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Algèbres de Hall cohomologiques et variétés de Nakajima
associées à des courbes

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Titre : Algèbres de Hall cohomologiques et variétés de Nakajima associées à des courbes

Mots Clefs : Algèbres de Hall, espaces de modules, théorie de représentations, fibrés de Higgs, algèbres de battage

Résumé : Pour toute courbe projective lisse C et théorie homologique orientée de Borel-Moore libre A , on construit un produit associatif de type Hall sur les A -groupes du champ de modules des faisceaux de Higgs de torsion sur C . On montre que l'algèbre $A\mathbf{Ha}_C^0$ qu'on obtient admet une présentation de battage naturelle, qui est fidèle dans le cas où A est l'homologie de Borel-Moore usuelle. On introduit de plus les espaces de modules des triplets stables $\mathcal{M}(d, n)$, fortement inspirés par les variétés de carquois de Nakajima. Ces espaces de modules sont des variétés lisses symplectiques, et admettent une autre caractérisation comme les espaces de modules de faisceaux sans torsion stables encadrés sur $\mathbb{P}(T^*C)$. De plus, on munit leurs A -groupes avec une action de $A\mathbf{Ha}_C^0$, qui généralise les opérateurs de modification ponctuelle de Nakajima sur l'homologie des schémas de Hilbert de T^*C .

Title : Cohomological Hall algebras and Nakajima varieties associated to curves

Key words : Hall algebras, moduli spaces, representation theory, Higgs bundles, shuffle algebras

Abstract : For a smooth projective curve C and a free oriented Borel-Moore homology theory A , we construct a Hall-like associative product on the A -theory of the moduli stack of Higgs torsion sheaves on C . We show that the resulting algebra $A\mathbf{Ha}_C^0$ admits a natural shuffle presentation, and prove it is faithful when A is replaced with usual Borel-Moore homology groups. We also introduce moduli spaces of stable triples $\mathcal{M}(d, n)$, heavily inspired by Nakajima quiver varieties. These moduli spaces are shown to be smooth symplectic varieties, which admit another characterization as moduli of framed stable torsion-free sheaves on $\mathbb{P}(T^*C)$. Moreover, we equip their A -theory with an $A\mathbf{Ha}_C^0$ -action, which generalizes Nakajima's raising operators on the homology of Hilbert schemes of points on T^*C .

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À mon grand-père

Table des matières

1	Introduction	9
1.1	Motivation	9
1.2	Les algèbres de Hall	11
1.3	Une version cohomologique	14
1.4	Espaces de modules et représentations	15
1.5	Carquois à une boucle	17
1.6	Théories homologiques orientées	19
1.7	Faisceaux de Higgs	19
1.8	Les résultats	20
1.9	Développements futurs	23
2	CoHAs for Higgs torsion sheaves	25
2.1	Introduction	25
2.2	Oriented Borel-Moore homology theories	28
2.3	Coherent sheaves and <i>Quot</i> -schemes	35
2.4	The product	39
2.5	Global shuffle algebra	44
2.6	Moduli of stable Higgs triples	52
2.7	Action on stable triples	59
2.8	Injectivity of shuffle presentation	62
2.9	Torsion-free sheaves on $\mathbb{P}(T^*C)$	65
	Références	70

Chapitre 1

Introduction

Cette thèse est consacrée à l'étude d'une classe d'algèbres, appelées *algèbres de Hall cohomologiques*. Nous associons une telle algèbre à toute courbe lisse projective, étudions leurs propriétés et construisons une action sur les groupes d'homologies des certains espaces de modules des faisceaux cohérents sur des surfaces.

1.1 Motivation

Soit S une surface quasi-projective lisse. On s'intéresse à l'ensemble des d -uplets des points de S , i.e. aux puissances symétriques $\mathrm{Sym}^d S$ pour $d \geq 2$. On peut regarder ces puissances comme les espaces de configuration de d points dans S . Or, il se trouve que $\mathrm{Sym}^d S$ est singulier dès que $d > 1$ (contrairement au cas des courbes lisses). Une manière de remédier à ceci est de considérer à la place l'ensemble des sous-variétés fermées $Z \subset S$ de dimension 0 et de longueur fixée d . De façon équivalente, on peut considérer l'ensemble $\mathrm{Hilb}_d S$ des faisceaux d'idéaux $\mathcal{I} \subset \mathcal{O}_S$ dont le quotient $\mathcal{O}_S/\mathcal{I}$ est de longueur d .

Théorème 1.1.1 (Grothendieck [31], Fogarty [20]). *Il existe une variété algébrique lisse $\mathrm{Hilb}_d S$ dont les \mathbb{C} -points sont en bijection avec $\mathrm{Hilb}_d S$. De plus, le morphisme*

$$\pi : \mathrm{Hilb}_d S \rightarrow \mathrm{Sym}^d S, \quad \mathcal{I} \mapsto \mathrm{supp}(\mathcal{O}_S/\mathcal{I})$$

est une résolution de singularités.

La définition précise de $\mathrm{Hilb}_d S$ se formule en termes de foncteurs explicites, cf [54]. On appelle $\mathrm{Hilb}_d S$ le *schéma de Hilbert*, et π le *morphisme de Hilbert-Chow*.

Exemple 1.1.1. Soit $d = 2$. Alors $\mathrm{Sym}^2 S \simeq S^2/\mathfrak{S}_2$ est lisse en dehors de la diagonale, et le schéma de Hilbert $\mathrm{Hilb}_2 S$ est isomorphe à l'éclatement de $\mathrm{Sym}^2 S$ le long de la diagonale.

Exemple 1.1.2. Soit $S = \mathbb{C}^2$. Dans ce cas, le morphisme de Hilbert-Chow $\mathrm{Hilb}_d \mathbb{C}^2 \rightarrow \mathrm{Sym}^d \mathbb{C}^2$ possède beaucoup de propriétés similaires à la résolution de Springer $\pi : T^*(G/B) \rightarrow \mathcal{N}_{\mathfrak{g}}$, et peut être considéré comme un analogue. En particulier, on a $\mathrm{Hilb}_2 \mathbb{C}^2 \simeq T^*\mathbb{P}^1 \times \mathbb{C}^2$, $\mathrm{Sym}^2 \mathbb{C}^2 \simeq \mathcal{N}_{\mathfrak{sl}_2} \times \mathbb{C}^2$, et le morphisme de Hilbert-Chow est égal à π au facteur \mathbb{C}^2 près.

Une question naturelle qu'on peut se poser est le calcul des groupes de cohomologie (singulière, à coefficients rationnels) $H^*(\mathrm{Hilb}_d S)$. Pour $S = \mathbb{C}^2$, la réponse a été obtenue par Ellingsrud et Strømme en étudiant les points fixes sous l'action naturelle de $\mathbb{C}^* \times \mathbb{C}^*$.

Théorème 1.1.2 (Ellingsrud-Strømme [18]). *Le polynôme de Poincaré de $\mathrm{Hilb}_d \mathbb{C}^2$ est donné par l'expression suivante :*

$$\sum_{n \geq 0} t^n \dim H^n(\mathrm{Hilb}_d \mathbb{C}^2) = \sum_{\eta \in \mathcal{P}_d} t^{2(n-l(\eta))},$$

où \mathcal{P}_d est l'ensemble des partitions de d et $l(\eta)$ est la longueur de η .

Cette formule a été généralisée par Göttsche [27] à toute surface projective en comptant les \mathbb{F}_q -points, et à toute surface quasi-projective par Göttsche et Soergel en utilisant le théorème de décomposition pour les faisceaux pervers.

Théorème 1.1.3 (Göttsche-Soergel [28]). *Soit S une surface lisse quasi-projective. La cohomologie de $\text{Hilb}_d \mathbb{C}^2$ est déterminée par la formule suivante :*

$$\sum_{d,n \geq 0} t^n q^d \dim H^n(\text{Hilb}_d S) = \prod_{m \geq 1} \frac{(1 + t^{2m-1} q^m)^{b_1} (1 + t^{2m+1} q^m)^{b_3}}{(1 - t^{2m-2} q^m)^{b_0} (1 - t^{2m} q^m)^{b_2} (1 - t^{2m+2} q^m)^{b_4}}, \quad (1.1)$$

où $b_i = b_i(S)$ est le i -ème nombre de Betti de S .

Après avoir calculé l'espace vectoriel $H^*(\text{Hilb}_d S)$, la prochaine question naturelle est de savoir comment on peut y retrouver les classes cohomologiques naturelles, par exemples les classes de Chern des fibrés en droites sur $\text{Hilb}_d S$. L'apport décisif de Nakajima a été de voir que la formule (1.1) ressemble à la formule de caractères pour une représentation de l'algèbre de Heisenberg.

On commence par rappeler quelques définitions. Soit $V = V_0 \oplus V_1$ un super-espace vectoriel (de manière équivalente, un espace $\mathbb{Z}/2\mathbb{Z}$ -gradué), muni d'une forme bilinéaire super-symétrique \langle, \rangle , c'est-à-dire telle que

$$\langle v, w \rangle = (-1)^{\deg v \deg w} \langle w, v \rangle.$$

On peut étendre cette forme à l'espace $W := (V \otimes t\mathbb{C}[t]) \oplus (V \otimes t^{-1}\mathbb{C}[t^{-1}])$ par la formule

$$\langle v \otimes t^i, w \otimes t^j \rangle = i \delta_{i,-j} \langle v, w \rangle.$$

On voit W comme une (super-)algèbre de Lie abélienne.

Définition 1.1.3. L'algèbre de Heisenberg $\mathcal{H}eis(V)$ associée à V est l'extension centrale

$$0 \rightarrow \mathbb{C} \cdot 1 \rightarrow \mathcal{H}eis(V) \rightarrow W \rightarrow 0$$

en tant qu'une super-algèbre de Lie, donnée par $[w_1, w_2] = \langle w_1, w_2 \rangle 1$, $w_1, w_2 \in W$.

Cette algèbre admet une représentation naturelle.

Définition 1.1.4. L'espace vectoriel

$$F(V) = S^*(V_0 \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes \Lambda^*(V_1 \otimes t^{-1}\mathbb{C}[t^{-1}])$$

est appelé *l'espace de Fock*.

Pour tout $v \in V$ et $i \in \mathbb{N}^+$, on note $v_{-i} = v \otimes t^{-i} \in F(V)$. Les éléments de l'espace de Fock peuvent alors être vus comme des produits $P_0 \cdot P_1$, où P_0 est un polynôme symétrique en v_{-i} , $v \in V^0$, et P_1 est un polynôme anti-symétrique en v'_{-i} , $v' \in V_1$. L'algèbre de Heisenberg agit sur $F(V)$ comme suit :

$$(v \otimes t^k)(P_0 P_1) = \begin{cases} k \frac{\partial}{\partial v_{-k}} (P_0 P_1), & k > 0, \\ v_k \cdot P_0 P_1, & k < 0. \end{cases}$$

Afin de formuler le résultat de Nakajima, on définit pour tout $i > 0$ la sous-variété $P[i] \subset \text{Hilb}_{d-i} S \times \text{Hilb}_d S$ de drapeaux d'idéaux :

$$P[i] := \coprod_d \left\{ (\mathcal{I}_1, \mathcal{I}_2) \in \text{Hilb}_d S \times \text{Hilb}_{d+i} S \mid \mathcal{I}_1 \supset \mathcal{I}_2, |\text{supp}(\mathcal{I}_1/\mathcal{I}_2)| = 1 \right\}.$$

Pour $i < 0$, on définit $P[i]$ en échangeant les rôles de \mathcal{I}_1 et \mathcal{I}_2 . Notons qu'on dispose de projections naturelles

$$p_k : P[i] \rightarrow \coprod_d \text{Hilb}_d S, \quad p_k(\mathcal{I}_1, \mathcal{I}_2) = \mathcal{I}_k, \quad k = 1, 2,$$

ainsi que de l'application $\Pi : P[i] \rightarrow S$, qui envoie $(\mathcal{I}_1, \mathcal{I}_2)$ à $\text{supp}(\mathcal{I}_1/\mathcal{I}_2)$. Ensuite, pour tout $\alpha \in H_*(S)$ et $i \in \mathbb{Z}$, on introduit l'opérateur

$$P_\alpha[i] : H_*(\text{Hilb}_d S) \rightarrow H_*(\text{Hilb}_{d+i} S), \quad c \mapsto p_{2*}(p_1^* c \cdot \Pi^* \alpha).$$

Théorème 1.1.4 (Nakajima [54]). *Les opérateurs $P_\alpha[i]$, $\alpha \in H_*(S)$, $i \in \mathbb{Z}$ satisfont les relations suivantes :*

$$[P_\alpha[i], P_\beta[j]] := P_\alpha[i]P_\beta[j] - (-1)^{\deg \alpha \deg \beta} P_\beta[j]P_\alpha[i] = (-1)^{i-1} i \delta_{i,-j} \langle \alpha, \beta \rangle \text{id},$$

et donc définissent une action de $\mathcal{H}eis(H_(S))$ sur $\bigoplus_d H_*(\text{Hilb}_d S)$. De plus, $\mathbb{H}_S := \bigoplus_d H_*(\text{Hilb}_d S)$ est isomorphe à l'espace de Fock comme $\mathcal{H}eis(V)$ -module.*

Remarque 1.1.5. La valeur précise de coefficient $(-1)^{i-1} i$ a été calculé par Ellingsrud et Strømme dans [19].

Cette description de \mathbb{H}_S a été utilisée par Lehn pour calculer les classes de Chern de certains faisceaux tautologiques. Soit \mathcal{L} un fibré en droites sur S , et $\mathcal{L}^{[d]}$ le faisceau localement libre sur $\text{Hilb}_d S$ dont la fibre au point \mathcal{I} est égale à $H^0(\mathcal{L} \otimes (\mathcal{O}_S/\mathcal{I}))$.

Théorème 1.1.5 (Lehn [45]). *Les classes de Chern totales $c(\mathcal{L}^{[n]})$ sont exprimées par la formule suivante :*

$$\sum_{d=0}^{+\infty} c(\mathcal{L}^{[d]}) z^d = \exp \left(\sum_{m=1}^{+\infty} \frac{(-1)^{m-1}}{m} P_{c_1(\mathcal{L})}[m] z^m \right) \cdot 1.$$

Pour $S = \mathbb{C}^2$, le schéma de Hilbert $\text{Hilb}_d \mathbb{C}^2$ s'interprète en termes de faisceaux sans torsion de rang 1 sur \mathbb{P}^2 . Il existe de tels espaces de modules $\mathcal{M}(d, r)$ en rang r quelconque, voir section 1.4. Il se trouve que l'action de l'algèbre de Heisenberg peut s'étendre à ce cadre, mais ne constitue qu'une partie d'une algèbre bien plus grosse – l'algèbre $W_k(\mathfrak{gl}_r)$. Cette action est construite à l'aide de la théorie des algèbres de Hall cohomologiques (CoHA), voir section 1.5.

Le but de cette thèse est de développer une théorie analogue pour les surfaces de la forme $\mathbb{P}(T^*C)$, où C est une courbe algébrique projective lisse. Cette généralisation fait naturellement intervenir les algèbres de Hall (cohomologiques ou pas) associées aux courbes. Dans ce qui suit, on rappellera les définitions de ces algèbres. Puis, nous énoncerons quelques résultats sur les représentations de carquois qui nous ont servis d'inspiration, et enfin, résumerons les résultats obtenus dans cette thèse.

1.2 Les algèbres de Hall

Commençons par rappeler la définition des algèbres de Hall. Soit \mathcal{C} une catégorie \mathbb{F}_q -linéaire de dimension homologique finie. On note $\text{Iso } \mathcal{C}$ l'ensemble des classes d'isomorphisme d'objets de \mathcal{C} , et pour tout objet $M \in \text{Ob } \mathcal{C}$ on note $[M]$ la classe d'isomorphisme correspondante. Dans ce qui suit, on suppose que les espaces $\text{Hom}(M, N)$, $\text{Ext}^i(M, N)$ sont de dimension finie pour tout $M, N \in \text{Ob } \mathcal{C}$.

Définition 1.2.1. *L'algèbre de Hall $H_{\mathcal{C}}$ de \mathcal{C} est engendrée par $\text{Iso } \mathcal{C}$ en tant que \mathbb{C} -espace vectoriel ; pour tout $M, N \in \text{Ob } \mathcal{C}$ le produit est donné par la formule suivante :*

$$[M] \cdot [N] = \langle M, N \rangle \sum_{[R] \in \text{Iso } \mathcal{C}} \frac{\#\{0 \rightarrow N \rightarrow R \rightarrow M \rightarrow 0\}}{|\text{Aut } M| \cdot |\text{Aut } N|} [R],$$

où $\langle M, N \rangle := q^{\frac{1}{2} \sum_{i=0}^{\infty} (-1)^i \dim \text{Ext}^i(M, N)}$ désigne la forme d'Euler.

Cette notion est apparu, bien que sous un autre nom, pour la première fois dans les travaux de Steinitz dans le contexte des p -groupes abéliens [72]. Par la suite, Ringel a démontré [63] que $H_{\mathcal{C}}$ est une algèbre associative pour toute catégorie \mathcal{C} .

Les algèbres de Hall ont gagné en importance après la découverte de leur lien avec les groupes quantiques par Ringel et Green. Afin de pouvoir donner l'énoncé précis de ce résultat, fixons un carquois (c'est-à-dire un graphe orienté) $Q = (I, E)$. On supposera que Q n'a pas de boucles. Soit $H_Q = H_{\text{Rep}_{\mathbb{F}_q} Q}$ l'algèbre de Hall associée à la catégorie de représentations de dimension

finie sur \mathbb{F}_q de Q . Pour tout sommet $i \in I$, l'élément $S_i \in H_Q$ est donné par la représentation de Q de dimension 1 supportée en i . Notons

$$A_Q = (a_{ij})_{i,j \in I}, \quad a_{ij} = \#\{i \rightarrow j\} + \#\{j \rightarrow i\},$$

et $C = 2 \text{Id} - A_Q$. Si l'on suppose Q sans boucle, alors C est une matrice de Cartan (symétrique) généralisée. Soit $\mathfrak{g}_Q = \mathfrak{n}_Q^- \oplus \mathfrak{h} \oplus \mathfrak{n}_Q^+$ l'algèbre de Kac-Moody associée. Par le théorème de Gabber-Kac [22], l'algèbre enveloppante $U(\mathfrak{n}_Q^+)$ est engendrée par les éléments e_i , $i \in I$ et les relations de Serre

$$\sum_{p=0}^{1-c_{ij}} (-1)^p \binom{1-c_{ij}}{p} e_j^p e_i e_j^{1-c_{ij}-p} = 0.$$

Drinfeld et Jimbo ont introduit une déformation quantique $U_\nu(\mathfrak{g}_Q)$ de l'algèbre associative $U(\mathfrak{g}_Q)$ [15]. En particulier, $U_\nu(\mathfrak{g}_Q)$ contient une sous-algèbre $U_\nu(\mathfrak{n}_Q^+)$, engendrée par les éléments E_i , $i \in I$ sujets aux relations suivantes :

$$\sum_{p=0}^{1-c_{ij}} (-1)^p \begin{bmatrix} 1-c_{ij} \\ p \end{bmatrix} E_j^p E_i E_j^{1-c_{ij}-p} = 0,$$

où les coefficients sont donnés par

$$\begin{bmatrix} t \\ r \end{bmatrix} = \prod_{i=1}^r \frac{(\nu^{t-r+i} - \nu^{-t+r-i})}{(\nu^i - \nu^{-i})}.$$

Théorème 1.2.1 (Ringel [63], Green [29]). *Soit $\nu = q^{\frac{1}{2}}$. L'application $E_i \mapsto [S_i]$, $i \in I$ se prolonge en une injection $\Psi : U_\nu(\mathfrak{n}_Q^+) \hookrightarrow H_Q$. C'est un isomorphisme si et seulement si Q est de type fini, c'est-à-dire si \mathfrak{g}_Q est une algèbre de Lie semisimple.*

Remarque 1.2.2. Un raffinement du théorème 1.2.1 a été utilisé par Lusztig afin de définir la base canonique de $U_\nu(\mathfrak{n}_Q^+)$.

On suppose maintenant que \mathcal{C} est *héréditaire*, c'est-à-dire que les espaces $\text{Ext}^i(M, N)$ sont triviaux pour $i \geq 2$ et tout $M, N \in \text{Ob } \mathcal{C}$. Dans ce cas, Green a montré que $H_{\mathcal{C}}$ peut être munie d'une structure de bigèbre topologique (et même d'algèbre de Hopf, voir Xiao [75]). Comme l'a montré Drinfeld [15], il existe un moyen purement algébrique de reconstruire le groupe quantique $U_\nu(\mathfrak{g}_Q)$ en entier à partir de l'algèbre de Hopf $U_\nu(\mathfrak{n}_Q^+)$ – le double de Drinfeld [15]. Ceci permet de donner une réalisation de $U_\nu(\mathfrak{g}_Q)$ en terme d'algèbre de Hall de carquois.

Dans le cadre géométrique qui est celui de cette thèse, nous nous intéressons plutôt à une autre classe de catégories héréditaires : les catégories $\text{Coh } C$ de faisceaux cohérents sur une courbe projective lisse C sur \mathbb{F}_q . On a malheureusement beaucoup moins de résultats structurels sur les algèbres $H_C := H_{\text{Coh } C}$, et ils sont largement limités aux courbes de petit genre. Nous considérons ici brièvement les cas de courbes de genre inférieure à 2.

Genre 0. Soit $C = \mathbb{P}^1$. Rappelons que le groupe quantique de lacets $U_\nu(\mathcal{L}\mathfrak{b}^+)$ associé au Borel $\mathfrak{b}^+ \subset \mathfrak{sl}_2$ est engendré par les éléments $E_l, l \in \mathbb{Z}$ et $H_k, k \in \mathbb{N}^+$ avec les relations suivantes :

$$\begin{aligned} [H_n, H_k] &= 0, & [H_k, E_l] &= \frac{\nu^{2k} - \nu^{-2k}}{k(\nu - \nu^{-1})} E_{l+k}, \\ E_{k+1} E_l - \nu^2 E_l E_{k+1} &= \nu^2 E_k E_{l+1} - E_{l+1} E_k. \end{aligned}$$

Remarque 1.2.3. Ceci n'est pas la définition la plus courante. En effet, on ajoute d'habitude les générateurs $K^{\pm 1}$ et un élément central $C^{\pm \frac{1}{2}}$ à $U_\nu(\mathcal{L}\mathfrak{b}^+)$. Nous la modifions ainsi pour éviter les détails superflus à notre présentation.

Dans la catégorie $\text{Coh } \mathbb{P}^1$, pour tout $d \in \mathbb{Z}$ on a un unique fibré en droites $\mathcal{O}(d)$ de degré d . De plus, pour tout point fermé $x \in \mathbb{P}^1$ on a un unique faisceau irréductible $\mathcal{O}_{x,l}$ de longueur l , qui est supporté en x . On définit les éléments suivants :

$$T_{k,x} = \begin{cases} \frac{\nu^k - \nu^{-k}}{k(\nu - \nu^{-1})} \deg(x) \left(\sum_{l \geq 0} \left[\mathcal{O}_{x,l}^{\oplus \frac{k}{\deg(x)}} \right] \right) & \text{si } \deg(x) | k, \\ 0 & \text{sinon,} \end{cases}$$

où $\deg(x) := [k_x : \mathbb{F}_q]$, et on note $T_k := \sum_x T_{k,x}$.

Théorème 1.2.2 (Kapranov [36]). *L'application $E_l \mapsto [\mathcal{O}(l)]$, $H_k \mapsto T_k$ s'étend en un morphisme injectif d'algèbres $\Psi : U_\nu(\mathcal{Lb}^+) \hookrightarrow H_{\mathbb{P}^1}$.*

Comme avant, on peut élever Ψ au rang d'un morphisme entre algèbres de Hopf.

Genre 1. Si C est une courbe elliptique, la catégorie $\text{Coh } C$ a une structure particulièrement simple. On dira que $\mathcal{E} \in \text{Coh } C$ est *semi-stable* si $\deg \mathcal{E}' \text{ rg } \mathcal{E} \leq \deg \mathcal{E} \text{ rg } \mathcal{E}'$ pour tout $\mathcal{E}' \subset \mathcal{E}$, et on va noter par \mathcal{C}_η la sous-catégorie des faisceaux semi-stables avec $\frac{\deg \mathcal{E}}{\text{rg } \mathcal{E}} = \eta$. En particulier, \mathcal{C}_∞ est précisément la sous-catégorie des faisceaux de torsion.

Théorème 1.2.3 (Atiyah [3]). *Pour tout $\eta \in \mathbb{Q}$ il existe une équivalence de catégories $\varphi_\eta : \mathcal{C}_\infty \xrightarrow{\sim} \mathcal{C}_\eta$. De plus, tout faisceau $\mathcal{E} \in \text{Coh } C$ peut être décrit comme $\bigoplus_\eta E_\eta$ avec $E_\eta \in \mathcal{C}_\eta$.*

Grâce à ce théorème, on pose $T_{\eta,k} = \varphi_\eta(T_k)$ et $T_{p,q} = T_{\frac{q}{p}, \text{pgcd}(p,q)}$. La sous-algèbre $\mathbf{U}_C^+ \subset H_C$ engendrée par les éléments $T_{p,q}$ a une description explicite :

Théorème 1.2.4 (Burban-Schiffmann [10]). *Pour tout $n \in \mathbb{N}$, $a, b \in \mathbb{Z}$ avec $\text{pgcd}(a, b) = 1$ on définit $\theta_{na, nb}$ par la formule suivante :*

$$\sum_{n \geq 0} \theta_{na, nb} x^n = \exp \left(\sum_{n \geq 1} \frac{(q^n - 1)(1 - \sigma^n)(1 - \bar{\sigma}^n)}{n} T_{na, nb} x^n \right),$$

où $\sigma, \bar{\sigma} \in \mathbb{C}$ sont les nombres de Weil de C . Alors le noyau du morphisme $\mathbb{C}\langle T_{p,q} \rangle \rightarrow \mathbf{U}_X^+$ est engendré par des relations de deux types :

1. si les vecteurs (p, q) , (p', q') sont colinéaires, alors $[T_{p,q}, T_{p',q'}] = 0$;
2. si $\text{pgcd}(p, q) = 1$, et si le triangle avec les sommets $(0, 0)$, (p', q') , $(p+p', q+q')$ ne contient pas de points entiers à l'intérieur, alors

$$[T_{p,q}, T_{p',q'}] = \frac{\theta_{p+p', q+q'}}{(q-1)(1-\sigma)(1-\bar{\sigma})}.$$

Cette algèbre est extrêmement importante dans la théorie des représentations, notamment dans le contexte de la correspondance AGT ; voir section 1.5.

Dans les genres supérieurs à 1, on est loin de comprendre l'ensemble de relations dans H_C . Il existe des résultats partiels, par exemple les relations quadratiques prouvées par Kapranov [36]. Pour illustrer la difficulté de cette question, l'auteur ne peut s'empêcher de faire une comparaison avec le cadre arithmétique. Dans ce contexte on peut définir une version arithmétique de l'algèbre de Hall en remplaçant la courbe lisse par $\text{Spec}(\mathbb{Z})$. Alors les relations cubiques dans $H_{\text{Spec}(\mathbb{Z})}$ sont gouvernées par les zéros de la fonction zêta de Riemann [38].

1.3 Une version cohomologique

Comme le montrent les exemples de la droite projective et des courbes elliptiques, il est souvent utile de considérer des fonctions caractéristiques de sous-catégories naturelles (faisceaux de torsion, semi-stables, etc.) pour étudier l'algèbre de Hall d'une catégorie \mathcal{C} . Dans la même logique, on peut s'intéresser à une version plus topologique des algèbres de Hall, modelées sur la cohomologie des espaces de modules d'objets de \mathcal{C} . C'est la théorie des algèbres de Hall cohomologiques.

À partir de maintenant, on remplace \mathbb{F}_q par \mathbb{C} dans le rôle du corps de base. Commençons par décrire l'exemple le plus simple d'une telle algèbre, associée à la catégorie des représentations d'un carquois $Q = (I, E)$, où I est l'ensemble de sommets, et E est l'ensemble de flèches. Pour tout arête $e \in E$, on note sa source $s(e)$ et son but $t(e)$. L'ensemble des représentations de Q dans $\mathbb{V} = \bigoplus_i \mathbb{C}^{\mathbf{v}_i}$,

$$\text{Rep}(Q, \mathbf{v}) := \bigoplus_{e \in E} \text{Hom}(\mathbb{C}^{\mathbf{v}_{s(e)}}, \mathbb{C}^{\mathbf{v}_{t(e)}}),$$

admet une action naturelle de $GL_{\mathbf{v}} = \prod_{i \in I} GL_{\mathbf{v}_i}(\mathbb{C})$. Ainsi, les objets de $\text{Rep } Q$ de dimension \mathbf{v} sont paramétrés par les points du champ quotient $[\text{Rep}(Q, \mathbf{v})/GL_{\mathbf{v}}]$. On va fixer un Borel $B_{\mathbf{v}} \subset GL_{\mathbf{v}}$, qui équivaut au choix de drapeaux complets $\mathbb{C} \subset \mathbb{C}^2 \subset \dots \subset \mathbb{C}^{\mathbf{v}^{(i)}}$ pour tout $i \in I$. Alors pour tout $\mathbf{v}_1, \mathbf{v}_2$ on a le sous-espace

$$\widetilde{\text{Rep}}(Q; \mathbf{v}_1, \mathbf{v}_2) = \{(X_e)_{e \in E} \in \text{Rep}(Q, \mathbf{v}_1 + \mathbf{v}_2) \mid X_e(\mathbb{C}^{\mathbf{v}_{1,s(e)}}) \subset \mathbb{C}^{\mathbf{v}_{1,t(e)}} \forall e \in E\},$$

qui est préservé sous l'action du sous-groupe parabolique $P_{\mathbf{v}_1, \mathbf{v}_2} = B_{\mathbf{v}} \cdot (GL_{\mathbf{v}_1} \times GL_{\mathbf{v}_2})$.

Définition 1.3.1 (Kontsevich-Soibelman [40]). *L'algèbre de Hall cohomologique* associée à Q est l'espace vectoriel $\mathbf{Ha}_Q := \bigoplus_{\mathbf{v}} H_*^{GL_{\mathbf{v}}}(\text{Rep}(Q, \mathbf{v}))$, avec le produit défini par

$$\alpha_1 * \alpha_2 = i_{\mathbf{v}_1, \mathbf{v}_2} \circ \Psi_* \circ \Phi^*(\alpha_1 \boxtimes \alpha_2) \quad \alpha_i \in H_*^{GL_{\mathbf{v}_i}}(\text{Rep}(Q, \mathbf{v}_i)), i = 1, 2,$$

où Φ^*, Ψ_* sont les applications

$$\begin{aligned} \Phi^* &: H_*^{GL_{\mathbf{v}_1}}(\text{Rep}(Q, \mathbf{v}_1)) \otimes H_*^{GL_{\mathbf{v}_2}}(\text{Rep}(Q, \mathbf{v}_2)) \rightarrow H_*^{P_{\mathbf{v}_1, \mathbf{v}_2}}(\widetilde{\text{Rep}}(Q; \mathbf{v}_1, \mathbf{v}_2)), \\ \Psi_* &: H_*^{P_{\mathbf{v}_1, \mathbf{v}_2}}(\widetilde{\text{Rep}}(Q; \mathbf{v}_1, \mathbf{v}_2)) \rightarrow H_*^{P_{\mathbf{v}_1, \mathbf{v}_2}}(\text{Rep}(Q, \mathbf{v})), \end{aligned}$$

associées aux morphismes naturels

$$\begin{array}{ccc} & \widetilde{\text{Rep}}(Q; \mathbf{v}_1, \mathbf{v}_2) & \\ \Phi \swarrow & & \searrow \Psi \\ \text{Rep}(Q, \mathbf{v}_1) \times \text{Rep}(Q, \mathbf{v}_2) & & \text{Rep}(Q, \mathbf{v}) \end{array}$$

et $i_{\mathbf{v}_1, \mathbf{v}_2}$ est donnée par la composition

$$H^{P_{\mathbf{v}_1, \mathbf{v}_2}}(\text{Rep}(Q, \mathbf{v})) \xrightarrow{\text{Ind}_{P_{\mathbf{v}_1, \mathbf{v}_2}}^{GL_{\mathbf{v}}}} H^{GL_{\mathbf{v}}}(GL_{\mathbf{v}} \times_{P_{\mathbf{v}_1, \mathbf{v}_2}} \text{Rep}(Q, \mathbf{v})) \rightarrow H^{GL_{\mathbf{v}}}(\text{Rep}(Q, \mathbf{v})).$$

L'algèbre \mathbf{Ha}_Q admet une présentation de battage, qui peut être obtenue par localisation équivariante. Les formules étant assez complexes, on expliquera cette idée plus en détail dans un autre contexte (cf. Théorème B).

Au premier regard, l'algèbre \mathbf{Ha}_Q ressemble formellement à H_Q . Cependant, cette construction ne produit pas un objet proche des algèbres de Hall classiques. Par exemple, si on prend le carquois $Q = \bullet \rightleftarrows \bullet$, l'algèbre H_Q contient (la partie positive de) $U_{\nu}(\hat{\mathfrak{sl}}_2)$. En même temps, on a le théorème suivant :

Théorème 1.3.1 (Efimov [17]). *Si Q est un carquois symétrique, l'algèbre de Hall cohomologique \mathbf{Ha}_Q est libre super-symétrique.*

Cette divergence finalement n'est pas très surprenante, parce que les algèbres de Hall sont liées au comptage de \mathbb{F}_q -points, qui sont gouvernés par les faisceaux constructibles sur les champs de modules. D'après la correspondance de Riemann-Hilbert on sait que ces objets sont plutôt liés aux \mathcal{D} -modules, et donc aux faisceaux cohérents sur le *fibré cotangent* au champ de modules. Ceci motivera l'introduction d'une algèbre de Hall cohomologique plus compliquée (en dimension 2).

