))_t$ are averaged admissible observations for $(A(t))_{t \in [0, \tau]}$ if there exists a constant $M_\tau > 0$ such that

$$\int_s^\tau \|C(t)U(t, s)x\|^2 \, dt \leq M_\tau^2 \|x\|^2 \quad \forall x \in \mathcal{D}, s \in [0, \tau].$$

(one can also consider a weaker admissibility notion by requiring the above inequality for $s = 0$, only). For a single operator $C(t_0)$ such that

$$\int_0^\tau \|C(t_0)U(t, s)x\|^2 \, dt \leq M_\tau \|x\|^2 \quad \forall x \in \mathcal{D}$$

we say that $C(t_0)$ is admissible for $(A(t))_{t \in [0, T]}$.

For averaged admissible observations, $\Psi_{s,\tau}$ extends to a bounded operator from H to $L_2(s, \tau; Y)$ which we denote again by $\Psi_{s,\tau}$.

In this definition the norm inside the integral is taken in Y and the norm of x is taken in H . We always use the same notation $\|\cdot\|$ for both, the difference will be clear from the context.

Definition 5.2.2. Suppose that $(C(t))_t$ is an averaged admissible observation for $(A(t))_t$. We say that the system (A, C) is

- a) *exactly averaged observable in time τ* if the map $\Psi_{s,\tau}$ is bounded from below in the sense that there exists a constant $\kappa_\tau > 0$ such that for all $x \in \mathcal{D}$

$$\int_s^\tau \|C(t)U(t, s)x\|^2 \, dt \geq \kappa_\tau \|x\|^2.$$

For a given $t_0 \in [0, \tau]$, the system $(A, C(t_0))$ is exactly observable at time τ if

$$\int_0^\tau \|C(t_0)U(t, 0)x\|^2 \, dt \geq \kappa_\tau \|x\|^2.$$

- b) *final-time averaged observable in time τ* if there exists a constant $\kappa_\tau > 0$ such that

$$\int_0^\tau \|C(t)U(t,0)x\|^2 dt \geq \kappa_\tau \|U(\tau,0)x\|^2 \quad \forall x \in \mathcal{D}.$$

As above we define final observability for the simple operator $C(t_0)$ for some t_0 as

$$\int_0^\tau \|C(t_0)U(t,0)x\|^2 dt \geq \kappa_\tau \|U(\tau,0)x\|^2.$$

- c) *approximately averaged-observable in time τ* if $\ker \Psi_{s,\tau} = \{0\}$ for all $0 \leq s < \tau$. Again we define approximate observability for a single operator $C(t_0)$ if $(A, C(t_0))$ is approximate observable in average as above.

In order to justify the use of the term "averaged" in the previous notions of observability, we note that it might be possible that $(A, C(t_0))$ is not exactly (or final or approximately) observable for some $C(t_0)$ or even for all $t_0 \in J$ for some subset J of $[0, \tau]$ but (A, C) is exactly (or final or approximately) observable in average. In order to see this, we consider the autonomous case $A(t) = A$ and an observation operator C such that the autonomous system is exactly (or null or approximately) observable at time τ_0 . Define

$$C(t) = \begin{cases} C, & t \in [0, \tau_0] \\ 0, & t \in (\tau_0, \tau]. \end{cases}$$

Then

$$\int_0^\tau \|C(t)e^{-tA}x\|^2 dt \geq \int_0^{\tau_0} \|C(t)e^{-tA}x\|^2 dt \geq \kappa_\tau \|x\|^2.$$

Hence the averaged observability property for $(A, C(t))$ at time τ holds but the system $(A, C(t_0))$ is not observable for $t_0 \in (\tau_0, \tau]$ at any time. The same observation is valid for null and approximate average observability.

Along with (A,C) we consider a controlled evolution equation. First, we recall the following: one can construct an extrapolation space H_{-1} and extrapolated operators $A_{-1}(t)$ such that the following diagram commutes

$$\begin{array}{ccc} H & \xrightarrow{A_{-1}(t)} & H_{-1}(t) \\ \uparrow i & & \uparrow i \\ \mathcal{D} & \xrightarrow{A(t)} & H \end{array}$$

One way to realize $H_{-1}(t)$ is to take the completion of H with respect to a resolvent norm $\|(\lambda - A(t))^{-1}x\|_H$ or via its identification with $\mathcal{D}(A(t)^*)'$. For all this we refer to [65, Chapter II.5].

In order to keep the abstract setting simple we will suppose for the rest of this section that $\mathcal{D}(A(t)^*) =: \mathcal{D}^*$ is independent of time as well and equivalent norms with constants independent of t . Note that if for all $t \in [0, \tau]$, $A(t) = A(0) + R_t$ with a bounded operator on H , then $A(t)^* = A(0)^* + R_t^*$ with domain $\mathcal{D}^* := \mathcal{D}(A(0)^*)$ independent of t . In the setting of the averaged Hautus test we consider later, we

will make the assumption $A(t) = A(0) + R_t$ with a family of uniformly bounded operators R_t on H . In this case $H_{-1}(t) = H_{-1}$ and have equivalent norms with constants independent of t .

Let U be another Hilbert space and let $B(t) : U \rightarrow H_{-1}$ is bounded for each $t \in [0, \tau]$. We consider in H_{-1} the evolution equation

$$\begin{cases} x'(t) + A(t)x(t) &= B(t)u(t) & t \in [0, \tau] \\ x(s) &= 0. \end{cases} \quad (\text{A,B})$$

Since the mild solution is of the form (5.2.1), we have the naturally associated operator

$$\Phi_{s,\tau}u = \int_s^\tau U(\tau, r)B(r)u(r) \, dr \quad (\tau \leq \tau) \quad (5.2.2)$$

to (A,B).

Definition 5.2.3 (Averaged admissible controls). Let $(B(t))_{t \in [0, \tau]}$ be a family of bounded operators in $\mathcal{L}(U; H_{-1})$. We say that $(B(t))_t$ are averaged admissible controls for $(A(t))_{t \in [0, \tau]}$ if there exists a constant $M_\tau > 0$ such that the solution x to (A,B) satisfies $x(t) \in H$ and for all $s \in [0, \tau]$

$$\left\| \int_s^\tau U(\tau, r)B(r)u(r) \, dr \right\|^2 \leq M_\tau^2 \|u\|_{L_2(s, \tau; U)}^2$$

for all $u \in \mathcal{D}(0, \tau; U)$ (one can also consider a weaker admissibility notions by requiring the above inequality for $s = 0$, only).

Let us consider the retrograde final-value problem

$$\begin{cases} z'(t) - A(t)^*z(t) &= 0 \\ z(\tau) &= z_\tau. \end{cases} \quad (5.2.3)$$

Observe that for $x \in \mathcal{D}$ and $x^* \in \mathcal{D}^*$,

$$\frac{d}{dt} \langle x, U(\tau, t)^*x^* \rangle = \frac{d}{dt} \langle U(\tau, t)x, x^* \rangle = -\langle U(\tau, t)A(t)x, x^* \rangle = \langle x, -A(t)^*U(\tau, t)^*x^* \rangle$$

so that $z(t) = U(\tau, t)^*z_\tau$ solves the retrograde equation (5.2.3) on $[s, \tau]$ for all $0 \leq s < \tau$.

Lemma 5.2.4. *The family $(B(t))_{t \in [0, \tau]}$ are admissible controls for $(A(t))_{t \in [0, \tau]}$ if and only if the family $(B(t)^*)_{t \in [0, \tau]}$ are admissible observations for the retrograde equation (5.2.3).*

Proof. The following calculation is standard.

$$\begin{aligned} \sup_{\|u\|_2 \leq 1} \left\| \int_s^\tau U(\tau, r)B(r)u(r) \, dr \right\| &= \sup_{\|u\|_2 \leq 1} \sup_{\|x^*\| \leq 1} \left| \int_s^\tau \langle U(\tau, r)B(r)u(r), x^* \rangle \, dr \right| \\ &= \sup_{\|x^*\| \leq 1} \sup_{\|u\|_2 \leq 1} \left| \int_s^\tau \langle u(r), B(r)^*U(\tau, r)^*x^* \rangle \, dr \right| \\ &= \sup_{\|x^*\| \leq 1} \left(\int_s^\tau \|B(r)^*U(\tau, r)^*x^*\|^2 \, dr \right)^{1/2}. \quad \square \end{aligned}$$

Definition 5.2.5. Let $(B(t))_t$ be averaged admissible controls for $(A(t))_{t \in [0, \tau]}$. We say that (A, B) is

- a) *Exactly averaged controllable in time τ* if for any $s \in [0, \tau)$ and $x_s, x_\tau \in H$, there exist $u \in L^2(s, \tau; U)$ such that the mild solution x satisfies $x(s) = x_s$ and $x(\tau) = x_\tau$.
This definition coincides with the usual one in the autonomous case, that is, given two states $x_s, x_\tau \in H$ we find a control u such that the solution takes the value x_s at the initial time $t = s$ and the value x_τ at time $t = \tau$.
- b) *approximately averaged controllable in time τ* if for any $0 \leq s < \tau$ and any $x_s, x_\tau \in H$ and $\varepsilon > 0$, there exist $u \in L^2(0, \tau; U)$ such that $x(s) = x_s$ and $\|x(\tau) - x_\tau\| < \varepsilon$.
- c) *averaged null controllable in time τ* if for every $0 \leq s < \tau$ and every $x_s \in H$, there exist $u \in L^2(s, \tau; U)$ such that the mild solution x satisfies $x(s) = x_s$ and $x(\tau) = 0$.

