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Quang Hoa Tran

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présentée par

TRAN Quang Hoa

Images et fibres des applications rationnelles et algèbres d'éclatement

---

dirigée par Marc Chardin

Soutenue le 17 novembre 2017 devant le jury composé de :

M. Laurent BUSÉ	INRIA Sophia Antipolis	Examinateur
M. Marc CHARDIN	IMJ-PRG, UPMC	Directeur de thèse
M. Alexandru DIMCA	Université de Nice-Sophia Antipolis	Rapporteur
M. Antoine DUCROS	IMJ-PRG, UPMC	Examinateur

Rapporteur absent lors de la soutenance :

M. Frank-Olaf SCHREYER Universität des Saarlandes

Institut de Mathématiques de
Jussieu–Paris Rive Gauche
UMR 7586
Boite courrier 247
4, place Jussieu
75252 Paris Cedex 05

Université Pierre et Marie Curie
École Doctorale de Sciences
Mathématiques de Paris Centre
Boite courrier 290
4 place Jussieu
75252 Paris Cedex 05

Cette thèse est dédiée à ma femme, Ngọc Phương
et ma fille, Ngọc Anh
sans lesquelles je n'aurais jamais vu le jour.

Images et fibres des applications rationnelles et algèbres d'éclatement

Résumé

Les applications rationnelles sont des objets fondamentaux en géométrie algébrique. Elles sont utilisées pour décrire certains objets géométriques, tels que la représentation paramétrique d'une variété algébrique rationnelle. Plus récemment, les applications rationnelles sont apparues dans des contextes d'informatique pour l'ingénierie, dans le domaine de la modélisation de formes, en utilisant des méthodes de conception assistée par ordinateur pour les courbes et les surfaces. Des paramétrisations des courbes et des surfaces sont utilisées de manière intensive afin décrire des objets dans la modélisation géométrique, tel que structures des voitures, des avions. Par conséquent, l'étude des applications rationnelles est d'intérêt théorique dans la géométrie algébrique et l'algèbre commutative, et d'une importance pratique dans la modélisation géométrique. *Ma thèse étudie les images et les fibres des applications rationnelles en relation avec les équations des algèbres de Rees et des algèbres symétriques.*

Dans la modélisation géométrique, il est important d'avoir une connaissance détaillée des propriétés géométriques de l'objet et de la représentation paramétrique avec lesquels on travaille. La question de savoir combien de fois le même point est peint (c'est-à-dire, correspond à des valeurs distinctes du paramètre), ne concerne pas seulement la variété elle-même, mais également la paramétrisation. Il est utile pour les applications de déterminer les singularités des paramétrisations. Dans les chapitres 2 et 3, on étudie des fibres d'une application rationnelle $\phi : \mathbb{P}_k^m \dashrightarrow \mathbb{P}_k^n$ qui est génériquement finie sur son image. Une telle application ϕ est définie par un ensemble ordonné de polynômes homogènes $\mathbf{f} := f_0, \ldots, f_n$, de même degré d, dans l'anneau de coordonnées homogènes $R = k[X_0, \ldots, X_m]$, tel que $\gcd(f_0, \ldots, f_n) = 1$. Plus précisément, dans le chapitre 2, nous traiterons le cas des paramétrisations $\phi : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^3$ de surfaces rationnelles, et y donnons une borne quadratique en d pour le nombre de fibres de dimension 1 de la projection canonique du graphe de ϕ sur son image. Nous déduisons ce résultat d'une étude de la différence du degré initial entre les puissances ordinaires et les puissances saturées. Dans le chapitre 3, on affine et généralise les résultats sur les fibres du chapitre précédent. Plus généralement, nous établissons une borne linéaire en d pour le nombre de fibres $(m-1)$-dimensionnelles de la projection canonique du graphe de $\phi : \mathbb{P}_k^m \dashrightarrow \mathbb{P}_k^n$ sur son image, en utilisant des idéaux de mineurs de la matrice jacobienne de $\mathbf{f}$.

Dans le chapitre 4, nous considérons des applications rationnelles $\mathcal{F} : X \times_k Y \dashrightarrow Z$, où $X \subset \mathbb{P}_k^n$, $Y \subset \mathbb{P}_k^m$ and $Z \subset \mathbb{P}_k^s$ sont des sous-variétés irréductibles. Notre principal objectif est d'étudier les critères de birationalité pour ces applications. Tout d'abord, un critère général est donné en termes du rang d'une couple de matrices connues sous le nom *matrices jacobiennes duales*. Ensuite, nous nous concentrons sur des applications rationnelles de $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ vers $\mathbb{P}_k^2$ en bidegré bas et fournissons de nou-

veaux critères de birationalité en analysant les syzygies des équations de définition de l'application ; en particulier en examinant la dimension de certaines parties bigraduées du module de syzygies. Enfin, les applications de nos résultats au contexte de la modélisation géométrique sont discutées à la fin du chapitre.

Dans le dernier chapitre, nous étudions les intersections résiduelles. Nous montrons la propriété de Cohen-Macaulay et décrivons le module canonique d'une intersection résiduelle $J = \mathfrak{a} \colon {}_R I$ d'un idéal I qui satisfait des conditions de profondeur sur les homologies de Koszul de I, de type "sliding depth", dans un anneau local de Cohen-Macaulay R. Dans ce travail, nous construisons et étudions, en utilisant un article récent de Hassanzadeh et Naéliton [HN16], une famille de complexes qui contient des informations importantes sur une intersection résiduelle et son module canonique. Nous déterminons également plusieurs invariants d'une intersection résiduelle comme le module canonique gradué, la série de Hilbert, la régularité de Castelnuovo-Mumford et le type. Enfin, supposons que I soit fortement de Cohen-Macaulay, alors nous montrons des résultats de dualité pour les intersections résiduelles qui sont liés aux résultats de Eisenbud et Ulrich dans [EU16], et établissons ainsi des relations étroites entre les séries de Hilbert de quelques puissances symétriques de $I/\mathfrak{a}$. Nous proposons également des formules fermées pour les types et pour les nombres de Bass de certaines puissances symétriques de $I/\mathfrak{a}$.

Mots-clés

Complexes d'approximation, applications rationnelles bigraduées, critères de birationalité, algèbres d'éclatement, fibres des paramétrisations de surfaces, fibres des applications rationnelles, accouplement parfait, intersection résiduelle, fortement de Cohen-Macaulay.

Images and fibers of rational maps and Blow-up algebras

Abstract

Rational maps are fundamental objects in algebraic geometry. They are used to describe some geometric objects, such as parametric representation of rational algebraic varieties. Lately, rational maps appeared in computer-engineering contexts, mostly applied to shape modeling using computer-aided design methods for curves and surfaces. Parameterized algebraic curves and surfaces are used intensively to describe objects in geometric modeling, such as car bodies, airplanes.Therefore, the study of rational maps is of theoretical interest in algebraic geometry and commutative algebra, and of practical importance in geometric modeling. *My thesis studies images and fibers of rational maps in relation with the equations of the symmetric and Rees algebras.*

In geometric modeling, it is of vital importance to have a detailed knowledge of the geometry of the object and of the parametric representation with which one is working. The question of how many times is the same point being painted (i.e., corresponds to distinct values of parameter), depends not only on the variety itself, but also on the parameterization. It is of interest for applications to determine the singularities of the parameterizations. In the chapters 2 and 3, we study the fibers of a rational map $\phi : \mathbb{P}_k^m \dashrightarrow \mathbb{P}_k^n$ that is generically finite onto its image. Such a map ϕ is defined by an ordered set of homogeneous polynomials $\mathbf{f} := f_0, \ldots, f_n$, of the same degree d, in the homogeneous coordinate ring $R = k[X_0, \ldots, X_m]$, such that $\gcd(f_0, \ldots, f_n) = 1$. More precisely, in the second chapter, we will treat the case of parameterizations $\phi : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^3$ of algebraic rational surfaces. In this case, we give a quadratic bound in d for the number of one-dimensional fibers of the canonical projection of the graph of ϕ onto its image, by studying of the difference between the initial degree of ordinary and saturated powers of the base ideal of ϕ. In the third chapter, we refine and generalize the results on fibers of the previous chapter. More generally, we establish a linear bound in d for the number of $(m-1)$-dimensional fibers of the canonical projection of the graph of $\phi : \mathbb{P}_k^m \dashrightarrow \mathbb{P}_k^n$ onto its image, by using ideals of minors of the Jacobian matrix of $\mathbf{f}$.

In the fourth chapter, we consider rational maps whose source is a product of two subvarieties, each one being embedded in a projective space. Our main objective is to investigate birationality criteria for such maps. First, a general criterion is given in terms of the rank of a couple of matrices that came to be known as *Jacobian dual matrices*. Then, we focus on rational maps from $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ to $\mathbb{P}_k^2$ in very low bidegrees and provide new matrix-based birationality criteria by analyzing the syzygies of the defining equations of the map, in particular by looking at the dimension of certain bigraded parts of the syzygy module. Finally, applications of our results to the context of geometric modeling are discussed at the end of the chapter.

In the last chapter, we study residual intersections. We show the Cohen-Macaulay property and describe the canonical module of residual intersections $J = \mathfrak{a} :_R I$ in a Cohen-Macaulay local ring R, under sliding depth type hypotheses. For this purpose, we construct and study, using a recent article of Hassanzadeh and Naéliton [HN16], a family of complexes that contains important informations on a residual intersection and its canonical module. We also determine several invariants of a residual intersection as the graded canonical module, the stability of Hilbert function, the Hilbert series, the Castelnuovo-Mumford regularity and the type. Finally, whenever I is strongly Cohen-Macaulay, we show duality results for residual intersections that are closely connected to results by Eisenbud and Ulrich in [EU16], and thus establish some tight relations between the Hilbert series of some symmetric powers of $I/\mathfrak{a}$. We also provide closed formulas for the types and for the Bass numbers of some symmetric powers of $I/\mathfrak{a}$.

Keywords

Approximation complexes, bigraded rational maps, birationality criteria, Blow-up algebras, fibers of parameterization surfaces, fibers of rational maps, perfect pairing, residual intersections, sliding depth, strongly Cohen-Macaulay.

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Introduction

Rational maps are fundamental objects in algebraic geometry. They are used to describe some geometric objects, such as *parametric representation of rational algebraic varieties.* More precisely, let k be a field and $V \subset \mathbb{P}^n_k$ be a rational algebraic subvariety of dimension m defined by a homogeneous ideal $J = (F_1, \ldots, F_k)$ in the homogeneous coordinate ring $B = k[T_0, \ldots, T_n]$ of $\mathbb{P}^n_k$. Then there exist $(n+1)$ homogeneous polynomials $f_0, \ldots, f_n$ in $R = k[X_0, \ldots, X_m]$ of the same degree d such that $\gcd(f_0, \ldots, f_n) = 1$, $F_j(f_0, \ldots, f_n) = 0$ for all $j = 1, \ldots, k$ and the kernel of the k-algebra homomorphism $\varphi : B \longrightarrow k[f_0, \ldots, f_n]$, which sends T_i to f_i for all $i = 0, \ldots, n$, is the prime ideal $J^{\text{sat}} := J\colon_B (T_0, \ldots, T_n)^\infty$. In other words, the closed image of rational map $\phi : \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k$ defined by $f_0, \ldots, f_n$ is the subvariety V of $\mathbb{P}^n_k$. Such a rational map ϕ is called a *parameterization of* V.

Lately, rational maps appear in computer-engineering contexts. Mostly they are applied to shape modeling using computer-aided design methods for curves and surfaces. Parameterized algebraic curves and surfaces are used intensively to describe objects in geometric modeling, such as car bodies, airplanes. Therefore, the study of rational maps is of theoretical interest in algebraic geometry and commutative algebra, and of practical importance in geometric modeling. *My thesis studies images and fibers of rational maps in relation with the equations of the symmetric and Rees algebras.*

Let k be a field. We investigate rational maps $\phi : \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k$. Such a map ϕ is defined by homogeneous polynomials $f_0, \ldots, f_n$, of the same degree d, in a standard graded polynomial ring $R = k[X_0, \ldots, X_m]$, such that $\gcd(f_0, \ldots, f_n) = 1$. The ideal of R generated by these polynomials is denoted by I and is called the *base ideal of* ϕ. The scheme $\mathcal{B} := \operatorname{Proj}(R/I) \subset \mathbb{P}^m_k$ is called the *base locus of* ϕ. Let $B = k[T_0, \ldots, T_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n_k$ with each T_i is of degree one. The map ϕ corresponds to the k-algebra homomorphism $\varphi : B \longrightarrow R$, which sends T_i to f_i for all $i = 0, \ldots, n$. The closed image X_ϕ of ϕ is defined by the kernel of this homomorphism. In other words, after degree renormalization, $k[f_0, \ldots, f_n] \simeq B/\operatorname{Ker}(\varphi)$ is the homogeneous coordinate ring of X_ϕ and the minimal set of generators of $\operatorname{Ker}(\varphi)$ is called its *implicit equations.* The *implicitization problem* is to find these implicit equations. There are several results in commutative algebra that describe implicit equations of parameterized algebraic varieties in special case, such as algebraic curves, surfaces or hypersurfaces. Nevertheless, the implicitization problem is known to be very hard in general.

The implicitization problem for curves or surfaces has been of increasing interest to commutative algebraists and algebraic geometers due to its applications in Com-

puter Aided Geometric Design as explained by Cox [Cox05]. In this context, knowing the implicit equation of the curve or surface is important to perform elementary operations with these objects. For example describing the curve of intersection of two surfaces, or the points of intersection of a curve and a surface, or testing if a given point lies in the image of a parameterization,.... For this reason, the implicitization problem is a classical problem and there are numerous approaches to its solution. For a good historical overview on this subject we refer the reader to [SC95] and [Cox01].

The implicitization problem amounts to eliminating variables in a system of polynomial equations. Therefore, standard tools such as Gröbner bases or resultants can be used to solve it. However, Gröbner bases algorithms provide a straightforward theoretical approach that in practice tends to be very slow, hence it is rarely used in geometric modeling (see, e.g., [Hof89]). Resultants provide a more convenient representation of the implicit equation as a determinant of a matrix (see, e.g., [Jou96, Bus01]). But in many applications, the resultant vanishes identically due to the presence of *base points*, which are the source points where the parameterization is not well defined. The best current methods rely on Rees algebra techniques. The Rees algebra of I is the graded R-algebra

$$\mathcal{R}_I = R \oplus I \oplus I^2 \oplus I^3 \oplus \cdots .$$

Let $S = R[T_0, \ldots, T_n]$ be a polynomial ring over R with the standard bigrading where $\deg(X_i) = (1, 0)$ and $\deg(T_j) = (0, 1)$. Since I is generated in a fixed degree, the Rees algebra

$$\mathcal{R}_I \simeq R[It] \subset R[t]$$

is a standard bigraded k-algebra. Mapping $T_j \longmapsto f_j t$ yields a presentation $S/\mathfrak{P} \simeq \mathcal{R}_I$, with $\mathfrak{P}$ a bihomogeneous ideal of S. It is known that $\mathfrak{P}$ depends only on the rational map defined by $\mathbf{f}$ and not on this particular representative, see, for example, [Sim04] or [DHS12]. The generators of $\mathfrak{P}$ are the *defining equations of the Rees algebra.*

We now blow up the base locus of ϕ and obtain the following commutative diagram of algebraic varieties

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[d, "\pi_1"'] & \mathbb{P}_k^m \times \mathbb{P}_k^n \arrow[d, "\pi_2"] \\ \mathbb{P}_k^m \arrow[r, dashed, "\phi"] & \mathbb{P}_k^n. \end{tikzcd}$$

The variety Γ is the blow-up of $\mathbb{P}_k^m$ at $\mathcal{B}$ and it is also the Zariski closure of the graph of ϕ in $\mathbb{P}_k^m \times \mathbb{P}_k^n$. Moreover, Γ is the geometric version of the Rees algebra of I, i.e. $\mathrm{Proj}(\mathcal{R}_I) = \Gamma$ (see [Har77, Chapter II, §7]). As $\mathcal{R}_I$ is the graded domain defining Γ, the projection $\pi_2(\Gamma) = X_\phi$ is defined by the graded domain $\mathcal{R}_I \cap k[T_0, \ldots, T_n]$. In other words,

$$\mathrm{Ker}(\varphi) = \mathfrak{P} \cap k[T_0, \ldots, T_n]$$

and we can thus obtain the implicit equations of X_ϕ from the defining equations of $\mathcal{R}_I$. However, Rees algebras are hard to study and finding explicit descriptions of

their defining equations is a difficult open problem. It has been studied by Huneke, Vasconcelos, Kustin, Polini, Ulrich among others, see for instance in [Hun82, Vas94, UV93, KPU15].

Fortunately, the *symmetric algebra* of an ideal I over R

$$\mathcal{S}_I := \mathrm{Sym}_R(I) = R \oplus I \oplus \mathrm{Sym}_R^2(I) \oplus \mathrm{Sym}_R^3(I) \oplus \cdots$$

is easier to understand than $\mathcal{R}_I$ and is closely related to it via a canonical surjection

$$\delta : \mathcal{S}_I \twoheadrightarrow \mathcal{R}_I.$$

If $\mathfrak{P}_1$ is the graded piece of $\mathfrak{P}$ in degree one with respect to the T_i, then

$$\begin{aligned}\mathfrak{P}_1 &= \{a_0T_0 + \cdots + a_nT_n \mid a_i \in R, a_0f_0 + \cdots + a_nf_n = 0\}\\ &\simeq \mathrm{Syz}(f_0, \ldots, f_n).\end{aligned}$$

Using $(\mathfrak{P}_1) \subset \mathfrak{P}$, one easily obtains the commutative diagram

$$\begin{tikzcd} S/(\mathfrak{P}_1) \arrow[r, two heads] \arrow[d] & S/\mathfrak{P} \arrow[d] \\ \mathcal{S}_I \arrow[r, two heads, "\delta"] & \mathcal{R}_I. \end{tikzcd}$$

The vertical map on the right is an isomorphism by definition, and the vertical map on the left is known to be an isomorphism (see [Vas94, Page 2]). Thus the syzygies of the f_i's define the symmetric algebra and give the degree one relations of the Rees algebra. If the base locus $\mathcal{B}$ of ϕ is assumed to be locally a complete intersection, then $\mathrm{Proj}(\mathcal{S}_I) = \mathrm{Proj}(\mathcal{R}_I) = \Gamma$. Hence, under this assumption, we can extract the implicit equations of X_ϕ from the defining equations of $\mathcal{S}_I$, i.e. the *syzygies* of I. The use of syzygies to find implicit equations for curves and surfaces was introduced by Chen and Sederberg [SC95] and Cox [Cox01, Cox03]. It was precisely formulated in terms of Rees algebra by Jouanolou, Busé, Chardin, Botbol and others [BJ03, BC05, Cha06, BCJ09, Bot11]. One of the key tools of this approach is the *approximation complexes*. These complexes were introduced by Herzog, Simis, and Vasconcelos [HSV82] to study the canonical surjection $\delta : \mathcal{S}_I \longrightarrow \mathcal{R}_I$ and in some cases they are free resolutions of $\mathcal{S}_I$.

Besides the computation of implicit representations of parameterized curves or surfaces, syzygies of parameterizations have been studied and analyzed because syzygies are related to many geometric properties, such as singularities. The importance of the first syzygy module of the coordinates of a parameterization goes back to the paper [CSC98] where the concept of a μ-basis was introduced. In the context of rational plane and space curves, μ-bases are deeply linked to their singularities [SCG07, BD12, WJG09, JG09, JG12]; in the case of surfaces, there also exist some results in the same direction [BBC14, SSV14] but a lot of problems, including the extraction of singularities, are still open in this case.

In geometric modeling, it is of vital importance to have a detailed knowledge of the geometry of the object and of the parametric representation one is working with.

The question of how many times is the same point being painted (i.e. corresponds to distinct values of parameter), depends not only on the variety itself, but also on the parameterization. It is of interest for applications to determine the singularities of the parameterizations. In the chapters 2 and 3, we study the fibers of rational maps, in particular parameterizations of surfaces embedded in the projective space of dimension 3. More precisely, we set

$$\pi := \pi_{2|\Gamma} : \Gamma \longrightarrow \mathbb{P}^n_k.$$

For every closed point $y \in \mathbb{P}^n_k$, we will denote by $k(y)$ its residue field, that is, $k(y) = (B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})_0$, where $\mathfrak{p}$ is the defining homogeneous prime ideal of y. If k is assumed to be algebraically closed, then $k(y) \simeq k$. The fiber of π at $y \in \mathbb{P}^n_k$ is the subscheme

$$\pi^{-1}(y) = \operatorname{Proj}(\mathcal{R}_I \otimes_B k(y)) \subset \mathbb{P}^m_{k(y)} \simeq \mathbb{P}^m_k.$$

Let $0 \leq \ell \leq m$. We set

$$\mathcal{Y}_\ell = \{y \in \mathbb{P}^n_k \mid \dim \pi^{-1}(y) = \ell\} \subset \mathbb{P}^n_k.$$

Chevalley's theorem shows that the subsets $\mathcal{Y}_\ell$ are constructible, that is, they can be written as

$$\mathcal{Y}_\ell = \bigsqcup_{i=1}^{s} (U_i \cap Z_i),$$

where U_i (respectively Z_i) are the open (respectively closed) subsets of $\mathbb{P}^n_k$. We are interested in studying the structure of these subsets.

In the second chapter, the map ϕ is assumed to be generically finite onto its image and $m \geq 2$, it follows that $\mathcal{Y}_{m-1}$ is a 0-dimensional subscheme of $\mathbb{P}^n_k$, that is, π only has a finite number of $(m-1)$-dimensional fibers. For each $y \in \mathcal{Y}_{m-1}$, $\pi^{-1}(y)$ is a subcheme of $\mathbb{P}^m_{k(y)} \simeq \mathbb{P}^m_k$ of dimension $m-1$, as k is algebraically closed. Thus the unmixed component of the fiber $\pi^{-1}(y)$ is defined by a homogeneous polynomial $h_y \in R$, as R is factorial. One of the purposes of my thesis is to bound $\sum_{y \in \mathcal{Y}_{m-1}} \deg(h_y)$ in terms of the degree d.

As the fibers of π are defined by the specialization of the Rees algebra, which is closely related to the symmetric algebra of I via the canonical surjection δ, it would be natural to study the fibers of the canonical projection $\pi' := \pi_{2|V} : V \longrightarrow \mathbb{P}^n_k$, where $V := \operatorname{Proj}(\mathcal{S}_I)$. Recall that for any closed point $y \in \mathbb{P}^n_k$, the fiber of π' at y is the subscheme

$$\pi'^{-1}(y) = \operatorname{Proj}(\mathcal{S}_I \otimes_B k(y)) \subset \mathbb{P}^m_{k(y)} \simeq \mathbb{P}^m_k.$$

As the symmetric algebra is defined by the syzygies of I, one can determine the unmixed components of $(m-1)$-dimensional fibers of π' as in Lemma 2.2.3, is a trivial generalization of [BBC14, Lemma 10]. Furthermore, the unmixed components of $(m-1)$-dimensional fibers of π and π' are the same, whenever $\gcd(f_0, \ldots, f_n) = 1$.

For simplicity, we summarize the following data.

Data 2.2.5. *Let $R = k[\mathbf{X}] = k[X_0, \ldots, X_m]$ $(m \geq 2)$ be the standard polynomial ring over an algebraically closed field k and $\mathbf{f} := f_0, \ldots, f_n$ be k-linearly independent forms of the same degree $d \geq 1$, not all zero, such that $\gcd(f_0, \ldots, f_n) = 1$. Suppose that $\phi : \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k$ is a rational map defined by these forms and π is the canonical projection of the graph of ϕ onto its image. Let $I = (f_0, \ldots, f_n)$ be the base ideal and $\mathcal{B} = \operatorname{Proj}(R/I)$ the base locus of ϕ. Assume that ϕ is generically finite onto its image and put*

$$\mathcal{Y}_{m-1} = \{y \in \mathbb{P}^n_k \mid \dim \pi^{-1}(y) = m - 1\} \subset \mathbb{P}^n_k.$$

For each $y \in \mathcal{Y}_{m-1}$, we denote by $h_y \in R$ a defining equation of the unmixed component of the fiber $\pi^{-1}(y)$.

We recall that the initial degree of a graded R-module M is defined by

$$\operatorname{indeg}(M) := \inf\{n \in Z : M_n \neq 0\},$$

with the convention that $\sup \emptyset = +\infty$ and the saturation of an ideal J of R is defined by $J^{\mathrm{sat}} := J :_R (\mathbf{X})^\infty$. The following theorem is a key tool for our purpose.

Theorem 2.2.6. *Adopt Data 2.2.5. If there exists an integer s such that $\nu = \operatorname{indeg}((I^s)^{\mathrm{sat}}) < sd$, then*

$$\sum_{y \in \mathcal{Y}_{m-1}} \deg(h_y) \leq \nu < sd.$$

The important assumption in the above theorem is the existence of an integer s such that $\operatorname{indeg}((I^s)^{\mathrm{sat}}) < sd$. Observe that $I^s \subset (I^s)^{\mathrm{sat}}$ and in general we have a strict inclusion, that is, I^s is not saturated. It is interesting to study the initial degree of saturated powers, namely, $\operatorname{indeg}((I^s)^{\mathrm{sat}})$. Note that if X is a zero-dimensional scheme in $\mathbb{P}^m_k$ defined by a saturated ideal J, then $(J^s)^{\mathrm{sat}} = J^{(s)}$, where $J^{(s)}$ is the s-th symbolic power of J. More generally, the symbolic power of a homogeneous ideal gives the saturated ideal for the scheme which is the scheme defined by ordinary power, having subtracted the embedded components. Therefore, we wish to establish a bound for the minimal degrees of elements in symbolic powers. There are some results and also several conjectures on the initial degree of symbolic power, see for example [HH13, Chu81, BCH14, CEHH16]. In other words, saturated powers have been of increasing interest to commutative algebraists and algebraic geometers. It is shown that there are several results on the regularity of saturated powers, see, for example [CEL01, Cha13a, Cha13b, Niu13]. In particular, in [Cha13a], Chardin proved that $\lim_{t\to\infty}(\operatorname{reg}((I^t)^{\mathrm{sat}}) - td)$ exists and this limit is a nonnegative integer (or equivalent, this limit is not $-\infty$) if and only if the projection π admits a fiber of positive dimension.

The main section of this chapter studies the one-dimensional fibers of a parameterization of an algebraic surface $\phi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^3_k$. More precisely, we give a bound for $\sum_{y \in \mathcal{Y}_1} \deg(h_y)$ in terms of the degree d. The idea is to show the existence an integer s as in Theorem 2.2.6. Indeed, we prove that $\operatorname{indeg}((I^s)^{\mathrm{sat}}) < sd$ for all integer $s \geq \frac{d}{2}$ under the assumption that $\operatorname{indeg}(I^{\mathrm{sat}}) = \operatorname{indeg}(I) = d$ and $\mathcal{B}$ is locally a complete intersection of dimension zero in Corollary 2.3.6. Our key

tool in the proof is the $\mathcal{Z}$-approximation complex associated to $\mathbf{f} := f_0, f_1, f_2, f_3$, which is a resolution of $\mathcal{S}_I$ under these assumptions. Theorem 2.2.6 implies that $\sum_{y\in\mathcal{Y}_1} \deg(h_y) < d$ if $\operatorname{indeg}(I^{\mathrm{sat}}) < d$, the delicate case is when the ideal I satisfies $\operatorname{indeg}(I^{\mathrm{sat}}) = \operatorname{indeg}(I) = d$. Moreover, in this case, we deduce immediately that $d \geq 3$. Theorem 2.3.7 prove that $\sum_{y\in\mathcal{Y}_1} \deg(h_y) \leq 4$ whenever $d = 3$. In the general case, our main result in this chapter is following.

Theorem 2.3.9. *Adopt Data 2.2.5. Assume further that $m = n - 1 = 2$, that $\operatorname{indeg}(I^{\mathrm{sat}}) = d \geq 4$ and that $\mathcal{B}$ is locally a complete intersection. Then*

$$\sum_{y\in\mathcal{Y}_1} \deg(h_y) \leq \left\lfloor \frac{d}{2} \right\rfloor d - 1,$$

where $\lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \leq x\}$.

In the third chapter, we refine and generalize the results on fibers of rational maps in the previous chapter, by using ideals of minors of the Jacobian matrix of a representative of ϕ. Recall that if $\mathbf{f} := f_0, \ldots, f_n$ are polynomials in the polynomial ring $R = k[X_0, \ldots, X_m]$ over a field k, then the Jacobian matrix of $\mathbf{f}$ is defined by

$$J(\mathbf{f}) = \begin{pmatrix} \frac{\partial f_0}{\partial X_0} & \cdots & \frac{\partial f_0}{\partial X_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial X_0} & \cdots & \frac{\partial f_n}{\partial X_m} \end{pmatrix}.$$

Denote by $I_s(J(\mathbf{f}))$ the ideal of R generated by the s-minors of $J(\mathbf{f})$, where $1 \leq s \leq \min\{m+1, n+1\}$. One of the main results in this chapter is following.

Theorem 3.2.2. *Adopt Data 2.2.5. Assume further that $I_3(J(\mathbf{f})) \neq 0$ and denote by F be the greatest common divisor of generators of $I_3(J(\mathbf{f}))$. Then*

$$\sum_{y\in\mathcal{Y}_{m-1}} \deg(h_y) \leq \sum_{y\in\mathcal{Y}_{m-1}} \sum_{i=1}^{r_y} (2e_i - 1)\deg(h_i) \leq \deg(F) \leq 3(d-1),$$

where $h_y = h_1^{e_1} \cdots h_{r_y}^{e_{r_y}}$ is an irreducible factorization of h_y in R.

If the field k is of characteristic zero, then the assumption $I_3(J(\mathbf{f})) \neq 0$ is always satisfied due to the hypothesis that ϕ is generically finite onto its image. Therefore, Theorem 3.2.2 is a significant improvement and generalization of the theorems 2.3.7 and 2.3.9 in the previous chapter, at least in the case where k is of characteristic zero. Theorem 3.2.2 holds for any $d \geq 1$, without any condition on the initial degree of I^{sat} and the assumption locally a complete intersection of $\mathcal{B}$. However, if $\operatorname{indeg}(I^{\mathrm{sat}}) < d$, then we have a good bound for $\sum_{y\in\mathcal{Y}_{m-1}} \deg(h_y) < d$ as in Theorem 2.2.6.

For a parameterization of surfaces, we studied the syzygies of f_i's in relation with the degree of F, the greatest common divisor of generators of $I_3(J(\mathbf{f}))$. We have a good bound for $\sum_{y\in\mathcal{Y}_1} \deg(h_y)$ as follows.

Corollary 3.3.4. *Adopt Data 2.2.5. Assume further that $m = n - 1 = 2$, that the characteristic of k does not divide d, and that $[k(\mathbf{f}) : k(\mathbf{X})]$ is separable. Then*

$$\sum_{y\in\mathcal{Y}_1} \deg(h_y) \leq 3(d-1) - \operatorname{indeg}(\operatorname{Syz}(I)) < 3(d-1).$$

Notice that the last two conditions in the above corollary are automatically satisfied if k is of characteristic zero.

Given a rational map $\phi : \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k$ between projective spaces that is defined by an ordered set of homogeneous polynomials $\mathbf{f} := f_0, \ldots, f_n$, of the same degree d, in the homogeneous coordinate ring $R = k[X_0, \ldots, X_m]$ of $\mathbb{P}^m_k$. One of the important problems is to determine necessary and sufficient conditions for such a map ϕ to be *birational* onto its image. This problem has been studied extensively in the past and it is still an active area of research. For computational purposes, methods based on the nature of the syzygies of $\mathbf{f}$ are the most suitable in the sense of effective results in the usual implementation of the Gröbner basis algorithm. For $n = m$ this syzygy-based approach appears already in the work of Hulek, Katz, and Schreyer [HKS92] where sufficient conditions for birationality were given, and for $n \geq m$ it has been further developed in [RS01]. In [Sim04, DHS12] the method has been advanced by emphasizing the role of the Rees algebra associated to the ideal $I = (f_0, \ldots, f_n)$ of R. This method has been introduced in relation with the fibers of ϕ in [EU08, KPU16].

In the fourth chapter, we aim to extend some of these methods and techniques to the context of rational maps whose source is a product of two projective spaces $\mathbb{P}^n_k \times \mathbb{P}^m_k$ instead. These maps are defined by an ordered set of bihomogeneous polynomials in two sets of $n+1$ and $m+1$ variables, respectively. In order to emphasize, we call them *bigraded rational maps*. Important modern motivation for considering bigraded rational maps comes from the field of geometric modeling. Indeed, the geometric modeling community uses almost exclusively bigraded rational maps for parameterizing surfaces, dubbing such maps *rational tensor-product Bézier parameterizations*.

First, we provide a general effective criterion for birationality of bigraded rational maps. This criterion is given in terms of the rank of a couple of matrices that came to be known as *Jacobian dual matrices*. Let $X \subset \mathbb{P}^n_k$, $Y \subset \mathbb{P}^m_k$ and $Z \subset \mathbb{P}^s_k$ denote nondegenerate irreducible projective varieties over an algebraically closed field k. Let $A = k[\mathbf{x}] = k[x_0, \ldots, x_n]/\mathfrak{a}$, $B = k[\mathbf{y}] = k[y_0, \ldots, y_m]/\mathfrak{b}$ and $S = k[\mathbf{z}] = k[z_0, \ldots, z_s]/\mathfrak{c}$ stand for the respective homogeneous coordinate rings. We also denote $R := A \otimes_k B \simeq k[\mathbf{x}, \mathbf{y}]/(\mathfrak{a}, \mathfrak{b})$. A rational map $\mathcal{F} : X \times Y \longrightarrow Z$ is defined by bihomogeneous polynomials $f_0(\mathbf{x}, \mathbf{y}), \ldots, f_s(\mathbf{x}, \mathbf{y})$ in R of fixed bidegree (a, b), not all zero. We say that $\mathcal{F}$ is *birational with image* Z if it is dominant and admits an inverse rational map with image $X \times_k Y$. Note that the inverse map is necessarily given by a pair of rational maps $Z \longrightarrow X$ and $Z \longrightarrow Y$ defined by homogeneous polynomials $\mathbf{g} := g_0, \ldots, g_n$ and $\mathbf{h} := h_0, \ldots, h_m$ of fixed degrees d_1 and d_2, respectively. Let $I := (f_0(\mathbf{x}, \mathbf{y}), \ldots, f_s(\mathbf{x}, \mathbf{y}))$ be the ideal of R and consider a polynomial presentation

$$R[\mathbf{z}] = \frac{k[\mathbf{x}, \mathbf{y}]}{(\mathfrak{a}, \mathfrak{b})}[\mathbf{z}] \twoheadrightarrow R[It] = \frac{k[\mathbf{x}, \mathbf{y}]}{(\mathfrak{a}, \mathfrak{b})}[It] = \mathcal{R}_R(I), \quad z_k \mapsto f_k t$$

whose restriction to $R = k[\mathbf{x}, \mathbf{y}]/(\mathfrak{a}, \mathfrak{b})$ is the identity. Let $\overline{\mathcal{J}} = (\mathfrak{a}, \mathfrak{b}, \mathcal{J})/(\mathfrak{a}, \mathfrak{b})$ denote the kernel, with $\mathcal{J}$ the ideal generated by the $\mathbf{z}$-homogeneous polynomials, with bihomogeneous polynomial coefficients in $k[\mathbf{x}, \mathbf{y}]$, vanishing on $f_0(\mathbf{x}, \mathbf{y}), \ldots, f_s(\mathbf{x}, \mathbf{y})$ modulo $(\mathfrak{a}, \mathfrak{b})$.

We will focus on the presentation ideal $(\mathfrak{a}, \mathfrak{b}, \mathcal{J}) \subset k[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ of the Rees algebra $\mathcal{R}_R(I)$. Consider the elements of degrees $(1, 0, *)$ and $(0, 1, *)$ in $(\mathfrak{a}, \mathfrak{b}, \mathcal{J})$, where $*$ denotes an arbitrary degree in $\mathbf{z}$. Since by assumption X and Y are nondegenerate, these elements belong to the graded pieces $\mathcal{J}_{(1,0,*)}$ and $\mathcal{J}_{(0,1,*)}$, respectively. Now, a form of degree $(1, 0, *)$ can be thought as a form of bidegree $(1, *)$ in $k[\mathbf{x}, \mathbf{z}]$. Moreover, since X is nondegenerate, each such form has a unique expansion of the shape $\sum_i Q_i(\mathbf{z})x_i$, where $Q_i(\mathbf{z}) \in k[\mathbf{z}]$ is homogeneous of degree $*$. Considering these expansions for a minimal set of generating forms of the ideal $(\mathcal{J}_{(1,0,*)})$ and taking the corresponding matrix of $\mathbf{x}$-derivatives, we obtain a weak Jacobian dual matrix $\Psi_{\mathbf{x}}$ in the sense of [DHS12, Section 2.3] – here dubbed an $\mathbf{x}$-*partial Jacobian dual matrix*. We similarly introduce an $\mathbf{y}$-*partial Jacobian dual matrix* $\Psi_{\mathbf{y}}$. Finally, thinking of these matrices as maps over $k[\mathbf{z}]$, we denote by $\Psi_{\mathbf{x}} \otimes_{k[\mathbf{z}]} S$ and $\Psi_{\mathbf{y}} \otimes_{k[\mathbf{z}]} S$ the respective maps obtained modulo $\mathfrak{c}$.

We obtain one of the main results of this chapter.

Theorem 4.2.2. *With the previous notations, the rational map $\mathcal{F} : X \times Y \dashrightarrow Z$ is birational with image Z if and only if* $\operatorname{rank}_S(\Psi_{\mathbf{x}} \otimes_{k[\mathbf{z}]} S) = n$ *and* $\operatorname{rank}_S(\Psi_{\mathbf{y}} \otimes_{k[\mathbf{z}]} S) = m$. *In addition, both halves of the expression of the inverse of $\mathcal{F}$ are given by (signed) ordered maximal minors of an $n \times (n+1)$ submatrix of $\Psi_{\mathbf{x}}$ and of an $m \times (m+1)$ submatrix of $\Psi_{\mathbf{y}}$, respectively.*

The above theorem yields an explicit criterion for deciding if a given bigraded rational is birational. In order to apply this criterion, we need to get the equations of a Rees algebra, which are usually done by using Gröbner basis computations. Therefore, it has certain disadvantages in practice. Moreover, in order to overcome these drawbacks, we investigate how birationality can be detected by means of syzygies of the ideal I generated by the coordinates of the rational maps, instead of the whole collection of equations of I. Syzygies are computable by using any computational algebraic geometry system, such as `Macaulay2` or `Singular`. The following proposition gives an effective criterion related to the syzygies of the ideal I for deciding if a given bigraded rational is birational.

Proposition 4.2.3. *Let $\mathcal{F} : \mathbb{P}^n_k \times \mathbb{P}^m_k \dashrightarrow \mathbb{P}^s_k$ stand for a rational map defined by bihomogeneous polynomials $f_0(\mathbf{x}, \mathbf{y}), \ldots, f_s(\mathbf{x}, \mathbf{y})$ in $R := k[\mathbf{x}, \mathbf{y}]$ and set $I := (f_0(\mathbf{x}, \mathbf{y}), \ldots, f_s(\mathbf{x}, \mathbf{y}))$. If the image of $\mathcal{F}$ has dimension $n+m$ and the submatrix of the syzygy matrix of I consisting of columns of bidegrees $(1, 0)$ and $(0, 1)$ has rank s (maximal possible), then $\mathcal{F}$ is birational onto its image.*

Then, in the main section of this chapter, we focus on the linear syzygies of bigraded rational maps from $\mathbb{P}^1_k \times \mathbb{P}^1_k$ to $\mathbb{P}^2_k$. More precisely, let k be an infinite field and $R := k[x, y; u, v]$ be the bigraded polynomial ring with weights defined by $\deg(x) = \deg(y) = (1, 0)$ and $\deg(u) = \deg(v) = (0, 1)$. Consider a rational map $\phi : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$ is defined by three bihomogeneous polynomials $f_0, f_1, f_2 \in R$, of bidegree (a, b). We assume throughout that ϕ is a dominant rational map, namely it is a bihomogeneous parameterization of a surface. These parameterizations are very important in Computer Aided Geometric Design applications because they are useful to model intricate geometry (see, for example [Bot11, DS16, SSV14]).

Motivated from the fact that the plane Cremona maps $\mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$ defined by three homogeneous polynomials in degree 2 or 3 are automatically de Jonquières maps. In both cases the base ideal is an ideal of 2-minors of a 3×2 matrix, with two linear syzygies or a linear syzyzy and a quadratic one, respectively [HS12]. We will generalize these results when bigraded rational maps $\phi : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$ of bidegree (a, b) with $a + b \leq 3$. More precisely, these bigraded rational maps of bidegree $(1, 1)$ or $(1, 2)$ are birational if and only if the polynomials f_0, f_1, f_2 have a nonzero syzygy of bidegree $(0, 1)$, or equivalent to, the base ideals I are an ideal of 2-minors of a 3×2 matrix, with two syzygies in bidegree $(1, 0)$ and $(0, 1)$ or a syzyzy in bidegree $(0, 1)$ and a syzyzy in bidegree $(1, 1)$, respectively. We list these criteria as follows:

Proposition 4.3.3. *Let $\phi : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$ be a dominant rational map given by bihomogeneous polynomials f_0, f_1, f_2 of bidegree $(1, 1)$ without common factor in $R \setminus k$. The following three assertions are equivalent:*
(i) *ϕ is birational,*
(ii) *the polynomials f_0, f_1, f_2 have a nonzero bidegree $(1, 0)$ syzygy,*
(iii) *the polynomials f_0, f_1, f_2 have a nonzero bidegree $(0, 1)$ syzygy.*

Theorem 4.3.5. *Let $\phi : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$ be a dominant rational map given by bihomogeneous polynomials $f_0, f_1, f_2 \in R$ of bidegree $(1, 2)$ without common factor in $R \setminus k$. Setting $I = (f_0, f_1, f_2) \subset R$, the following four assertions are equivalent:*
(i) *ϕ is birational,*
(ii) $\deg(\mathcal{B}) = 3$, *and hence I is generically a complete intersection,*
(iii) $\dim_k \operatorname{Syz}(I)_{(0,1)} = 1$,
(iv) $\dim_k \operatorname{Syz}(I)_{(1,1)} = 3$.

Unlike the cases of rational maps of bidegree $(1, 1)$ or $(1, 2)$, the linear syzygies associated to a given parameterization are not enough to give birational criterion in higher bidegrees. However, in the case of bidegree $(2, 2)$, by studying the multiplicity of the base locus of the parameterization, we are able to describe a complete list of possible birational maps.

Theorem 4.3.12. *Let $\phi : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$ be a dominant rational map given by bihomogeneous polynomials $f_0, f_1, f_2 \in R$ of bidegree $(2, 2)$ without common factor in $R\setminus k$. Assume that the point in $\mathcal{B}$ with the largest multiplicity is the point $\mathfrak{p} := (x, u)$. Let J be the ideal generated by $f_0(x, 1, u, 1), f_1(x, 1, u, 1), f_2(x, 1, u, 1)$. Then ϕ is birational if and only if $\deg \mathcal{B} = 6$ and $J = (x, u)^2$ or*

$$J = (x^2 + \lambda u^2 + \mu xu, x^2u, xu^2), \ \ \lambda \neq 0.$$

Furthermore, by Lemma 4.3.9, if $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ is a birational map defined by bihomogeneous polynomials $f_0, f_1, f_2 \in R$ of bidegree $(2, 2)$ without common factor in $R \setminus k$, then $I = (f_0, f_1, f_2)$ is perfect ideal with exactly two minimal syzygies, of bidegree $(1, 1)$, and the free resolution of I is of the form

$$0 \longrightarrow R(-3, -3)^2 \xrightarrow{M} R(-2, -2)^3 \longrightarrow I \longrightarrow 0.$$

By a suitable coordinate change, I is generated by 2-minors of the following matrices

$$\begin{pmatrix} u(x+y) & yu \\ -x(u+v) & -yu \\ 0 & xv \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} (ax+y)u & (a-1)xv \\ -x(\alpha u+v) & -(\alpha-1)xv \\ 0 & yu-xv \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} xu & 0 \\ -xv & yu \\ \lambda yu+\mu xv & xv \end{pmatrix}$$

where $a, \alpha, \lambda, \mu \in k$ with $(a, \alpha) \neq (1, 1)$, $a \neq \alpha$ and $\lambda \neq 0$.

Finally, in the last section of this chapter, we investigate applications of our results to the field of modeling. In particular, for bigraded plane rational maps of bidegree $(1, 1)$ and $(1, 2)$ we explain how some particular coefficients of the map, called the *weights* of the parameterization, can be tuned in order to obtain a birational map. It is important to notice that the inverse map is then given by explicit minors from the matrix characterizing the birationality of the map. In the bidegree $(1, 2)$ case, our new birationality criterion allows to assign the control of this tuning to a structured low-rank matrices approximation algorithm, in the context of numerical computations.

The last chapter of this thesis is not motivated by the same question in terms of syzygies. However, our objects are still the Rees and symmetric algebras, and our tools in proofs are the approximation complexes and, specially, the spectral sequence.

In the fifth chapter, we study residual intersection theory. In [AN72], Artin and Nagata introduced the concept of a residual intersection of an ideal; they did not define explicitly this concept but roughly speaking if X is an algebraic variety and $Y \subset Z \subset X$ is a closed embedding of schemes, then the *residual intersection of* Y *in* Z (with respect to X) is a closed subscheme W such that $W \cup Y = Z$. Later, Huneke and Ulrich [HU88] gave a precise definition of residual intersection as follows: if R is a Noetherian ring, I an ideal of height g and $s \geq g$ an integer, then:

- An *s-residual intersection* of I is a proper ideal J of R such that $\operatorname{ht}(J) \geq s$ and $J = (\mathfrak{a} \colon {}_R I)$ for some ideal $\mathfrak{a} \subset I$ generated by s elements.
- A *geometric s-residual intersection* of I is an algebraic s-residual intersection J of I such that $\operatorname{ht}(I + J) \geq s + 1$.

Based on a construction of Laksov for residual intersections, Fulton [Ful98, Definition 9.2.2] presents a formulation for residual intersection that, locally, can be expressed as follows: suppose that $X = \operatorname{Spec}(R)$ and that Y and Z are closed subschemes of X defined by the ideals I and $\mathfrak{a}$, respectively. Let $\widetilde{X} = \operatorname{Proj}(\mathcal{R}_I)$ be the blow-up of X along Y. Consider the natural map $\pi : \widetilde{X} \longrightarrow X$. Let $\widetilde{Y} = \pi^{-1}(Y)$ and $\widetilde{Z} = \pi^{-1}(Z)$. Then $\widetilde{Y} = \operatorname{Proj}(\mathcal{R}_I / I\mathcal{R}_I)$ and $\widetilde{Z} = \operatorname{Proj}(\mathcal{R}_I / \mathfrak{a}\mathcal{R}_I)$ are closed subschemes of $\widetilde{X}$, with ideal sheaves $\mathcal{I}_{\widetilde{Y}}$ and $\mathcal{I}_{\widetilde{Z}}$, where $\mathcal{I}_{\widetilde{Y}}$ is an invertible sheaf. Let $\widetilde{W} \subset \widetilde{X}$ be the closed subscheme defined by the ideal sheaf $\mathcal{I}_{\widetilde{W}} = \mathcal{I}_{\widetilde{Z}} . \mathcal{I}_{\widetilde{Y}}^{-1}$. Then $\widetilde{W}$ is called the *residual scheme to* $\widetilde{Y}$ *in* $\widetilde{Z}$, [Ful98, Definition 9.2.1]. Precisely, $\widetilde{W} = \operatorname{Proj}(\mathcal{R}_I / \gamma\mathcal{R}_I)$ in which $\gamma = \mathfrak{a} \subset (\mathcal{R}_I)_{[1]}$. Finally, the residual intersection to Y in Z [Ful98, Definition 9.2.2] is the direct image of $\mathcal{O}_{\widetilde{W}}$ i.e., $\pi_*(\mathcal{O}_{\widetilde{W}})$. Since $\mathcal{O}_{\widetilde{W}}$ is a coherent sheaf, by [Har77, Chapter III, Proposition 8.5],

$$\pi_*(\mathcal{O}_{\widetilde{W}}) = \left(H^0(\widetilde{X}, \mathcal{O}_{\widetilde{W}})\right)^{\sim} = \left(\Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{W}})\right)^{\sim}.$$

Note that $\Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{W}})$ equals $\Gamma_*(\widetilde{X}, \mathcal{O}_{\widetilde{W}}(0))$, which is equal to the ideal transform $D_{(\mathcal{R}_I)_+}(\mathcal{R}_I / \gamma\mathcal{R}_I)_{[0]}$, by an application of the Čech complex [BS13]. The latter is

closely related to the ideal

$$H^0_{(\mathcal{R}_I)_+}(\mathcal{R}_I/\gamma\mathcal{R}_I)_{[0]} = \bigcup(\mathfrak{a}I^i \colon {}_RI^{i+1}).$$

In many situations, two ideals are the same. In the current works of Hassanzadeh and Naéliton [Has12, HN16], they consider the symmetric algebra $\mathcal{S}_I$ instead of the Rees algebra $\mathcal{R}_I$. This generalization has already proved its usefulness in studying multiple-point formulas by Kleiman; see [Ful98, Example 17.6.2]. We then find a kind of *arithmetical* residual intersection to be

$$\bigcup(\gamma\mathrm{Sym}^i_R(I) \colon {}_R\mathrm{Sym}^{i+1}_R(I)),$$

where $\gamma = \mathfrak{a} \subset \mathrm{Sym}_R(I)_{[1]}$. Comparing the above three definitions for residual intersection, we have

$$J = (\mathfrak{a} \colon {}_RI) \subset \bigcup(\gamma\mathrm{Sym}^i_R(I) \colon {}_R\mathrm{Sym}^{i+1}_R(I)) \subset \bigcup(\mathfrak{a}I^i \colon {}_RI^{i+1}).$$

Interestingly, these ideals coincide if the residual intersection J does not share any associated primes with I, e.g., if J is unmixed and the residual is geometric. Determining the cases where the first inclusion above is an equality leads to define an arithmetic residual intersection. More precisely, an *arithmetic s-residual intersection of I* is an s-residual intersection $J = (\mathfrak{a} \colon {}_RI)$ such that $\mu((I/\mathfrak{a})_\mathfrak{p}) \leq 1$ for all prime ideals $\mathfrak{p} \supset (I + J)$ with $\mathrm{ht}(\mathfrak{p}) \leq s$, here μ denotes the minimum number of generators.

The theory of residual intersections is a center of interest since the 80's, after Huneke repaired in [Hun83] an argument of Artin and Nagata in [AN72], introducing the notion of strongly Cohen-Macaulay ideal: an ideal such that all its Koszul homlogy is Cohen-Macaulay. The notion of strong Cohen-Macaulayness is stable under even linkage, in particular ideals linked to a complete intersection satisfy this property.

In [Hun83] Huneke showed that if R is a Cohen-Macaulay local ring, J is a geometric s-residual intersection of a strongly Cohen-Macaulay ideal I of R satisfying G_s, then R/J is Cohen-Macaulay of codimension s. Following [AN72], one says that I satisfies G_s if the number of generators $\mu_{R_\mathfrak{p}}(I_\mathfrak{p})$ is at most $\dim(R_\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ with $I \subset \mathfrak{p}$ and $\dim(R_\mathfrak{p}) \leq s - 1$ and that I satisfies G_∞ if I satisfies G_s for all s. Later, Herzog, Vasconcelos, and Villarreal in [HVV85] replaced the assumption strong Cohen-Macaulayness by the weaker sliding depth condition, for geometric residuals, but they also showed that this assumption cannot be weakened any further. On the other hand, Huneke and Ulrich proved in [HU88] that the condition G_s is superfluous for ideals in the linkage class of a complete intersection, and more precisely:

Theorem. [HU88, Theorem 5.3] *Let R be a Gorenstein local ring and I be an ideal of height g that is evenly linked to a strongly Cohen-Macaulay ideal satisfying G_∞. If $J = \mathfrak{a} \colon {}_RI$ is an s-residual intersection of I, then R/J is Cohen-Macaulay of codimension s and the canonical module of R/J is the $(s - g + 1)$-th symmetric power of $I/\mathfrak{a}$.*

Let us notice that, in the proof of this statement, it is important to keep track of the canonical module of the residual along the deformation argument that they are using. The reason why G_∞ comes into the proof stems from the original proof in [Hun83] or even the incorrect proof in [AN72] which proves residual intersections are Cohen-Macaulay by an induction which necessitates the G_∞ hypothesis. However Huneke and Ulrich mentioned the following natural question.

Question. [HU88, Question 5.7] Suppose that R is a Cohen-Macaulay local ring and I is an ideal of R which is strongly Cohen-Macaulay (or even satisfies sliding depth). Let J be any residual intersection of I. Then is R/J Cohen-Macaulay?

Hassanzadeh [Has12] answered affirmatively the above question for arithmetic residual intersections, thus in particular for geometric residuals, under the sliding depth condition. One of the main results in this chapter is to answer affirmatively this question. To state our results, one needs to recall the following definition: let R be a Noetherian local ring of dimension d and $I = (\mathbf{f}) = (f_1, \ldots, f_r)$ an ideal of height g. For an integer $k \geq 0$, we say that I satisfies SD_k if

$$\operatorname{depth}(H_i(\mathbf{f}; R)) \geq \min\{d - g, d - r + i + k\}$$

for all i such that $H_i(\mathbf{f}; R) \neq 0$.

Notice that SD_0 is the sliding depth condition and that I satisfies SD_{r-g} if and only if I is a strongly Cohen-Macaulay ideal. We obtain the following result.

Theorem 5.4.6. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module ω and $\mathfrak{a} \subset I$ two ideals of R, with $\operatorname{ht}(I) = g$. Suppose that I satisfies SD_1 and $J = (\mathfrak{a} \colon {}_R I)$ is an s-residual intersection of I. Then R/J is Cohen-Macaulay of codimension s.*

The next question is if J is a residual intersection of I, what is the canonical module of R/J? This question was studied by Huneke and Ulrich [HU88, Ulr94] when R is Gorenstein and by Cumming [Cum07] when R is Cohen-Macaulay. More precisely, Cumming proved the following

Theorem. [Cum07, Theorem 4.2] *Let R be a local Cohen-Macaulay ring with canonical module ω and I an R-ideal of height g satisfying G_s, where s is an integer satisfying $s \geq g$. Suppose that $\operatorname{depth}(\omega/I^j\omega) \geq \dim\ R/I - j + 1$ for $1 \leq j \leq s - g + 1$ and $\operatorname{Ext}_R^{g+j}(R/I^j, \omega) = 0$ for all $1 \leq j \leq s - g - 1$. If $J = \mathfrak{a} \colon {}_R I$ is an i-residual intersection of I for $g \leq i \leq s$, then*

(a) *R/J is Cohen-Macaulay,*

(b) *the canonical module of R/J is $I^{i-g+1}\omega/\mathfrak{a}I^{i-g}\omega$.*

The above theorem is a generalization of [Ulr94, Theorem 2.9] where Ulrich assumed that R was a local Gorenstein ring. Let us notice that, in the proof of these statements, it is important to prove the Cohen-Macaulayness of residual intersections by an induction which necessitates the G_s hypothesis, hence the G_s condition has become an important assumption which is not avoidable. In the recent article of Hassanzadeh and Naéliton [HN16], they proved that $\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a})$ is the canonical

module of R/J, provided any arithmetic s-residual intersection $J = (\mathfrak{a} \colon {}_R I)$ of a strongly Cohen-Macaulay ideal I of height g in a Gorenstein local ring R. Our second main result in this chapter is to generalize this result. More precisely, we obtain the following.

Theorem 5.4.9. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module ω and $\mathfrak{a} \subset I$ two ideals of R, with $\operatorname{ht}(I) = g$. Suppose that I satisfies SD_2, $\operatorname{Tor}_1^R(R/I, \omega) = 0$ and $J = (\mathfrak{a} \colon {}_R I)$ is an s-residual intersection of I. Then the canonical module of R/J is $\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega$.*

Let us notice that, the hypothesis $\operatorname{Tor}_1^R(R/I, \omega) = 0$ is immediately satisfied if R is Gorenstein or I has finite projection dimension.

In the last section of chapter, and also to close this thesis, we prove a result on duality for residual intersections of strongly Cohen-Macaulay ideals. This could be compared to recent results of Eisenbud and Ulrich in [EU16] where similar dualities were obtained under slightly different hypotheses. In their work, conditions on the local number of generators are needed and depth conditions are asked for some of the first powers of the ideal I, along the lines of [Ulr94], and the duality occurs between powers $I^t/\mathfrak{a}I^{t-1}$ in place of symmetric powers $\operatorname{Sym}_R^t(I/\mathfrak{a})$. Although their results and ours coincide in an important range of situations, like for geometric residuals of strongly Cohen-Macaulay ideals satisfying G_s, the domains of validity of the two results are quite distinct. We prove the following.

Theorem 5.6.7. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring and $\mathfrak{a} \subset I$ two ideals of R, with $\operatorname{ht}(I) = g$. Suppose that $J = (\mathfrak{a} \colon {}_R I)$ is an s-residual intersection of I. If I is strongly Cohen-Macaulay, then $\omega_{R/J} \simeq \operatorname{Sym}_{R/J}^{s-g+1}(I/\mathfrak{a})$ and for all $0 \leq k \leq s-g+1$*

(i) *the R/J-module $\operatorname{Sym}_{R/J}^k(I/\mathfrak{a})$ is faithful and Cohen-Macaulay,*

(ii) *the multiplication*

$$\operatorname{Sym}_{R/J}^k(I/\mathfrak{a}) \otimes_{R/J} \operatorname{Sym}_{R/J}^{s-g+1-k}(I/\mathfrak{a}) \longrightarrow \operatorname{Sym}_{R/J}^{s-g+1}(I/\mathfrak{a})$$

is a perfect pairing,

(iii) *setting $A := \operatorname{Sym}_{R/J}(I/\mathfrak{a})$, the graded R/J-algebra*

$$\overline{A} := A/A_{>s-g+1} = \bigoplus_{i=0}^{s-g+1} \operatorname{Sym}_{R/J}^i(I/\mathfrak{a})$$

is Gorenstein.

In this chapter, we also determine several invariants of a residual intersection as the graded canonical module, the stability of Hilbert function, the Hilbert series, the Castelnuovo-Mumford regularity and the type. Furthermore, whenever I is strongly Cohen-Macaulay, via the duality results for residual intersections, we establish some tight relations between the Hilbert series of some symmetric powers of $I/\mathfrak{a}$ and provide closed formulas for the types and for the Bass numbers of some symmetric powers of $I/\mathfrak{a}$.

Chapter 1

Preliminaries

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In this chapter we collect the definitions, results and techniques that we require for the later chapters. As a consequence, most of the material in this chapter is well known. We refer to [BH98] as standard text book, in particular for the discussion of regular sequences and depth, Cohen-Macaulay modules, graded rings, graded modules, Koszul complexes and Hilbert functions. The reader can consult the notations and general facts about local cohomologies and canonical modules in [BS13]. We refer to [HSV83b] for studying approximation complexes and [Sim04] or [DHS12] for the notions introduced in the last section.

Throughout, all given rings are assumed to be commutative, associative, with identity, and all modules are assumed to be unital. By a Noetherian local ring $(R, \mathfrak{m}, k)$ we mean a Noetherian ring R with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

1.1 Cohen-Macaulay modules

1.1.1 Regular sequences and depth

Let M be a module over a ring R. We say that $x \in R$ is an *M-regular element* if $xz = 0$ for $z \in M$ implies $z = 0$, in other words, if x is not a zero-divisor on M. Regular sequences are composed of successively regular elements:

Definition 1.1.1. A sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of R is called an *M-regular sequence* if the following conditions are satisfied: (i) x_i is an $M/(x_1, \ldots, x_{i-1})M$-regular element for $i = 1, \ldots, n$, and (ii) $M/\mathbf{x}M \neq 0$.

A *regular sequence* is an R-regular sequence. Very often used in present tense R is a Noetherian local ring with maximal ideal $\mathfrak{m}$, and $M \neq 0$ a finitely generated R-module. If $\mathbf{x} \subset \mathfrak{m}$, then condition (ii) is automatically satisfied because of Nakayama's lemma.

The classical example of a regular sequence is the sequence $X_1, \ldots, X_n$ of indeterminates in a polynomial ring $R = S[X_1, \ldots, X_n]$.

We use $\widehat{M}$ to denote the $\mathfrak{m}$-adic completion of a module M over a Noetherian local ring $(R, \mathfrak{m})$.

Proposition 1.1.2. [BH98, 1.1.3] *Let R be a Noetherian ring, M a finitely generated R-module, and $\mathbf{x}$ an M-regular sequence.*

(a) *Suppose that a prime ideal $\mathfrak{p} \in \operatorname{Supp} M$ contains $\mathbf{x}$. Then $\mathbf{x}$ (as a sequence in $R_\mathfrak{p}$) is an $M_\mathfrak{p}$-regular sequence.*

(b) *Suppose that R is local with maximal ideal $\mathfrak{m}$. Then $\mathbf{x}$ (as a sequence in $\widehat{R}$) is an $\widehat{M}$-regular sequence.*

Proposition 1.1.3. [BH98, 1.2.4] *Let R be a ring, M, N be R-modules, and $\mathbf{x} = x_1, \ldots, x_n$ an M-regular sequence in $\operatorname{ann}_R(N)$. Then*

$$\operatorname{Hom}_R(N, M/\mathbf{x}M) \simeq \operatorname{Ext}^n_R(N, M).$$

Let R be a Noetherian ring and M an R-module. If $\mathbf{x} = x_1, \ldots, x_n$ is an M-regular sequence, then the sequence $(x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, \ldots, x_n)$ ascends strictly. Therefore an M-regular sequence can be extended to a maximal such sequence: an M-regular sequence $\mathbf{x}$ (contained in an ideal I) is *maximal* (in I), if $x_1, \ldots, x_{n+1}$ is not an M-regular sequence for any $x_{n+1} \in R$ ($x_{n+1} \in I$).

Theorem 1.1.4 (Rees)**.** [BH98, 1.2.5] *Let R be a Noetherian ring, M a finitely generated R-module, and I an ideal such that $IM \neq M$. Then all maximal M-regular sequence in I have the same length n given by*

$$n = \min\{i : \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

Definition 1.1.5. Let R be a Noetherian ring, M a finitely generated R-module, and I an ideal such that $IM \neq M$. Then the common length of the maximal M-regular sequence in I is called the *grade of I on M*, denoted by $\operatorname{depth}_I(M)$, hence

$$\operatorname{depth}_I(M) = \min\{i : \operatorname{Ext}^i_R(R/I, M) \neq 0\},$$

with the convention that $\operatorname{depth}_I(M) = \infty$ if $IM = M$.

If in addition R is a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$, then grade of $\mathfrak{m}$ on M is called the *depth* of M, denoted by $\operatorname{depth}(M)$.

Because of its importance we repeat the most often used special case of Rees's theorem.

Theorem 1.1.6. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, M a finitely generated nonzero R-module. Then*

$$\operatorname{depth}(M) = \min\{i : \operatorname{Ext}^i_R(k, M) \neq 0\}.$$

We now study the behaviour of $\operatorname{depth}_I(M)$ along exact sequences.

Proposition 1.1.7. [BH98, 1.2.9] *Let R be a Noetherian ring, $I \subset R$ an ideal, and*

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

an exact sequence of finitely generated R-modules. Then

$$\begin{aligned} \operatorname{depth}_I(N) &\geq \min\{\operatorname{depth}_I(M), \operatorname{depth}_I(P)\}, \\ \operatorname{depth}_I(M) &\geq \min\{\operatorname{depth}_I(N), \operatorname{depth}_I(P) + 1\}, \\ \operatorname{depth}_I(P) &\geq \min\{\operatorname{depth}_I(M) - 1, \operatorname{depth}_I(N)\}. \end{aligned}$$

In the case where $(R, \mathfrak{m})$ is local, consider $I = \mathfrak{m}$ in the above proposition, we obtain the following.

Corollary 1.1.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and*

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

an exact sequence of finitely generated R-modules. Then, either

(a) *if* $\operatorname{depth}(N) < \operatorname{depth}(P)$, *then* $\operatorname{depth}(M) = \operatorname{depth}(N)$ *or*

(b) *if* $\operatorname{depth}(N) > \operatorname{depth}(P)$, *then* $\operatorname{depth}(M) = \operatorname{depth}(P) + 1$, *or*

(c) *if* $\operatorname{depth}(N) = \operatorname{depth}(P)$, *then* $\operatorname{depth}(M) \geq \operatorname{depth}(N)$.