Pour la catégorie des faisceaux cohérents sur une courbe C , l'algèbre de Hall cohomologique est encore peu étudiée. Cependant, on peut montrer qu'elle contient, entre autres, information sur les relations dans l'anneau de cohomologie de l'espace de modules des fibrés stable sur C [66].

1.4 Espaces de modules et représentations

Pour pouvoir expliquer le lien entre les CoHAs et schémas de Hilbert de points sur $S = \mathbb{A}^2$, reprenons plus précisément le cadre de la section 1.1. Il existe une présentation plus algébrique du schéma de Hilbert. Tout d'abord, on peut voir \mathbb{A}^2 comme un ouvert de \mathbb{P}^2 , avec le complémentaire $l_\infty := \mathbb{P}^2 \setminus \mathbb{A}^2$. Alors $\text{Hilb}_d \mathbb{A}^2$ est isomorphe à $\mathcal{M}(d, 1)$, où $\mathcal{M}(d, n)$ dénote l'espace de modules de faisceaux sans torsion E sur \mathbb{P}^2 de rang n et $c_2(E) = d$ avec une trivialisatoin à l'infini $E|_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{l_\infty}^{\oplus n}$.

Théorème 1.4.1 (Barth [5]). *Il existe un isomorphisme*

$$\mathcal{M}(d, n) \simeq \left\{ (X, Y, i, j) \mid \begin{array}{l} [X, Y] + ij = 0; \\ \#V \subsetneq \mathbb{C}^d : \mathbb{C}[X, Y] \text{Im } i \subset V \end{array} \right\} / GL_d,$$

où $X, Y \in \text{End}(\mathbb{C}^d)$, $i \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^d)$, $j \in \text{Hom}(\mathbb{C}^d, \mathbb{C}^n)$, et l'action de GL_d est donnée par

$$g \cdot (X, Y, i, j) = (gXg^{-1}, gYg^{-1}, gi, jg^{-1}).$$

On peut donc voir $\mathcal{M}(d, n)$ comme un espace de modules de représentations du carquois suivant (vérifiant des conditions supplémentaires) :

$$\begin{array}{ccc} & Y & \\ & \curvearrowright & \\ & \mathbb{C}^d & \xrightleftharpoons[i]{j} \mathbb{C}^n \\ & \curvearrowleft & \\ X & & \end{array}$$

L'intérêt de considérer ici des CoHAs se voit plus aisément si on se place dans le cadre général des variétés de Nakajima, dont les espaces $\mathcal{M}(d, n)$ sont des cas particuliers. Plus précisément, étant donné un carquois $Q = (I, E)$, on considère le carquois $\overline{Q}^\heartsuit = (I', E')$, défini par

$$I' = \{i, i' \mid i \in I\}; \quad E' = \{e, \bar{e} \mid e \in E\} \sqcup \{i \rightarrow i', i' \rightarrow i \mid i \in I\},$$

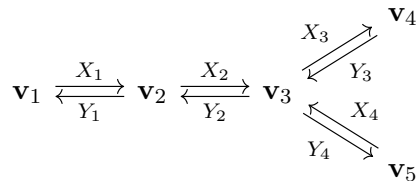
où \bar{e} est l'arête opposée à e . Pour toute paire de vecteurs de dimension $\mathbf{v}, \mathbf{w} \in \mathbb{N}^{I'}$, on considère l'espace $\text{Rep}(\overline{Q}^\heartsuit, \mathbf{v}, \mathbf{w})$ de représentations de \overline{Q}^\heartsuit :

$$\begin{aligned} \text{Rep}(\overline{Q}^\heartsuit, \mathbf{v}, \mathbf{w}) = & \bigoplus_{e \in E} \text{Hom}(\mathbb{C}^{\mathbf{v}_{s(e)}}, \mathbb{C}^{\mathbf{v}_{t(e)}}) \oplus \bigoplus_{e \in E} \text{Hom}(\mathbb{C}^{\mathbf{v}_{t(e)}}, \mathbb{C}^{\mathbf{v}_{s(e)}}) \\ & \oplus \bigoplus_{i \in I} \text{Hom}(\mathbb{C}^{\mathbf{w}_i}, \mathbb{C}^{\mathbf{v}_i}) \oplus \bigoplus_{i \in I} \text{Hom}(\mathbb{C}^{\mathbf{v}_i}, \mathbb{C}^{\mathbf{w}_i}). \end{aligned}$$

On notera un élément de $\text{Rep}(\overline{Q}^\heartsuit, \mathbf{v}, \mathbf{w})$ par $R = (X, Y, i, j)$. Par exemple, si Q est de la forme suivante :

$$\begin{array}{ccccc} & & & & \mathbf{v}_4 \\ & & & \nearrow & \\ \mathbf{v}_1 & \longrightarrow & \mathbf{v}_2 & \longrightarrow & \mathbf{v}_3 \\ & & & \nwarrow & \\ & & & & \mathbf{v}_5 \end{array}$$

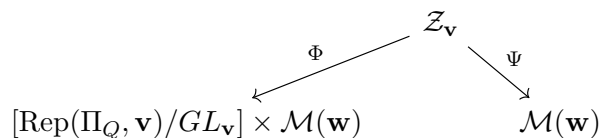
des représentations de l'algèbre préprojective Π_Q (de dimension ϵ_i), c'est-à-dire représentations (X, Y) du carquois doublé \overline{Q} avec $[X, Y] = 0$:



Ici, on a imposé la condition $\dim(R'/R) = \epsilon_i$ pour faire le lien avec le théorème 1.4.2. Rien n'interdit de considérer, en général, le champ

$$\mathcal{Z}_{\mathbf{v}} := \coprod_{\mathbf{u}} \left\{ R \subset R' \mid R \in \mathcal{M}(\mathbf{u}, \mathbf{w}), R' \in \mathcal{M}(\mathbf{u} + \mathbf{v}, \mathbf{w}) \right\},$$

qui donne lieu à une correspondance (où on note quotient champêtre par des crochets)



Par conséquent, pour tout $\mathbf{v} \in \mathbb{N}^I$ et $\alpha \in H_*^{GL_{\mathbf{v}}}(\text{Rep}(\Pi_Q, \mathbf{v}))$ on obtient un opérateur

$$E_{\mathbf{v}, \alpha} = \Psi_* \Phi^*(\alpha \boxtimes -) \in \text{End } H_*(\mathcal{M}(\mathbf{w})),$$

pour des foncteurs Φ^* , Ψ_* bien définis. En particulier, $E_{\epsilon_i, \mathbb{1}} = E_i$, où $\mathbb{1} \in H_*([\text{Rep}(\Pi_Q, \mathbf{v})/GL_{\mathbf{v}}])$ désigne la classe fondamentale. Ces opérateurs feront une partie de l'algèbre de Hall cohomologique \mathbf{Ha}_{Π_Q} de l'algèbre préprojective Π_Q .

Théorème 1.4.3 (Schiffmann-Vasserot [68], Yang-Zhao [76]).

Posons $\mathbf{Ha}_{\Pi_Q} = \bigoplus_{\mathbf{v}} H_*^{GL_{\mathbf{v}}}(\text{Rep}(\Pi_Q, \mathbf{v}))$. Alors

1. il existe une structure d'algèbre associative sur \mathbf{Ha}_{Π_Q} ;
2. il existe une action de \mathbf{Ha}_{Π_Q} sur $M_{\mathbf{w}}$, compatible avec l'action de $U(\mathfrak{g}_Q)$ du théorème 1.4.2.

Cette algèbre a été aussi étudié du point de vue de la théorie de Donaldson-Thomas par Davison et Meinhardt (par exemple, voir [13]). L'énoncé reste valable pour toute théorie d'homologie "raisonnable", voir section 1.6. En particulier, si on considère la sous-algèbre de \mathbf{Ha}_{Π_Q} engendrée par les opérateurs $E_{\epsilon_i, \alpha}$, le théorème 1.4.3 nous permet, dans le cas des carquois sans boucles, de retrouver l'action du Yangien $Y_h(\mathfrak{g}_Q)$ sur $M_{\mathbf{w}}$ et l'action de l'algèbre affine quantique sur $K(\mathcal{M}(\mathbf{w}))$, étudiées par Varagnolo [74] et Nakajima [55].

1.5 Carquois à une boucle

Explicitons le théorème 1.4.3 dans le cas correspondant au carquois de Jordan. Ici $Q = \bullet \curvearrowright$, $\Pi_Q = \mathbb{C}[X, Y]$, et

$$\text{Rep}(\Pi_Q, d) = \mathcal{C}_d = \{(X, Y) \in \mathfrak{gl}_d \mid [X, Y] = 0\}$$

est la variété commutante. Ainsi, il existe une structure d'algèbre associative sur

$$\mathbf{Ha}_{\mathbb{A}^2} = \bigoplus H_*^{GL_d}(\mathcal{C}_d).$$

Notons que $[\mathcal{C}_d/GL_d]$ est précisément le champ des faisceaux de torsion de longueur d sur \mathbb{C}^2 , par lesquels on effectue des modifications ponctuelles de faisceaux sans torsion sur \mathbb{P}^2 . On retrouve, en particulier, pour des bons choix de $\alpha \in H_*^{GL_d}(\mathcal{C}_d)$, les opérateurs de Nakajima.

On peut se poser la question : à quoi cette algèbre plus grosse ressemble-t-elle ? Une réponse est fournie par la correspondance originale de Alday-Gaiotto-Tachiwaka (AGT) [1], qui prédit une relation entre la théorie de jauge super-symétrique $\mathcal{N} = 2$ en dimension 4 et la théorie conforme des champs de Liouville en dimension 2. Sans rentrer dans les détails physiques (du fait de l'incompétence de l'auteur), cette correspondance implique une égalité de certaines quantités, à savoir le volume équivariant des espaces $\mathcal{M}(d, 2)$ d'un côté, et la dimension des blocs conformes de Liouville de l'autre. Ces derniers sont associés à l'algèbre de Virasoro.

Définition 1.5.1. L'algèbre de Lie $\mathcal{V}ir$, appelé *l'algèbre de Virasoro*, est engendrée par les générateurs L_n , $n \in \mathbb{Z}$ et c , modulo les relations

$$[c, L_n] = 0, \quad [L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m, -n} c.$$

On note $M_n := \bigoplus_d H_*^{\mathbb{C}^* \times \mathbb{C}^*}(\mathcal{M}(d, n))$, où la structure $\mathbb{C}^* \times \mathbb{C}^*$ -équivariante est induite par l'action naturelle $\mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathbb{A}^2$. On peut reformuler et renforcer la prédiction d'AGT, en disant que l'algèbre $\mathcal{V}ir$ doit agir sur M_2 de telle façon que M_2 est isomorphe à un module de Verma de $\mathcal{H}eis \times \mathcal{V}ir$. Cet énoncé peut être généralisé en remplaçant 2 par n et $\mathcal{H}eis \times \mathcal{V}ir$ par l'algèbre \mathcal{W} affine $\mathcal{W}_n := W_k(\mathfrak{gl}_n)$, $k \in \mathbb{C}$. On ne reproduira pas la définition des algèbres \mathcal{W}_n ici, car elle est très technique ; par exemple, aucune présentation de \mathcal{W}_n par générateurs et relations n'est connue pour $n \geq 4$.

Il s'avère que l'action de \mathcal{W}_n sur M_n provient de l'action de l'algèbre de Hall cohomologique préprojective \mathbf{Ha}_{Π_Q} . Plus précisément, l'action naturelle $\mathbb{C}^* \times \mathbb{C}^* \curvearrowright \mathbb{A}^2$ induit une action sur les variétés commutantes \mathcal{C}_d . Soit $\mathbf{Ha}_{\Pi_Q}^T$ la version $\mathbb{C}^* \times \mathbb{C}^*$ -équivariante de \mathbf{Ha}_{Π_Q} . Alors on a le théorème suivant.

Théorème 1.5.1 (Schiffmann-Vasserot [68]). *M_n admet une action du double de Drinfeld $\mathbb{D}(\mathbf{Ha}_{\Pi_Q}^T)$. De plus, cette action se factorise par l'algèbre enveloppante $U(\mathcal{W}_n)$, et M_n est isomorphe à un module de Verma explicite de \mathcal{W}_n .*

L'action de \mathcal{W}_n sur M_n a aussi été construite aussi par Maulik et Okounkov [47], en utilisant des méthodes différentes, fondées sur la notion d'enveloppe stable.

Bien qu'elle soit moins développée que la version mentionnée ci-dessus, il existe une version quantique de la correspondance AGT, impliquant à présent les algèbres \mathcal{W} quantiques (par exemple, l'algèbre de Virasoro quantique) et la K -théorie équivariante des espaces d'instantons $K^{\mathbb{C}^* \times \mathbb{C}^*}(\mathcal{M}(d, n))$. La première partie du théorème 1.5.1 est le sujet de [69], où le produit de Hall K -théorique est définie sur

$$K\mathbf{Ha}_{\mathbb{A}^2}^T := \bigoplus_d K^{GL_d \times \mathbb{C}^* \times \mathbb{C}^*}(\mathcal{C}_d),$$

et la deuxième assertion est démontrée dans [58]. L'analogie du théorème 1.4.3 est également vrai.

Il est naturel de se demander s'il existe une histoire analogue pour d'autres surfaces S . Cette question a été évoquée dans une série de papiers récents de Neguț [60, 57, 56] ; voir la section 1.8 pour la comparaison avec les résultats obtenus dans cette thèse.

Pour finir, notons le résultat — qui peut paraître surprenant — qui donne une autre caractérisation de l'algèbre $K\mathbf{Ha}_{\Pi_Q}^T$ en terme des algèbres de Hall classiques.

Théorème 1.5.2 (Schiffmann-Vasserot [69]). *Soit C une courbe elliptique dont les nombres de Weil sont $\sigma, \bar{\sigma}$. Alors il existe un isomorphisme*

$$K\mathbf{Ha}_{\mathbb{A}^2}^T|_{t=\bar{q}=\sigma} \simeq \mathbf{U}_C^+.$$

De plus, cette algèbre agit sur $\bigoplus_d K(\mathcal{M}(d, n))$ pour tout $n > 0$ comme avant.

1.6 Théories homologiques orientées

Pour traiter l'homologie et la K -théorie de façon homogène, on travaille avec les théories homologiques orientées [46]. En grandes lignes, une théorie homologique orientée de Borel-Moore (OBM) associe à chaque schéma X un groupe abélien gradué $A_*(X)$, de façon partiellement fonctorielle, et plus précisément :

- pour tout morphisme $f : X \rightarrow Y$ projectif, il y a une image directe $f_* : A_*(X) \rightarrow A_*(Y)$;
- pour tout morphisme localement d'intersection complète (l.c.i.) $g : X \rightarrow Y$ de dimension relative d , il y a une image inverse $f^* : A_*(Y) \rightarrow A_{*+d}(X)$.

Ces morphismes satisfont les conditions de compatibilité standards, une des plus remarquables étant *le changement de base*. Plus précisément, on a $f^* \circ g_* = g'_* \circ f'^*$ pour tout carré cartésien avec g projectif, qui est transversal à f l.c.i. :

$$\begin{array}{ccc} Z \times_X Y & \xrightarrow{f'} & Z \\ \downarrow g' & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

Soit $f : Y \rightarrow X$ un morphisme l.c.i., et $g : Z \rightarrow X$ un autre morphisme (sans aucune condition). Alors on peut définir *l'image inverse de Gysin*

$$f^! : A_*(Z) \rightarrow A_*(Z \times_X Y),$$

qui est fonctorielle en f et g , et coïncide avec f^* pour $Z = X$, $g = \text{Id}_X$. Il faut noter que malgré la notation ce morphisme dépend de g d'une manière importante.

Si X est lisse, le groupe $A^*(X) := A_{\dim X - *}(X)$ peut être muni d'un produit d'intersection, devenant un anneau commutatif. Pour tout fibré en droites $L \rightarrow X$ avec la section nulle $s : X \rightarrow L$, le morphisme $s^* \circ s_* : A_*(X) \rightarrow A_{*-1}(X)$ peut être réalisé par un élément $e(L) \in A^1(X)$, appelé *la classe d'Euler* (ou première classe de Chern) de L . Il s'avère qu'il existe toujours une loi de groupe formel $F_A \in A^*(pt)[[u, v]]$ sur $A^*(pt)$, qui gouverne la classe d'Euler du produit tensoriel pour tout schéma X :

$$e(L \otimes M) = F_A(e(L), e(M)).$$

On va supposer que la OBM A est *libre*, c'est-à-dire universelle parmi les autres OBM avec la même loi. Par exemple, les groupes de Chow CH_* et le groupe de Grothendieck de faisceaux cohérents K_0 sont des OBM libres ; on a $F_{CH}(u, v) = u + v$ et $F_{K_0}(u, v) = u + v - uv$.

Remarque 1.6.1. Une lectrice attentive peut noter que le foncteur standard de l'homologie de Borel-Moore H_* n'est ni un exemple de OBM libre, ni même de OBM tout court. Dans le cas des variétés de Nakajima ceci ne pose aucun problème, car leurs homologies de Borel-Moore sont isomorphes aux groupes de Chow. C'est faux pour une variété générale ; cependant, toutes les propriétés des OBM qu'on utilise ont des analogues homologiques.

1.7 Faisceaux de Higgs

Le point de départ est l'observation qu'on peut systématiquement remplacer les carquois par les courbes algébriques dans les constructions de la section 1.4. Par exemple, le champ des représentations de l'algèbre préprojective Π_Q peut être vu comme la réduction hamiltonienne de $T^* \text{Rep}(Q, \mathbf{v})$ sous l'action de $GL_{\mathbf{v}}$:

$$\begin{aligned} \text{Rep}(\Pi_Q, \mathbf{v}) &= \mu^{-1}(0); \\ \mu : T^* \text{Rep}(Q, \mathbf{v}) &= \text{Rep}(\overline{Q}, \mathbf{v}) \rightarrow \mathfrak{g}, \quad (X, Y) \mapsto [X, Y]. \end{aligned}$$

Donc $[\text{Rep}(\Pi_Q, \mathbf{v})/GL_{\mathbf{v}}] = T^*[\text{Rep}(Q, \mathbf{v})/GL_{\mathbf{v}}]$. Pour une courbe C , il est donc naturel de regarder le fibré cotangent $T^*\underline{\text{Coh}}C$ au champ des faisceaux cohérents.

Soit ω_C le fibré canonique de C . Les objets de $T^*\text{Coh } C$ sont décrits de la façon suivante.

Définition 1.7.1. Un *faisceau de Higgs* sur C est un faisceau $\mathcal{E} \in \text{Coh } C$ muni d'un morphisme $\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \omega_C$.

On note le champ des faisceaux de Higgs de rang r et degré d par $\underline{\text{Higgs}}_{r,d}C$. Ce champ n'est pas noethérien pour $r > 0$ (comme le champ des faisceaux cohérents $\underline{\text{Coh}}_{r,d}C$), et de plus il est très singulier. Pour illustrer la géométrie de $\underline{\text{Higgs}}_{r,d}C$, rappelons le théorème de Bozec sur les composantes irréductibles du sous-champ

$$\Lambda_{r,d} = \{(\mathcal{E}, \theta) \mid \theta \text{ est nilpotent}\} \subset \underline{\text{Higgs}}_{r,d}C.$$

On sait, par Laumon [42], que $\Lambda_{r,d}$ est un sous-champ Lagrangien de $\underline{\text{Higgs}}_{r,d}C$. Pour \mathcal{E} un faisceau cohérent sur C , on appellera *type* de \mathcal{E} , et on notera $\tau(\mathcal{E})$, la paire $\tau(\mathcal{E}) = (\text{rg } \mathcal{E}, \text{deg } \mathcal{E})$. Pour tout faisceau de Higgs nilpotent (\mathcal{E}, θ) , on définit son *type de Jordan* $J(\mathcal{E}, \theta)$ par

$$J(\mathcal{E}, \theta) = (J_i(\mathcal{E}, \theta))_i, \quad J_i(\mathcal{E}, \theta) = \tau(\text{Ker } \theta^i / (\text{Im } \theta \cap \text{Ker } \theta^i + \text{Ker } \theta^{i-1})).$$

Soit $\nu = (\nu_1, \nu_2, \dots)$ une collection finie d'éléments $\nu_i \in (\mathbb{N}^+ \times \mathbb{Z}) \sqcup (\{0\} \times \mathbb{N}^+)$ qui satisfait les conditions numériques

$$\sum_i i r_i = r, \quad \sum_i i d_i + i(i-1)(g-1)r_i = d,$$

où $\nu_i = (r_i, d_i)$ et $g = g(C)$ est le genre de C . On définit le champ

$$\mathcal{N}il_{\nu} = \{(\mathcal{E}, \theta) \in \Lambda_{r,d} : J(\mathcal{E}, \theta) = \nu\}.$$

Ces sous-champs définissent une stratification $\Lambda_{r,d} = \coprod_{\nu} \mathcal{N}il_{\nu}$.

Théorème 1.7.1 (Bozec [8]). *Les champs $\overline{\mathcal{N}il}_{\nu}$ sont les composantes irréductibles de $\Lambda_{r,d}$.*

En particulier, l'ensemble des composantes irréductibles de $\Lambda_{r,d}$ est infini dénombrable pour $r > 0$.

1.8 Les résultats

Dans cette thèse, on traite le cas des faisceaux de Higgs de rang 0. L'avantage de cette restriction est que le champ $\underline{\text{Higgs}}_{0,d}C$ peut être représenté comme un quotient $[X_d/GL_d]$, avec X_d un schéma noethérien. Explicitement, on sait que $\underline{\text{Coh}}_{0,d}C \simeq [\underline{\text{Quot}}_{0,d}/GL_d]$, où

$$\underline{\text{Quot}}_{0,d} = \left\{ \varphi : \mathbb{C}^d \otimes \mathcal{O}_C \rightarrow \mathcal{E} \mid \tau(\mathcal{E}) = (0, d), H^0(\varphi) \text{ est invertible} \right\},$$

donc $\underline{\text{Higgs}}_{0,d}C$ peut être obtenu comme la réduction hamiltonienne de $T^*\underline{\text{Quot}}_{0,d}$ par GL_d . Soit $d = d_1 + d_2$. De même, le champ des extensions

$$\widetilde{\underline{\text{Higgs}}}_{0,(d_1,d_2)} := \left\{ (\mathcal{E}_1 \subset \mathcal{E}, \theta) \mid (\mathcal{E}, \theta) \in \underline{\text{Higgs}}_{0,d}, \mathcal{E}_1 \in \underline{\text{Coh}}_{0,d_1}, \theta(\mathcal{E}_1) \subset \mathcal{E}_1 \right\}$$

est isomorphe à un quotient global $[Y_{d_1,d_2}/GL_d]$, et donc la correspondance

$$\begin{array}{ccc} & \widetilde{\underline{\text{Higgs}}}_{0,(d_1,d_2)} & \\ \Phi_{d_1,d_2} \swarrow & & \searrow \Psi_{d_1,d_2} \\ \underline{\text{Higgs}}_{0,d_1} \times \underline{\text{Higgs}}_{0,d_2} & & \underline{\text{Higgs}}_{0,d} \end{array}$$

est équivalente (dans toute A -théorie GL_d -équivariante) à la correspondance

$$\begin{array}{ccc} & Y_{d_1, d_2} & \\ \Phi_{d_1, d_2} \swarrow & & \searrow \Psi_{d_1, d_2} \\ GL_d \times_{H_{d_1, d_2}} (X_{d_1} \times X_{d_2}) & & X_d \end{array}$$

Notons que même si Ψ_{d_1, d_2} est propre, Φ_{d_1, d_2} n'est pas lisse, ou même l.c.i. Malgré tout, on peut prolonger ce morphisme à un morphisme entre des variétés lisses, qui sont construites à partir des plongements $X_d \hookrightarrow T^* \mathcal{Q}uot_{0, d}$. De cette façon, on obtient un morphisme d'image inverse de Gysin $\Phi_{d_1, d_2}^!$ d'après la section 1.6. En utilisant les propriétés formelles de A , on obtient alors le théorème suivant.

Théorème A. *Les applications $(\Psi_{d_1, d_2})_* \circ \Phi_{d_1, d_2}^!$ définissent un produit associatif sur $A\mathbf{Ha}_{0, C} := \bigoplus_d A(\mathcal{H}iggs_{0, d} C)$.*

Remarque 1.8.1. On peut généraliser cette construction de façon assez immédiate pour avoir un produit sur la somme des A -groupes de $\mathcal{H}iggs_{r, d} C$ pour tout $r \geq 0$. Dans ce cas, l'espace de modules des fibrés de Higgs n'est plus un champ quotient global; il faut donc vérifier que la procédure décrite ci-dessus ne dépend pas du choix d'atlas.

Soit $A\mathbf{Ha}_{0, C}^T := \bigoplus_d A^T(\mathcal{H}iggs_{0, d} C)$ la version équivariante de $A\mathbf{Ha}_{0, C}$, où le groupe $T = \mathbb{C}^*$ agit sur les faisceaux de Higgs par dilation de θ . Un autre avantage de se restreindre en rang 0 est l'existence d'une présentation combinatoire de $A\mathbf{Ha}_{0, C}^T$.

Définition 1.8.2. *L'algèbre de battage globale est l'espace vectoriel*

$$A\mathbf{Sh}_C = \bigoplus_d A\mathbf{Sh}_C[d] = \bigoplus_d \left(A(C^d)((e(t); e(t_1), \dots, e(t_d))) \right)^{\mathfrak{S}_d}$$

muni du produit

$$\Xi : A\mathbf{Sh}_C[d_1] \times A\mathbf{Sh}_C[d_2] \rightarrow A\mathbf{Sh}_C[d_1 + d_2],$$

$$\Xi(f, h) = \sum_{\sigma \in Sh(d_1, d_2)} \sigma \cdot \left(\prod_{\substack{1 \leq i \leq d_1 \\ 1 \leq j \leq d_2}} g \left(\frac{t_{i+j}}{t_i} \right) f(t_1, \dots, t_{d_1}) h(t_{d_1+1}, \dots, t_{d_1+d_2}) \right),$$

où $g \in A^*(C \times C)((e(t)))(e(z))$ est une série formelle de Laurent donnée par

$$g = \frac{e(z\mathcal{O}(-\Delta))e(tz\mathcal{O}(\Delta))}{e(z)e(tz)}.$$

Pour tout choix d'une base dans \mathbb{C}^d il existe une application $C^d \simeq \mathcal{Q}uot_{0, 1}^d \rightarrow \mathcal{C}^d$, donnée par la somme directe. C'est une immersion fermée, qui définit une image inverse en A -théorie

$$\rho_d : A^{G_d \times T}(\mathcal{C}_d) \rightarrow A^{\mathbb{T}}(C^d)^{\mathfrak{S}_d} \subset \mathbf{Sh}_C[d].$$

Théorème B. *L'ensemble des applications ρ_d , $d \in \mathbb{N}$ définit un morphisme d'algèbres associatives graduées*

$$\rho : A\mathbf{Ha}_C^0 \rightarrow A\mathbf{Sh}_C.$$

Théorème C. *Soit $A = H$ l'homologie de Borel-Moore. Alors l'application $\rho : H\mathbf{Ha}_{0, C}^T \rightarrow H\mathbf{Sh}_C$ est injective.*

L'analyse qui nous permet de démontrer le théorème C est spécifique au cas de l'homologie de Borel-Moore standard. En particulier, on utilise la notion de pureté homologique pour considérer chaque strate $\mathcal{N}il_\nu$ séparément. Néanmoins, on s'attend à ce qu'un résultat analogue soit vraie pour toute OBM A .

Pour construire des représentations de cette algèbre, on cherche un analogue aux variétés de Nakajima. Rappelons qu'ils paramètrent

- (i) les points stables
- (ii) dans le fibré cotangent
- (iii) du champ des représentations encadrées

d'un carquois fixe. Si on remplace les représentations de carquois par les faisceaux cohérents sur notre courbe C et si on applique les étapes (ii) et (iii), on arrive à la définition suivante :

Définition 1.8.3. Un *triplet de Higgs* de rang r , degré d et cadre \mathcal{F} est la donnée $(\mathcal{E}, \alpha, \theta)$ d'un faisceau $\mathcal{E} \in \text{Coh}_{r,d} C$, d'un morphisme $\alpha : \mathcal{F} \rightarrow \mathcal{E}$, et d'une autre application $\theta \in \mathbb{E}xt^1(\mathcal{E}, (\mathcal{F} \xrightarrow{\alpha} \mathcal{E}) \otimes \omega)$.

Comme dans le cas de carquois, il existe une famille de conditions de stabilité des triplets.

Définition 1.8.4. Soit $(\mathcal{E}, \alpha, \theta)$ est un triplet de Higgs, et $\delta > 0$.

- Un sous-faisceau $\mathcal{E}' \subset \mathcal{E}$ est *compatible*, si $\text{Im } \alpha \subset \mathcal{E}'$ et $a(\theta) \in \text{Im } b$ pour les applications naturelles

$$\mathbb{E}xt^1(\mathcal{E}, (\mathcal{F} \xrightarrow{\alpha} \mathcal{E}) \otimes \omega) \xrightarrow{a} \mathbb{E}xt^1(\mathcal{E}', (\mathcal{F} \xrightarrow{\alpha} \mathcal{E}) \otimes \omega) \xleftarrow{b} \mathbb{E}xt^1(\mathcal{E}', (\mathcal{F} \xrightarrow{\alpha} \mathcal{E}') \otimes \omega).$$

- $(\mathcal{E}, \alpha, \theta)$ est δ -*(semi-)stable*, si pour tout sous-faisceau compatible $\mathcal{E}' \subset \mathcal{E}$ on a

$$\frac{\deg \mathcal{E}'}{\text{rg } \mathcal{E}'} (\leq) \frac{\deg \mathcal{E} - \delta}{\text{rg } \mathcal{E}} \text{ si } \text{Im } \alpha \subset \mathcal{E}'; \quad \frac{\deg \mathcal{E}' - \delta}{\text{rg } \mathcal{E}'} (\leq) \frac{\deg \mathcal{E} - \delta}{\text{rg } \mathcal{E}} \text{ sinon.}$$

- $(\mathcal{E}, \alpha, \theta)$ est ∞ -*stable*, s'il n'existe aucun sous-faisceau compatible $\mathcal{E}' \subset \mathcal{E}$, tel que $\text{Im } \alpha \subset \mathcal{E}'$.

Dans cette thèse, on traite le cas particulier de rang 0 et cadre $\mathcal{O}^{\oplus n}$. Avec telles restrictions, il n'y a qu'un seul choix de δ qui est bien défini, à savoir $\delta \rightarrow +\infty$. On appelle simplement ces triplets *stables*.

Théorème D. *L'espace de modules des triplets stables de degré d et de cadre $\mathcal{O}^{\oplus n}$ est représentée par une variété lisse $\mathcal{B}(d, n)$.*

Remarque 1.8.5. Il y a une autre raison de se restreindre à $\delta = +\infty$: c'est la seule condition de stabilité telle que les sous-objets des objets stables soient stables. Sans cette restriction, l'analogue de l'application $(\Psi_{d_1, d_2})_*$ utilisé dans la définition de l'action cesse d'être propre.

Théorème E. *Pour tout $n > 0$, l'espace $A.\mathcal{M}_n = \bigoplus_d A(\mathcal{B}(d, n))$ admet une action de $A\text{H}a_{0,C}$.*

Il se trouve que les variétés $\mathcal{B}(d, n)$ admettent une autre description, dans l'esprit de la correspondance de Beauville-Narasimhan-Ramanan [6] et du théorème 1.4.1. Soit $S = \mathbb{P}(T^*C)$, et $\pi : S \rightarrow C$ la projection naturelle. On dit qu'un faisceau sans torsion E sur S est *f-semi-stable* si sa restriction à la fibre générique de π est isomorphe à $\mathcal{O}(i)^{\oplus k}$ pour $i \in \mathbb{Z}$, $k \in \mathbb{N}$.

Théorème F. *La variété $\mathcal{B}(d, n)$ est isomorphe à l'espace de modules $M_f(n, 0, d)$ qui paramètre les faisceaux sans torsion f -semistables E sur S munis d'une trivialisaton à l'infini, tels que $\text{rg } E = n$, $c_1(E) = 0$, et $c_2(E) = d$. En particulier, $\mathcal{B}(d, 1) \simeq \text{Hilb}_d T^*C$.*

L'algèbre $A\mathbf{Ha}_{0,C}$ contient les opérateurs de Nakajima $P_\alpha[1]$ pour les schémas de Hilbert, et conjecturalement les autres opérateurs $P_\alpha[i]$, $i \in \mathbb{N}$. La difficulté de cette conjecture réside essentiellement dans la géométrie des champs

$$[\{(X, Y) \in \mathcal{N}_{\text{gl}_d} \mid [X, Y] = 0\} / GL_d]$$

pour $d > 1$.

Cette description nous permet d'établir le rapport entre cette thèse et les travaux récents de Neguț [60]. Pour toute surface projective lisse S et H un diviseur nef sur S , il considère l'espace de modules \mathcal{M} des faisceaux H -stables sur S avec la seconde classe de Chern qui varie, et définit les opérateurs $e_n : K(\mathcal{M}) \rightarrow K(\mathcal{M} \times S)$ par les correspondances de Hecke pour tout $n \in \mathbb{Z}$. Ces opérateurs engendrent une sous-algèbre \mathcal{A} dans $\bigoplus_{k>0} \text{Hom}(K(\mathcal{M}) \rightarrow K(\mathcal{M} \times S^k))$, qui se projette dans une algèbre de battage $\mathcal{V}_{sm}(S)$. On conjecture que l'action de \mathcal{A} sur $K(\mathcal{M})$ se factorise par cette projection. L'algèbre $\mathcal{V}_{sm}(S)$ est définie pour toute S surface lisse *quasi*-projective, et coïncide avec $K\mathbf{Sh}_C$ pour $S = T^*C$. Alors les résultats des théorèmes C, E, F impliquent une version "homologique non-compacte" de cette conjecture pour un certain choix de H .

