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K-stability and Kähler manifolds with transcendental cohomology class

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de Toulouse

THÈSE

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Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*

Présentée et soutenue le 15/09/2017 par :

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K-stabilité et variétés kähleriennes avec classe transcendante

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Résumé

Dans cette thèse nous étudions des questions de stabilité géométrique pour des variétés kähleriennes à courbure scalaire constante (cscK) avec classe de cohomologie transcendante. En tant que point de départ, nous introduisons des notions généralisées de K-stabilité, étendant une image classique introduite par G. Tian et S. Donaldson dans le cadre des variétés polarisées. Contrairement à la théorie classique, ce formalisme nous permet de traiter des questions de stabilité pour des variétés kähleriennes compactes non projectives ainsi que des variétés projectives munis de polarisations non rationnelles.

Dans une première partie, nous étudions les rayons sous-géodésiques associés aux configurations tests dites *cohomologiques*, objets introduits dans cette thèse. Nous établissons ainsi des formules fondamentales pour la pente asymptotique d'une famille de fonctionnelles d'énergie, le long de ces rayons géodésiques. Ceci est lié au couplage de Deligne en géométrie algébrique, et ce formalisme permet en particulier de comprendre le comportement asymptotique d'un grand nombre de fonctionnelles d'énergie classiques en géométrie kählerienne, y compris la fonctionnelle d'Aubin-Mabuchi et la K-énergie. En particulier, ceci fournit une approche pluripotentielle naturelle pour étudier le comportement asymptotique des fonctionnelles d'énergie dans la théorie de K-stabilité.

En s'appuyant sur cette première partie, nous démontrons ensuite un certain nombre de résultats de stabilité pour les variétés cscK. Tout d'abord, nous prouvons que les variétés cscK sont K-semistables dans notre sens généralisé, prolongeant ainsi un résultat dû à Donaldson dans le cadre projectif. En supposant que le groupe d'automorphisme est discret, nous montrons en outre que la K-stabilité est une condition nécessaire pour l'existence des métriques cscK sur des variétés kähleriennes compactes. Plus précisément, nous prouvons que la coercivité de la K-énergie implique la K-stabilité uniforme, ainsi généralisant des résultats de Mabuchi, Stoppa, Berman, Dervan et Boucksom-Hisamoto-Jonsson pour des variétés polarisées. Cela donne une preuve nouvelle et plus générale *d'une direction* de la conjecture Yau-Tian-Donaldson dans ce contexte. L'autre direction (suffisance de K-stabilité) est considérée comme l'un des problèmes ouverts les plus importants en géométrie kählerienne.

Nous donnons enfin des résultats partiels dans le cas des variétés kähleriennes compactes qui admettent des champs de vecteurs holomorphes non triviaux. Nous discutons également autour des perspectives et applications de notre théorie de K-stabilité pour les variétés kähleriennes avec classe transcendante, notamment à l'étude des lieux de stabilité dans le cône de Kähler.

Abstract

In this thesis we are interested in questions of geometric stability for constant scalar curvature Kähler (cscK) manifolds with transcendental cohomology class. As a starting point we develop generalized notions of K-stability, extending a classical picture for polarized manifolds due to G. Tian, S. Donaldson, and others, to the setting of *arbitrary* compact Kähler manifolds. We refer to these notions as *cohomological K-stability*. By contrast to the classical theory, this formalism allows us to treat stability questions for non-projective compact Kähler manifolds as well as projective manifolds endowed with non-rational polarizations.

As a first main result and a fundamental tool in this thesis, we study subgeodesic rays associated to test configurations in our generalized sense, and establish formulas for the asymptotic slope of a certain family of energy functionals along these rays. This is related to the Deligne pairing construction in algebraic geometry, and covers many of the classical energy functionals in Kähler geometry (including Aubin's J-functional and the Mabuchi K-energy functional). In particular, this yields a natural potential-theoretic approach to energy functional asymptotics in the theory of K-stability.

Building on this foundation we establish a number of stability results for cscK manifolds: First, we show that cscK manifolds are K-semistable in our generalized sense, extending a result due to S. Donaldson in the projective setting. Assuming that the automorphism group is discrete we further show that K-stability is a necessary condition for existence of constant scalar curvature Kähler metrics on compact Kähler manifolds. More precisely, we prove that coercivity of the Mabuchi functional implies *uniform* K-stability, generalizing results of T. Mabuchi, J. Stoppa, R. Berman, R. Dervan as well as S. Boucksom, T. Hisamoto and M. Jonsson for polarized manifolds. This gives a new and more general proof of *one direction* of the Yau-Tian-Donaldson conjecture in this setting. The other direction (sufficiency of K-stability) is considered to be one of the most important open problems in Kähler geometry.

We finally give some partial results in the case of compact Kähler manifolds admitting non-trivial holomorphic vector fields, discuss some further perspectives and applications of the theory of K-stability for compact Kähler manifolds with transcendental cohomology class, and ask some questions related to stability loci in the Kähler cone.

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Chapter 1

Introduction

This thesis is concerned with questions of *K-stability* and *existence of canonical metrics* on Kähler manifolds, which is a subject at the intersection of Riemannian geometry, complex differential geometry, and complex algebraic geometry. K-stability is a very active area of research, especially following the recent resolution of the important Yau-Tian-Donaldson conjecture in the case of Fano manifolds, see [CDS15a, CDS15b, CDS15c], [Tia15] and also [DS16, BBJ15].

The present work includes developing a "transcendental" approach to K-stability which is valid for *arbitrary* compact Kähler manifolds, extending the formalism for projective varieties that is typically considered, following [Don02]. There are several reasons why such a more general framework is both natural and important, even if one is only interested in studying projective manifolds with rational polarizations. As a main application of the developed techniques we prove that K-stability, in our generalized sense, is a necessary condition for existence of constant scalar curvature Kähler metrics on compact Kähler manifolds, whenever the automorphism group is discrete. We also show that coercivity of the Mabuchi functional implies *uniform* K-stability, generalizing a result of [BHJ15, BHJ16] (cf. also [Don02, Mab08, Sto09b, Ber16, Der16b]) for polarized manifolds. This gives a new and more general proof of *one direction* of the Yau-Tian-Donaldson conjecture in this setting. The other direction (sufficiency of K-stability) is considered to be one of the most important open problems in Kähler geometry.

In the following paragraphs we give some context to the research questions treated and state a number of selected results.

Canonical metrics and K-stability of Kähler manifolds

A fundamental result in complex analysis is the *uniformization theorem*, stating that any simply connected Riemann surface is conformally equivalent to the punctured disc, the complex plane or the Riemann sphere. This yields a classification of Riemann surfaces according to whether they can be endowed with a hyperbolic, flat or positively curved metric respectively. With the ultimate goal of finding an analog for complex manifolds of dimension ≥ 2 , it is natural to ask the following question: *given a compact complex manifold X and a Kähler form ω_0 on X , is*

there a "best" Kähler metric ω cohomologous to ω_0 ? In his 1954 and 1957 papers E. Calabi made this question precise by introducing the notion of *extremal metrics*, suggesting that this is a natural candidate for a "best metric" on a given manifold X . Important classes of extremal metrics include the constant scalar curvature Kähler (cscK) metrics and the Kähler-Einstein metrics.

$$\text{Kähler-Einstein} \Rightarrow \text{cscK} \Rightarrow \text{Extremal}$$

This thesis is primarily concerned with the study of cscK metrics.

A common feature of all extremal metrics is that they, by definition, can be characterized variationally as the *minima* of certain functionals on the space of Kähler metrics. In particular, cscK metrics can be understood variationally as the critical points of a certain functional called the *Mabuchi energy* (or *K-energy*), introduced by T. Mabuchi in [Mab86]. An important and difficult open question is then that of *existence* of cscK metrics (i.e. of minima of the Mabuchi functional) in a given Kähler class on a given compact Kähler manifold. Due to various obstructions it is known that cscK metrics cannot always exist. For instance, it was shown by Matsushima and Lichnerowicz [Mat57, Lic57] that the non-reductivity of the automorphism group is an obstruction to existence of cscK metrics, so e.g. the blow up $X := \text{Bl}_p \mathbb{P}^2$ in a point does not admit any cscK metrics in any Kähler class. Another classical obstruction is given by the *Futaki invariant*, introduced in [Fut83]. We refer to the surveys [Tia00b, Sze14, Don17] and references therein for details on a large number of other obstructions to existence of cscK metrics.

A major open problem is to find a *necessary* condition that is also *sufficient*. The expectation is that such a condition is provided by *K-stability*, which is an algebro-geometric notion with roots in Geometric Invariant Theory (GIT). The concept of K-stability was traditionally considered only for a special class of Kähler manifolds called *polarized manifolds* (i.e. pairs (X, L) of a compact Kähler manifold X and an ample line bundle L over X). In this setting, the following guiding conjecture was introduced by Tian [Tia97] and refined by Donaldson [Don02]:

Conjecture 1.0.1. (Yau-Tian-Donaldson conjecture, [CDS15a, CDS15b, CDS15c, Tia15]) *A polarized manifold (X, L) admits a constant scalar curvature Kähler (cscK) metric if and only if it is K-stable.*

Remark 1.0.2. *There are in fact several versions of this conjecture, depending on slight differences in the meaning given to the term "K-stable".*

The Yau-Tian-Donaldson (YTD) conjecture was recently proven in the special case when X is a Fano manifold, see [CDS15a, CDS15b, CDS15c, Tia15], when cscK metrics are nothing but Kähler-Einstein metrics. A strongly related topic is existence of *twisted* Kähler-Einstein metrics, see e.g. [Rub08, Don12, JMR16, CS14, CSW]. The direction "existence of cscK metric implies K-stability" of the conjecture has been solved, due to [Don05a, Mab08, Sto09b, Ber16]. The other direction is completely open in general.

The YTD conjecture implies and is closely related to Tian's properness conjecture [Tia94, Remark 5.2], [Tia00b, Conjecture 7.12] (see [DR17] for a recent treatment and modification of

the statement) stating that a cscK metric exists in a given Kähler class $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ if and only if the Mabuchi functional is *coercive* (or *proper*) on the space \mathcal{H}_α of all Kähler metrics cohomologous to ω . It is heuristically not surprising to expect that the existence of minima of the Mabuchi functional in the space of Kähler metrics (which is an infinite dimensional Riemannian symmetric space of negative curvature, cf. [Mab87]) should be equivalent to the properness of the Mabuchi functional. One direction of Tian's properness conjecture was recently solved [DR17, BDL16], see also the work of Ding-Tian [DT92], Tian [Tia97] and Tian-Zhu [TZ00] in the context of Kähler-Einstein metrics. The other direction remains a main open problem in complex geometry. The added information in the YTD conjecture is the expectation that it suffices to test coercivity along certain geodesic rays that can be constructed *algebra-geometrically* (using so called *test configurations*, see below).

In order to go beyond the setting of projective manifolds with rational polarizations we recall the fundamental consequence of the work of Kodaira (cf. Kodaira embedding theorem [Kod54]) that we can consider real $(1,1)$ -classes (so called *transcendental* classes) to be the compact Kähler analogue of line bundles in complex projective geometry.

For transcendental classes very little is currently known about the validity of a correspondence between *existence of cscK metrics* and *stability* in the spirit of the YTD conjecture. This thesis is concerned with the study of K-stability in this more general setting, i.e. for compact Kähler manifolds that do *not* necessarily admit ample line bundles. This amounts to studying existence of cscK metrics in *arbitrary* "transcendental" Kähler classes $[\omega]$ (the polarized setting corresponds to demanding that the $(1,1)$ -class $[\omega]$ has integer, or possibly rational, coefficients). In particular there are two important classes of examples that are covered by the material in this thesis, that is not within reach of the algebraic approach to K-stability for polarized manifolds due to Tian [Tia97], Donaldson [Don02] and many others.

1. Projective compact Kähler manifolds endowed with a *non-rational* Kähler class $\alpha_X \in H^{1,1}(X, \mathbb{R}) \setminus H^2(X, \mathbb{Q})$. By a celebrated result of Kodaira [Kod54] the Kähler cone then contains rational classes, to which the classical algebraic theory of K-stability does apply. This situation is closely linked to the idea of extending the algebraic theory to \mathbb{R} -divisors.
2. Non-projective compact Kähler manifolds X . In this situation the Kähler cone does *not* contain any rational classes at all. This leads to a situation which is much more different from the algebraic theory than in 1) above. We will not list explicit examples here, but note the seminal work of C. Voisin [Voi04a, Voi06] proving existence of non-projective compact Kähler manifolds that are not even deformation equivalent to any projective manifold.

As a first application of the main results of this thesis a large number of examples of cscK manifolds that fall under the above categories can now be verified to be (uniformly) K-stable, or at least K-semistable. In particular this applies to the examples of Voisin mentioned above.

Moreover, the study of K-stability in this more general situation is natural, for several reasons: First of all, from a differential-geometric point of view there is no reason why one should study existence of cscK metrics only in the case of compact Kähler manifolds (X, ω)

whose associated Kähler class $[\omega]$ belongs to the integer lattice $H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$. In addition, even if one were interested only in the polarized case, there are indications due to an influential example of Apostolov, Calderbank, Gauduchon and Tønnesen-Friedman [ACGTF08] that, at least in the setting of extremal metrics, one may need to consider a generalized notion of K-stability allowing for test configurations with *irrational* polarizations. It is also interesting in its own right to develop a formalism for K-stability from a differential-geometric point of view (this is similar in spirit also to [Tel04] and related work on GIT from an analytic point of view). Such a perspective is in many ways more natural than the more classical algebraic geometric approach. As already mentioned it is also more general, as it applies to *arbitrary* compact Kähler manifolds.

Finally, one would hope that the formalism introduced in this thesis could ultimately be applied to advance the classical (algebraic) YTD conjecture by means of *analytic* methods (e.g. methods from pluripotential theory).

1.1 Selected results

A notion of K-stability in terms of cohomology

We briefly explain the transcendental framework we have in mind, using the intersection theoretic description of the Donaldson-Futaki invariant, cf. [Wan12] and [Oda13]. As first pointed out by Berman [Ber13], this leads to a straightforward generalized notion of K-stability in terms of cohomology. We refer to Section 3.1 or the surveys [Tia00b, Sze14, Don17] for definitions concerning the classical algebraic theory.

In order to extend the study of stability questions to a transcendental setting (i.e. the settings (1) and (2) above) we proceed as follows: As a starting point, there are natural generalisations of certain key concepts to the transcendental setting, a central notion being that of *test configurations*. First recall that a test configuration for a polarized manifold (X, L) , in the sense of Donaldson, cf. [Don02], is given in terms of a \mathbb{C}^* -equivariant degeneration of (X, L) as follows:

$$\begin{array}{ccc} \mathbb{C}^* \curvearrowright (\mathcal{X}, \mathcal{L}) & \longleftarrow & (X, L) := \pi^{-1}(\tau), \tau \neq 0 \\ & & \downarrow \pi \\ & & \mathbb{P}^1 \end{array}$$

Here each generic fiber $\pi^{-1}(\tau)$, $\tau \neq 0$, is isomorphic to (X, L) (but the central fiber $\pi^{-1}(0)$ is singular in general). A test configuration can be seen as an algebro-geometric way of compactifying the product $X \times \mathbb{C}^* \hookrightarrow \mathcal{X}$.

A basic observation is then that if (X, L) is K-stable and (X, L') is another polarized manifold so that $c_1(L) = c_1(L')$, then also (X, L') is K-stable (one way to see this is due to the intersection theoretic formulation of the Donaldson-Futaki invariant, due to [Wan12], [Oda13]). In other words, K-stability is a condition that bears only on the Kähler cohomology class considered.

Based on such considerations it was remarked in [Ber13] that a straightforward generalisation to the transcendental setting can be given by replacing the line bundles with $(1, 1)$ -cohomology classes. In the polarized setting we would thus consider $(\mathcal{X}, c_1(\mathcal{L}))$ as a "test configuration" for $(X, c_1(L))$, by simply replacing \mathcal{L} and L with their respective first Chern classes. The *details* of how to formulate a good definition of such a generalized test configuration have, however, not yet been completely clarified.

The definition of "cohomological K-stability" given in this thesis is motivated by a careful comparison to the usual polarized case, where we ensure that a number of basic but convenient tools still hold, cf. Section 3.2. In particular, restricting to the case of an integral class one may note that the definitions of stability in terms of cohomology are stronger than their usual algebraic counterparts. However, in the case of semistability and uniform stability the cohomological and algebraic notions turn out to be *equivalent* (this is proven in Propositions 3.2.23 and 3.2.24). Whether or not the notions of K-polystability are equivalent is an open question (we expect that the answer to this question may be negative, due to the aforementioned examples of [ACGTF08]).

The formal definition of the test configurations used in this thesis involves *Bott-Chern cohomology groups* (sometimes also referred to as $\partial\bar{\partial}$ -cohomology groups) on normal complex spaces. The Bott-Chern cohomology group is defined as the space of $(1, 1)$ -forms with local potentials modulo $dd^c C^\infty(X)$, i.e.

$$H_{BC}^{1,1}(\mathcal{X}) := \{(1, 1) \text{ - forms with local potentials}\} / dd^c C^\infty(\mathcal{X})$$

The same definition goes through for $(1, 1)$ -currents with local potentials, by instead taking the quotient with $dd^c \mathcal{D}'(\mathcal{X})$. We recall some background on Bott-Chern cohomology in Section 2.2.

Definition 1.1.1. (Cohomological test configuration) *A cohomological test configuration for (X, α) is a pair $(\mathcal{X}, \mathcal{A})$ where \mathcal{X} is a test configuration for X (see Definition 3.2.2) and $\mathcal{A} \in H_{BC}^{1,1}(\mathcal{X}, \mathbb{R})^{\mathbb{C}^*}$ is a \mathbb{C}^* -invariant $(1, 1)$ -Bott-Chern cohomology class whose image under the canonical \mathbb{C}^* -equivariant isomorphism*

$$\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{P}^1 \setminus \{0\})$$

is $p_1^ \alpha$, see (3.4). Here $p_1 : X \times \mathbb{P}^1 \rightarrow X$ denotes the first projection.*

Remark 1.1.2. A few remarks are in order:

- Note that the definition is given directly over \mathbb{P}^1 so that we consider the Bott-Chern cohomology on a compact Kähler normal complex space. In the polarized case, defining a test configuration over \mathbb{C} or over \mathbb{P}^1 is indeed equivalent, due to the existence of a natural \mathbb{C}^* -equivariant compactification over \mathbb{P}^1 .
- In order to study K-semistability or uniform K-stability it will in practice (by a resolution of singularities argument) be enough to consider the situation when the total space \mathcal{X} is smooth and dominates $X \times \mathbb{P}^1$, with $\mu : \mathcal{X} \rightarrow X \times \mathbb{P}^1$ the corresponding canonical \mathbb{C}^* -equivariant bimeromorphic morphism. This significantly simplifies matters and plays an important technical role in certain proofs. Note in particular the following:

- The Bott-Chern cohomology group $H_{BC}^{1,1}(\mathcal{X})$ can then be identified with the usual Dolbeault cohomology $H^{1,1}(X)$.
- If $(\mathcal{X}, \mathcal{A})$ is a cohomological test configuration for (X, α) with \mathcal{X} as above, then \mathcal{A} is always of the form $\mathcal{A} = \mu^* p_1^* \alpha + [D]$, for a unique \mathbb{R} -divisor D supported on the central fiber \mathcal{X}_0 , cf. Proposition 3.2.18. A cohomological test configuration can thus be characterised by an \mathbb{R} -divisor, clarifying the relationship between the point of view of \mathbb{R} -divisors and our cohomological approach to "transcendental K-stability".

A straightforward generalisation of the Donaldson-Futaki invariant can be defined based on the intersection theoretic characterisation of [Wan12] and [Oda13]. Indeed, we define the *Donaldson-Futaki invariant* associated to a cohomological test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) as the following intersection number

$$\text{DF}(\mathcal{X}, \mathcal{A}) := \frac{\bar{S}}{n+1} V^{-1}(\mathcal{A}^{n+1})_{\mathcal{X}} + V^{-1}(K_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{A}^n)_{\mathcal{X}}, \quad (1.1)$$

computed on the (compact) total space \mathcal{X} . Here V and \bar{S} are cohomological constants denoting the Kähler volume and mean scalar curvature of (X, α) respectively.

With the exception of the definition of product test configurations, the notions of K-stability are introduced in precise analogy with the usual setting of polarized manifolds. We will study the notions of K-semistability and K-polystability, as well as notions of *K-stability* and *uniform K-stability* (with respect to the *non-archimedean J-functional* J^{NA}) in case the automorphism group $\text{Aut}(X)$ is finite. Indeed, we say that (X, α) is *K-semistable* if $\text{DF}(\mathcal{X}, \mathcal{A}) \geq 0$ for all cohomological test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α) where the class \mathcal{A} is *relatively Kähler*, i.e. there is a Kähler form β on \mathbb{P}^1 such that $\mathcal{A} + \pi^* \beta$ is Kähler on \mathcal{X} . We say that (X, α) is *K-polystable* if moreover $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$ precisely when $(\mathcal{X}, \mathcal{A})$ is a *product configuration*, i.e. $\mathcal{X}_{\pi^{-1}(\mathbb{C})}$ is \mathbb{C}^* -equivariantly isomorphic to $X \times \mathbb{C}$. Finally, (X, α) is said to be *uniformly K-stable* if $\text{DF}(\mathcal{X}, \mathcal{A})$ is bounded below by a positive multiple of a certain *norm* of the test configuration, in this case if there is a $\delta > 0$ such that

$$\text{DF}(\mathcal{X}, \mathcal{A}) \geq \delta J^{\text{NA}}(\mathcal{X}, \mathcal{A})$$

for all relatively Kähler (cohomological) test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α) . It can be proven (see Section 5 that $J^{\text{NA}}(\mathcal{X}, \mathcal{A}) = 0$ if and only if $(\mathcal{X}, \mathcal{A})$ is trivial, i.e. \mathcal{X} is equivariantly isomorphic to $X \times \mathbb{C}$ endowed with the trivial \mathbb{C}^* -action, see Section 5.4 for precise definitions. Note that a necessary condition for uniform K-stability to hold is that the automorphism group $\text{Aut}(X)$ is finite. Moreover, in this situation uniform K-stability implies K-polystability, often just called K-stability in order to underline that we restrict to the situation when $\text{Aut}(X)$ is finite.

Asymptotics for energy functionals in Kähler geometry

A fundamental tool and a central part of this thesis consists in establishing a Kempf-Ness type formula connecting the *Donaldson-Futaki invariant* (in the sense of (1.1) above) with

the *asymptotic slope of the K-energy* along certain weak geodesic rays that are *compatible* in the sense that their singularity type is prescribed by a given test configuration. In fact, we first prove the following result, which is concerned with asymptotics of a certain multivariate analogue of the Monge-Ampère energy, cf. Section 2.4 for its definition. It turns out to be very useful for establishing a similar formula for the K-energy (cf. Remark 1.1.3), but may also be of independent interest.

In what follows, we will work on the level of potentials and refer the reader to Section 4 for precise definitions (in particular of the compatibility notions introduced).

Theorem A. *Let X be a compact Kähler manifold of dimension n and let θ_i , $0 \leq i \leq n$, be closed $(1,1)$ -forms on X . Set $\alpha_i := [\theta_i] \in H^{1,1}(X, \mathbb{R})$. Consider smooth cohomological test configurations $(\mathcal{X}_i, \mathcal{A}_i)$ for (X, α_i) dominating $X \times \mathbb{P}^1$. For each collection of smooth rays $(\varphi_i^t)_{t \geq 0}$, \mathcal{C}^∞ -compatible with $(\mathcal{X}_i, \mathcal{A}_i)$ respectively, the asymptotic slope of the multivariate energy functional $\langle \cdot, \dots, \cdot \rangle := \langle \cdot, \dots, \cdot \rangle_{(\theta_0, \dots, \theta_n)}$ is well-defined and satisfies*

$$\frac{\langle \varphi_0^t, \dots, \varphi_n^t \rangle}{t} \longrightarrow (\mathcal{A}_0 \cdots \mathcal{A}_n) \quad (1.2)$$

as $t \rightarrow +\infty$. See (4.1.2) for the definition of the above intersection number.

Remark 1.1.3. In the setting of Hermitian line bundles, the above multivariate energy functional naturally appears as the difference (or quotient) of metrics on Deligne pairings. Moreover, note that the above theorem applies to e.g. the Monge-Ampère energy functional E and its 'twisted' version $E^{\text{Ric}(\omega)}$ but *not* to the K-energy M . Indeed, the expression for $M(\varphi_t)$ on the form $\langle \varphi_0^t, \dots, \varphi_n^t \rangle_{(\theta_0, \dots, \theta_n)}$ involves the metric $\log(\omega + dd^c \varphi_t)$ on the relative canonical bundle $K_{\mathcal{X}/\mathbb{P}^1}$, which blows up close to \mathcal{X}_0 , cf. Section 4.3. As observed in [BHJ16], it is however possible to find functionals of the above form that 'approximate' M in the sense that their asymptotic slopes coincide, up to an explicit correction term that vanishes precisely when the central fiber \mathcal{X}_0 is reduced.

We further remark that such a formula (1.2) cannot be expected to hold unless the test configurations $(\mathcal{X}_i, \mathcal{A}_i)$ and the rays (φ_i^t) are compatible in a certain sense. This is the role of the notion of \mathcal{C}^∞ -compatibility (as well as the $\mathcal{C}^{1,\bar{1}}$ -compatibility used in Theorem C below). These notions may seem technical, but in fact mimic the case of a polarized manifolds, where the situation is well understood in terms of extension of metrics on line bundles, cf. Section 4.1.

As an example, the above formalism applies to give asymptotic expansions for the Monge-Ampère energy functional E (sometimes also called the *Aubin-Mabuchi functional*) and for Aubin's J-functional. Indeed, assume that $(\mathcal{X}, \mathcal{A})$ is smooth and dominates $X \times \mathbb{P}^1$. For each smooth ray $(\varphi_t)_{t \geq 0}$ \mathcal{C}^∞ -compatible with $(\mathcal{X}, \mathcal{A})$, we then have (see Section 4.2 for explanations)

$$E(\varphi_t) := \frac{1}{n+1} V^{-1} \langle \varphi_t, \dots, \varphi_t \rangle_{(\omega, \dots, \omega)}$$

so that

$$\lim_{t \rightarrow +\infty} \frac{E(\varphi_t)}{t} = \frac{1}{n+1} V^{-1}(\mathcal{A}^{n+1}).$$

Similarly, we have

$$J(\varphi_t) = V^{-1}\langle \varphi_t, 0, \dots, 0 \rangle_{(\omega, \dots, \omega)} - E(\varphi_t),$$

and since the constant ray (0) can be seen to be compatible with $(\mathcal{X}, \mu^* p_1^* \alpha)$ it follows that

$$\lim_{t \rightarrow +\infty} \frac{J(\varphi_t)}{t} = V^{-1}(\mathcal{A} \cdot \mu^* p_1^* \alpha^n) - \frac{1}{n+1} V^{-1}(\mathcal{A}^{n+1}).$$

In the non-archimedean terminology of [BHJ15, BHJ16] we denote the above intersection numbers by $E^{\text{NA}}(\mathcal{X}, \mathcal{A})$ and $J^{\text{NA}}(\mathcal{X}, \mathcal{A})$ respectively. We may also note that with the above formalism it is conveniently simple to see what functionals correspond to what intersection theoretic expression for the asymptotic slope.

As an important consequence of the above Theorem A we deduce that if $(\mathcal{X}, \mathcal{A})$ is a smooth, relatively Kähler cohomological test configuration for (X, α) dominating $X \times \mathbb{P}^1$, then for each subgeodesic ray $(\varphi_t)_{t \geq 0}$, $\mathcal{C}^{1, \bar{1}}$ -compatible with $(\mathcal{X}, \mathcal{A})$, we have the inequality

$$\lim_{t \rightarrow +\infty} \frac{M(\varphi_t)}{t} \leq \text{DF}(\mathcal{X}, \mathcal{A}). \quad (1.3)$$

This is the content of Theorem 4.3.1, and should be compared to the discussion in the introduction of [PRS08]. As an important special case, this inequality can be seen to hold in the case of a weak geodesic ray associated to the given test configuration $(\mathcal{X}, \mathcal{A})$, cf. Section 4.1 for its construction. The inequality (1.3) is moreover enough to conclude certain stability results for cscK manifolds, see e.g. Theorem C below.

Using ideas from [BHJ16] adapted to the present setting, we may further improve on formula (1.3) and compute the precise asymptotic slope of the K-energy. In this context, it is natural to consider the *non-Archimedean Mabuchi functional*

$$M^{\text{NA}}(\mathcal{X}, \mathcal{A}) := \text{DF}(\mathcal{X}, \mathcal{A}) + V^{-1}((\mathcal{X}_{0, \text{red}} - \mathcal{X}_0) \cdot \mathcal{A}^n)_{\mathcal{X}},$$

cf. [BHJ15] and [BHJ16] for an explanation of the terminology. It is a modification of the Donaldson-Futaki invariant which is *homogeneous under finite base change* and that satisfies $M^{\text{NA}}(\mathcal{X}, \mathcal{A}) \leq \text{DF}(\mathcal{X}, \mathcal{A})$, with equality precisely when the central fiber \mathcal{X}_0 is reduced. We have the following result, special cases of which have been obtained by previous authors in various different situations and generality (as recalled below):

Theorem B. *Let $(\mathcal{X}, \mathcal{A})$ be a smooth, relatively Kähler cohomological test configuration for (X, α) that dominates $X \times \mathbb{P}^1$. For each subgeodesic ray $(\varphi_t)_{t \geq 0}$, $\mathcal{C}^{1, \bar{1}}$ -compatible with $(\mathcal{X}, \mathcal{A})$, we then have*

$$\frac{M(\varphi_t)}{t} \longrightarrow M^{\text{NA}}(\mathcal{X}, \mathcal{A})$$

as $t \rightarrow +\infty$.

In particular, this result holds when (φ_t) is the weak geodesic ray associated to $(\mathcal{X}, \mathcal{A})$, constructed in Section 4.1.

For polarized manifolds (X, L) and *smooth* subgeodesic rays $(\varphi_t)_{t \geq 0}$ this precise formula was proven in [BHJ16] using Deligne pairings, as pioneered by Phong-Ross-Sturm in [PRS08] (cf.

also Paul-Tian [PT06] [PT09]). A formula in the same spirit has also been obtained for the so called *Ding functional* when X is a Fano variety, see [Ber16]. However, it appears as though no version of this result is currently known in the case of non-polarized manifolds.

Stability results for cscK manifolds with transcendental cohomology class

Let as before X be a compact Kähler manifold, and $\alpha \in H^{1,1}(X, \mathbb{R})$ a (not necessarily rational) Kähler class on X . A first main goal of this thesis is to establish the following result:

Theorem C. *If the Mabuchi K-energy functional is bounded from below in the Kähler class $\alpha \in H^{1,1}(X, \mathbb{R})$, then (X, α) is K-semistable (in the generalised sense of Definition 3.2.13)*

As an immediate consequence of the above and results of [BB14, CLP16] we obtain the following result:

Theorem D. *The pair (X, α) is K-semistable if the Kähler class $\alpha \in H^{1,1}(X, \mathbb{R})$ admits a cscK representative.*

The corresponding statement in the case of a polarized manifold was first obtained by Donaldson in [Don05a], as an immediate consequence of the lower bound for the Calabi functional. See also [Sto09b] and [RT06] for related work on *slope semistability*. The approach taken in this thesis should be compared to e.g. [PRS08] and [Ber13, Ber16, BHJ15, BHJ16], where K-semistability can be derived using Kempf-Ness type formulas. By analogy to the above papers, our proof relies on establishing Kempf-Ness type formulas valid also for transcendental classes (see Theorems B and C below) relating the asymptotic slope of the K-energy along weak geodesic rays to a natural generalisation of the Donaldson-Futaki invariant.

Recently it was proven by Berman, Darvas and Lu [BDL16] that the Mabuchi functional is so called *G-coercive* on cscK manifolds. This establishes one direction of Tian's properness conjecture [Tia94] with modifications due to Darvas-Rubinstein [DR17]. In particular, in case the automorphism group is finite this result says that there are $\delta > 0$ and $C > 0$ such that

$$M(\varphi) \geq \delta J(\varphi) - C, \quad \varphi \in \mathcal{H}.$$

As a second application of Theorems A and B we show that if the Mabuchi functional is *G-coercive*, then (X, α) is (cohomologically) uniformly K-stable:

Theorem E. *Let (X, ω) be a cscK manifold, with $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ the associated Kähler class. Suppose that $\text{Aut}_0(X) = \{0\}$. If the Mabuchi functional is *G-coercive*, then (X, α) is uniformly K-stable, i.e. there is a $\delta > 0$ such that*

$$\text{DF}(\mathcal{X}, \mathcal{A}) \geq \delta J^{\text{NA}}(\mathcal{X}, \mathcal{A}).$$

for each relatively Kähler cohomological test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) .

Combining the above with the main result of [BDL16] we in particular obtain the following corollary:

Theorem F. *If (X, ω) is a cscK manifold with discrete automorphism group, then $(X, [\omega])$ is uniformly K-stable.*

We further check in Theorem 5.3.2 that $J^{\text{NA}}(\mathcal{X}, \mathcal{A})$ indeed vanishes if and only if $(\mathcal{X}, \mathcal{A})$ is trivial, i.e. if \mathcal{X} is \mathbb{C}^* -equivariantly isomorphic to $X \times \mathbb{P}^1$ endowed with the action $\tau \cdot (x, z) := (x, \tau z)$.

Theorem G. *Let (X, ω) be a cscK manifold, with $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ the associated Kähler class. Suppose that $\text{Aut}_0(X) = \{0\}$. Let $(\mathcal{X}, \mathcal{A})$ be a normal and relatively Kähler test configuration for (X, α) . Then the following are equivalent:*

1. $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$.
2. $J^{\text{NA}}(\mathcal{X}, \mathcal{A}) = 0$.
3. $(\mathcal{X}, \mathcal{A})$ is the trivial test configuration.

In particular, as the name suggests, it follows from Theorem G that uniform K-stability implies K-stability. The above result moreover gives a self-contained proof of the following K-stability result, different from the one of [DR16] (that extend the methods of Stoppa [Sto09a]).

Theorem H. *If (X, ω) is a cscK manifold with discrete automorphism group, then $(X, [\omega])$ is K-stable.*

Remark 1.1.4. *The above results prove one direction of the YTD conjecture (both its uniform and non-uniform version) in this setting of compact Kähler manifolds with possibly non-rational cohomology class.*

Further developments related to the case $\text{Aut}_0(X) \neq \{0\}$

The situation when X is a compact Kähler manifold admitting non-trivial holomorphic vector fields is much more involved. In this thesis we obtain some partial results in this direction, related to ongoing work, and formulate a strategy for proving cohomological K-polystability of cscK manifolds. This includes a number of equivalent characterizations of test configurations with vanishing Donaldson-Futaki invariant, established using a certain injectivity result (cf. Theorem I below) that is a cornerstone of our strategy of proof. As a by product we obtain a new proof of cohomological K-stability of cscK manifolds with discrete automorphism group, different from the one using Stoppa-like methods given in [DR16].

Remark 1.1.5. *One may note that cohomological K-stability is a priori stronger than algebraic K-stability, so these results may give something more even in the case of polarized manifolds, cf. Proposition 3.2.25.*

Geodesic rays and the isomorphism class of relatively Kähler test configurations

Following the seminal work of Mabuchi [Mab87] the interplay between test configurations and their associated geodesic rays has been a fundamental tool in the study of K-stability for polarized manifolds (cf. e.g. [PRS08, Ber16, BHJ15, DR17, BB14, CLP16] to name a few recent works). The same techniques can more generally be applied in the setting of compact Kähler manifolds with transcendental cohomology class. Indeed, the first thing to note is that, relying on theory for degenerate Monge-Ampère equation on manifolds with boundary, we can assign a unique geodesic ray $(\varphi_t)_{t \geq 0} := (\varphi_t)_{t \geq 0}^{(\mathcal{X}, \mathcal{A})}$ to each cohomological test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) , see Section 4.1 for details on the standard construction. In particular, many questions about test configurations can be approached by working with the corresponding rays.

One way of formalising this approach is the following: First recall that in the case of polarized manifolds there is a one-one correspondence between relatively ample test configurations and finitely generated \mathbb{Z} -filtrations of the graded algebra $R(X, L) := \bigoplus_{k \in \mathbb{N}} H^0(X, kL)$. This is due to so called *reverse Rees construction*, see [BHJ15, Section 2.5] for details. From this it follows that the map

$$(\mathcal{X}, \mathcal{A}) \mapsto (\varphi_t)_{t \geq 0}^{(\mathcal{X}, \mathcal{A})}$$

assigning to a relatively ample test configuration its associated geodesic ray is *injective*. In the Kähler setting we do not have any analog of the above one-one correspondence, but there is nonetheless a natural generalization of the above observation. The following result of independent interest is a Kähler analog to the concept of *unique ample model* for polarized manifolds (cf. [BHJ15]).

Theorem I. *Any subgeodesic ray $[0, +\infty) \ni t \mapsto \varphi_t \in \text{PSH}(X, \omega) \cap L^\infty(X)$ is L^∞ -compatible with at most one normal, relatively Kähler test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) .*

See Section 4.1 for the definition of compatibility of rays (in particular this condition is satisfied by the associated geodesic ray). This leads to the following strategy for proving K-polystability of cscK manifolds: Suppose that (X, α) is a cscK manifold and that one could then show that the geodesic ray of any test configuration for (X, α) satisfying $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$ is also compatible with some product configuration. Then it would follow that cscK manifolds are K-polystable (indeed, recall that the K-semistability is already proven without any discreteness assumption on the automorphism group, see Theorem C above).

As a partial result in the direction of K-polystability of cscK manifolds, we give "analytic" characterizations of the condition that a cohomological test configuration has vanishing Donaldson-Futaki invariant. For technical reasons we consider only test configurations whose associated geodesic ray $(\varphi_t)_{t \geq 0}$ is normalized in such a way so that $E(\varphi_t) = 0$ (i.e. vanishing Monge-Ampère energy). This is not a very serious restriction, as explained in Section 5.4. Subpoint (4) of the following result involves the action of the connected automorphism group $G := \text{Aut}_0(X)$ on the space of normalized Kähler potentials. This is simply the action corresponding to pullback of forms, i.e. $g \cdot \omega := g^* \omega$, transported via the isomorphism sending ω to its normalized Kähler potential. We refer to Section 5.4 for full definitions and explanations.

Theorem J. *Suppose that (X, ω) is a cscK manifold, with $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ the corresponding Kähler class. Let $(\mathcal{X}, \mathcal{A})$ be a normal and relatively Kähler test configuration for (X, α) whose associated geodesic ray $(\varphi_t)_{t \geq 0}$ satisfies $E(\varphi_t) = 0$ for each $t \in [0, +\infty)$. Let $J : TX \rightarrow TX$ be the complex structure and ω a cscK metric on X . Then the following statements are equivalent:*

1. $DF(\mathcal{X}, \mathcal{A}) = 0$.
2. The central fiber \mathcal{X}_0 is reduced and $M^{\text{NA}}(\mathcal{X}, \mathcal{A}) = 0$.
3. The central fiber \mathcal{X}_0 is reduced and the Mabuchi K-energy functional is constant along the geodesic ray $(\varphi_t)_{t \geq 0}$ associated to $(\mathcal{X}, \mathcal{A})$, i.e. we have $M(\varphi_t) = M(\varphi_0)$ for each $t \in [0, +\infty)$.
4. The central fiber \mathcal{X}_0 is reduced and the associated geodesic ray satisfies

$$\inf_{g \in G} J(g \cdot \varphi_t) = 0 \quad \text{and} \quad \inf_{g \in G} d_1(0, g \cdot \varphi_t) = 0.$$

5. The central fiber \mathcal{X}_0 is reduced and there is a real holomorphic Hamiltonian vector field V such that the geodesic ray $(\varphi_t)_{t \geq 0}$ associated to $(\mathcal{X}, \mathcal{A})$ satisfies $\exp(tV)^*\omega = \omega$ and $\exp(tJV)^*\omega = \omega_{\varphi_t}$.
6. The central fiber \mathcal{X}_0 is reduced and the associated geodesic ray (φ_t) consists entirely of cscK potentials. More precisely, if \bar{S} denotes the mean scalar curvature of ω_{φ_0} , then

$$\mathcal{S}(\omega_{\varphi_t}) = \bar{S}$$

for each $t \in [0, +\infty)$.

The central part of the proof is an adaptation of [BDL16, Proposition 3.1] combined with energy functional asymptotics for singular test configurations (an additional difficulty in the case of cohomological K-stability is indeed that, unlike for cohomological K-semistability or uniform K-stability, we may no longer restrict attention to smooth and dominating test configurations). Elements of the above result also seem closely related to several results in the seminal paper [Mab86, cf. Section 5] of Mabuchi.

From the above result one can deduce further partial results in the direction of a possible proof that cscK manifolds are K-polystable. At the time of writing, however, we have been unable to establish the result in full generality.

Perspectives and applications

As already mentioned, as a first application of the main results of this thesis a large number of examples of cscK manifolds can be verified to be (uniformly) K-stable if the automorphism group is discrete, and K-semistable independently of the automorphism group. We now discuss some further interesting questions where we expect the approach of this thesis to be useful.

Deformation theory and stability loci in the Kähler cone

We first discuss an interpretation of the main results of this thesis in the context of stability loci in the Kähler cone $\text{Kah}(X)$: For example, if we suppose that the automorphism group $\text{Aut}(X)$ is discrete, then Theorems C and E yield the following chain of inclusions:

$$\begin{aligned} \text{cscK locus} \subset \text{Uniformly K-stable locus} \subset \text{K-stable locus} \subset \\ \subset \text{K-semistable locus} \subset \text{Kah}(X). \end{aligned}$$

By the cscK locus we here mean the set of Kähler classes $\alpha \in \text{Kah}(X)$ such that (X, α) is cscK (with obvious adaptations for the definition of the other stability loci). Two general lines of inquiry are the following:

1. What can be said about the topology of the above stability loci? For instance, are they open or closed in the Kähler cone of X ? Do they have some additional structure?
2. Which of the above inclusions are strict in general? Under what conditions?

Such questions may of course be very difficult to answer. For instance, regarding 2), even the YTD conjecture can be reformulated as stating that the cscK locus *equals* the K-stable locus (or the uniformly K-stable locus, as is widely conjectured). On the other hand, it is known with regards to 2) that the inclusion of the cscK locus \subset K-semistable locus is strict in general, due to concrete counterexamples for ruled manifolds (see [Kel16] and [KR11]). With regards to the notions of stability introduced in this thesis it would also be very interesting to clarify whether the inclusion cohomologically K-stable locus \subset algebraically K-stable locus is strict in general (recall that the algebraic and cohomological stability notions coincide for semistability and uniform stability, see Propositions 3.2.23 and 3.2.24). As previously mentioned, one could imagine that the inclusion may in fact be strict, due to examples of [ACGTF08] in the setting of extremal metrics.