1.1.2 Cohen-Macaulay modules

Definition 1.1.9. Let R be a Noetherian local ring. A finitely generated R-module $M \neq 0$ is a *Cohen-Macaulay module* if $\operatorname{depth}(M) = \dim\ M$. It R itself is a Cohen-Macaulay module, then it is called a *Cohen-Macaulay ring*. A *maximal Cohen-Macaulay module* is a Cohen-Macaulay module M such that $\operatorname{depth}(M) = \dim\ R$.

If I is an ideal contained in $\operatorname{ann}_R(M)$, then it is irrelevant for the Cohen-Macaulay property whether we consider M as an R-module or an R/I-module. In particular, if R is local and M is a Cohen-Macaulay module, then M is a maximal Cohen-Macaulay module over $R/\operatorname{ann}_R(M)$.

The next theorem exhibits the fact that for a Cohen-Macaulay module, the grade of an arbitrary ideal is given by its codimension.

Theorem 1.1.10. [BH98, 2.1.2] *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and $M \neq 0$ a Cohen-Macaulay R-module. Then*

(a) $\dim\ R/\mathfrak{p} = \operatorname{depth}(M)$ *for all* $\mathfrak{p} \in \operatorname{Ass} M$,

(b) $\operatorname{depth}_I(M) = \dim\ M - \dim\ M/IM$ *for all ideals* $I \subset \mathfrak{m}$,

(c) $\mathbf{x} = x_1, \ldots, x_n$ *is an M-regular sequence if and only if* $\dim\ M/\mathbf{x}M = \dim\ M - n$,

(d) $\mathbf{x}$ *is an M-regular sequence if and only if it is part of a system of parameters of M.*

The Cohen-Macaulayness is stable under specialization and localization:

Theorem 1.1.11. [BH98, 2.1.3] *Let $(R, \mathfrak{m})$ be a Noetherian ring, and M a finitely generated R-module.*

(a) *Suppose $\mathbf{x}$ is an M-regular sequence. If M is a Cohen-Macaulay module, then $M/\mathbf{x}M$ is Cohen-Macaulay (over R or $R/(\mathbf{x})$). The converse holds if R is local.*

(b) *Suppose M is Cohen-Macaulay. Then for every multiplicatively closed set S in R the localized $S^{-1}M$ is also Cohen-Macaulay. In particular, $M_\mathfrak{p}$ is Cohen-Macaulay for every $\mathfrak{p} \in \operatorname{Spec}(R)$. If $M_\mathfrak{p} \neq 0$, then* $\operatorname{depth}(M_\mathfrak{p}) = \operatorname{depth}_\mathfrak{p}(M)$; *if in addition R is local, then* $\dim\ M = \dim\ M_\mathfrak{p} + \dim\ M/\mathfrak{p}M$.

An ideal generated by a regular sequence is said to be a *complete intersection ideal*. An important property of complete intersection ideals is following.

Proposition 1.1.12. [ZS60, Appendix 6, Lemma 5] *Let R be a Cohen-Macaulay local ring, and $\mathfrak{a}$ a complete intersection ideal of R. For every exponent n, the ideal $\mathfrak{a}^n$ is unmixed, and admits, therefore, the same associated prime ideals as $\mathfrak{a}$.*

Now we recall an important numerical invariant of modules.

Definition 1.1.13. Let R be a Noetherian ring, M a finitely generated nonzero R-module and $\mathfrak{p} \in \operatorname{Spec}(R)$. The number $\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$ is called the *i-th Bass number of M with respect to* $\mathfrak{p}$, where $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$.

These numbers have an interpretation in terms of the minimal injective resolution of M.

Proposition 1.1.14. [BH98, 3.2.9] *Let R be a Noetherian ring, M a finitely generated R-module, and $E^\bullet(M)$ the minimal injective resolution of M. Then*

$$E^i(M) \simeq \bigoplus_{\mathfrak{p} \in \operatorname{Spec}(R)} E(R/\mathfrak{p})^{\mu_i(\mathfrak{p}, M)},$$

where $E(R/\mathfrak{p})$ is the injective hull of $R/\mathfrak{p}$.

Definition 1.1.15. Let R be a ring and M be an R-module. The *injective dimension of* M is the smallest integer n for which there exists an injective resolution $E^\bullet(M)$ of M with $E^m(M) = 0$ for $m > n$. If there is no such n, the injective dimension of M is infinite. A Noetherian local ring R is a *Gorenstein ring* if its injective dimension is finite. A Noetherian ring R is *Gorenstein* if its localisation at every maximal ideal is a Gorenstein local ring.

Theorem 1.1.16. [Mat89, Theorem 18.8] *A necessary and sufficient condition for a ring R to be Gorenstein is that a minimal injective resolution $E^\bullet(R) : 0 \longrightarrow R \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots$ of R satisfies*

$$E^i = \bigoplus_{\mathrm{ht}\mathfrak{p}=i} E(R/\mathfrak{p}),$$

or, in other words, $\mu_i(\mathfrak{p}, R) = \delta_{i,\mathrm{ht}\mathfrak{p}}$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$.

When $(R, \mathfrak{m})$ is a Noetherian local ring, and M is a finitely generated R-module of depth t, the Bass number $\mu_t(\mathfrak{m}, M)$ is of particular importance.

Definition 1.1.17. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and M a finitely generated non-zero R-module of depth t. The number $r(M) = \dim_k \mathrm{Ext}^t_R(k, M)$ is called the *type of* M. In particular, $r(M) = \mu_t(\mathfrak{m}, M)$.

Theorem 1.1.18. [BH98, 3.2.10] *Let $(R, \mathfrak{m})$ be a Noetherian local ring. The following conditions are equivalent:*

(a) *R is a Gorenstein ring;*

(b) *R is a Cohen-Macaulay ring of type 1.*

Example 1.1.19. The polynomial ring $R = k[X_1, \ldots, X_n]$ over a field k is Gorenstein.

Definition 1.1.20. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and I, J two ideals of R.

(i) I and J are said to be *directly linked* if there is a regular sequence $\mathbf{x} := x_1, \ldots, x_g \in I \cap J$ such that $I = (\mathbf{x}) \colon_R J$ and $J = (\mathbf{x}) \colon_R I$.

(ii) We say I and J are in the *same linkage class* (or liaison class) if there exists a sequence of ideals $I = I_0, \ldots, I_n = J$ such that I_j is directly linked to I_{j+1} for $0 \leq j \leq n-1$.

Proposition 1.1.21. [Vas94, 4.1.1] *Let $(R, \mathfrak{m})$ be a Gorenstein local ring, I an unmixed ideal of height g, and $\mathbf{x} := x_1, \ldots, x_g$ a regular sequence contained in I. Then the ideal $J = (\mathbf{x}) \colon_R I$ is directly linked to I; more precisely, $I = (\mathbf{x}) \colon_R J$. Furthermore, R/I is Cohen-Macaulay if and only if R/J is Cohen-Macaulay.*

1.1.3 Graded Rings and Modules

Definition 1.1.22. An $\mathbb{N}$-*graded ring* R is a ring together with a decomposition (as abelian groups) $R = R_0 \oplus R_1 \oplus \ldots$ such that $R_m.R_n \subseteq R_{m+n}$ for all $m, n \in \mathbb{Z}_{\geq 0}$, and where R_0 is a subring (i.e. $1 \in R_0$). A $\mathbb{Z}$-*graded ring* is one where the decomposition is into $R = \oplus_{n \in \mathbb{Z}} R_n$. In either case, the elements of the subgroup R_n are called *homogeneous* of degree n.

Example 1.1.23. Let R be a ring and $X_1, \ldots, X_n$ indeterminates over R. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, set $\mathbf{X}^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then the polynomial ring $S = R[X_1, \ldots, X_n]$ is a graded ring, graded by the total degree of polynomials. In this case the graded pieces can be write as

$$S_d = \{ \sum_{|\alpha|=d} a_\alpha X^\alpha \mid a_\alpha \in R\}.$$

S is called the *standard polynomial ring over* R.

Definition 1.1.24. A *graded R-module* is an ordinary R-module M together with a decomposition $M = \oplus_{k\in\mathbb{Z}} M_k$ as abelian groups, such that $R_m.M_n \subseteq M_{m+n}$ for all $m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$. Elements in one of these pieces are called *homogeneous*. Any $m \in M$ is thus uniquely a finite sum $\sum m_{n_i}$ where each $m_{n_i} \in M_{n_i}$ is homogeneous of degree n_i and $n_i \neq n_j$ for $i \neq j$.

In the category of graded R-modules $\mathfrak{M}_0(R)$, the morphisms of R-modules are the ones that preserve the grading. In other words, morphisms of graded modules take homogeneous elements to homogeneous elements of the same degree. We say these morphisms are *homogeneous homomorphisms* or *graded homomorphisms of degree zero.*

Definition 1.1.25. If M is a graded module, we set $M(n)$ for the same R-module but with the grading $M(n)_k = M_{n+k}$.

Example 1.1.26. (a) If R is a graded ring, then R is a graded module over itself.

(b) If S is any ring, then S can be considered as a graded ring with $S_0 = S$ and $S_i = 0$ for $i > 0$. Then a graded S-module is just a $\mathbb{Z}$-indexed collection of (ordinary) S-modules.

(c) [The blow-up algebra, also called Rees algebra] Let R be any ring, and let I be an ideal of R. We can make $\mathcal{R}_I = R \oplus I \oplus I^2 \oplus \cdots$ (the blow-up algebra) into a graded ring, by defining the multiplication from the one of R after noticing that $I^i I^j \subseteq I^{i+j}$. Given any R-module M, there is a graded $\mathcal{R}_I$-module $M \oplus IM \oplus I^2M \oplus \cdots$, where multiplication is defined in the obvious way. We thus get a functor from R-modules to graded $\mathcal{R}_I$-modules.

Definition 1.1.27. Suppose that R is graded, and let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded R-module. We define the *end degree of* M by

$$\operatorname{end}(M) := \sup\{n \in \mathbb{Z} : M_n \neq 0\}$$

with the convention that $\sup \emptyset = -\infty$. We similarly define the *initial degree of* M by

$$\operatorname{indeg}(M) := \inf\{n \in \mathbb{Z} : M_n \neq 0\}$$

with the convention that $\sup \emptyset = +\infty$.

Definition 1.1.28. Let R be a graded ring, M be a graded R-module and $N \subseteq M$ an R-submodule. N is called a *graded submodule* if the homogeneous components of elements in N are in N. Furthermore, M/N is a graded R-module where $(M/N)_i = M_i/N_i$ for $i \in \mathbb{Z}$. In particular, if $M = R$, an ideal I of R is also called a *graded ideal* if it is a graded R-submodule.

The most important graded rings arise in algebraic geometry as the coordinate rings of projective varieties. They have the form $R = k[X_1, \ldots, X_n]/I$ where k is a field and I is an ideal generated by the homogeneous polynomials (in the usual sense). Then R is generated as k-algebra by elements of degree 1, namely the residue classes of the indeterminates. Graded rings R, which as R_0-algebras, are generated by 1-forms will be called *standard graded R_0-algebras or homogeneous R_0-algebras.* More generally, if R is a graded R_0-algebra generated by elements of positive degree, then we say that R is a *positively graded R_0-algebra.*

Proposition 1.1.29. [BH98, 1.5.4] *Let R be a positively graded R_0-algebra, and $x_1, \ldots, x_n$ homogeneous elements of positive degree. Then the following are equivalent:*

(a) *$x_1, \ldots, x_n$ generate the ideal $R_+ = \oplus_{i \geq 1} R_i$;*

(b) *$x_1, \ldots, x_n$ generate R as an R_0-algebra.*

In particular R is Noetherian if and only if R_0 is Noetherian and R is a finitely generated R_0-algebra.

Let I be an arbitrary ideal of R. Then the graded ideal I^* is defined to be the ideal generated by all homogeneous elements $a \in I$. It is clear that I^* is the largest graded ideal contained in I, and that R/I^* inherits a natural structure as a graded ring. Very often we shall derive properties of a graded ring or module from its localization with respect to graded prime ideals. The following lemma is basic for such arguments.

Lemma 1.1.30. [BH98, 1.5.6] *Let R be a graded ring.*

(a) *For every prime ideal $\mathfrak{p}$ the ideal $\mathfrak{p}^*$ is a prime ideal.*

(b) *Let M be a graded R-module.*

  (i) *If $\mathfrak{p} \in \operatorname{Supp} M$, then $\mathfrak{p}^* \in \operatorname{Supp} M$.*

  (ii) *If $\mathfrak{p} \in \operatorname{Ass} M$, then $\mathfrak{p}$ is graded; furthermore $\mathfrak{p}$ is the annihilator of a homogeneous element.*

Let $\mathfrak{p}$ be a prime ideal of R and S the set of homogeneous elements of R not belonging to $\mathfrak{p}$. Then S is a multiplicatively closed set and we put $M_{(\mathfrak{p})} = S^{-1}M$ for any graded R-module M. For $x/a \in M_{(\mathfrak{p})}$, x homogeneous, we set $\deg(x/a) = \deg(x) - \deg(a)$. We further define a grading on $M_{(\mathfrak{p})}$ by setting

$$(M_{(\mathfrak{p})})_n = \{x/a \in M_{(\mathfrak{p})} : x \quad \text{homogeneous}, \ \deg(x/a) = n\}.$$

Let M, N be graded R-modules. An R-module homomorphism $\varphi : M \longrightarrow N$ is called *homogeneous of degree i* if $\varphi(M_n) \subset N_{n+i}$ for all n. Denote by $\operatorname{Hom}_i(M, N)$ the group of homogeneous homomorphisms of degree i. The $\mathbb{Z}$-submodules $\operatorname{Hom}_i(M, N)$ of $\operatorname{Hom}_R(M, N)$ form a direct sum, and it is obvious that

$$^*\operatorname{Hom}_R(M, N) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_i(M, N)$$

is a graded R-submodule of $\operatorname{Hom}_R(M, N)$.

For any $N \in \mathfrak{M}_0(R)$ we define $^*\operatorname{Ext}^i_R(M, N)$ as the *i-th right derived functor* of $^*\operatorname{Hom}_R(-, N)$ in $\mathfrak{M}_0(R)$.

Proposition 1.1.31. [BH98, 1.5.11] *Let R be a Noetherian graded ring and I an ideal in R generated by homogeneous elements of positive degree. Set $h = \mathrm{ht}(I)$ and $g = \mathrm{depth}_I(M)$ where M is a finitely generated R-module. Then there exist sequences $\mathbf{x} = x_1, \ldots, x_h$ and $\mathbf{y} = y_1, \ldots, y_g$ of homogeneous elements of I such that $\mathrm{ht}(x_1, \ldots, x_i) = I$ for $i = 1, \ldots, h$ and $\mathbf{y}$ is an M-regular sequence.*

Often one needs a stronger version of Proposition 1.1.31.

Proposition 1.1.32. [BH98, 1.5.12] *In addition to the hypotheses of Proposition 1.1.31 assume that R_0 is a local ring with an infinite residue class field and that I is generated by elements of degree 1. Then the sequences $\mathbf{x} = x_1, \ldots, x_h$ and $\mathbf{y} = y_1, \ldots, y_g$ can be composed of elements of degree 1.*

Definition 1.1.33. Let R be a graded ring. A graded ideal $\mathfrak{m}$ of R is called **maximal*, if every graded ideal that properly contains $\mathfrak{m}$ equals R. The ring R is called **local*, if it has a unique *maximal ideal $\mathfrak{m}$. A graded *local ring with *maximal ideal $\mathfrak{m}$ will be denoted by $(R, \mathfrak{m})$.

Let $(R, \mathfrak{m})$ be a *local ring. All nonzero homogeneous elements of the graded ring $R/\mathfrak{m}$ are invertible, and so $R/\mathfrak{m}$ is either a field, or else $R/\mathfrak{m} \simeq k[t, t^{-1}]$, where k is a field and t is a homogeneous element of positive degree which is transcendental over k. In the first case $\mathfrak{m}$ is a maximal ideal, and in the second $\mathfrak{m}$ is a prime ideal with $\dim\ R/\mathfrak{m} = 1$. Note that R_0 is a local ring with maximal ideal $\mathfrak{m}_0 = \mathfrak{m} \cap R_0$, and that all homogeneous elements $a \in R \setminus \mathfrak{m}$ are units. We define **dimension of R* as the height of $\mathfrak{m}$ and denote it by $^*\dim\ R$.

Example 1.1.34. Let R be a graded ring.

(a) Let $\mathfrak{p}$ be a graded prime ideal. Then $R_{(\mathfrak{p})}$ is a *local ring.

(b) If R is a positively graded ring for which R_0 is a local ring with maximal ideal $\mathfrak{m}_0$, then R is a *local ring with *maximal ideal $\mathfrak{m} = \mathfrak{m}_0 \oplus \bigoplus_{n>0} R_n$. In particular a positively graded algebra over a field is *local.

1.2 Local cohomology and canonical module

1.2.1 Local cohomology

Definition 1.2.1. If R is a Noetherian ring, I is an ideal of R, and M is an R-module, then the 0-*th local cohomology module of M* is

$$H_I^0(M) := \{m \in M \mid I^n m = 0 \text{ for some } n\}.$$

$H_I^0(-)$ is a functor in an obvious way: if $\varphi : M \longrightarrow N$ is a map, the induced map $H_I^0(\varphi)$ is the restriction of φ to $H_I^0(M)$. It is immediate to see from this that the functor $H_I^0(-)$ is left exact, so it is natural to study its derived functors, which we call *the local cohomology functors*, denoted by $H_I^i(-)$, the right derived functors of $H_I^0(-)$.

Proposition 1.2.2. [Eis05, 10.1] *If R is a Noetherian ring, I is an ideal of R, and M is an R-module, then there is a canonical isomorphism*

$$H^i_I(M) \simeq \varinjlim \operatorname{Ext}^i_R(R/I^n, M),$$

where the limit is taken over the maps $\operatorname{Ext}^i_R(R/I^n, M) \longrightarrow \operatorname{Ext}^i_R(R/I^m, M)$ induced by the natural epimorphisms $R/I^m \twoheadrightarrow R/I^n$ for $m \geq n$.

Proposition 1.2.3 (Grothendieck)**.** [BH98, 3.5.7] *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. Then*

(a) *If $i < \operatorname{depth}(M)$ or $i > \dim M$ then $H^i_{\mathfrak{m}}(M) = 0$.*

(b) *If $i = \operatorname{depth}(M)$ or $i = \dim M$ then $H^i_{\mathfrak{m}}(M) \neq 0$.*

The next result implies immediately from the above proposition.

Corollary 1.2.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. A finitely generated R-module M is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq \dim\ M$.*

Another useful expression for the local cohomology is obtained from a Čech complex: Suppose that I is generated by elements $(x_1, \ldots, x_n)$. We write $[n] = 1, \ldots, n$ for the set of integers from 1 to n, and for any subset $J \subset [n]$ we set $x_J = \prod_{j \in J} x_j$. We denote by M_{x_J} the localization of M by inverting x_J. If $i \notin J$ we let $\sigma_J(i)$ denote the number of elements of J less than i. The *modified Čech complex of I over M*, denoted by $C^\bullet_I(M)$, is defined by

$$\begin{array}{ccccccccc} C^\bullet_I(M) : 0 & \longrightarrow & M & \xrightarrow{d} & \oplus_{i=1}^n M_{x_i} & \xrightarrow{d} & \cdots & \xrightarrow{d} & M_{x_1 \cdots x_n} & \longrightarrow & 0 \\ & & \| & & \| & & & & \| & & \\ & & C^0(M) & & C^1(M) & & & & C^n(M) & & \end{array} \tag{1.1}$$

whose differential takes an element

$$m_J \in M_{x_J} \subset \oplus_{\#J=s} M_{x_J} = C^s(M)$$

to the element

$$d(m_J) = \sum_{i \notin J} (-1)^{\sigma_J(i)} m_{J \cup \{i\}} \in C^{s+1}(M),$$

where $m_{J \cup \{i\}}$ denotes the image of m_J in the further localization $M_{x_{J \cup \{i\}}} = (M_{x_J})_{x_i}$.

Theorem 1.2.5. [Eis05, 10.2] *Suppose that R is a Noetherian ring and $I = (x_1, \ldots, x_n)$. For any R-module M the local cohomology $H^i_I(M)$ is the i-th cohomology of the Čech complex* (1.1)

Suppose that $\varphi : R \longrightarrow R'$ is a homomorphism of rings, I is an ideal of R, and M is an R'-module. Using the map φ we can also regard M as an R-module. In general, the relation between $\operatorname{Ext}^i_R(R/I^n, M)$ and $\operatorname{Ext}^i_{R'}(R'/I'^n, M)$, where $I' = IR'$, is mysterious. However, Theorem 1.2.5 gives the following easy consequence.

Corollary 1.2.6. *Suppose that $\varphi : R \longrightarrow R'$ is a homomorphism of Noetherian rings. With notation as above, there is a canonical isomorphism $H^i_I(M) \simeq H^i_{IR'}(M)$.*

Proposition 1.2.7. [Has12, Lemma 2.1] *Let R be a Noetherian ring and M an R-module. Suppose that $S = R[T_1, \dots, T_n]$ is a standard graded polynomial ring over R. Then*

(i) $H^i_{S_+}(M \otimes_R S) = 0$ *for all* $i \neq n$,

(ii) *there exists a functorial isomorphism* $\theta_M : H^n_{S_+}(M \otimes_R S) \longrightarrow M \otimes_R H^n_{S_+}(S)$.

Proof. (i) follows from the fact that $T_1, \dots, T_n$ is a regular sequence on $M \otimes_R S$ and (ii) from the computation of $H^n_{S_+}(-)$ via the Čech complex on $T_1, \dots, T_n$. □

We now start our approach to the Local Duality Theorem. From now on for any $(R, \mathfrak{m})$ is a local ring, we denote by $E(R/\mathfrak{m})$ (or E for simplicity) the injective envelope of the simple R-module $R/\mathfrak{m}$.

Lemma 1.2.8. [BS13, 11.2.3] *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d. Then $H^d_{\mathfrak{m}}(R) \simeq E(R/\mathfrak{m})$.*

Definition 1.2.9. The functor $\mathrm{Hom}_R(-, E)$ is called the *Matlis duality functor,* to be denoted as $-^\vee$, $M^\vee = \mathrm{Hom}_R(M, E)$ is called the Matlis dual of R-module M and $M^{\vee\vee}$ stands for $(M^\vee)^\vee$.

Theorem 1.2.10 (Local duality for a Gorenstein local ring)**.** [BS13, 11.2.5] *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d and M a finitely generated R-module. Then*

$$H^{d-i}_{\mathfrak{m}}(M) \simeq \mathrm{Ext}^i_R(M, R)^\vee$$

for all $i \in \mathbb{Z}$.

Theorem 1.2.11 (Local Duality Theorem)**.** [BS13, 11.2.6] *Suppose that $(R, \mathfrak{m})$ is a local ring which is a homomorphic image of a Gorenstein local ring A of dimension n. For each finitely generated R-module M,*

$$H^i_{\mathfrak{m}}(M) \simeq \mathrm{Ext}^{n-i}_A(M, A)^\vee$$

for all $i \in \mathbb{Z}$.

Note that if the ring R is graded, M is a graded R-module and I is generated by the homogeneous elements $x_1, \dots, x_n$ with $\deg(x_i) = d_i$, then the Čech complex is canonically graded by putting $\deg(\frac{m}{(x_{i_1}, \cdots x_{i_p})^\alpha}) := \deg(m) - \alpha(d_{i_1} + \cdots + d_{i_p})$, the differentials d^p are thus all of degree zero. Therefore, by Theorem 1.2.5, the local cohomology $H^i_I(M)$ is isomorphic to the i-th cohomology of the Čech complex, inherits a grading. Moreover, this grading is independent of the choice of homogeneous generators for I, see instance [BS13, Chapter 13] or [BH98, §3.6]. We use $H^i_I(M)_n$ to denote the n-th component of the graded R-module $H^i_I(M)$.

Proposition 1.2.12. [BS13, 16.1.5] *Assume that $R = \bigoplus_{n=0}^\infty R_n$ is positively graded and M is a finitely generated graded R-module. Set $R_+ = \bigoplus_{n>0} R_n$.*

(a) *For all $i \in \mathbb{Z}_{\geq 0}$ and all n, the R_0-module $H^i_{R_+}(M)_n$ is finitely generated.*

(b) *There exists $r \in \mathbb{Z}$ such that $H^i_{R_+}(M)_n = 0$ for all $i \in \mathbb{Z}_{\geq 0}$ and all $n \geq r$.*

Definition 1.2.13. Assume that $R = \bigoplus_{n=0}^{\infty} R_n$ is positively graded and M is a finitely generated graded R-module. Set $R_+ = \bigoplus_{n>0} R_n$. The *Castelnuovo-Mumford regularity of* M, denoted by $\operatorname{reg}(M)$, is defined by

$$\begin{aligned}\operatorname{reg}(M) :&= \sup\{\operatorname{end}(H^i_{R_+}(M)) + i : i \in \mathbb{Z}_{\geq 0}\}\\ &= \sup\{\operatorname{end}(H^i_{R_+}(M)) + i : \operatorname{depth}(M) \leq i \leq \dim\ M\}.\end{aligned}$$

The last equality implies from Proposition 1.2.3.

If M is any module over a Noetherian ring R and $I = (x_1, \ldots, x_n) \subset R$ is an ideal, then M gives rise by restriction to a sheaf $\mathcal{F}_M$ on the affine scheme $\operatorname{Spec} R \setminus V(I)$ whose i-th sheaf cohomology $H^i(\mathcal{F}_M)$ may be defined as the i-th cohomology of the Čech complex:

$$\begin{array}{ccccccccc}\check{\mathrm{C}}^\bullet_I(M) : 0 \longrightarrow & \oplus_{i=1}^n M_{x_i} & \xrightarrow{d^\vee} & \oplus_{i<j} M_{x_i x_j} & \xrightarrow{d^\vee} & \cdots & \xrightarrow{d^\vee} & M_{x_1\cdots x_n} & \longrightarrow 0\\ & \| & & \| & & & & \| & \\ & \check{\mathrm{C}}^0(M) & & \check{\mathrm{C}}^1(M) & & & & \check{\mathrm{C}}^{n-1}(M) & \end{array} \tag{1.2}$$

whose differential is defined as in (1.1). Note that $\check{\mathrm{C}}^i(M) = C^{i+1}(M)$ and $d_i^\vee = d_{i+1}$ for all $i \geq 0$.

The local cohomology is related to sheaf cohomology in a simple way:

Proposition 1.2.14. *With notations as above, then*

(a) *there is an exact sequence of R-modules*

$$0 \longrightarrow H^0_I(M) \longrightarrow M \longrightarrow H^0(\mathcal{F}_M) \longrightarrow H^1_I(M) \longrightarrow 0;$$

(b) *for every $i \geq 2$*

$$H^i_I(M) = H^{i-1}(\mathcal{F}_M).$$

Proof. This follows from the identification $\operatorname{Ker}(d_0^\vee) = H^0(\mathcal{F}_M)$ and Theorem 1.2.5. □

Henceforward we will restrict our attention to the case where R is the polynomial ring $S = k[X_0, \ldots, X_n]$ over a field k with its standard grading $\deg(X_i) = 1$ for all $i = 0, \ldots, n$, the ideal I is the homogeneous maximal ideal $\mathfrak{m} = (X_0, \ldots, X_n)$, and the module M is finitely generated and graded. It follows that all the cohomology is graded too. Following our usual convention we will write $H^i_{\mathfrak{m}}(M)_d$ for the d-th graded component of $H^i_{\mathfrak{m}}(M)$, and similarly for the sheaf cohomology of $\mathcal{F}_M$.

In this setting the sheaf cohomology has another interpretation: Any graded S-module M gives rise to a quasicoherent sheaf $\widetilde{M}$ on the projective space $\mathbb{P}^n_k$ (for the definition and properties of this construction see for example [Har77, II.5].) The Čech complex for $\widetilde{M}$ is the degree 0 part of the complex (1.2). In particular, the i-th cohomology of the sheaf $\widetilde{M}$ is the degree 0 part of the cohomology of $\mathcal{F}_M$, that is, $H^i(\widetilde{M}) = H^i(\mathcal{F}_M)_0$. If we shift the grading of M by d to get $M(d)$, then $\widetilde{M}(d)$ is the sheaf on $\mathbb{P}^n_k$ associated to $M(d)$, so in general $H^i(\widetilde{M}(d)) = H^i(\mathcal{F}_M)_d$. Thus Proposition 1.2.14 takes on the following form:

Proposition 1.2.15. *Let M be a graded S-module and $\widetilde{M}$ the corresponding quasi-coherent sheaf on $\mathbb{P}^n_k$.*

(a) *There is an exact sequence of graded S-modules*

$$0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow \bigoplus_{d\in\mathbb{Z}} H^0(\widetilde{M}(d)) \longrightarrow H^1_{\mathfrak{m}}(M) \longrightarrow 0.$$

(b) *For every $i \geq 2$*

$$H^i_{\mathfrak{m}}(M) = \bigoplus_{d} H^{i-1}(\widetilde{M}(d)).$$

Corollary 1.2.16. *If $S = k[X_0, \ldots, X_n]$ with $n \geq 1$ and $\mathfrak{m} = (X_0, \ldots, X_n)$ then*

$$H^i_{\mathfrak{m}}(S)_d = \begin{cases} 0 & \text{if } i \leq n \\ \mathrm{Hom}_k(S_{-n-1-d}, k) & \text{if } i = n+1. \end{cases}$$

Proof. It is an immediate consequence of Proposition 1.2.15, given the cohomology of $\mathcal{O}_{\mathbb{P}^n_k}(d) = \widetilde{S}(d)$. □

1.2.2 Canonical module

In this section, we will define and collect some properties of canonical modules for a local (not necessarily Cohen-Macaulay) ring. Recall that for a Noetherian local ring $(R, \mathfrak{m}, k)$, the $\mathfrak{m}$-adic completion $\varprojlim_t R/\mathfrak{m}^t$ of R will be denoted by $\widehat{R}$, and $\widehat{M}$ stands for $\varprojlim_t M/\mathfrak{m}^t M$ for each $\widehat{R}$-module M. We write E is the injective envelope $E_R(k)$ of the residue field k.

Proposition 1.2.17. [Str90, Proposition 3.4.2] *Matlis duality is a faithful, exact contravariant functor and $\mathrm{ann}_R(M^\vee) = \mathrm{ann}_R(M)$ for every module M.*

Proposition 1.2.18. [Str90, Corollary 3.4.7] *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-modules. Then $M^{\vee\vee} \simeq \widehat{M}$.*

Proposition 1.2.19. [HK71, Lemma 5.8] *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M, N be two finitely generated R-modules. If $\widehat{M} \simeq \widehat{N}$, then $M \simeq N$.*

We now define a canonical module for a local (not necessarily Cohen-Macaulay) ring. Here, we shall adopt the definition used by Hochester and Huneke in [HH94]. Many results in this subsection are due to Aoyama [Aoy80, Aoy83].

Definition 1.2.20. [BS13, 12.1.2] The canonical module of a d-dimensional local ring $(R, \mathfrak{m})$, if it exists, is a finitely generated R-module ω_R such that ${\omega_R}^\vee \simeq H^d_{\mathfrak{m}}(R)$.

Remark 1.2.21. [BS13, 12.1.3] We can make the following observations right away. Let $(R, \mathfrak{m})$ be a local ring with $\dim\ R = d$.

(a) Recall that the R-module $H^d_{\mathfrak{m}}(R)$ is Artinian, and that the Matlis dual of a finitely generated R-module is also Artinian.

(b) It is immediate from the Local Duality Theorem 1.2.11 that, if R is a homomorphic image of a Gorenstein local ring A of dimension n, then R has a canonical module, namely $\mathrm{Ext}_A^{n-d}(R, A)$.

(c) In particular, if R itself is Gorenstein, then R has a canonical module which is isomorphic to R. The converse holds if R is Cohen-Macaulay, by Theorem 1.1.18.

(d) In particular, when R is complete, so that it is a homomorphic image of a regular local ring by Cohen's Structure Theorem, R has a canonical module, and any canonical module is isomorphic to $H_{\mathfrak{m}}^d(R)^\vee$.

Theorem 1.2.22. [BS13, 12.1.6] *Any two canonical modules for R are isomorphic.*

It will be convenient to have a notation for the ideal that turns out to be the annihilator of the canonical module.

Definition 1.2.23. For a local ring R, we denote by $\mathbf{j}(R)$ the largest ideal which is a submodule of R of dimension smaller than $\dim(R)$. Then $\mathbf{j}(R)$ is nonzero if and only if some prime $\mathfrak{p}$ of $\mathrm{Ass}\ R$ is such that $\dim(R/\mathfrak{p}) < \dim\ R$, and then $\mathbf{j}(R) \supseteq \mathrm{ann}_R(\mathfrak{p})$. Thus, $\mathbf{j}(R) = (0)$ if and only if R is equidimensional and unmixed (where unmixed means that (0) has no embedded primes). Moreover, $\mathbf{j}(R)$ consists of all elements $r \in R$ such that $\dim(R/\mathrm{ann}_R(r)) < \dim(R)$.

Lemma 1.2.24. [BS13, 12.1.12] *A canonical module for R must be killed by $\mathbf{j}(R)$, and is also a canonical module for $R/\mathbf{j}(R)$, while any canonical module for $R/\mathbf{j}(R)$ is a canonical module for R. Thus, R has a canonical module if and only if $R/\mathbf{j}(R)$ has a canonical module.*

Theorem 1.2.25. [BS13, 12.1.15] *Let $(R, \mathfrak{m})$ be a local ring of dimension d. Suppose that canonical module ω_R exists. Then*

$$(0\colon {}_R\omega_R) = (0\colon {}_R H_{\mathfrak{m}}^d(R)) = \mathbf{j}(R).$$

Definition 1.2.26. Let R be a Noetherian ring and n a nonnegative integer. A finitely generated R-module M satisfies *Serre's condition $\mathcal{S}_n$* if

$$\mathrm{depth}(M_{\mathfrak{p}}) \geq \min\{n, \dim M_{\mathfrak{p}}\}$$

for every prime ideal $\mathfrak{p}$ of R.

Proposition 1.2.27. [BS13, 12.1.18] *Let $(R, \mathfrak{m})$ be a local ring. Suppose that ω_R exists.*

(i) *We have*

$$\mathrm{Supp}\ \omega_R = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathrm{ht}(\mathfrak{p}) + \dim\ R/\mathfrak{p} = \dim\ R\};$$

also the R-module ω_R and $\mathrm{Hom}_R(\omega, \omega)$ satisfy the condition $\mathcal{S}_2$.

(ii) *If R is a homomorphic image of a Gorenstein local ring, then, for each $\mathfrak{p} \in \mathrm{Supp}\ \omega_R$, the localization $(\omega_R)_{\mathfrak{p}}$ is a canonical module for $R_{\mathfrak{p}}$.*

Suppose that canonical module ω_R of a local ring $(R, \mathfrak{m})$ exists. we shall use

$$h_R : R \longrightarrow \mathrm{Hom}_R(\omega_R, \omega_R)$$

to denote the natural homomorphism for which $h_R(r) = r\mathrm{Id}_{\omega_R}$ for all $r \in R$.

Note that h_R is both an R-module homomorphism and a ring homomorphism, and that $\mathrm{Im}(h_R)$ is contained in the center of the ring $\mathrm{Hom}_R(\omega_R, \omega_R)$. Theorem 1.2.25 implies immediately the following.

Corollary 1.2.28. *Let $(R, \mathfrak{m})$ be a local ring with canonical module ω_R. The kernel of the map h_R is $\mathbf{j}(R)$. Thus, ω_R is faithful if and only if R is equidimensional and unmixed.*

Proposition 1.2.29. [BS13, 12.2.7] *Suppose that ω_R exists. Then*

$$h_R : R \longrightarrow \mathrm{Hom}_R(\omega_R, \omega_R)$$

has $\mathrm{Supp}(\mathrm{Coker}(h_R)) \subset \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathrm{ht}(\mathfrak{p}) \geq 2\}$. *Thus each element of* $\mathrm{Coker}(h_R)$ *has annihilator of height at least two.*

Proposition 1.2.30. [BS13, 12.2.9] *Suppose R is a homomorphic image of a Gorenstein local ring, so that ω_R exists. Then h_R is an isomorphism if and only if R satisfies the condition $\mathcal{S}_2$.*

The following variant of the Local Duality Theorem will be useful.

Theorem 1.2.31 (Local Duality Theorem)**.** [BS13, 12.1.20] *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d which has the canonical module ω_R. Then, for each finitely generated R-module M, there exist natural isomorphisms*

$$H^{d-i}_{\mathfrak{m}}(M) \simeq \mathrm{Ext}^i_R(M, \omega_R)^\vee$$

for all i.

Corollary 1.2.32. [BS13, 12.1.21] *Suppose that the local ring R is Cohen-Macaulay of dimension d, and that ω_R exists. Then ω_R has finite injective dimension equal to d and is a maximal Cohen-Macaulay R-module.*

Remark 1.2.33. [BS13, 12.1.27] Suppose that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension d. A finitely generated R-module C is a canonical module of R if and only if $\dim_k \mathrm{Ext}^i_R(R/\mathfrak{m}, C) = \delta_{id}$ for all $i \geq 0$. Thus, if R is a Cohen-Macaulay local ring, then the canonical module of R in our definition is a canonical module of R in the sense of [BH98, Definition 3.3.1], that is, a finitely generated R-module ω_R is a canonical module of R if it is a maximal Cohen-Macaulay R-module of type 1 and of finite injective dimension.

The next proposition lists some useful and often applied change of ring formulas for the canonical module.

Proposition 1.2.34. [BH98, 3.3.5] *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, with canonical module ω_R. Let $\mathbf{x} = x_1, \ldots, x_m \in \mathfrak{m}$. Then*

(i) $\mathbf{x}$ *is a regular sequence if and only if* $\mathbf{x}$ *is a* ω_R*-regular sequence. Moreover, when is the case,* $R/(\mathbf{x})$ *has a canonical module and* $\omega_{R/\mathbf{x}R} \simeq \omega_R/\mathbf{x}\omega_R$.

(ii) $(\omega_R)_{\mathfrak{p}} \simeq \omega_{R_{\mathfrak{p}}}$, *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$.

(iii) $\widehat{\omega}_R \simeq \omega_{\widehat{R}}$.

Proposition 1.2.35. [BH98, 3.3.3] *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring. If* ω_R *is a canonical module of* R *and* $\mathbf{x}$ *is a regular sequence, then*

$$\operatorname{Hom}_R(M, \omega_R) \otimes_R R/\mathbf{x}R \simeq \operatorname{Hom}_{R/\mathbf{x}R}(M/\mathbf{x}M, \omega_R/\mathbf{x}\omega_R)$$

for any maximal Cohen-Macaulay R*-module* M.

The next theorem clarify for which Cohen-Macaulay local rings the canonical module exists.

Theorem 1.2.36. [BH98, 3.3.6] *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring. The following conditions are equivalent:*

(a) R *admits a canonical module;*

(b) R *is the homomorphic image of a Gorenstein local ring.*

In the next propositions, some useful characterizations of a canonical module will be given.

Proposition 1.2.37. [BH98, 3.3.10] *Let* $(R, \mathfrak{m}, k)$ *be a Cohen-Macaulay local ring of dimension* d *and* C *a finitely generated* R*-module. Then the following conditions are equivalent:*

(a) C *is the canonical module of* R;

(b) $\mu_i(\mathfrak{p}, C) = \delta_{ih}$ *for all* $i \geq 0$ *and all* $\mathfrak{p} \in \operatorname{Spec}(R)$, *where* $h = \operatorname{ht}(\mathfrak{p})$;

(c) $\dim_k \operatorname{Ext}^i_R(k, C) = \delta_{id}$. *In particular,* $r_R(C) = 1$;

(d) *for all maximal Cohen-Macaulay* R*-module* M *one has*

  (i) $\operatorname{Hom}_R(M, C)$ *is a maximal Cohen-Macaulay* R*-module,*

  (ii) $\operatorname{Ext}^i_R(M, C) = 0$ *for* $i > 0$,

  (iii) *the natural homomorphism* $M \longrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(M, C), C)$ *is an isomorphism.*

Note that if C is the canonical module of R, then, by replacing M by R in the item (d)(iii), we have a natural isomorphism $R \longrightarrow \operatorname{Hom}_R(C, C)$.

Proposition 1.2.38. [BH98, 3.3.11] *Let* R *be a Cohen-Macaulay local ring of dimension* d*, with canonical module* ω_R*, and* M *a Cohen-Macaulay* R*-module of dimension* t*. Then*

(a) $\mu(M) = r(\operatorname{Ext}^{d-t}_R(M, \omega_R))$,

(b) $r(M) = \mu(\operatorname{Ext}^{d-t}_R(M, \omega_R))$,

(c) ω_R *is a faithful* R*-module, and*

  (i) $r(\omega_R) = 1$, $\mu(\omega_R) = r(R)$,

  (ii) $\operatorname{End}_R(\omega_R) \simeq R$.

1.2.3 Graded canonical module and the graded local duality theorem

Definition 1.2.39. [BS13, 14.5.1] Let $(R, \mathfrak{m})$ be a graded *local ring of *dimension d. A finitely generated R-module C is a **canonical module* (or a graded canonical module) of R if there exists a homogeneous isomorphism

$$^*\mathrm{Hom}_R(C,\ ^*E_R(R/\mathfrak{m})) \simeq H^d_{\mathfrak{m}}(R).$$

When R is Cohen-Macaulay, our definition is the same as the definition in [BH98, 3.6.8]

Proposition 1.2.40. [BS13, 14.5.12] *Let $(R, \mathfrak{m})$ be a graded Cohen-Macaulay *local ring of *dimension d. A finitely generated R-module C is a *canonical module of R if and only if there exists a homogeneous isomorphism*

$$^*\mathrm{Ext}^i_R(R/\mathfrak{m}, C) \simeq \begin{cases} 0 & \text{for} \quad i \neq d \\ R/\mathfrak{m} & \text{for} \quad i = d. \end{cases}$$

Proposition 1.2.41. [BH98, 3.6.9] *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay *local ring, and C be a *canonical module of R. Then*

(a) *C is a canonical module of R,*

(b) *C is uniquely determined up to homogeneous isomorphism, provided $\mathfrak{m}$ is maximal.*

Example 1.2.42. Let $R = k[X_0, \ldots, X_n]$ be a polynomial ring over a field, and assign to the indeterminates the degree $\deg(X_i) = a_i > 0$ for all $i = 0, \ldots, n$. The *maximal ideal of R is $\mathfrak{m} = (X_0, \ldots, X_n)$ and the Koszul complex of $X_0, \ldots, X_n$ yields a homogeneous free resolution of $R/\mathfrak{m}$ whose last term is $R(-\sum_{i=0}^n a_i)$. From this one concludes that

$$^*\mathrm{Ext}^i_R(R/\mathfrak{m}, R) \simeq \begin{cases} 0 & \text{for} \quad i \neq n+1 \\ (R/\mathfrak{m})(\sum_{i=0}^n a_i) & \text{for} \quad i = n+1. \end{cases}$$

In other words, the *canonical module of R is $R(-\sum_{i=0}^n a_i)$.

Proposition 1.2.43. [BH98, 3.6.11] *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay *local ring with *canonical module ω_R. The following conditions are equivalent:*

(a) *R is a Gorenstein ring;*

(b) *$\omega_R \simeq R(a)$ for some integer $a \in \mathbb{Z}$.*

The number a occurring in Proposition 1.2.43 is a numerical invariant of the Gorenstein *local ring $(R, \mathfrak{m})$, provided $\mathfrak{m}$ is maximal. In the case of a positively graded algebra over a field it will be given a special name.

Definition 1.2.44. Let k be a field, and R a Cohen-Macaulay positively graded k-algebra. Then

$$a(R) = -\min\{i : (\omega_R)_i \neq 0\}$$

is called the *a-invariant of R*.

Proposition 1.2.45. [BH98, 3.6.14] *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay *local ring with *canonical module ω_R and $\mathbf{x} = x_1, \ldots, x_n$ a regular sequence of homogeneous elements with $\deg(x_i) = a_i$ for $i = 1, \ldots, n$. Then*

$$\omega_{R/\mathbf{x}R} \simeq \big(\omega_R/\mathbf{x}\omega_R\big)\big(\sum_{i=1}^{n} a_i\big).$$

In particular, if k is a field and R is a Cohen-Macaulay positively graded k-algebra, then

$$a(R/\mathbf{x}R) = a(R) + \sum_{i=1}^{n} a_i.$$

Example 1.2.46. A graded polynomial ring $R = k[X_0, \ldots, X_n]$ over a field k with $\deg(X_i) = a_i > 0$ has the a-invariant $a(R) = -\sum_{i=0}^{n} a_i$.

We now state the graded version of the local duality theorem. This will concern the situation where $(R, \mathfrak{m})$ is a graded *local ring that can be expressed as a homomorphic image of a graded Gorenstein *local ring by means of a homogeneous homomorphism. Many of the graded rings that occur in applications are finitely generated algebras over a field, and so we are only imposing a mild restriction.

Theorem 1.2.47. [BS13, 14.4.1] *Let $(R, \mathfrak{m})$ be a graded *local ring of *dimension d, i.e. $\mathrm{ht}(\mathfrak{m}) = d$. Assume that there is a graded Gorenstein *local ring $(A, \mathfrak{m}')$ and a surjective homogeneous ring homomorphism $\varphi : A \longrightarrow R$. Let $\mathrm{ht}(\mathfrak{m}') = n$. Then*

(a) *there exists $a \in \mathbb{Z}$ such that $H^n_{\mathfrak{m}'}(A) \simeq {}^*E_A(A/\mathfrak{m}')(-a)$;*

(b) *for any such a, there are homogeneous isomorphisms*

$$H^i_{\mathfrak{m}}(M) \simeq {}^*\mathrm{Hom}_R({}^*\mathrm{Ext}^{n-i}_A(M, A(a)), {}^*E_R(R/\mathfrak{m}))$$

for all finitely generated graded R-module M and for all $i \geq 0$.

In the situation of the above theorem, the functor $^*D :=^* \mathrm{Hom}_R(-, {}^*E_R(R/\mathfrak{m}))$ would seem to be an obvious graded analogue of the Matlis functor. However, there is another approach to $^*D$ which is particularly useful when the 0-th component R_0 of the *local ring R is a field.

Let $(R, \mathfrak{m})$ be a Noetherian *local ring; then R_0 is local with maximal ideal $\mathfrak{m}_0$. We consider R_0 as a graded ring by defining $(R_0)_i = 0$ for $i \neq 0$. Similarly any R_0-module may be consider a graded R_0-module concentrated in degree 0. Moreover, if M is a graded R-module, it may be viewed as a graded R_0-module as well. Put $E_0 := E_{R_0}(R_0/\mathfrak{m}_0)$, we can define

$$M^* = {}^*\mathrm{Hom}_{R_0}(M, E_0).$$

A priori, M^* is a graded R_0-module whose grading is given by

$$(M^*)_i = \mathrm{Hom}_{R_0}(M_{-i}, E_0))$$

for all $i \in \mathbb{Z}$. But it is obvious that M^* has a natural structure as a graded R-module, and we have the identity

$$M^* = {}^*\mathrm{Hom}_R(M, E_0),$$

as a graded R-module.

Proposition 1.2.48. [BS13, 14.4.2] *There are the naturally homogeneous isomorphisms $M^* \simeq {}^*D(M)$, for all finitely generated graded R-module M.*

Thus, it is reasonable for us to regard the functor $-^* = {}^*\mathrm{Hom}_{R_0}(-, {}^*E_0)$ as the "graded Matlis duality functor".

Theorem 1.2.49 (Graded Local Duality Theorem)**.** [BS13, 14.5.10] *Let $(R, \mathfrak{m})$ be a graded Cohen-Macauley *local ring of *dimension d. Assume that R admits a *canonical module ω_R. For each finitely generated graded R-module M, there are homogeneous isomorphisms*

$$H^{d-i}_{\mathfrak{m}}(M) \simeq {}^*\mathrm{Ext}^i_R(M, \omega_R)^* := {}^*\mathrm{Hom}_{R_0}({}^*\mathrm{Ext}^i_R(M, \omega_R), {}^*E_0)$$

for all $i \in \mathbb{Z}$.

In the particular case in which R_0 is a field k, $E_{R_0}(R_0/\mathfrak{m}_0) = k$, and, for a graded R-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$, the grading of the "graded Matlis duality" is given by the attractively simple formula

$$M^* = {}^*\mathrm{Hom}_k(M, k) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_k(M_{-i}, k)$$

as a graded k-vector space or

$$M^* = {}^*\mathrm{Hom}_R(M, k)$$

as a graded R-module.

Let $R = k[X_0, \ldots, X_n]$ be a standard graded polynomial ring over a field k. Then R is a Gorenstein *local ring with *canonical module $R(-n-1)$. Therefore, we have the following.

Corollary 1.2.50. [BS13, 14.5.18] *Let $R = k[X_0, \ldots, X_n]$ be the standard graded polynomial ring over a field k and $\mathfrak{m} = (X_0, \ldots, X_n)$ the homogeneous maximal ideal. For all finitely generated graded R-modules M and for all $i \in \mathbb{Z}$, then there are naturally homogeneous homomorphisms*

$$H^i_{\mathfrak{m}}(M) \simeq {}^*\mathrm{Hom}_R\Big(\mathrm{Ext}^{n+1-i}_R(M, R(-n-1)), k\Big)$$

and

$${}^*\mathrm{Hom}_R(H^i_{\mathfrak{m}}(M), k) \simeq \mathrm{Ext}^{n+1-i}_R(M, R(-n-1)).$$

Proposition 1.2.51. [MN01, Lemma 10] *Let $R = k[X_0, \ldots, X_n]$ be the standard graded polynomial ring over a field k. Suppose that I is a homogeneous ideals of R of height g. Let $\mathbf{x} := x_1, \ldots, x_g$ be a regular sequence contained in I and $\deg(x_i) = d_i$ for all $i = 1, \ldots, g$. Put $J := (\mathbf{x}) \colon {}_R I$. Then there are exact sequences (of graded R-modules)*

$$0 \longrightarrow (\mathbf{x}) \longrightarrow J \longrightarrow \omega_{R/I}(n+1-\sum_{j=1}^{g} d_j) \longrightarrow 0$$

and

$$0 \longrightarrow \omega_{R/I}(n+1-\sum_{j=1}^{g} d_j) \longrightarrow R/(\mathbf{x}) \longrightarrow R/J \longrightarrow 0.$$

Proof. We have to show that $J/(\mathbf{x}) \simeq \omega_{R/I}(n+1-\sum_{j=1}^{g} d_j)$.
There are the following isomorphisms

$$J/(\mathbf{x}) \simeq ((\mathbf{x})\colon {}_R I)/(\mathbf{x}) \simeq \mathrm{Hom}_R(R/I, R/(\mathbf{x}))$$

and

$$\omega_{R/I}(n+1-\sum_{j=1}^{g} d_j) \simeq \mathrm{Ext}_R^g(R/I, R)(-\sum_{j=1}^{g} d_j).$$

Thus, our claim follows from the isomorphism

$$\mathrm{Hom}_R(R/I, R/(\mathbf{x})) \simeq \mathrm{Ext}_R^g(R/I, R)(-\sum_{j=1}^{g} d_j).$$

□

1.3 Hilbert functions and multiplicities

We will recall some definitions and notations of Hilbert function of graded modules over a graded ring and Hilbert-Samuel function of finitely generated modules over a local ring.

1.3.1 Hilbert functions for graded algebras

Our standard assumption in this section will be that R_0 is an Artinian local ring, and that R is finitely generated over R_0. Notice that for each finitely generated graded R-module, the homogeneous components M_n of M are finitely generated R_0-module, and hence have finite length.

Definition 1.3.1. Let M be a graded R-module whose graded components M_n have finite lenght for all n. The numerical function $HF_M(-) : \mathbb{Z} \longrightarrow \mathbb{Z}$ with $HF_M(n) = \mathrm{length}(M_n)$, for all $n \in \mathbb{Z}$, is the *Hilbert function* of M, and $HS_M(t) := \sum_{n\in\mathbb{Z}} HF_M(n)t^n$ is the *Hilbert series* of M.

In this section we consider positively graded algebras over an Artinian local ring R_0, that is, graded algebras of the form $R = \bigoplus_{i\geq 0} R_i$ where R_0 is an Artinian local ring and R is finitely generated over R_0. The generators of a positively graded algebra may be of arbitrarily high degree.

We first give a structure result about Hilbert series.

Proposition 1.3.2. [BH98, 4.4.1] *Let R be a positively graded algebra over an Artinian local ring R_0, and $M \neq 0$ be a finitely generated graded R-module of dimension d. Then there exist positive integers $a_1, \ldots, a_d$, and $Q(t) \in \mathbb{Z}[t, t^{-1}]$ such that*

$$HS_M(t) = \frac{Q(t)}{\prod_{i=1}^{d}(1-t^{a_i})} \quad \textit{with} \quad Q(1) > 0.$$

Let a be the least common multiple of degrees of the generators of the algebra R over R_0, we may write the Hilbert series of M in the above proposition as

$$HS_M(t) = \frac{Q(t)}{(1-t^a)^d} \quad \text{with} \quad Q(1) > 0.$$

The next results show that the Hilbert series of the canonical module of a Cohen-Macaulay positively graded algebra may be deduced from its Hilbert series. The automorphism $\varphi : \mathbb{Z}[t, t^{-1}] \longrightarrow \mathbb{Z}[t, t^{-1}]$, $\varphi(t) = t^{-1}$, can be extended to all rational functions $F(t)$, and we set $F(t^{-1}) = \varphi(F(t))$.

Theorem 1.3.3 (Stanley). [BH98, 4.4.6] *Let R be a d-dimensional Cohen-Macaulay positively graded algebra over an Artinian local ring R_0, with canonical module ω_R. Suppose that R has the Hilbert series*

$$HS_R(t) = \frac{\sum_{i=0}^{s} h_i t^i}{\prod_{j=1}^{d}(1-t^{a_j})}.$$

(a) *Then $HS_{\omega_R}(t) = (-1)^d HS_R(t^{-1})$, equivalently,*

$$HS_{\omega_R}(t) = \frac{t^{\sum a_j - s} \sum_{i=0}^{s} h_{s-i} t^i}{\prod_{j=1}^{d}(1-t^{a_j})}.$$

(b) *If R is Gorenstein, then $HS_R(t) = (-1)^d t^{a(R)} HS_R(t^{-1})$.*

(c) *Suppose R is a domain, and $HS_R(t) = (-1)^d t^q HS_R(t^{-1})$ for some integer q. Then R is Gorenstein.*

1.3.2 Hilbert functions for standard graded algebras

In this subsection we will assume that R is generated over R_0 by elements of degree 1, that is, $R = R_0[R_1]$. Recall that such a ring is said to be standard graded.

Theorem 1.3.4 (Hilbert). [BH98, 4.1.3] *Let M be a finitely generated graded R-module of dimension d. Then there exists a polynomial $HP_M(X) \in \mathbb{Q}[X]$ of degree $d-1$ such that $HF_M(n) = HP_M(n)$ for all $n \gg 0$.*

Definition 1.3.5. Let M be a finitely generated graded R-module of dimension d. The unique polynomial $HP_M(X) \in \mathbb{Q}[X]$ of degree $d-1$ for which $HF_M(n) = HP_M(n)$ for all $n \gg 0$ is called the *Hilbert polynomial* of M. We write

$$HP_M(X) = \sum_{i=0}^{d-1} (-1)^{d-1-i} e_{d-1-i} \binom{X+i}{i}.$$

Then the *multiplicity* of M is defined to be

$$e(M) = \begin{cases} e_0 & \text{if} \quad d > 0 \\ \text{length}(M) & \text{if} \quad d = 0. \end{cases}$$

Corollary 1.3.6. *Let $M \neq 0$ be a finitely generated graded R-module of dimension d. Then there exists a unique $Q_M(t) \in \mathbb{Z}[t, t^{-1}]$ with $Q_M(1) \neq 0$ such that*

$$HS_M(t) = \frac{Q_M(t)}{(1-t)^d}.$$

Moreover, if $Q_M(t) = \sum_i h_i t^i$, then $\min\{i : \ h_i \neq 0\}$ is the least number such that $M_i \neq 0$.

Proof. The first part of the assertion is a consequence of Proposition 1.3.2. In order to prove the second part, multiply both sides of $HS_M(t) = \frac{Q_M(t)}{(1-t)^d}$ by $(1-t)^d$ and compare coefficients. □

In the next proposition we show how one can recover the coefficients e_i of the Hilbert polynomial of a module M from Q_M. We will denote by $P^{(i)}$ the i-th formal derivative of an element $P \in \mathbb{Z}[t, t^{-1}]$.

Proposition 1.3.7. [BH98, 4.1.9] *Under the assumptions of Corollary 1.3.6 the following formulas hold:*

$$e_i = \frac{Q_M^{(i)}(1)}{i!}$$

for $i = 0, \ldots, d-1$. Moreover, $e(M) = Q_M(1)$.

Remark 1.3.8. The multiplicity $e(M)$ of M is also often called the *degree of M* because of its geometric meaning. Indeed, let k be a field, and $R = k[X_0, \ldots, X_n]$ with each X_i is of degree one. Suppose that I is a graded ideal of R and consider the quotient ring $A = R/I$. If d denotes the dimension of A then the subscheme $\operatorname{Proj}(A)$ of $\mathbb{P}^n_k$ is of dimension $d-1$. The *degree of* $\operatorname{Proj}(A)$ *over* $\mathbb{P}^n_k$ is defined to be the number of points obtained by cutting out $\operatorname{Proj}(A)$ by $d-1$ general linear forms. To be more precise, if $\ell_1, \ldots, \ell_{d-1}$ are general linear forms of R, then the scheme $\mathcal{B} = \operatorname{Proj}(A/(\ell_1, \ldots, \ell_{d-1}))$ is finite and we set

$$\deg_{\mathbb{P}^n_k}(\operatorname{Proj}(A)) := \dim_k \Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}) = \dim_k A_\nu/(\ell_1, \ldots, \ell_{d-1})_\nu$$

for all sufficiently large ν. It turns out that this geometric degree equals the multiplicity of A, i.e. we have

$$\deg_{\mathbb{P}^n_k}(\operatorname{Proj}(A)) = e(A).$$

Theorem 1.3.9 (Grothendieck-Serre)**.** [BH98, 4.4.3] *Let R be a positively standard graded algebra over a field $R_0 = k$, and $M \neq 0$ a finitely generated graded R-module of dimension d. Denote the homogenous maximal ideal of R by $\mathfrak{m} = \oplus_{i>0} R_i$. For every $d \in \mathbb{Z}$*

$$HP_M(d) = HF_M(d) - \sum_{i \geq 0} (-1)^i \dim_k H^i_{\mathfrak{m}}(M)_d.$$

Theorem 1.3.10. [BS13, 17.1.11] *Let R be a positively standard graded algebra such that R_0 is Artinian. Let $M \neq 0$ be a finitely generated graded R-module and i a nonnegative integer. Then there exists a polynomial $HP^i_M \in \mathbb{Q}[X]$ of degree less than i such that*

$$HF_{H^i_{R_+}(M)}(d) = HP^i_M(d) \quad \textit{for all} \quad d \ll 0.$$

Proposition 1.3.11. [MN01, Corollary 9] *Let k be a field, and $R = k[X_0, \dots, X_n]$ with each X_i is of degree one. Let I be a graded ideal of codimension $g \leq n$ and $\mathbf{x} := x_1, \dots, x_g$ a regular sequence contained in I. Put $J = (\mathbf{x}) \colon {}_R I$. Setting $X = \operatorname{Proj}(R/I)$, $Y = \operatorname{Proj}(R/J)$ and $Z = \operatorname{Proj}(R/(\mathbf{x}))$. Then we have*

$$\deg(X) + \deg(Y) = \deg(Z).$$

Proof. Suppose that $\deg(x_i) = d_i$ for all $i = 1, \dots, g$. By Theorem 1.3.10, the degree of the Hilbert polynomial of $H^i_{\mathfrak{m}}(R/I)$ is at most $i-1$. Therefore, by Theorem 1.3.9, we obtain for all $d \ll 0$

$$\begin{aligned} -HP_{R/I}(d) &= (-1)^{n+1-g} \dim_k H^{n+1-g}_{\mathfrak{m}}(R/I)_d + O(d^{n-1-g}) \\ &= (-1)^{n+1-g} \dim_k (\omega_{R/I})_{-d} + O(d^{n-1-g}). \end{aligned}$$

Combined with the standard exact sequence in Proposition 1.2.51, this provides

$$HP_{R/J}(d) = HP_{R/(\mathbf{x})}(d) + (-1)^{n-g} HP_{R/I}(\sum_{j=1}^{g} d_j - n - 1 - d) + O(d^{n-1-g}).$$

Comparing coefficients we get if $g \leq n$

$$\deg(X) = \deg(Z) - \deg(Y).$$

□

1.3.3 Macaulay's theorem for Hilbert functions

In this subsection we recall a beautiful theorem of Macaulay describing exactly those numerical functions which occur as the Hilbert function $HF_R(n)$ of a standard graded k-algebra R, k is a field.

Let $R = \bigoplus_{n \geq 0} R_n$ be a standard graded k-algebra, where $R_0 = k$ is a field. We will show that R has an k-basic consisting of monomials in basic $x_1, \dots, x_m$ of R_1. We are going to define this basic of monomials on the level of the polynomial ring. So let

$$\pi : k[X_1, \dots, X_m] \longrightarrow R$$

be the surjective k-algebra homomorphism with $\pi(X_i) = x_i$.

Definition 1.3.12. A nonempty set $\mathfrak{M}$ of monomials in the indeterminates $X_1, \dots, X_m$ is called an *order ideal of monomials* if the following holds: whenever $m \in \mathfrak{M}$ and a monomial m' divides m, then $m' \in \mathfrak{M}$. Equivalently, if $X_1^{a_1} \cdots X_m^{a_m} \in \mathfrak{M}$ and $0 \leq b_i \leq a_i$ for $i = 1, \dots, m$, then $X_1^{b_1} \cdots X_m^{b_m} \in \mathfrak{M}$.

Remark 1.3.13. An order ideal of monomials $\mathfrak{M}$ is not an k-basis of an ideal, let alone an ideal. Quite the contrary, if we let $C\mathfrak{M}$ be the complement of $\mathfrak{M}$ in the set of all monomials, then $C\mathfrak{M}$ is an k-basis of the ideal generated by the monomials $m \in C\mathfrak{M}$.

Theorem 1.3.14 (Macaulay). [BH98, 4.2.3] *Let R be a standard graded k-algebra, k is a field. Further let $x_1, \dots, x_m$ be an k-basis of R_1, and $\pi : k[X_1, \dots, X_m] \longrightarrow R$ be the surjective k-algebra homomorphism with $\pi(X_i) = x_i$ for $i = 1, \dots, m$. Then there exists an order ideal $\mathfrak{M}$ of monomials such that $\pi(\mathfrak{M})$ is an k-basis of R.*

Theorem 1.3.14 and Remark 1.3.13 immediately imply:

Corollary 1.3.15. *Let J be the ideal which is generated by the monomials in $C\mathfrak{M}$. Then the standard graded k-algebra R and $k[X_1, \dots, X_m]/J$ have the same Hilbert function. In particular, all Hilbert functions of standard graded rings arise as Hilbert function of standard graded rings whose defining ideal is generated by monomials.*

The set of monomials $C\mathfrak{M}$ associated with R can be described differently. First, we recall the following definition.

Definition 1.3.16. A monomial order $<$ on $k[X_1, \dots, X_m]$ is a total ordering on the set of monomials of $k[X_1, \dots, X_m]$ having two additional properties:

(i) $1 < m$ for every monomial $m \neq 1$.

(ii) whenever $m_1 < m_2$ and c is a monomial, then $cm_1 < cm_2$.