1.9 Développements futurs

Surface quelconque. En parallèle avec les travaux de Neguț, on peut étudier le champ des faisceaux de longueur finie sur une surface lisse S . Il apparaît qu'on peut munir les A -groupes de ce champ d'un produit associatif pour toute surface projective en utilisant les techniques de Álvarez-Cónsul et King [2]. De plus, on peut aussi obtenir une présentation de battage similaire au théorème B. Dans le cas $S = T^*C$ on espère prouver l'injectivité en étudiant l'action de $A\mathbf{Ha}_C^0$ sur les modules $A\mathcal{M}_n$. Par contre, pour une surface générale il n'y a pas de candidat clair pour l'analogie des variétés $\mathcal{M}(d, n)$ permettant de généraliser un tel argument.

CoHA totale et AGT. Comme mentionné plus en avant, l'algèbre $A\mathbf{Ha}_C^0$ n'est qu'une partie d'une algèbre plus grosse $A\mathbf{Ha}_C$, qui tient compte des faisceaux de Higgs dans tous les rangs. Par analogie avec le théorème E, on doit avoir une structure de $A\mathbf{Ha}_C$ -module sur les A -groupes des espaces de modules de triplets ∞ -stables de rang quelconque. De plus, on doit pouvoir munir $A\mathbf{Ha}_C$ d'une structure d'algèbre de Hopf (topologique) de telle façon que cette action s'étend à une action du double de Drinfeld $D(A\mathbf{Ha}_C)$. Il a été montré dans [56] que les opérateurs Ext, qui apparaissent dans la correspondance AGT, ne sont pas déterminés par son interaction avec $A\mathbf{Ha}_C^0$. On peut se demander alors si les interactions avec toute l'algèbre $D(A\mathbf{Ha}_C)$ fournissent assez de contrôle sur ces opérateurs.

Filtration perverse. Il a été montré dans [64] qu'on peut induire le produit de $A\mathbf{Ha}_C$ au sous-champ ouvert $\underline{\text{Higgs}}_{r,d}^{ss}$ des fibrés de Higgs semi-stables. Son espace de modules grossier admet une application vers l'espace affine $\bigoplus_{k=1}^r H^0(C, \omega_C^{\otimes k})$, obtenue en prenant le polynôme caractéristique du champ de Higgs. Cette application induit une filtration P_\bullet sur $H^*(\underline{\text{Higgs}}_{r,d}^{ss})$, appelée la *filtration perverse*. Heuristiquement, P_\bullet doit bien se comporter par rapport au produit de Hall. Une manière de formuler la conjecture $P = W$ de de Cataldo, Hausel et Migliorini est de dire que la filtration perverse est multiplicative par rapport au produit homologique. Sans rentrer dans des détails inutiles, l'existence d'un autre produit compatible avec P_\bullet peut nous servir pour déduire un résultat similaire concernant le produit homologique.

Chapitre 2

Cohomological Hall algebras for Higgs torsion sheaves

2.1 Introduction

Let \mathcal{C} be a hereditary abelian category over finite field \mathbb{F}_q , such that all Hom- and Ext-spaces have finite dimension. We have two important examples of such categories :

- for a finite quiver Q , the category of finite dimensional representations $\text{Rep } Q = \text{Rep}_{\mathbb{F}_q} Q$;
- for a smooth projective curve C over \mathbb{F}_q , the category of coherent sheaves $\text{Coh } C$.

Given a category \mathcal{C} satisfying the conditions above, one can associate to it the *Hall algebra* $\mathcal{H}(\mathcal{C})$, as defined in [65]. Broadly speaking, its basis is given by isomorphism classes of objects in \mathcal{C} , and the product is given by the sum of all non-isomorphic extensions. In the case $\mathcal{C} = \text{Rep } Q$, where Q is a quiver of Dynkin type, a famous theorem by Ringel [63] describes the Hall algebra $\mathcal{H}(\text{Rep } Q)$ as the positive half of the quantum group $U_\nu(\mathfrak{g}_Q)$, specialized at $\nu = q^{1/2}$. Moreover, one can upgrade $\mathcal{H}(\mathcal{C})$ into a (twisted, topological) bialgebra, such that the Drinfeld double $D(\mathcal{H}(\text{Rep } Q))$ is isomorphic to the quantum group itself.

By contrast, the Hall algebra $\mathcal{H}(\text{Coh } C)$ seems to be far less understood. For instance, an explicit description of (the spherical part of) $\mathcal{H}(\text{Coh } C)$ by generators and relations is known only when C is rational [36] or elliptic [10]. Our principal motivation is to get a better understanding of this algebra. One way to do it is to study its representation theory. Unfortunately, since we do not possess an explicit combinatorial description of $\mathcal{H}(\text{Coh } C)$ in terms of generators and relations (see, however, [65, Section 4.11] for partial results), we have to construct its representations indirectly.

We use an approach close in spirit to the well-known construction of Nakajima [53], which realizes irreducible representations of the universal enveloping algebra $U(\mathfrak{g})$ of a simple Lie algebra \mathfrak{g} as homology groups of certain varieties. Let us summarize a variant of this construction, following the point of view from [76]. Namely, for a finite type quiver $Q = (I, E)$ and a projective $\mathbb{C}Q$ -module P with top of graded dimension $\mathbf{w} \in \mathbb{Z}_+^I$, one considers the algebraic stack $T^* \text{Rep}_{\mathbf{v}}^{\leftarrow P} Q$, where $\text{Rep}_{\mathbf{v}}^{\leftarrow P} Q$ parametrizes pairs (V, φ) with $V \in \text{Rep } Q$, $\underline{\dim} V = \mathbf{v}$, and $\varphi \in \text{Hom}_{\mathbb{C}Q}(P, V)$. The \mathbb{C} -points of this stack can be identified with representations of a quiver \overline{Q}^\heartsuit , satisfying certain conditions [25, Section 5]. For every dimension vector $\mathbf{v} \in \mathbb{Z}_+^I$, one introduces a stability condition on these representations, such that subrepresentations of stable representations are stable, and the moduli stack of stable representations forms a smooth variety $\mathcal{M}(\mathbf{v}, \mathbf{w})$. Inside these varieties, one has Lagrangian subvarieties $\mathcal{L}(\mathbf{v}, \mathbf{w}) := \pi_{\mathbf{v}}^{-1}(0)$, where $\pi_{\mathbf{v}} : \mathcal{M}(\mathbf{v}, \mathbf{w}) \rightarrow \text{Spec } \Gamma(\mathcal{O}_{\mathcal{M}(\mathbf{v}, \mathbf{w})})$ is the affinization map. Finally, one considers a correspondence $Z_{\mathbf{v}} \subset (T^* \text{Rep}_{\mathbf{v}} Q \times M(\mathbf{v}_1, \mathbf{w})) \times \mathcal{M}(\mathbf{v} + \mathbf{v}_1, \mathbf{w})$, which parametrizes triples (V, V_1, V_2) with $V_2/V_1 \simeq V$. Denoting the projections on the first and second factor by $\Phi_{\mathbf{v}}$ and $\Psi_{\mathbf{v}}$ correspon-

dingly, we have the following operators in Borel-Moore homology :

$$\begin{aligned} e_{i,\mathbf{v}} &= (\Psi_{\epsilon_i})_*(\Phi_{\epsilon_i})^*([T^* \text{Rep}_{\epsilon_i} Q] \boxtimes -) : H(\mathcal{M}(\mathbf{v}, \mathbf{w})) \rightarrow \mathcal{M}(\mathbf{v} + \epsilon_i, \mathbf{w}), \\ f_{i,\mathbf{v}} &= \langle (\Phi_{\epsilon_i})_*(\Psi_{\epsilon_i})^*(-), [T^* \text{Rep}_{\epsilon_i} Q] \rangle : H(\mathcal{M}(\mathbf{v}, \mathbf{w})) \rightarrow \mathcal{M}(\mathbf{v} - \epsilon_i, \mathbf{w}), \end{aligned}$$

where ϵ_i is the dimension vector of the simple representation at vertex $i \in I$. Then $e_i := \sum_{\mathbf{v}} e_{i,\mathbf{v}}$, $f_i := \sum_{\mathbf{v}} f_{i,\mathbf{v}}$ give rise to an action of $U(\mathfrak{g}_Q)$ on $M_{\mathbf{w}} = \bigoplus_{\mathbf{v}} H(\mathcal{M}(\mathbf{v}, \mathbf{w}))$, and moreover its restriction to $\bigoplus_{\mathbf{v}} H(\mathcal{L}(\mathbf{v}, \mathbf{w}))$ is the irreducible highest module of weight \mathbf{w} .

In fact, this action can be extended to a much bigger algebra, so-called Yangian. This can be achieved by realizing it inside the *cohomological Hall algebra* [68, 76], isomorphic to $\bigoplus_{\mathbf{v}} H(T^* \text{Rep}_{\mathbf{v}} Q)$ as a vector space (see [47] for another perspective on Yangians). The latter algebra then acts on $M_{\mathbf{w}}$ by correspondences similar to the ones described above. The purpose of this paper is to begin investigation of analogous algebras and their representations in the context of curves.

In order to apply the same set of ideas to our situation, we have to introduce several modifications to our context. First, we have to consider $T^* \underline{\text{Coh}} C$ instead of $\underline{\text{Coh}} C$; note that the former stack is isomorphic to the stack of Higgs sheaves $\underline{\text{Higgs}} C$. Secondly, we will study a homological version of Hall algebra. It will be modeled on the vector space $A(\underline{\text{Higgs}} C)$, where A is either Borel-Moore homology or an arbitrary free oriented Borel-Moore homology theory (see [46, Chapter 5] for the definition of the latter).

Optimistically speaking, our program is as follows :

1. construct a (bi-)algebra structure $A\mathbf{Ha}_C$ on $A(\underline{\text{Higgs}} C)$;
2. define a suitable stability condition on $T^* \underline{\text{Coh}}^{\leftarrow \mathcal{F}} C$, where $\underline{\text{Coh}}^{\leftarrow \mathcal{F}} C$ is the stack of pairs (\mathcal{E}, α) with $\mathcal{E} \in \text{Coh} C$, $\alpha \in \text{Hom}(\mathcal{F}, \mathcal{E})$;
3. construct an action of the Drinfeld double $D(A\mathbf{Ha}_C)$ on the A -theory $A(\mathcal{M})$ of the moduli of stable objects.

In the present article, we treat a very particular case of the plan above. Namely, we restrict our attention to the category of *torsion* sheaves on C . Then, we have the following result :

Theorem 2.1.1. *There exists an associative product on $\bigoplus_d A(\underline{\text{Higgs}}_{0,d} C)$, which makes it into an algebra $A\mathbf{Ha}_{0,C}$ (Theorem 2.4.2).*

Furthermore, we choose our framing to be a free sheaf : $\mathcal{F} = \mathcal{O}^{\oplus n}$.

Definition. A *stable Higgs triple* of rank 0, degree d and weight n is a triple $(\mathcal{E}, \alpha, \theta)$ with $\mathcal{E} \in \text{Coh}_{0,d} C$, $\alpha \in \text{Hom}(\mathbb{k}^n \otimes \mathcal{O}, \mathcal{E})$, $\theta \in \text{Ext}^1(\mathcal{E}, (\mathbb{k}^n \otimes \mathcal{O} \xrightarrow{\alpha} \mathcal{E}) \otimes \omega)$, such that for each proper subsheaf $\mathcal{E}' \subset \mathcal{E}$ for which θ restricts (see Definition 2.3.10) to \mathcal{E}' , we have $\text{Im } \alpha \not\subset \mathcal{E}'$.

Theorem 2.1.2. *Let C be a smooth projective curve, and d, n positive integers.*

1. *The moduli of stable Higgs triples of degree d and weight n is represented by a smooth variety $\mathcal{B}(d, n)$ (Theorem 2.6.10);*
2. *for any n , the space $A\mathcal{M}_n = \bigoplus_d A(\mathcal{B}(d, n))$ is equipped with a structure of an $A\mathbf{Ha}_{0,C}$ -module (Corollary 2.7.2).*

Note that we do *not* construct a coproduct on $A\mathbf{Ha}_{0,C}$. However, if we denote by $A\mathbf{Ha}_{0,C}^T$ the version of $A\mathbf{Ha}_{0,C}$ equivariant with respect to the scaling action of \mathbb{G}_m on the cotangent fibers, one can define a certain algebra $A\mathbf{Sh}_C$ with explicit formulas for multiplication and construct a map $\rho : A\mathbf{Ha}_{0,C}^T \rightarrow A\mathbf{Sh}_C$. Roughly speaking, $A\mathbf{Sh}_C$ looks like the space of formal series with coefficients in $A(C)$, and the product is given by twisted symmetrization (see Definition 2.5.3). We expect ρ to be injective (Conjecture 2.8.11). This prediction is supported by the following theorem :

Theorem 2.1.3. *If $A = H$ are the usual Borel-Moore homology groups, the map $\rho : H\mathbf{Ha}_{0,C}^T \rightarrow H\mathbf{Sh}_C$ is injective (Corollary 2.8.5).*

If the conjecture is true, this map can be used to find relations in $A\mathbf{Ha}_{0,C}^T$ via direct computations, and also to transport a natural coproduct from $A\mathbf{Sh}_C$. Unfortunately, it is not entirely clear how to extend a $A\mathbf{Ha}_{0,C}^T$ -module structure on $A\mathcal{M}_n^T$ to a Yetter-Drinfeld module [62] with respect to such coproduct. Still, the isomorphism $\mathcal{B}(d, 1) \simeq \text{Hilb}_d(T^*C)$ suggests that $A\mathcal{M}_n^T$ should admit an action of $A\mathbf{Ha}_{0,C}^T$, similar to [54, Chapter 8].

In order to prove the results stated above, we adapt the techniques found in [69, 76]. Because of our restrictions on rank of sheaves and a particular choice of \mathcal{F} , all stacks we consider can be explicitly realized as global quotients, and thus we can forget about their stacky nature and work with equivariant A -theory of their atlases instead. In positive rank the stack $\underline{\text{Coh}}_{r,d}$ is only *locally* a quotient stack, which entails certain technical difficulties, and thus renders the application of analogous methods much less direct. Still, this can be done with enough determination, see [64]. Unfortunately, our geometric methods break down completely if we try to define a coproduct.

In higher rank, we expect the moduli of stable Higgs triples to be closely related to the moduli of sheaves on $\mathbb{P}(T^*C)$ framed at infinity. For instance, we have the following theorem :

Theorem 2.1.4. *The variety $\mathcal{B}(d, n)$ is isomorphic to the moduli space of f -semisimple torsion-free sheaves, equipped with a trivialization at infinity and satisfying certain numerical conditions. In particular, $\mathcal{B}(d, 1)$ is isomorphic to the Hilbert scheme of points $\text{Hilb}_d T^*C$ (Corollary 2.9.5).*

We refer the reader to Section 2.9 for definitions and a precise statement. Similar objects appear in the works of Neğüt [60, 57], where for any smooth projective surface S he defines an action of a certain \mathcal{W} -algebra on K -theory of moduli of stable sheaves on S . We expect that for $S = T^*C$, these algebras get embedded into a suitable completion of $K\mathbf{Ha}_{0,C}$. In general, since Higgs sheaves on C can be thought of as coherent sheaves with proper support on T^*C via BNR-correspondence [6], one can imagine a much more general picture :

Guiding principle. *Let S be a smooth projective surface together with a smooth divisor $D \subset S$. Denote by $\underline{\text{Coh}}(S, D)$ the stack of \mathcal{O}_S -modules with support disjoint from D , and by $\underline{\text{Coh}}_0(S, D)$ its substack of \mathcal{O}_S -modules of finite length. Then $A\mathbf{Ha}_S = A(\underline{\text{Coh}}(S, D))$ should admit a Hall-like structure of an associative algebra, such that $A\mathbf{Ha}_{0,S} = A(\underline{\text{Coh}}_0(S, D))$ is a subalgebra containing A -theoretic \mathcal{W} -algebra. Furthermore, $A\mathbf{Ha}_S$ should act on the A -theory of stable sheaves on S framed at D , for a certain stability condition.*

For this principle to hold true, one will certainly need additional technical assumptions, such as transversality of the divisor defining stability condition with D . However, this discussion reaches far beyond the scope of this article.

Let us finish the introduction with a brief outline of the structure of the paper. In Section 2.2 we recall the notion of oriented Borel-Moore homology functor, following the monograph by Levine and Morel [46], and gather the statements necessary for our proofs. In particular, we adapt the localization theorem of Borel-Atiyah-Segal to this framework. In Section 2.3 we choose explicit presentations of $\underline{\text{Coh}}_{0,d}$ and $\underline{\text{Higgs}}_{0,d}$ as global quotient stacks, given by certain *Quot*-schemes. We also recollect basic facts about these schemes. In Section 2.4 we recall a construction introduced in works of Schiffmann and Vasserot, which permits us to define an associative product on $\bigoplus_d A(\underline{\text{Higgs}}_{0,d})$. In Section 2.5 we introduce *global shuffle algebras* $A\mathbf{Sh}_g$, prove that these algebras satisfy some quadratic relations, and obtain a shuffle presentation ρ of $A\mathbf{Ha}_C^{0,T}$ for a certain choice of g . The map ρ is obtained by localizing our product diagrams to the fixed point sets of a certain torus \mathbb{T} . In passing, we also propose a geometric interpretation of the difference between two types of shuffle product, appearing in literature in similar context (Corollary 2.5.10). In Section 2.6 we introduce the moduli stack of Higgs triples, and prove that an open substack of stable triples can be represented by a smooth variety $\mathcal{B}(d, n)$. In Section 2.7 we construct an action of $A\mathbf{Ha}_C^0$ on $\bigoplus_d A(\mathcal{B}(d, n))$, and discuss how it can be related to the classical action of Heisenberg algebra on cohomology groups of Hilbert schemes of points on T^*C [54]. In Section 2.8, we prove that for $A = H$ the shuffle presentation ρ is faithful. The

proof uses the scaling torus action and weight filtration in a crucial way, so that it cannot be easily translated to other homology theories. However, we conjecture that ρ is faithful for general A . Finally, in Section 2.9 we provide an alternative description of $\mathcal{B}(d, n)$ as a moduli of sheaves on $\mathbb{P}(T^*C)$. We also briefly describe the relation of our work to the \mathcal{W} -algebras of Neguț.

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Conventions :

We denote by Sch/\mathbb{k} the category of \mathbb{k} -schemes of finite type over an algebraically closed base field \mathbb{k} of characteristic 0; pt stands for the terminal object $\text{Spec } \mathbb{k} \in \text{Sch}/\mathbb{k}$. For any $X \in \text{Sch}/\mathbb{k}$, the category of coherent \mathcal{O}_X -sheaves is denoted by $\text{Coh } X$. We will usually denote coherent sheaves by calligraphic letters, and implicitly identify locally free sheaves with corresponding vector bundles. For any $\mathcal{E}, \mathcal{F} \in \text{Coh } X$, we write $\text{Ext}^i(\mathcal{E}, \mathcal{F})$ for Ext-functors, $\text{Hom} := \text{Ext}^0$, and $\mathcal{E}xt^i(\mathcal{E}, \mathcal{F})$ for Ext-sheaves, $\mathcal{H}om := \mathcal{E}xt^0$. More generally, for any two complexes of sheaves $\mathcal{E}^\bullet, \mathcal{F}^\bullet$ we denote by $\mathbb{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ the space of morphisms in the derived category $\mathcal{D}(\text{Coh } X)$, and $\mathbb{E}xt^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet) := \mathbb{H}om(\mathcal{E}^\bullet, \mathcal{F}^\bullet[i])$. All unshifted complexes throughout the text begin in degree 0 unless otherwise stated. Finally, we will liberally use the language of stacks; see [43] or [61] for background.

2.2 Oriented Borel-Moore homology theories

In this section we recall the notion of equivariant oriented Borel-Moore functor and recollect some of its properties. For a more detailed exposition, we refer the reader to the monograph [46] for a treatment of non-equivariant version, and to [32] for the equivariant case.

Definition 2.2.1. An *oriented Borel-Moore homology theory* A on Sch/\mathbb{k} (or OBM for short) is the data of :

1. for every object in $X \in \text{Sch}/\mathbb{k}$, a graded abelian group $A_*(X)$;
2. for every projective morphism $f : X \rightarrow Y$, a homomorphism $f_* : A_*(X) \rightarrow A_*(Y)$;
3. for every locally complete intersection (l.c.i. for short) morphism $g : X \rightarrow Y$ of relative dimension d , a homomorphism $f^* : A_*(Y) \rightarrow A_{*+d}(X)$;
4. an element $\mathbb{1} \in A_0(pt)$, and for any $X, Y \in \text{Sch}/\mathbb{k}$ a bilinear pairing

$$\begin{aligned} \times : A_*(X) \otimes A_*(Y) &\rightarrow A_*(X \times Y), \\ u \otimes v &\mapsto u \times v, \end{aligned}$$

which is associative, commutative and has $\mathbb{1}$ as unit;

satisfying the following conditions :

$$(BM0) \quad A_*(X_1 \sqcup X_2) = A_*(X_1) \oplus A_*(X_2);$$

$$(BM1) \quad \text{Id}_X^* = \text{Id}_{A_*(X)}, (f \circ g)^* = g^* \circ f^*;$$

$$(BM2) \quad g^* \circ f_* = f'_* \circ g'^*$$
 for any cartesian diagram with projective f , transversal to l.c.i. g :

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ \downarrow f' & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

(BM3) $(f \times g)_*(u' \times v') = f_*(u') \times g_*(v')$, $(f \times g)^*(u \times v) = f^*(u) \times g^*(v)$;

(PB) let $E \rightarrow X$ be a vector bundle of rank $n + 1$, $q : \mathbb{P}E \rightarrow X$ its projectivization, $\mathcal{O}(1) \rightarrow \mathbb{P}E$ tautological line bundle, and $s : \mathbb{P}E \rightarrow \mathcal{O}(1)$ its zero section. Then the map

$$\sum_{i=0}^n (s^* \circ s_*)^i \circ q^* : \bigoplus_{i=0}^n A_{*+i-n}(X) \rightarrow A_*(\mathbb{P}E)$$

is an isomorphism ;

(EH) let $p : V \rightarrow X$ be an affine fibration of rank n . Then $p^* : A_*(X) \rightarrow A_{*+n}(V)$ is an isomorphism ;

(CD) for $r, N > 0$, let $W = (\mathbb{P}^N)^r$, with $p_i : W \rightarrow \mathbb{P}^N$ being i -th projection. Let also $[X_0 : \dots : X_N]$ be homogeneous coordinates on \mathbb{P}^N , $n_1, \dots, n_r \in \mathbb{Z}$, and $i : E \rightarrow W$ the subscheme defined by $\prod_{i=1}^r p_i^*(X_N)^{n_i} = 0$. Then $i_* : A_*(E) \rightarrow A_*(W)$ is injective.

Given an OBM A , we can further define :

- for any smooth variety Y , set $A^*(Y) := A_{\dim Y - *}(Y)$. The map $\Delta_Y^* \circ \times : A^*(Y) \otimes A^*(Y) \rightarrow A^*(Y)$ defines an associative product on $A^*(Y)$;
- for any $f : Z \rightarrow Y$, we have the graph morphism $\Gamma_f = (f, \text{Id}_Z) : Z \rightarrow Y \times Z$, which is always a regular embedding. The map $\Gamma_f^* \circ \times : A^*(Y) \otimes A_*(Z) \rightarrow A_*(Z)$ defines a $A^{-*}(Y)$ -module structure on $A_*(Z)$;
- for any l.c.i. morphism $f : Y \rightarrow X$ and arbitrary morphism $g : Z \rightarrow X$, we have a *Gysin pullback* map

$$f^! = f_g^! : A_*(Z) \rightarrow A_*(Z \times_X Y).$$

It coincides with the usual pullback for $Z = X$, $g = \text{Id}_X$. We will also liberally replace f by its pullback $f' : Z \times_X Y \rightarrow Z$ in the notations ;

- for any line bundle $L \rightarrow X$, denote by $s : X \rightarrow L$ its zero section. We have a graded homomorphism

$$c_1(L) := s^* \circ s_* : A_*(X) \rightarrow A_{*-1}(X).$$

Moreover, let us consider any vector bundle $E \rightarrow X$ of rank n together with its projectivization $q : \mathbb{P}E \rightarrow X$ and the tautological line bundle $\mathcal{O}(1) \rightarrow \mathbb{P}E$. Then there exist unique homomorphisms $c_i(E) : A_*(X) \rightarrow A_{*-i}(X)$ for $i = \{0, \dots, n\}$, called *i -th Chern classes*, such that $c_0(E) = 1$ and

$$\sum_{i=0}^n (-1)^i c_1(\mathcal{O}(1))^{n-i} \circ q^* \circ c_i(E) = 0.$$

They satisfy all the usual properties of Chern classes (see [46, Proposition 4.1.15]). For a smooth variety X , the classes $c_i(E)$ can be realized by elements of $A^i(X)$;

- there exists a formal group law $F_A \in (u + v) + uvA^*(pt)[[u, v]]$ on $A^*(pt)$ such that for any $X \in \text{Sch}_{\mathbb{k}}$ and two line bundles L, M on X , we have

$$c_1(L \otimes M) = F_A(c_1(L), c_1(M)).$$

Since we never consider two OBMs at the same time, we will use multiplicative notation for group laws, and write $F_A(c_1(L), c_1(M)) = c_1(L) \star c_1(M)$.

In [46], Levine and Morel define and study algebraic cobordism theory Σ_* associated to the universal formal group law $(\mathbb{L}, F_{\mathbb{L}})$ on the Lazard ring \mathbb{L} . Let us call an OBM A *free* if the natural map $\Sigma_* \otimes_{\mathbb{L}_*} A^*(pt) \rightarrow A_*$ is an isomorphism. For this class of OBMs many properties will follow immediately after establishing them for Σ_* .

Example 2.2.2. Chow group functor CH_* and the Grothendieck group of coherent sheaves K_0 are free OBMs.

Note that usual Borel-Moore homology is *not* an OBM, because of the presence of odd-dimensional part. Moreover, even the even-dimensional part fails to be a free OBM, which prevents us from translating results found in [46] in a straightforward way. Still, all of the results we need can be proved in a similar way for the usual Borel-Moore homology. We will thus abuse the language somewhat and allude to it as a free OBM in the propositions below, giving separate proofs where needed; in the case of omitted proof, we give a separate reference.

For any reductive group G , free OBM A , and a G -variety X Heller and Malagón-López [32] define equivariant homology groups $A_*^G(X)$. Roughly speaking, the group G has a classifying space represented by a projective system $\{EG_N\}_{N \in \mathbb{N}}$ of G -varieties, and we set

$$A_*^G(X) = \varprojlim_N A_*(X \times_G EG_N).$$

For example, if $G = GL_d$, the varieties EG_N are just the Grassmanians $\text{Grass}^d(d, N)$. Most of the constructions mentioned above for ordinary OBMs can be extended to the equivariant ones.

Example 2.2.3. If $G = T$ is an algebraic torus of rank d , then we have

$$A_*^T(pt) = A_*(pt)[[c_1(t_1), \dots, c_1(t_d)]],$$

for some choice of basis t_1, \dots, t_d of the character lattice of T [76, Lemma 1.3].

Remark 2.2.4. One can observe that in the case of algebraic K -theory we get $K^T(pt) = \mathbb{Z}[1 - t_1^{-1}, \dots, 1 - t_d^{-1}]$, which is different from the usual ring of Laurent polynomials $\mathbb{Z}[[t_1^{\pm 1}, \dots, t_d^{\pm 1}]]$. However, the two become isomorphic after passing to completion. One can prove that this happens for any T -scheme X , using the argument in [4, Lemma 3.1] (the author would like to thank Gufang Zhao for pointing out this article).

From now on until the end of this section, let us fix a free OBM A . Moreover, since we're concerned neither with questions of integrality nor with homological grading, we will only consider the groups

$$A(X) := \sum_i A_i(X) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad A^G(X) := \sum_i A_i^G(X) \otimes_{\mathbb{Z}} \mathbb{Q}, \quad A_G(X) := \sum_i A_G^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 2.2.5. To the best of author's knowledge, the notion of oriented Borel-Moore homology theory isn't yet fully developed for arbitrary algebraic stacks. However, since all the stacks of interest in our paper are quotient stacks, we usually slightly abuse the notation and write

$$A([X/G]) := A^G(X)$$

for any quotient stack $[X/G]$ (see also [32, Proposition 27]).

Proposition 2.2.6 ([32, Theorem 26]). *Let $H \subset G$ be a closed subgroup, and X a G -variety. Then there exists a natural isomorphism*

$$\text{ind}_H^G : A^H(X) \xrightarrow{\sim} A^G((X \times G)/H),$$

where H acts on $X \times G$ diagonally.

Proposition 2.2.7 ([32, Theorem 33]). *Let G be a reductive simply connected algebraic group, $T \subset G$ a maximal torus with normalizer N , $W = N/T$ the Weyl group, and X a G -variety. Then W acts on $A^T(X)$, and we have a natural isomorphism*

$$A^G(X) \simeq A^T(X)^W.$$

Proposition 2.2.8. *Let Z be a non-reduced G -scheme, and denote by Z^{red} its reduction. Then the pushforward map along the natural embedding*

$$A^G(Z^{\text{red}}) \rightarrow A^G(Z)$$

is an isomorphism.

Proof. Follows from the definition of algebraic cobordism theory. For an explicit mention of this fact, see the proof of Proposition 3.4.1 in [46]. \square

Proposition 2.2.9 (Projection formula). *Let $f : X \rightarrow Y$ be a G -equivariant projective morphism of smooth varieties, $\beta \in A_*^G(X)$, and $\alpha \in A_G^*(Y)$. We have the following identity :*

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta.$$

Proof. See [46, Proposition 5.2.1] for non-equivariant version ; equivariant proof is completely analogous. \square

The following proposition holds only for universal OBMs, usual Borel-Moore homology not included.

Proposition 2.2.10 ([32, Theorem 20]). *Let $i : Z \rightarrow X$ be a closed equivariant embedding of G -varieties, and $j : U \rightarrow X$ the complementary open embedding. Then the sequence*

$$A^G(Z) \xrightarrow{i_*} A^G(X) \xrightarrow{j^*} A^G(U) \rightarrow 0$$

is exact.

For a G -equivariant vector bundle $E \rightarrow X$, we define its *Euler class* $e(E)$ as the top Chern class :

$$e(E) := c_{\text{rk } E}(E) \in A^G(X).$$

Proposition 2.2.11 (Self-intersection formula). *Let $i : N \hookrightarrow M$ be a G -equivariant regular embedding of smooth G -varieties. Then*

$$i^*i_*(c) = e(T_N M) \cdot c$$

for any $c \in A^G(N)$.

Proof. Follows from [46, Theorem 6.6.9]. For $A = H$, see [12, Corollary 2.6.44]. \square

Given G -equivariant regular embeddings $j : P \hookrightarrow N$, $i : N \hookrightarrow M$, Whitney product formula applied to the short exact sequence

$$0 \rightarrow T_P N \rightarrow T_P M \rightarrow j^*T_N M \rightarrow 0$$

tells us that

$$e(T_P M) = e(T_P N) \cdot j^*e(T_N M). \quad (2.1)$$

For our purposes, one of the most important pieces of data coming from an OBM is the Gysin pullback. Let us state several compatibility results about it.

Lemma 2.2.12. *The following properties of Gysin pullback are verified :*

1. *Gysin pullback commutes with composition, that is for any diagram with cartesian squares*

$$\begin{array}{ccccc} Z' & \longrightarrow & Y' & \longrightarrow & X' \\ \downarrow & & \downarrow g' & & \downarrow g \\ Z & \xrightarrow{f_2} & Y & \xrightarrow{f_1} & X \end{array}$$

one has $(f_1 \circ f_2)_g^! = (f_2)_{g'}^! \circ (f_1)_g^!$, provided that f_1 and f_2 are locally complete intersections ;

2. let $F : Y \rightarrow X$, $G : X' \rightarrow X$, $\iota : Z \rightarrow X$ be morphisms of schemes such that F is l.c.i, G and ι are proper, and F and G are transversal. Consider the following diagram, where all squares are cartesian :

$$\begin{array}{ccccc}
 Y' & \xrightarrow{\quad} & & \xrightarrow{\quad} & X' \\
 & \swarrow & & \searrow & \\
 & & W' & \xrightarrow{f'} & Z' \\
 & & \downarrow g' & & \downarrow g \\
 & & W & \xrightarrow{f} & Z \\
 & \swarrow & & \searrow & \\
 Y & \xrightarrow{\quad} & & \xrightarrow{F} & X \\
 & & & & \downarrow G \\
 & & & & X
 \end{array}$$

Then we have an equality

$$g'_* \circ f'_! = f_! \circ g_*.$$

Proof. See [46, Theorem 6.6.6(3)] for (1) and [76, Lemma 1.14] for (2). \square

Proposition 2.2.13. *Let $i : Y \hookrightarrow X$ be a closed embedding of T -varieties, and $\{\chi_1, \dots, \chi_k\} \subset T^\vee$ a finite set of characters. Suppose that X^T is not empty, $X^T \subset Y$, and for any point $x \in X \setminus Y$ its stabilizer under the action of T is contained in $\bigcup_{i=1}^k \text{Ker}(\chi_i)$. Then the pushforward along i induces an isomorphism*

$$i_* : A_*^T(Y)[c_1(\chi_1)^{-1}, \dots, c_1(\chi_k)^{-1}] \xrightarrow{\sim} A_*^T(X)[c_1(\chi_1)^{-1}, \dots, c_1(\chi_k)^{-1}].$$

Proof. In the interest of brevity, we will abuse the notation and write χ_i instead of $c_1(\chi_i)$. Let us start with surjectivity. By Proposition 2.2.10 we have an exact sequence

$$A^T(Y) \xrightarrow{i_*} A^T(X) \rightarrow A^T(X \setminus Y) \rightarrow 0.$$

Thus it suffices to prove that $A_*^T(X \setminus Y)[c_1(\chi_1)^{-1}, \dots, c_1(\chi_k)^{-1}] = 0$. By Lemma 2 in [16], there exists an open subvariety $U \subset X \setminus Y$ and a subgroup $T_1 \subset T$ such that $U \simeq \tilde{U} \times T/T_1$ as T -variety, where \tilde{U} is equipped with a trivial action of T . In particular, $A_*^T(U) \simeq A_*(\tilde{U}) \otimes_{A_*(pt)} A_*^T(T/T_1)$. Because of our hypotheses, one has $T_1 \subset \text{Ker}(\chi_i)$ for some i , and thus $c_1(\chi_i)A_*^T(U) = 0$. We conclude by Noetherian induction. Namely, let Z be the complement of U in $X \setminus Y$. We have the following exact sequence :

$$A^T(Z) \xrightarrow{i_*} A^T(X \setminus Y) \rightarrow A^T(U).$$

By induction $pA^T(Z) = 0$, where p is a monomial in χ_1, \dots, χ_k . Therefore $A^T(X)$ is annihilated by $\chi_i p$, and thus $A^T(X \setminus Y)[\chi_1^{-1}, \dots, \chi_k^{-1}] = 0$.