Since the mild solution is given by

$$x(t) = U(t, s)x_s + \int_s^t U(t, r)B(r)u(r) dr$$

it is clear that in order to obtain exact averaged controllability it suffices to consider the case where $x(s) = 0$.

Proposition 5.2.6. Let $B(t) \in \mathcal{L}(U, H_{-1})$ be a family of averaged admissible controls for $(A(t))_{t \in [0, \tau]}$. Then

- a) *Exact averaged controllability for (A, B) in time τ is equivalent to exact averaged observability of the retrograde final-value problem (5.2.3) with the observation operators $C(t) = B(t)^*$.*
- b) *Approximate averaged controllability for (A, B) in time τ is equivalent to approximate averaged observability of the retrograde final-value problem (5.2.3) with the observation operators $C(t) = B(t)^*$.*
- c) *Averaged null controllability for (A, B) in time τ is equivalent to averaged observability of $z(s)$, $0 \leq s < \tau$ where z is the solution of the retrograde final-value problem (5.2.3) with the observation operators $C(t) = B(t)^*$.*

Proof. First note that $(\Phi_{s, \tau}^* z_s)(t) = B(t)^* U^*(\tau, t) z_s$ for $t \in [s, \tau]$. For simplicity we extend this function by zero for other values of t . Exact averaged controllability for (A, B) at τ is equivalent to $\text{range}(\Phi_{s, \tau}) = H$ for all s . Since these operators are bounded, the latter property is equivalent to the fact that their adjoints $\Phi_{s, \tau}^*$ is bounded from below on $L^2(s, \tau; H)$, i.e., there exists $\kappa_{s, \tau}$ such that

$$\int_s^\tau \|B(t)^* U(\tau, t)^* z_s\|^2 dt \geq \kappa_{s, \tau} \|z_s\|^2$$

for all $z_s \in \mathcal{D}^*$. Approximate averaged controllability is equivalent to $\text{range}(\Phi_{s, \tau})$ being dense for all $s \in [0, \tau)$, or, equivalently, the respective adjoints being injective. Finally, averaged null controllability in time τ is equivalent to $\text{range}(U(\tau, s)) \subset$

range($\Phi_{s,\tau}$) for all $0 \leq s < \tau$. Applying [123, Proposition 12.1.2], averaged null controllability is equivalent to

$$\|U(\tau, s)^* z_\tau\|^2 \leq \delta^2 \|\Phi_{s,\tau}^* z_\tau\|^2 = \delta^2 \int_s^\tau \|B(t)^* U(\tau, t)^* z_\tau\|^2 dt$$

for some constant $\delta > 0$. But $U(\tau, s)^* z_\tau = z(s)$ where $z(\cdot)$ is the solution of the retrograde equation (5.2.3). \square

5.3 The averaged Hautus test: skew-adjoint operators

Throughout this section, the family of operators $\{A(t)\}_{0 \leq t \leq \tau}$ is as before. Let $\{C(t)\}_{0 \leq t \leq \tau}$ be a family of bounded operators from \mathcal{D} to a Hilbert space Y . In the autonomous case $A(t) = A$ and $C(t) = C$ for all t , it is well known that for admissible C the exact observability of the system (A, C) implies the so-called Hautus test (or spectral condition)

$$\|x\|^2 \leq m^2 \|Cx\|^2 + M^2 \|(i\xi + A)x\|^2 \quad (5.3.1)$$

for some positive constants m and M and all $\xi \in \mathbb{R}$ and $x \in \mathcal{D}(A)$. There is also another condition with $\lambda \in \mathbb{C}$ in place of $i\xi$, see below. In the general non-autonomous situation we introduce an integrated (or averaged) version of this test. We also study, as in the autonomous case, when the averaged Hautus test is necessary and/or sufficient for averaged observability. We start with the "necessary" part.

Proposition 5.3.1. *Suppose that $(C(t))$ is averaged admissible for $(A(t))$. If the system (A, C) is exactly averaged observable at time $\tau > 0$ then there exist positive constants m and M such that:*

$$\|x\|^2 \leq m^2 \left(\frac{1}{\tau} \int_0^\tau \|C(s)e^{\lambda s} x\|^2 ds \right) + M^2 \left(\frac{1}{\tau} \int_0^\tau e^{\operatorname{Re} \lambda s} \|(\lambda + A(s))x\| ds \right)^2 \quad (\text{AH.1})$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{D}$,

$$\|x\|^2 \leq m^2 \left(\frac{1}{\tau} \int_0^\tau \|C(s)x\|^2 ds \right) + M^2 \left(\frac{1}{\tau} \int_0^\tau \|(i\xi + A(s))x\|^2 ds \right) \quad (\text{AH.2})$$

for all $\xi \in \mathbb{R}$ and $x \in \mathcal{D}$.

Remark 5.3.2. If $C(s) = C$ for all s then (AH.1) can be written as:

$$\|x\|^2 \leq \frac{e^{2\tau \operatorname{Re}(\lambda)} - 1}{2\tau \operatorname{Re}(\lambda)} m^2 \|Cx\|^2 + M^2 \left(\frac{1}{\tau} \int_0^\tau e^{\operatorname{Re} \lambda s} \|(\lambda + A(s))x\| ds \right)^2 \quad (\text{AH.3})$$

If, in addition, $A(s) = A$ then both assertions coincide with the classical Hautus (or spectral) conditions. We call the conditions (AH.1) and (AH.2) *averaged Hautus tests*.

Proof. The proof is similar to the autonomous case. We start from $\frac{d}{ds}(e^{\lambda s} C(t) U(t, s)x) = \lambda e^{\lambda s} C(t) U(t, s)x + e^{\lambda s} C(t) U(t, s) A(s)x$ for $x \in \mathcal{D}$. Integrating on $[0, \tau]$ yields

$$e^{\lambda \tau} C(\tau)x - C(0)U(\tau, 0)x = \int_0^\tau C(t)U(t, s)(A(s) + \lambda)x e^{\lambda s} ds.$$

Hence,

$$\int_0^\tau \|C(t)U(t,0)x\|^2 dt \leq 2 \int_0^\tau \|C(t)xe^{\lambda t}\|^2 dt + 2 \int_0^\tau \left\| \int_0^t C(t)U(t,s)(\lambda+A(s))xe^{\lambda s} ds \right\|^2 dt$$

Since (A,C) is exactly averaged observable on $[0, \tau]$, the left hand side is bounded below by $m_0\|x\|^2$ for some constant $m_0 > 0$. We estimate the second term on the right hand side

$$\begin{aligned} I &:= \left(\int_0^\tau \left\| \int_0^t C(t)U(t,s)(\lambda+A(s))xe^{\lambda s} ds \right\|^2 dt \right)^{1/2} \\ &= \sup \left\{ \left| \int_0^\tau \int_0^t \left\langle C(t)U(t,s)(\lambda+A(s))xe^{\lambda s}, g(t) \right\rangle_H ds dt \right| : \|g\|_{L_2(0,\tau;H)} \leq 1 \right\} \\ &= \sup_{\|g\|_{L_2} \leq 1} \left| \int_0^\tau \left\langle (\lambda+A(s))xe^{\lambda s}, \int_s^\tau U(t,s)^*C(t)^*g(t) dt \right\rangle_H ds \right| \\ &\leq \sup_{\|g\|_{L_2} \leq 1} \left(\int_0^\tau \|(\lambda+A(s))xe^{\lambda s}\|_H \left\| \int_s^\tau U(t,s)^*C(t)^*g(t) dt \right\|_H ds \right). \end{aligned}$$

By Lemma 5.2.4 and the admissibility assumption of $(C(t))$, there exists a constant $K_\tau > 0$ such that

$$I \leq K_\tau \int_0^\tau \|(\lambda+A(s))xe^{\lambda s}\| ds = K_\tau \int_0^\tau \|(\lambda+A(s))x\| e^{\operatorname{Re} \lambda s} ds.$$

and (AH.1) follows. The second assertion is obtained from the first one by taking $\lambda = i\xi$ and using the Cauchy-Schwarz inequality. \square

Now we study the converse. In the autonomous case i.e., $A(s) = A$ and $C(t) = C$, it is well known that condition (AH.2) implies the exact observability if the single operator A is skew-adjoint. We extend this result to our more general situation.