Now let $<$ be a monomial order on $k[X_1, \dots, X_n]$. Let $I = \mathrm{Ker}(\pi)$, and set $\mathrm{LT}(I) = \{\mathrm{LT}(f) : f \in I\}$, where $\mathrm{LT}(f)$ denotes the leading monomial of f, that is, the monomial occurring in f which is maximal with respect to $<$. Then $\mathrm{LT}(I) = C\mathfrak{M}$. Indeed, let $v \in \mathrm{LT}(I)$, and choose $f \in I$, $f = \sum_{i=1}^n \lambda_i v_i$ with monomials v_i such that $v = \mathrm{LT}(f) = v_n$. Assume $v_n \notin C\mathfrak{M}$, then $v_n \in \mathfrak{M}$, and so

$$0 \neq \pi(v_n) = -\sum_{i=1}^{n-1} \lambda_n^{-1} \lambda_i \pi(v_i).$$

Each $\pi(v_i)$ is a linear combination $\sum a_{ij}\pi(u_j), a_{ij} \in k, u_j \in \mathfrak{M}, u_j \leq v_i < v_n$. Replacing the $\pi(v_i)$ in the above equation by their linear combinations gives a representation as a nontrivial combination of elements in $\pi(\mathfrak{M})$. This contradicts Theorem 1.3.14.

Conversely, suppose $v \in C\mathfrak{M}$. Then $\pi(v) = \sum \lambda_i \pi(u_i)$ with $u_i \in \mathfrak{M}, u_i < v$. Hence, if we set $f = v - \sum \lambda_i \pi(u_i)$, then $\pi(f) = 0$ and $\mathrm{LT}(f) = v$.

Set $\mathrm{in}(I)$ is the ideal of $k[X_1, \dots, X_m]$ generated by $\mathrm{LT}(I)$. Then $\mathrm{in}(I)$ is finitely generated. Therefore there exist polynomials $f_1, \dots, f_n \in I$ such that $\mathrm{in}(I) = (\mathrm{LT}(f_1), \dots, \mathrm{LT}(f_n))$. Any such subset of I is called a *Gröbner or standard basis* of I. Note that any Gröbner basis of I generates I.

Let $n, d \geq 0$ be two integers. We write

$$\binom{n}{d} = \begin{cases} \frac{(d+1)(d+2)\cdots n}{1.2\cdots(n-d)} & \text{if } 0 \leq d \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.3.17. [BH98, 4.2.6] *Let d be a positive integer. Any $a \in \mathbb{N}$ can be written uniquely in the form*

$$a = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_1}{1}, \tag{1.3}$$

where $m_d > m_{d-1} > \cdots > m_1 \geq 0$.

Skipping the summands which are zero in (1.3) we get the following definition.

Definition 1.3.18. Let a and d two positive integers.

(a) The *d-binomial expansion* of a is the unique expression

$$a = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \cdots + \binom{m_j}{j}, \tag{1.4}$$

where $m_d > m_{d-1} > \cdots > m_j \geq j \geq 1$.

(b) If a has d-binomial expansion as in (1.4), then we set

$$a^{<d>} = \binom{m_d+1}{d+1} + \binom{m_{d-1}+1}{d} + \cdots + \binom{m_j+1}{j+1}.$$

For example, the 5-binomial expansion of 76 is

$$76 = \binom{8}{5} + \binom{6}{4} + \binom{4}{3} + \binom{2}{2},$$

so

$$76^{<5>} = \binom{9}{6} + \binom{7}{5} + \binom{5}{4} + \binom{3}{3} = 111.$$

Definition 1.3.19. A sequence of non-negative integers $\{c_i : i \geq 0\}$ is called an *O-sequence* if

$$c_0 = 1 \text{ and } c_{i+1} \leq c_i^{<i>},$$

for all $i \geq 1$.

The importance of the binomial expansions described above becomes apparent from the following beautiful theorem of Macaulay:

Theorem 1.3.20 (Macaulay)**.** [BH98, 4.2.10] *Let k be a field and $h : \mathbb{N} \longrightarrow \mathbb{N}$ a numerical function. The following conditions are equivalent:*

(i) *there exists a standard graded k-algebra R with Hilbert function $HF_R(n) = h(n)$ for all $n \geq 0$;*

(ii) *there exists a standard graded k-algebra R with monomial relations and with Hilbert function $HF_R(n) = h(n)$ for all $n \geq 0$;*

(iii) *$\{h(n) : n \geq 0\}$ is an O-sequence;*

(iv) *let $m = h(1)$, and for each $n \geq 0$ let $\mathfrak{M}_{\mathfrak{n}}$ be the first $h(n)$ monomials in the variables $X_1, \ldots, X_m$ of degree n in the reverse degree-lexicographical order; set $\mathfrak{M} = \bigcup_{n \geq 0} \mathfrak{M}_{\mathfrak{n}}$; then $\mathfrak{M}$ is an order ideal of monomials.*

1.3.4 Hilbert-Samuel functions and multiplicities

Let R be a ring and I an ideal of R. Set

$$\mathrm{gr}_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}.$$

An element I^n/I^{n+1} can be expressed as a linear combination of products of n elements of I/I^2, so that $\mathrm{gr}_I(R)$ is generated over the subring $\mathrm{gr}_I(R)_0 = R/I$ by elements of I/I^2. If $I = (x_1, \ldots, x_r)$ and ξ_i denotes the image of x_i in I/I^2 then

$$\mathrm{gr}_I(R) = (R/I)[\xi_1, \ldots, \xi_r],$$

and $\mathrm{gr}_I(R)$ is a quotient of the polynomial $(R/I)[X_1, \ldots, X_r]$ as graded algebras. Therefore $\mathrm{gr}_I(R)$ is a standard graded algebra.

Definition 1.3.21. Let R be a ring and I an ideal of R. The *associated graded ring of R with respect to I* is defined by

$$\mathrm{gr}_I(R) = \bigoplus_{i\geq 0} I^i/I^{i+1}.$$

Let M be an R-module. If we set

$$\mathrm{gr}_I(M) = \bigoplus_{i\geq 0} I^iM/I^{i+1}M$$

then $\mathrm{gr}_I(M)$ is in a natural way a graded module over $\mathrm{gr}_I(R)$.

Let $(R, \mathfrak{m})$ be a Noetherian local ring. If I is a ideal of R such that for some $\nu > 0$ we have $\mathfrak{m}^\nu \subset I \subset \mathfrak{m}$, we call I an *ideal of definition*. As R/I is Artinian local ring, the homogeneous components $I^nM/I^{n+1}M$ of $\mathrm{gr}_I(M)$ are finitely generated R/I-module, and hence have finite length.

Definition 1.3.22. The first iterated Hilbert function

$$\begin{aligned}\chi_M^I(n) &= \sum_{i=0}^{n} HF_{\mathrm{gr}_I(M)}(n)\\ &= \sum_{i=0}^{n} \mathrm{length}(I^iM/I^{i+1}M) = \mathrm{length}(M/I^{n+1}M)\end{aligned}$$

is called the *Hilbert-Samuel function of M*, and $e(I, M) := e(\mathrm{gr}_I(M))$ the *multiplicity of I in M*.

As an immediate consequence of Theorem 1.3.4, there exists a polynomial $S_M^I(X) \in \mathbb{Q}[X]$ of degree $d = \dim(M)$ such that $\chi_M^I(n) = S_M^I(n)$ for $n \gg 0$. This polynomial is called the *Hilbert-Samuel polynomial of M with respect to I* and it can be written of form:

$$S_M^I(X) = \frac{e(I, M)}{d!}X^d + \text{terms of lower powers in } X.$$

The multiplicity $e(I, M)$ of I in M is also called the algebraic multiplicity one can define the algebraic multiplicity of a zero-dimensional subscheme as follows: Let J be

a graded ideal of a $\mathbb{N}$-graded ring R such that $\mathcal{B} = \mathrm{Proj}(R/J)$ is a finite subscheme of $\mathrm{Proj}(R)$. For each $x \in \mathcal{B}$, we denote $\mathfrak{p}$ by the defining ideal of x. Then its *algebraic multiplicity* is defined by

$$e(\mathcal{B}, \mathrm{Proj}(R)) := \sum_{x \in \mathcal{B}} e(J_{\mathfrak{p}}, R_{\mathfrak{p}}).$$

Definition 1.3.23. Let R be a Noetherian local ring, I a proper ideal, and M a finitely generated R-module. An ideal $J \subset I$ is called a *reduction ideal of I with respect to M* if $JI^nM = I^{n+1}M$ for some (or equivalently all) $n \gg 0$.

Proposition 1.3.24. [BH98, 4.6.10] *Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue class field, M a finitely generated R-module, and I an ideal of definition of M. Then there exists a system of parameters $\mathbf{x}$ of M such that $(\mathbf{x})$ is a reduction ideal of I with respect to M. Moreover, $e(I, M) = e((\mathbf{x}), M)$.*

1.4 Spectral sequences

In this section, we are going to collect some necessary notations and terminologies about spectral sequences.

1.4.1 Definition of spectral sequences

Definition 1.4.1. A *spectral sequence* (starting with ${}^0\mathbf{E}$) in an abelian category $\mathfrak{C}$ consists of the following data:

(i) A family $\{{}^r\mathbf{E}^{p,q}\}$ defined for all $p, q \in \mathbb{Z}$ and $r \geq 0$.

(ii) Maps $d_r^{p,q} : {}^r\mathbf{E}^{p,q} \longrightarrow {}^r\mathbf{E}^{p+r,q-r+1}$ are differentials in the sense that $d_r d_r = 0$ for all $r \geq 0$.

(iii) There is isomorphisms between ${}^{r+1}\mathbf{E}^{p,q}$ and homology of ${}^r\mathbf{E}^{*,*}$ at the spot ${}^r\mathbf{E}^{p,q}$:

$${}^{r+1}\mathbf{E}^{p,q} \simeq \frac{\mathrm{Ker}(d_r^{p,q})}{\mathrm{Im}(d_r^{p-r,q+r-1})}.$$

Note that ${}^{r+1}\mathbf{E}^{p,q}$ is a subquotient of ${}^r\mathbf{E}^{p,q}$. The *total degree* of the term ${}^r\mathbf{E}^{p,q}$ is $n = p + q$; the terms of total degree n lie on a line of slop 1, and each differential $d_r^{p,q}$ increases the total degree by one.

Definition 1.4.2.

(a) A spectral sequence is called *bounded* if for each n there are only finitely many nonzero terms of total degree degree n in ${}^0\mathbf{E}^{*,*}$; more precisely there is an r_0 such that ${}^r\mathbf{E}^{p,q} = {}^{r+1}\mathbf{E}^{p,q}$ for all $p, q \in \mathbb{Z}$ and $r \geq r_0$. We write ${}^\infty\mathbf{E}^{p,q}$ for this stable value of ${}^r\mathbf{E}^{p,q}$.

(b) A bounded spectral sequence is called *converges* to a given family $\{H^n\}$ of objects of an abelian category $\mathfrak{C}$, if we have a finite filtration for each H^n :

$$0 = F^sH^n \subset \cdots \subset F^{p+1}H^n \subset F^pH^n \subset F^{p-1}H^n \subset \cdots \subset F^tH^n = H^n,$$

and we are given isomorphisms

$$^{\infty}\mathbf{E}^{p,q} = \frac{F^p H^{p+q}}{F^{p+1} H^{p+q}}$$

and write $^{r}\mathbf{E}^{p,q} \Rightarrow H^{p+q}$ to represent this fact.

Definition 1.4.3. A spectral sequence collapses at $^{r}\mathbf{E}$ ($r \geq 2$) if there is exactly one nonzero row or column in the lattice $\{^{r}\mathbf{E}^{p,q}\}$. If a collapsing spectral sequence converges to H^*, we can read the H^n off: H^n is the unique nonzero $^{r}\mathbf{E}^{p,q}$ with $p+q = n$. The overwhelming majority of all applications of spectral sequence involve spectral sequence that collapses at $^{1}\mathbf{E}$ or $^{2}\mathbf{E}$.

For a spectral sequence, each $^{r+1}\mathbf{E}^{p,q}$ is a subquotient of the previous term $^{r}\mathbf{E}^{p,q}$. By induction on r, we see that there is a nested family of subobjects of $^{0}\mathbf{E}^{p,q}$:

$$0 = B_0^{p,q} \subset \cdots \subset B_r^{p,q} \subset B_{r+1}^{p,q} \subset \cdots \subset Z_{r+1}^{p,q} \subset Z_r^{p,q} \subset \cdots \subset Z_0^{p,q} = {}^{0}\mathbf{E}^{p,q}$$

such that $^{r}\mathbf{E}^{p,q} \simeq Z_r^{p,q}/B_r^{p,q}$. We introduce the intermediate objects

$$B_\infty^{p,q} = \bigcup_{r=0}^{\infty} B_r^{p,q} \quad \text{and} \quad Z_\infty^{p,q} = \bigcap_{r=0}^{\infty} Z_r^{p,q}$$

and define $^{\infty}\mathbf{E}^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$. In a bounded spectral sequence, both the union and intersection are finite, so $B_\infty^{p,q} = B_r^{p,q}$ and $Z_\infty^{p,q} = Z_r^{p,q}$ for large r. Thus we recover our earlier definition $^{\infty}\mathbf{E}^{p,q} = {}^{r}\mathbf{E}^{p,q}$ for large r.

1.4.2 Spectral sequences of a double complex

Where do interesting spectral sequences come from? Most of the applications in algebra have to do with a spectral sequence that arises from a double complex C. More precisely, there are two filtrations associated to every double complex, resulting in two spectral sequences related to homology of $\mathrm{Tot}(C)$.

Let $C = C^{*,*}$ be a double complex in an abelian category $\mathfrak{C}$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C^{p-1,q-1} \arrow[r] \arrow[d] & C^{p,q-1} \arrow[r] \arrow[d] & C^{p+1,q-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C^{p-1,q} \arrow[r] \arrow[d] & C^{p,q} \arrow[r] \arrow[d] & C^{p+1,q} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C^{p-1,q+1} \arrow[r] \arrow[d] & C^{p,q-1} \arrow[r] \arrow[d] & C^{p+1,q+1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

The *total complex* $G = \mathrm{Tot}(C)$ is defined by $G^n = \oplus_{i+j=n} C^{i,j}$. There are two natural filtrations on the total complex - vertical filtration and horizontal filtration. The

horizontal filtration is defined by subcomplex ${}^pF^\bullet_{\text{hor}}(G)$, where ${}^pF^\bullet_{\text{hor}}(G)$ comes from the rows of C where the second index $\geq p$. Similarly, the vertical filtration ${}^pF^\bullet_{\text{ver}}(G)$ is the subcomplex coming from the columns of C where the first index $\geq p$. More fomally, we let

$$
\begin{aligned}
{}^pF^n_{\text{hor}}(G) &= \bigoplus_{m\geq p} C^{n-m,m} \\
{}^pF^n_{\text{ver}}(G) &= \bigoplus_{m\geq p} C^{m,n-m}
\end{aligned}
$$

with differential defined as the restrcition of the differential of G.

Theorem 1.4.4. [Eis95, Theorem A.3.24] *Let $C = C^{p,q}$ be a double complex in an abelian category A. There are two spectral sequences ${}^r\boldsymbol{E}_{\text{hor}}$ and ${}^r\boldsymbol{E}_{\text{ver}}$ determined by the filtrations ${}^pF^\bullet_{\text{hor}}(\text{Tot}(C))$ and ${}^pF^\bullet_{\text{ver}}(\text{Tot}(C))$ with*

$$
\begin{array}{lll}
{}^0\boldsymbol{E}^{p,q}_{\text{hor}} = C^{p,q} & {}^1\boldsymbol{E}^{p,q}_{\text{hor}} = H^q_{\text{hor}}(C^{*,p}) & {}^2\boldsymbol{E}^{p,q}_{\text{hor}} = H^p_{\text{ver}}H^q_{\text{hor}}(C^{*,*}) \\
{}^0\boldsymbol{E}^{p,q}_{\text{ver}} = C^{p,q} & {}^1\boldsymbol{E}^{p,q}_{\text{ver}} = H^q_{\text{ver}}(C^{p,*}) & {}^2\boldsymbol{E}^{p,q}_{\text{ver}} = H^p_{\text{hor}}H^q_{\text{ver}}(C^{*,*}).
\end{array}
$$

If $C^{p,q} = 0$ for all $p < 0$ and $q < 0$ or $p > 0$ and $q > 0$, then both spectral sequences ${}^r\boldsymbol{E}_{\text{hor}}$ and ${}^r\boldsymbol{E}_{\text{ver}}$ converge to $H^{p+q}(Tot(C))$.

1.5 Koszul complexes and approximation complexes

We give here a discussion on the most useful complexes in commutative algebra: Koszul complexes and approximation complexes, in particular, their applications on the implicitization problem.

1.5.1 Koszul complexes

Let R be a ring and $\mathbf{x} := x_1, \ldots, x_n$ a sequence of elements in R. Suppose that $\{e_1, \ldots, e_n\}$ denotes the canonical basis of R^n, then we define $K_0(\mathbf{x}) = R$ and for all $p \geq 1$

$$
K_p(\mathbf{x}) = \bigoplus_{1\leq i_1\leq\cdots\leq i_p\leq n} Re_{i_1} \wedge \cdots \wedge e_{i_p} \simeq \bigwedge^p R^n,
$$

where $\bigwedge^p R^n$ is the p-th exterior power of R^n. Moreover, the differential

$$
\partial^{\mathbf{x}}_p : K_p(\mathbf{x}) \longrightarrow K_{p-1}(\mathbf{x})
$$

sends a basis element $e_{i_1} \wedge \cdots \wedge e_{i_p}$ to

$$
\partial^{\mathbf{x}}_p(e_{i_1} \wedge \cdots \wedge e_{i_p}) := \sum_{k=1}^{p} (-1)^{k+1} x_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_k} \wedge \cdots \wedge e_{i_p},
$$

where the symbol $\widehat{e}_i$ indicates that e_i has been omitted from the exterior product. The collection of the maps $\partial^{\mathbf{x}}_p$ defines a graded R-homomorphism

$$
\partial^{\mathbf{x}} : \bigwedge R^n \longrightarrow \bigwedge R^n
$$

of degree -1. By a straightforward calculation one verifies the following identities:

$$\partial^{\mathbf{x}} \circ \partial^{\mathbf{x}} = 0 \quad \text{and} \quad \partial^{\mathbf{x}}(a \wedge b) = \partial^{\mathbf{x}}(a) \wedge b + (-1)^{\deg a} a \wedge \partial^{\mathbf{x}}(b)$$

for all homogeneous $a, b \in \bigwedge R^n$. To say that $\partial^{\mathbf{x}} \circ \partial^{\mathbf{x}} = 0$ is to say that

$$K_\bullet(\mathbf{x}) : \ 0 \longrightarrow \bigwedge^n R^n \xrightarrow{\partial_n^{\mathbf{x}}} \bigwedge^{n-1} R^n \xrightarrow{\partial_{n-1}^{\mathbf{x}}} \cdots \longrightarrow \bigwedge^1 R^n \xrightarrow{\partial_1^{\mathbf{x}}} R \longrightarrow 0.$$

is a complex. The second equation expresses that $\partial^{\mathbf{x}}$ is an antiderivation (of degree -1).

Definition 1.5.1. The complex $K_\bullet(\mathbf{x})$ is called the *Koszul complex* of $\mathbf{x}$. More generally, if M is an R-module, then $K_\bullet(\mathbf{x}; M) := K_\bullet(\mathbf{x}) \otimes_R M$, is called the Koszul complex of $\mathbf{x}$ with coefficients in M; its differential is denoted by $\partial_M^{\mathbf{x}}$.

Proposition 1.5.2. *Let R be a ring, $\mathbf{x}$ a sequence of elements in R and M an R-module.*

(a) *The Koszul complex $K_\bullet(\mathbf{x})$ carries the structure of an associative graded alternating algebra, namely that $\bigwedge R^n$. Its differential $\partial^{\mathbf{x}}$ is an antiderivation of degree -1.*

(b) *The Koszul complex $K_\bullet(\mathbf{x}; M)$ is a $K_\bullet(\mathbf{x})$-module in a natural way.*

(c) *One has $\partial_M^{\mathbf{x}}(a.b) = \partial^{\mathbf{x}}(a).b + (-1)^{\deg(a)} a.\partial_M^{\mathbf{x}}(b)$ for all homogeneous element a of $K_\bullet(\mathbf{x})$ and all element $b \in K_\bullet(\mathbf{x}; M)$.*

Proof. Item (a) derives immediately from the construction of Koszul complex. Item (b) is obvious: if A is an R-algebra, then $A \otimes_R M$ is an A-module for every R-module M. For item (c), it is enough to verify the equation for elements $y = w \otimes z$ with $w \in K_\bullet(\mathbf{x}), z \in M$. Then $\partial_M^{\mathbf{x}}(x.w \otimes z) = \partial_M^{\mathbf{x}}((x \wedge w) \otimes z) = \partial^{\mathbf{x}}(x \wedge w) \otimes z$, and the rest follows from the fact that $\partial^{\mathbf{x}}$ is an antiderivation. $\square$

Set

$$\begin{array}{ll} Z_\bullet(\mathbf{x}) := \operatorname{Ker} \partial^{\mathbf{x}} & Z_\bullet(\mathbf{x}; M) := \operatorname{Ker} \partial_M^{\mathbf{x}} \\ B_\bullet(\mathbf{x}) := \operatorname{Im} \partial^{\mathbf{x}} & B_\bullet(\mathbf{x}; M) := \operatorname{Im} \partial_M^{\mathbf{x}} \\ H_\bullet(\mathbf{x}) := Z_\bullet(\mathbf{x})/B_\bullet(\mathbf{x}) & H_\bullet(\mathbf{x}; M) := Z_\bullet(\mathbf{x}; M)/B_\bullet(\mathbf{x}; M). \end{array}$$

Definition 1.5.3. $Z_\bullet(\mathbf{x})$, $B_\bullet(\mathbf{x})$ and $H_\bullet(\mathbf{x})$ are called the *Koszul cycle* of $\mathbf{x}$, the *Koszul boundary* of $\mathbf{x}$ and the *Koszul homology* of $\mathbf{x}$, respectively. We say that $Z_\bullet(\mathbf{x}; M)$, $B_\bullet(\mathbf{x}; M)$ and $H_\bullet(\mathbf{x}; M)$ are the Koszul cycle of $\mathbf{x}$ with coefficients in M, the Koszul boundary of $\mathbf{x}$ with coefficients in M, and the Koszul homology of $\mathbf{x}$ with coefficients in M, respectively.

It is also useful to introduce the *Koszul cohomology (with coeficients in M)*: we set

$$\begin{array}{ll} K^\bullet(\mathbf{x}) = \operatorname{Hom}_R(K_\bullet(\mathbf{x}), R) & K^\bullet(\mathbf{x}; M) = \operatorname{Hom}_R(K_\bullet(\mathbf{x}; M), R) \\ H^\bullet(\mathbf{x}) = H^\bullet(K^\bullet(\mathbf{x})) & H^\bullet(\mathbf{x}; M) = H^\bullet(K^\bullet(\mathbf{x}; M)). \end{array}$$

Let I be an ideal of R generated by these elements $x_1, \ldots, x_n$. Then, by construction, $H_0(\mathbf{x}) = R/I$ and $H_0(\mathbf{x}; M) = M/IM$.

Proposition 1.5.4. [BH98, 1.6.5] *Let R be a ring, $\mathbf{x} = x_1, \dots, x_n$ a sequence in R, and M an R-module. Set $I = (\mathbf{x})$.*

(a) *For every $a \in I$ multiplication by a on $K_\bullet(\mathbf{x}), K_\bullet(\mathbf{x}; M), K^\bullet(\mathbf{x}), K^\bullet(\mathbf{x}; M)$ is null-homotopic.*

(b) *In particular, I annihilates $H_\bullet(\mathbf{x}), H_\bullet(\mathbf{x}; M), H^\bullet(\mathbf{x}), H^\bullet(\mathbf{x}; M)$.*

(c) *If $I = R$, then the complexes $K_\bullet(\mathbf{x}), K_\bullet(\mathbf{x}; M), K^\bullet(\mathbf{x}), K^\bullet(\mathbf{x}; M)$ are null-homotopic.In particular their (co)homology vanishes.*

Proposition 1.5.5. *Let R be a ring and $\mathbf{x} = x_1, \dots, x_n$ a sequence R. If $I = (\mathbf{x}) = R$, then $Z_i(\mathbf{x}) \simeq \bigwedge^i R^{n-1}$.*

Proof. Since $I = R$, $H_i(\mathbf{x}) = 0$, for all i by Proposition 1.5.4(c). As $K_i(\mathbf{x}) \simeq \bigwedge^i R^n$, using induction on i and the sequence exact

$$0 \longrightarrow Z_i(\mathbf{x}) \longrightarrow K_i(\mathbf{x}) \longrightarrow Z_{i-1}(\mathbf{x}) \longrightarrow 0$$

is split for all $1 \leq i \leq n$. □

Lemma 1.5.6 (Koszul-Hilbert's lemma)**.** *Let R be a commutative ring and n a positive integer. Let ρ stand for an $(n+1) \times n$ matrix with arbitrary entries in R and $\Delta = \{\Delta_0, \dots, \Delta_n\}$ stand for its (ordered, signed) n-minors. Then the Koszul relations of Δ belong to the submodule*

$$\mathrm{Im}(\rho) \subset R^{n+1}.$$

Proof. Let $S = \mathbb{Z}[\mathbf{X}]$ be a polynomial ring in $(n+1) \times n$ indeterminates and let χ denote the corresponding $(n+1) \times n$ matrix whose entries are $\mathbf{X}$. Then the ideal of $\mathbb{Z}[\mathbf{X}]$ generate by the n-minors of χ has graded ≥ 2. Indeed, this is clear as, say, $\Delta_1 \in \mathbb{Z}[\mathbf{X}]$ generate a prime ideal and, say, Δ_2 is not a multiple of Δ_1.

As a consequence, the Hilbert-Burch theorem [Eis95, Theorem 20.15] says that the sequence

$$0 \longrightarrow \mathbb{Z}[\mathbf{X}]^n \xrightarrow{\chi} \mathbb{Z}[\mathbf{X}]^{n+1} \xrightarrow{\Delta} \mathbb{Z}[X]$$

is exact. Clearly, then the Koszul relations of Δ belong to the image of the map χ as this is exactly the matrix of the first syzygies of Δ.

Now specialize the variables $\mathbf{X}$ to the entries of ρ by means of the obvious homomorphism $\mathbb{Z}[\mathbf{X}] \longrightarrow R$ extending the inclusion $\mathbb{Z} \longrightarrow R$ $(n \mapsto n \cdot 1_R)$. Since the formation of determinants and the Koszul complex specialize, it is clear that the specialization preserves the above relation. □

Proposition 1.5.7. [BH98, 1.6.10] *Let $\mathbf{x} = x_1, \dots, x_n$ be a sequence in a ring R.*

(a) *The complexes $K_\bullet(\mathbf{x})$ and $K^\bullet(\mathbf{x})$ are isomorphic (we say that $K_\bullet(\mathbf{x})$ is self-dual).*

(b) *More generally, for every R-module M the complexes $K_\bullet(\mathbf{x}; M)$ and $K^\bullet(\mathbf{x}; M)$ are isomorphic, and*

(c) *$H_i(\mathbf{x}; M) \simeq H^{n-i}(\mathbf{x}; M)$ for $i = 0, \dots, n$.*

Proposition 1.5.8. [BH98, 1.6.16] *Let R be a ring, $\mathbf{x} = x_1, \ldots, x_n$ a sequence in R, and M an R-module. If $I = (\mathbf{x})$ contains a regular sequence $\mathbf{y} := y_1, \ldots, y_m$ on M, then*

$$H_{n+1-i}(\mathbf{x}; M) = 0 \quad \text{for } i = 1, \ldots, m, \quad \text{and}$$
$$H_{n-m}(\mathbf{x}; M) \simeq \operatorname{Hom}_R(R/I, M/\mathbf{y}M) \simeq \operatorname{Ext}^m_R(R/I, M).$$

Theorem 1.5.9. [BH98, 1.6.17] *Let R be a Noetherian ring, and M a finitely generated R-module. Suppose that I is an ideal of R generated by $\mathbf{x} := x_1, \ldots, x_n$.*

(i) *All the modules $H_i(\mathbf{x}; M)$, $i = 0, \ldots, n$, vanish if and only if $M = IM$.*

(ii) *Suppose that $H_i(\mathbf{x}; M) \neq 0$ for some i, and let*

$$h = \max\{i : \ H_i(\mathbf{x}; M) \neq 0\}.$$

Then every maximal M-regular sequence in I has length $g = n - h$; in other words, $\operatorname{depth}_I(M) = n - h$.

Proposition 1.5.10. [Vas05, 5.4] *Let R be a Noetherian ring and I an ideal of R generated by $\mathbf{x} := x_1, \ldots, x_n$. Suppose that M is a finitely generated R-module such that $IM \neq M$ and* $\operatorname{depth}_I(M) = g$. *If $\mathbf{a} = a_1, \ldots, a_g$ is a maximal regular sequence on M contained in I, then*

$$H_{n-g}(\mathbf{x}; M) \simeq \operatorname{Ext}^g_R(R/I, M) \simeq \operatorname{Hom}_R(R/I, M/\mathbf{a}M) \simeq (\mathbf{a}M :_R I)/\mathbf{a}M.$$

In the case that R is a Cohen-Macaulay ring with a canonical module ω_R, and $I = (\mathbf{x})$ is an ideal of codimension g, the last non-vanishing homology module of $K_\bullet(\mathbf{x}; \omega_R)$ is $\operatorname{Ext}^g_R(R/I, \omega_R)$, the canonical module of R/I.

Proposition 1.5.11. [Vas05, 5.17] *Let R be a Cohen-Macaulay local ring. If $I = (\mathbf{x})$ is an unmixed ideal then all nonzero homology modules of $K_\bullet(\mathbf{x}; R)$ have Krull dimension equal to* $\dim R/I$.

The next theorem gives a criterion for a sequence being M-regular sequence.

Theorem 1.5.12. [BH98, 1.6.19] *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $M \neq 0$ a finitely generated R-module, and $I \subset \mathfrak{m}$ an ideal generated by $\mathbf{x} := x_1, \ldots, x_n$. Then the following are equivalent:*

(i) $\operatorname{depth}_I M = n$;

(ii) $H_i(\mathbf{x}; M) = 0$ *for all* $i > 0$;

(iii) $H_1(\mathbf{x}; M) = 0$;

(iv) $\mathbf{x}$ *is an M-regular sequence.*

To close this subsection, we recall a result of Busé and Chardin on the cycles of certain Koszul complexes.

Proposition 1.5.13. [BC05, Lemma 1] *Let $R = k[X_0, \ldots, X_{n-1}]$ be the standard graded polynomial ring over a field k and $\mathbf{f} := f_0, \ldots, f_n \in R$ be homogeneous polynomials of positive degrees $d_0, \ldots, d_n$. Suppose that $\dim(R/I) \leq 1$, where I is the ideal of R generated by $f_0, \ldots, f_n$. Denote $\mathfrak{m} := (X_0, \ldots, X_{n-1})$, $\sigma := d_0 + \cdots + d_n$. Then*

(i) $H_i(\mathbf{f}) \neq 0$ *for* $i = 0, 1$, $H_i(\mathbf{f}) = 0$ *for* $i > 2$ *and* $H_2(\mathbf{f})$ *is zero if and only if* $\dim(R/I) = 0$. *If* $\dim(R/I) = 1$, $H_2(\mathbf{f}) \simeq \omega_{R/I}[n - \sigma]$.

(ii) *If* $n \geq 3$, *then*

$$H^i_{\mathfrak{m}}(Z_p(\mathbf{f})) \simeq \begin{cases} 0 & \textit{for} \quad i = 0, 1 \\ H^0_{\mathfrak{m}}(H_{i-p}(\mathbf{f}))^*[n - \sigma] & \textit{for} \quad i = 2 \\ H_{i-p}(\mathbf{f})^*[n - \sigma] & \textit{for} \quad 2 < i < n \\ Z_{n-p}(\mathbf{f})^*[n - \sigma] & \textit{for} \quad i = n \end{cases}$$

where $-^* := {}^*\mathrm{Hom}_R(-, k)$.

1.5.2 Approximation complexes

Let R be a Noetherian ring, $\mathbf{x} = x_1, \ldots, x_n$ a sequence in R, and M a finitely generated R-module. There is a double complex $(\mathfrak{L}_{r,s}, \partial, \partial') : \mathfrak{L}_{r,s} = M \otimes_R \wedge^r(F) \otimes_R \mathrm{Sym}^s_R(F)$, with $\wedge^r(F)$ the r-th exterior power of F and $\mathrm{Sym}^s_R(F)$ the s-th symmetric power of F, where F is a free R-module with basis $\{e_1, \ldots, e_n\}$ – one generator for each element in the sequence $\mathbf{x}$. We shall also denote $S = \mathrm{Sym}_R(F) = R[T_1, \ldots, T_n]$. The differential $\partial^{\mathbf{x}}_M$ and $\partial^{\mathbf{T}}_M$ are, respectively, obtained by viewing $\mathfrak{L}$ as either $K_\bullet(\mathbf{x}; M) \otimes_R S = K_\bullet(\mathbf{x}; M \otimes_R S)$, the Koszul complex of the sequence $\mathbf{x} \subset S$ with coefficients in $M \otimes_R S$; or as $K_\bullet(\mathbf{T}; M \otimes_R S)$, the Koszul complex of $\mathbf{T} := T_1, \ldots, T_n$ with coefficients in $M \otimes_R S$. Now we consider the following double complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \otimes_R S(-n) \arrow[r, "\partial^{\mathbf{x}}_M"] \arrow[d, "\partial^{\mathbf{T}}_M"] & \cdots \arrow[r, "\partial^{\mathbf{x}}_M"] & M^n \otimes_R S(-n) \arrow[r, "\partial^{\mathbf{x}}_M"] \arrow[d, "\partial^{\mathbf{T}}_M"] & M \otimes_R S(-n) \arrow[r] \arrow[d, "\partial^{\mathbf{T}}_M"] & 0 \\
0 \arrow[r] & M \otimes_R S^n(1-n) \arrow[r, "\partial^{\mathbf{x}}_M"] \arrow[d, "\partial^{\mathbf{T}}_M"] & \cdots \arrow[r, "\partial^{\mathbf{x}}_M"] & M^n \otimes_R S^n(1-n) \arrow[r, "\partial^{\mathbf{x}}_M"] \arrow[d, "\partial^{\mathbf{T}}_M"] & M \otimes_R S^n(1-n) \arrow[r] \arrow[d, "\partial^{\mathbf{T}}_M"] & 0 \\
 & \vdots \arrow[d, "\partial^{\mathbf{T}}_M"] & & \vdots \arrow[d, "\partial^{\mathbf{T}}_M"] & \vdots \arrow[d, "\partial^{\mathbf{T}}_M"] & \\
0 \arrow[r] & M \otimes_R S^n(-1) \arrow[r, "\partial^{\mathbf{x}}_M"] \arrow[d, "\partial^{\mathbf{T}}_M"] & \cdots \arrow[r, "\partial^{\mathbf{x}}_M"] & M^n \otimes_R S^n(-1) \arrow[r, "\partial^{\mathbf{x}}_M"] \arrow[d, "\partial^{\mathbf{T}}_M"] & M \otimes_R S^n(-1) \arrow[r] \arrow[d, "\partial^{\mathbf{T}}_M"] & 0 \\
0 \arrow[r] & M \otimes_R S \arrow[r, "\partial^{\mathbf{x}}_M"] \arrow[d] & \cdots \arrow[r, "\partial^{\mathbf{x}}_M"] & M^n \otimes_R S \arrow[r, "\partial^{\mathbf{x}}_M"] \arrow[d] & M \otimes_R S \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 &
\end{tikzcd}$$

It is easy to verify that

$$\partial^{\mathbf{x}}_M \circ \partial^{\mathbf{T}}_M + \partial^{\mathbf{T}}_M \circ \partial^{\mathbf{x}}_M = 0.$$

If we denote by $Z^M_\bullet := Z_\bullet(\mathbf{x}; M)$ the cycles of $K_\bullet(\mathbf{x}; M)$ and by $H^M_\bullet := H_\bullet(\mathbf{x}; M)$ its homologies, the (skew)-commutativity of $\partial^{\mathbf{x}}_M$ and $\partial^{\mathbf{T}}_M$ yield several new complexes among which we single out those defined as follows.

Definition 1.5.14. Let I be an ideal generated by the sequence $\mathbf{x} = x_1, \ldots, x_n$ in R. The *approximation complexes of* $\mathbf{x}$ are the following chain complexes of $S = R[T_1, \ldots, T_n]$-modules:

$$\mathcal{Z}_\bullet(\mathbf{x}; M) = \{Z_\bullet^M \otimes_R S, \partial_M^{\mathbf{T}}\}$$
$$\mathcal{M}_\bullet(\mathbf{x}; M) = \{H_\bullet^M \otimes_R S, \partial_M^{\mathbf{T}}\},$$

called, respectively, the $\mathcal{Z}$*-complex* and the $\mathcal{M}$*-complex* of I with coefficients in M.

We consider S as a standard graded algebra over R. The $\mathcal{Z}$-complex, as well the $\mathcal{M}$-complex, are graded complexes over S. The t-th homogeneous part $\mathcal{Z}_t(\mathbf{x}; M)$ of $\mathcal{Z}_\bullet(\mathbf{x}; M)$ is a complex of finitely generated R-modules of the form

$$0 \longrightarrow Z_n^M \otimes_R S_{t-n} \xrightarrow{\partial_M^{\mathbf{T}}} \cdots \xrightarrow{\partial_M^{\mathbf{T}}} Z_1^M \otimes_R S_{t-1} \xrightarrow{\partial_M^{\mathbf{T}}} Z_0^M \otimes_R S_t \longrightarrow 0$$

In a similar manner $\mathcal{M}_t(\mathbf{x}; M)$ is a complex of finitely generated R-modules. For certain uses, however, we must view $\mathcal{Z}_\bullet(\mathbf{x}; M)$ and $\mathcal{M}_\bullet(\mathbf{x}; M)$ as complexes over the polynomial ring S. Thus, for instance, the $\mathcal{Z}$-complex $\mathcal{Z}_\bullet(\mathbf{x}; M)$ is

$$0 \longrightarrow Z_n^M \otimes_R S(-n) \xrightarrow{\partial_M^{\mathbf{T}}} \cdots \xrightarrow{\partial_M^{\mathbf{T}}} Z_1^M \otimes_R S(-1) \xrightarrow{\partial_M^{\mathbf{T}}} Z_0^M \otimes_R S \longrightarrow 0$$

that can be written

$$0 \longrightarrow Z_n^M[\mathbf{T}](-n) \xrightarrow{\partial_M^{\mathbf{T}}} \cdots \xrightarrow{\partial_M^{\mathbf{T}}} Z_1^M[\mathbf{T}](-1) \xrightarrow{\partial_M^{\mathbf{T}}} M[\mathbf{T}] \longrightarrow 0$$

where $N(k)$ denotes the graded S-module with N_{k+m} for its component in degree m.

Proposition 1.5.15. *The homology modules of the complexes* $\mathcal{Z}_\bullet(\mathbf{x}; M)$ *and* $\mathcal{M}_\bullet(\mathbf{x}; M)$ *do not depend on the generating set* $\mathbf{x}$ *of the ideal* I.

Proof. See Proposition 3.2.6 and Corollary 3.2.7 of [Vas94] or [HSV83b, §3]. □

For this reason we shall often denote these complexes by $\mathcal{Z}_\bullet(I; M)$ and $\mathcal{M}_\bullet(I; M)$, or if I is fixed in a discussion, by $\mathcal{Z}_\bullet(M)$ and $\mathcal{M}_\bullet(M)$. If M is fixed as well, as above we shall denote the complexes simply by $\mathcal{Z}_\bullet$ and $\mathcal{M}_\bullet$.

A major source of interest for introducing these complexes occur when one takes $M = R$. The $\mathcal{Z}_\bullet$ and $\mathcal{M}_\bullet$ are then differential graded algebras with $H_0(\mathcal{Z}_\bullet) = \mathcal{S}_I$ and $H_0(\mathcal{M}_\bullet) = \mathrm{Sym}_R(I/I^2)$ with $I = (\mathbf{x})$. They permit comparisons of these rings to the more standard blowing-up algebras represented by the Rees algebra $\mathcal{R}_I := \oplus_{j\geq 0} I^j$ and the associated graded ring $\mathrm{gr}_I(R) := \oplus_{j\geq 0} I^j/I^{j+1}$.

There are natural surjections

$$\alpha : H_0(\mathcal{Z}_\bullet(I; M)) \longrightarrow \mathcal{R}_I(M) := \oplus_{j\geq 0} I^j M = \text{associated Rees module},$$

and

$$\beta : H_0(\mathcal{M}_\bullet(I; M)) \longrightarrow \mathrm{gr}_I(M) := \oplus_{j\geq 0} I^j M/I^{j+1}M = \text{associated graded module}.$$

Proposition 1.5.16. *Let R be a Noetherian ring and M a finitely generated R-module. Suppose that $I = (x_1, \dots, x_n)$ is an ideal of R. Then the homology modules of the $\mathcal{Z}$-complex $\mathcal{Z}_\bullet(\mathbf{x}; M)$ are $\mathcal{S}_I$-modules.*

Proof. We identify $\mathcal{S}_I$ with $S/\mathfrak{L}$, where $\mathfrak{L} = \partial^{\mathbf{T}}(Z_1(\mathbf{x}; R))$. The Koszul complex $K_\bullet^M := K_\bullet \otimes_S M[T_1, \dots, T_n]$ is a (homologically) graded S-module over the exterior algebra $K_\bullet = \bigwedge S^n$. It is a multiplication,

$$K_i \otimes_S K_j^M \longrightarrow K_{i+j}^M$$

sending $x \otimes (y \otimes m)$ to $(x \wedge y) \otimes m$, satisfying $\partial_M^{\mathbf{T}}(p.q) = \partial^{\mathbf{T}}(p).q + (-1)^i p.\partial_M^{\mathbf{T}}(q)$ and $\partial_M^{\mathbf{x}}(p.q) = \partial^{\mathbf{x}}(p).q + (-1)^i p.\partial_M^{\mathbf{x}}(q)$ by Proposition 1.5.2(c). This map restricts to a map

$$\mathcal{Z}_i(\mathbf{x}; S) \otimes_S \mathcal{Z}_j(\mathbf{x}; M) \longrightarrow \mathcal{Z}_{i+j}(\mathbf{x}; M)$$

(and similarly for cycles with repect to $\partial^{\mathbf{T}}$).

Let α be a j-th cycle of $\mathcal{Z}_\bullet(\mathbf{x}; M)$. We have to show that for any syzygy $b = (b_1, \dots, b_n) \in Z_1(\mathbf{x}; R) \subset R^n, \partial^{\mathbf{T}}(b).\alpha = (\sum_i b_i T_i).\alpha$ is a border.

By the above formula giving the action of $\partial^{\mathbf{T}}$ on a product (in the case $i = 1$),

$$\partial_M^{\mathbf{T}}(b.\alpha) = \partial^{\mathbf{T}}(b).\alpha - b.\partial_M^{\mathbf{T}}(\alpha) = \partial^{\mathbf{T}}(b).\alpha,$$

as α is a cycle for $\partial_M^{\mathbf{T}}$. Hence $\partial^{\mathbf{T}}(b).\alpha$ is the image of $b.\alpha \in \mathcal{Z}_{j+1}(\mathbf{x}; M)$. □

We next study the acyclicity of these complexes. To do it, we need some definitions of the various regular sequences we shall be interested in.

Definition 1.5.17. Let $\mathbf{x} = x_1, \dots, x_n$ be a sequence of elements of a ring R, $I = (x_1, \dots, x_n)$ an ideal of R and M a R-module. We say that $\mathbf{x}$ is a

1. *d-sequence* with respect to M, if
   a) $\mathbf{x}$ is a minimal generating set of the ideal I;
   b) $(x_1, \dots, x_i)M : {}_M x_{i+1} x_k = (x_1, \dots, x_i)M : {}_M x_k$, for $i = 0, \dots, n-1$ and $k \geq i+1$.
2. *proper sequence* with respect to M, if $x_{i+1} H_j(x_1, \dots, x_i; M) = 0$, for $i = 0, \dots, n_1,\ j > 0$.

These conditions are related in the following way:

$$\text{regular sequence} \Longrightarrow d\text{-sequence} \Longrightarrow \text{proper sequence}.$$

The acyclicity of the $\mathcal{Z}_\bullet(\mathbf{x}; M)$-complexes and $\mathcal{M}_\bullet(\mathbf{x}; M)$-complexes bears a striking resemblance to that of an ordinary Koszul complex $K_\bullet(\mathbf{x}; M)$, with the role of regular sequences being played by the proper sequences and d-sequences.

Theorem 1.5.18. [HSV83a, Theorem 4.1] *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field. Let $I = (\mathbf{x})$ be an ideal of R and M a finitely generated R-module. Then the following conditions are equivalent:*

(a) *$\mathcal{M}_\bullet(\mathbf{x}; M)$ is acyclic;*

(b) *I is generated by a d-sequence with respect to M.*

Theorem 1.5.19. [HSV83b, Theorem 12.9] *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field. Let $I = (\mathbf{x})$ be an ideal of R and M a finitely generated R-module. Then the following conditions are equivalent :*

(a) *$\mathcal{Z}_\bullet(\mathbf{x}; M)$ is acyclic;*

(b) *I is generated by a proper sequence with respect to M.*

Finally, we recall some results on implicitizing rational hypersurfaces using approximation complexes. Let $\mathbf{f} := f_0, \ldots, f_n$ be $n+1$ homogeneous polynomials in $R = k[X_0, \ldots, X_{n-1}]$ of the same degree $d \geq 1$. Denoting by Z_i the i-th cycles of the Koszul complex of the $\mathbf{f}$ on R, we set $\mathcal{Z}_i := Z_i[id] \otimes_R R[\mathbf{T}]$, where $\mathbf{T} := T_0, \ldots, T_n$ and $[-]$ (respectively $(-)$) stands for the degree shift in the X_i's (respectively in the T_i's). Note that $\mathcal{Z}_{n+1} = 0$, the $\mathcal{Z}_\bullet(\mathbf{f})$-complex is of form

$$\mathcal{Z}_\bullet : 0 \longrightarrow \mathcal{Z}_n(-n) \xrightarrow{\partial_n^{\mathbf{T}}} \cdots \longrightarrow \mathcal{Z}_2(-2) \xrightarrow{\partial_2^{\mathbf{T}}} \mathcal{Z}_1(-1) \xrightarrow{\partial_1^{\mathbf{T}}} \mathcal{Z}_0 = R[\mathbf{T}].$$

It is a naturally a bigraded complex, and $H_0(\mathcal{Z}_\bullet) \simeq \mathrm{Sym}_R(I)$. We now give some acyclicity criterions in the case where $I = (f_0, \ldots, f_n)$ define isolated points in $\mathrm{Proj}(R)$. Set $\mathfrak{m} = (X_0, \ldots, X_{n-1})$.

Proposition 1.5.20. [BC05, Lemma 2] *Suppose that $I = (f_0, \ldots, f_n)$ is of codimension at least $n-1$, k is infinite, and set $\mathcal{B} := \mathrm{Proj}(R/I)$. Then the following are equivalent:*

(1) *$\mathcal{Z}_\bullet$ is acyclic;*

(2) *$\mathcal{Z}_\bullet$ is acyclic outside $V(\mathfrak{m})$;*

(3) *I is generated by a proper sequence;*

(4) *$\mathcal{B}$ is locally defined by a proper sequence;*

(5) *$\mathcal{B}$ is locally defined by n equations.*

Theorem 1.5.21. [BC05, Theorem 4] *Let $n \geq 3$. Let $I = (f_0, \ldots, f_n)$ be an ideal of R, $\mathcal{B} := \mathrm{Proj}(R/I)$ and I^{sat} the saturation of I with respect to $\mathfrak{m}$. Assume that k is infinite and that $\dim \mathcal{B} \leq 0$, that is, $\mathcal{B}$ define a finite number of base points in $\mathrm{Proj}(R)$, possibly empty. Then we have*

(1) *$\mathcal{Z}_\bullet$ is acyclic if and only if $\mathcal{B}$ is locally defined by n equations.*

(2) *Let $\nu \geq \nu_0 := (n-1)(d-1) - \mathrm{indeg}(I^{\mathrm{sat}})$. Then $H^0_{\mathfrak{m}}(\mathrm{Sym}_R(I))_{[\nu]} = 0$. Moreover $(\mathcal{Z}_\bullet)_{[\nu]} \otimes_{k[\mathbf{T}]} k[\mathbf{T}]$ is acyclic if and only if $\mathcal{Z}_\bullet$ acyclic.*

(3) *Let $\nu \geq \nu_0$ and assume that $\mathcal{B}$ is locally defined by n equations. Then $D := \det((\mathcal{Z}_\bullet)_{[\nu]})$ is a non zero homogeneous element of $k[\mathbf{T}]$, independent of ν (modulo $k^\times$), of degree $d^{n-1} - \sum_{\mathfrak{p} \in \mathcal{B}} d_{\mathfrak{p}}$. Denoting by $\mathcal{H}$ the closed image of the rational map $\phi : \mathbb{P}^{n-1}_k \longrightarrow \mathbb{P}^n_k$, $D = H^{\deg(\phi)} G$ where H is an implicit equation of $\mathcal{H}$. Moreover $G \in k^\times$ if and only if $\mathcal{B}$ is locally of linear type, and if and only if $\mathcal{B}$ is locally a complete intersection.*

1.6 Rational maps and Jacobian dual criterion of birationality

In this section, we recall some definitions and notations on rational maps. Let k be an algebraically closed field. We are given nondegenerate irreducible projective variety $X \subset \mathbb{P}^m_k$, with homogeneous coordinate ring $R := k[X] = k[\mathbf{X}]/I(X) = k[X_0, \ldots, X_m]/I(V)$. A rational map $\mathcal{F} : X \dashrightarrow \mathbb{P}^n_k$ is defined by homogeneous polynomial $f_0, \ldots, f_n \in R$ of the same degree d, the tuple $\mathbf{f} := f_0, \ldots, f_n$ is called a *representative* of $\mathcal{F}$ and the f_j's are its coordinates.

There is nothing unique about the choice of $f_0, \ldots, f_n$. However, any other choice of such a representative of $\mathcal{F}$, say, $f'_0, \ldots, f'_n$, necessarily satisfies the condition that the (2×2)-minors of the matrix

$$\begin{pmatrix} f_0 & f_1 & \cdots & f_n \\ f'_0 & f'_1 & \cdots & f'_n \end{pmatrix}$$

vanish (as elements of the homogeneous coordinate ring R.)

If $\mathbf{f}$ and $\mathbf{f}'$ are two representatives of $\mathcal{F}$, then they induce a k-isomorphism of the respective representative algebras $k[\mathbf{f}]$ and $k[\mathbf{f}']$, such that $f_j \leftrightarrow f'_j$. Although this isomorphism is graded, it is not homogeneous (of degree zero). Nevertheless, one can renormalize the gradings of all these algebras so that they become standard k-algebras. What this actually means is that one can take a common polynomial representative $S = k[\mathbf{Y}]/J = k[Y_0, \ldots, Y_n]/J \simeq k[\mathbf{f}]$ for all the representative of $\mathcal{F}$, with S is standard. We call $Y = \mathrm{Proj}(S) \subset \mathbb{P}^n_k$ the *image* of $\mathcal{F}$.

Proposition 1.6.1. [Sim04, Proposition 1.1] *Let $\mathcal{F} : X \dashrightarrow \mathbb{P}^n_k$ be a rational map and $\mathbf{f} = f_0, \ldots, f_n$ a representative of $\mathcal{F}$. Set $I = (\mathbf{f}) \subset R$. Then the set of representatives of $\mathcal{F}$ correspond bijectively to the homogeneous vectors in the rank one graded R-module $\mathrm{Hom}_R(I, R)$. In particular, $\mathcal{F}$ is uniquely represented up to proportionality if and only if I has grade at least two.*

Proof. Given another representative $\mathbf{f}' = f'_0, \ldots, f'_n$ of $\mathcal{F}$, then by definition

$$I_2 \begin{pmatrix} f_0 & f_1 & \cdots & f_n \\ f'_0 & f'_1 & \cdots & f'_n \end{pmatrix} = 0,$$

i.e., the two sets $\mathbf{f}$ and $\mathbf{f}'$ are proportional by a factor which is a homogeneous element of the field of fractions K of R. This establishes a bijection between the set of representatives of $\mathcal{F}$ and the set of homogeneous elements of K that drive $I = (\mathbf{f})$ into R. The last one generates the fractional ideal $R\colon {}_K I \simeq \mathrm{Hom}_R(I, R)$. Finally, it is known that the natural inclusion $R \subset \mathrm{Hom}_R(I, R)$ is an equality if and only if I has grade at least two. Therefore, $\mathcal{F}$ is uniquely represented up to proportionality if and only if I has graded at least two. $\square$

The above proposition motivates the following notion.

Definition 1.6.2. The *degree sequence* of the rational map $\mathcal{F} : X \dashrightarrow \mathbb{P}^n_k$ is the sequence $d_1 \leq \cdots \leq d_r$ of degrees of a minimal set of generators of $\mathrm{Hom}_R((\mathbf{f}), R)$, where $\mathbf{f}$ is a representative of $\mathcal{F}$.

The previous proposition shows that the degree sequence of $\mathcal{F}$ depends solely on $\mathcal{F}$ and not on the choice of representative. In particular, if for one representative $\mathbf{f}$ the corresponding ideal has grade at least two - for example, if $X = \mathbb{P}^m_k$ - then every representative of $\mathcal{F}$ is a multiple of $\mathbf{f}$ by a factor which is a homogeneous element of $k[\mathbf{X}]$. Clearly, this is then the unique (up to a nonzero factor in the base field k) representative with this property. In this case, the degree sequence consists of an unique number, namely, the degree of $\mathbf{f}$. This degree will then be said the *degree* of $\mathcal{F}$ and the corresponding representative will generate the (uniquely defined) *base ideal* of $\mathcal{F}$.

Definition 1.6.3. Given a rational map $\mathcal{F} : X \dashrightarrow \mathbb{P}^n_k$ and let $Y \subset \mathbb{P}^n_k$ denote its image. One says that $\mathcal{F}$ is *birational onto its image* if there exists a rational map $\mathcal{G} : Y \dashrightarrow \mathbb{P}^m_k$ whose image is X such that $\mathcal{F}$ and $\mathcal{G}$ are inverse to each other.

Proposition 1.6.4. [Sim04, Proposition 2.1] *Let $X \subset \mathbb{P}^m_k$ and $Y \subset \mathbb{P}^n_k$ denote subvarieties of positive dimension and let $\mathcal{F} : X \dashrightarrow \mathbb{P}^n_k$ and $\mathcal{G} : Y \dashrightarrow \mathbb{P}^m_k$ stand for rational maps with image Y and X, respectively (so that, in particular,* $\dim\ X = \dim\ Y$*). Let $R = k[\mathbf{X}]/I(X)$ and $S = k[\mathbf{Y}]/I(Y)$ denote the respective homogeneous coordinate ring of X and Y. Fix sets of forms $\mathbf{f} = f_0, \ldots, f_n \in k[\mathbf{X}]$ and $\boldsymbol{g} = g_0, \ldots, g_m \in k[\mathbf{Y}]$ whose respective residues in R and S are representatives of $\mathcal{F}$ and $\mathcal{G}$. The following are equivalent conditions:*

(i) *$\mathcal{F}$ and $\mathcal{G}$ are inverse to each other;*

(ii) *the identity map of $k[\mathbf{X}, \mathbf{T}]/(I(X), I(Y))$ induces a bigraded isomorphism*

$$\mathcal{R}_R\Big(\frac{(\mathbf{f}, I(X))}{I(X)}\Big) \simeq \mathcal{R}_S\Big(\frac{(\boldsymbol{g}, I(Y))}{I(Y)}\Big)$$

of Rees algebras.

Consider the bigrading of the Rees algebra $\mathcal{R}_R(\mathbf{f})$ in more detail. Let $\mathcal{F} : X \dashrightarrow \mathbb{P}^n_k$ be a rational map and $\mathbf{f} = f_0, \ldots, f_n$ be a representative of $\mathcal{F}$. Let $R[\mathbf{Y}] = R[Y_0, \ldots, Y_n]$ be a polynomial ring over R with the standard bigrading where $\deg(X_i) = (1, 0)$ and $\deg(Y_i) = (0, 1)$. Since $I = (\mathbf{f})$ is generated in a fixed degree, the Rees algebra

$$\mathcal{R}_R(I) := R \oplus I \oplus I^2 \oplus \cdots \simeq R[It] \subset R[t]$$

is a standard bigraded k-algebra. The map $Y_j \longmapsto f_j t$ yields a presentation $R[\mathbf{Y}]/\mathcal{J}_{\mathbf{f}} \simeq \mathcal{R}_R(I)$, with $\mathcal{J}_{\mathbf{f}}$ a bihomogeneous *defining ideal.* Note that if $\mathbf{f}$ and $\mathbf{f}'$ denote two representatives of $\mathcal{F}$, then $\mathcal{J}_{\mathbf{f}} = \mathcal{J}_{\mathbf{f}'}$ so that $\mathcal{R}_R((\mathbf{f})) \simeq \mathcal{R}_R((\mathbf{f}'))$. Therefore, $\mathcal{J}_{\mathbf{f}}$ depends only on the rational map and not on any of its representative, we write $\mathcal{J}$ for the defining ideal of the Rees algebra $\mathcal{R}_R(I)$. Write

$$\mathcal{J} = \bigoplus_{(p,q)\in\mathbb{N}^2} \mathcal{J}_{(p,q)},$$

where $\mathcal{J}_{(p,q)}$ denotes the k-vector space of forms of bidegree (p,q).

First focus on $\mathcal{J}_{(0,*)}$, which is spanned by the forms of bidegree $(0,q)$ for all $q \geq 1$. Since these forms have coefficients only in k, one can view as elements of $k[\mathbf{Y}]$. As such they generate the ideal of all polynomial relations (over k) of $\mathbf{f}$. Thus,

$$k[\mathbf{Y}]/(\mathcal{J}_{(0,*)k[\mathbf{Y}]}) \simeq k[\mathbf{f}],$$

as standard graded k-algebras, after a degree renormalization in $k[\mathbf{f}]$. Thus, the ideal $\mathcal{J}_{(0,*)k[\mathbf{Y}]}$ recovers the ideal J of the image of $\mathcal{F}$.

For birationality, the following bihomogeneous piece is no less important:

$$\mathcal{J}_{(1,*)} := \bigoplus_{q \in \mathbb{N}} \mathcal{J}_{(1,q)}$$

with $\mathcal{J}_{(1,q)}$ denoting the bigraded piece of $\mathcal{J}$ spanned by the forms of bidegree $(1,q)$ for all $q \geq 0$.

Now, a form of bidegree $(1,*)$ can be written as $\sum_{i=0}^{m} Q_i(\mathbf{Y})X_i$, for suitable homogeneous $Q_i(\mathbf{Y}) \in k[\mathbf{Y}] \subset R[\mathbf{Y}]$ of the same degree. Moreover, since X is nondegenerate, each such form has unique expansion.

Next, one can pick a minimal set of generators of the ideal $(\mathcal{J}_{(1,*)})$ consisting of a finite number of forms of bidegree $(1,q)$ for various q's. Let $\{P_1, \ldots, P_s\} \subset k[\mathbf{X}, \mathbf{Y}]$ denote liftings of these biforms. Consider the Jacobian matrix $\Psi_{\mathbf{X}}$ of the polynomial $\{P_1, \ldots, P_s\}$ with respect to $\mathbf{X}$, hence $\Psi_{\mathbf{X}}$ is a graded matrix since its rows are homogeneous vectors in $k[\mathbf{Y}]$ and we say that $\Psi_{\mathbf{X}}$ will be a *weak Jacobian dual matrix* of associated to the given set of generators of $(\mathcal{J}_{(1,*)})$.

Remark 1.6.5. Though the definition of a weak Jacobian dual matrix depends only on $\mathcal{F}$ and not on a representative of $\mathcal{F}$, it is not uniquely given. However, for a fixed bidegree $(1,q)$ the number of $(1,q)$-biforms in any minimal set of generating biforms of $\mathcal{J} \subset R[\mathbf{Y}]$ is invariant and equals the dimension of the correspondingly spanned k-vector subspace of the whole space $(1,q)$-biforms of $\mathcal{J}$. This implies that any two weak Jacobian dual matrices of the same rational map $\mathcal{F}$ have the same sizes.

Finally, we will mainly consider $\Psi_{\mathbf{X}} \otimes_{k[\mathbf{Y}]} S$ as a matrix over the homogeneous coordinate ring $S = k[\mathbf{Y}]/I(Y)$ of the image $Y \subset \mathbb{P}^n_k$. Note that $\Psi_{\mathbf{X}} \otimes_{k[\mathbf{Y}]} S$ can be the zero matrix, as it happens in many cases.

Lemma 1.6.6. [DHS12, Lemma 2.13] *If the weak Jacobian dual matrix of associated to a set of bihomogeneous minimal generators of $(\mathcal{J}_{(1,*)})$ has a rank over S, then the weak Jacobian dual matrix of associated to any other set of bihomogeneous minimal generators of $(\mathcal{J}_{(1,*)})$ has a rank over S and two ranks coincide.*

Because of the previous lemma, we define the following.

Definition 1.6.7. Let $\mathcal{F} : X \dashrightarrow \mathbb{P}^n_k$ be a rational map.

(i) We say that any weak Jacobian dual matrix of associated to a minimal set of generators of $(\mathcal{J}_{(1,*)})$ is a *Jacobian dual matrix* of $\mathcal{F}$.

(ii) If the matrix $\Psi_{\mathbf{X}} \otimes_{k[\mathbf{Y}]} S$ has a rank over S, we will say that $\mathcal{F}$ has a *Jacobian dual rank* and write $\operatorname{jdrank}(\mathcal{F}) := \operatorname{rank}_S(\Psi_{\mathbf{X}} \otimes_{k[\mathbf{Y}]} S)$.

The next result shows that the rank just introduced is sensitive to the dimension difference between the source and the target.

Proposition 1.6.8. [DHS12, Proposition 2.15] *Let $\mathcal{F} : X \dashrightarrow \mathbb{P}^n_k$ be a rational map defined by $\mathbf{f} = f_0, \ldots, f_n \in R$. Set $S = k[\mathbf{Y}]/I(Y) \simeq k[\mathbf{f}] \subset R$. If $\mathcal{F}$ has a Jacobian dual rank then*

$$0 \leq \dim\ R - \dim\ S \leq m - \mathrm{jdrank}(\mathcal{F}).$$

As a consequence, $\mathrm{jdrank}(\mathcal{F}) \leq m$, *and if equality holds then* $\dim\ R = \dim\ S$.

Theorem 1.6.9. [DHS12, Theorem 2.18] *With the previous notations, the rational map $\mathcal{F} : X \subset \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k$ is birational with image Y if and only if* $\mathrm{jdrank}(\mathcal{F}) = m$. *Moreover, when $\mathcal{F}$ is birational onto its image, then the coordinates of the inverse to $\mathcal{F}$ are the (signed, ordered) m-minors of an arbitrary $m \times (m+1)$ submatrix rank m of a weak Jacobian dual matrix of $\mathcal{F}$.*

The above theorem gives a criterion for deciding if a given rational map is birational. This criterion relies on Gröbner basis computations in order to get the equations of a Rees algebra which is very hard. For computational purposes, we investigate how birationality can be detected by means of syzygies of the ideal I generated by the coordinates of the rational maps, instead of the whole collection of equations of I.

Theorem 1.6.10. [DHS12, Theorem 3.2] *Let $\mathcal{F} : \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k$ stand for a rational map defined by homogeneous polynomials $f_0, \ldots, f_n$ of a fixed degree in $R = k[\mathbf{X}]$ and set $I = (f_0, \ldots, f_n)$. If the image of $\mathcal{F}$ has dimension m and the submatrix of the syzygy matrix of I consisting of columns of degree one (linear syzygies) has rank n (maximal possible), then $\mathcal{F}$ is birational onto its image.*

Chapter 2

One-dimensional fibers of parameterization surfaces

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Abstract: Given a rational map $\phi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^3_k$, which is a parameterization of an algebraic surface $\mathscr{S} \subset \mathbb{P}^3_k$, defined by four homogeneous polynomial $\mathbf{f} := f_0, \ldots, f_3$, of the same degree d, in the standard graded polynomial ring $R = k[X_0, X_1, X_2]$, we give a quadratic bound in terms of d for the number of one-dimensional fibers of ϕ, by studying of the difference between the initial degree of ordinary and saturated powers of the base ideal of ϕ. The results in this chapter have been submitted [Tra15].