It is left to prove injectivity. If $A = H$, we may already conclude by invoking long exact sequence in homology. Otherwise, we follow an approach found in [9, 2.3, Corollary 2]. First, let us denote

$$Y_{\natural} = \overline{\left\{ y \in Y : \text{Stab}(y) \not\subset \bigcup_{i=1}^k \text{Ker}(\chi_i) \right\}}.$$

Note that Y_{\natural} is non-empty, since $X^T \subset Y_{\natural}$. We have the following commutative triangle

$$\begin{array}{ccc}
 A_*^T(Y_{\natural})[\chi_1^{-1}, \dots, \chi_k^{-1}] & \xrightarrow{\quad} & A_*^T(X)[\chi_1^{-1}, \dots, \chi_k^{-1}] \\
 & \searrow & \nearrow \\
 & A_*^T(Y)[\chi_1^{-1}, \dots, \chi_k^{-1}] &
 \end{array}$$

where the diagonal arrows are surjective by the first part of the proof. If the horizontal arrow is injective, the same can be said of the map $A_*^T(Y)[\chi_1^{-1}, \dots, \chi_k^{-1}] \rightarrow A_*^T(X)[\chi_1^{-1}, \dots, \chi_k^{-1}]$. Thus, from now on we'll assume that $Y = Y_{\natural}$.

Let $U \simeq \tilde{U} \times T/T_1 \subset X \setminus Y$ be as in the proof of surjectivity. Since $T_1 \subset \text{Ker}(\chi_i)$ for some i , we get a regular function

$$f : \tilde{U} \times T/T_1 \rightarrow \mathbb{k}^*, \quad (u, t) \mapsto \chi_i(t).$$

It extends to a rational function $f : X \rightarrow \mathbb{P}_{\mathbb{k}}^1$ with the property that $f(t.x) = \chi_i(t)f(x)$ for any $x \in X$ and $t \in T$. In particular, if $y \in Y$ and $t \in \text{Ker}(\chi_i) \setminus \text{Stab}_T(y)$ this equality becomes $\chi_i(t)f(y) = f(y)$, so that y belongs to the support of the divisor associated to f . Let us denote this support by D . Thus $Y \subset D$, and the map $A_*^T(D) \rightarrow A_*^T(X)$ becomes injective after inverting χ_i ; see [9] for details. We conclude by Noetherian induction on D . \square

For any commutative \mathbb{k} -algebra R and an R -module M , let M_{loc} be the localized $\text{Frac}(R)$ -module $\text{Frac}(R) \otimes_R M$.

Theorem 2.2.14 (Localization theorem). *Let T be an algebraic torus, $R = A_*^T(pt)$, X a T -variety, and $i_T : X^T \rightarrow X$ inclusion of the fixed point set. Suppose that X^T is not empty. Then the $\text{Frac}(R)$ -linear map*

$$i_{T*} : A_*^T(X^T)_{loc} \rightarrow A_*^T(X)_{loc}$$

is an isomorphism. Moreover, if X is smooth, then the map

$$i_T^* : A_*^T(X)_{loc} \rightarrow A_*^T(X^T)_{loc}$$

is an isomorphism as well.

Proof. Any action of an algebraic torus has finitely many distinct stabilizers. One can therefore assume that

$$\bigcup_{x \in X \setminus X^T} \text{Stab}_T(x) \subset \bigcup_{i=1}^k \text{Ker}(\chi_i)$$

for some choice of characters $\chi_1, \dots, \chi_k \in T^\vee$. Applying Proposition 2.2.13 to the embedding $X^T \hookrightarrow X$ and localizing Chern classes of all characters instead of chosen ones, we see that i_{T*} becomes an isomorphism after localization. In view of Proposition 2.2.11, it remains to prove that if X is smooth, then the Euler class $e(T_{X^T}X)$ is not a zero-divisor in $A_*^T(X^T)$. Since the T -action on X^T is trivial, we can decompose $T_{X^T}X$ into isotypical components :

$$T_{X^T}X = \bigoplus_i p_i \otimes E_i,$$

where p_i are non-trivial characters of T , and E_i are vector bundles on X^T . It suffices to assume $T_{X^T}X = p_1 \otimes E_1$, because Euler class is multiplicative with respect to direct sums. In the case when E_1 is a line bundle, we have

$$e(p_1 \otimes E_1) = c_1(p_1) \star c_1(E_1) = c_1(p_1) + c_1(E_1)(1 + c_1(p_1 + \dots)).$$

The class $c_1(p_1)$ is not a zero-divisor by Example 2.2.3, $c_1(E_1)$ is nilpotent by [46, Remark 5.2.9], therefore $e(p_1 \otimes E_1)$ is not a zero-divisor as well. Finally, the case when rank of E_1 is bigger than 1 can be reduced to the former by using axiom (PB) and the usual technique of Chern roots. \square

The following proposition describes the behavior of localization map with respect to pullbacks and pushforwards.

Proposition 2.2.15. *Let $f : X \rightarrow Y$ be a morphism of smooth T -varieties. Assume that the fixed point sets X^T, Y^T are non-empty, and consider the natural commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_X \uparrow & & \uparrow i_Y \\ X^T & \xrightarrow{f_T} & Y^T \end{array}$$

1. if f is l.c.i., then $i_X^* \circ f^* = f_T^* \circ i_Y^*$;
2. if f is projective, then the following diagram commutes :

$$\begin{array}{ccc} A^T(X)_{loc} & \xrightarrow{f^*} & A^T(Y)_{loc} \\ e(T_{X^T}^* X)^{-1} \otimes i_X^*(-) \downarrow & & \downarrow e(T_{Y^T}^* Y)^{-1} \otimes i_Y^*(-) \\ A^T(X^T)_{loc} & \xrightarrow{f_{T^*}} & A^T(Y^T)_{loc} \end{array}$$

Proof. First claim is obvious, and Proposition 2.2.11 coupled with the localization theorem proves the second claim as well. \square

Corollary 2.2.16. *Under conditions of Proposition 2.2.15(2), we have*

$$i_Y^* \circ f_*(c) = f_{T^*} (f_T^*(e(T_{Y^T} Y)) \cdot e(T_{X^T} X)^{-1} \cdot i_X^*(c))$$

for any $c \in A^T(X)_{loc}$. If, moreover, f is a regular embedding such that $X^T = Y^T$, then

$$i_Y^* \circ f_*(c) = i_X^*(e(T_X Y)c).$$

For the following proposition we fix a reductive group G , and let $T \subset H \subset P$ be a maximal torus, Levi and parabolic subgroup of G respectively. Denote by W and W_H the Weyl groups of T in G and H respectively; we also fix a representative σ for each class in W/W_H .

Proposition 2.2.17. *Let X be an H -variety, and denote $Y = G \times_P X$, where the action of P on X is given by the natural projection $P \rightarrow H$. Then $Y^T = W \times_{W_H} X^T$, and we have a commutative diagram*

$$\begin{array}{ccc} A^H(X) & \xrightarrow{\text{ind}_H^G} & A^G(Y) \\ i_X^* \downarrow & & \downarrow i_Y^* \\ A^T(X^T) & \xrightarrow{s^*} & A^T(W \times_{W_H} X^T) \end{array}$$

where $s : W \times_{W_H} X^T \rightarrow X^T$ is the projection associated to the choice of representatives σ .

Remark 2.2.18. For an arbitrary H -variety X , the action of normalizer $N_H(T)$ can be restricted to X^T , and thus induces an action of Weyl group W_H on $A^T(X^T)$. Moreover, the restriction map $A^T(X) \rightarrow A^T(X^T)$ can be seen to be W_H -equivariant. With that in mind, note that even though s^* depends on the choice of representatives σ , its restriction to the W_H -equivariant part $A^T(X^T)^{W_H}$, which contains the image of i_X^* , does not.

Proof. First, let us compute T -fixed points of Y . Let (g, x) be a point in $G \times X$. Then we have :

$$\begin{aligned} t.(g, x) = (g, x) \bmod P \quad \forall t \in T &\Leftrightarrow \forall t \in T \exists p \in P : tg = gp^{-1}, p.x = x \\ &\Leftrightarrow g^{-1}Tg \subset P, x \in X^{g^{-1}Tg} \\ &\Leftrightarrow g \in N_G(T) \cdot P, x \in X^{g^{-1}Tg} \\ &\Leftrightarrow \exists p' \in P : gp'^{-1} \in N_G(T), x \in p'.X^T \end{aligned}$$

Therefore $(g, x) \bmod P$ is T -stable iff $(g, x) \in (N_G(T)P) \times (p'.X^T)/P = W \times_{W_H} X^T$, which proves the first claim. Next, let $i_\xi : X \hookrightarrow Y$ be the inclusion of the fiber over $\xi \in G/P$. Note that by definition of ind_H^G , it is a right inverse to i_e^* . Therefore, G -equivariance implies that

$$i_{gP}^*(\text{ind}_H^G c) \simeq g.c \in A^{Hg}(X) \text{ for all } g \in G.$$

If we restrict all our structure groups to T and suppose that $g \in N(T)$, we get

$$i_\xi^*(\text{ind}_H^G c) \simeq g.c \in A^T(X) \text{ for all } \xi \in G/P$$

with the action of g on $A^T(X)$ is as in Proposition 2.2.7. Moreover, we have the following commutative square

$$\begin{array}{ccc} A^T(X) & \xrightarrow{g} & A^T(X) \\ \downarrow i_X^* & & \downarrow i_X^* \\ A^T(X^T) & \xrightarrow{g} & A^T(X^T) \end{array}$$

since the action of g on $A^T(X^T)$ is just the restriction of the action above. Finally,

$$\begin{aligned} i_Y^*(\text{ind}_H^G c) &= i_X^* \left(\sum_{\xi \in N_G(T)/N_H(T)} i_{\xi P}^* \right) (\text{ind}_H^G c) = i_X^* \left(\sum_{\xi \in N_G(T)/N_H(T)} \xi.c \right) \\ &= \sum_{\sigma \in W/W_H} \sigma.i_X^*(c) = s^*.i_X^*(c) \end{aligned}$$

for all $c \in A^H(X)$, and the second claim follows. \square

Remark 2.2.19. The same proof as above shows that $(G \times_P X)^T = W \times_{W_H} X^T$ for a P -variety X .

2.3 Coherent sheaves and Quot-schemes

Let C be a smooth projective curve defined over \mathbb{k} , and \mathcal{O} its structure sheaf. Then one can define the following algebraic stacks (over Sch/\mathbb{k} in étale topology) :

- $\underline{\text{Coh}}_{0,d}$, the stack of torsion sheaves on C of degree d [43, Théorème 4.6.2.1];
- for any $\mathcal{F} \in \text{Coh } C$, the stack $\underline{\text{Coh}}_{0,d}^{\leftarrow \mathcal{F}}$ of pairs $(\mathcal{E} \in \text{Coh}_{0,d} C, \alpha \in \text{Hom}(\mathcal{F}, \mathcal{E}))$ [24, Section 4.1];
- the cotangent stack $\underline{\text{Higgs}}_{0,d} := T^*\underline{\text{Coh}}_{0,d}$. It is defined as the relative Spec of the symmetric algebra of the tangent sheaf; see [43, Chapitre 14,17] for the relevant definitions.

All of the stacks above can be realized as global quotient stacks. Below we will make an explicit choice of such presentation for computational purposes.

Definition 2.3.1. Let $\underline{\text{Quot}}_{0,d}$ be the following functor :

$$\begin{aligned} \underline{\text{Quot}}_{0,d} : \text{Sch}/\mathbb{k} &\rightarrow \text{Set}^{op}, \\ S &\mapsto \left\{ \varphi : \mathbb{k}^d \otimes \mathcal{O}_{S \times C} \twoheadrightarrow \mathcal{E}_S \mid \begin{array}{l} \mathcal{E}_S \in \text{Coh}(S \times C), \varphi \text{ flat over } S, \\ \text{for any } s \in S, \text{rk } \mathcal{E}_s = 0 \text{ and } \deg \mathcal{E}_s = d. \end{array} \right\}, \\ (S' \xrightarrow{f} S) &\mapsto (\varphi \mapsto f^*\varphi). \end{aligned}$$

Moreover, let us consider its open subfunctor $\underline{\text{Quot}}_{0,d}^\circ \subset \underline{\text{Quot}}_{0,d}$, consisting of quotients

$$\varphi : \mathbb{k}^d \otimes \mathcal{O}_{S \times C} \twoheadrightarrow \mathcal{E}_S$$

such that the map $H^0(\varphi_s) : \mathbb{k}^d \rightarrow H^0(\mathcal{E}_s)$ is an isomorphism for all $s \in S$.

Let $G_d := GL_d(\mathbb{k})$. Note that G_d acts on $\underline{Quot}_{0,d}^\circ$ via linear transformations of \mathbb{k}^d .

Proposition 2.3.2. *Let $d > 0$ be an integer.*

1. $\underline{Quot}_{0,d}$ and $\underline{Quot}_{0,d}^\circ$ are representable by smooth schemes $Quot_{0,d}$ and $Quot_{0,d}^\circ$ respectively, and $Quot_{0,d}$ is a projective variety of dimension d^2 ;
2. we have an isomorphism $[Quot_{0,d}^\circ/G_d] \simeq \underline{Coh}_{0,d}$;
3. let $\mathcal{F} \simeq V \otimes \mathcal{O}$ for some finite dimensional vector space V . Then we have an isomorphism $[(\mathrm{Hom}(V, \mathbb{k}^d) \times Quot_{0,d}^\circ)/G_d] \simeq \underline{Coh}_{0,d}^{\leftarrow \mathcal{F}}$.

Proof. For (1), see [44]. The claim (2) follows from the observation that any torsion sheaf of degree d is generated by its global sections, and every isomorphism of torsion sheaves is completely determined by its action on global sections. Finally, for (3) let us consider the natural map $\underline{Coh}_{0,d}^{\leftarrow V \otimes \mathcal{O}} \rightarrow \underline{Coh}_{0,d}$. This is a vector bundle, which is trivialized in the atlas given by $Quot_{0,d}^\circ$:

$$\underline{Coh}_{0,d}^{\leftarrow V \otimes \mathcal{O}} \times_{\underline{Coh}_{0,d}} Quot_{0,d}^\circ \simeq \mathrm{Hom}(V, \mathbb{k}^d) \times Quot_{0,d}^\circ.$$

The statement of (2) then implies the desired isomorphism. \square

Remark 2.3.3. Note that (3) fails for other sheaves $\mathcal{F} \neq V \otimes \mathcal{O}$. In general we'd have to pick a certain closed subvariety out of $\mathrm{Hom}(\mathcal{F}, \mathbb{k}^d \otimes \mathcal{O}) \times Quot_{0,d}^\circ$; see [34] for more details.

Recall that for any algebraic group G and any smooth G -variety X the cotangent bundle T^*X is naturally equipped with a Hamiltonian G -action. Let $\mu : T^*X \rightarrow \mathfrak{g}^*$ be the corresponding moment map, where \mathfrak{g} is the Lie algebra of G , and put $T_G^*X := \mu^{-1}(0)$. Note that the infinitesimal G -action provides a morphism $\mathfrak{g} \otimes \mathcal{O}_X \xrightarrow{\mu^*} \mathcal{T}_X$, where \mathcal{T}_X is the tangent sheaf of X .

Lemma 2.3.4. *Let X be a smooth variety equipped with an action of G . Then we have a natural isomorphism of stacks $T^*[X/G] \simeq [T_G^*X/G]$.*

Proof. It follows from the definition of the moment map that the composition

$$T^*X = \mathrm{Spec} \mathrm{Sym}(\mathcal{T}_X) \xrightarrow{\mu^*} \mathrm{Spec} \mathrm{Sym}(\mathfrak{g} \otimes \mathcal{O}_X) = \mathfrak{g}^* \times X \xrightarrow{pr_1} \mathfrak{g}^*$$

coincides with μ . Therefore $\mathrm{Spec} \mathrm{Sym}(\mathrm{Coker} \mu^*) \simeq \mu^{-1}(0) = T_G^*X$, and we obtain the desired isomorphism after descending to $[X/G]$. \square

The lemma above implies that

$$\underline{Higgs}_{0,d} = T^*\underline{Coh}_{0,d} \simeq [T_G^*Quot_{0,d}^\circ/G].$$

Example 2.3.5. Let $d = 1$. As a set, $Quot_{0,1}(\mathbb{k}) = \{\mathcal{O} \xrightarrow{\varphi} \mathcal{F} \mid \mathcal{F} \in \mathrm{Coh}_{0,1}, \varphi \neq 0\}$. Note that since the $G_1 = \mathbb{G}_m$ -action is trivial here and $\mathrm{Hom}(\mathcal{O}, \mathcal{F}) = \mathbb{k}$ by Riemann-Roch, we actually have $Quot_{0,1} \simeq Quot_{0,1}^\circ \simeq C$, and $\underline{Coh}_{0,1} \simeq C \times \mathrm{BG}_m$. The corresponding universal family is given by $\mathcal{O}_\Delta \in \mathrm{Coh}(C \times C)$, where \mathcal{O}_Δ is the structure sheaf of the diagonal $\Delta \subset C \times C$. Moreover, $T_G^*Quot_{0,1} \simeq T^*Quot_{0,1} \simeq T^*C$, and thus $\underline{Higgs}_{0,1} \simeq T^*C \times \mathrm{BG}_m$.

Let us also define the following filtered version of $Quot_{0,d}^\circ$.

Definition 2.3.6. For any $d_\bullet = \{0 = d_0 \leq d_1 \leq \dots \leq d_k = d\}$, fix a filtration $\mathbb{k}^{d_1} \subset \dots \subset \mathbb{k}^d$. Denote by $\widetilde{Quot}_{0,d_\bullet}$ the subset of $Quot_{0,d}$ consisting of quotients $\mathbb{k}^d \otimes \mathcal{O} \xrightarrow{\varphi} \mathcal{E}$ such that the map $H^0(\varphi)|_{\mathbb{k}^{d_i} \otimes \mathcal{O}} : \mathbb{k}^{d_i} \rightarrow H^0(\mathrm{Im} \varphi|_{\mathbb{k}^{d_i} \otimes \mathcal{O}})$ is an isomorphism for any i .

We introduce the following notations for the later use :

$$G_{d_\bullet} = \prod_{i=1}^k G_{d_i - d_{i-1}}, \quad Quot_{0,d_\bullet} = \prod_{i=1}^k Quot_{0,d_i - d_{i-1}}, \quad Quot_{0,d_\bullet}^\circ = \prod_{i=1}^k Quot_{0,d_i - d_{i-1}}^\circ.$$

We also fix isomorphisms $\mathbb{k}^{d_i}/\mathbb{k}^{d_{i-1}} \simeq \mathbb{k}^{d_i - d_{i-1}}$ for each $i \in [1, k]$, so that $G_{d_i - d_{i-1}}$ gets identified with invertible maps in $\mathrm{Hom}(\mathbb{k}^{d_i - d_{i-1}}, \mathbb{k}^{d_i}/\mathbb{k}^{d_{i-1}})$, and quotients of $(\mathbb{k}^{d_i}/\mathbb{k}^{d_{i-1}}) \otimes \mathcal{O}$ are parametrized by the variety $Quot_{0,d_i - d_{i-1}}$.

Proposition 2.3.7. $\widetilde{\mathcal{Q}uot}_{0,d_\bullet}$ is a smooth closed subvariety in $\mathcal{Q}uot_{0,d}^\circ$.

Proof. In order to prove that $\widetilde{\mathcal{Q}uot}_{0,d_\bullet}$ is a closed subvariety of $\mathcal{Q}uot_{0,d}^\circ$, let us recall the construction of $\mathcal{Q}uot$ -schemes in [44, Chapter 4]. Namely, fix $n \gg 0$ and an ample line bundle $\mathcal{O}(1)$ on C . Denote by $H = H^0(\mathcal{O}(n))$ the space of global sections of $\mathcal{O}(n)$, $h = \dim H$. Let $\text{Grass}^d(\mathbb{k}^d \otimes H)$ be the Grassmanian of subspaces of codimension d in $\mathbb{k}^d \otimes H$, and consider the following map :

$$\begin{aligned} \mathcal{Q}uot_{0,d} &\rightarrow \text{Grass}^d(\mathbb{k}^d \otimes H), \\ \left(0 \rightarrow \mathcal{K} \rightarrow \mathbb{k}^d \otimes \mathcal{O} \xrightarrow{\varphi} \mathcal{E} \rightarrow 0\right) &\mapsto \left(H^0(\mathcal{K} \otimes \mathcal{O}(n)) \subset \mathbb{k}^d \otimes H\right). \end{aligned}$$

It is a closed embedding for n big enough. Now, for each quotient $\mathbb{k}^d \otimes \mathcal{O} \xrightarrow{\varphi} \mathcal{E}$ and for each $i \in [1, k]$ we have the restricted short exact sequence

$$0 \rightarrow \mathcal{K}_i \rightarrow \mathbb{k}^{d_i} \otimes \mathcal{O} \xrightarrow{\varphi_i} \mathcal{E}_i \rightarrow 0 \quad (2.2)$$

with $\mathcal{K}_i := \mathcal{K} \cap (\mathbb{k}^{d_i} \otimes \mathcal{O})$, $\varphi_i = \varphi|_{\mathbb{k}^{d_i} \otimes \mathcal{O}}$, and $\mathcal{E}_i = \text{Im } \varphi_i$. Since $H^0(\varphi)$ is an isomorphism, $H^0(\varphi_i)$ is injective for all i , and thus $h^0(\mathcal{E}_i) \geq d_i$ for all i ; moreover, φ belongs to $\widetilde{\mathcal{Q}uot}_{0,d_\bullet}$ precisely when all the previous inequalities turn into equalities. Tensoring (2.2) by $\mathcal{O}(n)$ and taking global sections, we get an exact sequence

$$0 \rightarrow H^0(\mathcal{K}_i(n)) \rightarrow \mathbb{k}^{d_i} \otimes H \rightarrow H^0(\mathcal{E}_i(n)) \rightarrow 0$$

for n big enough and all \mathcal{K}_i . Since \mathcal{E}_i is torsion sheaf, there exists an isomorphism $\mathcal{E}_i(n) \simeq \mathcal{E}_i$, and the exact sequence above implies that

$$\dim H^0(\mathcal{K}_i(n)) = \dim H^0(\mathcal{K}(n)) \cap (\mathbb{k}^{d_i} \otimes H) \leq d_i(h-1),$$

where the equality holds if and only if φ belongs to $\widetilde{\mathcal{Q}uot}_{0,d_\bullet}$. Therefore

$$\widetilde{\mathcal{Q}uot}_{0,d_\bullet} = \mathcal{Q}uot_{0,d}^\circ \cap \left\{ V \subset \mathbb{k}^d \otimes H \mid \begin{array}{l} \dim V = d(h-1), \\ \dim V \cap (\mathbb{k}^{d_i} \otimes H) \geq d_i(h-1) \end{array} \right\} \subset \text{Grass}^d(\mathbb{k}^d \otimes H).$$

The second set is closed in $\text{Grass}(\mathbb{k}^d \otimes H)$, and thus $\widetilde{\mathcal{Q}uot}_{0,d_\bullet}$ is closed in $\mathcal{Q}uot_{0,d}^\circ$ as well.

In order to prove that $\widetilde{\mathcal{Q}uot}_{0,d_\bullet}$ is smooth, consider the following diagram :

$$\begin{array}{ccc} & \widetilde{\mathcal{Q}uot}_{0,d_\bullet} \times G_{d_\bullet} & \\ & \swarrow p & \searrow q \\ \mathcal{Q}uot_{0,d_\bullet} \times G_{d_\bullet} & & \widetilde{\mathcal{Q}uot}_{0,d_\bullet} \end{array}$$

where we identify $G_{d_i-d_{i-1}}$ with invertible maps in $\text{Hom}(\mathbb{k}^{d_i-d_{i-1}}, \mathbb{k}^{d_i}/\mathbb{k}^{d_{i-1}})$, the map p sends $(\varphi, (g_i)_{i=1}^k)$ to $\left((g_i^*(\varphi|_{\mathbb{k}^{d_i}/\mathbb{k}^{d_{i-1}}}))_{i=1}^k, (g_i)_{i=1}^k\right)$, and q is the projection on the first coordinate. Note that we have the following map between short exact sequences for any point in $\widetilde{\mathcal{Q}uot}_{0,d_\bullet}$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{k}^{d_{i-1}} & \longrightarrow & \mathbb{k}^{d_i} & \longrightarrow & \mathbb{k}^{d_i}/\mathbb{k}^{d_{i-1}} \longrightarrow 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow H^0(\varphi|_{\mathbb{k}^{d_i}/\mathbb{k}^{d_{i-1}}}) \\ 0 & \longrightarrow & H^0(\mathcal{E}_{i-1}) & \longrightarrow & H^0(\mathcal{E}_i) & \longrightarrow & H^0(\mathcal{E}_i/\mathcal{E}_{i-1}) \longrightarrow 0 \end{array}$$

It follows that the map $\varphi|_{\mathbb{k}^{d_i}/\mathbb{k}^{d_{i-1}}} : (\mathbb{k}^{d_i}/\mathbb{k}^{d_{i-1}}) \otimes \mathcal{O} \rightarrow \mathcal{E}_i/\mathcal{E}_{i-1}$ induces an isomorphism on global sections, and thus the image of p belongs to $\mathcal{Q}uot_{0,d_\bullet}^\circ \times G_{d_\bullet}$. Moreover, the diagram above also implies that p is an affine fibration over $\mathcal{Q}uot_{0,d_\bullet}^\circ$ with fiber

$$\bigoplus_{i=1}^k \text{Hom}(\mathbb{k}^{d_i-d_{i-1}} \otimes \mathcal{O}, \mathcal{E}_{i-1}) = \bigoplus_{i=1}^k \mathbb{k}^{d_i-d_{i-1}} \otimes H^0(\mathcal{E}_{i-1})$$

of dimension $(d_i - d_{i-1})d_{i-1}$. Since $\mathcal{Q}uot_{0,d_\bullet}^\circ$ is smooth and q is a trivial G_{d_\bullet} -torsor, this observation implies the smoothness of $\widetilde{\mathcal{Q}uot}_{0,d_\bullet}$. \square

Remark 2.3.8. In the proof above we chose an n big enough so that all \mathcal{K} 's and \mathcal{K}_i 's cease to have higher cohomologies and become generated by global sections after tensoring by $\mathcal{O}(n)$. It is possible because all our sheaves live in bounded families (see [44]).

Notation. Throughout the paper, for any quotient $\mathbb{k}^d \otimes \mathcal{O} \xrightarrow{\varphi} \mathcal{E}$ in $\mathcal{Q}uot_{0,d}$ we will denote $\text{Ker } \varphi$ by \mathcal{K} , and the inclusion $\mathcal{K} \hookrightarrow \mathbb{k}^d \otimes \mathcal{O}$ by ι . We will also decorate \mathcal{K} , \mathcal{E} , φ and ι with appropriate indices and markings.

Next, we recall the description of tangent spaces of $\mathcal{Q}uot$ -schemes.

Proposition 2.3.9. *Let $\mathbb{k}^d \otimes \mathcal{O} \xrightarrow{\varphi} \mathcal{E}$ be a point in $\mathcal{Q}uot_{0,d}$, and let $\mathcal{K} = \text{Ker } \varphi$. Then the tangent space $T_\varphi \mathcal{Q}uot_{0,d}$ at φ is naturally isomorphic to $\text{Hom}(\mathcal{K}, \mathcal{E})$. Moreover, if $\varphi \in \widetilde{\mathcal{Q}uot}_{0,d_\bullet}$ we have*

$$T_\varphi \widetilde{\mathcal{Q}uot}_{0,d_\bullet} = \{v \in \text{Hom}(\mathcal{K}, \mathcal{E}) : v(\mathcal{K}_i) \subset \mathcal{E}_i \quad \forall i\}.$$

Proof. The proof of the first claim can be found in [44, ch. 8]. The second claim can be proved in a similar fashion, keeping track of the condition of admitting a sub-quotient throughout the proof of the first claim. \square

Because of this proposition, we will usually regard $T^* \mathcal{Q}uot_{0,d}^\circ$ as a variety, whose \mathbb{k} -points are identified with pairs

$$T^* \mathcal{Q}uot_{0,d}^\circ = \{(\varphi, \beta) \mid \varphi \in \mathcal{Q}uot_{0,d}^\circ, \beta \in \text{Hom}(\mathcal{K}, \mathcal{E})^*\}.$$

Definition 2.3.10. Let $0 \rightarrow \mathcal{K} \rightarrow \mathbb{k}^d \otimes \mathcal{O} \rightarrow \mathcal{E} \rightarrow 0$ be a point in $\mathcal{Q}uot_{0,d}$, and $\mathcal{E}' \subset \mathcal{E}$ be a subsheaf. An element $\beta \in \text{Hom}(\mathcal{K}, \mathcal{E})^*$ is said to *restrict to \mathcal{E}'* if $a(\beta) \subset \text{Im } b$ for the natural maps

$$\text{Hom}(\mathcal{K}, \mathcal{E})^* \xrightarrow{a} \text{Hom}(\mathcal{K}, \mathcal{E}')^* \xleftarrow{b} \text{Hom}(\mathcal{K}', \mathcal{E}')^*.$$

Consider the following commutative diagram for visual aid :

$$\begin{array}{ccccc} \text{Hom}(\mathcal{K}', \mathcal{E}/\mathcal{E}')^* & \longrightarrow & \text{Hom}(\mathcal{K}, \mathcal{E}/\mathcal{E}')^* & \longrightarrow & \text{Hom}(\mathcal{K}/\mathcal{K}', \mathcal{E}/\mathcal{E}')^* \\ \downarrow & & \downarrow & & \downarrow \\ \text{Hom}(\mathcal{K}', \mathcal{E})^* & \longrightarrow & \text{Hom}(\mathcal{K}, \mathcal{E})^* & \xrightarrow{c} & \text{Hom}(\mathcal{K}/\mathcal{K}', \mathcal{E})^* \\ \downarrow & & \downarrow \scriptstyle a & & \downarrow \\ \text{Hom}(\mathcal{K}', \mathcal{E}')^* & \xrightarrow{b} & \text{Hom}(\mathcal{K}, \mathcal{E}')^* & \longrightarrow & \text{Hom}(\mathcal{K}/\mathcal{K}', \mathcal{E}')^* \end{array}$$

Since locally free sheaves cannot be extended by torsion sheaves, all rows and columns are short exact sequences. Therefore we see that β restricts to \mathcal{E}' if and only if its image in $\text{Hom}(\mathcal{K}/\mathcal{K}', \mathcal{E}')^*$ vanishes. Moreover, for any such β we have $\text{Im } c \in \text{Hom}(\mathcal{K}/\mathcal{K}', \mathcal{E}/\mathcal{E}')^*$. We will denote $(b(\beta), c(\beta)) \in \text{Hom}(\mathcal{K}', \mathcal{E}')^* \times \text{Hom}(\mathcal{K}/\mathcal{K}', \mathcal{E}/\mathcal{E}')^*$ by $\text{gr}(\beta)$.