Generalizing an approach similar to that of the 'convex combinations of test configurations' considered in A. Isopoussu's thesis [Iso13] to the Kähler setting we propose some first attempts to describe different stability loci in the Kähler cone of X (and relate it to the cscK locus). For instance, we show that, whereas the cscK locus is open, the K-semistable locus is "closed along line segments" in the Kähler cone (see 6.1.8 for the precise formulation). In particular these inclusion of the cscK locus \subset K-semistable locus must be strict in general. We also discuss implications for the classical Futaki invariant. These investigations are natural applications of the techniques introduced in this thesis. We refer to section 6 for full explanations and further results in this direction.

In the very last paragraphs of this thesis we discuss implications of Theorem J for uniform K-polystability and mention further applications related to the study of existence of metrics twisted with a *transcendental* class. For instance, it seems likely that the formalism introduced in this thesis can be used to generalize the variational proof of Berman, Boucksom and Jönsson [BBJ15] of the YTD conjecture for Fano manifolds to the setting of Kähler-Einstein metrics twisted by a transcendental class.

Chapter 2

Preliminaries

In this section we introduce notation and definitions and recall some background on constant scalar curvature Kähler (cscK) manifolds, Kähler geometry on normal complex spaces, and the variational approach involving the study of energy functionals on spaces of Kähler potentials. In Section 2.4 we introduce our notion of 'Deligne energy functionals', which is a key object of study for the main results presented in Section 4.

2.1 Background on Kähler and complex projective geometry

The notions of K-stability introduced in this thesis lead to a study of compact Kähler manifolds from an analytic point of view, while the classical notions of K-stability originating from Tian [Tia97] and [Don02] are typically studied for complex projective manifolds. There are close links between the Kähler and the complex projective geometries, but also some important differences that are worth pointing out. We briefly review some of the fundamentals and explain how the compact Kähler setting can be seen as a generalization of the compact projective setting. For more details on the topic of this section we refer to the survey [Voi04b], and references therein.

Compact Kähler manifolds

First recall the notion of complex valued differential form of type (p, q) on a finite dimensional complex vector space V of complex dimension $n := \dim_{\mathbb{C}}(V)$: The space $V^* \otimes \mathbb{C}$ of complex valued forms on V has a decomposition as a direct sum $V^{*1,0} \oplus V^{*0,1}$ into the space of \mathbb{C} -linear forms and its complex conjugate. The (p, q) -forms are generated by forms of the type $a_1 \wedge \dots \wedge a_p \wedge b_1 \wedge \dots \wedge b_q$, where $a_i \in V^{*1,0}$ and $b_i \in V^{*0,1}$. If V is endowed with a Hermitian bilinear form h , we have a decomposition

$$h = g - i\omega, \tag{2.1}$$

where g is a symmetric real bilinear form on X and ω is a real 2-form of type $(1, 1)$ for the complex structure on V . The notion of (semi)-positivity of hermitian forms h can be transferred to a

corresponding notion for real $(1, 1)$ -forms, via the bijection $h \mapsto \omega$. When h is a positive definite hermitian form, the corresponding 2-form ω is non-degenerate, i.e. the top intersection $\omega^n \neq 0$.

Now suppose that X is a complex manifold. The tangent space $T_{X,x}$ of X at a point $x \in X$ is then endowed with a complex structure, and we have a one to one correspondence between Hermitian bilinear forms on T_X and real 2-forms of type $(1, 1)$ on X . As above, if h is a hermitian metric on T_X , then we can decompose it into $h = g - i\omega$, where g is a Riemannian metric which is compatible with the complex structure on X , and ω is a real positive $(1, 1)$ -form on X .

Definition 2.1.1. *The hermitian metric h is said to be Kähler if the corresponding 2-form ω is closed.*

Remark 2.1.2. *By a standard abuse of notation we will often identify the Kähler form ω with the Kähler metric h it represents.*

Since a positive multiple $\lambda\omega$, $\lambda > 0$, of a Kähler form ω on X remains Kähler we see that the set of Kähler forms ω on X forms an open cone. It is referred to as the *Kähler cone of X* . Further note that, by non-degeneracy, the Kähler form ω provides a symplectic structure on X . In other words, any Kähler manifold is in particular also a symplectic manifold. One of the main consequences of the compact Kähler assumption is that the *Hodge decomposition theorem* holds, i.e. the de Rham cohomology spaces

$$H^k(X, \mathbb{C}) := \frac{\{\text{closed complex valued } k\text{-forms}\}}{\{\text{exact complex valued } k\text{-forms}\}} \quad (2.2)$$

splits as

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \quad (2.3)$$

where $H^{p,q}(X)$ denotes the space of classes admitting a representative which is a closed form of type (p, q) .

Complex projective manifolds

A complex manifold X is said to be *complex projective* if it is a submanifold of complex projective space $\mathbb{P}^1(\mathbb{C})$. By Chow's theorem (with a generalisation by Serre [Ser56]) complex projective manifolds are precisely the smooth algebraic subvarieties of $\mathbb{P}^1(\mathbb{C})$. The projective space carries a natural Kähler metric (the *Fubini-Study metric* ω_{FS}) and is thus a Kähler manifold. By restriction of ω_{FS} any complex projective manifold is also Kähler. The projective space $\mathbb{P}^n(\mathbb{C})$ furthermore carries a natural line bundle $\mathcal{O}(1)$ (defined as the dual to the hyperplane bundle on $\mathbb{P}^n(\mathbb{C})$) which is *ample*. By restriction any projective complex manifold admits an ample line bundle $\mathcal{O}(1)|_X$. In classical terminology we say that any complex projective manifold is *polarized*:

Definition 2.1.3. *A polarized manifold is a pair (X, L) where X is a compact Kähler manifold and L is an ample line bundle on X .*

By definition L is said to be ample if there exists a $k \gg 1$ large enough so that L^k is *very ample*, meaning that there is an embedding

$$X \hookrightarrow \mathbb{P}(H^0(X, L^k))^*$$

into projective space, given by

$$x \mapsto [s_0(x) : s_1(x) : \cdots : s_{N_k}(x)]$$

where $\{s_i\}$ is a basis of the space of holomorphic sections $H^0(X, L^k)$ and $N_k + 1$ its dimension. In other words a compact Kähler manifold admits a polarization precisely if it is projective. In fact, we have the following consequence of the Kodaira embedding theorem:

Theorem 2.1.4 ([Kod54]). *A compact Kähler manifold X is projective if and only if it carries a Kähler form ω whose cohomology class is rational, i.e. $[\omega] \in H^2(X, \mathbb{Q})$.*

In particular, a fundamental consequence of the work of Kodaira is that we can consider real $(1, 1)$ -classes to be the compact Kähler analogue of line bundles in complex projective geometry. This point of view will be systematically adopted in the sequel.

Compact Kähler non-polarized manifolds

One of the main contributions of this thesis is the development of an *analytic* approach to questions of K-stability, which is also valid for *arbitrary* compact Kähler manifolds. In particular, we are interested in the following two situations (see Section 3) not covered by the classical algebraic theory:

1. (X, ω) is a Kähler manifold with X projective and $[\omega] \in H^{1,1}(X, \mathbb{R})$ a non-rational (i.e. *transcendental*) Kähler class.
2. (X, ω) is a Kähler manifold with X not projective.

These two cases are very different in nature. The first situation is particularly suited to examples, since it falls under the situation of any complex projective manifold, but with an added flexibility in choosing the polarisation (so that it may be any $(1, 1)$ -Kähler class). One way of obtaining non-projective compact Kähler manifolds is as deformations of projective ones. There are however also examples that cannot be obtained in this way, as showed by C. Voisin [Voi04a]. See also [Voi06].

2.2 Kähler geometry and potential theory on normal complex spaces

In order to introduce K-stability notions for arbitrary compact Kähler manifolds (see Section 3) one is naturally led to consider (possibly singular) normal complex analytic spaces. Such objects replace the schemes in the algebraic theory of K-stability for polarized manifolds, due to [Don02] and others. In this section we recall the necessary tools from Kähler geometry for normal complex spaces. For more details we refer the reader to the book [Dem12].

Positive currents and quasi-plurisubharmonic functions

Local preliminaries

We first recall some basic definitions concerning the local theory of plurisubharmonic functions and operators on such functions. To do this, first consider an open subset $\Omega \subset \mathbb{C}^N$.

Definition 2.2.1. *An upper semi-continuous function $u : \Omega \rightarrow [-\infty, +\infty)$ is said to be plurisubharmonic (psh for short) if it is $\not\equiv -\infty$ and if for all complex lines $\Lambda \subset \Omega$ the restriction $u|_{\Omega \cap \Lambda}$ is subharmonic, i.e. if for all $a \in \Omega$, $\xi \in \mathbb{C}^N$ with $|\xi| = 1$ and $r > 0$ such that $\bar{B}(0, r) \subset \Omega$, we have*

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}\xi) d\theta.$$

We denote the space of plurisubharmonic functions on Ω by $\text{PSH}(\Omega)$.

Example 2.2.2. *A prototypical example of a plurisubharmonic (psh) function is $\log |f|$ where $f \in \mathcal{O}(\Omega)$ is a holomorphic function on Ω .*

Define the real operator $dd^c := \frac{1}{2i\pi}(\partial - \bar{\partial})$ on psh functions, so that $dd^c = \frac{1}{2\pi}\partial\bar{\partial} = \frac{1}{2\pi}\Delta$, where the latter is the complex Laplacian. If $u \in \text{PSH}(\Omega) \cap C^\infty(\Omega)$ then $dd^c u$ is simply the Hessian of u . More generally, if $u \in \text{PSH}(\Omega) \cap L^1_{\text{loc}}(\Omega)$, then we may interpret $dd^c u$ as a closed positive $(1, 1)$ -current as follows. Let $\mathcal{D}_{(n-1, n-1)}(\Omega)$ be the space of smooth $(n-1, n-1)$ -forms of compact support in Ω . Then

$$dd^c u : \psi \mapsto \int_{\Omega} u dd^c \psi, \quad \psi \in \mathcal{D}_{(n-1, n-1)}(\Omega) \quad (2.4)$$

is closed and positive, i.e. for each ψ of the form $i\alpha_1 \wedge \bar{\alpha}_1 \wedge \dots \wedge i\alpha_{n-1} \wedge \bar{\alpha}_{n-1}$ we have

$$\langle dd^c u, \psi \rangle \geq 0.$$

It is a well-known fact that $dd^c u$ is of order 0, so it has measure coefficients. An equivalent formulation of positivity of the $(1, 1)$ -current $dd^c u$ is to ask that

$$\sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi}_k$$

is a positive measure on Ω for all $\xi \in \mathbb{C}^N$.

Quasi-psh functions on manifolds

The definition can further be extended to complex manifolds with no boundary, in particular to compact complex manifolds. To see this, let (M, ω_M) be a complex manifold with no boundary, with ω_M a closed and positive smooth $(1, 1)$ -form on M . Prototypical and useful examples include taking $(M, \omega_M) = (X, \omega_X)$ for a compact Kähler manifold (X, ω_X) , but also $(M, \omega_M) = (X \times A, p_1^* \omega_X)$ where (X, ω_X) is a compact Kähler manifold, $A := \{z \in \mathbb{C} : r < |z| < R\}$ is an open annulus in \mathbb{C} , and $p_1 : X \times A \rightarrow X$ is the first projection.

Definition 2.2.3. A function $\varphi \in L^1(M, [-\infty, +\infty))$ is said to be ω_M -psh if it can be locally written as the sum of a smooth and a psh function, and such that

$$\omega_\varphi := \omega + dd^c\varphi \geq 0$$

in the weak sense of currents. Denote the space of ω_M -psh functions on M by $\text{PSH}(M, \omega_M)$.

In other words, locally on some open set Ω we can write $\omega_M = dd^c\varphi$ with $\varphi \in C^\infty(\Omega)$ and $\varphi + u \in \text{PSH}(\Omega)$, so that

$$\omega_M + dd^c u = dd^c(\varphi + u) \geq 0$$

in the sense made precise above.

Bedford-Taylor's monotonicity theorem and $L_{loc}^\infty(X)$ potentials

An important role in pluripotential theory is played by the work of Bedford-Taylor [BT76, BT82], which is used to work with continuous and locally bounded potentials respectively. In particular, this theory lays the foundation for the subsequent discussion on energy functionals, see Section 2.4.

Let Ω be an open subset of \mathbb{C}^N and let u_1, \dots, u_q be locally bounded psh functions on Ω . Then $dd^c u_1$ has measure coefficients, so $u_1 dd^c u_2$ is well-defined as a distribution. We may then define

$$dd^c u_1 \wedge dd^c u_2 := dd^c(u_2 dd^c u_1).$$

The following foundational result is crucial in order to work with psh functions that are merely locally bounded:

Theorem 2.2.4 ([BT82]). *Let $u_{1,j}, \dots, u_{q,j}$ be decreasing sequences of locally bounded psh functions on an open subset $\Omega \subset \mathbb{C}^N$, with respective limits u_1, \dots, u_q that are locally bounded and psh. Then $dd^c u_{1,j} \wedge \dots \wedge dd^c u_{q,j}$ converges weakly to the measure $dd^c u_1 \wedge \dots \wedge dd^c u_q$.*

In particular we can then make sense of the Monge-Ampère operator $(dd^c u)^n$ on locally bounded psh functions in Ω .

Forms and currents on complex spaces with singularities

In subsequent sections we will be concerned with "mildly singular" (i.e. normal) complex analytic spaces, that we shall write in *cursive*, such as \mathcal{X} or \mathcal{Y} . If \mathcal{X} is a complex space we denote respectively by \mathcal{X}_{reg} and $\mathcal{X}_{\text{sing}}$ the regular and singular locus of \mathcal{X} . The normality assumption in particular entails that the singular locus is of codimension at least two in \mathcal{X} . We refer to [Dem84] or [Dem12] for a detailed presentation of these concepts.

Forms and currents on normal complex spaces

The first thing to note is that the notion of Kähler form can be defined on normal complex spaces. In order to recall the definitions, let $j : \mathcal{X} \rightarrow \Omega$ be an embedding of \mathcal{X} into an open subset $\Omega \subset \mathbb{C}^N$. A (p, q) -form on \mathcal{X} is then defined as the image of the restriction map

$$j^* : \mathcal{A}^{p,q}(\Omega) \rightarrow \mathcal{A}^{p,q}(\mathcal{X}).$$

It can be checked that this notion is well-defined, see [Dem84] for details (briefly, if $j' : \mathcal{X} \rightarrow \Omega' \subset \mathbb{C}^{N'}$ is another embedding then there exist (locally) holomorphic functions $f : \Omega \rightarrow \mathbb{C}^{N'}$ and $g : \Omega' \rightarrow \mathbb{C}^N$ so that $j' = fj$ and $j = gj'$, and it can then be seen that the respective images of j^* and j'^* coincide).

Definition 2.2.5. *A (p, q) -form on \mathcal{X} is said to be smooth if it is given locally by restriction of a smooth (p, q) -form under an embedding of \mathcal{X} into an open subset of \mathbb{C}^N . A smooth $(1, 1)$ -form ω is a Kähler form on \mathcal{X} if it is locally the restriction of a smooth Kähler form under an embedding of \mathcal{X} as above.*

The notion of currents can then be defined by means of duality as in the smooth case [Dem84]. It is often convenient to introduce Kähler forms and Kähler currents in terms of potentials, as explained below.

Plurisubharmonic functions and potentials

Along the lines described above, we can also define the notion of plurisubharmonic function on normal complex spaces by means of local embeddings, in the following way:

Definition 2.2.6. *A function $f : \mathcal{X} \rightarrow [-\infty, +\infty[$ on the normal analytic space \mathcal{X} is said to be plurisubharmonic (psh) if it is upper semi-continuous on \mathcal{X} , not locally $-\infty$, and extends to a plurisubharmonic function in some local embedding of \mathcal{X} into \mathbb{C}^N .*

A classical result of Fornaess and Narasimhan [FN80] ensures that a continuous function is psh if and only if its restriction to \mathcal{X}_{reg} is psh. Furthermore, bounded psh functions on \mathcal{X}_{reg} extend to \mathcal{X} .

Remark 2.2.7. *In fact, it should be noted that there are several notions of plurisubharmonicity on complex spaces, but they all coincide in case the space is locally irreducible. In particular, it is sufficient for our purposes to consider the notion introduced in Definition 2.2.6 above. We refer to [FN80] for a discussion of the equivalent notions of plurisubharmonic functions on analytic spaces, as well as proofs of the equivalence.*

We can use the notion of psh functions in order to view Kähler metrics (and currents) as equivalence classes of potentials as in [GZ17, p. 413]:

Definition 2.2.8. *A Kähler potential on \mathcal{X} is a family $(U_i, \varphi_i)_{i \in I}$ where $(U_i)_{i \in I}$ is an open covering of \mathcal{X} and φ_i a C^∞ -smooth strictly psh function such that $\varphi_i - \varphi_j$ is pluriharmonic on $U_i \cap U_j$.*

We say that two potentials (U_i, φ_i) and (V_j, ψ_j) are *equivalent* iff $\varphi_i - \psi_j$ is pluriharmonic on $U_i \cap V_j$.

Definition 2.2.9. A Kähler metric is an equivalence class of Kähler potentials. A positive current on \mathcal{X} is an equivalence class of plurisubharmonic potentials¹.

If a positive current T has locally bounded potentials, it is fully determined by the closed $(1, 1)$ -current T_{reg} on \mathcal{X}_{reg} defined on U_i by $T_{\text{reg}} = dd^c \varphi_i$. The notion of quasi-psh function is defined as in the smooth case:

Definition 2.2.10. Let Ω be a Kähler metric on \mathcal{X} with potential (U_i, φ_i) . An upper semi-continuous function $\varphi : \mathcal{X} \rightarrow [-\infty, +\infty[$ is Ω -psh iff $\varphi_i + \varphi$ is psh on U_i , for all i .

We denote by $\Omega + dd^c \varphi$ the positive $(1, 1)$ -current whose potential is $(U_i, \varphi_i + \varphi)$.

Remark 2.2.11. We also recall that the Monge-Ampère operator $(dd^c \psi)^n$ can be defined on a complex space, via e.g. Bedford-Taylor's study of Monge-Ampère measures for locally bounded psh functions. The definition uses the fact that these measures do not charge proper analytic subsets, see [GZ17, Proposition 16.42].

Bott-Chern cohomology on normal complex spaces

The purpose of this section is to recall the definition of Bott-Chern cohomology for normal spaces. We begin by recalling some basic facts and definitions from the smooth setting (see e.g. [Dem12] or [Dem84]):

Definition 2.2.12. The Bott-Chern cohomology groups of a compact complex manifold X are defined to be the groups

$$H_{BC}^{p,q}(X, \mathbb{C}) := \frac{\{d\text{-closed } (p, q)\text{-forms}\}}{\{dd^c\text{-exact } (p, q)\text{-forms}\}}. \quad (2.5)$$

Remark 2.2.13. Some authors denote the Bott-Chern cohomology space by $H_{\partial\bar{\partial}}^{1,1}(X)$.

If the given manifold X is moreover compact Kähler then the Bott-Chern cohomology is finite dimensional, as a consequence of Proposition 2.2.14 below. The proof of Proposition 2.2.14 relies on the Hodge decomposition theorem.

Proposition 2.2.14. ([Dem12, Corollary 8.7]) Let X be a compact Kähler manifold. Then the natural morphisms

$$H_{BC}^{p,q}(X, \mathbb{C}) \rightarrow H^{p,q}(X, \mathbb{C}), \quad \bigoplus_{p+q=k} H_{BC}^{p,q}(X, \mathbb{C}) \rightarrow H_{DR}^k(X, \mathbb{C})$$

are isomorphisms.

¹Defined analogously to Kähler potentials, in the obvious way.

More generally, let \mathcal{X} be a (possibly singular) normal complex space. Then the local $\partial\bar{\partial}$ -lemma fails in general, so it is useful to consider the notion of $(1,1)$ -forms with *local potentials* here. We reproduce the discussion of [BG13, Section 4.6.1]:

Definition 2.2.15. *A $(1,1)$ -form (or current) with local potentials on \mathcal{X} is defined to be a section of the quotient sheaf $C^\infty(\mathcal{X})/\Re\mathcal{O}_{\mathcal{X}}$ (resp. $\mathcal{D}'(\mathcal{X})/\Re\mathcal{O}_{\mathcal{X}}$). Equivalently a $(1,1)$ -form θ on \mathcal{X} has local potentials if it is closed and locally of the form $\theta = dd^c u$ for a smooth function u . If u is strictly psh we say that θ is Kähler. A $(1,1)$ -current T with local potentials is locally of the form $dd^c\varphi$ where φ is a distribution.*

The idea is then to define the Bott-Chern cohomology space as the quotient of the space of $(1,1)$ -forms with local potentials by the space $dd^c C^\infty(X)$ of dd^c -exact $(1,1)$ -forms:

Definition 2.2.16. *The Bott-Chern cohomology space is defined as the space of $(1,1)$ -forms with local potentials modulo $dd^c C^\infty(\mathcal{X})$, i.e.*

$$H_{BC}^{1,1}(\mathcal{X}) := \frac{\{(1,1)\text{-forms with local potentials}\}}{dd^c C^\infty(\mathcal{X})}$$

The same definition goes through for $(1,1)$ -currents with local potentials, by instead taking the quotient with $dd^c \mathcal{D}'(\mathcal{X})$.

To make sense of this definition, note that any pluriharmonic distribution is locally the real part of a holomorphic function, see [BG13, Lemma 4.6.1] for a proof of this fact. Hence $\Re\mathcal{O}_{\mathcal{X}}$ can be identified with the kernel of dd^c on $\mathcal{D}'(\mathcal{X})$. Furthermore, any representative θ of $H_{BC}^{1,1}(\mathcal{X})$ may by definition be written *locally* as $\theta =_{loc} dd^c\varphi$ for some local smooth $(1,1)$ -form (or distribution) φ . By definition of forms and currents on normal complex spaces θ is identified with the data (U_i, φ_i) where $\{U_i\}_i$ is an open cover of \mathcal{X} and φ_i are local representatives satisfying the condition that $\varphi_i - \varphi_j$ is plurharmonic on $U_i \cap U_j$. In other words,

$$\varphi_i - \varphi_j \in \Re\mathcal{O}_{\mathcal{X}},$$

since any plurharmonic function can be realized as the real part of a holomorphic function (and vice versa). It follows that

$$H_{BC}^{1,1}(\mathcal{X}) := H^1(\mathcal{X}, \Re\mathcal{O}_{\mathcal{X}}). \quad (2.6)$$

We may further note that if T is a $(1,1)$ -current T with local potentials, then it can be written (globally) as

$$T = \theta + dd^c\varphi,$$

where θ is a $(1,1)$ -form with local potentials and φ is a distribution. We state some additional properties of the Bott-Chern cohomology:

1. The Bott-Chern cohomology group $H_{BC}^{1,1}(\mathcal{X})$ is finite dimensional. To see this, consider the short exact sequence

$$0 \longrightarrow i\mathbb{R} \longrightarrow \mathcal{O}_{\mathcal{X}} \longrightarrow \Re\mathcal{O}_{\mathcal{X}} \longrightarrow 0.$$

Since \mathcal{X} is compact the cohomology groups $H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $H^2(\mathcal{X}, \mathbb{R})$ are both finite dimensional, hence $H^1(\mathcal{X}, \Re \mathcal{O}_{\mathcal{X}})$ is as well.

2. If $\mathcal{Y} \rightarrow \mathcal{X}$ is a morphism between complex spaces and $\alpha \in H_{BC}^{1,1}(\mathcal{X})$ then we have a well-defined pullback $\mu^* \alpha$ which lives in $H_{BC}^{1,1}(\mathcal{Y})$. It is defined by pulling back the local potentials.
3. Let $\alpha \in H_{BC}^{1,1}(\mathcal{X})$ on a normal complex space \mathcal{X} and let T be a closed positive $(1, 1)$ -current on \mathcal{X}_{reg} representing the restricting $\alpha|_{\mathcal{X}_{reg}}$ of α to the regular part of \mathcal{X} . Then
 - a) T uniquely extends as a positive $(1, 1)$ -current with local potentials on \mathcal{X} , and the dd^c -class of the extension coincides with α ,
 - b) If \mathcal{X} is compact Kähler and T has locally bounded potentials on an open subset U of \mathcal{X}_{reg} , then the positive measure T^n on U defined in the sense of Bedford-Taylor [BT82], has finite total mass on U .

See [BG13, Proposition 4.6.3].

4. Finally note that if (X, L) is a polarized manifold and h is a hermitian metric on L , then the curvature form $\Theta_L(h)$ of h naturally lives in $H_{BC}^{1,1}(X)$.

Remark 2.2.17. *We finally remark that, while it will be important to us to consider intersection of Bott-Chern cohomology classes, we in practice are always able to pass to a resolution and compute the intersection numbers "upstairs". Indeed, if X is a compact Kähler manifold, then the Bott-Chern cohomology groups are isomorphic to the Dolbeault cohomology groups, and we may define the intersection number by integration over X : If $\alpha_i = [\omega_i]$ where ω_i are any smooth representatives of the Kähler classes α_i , then the top intersection number*

$$(\alpha_1 \cdots \alpha_n)_X := \int_X \omega_1 \wedge \cdots \wedge \omega_n.$$

This intersection product is well-defined, multilinear and symmetric. Since X is compact it depends only on the Kähler classes α_i . We refer to e.g. [Dem12] for details.

2.3 Canonical metrics

The purpose of this section is to introduce some of the framework for extremal metrics, with a focus on the particular case of constant scalar curvature (cscK) metrics.

Constant scalar curvature Kähler manifolds

This section serves to introduce basic notation and definitions related to constant scalar curvature Kähler (cscK) manifolds, which is the central object of study in this thesis.

Scalar curvature

Let X be a compact complex manifold of $\dim_{\mathbb{C}} X = n$ equipped with a given Kähler form $\omega \in H^{1,1}(X, \mathbb{R})$, i.e. a smooth real closed positive $(1,1)$ -form on X . In local coordinates (z_1, \dots, z_n) , we may then write

$$\omega :=_{loc} i \sum_{j,k} g_{j\bar{k}} dz_j \wedge d\bar{z}_k$$

In order to further fix notation, let $\text{Ric}(\omega) = -dd^c \log \omega^n$ be the Ricci curvature form, where $dd^c := \frac{i}{2\pi} \partial \bar{\partial}$ is normalised so that $\text{Ric}(\omega)$ represents the first Chern class $c_1(X)$. The notation $-dd^c \log \omega^n$ is here shorthand to state that the Ricci form can be *locally* written

$$\text{Ric}(\omega) =_{loc} -i \sum_{j,k} \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log(\det(g_{p\bar{q}})) dz_j \wedge d\bar{z}_k.$$

The trace

$$S(\omega) := \text{Tr}_{\omega} \text{Ric}(\omega) = n \frac{\text{Ric}(\omega) \wedge \omega^{n-1}}{\omega^n}$$

is the scalar curvature of ω . Denote the Kähler class $[\omega] \in H^{1,1}(X, \mathbb{R})$ by α . The mean scalar curvature is the cohomological constant given by

$$\bar{S} := V^{-1} \int_X S(\omega) \omega^n = n \frac{\int_X c_1(X) \cdot \alpha^{n-1}}{\int_X \alpha^n} := n \frac{(c_1(X) \cdot \alpha^{n-1})_X}{(\alpha^n)_X}, \quad (2.7)$$

where $V := \int_X \omega^n := (\alpha^n)_X$ is the Kähler volume.

Definition 2.3.1. [Cal82, Cal85] A Kähler metric ω is said to be extremal if

$$\bar{\partial} \nabla^{1,0} S(\omega) = 0$$

i.e. if the $(1,0)$ -part of the gradient of the scalar curvature is a holomorphic vector field.

In this thesis we are mainly considered with the important subclass of extremal Kähler metrics of constant scalar curvature:

Definition 2.3.2. We say that ω is a constant scalar curvature Kähler (cscK) metric if $S(\omega)$ is constant (equal to \bar{S}) on X , i.e. iff the Kähler form ω satisfies the equation

$$S(\omega) = \bar{S}, \quad (2.8)$$

referred to as the cscK equation.

Remark 2.3.3. The above cscK equation is a 4th order highly non-linear elliptic partial differential equation, which is currently intractable from a PDE point of view. One may note that in the Kähler-Einstein one can reduce equation 2.8 to a 2nd order equation.

A first thing to note is that cscK metrics do not always exist:

Example 2.3.4. Denote by X_k the projective space \mathbb{P}^2 blown up in $0 \leq k \leq 8$ points. Then X_k is a Fano manifold, i.e. the anticanonical bundle $-K_{X_k}$ is ample. If $3 \leq k \leq 8$, then X has polarisations which admit cscK metrics and polarisations that do not, see e.g. [Shu09]. If $4 \leq k \leq 8$ then the automorphism group $\text{Aut}_0(X_k)$ is discrete.

Some further useful remarks include:

- Any cscK metric is extremal and any Kähler-Einstein metric has constant scalar curvature.
- Conversely, an extremal metric on a compact Kähler manifold X has constant scalar curvature if and only if the *Futaki character* [Fut83] vanishes (see Section 5.4 for the definition). As particular cases of special interest, the Futaki character vanishes if the automorphism group $\text{Aut}(X)$ is discrete, or more generally if $\text{Aut}(X)$ is *semisimple* as a Lie group (this is however not a complete characterization of the vanishing of the Futaki character). Varieties with semisimple automorphism group include flag varieties (\mathbb{P}^n , Grassmanians etc), wonderful compactifications and the Mukai-Umemura variety to name a few.
- Kähler-Einstein metrics can only exist if the class $[\omega]$ has a definite sign, i.e. if it is a multiple $\lambda c_1(X)$ of the first Chern class of $-K_X$. If $\lambda < 0$ or $\lambda = 0$ (the Calabi-Yau case) then Kähler-Einstein metrics always exist [Aub76, Yau78]. When $\lambda > 0$ the manifold X is *Fano*, i.e. the anticanonical bundle $-K_X$ is ample. An important difference compared to the case $\lambda \leq 0$ is that there are obstructions to existence of Kähler-Einstein metrics in general. In principle this can be seen through the obstructions introduced in this thesis, but for most purposes it suffices with much less general notions in order to study Fano manifolds (which are automatically projective). The simplest example of a Fano manifold admitting a Kähler-Einstein metric is the complex projective space \mathbb{P}^1 endowed with the Fubini-Study metric ω_{FS} . A simple example of a Fano manifold that does *not* admit Kähler-Einstein metrics is provided by e.g. \mathbb{P}^2 blown up in 1 or 2 points. This can be seen by noting that the Lie group of automorphisms is *non-reductive*. Hence, by the obstruction of Matsushima [Mat57] and Lichnerowicz [Lic57] (see Section 2.4) cscK metrics cannot exist in any Kähler class on either of these manifolds.

In view of the above discussion it is natural to attempt to clarify the situation in the Fano case $\lambda > 0$, namely one would wish to *characterize the Kähler classes on a given Fano manifold X that admits Kähler-Einstein metrics*. This question has attracted a lot of attention, and a complete answer was recently provided by the important *Yau-Tian-Donaldson correspondence*, which was conjectured in [Tia97, Don02] and finally proven by Chen-Donaldson-Sun [CDS15a, CDS15b, CDS15c] and Tian [Tia15]. See also [DS16] and [BBJ15] for related work.

Theorem 2.3.5 (YTD correspondence, Fano version: [CDS15a, CDS15b, CDS15c, Tia15]). *Suppose that X is a Fano manifold. Then $c_1(X)$ admits a Kähler-Einstein representative if and only if the polarized manifold $(X, -K_X)$ is K -polystable.*

- More general versions of the Yau-Tian-Donaldson correspondence conjecture that existence of cscK metrics (or extremal metrics) should be equivalent to certain algebro-geometric K-stability notions (see Section 3).

2.4 Energy functionals and G-coercivity

In this section we recall that canonical metrics, cscK metrics in particular, can be characterized as critical points of certain energy functionals on the space \mathcal{H} of smooth Kähler potentials.

The space of Kähler potentials

In what follows we rely on the variational approach developed by Berman, Boucksom, Guedj and Zeriahi in [BBGZ13]. In particular we make frequent use of energy functionals on the space of (normalized) smooth Kähler potentials, first studied by Mabuchi in [Mab87]: By the dd^c -lemma we may write

$$\mathcal{H} := \{\varphi \in C^\infty(X) : \omega_\varphi := \omega + dd^c\varphi > 0\}, \quad (dd^c := \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}) \quad (2.9)$$

for the space of smooth Kähler potentials in α . The space of Kähler metrics representing α is then given by

$$\mathcal{K} = \mathcal{H}/\mathbb{R}, \quad (2.10)$$

noting that potentials are defined up to constants. It is an easy consequence of the definition that \mathcal{H} is an *open* and *convex* subset of $C^\infty(X)$. Moreover, for each $\varphi \in C^\infty(X)$ we have $T_\varphi(\mathcal{H}) = C^\infty(X)$.

As first introduced by Mabuchi [Mab87] the spaces \mathcal{H} and \mathcal{K} can be endowed with a natural Riemannian L^2 -metric defined as follows:

$$(u, v)_\varphi := \int_X uv\omega_\varphi^n. \quad (2.11)$$

It is often referred to as the *Mabuchi metric*. It admits a Levi-Cevita connection \mathcal{D} and the geodesic equation is given by

$$\ddot{\varphi} - |d_x \dot{\varphi}|_{\omega_\varphi}^2 \equiv 0. \quad (2.12)$$

We refer to [Don99] for proofs of these claims. The Mabuchi metric makes \mathcal{H} into an infinite dimensional symmetric space of negative curvature [Mab87]. We further remark that (\mathcal{H}, d_2) is *not* geodesically complete. In fact, due to Chen [Che00b] with complements by Blocki [Blo13], any two potentials $\varphi_0, \varphi_1 \in \mathcal{H}$ can be joined by a so called *weak geodesic*, but the obtained geodesic is not in general contained in \mathcal{H} [DL12]. The definition of weak geodesics (and weak subgeodesics) is recalled below.

Weak geodesic rays in the space of Kähler potentials

The geodesic equation can be made sense of due to a convenient reformulation in terms of homogeneous complex Monge-Ampère equations, as follows: Let $(\varphi_t)_{t \geq 0} \subset \text{PSH}(X, \omega)$ be a ray of ω -psh functions. As noted by Donaldson [Don02] and Semmes [Sem92], there is a useful correspondence between the family $(\varphi_t)_{t \geq 0}$ and an associated S^1 -invariant function Φ on $X \times \bar{\Delta}^*$, where $\bar{\Delta}^* \subset \mathbb{C}$ denotes the pointed unit disc. We denote by τ the coordinate on Δ . Explicitly, the correspondence is given by

$$\Phi(x, e^{-t+is}) = \varphi^t(x),$$

where the sign is chosen so that $t \rightarrow +\infty$ corresponds to $\tau := e^{-t+is} \rightarrow 0$. The function Φ restricted to a fiber $X \times \{\tau\}$ thus corresponds precisely to φ_t on X . In the direction of the fibers we thus have $p_1^* \omega + dd_x^c \Phi \geq 0$ (in the sense of currents, letting $p_1 : X \times \Delta \rightarrow X$ denote the first projection).

Definition 2.4.1. *Viewing the family $(\varphi_t)_{t \geq 0}$ as a map $(0, +\infty) \rightarrow \text{PSH}(X, \omega)$, we say that $(\varphi^t)_{t \geq 0}$ is continuous (resp. locally bounded, smooth) if the corresponding S^1 -invariant function Φ is continuous (resp. locally bounded, smooth).*

We will use the following standard terminology, motivated by the extensive study of (weak) geodesics in the space \mathcal{H} , see e.g. [Blo13], [Che00b], [Dar14], [Don02], [Sem92].

Definition 2.4.2. *We say that $(\varphi_t)_{t \geq 0}$ is a subgeodesic ray if the associated S^1 -invariant function Φ on $X \times \bar{\Delta}^*$ is $p_1^* \omega$ -psh. Furthermore, a locally bounded family of functions $(\varphi_t)_{t \geq 0}$ in $\text{PSH}(X, \omega)$ is said to be a weak geodesic ray if the associated S^1 -invariant function $\Phi \in \text{PSH}(X \times \bar{\Delta}^*, p_1^* \omega)$ satisfies*

$$(p_1^* \omega + dd_{(x,\tau)}^c \Phi)^{n+1} = 0$$

on $X \times \Delta^*$.

The existence of geodesics with bounded Laplacian was proven by Chen [Che00b] with complements by Blocki [Blo13], see also e.g. [Dar14], [DL12]. We will refer to such a geodesic as being $\mathcal{C}^{1,\bar{1}}$ -regular, cf. Lemma 4.1.10 below.

Definition 2.4.3. *We say that a function φ is $\mathcal{C}^{1,\bar{1}}$ -regular if $dd^c \varphi \in L_{\text{loc}}^\infty$, and we set $\mathcal{H}^{1,\bar{1}} := \text{PSH}(X, \omega) \cap \mathcal{C}^{1,\bar{1}}$.*

A standard remark at this point is that a $\mathcal{C}^{1,\bar{1}}$ -regular function is automatically $\mathcal{C}^{1,a}$ -regular for all $0 < a < 1$. This follows from usual regularity theory for second order linear elliptic partial differential equations, as follows: First note that $dd^c \varphi \in L_{\text{loc}}^\infty$ is equivalent to φ having locally bounded Laplacian on X (since φ is quasi-psh). By [GT83] it then follows that $\varphi \in L_2^p$ for each finite $p \geq 1$, where L_k^p denotes the Sobolev space of functions whose derivatives of order at most k belongs to L^p (locally). But $L_2^p \subset C^{1,a}$ for each $a < 1 - d/p$ by Sobolev's embedding theorem, so the result follows.

Remark 2.4.4. *A priori the condition of being $C^{1,\bar{1}}$ is weaker than $C^{1,1}$ -regularity (i.e. bounded real Hessian). However, in light of new developments due to J. Chu, V. Tossatti and B. Weinkove [CTW] it turns out that any two Kähler potentials on a compact Kähler manifold can be connected by a geodesic segment of $C^{1,1}$ regularity.*

The d_1 -Finsler metric and completions of \mathcal{H}

More recently it has been understood that instead of considering \mathcal{H} as a Riemannian space with the Mabuchi metric, it is often useful to consider \mathcal{H} as a path metric space endowed with a certain *Finsler metric* d_1 . This point of view has been argued by e.g. [DR17], [Dar17b, Dar14, Dar15] and others (see also the survey article [Dar17a] for a detailed treatment and background). The Mabuchi metric (2.11) is then often denoted by d_2 . To introduce it, let $d_1 : \mathcal{H}_\omega \times \mathcal{H}_\omega \rightarrow \mathbb{R}_+$ be the path length pseudometric associated to the weak Finsler metric on \mathcal{H}_ω defined by

$$\|\xi\|_\varphi := V^{-1} \int_X |\xi| \omega_\varphi^n, \quad \xi \in T_\varphi \mathcal{H}_\omega = C^\infty(X).$$

More explicitly, if $[0, 1] \ni t \mapsto \phi_t$ is a smooth path in X , then let

$$l_1(\phi_t) := \int_0^1 \|\dot{\phi}_t\|_{\phi_t} dt$$

be its length, and define

$$d_1(\varphi, \psi) = \inf \{l_1(\phi_t), (\phi_t)_{0 \leq t \leq 1} \subset \mathcal{H}_\omega, \phi_0 = \varphi, \phi_1 = \psi\},$$

the infimum being taken over smooth paths $t \mapsto \phi_t$ as above. It can be seen (see [Dar17b, Theorem 2]) that $(\mathcal{H}_\omega, d_1)$ is a metric space.

As a matter of notation, we write

$$\mathcal{H}_0 := \mathcal{H}_\omega \cap E^{-1}(0)$$

for the space of Kähler potentials normalized so that the Monge-Ampère energy vanishes. The level set \mathcal{H}_0 is isomorphic to the space \mathcal{K} of Kähler metrics cohomologous to ω via the natural map $\varphi \mapsto \omega_\varphi$, and we have $\mathcal{H} = \mathcal{H}_0 \times \mathbb{R}$.

Basic energy functionals

We here work on the level of potentials, using the notation of quasi-plurisubharmonic (quasi-psh) functions. We briefly recall the setup: Let θ be a closed $(1, 1)$ -form on X and denote, as usual, by $\text{PSH}(X, \theta)$ the space of θ -psh functions φ on X , i.e. the set of functions that can be locally written as the sum of a smooth and a plurisubharmonic function, and such that

$$\theta_\varphi := \theta + dd^c \varphi \geq 0$$

in the weak sense of currents. In particular, if ω is our fixed Kähler form on X , then we write

$$\mathcal{H}_\omega := \{\varphi \in C^\infty(X) : \omega_\varphi := \omega + dd^c \varphi > 0\} \subset \text{PSH}(X, \omega)$$

for the space of Kähler potentials on X . As a subset of $\mathcal{C}^\infty(X)$ it is convex and consists of strictly ω -psh functions. It has been extensively studied (for background we refer the reader to e.g. [Bou12] and references therein).

Recall that a θ -psh function is always upper semi-continuous (usc) on X , thus bounded from above by compactness. Moreover, if $\varphi_i \in \text{PSH}(X, \theta) \cap L_{\text{loc}}^\infty$, $1 \leq i \leq p \leq n$, it follows from the work of Bedford-Taylor [BT82] that we can give meaning to the product $\bigwedge_{i=1}^p (\theta + dd^c \varphi_i)$, which then defines a closed positive (p, p) -current on X . As usual, we then define the *Monge-Ampère measure* as the following probability measure, given by the top wedge product

$$\text{MA}(\varphi) := V^{-1}(\omega + dd^c \varphi)^n.$$

Energy functionals and a Deligne functional formalism

We now introduce the notation for energy functionals that we will use. Let $\varphi_i \in \text{PSH}(X, \omega) \cap L_{\text{loc}}^\infty$. The *Monge-Ampère energy functional* (or *Aubin-Mabuchi functional*) $E := E_\omega$ is defined by

$$E(\varphi) := \frac{1}{n+1} \sum_{j=0}^n V^{-1} \int_X \varphi (\omega + dd^c \varphi)^{n-j} \wedge \omega^j$$

(such that $E' = \text{MA}$ and $E(0) = 0$). Similarly, if θ is any closed $(1, 1)$ -form, we define a functional $E^\theta := E_\omega^\theta$ by

$$E^\theta(\varphi) := \sum_{j=0}^{n-1} V^{-1} \int_X \varphi (\omega + dd^c \varphi)^{n-j-1} \wedge \omega^j \wedge \theta,$$

and we will also have use for the Aubin J-functional $J : \text{PSH}(X, \omega) \cap L_{\text{loc}}^\infty \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$J(\varphi) := V^{-1} \int_X \varphi \omega^n - E(\varphi),$$

which essentially coincides with the *minimum norm* of a test configuration (see [Der16b, BHJ15]).