2.1 Introduction

Rational algebraic variety can be described in several ways, the most common being parametric and implicit representations. Parametric representations describe such variety as the closed image of a rational map, whereas implicit representations describe it as the zero locus of some algebraic equations, e.g. as the zeros of a system of polynomials. Both representations have a wide range of applications in Computer Aided Geometric Design, and depending on the problem one needs to solve, one or the other might be better suited. It is thus interesting to be able to pass from parametric representations to implicit representations. This is a classical problem and there are numerous approaches to its solution. For a good historical overview on this subject we refer the reader to [SC95] and [Cox01].

Find the implicit equations for a rational algebraic variety, in particular for curve or surface, starting from a parameterization is a central problem in geometric modeling, it is called the *implicitization problem*. Computing the implicit equations can be solved by computing a suitable Gröbner basis. However, it is known to be

quite slow in practice and hence is rarely used in geometric modeling [Hof89]. An other method for finding the implicit equations is to eliminate by computing the resultant of the polynomials (see, e.g., [Bus01, Jou96]). But in many applications, the resultant vanishes identically due to the presence of base points, which are the source points where the parameterization is not well defined. Instead, the method of moving curves and surfaces was introduced by Sederberg and Chen [SC95] for the case of parameterized curves, and then was regularly extended for the case of surfaces, such as in [BCD03, CGZ00, LC16]. It was precisely formulated in terms of Rees algebra by Jouanolou, Busé, Chardin, Botbol and others [BJ03, BC05, Cha06, BCJ09, Bot11]. One of the key tools of this approach is the *approximation complexes* that were introduced by Herzog, Simis, and Vasconcelos [HSV82].

Besides the computation of implicit representations of parameterizations, in geometric modeling it is of vital importance to have a detailed knowledge of the geometry of the object and of the parametric representation one is working with. The question of how many times is the same point being painted (i.e., corresponds to distinct values of parameter), depends not only on the variety itself but also on the parameterization. It is of interest for applications to determine the singularities of the parameterizations. The main goal of this chapter is to study the fibers of parameterizations, in particular, parameterizations of surfaces.

The chapter is organized as follows. In Section 2.2, we study fibers of a rational map $\phi : \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k$ defined by an ordered set of homogeneous polynomials $\mathbf{f} := f_0, \ldots, f_n$, of the same degree d, in the standard graded polynomial ring $R = k[X_0, \ldots, X_m]$ over an algebraically closed field k, such that $\gcd(f_0, \ldots, f_n) = 1$. Let $\Gamma \subset \mathbb{P}^m_k \times \mathbb{P}^n_k$ be the closure of the graph of ϕ and π be the canonical projection of Γ onto its image. Let $\mathcal{R}_I$ be the Rees algebra of the ideal $(\mathbf{f})$ which is the homogeneous coordinate ring of Γ and $B = k[T_0, \ldots, T_n]$ the homogeneous coordinate ring of $\mathbb{P}^n_k$. For every $y \in \mathbb{P}^n_k$, the fiber of π at y is the subscheme

$$\pi^{-1}(y) := \operatorname{Proj}(\mathcal{R}_I \otimes_B k(y)) \subset \mathbb{P}^m_{k(y)} \simeq \mathbb{P}^m_k,$$

where $k(y)$ is its residue field. Assume that ϕ is generically finite onto its image. Then the subset

$$\mathcal{Y}_{m-1} := \{y \in \mathbb{P}^n_k \mid \dim \pi^{-1}(y) = m-1\}$$

is finite. Therefore, in this chapter and the next one, we will propose some methods to estimate the cardinality of $\mathcal{Y}_{m-1}$. More precisely, for any $y \in \mathcal{Y}_{m-1}$, we denote by $h_y \in R$ the defining equation of the unmixed components of the fiber $\pi^{-1}(y)$, then our purpose is to bound for $\sum_{y \in \mathcal{Y}_{m-1}} \deg(h_y)$ in the terms of d.

In this section, we propose the following key tool – recall that the saturation of an ideal J of R is denoted by $J^{\mathrm{sat}} := J :_R (X_0, \ldots, X_m)^\infty$.

Theorem (Theorem 2.2.6)**.** *Adopt Data 2.2.5. If there exists an integer s such that $\nu = \operatorname{indeg}((I^s)^{\mathrm{sat}}) < sd$, then*

$$\sum_{y \in \mathcal{Y}_{m-1}} \deg(h_y) \leq \nu < sd.$$

In Section 2.3, we will restrict to the case where ϕ is a parameterization of a surface. More precisely, one studies the one-dimensional fibers of a parameterization

of an algebraic surface $\phi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^3_k$. The idea is to show the existence an integer s as in Theorem 2.2.6 and our key tool in the proof is to use the $\mathcal{Z}$-approximation complex associated to $\mathbf{f} := f_0, f_1, f_2, f_3$, which is a resolution of $\mathcal{S}_I$ under the assumption that $\mathcal{B}$ is locally a complete intersection of dimension zero. The main results are the following.

Theorem (Theorems 2.3.7 and 2.3.9)**.** *Adopt Data 2.2.5. Assume further that $m = n - 1 = 2$ and set $\mu = \inf\{\nu \mid I_\nu^{\mathrm{sat}} \neq 0\}$.*

(i) *If $\mu < d$, then $\sum_{y \in \mathcal{Y}_1} \deg(h_y) \leq \mu$.*

(ii) *If $\mu = d$ and $\mathcal{B} = \mathrm{Proj}(R/I)$ is locally a complete intersection of dimension zero, then*

$$\sum_{y \in \mathcal{Y}_1} \deg(h_y) \leq \begin{cases} 4 & \text{if} \quad d = 3 \\ \left\lfloor \frac{d}{2} \right\rfloor d - 1 & \text{if} \quad d \geq 4 \end{cases}$$

where $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.

2.2 Fibers of rational maps

Let $R := k[X_0, \ldots, X_m]$ $(m \geq 2)$ be the standard graded polynomial ring over an algebraically closed field k. Suppose we are given an integer $d \geq 1$ and $n+1$ $(n \geq m)$ homogeneous polynomials $f_0, \ldots, f_n \in R_d$, not all zero. We may further assume that $\gcd(f_0, \ldots, f_n) = 1$, replacing the $f_i's$ by their quotient by the greatest common divisor of $f_0, \ldots, f_n$ if needed, hence the ideal I of R generated by these polynomials is of codimension at least two. Set $\mathcal{B} := \mathrm{Proj}(R/I) \subseteq \mathbb{P}^m_k := \mathrm{Proj}(R)$ and consider the rational map

$$\begin{aligned} \phi : \quad & \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k \\ & x \longmapsto (f_0(x) : \cdots : f_n(x)) \end{aligned}$$

whose closed image is the subvariety $\mathscr{S}$ in $\mathbb{P}^n_k$. Let $\Gamma_0 \subset \mathbb{P}^m_k \times \mathbb{P}^n_k$ be the graph of $\phi : \mathbb{P}^m_k \setminus \mathcal{B} \longrightarrow \mathbb{P}^n_k$ and Γ the Zariski closure of Γ_0. We have the following diagram

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[d, "\pi_1"'] & \mathbb{P}^m_k \times \mathbb{P}^n_k \arrow[d, "\pi_2"] \\ \mathbb{P}^m_k \arrow[r, dashed, "\phi"'] & \mathbb{P}^n_k \end{tikzcd}$$

where the maps π_1, π_2 are the canonical projections. One has

$$\mathscr{S} = \overline{\pi_2(\Gamma_0)} = \pi_2(\Gamma),$$

where the bar denotes the Zariski closure. The first equality directly follows from the definition of $\mathscr{S}$, and the second from the fact that π_2 is a projective morphism (so that the image of a closed variety is a closed variety).

Furthermore, Γ is the irreducible subscheme of $\mathbb{P}^m_k \times \mathbb{P}^n_k$ defined by the Rees algebra $\mathcal{R}_I := \mathrm{Rees}_R(I)$ (see [Har77, Chapter II, §7]). Let $B := k[T_0, \ldots, T_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n_k$ and $S := R \otimes_k B = R[T_0, \ldots, T_n]$ with the

standard bigraded structure by the canonical grading $\deg(X_i) = (1,0)$ and $\deg(T_j) = (0,1)$ for all $i = 0,\dots,m$ and $j = 0,\dots,n$. The natural bigraded morphism of bigraded k-algebras

$$\begin{aligned} \alpha : \quad & S \longrightarrow \mathcal{R}_I = \oplus_{s\geq 0} I(d)^s = \oplus_{s\geq 0} I^s(sd) \\ & T_i \longmapsto f_i \end{aligned}$$

is onto and corresponds to the embedding $\Gamma \subset \mathbb{P}^m_k \times \mathbb{P}^n_k$.

Let $\mathfrak{P}$ be the kernel of α. Then it is a bihomogeneous ideal of S and the part of degree one of $\mathfrak{P}$ in T_i, denoted by $\mathfrak{P}_1 = \mathfrak{P}_{(*,1)}$, is the module of syzygies of the f_i

$$a_0T_0 + \cdots + a_nT_n \in \mathfrak{P}_1 \Longleftrightarrow a_0f_0 + \cdots + a_nf_n = 0.$$

Set $\mathcal{S}_I := \mathrm{Sym}_R(I)$ for the symmetric algebra of I. The natural bigraded epimorphisms

$$S \longrightarrow S/(\mathfrak{P}_1) \simeq \mathcal{S}_I \quad \text{and} \quad \delta : \mathcal{S}_I \simeq S/(\mathfrak{P}_1) \longrightarrow S/\mathfrak{P} \simeq \mathcal{R}_I$$

correspond to the embeddings of schemes $\Gamma \subset V \subset \mathbb{P}^m_k \times \mathbb{P}^n_k$, where V is the projective scheme defined by $\mathcal{S}_I$.

Let $\mathcal{K}$ be the kernel of δ, one has the following exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{S}_I \longrightarrow \mathcal{R}_I \longrightarrow 0.$$

Notice that the module $\mathcal{K}$ is supported in $\mathcal{B}$ because I is locally trivial outside $\mathcal{B}$.

As the construction of symmetric and Rees algebras commute with localization, and both algebras are the quotient of a polynomial extension of the base ring by the Koszul syzygies on a minimal set of generators in the case of a complete intersection ideal, it follows that Γ and V coincide on $(\mathbb{P}^m_k \setminus X) \times \mathbb{P}^n_k$, where X is the (possibly empty) set of points where $\mathcal{B}$ is not locally a complete intersection.

Now we set $\pi := \pi_{2|\Gamma} : \Gamma \longrightarrow \mathbb{P}^n_k$. For every closed point $y \in \mathbb{P}^n_k$, we will denote by $k(y)$ its residue field, that is, $k(y) = (B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})_0$, where $\mathfrak{p}$ is the defining prime ideal of y. As k is algebraically closed, $k(y) \simeq k$. The fiber of π at $y \in \mathbb{P}^n_k$ is the subscheme

$$\pi^{-1}(y) = \mathrm{Proj}(\mathcal{R}_I \otimes_B k(y)) \subset \mathbb{P}^m_{k(y)} \simeq \mathbb{P}^m_k.$$

Let $0 \leq \ell \leq m$, we define

$$\mathcal{Y}_\ell = \{y \in \mathbb{P}^n_k \mid \dim \pi^{-1}(y) = \ell\} \subset \mathbb{P}^n_k.$$

One is interested in studying the structure of $\mathcal{Y}_\ell$. First, Chevalley's theorem shows that the subsets $\mathcal{Y}_\ell$ are constructible, that is, they can be written as

$$\mathcal{Y}_\ell = \bigsqcup_{i=1}^{s} (U_i \cap Z_i),$$

where U_i (respectively Z_i) are the open (respectively closed) subsets of $\mathbb{P}^n_k$.

Lemma 2.2.1. *Let $\phi : \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k$ be a rational map and Γ be the closure of the graph of ϕ. Consider the canonical projection $\pi : \Gamma \longrightarrow \mathbb{P}^n_k$. Then*

$$\dim\ \overline{\mathcal{Y}_\ell} + \ell \leq m.$$

Furthermore, this inequality is strict for any $\ell > m - \dim\ \mathscr{S}$, where $\mathscr{S}$ is the closed image of ϕ.

Proof. Set $V_\ell := \overline{\pi^{-1}(\mathcal{Y}_\ell)}$, a subvariety of Γ. For the first statement

$$\dim\ \overline{\mathcal{Y}_\ell} + \ell = \dim\ V_\ell \leq \dim\ \Gamma = \dim\ \mathscr{S} \leq m.$$

Moreover, if $\dim\ \overline{\mathcal{Y}_\ell} + \ell = m$, then $\dim\ V_\ell = \dim\ \Gamma = m$. It implies that $\overline{\mathcal{Y}_\ell} = \mathscr{S}$ and proves the second assertion. □

From now on, we will always assume throughout that ϕ is generically finite onto its image, or equivalently that the closed image of ϕ is a subvariety $\mathscr{S}$ in $\mathbb{P}^n_k$ of dimension m. Therefore, by Lemma 2.2.1, $\dim\ \overline{\mathcal{Y}_m} < 0$, which shows that $\mathcal{Y}_m = \emptyset$. This was noticed in [BBC14, Lemma 14]. Now if $\ell = m - 1 \geq 1$, as $m \geq 2$, then $\overline{\mathcal{Y}}_{m-1}$ consists of only a finite number of points in $\mathbb{P}^n_k$. In other words, π only has a finite number of $(m-1)$-dimensional fibers.

For any $y \in \mathcal{Y}_{m-1}$, $\pi^{-1}(y)$ is a subcheme of $\mathbb{P}^m_{k(y)} \simeq \mathbb{P}^m_k$ of dimension $m-1$, as k is algebraically closed. Thus the unmixed part of the fiber $\pi^{-1}(y)$ is defined by a homogeneous polynomial $h_y \in R$, as R is factorial. Our purpose is then to bound $\sum_{y \in \mathcal{Y}_{m-1}} \deg(h_y)$ in the terms of the degree d.

The fibers of π are defined by the specialization of the Rees algebra. However, Rees algebras are hard to study. Fortunately, the symmetric algebra of I is easier to understand than $\mathcal{R}_I$ and the fibers of π are closely related to the fibers of

$$\pi' := \pi_{2|V} : V \longrightarrow \mathbb{P}^n_k.$$

Recall that for any closed point $y \in \mathbb{P}^n_k$, the fiber of π' at y is the subscheme

$$\pi'^{-1}(y) = \operatorname{Proj}(\mathcal{S}_I \otimes_B k(y)) \subset \mathbb{P}^m_{k(y)} \simeq \mathbb{P}^m_k.$$

The next result gives a relation between fibers of π and π' –recall that X is the (possible empty) set of points where $\mathcal{B}$ is not locally a complete intersection.

Lemma 2.2.2. *The fibers of π and π' agree outside X, hence they have the same $(m-1)$-dimensional components.*

Proof. The first statement implies directly from the fact that Γ and V coincide on $(\mathbb{P}^m_k \setminus X) \times \mathbb{P}^n_k$. Moreover, as I is assumed to be of codimension at least two, $\dim\ \mathcal{B} \leq m-2$, showing that $\dim X \leq m-2$. The second statement follows. □

As the symmetric algebra is defined by the syzygies of I. We have the following lemma that is a simple generalization of [BBC14, Lemma 10]. Let us denote by I^{sat} the saturation of I, that is,

$$I^{\text{sat}} := I :_R \mathfrak{m}^\infty = \bigcup_{n \in \mathbb{N}} I :_R \mathfrak{m}^n,$$

where $\mathfrak{m} := (X_0, \ldots, X_m)$ is the irrelevant maximal ideal of R.

Lemma 2.2.3. *Assume that* $\mathbf{f} := f_0, \dots, f_n$ *are* k*-linearly independent such that* $\gcd(f_0, \dots, f_n) = 1$. *Suppose that the fiber of* π' *over a closed point* y *with coordinates* $(p_0 : \cdots : p_n)$ *is of dimension* $m-1$, *and its unmixed component is defined by* $h_y \in R$. *Let* ℓ_y *be a linear form in* $\mathbf{T} := T_0, \dots, T_n$ *such that* $\ell_y(p_0, \dots, p_n) = 1$ *and set* $\ell_i(\mathbf{T}) := T_i - p_i \ell_y(\mathbf{T})$ $(i = 0, \dots, n)$. *Then,* $h_y = \gcd(l_0(\mathbf{f}), \dots, l_n(\mathbf{f}))$ *and*

$$I = (\ell_y(\mathbf{f})) + h_y(g_0, \dots, g_n)$$

with $\ell_i(\mathbf{f}) = h_y g_i$ *and* $\ell_y(g_0, \dots, g_n) = 0$. *In particular,*

$$(\ell_y(\mathbf{f})) + h_y(g_0, \dots, g_n)^{\mathrm{sat}} \subset I^{\mathrm{sat}} \subset (\ell_y(\mathbf{f}), h_y).$$

Proof. A syzygy $L = \sum_{i=0}^n a_i T_i = \sum_{i=0}^n a_i(p_i \ell_y(\mathbf{T}) + \ell_i(\mathbf{T}))$ provides an equation for the fiber: $\overline{L} = \sum_{i=0}^n a_i p_i$. Recall that $h_y = \gcd(\overline{L}_1, \dots, \overline{L}_t)$ where the L_j are generators of the syzygiez of I.

The particular syzygy $L_{(i)} := \ell_i(\mathbf{f})\ell_y(\mathbf{T}) - \ell_y(\mathbf{f})\ell_i(\mathbf{T})$ satisfies $\overline{L}_{(i)} = \ell_i(\mathbf{f})$. It follows that h_y divides $h := \gcd(\ell_0(\mathbf{f}), \dots, \ell_n(\mathbf{f}))$. Set $\ell_i(\mathbf{f}) = h g_i$ for some $g_i \in R$. The ℓ_i's span the linear forms vanishing at y, and are related by the equation $\ell_y(\ell_0, \dots, \ell_n) = 0$. It follows that

$$I = (\ell_y(\mathbf{f}), \ell_0(\mathbf{f}), \dots, \ell_n(\mathbf{f})) = (\ell_y(\mathbf{f})) + h_y(g_0, \dots, g_n),$$

and $\ell_y(g_0, \dots, g_n) = 0$. Now if one has a relation $a\ell_y(\mathbf{f}) + \sum_i b_i h g_i = 0$, then h divides a, as the f_i's have no common factor. This in turn shows that h divides h_y and completes the proof. □

Notice that if $p_i \neq 0$, g_i is a linear combination of the g_j's for $j \neq i$ as $\ell_y(g_0, \dots, g_n) = 0$. For instance if $p_0 \neq 0$, $I = (\ell_y(\mathbf{f})) + h_y(g_1, \dots, g_n)$.

Remark 2.2.4. The above lemma shows that fibers of dimension $m-1$ can only occur when $V(I) \neq \emptyset$ as $V(I) \supset V((\ell_y(\mathbf{f}), h_y))$. It also shows that

$$d \deg(h_y) \leq \deg(I),$$

if there is a $(m-1)$-dimensional fiber with unmixed part given by h_y.

For simplicity, we summarize the following data.

Data 2.2.5. *Let* $R = k[\mathbf{X}] = k[X_0, \dots, X_m]$ $(m \geq 2)$ *be the standard polynomial ring over an algebraically closed field* k *and* $\mathbf{f} := f_0, \dots, f_n$ *be* k*-linearly independent forms of the same degree* $d \geq 1$, *not all zero, such that* $\gcd(f_0, \dots, f_n) = 1$. *Suppose that* $\phi : \mathbb{P}_k^m \dashrightarrow \mathbb{P}_k^n$ *is a rational map defined by these forms and* π *is the canonical projection of the graph of* ϕ *onto its image. Let* $I = (f_0, \dots, f_n)$ *be the base ideal and* $\mathcal{B} = \mathrm{Proj}(R/I)$ *the base locus of* ϕ. *Assume that* ϕ *is generically finite onto its image and put*

$$\mathcal{Y}_{m-1} = \{y \in \mathbb{P}_k^n \mid \dim \pi^{-1}(y) = m-1\} \subset \mathbb{P}_k^n.$$

For each $y \in \mathcal{Y}_{m-1}$, *we denote by* $h_y \in R$ *a defining equation of the unmixed component of the fiber* $\pi^{-1}(y)$.

The following theorem gives an approach for our purpose.

Theorem 2.2.6. *Adopt Data 2.2.5. If there exists an integer s such that $\nu = \operatorname{indeg}((I^s)^{\mathrm{sat}}) < sd$, then*

$$\sum_{y \in \mathcal{Y}_{m-1}} \deg(h_y) \leq \nu < sd.$$

Proof. By Lemma 2.2.2, the unmixed components of $\pi^{-1}(\mathfrak{p})$ and $\pi'^{-1}(\mathfrak{p})$ are the same for every closed point $\mathfrak{p} \in \mathcal{Y}_{m-1}$. By Lemma 2.2.3, there exists a homogeneous polynomial $f \in I$ of degree d such that, for any $y \in \mathcal{Y}_{m-1}$,

$$I = (f) + h_y(g_{1y}, \ldots, g_{ny}) \quad \text{and} \quad I^{\mathrm{sat}} \subseteq (f, h_y)$$

for some $g_{1y}, \ldots, g_{ny} \in R$. Since (f, h_y) is a complete intersection ideal, it follows from Proposition 1.1.12 that $(f, h_y)^s$ is unmixed, hence saturated. Therefore, for all $y \in \mathcal{Y}_{m-1}$,

$$(I^s)^{\mathrm{sat}} \subseteq ((I^{\mathrm{sat}})^s)^{\mathrm{sat}} \subseteq ((f, h_y)^s)^{\mathrm{sat}} = (f, h_y)^s = (f^s, f^{s-1}h_y, \ldots, h_y^s).$$

Now, let $0 \neq F \in (I^s)^{\mathrm{sat}}$ such that $\deg(F) = \nu < sd$, then h_y is a divisor of F. Moreover, if $y \neq y'$ in $\mathcal{Y}_{m-1}$, then $\gcd(h_y, h_{y'}) = 1$. We deduce that

$$\prod_{y \in \mathcal{Y}_{m-1}} h_y \mid F$$

which gives

$$\sum_{y \in \mathcal{Y}_{m-1}} \deg(h_y) \leq \deg(F) = \nu < sd.$$

□

In particular, if $\operatorname{indeg}(I^{\mathrm{sat}}) = \delta < d$, Theorem 2.2.6 shows that

$$\sum_{y \in \mathcal{Y}_{m-1}} \deg(h_y) \leq \delta < d.$$

Example 2.2.7. Let $\phi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^3_k$ be a parameterization of surface defined by

$$\begin{array}{ll} f_0 = X_1^2 X_2^2 (X_1^2 - X_2^2) & f_2 = X_0^2 X_1^2 (X_1^2 - X_2^2) \\ f_1 = X_2^2 (X_0^4 - X_2^4) & f_3 = X_1^2 (X_0^4 - X_2^4). \end{array}$$

Using `Macaulay2` [GS], we see that I is a saturated ideal with $\mathcal{B}$ locally a complete intersection of dimension zero. Since $\operatorname{indeg}((I^2)^{\mathrm{sat}}) = 8 < 2.6 = 12$, Theorem 2.2.6 shows that

$$\sum_{y \in \mathcal{Y}_1} \deg(h_y) \leq 8.$$

Precisely, I admits a minimal free resolution of the form

$$0 \longrightarrow R(-8)^3 \xrightarrow{M} R(-6)^4 \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

where the matrix M is given by

$$\begin{pmatrix} -X_2^2 & 0 & X_0^2 \\ 0 & X_1^2 & 0 \\ X_0^2 & 0 & -X_2^2 \\ -X_1^2+X_2^2 & -X_2^2 & 0 \end{pmatrix}.$$

We thus get $\mathcal{Y}_1 = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4\}$ with

$$\begin{array}{llll} y_1 = (1:0:1:0) & h_{y_1} = X_0^2 - X_2^2 & y_2 = (0:1:0:1) & h_{y_2} = X_1^2 - X_2^2 \\ y_3 = (1:0:-1:0) & h_{y_3} = X_0^2 + X_2^2 & y_4 = (0:1:0:0) & h_{y_4} = X_1^2 \end{array}$$

which gives

$$\sum_{y \in \mathcal{Y}_1} \deg(h_y) = 8.$$

2.3 Estimation of number of one-dimensional fibers of parameterization surfaces

In this section, we consider a rational map $\phi : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^3$ given by four homogeneous polynomials $f_0, \ldots, f_3 \in R = k[X_0, X_1, X_2]$ of the same degree $d \geq 1$. We will always assume throughout that ϕ is generically finite onto its image, or equivalently that the closed image of ϕ is a surface $\mathscr{S}$ in $\mathbb{P}_k^3$, and that the base locus $\mathcal{B} = \mathrm{Proj}(R/I) \subset \mathbb{P}_k^2$ of ϕ is supported on a finite set of points.

From now on we assume that the ideal I is locally a complete intersection outside $V(\mathfrak{m})$, that is, $\mathcal{B}$ is locally a complete intersection. Under this hypothesis, the module $\mathcal{K}$ is supported in $\mathfrak{m}S$, hence $H_{\mathfrak{m}}^i(\mathcal{K}) = 0$ for any $i \geq 1$. We deduce that

$$H_{\mathfrak{m}}^i(\mathcal{S}_I) \simeq H_{\mathfrak{m}}^i(\mathcal{R}_I), \ \forall i \geq 1.$$

Our main purpose in this section is to give a bound for $\sum_{y \in \mathcal{Y}_1} \deg(h_y)$ in the terms of d. First, we recall a result of Botbol, Busé, and Chardin in [BBC14, Lemma 4].

Lemma 2.3.1. *Let J be a graded ideal in R generated in degree d such that* $\dim(R/J) \leq 1$ *and $\mu(J_{\mathfrak{p}}) \leq 3$ for every prime ideal $\mathfrak{m} \supsetneq \mathfrak{p} \supset J$. Then,*

$$\mathrm{reg}(R/J^{\mathrm{sat}}) \leq 2d - 3$$

and

$$\mathrm{reg}(R/J) \leq 3d - 3 - \mathrm{indeg}(J^{\mathrm{sat}})$$

unless $\mu(J) = 2$, in which case $J = J^{\mathrm{sat}}$ and $\mathrm{reg}(R/J) = 2d - 2$.

Let $K_\bullet := K_\bullet(\mathbf{f}; R)$ be the Koszul complex associated to the sequence of polynomials $\mathbf{f} := f_0, f_1, f_2, f_3$ in R. We denote by $Z_i := Z_i(\mathbf{f}; R), B_i := B_i(\mathbf{f}; R), H_i := H_i(\mathbf{f}; R) = Z_i/B_i$ the Koszul cycle, the Koszul boundary and the Koszul homology in degree i, respectively. Since the ideal I is homogeneous, these modules inherit a natural structure of graded R-modules.

Let $\mathcal{Z}_\bullet$ be the $\mathcal{Z}$-complex associated to the ideal $I = (\mathbf{f})$. By definition

$$\mathcal{Z}_q = Z_q[qd] \otimes_R R[T_0, \dots, T_3](-q)$$

for all $q = 0, \dots, 3$. This complex is of the form

$$\mathcal{Z}_\bullet : 0 \longrightarrow \mathcal{Z}_3 \xrightarrow{\partial_3^{\mathbf{T}}} \mathcal{Z}_2 \xrightarrow{\partial_2^{\mathbf{T}}} \mathcal{Z}_1 \xrightarrow{\partial_1^{\mathbf{T}}} \mathcal{Z}_0 = R[T_0, \dots, T_3] \longrightarrow 0,$$

where $\partial_1^{\mathbf{T}}(a_0, a_1, a_2, a_3) = a_0T_0 + \cdots + a_3T_3$. As $\mathcal{B}$ is assumed to be locally a complete intersection, it follows from Theorem 1.5.21 that the complex $\mathcal{Z}_\bullet$ is acyclic, and it is thus a resolution of $H_0(\mathcal{Z}_\bullet) \simeq \mathcal{S}_I$.

As we already noticed that $\sum_{y \in \mathcal{Y}_1} \deg(h_y) < d$ if $\operatorname{indeg}(I^{\mathrm{sat}}) < d$, the delicate case is when the ideal I satisfies $\operatorname{indeg}(I^{\mathrm{sat}}) = \operatorname{indeg}(I) = d$. This condition implies $d \geq 3$. Indeed, by Lemma 2.3.1, $d = \operatorname{indeg}(I^{\mathrm{sat}}) \leq \operatorname{reg}(I^{\mathrm{sat}}) = \operatorname{reg}(R/I^{\mathrm{sat}}) + 1 \leq 2d - 2$, hence $d \geq 2$. Moreover, it is known that in [HW14, Proposition 3.5] that if $d = 2$, then $\operatorname{indeg}(I^{\mathrm{sat}}) = 1 < 2 = \operatorname{indeg}(I)$.

Theorem 2.3.2. *Let $B = k[T_0, \dots, T_3]$ be the standard graded polynomial ring over a field k and $I = (f_0, \dots, f_3)$ be a homogeneous ideal of R. Suppose that $\mathcal{B} = \operatorname{Proj}(R/I)$ is locally a complete intersection of dimension zero and $\operatorname{indeg}(I^{\mathrm{sat}}) = \operatorname{indeg}(I) = d$. Then there exists a complex $(C_\bullet)$ of free B-modules*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B(-3)^l & \longrightarrow & B(-2)^m & \longrightarrow & B(-1)^n & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & C_3 & & C_2 & & C_1 & & \end{array}$$

where $n = \dim_k H^3_{\mathfrak{m}}(Z_1)_{d-1}$, $m = \dim_k H^3_{\mathfrak{m}}(Z_2)_{2d-1}$, $l = \dim_k H^3_{\mathfrak{m}}(Z_3)_{3d-1}$ with homologies

$$H_1(C_\bullet) \simeq \oplus_{s \geq 0} H^2_{\mathfrak{m}}(I^s)_{sd-1}, \quad H_2(C_\bullet) \simeq \oplus_{s \geq 0} (I^s)^{\mathrm{sat}}_{sd-1}, \quad H_3(C_\bullet) = 0.$$

Proof. By Proposition 1.5.13, one has the following graded (degree zero) isomorphisms of R-modules:

$$H^p_{\mathfrak{m}}(Z_q) \simeq \begin{cases} 0 & \text{for } p = 0, 1 \\ H^0_m(H_{p-q})^*[3 - 4d] & \text{for } p = 2 \\ Z^*_{p-q}[3 - 4d] & \text{for } p = 3 \end{cases} \tag{2.1}$$

where $-^* := {}^*\mathrm{Hom}_R(-, k)$.

Let $C^\bullet_{\mathfrak{m}}(M)$ denote the Čech complex on M relatively to the ideal $\mathfrak{m}$. We consider

the two spectral sequences associated to the double complex $C^\bullet_{\mathfrak{m}}(\mathcal{Z}_\bullet)$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C^0_{\mathfrak{m}}(\mathcal{Z}_3) \arrow[r] \arrow[d] & C^0_{\mathfrak{m}}(\mathcal{Z}_2) \arrow[r] \arrow[d] & C^0_{\mathfrak{m}}(\mathcal{Z}_1) \arrow[r] \arrow[d] & C^0_{\mathfrak{m}}(\mathcal{Z}_0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^1_{\mathfrak{m}}(\mathcal{Z}_3) \arrow[r] \arrow[d] & C^1_{\mathfrak{m}}(\mathcal{Z}_2) \arrow[r] \arrow[d] & C^1_{\mathfrak{m}}(\mathcal{Z}_1) \arrow[r] \arrow[d] & C^1_{\mathfrak{m}}(\mathcal{Z}_0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^2_{\mathfrak{m}}(\mathcal{Z}_3) \arrow[r] \arrow[d] & C^2_{\mathfrak{m}}(\mathcal{Z}_2) \arrow[r] \arrow[d] & C^2_{\mathfrak{m}}(\mathcal{Z}_1) \arrow[r] \arrow[d] & C^2_{\mathfrak{m}}(\mathcal{Z}_0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^3_{\mathfrak{m}}(\mathcal{Z}_3) \arrow[r] \arrow[d] & C^3_{\mathfrak{m}}(\mathcal{Z}_2) \arrow[r] \arrow[d] & C^3_{\mathfrak{m}}(\mathcal{Z}_1) \arrow[r] \arrow[d] & C^3_{\mathfrak{m}}(\mathcal{Z}_0) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Since $\mathcal{Z}_\bullet$ is acyclic, one of them abuts at step two with:

$$^{\infty}\mathbf{E}^{-p,-q}_{\text{hor}} = {}^{2}\mathbf{E}^{-p,-q}_{\text{hor}} = \begin{cases} H^p_{\mathfrak{m}}(\mathcal{S}_I) & \text{for} \quad q = 0 \\ 0 & \text{for} \quad q \neq 0. \end{cases} \tag{2.2}$$

The other one gives at step one:

$$^{1}\mathbf{E}^{-p,-q}_{\text{ver}} = H^p_{\mathfrak{m}}(\mathcal{Z}_q) = H^p_{\mathfrak{m}}(Z_q)[qd] \otimes_R R[T_0, \ldots, T_3](-q) = H^p_{\mathfrak{m}}(Z_q)[qd] \otimes_k B(-q).$$

Notice that $H^p_{\mathfrak{m}}(Z_q) = 0$ for $p = 0, 1$; $Z_3 \simeq R[-4d]$ and $Z_0 = R$. Therefore the first page of the vertical spectral sequence has only two nonzero lines

$$0 \longrightarrow H^2_{\mathfrak{m}}(Z_2)[2d] \otimes_k B(-2) \to H^2_{\mathfrak{m}}(Z_1)[d] \otimes_k B(-1) \longrightarrow 0$$

$$H^3_{\mathfrak{m}}(Z_3)[3d] \otimes_k B(-3) \to H^3_{\mathfrak{m}}(Z_2)[2d] \otimes_k B(-2) \to H^3_{\mathfrak{m}}(Z_1)[d] \otimes_k B(-1) \to H^3_{\mathfrak{m}}(Z_0) \otimes_k B.$$

In bidegree $(-1, *)$, we have $H^3_{\mathfrak{m}}(Z_0)_{-1} \otimes_k B = H^3_{\mathfrak{m}}(R)_{-1} \otimes_k B = 0$. Therefore, we obtain the complex $(C_\bullet)$ of free B-modules

$$\begin{tikzcd}
0 \arrow[r] & B(-3)^l \arrow[r] \arrow[d, equal] & B(-2)^m \arrow[r] \arrow[d, equal] & B(-1)^n \arrow[r] \arrow[d, equal] & 0. \\
 & C_3 & C_2 & C_1 &
\end{tikzcd}$$

Let us compute the homology of $(C_\bullet)$. The exact sequence

$$0 \longrightarrow Z_1 \longrightarrow R[-d]^4 \longrightarrow I \longrightarrow 0 \tag{2.3}$$

gives

$$H^2_{\mathfrak{m}}(Z_1) \simeq H^1_{\mathfrak{m}}(I) \simeq H^0_{\mathfrak{m}}(R/I) \simeq I^{\text{sat}}/I.$$

Since $\operatorname{indeg}(I^{\mathrm{sat}}) = \operatorname{indeg}(I) = d$, we have

$$H^2_{\mathfrak{m}}(Z_1)_{d-1} \simeq (I^{\mathrm{sat}}/I)_{d-1} = 0. \tag{2.4}$$

It follows from (2.1) that $H^2_{\mathfrak{m}}(Z_2)_{2d-1} \simeq H^0_{\mathfrak{m}}(H_0)^*_{2-2d} \simeq \operatorname{Hom}_R((I^{\mathrm{sat}}/I)_{2d-2}, k)$ as $H^0_{\mathfrak{m}}(H_0) \simeq I^{\mathrm{sat}}/I$. It implies from Lemma 2.3.1 that $\operatorname{reg}(R/I) \leq 2d-3$, therefore $I_{2d-2} = (I^{\mathrm{sat}})_{2d-2}$, thus

$$H^2_{\mathfrak{m}}(Z_2)_{2d-1} = 0. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$({}^1\mathbf{E}^{-p,-q}_{\mathrm{ver}})_{(-1,*)} = 0 \quad \text{unless} \quad p = 3 \ \text{ and } q \in \{1,2,3\}.$$

Therefore, in bidegree $(-1,*)$,

$$ {}^{\infty}\mathbf{E}^{-p,-q}_{\mathrm{ver}} = {}^{2}\mathbf{E}^{-p,-q}_{\mathrm{ver}} = \begin{cases} H_q(C_\bullet) & \text{if } p = 3 \text{ and } q = 1,2,3 \\ 0 & \text{else.} \end{cases} \tag{2.6}$$

By comparing the two spectral sequences (2.2) and (2.6), one has:

$$\begin{aligned}
H_1(C_\bullet) &= H^2_{\mathfrak{m}}(\mathcal{S}_I)_{(-1,*)} = \oplus_{s\geq 0} H^2_{\mathfrak{m}}(I^s)_{sd-1}, \\
H_2(C_\bullet) &= H^1_{\mathfrak{m}}(\mathcal{S}_I)_{(-1,*)} = \oplus_{s\geq 0} H^1_{\mathfrak{m}}(I^s)_{sd-1} = \oplus_{s\geq 0} H^0_{\mathfrak{m}}(R/I^s)_{sd-1} = \oplus_{s\geq 0}((I^s)^{\mathrm{sat}}/I^s)_{sd-1} \\
&= \oplus_{s\geq 0}(I^s)^{\mathrm{sat}}_{sd-1} \text{ since } I^s_{sd-1} = 0, \\
H_3(C_\bullet) &= H^0_{\mathfrak{m}}(\mathcal{S}_I)_{(-1,*)} \subseteq (\mathcal{S}_I)_{(-1,*)} \subseteq R_{-1} \otimes_k B = 0.
\end{aligned}$$

□

We now give an expression of m, n, l in terms of the degree d and $\deg(\mathcal{B})$.

Lemma 2.3.3. *Under the assumptions of Theorem 2.3.2,*

$$n = \deg(\mathcal{B}) - \frac{1}{2}d(d+1), \ m = \deg(\mathcal{B}) - d \text{ and } l = \frac{1}{2}d(d-1).$$

As a consequence, $n - m + l = 0$.

Proof. We first compute $n = \dim_k H^3_{\mathfrak{m}}(Z_1)_{d-1}$. From (2.3), we derive an exact sequence

$$0 \longrightarrow H^2_{\mathfrak{m}}(I) \longrightarrow H^3_{\mathfrak{m}}(Z_1) \longrightarrow H^3_{\mathfrak{m}}(R)[-d]^4 \longrightarrow 0,$$

hence $H^3_{\mathfrak{m}}(Z_1)_{d-1} \simeq H^2_{\mathfrak{m}}(I)_{d-1}$. We get

$$n = \dim_k H^2_{\mathfrak{m}}(I)_{d-1} = \dim_k H^1_{\mathfrak{m}}(R/I)_{d-1}.$$

We now consider an exact sequence

$$0 \longrightarrow R/I^{\mathrm{sat}} \longrightarrow \bigoplus_{\mu\in\mathbb{Z}} H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathcal{B}}(\mu)) \longrightarrow H^1_{\mathfrak{m}}(R/I) \longrightarrow 0,$$

hence

$$\dim_k H^1_{\mathfrak{m}}(R/I)_{d-1} = \dim_k H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathcal{B}}(d-1)) - \dim_k (R/I^{\mathrm{sat}})_{d-1} = \deg(\mathcal{B}) - \dim_k (R/I^{\mathrm{sat}})_{d-1}.$$

Since $\operatorname{indeg}(I^{\mathrm{sat}}) = d$, $\dim_k (R/I^{\mathrm{sat}})_{d-1} = \dim_k R_{d-1} = \binom{d+1}{2} = \frac{1}{2}d(d+1)$. It follows that

$$n = \deg(\mathcal{B}) - \frac{1}{2}d(d+1).$$

We next compute $m = \dim_k H^3_{\mathfrak{m}}(Z_2)_{2d-1}$. It follows from (2.1) that

$$H^3_{\mathfrak{m}}(Z_2)_{2d-1} \simeq (Z_1)^*_{2-2d} \simeq \operatorname{Hom}_R((Z_1)_{2d-2}, k).$$

Therefore, by (2.3),

$$m = \dim_k (Z_1)_{2d-2} = \dim_k R^4_{d-2} - \dim_k I_{2d-2} = 4\binom{d}{2} - \dim_k I_{2d-2}.$$

Since $\dim_k I_{2d-2} = \dim_k R_{2d-2} - \dim_k (R/I)_{2d-2} = \binom{2d}{2} - \deg(\mathcal{B})$, it follows that

$$m = 4\binom{d}{2} - \binom{2d}{2} + \deg(\mathcal{B}) = \deg(\mathcal{B}) - d.$$

Finally, we have

$$l = \dim_k H^3_{\mathfrak{m}}(Z_3)_{3d-1} = \dim_k H^3_{\mathfrak{m}}(R)_{-d-1} = \binom{d}{2} = \frac{1}{2}d(d-1).$$

□

Now we will give a bound for the degree of the projective variety $\mathcal{B} \subset \mathbb{P}^2_k$ in terms of the degree d.

Lemma 2.3.4. *Under the assumptions of Theorem 2.3.2,*

$$\frac{1}{2}d(d+1) \leq \deg(\mathcal{B}) \leq d^2 - 2d + 3.$$

Proof. The Hilbert function of R/I^{sat} is weakly increasing, hence for any μ

$$\dim_k (R/I^{\mathrm{sat}})_\mu \leq \deg(\mathcal{B}).$$

Since $\operatorname{indeg}(I^{\mathrm{sat}}) = d, \deg(\mathcal{B}) \geq \dim_k (R/I^{\mathrm{sat}})_{d-1} = \dim_k R_{d-1} = \frac{1}{2}d(d+1)$.

We can assume that k is infinite. Since $\operatorname{codim}(I) = 2$, there are homogeneous polynomials $g_1, g_2 \in I$ of degree d which form a regular sequence. Therefore $\mathfrak{b} := (g_1, g_2)$ is a complete intersection ideal. Setting $J := \mathfrak{b} :_R I$, J is saturated and $I^{\mathrm{sat}} = \mathfrak{b} :_R J$ since R is Gorenstein, by Proposition 1.1.21.

We set $\mathcal{Q} = \operatorname{Proj}(R/J)$ and $\mathcal{X} = \operatorname{Proj}(R/\mathfrak{b})$. By Proposition 1.3.11

$$\deg(\mathcal{B}) + \deg(\mathcal{Q}) = \deg(\mathcal{X}),$$

hence $\deg(\mathcal{B}) = d^2 - \deg(\mathcal{Q})$.

Proposition 1.1.3 and Corollary 1.2.50 show that

$$\begin{aligned} H^1_{\mathfrak{m}}(R/J) &\simeq \operatorname{Ext}^2_R(R/J, R[-3])^* \simeq \Big(\operatorname{Hom}_R(R/J, R/\mathfrak{b})[2d-3]\Big)^* \\ &\simeq \Big((\mathfrak{b} :_R J)/\mathfrak{b}\Big)^*[3-2d] \simeq (I^{\mathrm{sat}}/\mathfrak{b})^*[3-2d]. \end{aligned}$$

We thus get

$$H^1_{\mathfrak{m}}(R/J)_\mu \simeq \operatorname{Hom}_R((I^{\mathrm{sat}}/\mathfrak{b})_{2d-3-\mu}, k).$$

Since $\operatorname{indeg}(I^{\mathrm{sat}}) = \operatorname{indeg}(\mathfrak{b}) = d$, we obtain

$$H^1_{\mathfrak{m}}(R/J)_\mu = \begin{cases} 0 & \text{if } \mu > d-3 \\ \operatorname{Hom}_R((I^{\mathrm{sat}}/\mathfrak{b})_d, k) \neq 0 & \text{if } \mu = d-3. \end{cases}$$

Moreover, we have $H^0_{\mathfrak{m}}(R/J) = J^{\mathrm{sat}}/J = 0$. We therefore conclude that

$$\operatorname{reg}(R/J) = d-2,$$

and hence $\dim_k(R/J)_{d-2} = \deg(\mathcal{Q})$, by Theorem 1.3.9.

On the other hand, the Hilbert function $HF_{R/J}(\mu)$ of R/J strictly increases from 1 in degree 0 until $\deg(\mathcal{Q})$ in degree $d-2$. Therefore one has $\deg(\mathcal{Q}) \geq d-1$. By Theorem 1.3.9

$$HF_{R/J}(\mu) = \deg(\mathcal{Q}) - \dim_k H^1_{\mathfrak{m}}(R/J)_\mu,$$

for any μ. Since $\dim_k H^1_{\mathfrak{m}}(R/J)_{d-3} = \dim_k(I^{\mathrm{sat}}/\mathfrak{b})_d \geq 2$, we deduce $\deg(\mathcal{Q}) \geq d$.

The proof will be completed by showing that $\deg(\mathcal{Q}) \geq 2d-3$. Set $c_i = \dim_k(R/J)_i$ for all $i \geq 0$. The sequence $\underline{c} = \{c_i : i \geq 0\}$ satisfies the following properties:

(i) $c_0 = 1$ and $c_1 = 3$. Indeed, if $c_1 < 3$, then $J_1 \neq 0$, there is therefore a linear form $\ell \in J_1$. We assume that $J = (\ell, w_1, w_2, \ldots, w_s)$. Since $\operatorname{reg}(J) = d-1$, we have $\deg(w_i) \leq d-1$ for all $i = 1, \ldots s$. Fix $i \in \{1, \ldots, s\}$ such that $\gcd(\ell, w_i) = 1$. Since $\mathcal{Q} \subseteq V(\ell, w_i)$, hence $\deg(\mathcal{Q}) \leq d-1$, a contradiction.

(ii) $c_i = \deg(\mathcal{Q})$ for any $i \geq d-2$ and $c_{d-2} - c_{d-3} = \dim_k H^1_{\mathfrak{m}}(R/J)_{d-3} \geq 2$.

Since J is saturated and k is infinite, there is a linear form ℓ in R such that $J : \ell \simeq J(-1)$ as graded R-modules. It follows that ℓ induces an injection $\times\ell : (R/J)[-1] \longrightarrow R/J$. Therefore the Hilbert function of $R/(J+\ell)$ is $\underline{b} = \{b_i : i \geq 0\}$ with $b_0 = 1$ and $b_i = c_i - c_{i-1}$ for every $i \geq 1$.

By (ii), we have $b_{d-2} \geq 2$. We will show that $b_i \geq 2$, for any $i = 1, \ldots, d-2$. Indeed, suppose that there exists an integer $i \geq 1$ such that $b_i = 1$, and let

$$\nu = \sup_{i \geq 1}\{i \mid b_i = 1\}.$$

One has $2 \leq \nu \leq d-3$ and since $R/(J+\ell)$ is a standard graded k-algebra, Theorem 1.3.20 shows that $\underline{b} = \{b_i : i \geq 0\}$ is a O-sequence. Therefore

$$2 \leq b_{\nu+1} \leq b_\nu^{<\nu>} = 1^{<\nu>} = 1.$$

This contradiction shows that $b_i \geq 2$, for any $i = 1, \ldots, d-2$, which is equivalent to $c_{i+1} \geq c_i + 2$ for all $i = 0, \ldots, d-3$, hence

$$\deg(\mathcal{Q}) \geq 2(d-2) + 1 = 2d - 3.$$

□

Proposition 2.3.5. *Under the assumptions of Theorem 2.3.2, the following holds for any integer s:*

$$\dim_k H^2_{\mathfrak{m}}(I^s)_{sd-1} = \frac{1}{2}s(s+1)\deg(\mathcal{B}) - \frac{1}{2}sd(sd+1) + \dim_k (I^s)^{\text{sat}}_{sd-1}.$$

Proof. By Theorem 2.3.2, for any integer s, one has

$$\begin{aligned}
\dim_k H^2_{\mathfrak{m}}(I^s)_{sd-1} &= n.\dim_k B_{s-1} - m.\dim_k B_{s-2} + l.\dim_k B_{s-3} + \dim_k (I^s)^{\text{sat}}_{sd-1} \\
&= n\binom{s+2}{3} - (n+l)\binom{s+1}{3} + l\binom{s}{3} + \dim_k (I^s)^{\text{sat}}_{sd-1} \\
&= n\binom{s+1}{2} - l\binom{s}{2} + \dim_k (I^s)^{\text{sat}}_{sd-1}.
\end{aligned}$$

Using Lemma 2.3.3, we have

$$\begin{aligned}
\dim_k H^2_{\mathfrak{m}}(I^s)_{sd-1} &= \Big(\deg(\mathcal{B}) - \frac{1}{2}d(d+1)\Big)\binom{s+1}{2} - \frac{1}{2}d(d-1)\binom{s}{2} + \dim_k (I^s)^{\text{sat}}_{sd-1}, \\
&= \frac{1}{2}s(s+1)\deg(\mathcal{B}) - \frac{1}{2}sd(sd+1) + \dim_k (I^s)^{\text{sat}}_{sd-1}.
\end{aligned}$$

□

Corollary 2.3.6. *Under the assumptions of Theorem 2.3.2,* $\operatorname{indeg}((I^s)^{\text{sat}}) < sd$ *for all integer* $s \geq \frac{d}{2}$.

Proof. If there exist an integer $s \geq \frac{d}{2}$ such that $\operatorname{indeg}((I^s)^{\text{sat}}) = sd$. Then $(I^s)^{\text{sat}}_{sd-1} = 0$. By Proposition 2.3.5,

$$\begin{aligned}
\dim_k H^2_{\mathfrak{m}}(I^s)_{sd-1} &= \frac{1}{2}s(s+1)\deg(\mathcal{B}) - \frac{1}{2}sd(sd+1) \\
&\leq \frac{1}{2}s(s+1)(d^2 - 2d + 3) - \frac{1}{2}sd(sd+1) \quad \text{by Lemma 2.3.4} \\
&\leq \frac{1}{2}s(d^2 - 2sd - 3d + 3s + 3) \\
&\leq \frac{1}{2}s[(d - \frac{3}{2})(d - 2s) - \frac{3}{2}(d-2)].
\end{aligned}$$

We already noticed at beginning of this section that $d \geq 3$ and by hypothesis $s \geq \frac{d}{2}$, $\dim_k H^2_{\mathfrak{m}}(I^s)_{sd-1} < 0$, which is impossible. □

As we already noticed that the interested cases are that the locus ideal of ϕ satisfying $\operatorname{indeg}(I) = \operatorname{indeg}(I^{\text{sat}}) = d \geq 3$. In the case $d = 3$, using Theorem 2.2.6 and Corollary 2.3.6, we have $\sum_{y \in \mathcal{Y}_1} \deg(h_y) < 6$. However, the following result gives a sharp bound for $\sum_{y \in \mathcal{Y}_1} \deg(h_y)$ in this case.

Theorem 2.3.7. *Adopt Data 2.2.5. Assume further that* $m = n - 1 = 2$, *that* $\operatorname{indeg}(I^{\mathrm{sat}}) = d = 3$ *and that* $\mathcal{B}$ *is locally a complete intersection. Then*

$$\sum_{y \in \mathcal{Y}_1} \deg(h_y) \leq 4.$$

Proof. We will prove first that $I = I^{\mathrm{sat}}$. In this case, $\deg(\mathcal{B}) = 6$ by Lemma 2.3.4. Moreover, with the notations in the proof of Lemma 2.3.4, we have

$$H^1_{\mathfrak{m}}(R/I^{\mathrm{sat}}) \simeq (J/\mathfrak{b})^*[-3]$$

and $\operatorname{reg}(R/J) = 1$. Hence $\dim_k (R/J)_\mu = \deg(\mathcal{Q}) = 3$ for all $\mu \geq 1$. It follows that

$$\dim_k H^1_{\mathfrak{m}}(R/I^{\mathrm{sat}})_\mu = 0, \ \forall \mu \geq 2, \ \dim_k H^1_{\mathfrak{m}}(R/I^{\mathrm{sat}})_1 = 3.$$

Therefore $\operatorname{reg}(R/I^{\mathrm{sat}}) = 2$, it follows that I^{sat} is generated in degree at most 3. Since $\operatorname{indeg}(I^{\mathrm{sat}}) = 3$, it is immediate that I^{sat} is exactly generated in degree 3. Furthemore,

$$\dim_k (I^{\mathrm{sat}})_3 = \dim_k R_3 - \dim_k (R/I^{\mathrm{sat}})_3 = 10 - 6 = 4,$$

which implies $I^{\mathrm{sat}} = I$.

By the Hilbert-Burch Theorem, I admits a free resolution of the form

$$0 \longrightarrow R(-4)^3 \xrightarrow{M} R(-3)^4 \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

By changing the coordinates, there is no loss of generality in assuming the matrix M be of the form

$$M = \begin{pmatrix} X_0 & 0 & 0 \\ a_1 X_1 & X_1 & 0 \\ b_1 X_2 & b_2 X_2 & X_2 \\ c_1 L & c_2 L & c_3 L \end{pmatrix}$$

where $L = \alpha_1 X_0 + \alpha_2 X_1 + \alpha_3 X_2$. One has $c_3 \neq 0$ since $f_i \neq 0$ for all $i = 0, 1, 2, 3$. Since I is locally a complete intersection, $V(I_2(M)) = \emptyset$, where $I_2(M)$ is the ideal of 2-minors of M. It follows that $\alpha_1 \alpha_2 \alpha_3 \neq 0$.

Let now a fiber over a closed point y of coordinates $(\lambda_1 : \lambda_2 : \lambda_3 : \lambda_4) \in \mathbb{P}^3_k$ be of dimension 1 and its unmixed component be defined by $h_y \in R$. It follows from Lemma 2.2.3 that

$$h_y = \gcd(\ell_1, \ell_2, \ell_3),$$

where

$$\begin{aligned} \ell_1 &= (\lambda_1 + c_1 \alpha_1 \lambda_4) X_0 + (a_1 \lambda_2 + c_1 \alpha_2 \lambda_4) X_1 + (b_1 \lambda_3 + c_1 \alpha_3 \lambda_4) X_2, \\ \ell_2 &= c_2 \alpha_1 \lambda_4 X_0 + (\lambda_2 + c_2 \alpha_2 \lambda_4) X_1 + (b_2 \lambda_3 + c_2 \alpha_3 \lambda_4) X_2, \\ \ell_3 &= c_3 \alpha_1 \lambda_4 X_0 + c_3 \alpha_2 \lambda_4 X_1 + (\lambda_3 + c_3 \alpha_3 \lambda_4) X_2. \end{aligned}$$

• If $\lambda_4 = 0$, then there are the following cases:

Case 1: $\lambda_3 \neq 0$. In this case we have $\lambda_1 = \lambda_2 = 0$. Therefore $y = (0 : 0 : 1 : 0)$ and $h_y = X_2$.

Case 2: $\lambda_3 = 0$ and $\lambda_2 \neq 0$. It follows that $\lambda_1 = 0$. Thus $y = (0:1:0:0)$ and $h_y = X_1$.

Case 3: $\lambda_3 = 0$ and $\lambda_2 = 0$. We must have $\lambda_1 \neq 0$. Hence $y = (1:0:0:0)$ and $h_y = X_0$.

- If $\lambda_4 = 1$, then from three polynomials ℓ_1, ℓ_2, ℓ_3 have a common divisor, we have

$$\frac{\lambda_1 + c_1\alpha_1}{c_3\alpha_1} = \frac{a_1\lambda_2 + c_1\alpha_2}{c_3\alpha_2} = \frac{b_1\lambda_3 + c_1\alpha_3}{\lambda_3 + c_3\alpha_3},$$
$$\frac{c_2}{c_3} = \frac{\lambda_2 + c_2\alpha_2}{c_3\alpha_2} = \frac{b_2\lambda_3 + c_2\alpha_3}{\lambda_3 + c_3\alpha_3}.$$

These equalities imply that $\lambda_1 = \lambda_2 = 0$. Furthermore, we also have $\lambda_3 = 0$. Indeed, if $\lambda_3 \neq 0$, then $c_1 = b_1c_3$ and $c_2 = b_2c_3$, contrary to $f_i \neq 0$ for all $i = 0, \ldots, 3$. Therefore $y = (0:0:0:1)$ and $h_y = L$. □

Example 2.3.8. Let $\phi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^3_k$ be a parameterization of surface defined by

$$f_0 = X_1X_2(X_0 + X_1 + X_2) \qquad f_2 = X_0X_1(X_0 + X_1 + X_2)$$
$$f_1 = X_0X_2(X_0 + X_1 + X_2) \qquad f_3 = X_0X_1X_2.$$

Using `Macaulay2` [GS], we see that $\mathcal{B}$ is locally a complete intersection of degree 6 and that I is a saturated ideal with free resolution

$$0 \longrightarrow R(-4)^3 \xrightarrow{M} R(-3)^4 \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

where the matrix M is given by

$$\begin{pmatrix} 0 & 0 & X_0 \\ X_1 & -X_1 & 0 \\ -X_2 & 0 & 0 \\ 0 & X_0 + X_1 + X_2 & -(X_0 + X_1 + X_2) \end{pmatrix}.$$

Thus we obtain $\mathcal{Y}_1 = \{y_1, y_2, y_3, y_4\}$ with

$$y_1 = (1:0:0:0) \quad h_{y_1} = X_0 \quad y_2 = (0:1:0:0) \quad h_{y_2} = X_1$$
$$y_3 = (0:0:1:0) \quad h_{y_3} = X_2 \quad y_4 = (0:0:0:1) \quad h_{y_4} = X_0 + X_1 + X_2.$$

Consequently, we have

$$\sum_{y \in \mathcal{Y}_1} \deg(h_y) = 4.$$

Now, we prove the main result of this chapter.

Theorem 2.3.9. *Adopt Data 2.2.5. Assume further that $m = n - 1 = 2$, that $\operatorname{indeg}(I^{\mathrm{sat}}) = d \geq 4$ and that $\mathcal{B}$ is locally a complete intersection. Then*

$$\sum_{y \in \mathcal{Y}_1} \deg(h_y) \leq \left\lfloor \frac{d}{2} \right\rfloor d - 1,$$

where $\lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \leq x\}$.

Proof. Set $\nu = \lfloor \frac{d}{2} \rfloor$. By Theorem 2.2.6, it suffies to show that $(I^\nu)^{\mathrm{sat}}_{\nu d-1} \neq 0$. Suppose on the contrary that $(I^\nu)^{\mathrm{sat}}_{\nu d-1} = 0$. By Proposition 2.3.5, one has

$$\dim_k H^2_{\mathfrak{m}}(I^\nu)_{\nu d-1} = \frac{1}{2}\nu(\nu+1)\deg(\mathcal{B}) - \frac{1}{2}\nu d(\nu d+1).$$

We now show that this is not possible by using Lemma 2.3.4: $\deg(\mathcal{B}) \leq d^2 - 2d + 3$. Indeed, consider two cases:

(i) If $d = 2s$ with $s \geq 2$, then $\nu = \lfloor \frac{d}{2} \rfloor = s$. Therefore

$$\begin{aligned}\dim_k H^2_{\mathfrak{m}}(I^s)_{sd-1} &= \frac{1}{2}s\Big((s+1)\deg(\mathcal{B}) - 2s(2s^2+1)\Big)\\ &\leq \frac{1}{2}s\Big((s+1)(4s^2-4s+3) - 2s(2s^2+1)\Big)\\ &\leq -\frac{3}{2}s(s-1) < 0,\end{aligned}$$

which is impossible.

(ii) Similarly, if $d = 2s+1$ with $s \geq 2$, then $\nu = \lfloor \frac{d}{2} \rfloor = s$. Thus $\dim_k H^2_{\mathfrak{m}}(I^s)_{sd-1} \leq -\frac{1}{2}s(s-1) < 0$, which is impossible.

□

Example 2.3.10. Let $d \geq 4$ be an integer and $\phi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^3_k$ a parameterization of surface defined by

$$\begin{array}{ll} f_0 = X_0^{d-3}X_1(X_0^2 - X_1^2) & f_2 = X_0^{d-3}X_2(X_1^2 - X_2^2)\\ f_1 = X_0^{d-3}X_2(X_0^2 - X_1^2) & f_3 = X_1^{d-3}X_2(X_1^2 - X_2^2).\end{array}$$

Using `Macaulay2` [GS], we see that $\mathcal{B}$ is locally a complete intersection of degree $d^2 - 3d + 7$. Moreover, I is a saturated ideal with free resolution

$$0 \longrightarrow R(-d-1) \oplus R(-d-2) \oplus R(-2d+3) \xrightarrow{M} R(-d)^4 \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

where the matrix M is given by

$$\begin{pmatrix} -X_2 & 0 & 0\\ X_1 & X_1^2 - X_2^2 & 0\\ 0 & -X_0^2 + X_1^2 & X_1^{d-3}\\ 0 & 0 & -X_0^{d-3}\end{pmatrix}.$$

We consider the following cases.

<u>Case 1:</u> if $d-3$ is odd, we obtain $\mathcal{Y}_1 = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ with

$$\begin{array}{llll} y_1 = (1:0:0:0) & h_{y_1} = X_2 & y_2 = (0:0:0:1) & h_{y_2} = X_0^{d-3}\\ y_3 = (1:1:0:0) & h_{y_3} = X_1 - X_2 & y_4 = (1:-1:0:0) & h_{y_4} = X_1 + X_2\\ y_5 = (0:0:1:1) & h_{y_5} = X_0 - X_1 & y_6 = (0:0:1:-1) & h_{y_6} = X_0 + X_1.\end{array}$$

<u>Case 2:</u> if $d-3$ is even, we have $\mathcal{Y}_1 = \{y_1, y_2, y_3, y_4, y_5\}$ with

$$\begin{array}{llll} y_1 = (1:0:0:0) & h_{y_1} = X_2 & y_2 = (0:0:0:1) & h_{y_2} = X_0^{d-3}, \\ y_3 = (1:1:0:0) & h_{y_3} = X_1 - X_2 & y_4 = (1:-1:0:0) & h_{y_4} = X_1 + X_2, \\ y_5 = (0:0:1:1) & h_{y_5} = X_0^2 - X_1^2. & & \end{array}$$

Therefore

$$\sum_{y \in \mathcal{Y}_1} \deg(h_y) = d + 2.$$

Question 2.3.11. Adopt Data 2.2.5. Assume further that $m = n - 1 = 2$, that $\operatorname{indeg}(I^{\mathrm{sat}}) = d$ and that $\mathcal{B}$ is locally a complete intersection. Is $\sum_{y \in \mathcal{Y}_1} \deg(h_y)$ bounded linearly in the degree d?

Chapter 3

Fibers of rational maps and Jacobian matrices

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Abstract: A rational map $\phi : \mathbb{P}_k^m \dashrightarrow \mathbb{P}_k^n$ is defined by homogeneous polynomials of a common degree d. We establish a linear bound in terms of d for the number of $(m-1)$-dimensional fibers of ϕ, by using ideals of minors of the Jacobian matrix. In particular, we answer affirmatively the question in the end of the previous chapter. The results in this chapter are joint work with M. Chardin and S. D. Cutkosky and have been submitted [CCT17].

3.1 Tangent space maps and Jacobian matrices

Suppose that X is a k-scheme where k is an algebraically closed field, and $q \in X$ is a closed point. The tangent space $T(X)_q$ of X at q is the k-vector space $T(X)_q = \mathrm{Hom}_k(\mathfrak{m}_q/\mathfrak{m}_q^2, k)$ where $\mathfrak{m}_q$ is the maximal ideal of $\mathcal{O}_{X,q}$. Suppose that Y is another k-scheme and $\phi : X \to Y$ is a morphism of k-schemes. Then $\phi^* : \mathcal{O}_{Y,\phi(q)} \to \mathcal{O}_{X,q}$ induces a homomorphism of k-vector spaces $d\phi_q : T(X)_q \to T(Y)_{\phi(q)}$. If V is a subscheme of X and W is a subscheme of Y such that $\phi(V) \subset W$, then we have a natural commutative diagram of homomorphisms of k-vector spaces

$$\begin{array}{ccc} T(V)_q & \subset & T(X)_q \\ \downarrow & & \downarrow \\ T(W)_{\phi(q)} & \subset & T(Y)_{\phi(q)}. \end{array} \tag{3.1}$$

From now on, we will consider the following situation. Suppose that k is an algebraically closed field of characteristic $p \geq 0$. Consider $f_0, \ldots, f_n$ homogeneous polynomials of a common degree d in the standard polynomial ring $k[X_0, \ldots, X_m]$, such

that $\gcd(f_0, \ldots, f_n) = 1$. Let $\phi : \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k$ be a rational map defined by $f_0, \ldots, f_n$. The maximal open set on which ϕ is a morphism is $\Omega_\phi = \mathbb{P}^m_k \setminus Z(f_0, \ldots, f_n)$. Let

$$J(\mathbf{f}) = \begin{pmatrix} \frac{\partial f_0}{\partial X_0} & \cdots & \frac{\partial f_0}{\partial X_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial X_0} & \cdots & \frac{\partial f_n}{\partial X_m} \end{pmatrix}$$

be the Jacobian matrix of $\mathbf{f} = f_0, \ldots, f_n$. For any closed point $q = (q_0 : \cdots : q_m) \in \mathbb{P}^m_k$, we denote by $J(q)$ the matrix obtained from $J(\mathbf{f})$ by mapping X_i to q_i for all $i = 0, \ldots, m$. The entries of this matrix are defined by q up to multiplication by a common non zero scalar.