Additionally, let us define for later purposes the *nilpotent* part $(T_\varphi \widetilde{\mathcal{Q}uot}_{0,d_\bullet})^{\text{nilp}}$ of $T_\varphi \widetilde{\mathcal{Q}uot}_{0,d_\bullet}$:

$$(T_\varphi \widetilde{\mathcal{Q}uot}_{0,d_\bullet})^{\text{nilp}} := \{v \in \text{Hom}(\mathcal{K}, \mathcal{E}) : v(\mathcal{K}_i) \subset \mathcal{E}_{i-1} \quad \forall i\}. \quad (2.3)$$

Note that

$$\text{Hom}(\mathbb{k}^d \otimes \mathcal{O}, \mathcal{E}) \simeq \text{Hom}(\mathbb{k}^d, H^0(\mathcal{E})) \simeq \text{Hom}(\mathbb{k}^d, \mathbb{k}^d) =: \mathfrak{g}_d,$$

with the second isomorphism being induced by $H^0(\varphi) : \mathbb{k}^d \xrightarrow{\sim} H^0(\mathcal{E})$. In these terms the moment map for the G_d -action on $T^* \mathcal{Q}uot_{0,d}^\circ$ can be written as follows :

$$\begin{aligned} \mu : T^* \mathcal{Q}uot_{0,d}^\circ &\rightarrow \mathfrak{g}_d^*, \\ (\varphi, \theta) &\mapsto \iota^* \theta \in \text{Hom}(\mathbb{k}^d \otimes \mathcal{O}, \mathcal{E})^* \simeq \mathfrak{g}_d^*. \end{aligned}$$

Since \mathcal{E} is a torsion sheaf, we have $\text{Ext}^1(\mathbb{k}^d \otimes \mathcal{O}, \mathcal{E}) = 0$, and thus over each $\varphi \in \text{Quot}_{0,d}^\circ$ the restriction μ_φ of the map μ to $T_\varphi^* \text{Quot}_{0,d}^\circ$ can be embedded in a long exact sequence :

$$0 \rightarrow \text{Ext}^1(\mathcal{E}, \mathcal{E})^* \rightarrow \text{Hom}(\mathcal{K}, \mathcal{E})^* \xrightarrow{\mu_\varphi} \text{Hom}(\mathbb{k}^d \otimes \mathcal{O}, \mathcal{E})^* \rightarrow \text{Hom}(\mathcal{E}, \mathcal{E})^* \rightarrow 0.$$

This implies that $\mu_\varphi^{-1}(0) \simeq \text{Ext}^1(\mathcal{E}, \mathcal{E})^*$, and we get an identification on the level of \mathbb{k} -points

$$T_{G_d}^* \text{Quot}_{0,d}^\circ = \{(\varphi, \theta) : \varphi \in \text{Quot}_{0,d}^\circ, \theta \in \text{Ext}^1(\mathcal{E}, \mathcal{E})^*\} \subset T^* \text{Quot}_{0,d}^\circ.$$

Example 2.3.11. Let $C = \mathbb{A}^1$. Even though this curve is not projective, we can fix an isomorphism $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$, and define

$$\text{Quot}_{0,d}(\mathbb{A}^1) := \{\mathbb{k}^d \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\alpha} \mathcal{E} : \text{supp } \mathcal{E} \subset \mathbb{A}^1\} \subset \text{Quot}_{0,d}(\mathbb{P}^1).$$

Then the open subvariety

$$\text{Quot}_{0,d}^\circ(\mathbb{A}^1) = \text{Quot}_{0,d}(\mathbb{A}^1) \cap \text{Quot}_{0,d}^\circ(\mathbb{P}^1)$$

parametrizes quotients $\mathbb{k}^d[t] \xrightarrow{\alpha} V$, where V is a torsion $\mathbb{k}[t]$ -module of length d , and α induces an isomorphism of \mathbb{k} -vector spaces $\mathbb{k}^d t^0 \simeq V$. Equivalently, $\text{Quot}_{0,d}^\circ(\mathbb{A}^1)$ parametrizes linear operators on \mathbb{k}^d , that is

$$\text{Quot}_{0,d}^\circ(\mathbb{A}^1) \simeq \mathfrak{g}_d,$$

where the G_d -action on the left gets identified with the adjoint action on the right. Thus $T^* \text{Quot}_{0,d}^\circ(\mathbb{A}^1) \simeq (\mathfrak{g}_d)^2$, and $T_{G_d}^* \text{Quot}_{0,d}^\circ(\mathbb{A}^1)$ is isomorphic to the commuting variety $\mathcal{C}(\mathfrak{g}_d) = \{(u, v) \in (\mathfrak{g}_d)^2 : [u, v] = 0\}$.

In light of the example above, we will refer to $T_{G_d}^* \text{Quot}_{0,d}^\circ$ as the *commuting variety* of C , and denote it by $\mathcal{C}_d = \mathcal{C}_d(C)$. We will also write $\mathcal{C}_{d\bullet} = \mathcal{C}_{d\bullet}(C) := T_{G_d\bullet}^* \text{Quot}_{0,d}^\circ$.

Example 2.3.12. Let us fix a geometric point $x \in C(\mathbb{k})$, and consider the *punctual Quot*-scheme

$$\text{Quot}_{0,d}^\circ(x) := \{\mathbb{k}^d \otimes \mathcal{O} \xrightarrow{\alpha} \mathcal{E} : \text{supp } \mathcal{E} = x\} \subset \text{Quot}_{0,d}^\circ.$$

Such quotient is completely determined by its localization at x . More explicitly, since C is smooth, the completion $\hat{\mathcal{O}}_x$ is (non-canonically) isomorphic to $\mathbb{k}[[t]]$. The stalk of α at x is thus of the form $\mathbb{k}^d[[t]] \xrightarrow{\alpha_1} \mathcal{E}$, where \mathcal{E} is a $\mathbb{k}[[t]]$ -module, and α_1 induces an isomorphism of \mathbb{k} -vector spaces $\mathbb{k}^d t^0 \simeq \mathcal{E}$. Such quotient is in its turn uniquely determined by a nilpotent operator T on \mathbb{k}^d , the correspondence given by

$$T \in \text{End}(\mathbb{k}^d) \mapsto (0 \rightarrow \mathbb{k}^d[[t]] \xrightarrow{\iota} \mathbb{k}^d[[t]] \rightarrow \mathcal{E} \rightarrow 0), \quad \iota(vt^i) = (T.v)t^i - vt^{i+1}.$$

Thus we see that $\text{Quot}_{0,d}^\circ(x)$ is isomorphic to the nilpotent cone $\mathcal{N}_d \subset \mathfrak{g}_d$ together with the adjoint action of G_d . Moreover, under this identification the cotangent space $\text{Hom}(\mathcal{K}, \mathcal{E})^*$ in $\text{Quot}_{0,d}^\circ$ of a point α gets identified with \mathfrak{g}_d , and the restriction of the moment map $\mu : T^* \text{Quot}_{0,d}^\circ \rightarrow \mathfrak{g}_d$ to $\mathcal{N}_d \times \mathfrak{g}_d$ is the commutator. In particular,

$$\mathcal{C}_d \cap T^* \text{Quot}_{0,d}^\circ|_{\text{Quot}_{0,d}^\circ(x)} \simeq \mathcal{C}_{d\bullet}^{\text{n}\bullet} := \{(u, v) \in (\mathfrak{g}_d)^2 : [u, v] = 0, u \text{ nilpotent}\}.$$

2.4 The product

Let us once and for all fix a free oriented Borel-Moore homology theory (OBM) A ; for the definition and basic facts about this notion, see Appendix 2.2. As explained there, we abuse the notation somewhat and consider the usual Borel-Moore homology H_* as if it were a free OBM.

We also equip the cotangent bundle T^*C with an action of \mathbb{G}_m , given by dilations along the fibers; let us denote this torus by T .

We begin by recalling a general construction from [67]. Let G be an algebraic group with fixed Levi and parabolic subgroups $H \subset P$. Assume we are given smooth quasi-projective varieties X', Y, V , equipped with actions of G, H, P respectively, and H -equivariant morphisms

$$\begin{array}{ccc} & V & \\ p \swarrow & & \searrow q \\ Y & & X' \end{array}$$

such that p is an affine fibration and q is a closed embedding. Set $W = G \times_P V$, $X = G \times_P Y$, where the P -action on Y is induced by the natural projection $P \rightarrow H$, and consider the following maps of G -varieties :

$$\begin{array}{ccc} & W & \\ f \swarrow & & \searrow g \\ X & & X' \end{array} \quad \begin{array}{l} f : (g, v) \bmod P \mapsto (g, p.v) \bmod P, \\ g : (g, v) \bmod P \mapsto g.q(v). \end{array}$$

The map $(f, g) : W \rightarrow X \times X'$ is a closed embedding, so from now on we'll identify the smooth variety W with its image in $X \times X'$. Let $Z = T_W^*(X \times X')$ be the conormal bundle. Projections on factors define two maps :

$$\begin{array}{ccc} & Z & \\ \Phi \swarrow & & \searrow \Psi \\ T^*X & & T^*X' \end{array}$$

We denote $Z_G = Z \cap (T_G^*X \times T_G^*X')$. Then $\Phi^{-1}(T_G^*X) = \Psi^{-1}(T_G^*X') = Z_G$ [69, Lemma 7.3(b)], and we have the following induced diagram :

$$\begin{array}{ccccc} & & Z_G & & \\ & \Phi_G \swarrow & \downarrow & \searrow \Psi_G & \\ T_G^*X & & Z & & T_G^*X' \\ & \downarrow & \Phi \swarrow & \searrow \Psi & \downarrow \\ T^*X & & & & T^*X' \end{array} \tag{2.4}$$

Now, Ψ and Ψ_G are projective, Φ is an l.c.i. map, so that we get the following morphisms in A -groups :

$$\begin{aligned} (\Psi_G)_* &: A^G(Z_G) \rightarrow A^G(T_G^*X'), \\ \Phi^! &: A^G(T_G^*X) \rightarrow A^G(Z_G). \end{aligned}$$

By composing these two maps and using the induction isomorphism $A^H(T_H^*Y) \xrightarrow{\sim} A^G(T_G^*X)$ (see Proposition 2.2.6), we obtain a map

$$\Upsilon = (\Psi_G)_* \circ \Phi^! \circ \text{ind}_H^G : A^H(T_H^*Y) \rightarrow A^G(T_G^*X'). \tag{2.5}$$

Let us apply this general construction to a particular case of *Quot*-schemes of rank 0. Namely, let $d_\bullet = \{0 = d_0 \leq d_1 \leq \dots \leq d_k = d\}$, denote $G = G_d$, $H = G_{d_\bullet}$, and let $P = P_{d_\bullet}$ be the parabolic group preserving the flag $\mathbb{k}^{d_1} \subset \dots \subset \mathbb{k}^d$. We use Gothic letters $\mathfrak{g}, \mathfrak{p}, \mathfrak{h}$ for corresponding Lie algebras, and \mathfrak{p}_- for the parabolic algebra opposite to \mathfrak{p} . Next, put

$$Y = \text{Quot}_{0,d_\bullet}^\circ, \quad V = \widetilde{\text{Quot}}_{0,d_\bullet}, \quad X' = \text{Quot}_{0,d}^\circ.$$

By Proposition 2.3.7 we have a closed embedding $g : V \hookrightarrow X'$ and an affine fibration $f = \text{gr} : V \twoheadrightarrow Y$.

The following lemma will help us to identify all the terms in diagram (2.4).

Lemma 2.4.1. *Let G, P, H, X', X, V, Y be as above.*

1. *There exist natural isomorphisms of G -varieties*

$$T^*X' = T^*\text{Quot}_{0,d}^\circ, \quad Z = G \times_P \left\{ (\varphi, \theta) \in T^*\text{Quot}_{0,d}^\circ|_{\widetilde{\text{Quot}}_{0,d_\bullet}} : \theta|_{(T_\varphi\widetilde{\text{Quot}}_{0,d_\bullet})^{\text{nilp}}} = 0 \right\},$$

$$T^*X = G \times_P (\mathfrak{p}_-^* \times_{\mathfrak{h}^*} T^*\text{Quot}_{0,d_\bullet}^\circ) = G \times_P \{(x, (\varphi_\bullet, \theta_\bullet)) \in \mathfrak{p}_-^* \times T^*\text{Quot}_{0,d_\bullet}^\circ : x|_{\mathfrak{h}} = \mu(\varphi_\bullet, \theta_\bullet)\},$$

where $(T_\varphi\widetilde{\text{Quot}}_{0,d_\bullet})^{\text{nilp}}$ is defined as in (2.3). For each $(\varphi, \theta) \in T^*\widetilde{\text{Quot}}_{0,d_\bullet}$ we have

$$\Phi((g, \varphi, \theta) \bmod P) = (g, \mu(\varphi, \theta), \text{gr}(\varphi, \theta)) \bmod P,$$

$$\Psi((g, \varphi, \theta) \bmod P) = g \cdot (\varphi, \theta).$$

2. *There are isomorphisms of G -varieties*

$$T_G^*X' = \mathcal{C}_d, \quad T_G^*X = G \times_P \mathcal{C}_{d_\bullet}.$$

Proof. (1) The first isomorphism is obvious, so we start with T^*X :

$$T^*X = T^*(G \times_P \text{Quot}_{0,d_\bullet}^\circ) = T_P^*(G \times \text{Quot}_{0,d_\bullet}^\circ)/P$$

$$= \{(g, x, (\varphi_\bullet, \theta_\bullet)) \in G \times \mathfrak{g}^* \times T^*\text{Quot}_{0,d_\bullet}^\circ : g \cdot x|_{\mathfrak{p}} - \mu(\varphi_\bullet, \theta_\bullet) = 0 \in \mathfrak{p}^*/P\}$$

$$= G \times_P \{(x, (\varphi_\bullet, \theta_\bullet)) \in \mathfrak{p}_-^* \times T^*\text{Quot}_{0,d_\bullet}^\circ : x|_{\mathfrak{h}} = \mu(\varphi_\bullet, \theta_\bullet)\}.$$

Let us also note that the moment map $\mu : T^*X \rightarrow \mathfrak{g}^*$ is given by

$$\mu((g, x, \varphi_\bullet, \theta_\bullet) \bmod P) = g \cdot x.$$

By the same reasoning,

$$T^*(X \times X') = T^*(G \times_P (\text{Quot}_{0,d_\bullet}^\circ \times \text{Quot}_{0,d}^\circ)) = T_P^*(G \times \text{Quot}_{0,d_\bullet}^\circ \times \text{Quot}_{0,d}^\circ)/P$$

$$= G \times_P \{(x, (\varphi_\bullet, \theta_\bullet), (\varphi, \theta)) \in \mathfrak{g}^* \times T^*\text{Quot}_{0,d_\bullet}^\circ \times T^*\text{Quot}_{0,d}^\circ : x|_{\mathfrak{p}} = \mu(\varphi_\bullet, \theta_\bullet) + \mu(\varphi, \theta)|_{\mathfrak{p}}\}.$$

Next, let us compute $Z = T_W^*(X \times X')$. We have $W = G \times_P \widetilde{\text{Quot}}_{0,d_\bullet}$, and therefore

$$T^*W = T_P^*(G \times \widetilde{\text{Quot}}_{0,d_\bullet})/P$$

$$= \{(g, x, \varphi, \theta) \in G \times \mathfrak{g}^* \times T^*\widetilde{\text{Quot}}_{0,d_\bullet} : g \cdot x|_{\mathfrak{p}} - \mu(\varphi, \theta) = 0\}/P$$

$$= G \times_P \{(x, \varphi, \theta) \in \mathfrak{g}^* \times T^*\widetilde{\text{Quot}}_{0,d_\bullet} : x|_{\mathfrak{p}} = \mu(\varphi, \theta)\},$$

so that

$$T^*(X \times X')|_W = G \times_P \left\{ \begin{array}{l} (x, \varphi, \theta_\bullet, \theta) \in \mathfrak{g}^* \times \widetilde{\text{Quot}}_{0,d_\bullet} \times T_{\text{gr}\varphi}^*\text{Quot}_{0,d_\bullet}^\circ \times T_\varphi^*\text{Quot}_{0,d}^\circ \\ \text{such that } x|_{\mathfrak{p}} = \mu(\varphi_\bullet, \theta_\bullet) + \mu(\varphi, \theta)|_{\mathfrak{p}} \end{array} \right\}.$$

But the conormal bundle $T_W^*(X \times X')$ can be expressed as the kernel of the following map of vector bundles :

$$T^*(X \times X')|_W \rightarrow T^*W,$$

$$(g, x, \varphi, \theta_\bullet, \theta) \bmod P \mapsto (g, x, \varphi, \theta_\bullet - \theta|_{T_\varphi\widetilde{\text{Quot}}_{0,d_\bullet}}) \bmod P.$$

Therefore, we finally obtain

$$\begin{aligned} T_W^*(X \times X') &= G \times_P \{(\varphi, \theta_\bullet, \theta) \in \widetilde{\mathcal{Q}uot}_{0,d_\bullet} \times T_{\text{gr } \varphi}^* \mathcal{Q}uot_{0,d_\bullet}^\circ \times T_\varphi^* \mathcal{Q}uot_{0,d}^\circ : \theta|_{T_\varphi \widetilde{\mathcal{Q}uot}_{0,d_\bullet}} = \theta_\bullet\} \\ &\simeq G \times_P \left\{ (\varphi, \theta) \in \widetilde{\mathcal{Q}uot}_{0,d_\bullet} \times T_\varphi^* \mathcal{Q}uot_{0,d}^\circ : \theta|_{(T_\varphi \widetilde{\mathcal{Q}uot}_{0,d_\bullet})^{\text{nilp}}} = 0 \right\}. \end{aligned}$$

Note that the desired formula for Φ follows from the first equality, and the formula for Ψ is evident. The claim (2) follows from the explicit descriptions of moment maps $T^*X' \rightarrow \mathfrak{g}^*$, $T^*X \rightarrow \mathfrak{g}^*$. \square

The general construction thus produces a map

$$\Upsilon : A^H(\mathcal{C}_{d_\bullet}) \rightarrow A^G(\mathcal{C}_d).$$

For instance, if $k = 2$, then $H = G_{d'} \times G_{d''}$, $d = d' + d''$, and we get a bilinear map

$$m : A\mathbf{Ha}_C^0[d'] \otimes A\mathbf{Ha}_C^0[d''] \rightarrow A\mathbf{Ha}_C^0[d], \quad A\mathbf{Ha}_C^0[d] := A(\mathcal{H}iggs_{0,d}) = A^{G^d}(\mathcal{C}_d).$$

We denote $A\mathbf{Ha}_C^0 = \bigoplus_{d \geq 0} A\mathbf{Ha}_C^0[d]$, where $A\mathbf{Ha}_C^0[0] := A(pt)$.

Theorem 2.4.2. *($A\mathbf{Ha}_C^0, m$) is an associative algebra.*

Proof. We begin by introducing some notations. Let $d_1 \leq d_2 \leq d$, $d_\bullet = (d_1, d_2, d)$, $d'_\bullet = (d_1, d)$, $d''_\bullet = (d_2 - d_1, d - d_1)$, $G = G_d$, $P = P_{d_\bullet}$, $P' = P_{d'_\bullet}$. Define the following varieties :

$$\begin{aligned} X_1 &= G \times_P \mathcal{Q}uot_{0,d_\bullet}^\circ; & W_1 &= G \times_{P'} \widetilde{\mathcal{Q}uot}_{0,d'_\bullet}; \\ X_2 &= G \times_{P'} \mathcal{Q}uot_{0,d'_\bullet}^\circ; & W_2 &= G \times_P \widetilde{\mathcal{Q}uot}_{0,d_\bullet}; \\ X_3 &= \mathcal{Q}uot_{0,d}^\circ; & W_3 &= G \times_P (\mathcal{Q}uot_{0,d_1}^\circ \times \widetilde{\mathcal{Q}uot}_{0,d''_\bullet}). \end{aligned}$$

These varieties are $\mathcal{Q}uot$ - and $\widetilde{\mathcal{Q}uot}$ -bundles over certain partial flag varieties, so we may identify their \mathbb{k} -points as pairs (flag, quotient). Adopting “mod 3”-notation for indices, we have obvious inclusions $W_i \hookrightarrow X_{i-1} \times X_{i+1}$.

Lemma 2.4.3. *Using the notations above,*

1. *we have an isomorphism $W_2 \rightarrow W_1 \times_{X_2} W_3$;*
2. *the intersection $(W_1 \times X_1) \cap (X_3 \times W_3)$ in $X_3 \times X_2 \times X_1$ is transversal.*

Proof. First of all, we introduce a small abuse of notation. Namely, for any morphism of sheaves $\mathcal{E} \xrightarrow{f} \mathcal{F}$ and for any subsheaf $\mathcal{E}' \hookrightarrow \mathcal{E}$, the codomain of $H^0(f)|_{\mathcal{E}'}$ is assumed to be $H^0(f(\mathcal{E}'))$. With that in mind, we have

- $X_1 = \{(D_\bullet, (\varphi_i)_{i=1}^3) \mid \dim D_\bullet = d_\bullet, \varphi_i : D_i/D_{i-1} \otimes \mathcal{O} \rightarrow \mathcal{E}_i, H^0(\varphi_i) \text{ is an iso}, i \in \{1, 2, 3\}\}$;
- $X_2 = \{(D'_\bullet, (\varphi_i)_{i=1}^2) \mid \dim D'_\bullet = d'_\bullet, \varphi_i : D_i/D_{i-1} \otimes \mathcal{O} \rightarrow \mathcal{E}_i, H^0(\varphi_i) \text{ is an iso}, i \in \{1, 2\}\}$;
- $X_3 = \{\varphi \mid \mathbb{k}^d \otimes \mathcal{O} \rightarrow \mathcal{E}, H^0(\varphi) \text{ is an iso}\}$;
- $W_1 = \{(D'_\bullet, \varphi) \mid \dim D'_\bullet = d'_\bullet, \varphi : \mathbb{k}^d \otimes \mathcal{O} \rightarrow \mathcal{E}, H^0(\varphi)|_{D_i \otimes \mathcal{O}} \text{ is an iso}, i \in \{1, 2\}\}$;
- $W_2 = \{(D_\bullet, \varphi) \mid \dim D_\bullet = d_\bullet, \varphi : \mathbb{k}^d \otimes \mathcal{O} \rightarrow \mathcal{E}, H^0(\varphi)|_{D_i \otimes \mathcal{O}} \text{ is an iso}, i \in \{1, 2, 3\}\}$;
- $W_3 = \left\{ (D_\bullet, \varphi_1, \varphi_2) \mid \begin{array}{l} \dim D_\bullet = d_\bullet, \varphi_1 : D_1 \otimes \mathcal{O} \rightarrow \mathcal{E}_1, \varphi_2 : \mathbb{k}^d/D_1 \otimes \mathcal{O} \rightarrow \mathcal{E}_2, \\ H^0(\varphi_i), i \in \{1, 2\}, H^0(\varphi_2)|_{D_2/D_1 \otimes \mathcal{O}} \text{ are iso} \end{array} \right\}$.

Next, consider a commutative diagram

$$\begin{array}{ccc}
 X_1 & \xleftarrow{p_{31}} & W_3 \xrightarrow{p_{32}} X_2 \\
 & \searrow p_{21} & \uparrow \alpha_{23} \\
 & & W_2 \xrightarrow{\alpha_{21}} W_1 \\
 & & \searrow p_{23} \\
 & & X_3
 \end{array}
 \quad
 \begin{array}{l}
 p_{12} : (D'_\bullet, \varphi) \mapsto (D'_\bullet, \text{gr}' \varphi); \\
 p_{13} : (D'_\bullet, \varphi) \mapsto \varphi; \\
 p_{21} : (D_\bullet, \varphi) \mapsto (D_\bullet, \text{gr} \varphi); \\
 p_{23} : (D_\bullet, \varphi) \mapsto \varphi; \\
 p_{31} : (D_\bullet, \varphi_1, \varphi_2) \mapsto (D_\bullet, (\varphi_1, \text{gr}'' \varphi_2)); \\
 p_{32} : (D_\bullet, \varphi_1, \varphi_2) \mapsto (\text{fgt}(D_\bullet), (\varphi_1, \varphi_2)); \\
 \alpha_{21} : (D_\bullet, \varphi) \mapsto (\text{fgt}(D_\bullet), \varphi); \\
 \alpha_{23} : (D_\bullet, \varphi) \mapsto (D_\bullet, \text{gr}' \varphi),
 \end{array}$$

where $\text{gr} = \text{gr}_{d_\bullet} : \widetilde{\text{Quot}}_{0,d_\bullet} \rightarrow \text{Quot}_{0,d_\bullet}^\circ$ is the natural affine fibration, and

$$\begin{aligned}
 \text{fgt} : (D_1 \subset D_2 \subset E) &\mapsto (D_1 \subset E), \\
 \text{gr}' &= \text{gr}_{d'_\bullet}, \text{gr}'' = \text{gr}_{d''_\bullet}.
 \end{aligned}$$

(1) We have the following equalities on the level of \mathbf{k} -points :

$$\begin{aligned}
 W_1 \times_{X_2} W_3 &= \{((D'_\bullet, \varphi), (D_\bullet, \varphi_1, \varphi_2)) \mid \text{fgt } D_\bullet = D'_\bullet, (\varphi_1, \varphi_2) = \text{gr}' \varphi\} \\
 &= \{((\text{fgt } D_\bullet, \varphi), (D_\bullet, \text{gr}' \varphi))\} \subset W_1 \times W_3.
 \end{aligned}$$

The natural map

$$\begin{aligned}
 p &= (p_{23}, p_{21}) : W_2 \rightarrow W_1 \times_{X_2} W_3, \\
 (D_\bullet, \varphi) &\mapsto ((\text{fgt } D_\bullet, \varphi), (D_\bullet, \text{gr}' \varphi))
 \end{aligned}$$

can be thus seen to be a bijection. The fiber product $W_1 \times_{X_2} W_3$ is normal by [30, Proposition 6.14.1], W_2 is connected, therefore Zariski's main theorem implies that p is an isomorphism.

(2) To prove that our intersection is transversal, we need to show that for any $x \in (W_1 \times X_1) \cap (X_3 \times W_3) \simeq W_2$ there is an equality

$$T_x(W_1 \times X_1) \cap T_x(X_3 \times W_3) = T_x W_2.$$

By Proposition 2.3.9 we have the following isomorphisms :

$$T_\varphi \text{Quot}_{0,d} = \text{Hom}(\mathcal{K}, \mathcal{E}); \quad (2.6)$$

$$T_\varphi \text{Quot}_{0,d_\bullet} = \text{Hom}(\mathcal{K}_\bullet, \mathcal{E}_\bullet) := \bigoplus_i \text{Hom}(\mathcal{K}_i / \mathcal{K}_{i-1}, \mathcal{E}_i / \mathcal{E}_{i-1}); \quad (2.7)$$

$$T_\varphi \widetilde{\text{Quot}}_{0,d_\bullet} = \widetilde{\text{Hom}}_{d_\bullet}(\mathcal{K}, \mathcal{E}) := \{\theta \in \text{Hom}(\mathcal{K}, \mathcal{E}) : \theta(\mathcal{K}_i) \subset \mathcal{E}_i \text{ for all } i\}. \quad (2.8)$$

Let us fix a flag D_\bullet of dimension d_\bullet , and a quotient $\varphi \in \text{Quot}_{0,d}$ such that $(D_\bullet, \varphi) \in W_2$. Then the equalities above allow us to compute all the tangent spaces in question :

$$\begin{array}{ccc}
 T_{(D_\bullet, \text{gr}' \varphi)} X_1 & \leftarrow T_{(D_\bullet, \text{gr}' \varphi)} W_3 \rightarrow T_{(D'_\bullet, \text{gr}' \varphi)} X_2 \\
 & \searrow & \uparrow \\
 & & T_{(D'_\bullet, \varphi)} W_1 \\
 & \uparrow & \downarrow \\
 T_{(D_\bullet, \varphi)} W_2 & \longrightarrow & T_\varphi X_3 \\
 & \searrow & \\
 & &
 \end{array}
 \quad
 \begin{array}{l}
 T_{(D'_\bullet, \varphi)} W_1 = \mathfrak{g}/\mathfrak{p}' \oplus \widetilde{\text{Hom}}_{d'_\bullet}(\mathcal{K}, \mathcal{E}); \\
 T_{(D_\bullet, \varphi)} W_2 = \mathfrak{g}/\mathfrak{p} \oplus \widetilde{\text{Hom}}_{d_\bullet}(\mathcal{K}, \mathcal{E}); \\
 T_{(D_\bullet, \text{gr}' \varphi)} W_3 = \mathfrak{g}/\mathfrak{p} \oplus \text{Hom}(\mathcal{K}_1, \mathcal{E}_1) \\
 \oplus \widetilde{\text{Hom}}_{d_\bullet}(\mathcal{K}/\mathcal{K}_1, \mathcal{E}/\mathcal{E}_1),
 \end{array}$$

where $\mathfrak{p}', \mathfrak{p} \subset \mathfrak{g}$ are parabolic subalgebras associated to flags D'_\bullet, D_\bullet respectively, and $\mathcal{K}_1 = \text{Ker } \varphi|_{D_1 \otimes \mathcal{O}_X}$. A straightforward computation shows that at a point $x = (D_\bullet, \varphi) \in W_2$ we have

$$T_x(W_1 \times X_1) \cap T_x(X_3 \times W_3) = \{(\theta, (\xi^{\text{red}}, \text{gr}' \theta), (\xi, \text{gr} \theta)) \mid \theta \in \widetilde{\text{Hom}}_{d_\bullet}(\mathcal{K}, \mathcal{E}), \xi \in \mathfrak{g}/\mathfrak{p}\},$$

where ξ^{red} denotes the image of ξ under the quotient map $\mathfrak{g}/\mathfrak{p} \rightarrow \mathfrak{g}/\mathfrak{p}'$. The space above is isomorphic to $T_x W_2$ by means of

$$\begin{aligned} T_x W_2 &\rightarrow T_x(W_1 \times X_1) \cap T_x(X_3 \times W_3), \\ (\xi, \theta) &\mapsto (\theta, (\xi^{red}, \text{gr}' \theta), (\xi, \text{gr} \theta)). \end{aligned}$$

This proves the transversality. \square

Let us now put $Z_i = T_{W_i}^*(X_{i-1} \times X_{i+1})$. The lemma above, combined with Theorem 2.7.26 in [12], tells us that the projection $T^*(X_3 \times X_2 \times X_1) \rightarrow T^*(X_3 \times X_1)$ gives rise to an isomorphism

$$Z_1 \times_{T^* X_2} Z_3 \xrightarrow{\sim} Z_2.$$

Therefore, we obtain the following diagrams with cartesian squares :

$$\begin{array}{ccc} T^* X_1 & \xleftarrow{\Phi_3} & Z_3 & \xrightarrow{\Psi_3} & T^* X_2 \\ & \searrow \Phi_2 & \uparrow \alpha & & \uparrow \Phi_1 \\ & & Z_2 & \xrightarrow{\beta} & Z_1 \\ & & \searrow \Psi_2 & & \downarrow \Psi_1 \\ & & & & T^* X_3 \end{array} \qquad \begin{array}{ccc} T_G^* X_1 & \xleftarrow{\Phi_{3G}} & Z_{3G} & \xrightarrow{\Psi_{3G}} & T_G^* X_2 \\ & \searrow \Phi_{2G} & \uparrow \alpha_G & & \uparrow \Phi_{1G} \\ & & Z_{2G} & \xrightarrow{\beta_G} & Z_{1G} \\ & & \searrow \Psi_{2G} & & \downarrow \Psi_{1G} \\ & & & & T_G^* X_3 \end{array}$$

where $Z_{iG} := Z_i \cap (T_G^* X_{i-1} \times T_G^* X_{i+1})$. In view of Lemma 2.2.12, the second diagram implies that

$$m \circ (\text{id} \otimes m) = ((\Psi_{1G})_* \circ \Phi_{1G}^!) \circ ((\Psi_{3G})_* \circ \Phi_{3G}^!) = (\Psi_{1G})_* \circ (\beta_G)_* \circ \alpha_G^! \circ \Phi_{3G}^! = (\Psi_{2G})_* \circ \Phi_{2G}^!.$$

In a similar way we may prove that $m \circ (m \otimes \text{id}) = (\Psi_{2G})_* \circ \Phi_{2G}^!$, so that the associativity of multiplication in $A\mathbf{Ha}_C^0$ follows. \square

Note that all varieties in the definition of $A\mathbf{Ha}_C^0$ admit a $T = \mathbb{G}_m$ -action (by dilation along the fibers of cotangent and conormal bundles) and all maps we consider are T -equivariant. Therefore, the construction above also defines an associative product on

$$A\mathbf{Ha}_C^{0,T} = \bigoplus_d A\mathbf{Ha}_C^{0,T}[d] = \bigoplus_d A^T(\mathcal{Higgs}_{0,d}).$$

Example 2.4.4. Let $C = \mathbb{A}^1$, and equip it with a natural action of weight 1 of another torus $T' = \mathbb{G}_m$. In this case $K\mathbf{Ha}_C^{0,T \times T'}$ and $H\mathbf{Ha}_C^{0,T \times T'}$ are precisely the K -theoretic and cohomological Hall algebras studied in [69] and [68] respectively.

2.5 Global shuffle algebra

In this section we focus our attention on the algebra $A\mathbf{Ha}_C^{0,T}$. In order to study its product, we will utilize the localization theorem 2.2.14. Let $d_\bullet = \{0 = d_0 \leq d_1 \leq \dots \leq d_k = d\}$, and denote $G = G_d$, $P = P_{d_\bullet}$, $H = G_{d_\bullet}$ as before. We will also denote by $T_d \subset H$ the maximal torus of operators, which are diagonal with respect to the standard basis v_1, \dots, v_d of \mathbb{k}^d . The Weyl group of G is then naturally isomorphic to \mathfrak{S}_d , and the Weyl group of H is isomorphic to $\mathfrak{S}_{d_\bullet} = \prod_i \mathfrak{S}_{d_i - d_{i-1}}$.

Denote $\mathbb{T} = T_d \times T$, and write t_1, \dots, t_d for the basis of character lattice of T_d corresponding to the standard basis of \mathbb{k}^d . In the same way, let t be the character of T of weight 1. One can think about characters of \mathbb{T} as equivariant line bundles over a point. In this fashion, we identify $A^\mathbb{T}(pt)_{loc} := \text{Frac}(A^\mathbb{T}(pt))$ with $A(pt)((e(t_1), \dots, e(t_d), e(t)))$ (see Example 2.2.3).