More generally, it is possible to define a natural *multivariate* version of the Monge-Ampère energy, of which all of the above functionals are special cases. As a motivation, we first recall the Deligne pairing of line bundles.

Deligne metrics and energy functionals in Kähler geometry

Deligne pairings of line bundles, and the Deligne metrics they carry, form a fundamental tool in proving many results regarding K-stability for polarized manifolds, cf. [PRS08, Ber16, BHJ15, BHJ16] and others. Following [Elk90] we now discuss Deligne metrics, as an introduction to the closely related "Deligne functionals" that we here introduce. The latter play an analogous role in the Kähler setting, to the Deligne metrics in the polarized setting.

Remark 2.4.5. *A for our purposes equivalent approach is given by Bott-Chern forms, see Example 2.4.9 below.*

Deligne pairing of line bundles

The Deligne pairing is a powerful and general technique from algebraic geometry, which we will here apply to our specific setting in Kähler geometry. We refer the interested reader to [Elk90, Zha96, Mor99] for a general treatment of Deligne pairings. Let $\pi : Y \rightarrow T$ be a flat projective morphism between smooth complex algebraic varieties, of relative dimension $n \geq 0$. Let the line bundles L_0, L_1, \dots, L_n on Y be given. The *Deligne pairing* of $L_0 \dots L_n$ will be a way of producing a single line bundle $\langle L_0, \dots, L_n \rangle$ on the base space T , depending in a multilinear way on the L_i and satisfies

$$c_1(\langle L_0, \dots, L_n \rangle) = \pi_*(c_1(L_0) \wedge \dots \wedge c_1(L_n)).$$

In order to introduce the construction, first note that the intersection product

$$L_0 \cdot \dots \cdot L_n \cdot [Y] \in \text{CH}_{\dim Y - (n+1)}(Y) = \text{CH}_{\dim T - 1}(Y).$$

Hence the pushforward $\pi_*(L_0 \cdot \dots \cdot L_n \cdot [Y]) \in \text{CH}_{\dim T - 1}(T) = \text{Pic}(T)$, where we used in the last equality that T is smooth. This data then defines an isomorphism class of line bundles on T . In order to say something about its construction, note that the Deligne pairing is a canonical choice of representative of this isomorphism class that is multilinear, functorial, commutes with base change and moreover satisfies the following inductive property: For $n = 0$ the Deligne pairing $\langle L_0 \rangle$ is nothing but the norm of L_0 with respect to the finite flat morphism $Y \rightarrow T$. Moreover, if Z_0 is any non-singular divisor on Y which is flat over T and defined by a section $\sigma_0 \in H^0(Y, L_0)$, then there is a canonical identification

$$\langle L_0, \dots, L_n \rangle_{Y/T} = \langle L_1|_{Z_0}, \dots, L_n|_{Z_0} \rangle_{Z_0/T}.$$

The point is that the right hand side now is a pairing of n line bundles (rather than $n + 1$), and the above properties in fact characterize the Deligne pairing uniquely. Indeed, as explained in e.g. [BHJ16] we may first note that the definition reduces to the situation when L_i are all very ample, since indeed any line bundle can be written as a difference of very ample ones, and then use multilinearity of the pairing. In case the L_i are very ample we may find such non-singular divisors Z_i in the linear system of L_i such that the intersection $\cap_{i \in I} Z_i$ is flat over T for each set of indices $I \subset \{0, 1, \dots, n\}$. By the inductive property we then see that

$$\langle L_0, \dots, L_n \rangle_{Y/T} = \langle L_n|_{Z_0 \cap \dots \cap Z_{n-1}} \rangle_{Z_0 \cap \dots \cap Z_{n-1}/T},$$

which then equals the norm of $L_n|_{Z_0 \cap \dots \cap Z_{n-1}}$ with respect to $Z_0 \cap \dots \cap Z_{n-1} \rightarrow T$, by the case $n = 0$ of the above. In practice we wish to apply this to the case of $\dim(T) = 1$.

Deligne metrics

We use so called additive notation for line bundles and metrics, meaning that a hermitian metric $\|\cdot\|$ on a line bundle $L \rightarrow X$ is represented by a collection of local functions $\phi := \{\phi_U\}$, defined

as follows: If $U \subset X$ is an open subset and s_U is a trivializing section of L (i.e. a local generator of the invertible sheaf $\mathcal{O}(L)$), then we set

$$\phi_U := -\log \|s_U\|^2,$$

Here ϕ_U depends on s_U , but the curvature current $dd^c\phi$ is globally well-defined and represents the first Chern class $c_1(L)$. In the sequel we identify the hermitian metric with the additive object ϕ .

Now let $L_i \rightarrow X$, $i = 0, 1, \dots, n$ be holomorphic line bundles on X . Let ϕ_i be hermitian metrics on L_i for $i = 0, 1, \dots, n$. We then obtain an induced metric $\langle \phi_0, \phi_1, \dots, \phi_n \rangle$ on the Deligne pairing $\langle L_0, \dots, L_n \rangle$ that satisfies the following key properties, that should be compared with the properties of the 'Deligne functional introduced in 2.4:

1. The curvature form of the induced hermitian metric on the Deligne pairing is given by $dd^c \langle \phi_0, \phi_1, \dots, \phi_n \rangle = \pi_*(dd^c \phi_0 \wedge \dots \wedge dd^c \phi_n)$.
2. If ϕ_i , $i = 0, 1, \dots, n$, are *bounded* hermitian metrics L_i respectively, and ϕ'_0 is a *bounded* hermitian metric on L_0 , then we have the following 'change of metric' property

$$\langle \phi_0, \phi_1, \dots, \phi_n \rangle - \langle \phi'_0, \phi_1, \dots, \phi_n \rangle = (\phi_0 - \phi'_0) dd^c \phi_1 \wedge \dots \wedge dd^c \phi_n.$$

Note that in order to define $\langle \phi_0, \phi_1, \dots, \phi_n \rangle$ in case ϕ is merely *locally* bounded one may proceed by considering the problem from the point of view of the energy functional in the left hand side, i.e. if a given ϕ_0 is locally bounded we may *define* the induced Deligne metric on the Deligne pairing $\langle L_0, \dots, L_n \rangle$ by

$$\langle \phi_0, \phi_1, \dots, \phi_n \rangle - \langle \phi'_0, \phi_1, \dots, \phi_n \rangle = (\phi_0 - \phi'_0) dd^c \phi_1 \wedge \dots \wedge dd^c \phi_n. \quad (2.13)$$

where ϕ'_0 is any *bounded* hermitian metric on L . The later energy functional is then defined by invoking Bedford-Taylor [BT82]. By multilinearity the argument can further be extended to define $\langle \phi_0, \phi_1, \dots, \phi_n \rangle$ in case all ϕ_i are locally bounded, yielding precisely the expression below (see Definition 2.4.6).

Arguably, in view of the above discussion it seems natural to consider the energy functional expressions immediately. Using the correspondence $\phi \mapsto \phi - \phi_{ref}$ between metrics and potentials the above indeed translates to the discussion on Deligne functionals below.

Deligne functionals

Let $\theta_0, \dots, \theta_n$ be closed $(1,1)$ -forms on X . Motivated by corresponding properties for the *Deligne pairing* (cf. e.g. [Ber16], [Elk90] for background) we would like to consider a *functional* $\langle \cdot, \dots, \cdot \rangle_{(\theta_0, \dots, \theta_n)}$ on the space $\text{PSH}(X, \theta_0) \cap L_{loc}^\infty \times \dots \times \text{PSH}(X, \theta_n) \cap L_{loc}^\infty$ that is

- symmetric, i.e. for any permutation σ of the set $\{0, 1, \dots, n\}$, we have

$$\langle \varphi_{\sigma(0)}, \dots, \varphi_{\sigma(n)} \rangle_{(\theta_{\sigma(0)}, \dots, \theta_{\sigma(n)})} = \langle \varphi_0, \dots, \varphi_n \rangle_{(\theta_0, \dots, \theta_n)}.$$

- if φ'_0 is another θ_i -psh function in $\text{PSH}(X, \theta) \cap L_{\text{loc}}^\infty$, then we have a 'change of function' property

$$\begin{aligned} & \langle \varphi'_0, \varphi_1, \dots, \varphi_n \rangle - \langle \varphi_0, \varphi_1, \dots, \varphi_n \rangle = \\ & \int_X (\varphi'_0 - \varphi_0) (\omega_1 + dd^c \varphi_1) \wedge \dots \wedge (\omega_n + dd^c \varphi_n). \end{aligned}$$

Demanding that the above properties hold necessarily leads to the following definition of *Deligne functionals*, that will provide a useful terminology for this thesis;

Definition 2.4.6. *Let $\theta_0, \dots, \theta_n$ be closed $(1, 1)$ -forms on X . Define a multivariate energy functional $\langle \cdot, \dots, \cdot \rangle_{(\theta_0, \dots, \theta_n)}$ on the space $\text{PSH}(X, \theta_0) \cap L_{\text{loc}}^\infty \times \dots \times \text{PSH}(X, \theta_n) \cap L_{\text{loc}}^\infty$ ($n + 1$ times) by*

$$\begin{aligned} & \langle \varphi_0, \dots, \varphi_n \rangle_{(\theta_0, \dots, \theta_n)} := \int_X \varphi_0 (\theta_1 + dd^c \varphi_1) \wedge \dots \wedge (\theta_n + dd^c \varphi_n) \\ & + \int_X \varphi_1 \theta_0 \wedge (\theta_2 + dd^c \varphi_2) \wedge \dots \wedge (\theta_n + dd^c \varphi_n) + \dots + \int_X \varphi_n \theta_0 \wedge \dots \wedge \theta_{n-1}. \end{aligned}$$

Remark 2.4.7. *The multivariate energy functional $\langle \cdot, \dots, \cdot \rangle_{(\theta_0, \dots, \theta_n)}$ can also be defined on $\mathcal{C}^\infty(X) \times \dots \times \mathcal{C}^\infty(X)$ by the same formula. In Sections 4 and 5 it will be interesting to consider both the smooth case and the case of locally bounded θ_i -psh functions.*

Proposition 2.4.8. *The functional $\langle \cdot, \dots, \cdot \rangle_{(\theta_0, \dots, \theta_n)}$ is symmetric.*

Proof. Since every permutation is a composition of transpositions it suffices to check the sought symmetry property for transpositions $\sigma := \sigma_{j,k}$ exchanging the position of $j, k \in \{0, 1, \dots, n\}$. Suppose for simplicity of notation that $j < k$ and write $\theta_i^t := \theta_i + dd^c \varphi_i$. A straightforward computation then yields

$$\begin{aligned} & \langle \varphi_0, \dots, \varphi_j, \varphi_k, \dots, \varphi_n \rangle_{(\theta_0, \dots, \theta_j, \theta_k, \dots, \theta_n)} - \langle \varphi_0, \dots, \varphi_k, \varphi_j, \dots, \varphi_n \rangle_{(\theta_0, \dots, \theta_k, \theta_j, \dots, \theta_n)} = \\ & = \int_X \varphi_j dd^c \varphi_k \wedge \Theta_{j,k} - \int_X \varphi_k dd^c \varphi_j \wedge \Theta_{j,k} = 0, \end{aligned}$$

where in the last step we used integration by parts and

$$\Theta_{j,k} := \theta_0 \wedge \dots \wedge \theta_{j-1} \wedge \theta_{j+1}^t \wedge \dots \wedge \theta_{k-1}^t \wedge \theta_{k+1}^t \wedge \theta_n^t,$$

(with factors θ_j and θ_k^t omitted). The case $j > k$ follows in the exact same way, with obvious modifications to the above proof. \square

Example 2.4.9. *The functionals E , E^θ and J can be written using the above multivariate energy functional formalism. Indeed, if θ is a closed $(1, 1)$ -form on X , ω is a Kähler form on X and φ is an ω -psh function on X , then*

$$E(\varphi) = \frac{1}{n+1} V^{-1} \langle \varphi, \dots, \varphi \rangle_{(\omega, \dots, \omega)}, \quad E^\theta(\varphi) = V^{-1} \langle 0, \varphi, \dots, \varphi \rangle_{(\theta, \omega, \dots, \omega)}$$

and

$$J(\varphi) = V^{-1} \langle \varphi, 0, \dots, 0 \rangle_{(\omega, \dots, \omega)} - E(\varphi).$$

Compare also [Rub14, Example 5.6] on Bott-Chern forms.

Second order variation of Deligne functionals

We have the following identity for the second order variations of the multivariate energy functional $\langle \cdot, \dots, \cdot \rangle_{(\theta_0, \dots, \theta_n)}$.

Proposition 2.4.10. *Let $\theta_0, \dots, \theta_n$ be closed $(1, 1)$ -forms on X and let $(\varphi_i^t)_{t \geq 0}$ be a smooth ray of smooth functions. Let $\tau := e^{-t+is}$ and consider the reparametrised ray $(\varphi_i^\tau)_{\tau \in \bar{\Delta}^*}$. Denoting by Φ_i the corresponding S^1 -invariant function on $X \times \Delta^*$, we have*

$$dd_\tau^c \langle \varphi_0^\tau, \dots, \varphi_n^\tau \rangle_{(\theta_0, \dots, \theta_n)} = \int_X (p_1^* \theta_0 + dd_{(x, \tau)}^c \Phi_0) \wedge \dots \wedge (p_1^* \theta_n + dd_{(x, \tau)}^c \Phi_n)$$

where \int_X denotes fiber integration, i.e. pushforward of currents.

Proof. In case all Φ_i are smooth, consider

$$\langle \Phi_0, \dots, \Phi_n \rangle_{(\theta_0, \dots, \theta_n)} := (p_2)_* (\Phi_0 (p_1^* \theta_0 + dd^c \Phi_1) \wedge \dots \wedge (p_1^* \theta_n + dd^c \Phi_n))$$

$$+ (p_2)_* (\Phi_1 p_1^* \theta_0 \wedge (p_1^* \theta_2 + dd^c \Phi_2) \wedge \dots \wedge (p_1^* \theta_n + dd^c \Phi_n)) + \dots + (p_2)_* (\Phi_n p_1^* \theta_0 \wedge \dots \wedge p_1^* \theta_{n-1}).$$

(cf. Definition 2.4.6). The result

$$dd_\tau^c \langle \Phi_0, \dots, \Phi_n \rangle_{(\theta_0, \dots, \theta_n)} = (p_2)_* \left((p_1^* \theta_0 + dd_{(x, \tau)}^c \Phi_0) \wedge \dots \wedge (p_1^* \theta_n + dd_{(x, \tau)}^c \Phi_n) \right)$$

then follows from an elementary but tedious computation relying on integration by parts, using in particular the S^1 -invariance of the functions Φ_i : Indeed, let $\tau := e^{-t+is}$ so that $t = -\log |\tau|$. We write $\varphi_i^t = \Phi_i(x, e^{-t+is})$ and use the shorthand $\theta_j^t := \theta_j + dd^c \varphi_j^t$, $j = 0, 1, \dots, n$. Set

$$g(t) := \langle \varphi_0^t, \dots, \varphi_n^t \rangle_{(\theta_0, \dots, \theta_n)}.$$

Denoting by $\dot{\varphi}_j^t$ the time derivative of φ_j^t , we then see (using integration by parts and the observation that $dd^c \varphi_j^t = \theta_j^t - \theta_j$ to cancel terms) that

$$g'(t) = \sum_{j=0}^n \int_X \dot{\varphi}_j^t \text{MA}_j(\varphi_0^t, \dots, \varphi_n^t),$$

where

$$\text{MA}_j(\varphi_0^t, \dots, \varphi_n^t) := V^{-1} \theta_0^t \wedge \dots \wedge \theta_{j-1}^t \wedge \theta_{j+1}^t \wedge \dots \wedge \theta_n^t,$$

(with the factor θ_j^t being omitted). Furthermore, for each fixed $j \in \{0, 1, \dots, n\}$ we have

$$\begin{aligned} g''(t) &= \sum_{j=0}^n \frac{d}{dt} \int_X \dot{\varphi}_j^t \text{MA}_j(\varphi_0^t, \dots, \varphi_n^t) = \\ &= \sum_{j=0}^n \left\{ \int_X \ddot{\varphi}_j^t \text{MA}_j(\varphi_0^t, \dots, \varphi_n^t) - \int_X R_j \right\}, \end{aligned}$$

where

$$R_j := \sum_{\substack{l=0 \\ l \neq j}}^n d\dot{\varphi}_j^t \wedge d^c \dot{\varphi}_l^t \wedge \text{MA}_{(l,j)}(\varphi_0^t, \dots, \varphi_n^t),$$

having used again integration by parts and the notation

$$\text{MA}_{(l,j)}(\varphi_0^t, \dots, \varphi_n^t) := \theta_0^t \wedge \dots \wedge \theta_{j-1}^t \wedge \theta_{j+1}^t \wedge \dots \wedge \theta_{l-1}^t \wedge \theta_{l+1}^t \wedge \dots \wedge \theta_n^t,$$

(i.e. both terms θ_j^t and θ_l^t omitted).

In view of the standard expression for the Laplacian in log polar coordinates of a radial function, we then have

$$dd_{\bar{\tau}}^c g(t) = g''(t) d\tau \wedge d\bar{\tau} = \int_X (p_1^* \theta_0 + dd_{(x,\tau)}^c \Phi_0) \wedge \dots \wedge (p_1^* \theta_n + dd_{(x,\tau)}^c \Phi_n).$$

where the last equality follows from an immediate adaptation of e.g. [GZ17, Proposition 15.16] (following [Sem92]). This finishes the proof in case all Φ_i are smooth.

In case the Φ_i are arbitrary $\pi_1^* \omega$ -psh functions the proof follows using Bedford-Taylor's monotonic continuity theorem, as detailed in [BBGZ13, Proposition 6.2]: In case the Φ_i are bounded the proof goes through using integration by parts as in the smooth setting. In general, if Φ_i are arbitrary $p_1^* \omega_i$ -psh functions, then write each Φ_i respectively as the decreasing limit as $k \rightarrow +\infty$ of the bounded functions $\Psi_i^{(k)} := \max(\Phi_i, -k)$. By [BBGZ13, Proposition 2.4] $\langle \Phi_0, \dots, \Phi_n \rangle_{(\theta_0, \dots, \theta_n)}$ is the pointwise decreasing limit of the functions $\langle \Psi_0^{(k)}, \dots, \Psi_n^{(k)} \rangle_{(\theta_0, \dots, \theta_n)}$, and

$$\begin{aligned} (p_1^* \theta_0 + dd_{(x,\tau)}^c \Psi_0^{(k)}) \wedge \dots \wedge (p_1^* \theta_n + dd_{(x,\tau)}^c \Psi_n^{(k)}) &\longrightarrow \\ (p_1^* \theta_0 + dd_{(x,\tau)}^c \Phi_0) \wedge \dots \wedge (p_1^* \theta_n + dd_{(x,\tau)}^c \Phi_n), \end{aligned}$$

by Bedford-Taylor's monotonic continuity theorem (see 2.2). \square

As a particular case of the above, we obtain the familiar formulas for the second order variation of E and E^θ , given by

$$dd_{\bar{\tau}}^c E(\varphi_\tau) = \frac{1}{n+1} V^{-1} \int_X (p_1^* \omega + dd_{(x,\tau)}^c \Phi)^{n+1}$$

and

$$dd_{\bar{\tau}}^c E^\theta(\varphi_\tau) = V^{-1} \int_X (p_1^* \omega + dd_{(x,\tau)}^c \Phi)^n \wedge \theta$$

respectively. In particular, note that $E(\varphi_\tau) := E \circ \Phi$ is a subharmonic function on $\bar{\Delta}^*$. The function $t \mapsto E(\varphi^\tau)$ is *affine* along weak geodesics, and *convex* along subgeodesics.

The Mabuchi K-energy functional and cscK metrics

Let ω be a Kähler form on X and consider any path $(\varphi_t)_{t \geq 0}$ in the space \mathcal{H} of Kähler potentials on X . The *Mabuchi functional* (or *K-energy functional*) $M : \mathcal{H} \rightarrow \mathbb{R}$, first introduced in [Mab85, Mab86], is defined by its Euler-Lagrange equation

$$\frac{d}{dt} M(\varphi_t) = -V^{-1} \int_X \dot{\varphi}_t (\mathcal{S}(\omega_{\varphi_t}) - \bar{\mathcal{S}}) \omega_{\varphi_t}^n$$

Indeed, it was shown by Mabuchi [Mab86] that the $(1, 1)$ -form

$$\sigma : (\varphi, \nu) \mapsto \int_X \nu(\bar{S} - S_\varphi) \omega_\varphi^n, \quad \nu \in T_\varphi \mathcal{H} \simeq C^\infty(X),$$

is *closed*. Since \mathcal{H} is star shaped (even convex) we may find a functional $M : \mathcal{H} \rightarrow \mathbb{R}$ satisfying $M(0) = 0$ and $dM = \sigma$. The Mabuchi functional is independent of the path chosen and its critical points are precisely the cscK metrics, when they exist. It is also possible to give an explicit formula for the Mabuchi functional as a sum of an "energy" and an "entropy" part, called the *Chen-Tian formula* [Che00a]. Indeed, with our normalisations we have

$$M(\varphi) = \left(\bar{S}E(\varphi) - E^{\text{Ric}(\omega)}(\varphi) \right) + V^{-1} \int_X \log \left(\frac{(\omega + dd^c \varphi)^n}{\omega^n} \right) (\omega + dd^c \varphi)^n, \quad (2.14)$$

where the latter term is the *relative entropy* of the probability measure $\mu := \omega_\varphi^n/V$ with respect to the reference measure $\mu_0 := \omega^n/V$. The entropy is always lower semi-continuous in μ , takes values in $[0, +\infty]$ and is finite if μ/μ_0 is bounded (which is the case here, but not in general for less regular φ).

Following Chen [Che00a] (using the formula (2.14)) we will often work with the extension $M : \mathcal{H}^{1, \bar{1}} \rightarrow \mathbb{R}$ of the Mabuchi functional to the space of ω -psh functions with bounded Laplacian. This is a natural setting to consider, since (\mathcal{H}, d_2) is *not* geodesically complete, while weak geodesic rays with bounded Laplacian are known to always exist, cf. [Che00b, Blo13, Dar14, DL12] as well as Lemma 4.1.10.

We further recall that the Mabuchi functional is convex along weak geodesic rays. In case $(\varphi_t)_{t \geq 0}$ is a smooth geodesic this can be seen from a straightforward computation:

$$\begin{aligned} \frac{d^2}{dt^2} M(\varphi_t) &= \frac{d}{dt} \int_X \dot{\varphi}_t (\mathcal{S}(\omega_{\varphi_t}) - \bar{S}) \frac{\omega_{\varphi_t}^n}{V} = \\ &= - \int_X \ddot{\varphi}_t (\mathcal{S}(\omega_{\varphi_t}) - \bar{S}) \frac{\omega_{\varphi_t}^n}{V} - n \int_X (\mathcal{S}(\omega_{\varphi_t}) - \bar{S}) dd^c \dot{\varphi}_t \frac{\omega_{\varphi_t}^n}{V} - \int_X \dot{S}_{\varphi_t}(\dot{\varphi}_t) \dot{\varphi}_t \frac{\omega_{\varphi_t}^n}{V}. \end{aligned}$$

Here $\dot{S}_\varphi(v) = -2\mathcal{D}_\varphi^* \mathcal{D}_\varphi v - (dS_\varphi, dv)_\varphi$, so by integration by parts the second time derivative of $t \mapsto M(\varphi_t)$ equals

$$2 \int_X |\mathcal{D}_{\varphi_t}(\varphi_t)|_\varphi^2 \omega_\varphi^n - \int_X (\mathcal{S}(\omega_{\varphi_t}) - \bar{S})(\ddot{\varphi}_t - |d\dot{\varphi}_t|_\varphi^2) \frac{\omega_{\varphi_t}^n}{V}$$

which is ≥ 0 since the last term vanishes due to the geodesic equation (2.12).

In general, proving convexity of the Mabuchi functional along geodesics is much more involved and was recently established by [BB14], see also [CLP16]. As before one makes sense of such a convexity in the weak sense, relying on the foundational work of Bedford-Taylor [BT82].

Theorem 2.4.11 ([BB14, CLP16]). *For any Kähler class $[\omega]$ the Mabuchi functional is convex along the weak geodesic ray u_t connecting any two points u_1 and u_2 in the space \mathcal{H}_ω of smooth ω -Kähler potentials.*

As a consequence of this convexity, the Mabuchi functional is bounded from below (in the given Kähler class) whenever α contains a cscK metric, see [Don05b], [Li11] for a proof in the polarized case and [BB14] for the general Kähler setting.

Proposition 2.4.12 ([BB14, CLP16]). *If X is a compact Kähler manifold and $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ admits a cscK representative, then there is a constant $C > 0$ such that*

$$M(\varphi) \geq -C, \quad \varphi \in \mathcal{H}.$$

Remark 2.4.13. *An improved result in this direction was recently proven in [BDL16]. It states that the Mabuchi functional is so called $\text{Aut}_0(X)$ -coercive on cscK manifolds, as explained below.*

G-action functionals

Denote by $G := \text{Aut}_0(X)$ the connected complex Lie group of automorphisms of X . For later use we recall the following definitions:

Definition 2.4.14. *A complex Lie group H is said to be*

- *semisimple if its Lie algebra \mathfrak{h} is the sum of simple Lie algebras, i.e. non-abelian Lie algebras \mathfrak{k} whose only ideals are $\{0\}$ and \mathfrak{k} itself.*
- *reductive if \mathfrak{h} decomposes as the direct sum of a semisimple and an abelian Lie algebra.*

In practice it is useful to note that, equivalently, a linear complex Lie group H is *reductive* if and only if it is the complexification of a maximal compact subgroup $K \subset H$. In particular, examples of reductive subgroups include e.g. the complexification $K^c = \text{SL}(n, \mathbb{C})$ of $K = \text{SU}(n)$. A particularly important example for us is \mathbb{C}^* whose maximal compact subgroup is $U(1)$.

As previously mentioned, in the study of cscK manifolds a key result involving reductive subgroups is the following obstruction:

Proposition 2.4.15. ([Lic57, Mat57]) *If X is a cscK manifold, then $\text{Aut}(X)$ is reductive.*

The proof uses the Matsushima-Lichnerowicz decomposition [Lic57, Mat57] in a central way. We refer also to e.g. [DR17] for details.

Group actions

Let $G \subset \text{Aut}_0(X)$ be any reductive subgroup of the connected component of the complex Lie group of automorphisms (i.e. biholomorphisms) of X . We denote its Lie algebra by $\mathfrak{aut}(X)$. It consists of infinitesimal automorphisms composed of real vector fields V satisfying $\mathcal{L}_V J = 0$. We now recall the definition of the action of G on the space \mathcal{H}_0 of normalized Kähler potentials, following [DR17, Section 5.2] as a reference.

First recall that \mathcal{H}_0 is in one-one correspondence with the set $\mathcal{H} := \{\omega_\varphi := \omega + dd^c \varphi : \varphi \in \mathcal{C}^\infty(X), \omega_\varphi > 0\}$. The group G acts on \mathcal{H} by pullback, i.e.

$$g \cdot \xi := g^* \xi, \quad g \in G, \quad \xi \in \mathcal{H}.$$

Due to the one-one correspondence with \mathcal{H} , the group G also acts on potentials in \mathcal{H}_0 , so that $g \cdot \varphi$ is the unique element in \mathcal{H}_0 satisfying $g \cdot \omega_\varphi = \omega_{g \cdot \varphi}$. As in [DR17, Lemma 5.8]) one may show that

$$g \cdot \varphi = g \cdot 0 + \varphi \circ g. \quad (2.15)$$

For future use, we emphasize that the function $g \cdot 0$ is smooth (hence bounded) on X , as follows from the dd^c -lemma since $g^*\omega$ is always a Kähler form cohomologous to ω .

Proposition 2.4.16. *Let (X, ω) be a given compact Kähler manifold. Then for any $g \in G$, the function $g \cdot 0$ is smooth on X .*

The J_G -functional and the Finsler metric $d_{1,G}$

Let $d_1 : \mathcal{H}_\omega \times \mathcal{H}_\omega \rightarrow \mathbb{R}_+$ be the path length pseudometric associated to the weak Finsler metric on \mathcal{H}_ω defined by

$$\|\xi\|_\varphi := V^{-1} \int_X |\xi| \omega_\varphi^n, \quad \xi \in T_\varphi \mathcal{H}_\omega = \mathcal{C}^\infty(X).$$

More explicitly, if $[0, 1] \ni t \mapsto \phi_t$ is a smooth path in X , then let

$$l_1(\phi_t) := \int_0^1 \|\dot{\phi}_t\|_{\phi_t} dt$$

be its length, and define

$$d_1(\varphi, \psi) = \inf \{l_1(\phi_t), (\phi_t)_{0 \leq t \leq 1} \subset \mathcal{H}_\omega, \phi_0 = \varphi, \phi_1 = \psi\},$$

the infimum being taken over smooth paths $t \mapsto \phi_t$ as above.

It can be seen ([DR17, Theorem 4.3]) that $(\mathcal{H}_\omega, d_1)$ is a metric space (whose d_1 -metric completion is (\mathcal{E}^1, d_1)). Moreover, the action of $G := \text{Aut}_0(X)$ on \mathcal{H}_0 is a d_1 -isometry [DR17, Section 5]. It thus induces a pseudometric $d_{1,G} : \mathcal{H}/G \rightarrow \mathbb{R}_+$ on the orbit space, given by

$$d_{1,G}(G\varphi, G\psi) := \inf_{f, g \in G} d_1(f \cdot \varphi, g \cdot \psi)$$

Note that $d_{1,G}(G0, G\varphi) = \inf_{g \in G} d_1(0, g \cdot \varphi)$. Following [ZZ08, Definition 2.1] and [Tia12, Definition 2.5] we furthermore consider a version of Aubin's J-functional defined on the orbit space \mathcal{H}/G by

$$J_G(G\varphi) := \inf_{g \in G} J(g \cdot \varphi), \quad \varphi \in \mathcal{H}_0.$$

Definition 2.4.17. *We say that M is G -coercive if there are constants $\delta, C > 0$ such that*

$$M(\varphi) \geq \delta J_G(G\varphi) - C$$

for all $\varphi \in \mathcal{H}_0$.

Remark 2.4.18. *Note in particular that the K-energy $M : \mathcal{H} \rightarrow \mathbb{R}$ (as well as its extension to $\mathcal{H}^{1,1}$ via the Chen-Tian formula) is G -invariant.*

It is important for our purposes to recall that the J_G -functional is comparable to $d_{1,G}$:

Lemma 2.4.19. ([DR17, Lemma 5.11]) *If $\varphi \in \mathcal{H}_0$, then*

$$\frac{1}{C}J_G(G\varphi) - C \leq d_{1,G}(G0, G\varphi) \leq CJ_G(G\varphi) + C,$$

for some $C > 0$.

See also [DR17, Proposition 5.5].

Tian's properness conjecture and G-coercivity of cscK manifolds

We here recall Tian's properness conjecture, with modifications due to T. Darvas and Y. Rubinstein [DR17]: Let $G \subset \text{Aut}(X)_0$ be any reductive subgroup of the connected component of the automorphism group. As above, following [ZZ08, Definition 2.1] and [Tia12, Definition 2.5] we furthermore consider a version of Aubin's J-functional defined on the orbit space \mathcal{H}/G by

$$J_G(G\varphi) := \inf_{g \in G} J(g.\varphi), \quad \varphi \in \mathcal{H}_0.$$

Definition 2.4.20. *We say that M is G-coercive if there are constants $\delta, C > 0$ such that*

$$M(\varphi) \geq \delta J_G(G\varphi) - C$$

for all $\varphi \in \mathcal{H}_0$.

It was proven in [BDL16, Theorem 1.1] that M is G-coercive iff the Kähler class α admits a constant scalar curvature representative.

Theorem 2.4.21. ([BDL16]) *Suppose that (X, ω) is a Kähler manifold. If $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ admits a constant scalar curvature representative, then M is $\text{Aut}_0(X)$ -coercive.*

This deep result is a version of Tian's properness conjecture (with a modification by T. Darvas and Y. Rubinstein [DR17], following a counterexample [DR17, Example 2.2] to the original conjecture [Tia00b, Conjecture 7.12], [Tia94, Remark 5.2]):

Conjecture 2.4.22. ([DR17, Conjecture 2.8]) *Let (X, ω) be a compact Kähler manifold and let $G := \text{Aut}_0(X)$ be the connected automorphism group of X . Then the following are equivalent:*

1. *There exists a cscK metric in the space \mathcal{H} of smooth Kähler potentials of $[\omega]$.*
2. *The Mabuchi K-energy functional M is G-invariant and its descent to the quotient space \mathcal{H}/G satisfies $M \geq \delta J_G - C$ for some $\delta, C > 0$.*
3. *The Mabuchi K-energy functional M is G-invariant and its descent to the quotient space \mathcal{H}/G satisfies $M \geq \delta d_{1,G} - C$ for some $\delta, C > 0$.*

Remark 2.4.23. We give some brief comments about the proof: The equivalence between (2) and (3) follows immediately from the fact that J_G and $d_{1,G}$ are comparable, see Lemma 2.4.19. The implication (1) \Rightarrow (2) is precisely Theorem 2.4.21, whose proof consists mainly in showing

that *a*) the K-energy can be extended to the larger space (\mathcal{E}^1, d_1) of finite energy potentials [BDL15, Theorem 1.2], *b*) In case a smooth minimizer exists, any minimizer of the extended K-energy $\bar{M} : \mathcal{E}^1 \rightarrow [-\infty, +\infty)$ is a smooth cscK metric [BDL16, Theorem 1.4], and *c*) by [DR17, Theorem 2.10] this suffices to conclude that (2) holds. The converse (2) \Rightarrow (1) is an important open problem in complex geometry, closely related to the YTD conjecture.

Chapter 3

Notions of K-stability for Kähler manifolds

3.1 A review of Geometric Invariant Theory and K-stability for polarized manifolds

The notion of K-stability has its historical roots in Mumford's Geometric Invariant Theory (GIT). Of particular interest here is the interplay between K-stability and metric geometry, via the question of existence of canonical metrics. As motivation and as an introduction to our more general notions we thus briefly recall the broad strokes of this finite dimensional analogy to K-stability, for which the standard references are [KKMSD73, MFK94]. As a reference for this section we have also used the expository notes [Sze14] and [Don17].

The GIT quotient and complex moduli

The original motivation of Geometric Invariant Theory (GIT) was to study certain complex moduli spaces $X//G$ called the *GIT quotient* of a complex projective variety X by the action of a reductive group G . The GIT quotient (defined below) is a projective variety, but a major problem is that it is not always clear what it represents geometrically. For instance, it is not always Hausdorff separable.

To discuss these questions, suppose that $G := K^c$ is a reductive complex Lie group acting on a polarized manifold (X, L) , with K a maximal compact subgroup of G by definition (see Section 2.4). Then for each $k \geq 1$ the group G acts also on the space $V_k := H^0(X, L^k)$ of sections of $L^k \rightarrow X$. Choosing k large enough we thus get an embedding $\iota : X \hookrightarrow \mathbb{P}(V_k^*)$ of X into complex projective space, coming with a morphism $G \rightarrow GL(V_k)$ in such a way so that the action of G on X is induced by the action of $GL(V_k)$ on $\iota(X)$. In the subsequent presentation we restrict to considering a complex projective variety X with a G -action induced by a representation $G \rightarrow SL(V)$.

In order to define the GIT-quotient $X//G$ we proceed by algebraic methods, and consider the graded ring R of G -invariant polynomials on V . It can then be shown that R is finitely

generated (using the assumption that G is reductive). Hence we may consider the projective variety $\text{Proj}(R)$, and define

$$X//G := \text{Proj}(R).$$

An important observation is that this quotient does *not* correspond precisely to the orbits of the action of G on X , but rather there are some potential issues that may occur. Indeed, it should be noted (see e.g. [Sze14, Section 5.3]) that a) $X//G$ parametrises orbits on which there is at least one non-vanishing G -invariant orbit function in R , and b) the quotient map $X \dashrightarrow X//G$ identifies any two orbits that cannot be distinguished by G -invariant functions in R .

In order to circumvent the issue of two orbits being identified under the quotient map $X \dashrightarrow X//G$, one may restrict attention to so called (semi/poly)-stable points in X , defined as follows:

Definition 3.1.1. *A point $p \in X$ is said to be*

- *semistable if there exists a non-constant homogeneous polynomial $f \in R(X)^G$ such that $f(p) \neq 0$. Denote the set of semistable points in X by X^{ss} .*
- *stable if it is semistable, and moreover the stabiliser of p in G is finite, and the orbit $G \cdot p$ is closed in X^{ss} . Denote the set of stable points in X by X^s .*

Remark 3.1.2. *By definition we have $X^s \subset X^{ss} \subset X$ and the sets of stable and semistable points are both open in X . When comparing the terminology naively to that of K -stability (see Section 3.2) one should note that X^{ss} being open is in stark contrast to the fact that the K -semistable locus is closed in the Kähler cone of X , see Section 6.*

In view of the above definitions we may identify $X//G$ with the quotient of X^{ss} by the equivalence relation defined by $p \sim q$ iff $\overline{G \cdot p} \cap \overline{G \cdot q}$ is non-empty in X^{ss} . Furthermore, since G by construction has closed orbits on X^s , we may consider the quotient X^s/G , sitting inside the GIT quotient $X//G$, which by contrast admits several nice geometric properties (it is a so called *geometric quotient* in GIT terminology).

It will be convenient to note the following equivalent characterization of stable and semistable points in X :

Proposition 3.1.3. ([Sze14, Section 5.4]) *Let $p \in X$. Write \hat{p} for any given lift of p with respect to the projection map $\mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$. Denote the G -orbit of \hat{p} in \mathbb{C}^{n+1} by $G \cdot \hat{p}$. The point $p \in X$ is then*

- *semistable if and only if 0 is not in the closure $\overline{G \cdot \hat{p}}$ of the orbit $G \cdot \hat{p}$.*
- *stable if and only if the orbit $G \cdot \hat{p}$ is closed in \mathbb{C}^{n+1} and the stabilizer of p in G is finite.*

We refer to [Sze14, Section 5.4] for a proof of this result. We also introduce a notion of *polystability* that is stronger than semistability but weaker than stability:

Definition 3.1.4. *We say that a point $p \in X$ is polystable if the orbit $G \cdot \hat{p}$ is closed in \mathbb{C}^{n+1} . Denote by X^{ps} the set of polystable points in X .*

The Hilbert-Mumford criterion and the Kempf-Ness theorem

In view of the above discussion it is important to determine whether a given point $p \in X$ is (semi)-stable. In the setup of Section 3.1, the Hilbert-Mumford criterion provides a tool for doing so. It essentially states that it suffices to test closedness of orbits $G \cdot \hat{p}$ for one-parameter subgroups $\mathbb{C}^* \hookrightarrow G$. In order to formulate the Hilbert-Mumford criterion we need to introduce the concept of weights of the pair (p, λ) , where $p \in X$ is a point and $\lambda : \mathbb{C}^* \hookrightarrow G$ is a 1-parameter subgroup of G : First note that since G acts on \mathbb{C}^{n+1} and \mathbb{C}^* is a 1-parameter subgroup of G , we obtain an induced \mathbb{C}^* -action on \mathbb{C}^{n+1} and with it a decomposition

$$\mathbb{C}^{n+1} = \bigoplus_{i=1}^k V(w_i) \quad (3.1)$$

into weight spaces such that each $w_i \in \mathbb{Z}$, $k \leq n+1$, and $\lambda(t) \cdot v = t^{w_i} v$ for $v \in V(w_i)$. Without loss of generality we may (up to reordering) assume that $w_1 > w_2 > \dots > w_k$ and the lift \hat{p} to \mathbb{C}^{n+1} of the given point $p \in X$ decomposes as $\hat{p} := \sum_{i=1}^k \hat{p}_i$ where each $\hat{p}_i \in V(w_i)$.

Definition 3.1.5. Let $l \in \{1, \dots, k\}$ be the smallest index such that $\hat{p}_l \neq 0$. Then we define the weight $\mu(p, \lambda)$ of $\lambda : \mathbb{C}^* \hookrightarrow G$ at $p \in X$ as $\mu(p, \lambda) := -w_l$.

Theorem 3.1.6. (Hilbert-Mumford Criterion). Let $p \in X$. Then p is

- *semistable* $\Leftrightarrow \mu(p, \lambda) \geq 0$ for all 1-parameter subgroups λ .
- *stable* $\Leftrightarrow \mu(p, \lambda) > 0$ for all 1-parameter subgroups λ .
- *polystable* $\Leftrightarrow \mu(p, \lambda) > 0$ for all 1-parameter subgroups λ for which $\lim_{t \rightarrow 0} \lambda(t) \cdot p \notin G \cdot p$.

See for instance [Sze14] for details.

Remark 3.1.7. This criterion has many similarities with the definition of K-stability, cf. Section 3.2. The role of the 1-parameter subgroups in GIT are then played by the *test configurations*, and the analogy of the weights $\mu(p, \lambda)$ are the *Donaldson-Futaki invariants*.

Background on K-stability for polarized manifolds

The purpose of this section is to introduce the classical notion of K-stability for polarized manifolds (X, L) , following Donaldson [Don02]). In order to distinguish it from the alternative 'cohomological approach' introduced in Section 3.2 we sometimes refer to the present notion as 'algebraic K-stability'.

The following review of the notion of *test configurations* and K-stability for polarized manifolds largely follows the point of view taken by Donaldson in [Don02]. Earlier, a notion of K-stability had been introduced by Tian [Tia97] and which was valid under a normality assumption. A main point of Donaldson's algebraic formalism was to bypass this assumption. However, it was shown by Li and Xu [LX14] that it suffices to consider test configurations whose total space is normal, so the respective approaches of Donaldson and Tian can ultimately be thought of as equivalent. As a main reference for the following exposition we use [BHJ15].

Donaldson's algebraic test configurations for polarized manifolds

In this section, let (X, L) be a polarized manifold, i.e. X is a compact Kähler manifold and $L \rightarrow X$ an ample line bundle on X . We write $n := \dim_{\mathbb{C}}(X)$ for the dimension of X .