Theorem 5.3.3. *Suppose that $A(t) \in \mathcal{L}(\mathcal{D}; H)$ be a family of skew-adjoint operators generating an evolution family $U(t, s)_{0 \leq s \leq t \leq \tau}$. Suppose that the differences of the operators $A(t)$ are bounded and satisfy the estimate*

$$\|A(t) - A(s)\|_{\mathcal{L}(H)} \leq L \quad \forall t, s \in [0, \tau]$$

for some constant $L < \frac{1}{\sqrt{2}M}$. Assume that $C(t) \in \mathcal{L}(\mathcal{D}; Y)$ is a family of averaged admissible observation operators and that the second averaged Hautus condition (AH.2) holds with positive constants m and M . Then, for all $\tau > \tau^* := \frac{2\pi M}{\sqrt{1-2L^2M^2}}$ there exists $\kappa_\tau > 0$ depending on M, L and τ such that, for all $x \in \mathcal{D}$ the exact averaged observability estimate

$$\frac{1}{\tau} \int_0^\tau \int_0^\tau \|C(s)U(t,0)x\|^2 dt ds \geq \frac{\kappa_\tau}{m^2} \|x\|^2 \quad (5.3.2)$$

holds. In particular, if $C(s) = C$ is constant, then the system (A,C) is exactly averaged observable for $\tau > \tau^*$, i.e., for all $x \in \mathcal{D}$,

$$\int_0^\tau \|CU(t,0)x\|^2 dt \geq \frac{\kappa_\tau}{m^2} \|x\|^2.$$

Proof. We proceed in a similar way as in the autonomous case. Let $\tau > 0$, $\varphi \in H_0^1(0, \tau)$ and $x \in \mathcal{D}$. For $t, s \in [0, \tau]$, let $h(t) := \varphi(t)U(t, 0)x$ and $f(t, s) := h'(t) + A(s)h(t)$. Note that h and $f(\cdot, s)$ can be extended continuously by zero outside $(0, \tau)$ since $\varphi \in H_0^1(0, \tau)$. We write $\widehat{f}(\xi, s)$ for the partial Fourier transform of f with respect to the first variable, and observe that

$$\widehat{f}(\xi, s) = \int_{\mathbb{R}} e^{-it\xi} f(t, s) dt = \int_{\mathbb{R}} e^{-it\xi} h'(t) dt + \int_{\mathbb{R}} e^{-it\xi} A(s)h(t) dt = i\xi \widehat{h}(\xi) + A(s)\widehat{h}(\xi)$$

where we use the fact that each operator $A(s)$ is closed in order to have $\widehat{A(s)h}(\xi) = A(s)\widehat{h}(\xi)$. We apply (AH.2) with $z_0 = \widehat{h}(\xi)$ to obtain

$$\begin{aligned} \|\widehat{h}(\xi)\|^2 &\leq \frac{m^2}{\tau} \int_0^\tau \|C(s)\widehat{h}(\xi)\|^2 ds + \frac{M^2}{\tau} \int_0^\tau \|(i\xi + A(s))\widehat{h}(\xi)\|^2 ds \\ &= \frac{m^2}{\tau} \int_0^\tau \|C(s)\widehat{h}(\xi)\|^2 ds + \frac{M^2}{\tau} \int_0^\tau \|\widehat{f}(\xi, s)\|^2 ds. \end{aligned}$$

We integrate over all $\xi \in \mathbb{R}$ and use Plancherel's theorem together with the fact that $C(s)\widehat{h}(\xi) = \widehat{C(s)h}(\xi)$ to deduce

$$\int_0^\tau \|h(t)\|^2 dt \leq \frac{m^2}{\tau} \int_0^\tau \int_0^\tau \|C(s)h(t)\|^2 dt ds + \frac{M^2}{\tau} \int_0^\tau \int_0^\tau \|f(t, s)\|^2 dt ds. \quad (5.3.3)$$

We estimate the last term on the right hand side as follows

$$\begin{aligned} &\int_0^\tau \int_0^\tau \|f(t, s)\|^2 dt ds \\ &= \int_0^\tau \int_0^\tau \|h'(t) + A(s)h(t)\|^2 dt ds \\ &= \int_0^\tau \int_0^\tau \|\varphi'(t)U(t, 0)x - \varphi(t)A(t)U(t, 0)x + \varphi(t)A(s)U(t, 0)x\|^2 dt ds \\ &\leq 2\tau \int_0^\tau \|U(t, 0)x\|^2 |\varphi'(t)|^2 dt + 2 \int_0^\tau \int_0^\tau \|(A(t) - A(s))U(t, 0)x\|^2 |\varphi(t)|^2 dt ds. \end{aligned} \quad (5.3.4)$$

By skew-adjointness,

$$\frac{d}{dt} \|U(t, s)x\|^2 = -2\operatorname{Re}\langle A(t)U(t, s)x, U(t, s)x \rangle = 0$$

for $x \in \mathcal{D}$ and so $U(t, s)$ is unitary for $0 \leq s \leq t \leq \tau$. Therefore (5.3.3) can be rewritten as

$$\begin{aligned} \|x\|^2 \int_0^\tau |\varphi(t)|^2 dt &\leq \frac{m^2}{\tau} \int_0^\tau \int_0^\tau \|C(s)U(t, 0)x\|^2 |\varphi(t)|^2 dt ds + 2M^2 \|x\|^2 \int_0^\tau |\varphi'(t)|^2 dt \\ &\quad + 2L^2 M^2 \|x\|^2 \int_0^\tau |\varphi(t)|^2 dt. \end{aligned}$$

Hence

$$\kappa(\varphi) \|x\|^2 \leq \frac{m^2}{\tau} \int_0^\tau \int_0^\tau \|C(s)U(t, 0)x\|^2 |\varphi(t)|^2 dt ds$$

where

$$\kappa(\varphi) = \left((1 - 2L^2M^2) \int_0^\tau |\varphi(t)|^2 dt - 2M^2 \int_0^\tau |\varphi'(t)|^2 dt \right).$$

We have to chose φ such that the constant $\kappa(\varphi)$ is positive. Taking the first eigenfunction of the Dirichlet Laplacian on $(0, \tau)$, i.e., $\varphi(t) := \sin\left(\frac{t\pi}{\tau}\right)$, we maximize $\kappa(\varphi)$ and obtain from $\|\varphi\|_\infty = 1$

$$\frac{\kappa\tau}{m^2} \|x\|^2 \leq \int_0^\tau \int_0^\tau \|C(s)U(t, 0)x\|^2 dt ds$$

where $\kappa = \left((1 - 2L^2M^2)\frac{\tau}{2} - \frac{\pi^2M^2}{\tau} \right)$. To ensure $\kappa > 0$ we need $L^2 < \frac{1}{2M^2}$ and $\tau > \tau^*$. \square

Remark 5.3.4. 1. In (5.3.4) we have used for simplicity the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ but we could instead use $(a + b)^2 \leq (1 + r)a^2 + (1 + r^{-1})b^2$ for any $r > 0$. In this case, we obtain the theorem (with the same proof) with the conditions $L < \frac{1}{M\sqrt{1+r}}$ and $\tau^* = \frac{\pi M\sqrt{1+r^{-1}}}{\sqrt{1-M^2(1+r)L^2}}$.

2. If $A(t) = A$ and hence $L = 0$ we obtain (from the previous remark) as minimal control time $\tau^* = \pi M$. This is the usual minimal time in the case of unitary groups.

3. In the last assertion of theorem, if instead of $C(s) = C$, we assume that

$$\|C(s) - C(t)\| \leq L_0 |t - s|^\alpha$$

for some positive constants α and L_0 we obtain that for L_0 small enough, the system (A,C) is exactly averaged observable. Indeed, we have from (5.3.2)

$$\begin{aligned} \kappa \|x\|^2 &\leq 2 \int_0^\tau \int_0^\tau \|(C(t) - C(s))U(t, 0)x\|^2 ds dt + 2 \int_0^\tau \int_0^\tau \|C(t)U(t, 0)x\|^2 ds dt \\ &\leq 2L_0 \int_0^\tau \int_0^\tau |t - s|^{2\alpha} ds dt \|x\|^2 + 2\tau \int_0^\tau \|C(t)U(t, 0)x\|^2 dt \\ &= \frac{2L_0\tau^{2\alpha+2}}{(2\alpha+1)(\alpha+1)} \|x\|^2 + 2\tau \int_0^\tau \|C(t)U(t, 0)x\|^2 dt. \end{aligned}$$

4. If we define

$$\tilde{C}x := \frac{1}{\tau} \int_0^\tau C(s)x ds$$

then we can apply Proposition 5.3.1 and Theorem 5.3.3 to the time independent operator \tilde{C} . We obtain equivalence between

$$\kappa_\tau \|x\|^2 \leq \int_0^\tau \left\| \int_0^\tau C(s)U(t, 0)x ds \right\|^2 dt$$

and

$$\|x\|^2 \leq m^2 \left\| \left(\frac{1}{\tau} \int_0^\tau C(s) ds \right) x \right\|^2 + M^2 \left(\frac{1}{\tau} \int_0^\tau \|(i\xi - A(s))x\|^2 ds \right).$$

5. We have assumed in the theorem that $A(t)$ are skew-adjoint operators in order to have $U(t, s)$ is a unitary operator on H . The previous proof works under the assumption that

$$K_0\|x\| \leq \|U(t, 0)x\| \leq K_1\|x\|, \quad x \in H$$

for some positive constants K_0 and K_1 . The statement of the theorem holds with different conditions L and τ^* (depending on K_0 and K_1).

5.4 The averaged Hautus test: a more general class of operators

In this section we extend Theorem 5.3.3 to a more general class of operators. More precisely, we consider operators $A(t)$ for which the corresponding evolution family $U(t, s)$ is not necessarily an isometry but satisfies an estimate of the form

$$ke^{\alpha(t-s)}\|x\| \leq \|U(t, s)x\| \leq Ke^{\beta(t-s)}\|x\|, \quad x \in H \quad (5.4.1)$$

for some constants k, K, α and β . This question was considered in the autonomous case $A(t) = A$ and $C(t) = C$ by Jacob and Zwart [52]. We shall follow similar ideas as in their paper. Note however, even in this autonomous case, the result is very much less precise than in the case of unitary groups. In particular, the minimal time for observability obtained in [52] is $\frac{1}{\beta-\alpha}$. This value becomes large as α and β are close and this is not consistent with the result on unitary groups.

The main tool is the following optimal Hardy inequality.