Lemma 2.5.1. *We have $(T^* \text{Quot}_{0,d}^\circ)^\mathbb{T} = \text{Quot}_{0,1}^d$, where the embedding $\text{Quot}_{0,1}^d \hookrightarrow \text{Quot}_{0,d}^\circ$ is defined by the basis $\{v_1, \dots, v_d\}$ associated to T_d .*

Proof. First of all, $(T^* \text{Quot}_{0,d}^\circ)^{T_d \times T} = (\text{Quot}_{0,d}^\circ)^{T_d}$. Next, a point $(\mathbb{k}^d \otimes \mathcal{O} \xrightarrow{\varphi} \mathcal{E}) \in \text{Quot}_{0,d}$ is T_d -stable if and only if for any $g \in T_d$ we have $\mathcal{K}_\varphi := \text{Ker } \varphi = \text{Ker}(\varphi \circ (g \otimes \text{id})) = g \cdot \mathcal{K}_\varphi$. We choose an integer N and an ample line bundle \mathcal{L} such that for any φ the sheaf $\mathcal{L}^{\otimes N} \otimes \mathcal{K}_\varphi$ is generated by its global sections. Then \mathcal{K}_φ is uniquely determined by the subspace $V := H^0(\mathcal{L}^{\otimes N} \otimes \mathcal{K}_\varphi) \subset \mathbb{k}^d \otimes H^0(\mathcal{L}^{\otimes N})$ of codimension d , that is by a point in certain Grassmanian. Let us denote $H = H^0(\mathcal{L}^{\otimes N})$. Then \mathcal{K}_φ is T -stable if and only if V is T -stable. But we know that the only subspaces stable under the torus actions are direct sums of subspaces of weight spaces. Therefore $V = \bigoplus_{i=1}^n V \cap (\mathbb{k}v_i \otimes H)$, and thus \mathcal{K}_φ is T -stable iff $\mathcal{K}_\varphi = \bigoplus_{i=1}^d \mathcal{K}_\varphi \cap (\mathbb{k}v_i \otimes \mathcal{O})$. Finally, this is equivalent to saying that $\mathcal{E} = \bigoplus_{i=1}^d \varphi(\mathbb{k}v_i \otimes \mathcal{O})$, that is $\varphi \in \text{Quot}_{0,1}^d$. \square

Recall that $\text{Quot}_{0,1} \simeq C$ (see Example 2.3.5), and let $p_{ij} : \text{Quot}_{0,1} \times \text{Quot}_{0,1} \times C \rightarrow C \times C$ denote the projection along the unnamed factor.

Lemma 2.5.2. *Let $\mathcal{K}, \mathcal{E} \in \text{Coh}(\text{Quot}_{0,1} \times C) \simeq \text{Coh}(C \times C)$ be the universal families of kernels and images of quotients $\mathcal{O} \rightarrow \mathcal{E}$ respectively. Then*

$$\begin{aligned} p_{12*} \mathcal{H}om(p_{13}^* \mathcal{K}, p_{23}^* \mathcal{E}) &\simeq \mathcal{O}_{C \times C}(\Delta), \\ p_{12*} \mathcal{H}om(p_{13}^* \mathcal{O}, p_{23}^* \mathcal{E}) &\simeq \mathcal{O}_{C \times C}, \end{aligned}$$

where $\Delta \subset C \times C$ is the diagonal.

Proof. It's easy to see that $\mathcal{E} \simeq \mathcal{O}_\Delta$, $\mathcal{K} \simeq \mathcal{O}(-\Delta)$. Since \mathcal{K} is locally free and \mathcal{E} is torsion sheaf over any point $\varphi \in \text{Quot}_{0,1}$, the higher Ext-sheaves $\mathcal{E}xt^i(p_{13}^* \mathcal{K}, p_{23}^* \mathcal{E})$ vanish for all $i > 0$. We denote $\Delta_{ij} = p_{ij}^{-1} \Delta$, and $\Delta_{123} \subset C \times C \times C$ the small diagonal. Then

$$\begin{aligned} p_{12*} \mathcal{H}om(p_{13}^* \mathcal{K}, p_{23}^* \mathcal{E}) &\simeq p_{12*}((p_{13}^* \mathcal{K})^* \otimes p_{23}^* \mathcal{E}) \simeq p_{12*}(\mathcal{O}(\Delta_{13}) \otimes \mathcal{O}_{\Delta_{23}}) \simeq p_{12*}(\mathcal{O}_{\Delta_{23}}(\Delta_{123})) \simeq \\ &\simeq \mathcal{O}_{C \times C}(\Delta). \end{aligned}$$

This proves the first equality. For the second one, we conclude by a similar computation :

$$p_{12*} \mathcal{H}om(p_{13}^* \mathcal{O}, p_{23}^* \mathcal{E}) \simeq p_{12*}(\mathcal{O}_{\Delta_{23}}) \simeq \mathcal{O}_{C \times C}. \quad \square$$

Notation. In order to keep notation concise, for any two sheaves $\mathcal{A}, \mathcal{B} \in \text{Coh}(\text{Quot}_{0,1} \times C)$ we'll write $\text{Hom}(\mathcal{A}, \mathcal{B})$ instead of $p_{12*} \mathcal{H}om(p_{13}^* \mathcal{A}, p_{23}^* \mathcal{B})$ (here the pushforward p_{12*} is underived).

Let $j_H : \mathcal{C}_d \hookrightarrow T^* \text{Quot}_{0,d}^\circ$ denote the closed embedding, and let $i_H : \text{Quot}_{0,1}^d \hookrightarrow T^* \text{Quot}_{0,d}^\circ$ be the inclusion of the fixed point set. Recall that by localization theorem 2.2.14 the map i_H^* becomes an isomorphism upon tensoring with the fraction field of $A^\mathbb{T}(pt)$. Consider the following composition :

$$\rho_H = i_H^* \circ j_{H*} : A^{H \times T}(\mathcal{C}_d) \simeq A^\mathbb{T}(\mathcal{C}_d)^{W_d} \rightarrow A^\mathbb{T}(C^d)^{W_d},$$

where the isomorphism on the left is given by Proposition 2.2.7, Weyl group acts on the right-hand side by Remark 2.2.18, and $\text{Quot}_{0,1}^d$ is identified with C^d . In the same way, we can define a map

$$\rho_d = \rho_G : A^{G \times T}(\mathcal{C}_d) \rightarrow A^\mathbb{T}(C^d)^{W_d}.$$

The goal of this section is to construct a map Υ_{loc} (at the level of localized A -groups), such that the following diagram commutes :

$$\begin{array}{ccc} A^{H \times T}(\mathcal{C}_d) & \xrightarrow{\Upsilon} & A^{G \times T}(\mathcal{C}_d) \\ \downarrow \rho_H & & \downarrow \rho_G \\ A^{T_d \times T}(C^d)_{loc}^{\mathfrak{S}_d} & \xrightarrow{\Upsilon_{loc}} & A^{T_d \times T}(C^d)_{loc}^{\mathfrak{S}_d} \end{array}$$

As in [69, Section 10], one expects Υ_{loc} to be some incarnation of shuffle product. Let z be a formal variable, and let $g \in (A^*(C \times C)((t)))((e(z)))$ be a $A_T^*(C \times C)_{loc}$ -valued formal Laurent series in $e(z)$, where we interpret the latter as a formal symbol. We will also abuse the notations and write g as a function of z . For any positive d_1, d_2 with $d_1 + d_2 = d$, put

$$g_{d_1, d_2}(z_1, \dots, z_d) = \prod_{i=1}^{d_1} \prod_{j=d_1+1}^d g\left(\frac{z_j}{z_i}\right).$$

Let us also fix the following set of representatives of classes in $\mathfrak{S}_d/(\mathfrak{S}_{d_1} \times \mathfrak{S}_{d_2})$:

$$Sh(d_1, d_2) = \{\sigma \in \mathfrak{S}_d \mid \sigma(i) < \sigma(j) \text{ if } 1 \leq i < j \leq d_1 \text{ or } d_1 + 1 \leq i < j \leq d\}.$$

Definition 2.5.3. The *shuffle algebra associated to g* is the vector space

$$A\mathbf{Sh}_g = \bigoplus_d A\mathbf{Sh}_g[d] = \bigoplus_d \left(A(C^d)((e(t); e(t_1), \dots, e(t_d))) \right)^{\mathfrak{S}_d}$$

equipped with the product

$$\begin{aligned} \Xi : A\mathbf{Sh}_g[d_1] \times A\mathbf{Sh}_g[d_2] &\rightarrow A\mathbf{Sh}_g[d_1 + d_2], \\ \Xi(f, h) &= \sum_{\sigma \in Sh(d_1, d_2)} \sigma. (g_{d_1, d_2}(t_1, \dots, t_{d_1+d_2}) f(t_1, \dots, t_{d_1}) h(t_{d_1+1}, \dots, t_{d_1+d_2})). \end{aligned} \quad (2.9)$$

The formula (2.9) requires some explanation. First of all, the product between f and h is given by the map $A(C^{d_1}) \otimes A(C^{d_2}) \xrightarrow{\times} A(C^{d_1+d_2})$. Next, after replacing z_i 's by t_i 's and taking Euler classes, the function g_{d_1, d_2} becomes an honest cohomology class, which then operates on the product $f \cdot h$. Finally, the natural action of $\sigma \in Sh(d_1, d_2)$ (see Remark 2.2.18) simultaneously permutes t_i 's and factors in the direct product $C^{d_1+d_2}$.

It's easy to check that this product is associative. We will be mostly concerned with two specific choices of g :

— the *global shuffle algebra*, denoted by $A\mathbf{Sh}_C$, is the shuffle algebra associated to

$$g_C = \frac{e(tz^{-1})e(z\mathcal{O}(-\Delta))e(tz\mathcal{O}(\Delta))}{e(z)};$$

— the *normalized global shuffle algebra*, denoted by $A\mathbf{Sh}_C^{norm}$, is the shuffle algebra associated to

$$g_C^{norm} = \frac{e(z\mathcal{O}(-\Delta))e(tz\mathcal{O}(\Delta))}{e(z)e(tz)}.$$

By invoking formal group \star associated to A , we can deduce that both functions are Laurent series in $e(z)$ (see also discussion before Proposition 2.5.12).

Remark 2.5.4. Two algebras above are isomorphic under the map $RN = \bigoplus_d RN[d]$, where :

$$\begin{aligned} RN[d] : A\mathbf{Sh}_C^{norm}[d] &\rightarrow A\mathbf{Sh}_C[d], \\ f(t_1, \dots, t_d) &\mapsto \left(\prod_{i \neq j} e\left(\frac{t_j}{t_i}\right) \right) f(t_1, \dots, t_d). \end{aligned}$$

Theorem 2.5.5. The collection of maps $\rho_d : A^{G_d \times T}(\mathcal{C}_d) \rightarrow A^{\mathbb{T}}(C^d)^{\mathfrak{S}_d} \subset A^{\mathbb{T}}(C^d)_{loc}^{\mathfrak{S}_d}$ defines a morphism of graded associative algebras

$$\rho : A\mathbf{Ha}_C^0 \rightarrow A\mathbf{Sh}_C.$$

Proof. Let us first introduce some notations. Define

$$I_d^2 = [1; d] \times [1; d] \subset \mathbb{Z}^2;$$

$$T_{\mathfrak{p}} = \bigcup_{i=1}^k [d_{i-1} + 1; d_i] \times [d_{i-1} + 1; d_i]; \quad T_{\mathfrak{n}} = \bigcup_{i=1}^{k-1} [d_{i-1} + 1; d_i] \times [d_i + 1; d].$$

Also, let us denote the space $W \times_{W_H} C^d$ from Proposition 2.2.17 by \hat{C}^d . Recall that we have a projection $s : \hat{C}^d \rightarrow C^d$, given by shuffle permutations $Sh(d_1, d_2)$.

Recall the notations of Section 2.4, c.f. Lemma 2.4.1(2). Our proof will proceed in two steps. First, consider the following diagram :

$$\begin{array}{ccccc} A^{H \times T}(\mathcal{C}_{d_\bullet}) & \xrightarrow{\text{ind}_H^G} & & & A^{G \times T}(T_G^* X) \\ \downarrow j_* & \searrow & & & \downarrow j_* \\ A^{H \times T}(T^* \text{Quot}_{0, d_\bullet}^\circ) & \longrightarrow & A^{H \times T}(\mathfrak{p}_-^* \times_{\mathfrak{h}^*} T^* \text{Quot}_{0, d_\bullet}^\circ) & \xrightarrow{\text{ind}_H^G} & A^{G \times T}(T^* X) \\ \downarrow i^* & & \downarrow i^* & & \downarrow i^* \\ A^{\mathbb{T}}(C^d)^{\mathfrak{S}_d} & \xlongequal{\quad} & A^{\mathbb{T}}(C^d)^{\mathfrak{S}_d} & \xrightarrow{s^*} & A^{\mathbb{T}}(\hat{C}^d)^{\mathfrak{S}_d} \end{array}$$

Let us denote all vertical compositions by ρ , leaving out the subscripts.

Lemma 2.5.6. *We have*

$$\rho \circ \text{ind}_H^G(c) = s^* \left(\prod_{(i,j) \in T_{\mathfrak{n}}} e \left(t \frac{t_i}{t_j} \right) \rho(c) \right)$$

for any $c \in A^{H \times T}(\mathcal{C}_{d_\bullet})$.

Proof. Everything in this diagram commutes, except for the lower left square. Moreover, by Proposition 2.2.15 this square becomes commutative after multiplying by an appropriate Euler class. Note that since $\mathfrak{p}_-^* = \mathfrak{n}_-^* \oplus \mathfrak{h}^*$, we have an isomorphism

$$\mathfrak{p}_-^* \times_{\mathfrak{h}^*} T^* \text{Quot}_{0, d_\bullet}^\circ \simeq \mathfrak{n}_-^* \times T^* \text{Quot}_{0, d_\bullet}^\circ.$$

This (trivial) vector bundle has the same T -fixed points as its zero section $T^* \text{Quot}_{0, d_\bullet}^\circ$. Therefore, the Proposition 2.2.15 tells us that the required Euler class is

$$e \left(T_{T^* \text{Quot}_{0, d_\bullet}^\circ} \mathfrak{n}_-^* \times T^* \text{Quot}_{0, d_\bullet}^\circ \Big|_{C^d} \right) = e(\mathfrak{n}_-^*) = \prod_{(i,j) \in T_{\mathfrak{n}}} e \left(t \frac{t_i}{t_j} \right),$$

and we're done. \square

Next, consider another diagram :

$$\begin{array}{ccccc} A^{G \times T}(T_G^* X) & \xrightarrow{\Phi^!} & A^{G \times T}(Z_G) & \xrightarrow{\Psi_*} & A^{G \times T}(\mathcal{C}_d) \\ \downarrow j_* & & \downarrow j_* & & \downarrow j_* \\ A^{G \times T}(T^* X) & \xrightarrow{\Phi^*} & A^{G \times T}(Z) & \xrightarrow{\Psi_*} & A^{G \times T}(T^* \text{Quot}_{0, d}^\circ) \\ \downarrow i^* & & \downarrow i^* & & \downarrow i^* \\ A^{\mathbb{T}}(\hat{C}^d)^{\mathfrak{S}_d} & \xlongequal{\quad} & A^{\mathbb{T}}(\hat{C}^d)^{\mathfrak{S}_d} & \xrightarrow{s_*} & A^{\mathbb{T}}(C^d)^{\mathfrak{S}_d} \end{array}$$

Let $p_{ij} : C^d \rightarrow C \times C$ be the projection to i -th and j -th components, and $\Delta_{ij} = p_{ij}^{-1}(\Delta)$.

Lemma 2.5.7. *We have*

$$\rho \circ \Psi_* \circ \Phi^!(c) = s_* \left(s^* \left(\prod_{(i,j) \in T_n} e \left(\frac{t_j}{t_i} \right)^{-1} e \left(\frac{t_j}{t_i} \mathcal{O}(-\Delta_{ij}) \right) e \left(\frac{t_j}{t_i} \mathcal{O}(\Delta_{ij}) \right) \right) \rho(c) \right)$$

for any $c \in A^{G \times T}(T_G^* X)$.

Proof. Once again, all squares in this diagram commute, except for the lower right one, which commutes up to multiplication by a certain Euler class (see Lemma 2.2.12(2) and Proposition 2.2.15). Therefore, we have :

$$\rho \circ \Psi_* \circ \Phi^!(c) = s_* \circ \rho \left(e(T_{\hat{C}^d} Z)^{-1} s^*(e(T_{C^d} T^* \text{Quot}_{0,d}^\circ)) \cdot c \right).$$

It is left to compute the product of Euler classes in parentheses. We have the following chain of equalities :

$$\begin{aligned} & e(T_{\hat{C}^d} Z)^{-1} s^*(e(T_{C^d} T^* \text{Quot}_{0,d}^\circ)) \\ &= s^* \left(e(T_{C^d} \widetilde{\text{Quot}}_{0,d,\bullet})^{-1} e(T_{\widetilde{\text{Quot}}_{0,d,\bullet}} Z|_{C^d})^{-1} e(T_{C^d} \text{Quot}_{0,d}^\circ) e(T_{\text{Quot}_{0,d}^\circ} T^* \text{Quot}_{0,d}^\circ|_{C^d}) \right) \\ &= s^* \left(e(T_{C^d} \widetilde{\text{Quot}}_{0,d,\bullet})^{-1} e(\mathfrak{n}_-)^{-1} e \left(\frac{T^* \text{Quot}_{0,d}^\circ}{(T \widetilde{\text{Quot}}_{0,d,\bullet})^{\text{nilp},*}} \Big|_{C^d} \right)^{-1} \times \right. \\ & \quad \times e(T_{C^d} \text{Quot}_{0,d}^\circ) e(T^* \text{Quot}_{0,d}^\circ|_{C^d}) \\ & \left. = s^* \left(e(\mathfrak{n}_-)^{-1} e((T \widetilde{\text{Quot}}_{0,d,\bullet})^{\text{nilp},*}|_{C^d}) e(T_{C^d} \widetilde{\text{Quot}}_{0,d,\bullet})^{-1} e(T_{C^d} \text{Quot}_{0,d}^\circ) \right) \right). \end{aligned}$$

Let $\varphi = \bigoplus_{i=1}^d \varphi_i$ be a point in $C^d \simeq \text{Quot}_{0,1}^d$, where each φ_i produces a short exact sequence $0 \rightarrow \mathcal{K}_i \rightarrow \mathcal{O}_C \xrightarrow{\varphi_i} \mathcal{E}_i \rightarrow 0$. Formulas (2.6)-(2.8) provide us with explicit expressions for the tangent spaces of various *Quot*-schemes :

$$\begin{aligned} T_\varphi \text{Quot}_{0,1}^d &= \bigoplus_{i=1}^d \text{Hom}(\mathcal{K}_i, \mathcal{E}_i); & T_\varphi \text{Quot}_{0,d}^\circ &= \bigoplus_{i,j=1}^d \text{Hom}(\mathcal{K}_j, \mathcal{E}_i); \\ T_\varphi \widetilde{\text{Quot}}_{0,d,\bullet} &= \bigoplus_{(i,j) \in T_p} \text{Hom}(\mathcal{K}_j, \mathcal{E}_i). \end{aligned}$$

Therefore by lemma 2.5.2

$$\begin{aligned} (T \widetilde{\text{Quot}}_{0,d,\bullet})^{\text{nilp},*}|_{C^d} &= t \bigoplus_{(i,j) \in T_n} \text{Hom}(\mathcal{K}_j, \mathcal{E}_i)^* \simeq \bigoplus_{(i,j) \in T_n} t \frac{t_j}{t_i} \mathcal{O}(\Delta_{ij}), \\ T_{C^d} \widetilde{\text{Quot}}_{0,d,\bullet} &= \bigoplus_{(i,j) \in T_p, i \neq j} \text{Hom}(\mathcal{K}_j, \mathcal{E}_i) \simeq \bigoplus_{(i,j) \in T_p, i \neq j} \frac{t_i}{t_j} \mathcal{O}(-\Delta_{ij}), \\ T_{C^d} \text{Quot}_{0,d}^\circ &= \bigoplus_{i \neq j} \text{Hom}(\mathcal{K}_j, \mathcal{E}_i) \simeq \bigoplus_{i \neq j} \frac{t_i}{t_j} \mathcal{O}(-\Delta_{ij}), \end{aligned}$$

and a straightforward computation shows that

$$e(\mathfrak{n}_-) = \prod_{(i,j) \in T_n} e \left(\frac{t_j}{t_i} \right),$$

$$e((T \widetilde{\text{Quot}}_{0,d,\bullet})^{\text{nilp},*}|_{C^d}) e(T_{C^d} \widetilde{\text{Quot}}_{0,d,\bullet})^{-1} e(T_{C^d} \text{Quot}_{0,d}^\circ) = \prod_{(i,j) \in T_n} e \left(\frac{t_j}{t_i} \mathcal{O}(-\Delta_{ij}) \right) e \left(\frac{t_j}{t_i} \mathcal{O}(\Delta_{ij}) \right).$$

The statement of lemma follows. \square

Combining the results of two lemmas, we get :

$$\rho_G \circ m(c) = \rho \circ \Psi_* \circ \Phi^! \circ \text{ind}_H^G(c) = s_* \circ s^* \circ (g_{C,d_\bullet}(z)\rho(c)) = \Xi \circ \rho_H(c),$$

which proves the theorem. \square

Remark 2.5.8. In order to recover the shuffle presentation in [69, Theorem 10.1], we can add an action of another torus as in Example 2.4.4. If we denote by q the T' -character of weight by -1 , we get $\mathcal{O}(\Delta) = q^{-1}$, and we obtain the desired presentation after further replacing t by qt . Unfortunately, we don't have a succinct explanation of the latter change of variables. Morally speaking, it occurs because in the natural compactification $\mathbb{A}^2 \subset \mathbb{P}^2$ the divisor at infinity is “diagonal”, and for $T^*C \subset \mathbb{P}(T^*C)$ it is “horizontal”.

Even though we've got an explicit formula, the morphism ρ depends on the embedding of \mathcal{C}_d into a smooth ambient variety $T^*\mathcal{Q}uot_{0,d}^\circ$. Unfortunately, the scheme \mathcal{C}_d is highly singular; for instance, the inclusion $C^d \hookrightarrow \mathcal{C}_d$ is not known to be l.c.i., so that we can not localize to \mathbb{T} -fixed points directly. Still, we can do a little better. Let $\tilde{\mu}$ be the composition $T^*\mathcal{Q}uot_{0,d}^\circ \xrightarrow{\mu} \mathfrak{g}_d^* \simeq \mathfrak{g}_d \rightarrow \mathfrak{g}_d/\mathfrak{t}_d$, where $\mathfrak{t}_d \subset \mathfrak{g}_d$ is the tangent algebra of T_d . We introduce the following auxiliary variety, analogous to the one in [39] :

$$\mathcal{C}_d^\Delta := \tilde{\mu}^{-1}(0) = \mu^{-1}(\mathfrak{t}_d).$$

Proposition 2.5.9. *The closed embedding $\mathcal{C}_d^\Delta \hookrightarrow T^*\mathcal{Q}uot_{0,d}^\circ$ is a complete intersection.*

Proof. First of all, the statement is true for $C = \mathbb{A}^1$ [39]. For general C , it suffices to prove that $\dim \mathcal{C}_d^\Delta \leq \dim \mathcal{Q}uot_{0,d}^\circ + \dim \mathfrak{t}_d = d(d+1)$. Let us consider the map

$$\begin{aligned} \sigma : T^*\mathcal{Q}uot_{0,d}^\circ &\rightarrow S^d C, \\ (\mathbb{k}^d \rightarrow \mathcal{E}) &\mapsto \text{supp } \mathcal{E}. \end{aligned}$$

For any $\nu = (\nu_1 \geq \dots \geq \nu_k)$ partition of d let

$$S^\nu C = \{\underline{x} = \nu_1 x_1 + \dots + \nu_k x_k : x_i \neq x_j \text{ for } i \neq j\} \subset S^d C$$

Then $S^d C = \coprod_{\nu \vdash d} S^\nu C$, and this defines us a stratification of \mathcal{C}_d^Δ :

$$\mathcal{C}_d^\Delta = \coprod_{\nu \vdash d} \mathcal{C}_\nu^\Delta, \quad \mathcal{C}_\nu^\Delta = \mathcal{C}_d^\Delta \cap \sigma^{-1}(S^\nu C).$$

Consider the restriction of σ to these strata. For any point $\underline{x} \in S^\nu C$, we have a G_d -equivariant map

$$\begin{aligned} \tau : \sigma^{-1}(\underline{x}) \cap \mathcal{C}_d^\Delta &\rightarrow \prod_i \text{Grass}^{\nu_i}(\mathbb{k}^d), \\ (\mathbb{k}^d \otimes \mathcal{O} \xrightarrow{p} \mathcal{E}, \beta) &\mapsto (H^0(\mathcal{E}|_{x_1}), \dots, H^0(\mathcal{E}|_{x_k})). \end{aligned}$$

The image of this map is an open subset where the vector subspaces defined by points in Grassmanians do not intersect. At each such point, the G_d -action induces an isomorphism between the preimage of τ and $\prod_i \mathcal{C}_{\nu_i}^{n, \bullet, \Delta}$ by Example 2.3.12, where

$$\mathcal{C}_k^{n, \bullet, \Delta} = \{(x, y) \in (\mathfrak{g}_k)^2 : [x, y] \in \mathfrak{t}_k, x \text{ nilpotent}\}.$$

Since in particular this applies to $C = \mathbb{A}^1$, we have

$$\dim \mathcal{C}_\nu^\Delta(C) = k + \sum_i (\nu_i(d - \nu_i) + \dim \mathcal{C}_{\nu_i}^{n, \bullet, \Delta}) = \dim \mathcal{C}_\nu^\Delta(\mathbb{A}^1) \leq d(d+1).$$

The dimension of \mathcal{C}_d^Δ is bounded above by the dimensions of stratas, therefore $\dim \mathcal{C}_d^\Delta \leq d(d+1)$. \square

Armed with the proposition above, let $i_d^\Delta : C^d \hookrightarrow \mathcal{C}_d^\Delta$, $j_d^\Delta : \mathcal{C}_d \hookrightarrow \mathcal{C}_d^\Delta$ be the natural closed embeddings, and consider the composition

$$\varrho_d = (i_d^\Delta)^* \circ (j_d^\Delta)_* : A^{G \times T}(\mathcal{C}_d) \rightarrow A^{\mathbb{T}}(C^d) \mathfrak{S}_d.$$

Corollary 2.5.10. *The collection of maps ϱ_d , $d \in \mathbb{N}^+$ defines a morphism of graded associative algebras*

$$\varrho : A\mathbf{Ha}_C^0 \rightarrow A\mathbf{Sh}_C^{\text{norm}}.$$

Proof. Denote the closed embedding $\mathcal{C}_d^\Delta \hookrightarrow T^* \text{Quot}_{0,d}^\circ$ by j'_d . By Corollary 2.2.16, we have the following identity :

$$\rho(c) = i_d^* \circ j_{d*}(c) = i_d^* \circ j'_{d*} \circ (j_d^\Delta)_*(c) = (i_d^\Delta)^* \circ (j_d^\Delta)_*(e_1 c) = \varrho(e_1 c), \quad e_1 = e(T_{\mathcal{C}_d^\Delta} T^* \text{Quot}_{0,d}^\circ).$$

Note that the map $\tilde{\mu}$ is T -equivariant. Since T contracts $T^* \text{Quot}_{0,d}^\circ$ to a subvariety of \mathcal{C}_d^Δ and $\dim \mathcal{C}_d^\Delta = \dim \tilde{\mu}$, the argument similar to the one found in [23, Proposition 2.3.2] shows that $\tilde{\mu}$ is flat. In particular,

$$e(T_{\mathcal{C}_d^\Delta} T^* \text{Quot}_{0,d}^\circ) = e((\mathfrak{g}_d/\mathfrak{t}_d)^*),$$

by base change ; see [21, B.7.4]. Therefore, we have

$$\varrho(c) = \rho(e_1^{-1} c) = \rho \left(\left(\prod_{i \neq j} e \left(\begin{smallmatrix} t_j \\ t_i \end{smallmatrix} \right) \right)^{-1} c \right) = \rho \circ RN^{-1}(c).$$

Since both ρ and RN are morphisms of algebras, ϱ is as well. \square

Remark 2.5.11. For any function g , one can equip the algebra $A\mathbf{Sh}_g$ with a topological coproduct, analogous to the coproduct in [59, Section 4]. If the morphism ρ is injective (see Section 2.8 for discussion and partial results), it can be used to induce a coproduct on $A\mathbf{Ha}_C^0$. However, it is less clear how to construct such coproduct without using shuffle presentation.

Let us conclude this chapter by computing some relations in the algebra $A\mathbf{Sh}_g$, for arbitrary Laurent series g . Given a line bundle \mathcal{L} on C , define a bi-infinite series

$$E_{\mathcal{L}}(z) = \sum_{i \in \mathbb{Z}} e(\mathcal{L})^{-i} e(z)^i,$$

where z is a formal variable, and we consider $e(z)$ to be a formal expression. Using the formal group law \star associated to A (see Appendix 2.2), we have the following equality for some $f \in A^*(pt)[[u, v]]$:

$$\begin{aligned} 0 = e(1) &= e \left(\frac{z}{z} \right) = e(z) \star e(z^{-1}) = e(z) + e(z^{-1}) + e(z)e(z^{-1})f(e(z), e(z^{-1})) \\ &\Rightarrow e(z^{-1}) = -e(z) (1 + e(z^{-1})f(e(z), e(z^{-1}))). \end{aligned}$$

Therefore, by implicit function theorem for formal series [71, Exercise 5.59] $e(z^{-1})$ can be interpreted as formal series in $e(z)$. In particular, $g(w/z)$ is a formal series in $e(w)$, $e(z)$.

Proposition 2.5.12. *Let $\mathcal{L}_1, \mathcal{L}_2$ be two line bundles on C . Suppose that $e(zw)$ is a rational function in $e(z)$ and $e(w)$, and $e(z^{-1})$ is a rational function of $e(z)$. Then the following equality holds :*

$$g \left(\frac{t_1 \mathcal{L}_1 w}{t_2 \mathcal{L}_2 z} \right) E_{\mathcal{L}_1}(z) E_{\mathcal{L}_2}(w) = g \left(\frac{t_2 \mathcal{L}_2 z}{t_1 \mathcal{L}_1 w} \right) E_{\mathcal{L}_2}(w) E_{\mathcal{L}_1}(z), \quad (2.10)$$

where product between $E_{\mathcal{L}_1}$ and $E_{\mathcal{L}_2}$ is taken in $A\mathbf{Sh}_g$, and we consider both sides as bi-infinite series in $e(z)$, $e(w)$ with coefficients in $A\mathbf{Sh}_g[2]$.

Proof. In order to unburden the notation, denote $Z = e(z)$, $W = e(w)$, $T_i = e(t_i)$, $L_i = e(\mathcal{L}_i)$. Let us also introduce bi-infinite series $\delta(z) = \sum_{i \in \mathbb{Z}} z^i$. Note that for any Laurent polynomial $f(z)$ the following identity, which we call *change of variables*, is satisfied :

$$\delta(w/z)f(z) = \delta(w/z)f(w). \quad (2.11)$$

We have :

$$\begin{aligned} E_{\mathcal{L}_1}(z)E_{\mathcal{L}_2}(w) &= \left(\sum_{i \in \mathbb{Z}} L_1^{-i} Z^i \right) \left(\sum_{j \in \mathbb{Z}} L_2^{-j} W^j \right) = \sum_{i,j \in \mathbb{Z}} \left(L_1^{-i} L_2^{-j} g \left(\frac{t_2}{t_1} \right) + L_1^{-j} L_2^{-i} g \left(\frac{t_1}{t_2} \right) \right) Z^i W^j \\ &= \delta(L_1^{-1} Z) \delta(L_2^{-1} W) g \left(\frac{t_2}{t_1} \right) + \delta(L_1^{-1} W) \delta(L_2^{-1} Z) g \left(\frac{t_1}{t_2} \right). \end{aligned}$$

Therefore, the equality (2.10) is equivalent to the following :

$$\begin{aligned} \delta(L_1^{-1} Z) \delta(L_2^{-1} W) \left(g \left(\frac{t_1 \mathcal{L}_1 w}{t_2 \mathcal{L}_2 z} \right) g \left(\frac{t_2}{t_1} \right) - g \left(\frac{t_2 \mathcal{L}_2 z}{t_1 \mathcal{L}_1 w} \right) g \left(\frac{t_1}{t_2} \right) \right) \\ = \delta(L_1^{-1} W) \delta(L_2^{-1} Z) \left(g \left(\frac{t_2 \mathcal{L}_2 z}{t_1 \mathcal{L}_1 w} \right) g \left(\frac{t_2}{t_1} \right) - g \left(\frac{t_1 \mathcal{L}_1 w}{t_2 \mathcal{L}_2 z} \right) g \left(\frac{t_1}{t_2} \right) \right) \end{aligned} \quad (2.12)$$

Note that getting rid of denominators we can assume that the expressions in brackets are polynomials in $e(z)$ and $e(w)$, so that the equality above makes sense. Using change of variables (2.11) for LHS, we get :

$$\begin{aligned} \delta(L_1^{-1} Z) \delta(L_2^{-1} W) \left(g \left(\frac{t_1 \mathcal{L}_1 w}{t_2 \mathcal{L}_2 z} \right) g \left(\frac{t_2}{t_1} \right) - g \left(\frac{t_2 \mathcal{L}_2 z}{t_1 \mathcal{L}_1 w} \right) g \left(\frac{t_1}{t_2} \right) \right) \\ = \left(\delta(L_1^{-1} Z) g \left(\frac{t_1 \mathcal{L}_1 w}{t_2 \mathcal{L}_2 z} \right) \right) \left(\delta(L_2^{-1} W) g \left(\frac{t_2}{t_1} \right) \right) \\ - \left(\delta(L_1^{-1} Z) g \left(\frac{t_2 \mathcal{L}_2 z}{t_1 \mathcal{L}_1 w} \right) \right) \left(\delta(L_2^{-1} W) g \left(\frac{t_1}{t_2} \right) \right) \\ = \delta(L_1^{-1} Z) \delta(L_2^{-1} W) \left(g \left(\frac{t_1 w}{t_2 \mathcal{L}_2} \right) g \left(\frac{t_2 \mathcal{L}_2}{t_1 w} \right) - g \left(\frac{t_2 \mathcal{L}_2}{t_1 w} \right) g \left(\frac{t_1 w}{t_2 \mathcal{L}_2} \right) \right) = 0. \end{aligned}$$

By the same reasoning RHS is also equal to zero. Therefore (2.12) is satisfied, and we may conclude. \square

In particular, if we set $\mathcal{L}_1 = t_1^{-1}$ and $\mathcal{L}_2 = t_2^{-1}$, the equality (2.10) assumes an easier form :

$$g \left(\frac{w}{z} \right) E(z)E(w) = g \left(\frac{z}{w} \right) E(w)E(z), \quad (2.13)$$

where $E(z) = \sum_{i \in \mathbb{Z}} e(t_1^{-1})^{-i} e(z)^i$.