Definition 3.1.8. *A test configuration for (X, L) is a normal polarized manifold $(\mathcal{X}, \mathcal{L})$ with a \mathbb{C}^* -action on \mathcal{X} lifting to \mathcal{L} , and a flat \mathbb{C}^* -equivariant projective morphism $\pi : \mathcal{X} \rightarrow \mathbb{C}$ where \mathbb{C} is given the standard \mathbb{C}^* -action, such that the fiber $\pi^{-1}(t)$ is isomorphic to (X, L^r) for all $t \in \mathbb{C}^*$ and some $r \in \mathbb{N}$, called the exponent of the test configuration.*

Remark 3.1.9. *In order to distinguish this definition from the alternative notion of "test configurations in terms of cohomology" introduced in Section 3.2, we sometimes refer to the above test configurations as "algebraic test configurations".*

If $(\mathcal{X}, \mathcal{L})$ is a test configuration for (X, L) , it is convenient to refer to the total space \mathcal{X} simply as a *test configuration for X* . We say that $(\mathcal{X}, \mathcal{L})$ is ample (respectively nef) if \mathcal{L} is.

Since any polarized manifold (X, L) can be embedded into projective space $\mathbb{P}(H^0(X, kL)^*)$ for some $k \gg 1$ large enough (so that kL is very ample, cf. Kodaira embedding theorem) all algebraic test configurations can in fact be constructed according to the following principle: Suppose that $X \hookrightarrow \mathbb{P}^r$ is embedded into complex projective space of dimension r . Any one-parameter subgroup $\rho : \mathbb{C}^* \rightarrow \mathbb{P}^r$ then induces a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) , whose central fiber is the *flat limit* as $t \rightarrow 0$ of the image of X under $\rho(t)$ (cf. Hartshorne, Prop. 9.8]).

Example 3.1.10. *We give two simple and standard examples of test configurations that serve as useful references in the sequel:*

1. (Product configurations) *Let $\lambda : \mathbb{C}^* \rightarrow \text{Aut}(X)$ be a 1-parameter subgroup and consider the induced diagonal \mathbb{C}^* -action $\tau \cdot (x, z) := (\lambda(\tau) \cdot x, \tau z)$ on $X \times \mathbb{C}$. This yields a so called product configuration. Precisely when the 1-parameter subgroup is trivial we obtain the trivial test configuration this way.*
2. (Deformation to the normal cone, cf. [RT06, RT07]) *Let Z be any (reduced, for simplicity) divisor on X and consider the blow up \mathcal{X} of $X \times \mathbb{C}$ along the subscheme $Z \times \{0\}$. The construction comes with a corresponding flat (i.e. surjective) morphism $\pi : \mathcal{X} \rightarrow \mathbb{C}$ which factors through the blow down map $p : \mathcal{X} \rightarrow X \times \mathbb{C}$. Moreover, there is a natural embedding of $X \times \mathbb{C}^*$ into \mathcal{X} . Given an ample line bundle L on X and a positive number $c > 0$ let $\mathcal{L}_c := p^*L - cE$, where E is the exceptional divisor and we are assuming that c is small enough so that \mathcal{L}_c is π -relatively ample. This yields a test configuration $(\mathcal{X}, \mathcal{L}_c)$ for (X, L) , and it can be seen (see e.g. [Ber13, Section 4]) that $0 \leq \text{DF}(\mathcal{X}, \mathcal{L}_c) \rightarrow 0$ as $c \rightarrow 0$.*

Compactification of a test configuration

An important object is the *compactification* $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ of a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) . This notion is used to express the important Donaldson-Futaki invariant (cf. Section 3.1 below)

and can thus be used to indirectly define K-stability. Given the bimeromorphic morphism $\mathcal{X} \rightarrow X \times \mathbb{C}$ induced by any test configuration \mathcal{X} for X , a compactification can be defined in a canonical way as follows:

Definition 3.1.11. *The compactification $\bar{\mathcal{X}}$ of a test configuration \mathcal{X} for X is defined by gluing together \mathcal{X} and $X \times \mathbb{P}^1 \setminus \{0\}$ along the open subsets $\mathcal{X} \setminus \mathcal{X}_0$ and $X \times (\mathbb{C} \setminus \{0\})$, using the canonical \mathbb{C}^* -equivariant isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{C} \setminus \{0\})$.*

Remark 3.1.12. *The compactification $\bar{\mathcal{X}}$ comes with a \mathbb{C}^* -equivariant flat morphism $\pi : \bar{\mathcal{X}} \rightarrow \mathbb{P}^1$; in fact making it a test configuration for X in the generalised sense of Section 3.2 below.*

The compactification of a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) is defined similarly, where $\bar{\mathcal{L}}$ is a \mathbb{C}^* -linearized (\mathbb{Q}) -line bundle on $\bar{\mathcal{X}}$. The polarization $\bar{\mathcal{L}}$ is relatively (semi)ample iff \mathcal{L} is.

Remark 3.1.13. *It is important to note that the compactification of a product configuration $X \times \mathbb{C}$ is not a product over \mathbb{P}^1 in general. This can be illustrated for e.g. by the following example (see [BHJ15, Example 2.8]): Consider $X = \mathbb{P}^1$ with the \mathbb{C}^* -action defined by $t \cdot [x : y] := [t^d x : y]$, $d \in \mathbb{N}$. This \mathbb{C}^* -action induces a product configuration $\mathcal{X} := X \times \mathbb{C}$ for X endowed with the induced diagonal action $t \cdot ([x : y], z) = ([t^d x : y], tz)$. The compactification $\bar{\mathcal{X}}$ can then be identified with the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$.*

Equivariant Riemann-Roch and the Donaldson-Futaki invariant

Following the classical approach of Donaldson [Don02] the definition of the Donaldson-Futaki invariant of a test configuration $(\mathcal{X}, \mathcal{L})$ for a polarized manifold (X, L) involves the equivariant Riemann-Roch theorem for schemes, which we now recall.

Theorem 3.1.14. (Equivariant Riemann-Roch theorem, [BHJ15, Theorem 3.1]) *Let $(\mathcal{X}, \mathcal{L})$ be a polarized scheme endowed with a \mathbb{G}_m -action. Let $n + 1 := \dim_{\mathbb{C}}(\mathcal{X})$. Then for each $k \in \mathbb{Z}$ the finite sum*

$$\sum_{\lambda \in \mathbb{Z}} \lambda \dim H^0(X, kL)_{\lambda}$$

is a polynomial function in k for all $k \gg 1$ large enough. The degree of the polynomial is at most $n + 1$ and the coefficient of k^{n+1} equals the top intersection number $(\mathcal{B}^{n+1})/(n + 1)!$.

The details of the proof are explicitly clarified in [BHJ15, Appendix B].

The Donaldson-Futaki invariant of $(\mathcal{X}, \mathcal{L})$ is then defined as follows: Let $w_k \in \mathbb{Z}$ be the weight of the \mathbb{C}^* -action on the determinant line bundle $\det H^0(X, kL)$, and $N_k := \dim H^0(X, kL)$. Then by Theorem 3.1.14 $w_k = \sum_{\lambda \in \mathbb{Z}} \lambda \dim H^0(X, kL)_{\lambda}$ is a polynomial in k for $k \gg 1$ large enough, of degree at most $n + 1$. Moreover, N_k is a polynomial of degree n by the ordinary Riemann-Roch formula. Note that by flatness $\dim H^0(X, kL)$ equals the Hilbert polynomial of (X, rL) for all $k \gg 1$ large enough. We write

$$w_k = a_0 k^{n+1} + a_1 k^n + O(k^{n-1}) \tag{3.2}$$

$$N_k = b_0 k^n + b_1 k^{n-1} + O(k^{n-2}) \quad (3.3)$$

In particular, there is then an asymptotic expansion

$$\frac{w_k}{kN_k} = F_0 + k^{-1}F_1 + k^{-2}F_2 + \dots,$$

where $F_0 = (\bar{\mathcal{L}}^{n+1}/(n+1)!)$ and $\text{DF}(\mathcal{X}, \mathcal{L}) = -2F_1$.

Definition 3.1.15. *The Donaldson-Futaki invariant of the test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) is defined as*

$$\text{DF}(\mathcal{X}, \mathcal{L}) = -2F_1.$$

The factor 2 is there for convenience and the minus sign is there to agree with our sign convention in the definition of K-stability (which differs throughout the literature).

Definition 3.1.16. *We say that a polarized manifold (X, L) is algebraically K-semistable if and only if $\text{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for all normal and ample test configurations $(\mathcal{X}, \mathcal{L})$ for (X, L) .*

Definition 3.1.17. *The polarized manifold (X, L) is algebraically K-stable if it is K-semistable and, moreover, $\text{DF}(\mathcal{X}, \mathcal{L})$ vanishes precisely when $(\mathcal{X}, \mathcal{L})$ is the trivial test configuration.*

Note that (X, L) can be K-stable only if the automorphism group of X is discrete. Indeed, if not, then X admits a non-trivial \mathbb{C}^* -action and we obtain induced non-trivial product configurations as in Example 3.1.10. On the other hand, if $c_1(L)$ admits a cscK representative, then the Donaldson-Futaki invariant of a product configuration always vanishes. In case the automorphism group is not discrete the appropriate corresponding stability notion is denoted K-polystability:

Definition 3.1.18. *The polarized manifold (X, L) is algebraically K-polystable if it is K-semistable and, moreover, $\text{DF}(\mathcal{X}, \mathcal{L})$ vanishes if and only if $(\mathcal{X}, \mathcal{L})$ is a product configuration.*

Remark 3.1.19. To avoid confusion, we point out that the above notion of K-polystability is what is referred to as simply "K-stability" in the work of Donaldson [Don02]. The terminology K-stability and K-polystability are respectively employed to signal to the reader whether we restrict attention to the case of manifolds with discrete automorphism group, or whether we consider the (significantly more involved) general case with automorphisms.

Uniform K-stability for polarized manifolds

Suppose that (X, L) is a polarized manifold with discrete automorphism group. In order for the YTD conjecture to hold in general (beyond the Fano case and other known cases) it is believed that one may need a stronger notion of K-stability. For this, the first thing to note is that the Donaldson-Futaki invariant *cannot* in general be bounded below by a positive constant (take for instance the test configuration given by deformation to the normal cone with parameter $c > 0$ and let $c \rightarrow 0$, see Example 3.1.10). However, a candidate for a stronger notion of K-stability that has attracted a lot of attention is the notion of *uniform K-stability* introduced in G.

Szekelyhidi's thesis (cf. also e.g. [Der16b, BHJ15]) defined by demanding that the Donaldson-Futaki invariant is always bounded below by some (uniform) multiple of a certain norm of the test configuration. There are several ways of introducing such norms, some of which turn out to be equivalent. In this thesis we study a notion of uniform K-stability where the role of the "norm" is played by the so called *non-archimedean J-functional* J^{NA} , in the terminology of [BHJ15, BHJ16]. We do not attempt to explain the reason for this terminology here (the interested reader may consult [BHJ15, Section 6.8] and references therein). However, we clarify in Section 4 that non-archimedean functionals arise as the limit slope classical energy functionals in Kähler geometry (in this case Aubin's J-functional) along certain geodesic rays associated to the given test configuration.

Definition 3.1.20. *We say that a polarized manifold (X, L) is uniformly K-stable if there is a $\delta > 0$ such that*

$$\text{DF}(\mathcal{X}, \mathcal{L}) \geq \delta J^{\text{NA}}(\mathcal{X}, \mathcal{L})$$

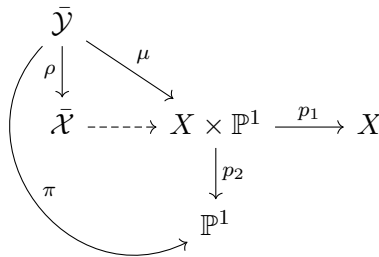
for all relatively Kähler cohomological test configurations $(\mathcal{X}, \mathcal{L})$ for (X, L) .

The approach taken is equivalent to that of the L^1 -norm (see e.g. [BHJ15]) and that of the *minimum norm* introduced in [Der16b]. Indeed, all these norms are comparable.

The non-archimedean J-functional $J^{\text{NA}}(\mathcal{X}, \mathcal{L})$ of a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) can be defined as an intersection number as follows: In case the compactification $\bar{\mathcal{X}}$ of the total space \mathcal{X} is smooth and dominates the product $X \times \mathbb{P}^1$ via a morphism $\mu : \bar{\mathcal{X}} \rightarrow X \times \mathbb{P}^1$ we write

$$J^{\text{NA}}(\mathcal{X}, \mathcal{L}) := V^{-1}(\bar{\mathcal{L}} \cdot \mu^* p_1^* L^n)_{\bar{\mathcal{X}}} - \bar{S} V^{-1}(\bar{\mathcal{L}}^{n+1})_{\bar{\mathcal{X}}}.$$

In general, in view of the projection formula, we may resolve singularities $\rho : \bar{\mathcal{Y}} \rightarrow \bar{\mathcal{X}}$ and compute the above intersection number by pulling back the respective line bundles to the compactification $\bar{\mathcal{Y}}$ of the resolution, see the figure below.



Intersection theoretic interpretation of the Donaldson-Futaki invariant

We here explain a result due to Wang [Wan12] and Odaka [Oda13], which is the foundation for the "cohomological approach" to K-stability that we will introduce in the following section 3.2. The result in question interprets the Donaldson-Futaki invariant (as introduced above in Definition 3.1.15) as an intersection number involving the compactification $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ of $(\mathcal{X}, \mathcal{L})$. In

particular, it follows that the Donaldson-Futaki invariant can be computed in cohomology and that K-stability can naturally be viewed as a condition on the level of cohomology.

Proposition 3.1.21. ([Wan12, Oda13]) *Suppose that (X, L) is a polarized manifold. Let $(\mathcal{X}, \mathcal{L})$ be a normal (algebraic) test configuration for (X, L) . Then*

$$\text{DF}(\mathcal{X}, \mathcal{L}) := \frac{\bar{S}}{n+1} V^{-1}(\bar{\mathcal{L}}^{n+1})_{\hat{\mathcal{X}}} + V^{-1}(K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n)_{\hat{\mathcal{X}}}.$$

Here the intersection number is computed on the compactification $\bar{\mathcal{X}}$ of \mathcal{X} , and the relative canonical class is now given by $K_{\bar{\mathcal{X}}/\mathbb{P}^1} := K_{\bar{\mathcal{X}}} - \pi^* K_{\mathbb{P}^1}$.

Remark 3.1.22. Due to [LX14] it is now known that the Donaldson-Futaki invariant can only decrease under normalisation, whence it suffices to consider test configurations whose total space is normal.

Proof of Theorem 3.1.21. This can be proven using the two-term asymptotic Riemann-Roch theorem on normal varieties as explained in [BHJ15, Proposition 3.12 (iv)]. This yields the expansions

$$N_k = V \frac{k^n}{n!} \left[1 + \frac{\bar{S}}{2} k^{-1} + O(k^{-2}) \right],$$

$$w_k = \frac{\bar{\mathcal{L}}^{n+1}}{(n+1)!} k^{n+1} - \frac{K_{\bar{\mathcal{X}}} \cdot \bar{\mathcal{L}}^n}{2n!} k^n + O(k^{n-1}).$$

By a straightforward computation one may then check that -2 times the second factor in the expansion of w_k/kN_k is given precisely by the above intersection number. By Definition 3.1.15 this is what we wanted to prove. \square

3.2 Definition and properties of K-stability for Kähler manifolds with transcendental cohomology class

In this section we introduce generalised K-stability notions of (X, α) that has meaning even when the class $\alpha \in H^{1,1}(X, \mathbb{R})$ is non-integral (or non-rational), i.e. when α is not necessarily of the form $c_1(L)$ for some ample (\mathbb{Q}) -line bundle L on X . As remarked by Berman in [Ber13], in view of the intersection theoretic formula 3.1.21 it is natural to generalise the notion of test configuration in terms of cohomology classes. In the polarized setting the idea is then to consider $(\bar{\mathcal{X}}, c_1(\bar{\mathcal{L}}))$ as a "test configuration" for $(X, c_1(L))$, by simply replacing \mathcal{L} and L with their respective first Chern classes. This approach is motivated in detail below. Moreover, a number of basic and useful properties will be established, and throughout, this generalisation will systematically be compared to the original notion of algebraic test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) , introduced by Donaldson in [Don02].

Remark 3.2.1. *Much of the following exposition goes through even when the cohomology class α is not Kähler. Unless explicitly stated otherwise, we thus assume that $\alpha = [\theta]$ for some closed $(1, 1)$ -form θ on X .*

Cohomological K-stability: Definitions

The main difficulty compared to the usual projective algebraic setting consists in making sense of test configurations for (X, α) , where X is *any* Kähler compact manifold and $\alpha \in H^{1,1}(X, \mathbb{R})$ is a Kähler class. It is often convenient to introduce the intermediate terminology of test configurations for X before introducing test configurations for (X, α) .

Test configurations for X

We first introduce the notion of test configuration \mathcal{X} for X , working directly over \mathbb{P}^1 . For the sake of comparison, recall the usual concept of test configuration for polarized manifolds, see e.g. [BHJ15] and [Sze14]. In what follows, we refer to Section 2.2 and [Fis76] for background on normal complex spaces.

Definition 3.2.2. *A test configuration \mathcal{X} for X consists of*

- a normal compact Kähler complex space \mathcal{X} with a flat (i.e. surjective) morphism $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$
- a \mathbb{C}^* -action λ on \mathcal{X} lifting the canonical action on \mathbb{P}^1
- a \mathbb{C}^* -equivariant isomorphism

$$\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{P}^1 \setminus \{0\}). \quad (3.4)$$

Note that since π is flat the central fiber $\mathcal{X}_0 := \pi^{-1}(0)$ is a Cartier divisor, so $\mathcal{X} \setminus \mathcal{X}_0$ is dense in \mathcal{X} in Zariski topology.

Definition 3.2.3. *We say that two test configurations \mathcal{X} and \mathcal{Y} for X are equal, and write $\mathcal{X} = \mathcal{Y}$, if the canonical isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \rightarrow \mathcal{Y} \setminus \mathcal{Y}_0$ extends to an isomorphism $\mathcal{X} \rightarrow \mathcal{Y}$. The isomorphism is then automatically \mathbb{C}^* -equivariant.*

The isomorphism (3.4) gives an open embedding of $X \times (\mathbb{P}^1 \setminus \{0\})$ into \mathcal{X} , hence induces a canonical \mathbb{C}^* -equivariant bimeromorphic map $\mu : \mathcal{X} \dashrightarrow X \times \mathbb{P}^1$. We say that \mathcal{X} dominates $X \times \mathbb{P}^1$ if the above bimeromorphic map μ is a morphism. Taking \mathcal{X}' to be the normalisation of the graph of $\mathcal{X} \dashrightarrow X \times \mathbb{P}^1$ we obtain a \mathbb{C}^* -equivariant bimeromorphic morphism $\rho : \mathcal{X}' \rightarrow \mathcal{X}$ with \mathcal{X}' normal and dominating $X \times \mathbb{P}^1$. In the terminology of [BHJ15] such a morphism ρ is called a *determination* of \mathcal{X} . In particular, a determination of \mathcal{X} always exists. By the above considerations we will often, up to replacing \mathcal{X} by \mathcal{X}' , be able to assume that the given test configuration for X dominates $X \times \mathbb{P}^1$.

Moreover, any test configuration \mathcal{X} for X can be dominated by a smooth test configuration \mathcal{X}' for X (where we may even assume that \mathcal{X}'_0 is a divisor of simple normal crossings). Indeed, by Hironaka (see [Kol07, Theorem 45] for the precise statement concerning normal complex spaces) there is a \mathbb{C}^* -equivariant proper bimeromorphic map $\mu : \mathcal{X}' \rightarrow \mathcal{X}$, with \mathcal{X}' smooth, such that \mathcal{X}'_0 has simple normal crossings and μ is an isomorphism outside of the central fiber \mathcal{X}_0 .

Remark 3.2.4. *When X is projective (hence algebraic), the GAGA principle shows that the usual (i.e. algebraic, and normal) test configurations of X correspond precisely to the test configurations (in our sense of Definition 3.2.2) with \mathcal{X} projective.*

Cohomological test configurations for (X, α)

We now introduce a natural generalisation of the usual notion of *algebraic test configuration* $(\mathcal{X}, \mathcal{L})$ for a polarized manifold (X, L) . This following definition involves the Bott-Chern cohomology on normal complex spaces, i.e. the space of locally dd^c -exact $(1, 1)$ -forms (or currents) modulo globally dd^c -exact $(1, 1)$ -forms (or currents). The Bott-Chern cohomology is finite dimensional and the cohomology classes can be pulled back. Moreover, $H_{\text{BC}}^{1,1}(\mathcal{X}, \mathbb{R})$ coincides with the usual Dolbeault cohomology $H^{1,1}(\mathcal{X}, \mathbb{R})$ whenever \mathcal{X} is smooth. See e.g. [BG13] for background.

Definition 3.2.5. *A cohomological test configuration for (X, α) is a pair $(\mathcal{X}, \mathcal{A})$ where \mathcal{X} is a test configuration for X and $\mathcal{A} \in H_{\text{BC}}^{1,1}(\mathcal{X}, \mathbb{R})^{\mathbb{C}^*}$ is a \mathbb{C}^* -invariant $(1, 1)$ -Bott-Chern cohomology class whose image under the \mathbb{C}^* -equivariant isomorphism*

$$\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{P}^1 \setminus \{0\}).$$

is $p_1^ \alpha$. Here $p_1 : X \times \mathbb{P}^1 \rightarrow X$ is the first projection.*

The *trivial* test configuration for (X, α) is given by $(\mathcal{X} := X \times \mathbb{P}^1, p_1^* \alpha, \lambda_{\text{triv}}, p_2)$, where $p_1 : X \times \mathbb{P}^1 \rightarrow X$ and $p_2 : X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ are the projections on the 1st and 2nd factor respectively, and $\lambda_{\text{triv}} : \mathbb{C}^* \times \mathcal{X} \rightarrow \mathcal{X}$, $(\tau, (x, z)) \mapsto (x, \tau z)$ is the \mathbb{C}^* -action that acts trivially on the first factor. If we instead let $\sigma : \mathbb{C}^* \times X \rightarrow X$ be any \mathbb{C}^* -action on X , then we obtain an induced test configuration as above with $\lambda(\tau, (x, z)) := (\sigma(\tau, x), \tau z)$ (by also taking the compactification so that the fiber at infinity is trivial). Such test configurations are called *product* test configurations of (X, α) .

In either case, we identify X with $X \times \{1\}$ and the canonical equivariant isomorphism (3.4) is then explicitly induced by the isomorphisms $X \cong X \times \{1\} \rightarrow X \times \{\tau\}$ given by $x \mapsto \lambda(\tau, (x, 1)) =: \lambda(\tau) \cdot x$. As a consequence of the isomorphism (3.4), note that if Φ is a function on \mathcal{X} , then its restriction to each fibre $\mathcal{X}_\tau \simeq X$, $\tau \in \mathbb{P}^1 \setminus \{0\}$ identifies with a function on X . The function Φ thus gives rise to a family of functions $(\varphi_t)_{t \geq 0}$ on X , recalling our convention of reparametrising so that $t := -\log |\tau|$.

Definition 3.2.6. *A cohomological test configuration $(\mathcal{X}, \mathcal{A})$ is said to be smooth if the total space \mathcal{X} is smooth. It is said to be dominating $X \times \mathbb{P}^1$ if the canonical isomorphism $\mu : \mathcal{X} \setminus \mathcal{X}_0 \rightarrow X \times \mathbb{P}^1 \setminus \{0\}$ extends to a morphism $\mu : \mathcal{X} \rightarrow X \times \mathbb{P}^1$. It is said to be relatively Kähler if \mathcal{A} is a relatively Kähler class, i.e. if there is a Kähler form $\beta \in H^{1,1}(\mathbb{P}^1)$ such that $\mathcal{A} + \pi^* \beta$ is Kähler on \mathcal{X} .*

Definition 3.2.7. *A pull-back of a cohomological test configuration $(\mathcal{X}, \mathcal{A}, \lambda, \pi)$ for (X, α) via a morphism $\mu : \hat{\mathcal{X}} \rightarrow \mathcal{X}$ is any test configuration $(\hat{\mathcal{X}}, \mu^* \mathcal{A}, \hat{\lambda}, \pi \circ \mu)$, where the \mathbb{C}^* -action $\hat{\lambda}$ may be taken arbitrarily provided that the data still defines a test configuration.*

By the discussion preceding Definition 3.2.2, we *in practice* restrict attention to the situation when $(\mathcal{X}, \mathcal{A})$ is a smooth (cohomological) test configuration for (X, α) dominating $X \times \mathbb{P}^1$, with $\mu : \mathcal{X} \rightarrow X \times \mathbb{P}^1$ the corresponding \mathbb{C}^* -equivariant bimeromorphic morphism. This situation is studied in detail in Section 3.2, where we in particular show that the class $\mathcal{A} \in H^{1,1}(\mathcal{X}, \mathbb{R})$ is always of the form $\mathcal{A} = \mu^* p_1^* \alpha + [D]$ for a unique \mathbb{R} -divisor D supported on the central fiber, cf. Proposition 3.2.18.

It is further natural to ask how the above notion of cohomological test configurations compares to the algebraic test configurations introduced by Donaldson in [Don02]:

Remark 3.2.8. We give a few remarks and examples on how to compare cohomological test configurations with *algebraic test configurations* $(\mathcal{X}, \mathcal{L})$ for a polarized manifold (X, L) . The latter refers to the well-known concept first introduced in [Don02] (which due to [LX14] is known to be equivalent to the notion due to Tian [Tia97]).

1. If (X, L) is any compact Kähler manifold endowed with an ample line bundle L (so X is projective) and $(\mathcal{X}, \mathcal{L})$ is a test configuration for (X, L) in the usual algebraic sense, cf. e.g. [Don02], then $(\tilde{\mathcal{X}}, c_1(\tilde{\mathcal{L}}))$ is a cohomological test configuration for $(X, c_1(L))$. In particular, every example of algebraic test configurations $(\mathcal{X}, \mathcal{L})$ for polarized manifolds (X, L) (see e.g. [Sze14, Tia00b] and references therein for concrete examples) provides an example of a cohomological test configuration for $(X, c_1(L))$.
2. There are more cohomological test configurations for $(X, c_1(L))$ than there are algebraic test configurations for (X, L) (take for instance $(\mathcal{X}, \mathcal{A})$ with \mathcal{A} a transcendental class as in the above definition), but it was shown in [SD16] that the notions of cohomological- and algebraic K-(semi)stability coincide when X is projective.

The Donaldson-Futaki invariant and the non-Archimedean Mabuchi functional

The following generalisation of the Donaldson-Futaki invariant is straightforward, with the exception of potential subtleties concerning the relative canonical class $K_{\mathcal{X}/\mathbb{P}^1}$ and intersection numbers whenever the test configuration is singular. The latter issues are addressed below.

Definition 3.2.9. *To any cohomological test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) we may associate its Donaldson-Futaki invariant $\text{DF}(\mathcal{X}, \mathcal{A})$ and its non-Archimedean Mabuchi functional $\text{M}^{\text{NA}}(\mathcal{X}, \mathcal{A})$. The latter refers to the modification of the Donaldson-Futaki invariant first introduced in [BHH15]. They are given respectively by the following intersection numbers*

$$\text{DF}(\mathcal{X}, \mathcal{A}) := \frac{\bar{S}}{n+1} V^{-1}(\mathcal{A}^{n+1})_{\tilde{\mathcal{X}}} + V^{-1}(K_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{A}^n)_{\tilde{\mathcal{X}}}$$

and

$$\text{M}^{\text{NA}}(\mathcal{X}, \mathcal{A}) := \text{DF}(\mathcal{X}, \mathcal{A}) + ((\mathcal{X}_{0, \text{red}} - \mathcal{X}_0) \cdot \mathcal{A}^n)_{\tilde{\mathcal{X}}}$$

computed on any smooth and dominating model $\tilde{\mathcal{X}}$ of \mathcal{X} (due to the projection formula it does not matter which one). Note that $\text{DF}(\mathcal{X}, \mathcal{A}) \geq \text{M}^{\text{NA}}(\mathcal{X}, \mathcal{A})$ with equality precisely when \mathcal{X}_0 is reduced.

In case \mathcal{X} is smooth, $K_{\mathcal{X}/\mathbb{P}^1} := K_{\mathcal{X}} - \pi^*K_{\mathbb{P}^1}$ denotes the relative canonical class taken with respect to the flat morphism $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$. In the general case of a normal (possibly singular) test configuration \mathcal{X} for X , we need to give meaning to the intersection number $K_{\mathcal{X}} \cdot \mathcal{A}_1 \cdots \mathcal{A}_n$, for $\mathcal{A}_i \in H_{\text{BC}}^{1,1}(\mathcal{X}, \mathbb{R})$. Write $j : \mathcal{X}_{\text{reg}} \hookrightarrow \mathcal{X}$ for the inclusion of the regular part of \mathcal{X} , and recall the one-one correspondence between reflexive sheaves of rank 1 and invertible rank 1 sheaves on the regular part \mathcal{X}_{reg} of \mathcal{X} , given by

$$\mathcal{L} \mapsto \mathcal{L}|_{\mathcal{X}_{\text{reg}}},$$

with inverse

$$\mathfrak{m} \mapsto (j_*\mathfrak{m})^{**},$$

(i.e. the reflexive hull of $j_*\mathfrak{m}$).

Having been unable to find a suitable reference for an intersection theory of one reflexive sheaf against Weil divisors, we for the moment content ourselves with an "ad hoc" definition of the intersection number $K_{\mathcal{X}} \cdot \mathcal{A}_1 \cdots \mathcal{A}_n$ in question, by means of resolution of singularities as follows: Suppose that $\tilde{\mathcal{X}}$ is a smooth model for \mathcal{X} , with $\pi' : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ the associated morphism. Since $\tilde{\mathcal{X}}$ is smooth the canonical class $K_{\tilde{\mathcal{X}}} := \omega_{\tilde{\mathcal{X}}}$ is a line bundle. Consider $\omega_{\mathcal{X}} := \mathcal{O}(K_{\mathcal{X}}) := (\pi'_*\omega_{\tilde{\mathcal{X}}})^{**}$, i.e. the unique rank 1 reflexive sheaf on \mathcal{X} whose restriction to \mathcal{X}_{reg} coincides with $\mathcal{O}(K_{\mathcal{X}_{\text{reg}}})$. Then set

$$(\omega_{\mathcal{X}} \cdot \mathcal{A}_1 \cdots \mathcal{A}_n) := (c_1(K_{\tilde{\mathcal{X}}}) \cdot \pi'^*\mathcal{A}_1 \cdots \pi'^*\mathcal{A}_n). \quad (3.5)$$

Using the projection formula, it is straightforward to see that the above intersection number (hence also DF and M^{NA}) is independent of the choice of resolution $\pi' : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$:

Proposition 3.2.10. *The intersection number (3.5) is independent of the choice of resolution $\pi' : \mathcal{X}' \rightarrow \mathcal{X}$.*

Proof. Let $\pi_1 : \mathcal{X}_1 \rightarrow \mathcal{X}$ and $\pi_2 : \mathcal{X}_2 \rightarrow \mathcal{X}$ be two such resolutions. Then there is a \mathcal{X}_3 simultaneously dominating \mathcal{X}_1 and \mathcal{X}_2 via morphisms μ_1 and μ_2 respectively, as in the following commutative diagram:

$$\begin{array}{ccc} \mathcal{X}_3 & \xrightarrow{\mu_2} & \mathcal{X}_2 \\ \mu_1 \downarrow & & \downarrow \pi_2 \\ \mathcal{X}_1 & \xrightarrow{\pi_1} & \mathcal{X} \end{array}$$

Since the \mathcal{X}_i are smooth we then have

$$(\mu_i)_*c_1(K_{\mathcal{X}_3}) = c_1(K_{\mathcal{X}_i}), \quad i = 1, 2,$$

concluding the argument. □

Remark 3.2.11. (Weil divisor classes and pullback) *In view of the above definition of the Donaldson-Futaki invariant, it is useful to remark further on the definition and interpretation of the canonical divisor $K_{\mathcal{X}}$ and its pullback by finite surjective morphisms $f : \mathcal{Y} \rightarrow \mathcal{X}$ between normal*

complex spaces. Indeed, we wish to interpret $K_{\mathcal{X}}$ as a Weil divisor class, i.e. an element of

$$Cl(\mathcal{X}) := \{\text{Weil divisors on } \mathcal{X}\} / \{\text{principal divisors}\},$$

but one should address the potential issue that in the analytic setting it can happen that there are very few meromorphic functions on X (we can indeed even have e.g. $\mathbb{C}(\mathcal{X}) = \mathbb{C}$). In order to define the pullback of a Weil divisor class we then define the pullback f^*D of a Weil divisor $D := \sum_i a_i D_i$ as the divisor $\sum_i a_i f^{-1}(D_i)$. The pullback of principal divisors $\text{div}(g)$ are in turn given by simply pulling back functions. In this way, intersection numbers such as the Donaldson-Futaki invariant can be interpreted via resolution of singularities as explained above.

Equivalence classes of test configurations

It is a useful point of view and notationally convenient to introduce an equivalence relation on the set of test configurations as follows:

Definition 3.2.12. (Equivalent test configurations) *Let $(\mathcal{X}_1, \mathcal{A}_1)$ and $(\mathcal{X}_2, \mathcal{A}_2)$ be two cohomological test configurations for (X, α) . We say that they are equivalent, and write $(\mathcal{X}_1, \mathcal{A}_1) \sim (\mathcal{X}_2, \mathcal{A}_2)$, iff they are both dominated by a third test configuration \mathcal{Y} for X such that $\mu_1^* \mathcal{A}_1 = \mu_2^* \mathcal{A}_2$ on \mathcal{Y} (see figure below).*

$$\begin{array}{ccc} \mathcal{Y} & & \\ \mu_1 \downarrow & \searrow \mu_2 & \\ \mathcal{X}_1 & \dashrightarrow & \mathcal{X}_2 \end{array}$$

In the terminology of [BHJ15, BHJ16] such an equivalence class is called a non-archimedean metric. We show in Section 5.1 that each equivalence contains a *unique* representative $(\mathcal{X}, \mathcal{A})$ with \mathcal{A} relatively Kähler. This generalises the discussion in [BHJ15, Section 2.5] by means of different methods.

Due to the projection formula equivalent test configurations have the same Donaldson-Futaki invariant. It can be computed on the dominating test configuration \mathcal{Y} for X .

K-semistability

The analogue of K-semistability in the context of cohomological test configurations is defined as follows:

Definition 3.2.13. *We say that (X, α) is K-semistable if $\text{DF}(\mathcal{X}, \mathcal{A}) \geq 0$ for all relatively Kähler test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α) .*

With the study of K-semistability in mind, we emphasise that the Donaldson-Futaki invariant $\text{DF}(\mathcal{Y}, \mathcal{L})$ (cf. [Wan12, Oda13]) depends only on $\bar{\mathcal{Y}}$ and $c_1(\bar{\mathcal{L}})$. The notion of cohomological test configuration emphasises this fact.

K-stability and uniform K-stability

In the case when X is a Kähler manifold with discrete automorphism group, two important stability notions related to the YTD conjecture are *K-stability* and *uniform K-stability*.

Definition 3.2.14. *We say that (X, α) is K-stable iff it is K-semistable and $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$ precisely if $(\mathcal{X}, \mathcal{A})$ is the trivial configuration.*

As a natural generalisation of the point of view in [BHJ15], it is natural to define our uniform cohomological notion of K-stability as follows:

Definition 3.2.15. *We say that (X, α) is uniformly K-stable iff there is a $\delta > 0$ such that*

$$\text{DF}(\mathcal{X}, \mathcal{A}) \geq \delta \text{J}^{\text{NA}}(\mathcal{X}, \mathcal{A})$$

for each relatively Kähler test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) . Here $\text{J}^{\text{NA}}(\mathcal{X}, \mathcal{A})$ denotes the non-Archimedean J-functional. Denoting by $\rho : \hat{\mathcal{X}} \rightarrow \mathcal{X}$ a resolution of singularities of \mathcal{X} that is also dominating $X \times \mathbb{P}^1$ via a morphism $\mu : \hat{\mathcal{X}} \rightarrow X \times \mathbb{P}^1$ it is given by the following intersection number

$$\text{J}^{\text{NA}}(\mathcal{X}, \mathcal{A}) := (\rho^* \mathcal{A} \cdot \mu^* p_1^* \alpha^n)_{\hat{\mathcal{X}}} - (\rho^* \mathcal{A}^{n+1})_{\hat{\mathcal{X}}}$$

computed on $\hat{\mathcal{X}}$. In other words, if \mathcal{X} is singular we define $\text{J}^{\text{NA}}(\mathcal{X}, \mathcal{A})$ as the quantity $\text{J}^{\text{NA}}(\hat{\mathcal{X}}, \rho^* \mathcal{A})$ where we note that $(\hat{\mathcal{X}}, \rho^* \mathcal{A})$ is smooth and dominating $X \times \mathbb{P}^1$, and moreover $(\hat{\mathcal{X}}, \rho^* \mathcal{A}) \sim (\mathcal{X}, \mathcal{A})$. Such a definition is natural in view of the projection formula.

If (X, L) is a polarized manifold the above definition corresponds precisely to the usual notion of L^1 -uniform K-stability (see [Sze15, BHJ15]) as well as the notion using the minimum norm [Der16b].

We remark that if $\text{Aut}_0(X) \neq \{0\}$, then neither of these stability notions can hold. Indeed, in the presence of automorphism on X we can always construct so called *product test configurations* (that are non-trivial) in the following way: Let $\lambda : \mathbb{C}^* \rightarrow \text{Aut}(X)$ be a 1-parameter subgroup and consider the induced diagonal \mathbb{C}^* -action σ on $X \times \mathbb{C}^*$, i.e. $\sigma(\tau) \cdot (x, z) := (\lambda(\tau)x, \tau z)$. Moreover, product configurations always have vanishing Donaldson-Futaki invariant, see 5. In general a candidate notion of stability (with respect to the YTD conjecture) is that of K-polystability.

K-polystability

We now attempt to extend the definition of K-stability to the case of arbitrary compact Kähler manifolds X , with no hypothesis on the automorphism group $\text{Aut}(X)$. It is not completely clear what should be the "right" definition of triviality of a test configurations in this more general setting (this is indeed an important open point to clarify, as mentioned e.g. in [LX14]). We begin by discussing the case of polarized manifolds (X, L) such that the first Chern class $c_1(L)$ admits a cscK metric. In this setting, there are several natural candidates for the notion of K-polystability: Indeed, one may then require that $\text{DF}(\mathcal{X}, \mathcal{L}) = 0$ if and only if any one of the following *a priori non-equivalent* conditions hold:

1. $(\mathcal{X}, \mathcal{L})$ is \mathbb{C}^* -equivariantly isomorphic to $(X, L) \times \mathbb{C}$.
2. $(\mathcal{X}, \mathcal{L})$ is isomorphic to $(X, L) \times \mathbb{C}$, but we do not require \mathbb{C}^* -equivariance.
3. The central fiber $(\mathcal{X}_0, \mathcal{L}_0)$ is isomorphic to (X, L) .

Each of the above notions have been seen in previous work. As long as (X, α) is a cscK manifold we in all three cases have $\text{DF}(\mathcal{X}, \mathcal{L}) = 0$. In this thesis we adopt the strongest of these definitions of K-polystability, corresponding to subpoint (1) above. Its natural adaptation to the case of Kähler manifolds leads to the following definition:

Definition 3.2.16. *We say that (X, α) is K-polystable iff it is K-semistable and $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$ precisely if $(\mathcal{X}, \mathcal{A})$ is a product configuration, i.e. $\mathcal{X}|_{\pi^{-1}(\mathbb{C})}$ is \mathbb{C}^* -equivariantly isomorphic to $X \times \mathbb{C}$ via an extension of the canonical isomorphism $\mathcal{X}|_{\pi^{-1}(\mathbb{C}^*)} \simeq X \times \mathbb{C}^*$.*

Note that asking that \mathcal{X} is isomorphic to $X \times \mathbb{P}^1$ would be incorrect. Indeed, it is well-known from the polarized setting that compactifications of product test configurations are not necessarily products (see Remark 3.1.13).

Remark 3.2.17. Finally, it seems that the list of stability notions is not complete without a notion of uniform K-polystability. However, defining such a notion involves several subtleties and technical problems in making sense of the norm " $\text{J}_G^{\text{NA}}(\mathcal{X}, \mathcal{A})$ ", which one would hope could be defined as the limit

$$\lim_{t \rightarrow +\infty} \inf_{g \in G} \frac{\text{J}(g \cdot \varphi_t)}{t}$$

if this limit exists (here (φ_t) is the geodesic ray associated to $(\mathcal{X}, \mathcal{A})$, see Section 4.1). When $G := \text{Aut}_0(X) \neq \{0\}$ this is expected but not known, due to the fact that convexity properties analogous to the ones for $t \mapsto \text{J}^{\text{NA}}(\varphi_t)$ have not yet been established in this setting. By analogy to the fact that the infimum of convex functions is convex, one would however expect it to be possible to prove such a convexity.

Properties of cohomological test configurations and K-stability

In order to further motivate the above definitions, we now introduce a number of related concepts and basic properties that will be useful in the sequel.

Test configurations characterised by \mathbb{R} -divisors

Recall that if $(\mathcal{X}, \mathcal{L})$ is an algebraic test configuration for a polarized manifold (X, L) that dominates $(X, L) \times \mathbb{C}$, then $\mathcal{L} = \mu^* p_1^* L + D$ for a unique \mathbb{Q} -Cartier divisor D supported on \mathcal{X}_0 , see [BHJ15]. Similarly, the following result characterises the classes \mathcal{A} associated to smooth and dominating cohomological test configurations, in terms of \mathbb{R} -divisors D supported on the central fiber \mathcal{X}_0 .

Proposition 3.2.18. *Let $(\mathcal{X}, \mathcal{A})$ be a smooth cohomological test configuration for (X, α) dominating $X \times \mathbb{P}^1$, with $\mu : \mathcal{X} \rightarrow X \times \mathbb{P}^1$ the corresponding canonical \mathbb{C}^* -equivariant bimeromorphic morphism. Then there exists a unique \mathbb{R} -divisor D supported on the central fiber \mathcal{X}_0 such that*

$$\mathcal{A} = \mu^* p_1^* \alpha + [D]$$

in $H^{1,1}(\mathcal{X}, \mathbb{R})$.