Theorem 5.4.1 (Gurka [40], Opic-Kufner [101]). *Let $v, w \geq 0$ be weight functions on $[0, \tau]$. Then the weighted Hardy inequality*

$$\|\varphi\|_{L_2(0, \tau; w(x)dx)} \leq C_H \|\varphi'\|_{L_2(0, \tau; v(x)dx)} \quad (5.4.2)$$

holds for all $\varphi \in H_0^1(0, \tau)$ if and only if

$$B := \sup \left\{ \left(\int_x^y w(t) dt \right) \min \left(\int_0^x \frac{1}{v(t)} dt, \int_y^\tau \frac{1}{v(t)} dt \right) : 0 < x, y < \tau \right\}$$

is finite. In this case, the optimal constant C_H in (5.4.2) satisfies $\frac{B}{\sqrt{2}} \leq C_H \leq 4B$.

We make a basic remark on evolution families $U(t, s)_{0 \leq s \leq t}$. Given $U(t, s)$ which is exponentially bounded, i.e., $\|U(t, s)x\| \leq Ke^{\beta(t-s)}\|x\|$. If in addition each $U(t, s)$ is invertible then writing $V(t) := U(t, 0)$ gives

$$V(t) = U(t, 0) = U(t, s)U(s, 0) = U(t, s)V(s) \iff U(t, s) = V(t)V(s)^{-1}.$$

Then $I = V(t)V(t)^{-1}$ gives $\|x\| \leq Ke^{\beta t}\|V(t)^{-1}x\|$ and so $\|V(t)^{-1}x\| \geq \frac{1}{K}e^{-\beta t}\|x\|$ so that

$$ke^{\alpha(t-s)}\|x\| \leq \|U(t, s)x\| \leq Ke^{\beta(t-s)}\|x\|. \quad (5.4.3)$$

holds for $\alpha = -\beta$ and $k = \frac{1}{K}$. If A is 'shifted', i.e., replaced by $A + \omega$, this symmetry $\alpha = -\beta$ will break, and we will therefore use only (5.4.3) for some constants $k, K > 0$ and $\alpha \leq \beta$.

Theorem 5.4.2. Let $A(t)_{0 \leq t \leq \tau} \in \mathcal{L}(\mathcal{D}; H)$ be a family of operators generating an evolution family $U(t, s)$ and let $0 < k \leq K$ and $\alpha < \beta$ be such that (5.4.3) holds. We suppose that the differences $A(t) - A(s)$ are bounded operators with $\|A(t) - A(s)\| \leq L$ for some L such that $L < \frac{k}{\sqrt{2}KM e^{(\beta-\alpha)\tau}}$. Let $C \in \mathcal{L}(\mathcal{D}; Y)$. Then the averaged Hautus condition (AH.3) implies exact observability for all $\tau > \tau^{**}$, i.e.,

$$\int_0^\tau \|CU(t, 0)x\|^2 dt \geq \frac{\kappa}{m^2} \|x\|^2 \quad \forall x \in H$$

for some $\tau^{**} > 0$ provided that there exist $0 \leq x \leq y \leq \tau^{**}$ such that

$$f(x, y) := \left(\frac{k^2}{4K^2M^2(\beta-\alpha)} (e^{-2(\beta-\alpha)x} - e^{-2(\beta-\alpha)y}) + L^2(x-y) \right) \min(x, \tau-y) > 2.$$

Proof. Observe that exact (averaged) observability is invariant under spectral shifts (replacing A by $A+\omega$), which in turn allows to assume $\beta = 0$ and $\alpha = -\omega$ for $\omega = \beta - \alpha > 0$. We follow the lines of the proof of Theorem 5.3.3 until (5.3.4). Using (5.4.3) instead of unitarity leads to consider a new function

$$\kappa(\varphi) := \int_0^\tau |\varphi(t)|^2 (k^2 e^{-2\omega t} - 2K^2M^2L^2) dt - 2K^2M^2 \int_0^\tau |\varphi'(t)|^2 dt.$$

Then $\kappa(\varphi) > 0$ is equivalent to

$$\int_0^\tau |\varphi'(t)|^2 dt < \int_0^\tau |\varphi(t)|^2 \left(\frac{k^2}{2K^2M^2} e^{-2\omega t} - L^2 \right) dt. \quad (5.4.4)$$

This is an 'inverse Hardy inequality', when compared to (5.4.2). To establish such an estimate for at least one function $\varphi \in H_0^1(0, \tau)$, we consider on $[0, \tau]$ the weight function

$$w(t) = \frac{k^2}{2K^2M^2} e^{-2\omega t} - L^2.$$

Observe that w is positive if

$$0 \leq L < \frac{k}{\sqrt{2}KM e^{(\beta-\alpha)\tau}}. \quad (5.4.5)$$

In order to obtain (5.4.4) we use the optimality statement in Theorem 5.4.1 with $v(x) = 1$: if $\sqrt{2} < B < \infty$, the optimal constant guaranteeing (5.4.2) is larger than one. Hence, for any $C < 1$ there exists a $\varphi \in H_0^1([0, \tau])$ for which (5.4.2) fails. This function will then satisfy (5.4.4), and provides a strictly positive constant $\kappa(\varphi)$, yielding exact averaged observability with $\kappa := \kappa(\varphi)$, as in the proof of Theorem 5.3.3 (by rescaling we may suppose $\|\varphi\|_\infty = 1$). Clearly, $\sqrt{2} < B$ is equivalent to our condition on $f(x, y)$ to be larger than 2 for some $0 \leq x \leq y$. \square

On the compact set $T = \{0 \leq x \leq y \leq \tau\} \subset \mathbb{R}^2$ we consider the function

$$\begin{aligned} f(x, y) &:= \left(\int_x^y w(t) dt \right) \min \left(\int_0^x dt, \int_y^\tau dt \right) \\ &= \left(\frac{k^2}{4K^2M^2\omega} (e^{-2\omega x} - e^{-2\omega y}) + L^2(x-y) \right) \min(x, \tau-y). \end{aligned}$$

It is continuous and satisfies $f|_{\partial T} = 0$ so that the maximum is taken inside T . However, due to the many parameters and the mixture of power-type functions with exponentials it may be difficult to calculate explicitly the maximum of f in T . We therefore concentrate on a sufficient condition that ensures $f(x, y) > 2$ for some x and y . We consider for example the case where $f(\frac{1}{4}\tau, \frac{3}{4}\tau) > 2$, i.e.,

$$\tau e^{-\frac{\omega\tau}{2}} \left(\frac{1-e^{-\omega\tau}}{\omega\tau} \right) > \frac{1}{\tau} \left(\frac{32K^2M^2}{k^2} \right) + \tau \left(\frac{2L^2K^2M^2}{k^2} \right).$$

By numerical calculations*, we see that if $\omega\tau \leq \frac{1}{\sqrt{2}}$, then the left hand side is larger than $\frac{\tau}{2}$, so that for $\frac{2L^2K^2M^2}{k^2} < \frac{1}{4}$, $\tau^2 = \frac{128K^2M^2}{k^2}$ gives a concrete observation time. We obtain the following corollary.

Corollary 5.4.3. *Suppose that $L < \frac{k}{2\sqrt{2}KM}$ and $0 \leq \beta - \alpha \leq \frac{k}{16KM}$. Then we have exact observability at time $\tau > \tau^{**}$ where $\tau^{**} = \frac{8\sqrt{2}KM}{k}$. In particular, if $k=K=1$ and L, M are such that $8L^2M^2 < 1$ and $0 \leq \beta - \alpha \leq \frac{1}{16M}$, then we have exact observability at time $\tau > \tau^{**}$ where $\tau^{**} = 8\sqrt{2}M$.*

In the autonomous case $A(t) = A$ with A is a generator of a group we have $L = 0$, hence for $0 \leq \beta - \alpha \leq \frac{k}{16KM}$ we obtain exact observability at time $\tau > \tau^{**} = \frac{8\sqrt{2}KM}{k}$. This might be better than the observation time given in [52] which is $\frac{1}{\beta - \alpha}$.

5.5 Applications to the wave and Schrödinger equations with time dependent potentials

In this section we give applications of our results to observability of the Schrödinger and wave equations both with time dependent potentials. We also consider the damped wave equation with time dependent damped term. Before going into these examples we explain the general idea. It is based on a perturbation argument which shows that the Hautus test carries over from the time independent operator to time dependent ones. Once the Hautus test is satisfied by the perturbed operator we appeal to the results of the previous sections and obtain observability of the system.

Let A be the generator of unitary group on H . We assume that $C : \mathcal{D}(A) \rightarrow Y$ is an admissible operator and such that the system (A, C) is exactly observable at time τ_0 . Therefore the Hautus test is satisfied by the operators A and C . Now let $R(t)_{0 \leq t \leq \tau}$ be a family of uniformly bounded operators on H . By classical bounded perturbation argument (see, e.g., [65, Theorem 9.19]), the operators given by $A(t) = A + R(t)$, $t \in [0, \tau]$, generate an evolution family $U(t, s)$ on H . Note that for every $x \in H$

$$e^{-\beta(t-s)}\|x\| \leq \|U(t, s)x\| \leq e^{\beta(t-s)}\|x\| \quad (5.5.1)$$

with $\beta = \sup_{t \in [0, \tau]} \|R(t)\|$. Indeed, one has for every $x \in \mathcal{D}(A)$, $\operatorname{Re}\langle (A + R(t))x, x \rangle = \operatorname{Re}\langle R(t)x, x \rangle$ and hence

$$-\beta\|x\|^2 \leq \operatorname{Re}\langle (A + R(t))x, x \rangle \leq \beta\|x\|^2.$$

*The function $g(x) = e^{-x/2}(\frac{1-e^{-x}}{x})$ is larger than $1/2$ for $x \leq 0.7143$ and $\frac{1}{\sqrt{2}} \leq 0.70711$.