Remark 2.5.13. For $A = K$, there is a subtle point to address. Equivariant K -theory, considered in the context of OBM homology theories, has two dissimilarities with the usual equivariant K -theory, as defined for instance in [12, Chapter 5]. First and less serious issue is that our K -theory possesses a synthetic homological grading. Secondly, as per Remark 2.2.4, our K -theory has a different set of equivariant generators, and isomorphic to usual equivariant K -groups only up to certain completion. This fact has several undesired consequences. For example, our identity (2.13) doesn't exactly match the identity (3.4) in [60], since i -th power of Chern class of a line bundle \mathcal{L} is quite different from the $[\mathcal{L}^{\otimes i}]$ or the Chern class of $\mathcal{L}^{\otimes i}$. However, this is a sacrifice made for the sake of uniformity, and all of our methods and proofs can be adapted to the usual equivariant K -theory in a straightforward fashion.

2.6 Moduli of stable Higgs triples

In this section we construct a certain family of smooth varieties, which can be regarded as a generalization of the Hilbert schemes of points on T^*C (see Corollary 2.9.5).

We start with the stack $\underline{\mathcal{C}oh}_{0,d}^{\leftarrow \mathcal{F}}$, where $\mathcal{F} \in \text{Coh } C$ is a fixed coherent sheaf on C . The following proposition seems to be well-known (compare to [34, Theorem 4.1.(i)] and the entirety of [26]), but we didn't manage to find a precise reference.

Proposition 2.6.1. *Let $p = (\mathcal{E}, \alpha) \in \underline{\mathcal{C}oh}_{0,d}^{\leftarrow \mathcal{F}}$ be a pair. Then the tangent space $T_p \underline{\mathcal{C}oh}_{0,d}^{\leftarrow \mathcal{F}}$ at p is naturally isomorphic to $\mathbb{H}\text{om}(\mathcal{F} \xrightarrow{\alpha} \mathcal{E}, \mathcal{E})$.*

Proof. Let $D = \text{Spec } \mathbb{k}[\epsilon]/\epsilon^2$. By definition, the tangent space $T_p \underline{\mathcal{C}oh}_{0,d}^{\leftarrow \mathcal{F}}$ is given by the space of maps $D \rightarrow \underline{\mathcal{C}oh}_{0,d}^{\leftarrow \mathcal{F}}$, which restrict to p at origin. Again, by definition

$$\{D \rightarrow \underline{\mathcal{C}oh}_{0,d}^{\leftarrow \mathcal{F}}\} = \{\mathcal{F}[\epsilon] \xrightarrow{\tilde{\alpha}} \tilde{\mathcal{E}} : \tilde{\mathcal{E}} \in \mathcal{O}_C[\epsilon] - \text{mod}; \quad \tilde{\mathcal{E}}, \tilde{\alpha} \text{ flat over } D; \quad \tilde{\alpha} \text{ mod } \epsilon = \alpha\},$$

and since the infinitesimal deformations of a coherent sheaf over a scheme are given by its self-extensions, we see that the maps $D \rightarrow \underline{\mathcal{C}oh}_{0,d}^{\leftarrow \mathcal{F}}$ are parametrized by diagrams of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F} \oplus \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \tilde{\alpha} & & \downarrow \alpha & & \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \tilde{\mathcal{E}} & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \end{array}$$

Splitting off \mathcal{F} on the left, this data is equivalent to the following diagram :

$$\begin{array}{ccccccc} & & \mathcal{F} & & & & \\ & & \downarrow \tilde{\alpha} & \searrow \alpha & & & \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \tilde{\mathcal{E}} & \longrightarrow & \mathcal{E} \longrightarrow 0 \end{array} \quad (2.14)$$

On the other hand, let us compute $\mathbb{H}\text{om}(\mathcal{F} \xrightarrow{\alpha} \mathcal{E}, \mathcal{E})$. Fix an injective resolution \mathcal{I}^\bullet of \mathcal{E} . Applying Hom-functor, one produces a double complex

$$\begin{array}{ccccccc} \text{Hom}(\mathcal{E}, \mathcal{I}^0) & \xrightarrow{d_0 \circ -} & \text{Hom}(\mathcal{E}, \mathcal{I}^1) & \xrightarrow{d_1 \circ -} & \text{Hom}(\mathcal{E}, \mathcal{I}^2) & \longrightarrow & \dots \\ \downarrow -\circ \alpha & & \downarrow -\circ \alpha & & \downarrow -\circ \alpha & & \\ \text{Hom}(\mathcal{F}, \mathcal{I}^0) & \xrightarrow{d_0 \circ -} & \text{Hom}(\mathcal{F}, \mathcal{I}^1) & \xrightarrow{d_1 \circ -} & \text{Hom}(\mathcal{F}, \mathcal{I}^2) & \longrightarrow & \dots \end{array}$$

Taking cohomology of its total complex, we get

$$\begin{aligned} \mathbb{H}\text{om}(\mathcal{F} \xrightarrow{\alpha} \mathcal{E}, \mathcal{E}) &= \frac{\{(f, g) \in \text{Hom}(\mathcal{E}, \mathcal{I}^1) \oplus \text{Hom}(\mathcal{F}, \mathcal{I}^0) : d_1 \circ f = 0, d_0 \circ g = f \circ \alpha\}}{\{(d_0 \circ h, -h \circ \alpha) : h \in \text{Hom}(\mathcal{E}, \mathcal{I}_0)\}} \\ &= \frac{\{(f, g) \in \text{Hom}(\mathcal{E}, \text{Ker } d_1) \oplus \text{Hom}(\mathcal{F}, \mathcal{I}^0) : d_0 \circ g = f \circ \alpha\}}{\{(d_0 \circ h, -h \circ \alpha) : h \in \text{Hom}(\mathcal{E}, \mathcal{I}_0)\}}. \end{aligned}$$

But by Yoneda construction, pullback of the extension $0 \rightarrow \mathcal{E} \rightarrow \mathcal{I}^0 \rightarrow \text{Ker } d_1 \rightarrow 0$ gives a bijection between self-extensions of \mathcal{E} and morphisms $\mathcal{E} \rightarrow \text{Ker } d_1$ up to the ones factorizing through \mathcal{I}_0 . Associating to every element $\rho \in \text{Ext}^1(\mathcal{E}, \mathcal{E})$ the corresponding extension $0 \rightarrow \mathcal{E} \rightarrow \mathcal{E}_\rho \xrightarrow{\pi_\rho} \mathcal{E} \rightarrow 0$, we get

$$\mathbb{H}\text{om}(\mathcal{F} \xrightarrow{\alpha} \mathcal{E}, \mathcal{E}) = \{(\rho \in \text{Ext}^1(\mathcal{E}, \mathcal{E}), g : \mathcal{F} \rightarrow \mathcal{E}_\rho) : g \circ \pi_\rho = \alpha\},$$

which is precisely the space of infinitesimal deformations of (\mathcal{E}, α) as seen above in the diagram (2.14). \square

From now on, let us suppose that $\mathcal{F} \simeq \mathbb{k}^n \otimes \mathcal{O}$. Following [73], we call the pairs $(\mathcal{E}, \mathbb{k}^n \otimes \mathcal{O} \xrightarrow{\alpha} \mathcal{E})$ *Bradlow pairs of weight n* , and denote

$$\mathcal{B}r_{0,d}^n := \underline{\text{Coh}}_{0,d}^{\leftarrow \mathbb{k}^n \otimes \mathcal{O}}.$$

Remark 2.6.2. We use the word “weight” as in “highest weight representation” ; see Example 2.7.3 for an explanation of this choice.

Consider the cotangent stack $T^* \mathcal{B}r_{0,d}^n$. Thanks to Serre duality and the proposition above, we can describe the points of this stack rather explicitly. In what follows, we denote by $\omega = \omega_C$ the canonical bundle on C .

Definition 2.6.3. A *Higgs triple* of rank 0, degree d and weight n is the data $(\mathcal{E}, \alpha, \theta)$ of a torsion sheaf $\mathcal{E} \in \text{Coh}_{0,d} C$, a morphism $\alpha : \mathbb{k}^n \otimes \mathcal{O} \rightarrow \mathcal{E}$, and another morphism $\theta \in \mathbb{E}xt^1(\mathcal{E}, (\mathbb{k}^n \otimes \mathcal{O} \xrightarrow{\alpha} \mathcal{E}) \otimes \omega)$. Given two Higgs triples $T_1 = (\mathcal{E}_1, \alpha_1, \theta_1)$, $T_2 = (\mathcal{E}_2, \alpha_2, \theta_2)$, a *morphism* from T_1 to T_2 is a map $f \in \text{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ such that $\alpha_2 = f \circ \alpha_1$, and $\theta_2 \circ f = f \circ \theta_1$.

Remark 2.6.4. The same exact definition makes sense for positive rank and arbitrary framing sheaf. The author intends to study the properties of these more general objects in the future.

Definition 2.6.5. A Higgs triple is called *stable* if there doesn't exist a subsheaf $\mathcal{E}' \subset \mathcal{E}$ such that :

- $\text{Im } \alpha \subset \mathcal{E}'$, and
- $a(\theta) \in \text{Im}(b)$, where a, b are the maps below, induced by inclusion $\mathcal{E}' \subset \mathcal{E}$:

$$\mathbb{E}xt^1(\mathcal{E}, (\mathbb{k}^n \otimes \mathcal{O} \xrightarrow{\alpha} \mathcal{E}) \otimes \omega) \xrightarrow{a} \mathbb{E}xt^1(\mathcal{E}', (\mathbb{k}^n \otimes \mathcal{O} \xrightarrow{\alpha} \mathcal{E}) \otimes \omega) \xleftarrow{b} \mathbb{E}xt^1(\mathcal{E}', (\mathbb{k}^n \otimes \mathcal{O} \xrightarrow{\alpha} \mathcal{E}') \otimes \omega). \quad (2.15)$$

In other words, a triple is stable if the image of α generates \mathcal{E} under θ . Note that any subobject of a stable object is stable.

Recall that for an abelian category \mathcal{C} of homological dimension 1 every complex in the derived category $\mathcal{D}(\mathcal{C})$ is quasi-isomorphic to the direct sum of its shifted cohomology objects (see [37, Proposition 2.1.2]). Because of this observation, we can alternatively write Higgs triples as quadruples

$$(\mathcal{E}, \alpha, \theta_e, \theta_h) : \quad \mathcal{E} \in \text{Coh}_{0,d}, \quad \alpha : \mathbb{k}^n \otimes \mathcal{O} \rightarrow \mathcal{E}, \\ \theta_e \in \text{Ext}^1(\mathcal{E}, \text{Ker } \alpha \otimes \omega), \quad \theta_h \in \text{Hom}(\mathcal{E}, \text{Coker } \alpha \otimes \omega).$$

Lemma 2.6.6. *A Higgs triple is stable if and only if there are no non-trivial subsheaves $\mathcal{E}' \subset \mathcal{E}$ such that $\text{Im } \alpha \subset \mathcal{E}'$ and $\theta_h(\mathcal{E}') \subset (\mathcal{E}' / \text{Im } \alpha) \otimes \omega$.*

Remark 2.6.7. Note that the stability condition doesn't depend on θ_e in this form.

Proof. Replacing the complex $\mathbb{k}^n \otimes \mathcal{O} \xrightarrow{\alpha} \mathcal{E}$ by the sum of its kernel and cokernel, the diagram (2.15) splits into two :

$$\text{Ext}^1(\mathcal{E}, \text{Ker } \alpha \otimes \omega) \xrightarrow{a_e} \text{Ext}^1(\mathcal{E}', \text{Ker } \alpha \otimes \omega) \xleftarrow{b_e} \text{Ext}^1(\mathcal{E}', \text{Ker } \alpha \otimes \omega), \\ \text{Hom}(\mathcal{E}, \text{Coker } \alpha \otimes \omega) \xrightarrow{a_h} \text{Hom}(\mathcal{E}', \text{Coker } \alpha \otimes \omega) \xleftarrow{b_h} \text{Hom}(\mathcal{E}', (\mathcal{E}' / \text{Im } \alpha) \otimes \omega).$$

Note that the map b_e is an isomorphism. Therefore, the condition $a(\theta) \in \text{Im}(b)$ is equivalent to $a_h(\theta_h) \in \text{Im}(b_h)$, that is $\theta_h(\mathcal{E}') \subset (\mathcal{E}' / \text{Im } \alpha) \otimes \omega$. \square

We say that a morphism of triples is a *quotient*, if the underlying map of sheaves is surjective.

Lemma 2.6.8. *Let $T = (\mathcal{E}, \alpha, \theta)$, $T' = (\mathcal{E}', \alpha', \theta')$ be two triples, together with a quotient map $\pi : \mathcal{E} \rightarrow \mathcal{E}'$. If T is stable, then T' is stable as well.*

Proof. Suppose that T' is not stable, that is there exists a subsheaf $\mathcal{E}'_1 \subset \mathcal{E}'$, such that $\text{Im } \alpha' \subset \mathcal{E}'_1$ and $\theta'(\mathcal{E}'_1) \subset (\mathcal{E}'_1 / \text{Im } \alpha') \otimes \omega$. Consider its preimage $\mathcal{E}_1 := \pi^{-1}(\mathcal{E}'_1)$. Since $\alpha' = \pi \circ \alpha$ by definition, we get $\text{Im } \alpha \subset \mathcal{E}_1$.

Let us denote $U = \text{Ker } \pi$. Since $\theta' \circ \pi = \pi \circ \theta$, we have

$$\theta_h(U) \subset (U / (U \cap \text{Im } \alpha)) \otimes \omega \subset (\mathcal{E}_1 / \text{Im } \alpha) \otimes \omega. \quad (2.16)$$

Moreover, since $\mathcal{E}_1 / U \simeq \mathcal{E}'_1$, we get

$$\theta_h(\mathcal{E}_1) / (U + \text{Im } \alpha) = \theta'_h(\mathcal{E}'_1) \subset (\mathcal{E}'_1 / \text{Im } \alpha') \otimes \omega = (\mathcal{E}_1 / (U + \text{Im } \alpha)) \otimes \omega. \quad (2.17)$$

Combining (2.16) and (2.17), we conclude that $\theta_h(\mathcal{E}_1) \subset (\mathcal{E}_1 / \text{Im } \alpha) \otimes \omega$. Therefore $\mathcal{E}_1 \subset \mathcal{E}$ is a destabilizing subsheaf by Lemma 2.6.6, and thus instability of T' implies instability of T . \square

The following lemma can be viewed as an avatar of Schur's lemma.

Lemma 2.6.9. *Stables Higgs triples have no non-trivial automorphisms.*

Proof. Let $T = (\mathcal{E}, \alpha, \theta)$ be a stable Higgs triple, and suppose $f \in \text{End}(\mathcal{E})$ induces an automorphism of T . We pose $\mathcal{E}' = \text{Ker}(f - \text{id}_{\mathcal{E}}) \subset \mathcal{E}$. Since $f \circ \alpha = \alpha$, we have $\text{Im } \alpha \subset \mathcal{E}'$. Moreover, by definition $f|_{\mathcal{E}'} = \text{id}_{\mathcal{E}'}$. Therefore the equality $\theta_h \circ f = f \circ \theta_h$ implies that $(f - \text{id}_{\text{Coker } \alpha}) \circ \theta_h|_{\mathcal{E}'} = 0$, and thus $\theta_h(\mathcal{E}') \subset \mathcal{E}' / \text{Im } \alpha$. This means that \mathcal{E}' is a destabilizing subsheaf, which can only happen for $\mathcal{E}' = \mathcal{E}$. Thus $f = \text{id}_{\mathcal{E}}$. \square

Let us now state the main theorem of this section :

Theorem 2.6.10. *The moduli stack of stable Higgs triples of degree d and weight n can be represented by a smooth quasi-projective variety $\mathcal{B}(d, n)$.*

We will prove this theorem by relating stability of Higgs triples to Mumford's GIT stability [52]. Recall that by Lemma 2.3.4 and Proposition 2.3.2(3) the stack $T^* \mathcal{B}r_{0,d}^n$ is a global quotient stack, namely

$$T^* \mathcal{B}r_{0,d}^n \simeq [T_{G_d}^*(\text{Hom}(\mathbb{k}^n, \mathbb{k}^d) \times \text{Quot}_{0,d}^\circ) / G_d] \subset [T^*(\text{Hom}(\mathbb{k}^n, \mathbb{k}^d) \times \text{Quot}_{0,d}^\circ) / G_d]. \quad (2.18)$$

It is tempting to appeal to Hilbert-Mumford criterion [52, Theorem 2.1] in order to deduce a numerical description of stable points in the latter variety. However, it is not projective, so we cannot pursue this approach directly. Instead, we will consider a slight variation of Hilbert-Mumford criterion.

Let X be a smooth projective variety equipped with an action of a reductive group G , and $E \xrightarrow{\pi} X$ a G -equivariant vector bundle of rank r . Let us also fix an ample G -equivariant line bundle $L \rightarrow X$. Recall that we have an open immersion $\mathbb{G}_m \hookrightarrow \mathbb{G}_a \simeq \mathbb{A}^1$.

Definition 2.6.11. Let $v \in E$ and $\lambda : \mathbb{G}_m \rightarrow G$ be a cocharacter of G , such that there exists a \mathbb{G}_m -equivariant extension of the map $\lambda_v : \mathbb{G}_m \rightarrow E$, $t \mapsto \lambda(t).v$ to \mathbb{A}^1 :

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\lambda_v} & E \\ \downarrow & & \parallel \\ \mathbb{A}^1 & \xrightarrow{\tilde{\lambda}_v} & E \end{array}$$

In this case we say that λ_v admits a limit $l = \tilde{\lambda}_v(0)$, and define

$$\mu_{\pi^* L}(v, \lambda) = \mu(v, \lambda) = \text{wt}_{\mathbb{G}_m}(\pi^* L|_l).$$

Note that since E is separable, such limit is unique if it exists, therefore μ is well-defined. When such limit doesn't exist, we pose $\mu(v, \lambda) = -\infty$. The following lemma is immediate.

Lemma 2.6.12. *Let X_1, X_2 be two G -varieties equipped with G -equivariant line bundles $L_1 \rightarrow X_1, L_2 \rightarrow X_2$. Then*

$$\mu_{L_1 \boxtimes L_2}((x_1, x_2), \lambda) = \mu_{L_1}(x_1, \lambda) + \mu_{L_2}(x_2, \lambda)$$

for any $x_1 \in X_1, x_2 \in X_2, \lambda : \mathbb{G}_m \rightarrow G$.

Definition 2.6.13. A point $v \in E$ is (L) -stable if $\dim \text{Stab}(v) = 0$, and for any cocharacter $\lambda : \mathbb{G}_m \rightarrow G$ such that λ_v admits a limit, one has $\mu(v, \lambda) < 0$.

Let $i : E \hookrightarrow \tilde{E} = \mathbb{P}_X(E \oplus \mathbb{k})$ be the projectivization of E , and denote by $\Omega(1)$ the line bundle on \tilde{E} associated to the divisor at infinity.

Proposition 2.6.14. *Suppose that $\text{Stab}_G(x)$ is connected for all $x \in X$. For N big enough, a point $v \in E$ is L -stable if and only if $i(v) \in \tilde{E}$ is $(L \otimes \Omega(N))$ -stable in the sense of Mumford's GIT.*

Proof. Let us denote $L(N) := L \otimes \Omega(N)$. Recall that by Hilbert-Mumford criterion [52, Theorem 2.1] a point $x \in \tilde{E}$ is stable if for any cocharacter $\lambda : \mathbb{G}_m \rightarrow G$ one has $\mu(x, \lambda) < 0$, and $\dim \text{Stab}_G(x) = 0$. Let us fix a cocharacter λ , and compute $\mu(i(v), \lambda)$ for a point $v \in E$. We denote $v' = \lim_{t \rightarrow 0} i(\lambda(t).v)$:

$$\mu_{L(N)}(i(v), \lambda) = \mu_{L(N)}(v', \lambda) = \mu_L(\pi(v'), \lambda) + \mu_{\Omega(N)}(v', \lambda) = \mu_L(\pi(v), \lambda) + \mu_{\Omega(N)}(v', \lambda).$$

Note that the second summand can be computed directly on the fiber of \tilde{E} over $\pi(v')$. Thus, we're reduced to computing $\mu_{\mathcal{O}_{\tilde{E}}(N)}(v, \lambda)$ for $v \in E = \mathbb{k}^r, \tilde{E} = \mathbb{P}(E \oplus \mathbb{k})$. One can suppose that $E = \bigoplus_i E_i$, where $\lambda(t).w = t^i w$ for $w \in E_i$. Let us write $v = \sum_i v_i, v_i \in E_i$, and consider two cases :

- $v \in \bigoplus_{i \geq 0} E_i$. Then the limit of λ_v is $[v_0 : 1] \in E$, and $\mu(v, \lambda) = 0$;
- $v \notin \bigoplus_{i \geq 0} E_i$. Let $i_0 = \min\{i : v_i \neq 0\}$. Then the limit of λ_v is $[v_{i_0} : 0] \notin E$, and $\mu(v, \lambda) = N i_0 \leq -N$.

Putting everything together, we see that

$$\mu_{L(N)}(i(v), \lambda) = \mu_L(v, \lambda)$$

if $v' \in E$, and $\mu_{L(N)}(i(v), \lambda) \leq \mu_L(v, \lambda) - N$ otherwise. However, we know that there exists a finite set S of cocharacters of G , such that $\mu_L(\pi(v), \lambda) < N$ for any cocharacter λ if and only if $\mu_L(\pi(v), \lambda) < N$ for $\lambda \in S$ (see [70, Example 1.5.1.18]). Let $N \geq \max_{\lambda \in S} \mu_L(\pi(v), \lambda) + 1$. Then for all λ such that $v' \notin E$ we have

$$\mu_{L(N)}(i(v), \lambda) \leq \mu_L(\pi(v), \lambda) - N < 0,$$

and thus $i(v)$ is $L(N)$ -stable if and only if $\mu_L(v, \lambda) < 0$ for all λ such that λ_v admits a limit in E . □

Corollary 2.6.15. *The set E^s of stable points is open in E , and the stack $[E^s/G]$ is represented by a quasi-projective variety.*

Proof. Follows directly from 2.6.14, coupled with Theorem 1.10 in [52]. □

Remark 2.6.16. One may wonder if Definition 2.6.13 can be used to generalize Hilbert-Mumford criterion to any quasi-projective variety E . Unfortunately, this is not the case. For example, consider $E = \mathbb{k}^2 \setminus \{0\}$ equipped with a \mathbb{G}_m -action of trace 0. Then λ_v admits no limit for any point $v \in E$ and any cocharacter λ . This means that stability condition is trivially satisfied for all points; however, the stack $[E/\mathbb{G}_m]$ is not separated.

Let $l > 0$ be an integer, and write $H = H^0(\mathcal{O}(l))$. For l big enough, we have a closed G_d -equivariant embedding $i_l : \mathcal{Q}uot_{0,d} \hookrightarrow \text{Grass}^d(\mathbb{k}^d \otimes H)$, which sends any quotient $\mathbb{k}^d \otimes \mathcal{O} \xrightarrow{p} \mathcal{E}$ to $H^0(\mathcal{K} \otimes \mathcal{O}(l)) \subset \mathbb{k}^d \otimes H$ (see proof of Proposition 2.3.7). In order to analyze the G_d -action on $\mathcal{Q}uot_{0,d}$, let us recall several properties of Grassmanians. To that end, we fix a vector space V and a cocharacter $\lambda : \mathbb{G}_m \rightarrow GL(V)$. We have $V = \bigoplus_i V_i$, where $V_i = \{v \in V \mid \lambda(t).v = t^i v\}$; set $V_{\geq i} = \bigoplus_{j \geq i} V_j$.

Lemma 2.6.17 ([44], Lemma 6.6.2). *Let $D \rightarrow \text{Grass}^d(V)$ be the determinant line bundle, and $K \in \text{Grass}^d(V)$ a subspace of V of codimension d . Denote $K_i = K \cap V_{\geq i}$, $\bar{K}_i = K_i/K_{i+1} \subset V_i$. Then $\lim_{t \rightarrow 0} \lambda(t).K = \bigoplus_i \bar{K}_i$, and $\mu_D(K, \lambda) = \sum_i i \dim(\bar{K}_i)$.*

Notation. Until the end of this section, we will denote quotients in $\mathcal{Q}uot_{0,d}$ by letter p instead of φ , in order to avoid conflict of notations later on.

Now choose $l > 0$ big enough, so that for any point $\mathbb{k}^d \otimes \mathcal{O} \xrightarrow{p} \mathcal{E}$ and for any $V \subset \mathbb{k}^d$ the sheaf $\mathcal{K} \cap (V \otimes \mathcal{O})$ is generated by global sections and has no higher cohomology groups. Denote by L_l the pullback of the determinant line bundle under the closed embedding $i_l : \mathcal{Q}uot_{0,d} \hookrightarrow \text{Grass}^d(\mathbb{k}^d \otimes H)$. Let also $\lambda : \mathbb{G}_m \rightarrow G_d$ be a cocharacter, which defines a weight decomposition $\mathbb{k}^d = \bigoplus_i V_i$ as above.

Proposition 2.6.18. *For any $(\mathbb{k}^d \otimes \mathcal{O} \xrightarrow{p} \mathcal{E}) \in \mathcal{Q}uot_{0,d}$, let $\mathcal{E}_i = p(V_{\geq i} \otimes \mathcal{O})$, and $\bar{\mathcal{E}}_i = \mathcal{E}_i/\mathcal{E}_{i+1}$. Then we have $\lim_{t \rightarrow 0} \lambda(t).p = \bigoplus_i (V_i \otimes \mathcal{O} \xrightarrow{p_i} \bar{\mathcal{E}}_i)$, where p_i is the natural subquotient of p , and $\mu_{L_l}(p, \lambda) = \dim H \sum_i i \dim V_i - \sum_i i \deg(\bar{\mathcal{E}}_i)$.*

Proof. Let $\mathcal{K}_i = \mathcal{K} \cap (V_{\geq i} \otimes \mathcal{O})$. By Lemma 2.6.17, we have :

$$\begin{aligned} i_l(\lim_{t \rightarrow 0} \lambda(t).p) &= \lim_{t \rightarrow 0} \lambda(t).H^0(\mathcal{K}(l)) = \bigoplus_i H^0(\mathcal{K}_i(l))/H^0(\mathcal{K}_{i+1}(l)) \\ &= \bigoplus_i i_l(p_i). \end{aligned}$$

Since i_l is a closed embedding, this proves the first claim. For the second claim, recall that l is chosen so that $H^1(\mathcal{K}_i(l)) = 0$, and invoke Lemma 2.6.17 once again :

$$\begin{aligned} \mu_{L_l}(p, \lambda) &= \mu_D(i_l(p), \lambda) = \sum_i i (h^0(\mathcal{K}_i(l)) - h^0(\mathcal{K}_{i+1}(l))) = \sum_i i (\dim H \otimes V_i + h^0(\mathcal{E}_{i+1}) - h^0(\mathcal{E}_i)) \\ &= \dim H \sum_i i \dim V_i - \sum_i i \deg(\bar{\mathcal{E}}_i), \end{aligned}$$

which provides the desired expression. \square

Proposition 2.6.19. *Let $(\mathbb{k}^d \otimes \mathcal{O} \xrightarrow{p} \mathcal{E}) \in \mathcal{Q}uot_{0,d}$, and $\beta \in \text{Hom}(\mathcal{K}, \mathcal{E})^* = T_p^* \mathcal{Q}uot_{0,d}$. Then β admits a limit in $T^* \mathcal{Q}uot_{0,d}$ under λ if and only if β restricts to \mathcal{E}_i for any i .*

Proof. The λ -orbit of p in $\mathcal{Q}uot_{0,d}$ together with the limit at 0 gives us a closed \mathbb{G}_m -equivariant embedding $\mathbb{A}^1 \rightarrow \mathcal{Q}uot_{0,d}$, where the \mathbb{G}_m -action on $\mathcal{Q}uot_{0,d}$ is given by λ . Since the limit of β , if it exists, lies over the image of $0 \in \mathbb{A}^1$, we can work with the restriction of the cotangent bundle to \mathbb{A}^1 . The natural map

$$T_{ij} := \text{Hom}(\mathcal{K}_i, \mathcal{E}/\mathcal{E}_{j+1})^* \rightarrow \text{Hom}(\mathcal{K}, \mathcal{E})^*$$

is injective for any i, j . Therefore, $(T_{ij})_{i,j}$ defines a double filtration, descending in i and ascending in j , on the cotangent bundle restricted to \mathbb{A}^1 . This filtration is \mathbb{G}_m -equivariant, therefore existence of limits is preserved if we pass to associated bigraded bundle

$$\bigoplus_{i,j} \text{Hom}(\mathcal{K}_i/\mathcal{K}_{i+1}, \bar{\mathcal{E}}_j)^*.$$

Note that each of the bundles $\mathrm{Hom}(\mathcal{K}_i/\mathcal{K}_{i+1}, \bar{\mathcal{E}}_j)^*$ is canonically trivialized over \mathbb{A}^1 . Moreover, λ acts on $\mathrm{Hom}(\mathcal{K}_i/\mathcal{K}_{i+1}, \bar{\mathcal{E}}_j)^*$, regarded as a vector space, by t^{i-j} . Collecting all our considerations, the limit of β under λ exists if and only if $\beta \in \sum_i T_{ii}$. A simple computation shows that

$$\begin{aligned} \sum_i \mathrm{Hom}(\mathcal{K}_i, \mathcal{E}/\mathcal{E}_{i+1})^* &= \bigcap_i (\mathrm{Hom}(\mathcal{K}_i, \mathcal{E})^* \cup \mathrm{Hom}(\mathcal{K}, \mathcal{E}/\mathcal{E}_i)^*) \\ &= \bigcap_i \mathrm{Ker}(\mathrm{Hom}(\mathcal{K}, \mathcal{E})^* \rightarrow \mathrm{Hom}(\mathcal{K}/\mathcal{K}_i, \mathcal{E}_i)^*). \end{aligned}$$

But the remark after Definition 2.3.10 shows that the intersection of kernels on the right-hand side is precisely the set of elements in $\mathrm{Hom}(\mathcal{K}, \mathcal{E})^*$ which restrict to \mathcal{E}_i for any i . \square

Observe also that $\varphi \in \mathrm{Hom}(\mathbb{k}^n, \mathbb{k}^d)$ admits a limit under λ if and only if $\mathrm{Im} \varphi \subset V_{\geq 0}$, and $\psi \in \mathrm{Hom}(\mathbb{k}^d, \mathbb{k}^n)$ admits a limit if and only if $V_{\geq 1} \subset \mathrm{Ker} \psi$.

Proof of Theorem 2.6.10. Let us consider the G_d -variety $Y := T^*(\mathrm{Hom}(\mathbb{k}^n, \mathbb{k}^d) \times \mathrm{Quot}_{0,d})$, which is a vector bundle over $\mathrm{Quot}_{0,d}$. We can equip variety $\mathrm{Quot}_{0,d}$ with a G_d -equivariant ample line bundle $L_{l,h} := L_l \otimes L_{det}^{\otimes h}$, where L_{det} is the character of G_d given by determinant, and $h = -\dim H$. Let us compute the set of $L_{l,h}$ -stable points in Y . Let $\lambda : \mathbb{G}_m \rightarrow G_d$ be a cocharacter with corresponding weight decomposition $\mathbb{k}^d = \bigoplus_i V_i$. Proposition 2.6.19 and remarks after it show that for any given quadruple $x = (p, \beta, \varphi, \psi) \in Y$, λ_x admits a limit if and only if :

- β restricts to \mathcal{E}_i for all i ;
- $\mathrm{Im} \varphi \subset V_{\geq 0}$;
- $V_{\geq 1} \subset \mathrm{Ker} \psi$.