Proposition 3.1.1. *Suppose that p does not divide d and $q \in \Omega_\phi$ is a closed point. Then*

$$\textit{rank } J(q) = \textit{rank } d\phi_q + 1$$

where $d\phi_q : T(\mathbb{P}^m_k)_q \to T(\mathbb{P}^n_k)_{\phi(q)}$ is the tangent space map.

Proof. After possibly making linear changes of homogeneous coordinates in $\mathbb{P}^m_k$ and $\mathbb{P}^n_k$, we may assume that $q = (1 : 0 : \cdots : 0)$ and $\phi(q) = (1 : 0 : \cdots : 0)$. Let $\overline{X}_i = \frac{X_i}{X_0}$ for $1 \le i \le m$. Let $F_i = \frac{f_i}{X_0^d} \in k[\overline{X}_1, \ldots, \overline{X}_m]$, which is the affine coordinate ring of $\mathbb{P}^m_k \setminus Z(X_0)$. As ϕ is a regular map near q,

$$\phi = (f_0 : f_1 : \cdots : f_n) = (1 : g_1 : \cdots : g_n),$$

where $g_i = \frac{f_i}{f_0} = \frac{F_i}{F_0}$. Let $\alpha = (1, 0, \ldots, 0)$. We have that

$$\frac{\partial f_j}{\partial X_0}(\alpha) = df_j(\alpha)$$

for all j, by Euler's formula. Thus

$$\begin{aligned} \text{rank } J(p) &= \text{rank} \begin{pmatrix} df_0(\alpha) & \frac{\partial f_0}{\partial X_1}(\alpha) & \cdots & \frac{\partial f_0}{\partial X_m}(\alpha) \\ df_1(\alpha) & \frac{\partial f_1}{\partial X_1}(\alpha) & \cdots & \frac{\partial f_1}{\partial X_m}(\alpha) \\ \vdots & \vdots & & \vdots \\ df_n(\alpha) & \frac{\partial f_n}{\partial X_1}(\alpha) & \cdots & \frac{\partial f_n}{\partial X_m}(\alpha) \end{pmatrix} \\ &= \text{rank} \begin{pmatrix} df_0(\alpha) & \frac{\partial f_0}{\partial X_1}(\alpha) & \cdots & \frac{\partial f_0}{\partial X_m}(\alpha) \\ 0 & \frac{\partial f_1}{\partial X_1}(\alpha) & \cdots & \frac{\partial f_1}{\partial X_m}(\alpha) \\ \vdots & \vdots & & \vdots \\ 0 & \frac{\partial f_n}{\partial X_1}(\alpha) & \cdots & \frac{\partial f_n}{\partial X_m}(\alpha) \end{pmatrix} \\ &= \text{rank} \begin{pmatrix} \frac{\partial f_1}{\partial X_1}(\alpha) & \cdots & \frac{\partial f_1}{\partial X_m}(\alpha) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial X_1}(\alpha) & \cdots & \frac{\partial f_n}{\partial X_m}(\alpha) \end{pmatrix} + 1 \end{aligned}$$

Let $\overline{\alpha} = (0, \ldots, 0)$. As

$$\frac{\partial f_i}{\partial X_j}(\alpha) = \frac{\partial F_i}{\partial \overline{X}_j}(\overline{\alpha})$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$ (it suffices to check this on a monomial), so

$$\operatorname{rank}\begin{pmatrix} \frac{\partial f_1}{\partial X_1}(\alpha) & \cdots & \frac{\partial f_1}{\partial X_m}(\alpha) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial X_1}(\alpha) & \cdots & \frac{\partial f_n}{\partial X_m}(\alpha) \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \frac{\partial F_1}{\partial \overline{X}_1}(\overline{\alpha}) & \cdots & \frac{\partial F_1}{\partial \overline{X}_m}(\overline{\alpha}) \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial \overline{X}_1}(\overline{\alpha}) & \cdots & \frac{\partial F_n}{\partial \overline{X}_m}(\overline{\alpha}) \end{pmatrix}$$

$$= \operatorname{rank}\begin{pmatrix} \frac{\partial g_1}{\partial \overline{X}_1}(\overline{\alpha}) & \cdots & \frac{\partial g_1}{\partial \overline{X}_m}(\overline{\alpha}) \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial \overline{X}_1}(\overline{\alpha}) & \cdots & \frac{\partial g_n}{\partial \overline{X}_m}(\overline{\alpha}) \end{pmatrix}$$

since

$$\frac{\partial}{\partial \overline{X}_j}\left(\frac{F_i}{F_0}\right) = \frac{\partial F_i}{\partial \overline{X}_j} F_0^{-1} - F_0^{-2}\frac{\partial F_0}{\partial \overline{X}_j} F_i,$$

so

$$\frac{\partial g_i}{\partial \overline{X}_j}(\overline{\alpha}) = \frac{1}{F_0(\overline{\alpha})}\frac{\partial F_i}{\partial \overline{X}_j}(\overline{\alpha})$$

for $1 \leq i \leq n$ and $1 \leq j \leq m$, as $F_0(\overline{\alpha}) \neq 0$ and $F_i(\overline{\alpha}) = 0$ for $1 \leq i \leq n$. □

Remark 3.1.2. If p divides d the proof of Proposition 3.1.1 shows that we can either have

$$\operatorname{rank} J(q) = \operatorname{rank} d\phi_q + 1 \text{ or } \operatorname{rank} J(q) = \operatorname{rank} d\phi_q.$$

Both options are possible.

Proposition 3.1.3. *Suppose that $r \in \mathbb{N}$ and V is a subvariety of $\mathbb{P}^m_k$ such that $V \cap \Omega_\phi \neq \emptyset$ and $r = \dim V - \dim \phi(V)$. Then $V \subset Z(I_{m-r+2}(J))$, where $I_{m-r+2}(J)$ is the ideal generated by the $(m-r+2)$-minors of J.*

Proof. There exists a dense open subset U of V such that for any $q \in U$, V is smooth at q and $\phi(V)$ is smooth at $\phi(q)$ (take U to be the intersection of smooth locus of V with the preimage of the smooth locus of $\phi(V)$). We have that $\dim T(\phi(V))_{\phi(q)} = \dim V - r$ for $q \in U$, so by consideration of diagram (3.1), it follows that $\dim \operatorname{Ker} d\phi_q \geq r$ for $q \in U$, hence $\operatorname{rank} d\phi_q \leq m - r$ for $q \in U$. By Proposition 3.1.1 and Remark 3.1.2,

$$\operatorname{rank} J(q) \leq \operatorname{rank} d\phi_q + 1 \leq m - r + 1$$

for $q \in U$. Thus U is contained in the closed set $Z(I_{m-r+2}(J))$, so the closure V of U is contained in this set. □

3.2 Bound for the number of $(m-1)$-dimensional fibers of a rational map $\phi : \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k$

Let $R := k[X_0, \ldots, X_m]$ $(m \geq 2)$ be the standard graded polynomial ring over an algebraically closed field k. Suppose we are given an integer $d \geq 1$ and $n+1$ $(n \geq m)$ homogeneous polynomials $f_0, \ldots, f_n \in R_d$, not all zero. We denote by I the ideal of

R generated by these polynomials and set $\mathcal{B} := \mathrm{Proj}(R/I) \subseteq \mathbb{P}^m_k := \mathrm{Proj}(R)$. Let $B := k[T_0, \dots, T_n]$ with its standard grading and consider the rational map

$$\begin{aligned} \phi : \quad & \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k \\ & x \longmapsto (f_0(x) : \cdots : f_n(x)). \end{aligned}$$

From now on we assume that ϕ is generically finite onto its image, or equivalently that the closed image of ϕ is a subvariety $\mathscr{S}$ in $\mathbb{P}^n_k := \mathrm{Proj}(B)$, of dimension m. We may further assume that $\gcd(f_0, \dots, f_n) = 1$, replacing the $f_i's$ by their quotient by the greatest common divisor of $f_0, \dots, f_n$ if needed, hence the ideal I is of codimension at least 2.

For $\mathbf{f} = f_0, \dots, f_n$, set

$$J(\mathbf{f}) = \begin{pmatrix} \frac{\partial f_0}{\partial X_0} & \cdots & \frac{\partial f_0}{\partial X_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial X_0} & \cdots & \frac{\partial f_n}{\partial X_m} \end{pmatrix}$$

for the Jacobian matrix of $\mathbf{f}$ and $I_s(J(\mathbf{f}))$ for the ideal of R generated by the s-minors of $J(\mathbf{f})$, where $1 \leq s \leq m+1$.

Lemma 3.2.1. *Suppose that* $\dim_k I_d = n+1$ *and let* $\mathbf{f} = f_0, \dots, f_n$ *and* $\boldsymbol{g} = g_0, \dots, g_n$ *be two bases of* I_d. *Then* $I_s(J(\mathbf{f})) = I_s(J(\boldsymbol{g}))$, *for any* s.

Proof. Indeed, these are the Fitting ideals (with the same indices) of two matrices that are equal after change of basis over the base field. □

Now we state the main result in this section.

Theorem 3.2.2. *Adopt Data 2.2.5. Assume further that* $I_3(J(\mathbf{f})) \neq 0$ *and denote by* F *be the greatest common divisor of generators of* $I_3(J(\mathbf{f}))$. *Then*

$$\sum_{y \in \mathcal{Y}_{m-1}} \deg(h_y) \leq \sum_{y \in \mathcal{Y}_{m-1}} \sum_{i=1}^{r_y} (2e_i - 1) \deg(h_i) \leq \deg(F) \leq 3(d-1),$$

where $h_y = h_1^{e_1} \cdots h_{r_y}^{e_{r_y}}$ *is an irreducible factorization of* h_y *in* R.

Proof. By Lemma 2.2.2, the unmixed components of $\pi^{-1}(\mathfrak{p})$ and $\pi'^{-1}(\mathfrak{p})$ are the same for every closed point $\mathfrak{p} \in \mathcal{Y}_{m-1}$. By Lemma 2.2.3, there exists a homogeneous polynomial $f \in I$ of degree d such that, for any $\mathfrak{p} \in \mathcal{Y}_{m-1}$

$$I = (f) + h_y(g_{1y}, \dots, g_{ny}),$$

for some $g_{1y}, \dots, g_{ny} \in R$.

The Jacobian matrix of $\mathbf{f}' = (f, h_y g_{1y}, \dots, h_y g_{ny})$ is

$$J(\mathbf{f}') = \begin{pmatrix} \frac{\partial f}{\partial X_0} & \cdots & \frac{\partial f}{\partial X_m} \\ h_y \frac{\partial g_{1y}}{\partial X_0} + g_{1y} \frac{\partial h_y}{\partial X_0} & \cdots & h_y \frac{\partial g_{1y}}{\partial X_m} + g_{1y} \frac{\partial h_y}{\partial X_m} \\ \vdots & & \vdots \\ h_y \frac{\partial g_{ny}}{\partial X_0} + g_{ny} \frac{\partial h_y}{\partial X_0} & \cdots & h_y \frac{\partial g_{ny}}{\partial X_m} + g_{ny} \frac{\partial h_y}{\partial X_m} \end{pmatrix}.$$

For all $j = 0, \dots, m$

$$\frac{\partial h_y}{\partial X_j} = \sum_{i=1}^{r_y} e_i \frac{h_y}{h_i} \frac{\partial h_i}{\partial X_j},$$

therefore the i-th row of $J(\mathbf{f}')$ has a common factor $h_1^{e_1-1} \cdots h_{r_y}^{e_{r_y}-1}$, for all $i = 2, \dots, n+1$. It follows that, for any subset $\mathcal{I}$ of $\{1, \dots, n+1\}$ with 3 elements and a subset $\mathcal{J}$ of $\{1, \dots, m+1\}$ with 3 elements,

$$[J(\mathbf{f}')]_{\mathcal{I},\mathcal{J}} = h_1^{2(e_1-1)} \cdots h_{r_y}^{2(e_{r_y}-1)} [M]_{\mathcal{I},\mathcal{J}}, \tag{3.2}$$

where $[Q]_{\mathcal{I},\mathcal{J}}$ denotes the 3-minor of a $(n+1) \times (m+1)$-matrix Q that corresponds to the rows with index in $\mathcal{I}$ and the columns with index in $\mathcal{J}$ and M is the $(n+1) \times (m+1)$-matrix

$$\begin{pmatrix} \frac{\partial f}{\partial X_0} & \cdots & \frac{\partial f}{\partial X_m} \\ \widehat{h}_y \frac{\partial g_{1y}}{\partial X_0} + g_{1y}\sigma_0 & \cdots & \widehat{h}_y \frac{\partial g_{1y}}{\partial X_m} + g_{1y}\sigma_m \\ \vdots & & \vdots \\ \widehat{h}_y \frac{\partial g_{ny}}{\partial X_0} + g_{ny}\sigma_0 & \cdots & \widehat{h}_y \frac{\partial g_{ny}}{\partial X_m} + g_{ny}\sigma_m \end{pmatrix},$$

here $\widehat{h}_y = h_1 \cdots h_{r_y}$ and $\sigma_j = \sum_{i=1}^{r_y} e_i \frac{\widehat{h}_y}{h_i} \frac{\partial h_i}{\partial X_j}$, $(j = 0, \dots, m)$. Thus there is a homogeneous polynomial P such that $[M]_{\mathcal{I},\mathcal{J}} = \widehat{h}_y P + [N]_{\mathcal{I},\mathcal{J}}$, where N is the $(n+1)\times(m+1)$-matrix

$$\begin{pmatrix} \frac{\partial f}{\partial X_0} & \cdots & \frac{\partial f}{\partial X_m} \\ g_{1y}\sigma_0 & \cdots & g_{1y}\sigma_m \\ \vdots & & \vdots \\ g_{ny}\sigma_0 & \cdots & g_{ny}\sigma_m \end{pmatrix}$$

which shows that $[N]_{\mathcal{I},\mathcal{J}} = 0$, as rank $N \leq 2$. By (3.2), we obtain

$$[J(\mathbf{f}')]_{\mathcal{I},\mathcal{J}} = h_1^{2(e_1-1)} \cdots h_{r_y}^{2(e_{r_y}-1)} \widehat{h}_y P = h_1^{2e_1-1} \cdots h_{r_y}^{2e_{r_y}-1} P, \tag{3.3}$$

for all $\mathcal{I}, \mathcal{J}$. Let G be the greatest common divisor of generators of $I_3(J(\mathbf{f}'))$. Then $h_1^{2e_1-1} \cdots h_{r_y}^{2e_{r_y}-1}$ is a divisor of G by (3.3). By Lemma 3.2.1, $h_1^{2e_1-1} \cdots h_{r_y}^{2e_{r_y}-1}$ is a divisor of F.

Moreover, if $y \neq y'$ in $\mathcal{Y}_{m-1}$, then $\gcd(h_y, h_{y'}) = 1$, hence $\gcd(h_i, h'_j) = 1$, for every factor h_i (res. h'_j) of h_y (res. $h_{y'}$). We deduce that

$$\prod_{y \in \mathcal{Y}_{m-1}} h_1^{2e_1-1} \cdots h_{r_y}^{2e_{r_y}-1} \mid F$$

which gives

$$\sum_{y \in \mathcal{Y}_{m-1}} \sum_{i=1}^{r_y} (2e_i - 1) \deg(h_i) \leq \deg(F) \leq 3(d-1).$$

□

Remark 3.2.3. Let $p = \mathrm{char}(k)$ be the characteristic of the field k. Then there are two cases:

(i) Case 1: p does not divide d. Then $I_{m+1}(J(\mathbf{f})) \neq 0$ if and only if $[k(\mathbf{f}) : k(\mathbf{X})]$ is separable, where $\mathbf{X} := X_0, \ldots, X_m$. In particular, if $p = 0$, then the condition $I_{m+1}(J(\mathbf{f})) \neq 0$ always holds.

(ii) Case 2: p divides d. Then $I_{m+1}(J(\mathbf{f})) \neq 0$ only if $[k(\mathbf{f}) : k(\mathbf{X})]$ is separable.

Note that if $I_{m+1}(J(\mathbf{f})) \neq 0$, then $I_j(J(\mathbf{f})) \neq 0$, for all $1 \leq j \leq m+1$. In particular, if the characteristic of k is 0, then the assumption $I_3(J(\mathbf{f})) \neq 0$ is always satisfied.

Remark 3.2.4.

(i) The inequality

$$\sum_{y \in \mathcal{Y}_{m-1}} \deg(h_y) \leq \sum_{y \in \mathcal{Y}_{m-1}} \sum_{i=1}^{r_y} (2e_i - 1) \deg(h_i)$$

becomes an equality if and only if the defining equation of the unmixed component of the fiber $\pi^{-1}(y)$ has no multiple factors, for every $y \in \mathcal{Y}_{m-1}$.

(ii) The bound

$$\sum_{y \in \mathcal{Y}_{m-1}} \sum_{i=1}^{r_y} (2e_i - 1) \deg(h_i) \leq \deg(F)$$

is optimal as the following example shows.

Example 3.2.5. Let $d \geq 4$ be an integer. Consider the parameterization given by $\mathbf{f} = f_0, \ldots, f_3$, with

$$\begin{aligned} f_0 &= X_0^{d-3} X_1 (X_0^2 - X_1^2), & f_2 &= X_0^{d-3} X_2 (X_1^2 - X_2^2), \\ f_1 &= X_0^{d-3} X_2 (X_0^2 - X_1^2), & f_3 &= X_1^{d-3} X_2 (X_1^2 - X_2^2) \end{aligned}$$

in Example 2.3.10. Using `Macaulay2` [GS], the greatest common divisor of generators of $I_3(J(\mathbf{f}))$ is

$$F = X_0^{2d-7} X_2 (X_0^2 - X_1^2)(X_1^2 - X_2^2).$$

It is known as in Example 2.3.10 that

$$\sum_{y \in \mathcal{Y}_1} \deg(h_y) = d + 2$$

and

$$\sum_{y \in \mathcal{Y}_1} \sum_{i=1}^{r_y} (2e_i - 1) \deg(h_i) = 2(d-1) = \deg(F) < 3(d-1).$$

Furthermore, if $d = 4$, then

$$\sum_{y \in \mathcal{Y}_1} \deg(h_y) = \sum_{y \in \mathcal{Y}_1} \sum_{i=1}^{r_y} (2e_i - 1) \deg(h_i) = \deg(F).$$

3.3 Bound for the number of one-dimensional fibers of a parameterization surface

In this section, we will treat the case of parameterizations $\phi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^3_k$ of algebraic rational surfaces. Such a map ϕ is defined by four homogeneous polynomials $f_0, \ldots, f_3$, of a common degree d, in the standard graded polynomial ring $R = k[X_0, X_1, X_2]$. Our objective is to refine the bound $\sum_{y \in \mathcal{Y}_1} \deg(h_y)$ which gave in the previous chapter.

The following result is a direct consequence of Theorem 3.2.2.

Corollary 3.3.1. *Adopt Data 2.2.5. Assume further that $m = n - 1 = 2$, that $I_3(\mathbf{f}) \neq 0$ and let F be the greatest common divisor of generators of $I_3(\mathbf{f})$. Then*

$$\sum_{y \in \mathcal{Y}_1} \deg(h_y) \leq \sum_{y \in \mathcal{Y}_1} \sum_{i=1}^{r_y} (2e_i - 1) \deg(h_i) \leq \deg(F) \leq 3(d-1),$$

where $h_y = h_1^{e_1} \cdots h_{r_y}^{e_{r_y}}$ is an irreducible factorization of h_y in R.

Notice that the condition $I_3(J(\mathbf{f})) \neq 0$ is always satisfied if the characteristic of k is zero. Thus, the above corollary gives an affirmative answer for Question 2.3.11, at least in the case where k is of characteristic zero. Note that we do not need the assumption locally a complete intersection of $\mathcal{B}$.

Now we study the syzygies of f_i's in relation with the degree of the greatest common divisor of the generators of $I_3(J(\mathbf{f}))$.

Proposition 3.3.2. *Let $\mathbf{f} := f_0, \ldots, f_3$ be the homogeneous polynomials of the same degree d and F the greatest common divisor of generators of $I_3(J(\mathbf{f}))$. Suppose that p does not divide d. If $\deg(F) = 3(d-1) - \delta$, then $I = (f_0, \ldots, f_3)$ has a syzygy of degree δ : there exist the homogeneous polynomials $a_0, \ldots, a_3 \in R$, not all 0, of degree δ, such that*

$$a_0 f_0 + \cdots + a_3 f_3 = 0.$$

Proof. Suppose that $\deg(F) = 3(d-1) - \delta$. Then there exist the homogeneous polynomials $a_0, \ldots, a_3 \in R$, not all 0, of the same degree δ such that $F_i = a_{i-1} F$ for all $i = 1, \ldots, 4$. Without loss of the generatily, we assume that $a_0 \neq 0$. For simplicity, we set $P_{ij} = \frac{\partial f_i}{\partial X_j}$, for all $i = 0, \ldots, 3$ and $j = 0, 1, 2$. Then $(\frac{a_1}{a_0}, -\frac{a_2}{a_0}, \frac{a_3}{a_0})$ is a root of the system of equations

$$\begin{cases} P_{10}Y_1 + P_{20}Y_2 + P_{30}Y_3 & = P_{00} \\ P_{11}Y_1 + P_{21}Y_2 + P_{31}Y_3 & = P_{01} \\ P_{12}Y_1 + P_{22}Y_2 + P_{32}Y_3 & = P_{02} \end{cases}$$

which implies that

$$P_{0j} = \frac{a_1}{a_0} P_{1j} - \frac{a_2}{a_0} P_{2j} + \frac{a_3}{a_0} P_{3j}, \text{ for all } j = 0, 1, 2.$$

It follows that

$$a_0 \sum_{j=0}^{2} X_j P_{0j} = a_1 \sum_{j=0}^{2} X_j P_{1j} - a_2 \sum_{j=0}^{2} X_j P_{2j} + a_3 \sum_{j=0}^{2} X_j P_{3j}$$

gives $da_0f_0 = d(a_1f_1 - a_2f_2 + a_3f_3)$, by Euler's formula. Thus $(a_0, -a_1, a_2, -a_3)$ is a syzygy of $f_0, \dots, f_3$. □

Corollary 3.3.3. *Under the assumptions of Proposition 3.3.2. Then* $\deg(F) = 3(d-1)$ *if and only if* $f_0, \dots, f_3$ *are linearly dependent over* k.

Proof. Suppose that $\deg(F) = 3(d-1)$. By Proposition 3.3.2, there exist $a_0, \dots, a_0 \in k$, not all 0, such that

$$a_0f_0 + a_1f_1 + a_2f_2 + a_3f_3 = 0.$$

Suppose that $f_0, \dots, f_3$ are linearly dependent over k. Then there are $\lambda_0, \dots, \lambda_3 \in k$, not all 0, such that $\lambda_0f_0 + \cdots + \lambda_3f_3 = 0$. Without loss of the generality, we assume that $\lambda_0 = -1$, hence $f_0 = \lambda_1f_1 + \lambda_2f_2 + \lambda_3f_3$. It follows that

$$\frac{\partial f_0}{\partial X_j} = \lambda_1\frac{\partial f_1}{\partial X_j} + \lambda_2\frac{\partial f_2}{\partial X_j} + \lambda_3\frac{\partial f_3}{\partial X_j}, \text{ for all } j = 0, 1, 2.$$

Thus, we obtain $I_3(\mathbf{f}) = (F_1, \lambda_1F_1, \lambda_2F_1, \lambda_3F_1)$, which shows that $F = F_1$. □

We denote by $\mathrm{Syz}(I) \subseteq R^4$ the module of syzygies of I. It is a graded module and in the structural graded exact sequence

$$0 \longrightarrow Z_1 \longrightarrow R^4(-d) \xrightarrow{(f_0,f_1,f_2,f_3)} I \longrightarrow 0,$$

we have the identification $\mathrm{Syz}(I) = Z_1(d)$. Recall that for a finitely generated graded R-module M, its initial degree is defined by

$$\mathrm{indeg}(M) := \inf\{\nu \mid M_\nu \neq 0\},$$

with the convention $\mathrm{indeg}(M) = +\infty$ when $M = 0$.

Corollary 3.3.4. *Adopt Data 2.2.5. Assume further that* $m = n - 1 = 2$, *that the characteristic of* k *does not divide* d, *and that* $[k(\mathbf{f}) : k(\mathbf{X})]$ *is separable. Then*

$$\sum_{y\in\mathcal{Y}_1} \deg(h_y) \leq 3(d-1) - \mathrm{indeg}(\mathrm{Syz}(I)) < 3(d-1).$$

Proof. By Proposition 3.3.2,

$$\deg(F) \leq 3(d-1) - \mathrm{indeg}(\mathrm{Syz}(I)).$$

The last inequality implies from the fact that $\mathrm{indeg}(\mathrm{Syz}(I)) = 0$ if and only if $f_0, \dots, f_3$ are linearly dependent over k. □

Notice that the two last condition in the above corollary are automatically satisfied if k is of characteristic zero.

Example 3.3.5. Consider the parameterization given by $\mathbf{f} = f_0, f_1, f_2, f_3$, with

$$\begin{aligned} f_0 &= X_1^2X_2^4 - X_1^4X_2^2, & f_2 &= X_0^2X_1^2X_2^2 - X_0^2X_1^4, \\ f_1 &= X_0^4X_2^2 - X_2^6, & f_3 &= X_0^4X_1^2 - X_1^2X_2^4 \end{aligned}$$

in Example 2.2.7. Using `Macaulay2` [GS], the greatest common divisor of generators of $I_3(J(\mathbf{f}))$ is

$$F = X_0X_1^3X_2(X_0^4 - X_2^4)(X_1^2 - X_2^2).$$

It is known as in Example 2.2.7 that

$$\sum_{y\in\mathcal{Y}_1} \deg(h_y) = 8 \leq \deg(F) = 11 \leq 3.5 - \mathrm{indeg}(\mathrm{Syz}(\mathbf{f})) = 13.$$

Chapter 4

Effective criteria for bigraded birational maps

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Abstract: In this chapter, we consider rational maps whose source is a product of two subvarieties, each one being embedded in a projective space. Our main objective is to investigate birationality criteria for such maps. First, a general criterion is given in terms of the rank of a couple of matrices that came to be known as *Jacobian dual matrices*. Then, we focus on rational maps from $\mathbb{P}^1_k \times \mathbb{P}^1_k$ to $\mathbb{P}^2_k$ in very low bidegrees and provide new matrix-based birationality criteria by analyzing the syzygies of the defining equations of the map, in particular by looking at the dimension of certain bigraded parts of the syzygy module. Finally, applications of our results to the context of geometric modeling are discussed at the end of the chapter. The results in this chapter are joint work with N. Botbol, L. Busé, M. Chardin, S. H. Hassanzadeh and A. Simis and have been published [BBC+17].

4.1 Introduction

A rational map $\mathcal{F} : \mathbb{P}_k^m \dashrightarrow \mathbb{P}_k^n$ between projective spaces is defined by an ordered set of homogeneous polynomials $\mathbf{f} := f_0, \ldots, f_n$ in $m+1$ variables, of the same degree and not all zero. The problem of providing sufficient conditions for such a map $\mathcal{F}$ to be birational has attracted much interest in the past and it is still an active area of research. For computational purposes, methods based on the nature of the syzygies of $\mathbf{f}$ are the most suitable in the sense of effective results in the usual implementation of the Gröbner basis algorithm. This syzygy-based approach goes back to [HKS92] where sufficient conditions for birationality were given in the case $m = n$. Then, several improvements have been introduced in relation with the equations of the symmetric and the Rees algebras of the ideal generated by $\mathbf{f}$ (see, e.g., [RS01, Sim04]), including in arbitrary characteristic [DHS12], and also in relation with the fibers of $\mathcal{F}$ in [EU08, KPU16].

In this chapter, we aim to extend some of these methods and techniques to the context of rational maps whose source is a product of two projective spaces $\mathbb{P}_k^n \times \mathbb{P}_k^m$ instead of. These maps are defined by an ordered set of bihomogeneous polynomials in two sets of $n+1$ and $m+1$ variables, respectively. In order to of emphasize, we call them *bigraded rational maps*. Important modern motivation for considering bigraded rational maps comes from the field of geometric modeling. Indeed, the geometric modeling community uses almost exclusively bigraded rational maps for parameterizing surfaces, dubbing such maps *rational tensor-product Bézier parameterizations*. It turns out that an important property is to guarantee the birationality of these parameterizations onto their images. An even more important property is to preserve this birationality property during a design process, that is to say when the coefficients of the defining polynomials are continuously modified. As a first attempt to tackle these difficult problems, we will analyze in detail birational maps from $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ to $\mathbb{P}_k^2$ in low bidegrees by means of syzygies.

Through its various sections, this chapter traverses topics from algebra to geometry and to modeling. In Section 4.2, a general criterion for characterizing bigraded birational maps is proved by means of algebraic tools. It is based on the rank of two matrices, called *Jacobian dual matrices*, that are built from some particular equations of the Rees algebra of the bihomogeneous equations defining the rational map. This criterion is actually an analogue of the existing Jacobian dual criterion of rational maps between varieties embedded in projective spaces [RS01, Sim04, DHS12].

In Section 4.3, we turn to a more geometric language since the bigraded rational maps are investigated through the properties of their base locus. By focussing on bigraded birational maps from $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ to $\mathbb{P}_k^2$, we obtain very simple birationality criteria in bidegree $(1,1)$ and bidegree $(1,2)$ in terms of the dimension of some bigraded parts of the syzygies of the equations defining the rational map. Another important contribution of our work is a detailed study of the case of bidegree $(2,2)$ maps for which we provide a complete list of possible birational maps.

Finally, in Section 4.4 we investigate applications of our results to the field of modeling. In particular, for bigraded plane rational maps of bidegree $(1,1)$ and $(1,2)$ we explain how some particular coefficients of the map, called the *weights* of the parameterization, can be tuned in order to obtain a birational map. It is important

to notice that the inverse map is then given by explicit minors from the matrix characterizing the birationality of the map. In the bidegree $(1,2)$ case, our new birationality criterion allows to assign the control of this tuning to a structured low-rank matrices approximation algorithm, in the context of numerical computations.

4.2 General birationality criterion

In this section, we provide a general effective criterion for birationality of a bigraded rational map with source a biprojective space $\mathbb{P}^n_k \times_k \mathbb{P}^m_k$. We will state the results under more general hypotheses, namely, when the source is a product $X \times_k Y$, where $X \subset \mathbb{P}^n_k$, $Y \subset \mathbb{P}^m_k$ denote nondegenerate irreducible projective varieties over an algebraically closed field k. The criterion is an analogue of the so-called Jacobian dual criterion which has been studied so far in the context of a rational maps between varieties embedded in projective spaces [RS01, Sim04, DHS12]. For detail, see Section 1.6.

4.2.1 Birationality and bigraded Rees algebras

As in the case of a rational map between projectively embedded varieties, where the notion of the graph of the map is encoded in taking the Rees algebra of an equigenerated homogeneous base ideal, a rational map with source a multi-projectively embedded variety and target a projectively embedded variety has a graph encoded in taking the Rees algebra of an equigenerated multihomogeneous ideal.

As for a rational map with source a projectively embedded variety and target a multi-projectively embedded variety, the algebraic object that conveniently encodes the graph is a multi-Rees algebra – i.e., the Rees algebra of a module which is the direct sum of a finite set of equigenerated homogeneous ideals of various degrees.

Although valid in the arbitrary multigraded case, for simplicity, we state it in the biprojective case. Thus, let $X \subset \mathbb{P}^n_k$, $Y \subset \mathbb{P}^m_k$ and $Z \subset \mathbb{P}^s_k$ denote nondegenerate irreducible projective varieties over an algebraically closed field k. Let $A = k[\mathbf{x}] = k[x_0, \ldots, x_n]/\mathfrak{a}$, $B = k[\mathbf{y}] = k[y_0, \ldots, y_m]/\mathfrak{b}$ and $S = k[\mathbf{z}] = k[z_0, \ldots, z_s]/\mathfrak{c}$ stand for the respective homogeneous coordinate rings. We also denote $R := A \otimes_k B \simeq k[\mathbf{x}, \mathbf{y}]/(\mathfrak{a}, \mathfrak{b})$. A rational map $\mathcal{F} : X \times Y \longrightarrow Z$ is defined by bihomogeneous polynomials $f_0(\mathbf{x}, \mathbf{y}), \ldots, f_s(\mathbf{x}, \mathbf{y})$ in R of fixed bidegree (a, b), not all zero. We say that $\mathcal{F}$ is birational with image Z if it is dominant and admits an inverse rational map with image $X \times_k Y$. Note that the inverse map is necessarily given by a pair of rational maps $Z \longrightarrow X$ and $Z \longrightarrow Y$ defined by homogeneous polynomials $\mathbf{g} := \{g_0, \ldots, g_n\}$ and $\mathbf{h} := \{h_0, \ldots, h_m\}$ of fixed degrees d_1 and d_2, respectively. Set $I := (f_0(\mathbf{x}, \mathbf{y}), \ldots, f_s(\mathbf{x}, \mathbf{y})) \subset R$ and $J_1 := (\mathbf{g}) \subset S$, $J_2 := (\mathbf{h}) \subset S$.

Lemma 4.2.1. *With the above notations. Then the identity map on $k[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ induces a k-algebra isomorphism between the Rees algebra $\mathcal{R}_R(I)$ and the multi-Rees algebra $\mathcal{R}_S(J_1 \oplus J_2)$.*

Proof. The proof is tailored on the one in [Sim04, Proposition 2.1] (see also [DHS12,

Theorem 2.18]). Consider a polynomial presentation

$$R[\mathbf{z}] = \frac{k[\mathbf{x},\mathbf{y}]}{(\mathfrak{a},\mathfrak{b})}[\mathbf{z}] \twoheadrightarrow R[\mathbf{f}t] = \frac{k[\mathbf{x},\mathbf{y}]}{(\mathfrak{a},\mathfrak{b})}[\mathbf{f}t] = \mathcal{R}_R(I), \quad z_k \mapsto f_k t$$

whose restriction to $R = k[\mathbf{x},\mathbf{y}]/(\mathfrak{a},\mathfrak{b})$ is the identity. Let $\overline{\mathcal{J}} = (\mathcal{J},\mathfrak{a},\mathfrak{b})/(\mathfrak{a},\mathfrak{b})$ denote the kernel, with $\mathcal{J}$ the ideal generated by the $\mathbf{z}$-homogeneous polynomials, with bihomogeneous polynomial coefficients in $k[\mathbf{x},\mathbf{y}]$, vanishing on $f_0(\mathbf{x},\mathbf{y}),\dots,f_s(\mathbf{x},\mathbf{y})$ modulo $(\mathfrak{a},\mathfrak{b})$.

Note that $\mathfrak{c} \subset \mathcal{J}$. Indeed, taking $\mathbf{f}$ as homogeneous polynomials for the total degree of their fixed bidegree, it is clear that the image Z is identified with $\mathrm{Proj}(k[\mathbf{f}])$ up to degree normalization. Since the two algebras $k[\mathbf{f}]$ and $k[\mathbf{f}t]$ are k-isomorphic as graded algebras and $\ker(k[\mathbf{z}] \twoheadrightarrow k[\mathbf{f}t]) \subset \mathcal{J}$, we are through. In particular, the Rees algebra $\mathcal{R}_R(I)$ is a residue k-algebra of $R[\mathbf{z}]/\overline{\mathfrak{c}} = k[\mathbf{x},\mathbf{y},\mathbf{z}]/(\mathfrak{a},\mathfrak{b},\mathfrak{c})$.

By the same token, one has

$$\mathcal{R}_S(J_1 \oplus J_2) \simeq S[\mathbf{g}u,\mathbf{h}v] \simeq k[\mathbf{z}][\mathbf{x},\mathbf{y}]/(\mathfrak{c},\mathcal{J}'),$$

where $\mathcal{J}'$ is generated by those $\mathbf{x},\mathbf{y}$-bihomogeneous polynomials with homogenous coefficients in $k[\mathbf{z}]$ vanishing on both sets $\mathbf{g}$ and $\mathbf{h}$ modulo $\mathfrak{c}$. Similarly, both $\mathfrak{a}$ and $\mathfrak{b}$ are contained in $\mathcal{J}'$ – for example, note for this, that a form of degree d in $k[\mathbf{x}]$ is a bihomogeneous polynomial in $k[\mathbf{x},\mathbf{y}]$ of bidegree $(d,0)$ with homogeneous coefficients in $k[\mathbf{z}]$ of degree 0. Thus, $\mathcal{R}_S(J_1 \oplus J_2)$ is a residue k-algebra of $S[\mathbf{x},\mathbf{y}]/(\mathfrak{a},\mathfrak{b}) = k[\mathbf{x},\mathbf{y},\mathbf{z}]/(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ as well.

We now claim that the identity map of $k[\mathbf{x},\mathbf{y},\mathbf{z}]/(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ induces the required k-algebra isomorphism, for which it suffices now to show that $\mathcal{J}' \subset (\mathfrak{a},\mathfrak{b},\mathcal{J})$ and that $\mathcal{J} \subset (\mathfrak{c},\mathcal{J}')$.

Let $F(\mathbf{z},\mathbf{x},\mathbf{y}) = \sum_i p_i(\mathbf{z})\mathbf{x}^{\alpha_i}\mathbf{y}^{\beta_i} \in \mathcal{J}'$, where $|\alpha_i| = p$ and $|\beta_i| = q$ for all i. By the definition of $\mathcal{J}'$, one has $F(\mathbf{z},\mathbf{g},\mathbf{h}) \in \mathfrak{c}$. Therefore, thus $F(\mathbf{f},\mathbf{g},\mathbf{h}) = 0 \in R$. On the other hand since the pair $(\mathbf{g};\mathbf{h})$ defines the inverse map to $\mathcal{F}$, there exist forms D and D' in $R\setminus 0$, perhaps of different degrees, such that $\mathbf{g}(\mathbf{f}) = \mathbf{x}\,D$ and $\mathbf{h}(\mathbf{f}) = \mathbf{y}\,D'$. It follows that $F(\mathbf{f},\mathbf{g},\mathbf{h}) = (\sum_i p_i(\mathbf{f})\mathbf{x}^{\alpha_i}\mathbf{y}^{\beta_i})D^aD'^b$. Since R is an integral domain, the vanishing of the latter shows that $F(\mathbf{f},\mathbf{x},\mathbf{y}) = 0$ on R. In particular $F \in (\mathfrak{a},\mathfrak{b},\mathcal{J})$.

The other inclusion is obtained by a similar argument. □

4.2.2 Bigraded Jacobian dual criterion

We are now ready to present a multiprojective version of the Jacobian dual criterion of birationality. For simplicity, we stick to the biprojective case, as the arbitrary multiprojective case requires only a small set of changes.

We will focus on the presentation ideal $(\mathfrak{a},\mathfrak{b},\mathcal{J}) \subset k[\mathbf{x},\mathbf{y},\mathbf{z}]$ of the Rees algebra $\mathcal{R}_R(I)$. Consider the elements of degrees $(1,0,*)$ and $(0,1,*)$ in $(\mathfrak{a},\mathfrak{b},\mathcal{J})$, where $*$ denotes an arbitrary degree in $\mathbf{z}$. Since by assumption X and Y are nondegenerate, these elements belong to the graded pieces $\mathcal{J}_{(1,0,*)}$ and $\mathcal{J}_{(0,1,*)}$, respectively. Now, a form of degree $(1,0,*)$ can be thought as a form of bidegree $(1,*)$ in $k[\mathbf{x},\mathbf{z}]$. Moreover, since X is nondegenerate, each such form has a unique expansion of the shape $\sum_i Q_i(\mathbf{z})x_i$, where $Q_i(\mathbf{z}) \in k[\mathbf{z}]$ is homogeneous of degree $*$. Considering these

expansions for a minimal set of generating forms of the ideal $(\mathcal{J}_{(1,0,*)})$ and taking the corresponding matrix of $\mathbf{x}$-derivatives, we obtain a weak Jacobian dual matrix $\Psi_{\mathbf{x}}$ as in Section 1.6 – here dubbed an $\mathbf{x}$-*partial Jacobian dual matrix*. We similarly introduce an $\mathbf{y}$-*partial Jacobian dual matrix* $\Psi_{\mathbf{y}}$. Finally, thinking of these matrices as maps over $k[\mathbf{z}]$, we denote by $\Psi_{\mathbf{x}} \otimes_{k[\mathbf{z}]} S$ and $\Psi_{\mathbf{y}} \otimes_{k[\mathbf{z}]} S$ the respective maps obtained modulo $\mathfrak{c}$.

Theorem 4.2.2. *With the previous notations, the rational map $\mathcal{F} : X \times Y \dashrightarrow Z$ is birational with image Z if and only if* $\operatorname{rank}_S(\Psi_{\mathbf{x}} \otimes_{k[\mathbf{z}]} S) = n$ *and* $\operatorname{rank}_S(\Psi_{\mathbf{y}} \otimes_{k[\mathbf{z}]} S) = m$. *In addition, both halves of the expression of the inverse of $\mathcal{F}$ are given by (signed) ordered maximal minors of an $n \times (n+1)$ submatrix of $\Psi_{\mathbf{x}}$ and of an $m \times (m+1)$ submatrix of $\Psi_{\mathbf{y}}$, respectively.*

Proof. Suppose that $\mathcal{F}$ is birational with image Z. By the proof of Lemma 4.2.1, in particular $\mathcal{J}_{(1,0,*)} = \mathcal{J}'_{(*,1,0)}$ – notice that $*$, 1 and 0 are the respective degrees in $\mathbf{z}$,$\mathbf{x}$ and $\mathbf{y}$. This implies that $\Psi_{\mathbf{x}}$ can also be written in terms of $\mathcal{J}'_{(*,1,0)}$. But, as such and due to the definition of $\mathcal{J}'$, we get an equality

$$I_1([x_0 \cdots x_n] \cdot ({}^t\Psi_{\mathbf{x}} \otimes S)) = I_1([x_0 \cdots x_n] \cdot \operatorname{Syz}_S(\mathbf{g})),$$

where $\operatorname{Syz}_S(\mathbf{g})$ denotes the matrix of syzygies of $(\mathbf{g}) \subset S$. Since neither ${}^t\Psi_{\mathbf{x}} \otimes S$ nor $\operatorname{Syz}_S(\mathbf{g})$ involves any variables other than $\mathbf{z}$, it then follows that these matrices define the same column space and hence have the same rank. But, clearly $\operatorname{rank}_S(\operatorname{Syz}_S(\mathbf{g})) = n$. A similar argument applies to the $\mathbf{y}$ part.

The proof of the converse statement is the same as the proof for the projective varieties in [DHS12], with the obvious adaptation. Thus, let M denote an $n \times (n+1)$ submatrix of $\Psi_{\mathbf{x}}$ which is of rank n over S. Let $\Delta_0(\mathbf{z}), \cdots, \Delta_n(\mathbf{z})$ be its ordered signed minors. By the Hilbert-Koszul's lemma 1.5.6, the vector $\Delta_i(\mathbf{z})e_j - \Delta_j(\mathbf{z})e_i$ belongs to the column space of M and hence to that of $\Psi_{\mathbf{x}}$. The fact that $I_1([x_0 \cdots x_n] \cdot \Psi_{\mathbf{x}}) = (\mathcal{J}_{(1,0,*)})$ ensures that $x_i\Delta_j(\mathbf{z}) - x_j\Delta_i(\mathbf{z}) \in (\mathcal{J}_{(1,0,*)})$. In particular $x_i\Delta_j(\mathbf{f}) - x_j\Delta_i(\mathbf{f}) = 0$ in R. We claim that the $n+1$-tuple $(\Delta_0(\mathbf{f}) \cdots \Delta_n(\mathbf{f}))$ does not vanish on R. To see this, recall that the homogeneous coordinate ring of the image of $\mathcal{F}$ is $k[f_0, \cdots, f_s] \simeq S$ (up to degree normalization). Since $S = k[\mathbf{z}]/\mathfrak{c}$, then $\Delta_i(\mathbf{f}) = 0 \in R$ if and only if $\Delta_i(\mathbf{z}) \in \mathfrak{c}$, i.e., if and only if $\Delta_i(\mathbf{z}) = 0 \in S$. But this cannot happen for all i because $\operatorname{rank}(M \otimes_{k[\mathbf{z}]} S) = n$. It now follows that the rational map $Z \dashrightarrow X$ defined by $(\Delta_0(\mathbf{z}) : \cdots : \Delta_n(\mathbf{z}))$ gives the first half of the inverse to $\mathcal{F}$.

The second half is treated entirely in the same way. □

4.2.3 Linear syzygies and birationality

Theorem 4.2.2 yields an explicit criterion for deciding if a given bigraded rational is birational. In order to apply this criterion, we need to get the equations of a Rees algebra, which are usually done by using Gröbner basis computations. Therefore, it has certain disadvantages in practice. Moreover, in order to overcome these two drawbacks, we investigate how birationality can be detected by means of syzygies of the ideal I generated by the coordinates of the rational maps, instead of the whole collection of equations of I. Indeed, any criterion based on some syzygies of given bidegree of I will only rely on linear algebra computations.

We will need to consider not just Rees algebras of ideals or multi-Rees algebras, but the full notion of the Rees algebra of a module as discussed in [SUV03].

The following proposition gives an analogue of [DHS12, Theorem 3.2].

Proposition 4.2.3. *Let $\mathcal{F} : \mathbb{P}^n_k \times \mathbb{P}^m_k \dashrightarrow \mathbb{P}^s_k$ stand for a rational map defined by bihomogeneous polynomials $f_0(\mathbf{x},\mathbf{y}),\dots,f_s(\mathbf{x},\mathbf{y})$ in $R := k[\mathbf{x},\mathbf{y}]$ and set $I := (f_0(\mathbf{x},\mathbf{y}),\dots,f_s(\mathbf{x},\mathbf{y}))$. If the image of $\mathcal{F}$ has dimension $n+m$ and the submatrix of the syzygy matrix of I consisting of columns of bidegrees $(1,0)$ and $(0,1)$ has rank s (maximal possible), then $\mathcal{F}$ is birational onto its image.*

Proof. Note that the image of $\mathcal{F}$ is a projective subvariety of $\mathbb{P}^s_k$. Let $\mathbf{z}$ be homogeneous coordinates on $\mathbb{P}^s_k$ and set $S = k[\mathbf{z}]/\mathfrak{c}$ for the homogeneous coordinate ring of the image of $\mathcal{F}$. Since I is bihomogeneous, it admits a minimal syzygy matrix whose columns are bihomogeneous. Clearly, the independent syzygies of degrees either $(1,0)$ or $(0,1)$ will be columns of this matrix. Let M denote the submatrix with these columns. Then choose a matrix N with entries in $k[\mathbf{z}]$ such that $[\mathbf{z}]\cdot M = [\mathbf{x},\mathbf{y}]\cdot N$. Let $E = \operatorname{Coker}(N \otimes_{k[\mathbf{z}]} S)$.

We now introduce in the discussion the Rees algebras $\mathcal{R}_R(\operatorname{Coker} M)$ and $\mathcal{R}_S(E)$. Thus, one has

$$\mathcal{R}_R(\operatorname{Coker} M) = \frac{k[\mathbf{x},\mathbf{y},\mathbf{z}]}{([\mathbf{z}]M,\tau_1)} \quad \text{and} \quad \mathcal{R}_S(E) = \frac{k[\mathbf{x},\mathbf{y},\mathbf{z}]}{(\mathfrak{c},[\mathbf{x},\mathbf{y}]N,\tau_2)}$$

where τ_1 is the R-torsion of $\operatorname{Sym}_R(\operatorname{Coker} M)$ lifted to $k[\mathbf{x},\mathbf{y},\mathbf{z}]$ and, similarly, τ_2 is the S-torsion of $\operatorname{Sym}_S(E)$ lifted to $k[\mathbf{x},\mathbf{y},\mathbf{z}]$.

Note that, by definition, the Rees algebra $\mathcal{R}_S(E)$ of E and that of E modulo its torsion coincide. Since S is a domain, the latter module embeds into a free module over S. In particular, $\mathcal{R}_S(E)$ is a domain, i.e., $(\mathfrak{c},[\mathbf{x},\mathbf{y}]N,\tau_2)$ is a prime ideal.

We claim that $([\mathbf{z}]M,\tau_1) \subseteq (\mathfrak{c},[\mathbf{x},\mathbf{y}]N,\tau_2)$.

Indeed, let $G = G(\mathbf{x},\mathbf{y},\mathbf{z}) \in \tau_1$. Then there exists $F(\mathbf{x},\mathbf{y}) \in k[\mathbf{x},\mathbf{y}] \setminus \{0\}$ such that $F(\mathbf{x},\mathbf{y})G \subset (I_1(\mathbf{z}\cdot M)) \subset (\mathfrak{c},[\mathbf{x},\mathbf{y}]N,\tau_2)$. If $G \notin (\mathfrak{c},[\mathbf{x},\mathbf{y}]N,\tau_2)$ then $F(\mathbf{x},\mathbf{y}) \in (\mathfrak{c},[\mathbf{x},\mathbf{y}]N,\tau_2)$. By the definition of τ_2 there exists $H(\mathbf{z}) \in k[\mathbf{z}] \setminus \mathfrak{c}$ such that $H(\mathbf{z})F(\mathbf{x},\mathbf{y}) \in (\mathfrak{c},[\mathbf{x},\mathbf{y}]N)$. Recall that $[\mathbf{z}]\cdot M = [\mathbf{x},\mathbf{y}]\cdot N$. Evaluating $\mathbf{z} \mapsto \mathbf{f}$ would give $H(\mathbf{f})F(\mathbf{x},\mathbf{y}) = 0$ whence $F(\mathbf{x},\mathbf{y}) = 0$ since $H(\mathbf{f}) \neq 0$; this is a contradiction.

As a consequence, one has a surjective R-algebra map $\mathcal{R}_R(\operatorname{Coker} M) \twoheadrightarrow \mathcal{R}_S(E)$ and hence

$$\dim(\mathcal{R}_S(E)) \leq \dim(\mathcal{R}_R(\operatorname{Coker} M)).$$

Now $\dim(\mathcal{R}_S(E)) = \dim(S)+(n+1+m+1)-\operatorname{rank}(N\otimes S)$ and $\dim(\mathcal{R}_R(\operatorname{Coker} M)) = \dim(R)+s+1-\operatorname{rank}(M)$. Since $\dim(R) = \dim(\mathbb{P}^n_k \times \mathbb{P}^m_k)+2$ we have $\dim(R) - \dim(S) = 1$. Therefore the above inequality implies that

$$n+m+(\operatorname{rank}(M)-s) \leq \operatorname{rank}(N\otimes S).$$

Since $\operatorname{rank}(M) = s$ by assumption, we obtain $n+m \leq \operatorname{rank}(N\otimes S)$. Notice that $N\otimes S$ is a submatrix of the "concatenated" Jacobian dual matrix

$$\rho := \left(\frac{\Psi_{\mathbf{x}}\otimes S}{\Psi_{\mathbf{y}}\otimes S}\right)$$

in the notation introduced in the previous subsection.

Thus we have $\operatorname{rank}(\rho) \geq n + m$, whenever $\operatorname{rank}(M) = s$.

Claim: $\operatorname{rank}(\Psi_{\mathbf{x}} \otimes S) \leq n$ (and, similarly, $\operatorname{rank}(\Psi_{\mathbf{y}} \otimes S) \leq m$).

Assuming the claim, it follows that $\operatorname{rank}(\rho) \leq n+m$ and the equality happens if and only if $\operatorname{rank}(\Psi_{\mathbf{x}} \otimes S) = n$ and $\operatorname{rank}(\Psi_{\mathbf{y}} \otimes S) = m$. Therefore, the result follows from Theorem 4.2.2.

We now show that $\operatorname{rank}(\Psi_{\mathbf{x}} \otimes S) \leq n$. Indeed, consider the field $K := k(\mathbf{y})$ (the generic point of $\mathbb{P}^m_k$) and the rational map

$$\mathcal{F}' : \mathbb{P}^n_K \dashrightarrow \mathbb{P}^s_K$$

which is defined by the polynomials $f_0, \ldots, f_s$ viewed as polynomials in $K[\mathbf{x}]$. Let $S' := K[\mathbf{y}]/(\mathfrak{c})$ be the coordinate ring of the image of $\mathcal{F}'$ and consider the Jacobian dual matrix of $\mathcal{F}'$ over S' : $\Psi' \otimes S'$. Then, because of the field inclusion $k \hookrightarrow K$ the column space of $\Psi_{\mathbf{x}} \otimes S$ is contained in the column space of $\Psi' \otimes S$. Therefore, $\operatorname{rank}(\Psi_{\mathbf{x}} \otimes S) \leq \operatorname{rank}(\Psi' \otimes S') \leq n$ where the last inequality follows from Proposition 1.6.8. □

Remark 4.2.4. The mutual independence of the hypotheses in Proposition 4.2.3 has already been observed in [DHS12, bottom p. 409] in the case the source of $\mathcal{F}$ is a single projective space; likewise, in our setting. The most obvious situation where the number of linear syzygies of the required type is maximal and yet the image has smaller dimension is obtained as follows. We explain the projective version, the biprojective one being entirely similar.

Let $\mathcal{F} : \mathbb{P}^r_k \dashrightarrow \mathbb{P}^s_k$ be a birational map onto the image such that the linear syzygies of the defining forms $\mathbf{f}$ have maximal rank. Let $I \subset R$ denote the base ideal of $\mathcal{F}$. Consider the coordinate projection $\pi : \mathbb{P}^{r+1}_k \dashrightarrow \mathbb{P}^r_k$ defined by the first $r+1$ variables – thus, this corresponds to the ring extension $R = k[x_0, \ldots, x_r] \subset R[x_{r+1}]$. Since the latter is a faithfully flat extension, or directly, the module of syzygies of $\mathbf{f}$ on $R[x_{r+1}]$ is extended from the R-module of syzygies of $\mathbf{f}$, in particular the linear parts have the same rank as R-module or $R[x_{r+1}]$-module. At the other end the k-algebra $k[\mathbf{f}]$ is the same whether considered as a subalgebra of R or of $R[x_{r+1}]$. Therefore, the composite map $\mathcal{F} \circ \pi : \mathbb{P}^{r+1}_k \dashrightarrow \mathbb{P}^s_k$ has the same linear rank and the same image as $\mathcal{F}$. This shows that maximal linear rank does not imply maximal dimension of the image.

To get a biprojective analogue, it suffices to take a one-sided projection $\mathbb{P}^{n+1}_k \times \mathbb{P}^m_k \dashrightarrow \mathbb{P}^n_k \times \mathbb{P}^m_k$ to the source of a birational map $\mathbb{P}^n_k \times \mathbb{P}^m_k \dashrightarrow \mathbb{P}^s_k$ having maximal linear rank in the sense of the statement of the above proposition (e.g., an arbitrary Segre map).

It is of course clear that the full converse of the statement in the proposition is false. In the projective case, one can take a birational parameterization $\mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$ of a plane curve with parameters of degree ≥ 4 (hence, with linear rank 0). For example, take the parameters $x^4, y^4, x^3y + xy^3$ on $k[x, y]$. Since the image is a quartic curve, the map $\mathcal{F}$ defined by these parameters is automatically birational onto the curve.

To extract a biprojective example, compose the induced map

$$(id, \mathcal{F}) : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^1_k \times \mathbb{P}^2_k$$

with the Segre map $\mathbb{P}^1_k \times \mathbb{P}^2_k \dashrightarrow \mathbb{P}^5_k$. The result is clearly birational onto a subvariety of dimension 2 of the Segre embedding. However, a calculation with `Macaulay2` shows that the linear rank is only 3.

4.3 Syzygies of low degree of bigraded maps in the plane

In this section, we will focus on the linear syzygies of bigraded rational maps from $\mathbb{P}^1_k \times \mathbb{P}^1_k$ to $\mathbb{P}^2_k$. Under consideration will be the cases where the total degree of the biforms is 2 or 3. Note that in the projective case, the plane Cremona maps of these degrees are automatically de Jonquières maps. In both cases the base ideal is an ideal of 2-minors of a 3×2 matrix, with two linear syzygies or a linear syzyzy and a quadratic one, respectively [HS12].

In the case of a bigraded rational map defined by polynomials $\mathbf{f} := \{f_0, f_1, f_2\}$ of bidegree $(1, 1)$ it is very easy to see that, up to linear transformations in the source and target spaces, there are only two maps:

$$\begin{aligned} &\mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^2_k, \quad (x : y) \times (u : v) \mapsto (xu : yu : yv), \\ &\mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^2_k, \quad (x : y) \times (u : v) \mapsto (xu : yu : xv + yu). \end{aligned}$$

The first one is birational and $\mathbf{f}$ has two minimal syzygies of respective bidegrees $(1, 0)$ and $(0, 1)$, whereas the second one is not birational and $\mathbf{f}$ has exactly five linearly independant minimal syzygies. Therefore, birationality is here guaranteed by the existence of a linear syzygy. To understand to which extent such a result can be generalized to higher bidegree, some preliminary work is required. Our tools will be largely homologically oriented. Before going into the details, we first fix some notation.

We will switch from the previous notation $\mathcal{F}$ for a rational map to the symbol ϕ. Let k be an infinite field. Let $R := k[x, y; u, v]$ be the bigraded polynomial ring with weights defined by $\deg(x) = \deg(y) = (1, 0)$ and $\deg(u) = \deg(v) = (0, 1)$. Let f_0, f_1, f_2 be three bihomogeneous polynomials of bidegree (a, b) and set $I = (f_0, f_1, f_2) \subset R$. Consider the rational map defined by

$$\begin{array}{rcl} \phi : \mathbb{P}^1_k \times \mathbb{P}^1_k & \dashrightarrow & \mathbb{P}^2_k \\ ((\alpha : \beta), (\gamma : \delta)) & \mapsto & (f_0(\alpha, \beta, \gamma, \delta) : f_1(\alpha, \beta, \gamma, \delta) : f_2(\alpha, \beta, \gamma, \delta)). \end{array}$$

We assume throughout that ϕ is a dominant rational map and that the polynomials f_0, f_1, f_2 do not have a proper common factor in R, which, in a more geometric terminology, means that these polynomials define a zero-dimensional scheme in $\mathbb{P}^1_k \times \mathbb{P}^1_k$; let $\mathcal{B}$ denote this scheme – called the *base scheme* of ϕ. We note that the degree of $\mathcal{B}$, denoted $\deg(\mathcal{B})$ is equal to the bigraded Hilbert function of R/I for sufficiently high bidegree (μ, ν).

In analogy to a well-known degree formula in the projective case, one has the following degree formula in the biprojective counterpart (see, e.g., [Bot11, Lemma 7.4]):

$$\deg(\phi) = 2ab - \sum_{x \in \mathcal{B}} e_x(I), \tag{4.1}$$

where $\deg(\phi)$ stands for the field degree of the rational map ϕ and $e_x(I)$ stands for the Hilbert-Samuel multiplicity of I on the localization $R_{\mathfrak{p}}$ at the defining prime ideal $\mathfrak{p}$ of the point x. An important property of this latter multiplicity is that it is equal to the length of the residue of $R_{\mathfrak{p}}$ modulo the ideal generated by two general k-linear combinations of the polynomials f_0, f_1, f_2.

In particular, the degree formula (4.1) can be easily derived from this property as follows. Two linear forms in three variables define a point P in the target $\mathbb{P}^2_k$ and the corresponding linear combinations of f_0, f_1, f_2 define a subscheme Y_P in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ giving the inverse image of P by ϕ off $\mathcal{B}$. On an open subset of $\mathbb{P}^2_k$, or equivalently of the space of coefficients of the linear forms, the inverse image is a finite set. Hence, for a general point P, Y_P is a complete intersection of degree $2ab$ and is the union of a component X_P not meeting $\mathcal{B}$, of degree equal to the degree of the map (notice that X_P is reduced by Bertini's theorem) and a component with support $\mathcal{B}$ in which each point x has multiplicity equal to $e_x(I)$. Indeed, the multiplicity at a point x is constant and equal to its minimal value for two linear forms corresponding to a dense open subset of $\mathbb{P}^2_k$; this value is $e_x(I)$ by Proposition 1.3.24.

4.3.1 Counting linear syzygies

We denote by $\mathrm{Syz}(I) \subseteq R^3$ the module of syzygies of I. It is a bigraded module and the *linear syzygies* correspond to the graded parts $\mathrm{Syz}(I)_{(1,0)}$ and $\mathrm{Syz}(I)_{(0,1)}$. In other words, in the structural bigraded exact sequence

$$0 \longrightarrow Z_1 \longrightarrow R^3(-a,-b) \xrightarrow{(f_0,f_1,f_2)} I \longrightarrow 0,$$

we have the identification $\mathrm{Syz}(I) = Z_1(a,b)$. In the sequel, we will use the notation $K_\bullet$, $Z_\bullet$, $B_\bullet$ and $H_\bullet$ to refer to the terms, cycles, boundaries and homology modules of the Koszul complex of the sequence f_0, f_1, f_2. We set $\mathfrak{n} := (x,y) \cap (u,v) = (xu, xv, yu, yv) \subset R$ for the ideal generated by all monomials of bidegree $(1,1)$. Recall the following bigraded exact sequence in local cohomology

$$0 \longrightarrow H^0_{\mathfrak{n}}(R/I) \longrightarrow R/I \longrightarrow \bigoplus_{(\mu,\nu)\in\mathbb{Z}^2} H^0(\mathbb{P}^1_k \times \mathbb{P}^1_k, \mathcal{O}_{\mathcal{B}}(\mu,\nu)) \longrightarrow H^1_{\mathfrak{n}}(R/I) \longrightarrow 0. \tag{4.2}$$

In the following, an upper right star $*$ attached to an R-module will denote its Matlis dual.

Lemma 4.3.1. *Set* $\Omega := \{(\mu,\nu) \in \mathbb{Z}^2 \mid -2b < b\mu - a\nu < 2a\}$. *Then*

$$H^1_{\mathfrak{n}}(R/I)_{(\mu,\nu)} \simeq (H_1)^*_{(3a-\mu-2,3b-\nu-2)},$$

for every $(\mu,\nu) \in \Omega$

Proof. Let $K_\bullet$ be the Koszul complex of the sequence f_0, f_1, f_2. The argument hinges on the two spectral sequences associated to the double complex $C_q^p = C_{\mathfrak{n}}^p(K_q)$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_{\mathfrak{n}}^0(K_3) \arrow[r] \arrow[d] & C_{\mathfrak{n}}^0(K_2) \arrow[r] \arrow[d] & C_{\mathfrak{n}}^0(K_1) \arrow[r] \arrow[d] & C_{\mathfrak{n}}^0(K_0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{\mathfrak{n}}^1(K_3) \arrow[r] \arrow[d] & C_{\mathfrak{n}}^1(K_2) \arrow[r] \arrow[d] & C_{\mathfrak{n}}^1(K_1) \arrow[r] \arrow[d] & C_{\mathfrak{n}}^1(K_0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{\mathfrak{n}}^2(K_3) \arrow[r] \arrow[d] & C_{\mathfrak{n}}^2(K_2) \arrow[r] \arrow[d] & C_{\mathfrak{n}}^2(K_1) \arrow[r] \arrow[d] & C_{\mathfrak{n}}^2(K_0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{\mathfrak{n}}^3(K_3) \arrow[r] \arrow[d] & C_{\mathfrak{n}}^3(K_2) \arrow[r] \arrow[d] & C_{\mathfrak{n}}^3(K_1) \arrow[r] \arrow[d] & C_{\mathfrak{n}}^3(K_0) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

One of them abuts at step two with:

$$\begin{array}{cccc}
H_{\mathfrak{n}}^0(H_3) & H_{\mathfrak{n}}^0(H_2) & H_{\mathfrak{n}}^0(H_1) & H_{\mathfrak{n}}^0(H_0) \\
H_{\mathfrak{n}}^1(H_3) & H_{\mathfrak{n}}^1(H_2) & H_{\mathfrak{n}}^1(H_1) & H_{\mathfrak{n}}^1(H_0) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}$$

The other one gives at step one:

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
H_{\mathfrak{n}}^2(K_3) & \longrightarrow & H_{\mathfrak{n}}^2(K_2) & \longrightarrow & H_{\mathfrak{n}}^2(K_1) & \longrightarrow & H_{\mathfrak{n}}^2(K_0) \\
H_{\mathfrak{n}}^3(K_3) & \longrightarrow & H_{\mathfrak{n}}^3(K_2) & \longrightarrow & H_{\mathfrak{n}}^3(K_1) & \longrightarrow & H_{\mathfrak{n}}^3(K_0)
\end{array}$$

Notice that for every $(\mu, \nu) \in \Omega$, $H_{\mathfrak{n}}^2(R)_{(\mu,\nu)} = 0$, hence $H_{\mathfrak{n}}^2(K_j)_{(\mu,\nu)} = 0$ for all $j = 0, \ldots, 3$. Moreover $H_{\mathfrak{n}}^3(K_q) \simeq K_{3-q}^*[2-3a, 2-3b]$ for every $q = 0, \ldots, 3$. Therefore, this spectral sequence at step two in bidegree $(\mu, \nu) \in \Omega$ gives

$$\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
H_0^*[2-3a, 2-3b] & H_1^*[2-3a, 2-3b] & H_2^*[2-3a, 2-3b] & H_3^*[2-3a, 2-3b]
\end{array}$$

By comparing the two spectral sequences, one has $H_2 = H_3 = 0$ and for all $(\mu, \nu) \in \Omega$, we have

$$H_{\mathfrak{n}}^1(R/I)_{(\mu,\nu)} \simeq (H_1)^*_{(3a-\mu-2, 3b-\nu-2)} = {}^*\mathrm{Hom}_R((H_1)_{(3a-\mu-2, 3b-\nu-2)}, k)$$

as claimed. $\square$

Next, let throughout $I^{\mathrm{sat}} := I : \mathfrak{n}^\infty$ denote the saturation of I with respect to $\mathfrak{n} = (x, y) \cap (u, v)$.