Proof. Let $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$. We begin by proving existence: By hypothesis \mathcal{X} dominates $X \times \mathbb{P}^1$ via the morphism $\mu : \mathcal{X} \rightarrow X \times \mathbb{P}^1$, such that the central fiber decomposes into the strict transform of $X \times \{0\}$ and the μ -exceptional divisor. We write $\mathcal{X}_0 = \sum_i b_i E_i$, with E_i irreducible. Denoting by $[E]$ the cohomology class of E and by $p_1 : X \times \mathbb{P}^1 \rightarrow X$ the projection on the first factor, we then have the following formula:

Lemma 3.2.19. *In the notation of Proposition 3.2.18, we have*

$$H^{1,1}(\mathcal{X}) = \mu^* p_1^* H^{1,1}(X) \oplus \bigoplus_i \mathbb{R}[E_i].$$

Proof. Let Θ be a closed $(1, 1)$ -form on \mathcal{X} . Then $T := \Theta - \mu^*(\mu_* \Theta)$ is a closed $(1, 1)$ -current of order 0 supported on $\cup_i E_i = \text{Exc}(\mu)$. By Demailly's second theorem of support (see [Dem12]) it follows that $T = \sum_i \lambda_i \delta_{E_i}$ and hence $[\Theta] = \mu^*(\mu_* [\Theta]) + \sum_i \lambda_i [E_i]$ in $H^{1,1}(\mathcal{X})$.

Since $H^{1,1}(\mathbb{P}^1)$ is generated by $[0]$, we have $p_2^* H^{1,1}(\mathbb{P}^1) = \mathbb{R}[X \times \{0\}]$. By the Künneth formula, it thus follows that $H^{1,1}(\mathcal{X}) = \mu^* p_1^* H^{1,1}(X) \oplus \mu^*(\mathbb{R}[X \times \{0\}]) \oplus \bigoplus_i \mathbb{R}[E_i]$. \square

We now finish the proof of Proposition 3.2.18: If we decompose \mathcal{A} accordingly we obtain $\mathcal{A} = \mu^* p_1^* \eta + [D]$ with $D := \mu^*(c[X \times \{0\}]) + \bigoplus_i b_i [E_i]$ and η a class in $H^{1,1}(X)$. The restrictions of \mathcal{A} and $\mu^* p_1^* \alpha$ to $\pi^{-1}(1) \simeq X \times \{1\} \simeq X$ are identified with α and η respectively. Since D is supported on \mathcal{X}_0 it follows that $\eta = \alpha$. We thus have the sought decomposition, proving existence.

As for the uniqueness, we let \mathcal{D}_0 be the set of \mathbb{R} -divisors D with support contained in the central fiber \mathcal{X}_0 . Consider the linear map

$$R : \mathcal{D}_0 \rightarrow H^{1,1}(\mathcal{X})$$

$$D \mapsto [D]$$

The desired uniqueness property is equivalent to injectivity of R . To this end, assume that $[D] = 0$ in $H^{1,1}(\mathcal{X})$. In particular $D|_{E_i} \equiv 0$ and it follows from a corollary of Zariski's lemma (see e.g. [BHPdV04, Lemma 8.2]) that $D = c\mathcal{X}_0$, with $c \in \mathbb{R}$. But, letting β be any Kähler form on X , we see from the projection formula that

$$(\mathcal{X}_0 \cdot (\mu^* p_1^* \beta)^n)_{\mathcal{X}} = ((X \times \{0\}) \cdot (p_1^* \beta)^n)_{X \times \mathbb{P}^1} = \beta^n = V > 0,$$

since V is the Kähler volume. Hence $[\mathcal{X}_0]$ is a non-zero class in $H^{1,1}(\mathcal{X})$. It follows that $c = 0$, thus $D = 0$ as well. We are done. \square

This gives a very convenient characterisation of smooth cohomological test configurations for (X, α) that dominate $X \times \mathbb{P}^1$.

In what follows, we will make use of resolution of singularities to associate a new test configuration $(\mathcal{X}', \mathcal{A}')$ for (X, α) to a given one, noting that this can be done *without changing the Donaldson-Futaki invariant*. Indeed, by Hironaka [Kol07, Theorem 45] (see also Section 3.2.2) there is a \mathbb{C}^* -equivariant proper bimeromorphic map $\mu : \mathcal{X}' \rightarrow \mathcal{X}$, with \mathcal{X}' smooth and such that \mathcal{X}'_0 has simple normal crossings. Moreover, μ is an isomorphism outside of the central fiber \mathcal{X}'_0 . Set $\mathcal{A}' := \mu^* \mathcal{A}$. By the projection formula we then have

$$\begin{aligned} \text{DF}(\mathcal{X}', \mathcal{A}') &= \frac{\bar{\mathcal{S}}}{n+1} V^{-1}((\mathcal{A}')^{n+1})_{\mathcal{X}'} + V^{-1}(K_{\mathcal{X}'/\mathbb{P}^1} \cdot (\mathcal{A}')^n)_{\mathcal{X}'} \\ &= \frac{\bar{\mathcal{S}}}{n+1} V^{-1}(\mathcal{A}^{n+1})_{\mathcal{X}} + V^{-1}(K_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{A}^n)_{\mathcal{X}} = \text{DF}(\mathcal{X}, \mathcal{A}). \end{aligned}$$

The following result states that it suffices to test K-semistability for a certain class of cohomological test configurations 'characterised by an \mathbb{R} -divisor' (in the above sense of Proposition 3.2.18).

Proposition 3.2.20. *Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be Kähler. Then (X, α) is K-semistable (resp. uniformly K-stable) if and only if $\text{DF}(\mathcal{X}, \mathcal{A}) \geq 0$ (resp. $\text{DF}(\mathcal{X}, \mathcal{A}) \geq \lambda \text{J}^{\text{NA}}(\mathcal{X}, \mathcal{A})$) for all smooth, relatively Kähler cohomological test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α) dominating $X \times \mathbb{P}^1$.*

Proof. Let $(\mathcal{X}, \mathcal{A})$ be any cohomological test configuration for (X, α) that is relatively Kähler. By Hironaka (see [Kol07]) there is a sequence of blow ups $\rho : \mathcal{X}' \rightarrow X \times \mathbb{P}^1$ with smooth \mathbb{C}^* -equivariant centers such that \mathcal{X}' simultaneously dominates \mathcal{X} and $X \times \mathbb{P}^1$ via morphisms μ and ρ respectively. Moreover, there is a divisor E on \mathcal{X}' that is ρ -exceptional and ρ -ample (and antieffective, i.e. $-E$ is effective). By Proposition 3.2.18, we have

$$\mu^* \mathcal{A} = \rho^* p_1^* \alpha + [D],$$

where D is an \mathbb{R} -divisor on \mathcal{X}' supported on \mathcal{X}'_0 . Note that the class $\mu^* \mathcal{A} \in H^{1,1}(\mathcal{X}', \mathbb{R})$ is relatively nef.

We proceed by perturbation; Since α is Kähler on X , we may pick a Kähler class η on \mathbb{P}^1 such that $p_1^* \alpha + p_2^* \eta =: \beta$ is Kähler on $X \times \mathbb{P}^1$. Since E is ρ -ample one may in turn fix an $\varepsilon \in (0, 1)$ sufficiently small such that $\rho^* \beta + \varepsilon[E]$ is Kähler on \mathcal{X}' . It follows that $\rho^* p_1^* \alpha + \varepsilon[E]$ is *relatively* Kähler (with respect to \mathbb{P}^1) on \mathcal{X}' . Thus $\rho^* p_1^* \alpha + [D] + \delta(\rho^* p_1^* \alpha + \varepsilon[E])$ is relatively Kähler for all $\delta \geq 0$ small enough. In turn, so is $\mathcal{A}'_\delta := \rho^* p_1^* \alpha + [D_\delta]$, where D_δ denotes the convex combination $D_\delta := \frac{1}{1+\delta} D + \frac{\delta \varepsilon}{1+\delta} E$. Assuming that the DF-invariant of a smooth and dominating test configuration is always non-negative, it follows from the projection formula and continuity of the Donaldson-Futaki invariant, that

$$0 \leq \text{DF}(\mathcal{X}', \mathcal{A}'_\delta) \longrightarrow \text{DF}(\mathcal{X}', \mu^* \mathcal{A}) = \text{DF}(\mathcal{X}, \mathcal{A}).$$

as $\delta \rightarrow 0$. The other direction holds by definition, so this proves the first part of the lemma.

The same assertion holds for uniform K-stability. This is seen by repeating the above argument applied to $\text{DF}(\mathcal{X}, \mathcal{A}) - \lambda \text{J}^{\text{NA}}(\mathcal{X}, \mathcal{A})$ instead, where $\lambda > 0$ is a positive constant. Indeed, in the same way as above, by the projection formula and continuity of DF and J^{NA} , we have

$$0 \leq \text{DF}(\mathcal{X}', \mathcal{A}'_\delta) - \lambda \text{J}^{\text{NA}}(\mathcal{X}', \mathcal{A}'_\delta) \longrightarrow \text{DF}(\mathcal{X}, \mathcal{A}) - \lambda \text{J}^{\text{NA}}(\mathcal{X}, \mathcal{A})$$

as $\delta \rightarrow 0$. □

Remark 3.2.21. *With respect to testing K-semistability one can in fact restrict the class of test configurations that need to be considered even further, as explained in Section 4.3.*

Comparison of algebraic and cohomological K-stability for polarized manifolds

It is useful to compare cohomological- and algebraic K-semistability in the special case of a polarized manifold (X, L) . The foundation for the following results is the intersection theoretic characterization of the Donaldson-Futaki invariant, due to Wang [Wan12] and Odaka [Oda13].

In such a comparison one may note that there are more cohomological test configurations than there are algebraic ones. However, we show in this section that the corresponding stability notions are *equivalent*.

Remark 3.2.22. *Due to [LX14] we also note that the stability notion of Tian [Tia97] is equivalent to the above algebraic K-stability of Donaldson [Don02] (at least in the case of Fano manifolds).*

Cohomological K-semistability for polarized manifolds

Proposition 3.2.23. *Let (X, L) be a polarized manifold and let $\alpha := c_1(L)$. Then $(X, c_1(L))$ is cohomologically K-semistable if and only if (X, L) is algebraically K-semistable.*

Proof. Suppose that $(X, c_1(L))$ is cohomologically K-semistable. If $(\mathcal{X}, \mathcal{L})$ is an ample test configuration for (X, L) , let $\mathcal{A} := c_1(\bar{\mathcal{L}})$. By the intersection theoretic characterisation of the Donaldson-Futaki invariant (Definition 3.2.9) we then have $\text{DF}(\mathcal{X}, \mathcal{A}) = \text{DF}(\mathcal{X}, \mathcal{L}) \geq 0$. Hence (X, L) is algebraically K-semistable.

Conversely, suppose that (X, L) is algebraically K-semistable and let $(\mathcal{X}, \mathcal{A})$ be a cohomological test configuration for (X, α) . By Lemma 3.2.20 we may assume that $(\mathcal{X}, \mathcal{A})$ is a smooth, relatively Kähler test configuration for (X, α) dominating $X \times \mathbb{P}^1$, with $\mu : \mathcal{X} \rightarrow X \times \mathbb{P}^1$ the corresponding \mathbb{C}^* -equivariant bimeromorphic morphism. By Proposition 3.2.18 we further have $\mathcal{A} = \mu^* p_1^* c_1(L) + [D]$ for a uniquely determined \mathbb{R} -divisor D on \mathcal{X} supported on the central fiber \mathcal{X}_0 . Since \mathcal{A} is relatively Kähler, there is a Kähler form η on \mathbb{P}^1 such that $\mathcal{A} + \pi^* \eta$ is Kähler on \mathcal{X} . Approximating the coefficients of the divisor D by a sequence of rationals, we write $D = \lim D_j$ for \mathbb{Q} -divisors D_j on \mathcal{X} , all supported on \mathcal{X}_0 . As $j \rightarrow +\infty$, we then have

$$\mu^* p_1^* c_1(L) + [D_j] + \pi^* \eta \longrightarrow \mathcal{A} + \pi^* \eta,$$

which is a Kähler form on \mathcal{X} . Since the Kähler cone is open, it follows that $\mu^* p_1^* c_1(L) + [D_j] + \pi^* \eta$ is also Kähler for all j large enough.

Now let $\mathcal{L}_j := \mu^* p_1^* L + D_j$. By the above, \mathcal{L}_j is a relatively ample \mathbb{Q} -line bundle over \mathcal{X} and $c_1(\mathcal{L}_j) \rightarrow \mathcal{A}$. We thus conclude that $(\mathcal{X}, \mathcal{L}_j)$ (for all j large enough) is an ample test configuration for (X, L) . Hence

$$0 \leq \text{DF}(\mathcal{X}, \mathcal{L}_j) \longrightarrow \text{DF}(\mathcal{X}, \mathcal{A}).$$

as $j \rightarrow +\infty$, which is what we wanted to prove. \square

Cohomological K-polystability and uniform K-stability for polarized manifolds

Proposition 3.2.24. *Let (X, L) be a polarized manifold and let $\alpha := c_1(L)$. Then $(X, c_1(L))$ is cohomologically uniformly K-stable if and only if (X, L) is algebraically uniformly K-stable.*

Proof. The proof is an immediate adaptation of Proposition 3.2.23: Indeed, first suppose that $(X, c_1(L))$ is cohomologically uniformly K-stable. If $(\mathcal{X}, \mathcal{L})$ is an ample test configuration for (X, L) , let $\mathcal{A} := c_1(\bar{\mathcal{L}})$. By the intersection theoretic characterisation of the Donaldson-Futaki invariant and of the non-Archimedean J-functional J^{NA} we have $(\text{DF} - \delta J^{\text{NA}})(\mathcal{X}, \mathcal{A}) = (\text{DF} - \delta J^{\text{NA}})(\mathcal{X}, \mathcal{L}) \geq 0$, for some $\delta > 0$ given by the definition of (X, L) being uniformly K-stable. Moreover, the condition on the vanishing of the norm J^{NA} does not bare on the polarisation. Hence (X, L) is algebraically uniformly K-stable as well.

Conversely, suppose that (X, L) is algebraically uniformly K-stable and let $(\mathcal{X}, \mathcal{A})$ be a cohomological test configuration for (X, α) . By Lemma 3.2.20 we may assume that $(\mathcal{X}, \mathcal{A})$ is a smooth, relatively Kähler test configuration for (X, α) dominating $X \times \mathbb{P}^1$, with $\mu : \mathcal{X} \rightarrow X \times \mathbb{P}^1$ the corresponding \mathbb{C}^* -equivariant bimeromorphic morphism. Precisely as in the proof of Proposition 3.2.23 above we may then construct a sequence \mathcal{L}_j , $j \in \mathbb{N}$, of relatively ample \mathbb{Q} -line bundles over \mathcal{X} such that $c_1(\mathcal{L}_j) \rightarrow \mathcal{A}$ and such that $(\mathcal{X}, \mathcal{L}_j)$ (for all j large enough) are ample test configurations for (X, L) . By an identical continuity argument we then have

$$0 \leq (\text{DF} - \delta J^{\text{NA}})(\mathcal{X}, \mathcal{L}_j) \longrightarrow (\text{DF} - \delta J^{\text{NA}})(\mathcal{X}, \mathcal{A}),$$

for $j \rightarrow +\infty$. In exactly the same way as above this concludes the proof. \square

At this point one may remark that the uniform K-stability notion is in this sense much more similar to K-semistability than the notions of K-stability and K-polystability. A main difference is that in the latter case we cannot (at least this is not known) restrict attention to smooth and dominating test configurations as in Lemma 3.2.20. Therefore the above proof breaks down. However, one direction of the above proof of proposition 3.2.23 of course still holds:

Proposition 3.2.25. *Let (X, L) be a polarized manifold and let $\alpha := c_1(L)$. If $(X, c_1(L))$ is cohomologically K-polystable, then (X, L) is algebraically K-polystable.*

The K-stability is covered as a particular case of the above, whenever the automorphism group of X is discrete. On the other hand, if $\text{Aut}_0(X) \neq \{0\}$ no manifold can be K-stable, since there are then non-trivial product test configurations (whose Donaldson-Futaki invariant always vanishes under the cscK assumption, see Proposition 5.4.26).

Question 3.2.26. *It is an open question whether the converse to Proposition 3.2.25 holds.*

In the spirit of the examples of [ACGTF08] (albeit that this is in the setting of extremal metrics) it is not a given whether we should in fact expect cohomological and algebraic K-polystability to be equivalent notions for polarized manifolds. This would be an interesting question to clarify.

Chapter 4

Main results 1: Asymptotics for energy functionals in Kähler geometry

Energy functional asymptotics play an important role in the theory of K-stability for polarized manifolds, following [Tia00a, Don85, Rub14, PRS08, Ber16, BHJ15, BHJ16] and many other authors. In the aforementioned papers the approach taken uses either Deligne pairings or Bott-Chern forms. In this section we develop an analogous set of results in the setting of K-stability for *arbitrary* (possibly non-projective) Kähler manifolds, and prove Theorems A and B. By contrast to the previously cited works, the methods used are of a much more differential-geometric flavour. Moreover, the introduction of 'Deligne functionals' (Definition 2.4.6) gives rise to a formalism that brings together, in a more general setting, both the Deligne pairing and the Bott-Chern form approach to K-stability and energy functional asymptotics (see Example 2.4.9).

4.1 Compatibility of subgeodesic rays and cohomological test configurations

Let X be a compact Kähler manifold of dimension n and let θ_i , $0 \leq i \leq n$, be closed $(1, 1)$ -forms on X . Let $\alpha_i := [\theta_i] \in H^{1,1}(X, \mathbb{R})$ be the corresponding cohomology classes. In this section we aim to prove Theorem A. In other words, we establish a Kempf-Ness type formula which connects the asymptotic slope of the multivariate energy functional $\langle \varphi_0^t, \dots, \varphi_n^t \rangle_{(\theta_0, \dots, \theta_n)}$ (see Definition 2.4) with a certain intersection number depending on given *cohomological* test configurations $(\mathcal{X}_i, \mathcal{A}_i)$ for (X, α_i) . In order for such a result to hold, we need to ask that the rays $(\varphi_i^t)_{t \geq 0}$ are *compatible* with $(\mathcal{X}_i, \mathcal{A}_i)$ in a sense that has to do with extension across the central fiber, see Section 4.1 below.

Intersection numbers

Now recall the notion of test configurations being equivalent if they can be simultaneously dominated by a third test configuration, see Section 3.2 for the precise definition. For what follows it is important to note the following lemma, which is a direct consequence of equivariant

resolution of singularities in the analytic category, see [Hir74, AHV75, AHV77, BM97, Wlo09] and also [Kol07, Theorem 45] for references:

Lemma 4.1.1. *The intersection*

$$\bigcap_{i=0}^n (\mathcal{X}_i, \mathcal{A}_i) / \sim$$

of the equivalence classes $(\mathcal{X}_i, \mathcal{A}_i) / \sim$ of the $(\mathcal{X}_i, \mathcal{A}_i)$ is non-empty and contains a test configuration \mathcal{X} for X which is smooth and dominates $X \times \mathbb{P}^1$. This setup comes with canonical \mathbb{C}^ -equivariant bimeromorphic morphisms $\rho_i : \mathcal{X} \rightarrow \mathcal{X}_i$ respectively.*

We then define intersection numbers on the collection $(\mathcal{X}_i, \mathcal{A}_i)$, $i = 0, 1, \dots, n$ by means of pulling back to the smooth and dominating common representative:

Definition 4.1.2. *We define the intersection number*

$$(\mathcal{A}_0 \cdots \mathcal{A}_n) := (\rho_0^* \mathcal{A}_0 \cdots \rho_n^* \mathcal{A}_n)_{\mathcal{X}}$$

by means of pulling back the respective cohomology classes to \mathcal{X} .

Up to desingularising we can and we will in this section consider only smooth cohomological test configurations $(\mathcal{X}_i, \mathcal{A}_i)$ for (X, α_i) dominating $X \times \mathbb{P}^1$, with $\mu_i : \mathcal{X}_i \rightarrow X \times \mathbb{P}^1$ the corresponding \mathbb{C}^* -equivariant bimeromorphic morphisms respectively. It should however be noted that it is sometimes necessary to work directly with singular test configurations (e.g. when studying K-polystability). Certain parts of the following exposition can be defined in such a singular setting as well, as made precise in Section 5.4.

Compatibility of rays and cohomological test configurations

Let $(\mathcal{X}, \mathcal{A})$ be a smooth (cohomological) test configuration for (X, α) dominating $X \times \mathbb{P}^1$, with $\mu : \mathcal{X} \rightarrow X \times \mathbb{P}^1$ the corresponding canonical \mathbb{C}^* -equivariant bimeromorphic morphism. We then have

$$\mathcal{A} = \mu^* p_1^* \alpha + [D]$$

for a unique \mathbb{R} -divisor D supported on \mathcal{X}_0 , with $p_1 : X \times \mathbb{P}^1 \rightarrow X$ denoting the first projection, cf. Proposition 3.2.18. Fix a choice of S^1 -invariant function 'Green function' ψ_D for D , so that $\delta_D = \theta_D + dd^c \psi_D$, with θ_D a smooth S^1 -invariant closed $(1, 1)$ -form on \mathcal{X} . Locally, we thus have

$$\psi_D = \sum_j a_j \log |f_j| \pmod{\mathcal{C}^\infty},$$

where (writing $D := \sum_j a_j D_j$ for the decomposition of D into irreducible components) the f_j are local defining equations for the D_j respectively. In particular, the choice of ψ_D is uniquely determined modulo a smooth function.

We now introduce a number of *compatibility conditions*, the point of which is to establish some natural situations in which the asymptotic formula of Theorem A holds. Indeed, the aforementioned result aims to relate algebraic (intersection theoretic) quantities to asymptotic

slopes of Deligne functionals (e.g. E or J) along certain rays, but of course, such a formula can not hold for *any* such ray.

Technically, recall that a ray $(\varphi_t)_{t \geq 0}$ on X is in correspondence with an S^1 -invariant functions Φ on $X \times \bar{\Delta}^*$, via $\varphi_t(x) = \Phi(x, e^{-t+is})$. We then relate Φ to the given test configuration by pulling it back to the total space \mathcal{X} via the isomorphism $\mu : \pi^{-1}(\bar{\Delta}^*) \rightarrow X \times \bar{\Delta}^*$ (we here use the same notation for the restriction of $\mu : \mathcal{X} \dashrightarrow X \times \mathbb{P}^1$). Viewing $\bar{\Delta}^*$ as embedded in \mathbb{P}^1 this yields a function $\Phi \circ \mu$ on $\pi^{-1}(\bar{\Delta}^*)$ that can be extended to all of $\mathcal{X} \setminus \mathcal{X}_0$. The proof of Theorem A will then show that it is important to extend the function $\Phi \circ \mu$ on $\mathcal{X} \setminus \mathcal{X}_0$ also across the central fiber \mathcal{X}_0 . As an answer to the question of under what conditions such an extension exists, and with what regularity, we introduce the notions of \mathcal{C}^∞ -, L^∞ - and $\mathcal{C}^{1,\bar{1}}$ -compatibility between the ray $(\varphi_t)_{t \geq 0}$ and the test configuration $(\mathcal{X}, \mathcal{A})$. The purpose of introducing more than one version of compatibility is that we will distinguish between the following two situations of interest to us:

1. Smooth but not necessarily *subgeodesic* rays (cf. Definition 2.4.2) (φ_t) that are \mathcal{C}^∞ -compatible with the smooth test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) , dominating $X \times \mathbb{P}^1$. Here we can consider $\alpha = [\theta] \in H^{1,1}(X, \mathbb{R})$ for any closed $(1,1)$ -form θ on X .
2. Locally bounded subgeodesic rays (φ_t) that are L^∞ -compatible or (more restrictively) $\mathcal{C}^{1,\bar{1}}$ -compatible with the given smooth and *relatively Kähler* test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) , dominating $X \times \mathbb{P}^1$. Here we thus suppose that α is a Kähler class.

Theorem A has valid formulations in both these situations, as pointed out in Remark 4.2.4. The second situation is interesting notably with weak geodesic rays in mind, cf. Section 4.1. In certain arguments of Section 5 it is (for regularity reasons) a key technical detail that we may work with (say smooth) compatible subgeodesics rather than the weak geodesic rays, who are merely $\mathcal{C}^{1,\bar{1}}$ -regular.

Definition of \mathcal{C}^∞ -compatible and L^∞ -compatible rays

We first introduce the notion of smooth (not necessarily subgeodesic) rays that are \mathcal{C}^∞ -compatible with the given test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) . Recall the set up of $(\Omega, \Phi, \mu, \psi_D, \theta_D)$ described above.

Definition 4.1.3. *Let $(\varphi_t)_{t \geq 0}$ be a smooth ray in $\mathcal{C}^\infty(X)$, and denote by Φ the corresponding smooth S^1 -invariant function on $X \times \bar{\Delta}^*$. We say that (φ_t) and $(\mathcal{X}, \mathcal{A})$ are \mathcal{C}^∞ -compatible if $\Phi \circ \mu + \psi_D$ extends smoothly across \mathcal{X}_0 .*

This compatibility condition is indeed independent of the choice of ψ_D , as the latter is well-defined modulo a smooth function. In the case of a polarized manifold (X, L) with an (algebraic) test configuration $(\mathcal{X}, \mathcal{L})$ this condition amounts to demanding that the metric on \mathcal{L} associated to the ray $(\varphi_t)_{t \geq 0}$ extends smoothly across the central fiber.

Example 4.1.4. (cf. [DR16]) *As a useful 'model example' to keep in mind, let Ω be a smooth S^1 -invariant representative of \mathcal{A} and denote the restrictions $\Omega|_{\mathcal{X}_\tau} =: \Omega_\tau$. Note that Ω_τ and Ω_1 are cohomologous for each $\tau \in \mathbb{P}^1 \setminus \{0\}$, and hence we may define a ray $(\varphi_t)_{t \geq 0}$ on X , \mathcal{C}^∞ -compatible with $(\mathcal{X}, \mathcal{A})$, by the following relation $\lambda(\tau)^*\Omega_\tau - \Omega_1 = dd^c\varphi_\tau$, where $t = -\log|\tau|$ and $\lambda(\tau) : \mathcal{X}_\tau \rightarrow \mathcal{X}_1 \simeq X$ is the isomorphism induced by the \mathbb{C}^* -action λ on \mathcal{X} .*

We further establish existence of a smooth \mathcal{C}^∞ -compatible *subgeodesic* ray associated to a given relatively Kähler test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) . In particular, the proof gives a simple construction of such rays.

Lemma 4.1.5. *If \mathcal{A} is relatively Kähler, then $(\mathcal{X}, \mathcal{A})$ is \mathcal{C}^∞ -compatible with some smooth subgeodesic ray (φ_t) .*

Proof. Since \mathcal{A} is relatively Kähler, it admits a smooth S^1 -invariant representative Ω with $\Omega + \pi^*\eta > 0$ for some S^1 -invariant Kähler form η on \mathbb{P}^1 . By the dd^c -lemma on \mathcal{X} , we have $\Omega = \mu^*p_1^*\omega + \theta_D + dd^cu$ for some S^1 -invariant $u \in \mathcal{C}^\infty(X)$, which may be assumed to be 0 after replacing ψ_D with $\psi_D - u$. As a result, we get

$$\Omega = \mu^*p_1^*\omega + \delta_D - dd^c\psi_D.$$

We may also choose a smooth S^1 -invariant function f on a neighborhood U of $\bar{\Delta}$ such that $\eta|_U = dd^cf$, and a constant $A \gg 1$ such that $D \leq A\mathcal{X}_0$. Using the Lelong-Poincaré formula $\delta_{\mathcal{X}_0} = dd^c \log|\tau|$ we get

$$0 < \Omega + \pi^*\eta = \mu^*p_1^*\omega + \delta_{D-A\mathcal{X}_0} + dd^c(f \circ \pi + A \log|\tau| - \psi_D)$$

on $\pi^{-1}(U)$. Since $D - A\mathcal{X}_0 \leq 0$, it follows that $f \circ \pi + A \log|\tau| - \psi_D$ is $\mu^*p_1^*\omega$ -psh, and hence descends to an S^1 -invariant $p_1^*\omega$ -psh function $\tilde{\Phi}$ on $X \times U$ (because the fibers of μ are compact and connected, by Zariski's main theorem). The ray associated with the S^1 -invariant function $\Phi := \tilde{\Phi} - A \log|\tau|$ has the desired properties. \square

Remark 4.1.6. *Working with such smooth subgeodesic rays is often enough for our purposes, and this avoids relying on the more involved theory for degenerate complex Monge-Ampère equations for manifolds with boundary, needed in the below construction of geodesic rays associated to a test configuration $(\mathcal{X}, \mathcal{A})$.*

The geodesic ray associated to a cohomological test configuration

Relying on theory for degenerate complex Monge-Ampère equations on manifolds with boundary we now introduce a notion of weak geodesic rays associated to any given *smooth* cohomological test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) dominating $X \times \mathbb{P}^1$. As in the polarized setting it is shown to have regularity $\mathcal{C}^{1, \bar{1}}$ (i.e. bounded complex Laplacian). The proof of the latter comes from regularity of the solution to a certain degenerate complex Monge-Ampère equation, and there are seemingly no new phenomena in the transcendental setting with regards to regularity.

We also introduce a more flexible notion of subgeodesic rays that are so called $\mathcal{C}^{1,\bar{1}}$ -compatible or L^∞ -compatible with a given cohomological test configuration. The weak geodesic ray would then be a particular example of a ray satisfying the $\mathcal{C}^{1,\bar{1}}$ -compatibility condition.

Remark 4.1.7. *The case of test configurations with singular central fiber is considered in Section 5.*

Homogeneous complex Monge-Ampère equations on manifolds with boundary

In order to discuss weak geodesic rays associated to cohomological test configurations we very briefly recall the necessary theory for homogeneous complex Monge-Ampère equations on manifolds with boundary, on which the construction of such rays relies. In particular, we will make use of the following precise result stated in [Bou12], building on combinations of results and techniques from a large number authors [Yau78, CKNS85, Gua98, Che00b, Blo12, PS10]:

Theorem 4.1.8. *Let (M, η) be an m -dimensional compact Kähler manifold with boundary. Let $\varphi_0 \in C^\infty(\partial M)$ and assume that φ_0 admits a smooth η -psh extension $\tilde{\varphi}_0 \in C^\infty(M)$. Then we have*

1. *There exists a unique Lipschitz continuous η -psh function ψ such that*

$$(\star) \begin{cases} (\eta + dd^c \psi)^m = 0 & \text{on Int}(M) \\ \psi|_{\partial M} = \varphi_0 \end{cases}$$

2. *If moreover ∂M is weakly pseudoconcave then $dd^c \psi$ has L^∞_{loc} coefficients.*

We refer the reader to [Bou12] for the necessary background as well as for a proof of the above statement.

Definition of $\mathcal{C}^{1,\bar{1}}$ -compatible rays and construction of the weak geodesic ray associated to a cohomological test configuration

As before, let (X, α) be compact Kähler and let $(\mathcal{X}, \mathcal{A})$ be a smooth, relatively Kähler cohomological test configuration for (X, α) that dominates $X \times \mathbb{P}^1$. With this setup, it is interesting and sometimes necessary to consider not only C^∞ -compatible subgeodesic rays, but also the following weaker compatibility conditions referred to as L^∞ -compatibility and $\mathcal{C}^{1,\bar{1}}$ -compatibility respectively:

Definition 4.1.9. *Let $(\varphi_t)_{t \geq 0}$ be a locally bounded subgeodesic ray, and denote by Φ the corresponding S^1 -invariant locally bounded $p_1^* \omega$ -psh function on $X \times \bar{\Delta}^*$. We say that $(\varphi_t)_{t \geq 0}$ and $(\mathcal{X}, \mathcal{A})$ are L^∞ -compatible if $\Phi \circ \mu + \psi_D$ is locally bounded near \mathcal{X}_0 , resp. $\mathcal{C}^{1,\bar{1}}$ -compatible if $\Phi \circ \mu + \psi_D$ is of class $\mathcal{C}^{1,\bar{1}}$ on $\pi^{-1}(\Delta)$.*

It can be seen that the $\mathcal{C}^{1,\bar{1}}$ -compatibility condition is always satisfied for weak geodesic rays associated to $(\mathcal{X}, \mathcal{A})$. In particular, for any given test configuration, $\mathcal{C}^{1,\bar{1}}$ -compatible subgeodesics

always exist. This is the content of the following construction of the weak geodesic ray associated to $(\mathcal{X}, \mathcal{A})$, which is a direct consequence of the theory for degenerate Monge-Ampère equations on manifolds with boundary.

Lemma 4.1.10. *With the situation (2) in mind, let $(\mathcal{X}, \mathcal{A})$ be a smooth, relatively Kähler cohomological test configuration of (X, α) dominating $X \times \mathbb{P}^1$. Then $(\mathcal{X}, \mathcal{A})$ is $\mathcal{C}^{1,\bar{1}}$ -compatible with some weak geodesic ray $(\varphi_t)_{t \geq 0}$.*

Remark 4.1.11. The proof will show that the constructed ray is actually unique, up to having fixed some boundary data as made precise below.

Proof of Lemma 4.1.10. Let $M := \pi^{-1}(\bar{\Delta}) \subset \mathcal{X}$. It is a compact complex manifold with boundary $\partial M = \pi^{-1}(S^1)$.

Let D, θ_D, ψ_D and Ω be as above. Since $\mathcal{A} = [\Omega]$ is relatively Kähler there is an $[\eta] \in H^{1,1}(\mathbb{P}^1)$ such that $\mathcal{A} + \pi^*[\eta]$ is Kähler on \mathcal{X} . We may then write $\tilde{\Omega} := \Omega + \pi^*\eta + dd^c g$, where $\tilde{\Omega}$ is a Kähler form on \mathcal{X} and $g \in \mathcal{C}^\infty(\mathcal{X})$. In a neighbourhood of $\bar{\Delta}$ the form η is further dd^c -exact (cf. the Poincaré lemma and [Dem12, Lemma 8.6]) and so we write $\eta = dd^c(g' \circ \pi)$ for a smooth function $g' \circ \pi$ on $\bar{\Delta}$. In order to construct the sought geodesic ray, we consider the following degenerate complex Monge-Ampère equation;

$$(\star) \begin{cases} (\tilde{\Omega} + dd^c \tilde{\Psi})^{n+1} = 0 & \text{on Int}(M) \\ \tilde{\Psi}|_{\partial M} = \varphi_0 + \psi_D - g' - g \end{cases}$$

Since $\tilde{\Omega}$ is Kähler it follows from Theorem 4.1.8 that there exists a unique $\tilde{\Omega}$ -psh function $\tilde{\Psi}$ solving (\star) and that is moreover of class $\mathcal{C}^{1,\bar{1}}$, i.e. such that $dd^c \tilde{\Psi} \in L_{loc}^\infty(\mathcal{X})$. We then define a $p_1^* \omega$ -psh function on $X \times \bar{\Delta}^* \hookrightarrow \mathcal{X}$ by

$$\mu^* \Phi = \tilde{\Psi} - \psi_D + g' + g.$$

It is S^1 -invariant and we then have

$$\mu^*(p_1^* \omega + dd^c \Phi) = \tilde{\Omega} + dd^c \tilde{\Psi}$$

on $\pi^{-1}(\bar{\Delta}^*)$. In particular, Φ defines a weak geodesic ray $(\varphi_t)_{t \geq 0}$ on X . Moreover, the current

$$\mu^* dd^c \Phi + \delta_D = dd^c \tilde{\Psi} + \delta_D - dd^c \psi_D = dd^c \tilde{\Psi} + \theta_D$$

has locally bounded coefficients. Indeed, $dd^c \tilde{\Psi} \in L_{loc}^\infty$ (as solution of (\star)) and θ_D is a smooth $(1,1)$ -form on $\bar{\mathcal{X}}$. The constructed ray is thus $\mathcal{C}^{1,\bar{1}}$ -compatible with $(\mathcal{X}, \mathcal{A})$. \square

4.2 Asymptotics for Deligne functionals. Proof of Theorem A.

We now note that in order to compute the asymptotic slope of the Monge-Ampère energy functional E or its multivariate analogues $\langle \cdot, \dots, \cdot \rangle_{(\omega_0, \dots, \omega_n)}$ we may in fact replace L^∞ -compatible

rays (φ^t) with $(\mathcal{X}, \mathcal{A})$ by C^∞ -compatible ones. Indeed, note that any two locally bounded subgeodesic rays (φ_t) and (φ'_t) L^∞ -compatible with $(\mathcal{X}, \mathcal{A})$ satisfy $\Phi \circ \mu = \Phi' \circ \mu + O(1)$ near \mathcal{X}_0 , and hence $\varphi_t = \varphi'_t + O(1)$ as $t \rightarrow +\infty$. This leads to the following observation, which will be useful in view of proving Theorems A and B.

Lemma 4.2.1. *Let $(\mathcal{X}_i, \mathcal{A}_i)$ be smooth, relatively Kähler cohomological test configurations for (X, α_i) respectively, dominating $X \times \mathbb{P}^1$. Let $(\varphi_i^t)_{t \geq 0}$ and $(\varphi_i'^t)_{t \geq 0}$ be locally bounded subgeodesics that are L^∞ -compatible with $(\mathcal{X}_i, \mathcal{A}_i)$ respectively. Then*

$$\langle \varphi_0^t, \varphi_1^t, \dots, \varphi_n^t \rangle_{(\omega_0, \dots, \omega_n)} = \langle \varphi_0'^t, \varphi_1'^t, \dots, \varphi_n'^t \rangle_{(\omega_0, \dots, \omega_n)} + O(1)$$

as $t \rightarrow +\infty$.

Proof. For each i , $0 \leq i \leq n$, we have $\varphi_i^t = \varphi_i'^t + O(1)$ as $t \rightarrow +\infty$. Recall that the mass of the Bedford-Taylor product $\bigwedge (\omega_i + dd^c \varphi_i^t)$ is computed in cohomology, thus independent of t . Hence, the quantity

$$\begin{aligned} & \langle \varphi_0^t, \varphi_1^t, \dots, \varphi_n^t \rangle_{(\omega_0, \dots, \omega_n)} - \langle \varphi_0'^t, \varphi_1'^t, \dots, \varphi_n'^t \rangle_{(\omega_0, \dots, \omega_n)} \\ &= \int_X (\varphi_0^t - \varphi_0'^t)(\omega_1 + dd^c \varphi_1^t) \wedge \dots \wedge (\omega_n + dd^c \varphi_n^t) \end{aligned}$$

is bounded as $t \rightarrow +\infty$. By symmetry, the argument may be repeated for the remaining i , yielding the result. \square

Proof of the asymptotic formula

With the above formalism in place, we are ready to formulate the main result of this section (Theorem A of the introduction). It constitutes the main contribution towards establishing Theorem A, and may be viewed as a transcendental analogue of Lemma 4.3 in [BHJ16]. We here formulate and prove the theorem in the 'smooth but not necessarily Kähler' setting (see Section 4.1, situation (1)). However, one should note that there is also a valid formulation for L^∞ -compatible subgeodesics, as pointed out in Remark 4.2.4.

Theorem 4.2.2. *Let X be a compact Kähler manifold of dimension n and let θ_i , $0 \leq i \leq n$, be closed $(1, 1)$ -forms on X . Set $\alpha_i := [\theta_i] \in H^{1,1}(X, \mathbb{R})$. Consider smooth cohomological test configurations $(\mathcal{X}_i, \mathcal{A}_i)$ for (X, α_i) dominating $X \times \mathbb{P}^1$. For each collection of smooth rays $(\varphi_i^t)_{t \geq 0}$ C^∞ -compatible with $(\mathcal{X}_i, \mathcal{A}_i)$ respectively, the asymptotic slope of the multivariate energy functional $\langle \cdot, \dots, \cdot \rangle := \langle \cdot, \dots, \cdot \rangle_{(\theta_0, \dots, \theta_n)}$ is well-defined and satisfies*

$$\frac{\langle \varphi_0^t, \dots, \varphi_n^t \rangle}{t} \longrightarrow (\mathcal{A}_0 \cdots \mathcal{A}_n)$$

as $t \rightarrow +\infty$. See 4.1.2 for the definition of the above intersection number in case the \mathcal{X}_i are not all equal.

Proof. Fix any smooth S^1 -invariant $(1,1)$ -forms Ω_i on \mathcal{X}_i such that $[\Omega_i] = \mathcal{A}_i$ in $H^{1,1}(\mathcal{X}_i, \mathbb{R})$. Let $(\varphi_i^t)_{t \geq 0}$ be smooth and \mathcal{C}^∞ -compatible with $(\mathcal{X}_i, \mathcal{A}_i)$ respectively. Let \mathcal{X} be a smooth test configuration that simultaneously dominates the \mathcal{X}_i . By pulling back to \mathcal{X} we can assume that the \mathcal{X}_i are all equal (note that the notion of being \mathcal{C}^∞ -compatible is preserved under this pull-back).

In the notation of Section 4.1, the functions $\Phi_i \circ \mu + \psi_D$ are then smooth on the manifold with boundary $M := \pi^{-1}(\bar{\Delta})$, and may thus be written as the restriction of smooth S^1 -invariant functions Ψ_i on \mathcal{X} respectively.

Using the \mathbb{C}^* -equivariant isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{P}^1 \setminus \{0\})$ we view $(\Psi_i - \psi_D)|_{\mathcal{X}_\tau}$ as a function $\varphi_i^\tau \in \mathcal{C}^\infty(X)$. By Proposition 2.4.10 we then have

Lemma 4.2.3. *Over $\mathbb{P}^1 \setminus \{0\}$ we have*

$$dd_\tau^c \langle \varphi_0^t, \dots, \varphi_n^t \rangle = \pi_* \left(\bigwedge_i (\Omega_i + dd^c \Psi_i) \right).$$

Proof. Write Φ_i for the S^1 -invariant function on $X \times \bar{\Delta}^*$ corresponding to φ_i via the relation $\Phi_i(x, e^{-t+is}) = \varphi_{i,t}(x)$. By Proposition 2.4.10 we then have

$$dd_\tau^c \langle \varphi_0^t, \dots, \varphi_n^t \rangle = \pi_* \left(\bigwedge_i (p_1^* \theta_i + dd^c \Phi_i) \right).$$

interpreted in the weak sense of currents (see the proof of Proposition 2.4.10). Note that away from $\tau = 0$, the map μ is a biholomorphism and $\delta_D = 0$ (recalling that the \mathbb{R} -divisor D is supported on \mathcal{X}_0). Hence, away from $\tau = 0$ we may identify $p_1^* \theta_i + dd^c \Phi_i$ with $\Omega_i + dd^c \Psi_i$ via μ , i.e. $\mu^*(p_1^* \theta_i + dd^c \Phi_i) = \Omega_i + dd^c \Psi_i$. The result follows. \square

Denoting by $u(\tau) := \langle \varphi_0^\tau, \dots, \varphi_n^\tau \rangle$ the Green-Riesz formula then yields

$$\begin{aligned} \frac{d}{dt} \Big|_{t=-\log \varepsilon} u(\tau) &= \int_{\mathbb{P}^1 \setminus \Delta_\varepsilon} dd_\tau^c u(\tau) = \\ &= \int_{\pi^{-1}(\mathbb{P}^1 \setminus \Delta_\varepsilon)} \bigwedge_i (\Omega_i + dd^c \Psi_i), \end{aligned}$$

which converges to $(\mathcal{A}_0 \cdots \mathcal{A}_n)$ as $\varepsilon \rightarrow 0$.