We apply this with $U(t, s)x$ at the place of x and obtain

$$-\beta \|U(t, s)x\|^2 \leq \frac{1}{2} \frac{\partial}{\partial t} \|U(t, s)x\|^2 \leq \beta \|U(t, s)x\|^2.$$

We integrate and obtain (5.5.1). Note that if $\operatorname{Re}\langle R(t)x, x \rangle = 0$, then $U(t, s)$ is unitary.

Let now $x \in \mathcal{D}(A)$ and $\xi \in \mathbb{R}$. The Hautus test for (A, C) gives

$$\begin{aligned} \|x\|^2 &\leq m^2 \|Cx\|^2 + M^2 \|(i\xi + A)x\|^2 \\ &\leq m^2 \|Cx\|^2 + 2M^2 \|(i\xi + A + R(s))x\|^2 + 2M^2 \|R(s)\|^2 \|x\|^2. \end{aligned}$$

Integrating on $[0, \tau]$ with respect to s gives

$$\|x\|^2 \leq m^2 \|Cx\|^2 + 2M^2 \left(\frac{1}{\tau} \int_0^\tau \|(i\xi + A + R(s))x\|^2 ds \right) + 2M^2 \left(\frac{1}{\tau} \int_0^\tau \|R(s)\|^2 ds \right) \|x\|^2.$$

Suppose in addition that there exists $\tau_1 > 0$ and $\mu < 1$ such that for $\tau \geq \tau_1$

$$2M^2 \left(\frac{1}{\tau} \int_0^\tau \|R(s)\|^2 ds \right) \leq \mu. \quad (5.5.2)$$

Then we obtain

$$(1 - \mu) \|x\|^2 \leq m^2 \|Cx\|^2 + 2M^2 \left(\frac{1}{\tau} \int_0^\tau \|(i\xi + A + R(s))x\|^2 ds \right). \quad (5.5.3)$$

Note that we could also replace $i\xi$ by $\lambda \in \mathbb{C}$ and obtain the Hautus test (AH.3). Next we assume that C is admissible for the unitary group e^{tA} generated by A . That is there exists a constant $K_\tau > 0$ such that

$$\int_0^\tau \|Ce^{tA}x\|^2 dt \leq K_\tau \|x\|^2, \quad x \in \mathcal{D}(A). \quad (5.5.4)$$

We prove that C is admissible for $(A + R(t))$. In order to do so, we start from Duhamel's formula[†]

$$U(t, s)x - e^{(t-s)A}x = \int_s^t e^{(t-r)A} R(r) U(r, s)x dr. \quad (5.5.5)$$

We use (5.5.4) so that

$$\begin{aligned} \int_0^\tau \|CU(t, s)x\|^2 dt &\leq 2 \int_0^\tau \|Ce^{(t-s)A}x\|^2 dt + 2 \int_0^\tau \left\| \int_s^t Ce^{(t-r)A} R(r) U(r, s)x dr \right\|^2 dt \\ &\leq 2K_\tau \|x\|^2 + 2\tau \int_s^\tau \int_r^\tau \|Ce^{(t-r)A} R(r) U(r, s)x\|^2 dt dr \\ &\leq 2K_\tau \|x\|^2 + 2K_\tau \int_s^\tau \|R(r) U(r, s)x\|^2 dr \leq K'_\tau \|x\|^2, \end{aligned}$$

[†]in order to prove this formula one takes the derivative of $f(r) := e^{(t-r)A}U(r, s)x$ for $s \leq r \leq t$ and then integrate from s to t .

where we use the fact that the operators $R(r)$ are uniformly bounded and $U(t, s)$ is exponentially bounded.

We have admissibility of C and the averaged Hautus test (5.5.3). Now we conclude either by Theorem 5.3.3 or Corollary 5.4.3 that, as soon as $\|R(t) - R(s)\|$ are small enough, we have exact observability of the system $(A + R(\cdot), C)$ at time $\tau > \tau^*$ for some $\tau^* > 0$. Note that (5.5.2) holds if $R(t) = 0$ for $t \geq t_0$ for some $t_0 > 0$.

The Schrödinger equation. Let Ω be a bounded domain of \mathbb{R}^d with a C^2 -boundary Γ . Let Γ_0 be an open subset of Γ and $Y = L^2(\Gamma_0)$. It is known that for appropriate condition on Γ_0 , the Schrödinger equation

$$\begin{cases} z'(t, x) &= i\Delta z(t, x) & (t, x) \in [0, \tau] \times \Omega \\ z(0, \cdot) &= z_0 \in H^2(\Omega) \cap H_0^1(\Omega) \\ z(t, x) &= 0 & (t, x) \in [0, \tau] \times \Gamma \end{cases} \quad (5.5.6)$$

satisfies the observability inequality

$$\int_0^\tau \int_{\Gamma_0} \left| \frac{\partial z}{\partial \nu}(t, x) \right|^2 d\sigma dt \geq \kappa_\tau \|z_0\|_{H_0^1(\Omega)}^2 \quad (5.5.7)$$

for every $\tau > 0$, see for example [123, Chapter 7]. Let C be the normal derivative $\frac{\partial}{\partial \nu}$ on Γ_0 , $Y = L^2(\Gamma_0, d\sigma)$ and Δ_D the Laplacian with Dirichlet boundary conditions. The previous inequality means that the system $(i\Delta_D, C)$ is exactly observable at time τ . Let now $R(t)f = iV(t)f$ where $V(t, \cdot) \in W^{1,\infty}(\Omega)$ is a real-valued potential which depends on time. Then under appropriate conditions on V we obtain from the discussion above that the non-autonomous system $(i(\Delta_D + V(t)), C)$ is exactly observable at time $\tau > \tau^*$ for some $\tau^* > 0$. This means that (5.5.7) is satisfied for the solution of the Schrödinger equation with time dependent potential

$$\begin{cases} z'(t, x) &= i\Delta z(t, x) + iV(t)z(t, x) & (t, x) \in [0, \tau] \times \Omega \\ z(0, \cdot) &= z_0 \in H^2(\Omega) \cap H_0^1(\Omega) \\ z(t, x) &= 0 & (t, x) \in [0, \tau] \times \Gamma. \end{cases} \quad (5.5.8)$$

Note however that our method does not give observability at any time $\tau > 0$. If $V(t) = V$ is independent of t then observability for the Schrödinger equation perturbed by the potential V holds at any time $\tau > 0$, see [123, Chapter 7] and the references there.

The wave equation. Let again Ω be a bounded smooth domain of \mathbb{R}^d . We consider the wave equation

$$\begin{cases} z''(t, x) &= \Delta z(t, x) & (t, x) \in [0, \tau] \times \Omega \\ z(0, \cdot) &= z_0 \in H_0^1(\Omega), z'(0, \cdot) = z_1 \in L^2(\Omega) \\ z(t, x) &= 0 & (t, x) \in [0, \tau] \times \Gamma. \end{cases} \quad (5.5.9)$$

Let Γ_0 be a part of the boundary Γ . Observability for the wave equation with the observation operator $C = \frac{\partial}{\partial \nu}|_{\Gamma_0}$ have been intensively studied. Under appropriate

geometric conditions on Γ_0 , there exists $\tau_0 > 0$ such that for $\tau > \tau_0$ there exists a positive constant κ_τ such that

$$\kappa_\tau \left(\int_{\Omega} |z_1|^2 + \int_{\Omega} |\nabla z_0|^2 \right) \leq \int_0^\tau \int_{\Gamma_0} \left| \frac{\partial z}{\partial \nu} \right|^2 d\sigma dt. \quad (5.5.10)$$

We refer to [7, 79, 67] and the references therein. Let $A_0 = \begin{pmatrix} 0 & I \\ -\Delta_D & 0 \end{pmatrix}$ on $H := H_0^1(\Omega) \times L^2(\Omega)$. It is a standard fact that A_0 generates a unitary group $U(t)_{t \in \mathbb{R}}$ on H . Set $\tilde{C}(f, g) := (\frac{\partial f}{\partial \nu}|_{\Gamma_0}, 0)$. Then the energy estimate (5.5.10) is precisely the observability inequality

$$\kappa_\tau \|(z_0, z_1)\|_H^2 \leq \int_0^\tau \|\tilde{C}U(t)(z_0, z_1)\|_{L^2(\Gamma_0)}^2 dt. \quad (5.5.11)$$

Now we consider the damped wave equation without a potential

$$\begin{cases} z''(t, x) &= \Delta z(t, x) + b(t, x)z'(t, x) + V(t, x)z(t, x) \in [0, \tau] \times \Omega \\ z(0, \cdot) &= z_0 \in H_0^1(\Omega), z'(0, \cdot) = z_1 \in L^2(\Omega) \\ z(t, x) &= 0 \quad (t, x) \in [0, \tau] \times \Gamma. \end{cases} \quad (5.5.12)$$

Going to the first order system on H , the wave equation (5.5.12) can be rewritten as $Z' = A(t)Z$ with $A(t) = \begin{pmatrix} 0 & I \\ \Delta + V(t) & b(t) \end{pmatrix} = A_0 + R(t)$ where $R(t) = \begin{pmatrix} 0 & 0 \\ V(t) & b(t) \end{pmatrix}$. As in the case of the Schrödinger equation we can apply the previous discussion to see that the Hautus test for A_0 implies our averaged Hautus test for $(A(t))_t$. In order to do so we need to verify (5.5.2). This property holds if

$$\frac{1}{\tau} \int_0^\tau \left(\|V(t)\|_{W^{1,\infty}(\Omega)}^2 + \|b(t)\|_{L^\infty(\Omega)}^2 \right) dt$$

is small enough. The norms $\|R(t) - R(s)\|$ are small if the quantities $\|V(t) - V(s)\|_{W^{1,\infty}(\Omega)} + \|b(t) - b(s)\|_{L^\infty(\Omega)}$ are small. In this case, we obtain exact averaged observability for (5.5.12). That is, we obtain the energy estimate (5.5.10) for τ large enough for solution z to (5.5.12). If V and b are independent of t then observability results are known (see [123]). If $b(t) = 0$ and V depends on t , then a more precise result can be found in [106] for a special class of Γ_0 . The proof in [106] is different from ours and it is based on Carleman estimates.