Proposition 4.3.2. *With the above notation, one has*

(i) *If* $0 < b < 2a$ *then* $\dim_k \operatorname{Syz}(I)_{(1,0)} = \deg(\mathcal{B}) - \dim_k (R/I^{\mathrm{sat}})_{(2a-3,2b-2)}$.

(ii) *If* $0 < a < 2b$ *then* $\dim_k \operatorname{Syz}(I)_{(0,1)} = \deg(\mathcal{B}) - \dim_k (R/I^{\mathrm{sat}})_{(2a-2,2b-3)}$.

Proof. (i) Observe that since $(B_1)_{(a+1,b)} = 0$, Lemma 4.3.1 and (4.2) imply that

$$\begin{aligned}\dim_k \operatorname{Syz}(I)_{(1,0)} &= \dim_k (Z_1)_{(a+1,b)} = \dim_k (H_1)_{(a+1,b)} \\ &= \dim_k H^1_{\mathfrak{n}}(R/I)_{(2a-3,2b-2)} \\ &= \deg(\mathcal{B}) - \dim_k (R/I^{\mathrm{sat}})_{(2a-3,2b-2)}.\end{aligned}$$

(ii) is proved similarly. □

4.3.2 Birationality of bidegree $(1, 1)$ maps

As noticed at the beginning of Section 4.3, the birationnality of bidegree $(1, 1)$ maps can be easily characterized by means of linear syzygies. Below, we reprove this fact using Proposition 4.3.2.

Proposition 4.3.3. *Let* $\phi : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$ *be a dominant rational map given by bihomogeneous polynomials* f_0, f_1, f_2 *of bidegree* $(1, 1)$ *without common factor in* $R \setminus k$. *The following three assertions are equivalent:*

(i) ϕ *is birational,*

(ii) *the polynomials* f_0, f_1, f_2 *have a nonzero bidegree* $(1, 0)$ *syzygy,*

(iii) *the polynomials* f_0, f_1, f_2 *have a nonzero bidegree* $(0, 1)$ *syzygy.*

Proof. The map ϕ is birational if and only if $\deg(\phi) = 1$ and by the degree formula (4.1) this is equivalent to having $\deg(\mathcal{B}) = \sum_{x\in\mathcal{B}} e_x(I) = 1$. Now, by Proposition 4.3.2 with $a = b = 1$, we have

$$\begin{aligned}\dim_k \operatorname{Syz}(I)_{(1,0)} &= \deg(\mathcal{B}) - \dim_k (R/I^{\mathrm{sat}})_{(-1,0)} = \deg(\mathcal{B}), \\ \dim_k \operatorname{Syz}(I)_{(0,1)} &= \deg(\mathcal{B}) - \dim_k (R/I^{\mathrm{sat}})_{(0,-1)} = \deg(\mathcal{B}).\end{aligned}$$

Therefore, ϕ is birational if and only if $\dim_k \operatorname{Syz}(I)_{(1,0)} = 1$, or equivalently if and only if $\dim_k \operatorname{Syz}(I)_{(0,1)} = 1$. □

The above birationality criterion can be translated into a numerical effective test. For that purpose, set

$$f_i(x, y; u, v) := c_{i,0}xu + c_{i,1}xv + c_{i,2}yu + c_{i,3}yv.$$

We seek a triple of polynomials g_0, g_1, g_2 that are linear forms in x, y (or equivalently u, v) and such that $\sum g_i f_i \equiv 0$. Such a triple can be found as elements in the kernel of a matrix M whose columns are filled with the coefficients of the polynomials

$$xf_0, yf_0, xf_1, yf_1, xf_2, yf_2$$

in a basis of bihomogeneous polynomials of bidegree $(2,1)$, typically

$$x^2u, x^2v, y^2u, y^2v, xyu, xyv.$$

The matrix M is hence the following 6×6-matrix

$$M = \begin{bmatrix} c_{0,0} & 0 & c_{1,0} & 0 & c_{2,0} & 0 \\ c_{0,1} & 0 & c_{1,1} & 0 & c_{2,1} & 0 \\ 0 & c_{0,2} & 0 & c_{1,2} & 0 & c_{2,2} \\ 0 & c_{0,3} & 0 & c_{1,3} & 0 & c_{2,3} \\ c_{0,2} & c_{0,0} & c_{1,2} & c_{1,0} & c_{2,2} & c_{2,0} \\ c_{0,3} & c_{0,1} & c_{1,3} & c_{1,1} & c_{2,3} & c_{2,1} \end{bmatrix}.$$

As a consequence, in Proposition 4.3.3, we could add as a fourth item the statement that $\det(M) = 0$.

4.3.3 Birationality of bidegree $(1,2)$ maps

Before providing our birationality criteria in this case, we establish the following technical lemma.

Lemma 4.3.4. *Let $\phi : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$ be a dominant rational map defined by bihomogeneous polynomials $f_0, f_1, f_2 \in R$ of bidegree $(1,2)$ without common factor in $R \setminus k$. Set $I = (f_0, f_1, f_2) \subset R$. Then, we have*

(i) $\dim_k \operatorname{Syz}(I)_{(0,1)} = \deg(\mathcal{B}) - 2$,

(ii) $\dim_k \operatorname{Syz}(I)_{(1,1)} = \deg(\mathcal{B})$.

Proof. (i) Since $(a,b) = (1,2)$, Proposition 4.3.2 shows that

$$\dim_k \operatorname{Syz}(I)_{(0,1)} = \deg \mathcal{B} - \dim_k (R/I^{\mathrm{sat}})_{(0,1)}.$$

If $\deg(\mathcal{B}) = 1$, then the base scheme of ϕ consists of a single simple point. Therefore $I^{\mathrm{sat}} = (x,u)$ up to a coordinate change, hence $\dim_k(R/I^{\mathrm{sat}})_{(0,1)} = 1$ and we deduce that there is no nonzero syzygy of bidegree $(0,1)$, as claimed.

Now, we assume that $\deg(\mathcal{B}) \geq 2$. Since $\dim_k(R_{(0,1)}) = 2$, it suffices to show that $I^{\mathrm{sat}}_{(0,1)} = 0$. Thus, suppose that $I^{\mathrm{sat}}_{(0,1)} \neq 0$; without loss of generality we may assume that $u \in I^{\mathrm{sat}}$. Now, since $\deg(\mathcal{B}) \geq 2$, there exists a form $q(x,y)$ of bidegree $(2,0)$ such that $I^{\mathrm{sat}} \subset (q,u)$. But since $f_i \in I^{\mathrm{sat}}$, we have

$$f_i = a_i q + b_i u, \quad i = 0,1,2.$$

As $\deg(f_i) = (1,2)$, we deduce that $a_0 = a_1 = a_2 = 0$ and that u divides f_i for all $i = 0,1,2$; this is a contradiction.

(ii) By inspecting the shifts of bidegrees in the Koszul complex of the sequence f_0, f_1, f_2, and taking into account that the f_i's are of bidegree $(1,2)$, we observe that

$$\dim_k \operatorname{Syz}(I)_{(1,1)} = \dim_k (Z_1)_{(2,3)} = \dim_k (H_1)_{(2,3)}.$$

Applying Lemma 4.3.1 (we have $(-1,1) \in \Omega$), we get the equality

$$\dim_k (H_1)_{(2,3)} = \dim_k H^1_{\mathfrak{n}}(R/I)_{(-1,1)}.$$

Now, the exact sequence (4.2) restricted to bidegree $(-1,1)$ yields the equality

$$\dim_k H^1_{\mathfrak{n}}(R/I)_{(-1,1)} = \deg(\mathcal{B}) - \dim_k(R/I^{\mathrm{sat}})_{(-1,1)} = \deg(\mathcal{B})$$

and the claimed equality is proved. □

Theorem 4.3.5. *Let $\phi : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$ be a dominant rational map given by bihomogeneous polynomials $f_0, f_1, f_2 \in R$ of bidegree $(1,2)$ without common factor in $R \setminus k$. Setting $I = (f_0, f_1, f_2) \subset R$, the following four assertions are equivalent:*

(i) *ϕ is birational,*
(ii) $\deg(\mathcal{B}) = 3$, *and hence I is generically a complete intersection,*
(iii) $\dim_k \mathrm{Syz}(I)_{(0,1)} = 1$,
(iv) $\dim_k \mathrm{Syz}(I)_{(1,1)} = 3$.

Proof. Since (ii) is equivalent to both (iii) and (iv) by Lemma 4.3.4, it suffices to show that (i) and (ii) are equivalent.

Now, by the degree formula (4.1), we have

$$\sum_{x\in\mathcal{B}} e_x(I) = 4 - \deg(\phi) \leq 3. \tag{4.3}$$

Moreover, by property of the Hilbert-Samuel multiplicity we also have $\deg(\mathcal{B}) \leq \sum_{x\in\mathcal{B}} e_x(I)$ with equality if and only if I is generically a complete intersection, by Proposition 1.3.24. Therefore, if $\deg(\mathcal{B}) = 3$ then $\sum_{x\in\mathcal{B}} e_x(I) = 3$, so that I is generically a complete intersection, and from (4.3) we deduce that $\deg(\phi) = 1$, i.e. ϕ is birational. Thus, we have just proved that (ii) implies (i). To prove the converse, suppose that $\deg(\mathcal{B}) \neq 3$. Then, necessarily, $\deg(\mathcal{B}) \leq 2$ and this implies that I is generically a complete intersection. Therefore $\deg(\mathcal{B}) = \sum_{x\in\mathcal{B}} e_x(I) \leq 2$ and hence ϕ cannot be birational by (4.3). It follows that (i) is equivalent to (ii). □

Remark 4.3.6. Item (iii) provides us with a minimal syzygy of bidegree $(0,1)$ so that $u(\sum_{i=0}^2 a_i f_i) = v(\sum_{i=0}^2 b_i f_i)$ for some a_i's and b_i's in k. It follows that there exist three polynomials p, q, r of bidegree $(1,1)$ such that $I = (pu, pv, qu + rv)$. Therefore, I is a perfect ideal generated by the 2-minors of the matrix

$$M := \begin{pmatrix} v & q \\ -u & r \\ 0 & -p \end{pmatrix}.$$

Thus, one could add yet another equivalent condition to Theorem 4.3.5, namely that the ideal I has a free R-resolution of the form

$$0 \longrightarrow R(-1,-3) \oplus R(-2,-3) \xrightarrow{M} R(-1,-2)^3 \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Note that, in this format, three independent $(1,1)-$syzygies of I are

$$(xv, -xu, 0), (yv, -yu, 0), (q, r, -p),$$

the first two being non-minimal. Hence, in contrast to the spirit of Proposition 4.3.3, in item (iv) of the above theorem $\mathrm{Syz}(I)_{(1,1)}$ is not spanned by 3 minimal syzygies of bidegree $(1,1)$.

Corollary 4.3.7. *If ϕ is birational, then* $\dim_k \operatorname{Syz}(I)_{(1,0)} = 0$.

Proof. This is an immediate consequence of Remark 4.3.6, using the fact that p and $qu + rv$ have no proper common factor as I has codimension 2. □

Remark 4.3.8. Theorem 4.3.5 and Corollary 4.3.7 provide another illustration that the converse of Proposition 4.2.3 does not hold, here for some dominant rational maps from $\mathbb{P}^1_k \times \mathbb{P}^1_k$ to $\mathbb{P}^2_k$.

4.3.4 Birationality of bidegree $(2, 2)$ maps

Unlike the cases of rational maps of bidegree $(1, 1)$ or $(1, 2)$, the linear syzygies associated to a given parameterization are not enough to give birational criterion in higher bidegrees. Yet, in the case of bidegree $(2, 2)$, we are able to describe a complete list of such birational maps.

Let $\phi : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$ be a dominant rational map given by bihomogeneous polynomials $f_0, f_1, f_2 \in R$ of bidegree $(2, 2)$. We set $I = (f_0, f_1, f_2)$ and we denote by $\mathcal{B}$ the base scheme of ϕ which is assumed to be zero-dimensional (i.e. supported on a finite set of points). The degree formula yields the equality

$$\deg(\phi) = 8 - \sum_{x \in \mathcal{B}} e_x(I). \tag{4.4}$$

And since $\deg(\phi) \geq 1$, we deduce that

$$1 \leq \deg \mathcal{B} \leq \sum_{x \in \mathcal{B}} e_x(I) \leq 7.$$

For a codimension 2 bihomogeneous prime ideal $\mathfrak{p} \supset I$, we will set $d_{\mathfrak{p}} := \dim_k(R_{\mathfrak{p}}/I_{\mathfrak{p}})$ ("point degree") and let as before $e_{\mathfrak{p}}$ denote the Hilbert-Samuel multiplicity of I on $\mathfrak{p}$. As is well-known, $e_{\mathfrak{p}} \geq d_{\mathfrak{p}}$, with equality if and only if $I_{\mathfrak{p}}$ is a complete intersection (a fact we have already used in the proof of Theorem 4.3.5).

By abuse, one may think of $\mathfrak{p}$ as belonging to $\mathcal{B}$; as such it is the defining prime ideal of a point $p \in \mathbb{P}^1_k \times \mathbb{P}^1_k$.

First, we remark the following:

Lemma 4.3.9. *Assume the above notation. If ϕ is birational then* $\deg \mathcal{B} \leq 6$. *Moreover, if* $\deg \mathcal{B} = 6$ *then I is perfect with a minimal resolution of the form :*

$$0 \longrightarrow R(-3, -3)^2 \longrightarrow R(-2, -2)^3 \longrightarrow I \longrightarrow 0,$$

Proof. Two general k-linear combinations of the f_i's define a scheme on the support of $\mathcal{B}$ plus an additional simple point q that does not share any coordinate with the base points (the argument is similar to the one given in the last paragraph of the introduction of Section 4.3). If $\deg \mathcal{B} = 7$ then choose a point p in the support of $\mathcal{B}$, and if $\deg \mathcal{B} = 6$ then take p as the point for which the Hilbert-Samuel multiplicity is not equal to its degree. After a linear change of coordinates on the source and target spaces, we may and will assume that $q = (1, 0) \times (1, 0)$, $p = (0, 1) \times (0, 1)$, i.e. $\mathfrak{q} = (y, v)$, $\mathfrak{p} = (x, u)$ and $(f_0, f_1) = (y, v) \cap J \cap K \cap L$ where $J \subset (x, u)$ is unmixed

with associated primes corresponding to the support of $\mathcal{B}$, while K (respectively L) is (x,y)-primary (respectively (u,v)-primary) of degree 4 and generically a complete intersection (i.e. the image of K (respectively L) in $k(u,v)[x,y]$ (respectively $k(x,y)[u,v]$) is a complete intersection).

Now, we observe that the defining ideal of $\mathcal{B}$ is either J if $\deg \mathcal{B} = 7$, or either $J\colon {}_R(x,u)$ if $\deg \mathcal{B} = 6$. The latter is a consequence of liaison, see Proposition 1.3.11. Furthermore, we have that $K\colon {}_R(x,y) = (x,y)^2$ and $L\colon {}_R(u,v) = (u,v)^2$. Therefore

$$(f_0, f_1)\colon {}_R(xv, yu) = (J\colon {}_R(x,u)) \cap (x,y)^2 \cap (u,v)^2$$

and in particular $((f_0,f_1)\colon {}_R(xv,yu))_{(2,2)} = (J\colon {}_R(x,u))_{(2,2)}$. By rewriting $f_0 = Axv + Byu$ and $f_1 = Cxv + Dyu$, we get

$$(f_0, f_1)\colon {}_R(xv, yu) = (f_1, f_2, AD - BC).$$

Now, $f_2 \in I \subseteq (J\colon {}_R(x,u)) \cap (x,y)^2 \cap (u,v)^2$. As the f_i's are linearly independent, f_2 is a nonzero multiple of $AD - BC$ modulo f_0 and f_1, hence we should have $I = (f_0,f_1)\colon {}_R(xv,yu)$ which is unmixed of degree 6. This rules out the possibility of having $\deg \mathcal{B} = 7$ and concludes the proof. □

Fix $\mathfrak{p} \in \mathcal{B}$. By changing coordinates, there is no loss of generality in assuming $p = (0,1) \times (0,1)$, i.e., $\mathfrak{p} = (x,u)$. We now discuss how the strict inequality $e_{\mathfrak{p}} > d_{\mathfrak{p}}$ reflects in the form of the generators of I. For this, we resort to explicit computations on the affine piece $y = v = 1$. Now, one has $I \subset (x,u)_{(2,2)}$ and the latter is spanned by the monomials

$$xyv^2, y^2uv, x^2v^2, xyuv, y^2u^2, x^2uv, xyu^2, x^2u^2.$$

Therefore, for $j = 0,1,2$, $g_j := f_j(x,1,u,1)$ is a k-linear combination of the monomials $\{x, u, x^2, xu, u^2, x^2u, xu^2, x^2u^2\}$. Set $J := (g_0, g_1, g_2) \subset S := k[x,u]$.

Consider the total order $>$ on the monomials of S by decreeing

$$1 > x > u > x^2 > xu > u^2 > x^3 > x^2u > xu^2 > u^3 > x^4 > x^3u > x^2u^2 > \cdots .$$

Let $\mathrm{in}(J)$ be the initial ideal J with respect to the order $>$. Therefore $d_{\mathfrak{p}} = \dim(k[x,u]/\mathrm{in}(J))$.

Lemma 4.3.10. *With the above notation, the equality $d_{\mathfrak{p}} = e_{\mathfrak{p}}$ holds except in the following cases:*

(i) $J = (x,u)^2$, *in which case* $d_{\mathfrak{p}} = 3$, $e_{\mathfrak{p}} = 4$,

(ii) $J = (x^2 + \lambda u^2 + \mu xu, x^2u, xu^2), \lambda \neq 0$, *in which case* $d_{\mathfrak{p}} = 5$, $e_{\mathfrak{p}} = 6$.

(iii) $J = (xu + \mu u^2 + xu^2, x^2 + \alpha xu + \beta u^2, x^2u^2), \mu \neq 0$, *in which case one has* $4 \leq \deg \mathcal{B} \leq 5$ *and* $\sum_{x \in \mathcal{B}} e_x(I) \leq 6$.

Proof. Write $d_{\mathfrak{p}} = n$, $n \geq 1$. We will argue in terms of the initial ideal $\mathrm{in}(J)$.

We first consider the easy case where $x \in \mathrm{in}(J)$ or $u \in \mathrm{in}(J)$. The argument will be totally symmetric in the two cases, so it suffices to consider one of them, say, $x \in \mathrm{in}(J)$. Then $u^n \in \mathrm{in}(J)$ and $u^{n-1} \notin \mathrm{in}(J)$, hence $\mathrm{in}(J) = (x, u^n)$. Letting

then $h_1, h_2 \in J$ be polynomials such that $\mathrm{in}(h_1) = x, \mathrm{in}(h_2) = u^n$, $\{h_1, h_2\}$ will be a Gröbner basis of J. Therefore, J is a complete intersection, hence $d_{\mathfrak{p}} = e_{\mathfrak{p}}$.

Next consider the case where neither $x \in \mathrm{in}(J)$ nor $u \in \mathrm{in}(J)$. Notice that $x^2 \in \mathrm{in}(J)$, as otherwise $\mathrm{in}(J) \subset (u)$. We now analyse all possibilities: both xu and u^2 belong to $\mathrm{in}(J)$; $xu \in \mathrm{in}(J)$ and $u^2 \notin \mathrm{in}(J)$; $xu \notin \mathrm{in}(J)$ and $u^2 \in \mathrm{in}(J)$; and neither xu nor u^2 belongs to $\mathrm{in}(J)$, respectively.

Case 1: $xu, u^2 \in \mathrm{in}(J)$. By the chosen order of the monomials, one must have $\{x^2, xu, u^2\} \subset J$. But certainly $J \subset (x^2, xu, u^2)$ since it does not contain either x or u and further $\{x^2u, xu^2, x^2u^2\} \subset (x^2, xu, u^2)$. This shows that that $J = (x, u)^2$, in which case $d_{\mathfrak{p}} = 3$ and $e_{\mathfrak{p}} = 4$.

Case 2: $xu \in \mathrm{in}(J)$ and $u^2 \notin \mathrm{in}(J)$. Hence $e_{\mathfrak{p}} \geq d_{\mathfrak{p}} \geq 4$. Write

$$g_0 = x^2 + \lambda u^2 + xul,\ \ g_1 = xu + \mu u^2 + xul',\ \ g_2 = xul''$$

where $(\lambda, \mu) \neq (0, 0)$ and l, l', l''belong to the k-vector space spanned by x, u, xu.

If $\mu = 0$ (hence $\lambda \neq 0$) then $xu \in J$ and $x^2 + \lambda u^2 \in J$, thus showing that $e_{\mathfrak{p}} \leq 4$, and hence that $d_{\mathfrak{p}} = e_{\mathfrak{p}} = 4$.

If $\mu \neq 0$, pick explicit coefficients for l, l', l'':

$$\begin{aligned} g_0 &= x^2 + \lambda u^2 + a_1x^2u + b_1xu^2 + c_1x^2u^2, \\ g_1 &= xu + \mu u^2 + a_2x^2u + b_2xu^2 + c_2x^2u^2, \\ g_2 &= a_3x^2u + b_3xu^2 + c_3x^2u^2. \end{aligned}$$

We are led to consider the following sub-cases:

(a) If $(a_3, b_3) = (0, 1)$, then $xu^2 \in J$, hence $x^2u \in J$. It follows that $J \supset (x^2 + \lambda u^2, xu + \mu u^2)$ which show that $e_{\mathfrak{p}} = 4$.

(b) If $(a_3, b_3) = (1, \nu)$, then we write:

$$\begin{aligned} g_0 &= x^2 + \lambda u^2 + axu^2 + bx^2u^2, \\ g_1 &= xu + \mu u^2 + cxu^2 + dx^2u^2, \\ g_2 &= x^2u + \nu xu^2 + ex^2u^2. \end{aligned}$$

Therefore $xg_1 - g_2 = (\mu - \nu)xu^2 + (c - e)x^2u^2 + dx^3u^2$. If $\mu \neq \nu$ then $xu^2 \in J$. Hence $J \supset (x^2 + \lambda u^2, xu + \mu u^2)$ which shows that $e_{\mathfrak{p}} = 4$. Conversely, if $\mu = \nu = 1$, then we write:

$$\begin{aligned} g_0 &= x^2 - \lambda xu + \alpha xu^2 + \beta x^2u^2, \\ g_1 &= xu + u^2 + \gamma xu^2 + \delta x^2u^2, \\ g_2 &= x^2u + xu^2 + \xi x^2u^2. \end{aligned}$$

Taking $g_2 - ug_0 \in J$ gives $xu^2 \in J$, therefore $J \supset (x^2 + \lambda u^2, xu + u^2)$ which shows that $e_{\mathfrak{p}} = 4$.

(c) If $(a_3, b_3) = (0, 0)$, then we write:

$$\begin{aligned} g_0 &= x^2 + \lambda u^2 + ax^2u + bxu^2, \\ g_1 &= xu + \mu u^2 + cx^2u + dxu^2, \\ g_2 &= x^2u^2. \end{aligned}$$

Since $x^2u^2 \in I$, $\mathcal{B}$ has only one prime $\mathfrak{p}$, which shows that $\deg \mathcal{B} = d_{\mathfrak{p}}$ and $\sum_{x\in\mathcal{B}} e_x(I) = e_{\mathfrak{p}}$. It is easy to see that $x^4 \in J$, hence $4 \leq d_{\mathfrak{p}} \leq 5$. Moreover $(x,u)^4 \subset J$, hence $x^2u + \mu xu^2 \in J$. We can write

$$\begin{aligned} g_0 &= x^2 + \lambda u^2 + (b - a\mu)xu^2, \\ g_1 &= xu + \mu u^2 + (d - c\mu)xu^2, \\ g_2 &= x^2u^2. \end{aligned}$$

If $b - a\mu = d - c\mu = 0$ then $J \supset (x^2 + \lambda u^2, xu + \mu u^2)$ which shows that $d_{\mathfrak{p}} = e_{\mathfrak{p}} = 4$. Conversely, if $(b - a\mu)^2 + (d - c\mu)^2 \neq 0$ then

$$J \supset (xu + \mu u^2 + xu^2, x^2 + \alpha xu + \beta u^2),$$

therefore $e_{\mathfrak{p}} \leq 6$.

<u>Case 3:</u> $xu \notin \mathrm{in}(J)$ and $u^2 \in \mathrm{in}(J)$. Therefore $d_{\mathfrak{p}} \geq 4$. We write

$$\begin{aligned} g_0 &= x^2 + \lambda xu + xu(ax + bu + cxu), \\ g_1 &= u^2 + xu(a'x + b'u + c'xu), \\ g_2 &= xu(a''x + b''u + c''xu). \end{aligned}$$

Since $g_1 - a'ug_0 = u^2(1 + \alpha x + \beta x^2 + \gamma xu + \delta x^2u) \in J$, hence $u^2 \in J$, therefore $x^2 + \lambda xu + ax^2u \in J$. It follows that

$$(u^2, x^2 + \lambda xu + ax^2u) \subset J.$$

Consider the codimension 2 homogeneous ideal $G = (u^2, x^2z + \lambda xuz + ax^2u) \subset T := k[x,u,z]$ obtained by homogenizing the two generators of the leftmost ideal in the above inclusion. Then T/G is a complete intersection of degree 6 supported on two points in $\mathbb{P}^2_k$, namely, $q_1 = (0:0:1)$ and $q_2 = (1:0:0)$. Letting $\mathfrak{q}_1, \mathfrak{q}_2$ denote the respective defining prime ideals, one has $d_{\mathfrak{q}_2} = e_{\mathfrak{q}_2} = 2$, hence $d_{\mathfrak{q}_1} = e_{\mathfrak{q}_1} = 4$. Since $e_{\mathfrak{p}} \leq e_{\mathfrak{q}_1} = 4$, therefore $e_{\mathfrak{p}} = d_{\mathfrak{p}} = 4$.

<u>Case 4:</u> $xu, u^2 \notin \mathrm{in}(J)$. It is seen that $d_{\mathfrak{p}} \geq 5$. We can write

$$\begin{aligned} g_0 &= x^2 + \lambda u^2 + \mu xu + xu(ax + bu + cxu), \\ g_1 &= a_1x^2u + b_1xu^2 + c_1x^2u^2, \\ g_2 &= a_2x^2u + b_2xu^2 + c_2x^2u^2, \end{aligned}$$

where $\lambda \neq 0$. Again, consider the following sub-cases:

(a) If $a_1 = a_2 = 0$, then $(b_1, b_2) \neq (0,0)$. Therefore, we obtain

$$\begin{aligned} g_0 &= x^2 + \lambda u^2 + \mu xu + ax^2u, \\ g_1 &= xu^2, \\ g_2 &= x^2u^2. \end{aligned}$$

Since $g_2 = xg_1$, hence $J = (g_0, g_1)$ is a complete intersection, therefore $e_{\mathfrak{p}} = d_{\mathfrak{p}}$.

(b) If $a_1 = 1$ and $b_2 = 0$. We obtain

$$\begin{aligned} g_0 &= x^2 + \lambda u^2 + \mu xu + \alpha x^2 u + \beta x u^2, \\ g_1 &= x^2 u + \gamma x u^2, \\ g_2 &= x^2 u^2. \end{aligned}$$

If $\gamma = 0$, then $J = (g_0, g_1)$ is a complete intersection, hence $e_{\mathfrak{p}} = d_{\mathfrak{p}}$. We deduce that $\gamma \neq 0$ and hence $xu^3 = \gamma^{-1}(ug_1 - g_2) \in J$. Write

$$\begin{aligned} g_0 &= x^2 + \lambda u^2 + \mu xu + \alpha x^2 u, \\ g_1 &= x^2 u + \gamma x u^2, \\ g_2 &= x^2 u^2. \end{aligned}$$

It is easy to see that $u^4 \in J$. If $u^3 \notin \mathrm{in}(J)$, then $d_{\mathfrak{p}} = e_{\mathfrak{p}} = 7$. Conversely, if $u^3 \in \mathrm{in}(J)$, then $5 \leq d_{\mathfrak{p}} \leq 6$. Moreover, since $xu^2 \notin \mathrm{in}(J)$, we obtain $\mathrm{in}(J) = (x^2, u^3)$. Therefore, there exists a Gröbner basis of J of two polynomials, hence J is a complete intersection.

(c) If $a_1 = 1$ and $b_2 = 1$. We obtain

$$\begin{aligned} g_0 &= x^2 + \lambda u^2 + \mu xu + \alpha x^2 u^2, \\ g_1 &= x^2 u + \beta x^2 u^2 = x^2 u(1 + \beta u), \\ g_2 &= xu^2 + \gamma x^2 u^2 = xu^2(1 + \gamma x). \end{aligned}$$

It follows that $x^2 u, xu^2 \in J$. We write

$$ug_0 = x^2 u + \lambda u^3 + \mu x u^2 + a x^2 u^3 \in J,$$

which shows that $u^3 \in J$. It follows that $d_{\mathfrak{p}} = 5$ and $e_{\mathfrak{p}} = 6$. In this case

$$J = (x^2 + \lambda u^2 + \mu xu, x^2 u, xu^2), \lambda \neq 0.$$

□

Now, we derive consequences of the above technical lemma and the degree formula (4.4).

Corollary 4.3.11. *Let $\phi : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$ be a dominant rational map given by bihomogeneous polynomials $f_0, f_1, f_2 \in R$ of bidegree $(2, 2)$ without common factor in $R \setminus k$. If ϕ is birational then $\deg \mathcal{B} = 6$.*

Theorem 4.3.12. *Let $\phi : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$ be a dominant rational map given by bihomogeneous polynomials $f_0, f_1, f_2 \in R$ of bidegree $(2, 2)$ without common factor in $R \setminus k$. Assume that the point in $\mathcal{B}$ with the largest multiplicity is the point $\mathfrak{p} := (x, u)$. Let J be the ideal generated by $f_0(x, 1, u, 1), f_1(x, 1, u, 1), f_2(x, 1, u, 1)$. Then ϕ is birational if and only if $\deg \mathcal{B} = 6$ and $J = (x, u)^2$ or*

$$J = (x^2 + \lambda u^2 + \mu xu, x^2 u, xu^2), \ \lambda \neq 0.$$

Proof. First, assume that ϕ is birational. Corollary 4.3.11 shows that $\deg(\mathcal{B}) = 6$. Moreover, by (4.4) we have $\sum_{x\in\mathcal{B}} e_{\mathfrak{p}} = 7$ and hence Lemma 4.3.10 implies that $(d_{\mathfrak{p}}, e_{\mathfrak{p}}) = (3,4)$ or $(d_{\mathfrak{p}}, e_{\mathfrak{p}}) = (5,6)$, that is to say $J = (x,u)^2$ or $J = (x^2 + \lambda u^2 + \mu xu, x^2u, xu^2), \lambda \neq 0$, as claimed.

For the converse it suffices to prove that if $\deg \mathcal{B} = 6$ and $J = (x,u)^2$ or $J = (x^2 + \lambda u^2 + \mu xu, x^2u, xu^2), \lambda \neq 0$, then ϕ is birational, i.e. $\sum_{x\in\mathcal{B}} e_x(I) = 7$. We now analyse these two possibilities.

Case 1: Suppose that $\deg \mathcal{B} = 6$ and $J = (x,u)^2 \in \mathcal{B}$. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ denote the primes of $\mathcal{B}$ other than $\mathfrak{p} = (x,u)$. Since $d_{\mathfrak{p}} = 3$, then $\sum_{i=1}^r d_{\mathfrak{p}_i} = 3$. Thus, in order to have the total sum $\sum_{x\in\mathcal{B}} e_x(I) = 7$ is now tantamount to having $J_{\mathfrak{p}_i}$ a complete intersection for every $i = 1, \ldots, r$. But this is clear because Lemma 4.3.10 shows that otherwise $d_{\mathfrak{p}_i} = 3$ and $e_{\mathfrak{p}_i} = 4$, for every $i = 1, \ldots, r$.

Case 2: Suppose that $\deg \mathcal{B} = 6$ and $J = (x^2+\lambda u^2+\mu xu, x^2u, xu^2), \lambda \neq 0$. Since $\deg \mathcal{B} = 6$, Lemma 4.3.10 implies that $\mathcal{B}$ has only one prime $\mathfrak{q}$ other than $\mathfrak{p} = (x,u)$, with $d_{\mathfrak{q}} = 1$ and $e_{\mathfrak{q}} = 1$. Therefore $\sum_{x\in\mathcal{B}} e_x(I) = 7$ as required. □

By Lemma 4.3.9, if $\phi : \mathbb{P}^1_k \times \mathbb{P}^1_k \dashrightarrow \mathbb{P}^2_k$ is a birational map defined by bihomogeneous polynomials $f_0, f_1, f_2 \in R$ of bidegree $(2,2)$ without common factor in $R \setminus k$, then, $I = (f_0, f_1, f_2)$ is perfect ideal with exactly two minimal syzygies, of bidegree $(1,1)$. Indeed, the free resolution of I is of the form

$$0 \longrightarrow R(-3,-3)^2 \xrightarrow{M} R(-2,-2)^3 \longrightarrow I \longrightarrow 0.$$

To understand the shape of the matrix M, we consider three cases.

Case 1: Suppose that $\mathcal{B} = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\} \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$ with $d_{\mathfrak{p}_1} = 3$, $d_{\mathfrak{p}_2} = 2$ and $d_{\mathfrak{p}_3} = 1$. By a suitable coordinate change, one can assume without loss of generality that the three primes are $\mathfrak{p}_1 = (x,u)$, $\mathfrak{p}_2 = (y,v)$ and $\mathfrak{p}_3 = (x+y, u+v)$. Accordingly

$$I \subset (x,u)^2 \cap (y^2, v) \cap (x+y, u+v).$$

Now, I is generated by elements of bidegree $(2,2)$. A computation with `Macaulay2` gives that

$$\{x^2v(u+v),\ xuv(x+y),\ y^2u^2 - xyuv\}$$

are the only forms of bidegree $(2,2)$ in the variables x, y, u, v. In particular, I must be contained in the ideal generated by these three forms, and hence coincides with it. These three forms are the 2-minors of the following 3×2 matrix

$$M = \begin{pmatrix} u(x+y) & yu \\ -x(u+v) & -yu \\ 0 & xv \end{pmatrix}.$$

Case 2: Suppose that $\mathcal{B} = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4\} \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$ with $d_{\mathfrak{p}_1} = 3$ and $d_{\mathfrak{p}_i} = 1$, $i = 2,3,4$. By the same token as in the first case, we may assume that $\mathfrak{p}_1 = (x,u)$, $\mathfrak{p}_2 = (y,v)$, $\mathfrak{p}_3 = (x+y, u+v)$ and $\mathfrak{p}_4 = (ax+y, \alpha u+v)$, for suitable coefficients $a, \alpha \in k$ with $(a,\alpha) \neq (1,1)$. Accordingly,

$$I \subset (x,u)^2 \cap (y,v) \cap (x+y, u+v) \cap (ax+y, \alpha u+v).$$

Repeating the same computational device as in the first case, one obtains that I is generated by the following forms

$\{(yu-xv)(\alpha xu+xv), (axu+yu)(yu-xv), (\alpha-a)x^2uv+(\alpha-1)xyuv-(a-1)x^2v^2\}$.

Once again, one can verify that these forms are the 2-minors of the 3×2 matrix

$$M = \begin{pmatrix} (ax+y)u & (a-1)xv \\ -x(\alpha u+v) & -(\alpha-1)xv \\ 0 & yu-xv \end{pmatrix}.$$

If $\alpha \neq a$, the three minors have no factor in common and it follows that M provides the free resolution of I. If $\alpha = a$, then all elements of bidegree $(2,2)$ in $(x,u)^2 \cap (y,v) \cap (x+y, u+v) \cap (ax+y, au+v)$ are multiple of $yu-xv$, contradicting the hypothesis that the f_i's have no common factor.

Case 3: Suppose that $\mathcal{B} = \{\mathfrak{p}, \mathfrak{q}\} \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$ with $d_\mathfrak{p} = 5$ and $d_\mathfrak{q} = 1$. Always by the same token, we may assume that $\mathfrak{p} = (x,u)$ and $\mathfrak{q} = (y,v)$. Accordingly, one has $I \subset (x^2+\lambda u^2+\mu xu, x^2u, xu^2) \cap (y,v)$, for suitable $\lambda \in k \setminus 0$. Since then

$$I \subset (x^2v^2+\lambda y^2u^2+\mu xyuv, x^2uv, xyu^2),$$

it must be generated by these three forms of bidegree $(2,2)$.

As before, direct inspection shows that I is perfect with syzygy matrix

$$M = \begin{pmatrix} xu & 0 \\ -xv & yu \\ \lambda yu+\mu xv & xv \end{pmatrix}.$$

4.4 Modeling: tensor-product maps in the plane

In this section we will explore the consequences of our previous results to the field of geometric modeling. Indeed, in this field bigraded rational maps are intensively used to describe parameterizations of curves, surfaces and volumes, including plane parameterizations. For that purpose, the Bernstein basis is preferred to the usual power basis for representing polynomials. Recall that the homogeneous Bernstein polynomials are defined by the formula

$$B_i^n(x,y) = \binom{n}{i} y^i (x-y)^{n-i}.$$

They are homogeneous of degree n and any homogeneous polynomial of degree n can be written as a linear combination of them. Consequently, a bihomogeneous polynomial of bidegree (a,b) can be written as a linear combination of all the products $B_i^a(x,y)B_j^b(u,v)$, $i=0,\dots,a$ and $j=0,\dots,b$. Rational maps written in this basis are dubbed *tensor-product Bézier parameterizations*.

It turns out that an important property of tensor-product Bézier parameterization is to guarantee their birationality. Moreover, an even more important property is to preserve this birationality property during a design process, that is to say when the coefficients of the defining polynomials are continuously modified (see e.g. Figure 4.1). In what follows, we will show how Proposition 4.3.3 and Theorem 4.3.5 allow to translate the detection of birationality as rank decision problems in the case of tensor-product parameterizations of bidegree $(1,1)$ and $(1,2)$.

4.4.1 Plane tensor-product parameterizations

For defining a bigraded rational map of bidegree (a, b) in Bernstein form we need to introduce a collection of *control points* $P_{i,j} = (x_{i,j}, y_{i,j}) \in \mathbb{R}^2$ and their associated weights $w_{i,j} \in \mathbb{R}$. The map is then defined as

$$\begin{aligned} \phi : \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}} &\dashrightarrow \mathbb{P}^2_{\mathbb{R}} \\ (x : y) \times (u : v) &\mapsto \left(\sum_{i,j} w_{i,j} B^{a,b}_{i,j} : \sum_{i,j} w_{i,j} x_{i,j} B^{a,b}_{i,j} : \sum_{i,j} w_{i,j} y_{i,j} B^{a,b}_{i,j} \right) \end{aligned} \tag{4.5}$$

where $B^{a,b}_{i,j} := B^a_i(x, y) B^b_j(u, v)$. Observe that ϕ "interpolates" the control points, in the sense that

$$\phi\left((1 : i) \times (1 : j)\right) = (w_{i,j} : w_{i,j} x_{i,j} : w_{i,j} y_{i,j}), \;\; i = 0, 1, \;\; j = 0, 1.$$

In addition, if all the weights are equal to 1 then $\sum_{i,j} w_{i,j} B^{a,b}_{i,j} = 1$, so that the control points $P_{i,j}$ fully control the map ϕ.

In general, the control points are the only coefficients of the map ϕ that are modified during a hand-design process because they really provide an intuitive way to reshape the parameterization ϕ. The weights are hidden behind and not used as an intuitive design tool. When the control points of a given birational parameterization are moved, then the new parameterization is in general no longer a birational parameterization. Below, we will illustrate how the weights of the map ϕ can be changed in order to retrieve a birational map without touching again to the control points modified by the designer.

4.4.2 Bilinear tensor-product parameterizations

Consider a bilinear rational map as defined in (4.5) with $(a, b) = (1, 1)$. By Proposition 4.3.3, this rational map will be birational if and only if there exists a syzygy of bidegree $(1, 0)$, or equivalently a syzygy of bidegree $(0, 1)$. Writing this condition under a linear system in the Bernstein basis, we obtain the following matrix whose kernel yields those bidegree $(1, 0)$ syzygies:

$$M := \begin{pmatrix} x_{0,0}w_{0,0} & 0 & y_{0,0}w_{0,0} & 0 & w_{0,0} & 0 \\ x_{0,1}w_{0,1} & 0 & y_{0,1}w_{0,1} & 0 & w_{0,1} & 0 \\ \frac{1}{2}x_{1,0}w_{1,0} & \frac{1}{2}x_{0,0}w_{0,0} & \frac{1}{2}y_{1,0}w_{1,0} & \frac{1}{2}y_{0,0}w_{0,0} & \frac{1}{2}w_{1,0} & \frac{1}{2}w_{0,0} \\ \frac{1}{2}x_{1,1}w_{1,1} & \frac{1}{2}x_{0,1}w_{0,1} & \frac{1}{2}y_{1,1}w_{1,1} & \frac{1}{2}y_{0,1}w_{0,1} & \frac{1}{2}w_{1,1} & \frac{1}{2}w_{0,1} \\ 0 & x_{1,0}w_{1,0} & 0 & y_{1,0}w_{1,0} & 0 & w_{1,0} \\ 0 & x_{1,1}w_{1,1} & 0 & y_{1,1}w_{1,1} & 0 & w_{1,1} \end{pmatrix}.$$

As a consequence, the map is birational if and only if $\det(M) = 0$. Now, using the Laplace expansion formula of determinants by 3×3-blocks with respect to columns 1,3,5 and 2,4,6, we get the condition :

$$\left(w_{1,0}w_{0,1} \left|\tilde{P}_{0,0}\tilde{P}_{0,1}\tilde{P}_{1,0}\right| \cdot \left|\tilde{P}_{0,1}\tilde{P}_{1,0}\tilde{P}_{1,1}\right| - w_{1,1}w_{0,0} \left|\tilde{P}_{0,0}\tilde{P}_{0,1}\tilde{P}_{1,1}\right| \cdot \left|\tilde{P}_{0,0}\tilde{P}_{1,0}\tilde{P}_{1,1}\right| \right) \times \\ w_{0,0}w_{0,1}w_{1,0}w_{1,1} = 0$$

where $\tilde{P}_{i,j}$ is the vector $(x_{i,j}, y_{i,j}, 1) = (P_{i,j}, 1)$. Weights are in general assumed to be nonzero; therefore, under this assumption we recover the following condition that already appeared in the recent paper [SZ15]:

$$\frac{w_{1,0}w_{0,1}}{w_{1,1}w_{0,0}} = \frac{\left|\tilde{P}_{0,0}\tilde{P}_{0,1}\tilde{P}_{1,1}\right| \cdot \left|\tilde{P}_{0,0}\tilde{P}_{1,0}\tilde{P}_{1,1}\right|}{\left|\tilde{P}_{0,0}\tilde{P}_{0,1}\tilde{P}_{1,0}\right| \cdot \left|\tilde{P}_{0,1}\tilde{P}_{1,0}\tilde{P}_{1,1}\right|}. \tag{4.6}$$

From here, it appears clearly that given the control points, a suitable modification of a single weight so that (4.6) holds, allow to obtain a birational map [SZ15].

4.4.3 Bidegree $(1, 2)$ tensor-product parameterizations

Now, consider a rational map as defined in (4.5) with $(a, b) = (1, 2)$. By our previous results, this rational map will be birational if and only if there exists a syzygy of bidegree $(0, 1)$. Proceeding as in the previous case of bilinear maps, we obtain the following multiplication matrix

$$M := \begin{pmatrix} w_{0,0}x_{0,0} & 0 & w_{0,0}y_{0,0} & 0 & w_{0,0} & 0 \\ \frac{2}{3}w_{0,1}x_{0,1} & \frac{1}{3}w_{0,0}x_{0,0} & \frac{2}{3}w_{0,1}y_{0,1} & \frac{1}{3}w_{0,0}y_{0,0} & \frac{2}{3}w_{0,1} & \frac{1}{3}w_{0,0} \\ \frac{1}{3}w_{0,2}x_{0,2} & \frac{2}{3}w_{0,1}x_{0,1} & \frac{1}{3}w_{0,2}y_{0,2} & \frac{2}{3}w_{0,1}y_{0,1} & \frac{1}{3}w_{0,2} & \frac{2}{3}w_{0,1} \\ 0 & w_{0,2}x_{0,2} & 0 & w_{0,2}y_{0,2} & 0 & w_{0,2} \\ w_{1,0}x_{1,0} & 0 & w_{1,0}y_{1,0} & 0 & w_{1,0} & 0 \\ \frac{2}{3}w_{1,1}x_{1,1} & \frac{1}{3}w_{1,0}x_{1,0} & \frac{2}{3}w_{1,1}y_{1,1} & \frac{1}{3}w_{1,0}y_{1,0} & \frac{2}{3}w_{1,1} & \frac{1}{3}w_{1,0} \\ \frac{1}{3}w_{1,2}x_{1,2} & \frac{2}{3}w_{1,1}x_{1,1} & \frac{1}{3}w_{1,2}y_{1,2} & \frac{2}{3}w_{1,1}y_{1,1} & \frac{1}{3}w_{1,2} & \frac{2}{3}w_{1,1} \\ 0 & w_{1,2}x_{1,2} & 0 & w_{1,2}y_{1,2} & 0 & w_{1,2} \end{pmatrix}. \tag{4.7}$$

It is 8×6-matrix and $\operatorname{rank}(M) < 6$ if and only if the corresponding map is birational. Similarly, the analysis of bidegree $(1, 1)$ syzygies leads to a square 12×12-matrix with the property that its rank drops by 3 if and only if the corresponding map is birational. Therefore, the decision of birationality is not given by a single polynomial condition as in the previous case of bilinear maps. Nevertheless, our syzygy-based formulation of birationality by means of the rank of the matrix M translates birationality decision to a *rank decision problem*. This opens a bridge to the field of numerical linear algebra where a huge amounts of works on this problem have been done during the last decades. In the following, we illustrate this link on an example with the help of a recent algorithm for structured low-rank approximation [SS16].

We start with the canonical non-rational tensor-product parameterization of the plane, i.e. all the weights are set to 1 and the control points have a rectangular shape. More precisely, we set $P_{i,j} = (x_{i,j}, y_{i,j}) = (i, j)$ for all $i = 0, 1$ and $j = 0, 1, 2$, which is illustrated on the left side of Figure 4.1. This initial parameterization is birational, which can be checked by observing that the matrix M specialized to this

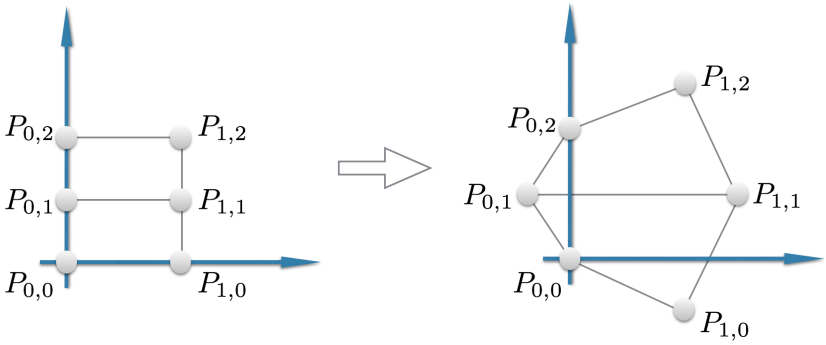

Figure 4.1 – Change of control points in a tensor-product parameterization of bidegree $(1, 2)$.

setting, that is to say the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2/3 & 0 & 2/3 & 1/3 \\ 0 & 0 & 2/3 & 2/3 & 1/3 & 2/3 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 2/3 & 1/3 & 2/3 & 0 & 2/3 & 1/3 \\ 1/3 & 2/3 & 2/3 & 2/3 & 1/3 & 2/3 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{bmatrix}$$

has rank 5.

Now, as illustrated in Figure 4.1, suppose that these control points are "moved" in order to reach the following new coordinates :

$$P_{0,0} = (0, 0), P_{0,1} = (-1/2, 1), P_{0,2} = (0, 2),$$

$$P_{1,0} = (2, -1/2), P_{1,1} = (5/2, 1), P_{1,2} = (2, 5/2).$$

If the weights are left unchanged, i.e. all equal to 1, then this new parameterization is no longer birational. Indeed, it is straightforward to check that the matrix M specialized with these new control points and all weights equal to 1

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ -1/3 & 0 & 2/3 & 0 & 2/3 & 1/3 \\ 0 & -1/3 & 2/3 & 2/3 & 1/3 & 2/3 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 2 & 0 & -1/2 & 0 & 1 & 0 \\ 5/3 & 2/3 & 2/3 & -1/6 & 2/3 & 1/3 \\ 2/3 & 5/3 & 5/6 & 2/3 & 1/3 & 2/3 \\ 0 & 2 & 0 & 5/2 & 0 & 1 \end{bmatrix}$$

has rank 6. So, we aim at changing the weights $w_{i,j}$, without changing the control points, so that the parameterization becomes birational. For that purpose, we will apply the structured low-rank approximation algorithm developed in [SS16].

Given a matrix M, the basic idea of structured low-rank approximation is to compute a matrix M' of given rank r in a linear subspace E of matrices such that the distance, in the sense of the Frobenius norm, between M and M' is small. Such an algorithm, based on Newton-like iterations, is given in [SS16]. In our context, by (4.7) the matrix M can be written as

$$M = w_{0,0}E_{0,0} + w_{0,1}E_{0,1} + w_{0,2}E_{0,2} + w_{1,0}E_{1,0} + w_{1,1}E_{1,1} + w_{1,2}E_{1,2}$$

where the $E_{i,j}$'s are matrices of size 8×6 whose entries only depend on the control points. These latter define a linear subspace of matrices and we are looking for a matrix M' such that M' belongs to this linear subspace and its rank is lower or equal to 5. Thus, applying the algorithm in [SS16], we find the following weights, up to numerical precision:

$$\begin{aligned} w_{0,0} &\approx 0.949726775368655, & w_{0,1} &\approx 1.0867765091791244, \\ w_{0,2} &\approx 0.9521336386754828, & w_{1,0} &\approx 1.0233828904581144, \\ w_{1,1} &\approx 0.9458573850234199, & w_{1,2} &\approx 1.0259764181881534, \end{aligned}$$

Therefore, by modifying the weights with the above values, the parameterization becomes birational "up to numerical precision", which means in practice that its 5-minors yield inversion formulas for almost all points, up to numerical precision.

Chapter 5

Cohen-Macaulayness and canonical module of residual intersections

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Abstract: We show the Cohen-Macaulayness and describe the canonical module of residual intersections $J = \mathfrak{a} \colon {}_R I$ in a Cohen-Macaulay local ring R, under sliding depth type hypotheses. For this purpose, we construct and study, using a recent article of Hassanzadeh and Naéliton [HN16], a family of complexes that contains important informations on a residual intersection and its canonical module. We also determine several invariants of residual intersections as the graded canonical module, the Hilbert series, the Castelnuovo-Mumford regularity and the type. Finally, whenever I is strongly Cohen-Macaulay, we show duality results for residual intersections that are closely connected to results by Eisenbud and Ulrich [EU16]. It establishes some tight relations between the Hilbert series of some symmetric powers of $I/\mathfrak{a}$. We also provide closed formulas for the types and for the Bass numbers of some symmetric powers of $I/\mathfrak{a}$. The results in this chapter are joint work with M. Chardin and J. Naéliton and have been submitted [CNT17].

5.1 Introduction

The concept of residual intersection was introduced by Artin and Nagata in [AN72], as a generalization of linkage; it is more ubiquitous, but also harder to understand. Geometrically, let X and Y be two irreducible closed subschemes of a scheme Z with $\operatorname{codim}_Z(X) \leq \operatorname{codim}_Z(Y) = s$ and $Y \nsubseteq X$, then Y is called a residual intersection of X if the number of equations needed to define $X \cup Y$ as a subscheme of Z is the smallest possible, i.e. s. For a ring R and a finitely generated R-module M, let $\mu_R(M)$ denotes the minimum number of generators of M.

The precise definition of a residual intersection is the following.

Definition 5.1.1. Let R be a Noetherian ring, I be an ideal of height g and $s \geq g$ be an integer.

1. An *s-residual intersection* of I is a proper ideal J of R such that $\operatorname{ht}(J) \geq s$ and $J = (\mathfrak{a} \colon {}_R I)$ for some ideal $\mathfrak{a} \subset I$ which is generated by s elements.
2. An *arithmetic s-residual intersection* of I is an s-residual intersection J of I such that $\mu_{R_\mathfrak{p}}((I/\mathfrak{a})_\mathfrak{p}) \leq 1$ for all prime ideal $\mathfrak{p}$ with $\operatorname{ht}(\mathfrak{p}) \leq s$.
3. A *geometric s-residual intersection* of I is an s-residual intersection J of I such that $\operatorname{ht}(I + J) \geq s + 1$.

Notice that an s-residual intersection is a direct link if I is unmixed and $s = \operatorname{ht}(I)$. Also any geometric s-residual intersection is arithmetic.

The theory of residual intersections is a center of interest since the 80's, after Huneke repaired in [Hun83] an argument of Artin and Nagata in [AN72], introducing the notion of strongly Cohen-Macaulay ideal: an ideal such that all its Koszul homlogy is Cohen-Macaulay. The notion of strong Cohen-Macaulayness is stable under even linkage, in particular ideals linked to a complete intersection satisfy this property.

In [Hun83] Huneke showed that if R is a Cohen-Macaulay local ring, J is a geometric s-residual intersection of a strongly Cohen-Macaulay ideal I of R satisfying G_s, then R/J is Cohen-Macaulay of codimension s. Following [AN72], one says that I satisfies G_s if the number of generators $\mu_{R_\mathfrak{p}}(I_\mathfrak{p})$ is at most $\dim(R_\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ with $I \subset \mathfrak{p}$ and $\dim(R_\mathfrak{p}) \leq s - 1$ and that I satisfies G_∞ if I satisfies G_s for all s. Later, Herzog, Vasconcelos, and Villarreal in [HVV85] replaced the assumption strong Cohen-Macaulayness by the weaker sliding depth condition, for geometric residuals, but they also showed that this assumption cannot be weakened any further. On the other hand, Huneke and Ulrich proved in [HU88] that the condition G_s is superfluous for ideals in the linkage class of a complete intersection, and more precisely:

Theorem. [HU88] *Let R be a Gorenstein local ring and I be an ideal of height g that is evenly linked to a strongly Cohen-Macaulay ideal satisfying G_∞. If $J = \mathfrak{a} \colon {}_R I$ is an s-residual intersection of I, then R/J is Cohen-Macaulay of codimension s and the canonical module of R/J is the $(s - g + 1)$-th symmetric power of $I/\mathfrak{a}$.*

Let us notice that, in the proof of this statement, it is important to keep track of the canonical module of the residual along the deformation argument that they are using.

A natural question is then to know if the G_s assumption is at all needed to assert that residuals of ideals that are strongly Cohen-Macaulay, or satisfy the weaker sliding depth condition, are always Cohen-Macaulay, and to describe the canonical module of the residual. In this direction, Hassanzadeh and Naéliton remarked in [HN16] that the following long-standing assertions were, explicitely or implicitly, conjectured:

Conjectures. [HU88, Ulr92, CEU01] *Let R be a Cohen-Macaulay local (or *local) ring. Suppose that I is strongly Cohen-Macaulay, or even just satisfies sliding depth. Then, for any s-residual intersection $J = (\mathfrak{a} \colon {}_R I)$ of I,*

(1) *R/J is Cohen-Macaulay.*

(2) *The canonical module of R/J is the $(s-g+1)$-th symmetric power of $I/\mathfrak{a}$, if R is Gorenstein, with $g = \operatorname{ht}(I) \leq s$.*

(3) *$\mathfrak{a}$ is minimally generated by s elements.*

(4) *J is unmixed.*

(5) *When R is positively graded over a field, the Hilbert series of R/J depends only upon I and the degrees of the generators of $\mathfrak{a}$.*

The first conjecture was shown by Hassanzadeh [Has12] for arithmetic residual intersections, thus in particular for geometric residual intersections, under the sliding depth condition. In the recent article [HN16], Hassanzadeh and Naéliton proved that the second and fifth conjectures hold for the arithmetic residual intersections of strongly Cohen-Macaulay ideals and that the third and fourth conjectures are true if $\operatorname{depth}(R/I) \geq \dim(R) - s$ and I satisfies the sliding depth condition.

In this chapter we will complete the picture, by showing that the first and fifth conjectures hold whenever I satisfies SD_1 and that the second conjecture is true if I satisfies SD_2 – recall that an ideal $I = (\mathbf{f}) = (f_1, \ldots, f_r)$ of height g in a Noetherian local ring R of dimension d satisfies SD_k $(k \geq 0)$ if $\operatorname{depth}(H_i(\mathbf{f}; R)) \geq \min\{d-g, d-r+i+k\}$ for all $i \geq 0$; note that SD_0 is the sliding depth condition and I satisfies SD_{r-g} if and only if I is strongly Cohen-Macaulay.

In particular all items in the conjecture holds for strongly Cohen-Macaulay ideals. The following puts together part of these results:

Theorem (Theorems 5.4.6, 5.4.9 and 5.6.2)**.** *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module ω. Assume that $J = (\mathfrak{a} \colon {}_R I)$ is an s-residual intersection of I with $\mathfrak{a} \subset I$ and $\operatorname{ht}(I) = g \leq s = \mu_R(\mathfrak{a})$. Then*

(i) *R/J is Cohen-Macaulay of codimension s if I satisfies SD_1.*

If furthermore $\operatorname{Tor}_1^R(R/I, \omega) = 0$, then

(ii) *$\omega_{R/J} \simeq \operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega$, provided I satisfies SD_2.*

(iii) *$\omega_{\operatorname{Sym}_R^k(I/\mathfrak{a})} \simeq \operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}) \otimes_R \omega$ for $1 \leq k \leq s-g$, provided I is strongly Cohen-Macaulay.*

Notice that $\operatorname{Tor}_1^R(R/I, \omega) = 0$ if R is Gorenstein or I has finite projective dimension.

A key ingredient of our proofs is a duality result between some of the first symmetric powers of $I/\mathfrak{a}$ together with a description of the canonical module of the residual as in items (ii) and (iii) above. This could be compared to recent results of Eisenbud and Ulrich in [EU16] where similar dualities were obtained under slightly different hypotheses. In their work, conditions on the local number of generators are needed and depth conditions are asked for some of the first powers of the ideal I, along the lines of [Ulr94], and the duality occurs between powers $I^t/\mathfrak{a}I^{t-1}$ in place of symmetric powers $\mathrm{Sym}^t_R(I/\mathfrak{a})$. Although their results and ours coincide in an important range of situations, like for geometric residuals of strongly Cohen-Macaulay ideals satisfying G_s, the domains of validity of the two results are quite distinct. We prove the following.

Theorem (Theorem 5.6.7)**.** *Let $(R,\mathfrak{m})$ be a Gorenstein local ring and let $\mathfrak{a} \subset I$ be two ideals of R, with $\mathrm{ht}(I) = g$. Suppose that $J = (\mathfrak{a} \colon {}_R I)$ is an s-residual intersection of I. If I is strongly Cohen-Macaulay, then $\omega_{R/J} \simeq \mathrm{Sym}^{s-g+1}_{R/J}(I/\mathfrak{a})$ and for all $0 \leq k \leq s-g+1$*

(i) *the R/J-module $\mathrm{Sym}^k_{R/J}(I/\mathfrak{a})$ is faithful and Cohen-Macaulay,*

(ii) *the multiplication*

$$\mathrm{Sym}^k_{R/J}(I/\mathfrak{a}) \otimes_{R/J} \mathrm{Sym}^{s-g+1-k}_{R/J}(I/\mathfrak{a}) \longrightarrow \mathrm{Sym}^{s-g+1}_{R/J}(I/\mathfrak{a})$$

is a perfect pairing,

(iii) *setting $A := \mathrm{Sym}_{R/J}(I/\mathfrak{a})$, the graded R/J-algebra*

$$\overline{A} := A/A_{>s-g+1} = \bigoplus_{i=0}^{s-g+1} \mathrm{Sym}^i_{R/J}(I/\mathfrak{a})$$

is Gorenstein.

The chapter is organized as follows.

In Section 5.2, we prove duality results for Koszul cycles in Propositions 5.2.1 and 5.2.2. We also describe the structure of the homology modules of the approximation complexes in Propositions 5.2.3 and 5.2.4.

In Section 5.3, we construct a family of residual approximation complex, all of same finite size, $\{{}^M_k\mathcal{Z}^+_\bullet\}_{k\in\mathbb{Z}}$. This family is a generalization of the family $\{{}_k\mathcal{Z}^+_\bullet\}_{k\in\mathbb{Z}}$ that is built in the recent article [HN16] by Hassanzadeh and Naéliton. We study the properties of these complexes, of particular complexes ${}^\omega_k\mathcal{Z}^+_\bullet$, where ω is the canonical module of R. The main results of this section are Propositions 5.3.2, 5.3.3 and 5.3.5.

In Section 5.4, we prove one of the main results of this chaper: the Cohen-Macaulayness and the description of the canonical module of residual intersections. Recall that in [Has12], Hassanzadeh proved that, under the sliding depth condition, $H_0({}_0\mathcal{Z}^+_\bullet) = R/K$ is Cohen-Macaulay of codimension s, with $K \subset J$, $\sqrt{K} = \sqrt{J}$, and further $K = J$ whenever the residual is arithmetic. First, we consider the height two case and show that under the SD_1 condition, there exist an epimorphism $\varphi : H_0({}^\omega_{s-1}\mathcal{Z}^+_\bullet) \twoheadrightarrow \omega_{R/K}$ which is an isomorphism if I satisfies SD_2 (Proposition 5.4.5). By exploring these complexes, we show that, under the SD_1 condition,

$K = J$; and therefore, under the SD_2 condition, the canonical module of R/J is $H_0({}_{s-1}^{\omega}\mathcal{Z}_\bullet^+)$. In a second step, we reduce the general case to the height two case. Our main results in this section are Theorems 5.4.6 and 5.4.9.

In Section 5.5, we study the stability of Hilbert functions and Castelnuovo-Mumford regularity of residual intersections. Using the acyclicity of ${}_0\mathcal{Z}_\bullet^+$, Proposition 5.5.1 says that the Hilbert function of R/J only depends on the degrees of the generators of $\mathfrak{a}$ and the Koszul homologies of I. The graded structure of the canonical module of R/J in Proposition 5.5.5 is the key to derive the Castelnuovo-Mumford regularity of residual intersection in Corollary 5.5.6.