It remains to show that

$$\lim_{t \rightarrow +\infty} \frac{u(\tau)}{t} = \lim_{t \rightarrow +\infty} \frac{d}{dt} u(\tau),$$

To see this, note that for each closed $(1,1)$ -form Θ on \mathcal{X} and each smooth function Φ on \mathcal{X} , there is a Kähler form η on \mathcal{X} and a constant C large enough so that $\Theta + C\eta + dd^c \Phi \geq 0$ on \mathcal{X} . Moreover, we have a relation

$$\langle \varphi_0^t, \varphi_1^t, \dots, \varphi_n^t \rangle_{(\omega - \omega', \theta_1, \dots, \theta_n)} =$$

$$\langle \varphi_0^t, \varphi_1^t, \dots, \varphi_n^t \rangle_{(\omega, \theta_1, \dots, \theta_n)} - \langle 0, \varphi_1^t, \dots, \varphi_n^t \rangle_{(\omega', \theta_1, \dots, \theta_n)}$$

and repeat this argument for each i , $0 \leq i \leq n$, by symmetry. It follows from the above 'multilinearity' that we can write $t \mapsto \langle \varphi_0^t, \dots, \varphi_n^t \rangle$ as a difference of convex functions, concluding the proof. \square

Remark 4.2.4. *The above proof in fact also yields a version of Theorem 4.2.2 for subgeodesics $(\varphi_i^t)_{t \geq 0}$ that are L^∞ -compatible with smooth test configurations $(\mathcal{X}_i, \mathcal{A}_i)$ for (X, α_i) dominating $X \times \mathbb{P}^1$. This follows from the observation that one may replace L^∞ -compatible subgeodesic rays with smooth C^∞ -compatible ones, using Lemma 4.1.5 and Lemma 4.2.1.*

Some examples of applications

As a special case of Theorem A we obtain transcendental versions of several previously known formulas (see for instance [BHJ16]). As an example, we may deduce the following formula for the asymptotics of the Monge-Ampère energy functional by recalling that if ω is a Kähler form on X and $(\varphi_t)_{t \geq 0}$ is a subgeodesic ray, then

$$E(\varphi_t) = \frac{1}{(n+1)V} \langle \varphi_t, \dots, \varphi_t \rangle_{(\omega, \dots, \omega)}.$$

Corollary 4.2.5. *Assume that $(\mathcal{X}, \mathcal{A})$ is smooth and dominates $X \times \mathbb{P}^1$. For each smooth ray $(\varphi_t)_{t \geq 0}$ C^∞ -compatible with $(\mathcal{X}, \mathcal{A})$, we then have*

$$\lim_{t \rightarrow +\infty} \frac{E(\varphi_t)}{t} = E^{\text{NA}}(\mathcal{X}, \mathcal{A})$$

with

$$E^{\text{NA}}(\mathcal{X}, \mathcal{A}) := \frac{(\mathcal{A}^{n+1})}{(n+1)V}.$$

Remark 4.2.6. *Here E^{NA} makes reference to the non-Archimedean Monge-Ampère energy functional, see [BHJ15] for an explanation of the terminology.*

To illustrate further the usefulness of Theorem 4.2.2 in order to relate intersection numbers computed on cohomological test configurations to asymptotic slopes of energy functionals, we state also the following immediate corollaries for the twisted Monge-Ampère energy functionals E^θ as well as for Aubin's J-functional (compare [DR16]):

Corollary 4.2.7. *Assume that $(\mathcal{X}, \mathcal{A})$ is a smooth cohomological test configuration for (X, α) that dominates $X \times \mathbb{P}^1$. Let θ be any closed $(1,1)$ -form on X . For each smooth ray $(\varphi_t)_{t \geq 0}$ C^∞ -compatible with $(\mathcal{X}, \mathcal{A})$, we then have*

$$\lim_{t \rightarrow +\infty} \frac{E^\theta(\varphi_t)}{t} = V^{-1}(\mu^* p_1^* \theta \cdot \mathcal{A}^n).$$

Proof. We can write

$$E^\theta(\varphi_t) = V^{-1} \langle 0, \varphi_t, \dots, \varphi_t \rangle_{(\theta, \omega, \dots, \omega)}$$

By hypothesis (φ_t) is compatible with $(\mathcal{X}, \mathcal{A})$. Moreover, the constant ray (0) is compatible with the cohomological test configuration $(\mathcal{X}, \mu^* p_1^* \theta)$ for (X, θ) . The result then follows by a direct application of Theorem 4.2.2 above. \square

For instance, this may be applied to $\theta = \text{Ric}(w)$. (Note in particular that we do not need any Kähler assumption on θ here).

Corollary 4.2.8. *Assume that $(\mathcal{X}, \mathcal{A})$ is a smooth cohomological test configuration for (X, α) that dominates $X \times \mathbb{P}^1$. For each smooth ray $(\varphi_t)_{t \geq 0}$ \mathcal{C}^∞ -compatible with $(\mathcal{X}, \mathcal{A})$, we then have*

$$\lim_{t \rightarrow +\infty} \frac{J(\varphi_t)}{t} = J^{\text{NA}}(\mathcal{X}, \mathcal{A}),$$

where

$$J^{\text{NA}}(\mathcal{X}, \mathcal{A}) := \frac{(\mathcal{A} \cdot \mu^* p_1^* \alpha^n)}{V} - E^{\text{NA}}(\mathcal{X}, \mathcal{A}).$$

Proof. The proof follows by noting that

$$J(\varphi_t) = V^{-1} \langle \varphi_t, 0, \dots, 0 \rangle_{(\omega, \dots, \omega)} - E(\varphi_t)$$

and applying Theorem 4.2.2, using that the constant ray 0 is compatible with the cohomological test configuration $(\mathcal{X}, \mu^* p_1^* \alpha)$ for (X, α) . \square

4.3 Asymptotics for the K-energy. Proof of Theorem B.

Let (X, ω) be a compact Kähler manifold and $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ a Kähler class on X . As before, let $(\mathcal{X}, \mathcal{A})$ be a smooth, relatively Kähler cohomological test configuration for (X, α) dominating $X \times \mathbb{P}^1$. In this section we explain how the above Theorem 4.2.2 can be used to compute the asymptotic slope of the Mabuchi (K-energy) functional along rays (φ^t) , $\mathcal{C}^{1, \bar{1}}$ -compatible with $(\mathcal{X}, \mathcal{A})$. It is useful to keep the case of weak geodesic rays (as constructed in Lemma 4.1.10) in mind, which in turn implies K-semistability of (X, α) (Theorem A).

Regarding the proof of Theorem B, we will see that the Mabuchi functional is in fact of the form $\langle \varphi_0^t, \dots, \varphi_n^t \rangle_{(\theta_0, \dots, \theta_n)}$ for the appropriate choice of closed $(1, 1)$ -forms θ_i on X and rays (φ_i^t) on X , but Theorem B does not directly apply in this situation. Indeed, the expression for the Mabuchi functional involves the metric $\log(\omega + dd^c \varphi_t)^n$ on $K_{\mathcal{X}/\mathbb{P}^1}$, which may blow up close to \mathcal{X}_0 (in particular, the compatibility conditions are not satisfied). However, a key point is that we can cook up a functional $M_{\mathcal{B}}$ of the above 'multivariate' form that satisfies the same asymptotic slope as the Mabuchi functional (up to an explicit error term), and to which we may apply Theorem B. More precisely, we show that

$$\lim_{t \rightarrow +\infty} \frac{M(\varphi_t)}{t} = \lim_{t \rightarrow +\infty} \frac{M_{\mathcal{B}}(\varphi_t)}{t} + V^{-1}((\mathcal{X}_{0, \text{red}} - \mathcal{X}_0) \cdot \mathcal{A}^n)_{\mathcal{X}},$$

and use Theorem 4.2.2 to choose $M_{\mathcal{B}}$ so that moreover $\lim_{t \rightarrow +\infty} M_{\mathcal{B}}(\varphi_t)/t = \text{DF}(\mathcal{X}, \mathcal{A})$. It follows that the asymptotic slope of the Mabuchi (K-energy) functional equals $\text{DF}(\mathcal{X}, \mathcal{A}) + V^{-1}((\mathcal{X}_{0, \text{red}} - \mathcal{X}_0) \cdot \mathcal{A}^n)_{\mathcal{X}} =: M^{\text{NA}}(\mathcal{X}, \mathcal{A})$.

Proof of the upper bound

We first explain how to obtain a weak version of Theorem B, as a direct consequence of Theorem A. This version is more direct to establish than the full Theorem B, and will in fact be sufficient in order to prove both K-semistability and uniform K-stability of (X, α) , as explained in Section 5.

Theorem 4.3.1. *Let $(\mathcal{X}, \mathcal{A})$ be a smooth, relatively Kähler cohomological test configuration for (X, α) dominating $X \times \mathbb{P}^1$. For each subgeodesic ray $(\varphi^t)_{t \geq 0}$, $\mathcal{C}^{1, \bar{1}}$ -compatible with $(\mathcal{X}, \mathcal{A})$, we have the inequality¹*

$$\overline{\lim}_{t \rightarrow +\infty} \frac{M(\varphi_t)}{t} \leq \text{DF}(\mathcal{X}, \mathcal{A}).$$

In view of the strong version (see Theorem 4.3.7) we actually know that the limit is well-defined and, moreover, we obtain this way the precise asymptotic slope of the Mabuchi functional, see Section 4.3.

Proof of Theorem 4.3.1. Let \mathcal{B} be any smooth metric on $K_{\mathcal{X}/\mathbb{P}^1} := K_{\mathcal{X}} - \pi^* K_{\mathbb{P}^1}$. Using the \mathbb{C}^* -action on \mathcal{X} we can associate to \mathcal{B} a ray of smooth metrics on K_X that we denote by $(\beta_t)_{t \geq 0}$ (or $(\beta_\tau)_{\tau \in \bar{\Delta}^*}$ for its reparametrisation by $t = -\log |\tau|$). Fix $\log \omega^n$ as a reference metric on K_X , and let

$$\xi_{\mathcal{B}}^t := \log \left(\frac{e^{\beta_\tau}}{\omega^n} \right), \quad (4.1)$$

i.e. the function given as the difference of metrics $\beta_\tau - \log \omega^n$ on X . The constructed ray $(\xi_{\mathcal{B}}^t)_{t \geq 0}$ is then \mathcal{C}^∞ -compatible with the cohomological test configuration $(\mathcal{X}, K_{\mathcal{X}/\mathbb{P}^1})$ for (X, K_X) .

Now let $(\varphi_t)_{t \geq 0}$ be any subgeodesic ray $\mathcal{C}^{1, \bar{1}}$ -compatible with $(\mathcal{X}, \mathcal{A})$. By Lemma 4.1.5, Lemma 4.2.1 and Theorem 4.2.2 it follows that

$$\frac{\langle \xi_{\mathcal{B}}^t, \varphi_t \dots, \varphi_t \rangle_{(-\text{Ric}(\omega), \omega, \dots, \omega)}}{t} \longrightarrow (K_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{A}^n)_{\mathcal{X}} \quad (4.2)$$

as $t \rightarrow +\infty$. Indeed, by Lemma 4.1.5 we may choose a smooth subgeodesic ray $(\varphi'_t)_{t \geq 0}$ in \mathcal{H} that is \mathcal{C}^∞ -compatible (and hence also L^∞ - and $\mathcal{C}^{1, \bar{1}}$ -compatible) with $(\mathcal{X}, \mathcal{A})$. Up to replacing (φ_t) with (φ'_t) we may thus assume that (φ_t) is smooth and \mathcal{C}^∞ -compatible with $(\mathcal{X}, \mathcal{A})$, using Lemma 4.2.1, so that Theorem 4.2.2 applies.

Motivated by the Chen-Tian formula (2.14) and the identity (4.2), we thus introduce the notation

$$M_{\mathcal{B}}(\varphi_t) := \bar{\mathcal{S}}\mathcal{E}(\varphi_\tau) + V^{-1} \langle \xi_{\mathcal{B}}^t, \varphi_t \dots, \varphi_t \rangle_{(-\text{Ric}(\omega), \omega, \dots, \omega)},$$

the point being that the asymptotic slope of this functional coincides with the Donaldson-Futaki invariant (even when the central fiber is not reduced).

Lemma 4.3.2.

$$\lim_{t \rightarrow +\infty} \frac{M_{\mathcal{B}}(\varphi_t)}{t} = \text{DF}(\mathcal{X}, \mathcal{A})$$

¹The limit is in fact well-defined, as shown in Section 4.3 below.

Proof. This result is an immediate consequence of (4.2), the Chen-Tian formula (2.14) and Corollary 4.2.5. \square

Hence, it suffices to establish the following inequality

$$\overline{\lim}_{t \rightarrow +\infty} \frac{M(\varphi_t)}{t} \leq \lim_{t \rightarrow +\infty} \frac{M_{\mathcal{B}}(\varphi_t)}{t}.$$

To do this, we set $\Gamma(\tau) := (M - M_{\mathcal{B}})(\varphi_t)$. By the Chen-Tian formula (2.14) and cancellation of terms we have

$$\begin{aligned} \Gamma(\tau) &= \bar{S}E(\varphi_t) - E^{\text{Ric}(\omega)}(\varphi_t) + V^{-1} \int_X \log \left(\frac{(\omega + dd^c \varphi_\tau)^n}{\omega^n} \right) (\omega + dd^c \varphi_\tau)^n \\ &\quad - \bar{S}E(\varphi_t) - V^{-1} \langle \xi_{\mathcal{B}}^t, \varphi_t \dots, \varphi_t \rangle_{(-\text{Ric}(\omega), \omega, \dots, \omega)} \\ &= -E^{\text{Ric}(\omega)}(\varphi_t) + V^{-1} \int_X \log \left(\frac{(\omega + dd^c \varphi_\tau)^n}{\omega^n} \right) (\omega + dd^c \varphi_\tau)^n - V^{-1} \int_X \xi_{\mathcal{B}}^t (\omega + dd^c \varphi_\tau)^n \\ &\quad + V^{-1} \sum_{j=0}^{n-1} \int_X \varphi_t \text{Ric}(\omega) \wedge \omega^j \wedge (\omega + dd^c \varphi_t)^{n-j-1} \\ &= V^{-1} \int_X \log \left(\frac{(\omega + dd^c \varphi_\tau)^n}{\omega^n} \right) (\omega + dd^c \varphi_\tau)^n - V^{-1} \int_X \log \left(\frac{e^{\beta_\tau}}{\omega^n} \right) (\omega + dd^c \varphi_\tau)^n \\ &= V^{-1} \int_X \log \left(\frac{(\omega + dd^c \varphi_\tau)^n}{e^{\beta_\tau}} \right) (\omega + dd^c \varphi_\tau)^n, \end{aligned}$$

recalling the definition (4.1) of $\xi_{\mathcal{B}}^t$ and Definition 2.4.6.

In view of Proposition 3.2.18, we as usual let D denote the unique \mathbb{R} -divisor supported on \mathcal{X}_0 such that $\mathcal{A} = \mu^* p_1^* \alpha + [D]$, with $p_1 : X \times \mathbb{P}^1 \rightarrow X$ the first projection. Fix a choice of an S^1 -invariant function 'Green function' ψ_D for D , so that $\delta_D = \theta_D + dd^c \psi_D$ with θ_D a smooth S^1 -invariant closed $(1, 1)$ -form on \mathcal{X} . Moreover, set $\Omega := \mu^* p_1^* \alpha + \theta_D$ (for which $[\Omega] = \mathcal{A}$ then holds) and let Φ denote the S^1 -invariant function on $X \times \mathbb{P}^1$ corresponding to the ray (φ_t) . In particular, the function $\Phi \circ \mu + \psi_D$ extends to a smooth Ω -psh function Ψ on \mathcal{X} , by \mathcal{C}^∞ -compatibility.

With the above notation in place, the integrand in the above expression for $\Gamma(\tau)$ can be written

$$\log \left(\frac{(\omega + dd^c \varphi_\tau)^n}{e^{\beta_\tau}} \right) = \mu_* \left(\log \left(\frac{(\Omega + dd^c \Psi)^n \wedge \pi^*(\sqrt{-1} d\tau \wedge d\bar{\tau})}{\lambda_{\mathcal{B}}} \right) \right),$$

where

$$\lambda_{\mathcal{B}} := e^{\mathcal{B} + \pi^* \log(\sqrt{-1} d\tau \wedge d\bar{\tau})}$$

is the volume form defined by the smooth metric $\mathcal{B} + \pi^* \log(\sqrt{-1} d\tau \wedge d\bar{\tau})$ on $K_{\mathcal{X}}$. Since Ψ is Ω -psh on \mathcal{X} and $\lambda_{\mathcal{B}}$ is a volume form on \mathcal{X} , this quantity is bounded from above. Moreover, we integrate against the measure $(\omega + dd^c \varphi_{\tau})^n$ which can be computed in cohomology, thus has mass independent of τ . Hence

$$\Gamma(\tau) = V^{-1} \int_X \log \left(\frac{(\omega + dd^c \varphi_{\tau})^n}{e^{\beta\tau}} \right) (\omega + dd^c \varphi_{\tau})^n \leq O(1).$$

Dividing by t and passing to the limit now concludes the proof. \square

Proof of the lower bound

Following [SD16] we now wish to explain how to obtain a precise asymptotic expansion of the K-energy. As one of many application, this will be a crucial tool for our study of (uniform) K-stability for Kähler manifolds, see Section 5.3.

In order to compute the asymptotic slope of the Mabuchi (K-energy) functional (even when the central fiber is not reduced), recall the definition of the non-Archimedean Mabuchi functional, i.e. the intersection number

$$M^{\text{NA}}(\mathcal{X}, \mathcal{A}) := \text{DF}(\mathcal{X}, \mathcal{A}) + V^{-1}((\mathcal{X}_{0,\text{red}} - \mathcal{X}_0) \cdot \mathcal{A}^n)_{\mathcal{X}},$$

discussed in Section 4.3. Note that it satisfies $M^{\text{NA}}(\mathcal{X}, \mathcal{A}) \leq \text{DF}(\mathcal{X}, \mathcal{A})$ with equality precisely when the central fiber is reduced. We first need to establish some preparatory results on the Mabuchi functional and base change, which are completely analogous to the ones introduced in [BHJ15].

The non-Archimedean Mabuchi functional and base change

Let $(\mathcal{X}, \mathcal{A})$ be a cohomological test configuration for (X, α) . A natural operation on $(\mathcal{X}, \mathcal{A})$ is that of *base change* (on \mathcal{X} and we pull back \mathcal{A}). Unlike resolution of singularities, however, the DF-invariant does not behave well under under base change. In this context, a more natural object of study is instead the *non-Archimedean Mabuchi functional* M^{NA} (first introduced in [BHJ16] and [BHJ15], where also an explanation of the terminology is given).

Definition 4.3.3. *The non-Archimedean Mabuchi functional is the modification of the Donaldson-Futaki invariant given by*

$$M^{\text{NA}}(\mathcal{X}, \mathcal{A}) := \text{DF}(\mathcal{X}, \mathcal{A}) + V^{-1}((\mathcal{X}_{0,\text{red}} - \mathcal{X}_0) \cdot \mathcal{A}^n)_{\mathcal{X}}.$$

Note that the 'correction term' $V^{-1}((\mathcal{X}_{0,\text{red}} - \mathcal{X}_0) \cdot \mathcal{A}^n)_{\mathcal{X}}$ is non-positive and vanishes precisely when the central fiber \mathcal{X}_0 is reduced. The point of adding to DF this additional term is that the resulting quantity $M^{\text{NA}}(\mathcal{X}, \mathcal{A})$ becomes *homogeneous under base change*, i.e. we have the following lemma.

Lemma 4.3.4. ([BHJ15]) *Let $(\mathcal{X}, \mathcal{A})$ be a cohomological test configuration for (X, α) and let $d \in \mathbb{N}$. Denote by \mathcal{X}_d the normalisation of the base change of \mathcal{X} , by $g_d : \mathcal{X}_d \rightarrow \mathcal{X}$ the corresponding morphism (of degree d) and set $\mathcal{A}_d := g_d^* \mathcal{A}$. Then*

$$M^{\text{NA}}(\mathcal{X}_d, \mathcal{A}_d) = d \cdot M^{\text{NA}}(\mathcal{X}, \mathcal{A}).$$

Proof. For the sake of completeness we briefly recall the argument in [BHJ15, Proposition 7.13] (cf. also [LX14, Section 3]), which goes through in the analytic case as well. The key point is that the pullback formula for log canonical divisors still holds for cohomological test configurations \mathcal{X} for X . More precisely, write

$$K_{\mathbb{P}^1}^{\log} := K_{\mathbb{P}^1} + [0] + [\infty],$$

and

$$K_{\mathcal{X}}^{\log} := K_{\mathcal{X}} + \mathcal{X}_{0,\text{red}} + \mathcal{X}_{\infty,\text{red}} = K_{\mathcal{X}} + \mathcal{X}_{0,\text{red}} + \mathcal{X}_{\infty}.$$

As a consequence the relative log canonical bundle is given by

$$K_{\mathcal{X}/\mathbb{P}^1}^{\log} := K_{\mathcal{X}}^{\log} - \pi^* K_{\mathbb{P}^1}^{\log} = K_{\mathcal{X}/\mathbb{P}^1} - (\mathcal{X}_0 - \mathcal{X}_{0,\text{red}}),$$

and we may note that the non-Archimedean Mabuchi functional of a cohomological test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) can be expressed as

$$\begin{aligned} M^{\text{NA}}(\mathcal{X}, \mathcal{A}) &:= \frac{\bar{\mathcal{S}}}{n+1} V^{-1}(\mathcal{A}^{n+1})_{\tilde{\mathcal{X}}} + V^{-1}(K_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{A}^n)_{\tilde{\mathcal{X}}} + ((\mathcal{X}_{0,\text{red}} - \mathcal{X}_0) \cdot \mathcal{A}^n)_{\tilde{\mathcal{X}}} = \\ &\frac{\bar{\mathcal{S}}}{n+1} V^{-1}(\mathcal{A}^{n+1})_{\tilde{\mathcal{X}}} + V^{-1}(K_{\mathcal{X}/\mathbb{P}^1}^{\log} \cdot \mathcal{A}^n)_{\tilde{\mathcal{X}}} \end{aligned}$$

where as before the intersection numbers are computed on any smooth and dominating model $\tilde{\mathcal{X}}$ of \mathcal{X} (due to the projection formula it does not matter which one). The first term is clearly homogeneous by the projection formula. As for the second term, by the projection formula it suffices to show that

$$K_{\mathcal{X}_d/\mathbb{P}^1}^{\log} = g_d^* K_{\mathcal{X}/\mathbb{P}^1}^{\log}$$

The first thing to note here is that the above pull back of the relatively canonical divisors by the finite morphism g_d makes sense even though they are not necessarily \mathbb{Q} -Cartier, see [Kol13, Section 2.40]. Secondly, just as in the projective setting (see [Kol13, Section 2.42]), if $f_d : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is the morphism of degree d induced by the base change, then we have

$$K_{\mathbb{P}^1}^{\log} = f_d^* K_{\mathbb{P}^1}^{\log}. \quad (4.3)$$

Finally, we claim that the pullback formula

$$K_{\mathcal{X}_d}^{\log} = g_d^* K_{\mathcal{X}}^{\log} \quad (4.4)$$

holds even in our analytic setting. We interpret the above statement in terms of pullback of Weil divisor classes as detailed in Remark 3.2.11. The claim (4.4) then follows from a local computation, working away from subvarieties in codimension 2 by normality. Indeed, we are in the situation of the following lemma:

Lemma 4.3.5. *Let $f : X \rightarrow Y$ be a finite surjective morphism between normal varieties X and Y , with ramification divisor $R \subset X$ and branching divisor $B \subset Y$ respectively. Then*

$$K_X + R = f^*(K_Y + B). \quad (4.5)$$

Proof. By normality, we reason outside of subvarieties of codimension 2. Let (x_1, \dots, x_n) be local coordinates in a neighbourhood of smooth point of R and let (y_1, \dots, y_n) be local coordinates in a neighbourhood of a smooth point of B , such that

$$R = \{x_1 = 0\}, \quad B = \{y_1 = 0\},$$

and

$$f(x_1, \dots, x_n) = (y_1^d, y_2, \dots, y_n),$$

where d is the degree of f . Then $K_X + R$ and $K_Y + B$ are generated respectively by the n -forms

$$\frac{dx_1}{x_1} \wedge dx_2 \wedge \dots \wedge dx_n$$

and

$$\frac{dy_1}{y_1} \wedge dy_2 \wedge \dots \wedge dy_n.$$

Moreover, we have

$$f^* \left(\frac{dy_1}{y_1} \wedge dy_2 \wedge \dots \wedge dy_n \right) = d \cdot \frac{dx_1}{x_1} \wedge dx_2 \wedge \dots \wedge dx_n,$$

and (4.5) follows. \square

A direct application of Lemma 4.3.5 yields the pullback formula (4.4). Finally, putting equations (4.3) and (4.4) together we conclude using the projection formula, which yields

$$(K_{\mathcal{X}_d/\mathbb{P}^1}^{\log} \cdot \mathcal{A}_d^n)_{\tilde{\mathcal{X}}} = (g_d^* K_{\mathcal{X}}^{\log} \cdot g_d^* \mathcal{A}^n)_{\mathcal{X}} = d \cdot (K_{\mathcal{X}_d/\mathbb{P}^1}^{\log} \cdot \mathcal{A}^n)_{\tilde{\mathcal{X}}}$$

and

$$(\mathcal{A}_d^{n+1})_{\tilde{\mathcal{X}}} = (g_d^* \mathcal{A}^{n+1})_{\tilde{\mathcal{X}}} = d \cdot (\mathcal{A}^{n+1})_{\tilde{\mathcal{X}}}.$$

This is what we wanted to prove. \square

As an application of the base change property of the non-archimedean Mabuchi functional, it follows from Mumford's semistable reduction theorem ([KKMSD73, p.53], see also [KNX17, §16, p.6] for a remark on the analytic case) that there is a $d \in \mathbb{N}$, a finite base change $f : \tau \mapsto \tau^d$ (for d 'divisible enough'), a smooth test configuration \mathcal{X}' and a diagram

$$\begin{array}{ccccc} \mathcal{X} & \xleftarrow{g_d} & \mathcal{X}_d & \xleftarrow{\rho} & \mathcal{X}' \\ \pi \downarrow & & \downarrow \pi_d & \swarrow \pi' & \\ \mathbb{P}^1 & \xleftarrow{f} & \mathbb{P}^1 & & \end{array}$$

such that \mathcal{X}' is semistable, i.e. smooth and such that \mathcal{X}'_0 is a reduced divisor with simple normal crossings. In particular, note that the correction term $V^{-1}((\mathcal{X}'_{0,\text{red}} - \mathcal{X}'_0) \cdot \mathcal{A}^n)_{\mathcal{X}'}$ vanishes. Here \mathcal{X}_d denotes the normalisation of the base change, which is dominated by the semistable test configuration \mathcal{X}' for X . Moreover, $g_d \circ \rho$ is an isomorphism over $\mathbb{P}^1 \setminus \{0\}$.

Letting $\mathcal{A}_d := g_d^* \mathcal{A}$ be the pullback of \mathcal{A} to \mathcal{X}_d , and $\mathcal{A}' := \rho^* \mathcal{A}_d$ the pullback to \mathcal{X}' , it follows from the above homogeneity of the M^{NA} that

$$\text{DF}(\mathcal{X}', \mathcal{A}') = M^{\text{NA}}(\mathcal{X}', \mathcal{A}') = M^{\text{NA}}(\mathcal{X}_d, \mathcal{A}_d) = d \cdot M^{\text{NA}}(\mathcal{X}, \mathcal{A}) \leq d \cdot \text{DF}(\mathcal{X}, \mathcal{A}),$$

where d is the degree of g_d . We have thus associated to $(\mathcal{X}, \mathcal{A})$ a new test configuration $(\mathcal{X}', \mathcal{A}')$ for (X, α) such that the total space \mathcal{X}' is semistable. Up to replacing \mathcal{X}' with a *determination* (see Section 3.2) we can moreover assume that \mathcal{X}' dominates $X \times \mathbb{P}^1$. Hence, the above shows that $\text{DF}(\mathcal{X}, \mathcal{A}) \geq \text{DF}(\mathcal{X}', \mathcal{A}')$. By an argument by perturbation much as the one in the proof of Proposition 3.2.20, we obtain the following stronger version of the aforementioned result.

Proposition 4.3.6. *Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be Kähler. Then (X, α) is K-semistable (Definition 3.2.13) if and only if $\text{DF}(\mathcal{X}, \mathcal{A}) \geq 0$ for all semistable, relatively Kähler cohomological test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α) dominating $X \times \mathbb{P}^1$.*

Proof of the lower bound

Adapting the techniques of [BHJ16] to the present setting we now obtain the following result, corresponding to Theorem C of the introduction.

Theorem 4.3.7. *Let X be a compact Kähler manifold and $\alpha \in H^{1,1}(X, \mathbb{R})$ a Kähler class. Suppose that $(\mathcal{X}, \mathcal{A})$ is a smooth, relatively Kähler cohomological test configuration for (X, α) dominating $X \times \mathbb{P}^1$. Then, for each subgeodesic ray $(\varphi_t)_{t \geq 0}$, $\mathcal{C}^{1,1}$ -compatible with $(\mathcal{X}, \mathcal{A})$, the asymptotic slope of the Mabuchi functional is well-defined and satisfies*

$$\frac{M(\varphi_t)}{t} \longrightarrow M^{\text{NA}}(\mathcal{X}, \mathcal{A})$$

as $t \rightarrow +\infty$.

Remark 4.3.8. *In particular, this result holds when $(\varphi_t)_{t \geq 0}$ is the weak geodesic ray associated to $(\mathcal{X}, \mathcal{A})$, constructed in Section 4.1.*

Proof of Theorem 4.3.7. Following ideas of [BHJ16] we associate to the given smooth, relatively Kähler and dominating test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) another test configuration $(\mathcal{X}', \mathcal{A}')$ for (X, α) which is semistable, i.e. smooth and such that \mathcal{X}'_0 is a reduced \mathbb{R} -divisor with simple normal crossings. As previously noted, we can also assume that \mathcal{X}' dominates the product. In the terminology of Section 4.3, this construction comes with a morphism $g_d \circ \rho : \mathcal{X}' \rightarrow \mathcal{X}$, cf. the diagram in Section 4.3. Pulling back, we set $\mathcal{A}' := g_d^* \rho^* \mathcal{A}$. Note that \mathcal{A}' is no longer relatively Kähler, but merely relatively semipositive (with the loss of positivity occurring along \mathcal{X}'_0).

On the one hand, Lemma 4.3.4 yields

$$M^{\text{NA}}(\mathcal{X}', \mathcal{A}') = d \cdot M^{\text{NA}}(\mathcal{X}, \mathcal{A}), \quad (4.6)$$

where $d > 0$ is the degree of the morphism g_d . On the other hand, we may consider the pull back by $g_d \circ \rho$ of the weak geodesic $(\varphi_t)_{t \geq 0}$ associated to $(\mathcal{X}, \mathcal{A})$. This induces a subgeodesic $(\varphi'_t)_{t \geq 0}$ which is $\mathcal{C}^{1, \bar{1}}$ -compatible with the test configuration $(\mathcal{X}', \mathcal{A}')$ for (X, α) (in particular, the boundedness of the Laplacian is preserved under pullback by $g_d \circ \rho$). Replacing τ by τ^d amounts to replacing t by $d \cdot t$, so that

$$\frac{M(\varphi'_t)}{t} = d \cdot \frac{M(\varphi_t)}{t}. \quad (4.7)$$

Combining equations (4.6) and (4.7) it thus follows that

$$\lim_{t \rightarrow +\infty} \frac{M(\varphi_t)}{t} = M^{\text{NA}}(\mathcal{X}, \mathcal{A})$$

if and only if

$$\lim_{t \rightarrow +\infty} \frac{M(\varphi'_t)}{t} = \text{DF}(\mathcal{X}', \mathcal{A}'). \quad (4.8)$$

In other words, it suffices to establish (4.8) above. By the asymptotic formula 4.3.2 it is in turn equivalent to show that

$$\lim_{t \rightarrow +\infty} \frac{M(\varphi'_t)}{t} = \lim_{t \rightarrow +\infty} \frac{M_{\mathcal{B}}(\varphi'_t)}{t}.$$

We use the notation of the proof of Theorem 4.3.1. In particular, we set $\Gamma(\tau) := (M - M_{\mathcal{B}})(\varphi'_\tau)$. As in the proof of Theorem 4.3.1 we have an upper bound $\Gamma(\tau) \leq O(1)$, using that the restriction of the relatively semipositive class \mathcal{A}' to $\mathcal{X}' \setminus \mathcal{X}'_0$ is in fact relatively Kähler.

To obtain a lower estimate of $\Gamma(\tau)$ we consider the Monge-Ampere measure $\text{MA}(\varphi'_\tau) := V^{-1}(\omega + dd^c \varphi'_\tau)^n$ and note that

$$\begin{aligned} V^{-1}\Gamma(\tau) &= V^{-1} \int_X \log \left(\frac{(\omega + dd^c \varphi'_\tau)^n}{e^{\beta_\tau}} \right) = \\ &= \int_X \log \left(\frac{\text{MA}(\varphi'_\tau)}{e^{\beta_\tau} / \int_X e^{\beta_\tau}} \right) \text{MA}(\varphi'_\tau) - \log \int_X e^{\beta_\tau} \geq -\log \int_X e^{\beta_\tau}, \end{aligned}$$

since the relative entropy of the two probability measures $\text{MA}(\varphi_\tau)$ and $e^{\beta_\tau} / \int_X e^{\beta_\tau}$ is non-negative. We now conclude by estimating this integral, using the following volume estimate lemma for simple normal crossing test configurations, see [BHJ16] and also [BJ17] for more precise results in this directions.

Lemma 4.3.9. ([BHJ16]). *Let $(\mathcal{X}, \mathcal{A})$ be a semistable and dominating test configuration for (X, α) and let \mathcal{B} be any smooth metric on K_{X/\mathbb{P}^1} . Let $(\beta^t)_{t \geq 0}$ be the family of smooth metrics on K_X induced by \mathcal{B} . Denote by $p \geq 1$ the largest integer such that $p - 1$ distinct irreducible components of \mathcal{X}_0 have a non-empty intersection. Then there are positive constants A and B such that*

$$At^{2(p-1)} \leq \int_X e^{\beta^t} \leq Bt^{2(p-1)}.$$

holds for all t .

We refer the reader to [BHJ16] for the proof. Recalling that $t = -\log |\tau|$, it follows from Lemma 4.3.9 that $\log \int_X e^{\beta\tau} = o(t)$. Hence we deduce from

$$\frac{-\log \int_X e^{\beta\tau}}{t} \leq \frac{\Gamma(\tau)}{t} \leq \frac{O(1)}{t}$$

that

$$\lim_{t \rightarrow +\infty} \frac{\Gamma(\tau)}{t} = 0,$$

completing the proof. □

Remark 4.3.10 (Remarks on the singular case). *It is further possible to compute the asymptotics along subgeodesic rays compatible with cohomological test configurations that are not necessarily smooth and dominating $X \times \mathbb{P}^1$. This is useful and perhaps, at least to the extent of the techniques used in this thesis, necessary to work with K -polystability as in Section 5.4. Indeed, we then establish a generalization of Theorem 4.2.2 to the singular setting of arbitrary normal test configurations.*

Chapter 5

Main results 2: Stability results for cscK manifolds with transcendental cohomology class

This chapter contains proofs of several main results. First, we show that cscK manifolds are K-semistable and uniformly K-stable whenever the automorphism group is discrete. In particular, this shows that cscK manifolds are K-stable (we also give an independent proof of this, different from the Stoppa-like methods used in [DR16], cf. also [Sto09a]). We further discuss extensions of these results to the case of cscK manifolds admitting holomorphic vector fields (so that $\text{Aut}_0(X) \neq \{0\}$).

A central role in the proof of these results is played by the energy functional asymptotics developed in previous sections. Indeed, as explained below, an application of the above Theorem 4.3.1 actually suffices to yield K-semistability of cscK manifolds. The same goes for uniform K-stability, see Section 5.3. Proving K-polystability is however more involved, see Section 5.4. In this case the energy functional asymptotics need to be strengthened somewhat before application and a number of additional tools are introduced. In particular, in order to study cohomological test configurations with vanishing Donaldson-Futaki invariant, we give a criterion, of independent interest, for when two relatively Kähler cohomological test configurations for (X, α) are isomorphic.

Notation 5.0.11. *In this chapter we always assume that (X, ω) is a compact Kähler manifold such that the Kähler class $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ admits a cscK representative.*

5.1 The isomorphism class of a cohomological test configuration.

Proof of Theorem I

Suppose that $(\mathcal{X}, \mathcal{A})$ is a normal and relatively Kähler (cohomological) test configuration for a given cscK manifold (X, α) . Fix a cscK potential $\varphi_0 \in \mathcal{H}_0 (= \mathcal{H} \cap E^{-1}(0))$. Let $[0, +\infty[\ni$

$t \mapsto \varphi_t \in \text{PSH}(X, \omega) \cap L^\infty$ be the associated geodesic ray emanating from φ_0 . In order to establish stability properties of (X, α) such as K-polytability or uniform K-stability one is led to characterize vanishing of certain quantities, e.g. in the case of K-polystability one must show that if (and only if) $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$, then $(\mathcal{X}, \mathcal{A})$ is a product configuration. In order to do so, we now explain that it in fact suffices to show that the associated geodesic $(\varphi_t)_{t \geq 0}$ is L^∞ -compatible with some product configuration. This useful result follows from the concept of *unique relatively Kähler model* of a given test configuration $(\mathcal{X}, \mathcal{A})$, which we now discuss.

Unique ample and relatively Kähler model

In order to motivate the concept of unique relatively Kähler model we first recall the arguments yielding a unique *ample model* for algebraic test configurations of a polarized variety (X, L) . To this end, consider the graded algebra

$$R(X, rL) := \bigoplus_{k \in \mathbb{N}} H^0(X, krL).$$

By the so called 'reverse Rees construction', see e.g. [Nys10] and [BHJ15, Section 1.2] it can be seen that every algebraic test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) induces a \mathbb{Z} -filtration of $R(X, rL)$ for r divisible enough. Indeed, letting $(\mathcal{X}, kr\mathcal{L})$ be a given test configuration for (X, krL) we set

$$F^\lambda H^0(X, krL) := \{s \in H^0(X, krL) : t^{-\lambda} \bar{s} \in H^0(\mathcal{X}, kr\mathcal{L})\},$$

where $\bar{s} \in H^0(\mathcal{X} \setminus \mathcal{X}_0, kr\mathcal{L})$ denotes the \mathbb{C}^* -invariant section defined by $s \in H^0(X, krL)$. The argument is then based on the existence of a one-to-one correspondence between ample test configurations for (X, L) and finitely generated \mathbb{Z} -filtrations of $R(X, rL)$ for r divisible enough (see [BHJ15, Proposition 2.15] for details). We briefly recall the idea of the proof: If $(\mathcal{X}, \mathcal{L})$ is an ample test configuration for (X, L) , then the induced \mathbb{Z} -filtration $R(X, rL)$ is finitely generated, since

$$\bigoplus_{k \in \mathbb{N}} \left(\bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} F^\lambda H^0(X, krL) \right) = R(\mathcal{X}, k\mathcal{L})$$

is finitely generated over $k[t]$. In fact, the same result holds if (X, L) is ample and $(\mathcal{X}, \mathcal{L})$ is merely semi-ample.

Conversely, let F^\cdot be a finitely generated \mathbb{Z} -filtration of $R(X, rL)$ for some r . Up to replacing r with a multiple we may assume that the graded $k[t]$ -algebra

$$\bigoplus_{k \in \mathbb{N}} \left(\bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} F^\lambda H^0(X, krL) \right)$$

is generated in degree $k = 1$. It follows that the projectivization

$$\text{Proj} \left(\bigoplus_{k \in \mathbb{N}} \left(\bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} F^\lambda H^0(X, krL) \right) \right)$$

defines an ample test configuration for (X, rL) , thus for (X, L) . One checks that these constructions are inverse to each other, see e.g. [BHJ15, Section 1.2].

Definition 5.1.1. *The ample model of a semi-ample algebraic test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L) is the unique ample test configuration corresponding to the finitely generated \mathbb{Z} -filtrations of $R(X, rL)$ induced by $(\mathcal{X}, \mathcal{L})$.*

One can prove that the ample model of $(\mathcal{X}, \mathcal{L})$ is normal precisely if $(\mathcal{X}, \mathcal{L})$ is normal [BHJ15, Proposition 2.17].

Uniqueness of relatively Kähler representatives - An injectivity lemma.

We now introduce the more general concept of *relatively Kähler model* and show that such an object is *unique*. The methods used are however completely different from the ones in the polarized case, using \mathbb{Z} -filtrations as described above in Section 5.1.

To introduce our methods, first suppose that X, Y, Z are normal compact Kähler spaces such that $\phi : X \dashrightarrow Y$ is a bimeromorphic map and $\mu : Z \rightarrow X, \rho : Z \rightarrow Y$ are bimeromorphic morphisms (modifications). Up to replacing Z by a Z' we can suppose (due to resolution of indeterminacy, cf. Hironaka¹) that μ is a sequence of blow-ups with smooth center. In particular, μ is a projective morphism. Importantly, this implies that the fibers $\mu^{-1}(x), x \in X$ are projective varieties, so they are covered by curves.

$$\begin{array}{ccc} Z & & \\ \downarrow \mu & \searrow \rho & \\ X & \dashrightarrow & Y \end{array}$$

Assume further that $\alpha \in H^{1,1}(X, \mathbb{R})$ and $\beta \in H^{1,1}(Y, \mathbb{R})$ are Kähler classes satisfying $\mu^*\alpha = \rho^*\beta$. We then claim that ϕ is in fact an isomorphism. Indeed, let $x \in X$, and C be a curve in $\mu^{-1}(x) \subset Z$. The projection formula then yields

$$0 = (\mu^*\alpha) \cdot [C] = (\rho^*\beta) \cdot [C] = \beta \cdot \rho_*[C].$$

Since β is Kähler we must have $\rho_*[C] = 0$, so that $\dim \rho(C) < 1$. Hence ρ contracts the curve C to a point in Y . Finally recall that ϕ is a morphism if and only if for all curves $C \subset Z, \mu(C) = \text{point}$ implies that $\rho(C) = \text{point}$. By symmetry in X and Y (in particular, ϕ is bimeromorphic and α is also Kähler) this concludes.