Conclusion

In this work, we consider the control problem for a general class of non-autonomous evolution equations. In application, the non-autonomous PDEs are used as models of various phenomena in a numerous different processes. The analysis of these models provides a fundamental basis for both the understanding the process dynamics and also the design of the control schemes. We also interested to the control problem with a class of PDEs defined on a time-varying domain, where the change in the spatial domain is due to the time-evolution of the boundary. The functional analytic description of the PDE with time-dependent coefficients, which is defined on the time-dependent domain with moving boundary, yields the associated representation as an abstract linear nonautonomous parabolic evolution system on an appropriately defined infinite-dimensional function space.

The modelling in terms of finite and infinite dimensional PDEs systems is very important in practice since it has quite different properties from the point of view on the control theory. In general, the solutions of non-autonomous systems are expressed in terms of two-parameter semigroups which inherit many of the properties of the standard one-parameter semigroups generated by time invariant parabolic operators, so that the optimal control problem can be handled by using the tools of time-varying infinite-dimensional systems theory. In fact, even the analysis varies according to the class of PDE's (parabolic or hyperbolic), its different characteristic properties play an important role in the controllability results.

While a comprehensive approach for non-autonomous PDEs is not yet to be developed fully, there are several ways which are applicable to a large number of problems. One such problem is infinite-dimensional non-autonomous systems control theory which provides a convenient method of control design for parabolic PDEs. In fact, we develop new methods that required in considering the application to particular classes of parabolic PDE systems of one dimensional wave and Schrödinger equation with moving boundary.

However, there are still some open questions. In chapter 3, we use a series solution to establish the boundary exact controllability. It is well-known that the moving boundary problem is equivalent to a variable coefficient one (depending on the material variable x and on the time variable t together) with the fixed domain. This means that the corresponding variable coefficient problems also have series solutions. Unfortunately, this approach is limited to 1D problems. For two dimensional problem, the series solution will not work if each coordinate is in time-dependent domain. Let us formulate the problem. We consider the curve $s_1, s_2 : [0, \tau] \rightarrow \mathbb{R}_+$ and $s_1(0) = s_2(0) = 1$. Let take the domain $\Omega = (0, s_1(t)) \times (0, s_2(t)) \times (0, \tau)$. We

consider the wave equation in non-cylindrical domain:

$$\left\{ \begin{array}{ll} u_{tt} - u_{xx} - u_{yy} = 0 & (x, y, t) \in \Omega \\ u(0, y, t) = u(s_1(t), y, t) = 0 & t \geq 0 \\ u(x, 0, t) = u(x, s_2(t), t) = 0 & t \geq 0 \\ u(x, y, 0) = g(x) & (x, y) \in [0, 1] \times [0, 1] \\ u_t(x, y, 0) = f(x) & (x, y) \in [0, 1] \times [0, 1] \end{array} \right. \quad (5.5.13)$$

where (g, f) are given initial conditions. If one of the coordinates is constant, for instance $s_1(t) = 1 \forall t$, we may use find again the series representation for the solution of 5.5.13. However, the interesting case arises when both $s_1(t)$ and $s_2(t)$ are not constant functions. We would like to investigate the relation of $s_1(t)$ and $s_2(t)$ such that 5.5.13 is exactly observable.

We raise the similar question for Schrödinger equation in chapter 4. In fact, we would like to generalize the moving boundary problem to n -dimension ($n \geq 2$). Considering the space variable $x = (x_1, \dots, x_n)$, let $\ell_i(t) : [0, \tau] \rightarrow \mathbb{R}_+$, $\ell_i(0) = 1$ for all $i = 1, \dots, n$. Let the time dependent domain $\Omega_t = \bigcup (0, \ell_i(t))$. We consider the Schrödinger equation:

$$\left\{ \begin{array}{ll} i \frac{\partial u}{\partial t} + \Delta u = 0 & x_i \in [0, \ell_i(t)] \\ u|_{\partial \Omega_t} = 0 & t \geq 0 \\ u(x, 0) = u_0 & x \in \Omega_t \end{array} \right. \quad (5.5.14)$$

where u_0 be the initial data. In fact, the multiplying method seem to be much more complicated when $n \geq 2$. We would like to investigate what is the relation of $\ell_i(t)$ such that 5.5.14 is exactly observable.

Bibliography

- [1] F.Alabau-Boussouira, J.-M.Coron, G.Olive, *Internal controllability of first order quasi-linear hyperbolic systems with a reduced number of controls*. SIAM J. Control Optim. 55 (2017), no. 1, 300–323.
- [2] K.Ammari, A.Bchatnia, and K.Mufti, *Stabilization of the nonlinear damped wave equation via linear weak observability*, Nonlinear Differential Equations Appl, Vol 23 (2016).
- [3] S.Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure. Appl. Math, 15, 119-147 (1962).
- [4] I.Aksikas, J.Winkin, D.Dochain, *Optimal LQ-feedback control for a class of first-order hyperbolic distributed parameter systems*. ESAIM Control Optim. Calc. Var. 14 (2008), no. 4, 897–908.
- [5] N.Balazs, *On the solution of the wave equation with moving boundaries*, Journal of Mathematical Analysis and Applications, vol 3, (1961), pp. 472 - 484.
- [6] C.Bardos and G.Chen, *Control and stabilization for the wave equation. III. Domain with moving boundary*, SIAM J. Control Optim., Vol 19 (1981), pp.123–138.
- [7] C.Bardos, G.Lebeau and J.Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim, Vol 30 (1992), pp.1024–1065.
- [8] K.Beauchard, *Controllability of a quantum particle in a 1D variable domain*, ESAIM, Vol 14, No.1, 105-147 (2008).
- [9] K.Beauchard, J.Coron, *Controllability of a quantum particle in a moving potential well*, J. Functional Analysis, 232, 328-389 (2006).
- [10] C.Bris, *Control theory applied to quantum chemistry*, ESAIM Proc, vol. 8 (2000), pp. 77–94.
- [11] N.Burq, *Contrôle de l'équation des plaques en présence d'obstacles strictement convexes*. Mem. Soc. Math. France, (N.S.) No. 55 (1993).
- [12] N.Burq and M.Zworski, *Geometric control in the presence of a black box*. J. Amer. Math. Soc. 17 (2004), no. 2, 443–471.

- [13] P.Cannarsa, G.D.Prato, and J-P. Zolesio, *Heat equation in non-cylindrical domains*, *Analisi matematica*, 08, 73-77, (1989).
- [14] C.Castro, *Exact controllability of the 1-D wave equation from a moving interior point*, *ESAIM Control Optim. Calc. Var*, Vol 19 (2013), pp. 301-316.
- [15] C.Castro, *Boundary controllability of the one-dimensional wave equation with rapidly oscillating density*, *Asymptot. Anal*, Vol 20 (1999), pp. 317-350.
- [16] C.Castro, N. Cîrdea and A. Münch, *Controllability of the linear one-dimensional wave equation with inner moving forces*, *SIAM J. Control Optim*, Vo 52(2014), pp. 4027-4056.
- [17] C.Chicone and Y.Latushkin, *Evolution semigroups in dynamical systems and differential equations*, Mathematical survey and monographs, AMS Vol 70, 1999.
- [18] J.Cooper and W.Strauss, *Energy boundedness and decay of waves reflecting off a moving obstacle*, *Indiana Univ. Math. J*, Vo 25(1976), pp.671-690.
- [19] N.Corbett, *A symmetry approach to an initial moving boundary value problem associated with the wave equation*, *Can. Appl. Math. Q*, Vol 18 (2010), pp. 351-360.
- [20] N.Corbett, *Initial moving-boundary value problems associated with the wave equation*, *University of Manitoba*, 1991.
- [21] J.Coron, *Control and Nonlinearity*, *Mathematical Surveys and Monographs*, vol. 136, American Mathematical Society, Providence, RI, 2007.
- [22] M.Cortez, A.Bernal, *PDEs in moving time dependent domains*, Without bounds: a scientific canvas of nonlinearity and complex dynamics, Springer, Berlin, Heidelberg, 2013.
- [23] L.Cui, *Exact controllability for a one-dimensional wave equation in non-cylindrical domains*, *Journal of Mathematical Analysis and Applications*, Vol 402 (2013), pp 612-625.
- [24] Diestel, J. and Uhl, J.J. *Vector measures*. Mathematical Surveys No. 15. American Mathematical Society, Providence, R.I., 1977. xiii+322 pp.
- [25] V.Dodonov, *Nonstationary Casimir effect and analytical solutions for quantum fields in cavities with moving boundaries*, *Modern nonlinear optics*, vol. 119, ch. pp. 309–394, Wiley, New York, 2002.
- [26] S.W.Doescher, M.H.Rice, *Infinite Square-Well Potential with a Moving Wall*, *Am. J. Phys*, 37, 1246, (1969).
- [27] S.Dolecki, D. L.Russell, *A general theory of observation and control*, *SIAM J. Control Optimization*, 15 (1977), no. 2, pp. 185–220.
- [28] O.Yu.Èmanuilov, *Controllability of parabolic equations* (Russian), *Mat. Sb.* 186 (1995), no. 6, 109–132; translation in *Sb. Math.* 186 (1995), no. 6, 879900.