Finally, in Section 5.6, we consider the case where I is strongly Cohen-Macaulay. The main results of this section are Theorems 5.6.2 and 5.6.7. In particular, for $1 \leq k \leq s-g$,

$$\omega_{\mathrm{Sym}_R^k(I/\mathfrak{a})} \simeq \mathrm{Sym}_R^{s-g+1-k}(I/\mathfrak{a}) \otimes_R \omega,$$

whenever $\mathrm{Tor}_1^R(R/I, \omega) = 0$. Consequently, we obtain some tight relations between the Hilbert series of the symmetric powers of $I/\mathfrak{a}$ in Corollary 5.6.8. We also give the closed formulas for the types and for the Bass number of some symmetric powers of $I/\mathfrak{a}$ in Corollaries 5.6.9 and 5.6.10, respectively.

5.2 Koszul cycles and approximation complexes

In this section, we give some results on the duality for Koszul cycles and describe the 0-th homology modules of approximation complexes with coefficients in a module.

Assume that R is a Noetherian ring, $I = (f_1, \ldots, f_r)$ is an ideal of R. Let M be a finitely generated R-module. The symmetric algebra of M is denoted by $\mathrm{Sym}_R(M)$ and the k-th symmetric power of M is denoted by $\mathrm{Sym}_R^k(M)$. We consider $S = R[T_1, \ldots, T_r]$ as a standard graded algebra over $S_0 = R$. For a graded S-module N, the k-th graded component of N is denoted by $N_{[k]}$. We make $\mathrm{Sym}_R(I)$ an S-algebra via the graded ring homomorphism $S \longrightarrow \mathrm{Sym}_R(I)$ sending T_i to f_i as an element of $\mathrm{Sym}_R(I)_{[1]} = I$, and write $\mathrm{Sym}_R(I) = S/\mathfrak{L}$.

For a sequence of elements $\mathbf{x}$ in R, we denote the Koszul complex by $K_\bullet(\mathbf{x}; M)$, its Koszul cycles by $Z_i(\mathbf{x}; M)$, its Koszul boundaries by $B_i(\mathbf{x}; M)$ and its Koszul homologies by $H_i(\mathbf{x}; M)$. If $M = R$, then we denote, for simplicity, K_i, Z_i, B_i, H_i. To set more notation, when we draw the picture of a double complex obtained from a tensor product of two complexes (in the sense of [Wei94, 2.7.1]) which at least one of them is finite, say $A \otimes B$ where B is finite, we always put A in the vertical and B in the horizontal one. We also label the module which is in the up-right corner by $(0, 0)$ and consider the labels for the rest, as the points in the third-quadrant.

5.2.1 Duality for Koszul cycles

Duality results for Koszul homology modules over Gorenstein rings have been obtained by several authors, for instance in [Her74, Cha04, MRS14]. For Koszul cycles, the following holds.

Proposition 5.2.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $I = (f_1, \ldots, f_r)$ an ideal of R. Suppose that R satisfies $\mathcal{S}_2$ and $\operatorname{ht}(I) \geq 2$. Then, for all $0 \leq i \leq r-1$,*

$$Z_i \simeq \operatorname{Hom}_R(Z_{r-1-i}, R).$$

Proof. The inclusions $Z_i \hookrightarrow K_i = \bigwedge^i R^r$ and $Z_{r-1-i} \hookrightarrow K_{r-1-i} = \bigwedge^{r-1-i} R^r$ induce a map

$$\varphi_i : Z_i \times Z_{r-1-i} \longrightarrow K_i \times K_{r-1-i} \longrightarrow K_{r-1},$$

where the last map is the multiplication of the Koszul complex, which is a differential graded algebra, and $\operatorname{Im}(\varphi_i) \subset Z_{r-1} \simeq K_r \simeq R$. It follows that φ_i induces a map

$$\psi_i : Z_i \longrightarrow \operatorname{Hom}_R(Z_{r-1-i}, R).$$

We induct on the height to show that for every $\mathfrak{p} \in \operatorname{Spec}(R)$, $(\psi_i)_\mathfrak{p}$ is an isomorphism. If $\operatorname{ht}(\mathfrak{p}) < 2$, then $I_\mathfrak{p} = R_\mathfrak{p}$, by Proposition 1.5.5

$$(Z_i)_\mathfrak{p} \simeq \bigwedge^i R_\mathfrak{p}^{r-1} \quad \text{and} \quad (Z_{r-1-i})_\mathfrak{p} \simeq \bigwedge^{r-1-i} R_\mathfrak{p}^{r-1}$$

and the self-duality of the Koszul complexes (see Proposition 1.5.7(a)) shows that $(\psi_i)_\mathfrak{p}$ is an isomorphism.

Suppose that $\operatorname{ht}(\mathfrak{p}) \geq 2$ and $(\psi_i)_\mathfrak{q}$ is an isomorphism for all prime contained properly in $\mathfrak{p}$. Replacing R by $R_\mathfrak{p}$ and $\mathfrak{m}$ by $\mathfrak{p}R_\mathfrak{p}$, we can suppose that ψ_i is an isomorphism on the punctured spectrum : the kernel and the corkernel of ψ_i are annihilated by a power of $\mathfrak{m}$. It follows that $H^j_\mathfrak{m}(\operatorname{Ker}(\psi_i)) = H^j_\mathfrak{m}(\operatorname{Coker}(\psi_i)) = 0$ for $j > 0$. Since R satisfies $\mathcal{S}_2$, $\operatorname{depth}(Z_i) \geq \min\{2, \operatorname{depth}(R)\} = 2$. The exact sequence

$$0 \longrightarrow \operatorname{Ker}(\psi_i) \longrightarrow Z_i \longrightarrow \operatorname{Im}(\psi_i) \longrightarrow 0$$

implies that $\operatorname{Ker}(\psi_i) = H^0_\mathfrak{m}(\operatorname{Ker}(\psi_i)) = 0$. Observing that

$$\operatorname{depth}(\operatorname{Hom}_R(Z_{r-1-i}, R)) \geq \min\{2, \operatorname{depth}(R)\} = 2,$$

the exact sequence

$$0 \longrightarrow Z_i \longrightarrow \operatorname{Hom}_r(Z_{r-1-i}, R) \longrightarrow \operatorname{Coker}(\psi_i) \longrightarrow 0$$

implies that $\operatorname{Coker}(\psi_i) = H^0_\mathfrak{m}(\operatorname{Coker}(\psi_i)) = 0$. □

If R is a Gorenstein local ring, $\omega_R \simeq R$, therefore, by Proposition 5.2.1,

$$\omega_{Z_p} \simeq Z_{r-1-p}$$

for all $0 \leq p \leq r-1$. The next proposition will generalize this result.

We are particularly interested in the case that R admits the canonical module, hence in the sequel we assume that R is a homomorphic image of a Gorenstein ring and write ω_R for the canonical module of R. We will denote by $Z_i^{\omega_R} := Z_i(\mathbf{f}; \omega_R)$ the module of i-th Koszul cycle, with $\mathbf{f} = f_1, \ldots, f_r$.

Proposition 5.2.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d which is a homomorphic image of a Gorenstein local ring. Suppose that $I = (f_1, \dots, f_r)$ is an ideal of R, with $\operatorname{ht}(I) \geq 2$. Then, for all $0 \leq p \leq r-1$,*

$$\omega_{Z_p} \simeq Z_{r-1-p}^{\omega_R}.$$

Moreover, if R satisfies $\mathcal{S}_2$, then

$$\omega_{Z_p^{\omega_R}} \simeq Z_{r-1-p}.$$

Proof. For simplicity, set $\omega := \omega_R$. First we consider the truncated complexes

$$\mathbf{K}_\bullet^{>p} : 0 \longrightarrow K_r \longrightarrow \cdots \longrightarrow K_{p+1} \longrightarrow Z_p \longrightarrow 0.$$

The double complex $C_{\mathfrak{m}}^\bullet(\mathbf{K}_\bullet^{>p})$

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_{\mathfrak{m}}^0(K_r) \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{\mathfrak{m}}^0(K_{p+1}) \arrow[r] \arrow[d] & C_{\mathfrak{m}}^0(Z_p) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{\mathfrak{m}}^1(K_r) \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{\mathfrak{m}}^1(K_{p+1}) \arrow[r] \arrow[d] & C_{\mathfrak{m}}^1(Z_p) \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_{\mathfrak{m}}^d(K_r) \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{\mathfrak{m}}^d(K_{p+1}) \arrow[r] \arrow[d] & C_{\mathfrak{m}}^d(Z_p) \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 &
\end{tikzcd}$$

gives rise to two spectral sequences. The second terms of the horizonal spectral are

$${}^2\mathbf{E}_{\text{hor}}^{-i,-j} = H_{\mathfrak{m}}^j(H_{i+p})$$

and the first terms of the vertical spectral are

$$\begin{tikzcd}
0 \arrow[r] & H_{\mathfrak{m}}^0(K_r) \arrow[r] & \cdots \arrow[r] & H_{\mathfrak{m}}^0(K_{p+1}) \arrow[r] & H_{\mathfrak{m}}^0(Z_p) \arrow[r] & 0 \\
 & \vdots & \cdots & \vdots & \vdots & \\
0 \arrow[r] & H_{\mathfrak{m}}^{d-1}(K_r) \arrow[r] & \cdots \arrow[r] & H_{\mathfrak{m}}^{d-1}(K_{p+1}) \arrow[r] & H_{\mathfrak{m}}^{d-1}(Z_p) \arrow[r] & 0 \\
0 \arrow[r] & H_{\mathfrak{m}}^d(K_r) \arrow[r] & \cdots \arrow[r] & H_{\mathfrak{m}}^d(K_{p+1}) \arrow[r] & H_{\mathfrak{m}}^d(Z_p) \arrow[r] & 0.
\end{tikzcd}$$

Since I annihilates H_i, $\dim(H_i) \leq \dim(R/I) \leq \dim(R) - \mathrm{ht}(I) \leq d-2$ if $H_i \neq 0$. Therefore, ${}^2\mathbf{E}_{\mathrm{hor}}^{-i,-j} = H_{\mathfrak{m}}^j(H_{i+p}) = 0$, for all $j > d-2$. The comparison of two spectral sequences gives a short exact sequence

$$H_{\mathfrak{m}}^d(K_{p+2}) \longrightarrow H_{\mathfrak{m}}^d(K_{p+1}) \longrightarrow H_{\mathfrak{m}}^d(Z_p) \longrightarrow 0. \tag{5.1}$$

By local duality

$$H_{\mathfrak{m}}^d(K_i) \simeq \mathrm{Hom}_R(K_i, \omega)^\vee \simeq (\mathrm{Hom}_R(K_i, R) \otimes_R \omega)^\vee \simeq (K_{r-i} \otimes_R \omega)^\vee = K_{r-i}(\mathbf{f}; \omega)^\vee.$$

Thus the exact sequence (5.1) provides an exact sequence

$$K_{r-p-2}(\mathbf{f}; \omega)^\vee \longrightarrow K_{r-p-1}(\mathbf{f}; \omega)^\vee \longrightarrow H_{\mathfrak{m}}^d(Z_p) \longrightarrow 0$$

that gives $H_{\mathfrak{m}}^d(Z_p) \simeq {Z_{r-1-p}^\omega}^\vee$.

Now one has

$$\widehat{\omega}_{Z_p} \simeq \omega_{Z_p}^{\vee\vee} \simeq H_{\mathfrak{m}}^d(Z_p)^\vee \simeq {Z_{r-1-p}^\omega}^{\vee\vee} \simeq \widehat{Z}_{r-1-p}^\omega.$$

Therefore the first isomorphism follows from Proposition 1.2.19.

The second assertion is proved similarly, by considering the truncated complexes

$$\mathbf{K}_\bullet^{\omega > p} : 0 \longrightarrow K_r(\mathbf{f}; \omega) \longrightarrow \cdots \longrightarrow K_{p+1}(\mathbf{f}; \omega) \longrightarrow Z_p^\omega \longrightarrow 0$$

and the double complex $C_{\mathfrak{m}}^\bullet(\mathbf{K}_\bullet^{\omega > p})$.

Since I annihilates $H_i(\mathbf{f}; \omega)$, $\dim(H_i(\mathbf{f}; \omega)) \leq \dim(R) - \mathrm{ht}(I) \leq d-2$, for all $0 \leq i \leq r-2$. Thus $H_{\mathfrak{m}}^j(H_i(\mathbf{f}; \omega)) = 0$, for all $j > d-2$ and $0 \leq i \leq r-2$. By comparing two spectral sequences, we also obtain a short exact sequence

$$H_{\mathfrak{m}}^d(K_{p+2}(\mathbf{f}; \omega)) \longrightarrow H_{\mathfrak{m}}^d(K_{p+1}(\mathbf{f}; \omega)) \longrightarrow H_{\mathfrak{m}}^d(Z_p^\omega) \longrightarrow 0. \tag{5.2}$$

By local duality

$$\begin{aligned} H_{\mathfrak{m}}^d(K_i(\mathbf{f}; \omega)) &\simeq H_{\mathfrak{m}}^d(K_i \otimes_R \omega) \simeq \mathrm{Hom}_R(K_i \otimes_R \omega, \omega)^\vee \\ &\simeq \mathrm{Hom}_R(K_i, \mathrm{Hom}_R(\omega, \omega))^\vee \simeq \mathrm{Hom}_R(K_i, R)^\vee \simeq K_{r-i}^\vee \end{aligned}$$

as $\mathrm{Hom}_R(\omega, \omega) \simeq R$ since R satisfies $\mathcal{S}_2$ by Proposition 1.2.30.

The exact sequence (5.2) provides an exact sequence

$$K_{r-p-2}^\vee \longrightarrow K_{r-p-1}^\vee \longrightarrow H_{\mathfrak{m}}^d(Z_p^\omega) \longrightarrow 0$$

which shows that $H_{\mathfrak{m}}^d(Z_p^\omega) \simeq Z_{r-1-p}^\vee$. $\square$

5.2.2 Homology modules of approximation complexes

Now we describe the 0-th homology module of *approximation complexes.* These complexes was introduced in [SV81] and systematically developed in [HSV82] and [HSV83a]. Recall that the approximation complex $\mathcal{Z}_\bullet(\mathbf{f}; M)$ is

$$0 \longrightarrow Z_r^M \otimes_R S(-r) \longrightarrow \cdots \longrightarrow Z_1^M \otimes_R S(-1) \xrightarrow{\partial_M^{\mathbf{T}}} Z_0^M \otimes_R S \longrightarrow 0$$

that can be written

$$0 \longrightarrow Z_r^M[\mathbf{T}](-r) \longrightarrow \cdots \longrightarrow Z_1^M[\mathbf{T}](-1) \xrightarrow{\partial_M^{\mathbf{T}}} M[\mathbf{T}] \longrightarrow 0$$

where $\mathbf{T} = T_1, \ldots, T_r$ and $Z_i^M = Z_i(\mathbf{f}; M)$ is the i-th Koszul cycle of $K_\bullet(\mathbf{f}; M)$. By the definition,

$$H_0(\mathcal{Z}_\bullet(\mathbf{f}; M)) \simeq M[T_1, \ldots, T_r]/\mathfrak{L}_M, \tag{5.3}$$

where $\mathfrak{L}_M$ is the submodule of $M[T_1, \ldots, T_r]$ generated by the linear forms $c_1T_1 + \cdots + c_rT_r$ with $(c_1, \ldots, c_r) \in Z_1^M$.

Let $\mathcal{F}_\bullet$ be a free resolution of R/I of the form

$$\cdots \longrightarrow F_1 \xrightarrow{\delta} R^r \longrightarrow R \longrightarrow 0\,,$$

where F_1 is the free R-module indexed by a generating set of Z_1. By the definition,

$$\mathrm{Tor}_1^R(R/I, M) = Z_1^M/\mathrm{Im}(\delta \otimes 1_M) \hookrightarrow M^r/\mathrm{Im}(\delta \otimes 1_M),$$

where 1_M denote the identity morphism on M. Note that δ is induced by the inclusion $Z_1 \hookrightarrow R^r$. Therefore, $\mathrm{Im}(\delta \otimes 1_M) = (\delta \otimes 1_M)(Z_1 \otimes_R M)$ and we obtain an exact sequence

$$Z_1 \otimes_R M \xrightarrow{\delta \otimes 1_M} Z_1^M \longrightarrow \mathrm{Tor}_1^R(R/I, M) \longrightarrow 0.$$

Let $\mathfrak{L}$ be the submodule of $S = R[T_1, \ldots, T_r]$ generated by the linear forms $c_1T_1 + \cdots + c_rT_r$ with $(c_1, \ldots, c_r) \in Z_1$. Then the exact sequence

$$0 \longrightarrow \mathfrak{L} \xrightarrow{\theta} S \longrightarrow \mathrm{Sym}_R(I) \longrightarrow 0$$

provides an exact sequence

$$\cdots \longrightarrow \mathfrak{L} \otimes_R M \xrightarrow{\theta \otimes 1_M} M[T_1, \ldots, T_r] \longrightarrow \mathrm{Sym}_R(I) \otimes_R M \longrightarrow 0.$$

The image of $\theta \otimes 1_M$ is denoted by $\mathfrak{L}M$. It follows that

$$\mathrm{Sym}_R(I) \otimes_R M \simeq M[T_1, \ldots, T_r]/\mathfrak{L}M. \tag{5.4}$$

Notice that $\mathfrak{L}M$ is the submodule of $M[T_1, \ldots, T_r]$ generated by the linear forms $c_1T_1 + \cdots + c_rT_r$ with $(c_1, \ldots, c_r) \in \mathrm{Im}(\delta \otimes 1_M) \subset Z_1^M$; thus $\mathfrak{L}M \subset \mathfrak{L}_M$.

Let $\mathfrak{L}'$ be the submodule of $M[T_1,\dots,T_r]/\mathfrak{L}M$ generated by the linear forms $c_1T_1+\cdots+c_rT_r+\mathfrak{L}M$ with $(c_1,\dots,c_r)+\mathrm{Im}(\delta\otimes 1_M)\in \mathrm{Tor}_1^R(R/I,M)$. Then

$$\mathfrak{L}'\simeq \mathfrak{L}_M/\mathfrak{L}M.$$

It follows that

$$\begin{aligned} M[T_1,\dots,T_r]/(\mathfrak{L}M+\mathfrak{L}') &\simeq (M[T_1,\dots,T_r]/\mathfrak{L}M)/\mathfrak{L}' \\ &\simeq (M[T_1,\dots,T_r]/\mathfrak{L}M)/(\mathfrak{L}_M/\mathfrak{L}M) \\ &\simeq M[T_1,\dots,T_r]/\mathfrak{L}_M. \end{aligned}$$

Thus we have already proved the following.

Proposition 5.2.3. *Let R be a Noetherian ring and $I=(f_1,\dots,f_r)$ an ideal of R. Assume that M is a finitely generated R-module. Then*

$$H_0(\mathcal{Z}_\bullet(\mathbf{f};M))\simeq M[T_1,\dots,T_r]/(\mathfrak{L}M+\mathfrak{L}'),$$

where $\mathfrak{L}\subset S$ is the defining ideal of $\mathrm{Sym}_R(I)$ and $\mathfrak{L}'$ is spanned by generators of $\mathrm{Tor}_1^R(R/I,M)$.

Proposition 5.2.4. *Let R be a Noetherian ring and $I=(f_1,\dots,f_r)$ an ideal of R. Assume that M is a finitely generated R-module. Then there exists a narural epimorphism*

$$\varphi:\mathrm{Sym}_R(I)\otimes_R M \twoheadrightarrow H_0(\mathcal{Z}_\bullet(\mathbf{f};M)),$$

that equals $H_0(\mathcal{Z}_\bullet(\mathbf{f};R))\simeq \mathrm{Sym}_R(I)$ when $M=R$. Furthermore, φ is an isomorphism if and only if $\mathrm{Tor}_1^R(R/I,M)=0$.

Proof. As $\mathfrak{L}M\subset\mathfrak{L}_M$, we can define an epimorphism

$$\varphi:\mathrm{Sym}_R(I)\otimes_R M \twoheadrightarrow H_0(\mathcal{Z}_\bullet(\mathbf{f};M)),$$

by (5.3) and (5.4). Moreover, the kernel of φ is isomorphic to $\mathfrak{L}_M/\mathfrak{L}M$. Thus $\mathrm{Tor}_1^R(R/I,M)=0$ if and only if φ is an isomorphism. □

5.3 Residual approximation complexes

Assume that R is a Noetherian ring of dimension d, $I=(\mathbf{f})=(f_1,\dots,f_r)$ is an ideal of height g. Let $\mathfrak{a}=(a_1,\dots,a_s)$ be an ideal contained in I with $s\geq g$. Set $J=\mathfrak{a}\colon_R I$, $S=R[T_1,\dots,T_r]$ and $\mathfrak{g}:=(T_1,\dots,T_r)$. We write $a_i=\sum_{j=1}^r c_{ji}f_j$, and $\gamma_i=\sum_{j=1}^r c_{ji}T_j$. Notice that the γ_i's depend on how one expresses the a_i's as a linear combination of the f_i's. Set $\gamma=\gamma_1,\dots,\gamma_s$. Finally, for a graded module N, we define $\mathrm{end}(N):=\sup\{\mu\mid N_\mu\neq 0\}$ and $\mathrm{indeg}(N):=\inf\{\mu\mid N_\mu\neq 0\}$.

Let M be a finitely generated R-module and we consider the approximation complex $\mathcal{Z}_\bullet(\mathbf{f};M)$

$$0\longrightarrow Z_r^M\otimes_R S(-r)\longrightarrow Z_{r-1}^M\otimes_R S(1-r)\longrightarrow\cdots\longrightarrow Z_0^M\otimes_R S\longrightarrow 0,$$

where $Z_i^M := Z_i(\mathbf{f}; M)$ the i-th Koszul cycle of $\mathbf{f}$ with coefficients in M. The Koszul complex $K_\bullet(\gamma; S)$ of $\gamma = \gamma_1, \ldots, \gamma_s$ with coefficients in S

$$0 \longrightarrow S^{\binom{s}{s}}(-s) \longrightarrow S^{\binom{s}{s-1}}(1-s) \longrightarrow \cdots \longrightarrow S^{\binom{s}{1}}(-1) \longrightarrow S \longrightarrow 0,$$

with $K_i(\gamma; S) = S^{\binom{s}{i}}(-i)$. We build a complex $\mathcal{D}_\bullet^M$ from $\mathcal{Z}_\bullet(\mathbf{f}; M)$ and $K_\bullet(\gamma; S)$ taking the total complex of the double complex

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{Z}_r(\mathbf{f}; M) \otimes_S K_s(\gamma; S) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{Z}_r(\mathbf{f}; M) \otimes_S K_1(\gamma; S) \arrow[r] \arrow[d] & \mathcal{Z}_r(\mathbf{f}; M) \otimes_S K_0(\gamma; S) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{Z}_{r-1}(\mathbf{f}; M) \otimes_S K_s(\gamma; S) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{Z}_{r-1}(\mathbf{f}; M) \otimes_S K_1(\gamma; S) \arrow[r] \arrow[d] & \mathcal{Z}_{r-1}(\mathbf{f}; M) \otimes_S K_0(\gamma; S) \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathcal{Z}_0(\mathbf{f}; M) \otimes_S K_s(\gamma; S) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{Z}_0(\mathbf{f}; M) \otimes_S K_1(\gamma; S) \arrow[r] \arrow[d] & \mathcal{Z}_0(\mathbf{f}; M) \otimes_S K_0(\gamma; S) \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 &
\end{tikzcd}$$

It follows that

$$\mathcal{D}_i^M = \bigoplus_{j=i-s}^{i} (Z_j^M \otimes_R S)^{\binom{s}{i-j}}(-i),$$

with $Z_j^M = 0$ for $j < 0$ or $j > r$, and for $j = r$ unless $\operatorname{depth}_M(I) = 0$.

In what follows, we assume that $\operatorname{depth}_M(I) > 0$, (hence $Z_r^M = 0$), in order that the complexes we will construct have length s.

For a graded S-module N, the k-th graded component of N is denoted by $N_{[k]}$. We have $(\mathcal{D}_i^M)_{[k]} = 0$ for all $k < i$. Consequently, the complex $(\mathcal{D}_\bullet^M)_{[k]}$ is

$$0 \longrightarrow (\mathcal{D}_k^M)_{[k]} \longrightarrow (\mathcal{D}_{k-1}^M)_{[k]} \longrightarrow \cdots \longrightarrow (\mathcal{D}_0^M)_{[k]} \longrightarrow 0$$

and therefore

$$H_k(\mathcal{D}_\bullet^M)_{[k]} = \operatorname{Ker}(\mathcal{D}_k^M \longrightarrow \mathcal{D}_{k-1}^M)_{[k]}.$$

Now the Čech complex of S with respect to the ideal $\mathfrak{g} = (T_1, \ldots, T_r)$ is denoted

by $C^\bullet_{\mathfrak{g}} = C^\bullet_{\mathfrak{g}}(S)$. We consider the double complex $C^\bullet_{\mathfrak{g}} \otimes_S \mathcal{D}^M_\bullet$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C^0_{\mathfrak{g}} \otimes D^M_{r+s-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C^0_{\mathfrak{g}} \otimes D^M_1 \arrow[r] \arrow[d] & C^0_{\mathfrak{g}} \otimes D^M_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^1_{\mathfrak{g}} \otimes D^M_{r+s-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C^1_{\mathfrak{g}} \otimes D^M_1 \arrow[r] \arrow[d] & C^1_{\mathfrak{g}} \otimes D^M_0 \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C^r_{\mathfrak{g}} \otimes D^M_{r+s-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C^r_{\mathfrak{g}} \otimes D^M_1 \arrow[r] \arrow[d] & C^r_{\mathfrak{g}} \otimes D^M_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 &
\end{tikzcd}$$

that gives rise to two spectral sequence. The second terms of the horizonal spectral are

$$^2\mathbf{E}^{-i,-j}_{\text{hor}} = H^j_{\mathfrak{g}}(H_i(\mathcal{D}^M_\bullet))$$

and the first terms of the vertical spectral are

$$^1\mathbf{E}^{-\bullet,-j}_{\text{ver}} = \begin{cases} 0 \longrightarrow H^r_{\mathfrak{g}}(\mathcal{D}^M_{r+s-1}) \longrightarrow \cdots \longrightarrow H^r_{\mathfrak{g}}(\mathcal{D}^M_1) \longrightarrow H^r_{\mathfrak{g}}(\mathcal{D}^M_0) \longrightarrow 0 & \text{if} \quad j = r \\ 0 \quad \text{otherwise} \end{cases}$$

and

$$H^r_{\mathfrak{g}}(\mathcal{D}^M_i) \simeq \bigoplus_{j=i-s}^{i} (Z^M_j \otimes_R H^r_{\mathfrak{g}}(S))^{\binom{s}{i-j}}(-i)$$

by Proposition 1.2.7. Since $\operatorname{end}(H^r_{\mathfrak{g}}(S)) = -r$, it follows that $\operatorname{end}(H^r_{\mathfrak{g}}(\mathcal{D}^M_i)) = i - r$ if $\mathcal{D}^M_i \neq 0$, thus $H^r_{\mathfrak{g}}(\mathcal{D}^M_i)_{[i-r+j]} = 0$, for all $j \geq 1$. Hence, the k-th graded component of $^1\mathbf{E}^{-\bullet,-r}_{\text{ver}}$ is the complex:

$$0 \longrightarrow H^r_{\mathfrak{g}}(\mathcal{D}^M_{r+s-1})_{[k]} \longrightarrow \cdots \longrightarrow H^r_{\mathfrak{g}}(\mathcal{D}^M_{r+k+1})_{[k]} \xrightarrow{\phi_k} H^r_{\mathfrak{g}}(\mathcal{D}^M_{r+k})_{[k]} \longrightarrow 0.$$

Since the vertical spectral collapses at the second step, the horizonal spectral converges to the homologies of $H^r_{\mathfrak{g}}(\mathcal{D}^M_\bullet)$. Since all of the homomorphisms are homogeneous of degree 0, the convergence will be case for any graded component. Therefore for any $k \geq 0$ there exists a filtration $\cdots \subset \mathcal{F}_{2k} \subset \mathcal{F}_{1k} \subset \operatorname{Coker}(\phi_k)$ such that

$$\frac{\operatorname{Coker}(\phi_k)}{\mathcal{F}_{1k}} \simeq \left({}^\infty\mathbf{E}^{-k,0}_{\text{hor}}\right)_{[k]}.$$

Notice that $H^{-t}_{\mathfrak{g}}(H_i(\mathcal{D}^M_\bullet)) = 0$ for all $t \geq 1$, one has $^l\mathbf{E}^{-k,0}_{\text{hor}} \subset H^0_{\mathfrak{g}}(H_k(\mathcal{D}^M_\bullet))$ for all $l \geq 2$. Hence we have the following chain of maps for which except the isomorphism

in the middle all are canonical

$$\begin{tikzcd}
H^r_{\mathfrak{g}}(\mathcal{D}^M_{r+k})_{[k]} \arrow[r, two heads] & \mathrm{Coker}(\phi_k) \arrow[r, two heads] & \frac{\mathrm{Coker}(\phi_k)}{\mathcal{F}_{1k}} \arrow[dll, "\simeq"'] \\
\left({}^{\infty}\mathbf{E}^{-k,0}_{\mathrm{hor}}\right)_{[k]} \arrow[r, hook] & H^0_{\mathfrak{g}}(H_k(\mathcal{D}^M_\bullet))_{[k]} \arrow[r, hook] & H_k(\mathcal{D}^M_\bullet)_{[k]} \arrow[r, hook] & (\mathcal{D}^M_\bullet)_{[k]}.
\end{tikzcd}$$

We denote the composition of the above chain of R-homomorphisms by τ_k.

Thus we may define a complex of lenght s, for any integer k,

$$ {}^M_k\mathcal{Z}^+_\bullet : \ 0 \longrightarrow {}^M_k\mathcal{Z}^+_s \longrightarrow \cdots \xrightarrow{\phi_k} {}^M_k\mathcal{Z}^+_{k+1} \xrightarrow{\tau_k} {}^M_k\mathcal{Z}^+_k \longrightarrow \cdots \longrightarrow {}^M_k\mathcal{Z}^+_0 \longrightarrow 0$$

wherein

$$ {}^M_k\mathcal{Z}^+_i = \begin{cases} (\mathcal{D}^M_i)_{[k]} & i \leq \min\{k, s\} \\ H^r_{\mathfrak{g}}(\mathcal{D}^M_{r-1+i})_{[k]} & i > k. \end{cases}$$

It is a trivial generalization of the family of complexes ${}_k\mathcal{Z}^+_\bullet$ in [HN16, Section 2.1].

Since $H^r_{\mathfrak{g}}(M \otimes_R S) \simeq M \otimes_R H^r_{\mathfrak{g}}(S)$ for any R-module M, ${}^M_k\mathcal{Z}^+_\bullet$ have, like graded strands of $\mathcal{D}^M_\bullet$, components that are direct sums of modules of Koszul cycles of $K_\bullet(\mathbf{f}; M)$.

The structure of ${}^M_k\mathcal{Z}^+_\bullet$ is depending on the generating sets of I, on the expression of the generators of $\mathfrak{a}$ in terms of the generators of I and on M. The complex ${}^R_k\mathcal{Z}^+_\bullet$ considered by Hassanzadeh and Naéliton in [HN16], will be denoted by ${}_k\mathcal{Z}^+_\bullet$ instead of ${}^R_k\mathcal{Z}^+_\bullet$.

Definition 5.3.1. The complex ${}^M_k\mathcal{Z}^+_\bullet$ is called the k-th *residual approximation complex* of $J = \mathfrak{a} \colon {}_RI$ with coefficients in M.

We consider the morphism

$$M[T_1, \ldots, T_r]^s(-1) \simeq M \otimes_R S^s(-1) \xrightarrow{1_M \otimes \partial^\gamma_1} M \otimes_R S \simeq M[T_1, \ldots, T_r],$$

where ∂^γ_1 is the first differential of $K_\bullet(\gamma; S)$, and denote by γM the image of $1_M \otimes \partial^\gamma_1$. It is the submodule of $M[T_1, \ldots, T_r]$ generated by the linear forms $\gamma_1, \ldots, \gamma_s$. Recall from Section 5.2.2 that we set $\mathfrak{L}$ for the defining ideal of $\mathrm{Sym}_R(I)$ in S and $\mathfrak{L}'$ for the module spanned by the linear forms correspond to generators of $\mathrm{Tor}^R_1(R/I, M)$.

Proposition 5.3.2. *Let R be a Noetherian ring and $\mathfrak{a} \subset I$ two ideals of R. Suppose that M is a finitely generated R-module. Then*

$$H_0(\mathcal{D}^M_\bullet) \simeq M[T_1, \ldots, T_r]/(\mathfrak{L}M + \mathfrak{L}' + \gamma M)$$

and for all $k \geq 1$,

$$H_0({}^M_k\mathcal{Z}^+_\bullet) \simeq M[T_1, \ldots, T_r]_{[k]}/(\mathfrak{L}M + \mathfrak{L}' + \gamma M)_{[k]}.$$

Proof. The first isomorphism follows from the definition of $\mathcal{D}_\bullet^M$ and Proposition 5.2.3. The last isomorphism is a consequence of the fact that, for all $k \geq 1$, $H_0({}_k^M\mathcal{Z}_\bullet^+) = H_0(\mathcal{D}_\bullet^M)_{[k]}$ is the k-th graded component of $H_0(\mathcal{D}_\bullet^M)$. □

Proposition 5.3.3. *Let R be a Noetherian ring and $\mathfrak{a} \subset I$ two ideals of R. Assume that M is a finitely generated R-module. Then, for all $k \geq 1$, there exists a natural epimorphism*

$$\psi : \mathrm{Sym}_R^k(I/\mathfrak{a}) \otimes_R M \twoheadrightarrow H_0({}_k^M\mathcal{Z}_\bullet^+).$$

Furthermore, if $\mathrm{Tor}_1^R(R/I, M) = 0$, then ψ is an isomorphism.

Proof. As $\mathrm{Sym}_R(I/\mathfrak{a}) \simeq \mathrm{Sym}_R(I)/\mathfrak{a}\mathrm{Sym}_R(I) \simeq S/(\mathfrak{L} + (\gamma))$, we have an exact sequence

$$\mathfrak{L} \oplus (\gamma) \xrightarrow{\alpha} S \longrightarrow \mathrm{Sym}_R(I/\mathfrak{a}) \longrightarrow 0$$

which provides a commutative diagram, with exact rows

$$\begin{tikzcd}
(\mathfrak{L} \oplus (\gamma)) \otimes_R M \arrow[r, "\alpha\otimes 1_M"] \arrow[d, "\simeq"'] & M[\mathbf{T}] \arrow[r] \arrow[d, "="] & \mathrm{Sym}_R(I/\mathfrak{a}) \otimes_R M \arrow[r] \arrow[d, "="] & 0 \\
\mathfrak{L} \otimes_R M \oplus (\gamma) \otimes_R M \arrow[r, "\theta\otimes 1_M \oplus \beta\otimes 1_M"] & M[\mathbf{T}] \arrow[r] & \mathrm{Sym}_R(I/\mathfrak{a}) \otimes_R M \arrow[r] & 0
\end{tikzcd}$$

where β is the inclusion $(\gamma) \hookrightarrow S$ and hence $\mathrm{Im}(\beta \otimes 1_M) = \gamma M$. It follows that

$$\mathrm{Sym}_R(I/\mathfrak{a}) \otimes_R M \simeq M[\mathbf{T}]/\mathrm{Im}(\alpha \otimes 1_M) \simeq M[T_1, \ldots, T_r]/(\mathfrak{L}M + \gamma M).$$

The natural onto map

$$M[T_1, \ldots, T_r]/(\mathfrak{L}M + \gamma M) \twoheadrightarrow M[T_1, \ldots, T_r]/(\mathfrak{L}M + \mathfrak{L}' + \gamma M)$$

provides an epimorphism, for all $k \geq 1$,

$$\psi : \mathrm{Sym}_R^k(I/\mathfrak{a}) \otimes_R M \twoheadrightarrow H_0({}_k^M\mathcal{Z}_\bullet^+)$$

by Proposition 5.3.2. Moreover, $\mathrm{Tor}_1^R(R/I, M) = 0$ is equivalent to $\mathfrak{L}_M = \mathfrak{L}M$ which gives the last assertion. □

Lemma 5.3.4. *Let M be a module over a ring R. Suppose that N is a quotient of $M[T_1, \ldots, T_r]$, with T_i's indeterminates of degree 1, by a graded submodule. Then, for all $k \geq 1$,*

$$\mathrm{ann}_R(N_k) \subset \mathrm{ann}_R(N_{k+1}).$$

Proof. We consider a graded S-homomorphism of degree zero

$$\begin{aligned} \vartheta : M[T_1, \ldots, T_r]^r(-1) &\longrightarrow M[T_1, \ldots, T_r] \,. \\ (g_1, \ldots, g_r) &\longmapsto \textstyle\sum_{i=1}^r g_i T_i \end{aligned}$$

Then ϑ provides the epimorphisms of R-modules, for all $k \geq 1$,

$$\widetilde{\vartheta}_k : N_k^r \twoheadrightarrow N_{k+1}.$$

We will show that $\operatorname{ann}_R(N_k) \subset \operatorname{ann}_R(N_{k+1})$, for all $k \geq 1$. Let $a \in \operatorname{ann}_R(N_k)$ and $u \in N_{k+1}$. We have to show that $au = 0$. Since $\widetilde{\vartheta}_k$ is surjective, there exist $g_1, \ldots, g_r \in N_k$ such that $u = \widetilde{\vartheta}_k(g_1, \ldots, g_r)$. Therefore,

$$au = a\widetilde{\vartheta}_k(g_1, \ldots, g_r) = \widetilde{\vartheta}_k(ag_1, \ldots, ag_r) = \widetilde{\vartheta}_k(0, \ldots, 0) = 0.$$

□

Proposition 5.3.5. *Let R be a Noetherian ring and $\mathfrak{a} \subset I$ two ideals of R. Assume that M is a finitely generated R-module. Then $J = \mathfrak{a} \colon {}_R I$ annihilates $H_0({}^M_k\mathcal{Z}^+_\bullet)$, for all $k \geq 1$.*

Proof. Fix $k \geq 1$. As in the proof of Lemma 5.3.4, the epimorphism ψ in Proposition 5.3.3 implies that

$$\operatorname{ann}_R(\operatorname{Sym}^k_R(I/\mathfrak{a}) \otimes_R M) \subset \operatorname{ann}_R(H_0({}^M_k\mathcal{Z}^+_\bullet)). \tag{5.5}$$

On the other hand, one always has

$$\operatorname{ann}_R(\operatorname{Sym}^k_R(I/\mathfrak{a})) \subset \operatorname{ann}_R(\operatorname{Sym}^k_R(I/\mathfrak{a}) \otimes_R M). \tag{5.6}$$

Notice that $\operatorname{Sym}_R(I/\mathfrak{a}) \simeq \operatorname{Sym}_R(I)/(\gamma)\operatorname{Sym}_R(I) \simeq S/(\mathfrak{L} + (\gamma))$. By Lemma 5.3.4,

$$J = \operatorname{ann}_R(I/\mathfrak{a}) \subset \operatorname{ann}_R(\operatorname{Sym}^k_R(I/\mathfrak{a})). \tag{5.7}$$

By (5.5), (5.6) and (5.7), $J \subset \operatorname{ann}_R(H_0({}^M_k\mathcal{Z}^+_\bullet))$. □

However, the structure of $H_0({}^M_0\mathcal{Z}^+_\bullet)$ is difficult to determine. We recall a definition of Hassanzadeh and Naéliton in [HN16, Definition 2.1].

Definition 5.3.6. Let R be a Noetherian ring and $\mathfrak{a} \subset I$ two ideals of R. The *disguised s-residual intersection of I w.r.t. $\mathfrak{a}$* is the unique ideal K such that $H_0({}_0\mathcal{Z}^+_\bullet) = R/K$.

To make use of the acyclicity of the ${}_k\mathcal{Z}^+_\bullet$ complexes, we recall the definition of classes of ideals that meet these requirements.

Definition 5.3.7. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and $I = (f_1, \ldots, f_r)$ an ideal of height g. Let $k \geq 0$ be an integer. Then

(i) I satisfies *the sliding depth condition,* SD_k, if

$$\operatorname{depth}(H_i(\mathbf{f}; R)) \geq \min\{d - g, d - r + i + k\}, \forall i;$$

also SD stands for SD_0;

(ii) I satisfies *the sliding depth condition on cycles,* SDC_k, if

$$\operatorname{depth}(Z_i(\mathbf{f}; R)) \geq \min\{d - r + i + k, d - g + 2, d\}, \forall i \leq r - g;$$

(iii) I is *strongly Cohen-Macaulay* if $H_i(\mathbf{f}; R)$ is Cohen-Macaulay, for all i.

Since the definition, we see that if I satisfies SD_k, then it satisfies SD_t for all $k \geq t$. Clearly I is strongly Cohen-Macaulay if and only if I satisfies SD_{r-g}. Some of the basic properties and relations between such conditions SD_k and SDC_k are given in [Has12, Remark 2.4, Proposition 2.5], [HN16, Proposition 2.4], also see [HSV83b, HVV85, Vas94]. It will be of importance to us the following proposition.

Proposition 5.3.8. [Has12, Proposition 2.5] *Let R be a Cohen-Macaulay local ring and $k \geq 0$ an integer. If I satisfies SD_k, then I satisfies SDC_{k+1}.*

Remark 5.3.9. Notice that adding an indeterminate x to the ring and to ideals I and $\mathfrak{a}$. One has $(\mathfrak{a} + (x)) \colon (I + (x)) = (\mathfrak{a} \colon I) + (x)$ in $R[x]$ and in its localization at $\mathfrak{m} + (x)$. Hence, for most statements, one may reduce to the case where the height of I is big enough, if needed.

We will recall some important results that were proved by Hassanzadeh and Naéliton. The Cohen-Macaulay hypothesis in this theorem is needed to show that if for an R-module M, $\operatorname{depth}(M) \geq d - t$ then for any prime $\mathfrak{p}$, $\operatorname{depth}(M_{\mathfrak{p}}) \geq \operatorname{ht}(\mathfrak{p}) - t$, see [Vas94, Section 3.3].

Theorem 5.3.10. [HN16, Theorem 2.6] *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and $I = (f_1, \ldots, f_r)$ an ideal of height g. Let $s \geq g$ and fix $0 \leq k \leq \min\{s, s - g + 2\}$. Suppose that one of the following hypotheses holds:*

(i) *$r + k \leq s$ and I satisfies SD, or*

(ii) *$r + k \geq s + 1$, I satisfies SD and $\operatorname{depth}(Z_i) \geq d - s + k$, for $0 \leq i \leq k$, or*

(iii) *I is strongly Cohen-Macaulay.*

Then for any s-residual intersection $J = (\mathfrak{a} \colon {}_R I)$, the complex ${}_k\mathcal{Z}_\bullet^+$ is acyclic. Furthermore, $\operatorname{Sym}_R^k(I/\mathfrak{a})$, for $1 \leq k \leq s - g + 2$, and the disguised residual intersection R/K are Cohen-Macaulay of codimension s.

Notice that the condition (iii) is stronger than (i) and (ii).

Theorem 5.3.11. [Has12, Theorem 2.11] *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and $\mathfrak{a} \subset I$ two ideals of R. Suppose $J = (\mathfrak{a} \colon {}_R I)$ is an s-residual intersection and K is the disguised s-residual intersection of I w.r.t. $\mathfrak{a}$. If I satisfies the sliding depth condition SD, then $K \subset J$ and $\sqrt{K} = \sqrt{J}$. Furthermore $K = J$, whenever the residual intersection J is arithmetic.*

5.4 Cohen-Macaulayness and canonical module of residual intersections

In this section we will prove two most important conjectures in the theory of residual intersections. They are the Cohen-Macaulayness and the canonical module of residual intersections.

In order to make reduction to lower height case and prove the Cohen-Macaulayness when $s = g$, we first state the following proposition, which is a trivial generalization of [HVV85, Lemma 3.5] that only treated the sliding depth condition SD. The proof goes along the same lines.

Proposition 5.4.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, I an ideal of height g and $k \geq 0$ an integer. Let $x_1, \ldots, x_\ell$ be a regular sequence in I. Let $'$ denote the canonical epimorphism $R \longrightarrow R' = R/(x_1, \ldots, x_\ell)$. Then I satisfies SD_k if and only if I' satisfies SD_k (in R'). In particular, I is strongly Cohen-Macaulay if and only if I' is also strongly Cohen-Macaulay (in R').*

Proof. It is suffice to prove the assertion for $\ell = 1$. Let I be an ideal generated by the sequence $\mathbf{x} = x_1, \ldots, x_r$. We have the exact sequence of Koszul complexes

$$0 \longrightarrow K_\bullet(\mathbf{x}; R) \xrightarrow{x_1} K_\bullet(\mathbf{x}; R) \longrightarrow K_\bullet(\mathbf{x}; R') \longrightarrow 0.$$

The last complex is the same as $K_\bullet(\mathbf{x}'; R')$ where $\mathbf{x}' = \{0, x_2', \ldots, x_r'\}$.

Taking the homology sequence, we obtain a long exact sequence

$$\cdots \to H_i(\mathbf{x}; R) \xrightarrow{\pm x_1} H_i(\mathbf{x}; R) \to H_i(\mathbf{x}'; R') \to H_{i-1}(\mathbf{x}; R) \xrightarrow{\pm x_1} H_{i-1}(\mathbf{x}; R) \to \cdots .$$

Since I annihilates the Koszul homologies, we obtain an exact sequence, for all i

$$0 \longrightarrow H_i(\mathbf{x}; R) \longrightarrow H_i(\mathbf{x}'; R') \longrightarrow H_{i-1}(\mathbf{x}; R) \longrightarrow 0.$$

By Corollary 1.1.8, we have

(i) if $\operatorname{depth}(H_i(\mathbf{x}'; R') < \operatorname{depth}(H_{i-1}(\mathbf{x}; R))$, then $\operatorname{depth}(H_i(\mathbf{x}; R)) = \operatorname{depth}(H_i(\mathbf{x}'; R'))$, or

(ii) if $\operatorname{depth}(H_i(\mathbf{x}'; R') > \operatorname{depth}(H_{i-1}(\mathbf{x}; R))$, then $\operatorname{depth}(H_i(\mathbf{x}; R)) = \operatorname{depth}(H_{i-1}(\mathbf{x}; R)) + 1$, or

(iii) if $\operatorname{depth}(H_i(\mathbf{x}'; R') = \operatorname{depth}(H_{i-1}(\mathbf{x}; R))$, then $\operatorname{depth}(H_i(\mathbf{x}; R)) \geq \operatorname{depth}(H_{i-1}(\mathbf{x}; R))$.

Notice that $\dim(R') = d - 1$ and $\operatorname{ht}(I') = g - 1$, hence I' of R' satisfies SD_k if

$$\operatorname{depth}(H_i(\mathbf{x}'; R')) \geq \min\{d - g, d - r + i + k - 1\}.$$

It is easy to prove the assertion by comparing (i), (ii) and (iii). □

Proposition 5.4.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and I an ideal of height g. Let $\mathbf{x} = x_1, \ldots, x_g$ be a regular sequence contained in I and $J = ((\mathbf{x}) \colon {}_R I)$. Suppose that R/I is Cohen-Macaulay and I satisfies SD. Then R/J is Cohen-Macaulay of codimension g.*

Proof. The proof goes along the same lines as in [HVV85]. By Proposition 5.4.1, we may reduce modulo $\mathbf{x} = x_1, \ldots, x_g$ and consider $R' = R/(\mathbf{x})$. Thus we can assume that $\operatorname{ht}(I) = g = 0$ and $J = (0 \colon {}_R I)$.

Suppose that I is an ideal generated by the sequence $x_1, \ldots, x_r$. Then $Z_r = (0 \colon {}_R I) = J$ and $K_r \simeq R$. The exact sequence

$$0 \longrightarrow Z_r \longrightarrow K_r \longrightarrow B_{r-1} \longrightarrow 0$$

shows that $B_{r-1} \simeq K_r/Z_r \simeq R/J$.

Since I satisfies SD, I satisfies SDC_1 by Proposition 5.3.8. It follows that Z_{r-1} is Cohen-Macaulay of dimension d. Moreover, I satisfies SD, $\operatorname{depth}(H_{r-1}) \geq d-1$. Therefore, the exact sequence

$$0 \longrightarrow B_{r-1} \longrightarrow Z_{r-1} \longrightarrow H_{r-1} \longrightarrow 0$$

implies that $H^i_{\mathfrak{m}}(B_{r-1}) = 0$, for all $i \neq d$, hence B_{r-1} is Cohen-Macaulay of dimension d. $\square$

To study the Cohen-Macaulayness of residual intersections in the general case, we will use the following lemmas.

Lemma 5.4.3. [Bou70, II, §4, no 4, Proposition 4] *Let R be a ring and M_1, M_2, N_1, N_2 four finitely generated R-modules. If one of couples $(M_1, M_2), (M_1, N_1), (M_2, N_2)$ are formed by projective R-modules, then the canonical homomorphism*

$$\lambda : \operatorname{Hom}_R(M_1, N_1) \otimes_R \operatorname{Hom}_R(M_2, N_2) \longrightarrow \operatorname{Hom}_R(M_1 \otimes_R N_1, M_2 \otimes_R N_2)$$

is an isomorphism.

Lemma 5.4.4. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, with canonical module ω. Suppose that $S = R[T_1, \ldots, T_r]$ is the standard graded polynomial ring over R and $\mathfrak{g} := S_+$. Let $\mathfrak{a} \subset I = (f_1, \ldots, f_r)$ be two ideals of R, with $\operatorname{ht}(I) = g$. If $J = (\mathfrak{a} :_R I)$ is an s-residual intersection of I, then*

(i) *There is a natural graded isomorphism*

$$H^r_{\mathfrak{g}}(S) \simeq {}^*\operatorname{Hom}_S(S(-r), R).$$

In particular, for all $\mu \in \mathbb{Z}$,

$$H^r_{\mathfrak{g}}(S)_\mu \simeq S^*_{-\mu-r} = \operatorname{Hom}_R(S_{-\mu-r}, R).$$

(ii) *If $g \geq 2$, then $\operatorname{depth}({}_k\mathcal{Z}^+_0) = \operatorname{depth}({}_k\mathcal{Z}^+_s) = d$, for all $0 \leq k \leq s-1$.*

(iii) *If $g = 2$ and I satisfies SD_ℓ, then*

$$\operatorname{depth}({}_0\mathcal{Z}^+_i) \geq \min\{d, d-s+i+\ell\},$$

for all $1 \leq i \leq s-1$.

(iv) *If $g \geq 2$, then the following diagram, where the vertical isomorphisms are induced by the identifications $H^d_{\mathfrak{m}}(Z_*) \simeq {Z^\omega_{r-1-*}}^\vee$ in Proposition 5.2.2, is commutative, for all $0 \leq k \leq s-2$,*

$$\begin{tikzcd}
H^d_{\mathfrak{m}}({}_k\mathcal{Z}^+_s) \arrow[r] \arrow[d, "\simeq"'] & H^d_{\mathfrak{m}}({}_k\mathcal{Z}^+_{s-1}) \arrow[d, "\simeq"] \\
({}_{s-k-1}^{\omega}\mathcal{Z}^+_0)^\vee \arrow[r] & ({}_{s-k-1}^{\omega}\mathcal{Z}^+_1)^\vee.
\end{tikzcd}$$

Proof. (i) is the graded local duality theorem.

(ii) Since $Z_{r-1} \simeq Z_0 = R$, $\operatorname{depth}({}_k\mathcal{Z}_0^+) = \operatorname{depth}({}_k\mathcal{Z}_s^+) = d$.

(iii) By Proposition 5.3.8, I satisfies $\mathsf{SDC}_{\ell+1}$, that is

$$\operatorname{depth}(Z_j) \geq \min\{d - r + j + \ell + 1, d\},$$

for all $0 \leq j \leq r-2$.

For any $1 \leq i \leq s-1$,

$${}_0\mathcal{Z}_i^+ = H_{\mathfrak{g}}^r(\mathcal{D}_{r-1+i})_{[0]} = \bigoplus_{j=r-1+i-s}^{r-1} \left(Z_j \otimes_R H_{\mathfrak{g}}^r(S)\right)^{\binom{s}{r-1+i-j}}_{[-r+1-i]}.$$

Thus ${}_0\mathcal{Z}_i^+$ is a direct sum of copies of modules $Z_\delta, \ldots, Z_{r-1}$, where $\delta = \max\{0, r-1+i-s\}$. Notice that $0 \leq \delta \leq r-2$. It follows that

$$\begin{aligned}\operatorname{depth}({}_0\mathcal{Z}_i^+) &= \min_{\delta \leq j \leq r-1}\{\operatorname{depth}(Z_j)\} = \min\{\min_{\delta \leq j \leq r-2}\{\operatorname{depth}(Z_j)\}, d\} \\ &\geq \min\{d, d-r+\delta+\ell+1\} \geq \min\{d, d-s+i+\ell\}.\end{aligned}$$

(iv) We have the following commutative diagrams, for all $0 \leq k \leq s-2$,

$$\begin{tikzcd}
H_{\mathfrak{m}}^d({}_k\mathcal{Z}_s^+) = H_{\mathfrak{m}}^d(H_{\mathfrak{g}}^r(\mathcal{D}_{r+s-1}))_{[k]} \arrow[r] \arrow[d, "\simeq"'] & H_{\mathfrak{m}}^d(H_{\mathfrak{g}}^r(\mathcal{D}_{r+s-2}))_{[k]} = H_{\mathfrak{m}}^d({}_k\mathcal{Z}_{s-1}^+) \arrow[d, "\simeq"] \\
H_{\mathfrak{m}}^d(Z_{r-1}) \otimes_R H_{\mathfrak{g}}^r(S)_{[k-r-s+1]} \arrow[r] \arrow[d, "\simeq"'] & H_{\mathfrak{m}}^d(Z_{r-1}) \otimes_R H_{\mathfrak{g}}^r(S)^s_{[k-r-s+2]} \oplus H_{\mathfrak{m}}^d(Z_{r-2}) \otimes_R H_{\mathfrak{g}}^r(S)_{[k-r-s+2]} \arrow[d, "\simeq"] \\
Z_0^{\omega\vee} \otimes_R S^*_{[s-k-1]} \arrow[r] \arrow[d, "\simeq"'] & Z_0^{\omega\vee} \otimes_R (S^s_{[s-k-2]})^* \oplus Z_1^{\omega\vee} \otimes_R S^*_{[s-k-2]} \arrow[d, "\simeq"] \\
(Z_0^\omega \otimes_R S_{[s-k-1]})^\vee \arrow[r] \arrow[d, "\simeq"'] & (Z_0^\omega \otimes_R S^s_{[s-k-2]} \oplus Z_1^\omega \otimes_R S_{[s-k-2]})^\vee \arrow[d, "\simeq"] \\
({}_{s-k-1}^{\omega}\mathcal{Z}_0^+)^\vee = ((\mathcal{D}_0^\omega)_{[s-k-1]})^\vee \arrow[r] & ((\mathcal{D}_1^\omega)_{[s-k-1]})^\vee = ({}_{s-k-1}^{\omega}\mathcal{Z}_1^+)^\vee
\end{tikzcd}$$

where the first diagram and the last diagram are commutative by the definitions, the second diagram is commutative by the natural isomorphisms in item (i) and Proposition 5.2.2, and the third diagram is commutative by the natural isomorphism

$$Z_i^{\omega\vee} \otimes_R S^*_{[\ell]} \simeq (Z_i^\omega \otimes_R S_{[\ell]})^\vee,$$

for all i, ℓ, by Lemma 5.4.3. □

Proposition 5.4.5. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, with canonical module ω, and $I = (f_1, \ldots, f_r)$ an ideal of height 2. Suppose that $J = (\mathfrak{a} \colon {}_RI)$ is an s-residual intersection of I and K is the disguised s-residual intersection of I w.r.t. $\mathfrak{a}$. If I satisfies SD_1, then there exists an epimorphism of R-modules*

$$\phi : H_0({}_{s-1}^{\omega}\mathcal{Z}_\bullet^+) \twoheadrightarrow \omega_{R/K}$$

and ϕ is an isomorphism if I satisfies SD_2.

Proof. Since I satisfies SD_1, ${}_0\mathcal{Z}^+_\bullet$ is acyclic and R/K is Cohen-Macaulay of dimension $d-s$ by Theorem 5.3.10. By local duality

$$H^{d-s}_{\mathfrak{m}}(R/K) \simeq \omega^{\vee}_{R/K}. \tag{5.8}$$

Now the double complex $C^\bullet_{\mathfrak{m}}({}_0\mathcal{Z}^+_\bullet)$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C^0_{\mathfrak{m}}({}_0\mathcal{Z}^+_s) \arrow[r] \arrow[d] & \cdots \arrow[r] & C^0_{\mathfrak{m}}({}_0\mathcal{Z}^+_1) \arrow[r] \arrow[d] & C^0_{\mathfrak{m}}({}_0\mathcal{Z}^+_0) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^1_{\mathfrak{m}}({}_0\mathcal{Z}^+_s) \arrow[r] \arrow[d] & \cdots \arrow[r] & C^1_{\mathfrak{m}}({}_0\mathcal{Z}^+_1) \arrow[r] \arrow[d] & C^1_{\mathfrak{m}}({}_0\mathcal{Z}^+_0) \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_s) \arrow[r] \arrow[d] & \cdots \arrow[r] & C^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_1) \arrow[r] \arrow[d] & C^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_0) \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 &
\end{tikzcd}$$

gives rise to two spectral sequences. The second terms of the horizonal spectral are

$${}^2\mathbf{E}^{-i,-j}_{\text{hor}} = \begin{cases} H^{d-s}_{\mathfrak{m}}(R/K) & \text{if} \quad j = d-s \text{ and } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

and the first terms of the vertical spectral are

$$\begin{tikzcd}
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & H^{d-s+2}_{\mathfrak{m}}({}_0\mathcal{Z}^+_1) & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & H^{d-1}_{\mathfrak{m}}({}_0\mathcal{Z}^+_{s-2}) \arrow[r] & \cdots \arrow[r] & H^{d-1}_{\mathfrak{m}}({}_0\mathcal{Z}^+_2) \arrow[r] & H^{d-1}_{\mathfrak{m}}({}_0\mathcal{Z}^+_1) & 0 \\
H^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_s) \arrow[r] & H^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_{s-1}) \arrow[r] & H^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_{s-2}) \arrow[r] & \cdots \arrow[r] & H^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_2) \arrow[r] & H^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_1) \arrow[r] & H^d_{\mathfrak{m}}({}_0\mathcal{Z}^+_0)
\end{tikzcd}$$

since $\operatorname{depth}({}_0\mathcal{Z}^+_0) = \operatorname{depth}({}_0\mathcal{Z}^+_s) = d$ and $\operatorname{depth}({}_0\mathcal{Z}^+_i) \geq d-s+i+1$, for all $1 \leq i \leq s-1$, by Lemma 5.4.4 (ii) and (iii), respectively.

By the convergence of the spectral sequences, we obtain

$$H_{\mathfrak{m}}^{d-s}(R/K) \simeq {}^{\infty}\mathbf{E}_{\mathrm{ver}}^{-s,-d} \subset {}^{2}\mathbf{E}_{\mathrm{ver}}^{-s,-d}. \tag{5.9}$$

By Lemma 5.4.4(iv), we have the following commutative diagram

$$\begin{tikzcd} H_{\mathfrak{m}}^{d}({}_0\mathcal{Z}_s^+) \arrow[r] \arrow[d, "\simeq"'] & H_{\mathfrak{m}}^{d}({}_0\mathcal{Z}_{s-1}^+) \arrow[d, "\simeq"] \\ ({}_{s-1}^{\omega}\mathcal{Z}_0^+)^\vee \arrow[r] & ({}_{s-1}^{\omega}\mathcal{Z}_1^+)^\vee. \end{tikzcd}$$

Therefore

$${}^{2}\mathbf{E}_{\mathrm{ver}}^{-s,-d} \simeq H_0({}_{s-1}^{\omega}\mathcal{Z}_\bullet^+)^\vee. \tag{5.10}$$

By (5.8), (5.9) and (5.10), we can define a monomorphism of R-modules by the compositions

$$\omega_{R/K}^\vee \xrightarrow{\simeq} H_{\mathfrak{m}}^{d-s}(R/K) \xrightarrow{\simeq} {}^{\infty}\mathbf{E}_{\mathrm{ver}}^{-s,-d} \hookrightarrow {}^{2}\mathbf{E}_{\mathrm{ver}}^{-s,-d} \xrightarrow{\simeq} H_0({}_{s-1}^{\omega}\mathcal{Z}_\bullet^+)^\vee$$

which provides an epimorphism

$$\phi : H_0({}_{s-1}^{\omega}\mathcal{Z}_\bullet^+) \longrightarrow \omega_{R/K}.$$

If I satisfies SD_2, then $\operatorname{depth}({}_0\mathcal{Z}_i^+) \geq \min\{d, d-s+i+2\}$, for all $1 \leq i \leq s-1$, by Lemma 5.4.4(iii). It follows that

$$H_{\mathfrak{m}}^{d-s}(R/K) \simeq {}^{\infty}\mathbf{E}_{\mathrm{ver}}^{-s,-d} = {}^{2}\mathbf{E}_{\mathrm{ver}}^{-s,-d}$$

and thus ϕ is an isomorphism. □

Now we state our main result that answers the question of Huneke and Ulrich in [HU88, Question 5.7] and also answers the conjecture of Hassanzadeh and Naéliton in [HN16, Conjecture 5.9].

Theorem 5.4.6. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module ω and $\mathfrak{a} \subset I$ two ideals of R, with $\operatorname{ht}(I) = g$. Suppose that I satisfies SD_1 and $J = (\mathfrak{a} :_R I)$ is an s-residual intersection of I. Then R/J is Cohen-Macaulay of codimension s.*

Proof. Let K be the disguised s-residual intersection of I w.r.t. $\mathfrak{a}$. Since I satisfies SD_1, hence SD, R/K is Cohen-Macaulay of dimension $d-s$ by Theorem 5.3.10 and $K \subset J$ by Theorem 5.3.11. The proof will be completed by showing that $J \subset K$.

We first consider the case where $g = 2$. By Proposition 5.4.5, there is the epimorphism

$$\phi : H_0({}_{s-1}^{\omega}\mathcal{Z}_\bullet^+) \longrightarrow \omega_{R/K}.$$

As R/K is Cohen-Macaulay, $\operatorname{ann}_R(\omega_{R/K}) = \operatorname{ann}_R(R/K) = K$. The epimorphism ϕ implies that

$$\operatorname{ann}_R(H_0({}_{s-1}^{\omega}\mathcal{Z}_\bullet^+)) \subset \operatorname{ann}_R(\omega_{R/K}) = K.$$

By Proposition 5.3.5, $J \subset \operatorname{ann}_R(H_0({}_{s-1}^{\omega}\mathcal{Z}_\bullet^+)) \subset K$.

We may always reduce to the case $g \geq 2$ by Remark 5.3.9. If $g > 2$, then we can choose a regular sequence $\mathbf{a}$ of length $g-2$ inside $\mathfrak{a}$ which is a part of a minimal generating set of $\mathfrak{a}$. Since R is Cohen-Macaulay, by Theorem 1.1.11 (a), $R/\mathbf{a}$ is a Cohen-Macaulay local ring of dimension $d-g+2$. Moreover, $J/\mathbf{a} = \mathfrak{a}/\mathbf{a} \colon I/\mathbf{a}$ and $\mu(\mathfrak{a}/\mathbf{a}) = \mu(\mathfrak{a}) - g + 2$, therefore $J/\mathbf{a}$ is an $(s-g+2)$-residual intersection of $I/\mathbf{a}$ which is of height 2. Furthermore, $I/\mathbf{a}$ satisfies SD_1 by Proposition 5.4.1. Hence, it follows from the height two case that $R/J \simeq (R/\mathbf{a})/(J/\mathbf{a})$ is Cohen-Macaulay of dimension $d-s$. □

It follows from the proof of Proposition 5.3.5 that $J \subset \operatorname{ann}_R(\operatorname{Sym}_R^k(I/\mathfrak{a}))$, for all $k \geq 1$. Then a natural question is: under what conditions one has

$$\operatorname{ann}_R(\operatorname{Sym}_R^k(I/\mathfrak{a})) = J?$$

It is known that $\operatorname{ann}_R(\operatorname{Sym}_R^k(I/\mathfrak{a})) = J$, for all $k \geq 1$ whenever J is arithmetic in [HN16, Corollary 2.8(iv)]. The next result answers this question.