The point is that, along the lines of the above, we obtain the following key lemma. It can be seen as a Kähler analogue of the notion of *unique ample model* introduced in [BHJ15].

Lemma 5.1.2. *Any subgeodesic ray $[0, +\infty) \ni t \mapsto \varphi_t \in \text{PSH}(X, \omega) \cap L^\infty(X)$ is L^∞ -compatible with at most one normal, relatively Kähler test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) .*

Proof. The proof follows the argument outlined above. To this end, let $(\mathcal{X}_i, \mathcal{A}_i, \lambda_i, \pi_i), i = 1, 2$, be normal relatively Kähler cohomological test configurations for (X, α) . By definition of a cohomological test configuration there are canonical isomorphisms $X \times (\mathbb{P}^1 \setminus \{0\}) \rightarrow \mathcal{X}_i \setminus \pi_i^{-1}(0)$, which induce bimeromorphic maps $\phi_i : X \times \mathbb{P}^1 \dashrightarrow \mathcal{X}_i, i = 1, 2$. Note also that we may choose a

¹See references in [Fujiki, p.230].

smooth manifold \mathcal{Z} that *simultaneously* dominates \mathcal{X}_1 , \mathcal{X}_2 and $X \times \mathbb{P}^1$. We thus have a resolution of indeterminacy as follows:

$$\begin{array}{ccccc}
 & & \mathcal{Z} & & \\
 & \swarrow & \downarrow r & \searrow & \\
 \mathcal{X}_1 & \xleftarrow{\mu_1} & X \times \mathbb{P}^1 & \xrightarrow{\mu_2} & \mathcal{X}_2 \\
 & \xleftarrow{\phi_1} & & \xrightarrow{\phi_2} &
 \end{array}$$

Up to replacing \mathcal{Z} by a \mathcal{Z}' we can moreover assume that the dominating morphisms are given by composition of blow-ups with smooth center, hence projective, so that the fibers can be covered by curves.

Now suppose that there is a subgeodesic ray $t \mapsto \varphi_t \in \text{PSH}(X, \omega) \cap L^\infty(X)$ that is L^∞ -compatible with both \mathcal{X}_1 and \mathcal{X}_2 . In other words, if ψ_1 and ψ_2 are Green functions for D_1 and D_2 respectively, and Φ is the S^1 -invariant function on $X \times \mathbb{P}^1$ corresponding to $(\varphi_t)_{t \geq 0}$, then $r^*\Phi + \psi_1$ and $r^*\Phi + \psi_2$ are both (locally) bounded on \mathcal{Z} . Hence so is their difference, i.e. we have

$$\psi_1 - \psi_2 \in L_{\text{loc}}^\infty(\mathcal{Z}).$$

In particular, the Green functions are of the same singularity type, so their respective divisors of singularities coincide, i.e. $[D_1] = [D_2]$. (One way to see this is to consider the associated multiplier ideal sheaves $\mathcal{I}(\psi_1)$ and $\mathcal{I}(\psi_2)$. Since ψ_1 and ψ_2 (locally) differ by a bounded function, we must have $\mathcal{I}(\psi_1) = \mathcal{I}(\psi_2)$. Pulling back via the morphism r we then get $\mathcal{O}_{\mathcal{Z}}(-D_1) = r^*\mathcal{I}(\psi_1) = r^*\mathcal{I}(\psi_2) = \mathcal{O}_{\mathcal{Z}}(-D_2)$, so that in particular $[D_2] = [D_1]$ holds).

We now wish to compare the respective pull-backs of \mathcal{A}_i to \mathcal{Z} . To do this, note that there are \mathbb{R} -divisors D_1 and D_2 supported on the central fiber \mathcal{Z}_0 such that

$$\mu_1^*\mathcal{A}_1 = r^*p_1^*\alpha + [D_1],$$

and

$$\mu_2^*\mathcal{A}_2 = r^*p_1^*\alpha + [D_2],$$

respectively, see Proposition 3.2.18. As a consequence, $\mu_1^*\mathcal{A}_1 = \mu_2^*\mathcal{A}_2$ if and only if $[D_1] = [D_2]$.

Finally, since the cohomology class \mathcal{A}_1 on \mathcal{X}_1 is relatively Kähler, there is a Kähler form η on \mathbb{P}^1 such that $\mathcal{A}_i + \pi_i^*\eta_i$ is Kähler on \mathcal{X}_i . As before, we conclude that the bimeromorphic map $\phi_2 \circ \phi_1^{-1} : \mathcal{X}_1 \dashrightarrow \mathcal{X}_2$ is in fact a morphism: Indeed, let $x \in \mathcal{X}_1$ and C be a curve in $\mu_1^{-1}(x) \subset \mathcal{Z}$. Since $\mu_i^*\pi_i^*\eta_i \cdot C = 0$, $i = 1, 2$, the projection formula yields

$$\begin{aligned}
 0 &= (\mu_1^*\mathcal{A}_1 + \mu_1^*\pi_1^*\eta_1) \cdot [C] = (\mu_2^*\mathcal{A}_2 + \mu_1^*\pi_1^*\eta_1) \cdot [C] \\
 &= (\mu_2^*\mathcal{A}_2 + \mu_2^*\pi_2^*\eta_2) \cdot [C] = (\mathcal{A}_2 + \pi_2^*\eta_2) \cdot (\mu_2)_*[C].
 \end{aligned}$$

Hence $\dim \mu_2(C) < 1$. It follows that the bimeromorphic morphism $\mathcal{X}_1 \dashrightarrow \mathcal{X}_2$ is in fact a morphism. By symmetry in \mathcal{X}_1 and \mathcal{X}_2 we see that if also \mathcal{A}_2 is relatively Kähler, then \mathcal{X}_1 and \mathcal{X}_2 are isomorphic (and the isomorphism is even given explicitly as the composition $\phi_2 \circ \phi_1^{-1}$). \square

Note that we may view the above Lemma 5.1.2 as an injectivity result. Another useful reformulation of the above is the following:

Corollary 5.1.3. *Two test configurations are isomorphic iff their associated geodesic rays are of same singularity type, i.e. if the difference of the associated S^1 -invariant functions is uniformly bounded.*

This is potentially very useful. For instance, we can compute the geodesic rays of cohomological product configurations (see Corollary 5.4.27). If we can show that the geodesic ray associated to any cohomological test configuration $(\mathcal{X}, \mathcal{A})$ with $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$ is compatible also with some product configuration $(\mathcal{X}', \mathcal{A}')$ (in fact, coincides with its uniquely associated geodesic), then by Lemma 5.1.2 $\mathcal{X} \equiv \mathcal{X}'$, so $(\mathcal{X}, \mathcal{A})$ is a product configuration itself. This is a rather general strategy that applies in several situations of interest.

Remark 5.1.4. A few remarks:

- It is possible to be compatible with other test configurations that are *not* relatively Kähler (e.g. relatively semipositive/nef). Indeed, given a relatively Kähler test configuration $(\mathcal{X}, \mathcal{A})$ we may use resolution of singularities and pullback to associate a new test configuration $(\hat{\mathcal{X}}, \hat{\mathcal{A}})$, and this is only relatively nef in general. In fact, it is relatively Kähler only if $\rho : \hat{\mathcal{X}} \rightarrow \mathcal{X}$ is an isomorphism. However, such modifications of \mathcal{X} do not change the associated geodesic ray, as can be seen e.g. from the techniques for solving the geodesic equation on a normal Kähler space. Hence it is not possible to extend this uniqueness result beyond the relatively Kähler case.
- Product test configurations $(X \times \mathbb{P}^1, p_1^* \alpha, \lambda, \pi)$ for (X, α) are automatically normal and relatively Kähler (since α is Kähler by assumption).

5.2 K-semistability of cscK manifolds. Proof of Theorem C.

Let (X, ω) be a compact Kähler manifold and let $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ be the corresponding Kähler class. We now explain how the above considerations apply to give a proof of K-semistability of cscK manifolds, i.e. Theorem C from the introduction.

The proof relies on the asymptotics in Section 4. For this, note the role of the preparatory results of section 3, showing that it suffices to test for test configurations whose total space is smooth and dominating the product $X \times \mathbb{P}^1$ (which is the precise situation where the results of Section 4 apply, cf. however Section 5.4 for an extension to the singular setting).

Theorem 5.2.1. *If the Mabuchi (K-energy) functional is bounded from below in α , then (X, α) is K-semistable (in the generalised sense of Definition 3.2.13).*

Proof of Theorem C. Let X be a compact Kähler manifold and ω a given Kähler form, with $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ the corresponding Kähler class. Let $(\mathcal{X}, \mathcal{A})$ be any (possibly singular) cohomological test configuration for (X, α) which by desingularisation and perturbation (see

Proposition 3.2.20) can be assumed to be smooth, relatively Kähler and dominating $X \times \mathbb{P}^1$. Consider any ray $(\varphi_t)_{t \geq 0}$ such that Theorem B applies; for instance one may take (φ_t) to be the associated weak geodesic ray emanating from ω (i.e. such that $\varphi_0 = 0$), which due to [Che00b] (cf. also [Blo13], [Dar14], [DL12]) is $\mathcal{C}^{1,\bar{1}}$ -compatible with $(\mathcal{X}, \mathcal{A})$. Now suppose that the Mabuchi functional is bounded from below (in the given class α). In particular, we then have

$$\text{DF}(\mathcal{X}, \mathcal{A}) \geq \overline{\lim}_{t \rightarrow +\infty} \frac{M(\varphi_t)}{t} \geq 0,$$

using the weak version of Theorem B, cf. Theorem 4.3.1. Since the cohomological test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) was chosen arbitrarily, this proves Corollary 5.2.3, i.e. shows that (X, α) is K-semistable. \square

Remark 5.2.2. *In fact, the proof even gives the slightly stronger statement that $M^{\text{NA}}(\mathcal{X}, \mathcal{A}) \geq 0$ for all relatively Kähler cohomological test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α) .*

In particular, due to [BB14, CLP16] it follows that the existence of cscK metrics implies that the Mabuchi functional is bounded below. As a consequence we obtain the following immediate corollary, which is a first main motivation for our work.

Corollary 5.2.3. *If the Kähler class $\alpha \in H^{1,1}(X, \mathbb{R})$ admits a constant scalar curvature representative, then (X, α) is K-semistable.*

We emphasize that this result holds for arbitrary cscK manifolds, with no restriction imposed on the automorphism group $\text{Aut}(X)$.

In the setting of polarized manifolds Corollary 5.2.3 follows from Donaldson's lower bound of the Calabi functional [Don05a]. Such a lower bound for the Calabi functional has recently also been generalised to hold in the context of extremal metrics, see [Der16a].

5.3 K-stability and uniform K-stability of cscK manifolds with discrete automorphism group. Proof of Theorem E.

Throughout this section we suppose that X is a compact Kähler manifold with discrete automorphism group. This is a necessary condition for K-stability and uniform K-stability to hold, since any non-trivial one-parameter subgroup $\mathbb{C}^* \rightarrow \text{Aut}(X)$ (which exists by assumption) induces a non-trivial product test configuration for X . By Proposition 5.4.26 any product test configuration for a cscK manifold has vanishing Donaldson-Futaki invariant. As a consequence, whenever $\text{Aut}(X)$ is non-discrete, it follows that (X, α) is K-unstable and uniformly K-unstable for all Kähler classes α on X .

Uniform K-stability and the Yau-Tian-Donaldson conjecture

We now explain how the energy functional asymptotics of Section 4 apply to give a proof of Theorem E and point out some immediate and important consequences regarding the Yau-Tian-Donaldson (YTD) conjecture. We also check that uniform K-stability indeed implies K-stability

(as expected from a good definition of these notions). Recall that the uniform notion of stability uses the non-archimedean J-functional as a norm, see Section 3.2 for definitions. It is comparable with the usual L^1 -norm and also with the minimum norm introduced in [Der16b].

Proposition 5.3.1. *Let (X, ω) be a cscK manifold, with $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ the associated Kähler class. Suppose that $\text{Aut}_0(X) = \{0\}$. Let $(\mathcal{X}, \mathcal{A})$ be a normal and relatively Kähler test configuration for (X, α) . Then*

$$\text{DF}(\mathcal{X}, \mathcal{A}) \geq \delta \text{J}^{\text{NA}}(\mathcal{X}, \mathcal{A}),$$

for some $\delta > 0$.

Proof of Theorem E. Let X be a compact Kähler manifold and ω a given Kähler form, with $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ the corresponding Kähler class. Let $(\mathcal{X}, \mathcal{A})$ be any (possibly singular) cohomological test configuration for (X, α) which by desingularisation and perturbation (see Proposition 3.2.20) can be assumed to be smooth, relatively Kähler and dominating $X \times \mathbb{P}^1$. Consider any ray $(\varphi_t)_{t \geq 0}$ such that Theorem B applies; for instance one may take (φ_t) to be the associated weak geodesic ray emanating from ω (i.e. such that $\varphi_0 = 0$), which due to [Che00b] (cf. also [Blo13], [Dar14], [DL12]) is $\mathcal{C}^{1,\bar{1}}$ -compatible with $(\mathcal{X}, \mathcal{A})$.

Since (X, ω) is assumed to be cscK, it follows from [BDL16] that the Mabuchi functional is coercive, so in particular $M(\varphi_t) \geq \delta \text{J}(\varphi_t) - C$ for some constants $\delta, C > 0$ uniform in t . Note that Corollary 4.2.8) and the (weak) Theorem B provides a link with the intersection theoretic quantities $\text{J}^{\text{NA}}(\mathcal{X}, \mathcal{A})$ and $M^{\text{NA}}(\mathcal{X}, \mathcal{A})$ respectively. More precisely, dividing by t and passing to the limit we have

$$0 \leq \overline{\lim}_{t \rightarrow +\infty} \frac{(M - \delta \text{J})(\varphi_t)}{t} \leq M^{\text{NA}}(\mathcal{X}, \mathcal{A}) - \delta \text{J}^{\text{NA}}(\mathcal{X}, \mathcal{A}).$$

Since $(\mathcal{X}, \mathcal{A})$ was chosen arbitrarily it follows that (X, α) is uniformly K-stable, concluding the proof. \square

This above shows something that one might call "uniform K-semistability". In order to prove the full uniform K-stability of cscK manifolds it remains to show that the norm J^{NA} vanishes precisely when the test configuration is trivial. Compare [Der16b, BHJ15] for the polarized case. See also [Der16a].

The vanishing loci of DF and J^{NA}

The aim of this section is to complete the picture of K-stability and uniform K-stability in the case when the automorphism group is discrete.

Theorem 5.3.2. *Let (X, ω) be a cscK manifold, with $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ the associated Kähler class. Suppose that $\text{Aut}_0(X) = \{0\}$. Let $(\mathcal{X}, \mathcal{A})$ be a normal and relatively Kähler test configuration for (X, α) . Then the following are equivalent:*

1. $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$.

2. $J^{\text{NA}}(\mathcal{X}, \mathcal{A}) = 0$.

3. $(\mathcal{X}, \mathcal{A})$ is the trivial test configuration.

Proof of Theorem 5.3.2. We begin by noting that (3) \Rightarrow (1) is easy and well-known. A non-standard but perhaps interesting way of proving this is by noting that the unique geodesic ray emanating from $\varphi_0 \in \mathcal{H}$ and associated to the trivial test configuration $(\mathcal{X}_{\text{triv}}, \mathcal{A}_{\text{triv}})$ is the constant ray (φ_0) . By Theorem B it follows that

$$M^{\text{NA}}(\mathcal{X}_{\text{triv}}, \mathcal{A}_{\text{triv}}) = \text{DF}(\mathcal{X}_{\text{triv}}, \mathcal{A}_{\text{triv}}) = \lim_{t \rightarrow +\infty} M(\varphi_0)/t = 0.$$

Hence (3) \Rightarrow (1). Moreover, due to [BDL16, Theorem 1.5] and Theorem B it follows that (1) \Rightarrow (2).

It remains to prove that $(\mathcal{X}, \mathcal{A})$ is trivial if $J^{\text{NA}}(\mathcal{X}, \mathcal{A}) = 0$. To see this, note that in this situation $t \mapsto J(\varphi_t)$ is convex, and moreover $J(\varphi_0) = 0$ and $0 \leq J(\varphi_t) < C$ for some $C > 0$ and all $t \in [0, +\infty)$. As a consequence we even have $J(\varphi_t) = 0$ for each t . But J is strictly convex along geodesics, so $\varphi_t = \varphi_0$ for each t . In other words, (φ_t) is the trivial ray, L^∞ -compatible with the trivial test configuration. Since $(\mathcal{X}, \mathcal{A})$ is normal and relatively Kähler it follows from the injectivity lemma 5.1.2 that $(\mathcal{X}, \mathcal{A})$ is trivial. This proves the equivalence of (1),(2) and (3). \square

Compare also with the independently obtained results [Der16a, Theorem 1.5]. Also note that the cscK assumption is only used to see that (1) \Rightarrow (2).

Corollary 5.3.3. *Let X be a compact Kähler manifold with discrete automorphism group, and let $\alpha \in H^{1,1}(X, \mathbb{R})$ a Kähler class on X . If α contains a cscK representative, then (X, α) is K-stable and uniformly K-stable.*

Indeed, the K-semistability is known (Corollary 5.2.3). From Proposition 5.3.1 we also have "uniform K-semistability", i.e.

$$\text{DF}(\mathcal{X}, \mathcal{A}) \geq \delta J^{\text{NA}}(\mathcal{X}, \mathcal{A})$$

for some $\delta > 0$. Combined with Theorem 5.4.16 we see that (X, α) is K-stable and uniformly K-stable, thus finishing the proof of Theorem C.

Remark 5.3.4. *It is interesting to point out the following:*

- *Because of Theorem 5.3.2 it is clear that uniform K-stability implies K-stability, as expected.*
- *The final part of the proof of Theorem 5.3.2, using the injectivity lemma 5.1.2, yields also a simple and direct proof of K-stability of cscK manifolds, different to the one in [DR16] that used Stoppa-like methods, see [Sto09a].*
- *Theorem E confirms one direction of the Yau-Tian-Donaldson (YTD) conjecture in this setting, here referring to its natural generalisation to the case of a general Kähler manifold (X, ω) .*

5.4 Further developments in the case $\text{Aut}_0(X) \neq \{0\}$. Proof of Theorem J.

We here treat the more involved case of cscK manifolds (X, ω) whose connected automorphism group $\text{Aut}_0(X) \neq \{0\}$ is non-trivial (so X admits holomorphic vector fields). As a main result of this section we give a number of equivalent characterizations of test configurations with vanishing Donaldson-Futaki invariant. In order to state and prove it, we first need some preparatory work, in particular extending the main results from Section 4 on energy functional asymptotics, and exploiting the flexibility given by the compatibility notion for subgeodesic rays.

We first recall some basic preliminaries on holomorphic vector fields and global flows on compact Kähler manifolds, see e.g. [For96] as a reference, and introduce the notation that we will use.

Background on holomorphic vector fields, flows and the Futaki invariant

Let (X, ω) be a compact Kähler manifold and denote by $J : TX \rightarrow TX$ the associated complex structure. We take here the viewpoint of considering *real* holomorphic vector fields on our compact complex manifold X . We denote the real tangent bundle of X by TX , which can be identified with the complex vector bundle $TX^{1,0}$. The former can be viewed as a complex vector bundle via the complex structure $J : TX \rightarrow TX$ and identifying J with multiplication by i . The latter is given as the eigensubbundle of $TX \otimes \mathbb{C}$ corresponding to the eigenvalue i of the extension of $J : TX \rightarrow TX$ to $TX \otimes \mathbb{C}$. There is an identification

$$TX \longrightarrow TX^{1,0} \tag{5.1}$$

$$V \mapsto V^{\mathbb{C}} := \frac{1}{2}(V - iJV).$$

By a real vector field we mean a section of TX .

Definition 5.4.1. *A real vector field V on X is said to be real holomorphic if the flow preserves the complex structure, i.e. it has vanishing Lie derivative $L_V J = 0$. Equivalently, a real vector field V on X is holomorphic if the corresponding $(1,0)$ -field $V^{\mathbb{C}}$ is a holomorphic section of the bundle $T^{1,0}X$.*

Recall that a holomorphic vector field on a *compact* manifold is automatically \mathbb{C} -complete. In case the vector field V is \mathbb{C} -complete, then its *flow* ϕ_t is an action of $(\mathbb{C}, +)$ on X by holomorphic automorphisms. Conversely, one may associate to every additive action $\phi : \mathbb{C} \times X \rightarrow X$ by holomorphic automorphisms on X the vector field

$$V_\phi(x) := \frac{d}{dt} \phi(t, x)|_{t=0},$$

called the *infinitesimal generator* of X . It is holomorphic and \mathbb{C} -complete on X , with the flow ϕ . In the sequel we will primarily privilege the point of view of working with real holomorphic vector fields V , keeping in mind the identification (5.1).

Definition 5.4.2. A real holomorphic vector field V on X is said to be Hamiltonian if it admits a Hamiltonian potential $h_\omega^V \in C^\infty(X, \mathbb{R})$ such that the contraction

$$i_V(\omega) := V \lrcorner \omega = \sqrt{-1} \bar{\partial} h_\omega^V.$$

Remark 5.4.3. Equivalently, a real holomorphic vector field admits a Hamiltonian potential if and only if it has a zero somewhere, see LeBrun-Simanca [BS94].

The Hamiltonian potential is unique up to constants. To relieve this ambiguity we impose the normalization

$$\int_X h_\omega^V \omega^n = 0.$$

Remark 5.4.4. For the purpose of comparing with the situation for polarized manifolds (X, L) it is interesting to recall that Hamiltonian vector fields are precisely those that lift to line bundles, see [Don02, Lemma 12]. This is a key property used e.g. in the proof [BDL16] of K -polystability for polarized cscK manifolds, due to R. Berman, T. Darvas. C. Lu.

Note that a real holomorphic Hamiltonian vector field is automatically a Killing field, since $L_V J = L_V \omega = 0$ implies that also $L_V g = 0$ for the Riemannian metric associated to the Kähler form ω . We further recall that since X is compact it follows that the Lie algebra of real holomorphic vector fields on X is finite dimensional. It is the (complex) Lie algebra of the complex Lie group $\text{Aut}(X)$. We refer to e.g the expository notes [Sze14] for details and further references.

We further recall the definition of the Futaki invariant, originally introduced in [Fut83].

Definition 5.4.5. Let (X, ω) be a compact Kähler manifold with $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ the corresponding Kähler class. Let \mathfrak{h} be the Lie algebra of real holomorphic Hamiltonian vector fields on X . The Futaki invariant is the Lie algebra character $\text{Fut}_\alpha : \mathfrak{h} \rightarrow \mathbb{R}$ defined by

$$\text{Fut}_\alpha(V) := \int_X h_\omega^V (S(\omega) - \bar{S}) \omega^n,$$

where V be a real holomorphic Hamiltonian vector field on X and $h_\omega^V \in C^\infty(X, \mathbb{R})$ is the associated Hamiltonian.

Note that the Futaki invariant is independent of the Kähler class considered:

Theorem 5.4.6. ([Fut83]) The above character is independent of representative of the Kähler class $\alpha := [\omega]$. In particular, if $[\omega]$ admits a cscK metric, then $\text{Fut}_\alpha(V) = 0$ for each $V \in \mathfrak{h}$.

Remark 5.4.7. There are several other common situations in which the Futaki character Fut_α vanishes identically: In particular, it suffices that (X, α) K -semistable, as explained in the proof of Proposition 5.4.26. More generally, if the connected Lie group $\text{Aut}_0(X)$ of automorphisms of X is semisimple, then $\text{Fut}_\alpha(V) = 0$ for each $V \in \mathfrak{h}$ and for each Kähler class α on X , see [BS94].

Energy functional asymptotics extended to singular test configurations

As remarked in Question 3.2.26, if we wish to say something about K-polystability of cscK manifolds, then we *cannot* a priori always restrict to the study of smooth and dominating test configurations. For this reason it is therefore an important tool to be able to consider energy functional asymptotics also along rays associated to arbitrary (normal and relatively Kähler) test configurations for (X, α) . Since such test configurations have possibly singular central fiber a pullback to a desingularisation will lead to considering the case of smooth and dominating test configurations that are merely *relatively nef* (with the loss of positivity occurring over the ample locus). We now extend some of the formalism from Section 4 to the general setting of test configurations whose central fiber is possibly singular.

Definition of compatibility for rays associated to singular test configurations

Let $(\mathcal{X}, \mathcal{A})$ be a normal and relatively Kähler test configuration, and let $(\varphi_t)_{t \geq 0}$ be a subgeodesic ray in \mathcal{H} . Let $\Phi(x, e^{-t+is}) = \varphi_t(x)$ be the corresponding S^1 -invariant $p_1^* \omega$ -psh function on $X \times \Delta^*$. We then have the following extension of Definition 4.1.3 to the case of arbitrary normal and relatively Kähler (or relatively nef) test configurations:

Definition 5.4.8. (General definition of compatibility notions) *A subgeodesic ray $(\varphi_t)_{t \geq 0}$ is L^∞ -compatible (resp. C^∞ -compatible) with $(\mathcal{X}, \mathcal{A})$ iff it is L^∞ -compatible (resp. C^∞ -compatible) with some smooth and dominating relatively nef test configuration $(\mathcal{Y}, \mathcal{B}) \sim (\mathcal{X}, \mathcal{A})$. See Section 3.2 for the definition of the equivalence relation on test configurations.*

In particular, note that there is always a smooth and relatively nef test configuration representing the equivalence class of $(\mathcal{X}, \mathcal{A})$ (e.g. consider resolution of singularities and pull back \mathcal{A}). Concretely, we can interpret Definition 5.4.8 as follows: Let Φ be the S^1 -invariant function on $X \times \mathbb{P}^1$ associated to the given ray $(\varphi_t)_{t \geq 0}$. By Proposition 3.2.18 we may write

$$\rho^* \mathcal{A} = \mu^* p_1^* \alpha + [D],$$

where $D = \sum_{j=1}^n a_j D_j$, $\rho : \mathcal{Y} \rightarrow \mathcal{X}$ is a modification, and $\mu : \mathcal{Y} \rightarrow X \times \mathbb{P}^1$ is a dominating morphism (see figure). We can decompose the current of integration $[D] = \theta_D + dd^c \psi_D$, where θ_D is a smooth S^1 -invariant $(1, 1)$ -form on \mathcal{Y} . Note that ψ_D is defined up to a smooth function on \mathcal{Y} . By definition $(\varphi_t)_{t \geq 0}$ is L^∞ -compatible with $(\mathcal{X}, \mathcal{A})$ if $\Psi := \Phi \circ \mu + \psi_D$ extends to a $\rho^* \Omega$ -psh function on \mathcal{Y} . In particular, the singularity type of $\Phi \circ \mu$ is determined by the Green function ψ_D .

$$\begin{array}{ccccc}
 \tilde{\mathcal{X}} & & & & \\
 \downarrow \rho & \searrow \mu & & & \\
 \mathcal{X} & \dashrightarrow & X \times \mathbb{P}^1 & \xrightarrow{p_1} & X
 \end{array}$$

Remark 5.4.9. *Note that we do not talk about $\mathcal{C}^{1,\bar{1}}$ -compatibility in the singular setting. The reason for this is that, unlike the L^∞ and C^∞ case, this regularity property is not preserved under pullback to the smooth model \mathcal{Y} .*

As an immediate corollary we obtain the following extension of Theorem C:

Theorem 5.4.10. *Theorem 4.2.2 holds for arbitrary normal and relatively Kähler test configurations for (X, α) .*

Note that the above discussion does *not* hold true for $\mathcal{C}^{1,\bar{1}}$ -compatibility (see Remark 5.4.9, so these results are not directly applicable to the K-energy. This is one of the difficulties that are circumvented in the characterization of cohomological test configurations with vanishing Donaldson-Futaki invariant, cf. Section 5.4. Note also that $\mu^*\mathcal{A}$ is not relatively Kähler, so the above is coherent with the statement of the injectivity lemma 5.1.2.

Asymptotics for the Mabuchi- and J-functional in the singular case

For later use, we first note that as a particular case of Theorem 5.4.10, we obtain the following generalization of Corollary 4.2.8:

Theorem 5.4.11. *Let $(\mathcal{X}, \mathcal{A})$ be a normal test configuration for (X, α) and let $t \mapsto \psi_t$ be a subgeodesic ray L^∞ -compatible with $(\mathcal{X}, \mathcal{A})$. Suppose that $\hat{\mathcal{X}}$ is a smooth and dominating test configuration, with $\rho : \hat{\mathcal{X}} \rightarrow \mathcal{X}$ the associated morphism. Then the following limit is well-defined and*

$$\frac{J(\psi_t)}{t} \longrightarrow J^{\text{NA}}(\mathcal{X}, \mathcal{A}),$$

as $t \rightarrow +\infty$. Here

$$J^{\text{NA}}(\mathcal{X}, \mathcal{A}) := V^{-1} \left\{ (\mathcal{A} \cdot \mu^* p_1^* \alpha^n) - \frac{(\mathcal{A}^{n+1})}{n+1} \right\}.$$

The intersection numbers are computed on $\hat{\mathcal{X}}$.

Along the same lines we see that the asymptotics of the K-energy functional may be estimated along C^∞ -compatible rays (or more generally along so called $\mathcal{C}^{1,\bar{1}}$ -compatible rays, cf. Section 5.1, but we will not need this here). What we will be using is the following version of [SD16, Theorem 5.1], valid also for mildly singular (i.e. normal) test configurations. As before, it should be understood that all intersection numbers are computed on any resolution, as in the definition of C^∞ -compatibility.

Theorem 5.4.12. *Let $(\mathcal{X}, \mathcal{A})$ be a normal and relatively Kähler test configuration for (X, α) . Let $\varphi_0 \in \mathcal{H}_0$. Then there is a smooth ray $[0, +\infty[\ni t \mapsto \psi_t \in \text{PSH}(X, \omega) \cap C^\infty(X)$ on X emanating from φ_0 , that is C^∞ -compatible with $(\mathcal{X}, \mathcal{A})$ and satisfies*

$$\overline{\lim}_{t \rightarrow +\infty} \frac{M(\psi_t)}{t} \leq M^{\text{NA}}(\mathcal{X}, \mathcal{A}).$$

Proof. We begin by constructing such a ray $r \mapsto \psi_t$. To do this, consider any smooth model $\hat{\mathcal{X}}$ for \mathcal{X} , with $\rho : \hat{\mathcal{X}} \rightarrow \mathcal{X}$ the associated morphism. Since \mathcal{A} is relatively Kähler on \mathcal{X} , note that $\hat{\mathcal{A}} := \rho^* \mathcal{A}$ is relatively nef on $\hat{\mathcal{X}}$ (with the loss of positivity occurring over the central fiber $\hat{\mathcal{X}}_0$). Now let Ω be any smooth S^1 -invariant $(1, 1)$ form on \mathcal{X} such that $[\Omega] = \mathcal{A}$. For each $\tau \in \mathbb{C}^*$ we write $\rho(\tau) : \hat{\mathcal{X}}_\tau \rightarrow \mathcal{X}_\tau$ for the isomorphism between the respective fibers, and $\lambda(\tau) : \mathcal{X}_\tau \rightarrow X$. In particular, we identify X with \mathcal{X}_1 and $\hat{\mathcal{X}}_1$, via the respective isomorphisms $\lambda(1)$ and $\lambda(1) \circ \rho(1)$, and write $\hat{\lambda}(\tau) : \hat{\mathcal{X}}_\tau \rightarrow X$ for the composition $\lambda(\tau) \circ \rho(\tau)$.

Now note that $\rho(\tau)^* \Omega_\tau := \hat{\Omega}_\tau$ and $\rho(1)^* \Omega_1 := \hat{\Omega}_1$ are cohomologous, so there is for each $\tau \neq 0$ a smooth real S^1 -invariant function ξ_τ on X , for which

$$\hat{\lambda}(\tau)^* \hat{\Omega}_\tau = \hat{\lambda}(1)^* \hat{\Omega}_1 + dd^c \xi_\tau.$$

Then $t \mapsto \psi_t := \xi_\tau$, with $\tau := e^{-t+is}$, defines a smooth ray that is moreover C^∞ -compatible with $(\hat{\mathcal{X}}, \hat{\mathcal{A}})$, thus also with $(\mathcal{X}, \mathcal{A})$ by Definition 5.4.8. We finally claim that the proof of [SD16, Theorem 5.1] can be seen to go through for this smooth ray (ψ_t) . Indeed $\hat{\Omega}_1 + dd^c \xi_\tau$ is strictly positive away from $\hat{\mathcal{X}}_0$, and due to the projection formula neither the lhs nor the rhs sees the central fiber. \square

Remark 5.4.13. In the case of polarized manifold (X, L) it is known that the asymptotic expansion of the Mabuchi functional holds also for rays compatible with non-ample test configurations $(\mathcal{X}, \mathcal{L})$, as long as the corresponding metric on \mathcal{L} is smooth on \mathcal{X} and strictly psh away from \mathcal{X}_0 . Note also that when $\mathcal{X}_{\text{sing}} \neq \emptyset$ it is not clear whether the asymptotic holds for the associated geodesic ray (since this ray is of regularity $\mathcal{C}^{1,1}$ only over the ample locus).

A characterization of test configurations with vanishing Donaldson-Futaki invariant.

Proof of Theorem J.

Let (X, ω) be a cscK manifold, with $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ the associated Kähler class. As an application of the techniques developed in the previous sections, we establish a number of conditions equivalent to the vanishing of the Donaldson-Futaki invariant. However, for technical reasons this result is formulated for test configurations with vanishing Donaldson-Futaki invariant whose associated geodesic rays are normalized so that $E(\varphi_t) = 0$ for each t . As it turns out, this is not a serious restriction. In order to discuss these points we introduce a certain projection operator on rays and on test configurations:

Let (X, ω) be a given cscK manifold with $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ the associated Kähler class. Fix $\varphi_0 \in \mathcal{H}_0$ a cscK potential. Consider the projection operator

$$\mathcal{P} : \mathcal{H} \simeq \mathcal{H}_0 \times \mathbb{R} \longrightarrow \mathcal{H}_0$$

$$\varphi \mapsto \varphi - E(\varphi),$$

where $E : \mathcal{H} \rightarrow \mathbb{R}$ is the Aubin-Mabuchi energy functional. If $(\mathcal{X}, \mathcal{A})$ is a test configuration for (X, α) , then there is a unique geodesic ray $(\varphi_t)_{t \geq 0}$ (emanating from $\varphi_0 \in \mathcal{H}_0$) that is also L^∞ -compatible with $(\mathcal{X}, \mathcal{A})$. The projection \mathcal{P} can also be defined on relatively Kähler test

configurations, by embedding the latter in the space of geodesic rays emanating from the given cscK potential $\varphi_0 \in \mathcal{H}_0$. This is possible due to the injectivity lemma, see Section 5.1.

Definition 5.4.14. *The projection $\mathcal{P}(\mathcal{X}, \mathcal{A})$ of a relatively Kähler test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α) is defined as the unique test configuration L^∞ -compatible with the geodesic ray $\mathcal{P}(\varphi_t)$.*

For later use, and in order to justify the above definition, we establish below some properties of the projection \mathcal{P} on rays and on test configurations.

Proposition 5.4.15. (Basic properties of the projection) *Let $\mathcal{P} : \mathcal{H} \rightarrow \mathcal{H}_0 \simeq \mathcal{K}$ be as above. Let $(\mathcal{X}, \mathcal{A})$ be a normal and relatively Kähler test configuration for (X, α) . Then*

1. $\mathcal{P}(\mathcal{X}, \mathcal{A})$ is a product iff $(\mathcal{X}, \mathcal{A})$ is a product.
2. $J^{\text{NA}}(\mathcal{P}(\mathcal{X}, \mathcal{A})) = J^{\text{NA}}(\mathcal{X}, \mathcal{A})$.
3. $(\mathcal{P}(\varphi_t))_{t \geq 0}$ is a geodesic iff $(\varphi_t)_{t \geq 0}$ is a geodesic.
4. There is a unique normal and relatively Kähler cohomological test configuration $\mathcal{P}(\mathcal{X}, \mathcal{A})$ for (X, α) with which $\mathcal{P}(\varphi_t)$ is compatible. It equals $(\mathcal{X}, \mathcal{A} - c[\mathcal{X}_0])$, where c is the slope of the linear function $t \mapsto E(\varphi_t)$.

Proof. The first subpoint (1) is an immediate consequence of (4), which in particular shows that $\mathcal{P}(\mathcal{X}) = \mathcal{X}$. The second point follows from the fact that $J(\varphi + c) = J(\varphi)$, $c \in \mathbb{R}$, so in particular $J(\mathcal{P}(\varphi_t)) = J(\varphi_t - E(\varphi_t)) = J(\varphi_t)$ for each $t \in [0, +\infty)$. Dividing by t and passing to the limit, assertion (2) follows. The subpoint (3) follows since $\mathcal{P}(\varphi_t) = \varphi_t - E(\varphi_t) = \varphi_t + at + b$ for some $a, b \in \mathbb{R}$, since the function $t \mapsto E(\varphi_t)$ is linear along geodesics. It follows that $E(\varphi_t)$ is harmonic for each t . Assertion (3) follows. Finally, in order to prove the last subpoint, recall that compatibility is determined by \mathcal{X} and the Green function ψ_D associated to the divisor D supported on \mathcal{X}_0 , satisfying $\rho^* \mathcal{A} = \mu^* p_1^* \alpha + \delta_D$ (see figure).

$$\begin{array}{ccc} \mathcal{X}' & & \\ \rho \downarrow & \searrow \mu & \\ \mathcal{X} & \dashrightarrow & X \times \mathbb{P}^1 \end{array}$$

But changing φ_t for $\mathcal{P}(\varphi_t)$ we preserve the compatibility relation (see Definition 5.4.8) by also changing ψ_D for $\psi_D - at - b$, where a is the slope of the linear function $t \mapsto E(\varphi_t)$. This corresponds precisely to the test configuration $(\mathcal{X}, \mathcal{A} - c[\mathcal{X}_0])$, which is relatively Kähler iff $(\mathcal{X}, \mathcal{A})$ is. \square

(It is immediately clear that $\mathcal{P}(\mathcal{X}, \mathcal{A})$ is normal and relatively Kähler, because $\mathcal{X}_0 = \pi^{-1}(0)$ is just a single fiber).

We now set out to prove the following analog of Theorem 5.3.2 in the case of $\text{Aut}_0(X) \neq \{0\}$ (cf. also [BDL16, Lemma 3.1] in the polarized case). It holds only for test configurations whose geodesic rays are normalized so that $E(\varphi_t) = 0$. Otherwise put, given a test configuration $(\mathcal{X}, \mathcal{A})$

we may instead consider $\mathcal{P}(\mathcal{X}, \mathcal{A})$ which then satisfies this property. Since the total space does not change under projection, this is often not a serious restriction (e.g. if one wishes to prove that test configurations with vanishing Donaldson-Futaki invariant are products, see Proposition 5.4.15).

Theorem 5.4.16. (cf. Theorem J) *Suppose that (X, ω) is a cscK manifold, with $\alpha := [\omega] \in H^{1,1}(X, \mathbb{R})$ the corresponding Kähler class. Let $(\mathcal{X}, \mathcal{A})$ be a normal and relatively Kähler test configuration for (X, α) whose associated geodesic ray $(\varphi_t)_{t \geq 0}$ satisfies $E(\varphi_t) = 0$ for each $t \in [0, +\infty)$. Let $J : TX \rightarrow TX$ be the complex structure and ω a cscK metric on X . Then the following statements are equivalent:*

1. $DF(\mathcal{X}, \mathcal{A}) = 0$.
2. The central fiber \mathcal{X}_0 is reduced and $M^{\text{NA}}(\mathcal{X}, \mathcal{A}) = 0$.
3. The central fiber \mathcal{X}_0 is reduced and the Mabuchi K-energy functional is constant along the geodesic ray $(\varphi_t)_{t \geq 0}$ associated to $(\mathcal{X}, \mathcal{A})$, i.e. we have $M(\varphi_t) = M(\varphi_0)$ for each $t \in [0, +\infty)$.
4. The central fiber \mathcal{X}_0 is reduced and the associated geodesic ray satisfies

$$\inf_{g \in G} J(g \cdot \varphi_t) = 0 \quad \text{and} \quad \inf_{g \in G} d_1(0, g \cdot \varphi_t) = 0.$$

5. The central fiber \mathcal{X}_0 is reduced and there is a real holomorphic Hamiltonian vector field V such that the geodesic ray $(\varphi_t)_{t \geq 0}$ associated to $(\mathcal{X}, \mathcal{A})$ satisfies $\exp(tV)^*\omega = \omega$ and $\exp(tJV)^*\omega = \omega_{\varphi_t}$.
6. The central fiber \mathcal{X}_0 is reduced and the associated geodesic ray (φ_t) consists entirely of cscK potentials. More precisely, if \bar{S} denotes the mean scalar curvature of ω_{φ_0} , then

$$\mathcal{S}(\omega_{\varphi_t}) = \bar{S}$$

for each $t \in [0, +\infty)$.

The structure of the proof is the following: The implications $(1) \Rightarrow (2) \Rightarrow (5)$ relies on the result B on asymptotics of the Mabuchi functional, extended to the setting of possibly singular cohomological test configurations. It is worth pointing out that, unlike in the polarized case, the asymptotics a priori yields only an upper bound $o(t)$ in (4). However, one can then improve this by other means to find the given statement above, namely by first proving (5) and noting that $(5) \Rightarrow (4)$. In order to establish $(4) \Leftrightarrow (5)$ we also rely on a very slight variation of the proof of [BDL16, Lemma 3.1], using the important G -coercivity result [BDL16, Theorem 1.5] and noting that J_G and $d_{1,G}$ are comparable [DR17, Lemma 5.11]. Further, we show $(5) \Rightarrow (2) \Rightarrow (1)$ by means of simple argument expressing the slope of the Mabuchi functional along the rays given by the flow of a holomorphic Hamiltonian vector field as in (5) in terms of the (original) Futaki invariant introduced in [Fut83]. Several of these arguments seem to be closely related to the seminal work of Mabuchi [Mab86, Section 5].

Remark 5.4.17. *The equivalence (1) \Leftrightarrow (2) holds without the assumption that $E(\varphi_t) = 0$ for each t .*

Proof of (1) \Rightarrow (2) \Rightarrow (5)

As a first application of the formalism developed in 5.4 we show that normal and relatively Kähler test configurations $(\mathcal{X}, \mathcal{A})$ with vanishing Donaldson-Futaki invariant automatically have reduced central fiber, and the $d_{1,G}$ -length of geodesics associated to such test configurations grows like a small $o(t)$. Recall that we view $\varphi_t(x) := \varphi(t, x)$ as a function on $X \times \mathbb{P}^1 \setminus \{0\}$, and $G := \text{Aut}(X)_0$.