- [29] K.Engel, R.Nagel, *A short course on operator semigroups*, Springer, 2005.
- [30] S.Ervedoza, E.Zuazua, *A systemetic method for building smooth control for smooth data*, *Discrete and continuous dynamical system*, Vol 14, No 4, (2010).
- [31] E.Evans, *Time dependent perturbations and scattering of strongly continuous groups on Banach spaces*, *Math. Ann.*, 221 (1976), 275–290.
- [32] C.Fabre and J.Puel, *Pointwise controllability as limit of internal controllability for the wave equation in one space dimension*, *Portugal. Math.*, Vol 51(1994), pp. 335-350.
- [33] H.Fattorini and S.Sritharan, *Existence of optimal controls for viscous flow problems*, *Proc. Royal Soc. London, Ser A*.
- [34] E.Fermi, S.Ulam, *Studies of the Nonlinear Problems*, *Lecture Notes in Applied Mathematics*, Vol. 15 (1974).
- [35] X.Fu, H.Rong, *Approximate Controllability of Semilinear Non-Autonomous Evolutionary Systems with Nonlocal Conditions*, *Automation and Remote Control*, 2016, Vol. 77, No. 3, pp. 428–442.
- [36] A.Fursikov, O.Imanuvilov, *Controllability of Evolution Equations*, *Lecture Notes Series*, Vol. 34, Research Institute of Mathematics, Seoul National University, Seoul, Korea, 1994.
- [37] L.Gaffour, *Analytical method for solving the one-dimensional wave equation with moving boundary*, *Journal of Electromagnetic Waves and Applications*, Vol 12(1998), pp. 1429-1430.
- [38] L.Gaffour and G.Grigorian, *Circular waveguide of moving boundary*, *Journal of Electromagnetic Waves and Applications*, Vol 10 (1996), pp. 97-108.
- [39] L.Glasser, J.Nergo, L.Nieto, *Quantum infinite square well with an oscillating wall*, *Chaos, Solitons, and Fractals*, Volume 41, Issue 4 (2009).
- [40] P.Gurka, Petr *Generalized Hardy’s inequality*. Časopis Pěst. Mat. 109 (1984), no. 2, 194–203.
- [41] B.Haak and E.M.Ouhabaz, *Exact observability, square functions and spectral theory*, *J. Funct. Anal.* 262 (2012), no. 6, 2903-2927.
- [42] B.Haak, D.T.Hoang, *Exact observability of a 1D wave equation on a non-cylindrical domain*, <https://arxiv.org/abs/1705.10652>, (2017).
- [43] B.Haak, D.T.Hoang, E-M. Ouhabaz, *Controllability and observability for non-autonomous evolution equations: the averaged Hautus test*, preprint.
- [44] L.F.Ho, *Observabilité frontière de l’équation des ondes*. C. R. Acad. Sci. Paris Sér. I Math. 302 (1986), no. 12, 443–446.

- [45] L.Hörmander, *Linear partial differential operator*, Springer-Verlag, Berlin, New York, 1969.
- [46] J.S.Howland, *Stationary scattering theory for time-dependent Hamiltonians*, *Math. Ann.*, 207 (1974), 315–335.
- [47] V.Isakov, *Carleman type estimates in an anisotropic case and applications*, *J. of Diff. Eq.*, Volume 105, Issue 2, October 1993, Pages 217-238.
- [48] B.Jacob, *Time-Varying Infinite Dimensional State-Space Systems*, Ph.D. thesis, Fachbereich Mathematik/Informatik, Universit at Bremen, Bremen, 1995.
- [49] B.Jacob, R.Schnaubelt, *Observability of polynomially stable systems*, *Control Lett.*, 56 (2007), pp. 277-284.
- [50] B.Jacob, H.Zwart, *Observability of diagonal systems with a finite-dimensional output operator*, *Control Lett.*, 43 (2001), pp. 101-109.
- [51] B.Jacob and H.Zwart, *Counterexamples concerning observation operators for C_0 -semigroups*. *SIAM J. Control Optim.* 43 (2004), no. 1, 137–153.
- [52] B.Jacob, H.Zwart, *On the Hautus test for exponentially stable C_0 -groups*, *SIAM J. Control Optim.*, vol. 48, No.3, (2009) pp 1275-1288.
- [53] S.Jaffard, *Controle interne exact des vibrations d’une plaque carree*, *C. R. Acad. Sci*, Paris Sr. I Math. 307 (1988), no. 14, 759–762.
- [54] S.Jaffard, M. Tucsnak and E. Zuazua, *Singular internal stabilization of the wave equation*, *J. Differential Equations*, Vol 145 (1998), pp. 184-215.
- [55] James Choon-sun Ng, *Applications of nonautonomous infinite-dimensional systems control theory for parabolic PDEs*, *PhD thesis*, Alberta, 2013.
- [56] D.T Hoang, *Observability of a 1D Schrödinger equation with time-varying boundaries*, <https://arxiv.org/pdf/1702.08059.pdf>
- [57] R.E.Kalman, *On the general theory of control systems*, *Proc. 1st IFAC Congress*, Moscow, 1960, vol. 1, Butterworth, London, 1961, 481–492.
- [58] R.Kalman, *Mathematical Description OP Linear Dynamical Systems*, *SIAM J. Control Optim.*, vol 1 (1963), pp. 152-192.
- [59] R.E.Kalman, Yu-Chi Ho, and K.S.Narendra, *Controllability of linear dynamical systems*, *Contributions to Differential Equations 1* (1963), 189–213.
- [60] T.Kato, *Linear evolution equations of hyperbolic type*, *J. Fac. Sci. Univ. Tokyo*, vol 25 (1970), pp. 241-258.
- [61] T.Kato, H.Tanabe, *On the abstract evolution equation*, *Osaka math*, vol 14 (1962), pp. 107-133.

- [62] A.Khapalov, *Controllability of the wave equation with moving point control*, *Appl. Math. Optim.*, Vol 31 (1995), pp. 155-175.
- [63] A.Khapalov, *Observability and stabilization of the vibrating string equipped with bouncing point sensors and actuators*, *Math. Methods Appl. Sci*, Vol 24 (2001), pp. 1055-1072.
- [64] F.Ammar Khodja, A.Benabdallah, C.Dupaix, M.González-Burgos, *A generalization of the Kalman rank condition for time-dependent coupled linear parabolic systems*. *Differ. Equ. Appl.* 1 (2009), no. 3, 427–457.
- [65] K.J.Engel and R.Nagel, *One-parameter semigroups for linear evolution equations*, *Graduate Texts in Mathematics*, 194. Springer-Verlag, New York, 2000. xxii+586 pp. ISBN: 0-387-98463-1.
- [66] E.Knobloch, R.Krechetnikov, *Problems on Time-Varying Domains: Formulation, Dynamics, and Challenges*, *Acta Applicandae Mathematicae*, Vol 137, Issue 1, pp. 123-157, (2005).
- [67] V.Komornik, *Exact controllability and stabilization (the multiplier method)*, *Wiley, Masson*, Paris 1995.
- [68] E.Kreindler, P.E.Sarachik, *On the concepts of controllability and observability of linear systems*, *IEEE Trans, Automatic Control*, AC-9 (1964), pp. 129-136.
- [69] M.Kristic, B.Guo. A.Smyshlyave, *Boundary controllers and observes for the linearized Schrödinger equation*, *SIAM J. Control Optim*, Vol 49, No.4, (2011).
- [70] M.Kuczma, *Functional equations in a single variable*, *Monografie Matematyczne*, Tom 46, Pan stwowe Wydawnictwo Naukowe, Warsaw, 1968.
- [71] M.Kuczma, B.Choczewski, and R.Ger, *Iterative functional equations*, *Encyclopedia of Mathematics and its Applications*, vol. 32, Cambridge University Press, Cambridge, 1990.
- [72] J.E.Lagnese, G.Leugering, and E.Schmidt, *Modeling, Analysis and control of dynamic elastic multi-link structures*, *Systems and Control: Foundations and Applications*, Birkhuser Boston, 1994.
- [73] I.Lasieck, R.Triggiani, *Dirichlet boundary control problems for parabolic equations with quadratic cost: Analyticity and riccati's feedback synthesis*, *SIAM J.Control Optim*, vol. 21 (1983), pp 41-67.
- [74] I.Lasieck, *Unified theory for abstract parabolic boundary problems: A semigroup approach*, *Appl. Math and Opti*, vol. 6 (1980), pp 287-333.
- [75] Y.Latushkin, T.Randolph, and R.Schnaubelt, *Exponential dichotomy and mild solutions of non-autonomous equations in Banach space*, *Journal of Dynamics and Differential Equations*, Volume 10, Issue 3, 1998, pp 489–510.