Corollary 5.4.7. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, with canonical module ω, and $\mathfrak{a} \subset I$ two ideals of R, with $\operatorname{ht}(I) = g$. Suppose that J is an s-residual intersection of I and let $1 \leq k \leq s-g+1$.*

(i) *If I satisfies SD_1, then $\operatorname{Sym}_R^k(I/\mathfrak{a})$ is a faithful R/J-module.*

(ii) *If I satisfies strongly Cohen-Macaulay, then $\operatorname{Sym}_R^k(I/\mathfrak{a})$ is a maximal Cohen-Macaulay faithful R/J-module.*

Proof. (i) The proof will be completed by showing that $\operatorname{ann}_R(\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a})) \subset J$. As in the proof of Theorem 5.4.6, it suffices to prove that $\operatorname{ann}_R(\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a})) \subset J$ in the case $g = 2$. The inclusions $\operatorname{ann}_R(\operatorname{Sym}_R^{s-1}(I/\mathfrak{a})) \subset \operatorname{ann}_R(H_0({}_{s-1}^{\omega}\mathcal{Z}_\bullet^+)) \subset K = J$ are demonstrated in the proofs of Proposition 5.3.5 and of Theorem 5.4.6.

(ii) follows immediately from Theorem 5.3.10, Theorem 5.4.6 and the first item. □

The following example shows that the above corollary does not hold for the $(s-g+2)$-th symmetric power of $I/\mathfrak{a}$.

Example 5.4.8. [HN16, Example 2.10] Let $R = \mathbb{Q}[x, y], I = (x, y)$ and $\mathfrak{a} = (x^2, y^2)$. We set $J = \mathfrak{a} \colon {}_R I$. Using `Macaulay2` [GS], we see that $J = (x^2, xy, y^2)$ is a 2-residual intersection (a link in this case) of I and

$$\operatorname{Sym}_R(I/\mathfrak{a}) \simeq R[T_1, T_2]/\big(xT_1, yT_2, -yT_1 + xT_2\big).$$

Thus a free resolution of $\operatorname{Sym}_R^2(I/\mathfrak{a})$ is

$$0 \longrightarrow R^3 \xrightarrow{N} R^6 \xrightarrow{M} R^3 \longrightarrow \operatorname{Sym}_R^2(I/\mathfrak{a}) \longrightarrow 0,$$

where

$$M = \begin{pmatrix} x & 0 & 0 & y & 0 & 0 \\ 0 & x & 0 & 0 & y & 0 \\ 0 & 0 & x & 0 & 0 & y \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} -y & 0 & 0 \\ 0 & -y & 0 \\ 0 & 0 & -y \\ x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}.$$

It follows that

$$\operatorname{ann}_R(\operatorname{Sym}_R^2(I/\mathfrak{a})) = (x, y) \supsetneq J.$$

We now give a description the canonical module of residual intersections.

Theorem 5.4.9. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module ω and $\mathfrak{a} \subset I$ two ideals of R, with $\operatorname{ht}(I) = g$. Suppose that I satisfies SD_2, $\operatorname{Tor}_1^R(R/I, \omega) = 0$ and $J = (\mathfrak{a} \colon {}_R I)$ is an s-residual intersection of I. Then the canonical module of R/J is $\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega$.*

Proof. We first consider the case where $g = 2$. By Proposition 5.4.5 and Theorem 5.4.6,

$$\omega_{R/J} \simeq H_0({}_{s-1}^{\omega}\mathcal{Z}_\bullet^+) \simeq \operatorname{Sym}_R^{s-1}(I/\mathfrak{a}) \otimes_R \omega.$$

The last isomorphism by Proposition 5.3.3.

We may always reduce to the case $g \geq 2$ by Remark 5.3.9. If $g > 2$, then we can choose a regular sequence $\mathbf{a}$ of length $g - 2$ inside $\mathfrak{a}$ which is a part of a minimal generating set of $\mathfrak{a}$ as in the proof of Theorem 5.4.6. As $\mathbf{a} \subset I$ is regular on ω,

$$\operatorname{Tor}_1^R(R/I, \omega) \simeq \operatorname{Tor}_1^{R/\mathbf{a}}(R/I, \omega/\mathbf{a}\omega) = 0.$$

Furthermore, observing that the canonical module of $R/\mathbf{a}$ is $\omega/\mathbf{a}\omega$ by Proposition 1.2.34 (i), it follows from the height two case that

$$\omega_{R/J} \simeq \operatorname{Sym}_{R/\mathbf{a}}^{(s-g+2)-1}\left(\frac{I/\mathbf{a}}{\mathfrak{a}/\mathbf{a}}\right) \otimes_{R/\mathbf{a}} \omega_{R/\mathbf{a}} \simeq \operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega.$$

□

Notice that the hypothesis $\operatorname{Tor}_1^R(R/I, \omega) = 0$ is always satisfied for ideals of finite projective dimension. In particular, if R is Gorenstein, then $\omega \simeq R$, hence $\operatorname{Tor}_1^R(R/I, \omega) \simeq \operatorname{Tor}_1^R(R/I, R) = 0$, therefore the canonical module of R/J is $(s - g + 1)$-th symmetric power of $I/\mathfrak{a}$. As a consequence, the second conjecture in the introduction is proved under the SD_2 condition.

Remark 5.4.10.

(i) Under assumptions of Theorem 5.4.9, but I only satisfies SD_1 instead of SD_2. Then there exists an epimorphism of R-modules

$$\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega \twoheadrightarrow \omega_{R/J}\,.$$

(ii) In the height two case, by using Propositon 5.4.5, we could omit the assumption $\mathrm{Tor}_1^R(R/I,\omega)=0$ in Theorem 5.4.9. In this case, the canonical module of R/J is the $(s-1)$-th graded component of

$$\omega[T_1,\dots,T_r]/(\mathfrak{L}\omega+\mathfrak{L}'+\gamma\omega)$$

by Proposition 5.3.2 and Theorem 5.4.6.

The following example shows that Theorem 5.4.9 does not hold if I only satisfies SD condition.

Example 5.4.11. [EU16, Example 2.9] Let $R=k[[x_1,\dots,x_5]]$ and let I be the ideal of 2×2 minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{pmatrix}.$$

Then I is of height 3. If we take $\mathfrak{a}$ to be the ideal generated by 4 sufficiently general cubic forms in I, then $J=\mathfrak{a}\colon_R I$ is a 4-residual intersection. Using `Macaulay2` [GS], it is easy to see that I satisfies SD. Moreover, we see that $I^2/\mathfrak{a}I$ requires 20 generators, whereas $\omega_{R/J}$ requires only 16. Thus there is no surjection $\omega_{R/J}\twoheadrightarrow I^2/\mathfrak{a}I$, therefore $\omega_{R/J}$ is not isomorphic to $\mathrm{Sym}_R^2(I/\mathfrak{a})$.

Computation of the initial degree of $\mathrm{Sym}_R^2(I/\mathfrak{a})$ and $\omega_{R/J}$ shows that there can be no surjection $\mathrm{Sym}_R^2(I/\mathfrak{a})\twoheadrightarrow\omega_{R/J}$. This shows that SD_1 condition in Remark 5.4.10 (i) is necessary.

Recall that in a Noetherian local ring $(R,\mathfrak{m})$, the *type* of a finitely generated R-module M is the dimension of the $R/\mathfrak{m}$-vector space $\mathrm{Ext}_R^{\mathrm{depth}(M)}(R/\mathfrak{m},M)$ and it is denoted by $r_R(M)$ or just $r(M)$. The minimal number of generators of the R-module M is the dimension of the $R/\mathfrak{m}$-vector space $R/\mathfrak{m}\otimes_R M$ and it is denoted by $\mu(M)$. Notice that if M,N are two finitely generated R-modules, then

$$\begin{aligned}\mu(M\otimes_R N)&=\dim_{R/\mathfrak{m}}(M\otimes_R N\otimes_R R/\mathfrak{m})\\&=\dim_{R/\mathfrak{m}}(M\otimes_R R/\mathfrak{m}\otimes_{R/\mathfrak{m}} N\otimes_R R/\mathfrak{m})\\&=\mu(M)\mu(N).\end{aligned}$$

Corollary 5.4.12. *Under the assumptions of Theorem 5.4.9,*

$$r(R/J)=\binom{\mu(I/\mathfrak{a})+s-g}{\mu(I/\mathfrak{a})-1}r(R).$$

Thus R/J is Gorenstein if and only if R is Goenstein and $\mu(I/\mathfrak{a})=1$.

Proof. Since the canonical module of R/J is $\mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a})\otimes_R\omega$ by Theorem 5.4.9, it follows from Proposition 1.2.38 that

$$\begin{aligned}r(R/J)=\mu(\omega_{R/J})&=\mu(\mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a})\otimes_R\omega)\\&=\mu(\mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a}))\mu(\omega)\\&=\dim_{R/\mathfrak{m}}(\mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a})\otimes_R R/\mathfrak{m})r(R)\\&=\dim_{R/\mathfrak{m}}(\mathrm{Sym}_{R/\mathfrak{m}}^{s-g+1}(I/\mathfrak{a}\otimes_R R/\mathfrak{m}))r(R).\end{aligned}$$

Since $I/\mathfrak{a} \otimes_R R/\mathfrak{m}$ is a $R/\mathfrak{m}$-vector space of dimension $\mu(I/\mathfrak{a})$,

$$\mathrm{Sym}_{R/\mathfrak{m}}(I/\mathfrak{a} \otimes_R R/\mathfrak{m}) \simeq (R/\mathfrak{m})[Y_1, \ldots, Y_{\mu(I/\mathfrak{a})}].$$

It follows that

$$r(R/J) = \binom{\mu(I/a) + s - g}{\mu(I/a) - 1} r(R).$$

□

5.5 Hilbert functions and Castelnuovo-Mumford regularity of residual intersections

One is based on the resolution of residual intersections ${}_0\mathcal{Z}_\bullet^+$, from which we could provide many informations concerning R/J, like the stability of Hilbert functions and the Castelnuovo-Mumford regularity of residual intersections.

First we study the stability of Hilbert functions of residual intersections. In [CEU01], Chardin, Eisenbud and Ulrich restated an old question of Stanley in [Sta80] asking for which open sets of ideals $\mathfrak{a}$ the Hilbert function of $R/\mathfrak{a}$ depends only on the degrees of the generators $\mathfrak{a}$. More precisely, they consider the following two conditions.

(A1) Is the Hilbert function of $R/\mathfrak{a}$ is constant on the open set of ideals $\mathfrak{a}$ generated by s forms of the given degrees such that $\mathrm{ht}(\mathfrak{a} \colon {}_R I) \geq s$;

(A2) Is the Hilbert function of $R/(\mathfrak{a} \colon {}_R I)$ is constant on this set.

It is shown in [CEU01, Theorem 2.1] that ideals with some sliding depth conditions in conjunction with G_{s-1} or G_s satisfy these two conditions. In [HN16, Proposition 3.1], Hassanzadeh and Naéliton proved that if $(R, \mathfrak{m})$ is a Cohen-Macaulay graded local ring of dimension d over an Artinian local ring R_0 and if $\mathfrak{a} \subset I$ are two homogeneous ideals, I satisfies SD, and $\mathrm{depth}(R/I) \geq d - s$, then the above condition (A1) is satisfied for any s-residual intersection $J = (\mathfrak{a} \colon {}_R I)$. It follows directly from [Has12, Theorem 2.11] and [HN16, Proposition 3.1] that if I satisfies SD, then, for any arithmetic s-residual intersection $J = (\mathfrak{a} \colon {}_R I)$, the above condition (A2) is satisfied.

The next proposition, we will show that the above condition (A2) is satisfied for any residual intersection under SD_1 condition.

Proposition 5.5.1. *Let $(R, \mathfrak{m})$ be a graded Cohen-Macaulay local ring over an Artinian local ring R_0 and $\mathfrak{a} \subset I$ two homogeneous ideals, with $\mathrm{ht}(I) = g$. Suppose that I satisfies SD_1 and $J = (\mathfrak{a} \colon {}_R I)$ is an s-residual intersection of I. Then the Hilbert function of R/J satisfies the above condition (A2).*

Proof. By Theorem 5.3.10 and Theorem 5.4.6, the complex ${}_0\mathcal{Z}_\bullet^+$ is a resolution of R/J. Hence, the Hilbert function of R/J can be written in terms of the Hilbert functions of the components of the complex ${}_0\mathcal{Z}_\bullet^+$ which, according to the definition of ${}_0\mathcal{Z}_\bullet^+$, are just some direct sums of Koszul cycles of I shifted by the twists appearing in the Koszul complex $K_\bullet(\gamma; S)$. Since the Hilbert functions of Koszul cycles are

inductively calculated in terms of those of the Koszul homology modules, the Hilbert function of R/J only depends on the Koszul homology modules of I and on the degrees of the generators of $\mathfrak{a}$. □

Next, the important numerical invariant associated an algebraic or geometric object is the Castelnuovo-Mumford regularity. Assume that $R = \bigoplus_{n\geq 0} R_n$ is a positively graded Noetherian *local ring of dimension d over a Noetherian local ring $(R_0, \mathfrak{m}_0)$. Set $\mathfrak{m} = \mathfrak{m}_0 + R_+$. Suppose that I and $\mathfrak{a}$ are two homogeneous ideals of R generated by homogeneous elements $f_1, \ldots, f_r$ and $a_1, \ldots, a_s$, respectively. For a homogeneous ideal $\mathfrak{b}$, the sum of the degrees of a minimal generating set of $\mathfrak{b}$ is denoted by $\sigma(\mathfrak{b})$. For a finitely generated graded R-module M, the Castelnuovo-Mumford regularity of M is defined as

$$\operatorname{reg}(M) := \max\{\operatorname{end}(H^i_{R_+}(M)) + i\}.$$

In [Has12], Hassanzadeh defined the regularity with respect to the maximal ideal $\mathfrak{m}$ as

$$\operatorname{reg}_{\mathfrak{m}}(M) := \max\{\operatorname{end}(H^i_{\mathfrak{m}}(M)) + i\}.$$

The following lemma establishes a relation between $\operatorname{reg}(M)$ and $\operatorname{reg}_{\mathfrak{m}}(M)$.

Lemma 5.5.2. [Has12, Proposition 3.4] *Assume that $(R, \mathfrak{m})$ is a positively graded Cohen-Macaulay *local ring and let M be a finitely generated graded R-module. Then*

$$\operatorname{reg}(M) \leq \operatorname{reg}_{\mathfrak{m}}(M) \leq \operatorname{reg}(M) + \dim(R_0).$$

The next proposition improves [Has12, Theorem 3.6] by removing the arithmetic hypothesis of residual intersections.

Proposition 5.5.3. *Let $(R, \mathfrak{m})$ be a positively graded Cohen-Macaulay *local ring over a Noetherian local ring $(R_0, \mathfrak{m}_0)$ and $\mathfrak{a} \subset I$ two homogeneous ideals, with $\operatorname{ht}(I) = g$. Suppose that I satisfies SD_1. Then, for any s-residual intersection $J = (\mathfrak{a} :_R I)$,*

$$\operatorname{reg}(R/J) \leq \operatorname{reg}(R) + \dim(R_0) + \sigma(\mathfrak{a}) - (s - g + 1)\operatorname{indeg}(I/\mathfrak{a}) - s.$$

Proof. The proof of this result goes along the same lines as in [Has12, Theorem 3.6]. Indeed, Theorem 5.4.6 implies that R/J is Cohen-Macaulay and is resolved by ${}_0\mathcal{Z}^+_\bullet$. □

We recall that in the case of linkage, that is when $s = g$, if in addition one has $\dim(R_0) = 0$, then the inequality in Proposition 5.5.3 is in fact an equality. However, when $\dim(R_0) \neq 0$, the next simple example shows that, in some cases, the regularity of residual intersections (or even linked ideals) may be strictly less than the proposed formula.

Example 5.5.4. Let $R_0 := k[x]_{(x)}$ and $R := R_0[y]$. In this case let $I = (y), \mathfrak{a} = (xy)$ and $J = (x)$ be ideals of R. It is now easy to see that I is linked to J by $\mathfrak{a}$, whence the invariants mentioned in Proposition 5.5.3 are determined as follows: $\operatorname{reg}(R) = \operatorname{reg}(R/I) = \operatorname{reg}(R/J) = 0, \dim(R_0) = 1, \sigma(\mathfrak{a}) = 1, s - g + 1 = 1, \operatorname{indeg}(I/\mathfrak{a}) = 1$, and $\operatorname{indeg}(J/\mathfrak{a}) = 0$. Therefore the formula is the equality for R/J and a strict inequality for R/I.

The next proposition improves the result of Hassanzadeh and Naéliton in [HN16, Proposition 3.3].

Proposition 5.5.5. *Let $(R, \mathfrak{m})$ be a positively graded Cohen-Macaulay *local ring of dimension d over a Noetherian local ring $(R_0, \mathfrak{m}_0)$, with canonical module ω. Let $\mathfrak{a} \subset I$ be two homogeneous ideals, with $\mathrm{ht}(I) = g$, and $J = (\mathfrak{a} \colon {}_R I)$ an s-residual intersection of I. Suppose that I satisfies SD_2 and $\mathrm{Tor}_1^R(R/I, \omega) = 0$. Then*

$$\omega_{R/J} = \mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega(\sigma(\mathfrak{a})).$$

Proof. The proof proceeds along the same lines as in the local case. □

The following result is already an improvement of [Has12, Proposition 3.15] and also of [HN16, Proposition 3.3]. We show the equality of the proposed upper bound for Castelnuovo-Mumford regularity of residual intersections in Proposition 5.5.3. This equality is showed by Hassanzadeh for perfect ideals of height 2 [Has12, Theorem 3.16(iii)].

Corollary 5.5.6. *Under the assumptions of Proposition 5.5.5,*

$$\mathrm{reg}_{\mathfrak{m}}(R/J) = \mathrm{reg}_{\mathfrak{m}}(R) + \sigma(\mathfrak{a}) - (s - g + 1)\mathrm{indeg}(I/\mathfrak{a}) - s.$$

In particular, if $\dim(R_0) = 0$ then

$$\mathrm{reg}(R/J) = \mathrm{reg}(R) + \sigma(\mathfrak{a}) - (s - g + 1)\mathrm{indeg}(I/\mathfrak{a}) - s.$$

Proof. By Theorem 5.4.6, R/J is Cohen-Macaulay of dimension $d - s$. By using the local duality theorem and Proposition 5.5.5,

$$\begin{aligned}
\mathrm{reg}_{\mathfrak{m}}(R/J) &= \mathrm{end}(H_{\mathfrak{m}}^{d-s}(R/J)) + d - s = -\mathrm{indeg}(\omega_{R/J}) + d - s \\
&= \sigma(\mathfrak{a}) - \mathrm{indeg}(\mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega) + d - s \\
&= \sigma(\mathfrak{a}) - \mathrm{indeg}(\mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a})) - \mathrm{indeg}(\omega) + d - s \\
&= \mathrm{reg}_{\mathfrak{m}}(R) + \sigma(\mathfrak{a}) - \mathrm{indeg}(\mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a})) - s,
\end{aligned}$$

since $\mathrm{reg}_{\mathfrak{m}}(R) = \mathrm{end}(H_{\mathfrak{m}}^d(R)) + d = -\mathrm{indeg}(\omega) + d$.

It remains to prove that $\mathrm{indeg}(\mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a})) = (s - g + 1)\mathrm{indeg}(I/\mathfrak{a})$.

Let $g_1, \ldots, g_\ell$ be a minimal set of generators of $I/\mathfrak{a}$. We have

$$\mathrm{Sym}_R(I/\mathfrak{a}) \otimes_R R/\mathfrak{m} \simeq (R/\mathfrak{m})[Y_1, \ldots, Y_\ell],$$

where Y_i is the class of g_i in $\mathrm{Sym}_R(I/\mathfrak{a}) \otimes_R R/\mathfrak{m}$.

Suppose that $\deg(g_1) = \mathrm{indeg}(I/\mathfrak{a})$. Since

$$\mathrm{Sym}_R(I/\mathfrak{a}) \otimes_R R/\mathfrak{m} \simeq (R/\mathfrak{m})[Y_1, \ldots, Y_\ell]$$

is a polynomial ring, we see that $Y_1^{s-g+1} \neq 0$, hence $g_1^{s-g+1} \neq 0$ (this product in $\mathrm{Sym}_R(I/\mathfrak{a})$) and $g_1^{s-g+1} \in \mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a})$. Thus

$$\mathrm{indeg}(\mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a})) \leq \deg(g_1^{s-g+1}) = (s - g + 1)\deg(g_1) = (s - g + 1)\mathrm{indeg}(I/\mathfrak{a}).$$

On the other hand, $\operatorname{indeg}(\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a})) \geq (s-g+1)\operatorname{indeg}(I/\mathfrak{a})$. Thus

$$\operatorname{indeg}(\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a})) = (s-g+1)\operatorname{indeg}(I/\mathfrak{a}).$$

The remaining part follows from $\operatorname{reg}(M) \leq \operatorname{reg}_m(M) \leq \operatorname{reg}(M) + \dim(R_0)$ for any finitely generated graded R-module M, by Lemma 5.5.2. □

Finally, we close this section by giving some tight relations between the Hilbert series of a residual intersection and the $(s-g+1)$-th symmetric power of I/a.

Corollary 5.5.7. *Let $(R, \mathfrak{m})$ be a positively graded Cohen-Macaulay *local algebra of dimension d over an Artinian local ring R_0, with canonical module ω. Suppose that $\mathfrak{a} \subset I$ are two homogeneous ideals of R, with $\operatorname{ht}(I) = g$, and $J = (\mathfrak{a} \colon {}_R I)$ is an s-residual intersection of I. Write*

$$H_{R/J}(t) = \frac{P(t)}{(1-t^a)^{d-s}}, \quad H_{\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a})\otimes_R\omega}(t) = \frac{Q(t)}{(1-t^a)^{d-s}},$$

with a the least common multiple of the degrees of the generators of the algebra R over R_0 and $P(t), Q(t) \in \mathbb{Z}[t, t^{-1}]$, with $P(1), Q(1) > 0$. If I satisfies SD_2 and $\operatorname{Tor}_1^R(R/I, \omega) = 0$, then

$$P(t) = t^{\sigma(\mathfrak{a})+a(d-s)}Q(t^{-1}).$$

In particular, if R is generated over R_0 by elements of degree 1, that is, $R = R_0[R_1]$, then

$$e(R/J) = e(\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega).$$

Proof. By Proposition 5.5.5,

$$\omega_{R/J} \simeq \operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega(\sigma(\mathfrak{a})).$$

It follows from Theorem 1.3.3 (a) that

$$H_{\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a})\otimes_R\omega(\sigma(\mathfrak{a}))}(t) = (-1)^{d-s}H_{R/J}(t^{-1})$$

is equivalent to

$$H_{\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a})\otimes_R\omega}(t) = (-1)^{d-s}t^{\sigma(\mathfrak{a})}H_{R/J}(t^{-1}).$$

Thus

$$Q(t) = t^{\sigma(\mathfrak{a})+a(d-s)}P(t^{-1})$$

gives

$$P(t) = t^{\sigma(\mathfrak{a})+a(d-s)}Q(t^{-1}).$$

In particular,

$$e(R/J) = P(1) = Q(1) = e(\operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega),$$

by Proposition 1.3.7. □

5.6 Duality for residual intersections of strongly Cohen-Macaulay ideals

The duality for residual intersetcions is a center of interest in during the development of the theory of residual. The first results of duality were proven by Peskine and Szpiro for the theory of liaison in [PS74]. Afterwards, around the works of Huneke and Ulrich in [HU88], generalizing the corresponding statement in the theory of linkage of Peskine and Szpiro. In particular, the recent works of Eisenbud and Ulrich in [EU16] give some results on the duality for residual intersections.

In this section, we provide the duality for residual intersections in the case where I is a strongly Cohen-Macaulay ideal. In this case, the structure of the canonical module of some symmetric powers of $I/\mathfrak{a}$ is given. Therefore, we may establish some tight relations between the Hilbert series of the symmetric powers of $I/\mathfrak{a}$ and we give the closed formulas for the type and for the Bass number of $\mathrm{Sym}_R^k(I/\mathfrak{a})$.

First we prove on the duality of residual approximation complexes in the height two case.

Proposition 5.6.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, with canonical module ω, and $\mathfrak{a} \subset I$ two ideals of R. Suppose that I is a strongly Cohen-Macaulay ideal of height 2 and $J = (\mathfrak{a} \colon {}_R I)$ is an s-residual intersection of I. Then, for all $0 \leq k \leq s-2$,*

$$\omega_{H_0({}_k\mathcal{Z}_\bullet^+)} \simeq H_0({}_{s-k-1}^{\omega}\mathcal{Z}_\bullet^+).$$

Proof. By Theorem 5.3.10, the complex ${}_k\mathcal{Z}_\bullet^+$ is acyclic and $H_0({}_k\mathcal{Z}_\bullet^+)$ is Cohen-Macaulay of dimension $d-s$. Therefore, by local duality,

$$\omega^\vee_{H_0({}_k\mathcal{Z}_\bullet^+)} \simeq H_{\mathfrak{m}}^{d-s}(H_0({}_k\mathcal{Z}_\bullet^+)). \tag{5.11}$$

As I is strongly Cohen-Macaulay of height 2, we have that $\mathrm{depth}(Z_i) = d$, for all $0 \leq i \leq r-1$. By the definition of ${}_k\mathcal{Z}_\bullet^+$, for all $0 \leq i \leq s$, ${}_k\mathcal{Z}_i^+$ is a direct sum of copies of modules $Z_0, Z_1, \ldots, Z_{r-1}$, therefore $\mathrm{depth}({}_k\mathcal{Z}_i^+) = d$.

We now consider the double complex $C_{\mathfrak{m}}^\bullet({}_k\mathcal{Z}_\bullet^+)$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_{\mathfrak{m}}^0({}_k\mathcal{Z}_s^+) \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{\mathfrak{m}}^0({}_k\mathcal{Z}_1^+) \arrow[r] \arrow[d] & C_{\mathfrak{m}}^0({}_k\mathcal{Z}_0^+) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{\mathfrak{m}}^1({}_k\mathcal{Z}_s^+) \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{\mathfrak{m}}^1({}_k\mathcal{Z}_1^+) \arrow[r] \arrow[d] & C_{\mathfrak{m}}^1({}_k\mathcal{Z}_0^+) \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_{\mathfrak{m}}^d({}_k\mathcal{Z}_s^+) \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{\mathfrak{m}}^d({}_k\mathcal{Z}_1^+) \arrow[r] \arrow[d] & C_{\mathfrak{m}}^d({}_k\mathcal{Z}_0^+) \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 &
\end{tikzcd}$$

that gives rise to two sequences. The second terms of the horizonal spectral are

$$ {}^{2}\mathbf{E}_{\text{hor}}^{-i,-j} = \begin{cases} H_{\mathfrak{m}}^{d-s}(H_0({}_k\mathcal{Z}_\bullet^+)) & \text{if} \quad j = d-s \text{ and } i = 0 \\ 0 & \text{otherwise} \end{cases} $$

and the first terms of the vertical spectral are

$$ {}^{1}\mathbf{E}_{\text{ver}}^{-i,-j} = \begin{cases} 0 \longrightarrow H_{\mathfrak{m}}^{d}({}_k\mathcal{Z}_s^+) \longrightarrow \cdots \longrightarrow H_{\mathfrak{m}}^{d}({}_k\mathcal{Z}_1^+) \longrightarrow H_{\mathfrak{m}}^{d}({}_k\mathcal{Z}_0^+) \longrightarrow 0 & \text{if } j = d \\ 0 \qquad \text{otherwise.} \end{cases} $$

By the convergence of the spectral sequences, we obtain

$$ H_{\mathfrak{m}}^{d-s}(H_0({}_k\mathcal{Z}_\bullet^+)) \simeq {}^{\infty}\mathbf{E}_{\text{ver}}^{-s,-d} = {}^{2}\mathbf{E}_{\text{ver}}^{-s,-d}. \tag{5.12} $$

By Lemma 5.4.4 (iv), we have a commutative diagram, for all $0 \leq k \leq s-2$

$$ \begin{tikzcd} H_{\mathfrak{m}}^{d}({}_k\mathcal{Z}_s^+) \arrow[r] \arrow[d, "\simeq"'] & H_{\mathfrak{m}}^{d}({}_k\mathcal{Z}_{s-1}^+) \arrow[d, "\simeq"] \\ ({}_{s-k-1}^{\omega}\mathcal{Z}_0^+)^\vee \arrow[r] & ({}_{s-k-1}^{\omega}\mathcal{Z}_1^+)^\vee. \end{tikzcd} $$

Therefore

$$ {}^{2}\mathbf{E}_{\text{ver}}^{-s,-d} \simeq H_0({}_{s-k-1}^{\omega}\mathcal{Z}_\bullet^+)^\vee. \tag{5.13} $$

By (5.11), (5.12), (5.13) and Proposition 1.2.19,

$$ \omega_{H_0({}_k\mathcal{Z}_\bullet^+)} \simeq H_0({}_{s-k-1}^{\omega}\mathcal{Z}_\bullet^+). $$

□

We now state the main result of this section. Let recall us that if M, N, L be three R-modules, then a morphism $\varphi : M \otimes_R N \longrightarrow L$ is a perfect pairing if $\psi_1 : M \longrightarrow \operatorname{Hom}_R(N, L)$, sending m to $\psi_1(m) : n \mapsto \varphi(m \otimes n)$ and $\psi_2 : N \longrightarrow \operatorname{Hom}_R(M, L)$, sending n to $\psi_2(n) : m \mapsto \varphi(m \otimes n)$ are two isomorphisms.

Theorem 5.6.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module ω and $\mathfrak{a} \subset I$ two ideals of R, with $\operatorname{ht}(I) = g$. Suppose that $J = (\mathfrak{a} :_R I)$ is an s-residual intersection of I. If I is strongly Cohen-Macaulay and $\operatorname{Tor}_1^R(R/I, \omega) = 0$, then, for all $1 \leq k \leq s-g$,*

(i) *the canonical module of $\operatorname{Sym}_R^k(I/\mathfrak{a})$ is $\operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}) \otimes_R \omega$;*

(ii) *there is a perfect pairing*

$$ (\operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega) \otimes_R \operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}) \longrightarrow \operatorname{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega. $$

Proof. (i) First we treat the case $g = 2$. By Proposition 5.6.1, for all $1 \leq k \leq s-2$,

$$ \begin{aligned} \omega_{\operatorname{Sym}_R^k(I/\mathfrak{a})} &\simeq H_0({}_{s-k-1}^{\omega}\mathcal{Z}_\bullet^+) \\ &\simeq \operatorname{Sym}_R^{s-k-1}(I/\mathfrak{a}) \otimes_R \omega. \end{aligned} $$

The last isomorphism follows from Proposition 5.3.3.

Now, we may suppose that $g \geq 2$ by Remark 5.3.9. If $g > 2$, then we choose a regular sequence $\mathbf{a}$ of length $g-2$ inside $\mathfrak{a}$ which is a part of a minimal generating set of $\mathfrak{a}$ as in the proof of Theorem 5.4.9. As $I/\mathbf{a}$ is strongly Cohen-Macaulay by Proposition 5.4.1, it follows from the height two case that

$$\begin{aligned}\omega_{\mathrm{Sym}_R^k(I/\mathfrak{a})} \simeq \omega_{\mathrm{Sym}^k_{R/\mathbf{a}}\left(\frac{I/\mathbf{a}}{\mathfrak{a}/\mathbf{a}}\right)} &\simeq \mathrm{Sym}_{R/\mathbf{a}}^{(s-g+2)-k-1}\left(\frac{I/\mathbf{a}}{\mathfrak{a}/\mathbf{a}}\right) \otimes_{R/\mathbf{a}} (\omega/\mathbf{a}\omega) \\ &\simeq \mathrm{Sym}_R^{s-g+1-k}(I/\mathfrak{a}) \otimes_R \omega.\end{aligned}$$

(ii) It suffices to prove that, for all $1 \leq k \leq s-g$,

$$\mathrm{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega \simeq \mathrm{Hom}_R\Big(\mathrm{Sym}_R^{s-g+1-k}(I/\mathfrak{a}), \mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega\Big).$$

As $\mathrm{Sym}_R^{s-g+1-k}(I/\mathfrak{a})$ is a maximal Cohen-Macaulay R/J-module by Corollary 5.4.7(ii) and $\mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega$ is the canonical module of R/J by Theorem 5.4.9,

$$\omega_{\mathrm{Sym}_R^{s-g+1-k}(I/\mathfrak{a})} \simeq \mathrm{Hom}_R\Big(\mathrm{Sym}_R^{s-g+1-k}(I/\mathfrak{a}), \mathrm{Sym}_R^{s-g+1}(I/\mathfrak{a}) \otimes_R \omega\Big).$$

The conclusion follows from (i). □

In particular, if the residual intersections are geometric, we obtain the following results that could be compared to one of [EU16, Theorem 2.2].

Corollary 5.6.3. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring and $\mathfrak{a} \subset I$ two ideals of R. Assume that I is a strongly Cohen-Macaulay ideal of height g and $J = (\mathfrak{a} \colon {}_R I)$ is a geometric s-residual intersection of I. Then, for all $1 \leq k \leq s-g$,*

(i) *the canonical module of $I^k/\mathfrak{a}I^{k-1}$ is $I^{s-g+1-k}/\mathfrak{a}I^{s-g-k}$;*

(ii) *there is a perfect pairing*

$$I^k/\mathfrak{a}I^{k-1} \otimes_R I^{s-g+1-k}/\mathfrak{a}I^{s-g-k} \longrightarrow I^{s-g+1}/\mathfrak{a}I^{s-g}.$$

Proof. It is an immediate translation from Theorem 5.6.2 with the fact that $\mathrm{Sym}_R^k(I/\mathfrak{a}) \simeq I^k/\mathfrak{a}I^{k-1}$ by [HN16, Corollary 2.11] and $\omega_R \simeq R$. □

Notice that the pairings in this corollary and in the main theorem above need not be given by multiplication. However, Eisenbud and Ulrich proved that, in many situations where our results apply, the multiplication indeed produces a perfect pairing. In this regards, an example they provide is interesting.

Example 5.6.4. [EU16, Example 2.8] Let $R = k[[x, y, z]]$, where k is an infinite field and $I = (x, y)^2$. If $\mathfrak{a}$ is generated by 3 sufficiently general elements of degree 3 in I, then $J = \mathfrak{a} \colon {}_R I$ is a 3-residual intersection. Using `Macaulay2` [GS], they verified that I is strongly Cohen-Macaulay, hence $\omega_{R/J} \simeq \mathrm{Sym}_R^2(I/\mathfrak{a})$. Moreover $\omega_{I/\mathfrak{a}} \simeq I/\mathfrak{a}$.

Computation shows that there is a unique (up to scalars) nonzero homogeneous map $I/\mathfrak{a} \otimes_R I/\mathfrak{a} \longrightarrow \omega_{R/J}$ of lowest degree, and this is a perfect pairing. But they notice that there can be no perfect pairing $I/\mathfrak{a} \otimes_R I/\mathfrak{a} \longrightarrow I^2/\mathfrak{a}I$ because the target is annihilated by $(x, y, z)^2$ while $I/\mathfrak{a}$ is not. This implies that $\omega_{R/J} \neq I^2/\mathfrak{a}I$ and J is not geometric.

However, the multiplication with value in the symmetric square $I/\mathfrak{a} \otimes_R I/\mathfrak{a} \longrightarrow \mathrm{Sym}_R^2(I/\mathfrak{a})$ is a perfect pairing.

Next, we will show that the perfect paring in Theorem 5.6.2 and also in Corollary 5.6.3 could be chosen by multiplication. Fisrt, we need the following lemmas.

Lemma 5.6.5. *Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring and S a Noetherian standard graded R-algebra. For any $s \geq t$, we consider*

$$\psi : S_t \longrightarrow \mathrm{Hom}_R(S_{s-t}, S_s)$$

the natural map given by the algebra structure of S. If $H^0_{S_+}(S \otimes_R k)_t = 0$, then $\psi \otimes k$ is into.

Proof. Let $L \in S_t$ be such that $0 \neq \overline{L} \in S_t \otimes_R k = (S \otimes_R k)_t$. The element L is sent to the class of the homomorphism $\times L$. We have to prove that this class is not zero. As

$$\mathfrak{m}\mathrm{Hom}_R(S_{s-t}, S_s) \subseteq \mathrm{Hom}_R(S_{s-t}, \mathfrak{m}S_s),$$

it suffices to show that the image of $\times L$ is not contained in $\mathfrak{m}S_s$. The assertion is obvious if $s = t$. If $s > t$, as $\overline{L} \notin H^0_{S_+}(S \otimes_R k)_t$ and $S_{s-t} = (S_+)^{s-t}$, there exist $u \in S_{s-t}$ such that $\overline{L}.\overline{u} \neq 0$. Hence the image of $\times L$ contains $L.u \notin \mathfrak{m}S_s$. □

Lemma 5.6.6. *Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring and M a finitely generated R-module. For any $s \geq t$, if there exists a R-module isomorphism*

$$\varphi : \mathrm{Hom}_R(\mathrm{Sym}_R^{s-t}(M), \mathrm{Sym}_R^s(M)) \longrightarrow \mathrm{Sym}_R^t(M),$$

then the natural map given by the algebra structure of $\mathrm{Sym}_R(M)$

$$\psi : \mathrm{Sym}_R^t(M) \longrightarrow \mathrm{Hom}_R(\mathrm{Sym}_R^{s-t}(M), \mathrm{Sym}_R^s(M))$$

is an isomorphism.

Proof. The assertion of the lemma is equivalent to show that $\varphi \circ \psi$ is onto, which in turn is equivalent to $\psi \otimes_R k$ being into (or equivalently onto).

Choose $\tau_1 : R^n \longrightarrow M$ onto with n minimal (equivalently such that $R^n \otimes_R k \simeq M \otimes_R k$ via τ_1). Then $\tau := \mathrm{Sym}_R(\tau_1) : \mathrm{Sym}_R(R^n) \longrightarrow \mathrm{Sym}_R(M)$ is onto and $\tau \otimes_R k$ is an isomorphism identifying $\mathrm{Sym}_R(M) \otimes_R k$ with a polynomial ring in n variables. It follows that $S = \mathrm{Sym}_R(M)$ satisfies the condition of Lemma 5.6.5, hence $\psi \otimes k$ is into. □

Note that $\mathrm{Sym}^0_{R/J}(I/\mathfrak{a}) = R/J$ and $\mathrm{Sym}^k_{R/J}(I/\mathfrak{a}) = \mathrm{Sym}^k_R(I/\mathfrak{a})$ for $k > 0$. We have the following results.

Theorem 5.6.7. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring and $\mathfrak{a} \subset I$ two ideals of R, with $\mathrm{ht}(I) = g$. Suppose that $J = (\mathfrak{a} :_R I)$ is an s-residual intersection of I. If I is strongly Cohen-Macaulay, then $\omega_{R/J} \simeq \mathrm{Sym}^{s-g+1}_{R/J}(I/\mathfrak{a})$ and for all $0 \leq k \leq s-g+1$*

(i) *the R/J-module $\mathrm{Sym}^k_{R/J}(I/\mathfrak{a})$ is faithful and Cohen-Macaulay,*

(ii) *the multiplication*

$$\mathrm{Sym}^k_{R/J}(I/\mathfrak{a}) \otimes_{R/J} \mathrm{Sym}^{s-g+1-k}_{R/J}(I/\mathfrak{a}) \longrightarrow \mathrm{Sym}^{s-g+1}_{R/J}(I/\mathfrak{a})$$

is a perfect pairing,

(iii) *setting* $A := \mathrm{Sym}_{R/J}(I/\mathfrak{a})$, *the graded* R/J*-algebra*

$$\overline{A} := A/A_{>s-g+1} = \bigoplus_{i=0}^{s-g+1} \mathrm{Sym}^i_{R/J}(I/\mathfrak{a})$$

is Gorenstein.

Proof. The first item is Corollary 5.4.7 (ii). The second and last items directly follow from Lemma 5.6.6 together with Theorem 5.6.2 (ii) and (i), respectively. □

Corollary 5.6.8. *Let* $(R, \mathfrak{m})$ *be a positively graded Cohen-Macaulay* **local algebra of dimension* d *over an Artinian local ring* R_0*, with canonical module* ω*. Suppose that* $\mathfrak{a} \subset I$ *are two homogeneous ideals of* R*, with* $\mathrm{ht}(I) = g$*, and* $J = (\mathfrak{a} \colon {}_R I)$ *is an* s*-residual intersection of* I*. Write*

$$H_{\mathrm{Sym}^k_R(I/\mathfrak{a})}(t) = \frac{P_k(t)}{(1-t^a)^{d-s}}, \quad H_{\mathrm{Sym}^k_R(I/\mathfrak{a})\otimes_R\omega}(t) = \frac{Q_k(t)}{(1-t^a)^{d-s}},$$

with a *the least common multiple of the degrees of the generators of the algebra* R *over* R_0 *and* $P_k(t), Q_k(t) \in \mathbb{Z}[t, t^{-1}]$*, with* $P_k(1), Q_k(1) > 0$*, for each* $1 \le k \le s-g$*. If* I *is strongly Cohen-Macaulay and* $\mathrm{Tor}_1^R(R/I, \omega) = 0$*, then*

$$P_k(t) = t^{\sigma(\mathfrak{a})+a(d-s)} Q_{s-g+1-k}(t^{-1}).$$

In particular, if R *is generated over* R_0 *by elements of degree 1, then*

$$e(\mathrm{Sym}^k_R(I/\mathfrak{a})) = e(\mathrm{Sym}^{s-g+1-k}_R(I/\mathfrak{a}) \otimes_R \omega).$$

Proof. The proof is analogous to one of Corollary 5.5.7. It follows from the fact that

$$H_{\mathrm{Sym}^k_R(I/\mathfrak{a})\otimes_R\omega}(t) = (-1)^{d-s} t^{\sigma(\mathfrak{a})} H_{\mathrm{Sym}^{s-g+1-k}_R(I/\mathfrak{a})}(t^{-1}).$$

□

The next corollary enables us to calculate the type of some symmetric powers of $I/\mathfrak{a}$. This is comparable with the results of Hassanzadeh and Naélitonr in [HN16, Theorem 2.12].

Corollary 5.6.9. *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring with canonical module* ω *and* $\mathfrak{a} \subset I$ *two ideals of* R*, with* $\mathrm{ht}(I) = g$*. Suppose that* $J = (\mathfrak{a} \colon {}_R I)$ *is an* s*-residual intersection of* I*. If* I *is strongly Cohen-Macaulay and* $\mathrm{Tor}_1^R(R/I, \omega) = 0$*, then, for each* $1 \le k \le s-g$*,*

$$r(\mathrm{Sym}^k_R(I/\mathfrak{a})) = \binom{\mu(I/\mathfrak{a}) + s - g - k}{\mu(I/\mathfrak{a}) - 1} r(R).$$

Proof. The proof is totally similar to one of Corollary 5.4.12. For all $1 \le k \le s-g$,

$$r(\mathrm{Sym}^k_R(I/\mathfrak{a})) = \mu(\mathrm{Sym}^{s-g+1-k}_R(I/\mathfrak{a}) \otimes_R \omega),$$

by Theorem 5.6.2 (i) and Proposition 1.2.38. □

The next corollary enables us to calculate the Bass numbers of some symmetric powers of $I/\mathfrak{a}$.

Corollary 5.6.10. *Under the assumptions of Corollary 5.6.9. If $\mathfrak{p}$ is a prime ideal containing J of R, with $\operatorname{ht}(\mathfrak{p}) = i$, then, for every $1 \leq k \leq s-g$,*

$$\mu_{i-s}(\mathfrak{p}, \operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega) = \binom{\mu((I/\mathfrak{a})_\mathfrak{p}) + s - g - k}{\mu((I/\mathfrak{a})_\mathfrak{p}) - 1}.$$

Proof. By Theorem 5.4.6, R/J is Cohen-Macaulay of codimension s and by Corollary 5.4.7(ii), $\operatorname{Sym}_R^k(I/\mathfrak{a})$ is a maximal Cohen-Macaulay faithful R/J-module, for all $1 \leq k \leq s-g+1$. Furthermore, by Theorem 5.6.2(i), $\operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega$ is a maximal Cohen-Macaulay faithful R/J-module, for all $1 \leq k \leq s-g$.

Suppose that $J \subset \mathfrak{p}_s \subsetneq \mathfrak{p}_{s+1} \subsetneq \cdots \subsetneq \mathfrak{p}_i = \mathfrak{p}$ is a maximal chain of primes of $\operatorname{Spec}(R/J)$ contained in $\mathfrak{p}$. Let $b_j \in \mathfrak{p}_j - \mathfrak{p}_{j-1}$, for all $s+1 \leq j \leq i$. Then $\mathfrak{b} = (b_{s+1}, \ldots, b_i)$ is a regular sequence over R/J and therefore also over $\operatorname{Sym}_R^k(I/\mathfrak{a})$ and $\operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega$, for all $1 \leq k \leq s-g$.

By Proposition 1.1.2, $(b_{s+1}, \ldots, b_i)$ is a regular sequence over $(\operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega)_\mathfrak{p}$ and annihilates $k(\mathfrak{p})$ for all $1 \leq k \leq s-g$, hence Proposition 1.1.3 gives

$$\begin{aligned}\operatorname{Ext}_{R_\mathfrak{p}}^{i-s}(k(\mathfrak{p}), (\operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega)_\mathfrak{p}) &\simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(k(\mathfrak{p}), (\operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega)_\mathfrak{p} \otimes_{R_\mathfrak{p}} R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\Big) \\ &\simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(k(\mathfrak{p}), \operatorname{Hom}_R(\operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}), \omega_{R/J}) \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\Big).\end{aligned}$$

The last isomorphism follows from Theorem 5.4.9 and Theorem 5.6.2 (ii). By Proposition 1.2.35

$$\begin{aligned}&\operatorname{Hom}_R\Big(\operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}), \omega_{R/J}\Big) \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p} \\ &\qquad \simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(\operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}) \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}, \omega_{R/J} \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\Big).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\operatorname{Ext}_{R_\mathfrak{p}}^{i-s}(k(\mathfrak{p}),(\operatorname{Sym}_R^k(I/\mathfrak{a}) \otimes_R \omega)_\mathfrak{p}) \\ &\qquad \simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(k(\mathfrak{p}), \operatorname{Hom}_{R_\mathfrak{p}}(\operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}) \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}, \omega_{R/J} \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p})\Big) \\ &\qquad \simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} \operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}) \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}, \omega_{R/J} \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\Big) \\ &\qquad \simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(k(\mathfrak{p}) \otimes_R \operatorname{Sym}_R^{s-g+1-k}(I/\mathfrak{a}), \omega_{R/J} \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\Big) \\ &\qquad \simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(\operatorname{Sym}_{k(\mathfrak{p})}^{s-g+1-k}(k(\mathfrak{p}) \otimes_R I/\mathfrak{a}), \omega_{R/J} \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\Big).\end{aligned}$$

Since $k(\mathfrak{p}) \otimes_R I/\mathfrak{a} \simeq k(\mathfrak{p}) \otimes_{R_\mathfrak{p}} (I/\mathfrak{a})_\mathfrak{p}$ is a $k(\mathfrak{p})$-vector space of dimension $\mu_\mathfrak{p} := \mu((I/\mathfrak{a})_\mathfrak{p})$,

$$\operatorname{Sym}_{k(\mathfrak{p})}(k(\mathfrak{p}) \otimes_R I/\mathfrak{a}) \simeq k(\mathfrak{p})[Y_1, \ldots, Y_{\mu_\mathfrak{p}}].$$

It follows that

$$\begin{aligned}\operatorname{Ext}^{i-s}_{R_\mathfrak{p}}(k(\mathfrak{p}), (\operatorname{Sym}^k_R(I/\mathfrak{a}) \otimes_R \omega)_\mathfrak{p}) &\simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(k(\mathfrak{p})^{\binom{\mu_\mathfrak{p}+s-g-k}{\mu_\mathfrak{p}-1}}, \omega_{R/J} \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\Big)\\ &\simeq \operatorname{Hom}_{R_\mathfrak{p}}\Big(k(\mathfrak{p}), \omega_{R/J} \otimes_R R_\mathfrak{p}/\mathfrak{b}R_\mathfrak{p}\Big)^{\binom{\mu_\mathfrak{p}+s-g-k}{\mu_\mathfrak{p}-1}}\\ &\simeq \operatorname{Ext}^{i-s}_{R_\mathfrak{p}}\Big(k(\mathfrak{p}), (\omega_{R/J})_\mathfrak{p}\Big)^{\binom{\mu_\mathfrak{p}+s-g-k}{\mu_\mathfrak{p}-1}}.\end{aligned}$$

The last isomorphism follows from the fact that $\mathfrak{b}R_\mathfrak{p}$ is regular over $(\omega_{R/J})_\mathfrak{p}$ and annihilates $k(\mathfrak{p})$. Therefore

$$\begin{aligned}\mu_{i-s}(\mathfrak{p}, \operatorname{Sym}^k_R(I/\mathfrak{a}) \otimes_R \omega) &= \binom{\mu_\mathfrak{p}+s-g-k}{\mu_\mathfrak{p}-1}\mu_{i-s}(\mathfrak{p}, \omega_{R/J})\\ &= \binom{\mu_\mathfrak{p}+s-g-k}{\mu_\mathfrak{p}-1},\end{aligned}$$

since $\operatorname{ht}(\mathfrak{p}) = i-s$ in R/J, $\mu_{i-s}(\mathfrak{p}, \omega_{R/J}) = 1$, by Proposition 1.2.37 (b). $\square$

Bibliography

[AN72] Michael Artin and Masayoshi Nagata. Residual intersections in Cohen-Macaulay rings. *J. Math. Kyoto Univ.*, 12:307–323, 1972.

[Aoy80] Yôichi Aoyama. On the depth and the projective dimension of the canonical module. *Japan. J. Math. (N.S.)*, 6(1):61–66, 1980.

[Aoy83] Yôichi Aoyama. Some basic results on canonical modules. *J. Math. Kyoto Univ.*, 23(1):85–94, 1983.

[BBC14] Nicolás Botbol, Laurent Busé, and Marc Chardin. Fitting ideals and multiple points of surface parameterizations. *J. Algebra*, 420:486–508, 2014.

[BBC+17] Nicolás Botbol, Laurent Busé, Marc Chardin, Seyed Hamid Hassanzadeh, Aron Simis, and Quang Hoa Tran. Effective criteria for bigraded birational maps. *J. Symbolic Comput.*, 81:69–87, 2017.

[BC05] Laurent Busé and Marc Chardin. Implicitizing rational hypersurfaces using approximation complexes. *J. Symbolic Comput.*, 40(4-5):1150–1168, 2005.

[BCD03] Laurent Busé, David Cox, and Carlos D'Andrea. Implicitization of surfaces in $\mathbb{P}^3$ in the presence of base points. *J. Algebra Appl.*, 2(2):189–214, 2003.

[BCH14] Cristiano Bocci, Susan M. Cooper, and Brian Harbourne. Containment results for ideals of various configurations of points in $\mathbf{P}^N$. *J. Pure Appl. Algebra*, 218(1):65–75, 2014.

[BCJ09] Laurent Busé, Marc Chardin, and Jean-Pierre Jouanolou. Torsion of the symmetric algebra and implicitization. *Proc. Amer. Math. Soc.*, 137(6):1855–1865, 2009.

[BD12] Laurent Busé and Carlos D'Andrea. Singular factors of rational plane curves. *J. Algebra*, 357:322–346, 2012.

[BH98] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Revised version, Cambridge University Press, Cambridge, 1998.

[BJ03] Laurent Busé and Jean-Pierre Jouanolou. On the closed image of a rational map and the implicitization problem. *J. Algebra*, 265(1):312–357, 2003.

[Bot11] Nicolás Botbol. The implicit equation of a multigraded hypersurface. *J. Algebra*, 348:381–401, 2011.

[Bou70] Nicolas Bourbaki. *Éléments de mathématique. Algèbre. Chapitres 1 à 3.* Hermann, Paris, 1970.

[BS13] Markus P. Brodmann and Rodney Y. Sharp. *Local cohomology*, volume 136 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, second edition, 2013. An algebraic introduction with geometric applications.

[Bus01] Laurent Busé. *Study of the resultant on an algebraic variety.* Theses, Université Nice Sophia Antipolis, December 2001.

[CCT17] Marc Chardin, Steven Dale Cutkosky, and Quang Hoa Tran. Fibers of rational maps and Jacobian matrices. *ArXiv e-prints*, October 2017.

[CEHH16] Susan M. Cooper, Robert J. D. Embree, Huy Tài Hà, and Andrew H. Hoefel. Symbolic powers of monomial ideals. *Proc. Edinb. Math. Soc. (2)*, 60(1):39–55, 2016.

[CEL01] Steven Dale Cutkosky, Lawrence Ein, and Robert Lazarsfeld. Positivity and complexity of ideal sheaves. *Math. Ann.*, 321(2):213–234, 2001.

[CEU01] Marc Chardin, David Eisenbud, and Bernd Ulrich. Hilbert functions, residual intersections, and residually S_2 ideals. *Compositio Math.*, 125(2):193–219, 2001.

[CGZ00] David Cox, Ronald Goldman, and Ming Zhang. On the validity of implicitization by moving quadrics of rational surfaces with no base points. *J. Symbolic Comput.*, 29(3):419–440, 2000.

[Cha04] Marc Chardin. Regularity of ideals and their powers. *Prépublication, Insitut de Mathématiques de Jussieu*, 364:1–30, 2004.

[Cha06] Marc Chardin. Implicitization using approximation complexes. In *Algebraic geometry and geometric modeling*, Math. Vis., pages 23–35. Springer, Berlin, 2006.

[Cha13a] Marc Chardin. Powers of ideals and the cohomology of stalks and fibers of morphisms. *Algebra Number Theory*, 7(1):1–18, 2013.

[Cha13b] Marc Chardin. Powers of ideals: Betti numbers, cohomology and regularity. In *Commutative algebra*, pages 317–333. Springer, New York, 2013.

[Chu81] Gregory V. Chudnovsky. Singular points on complex hypersurfaces and multidimensional Schwarz lemma. In *Seminar on Number Theory, Paris 1979–80*, volume 12 of *Progr. Math.*, pages 29–69. Birkhäuser, Boston, Mass., 1981.

[CNT17] Marc Chardin, José Naéliton, and Quang Hoa Tran. Cohen-Macaulayness and canonical module of residual intersections. *ArXiv e-prints*, January 2017.

[Cox01] David A. Cox. Equations of parametric curves and surfaces via syzygies. In *Symbolic computation: solving equations in algebra, geometry, and engineering (South Hadley, MA, 2000)*, volume 286 of *Contemp. Math.*, pages 1–20. Amer. Math. Soc., Providence, RI, 2001.

[Cox03] David Cox. Curves, surfaces, and syzygies. In *Topics in algebraic geometry and geometric modeling*, volume 334 of *Contemp. Math.*, pages 131–150. Amer. Math. Soc., Providence, RI, 2003.

[Cox05] David A. Cox. What is the role of algebra in applied mathematics? *Notices Amer. Math. Soc.*, 52(10):1193–1198, 2005.

[CSC98] David A. Cox, Thomas W. Sederberg, and Falai Chen. The moving line ideal basis of planar rational curves. *Comput. Aided Geom. Design*, 15(8):803–827, 1998.

[Cum07] Christine Cumming. Residual intersections in Cohen-Macaulay rings. *J. Algebra*, 308(1):91–106, 2007.

[DHS12] André V. Doria, Seyed Hamid Hassanzadeh, and Aron Simis. A characteristic-free criterion of birationality. *Adv. Math.*, 230(1):390–413, 2012.

[DS16] Eliana Duarte and Hal Schenck. Tensor product surfaces and linear syzygies. *Proc. Amer. Math. Soc.*, 144(1):65–72, 2016.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[Eis05] David Eisenbud. *The geometry of syzygies*, volume 229 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry.

[EU08] David Eisenbud and Bernd Ulrich. Row ideals and fibers of morphisms. *Michigan Math. J.*, 57:261–268, 2008. Special volume in honor of Melvin Hochster.

[EU16] David Eisenbud and Bernd Ulrich. Duality and Socle Generators for Residual Intersections. *ArXiv e-prints*, July 2016.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[GS] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Has12] Seyed Hamid Hassanzadeh. Cohen-Macaulay residual intersections and their Castelnuovo-Mumford regularity. *Trans. Amer. Math. Soc.*, 364(12):6371–6394, 2012.

[Her74] Jürgen Herzog. Komplexe auflösungen und dualität in der lokalen algebra. *Universität Regensburg, Habilitationsschrift*, 1974.

[HH94] Melvin Hochster and Craig Huneke. Indecomposable canonical modules and connectedness. In *Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992)*, volume 159 of *Contemp. Math.*, pages 197–208. Amer. Math. Soc., Providence, RI, 1994.

[HH13] Brian Harbourne and Craig Huneke. Are symbolic powers highly evolved? *J. Ramanujan Math. Soc.*, 28A:247–266, 2013.

[HK71] Jürgen Herzog and Ernst Kunz, editors. *Der kanonische Modul eines Cohen-Macaulay-Rings.* Lecture Notes in Mathematics, Vol. 238. Springer-Verlag, Berlin-New York, 1971. Seminar über die lokale Kohomologietheorie von Grothendieck, Universität Regensburg, Wintersemester 1970/1971.

[HKS92] Klaus Hulek, Sheldon Katz, and Frank-Olaf Schreyer. Cremona transformations and syzygies. *Math. Z.*, 209(3):419–443, 1992.

[HN16] Seyed Hamid Hassanzadeh and Jose Naéliton. Residual intersections and the annihilator of Koszul homologies. *Algebra Number Theory*, 10(4):737–770, 2016.

[Hof89] Christoph M. Hoffmann. *Geometric and Solid Modeling: An Introduction.* Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 1989.

[HS12] Seyed Hamid Hassanzadeh and Aron Simis. Plane Cremona maps: saturation and regularity of the base ideal. *J. Algebra*, 371:620–652, 2012.

[HSV82] Jürgen Herzog, Aron Simis, and Wolmer V. Vasconcelos. Approximation complexes of blowing-up rings. *J. Algebra*, 74(2):466–493, 1982.

[HSV83a] Jürgen Herzog, Aron Simis, and Wolmer V. Vasconcelos. Approximation complexes of blowing-up rings. II. *J. Algebra*, 82(1):53–83, 1983.

[HSV83b] Jürgen Herzog, Aron Simis, and Wolmer V. Vasconcelos. Koszul homology and blowing-up rings. In *Commutative algebra (Trento, 1981)*, volume 84 of *Lecture Notes in Pure and Appl. Math.*, pages 79–169. Dekker, New York, 1983.

[HU88] Craig Huneke and Bernd Ulrich. Residual intersections. *J. Reine Angew. Math.*, 390:1–20, 1988.

[Hun82] Craig Huneke. Linkage and the Koszul homology of ideals. *Amer. J. Math.*, 104(5):1043–1062, 1982.

[Hun83] Craig Huneke. Strongly Cohen-Macaulay schemes and residual intersections. *Trans. Amer. Math. Soc.*, 277(2):739–763, 1983.

[HVV85] Jürgen Herzog, Wolmer V. Vasconcelos, and Rafael H. Villarreal. Ideals with sliding depth. *Nagoya Math. J.*, 99:159–172, 1985.

[HW14] Jerome W. Hoffman and Haohao Wang. A note on quadratically parametrized surfaces. *Algebra Colloq.*, 21(3):461–476, 2014.

[JG09] Xiaohong Jia and Ron Goldman. μ-bases and singularities of rational planar curves. *Comput. Aided Geom. Design*, 26(9):970–988, 2009.

[JG12] Xiaohong Jia and Ron Goldman. Using Smith normal forms and μ-bases to compute all the singularities of rational planar curves. *Comput. Aided Geom. Design*, 29(6):296–314, 2012.

[Jou96] Jean-Pierre Jouanolou. Anisotropic resultant. complements and applications. (résultant anisotrope. compléments et applications.). *The Electronic Journal of Combinatorics [electronic only]*, 3(2), 1996.

[KPU15] Andrew R. Kustin, Claudia Polini, and Bernd Ulrich. The equations defining blowup algebras of height three Gorenstein ideals. *ArXiv e-prints*, May 2015.

[KPU16] Andrew R. Kustin, Claudia Polini, and Bernd Ulrich. Blowups and fibers of morphisms. *Nagoya Math. J.*, 224(1):168–201, 2016.

[LC16] Yisheng Lai and Falai Chen. Implicitizing rational surfaces using moving quadrics constructed from moving planes. *J. Symbolic Comput.*, 77:127–161, 2016.

[Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[MN01] J. C. Migliore and U. Nagel. Liaison and related topics: notes from the Torino workshop-school. *Rend. Sem. Mat. Univ. Politec. Torino*, 59(2):59–126 (2003), 2001. Liaison and related topics (Turin, 2001).

[MRS14] Claudia Miller, Hamidreza Rahmati, and Janet Striuli. Duality for Koszul homology over Gorenstein rings. *Math. Z.*, 276(1-2):329–343, 2014.

[Niu13] Wenbo Niu. Some results on asymptotic regularity of ideal sheaves. *J. Algebra*, 377:157–172, 2013.

[PS74] Christian Peskine and Lucien Szpiro. Liaison des variétés algébriques. I. *Invent. Math.*, 26:271–302, 1974.

[RS01] Francesco Russo and Aron Simis. On birational maps and Jacobian matrices. *Compositio Math.*, 126(3):335–358, 2001.

[SC95] Thomas W. Sederberg and Falai Chen. Implicitization using moving curves and surfaces. In *Proceedings of the 22Nd Annual Conference on Computer Graphics and Interactive Techniques*, SIGGRAPH '95, pages 301–308, New York, NY, USA, 1995. ACM.

[SCG07] Ning Song, Falai Chen, and Ron Goldman. Axial moving lines and singularities of rational planar curves. *Comput. Aided Geom. Design*, 24(4):200–209, 2007.

[Sim04] Aron Simis. Cremona transformations and some related algebras. *J. Algebra*, 280(1):162–179, 2004.

[SS16] Éric Schost and Pierre-Jean Spaenlehauer. A quadratically convergent algorithm for structured low-rank approximation. *Found. Comput. Math.*, 16(2):457–492, 2016.

[SSV14] Hal Schenck, Alexandra Seceleanu, and Javid Validashti. Syzygies and singularities of tensor product surfaces of bidegree $(2, 1)$. *Math. Comp.*, 83(287):1337–1372, 2014.

[Sta80] Richard P. Stanley. Weyl groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Algebraic Discrete Methods*, 1(2):168–184, 1980.

[Str90] Jan R. Strooker. *Homological questions in local algebra*, volume 145 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1990.

[SUV03] Aron Simis, Bernd Ulrich, and Wolmer V. Vasconcelos. Rees algebras of modules. *Proc. London Math. Soc. (3)*, 87(3):610–646, 2003.

[SV81] Aron Simis and Wolmer V. Vasconcelos. The syzygies of the conormal module. *Amer. J. Math.*, 103(2):203–224, 1981.

[SZ15] Thomas W. Sederberg and Jianmin Zheng. Birational quadrilateral maps. *Comput. Aided Geom. Design*, 32:1–4, 2015.

[Tra15] Quang Hoa Tran. Bound for the number of one-dimensional fibers of a projective morphism. *Accepted for publication in Journal of Algebra. ArXiv e-prints*, November 2015.

[Ulr92] Bernd Ulrich. Remarks on residual intersections. In *Free resolutions in commutative algebra and algebraic geometry (Sundance, UT, 1990)*, volume 2 of *Res. Notes Math.*, pages 133–138. Jones and Bartlett, Boston, MA, 1992.

[Ulr94] Bernd Ulrich. Artin-Nagata properties and reductions of ideals. In *Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992)*, volume 159 of *Contemp. Math.*, pages 373–400. Amer. Math. Soc., Providence, RI, 1994.

[UV93] Bernd Ulrich and Wolmer V. Vasconcelos. The equations of Rees algebras of ideals with linear presentation. *Math. Z.*, 214(1):79–92, 1993.

[Vas94] Wolmer V. Vasconcelos. *Arithmetic of blowup algebras*, volume 195 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[Vas05] Wolmer V. Vasconcelos. *Integral closure*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005. Rees algebras, multiplicities, algorithms.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[WJG09] Haohao Wang, Xiaohong Jia, and Ron Goldman. Axial moving planes and singularities of rational space curves. *Comput. Aided Geom. Design*, 26(3):300–316, 2009.

[ZS60] Oscar Zariski and Pierre Samuel. *Commutative algebra. Vol. II*. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-London-New York, 1960.