Lemma 5.4.18. *Suppose that $(\mathcal{X}, \mathcal{A})$ is a normal, relatively Kähler cohomological test configuration for (X, α) . Let $[0, +\infty) \ni t \mapsto \psi_t \in \text{PSH}(X, \omega) \cap C^\infty(X)$ be as in Theorem 5.4.12, emanating from a cscK potential $\psi_0 \in \mathcal{H}_0$. If $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$, then*

1. $\mathcal{X}_{0,red} = \mathcal{X}_0$.
2. $0 \leq d_{1,G}(G\psi_0, G\psi_t) \leq o(t)$.

Proof. If $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$ it follows from Theorem 5.4.12 that

$$0 \leq \lim_{t \rightarrow +\infty} \frac{M(\psi_t)}{t} \leq 0 + ((\mathcal{X}_{0,red} - \mathcal{X}_0) \cdot \mathcal{A}^n) \leq 0,$$

where the lower bound of M holds because φ_0 is a cscK potential. This forces $\mathcal{X}_{0,red} = \mathcal{X}_0$. As a consequence, we have

$$\overline{\lim}_{t \rightarrow +\infty} \frac{M(\psi_t)}{t} \leq 0,$$

so $M(\psi_t) \leq o(t)$. Since (X, α) is a cscK manifold, the Mabuchi functional is G -coercive [BDL16, Theorem 1.1]. Hence $0 \leq J_G(G\psi_t) \leq o(t)$. The conclusion (2) then follows immediately from the fact that the growth of J -functional is the same as that of the d_1 -metric [DR17, Proposition 5.5]. \square

Remark 5.4.19. *The above result also holds if one replaces the hypothesis $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$ with $M^{\text{NA}}(\mathcal{X}, \mathcal{A}) = 0$.*

Lemma 5.4.18 is useful in combination with the following observation:

Lemma 5.4.20. *Suppose that $|\psi_t - \varphi_t|$ is uniformly bounded in t . Then $J_G(G\varphi_t) \leq o(t)$ iff $J_G(G\psi_t) \leq o(t)$.*

Proof. Suppose that $\varphi_t - \psi_t$ is uniformly bounded in t . By [BDL16, Proposition 2.2] the infimum $J_G(G\psi_t) = \inf_{g \in G} J(g.\psi_t)$ is attained, so there is a constant $C > 0$ and a sequence $\{g_t\}_{t \geq 0} \subset G^{\mathbb{N}}$ such that $J_G(G\psi_t) = \int_X g_t.\psi_t \omega^n \leq C$. Since $0 \leq J_G(G\psi_t) \leq o(t)$ by assumption, by possible increasing the constant C , the expression (2.15) yields

$$0 \leq J_G(G\varphi_t) \leq J_G(G\psi_t) + \left\{ \int_X g_t.\varphi_t \omega^n - \int_X g_t.\psi_t \omega^n \right\}$$

$$= J_G(G\psi_t) + \int_X g_t^*(\varphi_t - \psi_t) \omega^n \leq o(t) + C = o(t).$$

Indeed, the function $g_t^*(\varphi_t - \psi_t)$ is uniformly bounded in t , because $\varphi_t - \psi_t$ is. \square

Note that any smooth subgeodesic ray $t \mapsto \psi_t$, C^∞ -compatible with $(\mathcal{X}, \mathcal{A}) \in \text{TC}$, is also L^∞ -compatible with the unique geodesic ray $t \mapsto \varphi_t$ associated to $(\mathcal{X}, \mathcal{A})$, emanating from φ_0 . As a consequence of the above Lemma 5.4.20 we then see that the conclusion of Lemma 5.4.18 holds for the unique geodesic ray $t \mapsto (\varphi_t)_{t \geq 0}$ associated to the test configuration $(\mathcal{X}, \mathcal{A})$, see Lemma 4.1.10 for the construction:

Corollary 5.4.21. *Suppose that $(\mathcal{X}, \mathcal{A})$ is a normal, relatively Kähler cohomological test configuration for (X, α) satisfying $\text{DF}(\mathcal{X}, \mathcal{A})$. Let $[0, +\infty) \ni t \mapsto \varphi_t \in \text{PSH}(X, \omega) \cap C^\infty(X)$ be the unique associated geodesic ray emanating from a cscK potential $\varphi_0 \in \mathcal{H}_0$. Then $\mathcal{X}_{0, \text{red}} = \mathcal{X}_0$ and $0 \leq d_{1,G}(G\varphi_0, G\varphi_t) \leq o(t)$.*

The following result is a very slight modification of the compactness argument of [BDL16, Proposition 3.1]. The only new observation is that we may replace the upper bound $\leq C$ by $\leq o(k)$. We give the necessary details showing that the proof then goes through in the same way. Note that the argument in question crucially relies on the assumption that the ray is a geodesic (cf. Proposition 5.4.15). See Section 2.4 for the definition of the action on potentials.

Proposition 5.4.22. (cf. [BDL16, Proposition 3.1]) *Suppose that $(\mathcal{X}, \mathcal{A})$ is a normal and relatively Kähler test configuration for (X, α) , with $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$. Let $\varphi_0 \in \mathcal{H}_0$ be a cscK potential and suppose that the unique associated geodesic ray $[0, +\infty) \ni t \mapsto \varphi_t \in \text{PSH}(X, \omega) \cap L^\infty(X)$ emanating from φ_0 is normalized so that $E(\varphi_t) = 0$ for each t . Then there is a real holomorphic Hamiltonian vector field $V \in \text{isom}(X, \omega_{\varphi_0})$ such that $\varphi_t = \exp(tJV) \cdot \varphi_0$.*

Proof. By Corollary 5.4.21 the $d_{1,G}$ -length of the geodesic (φ_t) is controlled by the inequality $0 \leq d_{1,G}(G\varphi_0, G\varphi_t) \leq o(t)$. We may thus find a sequence $\{g_k\}_{k \in \mathbb{N}} \subset G^{\mathbb{N}}$ such that $k^{-1}d_1(\varphi_0, g_k \cdot \varphi_k) \rightarrow 0$ as $k \rightarrow +\infty$. Because G is reductive we may for each k write $g_k = h_k \exp(-JV_k)$, where $h_k \in \text{Isom}_0(X, \omega_{\varphi_0})$ and $V_k \in \text{isom}(X, \omega_{\varphi_0})$ is a non-zero real holomorphic Hamiltonian vector field, see e.g. [DR17, Propositions 6.2 and 6.9]. Since G acts on \mathcal{H} by d_1 -isometries [DR17, Lemma 5.9] we see that

$$o(k) \geq d_1(\varphi_0, g_k \cdot \varphi_k) = d_1(g_k^{-1} \cdot \varphi_0, \varphi_k) = d_1(\exp(JV_k) \cdot \varphi_0, \varphi_k). \quad (5.2)$$

The geodesic $t \mapsto \varphi_t$ has constant speed which we may assume equal to t , i.e. $d_1(\varphi_0, \varphi_t) = t$, so the triangle inequality yields the double inequality

$$k - o(k) \leq d_1(\varphi_0, \exp(JV_k) \cdot \varphi_0) \leq k + o(k). \quad (5.3)$$

Now note that $t \mapsto \exp(tJV) \cdot \varphi_0$ is a d_1 -geodesic ray emanating from φ_0 (see [DR17, Section 7.2]). Since d_1 is linear along geodesics it follows from (5.3) that

$$1 - \frac{o(k)}{k} \leq d_1(\varphi_0, \exp\left(\frac{JV_k}{k}\right) \cdot \varphi_0) \leq 1 + \frac{o(k)}{k}.$$

Using (2.15) we conclude that there is a $D > 1$ and a $k_0 \in \mathbb{N}$ such that $1/D \leq \|JV_k/k\| \leq D$ for all $k \geq k_0$. Since the space of holomorphic Hamiltonian Killing vector fields of $(X, \omega_{\varphi_0}, J)$ is finite dimensional, it follows that there is a subsequential limit, i.e. there is a $0 \neq V \in \text{isom}(X, \omega_{\varphi_0})$ such that $V_{k_j}/k_j \rightarrow V$ as $k_j \rightarrow +\infty$. We finally argue that in fact $\varphi_t = \exp(tJV) \cdot \varphi_0$, following [BDL16]: For each k , consider the smooth d_1 -geodesic segments

$$[0, k] \ni t \mapsto \varphi_t^k := \exp\left(t \frac{V_k}{k}\right) \cdot \varphi_0 \in \mathcal{H}_0$$

Note that the function $t \mapsto h(t) := d_1(\varphi_t^k, \varphi_t)$ is convex (since (φ_t) and (φ_t^k) are both d_1 -geodesic rays), with $h(0) = 0$ and $h(k) \leq o(k)$. The above inequality (5.2) then implies that, for any fixed $t \in [0, k]$ we have $d_1(\varphi_t^k, \varphi_t) \leq t \frac{h(k)}{k} \rightarrow 0$, as $k \rightarrow +\infty$. Exactly as in [BDL16, Lemma 2.7] we then get that $\exp(tV_{k_j}/k_j) \cdot \varphi_0 \rightarrow \exp(tJV) \cdot \varphi_0$ smoothly, concluding the proof. \square

Putting these results together we see that if $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$, then \mathcal{X}_0 is reduced and so $((\mathcal{X}_{0,red} - \mathcal{X}_0) \cdot \mathcal{A}^n) = 0$. In particular $\text{M}^{\text{NA}}(\mathcal{X}, \mathcal{A}) = \text{DF}(\mathcal{X}, \mathcal{A}) = 0$ as well, so that (1) \Rightarrow (2). Furthermore, the above arguments show that (2) implies $d_{1,G}(\varphi_0, G \cdot \varphi_t) \leq o(t)$ and so in turn by Proposition 5.4.22 the statement (5) follows.

Proof of (5) \Leftrightarrow (4)

The implication (5) \Rightarrow (4) is immediate by definition, since then

$$0 \leq J_G(G\varphi_t) \leq J(G\varphi_0) = 0$$

for each t . The converse implication (4) \Rightarrow (5) follows precisely as in Proposition 5.4.22 above.

Proof of (5) \Rightarrow (2) \Rightarrow (1)

Lemma 5.4.23. *Suppose that h_ω^V is a Hamiltonian potential for the real holomorphic vector field $V \in \text{isom}(X, \omega)$ with respect to ω , i.e. $i_V(\omega) = i\bar{\partial}h_\omega^V$, where $h_\omega^V \in C^\infty(X, \mathbb{R})$. Then for all $t \geq 0$ we have $\exp(tV)^*\omega = \omega$ and $\exp(tJV)^*\omega = \omega_{\psi_t}$, where $(\psi_t)_{t \geq 0}$ is a smooth ray that can be chosen so that $\dot{\psi}_t = \exp(tJV)^*h_\omega^V$ and $E(\psi_t) = 0$ for each t .*

Proof. Write f_t for the flow $\exp(tJV)$ of JV . Then $i_V(\omega) = \sqrt{-1}\bar{\partial}h_\omega^V$ implies that

$$i_V(f_t^*h_\omega^V) = f_t^*i_V(\omega) = f_t^*(\sqrt{-1}\bar{\partial}h_\omega^V) = \sqrt{-1}\bar{\partial}(f_t^*h_\omega^V).$$

Hence

$$d \circ i_V(f_t^*\omega) = \sqrt{-1}\bar{\partial}f_t^*h_\omega^V = dd^c f_t^*h_\omega^V.$$

On the other hand there is a unique smooth ray $(\psi_t)_{t \geq 0}$ in \mathcal{H}_0 such that for all t we have $E(\psi_t) = 0$ and $f_t^*\omega = \omega_{\psi_t}$. As a consequence

$$d \circ i_V(f_t^*\omega) = d \circ i_V(\omega_{\psi_t}) = L_V(\omega_{\psi_t}) = dd^c \psi_t.$$

Since X is compact we then have $\dot{\psi}_t = f_t^* h_\omega^V + C$, for some constant $C = C(t)$. We then conclude by showing that with the normalization

$$\int_X h_\omega^V \omega^n = 0,$$

since we assume that $E(\psi_t) = 0$ for each t , we have

$$0 = \frac{d}{dt} E(\psi_t) = \int_X \dot{\psi}_t \omega_{\psi_t}^n = \int_X h_\omega^V \omega^n + \int_X C(t) \omega_{\psi_t}^n = \int_X C(t) \omega^n,$$

and hence $C(t) = 0$ for each t . \square

Proposition 5.4.24. *Let $(\varphi_t)_{t \geq 0}$ be the unique geodesic ray associated to $(\mathcal{X}, \mathcal{A})$. Then $t \mapsto M(\varphi_t)$ is linear with slope given by the Futaki invariant $\text{Fut}_\alpha(V)$. In particular, if $\varphi_0 \in \mathcal{H}_0$ is a cscK potential, then $M(\varphi_t) = M(\varphi_0)$ for all $t \geq 0$. If moreover the central fiber \mathcal{X}_0 is reduced, then $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$.*

Proof. Since the geodesic ray $(\varphi_t)_{t \geq 0}$ is smooth we may compute

$$\begin{aligned} V \cdot \frac{d}{dt} M(\varphi_t) &= \int_X \dot{\varphi}_t (\bar{S} - S(\omega_{\varphi_t})) \omega_{\varphi_t}^n \\ &= \int_X f_t^* h_\omega^V (\bar{S} - f_t^* S(\omega)) f_t^* \omega^n = \int_X h_\omega^V (\bar{S} - S(\omega)) \omega^n = V \cdot \text{Fut}_\alpha(V). \end{aligned}$$

We have here used that $f_t^* S(\omega) = S(f_t^* \omega)$. If we assume that $S(\omega) = \bar{S}$, i.e. that $0 \in \mathcal{H}$ is a cscK potential, then $\text{Fut}_\alpha(V)$ vanishes. By linearity $t \mapsto M(\varphi_t)$ is constant and $\text{Fut}_\alpha(V) = M^{\text{NA}}(\mathcal{X}, \mathcal{A}) = 0$. If the central fiber \mathcal{X}_0 is reduced we moreover have $\text{DF}(\mathcal{X}, \mathcal{A}) = M^{\text{NA}}(\mathcal{X}, \mathcal{A}) = 0$, which is what we wanted to prove. \square

Putting this together shows that (5) \Rightarrow (2) \Rightarrow (1).

Proof of (2) \Leftrightarrow (3)

We here wish to show that if $M^{\text{NA}}(\mathcal{X}, \mathcal{A}) = 0$ then M is constant along the associated geodesic ray emanating from the given cscK potential $\varphi_0 \in \mathcal{H}_0$. The proof is given precisely as in Proposition 5.4.24. Indeed, if $M^{\text{NA}}(\mathcal{X}, \mathcal{A}) = 0$ and the central fiber \mathcal{X}_0 is reduced, then (5) holds, so in particular we are in the situation of Proposition 5.4.24. As before

$$0 = \frac{d}{dt} M(\varphi_t) = \text{Fut}_\alpha(V).$$

and so $M(\varphi_t)$ is constant. The converse is immediate, since by Theorem 4.2.2 we have

$$M^{\text{NA}}(\mathcal{X}, \mathcal{A}) = \lim_{t \rightarrow +\infty} \frac{M(\varphi_t)}{t}.$$

Proof of (3) \Leftrightarrow (6)

The equivalence (3) \Leftrightarrow (6) is an immediate consequence of the characterization of cscK metrics as the minima of the Mabuchi functional: Indeed, on the one hand, if $M(\varphi_t) = M(\varphi_0)$ then since φ_0 is a minimum, so is φ_t . Hence φ_t is a cscK metric. Conversely, the Mabuchi functional is (tautologically) constant on the minimum set.

Product configurations and partial results on K-polystability of cscK manifolds

We here discuss product configurations and some partial results in the direction of a possible proof of K-polystability of cscK manifolds. Indeed, K-polystability of cscK manifolds follows if we can show that any of the conditions in Theorem J are equivalent to $(\mathcal{X}, \mathcal{A})$ being a product. We first recall the definition of product configuration used in this thesis, and establish some basic properties.

Definition 5.4.25. *A cohomological test configuration $(\mathcal{X}, \mathcal{A}, \pi)$ for (X, α) is said to be a product configuration if if the canonical morphism $\mu|_{\pi^{-1}(\mathbb{C})} : \mathcal{X}_{\pi^{-1}(\mathbb{C})} \rightarrow X \times \mathbb{C}$ is an isomorphism.*

As previously remarked (cf. Section 3 and example 3.1.13) note that the compactification of an algebraic test configuration is not always a product. Hence the YTD conjecture could not hold if we defined K-polystability by asking that $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$ iff $\mathcal{X} \simeq X \times \mathbb{P}^1$.

Cohomological product configurations and their associated geodesic rays

In the case of polarized manifolds it is well-known that the geodesic ray associated to a product test configuration is given by pullback via the \mathbb{C}^* -action (see e.g. [Ber16]). Moreover, up to a multiplicative constant the Donaldson-Futaki invariant equals the Futaki invariant. In particular, the Donaldson-Futaki invariant of a product configuration for a cscK manifold (or merely K-semistable) (X, α) must vanish. We here check the analog of these results for cohomological product configurations:

Proposition 5.4.26. *Let $(\mathcal{X}, \mathcal{A})$ be a product test configuration for any (not necessarily cscK) pair (X, α) , induced by a \mathbb{C}^* -action λ on X . Let V_λ be the infinitesimal generator of ρ . Then*

$$\text{DF}(\mathcal{X}, \mathcal{A}) = \text{Fut}_\alpha(V_\lambda).$$

In particular, if (X, α) is K-semistable, then $\text{Fut}_\alpha(V) = 0$ for each $V \in \mathfrak{h}$.

Proof. Any product test configuration is given by $X \times \mathbb{P}^1$ with the \mathbb{C}^* -action given as the diagonal action induced by a one-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow \text{Aut}(X)$, i.e. $\tau \cdot (x, z) := (\lambda(\tau) \cdot x, \tau z)$. The canonical morphism associated to the induced product configuration is given by $\mu : X \times \mathbb{P}^1 \rightarrow X \times \mathbb{P}^1$ is given by $(x, z) \mapsto (\lambda^{-1}(z) \cdot x, z)$.

We now compute the geodesic ray associated to $(\mathcal{X}, \mathcal{A})$: In the notation of Lemma 4.1.10 the solution of the homogeneous Monge-Ampère equation

$$(\star) \begin{cases} (\tilde{\Omega} + dd^c \tilde{\Psi})^{n+1} = 0 & \text{on Int}(M) \\ \tilde{\Psi}|_{\partial M} = \varphi_0 + \psi_D - g' - g \end{cases}$$

is given by the Perron envelope

$$\tilde{\Psi} = \sup\{\xi \in \text{PSH}(M, \Omega) : \xi|_{\partial M} \leq \varphi_0 + \psi_D - g' - g\}$$

which simply equals the constant function $\phi_0 + \psi_D - g' - g$ here. As explained in the construction (cf. Lemma 4.1.10) the S^1 -invariant function $\Phi(x, e^{-t+is}) = \varphi_t(x)$ associated to the geodesic ray (φ_t) satisfies

$$\mu^*(p_1^*\omega + dd^c\Phi) = \tilde{\Omega} + dd^c\tilde{\Phi}.$$

and the latter form identifies with $p_1^*\omega_{\varphi_0}$. Putting this together we see that $\lambda(\tau)^*\omega_{\varphi_0} = \omega_{\varphi_t}$. In particular, if the geodesic ray is normalized so that $E(\varphi_t) = 0$ for each t , then by definition of the action of $G := \text{Aut}_0(X)$ on the space $\mathcal{H} \cap E^{-1}(0)$ of normalized Kähler potentials (cf. Section 2.4) it follows that

$$\lambda(\tau)^*\omega_{\varphi_0} = \omega + dd^c\lambda(\tau).\varphi_0.$$

It then follows from Theorem 4.2.2 and a calculation identical to that in the proof of Proposition 5.4.24 above that

$$\text{DF}(\mathcal{X}, \mathcal{A}) = \lim_{t \rightarrow +\infty} \frac{M(\varphi_t)}{t} = \text{Fut}_\alpha(V_\lambda).$$

In particular, if (X, α) is cscK then $\text{DF}(\mathcal{X}, \mathcal{A}) = 0$. Moreover, if (X, α) is merely K-semistable, then we wish to show that the Futaki invariant vanishes identically. One way of seeing this is to note that \mathfrak{h} is generated by vector fields V_λ , so if the Futaki invariant vanishes on the generators it also vanishes everywhere. However, this can be seen also by noting that $\text{Fut}_\alpha(-V) = -\text{Fut}_\alpha(V)$, whence $\text{Fut}_\alpha(V)$ must vanish. This finishes the proof. \square

Remark 5.4.27. *Note that, in passing, we have also checked that the geodesic ray associated to a product test configuration $\mathcal{P}(\mathcal{X}, \mathcal{A})$ induced by a one-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow \text{Aut}(X)$ is given by $\mathcal{P}(\varphi_t) = \lambda(\tau).\varphi_0$ for each t . The non-normalized geodesic ray associated to $(\mathcal{X}, \mathcal{A})$ still satisfies $\lambda(\tau)^*\omega = \omega_{\varphi_t}$. One may note that these rays are precisely the ones studied already by Mabuchi in [Mab85].*

Partial results in the direction of K-polystability of cscK manifolds with transcendental cohomology class

We finally note that an adaptation of the proof of [Ber16, Lemma 3.4] yields the following partial result, giving evidence that a version of the K-polystability condition holds at least for smooth and dominating test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α) :

Proposition 5.4.28. *Let (X, α) be a cscK manifold and suppose that $(\mathcal{X}, \mathcal{A})$ is a smooth relatively Kähler test configuration for (X, α) , dominating $X \times \mathbb{P}^1$ via a morphism $\mu : \mathcal{X} \rightarrow X \times \mathbb{P}^1$. Then $\text{DF}(\mathcal{X}, \mathcal{A}) \geq 0$ with equality precisely when \mathcal{X}_0 is isomorphic to X . In particular, for such smooth and dominating relatively Kähler test configurations, the equivalent conditions in Theorem 5.4.16 hold if and only if $\mathcal{X}_0 = X$.*

Remark 5.4.29. *As an immediate consequence one may note that Theorem J holds even if one removes the assumption that $E(\varphi_t) = 0$ for each t .*

While the above result does not immediately or obviously extend to treat also the case of singular relatively Kähler test configurations (i.e. at the time of writing we are not able to do this) it is

interesting to confirm that also in the transcendental case the vanishing of the Donaldson-Futaki invariant (or the non-Archimedean J_G -functional) can be characterized using a condition that bares only on the total space \mathcal{X} . We expect this result to hold in general, with a proof adapting the argument in Proposition 5.4.28.

Question 5.4.30. *We end this section with two open questions regarding product configurations and K -polystability:*

- *Does Proposition 5.4.28 hold also for arbitrary (possibly singular) relatively Kähler test configurations for (X, α) ?*
- *Let $(\mathcal{X}, \mathcal{A}, \pi)$ be a relatively Kähler test configuration for (X, α) , whose central fiber \mathcal{X}_0 is isomorphic to X . Does it follow that $\mathcal{X}_{|\pi^{-1}(\mathbb{C})} = X \times \mathbb{C}$?*

We expect that both the above questions can be approached as an application of Theorem I, exploiting the uniqueness of the geodesic ray associated to a relatively Kähler cohomological test configuration. These and other questions are treated in the ongoing work [SD17].

Chapter 6

Perspectives and applications

As already mentioned, as a first application of the main results of this thesis a large number of examples of cscK manifolds can be verified to be (uniformly) K-stable if the automorphism group is discrete, and K-semistable independently of the automorphism group. We now discuss some further interesting questions where we expect the approach of this thesis to be useful.

6.1 Deformation theory and stability loci in the Kähler cone

In this thesis we develop a theory of K-stability for *arbitrary* compact Kähler manifolds. The main results of this thesis may be interpreted in the context of stability loci in the Kähler cone $\text{Kah}(X) := \{\alpha_X \in H^{1,1}(X, \mathbb{R}) : \alpha_X > 0\}$ of X . For example, if we suppose that the automorphism group $\text{Aut}(X)$ is discrete, then Theorems C and E yield the following chain of inclusions:

$$\begin{aligned} \text{cscK locus} &\subset \text{Uniformly K-stable locus} \subset \text{K-stable locus} \subset \\ &\subset \text{K-semistable locus} \subset \text{Kah}(X). \end{aligned}$$

By the cscK locus we here mean the set of Kähler classes $\alpha \in \text{Kah}(X)$ such that (X, α) is cscK (with obvious adaptations for the definition of the other stability loci). Two general lines of inquiry are the following:

1. What can be said about the topology of the above stability loci? For instance, are they open or closed in the Kähler cone of X ? Do they have some additional structure?
2. Which of the above inclusions are strict in general? Under what conditions?

Such questions may of course be very difficult to answer. For instance, regarding 2), even the YTD conjecture can be reformulated as stating that the cscK locus *equals* the K-stable locus (or the uniformly K-stable locus, as is widely conjectured). On the other hand, it is known with regards to 2) that the inclusion of the cscK locus \subset K-semistable locus is strict in general, due to concrete counterexamples for ruled manifolds (see [Kel16] and [KR11]). With regards to the notions of stability introduced in this thesis it would also be very interesting to clarify

whether the inclusion cohomologically K-stable locus \subset algebraically K-stable locus is strict in general (recall that the algebraic and cohomological stability notions coincide for semistability and uniform stability, see Propositions 3.2.23 and 3.2.24). As previously mentioned, one could imagine that the inclusion may in fact be strict, due to examples of [ACGTF08] in the setting of extremal metrics.

Under the assumption $\text{Aut}_0(X) = \{0\}$ on the automorphism group extremal metrics are automatically cscK, so it follows from the work of LeBrun and Simanca [BS94] that the cscK set is *open* (in the Euclidean topology). In fact, it was proven in [BS94] that

Theorem 6.1.1. [BS94] *The cscK locus is relatively open in the subset of the Kähler cone consisting of Kähler classes $\alpha := [\omega]$ such that the Futaki character*

$$\text{Fut}_\alpha(X, V) := \int_X h_\omega^V(S(\omega) - \bar{S}) \omega^n$$

vanishes identically.

In view of this result it is interesting to study the locus $\text{Fut}_0 := \{\alpha \in \mathcal{K}(X) : \text{Fut}_\alpha(X, V) \equiv 0\}$. It would further be interesting to be able to say something about properties and relationships between the above stability loci. Below we give some discussion on this topic, and some first results in this direction.

Variation of the Futaki invariant along affine lines in the Kähler cone

Consider the locus of Kähler classes on X admitting extremal metrics. We then clearly have an inclusion of the cscK locus \subset extremal locus. It is moreover well-known that in any given class either all extremal metrics are cscK or none are (this follows from a result of [Fut88, Cal85] since the Futaki character is independent of representative of the given class, and extremal Kähler metrics have constant scalar curvature if and only if the Futaki character vanishes identically). However, what can be said about the distribution of classes in the Kähler cone that admit extremal metrics of constant respectively non-constant scalar curvature? This depends on the Futaki invariant. In order to study, among other things, how the Futaki invariant varies along line segments in the Kähler cone, we make the following observation of independent interest:

Lemma 6.1.2. *Let α_X and β_X be Kähler classes on X and suppose that $(\mathcal{X}, \mathcal{A})$ is a relatively Kähler smooth and dominating test configuration for (X, α_X) . Then for each $t \in [0, 1]$ there is a relatively Kähler class A_t on \mathcal{X} , such that*

1. $A_0 = \mathcal{A}$
2. (\mathcal{X}, A_t) is a smooth and dominating relatively Kähler cohomological test configuration for $(X, (1-t)\alpha_X + t\beta_X)$.
3. The function $[0, 1] \ni t \mapsto \text{DF}(\mathcal{X}, A_t) \in \mathbb{R}$ is a polynomial of degree at most $n + 1 := \dim(X) + 1$.

4. For any given $\delta > 0$, the function $[0, 1] \ni t \mapsto (\text{DF} - \delta \text{J}_{(1-t)\alpha_X + t\beta_X}^{\text{NA}})(\mathcal{X}, \mathcal{A}_t) \in \mathbb{R}$ is a polynomial of degree at most $n + 1 := \dim(X) + 1$.

The reader may compare with the convex combinations of algebraic test configurations used in A. Isopoussus's thesis [Iso13].

Proof of Lemma 6.1.2. Let $(\mathcal{X}, \mathcal{A})$ be a relatively Kähler smooth and dominating test configuration for (X, α_X) , with $\mu : \mathcal{X} \rightarrow X \times \mathbb{P}^1$ the associated morphism. By [SD16, Proposition 3.11] we then have $\mathcal{A} = \mu^* p_1^* \alpha_X + [D]$ with D an \mathbb{R} -divisor supported on \mathcal{X}_0 . Moreover, there is a divisor E on \mathcal{X} which is μ -exceptional and μ -ample, in particular with support contained in \mathcal{X}_0 (see the proof of Proposition 3.2.20).

Now if $\beta_X \in \text{Kah}(X)$, let $\lambda > 0$ be sufficiently small so that $\mu^* p_1^* \beta_X + \lambda[E]$ is relatively Kähler on \mathcal{X} . Then $(\mathcal{X}, \mathcal{A}_t) := (\mathcal{X}, \mu^* p_1^*((1-t)\alpha_X + t\beta_X) + (1-t)[D] + t\lambda[E])$ is a relatively Kähler smooth and dominating test configuration for $(X, (1-t)\alpha_X + t\beta_X)$ for all $t \in [0, 1]$. The condition $\mathcal{A}_0 = \mathcal{A}$ is satisfied, and the subpoints (3) and (4) follow from the intersection theoretic interpretations of DF and J^{NA} . In particular, the function

$$[0, 1] \ni t \mapsto (\text{DF} - \delta \text{J}_{(1-t)\alpha_X + t\beta_X}^{\text{NA}})(\mathcal{X}, \mu^* p_1^*((1-t)\alpha_X + t\beta_X) + (1-t)[D] + t\lambda[E]) \in \mathbb{R}$$

is continuous for any given $\delta > 0$. □

Remark 6.1.3. The coefficients of the above polynomials are straightforward to compute explicitly, using the intersection theoretic expressions for DF and J^{NA} .

Vanishing of the Futaki invariant and extremal metrics

The above Lemma 6.1.2 has implications for the study of the locus of Kähler classes with identically vanishing Futaki invariant (recall in particular (see e.g. [Sze14, Section 4.2]) that the Futaki invariant

$$\mathfrak{h} \ni h_V \mapsto \int_X h_\omega^V(S(\omega) - \bar{S}) \omega^n$$

is independent of the representative ω of the Kähler class $\alpha := [\omega]$, where \mathfrak{h} denotes the space of holomorphic Hamiltonian vector fields on X . The following Proposition concerns the dependence of the Futaki invariant on the Kähler class as we move along line segments in the Kähler cone.

Proposition 6.1.4. *Let $\alpha_X, \beta_X \in \text{Kah}(X)$ be two Kähler classes and set $\alpha_t := (1-t)\alpha_X + t\beta_X$, $t \in [0, 1]$. Let $\lambda : \mathbb{C}^* \rightarrow \text{Aut}_0(X)$ be a one-parameter subgroup and let V_λ be the infinitesimal generator of the corresponding \mathbb{C}^* -action on X . Then*

$$t \mapsto \text{Fut}_{\alpha_t}(X, V_\lambda)$$

is a polynomial of degree at most $\dim_{\mathbb{C}}(X) + 1$.

Proof. Each V_λ as above induced a product test configuration whose total space is $X \times \mathbb{P}^1$ with the induced diagonal \mathbb{C}^* -action. Denote this product test configuration by $(\mathcal{X}^V, \mathcal{A}^V)$. Now consider a path $(\mathcal{X}^V, \mathcal{A}_t^V)$ as in Lemma 6.1.2, such that $\mathcal{A}_0^V = \mathcal{A}^V$ and

$$t \mapsto \text{DF}(\mathcal{X}^V, \mathcal{A}_t^V)$$

is a polynomial in at most n variables. But by a standard computation using the Hamiltonian as in the proof of Theorem 5.4.16 we see that

$$\text{DF}(\mathcal{X}^V, \mathcal{A}_t^V) = \text{Fut}_{\alpha_t}(X, V_\lambda),$$

which is then in particular also a polynomial of degree at most $\dim_{\mathbb{C}}(\mathcal{X}^V) = \dim_{\mathbb{C}}(X) + 1$ in t . \square

It follows that for a given V the Futaki invariant (or the corresponding Donaldson-Futaki invariant of a product configuration) can have at most $\dim_{\mathbb{C}}(X) + 1$ zeros along the affine line $\alpha_t := (1-t)\alpha + t\beta$ in the Kähler cone, unless these quantities vanish for every t . In particular, we have the following immediate corollary:

Corollary 6.1.5. *If the set of Kähler classes with vanishing Futaki invariant contains an open set, then the Futaki invariant $\text{Fut}_\alpha(X, V_\lambda)$ vanishes for each $\alpha \in \text{Kah}(X)$ and each V_λ as in Proposition 6.1.4.*

For example, we have inclusions $\text{cscK locus} \subset \text{K-semistable locus} \subset \text{locus of Kähler classes with vanishing Futaki invariant}$. Hence, if the K-semistable locus or the cscK locus contains an open set, this is enough to yield the conclusion above. It also leads to the following question:

Question 6.1.6. *If the set of Kähler classes with vanishing Futaki invariant contains an open set, then do all extremal metrics have constant scalar curvature?*

Remark 6.1.7. *These observations on the variation of the Futaki invariant in the Kähler cone seem likely to yield particularly interesting consequences in situations where it suffices to test K-stability for finitely many test configurations. This is an ongoing project.*

Structure of stability loci in the Kähler cone

There are a number of further interesting and immediate consequences following from the above Proposition 6.1.4, regarding the structure of the K-semistable and the uniformly K-stable loci.

The K-semistable locus

Since it suffices (by Proposition 3.2.20) to test K-semistability and uniform K-stability for relatively Kähler smooth and dominating test configurations for (X, α_X) , the above key lemma is very useful. In particular, as already mentioned, it can be used to establish the following result (by 'K-unstable' we here mean 'not K-semistable'):

Corollary 6.1.8. *Suppose that (X, α_X) is K-unstable. Then for any Kähler class β_X on X there is an $\epsilon_0 > 0$ such that $(X, (1-t)\alpha_X + t\beta_X)$ is K-unstable for each $t \in [0, \epsilon_0)$.*

Proof of Corollary 6.1.8. Suppose that (X, α_X) is K-unstable (i.e. not K-semistable). By Proposition 3.2.20 there is then a smooth and dominating relatively Kähler test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α_X) such that $\text{DF}(\mathcal{X}, \mathcal{A}) < 0$. Fix any Kähler class β_X on X . By Theorem 6.1.2 there is for each $t \in [0, 1]$ a normal and relatively Kähler test configuration $(\mathcal{X}, \mathcal{A}_t)$ for $(X, (1-t)\alpha_X + t\beta_X)$, such that $[0, 1] \ni t \mapsto \text{DF}(\mathcal{X}, \mathcal{A}_t)$ is continuous. As a consequence there is an $\epsilon_0 > 0$ small enough so that $\text{DF}(\mathcal{X}, \mathcal{A}_t) < 0$ for all $t \in [0, \epsilon_0)$. By definition this means that $(X, (1-t)\alpha_X + t\beta_X)$ is K-unstable for each $t \in [0, \epsilon_0)$, and we are done. \square

As a consequence, the above explains that the inclusion

$$\text{cscK locus} \subset \text{K-semistable locus}$$

is strict in general, and says something slightly more than simply having an example of when this happens. Indeed, the cscK locus is relatively open in the set of Kähler classes with vanishing Futaki invariant, while the K-semistable locus is 'closed' in the sense of Corollary 6.1.8. This argument moreover holds even for non-projective compact Kähler manifolds and for projective Kähler manifolds with possibly non-rational cohomology class.

The uniformly K-stable locus

With the purpose of studying the uniformly K-stable locus, where uniform K-stability is understood to be taken with respect to the non-archimedean J-functional J^{NA} (this norm is comparable to both the L^1 -norm and the minimum norm introduced [Der16b]) we introduce some auxiliary notation: If $\delta > 0$ is given, let us say that (X, α_X) is δ -uniformly K-stable if $(\text{DF} - \delta J_{\alpha_X}^{\text{NA}})(\mathcal{X}, \mathcal{A}) \geq 0$ for all smooth and dominating relatively Kähler test configurations $(\mathcal{X}, \mathcal{A})$ for (X, α_X) . It is clear that if $\delta \leq \delta'$ then the δ -uniformly K-stable locus \mathcal{U}_δ contains the δ' -uniformly K-stable locus $\mathcal{U}_{\delta'}$, i.e. $\mathcal{U}_{\delta'} \subseteq \mathcal{U}_\delta$. Denoting the uniformly K-stable locus by \mathcal{U} we thus have

$$\mathcal{U} = \bigcup_{\delta > 0} \mathcal{U}_\delta,$$

where the union is nested. One may note that each \mathcal{U}_δ is closed along line segments in the Kähler cone on X (in the same sense as the semistable locus).

Proposition 6.1.9. *Let $\delta > 0$ be given. Suppose that (X, α_X) is δ -uniformly K-unstable (i.e. not δ -uniformly K-stable). Then for any Kähler class β_X on X there is an $\epsilon_0 > 0$ such that $(X, (1-t)\alpha_X + t\beta_X)$ is δ -uniformly K-unstable for each $t \in [0, \epsilon_0)$.*

Proof of Proposition 6.1.9. Fix $\delta > 0$. Suppose that (X, α_X) is δ -uniformly K-unstable. By Proposition 3.2.20 there is then a smooth and dominating relatively Kähler test configuration $(\mathcal{X}, \mathcal{A})$ for (X, α_X) such that $(\text{DF} - \delta J_{\alpha_X}^{\text{NA}})(\mathcal{X}, \mathcal{A}) < 0$. Fix any Kähler class β_X on X . By Lemma 6.1.2 there is for each $t \in [0, 1]$ a normal and relatively Kähler test configuration $(\mathcal{X}, \mathcal{A}_t)$ such

that $[0, 1] \ni t \mapsto (\text{DF} - \delta \mathbf{J}_{(1-t)\alpha_X + t\beta_X}^{\text{NA}})(\mathcal{X}, \mathcal{A}_t)$ is continuous. As a consequence there is an $\varepsilon_0 > 0$ small enough so that

$$(\text{DF} - \delta \mathbf{J}_{(1-t)\alpha_X + t\beta_X}^{\text{NA}})(\mathcal{X}, \mathcal{A}_t) < 0$$

for all $t \in [0, \varepsilon_0)$. By definition this means that $(X, (1-t)\alpha_X + t\beta_X)$ is δ -uniformly K-unstable for each $t \in [0, \varepsilon_0)$. In other words, the δ -uniformly K-unstable locus is open, which is what we wanted to prove. \square

As a natural continuation one may, given a Kähler class α on X , study the quantity

$$\Delta(\alpha) := \sup\{\delta > 0 \mid (X, \alpha) \text{ is } \delta\text{-uniformly K-stable}\}.$$

By the above arguments one deduces that $\alpha \mapsto \Delta(\alpha)$ is upper semi-continuous as a function on the Kähler cone of X .

Question 6.1.10. *Is $\alpha \mapsto \Delta(\alpha)$ also lower semi-continuous?*

If the answer to this question is affirmative this implies that the uniformly K-stable locus is open, thus confirming a fundamental necessary condition for the YTD conjecture to hold.

6.2 Further directions

Remarks on uniform K-polystability

With the formalism and results proven in this thesis it seems very likely that uniform K-polystability of cscK manifolds would follow, if only one could introduce a "good" definition of the norm \mathbf{J}_G^{NA} . In the case when X has trivial connected automorphism group $G := \text{Aut}_0(X)$, the uniform K-polystability notion can be understood as in Section 3.2 where the "norm" $\mathbf{J}^{\text{NA}}(\mathcal{X}, \mathcal{A})$ of a test configuration can be defined as the asymptotic slope of Aubin's J-functional along the (unique) weak geodesic ray associated to $(\mathcal{X}, \mathcal{A})$, see [Der16b, BHJ15]. To show that $\mathbf{J}^{\text{NA}}(\mathcal{X}, \mathcal{A})$ is well-defined convexity is used, but when $G \neq \{0\}$ there is no known convexity result for the corresponding G -action functional $\mathbf{J}_G(\varphi) := \inf_{g \in G} J(g \cdot \varphi_t)$. In other words, in the presence of automorphisms it is non-trivial to show that the natural candidate for a norm of a test configuration is even well-defined. On the other hand, this would be an important point to clarify, since the uniform notion of K-stability is currently believed to be one of the most plausible candidates for the "correct" notion of stability to be used in the Yau-Tian-Donaldson conjecture (Conjecture 1.0.1 or its natural transcendental generalization). In this direction, the main aim would be to make sense of the definition of uniform K-polystability, and establish a result generalizing Theorem E.

Question 6.2.1. *Using the transcendental techniques of this thesis, can Theorem E be extended to a notion of uniform K-polystability?*

This would be a natural continuation of the results of this thesis.

The Yau-Tian-Donaldson conjecture for Kähler-Einstein metrics twisted by a transcendental class

The *twisted Kähler-Einstein equation* has been extensively studied. It is given by

$$\mathrm{Ric}(\omega) = \omega + \eta, \tag{6.1}$$

where η is a closed $(1,1)$ -current (possibly non-positive and singular) on X . A solution of this equation is called a twisted Kähler-Einstein metric, and is realized as critical points of a *twisted Mabuchi energy*, or alternatively, of the *twisted Ding energy* functional (see e.g. [Rub08] and Chi Li's thesis). When η is smooth, this equation was studied by e.g. [CS14]. In the case when $[\eta] = [D]$ is the current of integration of a divisor, the equation (6.1) was studied in [Don12, JMR16], and played a key role in the resolution of the YTD conjecture, see [CDS15a, CDS15b, CDS15c] and also [CSW].

In the case when $\eta = 0$, equation (6.1) is the ordinary Kähler-Einstein equation on Fano manifolds. In [BBJ15] it was then proven that uniformly K-stable Fano manifolds with discrete automorphism group admit cscK metric. This is the most difficult direction of the Yau-Tian-Donaldson conjecture, and yields a variational proof in the Fano case. Based on the variational nature of the proof in [BBJ15] one could expect to be able to make use of our transcendental formalism (cf. Section 4) in the case of a twisted Kähler-Einstein metrics.

In general, suppose that X is a compact Kähler manifold satisfying the condition $c_1(X) + [\eta] < 0$, where η is a closed $(1,1)$ -current on X . We then ask the following:

Question 6.2.2. *Can the variational proof of [BBJ15] be adapted to show that, if X is uniformly twisted K-stable (in an appropriate sense, which is part of the problem), then X admits a twisted Kähler-Einstein metric?*

We expect a positive answer to this question, and intend to study this problem in the near future.

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