- [76] G.Lebeau, *Contrôle d'équations de Schrödinger*, *J. Math. Pures Appl*, 71 (1992), pp. 267-291.
- [77] H.Li, Q.Lu, and X.Zhang, *Recent progress on controllability/observability for system governed by partial differential equations*, *J Syst Sci Comple*, Vol 23 (2010), pp. 527-545.
- [78] J.Lions, *Optimal control of systems governed by partial differential equations*, New York: Springer- Verlag, 1971.
- [79] J.Lions, *Controlabilite exacte, perturbations et stabilisation de systemes distribues*, Tome 1, *Recherches en Mathematiques Appliquees*, vol. 8, Masson, Paris, 1988.
- [80] J.Lions, *Control and Estimation in Distributed Parameter Systems (Frontiers in Applied Mathematics)*, *Pointwise Control for Distributed Systems*, 1987.
- [81] H.Li, Q.Lu, and X.Zhang, *Recent progress on controllability/observability for system governed by partial differential equations*, *J Syst Sci Comple*, Vol 23 (2010), pp. 527-545.
- [82] K.Liu, *Locally distributed control and damping for the conservative systems*. *SIAM J. Control Optim.* 35 (1997), no. 5, 1574–1590.
- [83] L.Lu, S.Li, G.Chen and P.Yao, *Control and stabilization for the wave equation with variable coefficients in domains with moving boundary*, *Systems Control Lett*, Vol 80 (2015), pp. 30-41.
- [84] E.Machtyngier, *Exact controllability for the Schrödinger equation*, *SIAM J. Control Optim*, Vol 3, No.1, (1994).
- [85] J.Makowski, *Two classes of exactly solvable quantum models with moving boundaries*, *J.Phys. Math. Gen*, Vol 25, pp.3419-3426, (1992).
- [86] F.Macia, G.Riviere, *Observability and quantum limits for the Schrödinger equation on the sphere*, *arXiv:1702.02066*, (2017).
- [87] P.Martin, L.Rosier, and P.Rouchon, *Controllability of the 1D Schrödinger equation by the flatness approach*, *IFAC Proceedings*, Vol 47, Issue 3, 2014, pp. 646-651.
- [88] L.Miller, *Controllability cost of conservative systems: resolvent condition and transmutation*. *J. Funct. Anal.* 218 (2005), no. 2, 425–444.
- [89] N.V.Minh, G.N.Guerekata, and C.Preda, *On the asymptotic behavior of the solutions of semilinear nonautonomous equation*, *Semigroup Forum*, Vol. 87 (2013), pp. 18-34.
- [90] M.Miranda, *Exact controllability for the wave equation in domains with variable boundary*, *Rev. Mat. Univ. Complut. Madrid*, 9 (1996), no. 2, 435–457.

- [91] M.Miranda, *Exact controllability for the wave equation in domains with variable boundary*, *Rev. Mat. Univ. Complut. Madrid*, Vol 09 (1996), pp. 435-457.
- [92] G.Moore, *Quantum Theory of the Electromagnetic Field in a variable-length one-dimensional Cavity*, *Journal of Mathematical Physics*, 11 (1970), no. 9, 2679–2691.
- [93] C.Morawetz, *Notes on time decay and scattering for some hyperbolic problems*. Regional Conference Series in Applied Mathematics, No. 19. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1975. v+81 pp.
- [94] S.V.Mousavi, *Quantum dynamics in a time-dependent hard-wall spherical trap*, *Europhysics Letters*, Vol 99, No 3, (2012).
- [95] I.Moyano, *Controllability of some kinetic equations, parabolic degenerate equations and of the Schrödinger equation.*, PhD thesis, Université Paris-Saclay, 2016.
- [96] I.Moyano, *Controllability of a 2D quantum particle in a time-varying disc with radial data*. *J. Math. Anal. Appl.* 455 (2017), no. 2, 1323–1350.
- [97] T.Myint and L.Debnath, *Linear Partial Differential Equations for Scientists and Engineers*, 2007.
- [98] R.Nagel, G.Nickel, *Well-posedness for Nonautonomous Abstract Cauchy Problems*, *Progress in Nonlinear PDEs*, vol 50 (2002), pp.279-293.
- [99] G.Nickel, *Evolution semigroups for non-autonomous Cauchy problems*, *Abstract and Applied Analysis*, Vol 2 (1997), Issue 1-2, Pp 73-95.
- [100] E.L.Nicolai, *On transverse vibrations of a portion of a string of uniformly variable length*, *Annals Petrograd Polytechn*, Inst. 28 (1921), 329–343.
- [101] B.Opic and A.Kufner, *Hardy-type inequalities*, Czechoslovak Academy of Sciences, Longman Sciences / Wiley (1990), ISBN 0-563-05135-3.
- [102] N.H.Pavel, *Nonlinear Evolution Operators and Semigroups: Applications to Partial Differential Equation*, *lecture Notes in Math*, no. 1260, Springer-Verlag, Berlin-New York, 1987.
- [103] A.Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, *Applied mathematical sciences AMS*, vol 44, 1983.
- [104] K-D.Phung, *Observability and control of Schrödinger equations*, *SIAM J. Control Optim*, Vol. 40, No.1, pp. 211–230, (2001).
- [105] D.N.Pinder, *The contracting square quantum well*, *American Journal of Physics*, 58, 54, (1990).
- [106] J.P.Puel, *Global Carleman inequalities for the wave equation and applications to controllability and inverse problems*, <http://www1.univ-ag.fr/aoc/pub/gdt/co/Puel-Inverse-problems.pdf>.

- [107] L.Rosier, B.Zhang, *Exact boundary controllability of the nonlinear Schrödinger equations*, *J. Differential Equations*, 246, pp. 4129-4153, (2009).
- [108] L.Rosier, B.Zhang, *Control and stabilization of the non linear Schrödinger equations on rectangles*, *Math. Models Methods Appl. Sci*, 20, 2293 (2010).
- [109] P.Rouchon, *Control of a quantum particle in a moving potential well*, *Journal of Functional Analysis*, 232, 328–389, (2006).
- [110] D.L.Russell, *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations*, *Stud. Appl. Math*, 52 (1973), 189–221.
- [111] D.L.Russell, G.Weiss, *A general necessary condition for exact observability*, *SIAM J. Control Optim*, Vol 32 (1), pp. 1–23, (1994).
- [112] D.Russell, *Exact boundary value controllability theorems for wave and heat processes in star-complemented regions*, *Lecture Notes in Pure Appl. Math*, Vol 10 (1974).
- [113] R.Schnaubelt, *Well-posedness and Asymptotic Behaviour of Non-autonomous Linear Evolution Equations*, *Evolution Equations, Semigroups and Functional Analysis*, Vol 50, pp. 311-338.
- [114] R.Schnaubelt, *Feedbacks for nonautonomous regular linear systems*, *SIAM J. Control Optim*, Vol. 41, No. 4, 2002, pp. 1141–1165.
- [115] L.M.Silverman, H.E.Meadowns, *Controllability and observability in time-variable linear systems*, *SIAM J. Control Optim*, vol. 5, no.1 , 64–73, 1967.
- [116] J.Simon, *Compact sets in the space $L^p(O,T;B)$* , *Annali di Matematica Pura ed Applicata*, Vol 146, Issue 1, pp 65–96, (1986).
- [117] H.Sun, H.Li, and L.Lu, *Exact controllability for a string equation in domains with moving boundary in one dimension*, *Electron. J. Differential Equations* (2015), No. 98, 7.
- [118] G.Szekeres, *Regular iteration of real and complex functions*, *Acta Mathematica*, 100 (1958), no. 3, 203–258.
- [119] L.Taar, *Quantum system with time dependent boundaries*, College thesis (2005).
- [120] L.Tartar, *Sur un lemme d’équivalence utilisée en Analyse Numerique*, *Calcolo*, Vol. XXIV, Fasc. II, (1987) pp. 129-140.
- [121] D.Tataru, *Carleman estimates and unique continuation for the Schrödinger equation*, *Differential Integral Equations*, 8 (1995) 901–905.
- [122] G.Tenenbaum, M.Tucsnak, *New blow-up rates for fast controls of Schrödinger and heat equations*, *AMS*, (2007).
- [123] M.Tucsnak, G.Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser Verlag, Basel, 2009.

- [124] L.Weiss, *The concepts of differential controllability and differential observability*, *J. Math. Anal. Appl.*, 10 (1965), pp. 442-449.
- [125] P.Yao, Peng-Fei, *On the observability inequalities for exact controllability of wave equations with variable coefficients*, *SIAM J. Control Optim.*, Vol 37 (1999), pp. 1568-1599.
- [126] E.Yilmaz, *One dimensional Schrödinger equation with two moving boundaries*, *arXiv:math-ph/0302006*, (2008).
- [127] Q.Zhou and M.Yamamoto, *Hautus condition on the exact controllability of conservative systems*. *Internat. J. Control* 67 (1997), no. 3, 371–379.
- [128] E.Zuazua, *Controllability of partial differential equations*, <https://cel.archives-ouvertes.fr/cel-00392196/document>, 2006.
- [129] E.Zuazua, *A remark on the observability of conservative linear systems*, DOI: 10.1090/conm/577/11462, (2003).
- [130] E.Zuazua, *Exact controllability for semilinear wave equations in one space dimension*, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, Vol 10 (1993), pp. 109-129.