By Proposition 2.6.18 and Lemma 2.6.12, for any such quadruple x and cocharacter λ we have

$$\mu_{L_{l,h}}(x, \lambda) = h \sum_i i \dim V_i - \sum_i i \deg(\bar{\mathcal{E}}_i) - h \sum_i i \dim V_i = - \sum_i i \deg(\bar{\mathcal{E}}_i).$$

Note that if $\mathcal{E}_0 = \mathcal{E}$, we have

$$\mu(x, \lambda) = - \sum_{i \geq 0} i \deg(\bar{\mathcal{E}}_i) < 0,$$

so that such cocharacter bears no relevance on stability. Let us therefore assume that \mathcal{E}_0 is strictly contained in \mathcal{E} . Given such quadruple, consider another cocharacter λ_2 , which acts trivially on $V_{\geq 0}$, and with weight -1 on V_i for all negative i . We have

$$\mu(x, \lambda_2) = - \sum_i i \deg(\bar{\mathcal{E}}_i) = \deg(\mathcal{E}) - \deg(\mathcal{E}_0) > 0.$$

This analysis shows that a quadruple $(p, \beta, \varphi, \psi)$ is stable if and only if

$$\exists \mathcal{E}' \subset \mathcal{E} \quad \text{such that} \quad \mathrm{Im} \varphi \subset \mathcal{E}', \quad \beta \text{ restricts to } \mathcal{E}'.$$

Recall that we have the following cartesian square :

$$\begin{array}{ccc} Z := T_{G_d}^*(\mathrm{Hom}(\mathbb{k}^n, \mathbb{k}^d) \times \mathrm{Quot}_{0,d}^\circ) & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ T^* \mathcal{B}r_{0,d}^n & \hookrightarrow & [Y/G_d] \end{array}$$

In order to compare the stability condition on Z induced from Y to the one for Higgs triples, we have to understand the map $\pi : Z \rightarrow T^* \mathcal{B}r_{0,d}^n$. Let us fix a quadruple $x = (\mathbb{k}^d \otimes \mathcal{O} \xrightarrow{p} \mathcal{E}, \beta, \varphi, \psi)$,

and compute its image $\pi(x) := (\mathcal{E}', \alpha, \theta)$. First, it's evident that $\mathcal{E}' = \mathcal{E}$, $\alpha = p \circ \varphi$. Next, given p and φ , let us look at the corresponding fibre of Z :

$$Z_{\mathcal{E}, \varphi} = \{(\beta, \psi) : \iota^* \beta = (\varphi \otimes \omega) \circ \psi\}.$$

This equality can be understood as commutativity of the following diagram in $\mathcal{D}(\text{Coh } C)$:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\psi} & \mathbb{k}^n \otimes \omega[1] \\ \downarrow \beta & & \downarrow \varphi \otimes \omega \\ \mathcal{K} \otimes \omega[1] & \longrightarrow & \mathbb{k}^d \otimes \omega[1] \end{array}$$

where a number in square brackets denotes cohomological shift. Providing two maps (β, ψ) is equivalent to giving a map from \mathcal{E} to a certain cone :

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\psi} & \mathbb{k}^n \otimes \omega[1] \\ \searrow \beta & & \downarrow \varphi \otimes \omega \\ \text{Cone} \left(\mathcal{K} \oplus (\mathbb{k}^n \otimes \mathcal{O}) \xrightarrow{(\iota, \varphi)} \mathbb{k}^d \otimes \mathcal{O} \right) \otimes \omega[1] & \longrightarrow & \mathbb{k}^n \otimes \omega[1] \\ \downarrow \beta & & \downarrow \varphi \otimes \omega \\ \mathcal{K} \otimes \omega[1] & \longrightarrow & \mathbb{k}^d \otimes \omega[1] \end{array}$$

However, this cone is quasi-isomorphic to the complex $\mathbb{k}^n \otimes \mathcal{O} \xrightarrow{\alpha} \mathcal{E}$, thus the data of (β, ψ) is equivalent to an element of $\mathbb{E}xt^1(\mathcal{E}, (\mathbb{k}^n \otimes \mathcal{O} \xrightarrow{\alpha} \mathcal{E}) \otimes \omega)$, which is precisely our θ .

Under this correspondence, we can also see that destabilizing subsheaves for a point $x \in Z$ get projected to destabilizing subsheaves for $\pi(x) = (\mathcal{E}, \alpha, \theta) \in T^* \mathcal{B}r_{0,d}^n$, and vice versa. Therefore, we have an isomorphism

$$(T^* \mathcal{B}r_{0,d}^n)^s \simeq [Z^s / G_d],$$

and Corollary 2.6.15 shows that $[Z^s / G_d]$ is represented by a variety $\mathcal{B}(d, n)$.

In order to finish the proof, it remains to show that the variety $\mathcal{B}(d, n)$ is smooth. Recall that for any hamiltonian action of a group G on a symplectic variety X , the cokernel of the differential of the moment map at a point $p \in X$ is dual to the Lie algebra of $\text{Stab}_G(p)$:

$$\text{Coker}(X \xrightarrow{d\mu_p} \mathfrak{g}^*) \simeq \text{Lie}(\text{Stab}_G(p))^*.$$

Therefore, all points with trivial stabilizer are smooth. In particular, all stable Higgs triples lie in the smooth locus by Lemma 2.6.9, and thus $\mathcal{B}(d, n)$ is smooth. \square

For another description of varieties $\mathcal{B}(d, n)$, see Section 2.9.

Remark 2.6.20. Our stability condition implicitly depends on a parameter, namely h in the beginning of the proof of Theorem 2.6.10. One could ask what happens when we vary h . It's easy to see that for any $h < -\dim H$ stability condition stays the same. However, if we increase h , the notion of stability *will* eventually change; for instance, Hilbert-Chow morphism $\text{Hilb}_d(T^*C) \rightarrow \text{Sym}^d(T^*C)$ can be recovered in this picture as degeneration of $\mathcal{B}(d, 1)$ toward the closest special stability condition (see [70, Corollary 1.4.3.23] for an example of such degenerations). This phenomenon becomes crucial when one considers moduli of Higgs triples of *positive* rank, which will be done in a future work [48].

2.7 Action on stable triples

In this section we will produce the action of $A\mathbf{Ha}_C^0$ on the A -theory of moduli spaces $\mathcal{B}(d, n)$. In order to do this, we will use the general machinery from the beginning of Section 2.4.

Let $d_\bullet = \{0 = d_0 \leq d_1 \leq \dots \leq d_k = d\}$, and F a vector space of dimension n . As always, we note $G = G_d$, $P = P_{d_\bullet}$. We put :

$$\tilde{Y} = \mathrm{Hom}(F, \mathbb{k}^d / \mathbb{k}^{d_{k-1}}) \times \mathcal{Q}uot_{0, d_\bullet}^\circ, \quad \tilde{V} = \mathrm{Hom}(F, \mathbb{k}^d) \times \widetilde{\mathcal{Q}uot}_{0, d_\bullet}, \quad \tilde{X}' = \mathrm{Hom}(F, \mathbb{k}^d) \times \mathcal{Q}uot_{0, d}^\circ.$$

We have a natural closed embedding $\tilde{g} : \tilde{V} \hookrightarrow \tilde{X}'$ and an affine fibration $\tilde{f} : \tilde{V} \rightarrow \tilde{Y}$. The formula (2.5) gives rise to a map in A -theory

$$m_{d_\bullet} : A_*^H(T_H^* \tilde{Y}) \rightarrow A_*^G(T_G^* \tilde{X}').$$

For instance, in the case $k = 2$ we get a map :

$$m_{d_1, d_2} : A\mathbf{Ha}_C^0[d_1] \otimes A_*(T^* \mathcal{B}r_{0, d-d_1}^n) \hookrightarrow A_*(\mathcal{H}iggs_{0, d_1} \times T^* \mathcal{B}r_{0, d-d_1}^n) \rightarrow A_*(T^* \mathcal{B}r_{0, d}^n).$$

Collecting these maps for all d_1, d_2 , we get a map

$$m : A\mathbf{Ha}_C^0 \otimes \mathbf{AM}_n \rightarrow \mathbf{AM}_n,$$

where $\mathbf{AM}_n = \bigoplus_d A_*(T^* \mathcal{B}r_{0, d}^n)$.

Proposition 2.7.1. *The map m defines a $A\mathbf{Ha}_C^0$ -module structure on \mathbf{AM}_n .*

Proof. The proof is mostly analogous to the proof of Theorem 2.4.2. Namely, using notations of that proof, let us consider the following varieties :

$$\begin{aligned} \tilde{X}_1 &= G \times_P (\mathrm{Hom}(F, \mathbb{k}^d / \mathbb{k}^{d_2}) \times \mathcal{Q}uot_{0, d_\bullet}^\circ); & \tilde{W}_1 &= G \times_{P'} (\mathrm{Hom}(F, \mathbb{k}^d) \times \widetilde{\mathcal{Q}uot}_{0, d_\bullet}); \\ \tilde{X}_2 &= G \times_{P'} (\mathrm{Hom}(F, \mathbb{k}^d / \mathbb{k}^{d_1}) \times \mathcal{Q}uot_{0, d_\bullet}^\circ); & \tilde{W}_2 &= G \times_P (\mathrm{Hom}(F, \mathbb{k}^d) \times \widetilde{\mathcal{Q}uot}_{0, d_\bullet}); \\ \tilde{X}_3 &= \mathrm{Hom}(F, \mathbb{k}^d) \times \mathcal{Q}uot_{0, d}^\circ; & \tilde{W}_3 &= G \times_P (\mathrm{Hom}(F, \mathbb{k}^d / \mathbb{k}^{d_1}) \times \mathcal{Q}uot_{0, d_1}^\circ \times \widetilde{\mathcal{Q}uot}_{0, d_\bullet}). \end{aligned}$$

Again, we have inclusions $\tilde{W}_i \hookrightarrow \tilde{X}_{i-1} \times \tilde{X}_{i+1}$. Taking into account Hom-terms, the proof of Lemma 2.4.3 easily implies that $\tilde{W}_2 = \tilde{W}_1 \times_{\tilde{X}_2} \tilde{W}_3$, and that the intersection $(\tilde{W}_1 \times \tilde{X}_1) \cap (\tilde{X}_3 \times \tilde{W}_3)$ inside $\tilde{X}_3 \times \tilde{X}_2 \times \tilde{X}_1$ is transversal. Finally, putting $\tilde{Z}_i = T_{\tilde{W}_i}^*(\tilde{X}_{i-1} \times \tilde{X}_{i+1})$ and contemplating the diagram with cartesian square below :

$$\begin{array}{ccccc} T^* \tilde{X}_1 & \longleftarrow & \tilde{Z}_3 & \longrightarrow & T^* \tilde{X}_2 \\ & \nearrow & \uparrow & & \uparrow \\ & & \tilde{Z}_2 & \longrightarrow & \tilde{Z}_1 \\ & & & \searrow & \downarrow \\ & & & & T^* \tilde{X}_3 \end{array} \quad (2.19)$$

we may conclude as in the proof of Theorem 2.4.2. \square

Recall that we have open embeddings $\mathcal{B}(d, n) \subset T^* \mathcal{B}r_{0, d}^n$. If we denote

$$A\mathcal{M}_n = \bigoplus_d A\mathcal{M}_n[d] := \bigoplus_d A(\mathcal{B}(d, n)),$$

the collection of these embeddings defines us a map of graded vector spaces

$$\mathbf{AM}_n \twoheadrightarrow A\mathcal{M}_n, \quad (2.20)$$

which is surjective if $A \neq H$ by Proposition 2.2.10.

Corollary 2.7.2. *There exists a \mathbf{AHa}_C^0 -module structure on $A\mathcal{M}_n$, such that the map (2.20) commutes with the action of \mathbf{AHa}_C^0 .*

Proof. Let us consider the following diagram :

$$\begin{array}{ccccc}
 (T_G^* \tilde{X})^{st} & \xleftarrow{\tilde{\Phi}'} & (\tilde{Z}_G)^{st} & \xrightarrow{\tilde{\Psi}'} & (T_G^* \tilde{X}')^{st} \\
 \downarrow i & & \downarrow i & & \downarrow i \\
 T_G^* \tilde{X} & \xleftarrow{\tilde{\Phi}} & \tilde{Z}_G & \xrightarrow{\tilde{\Psi}} & T_G^* \tilde{X}'
 \end{array} \tag{2.21}$$

where

$$\begin{aligned}
 (T_G^* \tilde{X}')^{st} &= T_G^* \tilde{X}' \times_{T^* \underline{\mathbf{Br}}_{0,d}^n} \mathcal{B}(d, n), \\
 (T_G^* \tilde{X})^{st} &= T_G^* \tilde{X} \times_{\underline{\mathbf{Higgs}}_{0,d_1} \times T^* \underline{\mathbf{Br}}_{0,d-d_1}^n} \left(\underline{\mathbf{Higgs}}_{0,d_1} \times \mathcal{B}(d-d_1, n) \right), \\
 (\tilde{Z}_G)^{st} &= \tilde{Z}_G \cap \left((T_G^* \tilde{X})^{st} \times (T_G^* \tilde{X}')^{st} \right).
 \end{aligned}$$

Recall that quotients of stable triples are stable by Lemma 2.6.8. Therefore we have an equality

$$\tilde{Z}_G \cap \left(T_G^* \tilde{X} \times (T_G^* \tilde{X}')^{st} \right) = \tilde{Z}_G \cap \left((T_G^* \tilde{X})^{st} \times (T_G^* \tilde{X}')^{st} \right),$$

which shows that the map $\tilde{\Psi}'$ is proper, and right square in the diagram above is cartesian. Hence, we have

$$i^* \circ \tilde{\Psi}_* = \tilde{\Psi}'_* \circ i^*$$

by Lemma 2.2.12. This shows us that the diagram (2.21) defines a commutative square

$$\begin{array}{ccc}
 \mathbf{AHa}_C^0 \otimes \mathbf{AM}_n & \xrightarrow{m} & \mathbf{AM}_n \\
 \downarrow \text{id} \otimes \pi & & \downarrow \pi \\
 \mathbf{AHa}_C^0 \otimes A\mathcal{M}_n & \xrightarrow{m'} & A\mathcal{M}_n
 \end{array}$$

where $m' = (\tilde{\Psi}')_* \circ (\tilde{\Phi}')^!$. Moreover, if we replace all varieties in diagram (2.19) by open subvarieties of stable points as above, we can equally see that the upper right square remains cartesian. Therefore the map m' defines a \mathbf{AHa}_C^0 -module structure on $A\mathcal{M}_n$. \square

Since the whole construction is T -equivariant, we also obtain an action of $\mathbf{AHa}_C^{0,T}$ on $\mathbf{AM}_n^T := \bigoplus_d A_*^T(T^* \underline{\mathbf{Br}}_{0,d}^n)$ and $A\mathcal{M}_n^T := \bigoplus_d A_*^T(\mathcal{B}(d, n))$.

Example 2.7.3. Suppose $C = \mathbb{A}^1$, and equip it with the natural action of \mathbb{G}_m of weight 1 as in Example 2.4.4. In this setting, for $A = H$ and $A = K$ we recover algebras and representations constructed in [68, Proposition 6.2] and [69, Proposition 7.9] respectively.

We finish this section by comparing our results with the classical construction by Grojnowski and Nakajima. Recall [54, Chapter 8] that for any smooth surface S there exists an action of Heisenberg algebra on $\bigoplus_d H_*(\text{Hilb}_d S)$. More precisely, for any positive k and any homology class $\alpha \in H_*(X)$ we possess an operator $P_\alpha[k]$, given as follows :

$$\begin{aligned}
 P_\alpha[i](\beta) &= q_* \circ p^!(\alpha \boxtimes \beta), \quad \beta \in H_*(\text{Hilb}_d S) \\
 Z^\Delta &= \{(\mathcal{I}_1, \mathcal{I}_2) \mid \mathcal{I}_1 \supset \mathcal{I}_2, |\text{supp}(\mathcal{I}_1/\mathcal{I}_2)| = 1\} \subset \text{Hilb}_d S \times \text{Hilb}_{d+k} S \\
 p : Z^\Delta &\rightarrow S \times \text{Hilb}_d S, \quad (\mathcal{I}_1, \mathcal{I}_2) \mapsto (\text{supp}(\mathcal{I}_1/\mathcal{I}_2), \mathcal{I}_1); \\
 q : Z^\Delta &\rightarrow \text{Hilb}_{d+k} S, \quad (\mathcal{I}_1, \mathcal{I}_2) \mapsto \mathcal{I}_2.
 \end{aligned}$$

We now suppose that $S = T^*C$. Let us compare this action with the $H\mathbf{Ha}_C^0$ -action on $H\mathcal{M}_1$. In view of Corollary 2.9.5, $H\mathcal{M}_1 = H_*(\mathrm{Hilb}_d T^*C)$. Recall that $\underline{\mathcal{Higgs}}_{0,k} \simeq \underline{\mathcal{Coh}}_{0,k}(T^*C)$, where the latter stack parametrizes coherent sheaves of length k on T^*C . Therefore, the correspondence defining the $H\mathbf{Ha}_C^0$ -module structure on $H\mathcal{M}_1$ can be identified with the lower row in the following diagram with cartesian square :

$$\begin{array}{ccccc}
 T^*C \times \mathrm{Hilb}_d(T^*C) & & & & \\
 \uparrow s \times \mathrm{id} & \swarrow p & & & \\
 \underline{\mathcal{Coh}}_{0,k}^\Delta(T^*C) \times \mathrm{Hilb}_d(T^*C) & \xleftarrow{\Phi^\Delta} & Z^\Delta & & \\
 \downarrow i \times \mathrm{id} & & \downarrow & \searrow q & \\
 \underline{\mathcal{Coh}}_{0,k}(T^*C) \times \mathrm{Hilb}_d(T^*C) & \xleftarrow{\tilde{\Phi}} & Z & \xrightarrow{\tilde{\Psi}} & \mathrm{Hilb}_{d+k}(T^*C)
 \end{array}$$

where

$$\begin{aligned}
 \underline{\mathcal{Coh}}_{0,k}^\Delta(T^*C) &= \{\mathcal{E} \in \mathrm{Coh}_{0,k}(T^*C) \mid |\mathrm{supp} \mathcal{E}| = 1\}, \\
 Z &= \{(\mathcal{I}_1, \mathcal{I}_2) \mid \mathcal{I}_1 \supset \mathcal{I}_2\} \subset \mathrm{Hilb}_d T^*C \times \mathrm{Hilb}_{d+k} T^*C,
 \end{aligned}$$

i is the natural closed embedding $\underline{\mathcal{Coh}}_{0,k}^\Delta(T^*C) \hookrightarrow \underline{\mathcal{Coh}}_0(T^*C)$, and $s : \underline{\mathcal{Coh}}_0^\Delta(T^*C) \rightarrow T^*C$ associates to each coherent sheaf its support. One would like to prove an equality of the form

$$q_* \circ p^! = (\tilde{\Psi}_* \circ \tilde{\Phi}^!) \circ ((i \times \mathrm{id})_* \circ (s \times \mathrm{id})^!), \quad (2.22)$$

so that the operators $P_\alpha[i]$ are realized by action of certain elements in $H\mathbf{Ha}_C^0$, supported at the diagonals $\underline{\mathcal{Coh}}_0^\Delta(T^*C)$. Unfortunately, the map s is too singular for a pullback to be well-defined. However, one can easily check that it's a locally trivial fibration with a fiber isomorphic to $[\mathcal{C}_k^{\mathrm{n},\mathrm{n}}/G_k]$, where

$$\mathcal{C}_k^{\mathrm{n},\mathrm{n}} := \{(x, y) \in (\mathfrak{g}_k)^2 : [x, y] = 0, x, y \text{ nilpotent}\}.$$

If the local system I_s^k on T^*C , given by homology groups of fibers of s , were trivial, $H_*(\underline{\mathcal{Coh}}_{0,k}^\Delta(T^*C))$ would be isomorphic to the direct product $H_*(T^*C) \otimes H_*^{G_k}(\mathcal{C}_k^{\mathrm{n},\mathrm{n}})$, and one would be able to define the pullback $s^!$ by $c \mapsto c \boxtimes \mathbf{1}$. After that, the identity (2.22) would follow once we proved that $p^! = (\Phi^\Delta)^! \circ (s^! \times \mathrm{id})$. In light of these considerations, let us state the following conjecture :

Conjecture 2.7.4. *The local system I_s^k is trivial, and the action of $P_\alpha[i]$ on $H\mathcal{M}_1 \simeq \bigoplus_d H_*(\mathrm{Hilb}_d T^*S)$ is given by $\iota_*(\alpha \boxtimes \mathbf{1}) \in H\mathbf{Ha}_C^0$.*

Note that Conjecture 2.7.4 is trivially satisfied for $k = 1$. Indeed, $\underline{\mathcal{Coh}}_{0,1}^\Delta(T^*C) \simeq \underline{\mathcal{Coh}}_{0,1}(T^*C) \simeq T^*C \times \mathrm{BG}_m$, thus the diagram above takes the following form :

$$\begin{array}{ccccc}
 T^*C \times \mathrm{Hilb}_d(T^*C) & & & & \\
 \uparrow s \times \mathrm{id} & \swarrow p & & & \\
 \mathrm{BG}_m \times T^*C \times \mathrm{Hilb}_d(T^*C) & \xleftarrow{\tilde{\Phi}} & Z = Z^\Delta & \xrightarrow{q = \tilde{\Psi}} & \mathrm{Hilb}_{d+1}(T^*C)
 \end{array}$$

Since the scheme Z is smooth by [11], pullbacks along all of the maps in triangle are well-defined, and therefore

$$q_* \circ p^! = q_* \circ p^* = (\tilde{\Psi}_* \circ \tilde{\Phi}^*) \circ (s \times \mathrm{id})^*,$$

which gives us a realization of operators $P_\alpha[1]$.

2.8 Injectivity of shuffle presentation

Definition 2.8.1. A Higgs sheaf (\mathcal{E}, θ) is called *nilpotent* if $\theta^N = 0$ for N big enough.

We denote the stack of nilpotent Higgs torsion sheaves by $\underline{\mathcal{Higgs}}_{0,d}^{nilp}$. It is a closed substack of $\underline{\mathcal{Higgs}}_{0,d}$, which has the following global quotient presentation :

$$\begin{aligned} \underline{\mathcal{Higgs}}_{0,d}^{nilp} &= [\mathcal{C}_d^{\bullet,n}/G_d], \\ \mathcal{C}_d^{\bullet,n} &= \left\{ (\mathbb{k}^d \otimes \mathcal{O} \xrightarrow{p} \mathcal{E}, \beta) : p \in \mathcal{Quot}_{0,d}^\circ, \beta \in \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega_X), \beta \text{ is nilpotent} \right\}, \end{aligned}$$

and the embedding $\underline{\mathcal{Higgs}}_{0,d}^{nilp} \hookrightarrow \underline{\mathcal{Higgs}}_{0,d}$ is given by the natural inclusion $\mathcal{C}_d^{\bullet,n} \hookrightarrow \mathcal{C}_d$.

Proposition 2.8.2. *Let G be a reductive group, and let $i : \Lambda \hookrightarrow X$ be a closed equivariant embedding of $G \times \mathbb{G}_m$ -varieties. Suppose that*

$$\{x \in X : \overline{G \cdot x} \cap \Lambda \neq \emptyset\} = \Lambda, \quad (2.23)$$

and assume that for any $x \in X$ the intersection $\overline{G \cdot x} \cap \Lambda$ is not empty. Then the pushforward along i induces an isomorphism of localized A -groups :

$$A^{G \times \mathbb{G}_m}(\Lambda) \otimes_{A_{\mathbb{G}_m}(pt)} \text{Frac}(A_{\mathbb{G}_m}(pt)) \xrightarrow{\sim} A^{G \times \mathbb{G}_m}(X) \otimes_{A_{\mathbb{G}_m}(pt)} \text{Frac}(A_{\mathbb{G}_m}(pt)).$$

Proof. Note that our assumptions imply $X^{G \times \mathbb{G}_m} \subset \Lambda$. Furthermore, by Proposition 2.2.7 we can assume that G is a torus. Take $x \in X$, and let $(g, t) \in G \times \mathbb{G}_m$ lie in the stabilizer of x . Suppose that t has infinite order. Then $t^{-1} \cdot x = g \cdot x$, and by consequence $\overline{G \cdot x} \cap \Lambda = \overline{(G \times T) \cdot x} \cap \Lambda$ is non-empty, so that $x \in \Lambda$. We conclude that for any $x \in X \setminus \Lambda$ there exists a positive number $N(x)$ such that $\text{Stab}(x) \subset G \times \mu_{N(x)}$. Since torus actions on finite type schemes always possess finitely many stabilizers, one can assume that $N = N(x)$ doesn't depend on x . Let us consider the following character of $G \times \mathbb{G}_m$:

$$\begin{aligned} \chi : G \times \mathbb{G}_m &\rightarrow \mathbb{G}_m, \\ (g, t) &\mapsto t^N. \end{aligned}$$

It is clear that for any $x \in X \setminus \Lambda$ one has $\text{Stab}_{G \times \mathbb{G}_m}(x) \subset \text{Ker } \chi$. Therefore by Proposition 2.2.13 one has an isomorphism

$$A^{G \times \mathbb{G}_m}(\Lambda)[c_1(\chi)^{-1}] \xrightarrow{\sim} A^{G \times \mathbb{G}_m}(X)[c_1(\chi)^{-1}],$$

which implies the desired result. □

Corollary 2.8.3. *The natural map*

$$A^T(\underline{\mathcal{Higgs}}_{0,d}^{nilp}) \rightarrow A^T(\underline{\mathcal{Higgs}}_{0,d})$$

becomes an isomorphism upon tensoring by $\text{Frac}(A_T(pt))$.

Proof. Take $X = \mathcal{C}_d$, $\Lambda = \mathcal{C}_d^{\bullet,n}$. Any point in $(p, \beta) \in X \setminus \Lambda$ is separated from Λ by the characteristic polynomial of β . Therefore condition (2.23) is verified. Moreover, the action of T contracts any Higgs field to zero, that is for any $x \in X$ the intersection $\overline{T \cdot x} \cap \mathcal{Quot}_{0,d}^\circ \subset \overline{T \cdot x} \cap \Lambda$ is not empty. We conclude by invoking Proposition 2.8.2. □

From now on till the end of the section we suppose that $\mathbb{k} = \mathbb{C}$, and A is the usual Borel-Moore homology H .

Theorem 2.8.4. *The group $H_*^T(\underline{\mathcal{Higgs}}_{0,d}^{nilp})$ is torsion-free as an $H_{G \times T}^*(pt)$ -module.*

Corollary 2.8.5. *The morphism $\rho : H\mathbf{Ha}_C^{0,T} \rightarrow H\mathbf{Sh}_C$ of Theorem 2.5.5 becomes injective after tensoring by $\text{Frac}(A_T(pt))$.*

We will prove Theorem 2.8.4 in three steps :

1. shrink localization set ;
2. reduce question to $\underline{\mathcal{C}oh}_{0,d} \subset \underline{\mathcal{H}iggs}_{0,d}^{nilp}$;
3. explicit computation for $\underline{\mathcal{C}oh}_{0,d}$.

First of all, let $I \subset H_{G \times T}^*(pt)$ be the ideal of functions $f(t_1, \dots, t_d, t)$ such that $f(0, \dots, 0, t) = 0$. It is clear that $\mathbb{Q}[t] \cap I = \{0\}$. For any $H_{G \times T}^*(pt)$ -module M we denote by $M_{loc,I}$ its localization with respect to $H_{G \times T}^*(pt) \setminus I$.

Since $H_T^*(pt) \setminus \{0\} \subset H_{G \times T}^*(pt) \setminus I$, the localization theorem 2.2.14 yields an isomorphism

$$H_*^{G_d \times T}((\mathcal{C}^{\bullet,n})^T)_{loc,I} \xrightarrow{\simeq} H_*^{G_d \times T}(\mathcal{C}^{\bullet,n})_{loc,I}.$$

Note, however, that

$$H_*^{G_d \times T}((\mathcal{C}^{\bullet,n})^T)_{loc,I} \simeq H_*^{G_d \times T}(\mathcal{Q}uot_{0,d}^{\circ})_{loc,I} \simeq H_*^T(\underline{\mathcal{C}oh}_{0,d})_{loc,I},$$

where the T -action on the latter stack is trivial. By Poincaré duality and a result of Laumon [41, Théorème 3.3.1],

$$H_*(\underline{\mathcal{C}oh}_{0,d}) \simeq H^*(\underline{\mathcal{C}oh}_{0,d}) \simeq S^d(H^*(C \times B\mathbb{G}_m)) = S^d(H^*(C)[z]).$$

The $H_G^*(pt)$ -action on the latter space is given as follows. The natural free $\mathbb{Q}[z]$ -module structure on $H^*(C)[z]$ defines embeddings of algebras :

$$\begin{array}{ccc} H_G^*(pt) & \xlongequal{\quad} & S^d(\mathbb{Q}[z]) \hookrightarrow S^d(H^*(C)[z]) \\ & & \downarrow \qquad \qquad \downarrow \\ & & T^d(\mathbb{Q}[z]) \hookrightarrow T^d(H^*(C)[z]) \end{array} \quad (2.24)$$

The upper horizontal map defines us the desired action. Note that since lower horizontal map turns $T^d(H^*(C)[z])$ into a free $T^d(\mathbb{Q}[z])$ -module, the same is true for a $H_G^*(pt)$ -module $S^d(H^*(C)[z])$. In particular, this implies that the $H_{G \times T}^*(pt)$ -module $H_*^T(\underline{\mathcal{C}oh}_{0,d})_{loc,I}$ is torsion-free. Putting together the arguments above, we get the following result :

Proposition 2.8.6. *The group $H_*^T(\underline{\mathcal{H}iggs}_{0,d}^{nilp})_{loc,I}$ is torsion-free as a $H_{G \times T}^*(pt)$ -module.*

Next, let us break down the stack of nilpotent Higgs sheaves into more manageable pieces. Recall the following stratification of $\underline{\mathcal{H}iggs}_{0,d}^{nilp}$ due to Laumon [42].

Definition 2.8.7. For any partition $\nu \vdash d$, $\nu = (1^{\nu_1} 2^{\nu_2} \dots)$, let

$$\underline{\mathcal{N}il}_{\nu} = \left\{ (\mathcal{E}, \theta) \in \underline{\mathcal{H}iggs}_{0,d}^{nilp} : \deg(\text{Ker } \theta^i / (\text{Im } \theta \cap \text{Ker } \theta^i + \text{Ker } \theta^{i-1})) = \nu_i \text{ for all } i \right\}.$$

These substacks are locally closed, disjoint, and cover the whole stack of nilpotent Higgs sheaves :

$$\underline{\mathcal{H}iggs}_{0,d}^{nilp} = \bigsqcup_{\nu \vdash d} \underline{\mathcal{N}il}_{\nu}.$$

Proposition 2.8.8. *For any $\nu \vdash d$, there exists a stack vector bundle $p_{\nu} : \underline{\mathcal{N}il}_{\nu} \rightarrow \prod_i \underline{\mathcal{C}oh}_{0,\nu_i}$.*

Proof. See the proof of Proposition 5.2 in [50]. □

As a consequence of this proposition, for any $\nu \vdash d$ we have an isomorphism

$$p_\nu^* : H_*^T\left(\prod_i \underline{\mathcal{C}oh}_{0,\nu_i}\right) \xrightarrow{\sim} H_*^T(\underline{\mathcal{N}il}_\nu). \quad (2.25)$$

Let us choose a total order \prec on the set of partitions of d such that for any two partitions ν, ν' the inclusion $\underline{\mathcal{N}il}_\nu \subset \underline{\mathcal{N}il}_{\nu'}$ implies $\nu \prec \nu'$. Denote

$$\underline{\mathcal{N}il}_{\prec\nu} = \coprod_{\nu' \prec \nu} \underline{\mathcal{N}il}_{\nu'}; \quad \underline{\mathcal{N}il}_{\prec\nu} = \underline{\mathcal{N}il}_{\prec\nu} \sqcup \underline{\mathcal{N}il}_\nu.$$

For each ν , this order gives rise to a long exact sequence in Borel-Moore homology :

$$\cdots \rightarrow H_k^T(\underline{\mathcal{N}il}_{\prec\nu}) \rightarrow H_k^T(\underline{\mathcal{N}il}_{\prec\nu}) \rightarrow H_k^T(\underline{\mathcal{N}il}_\nu) \rightarrow \cdots \quad (2.26)$$

The homology groups $H_*(\underline{\mathcal{C}oh}_{0,d})$ comprise the \mathfrak{S}_d -invariant part of $H_*(X \times \mathbf{B}\mathbb{G}_m)^{\otimes d}$. Since the latter group is pure as a mixed Hodge structure, the same is true for the former one as well, and for $H_*^T(\underline{\mathcal{N}il}_\nu)$ for any ν . Since natural applications in homology are strictly compatible with the weight filtration (see for example [14]), an easy induction shows that both $H_*^T(\underline{\mathcal{N}il}_{\prec\nu})$ and $H_*^T(\underline{\mathcal{N}il}_{\prec\nu})$ are also pure. Furthermore, the long exact sequence (2.26) splits into short exact sequences :

$$0 \rightarrow H_k^T(\underline{\mathcal{N}il}_{\prec\nu}) \rightarrow H_k^T(\underline{\mathcal{N}il}_{\prec\nu}) \rightarrow H_k^T(\underline{\mathcal{N}il}_\nu) \rightarrow 0$$

These short exact sequences yield a filtration F_\bullet of $H_*^T(\underline{\mathcal{H}iggs}_{0,d}^{nilp})$, such that F_i is of the form $H_k^T(\underline{\mathcal{N}il}_{\prec\nu})$, and

$$\mathrm{gr}_F H_*^T(\underline{\mathcal{H}iggs}_{0,d}^{nilp}) = \bigoplus_{\nu \vdash d} H_*^T(\underline{\mathcal{N}il}_\nu). \quad (2.27)$$

Proof of Theorem 2.8.4. The desired statement is equivalent to injectivity of the localization morphism, which can be written as a composition of two successive localizations :

$$H_*^T(\underline{\mathcal{H}iggs}_{0,d}^{nilp}) \xrightarrow{l_I} H_*^T(\underline{\mathcal{H}iggs}_{0,d}^{nilp})_{loc,I} \rightarrow H_*^T(\underline{\mathcal{H}iggs}_{0,d}^{nilp})_{loc}.$$

The second map being injective by Proposition 2.8.6, it suffices to prove injectivity of l_I . Taking into account isomorphisms (2.27), (2.25), and the fact that $H_*^T(\underline{\mathcal{C}oh}_{0,d})$ is a free $H_T^*(pt)$ -module, we are reduced to proving the injectivity of

$$l_{\mathfrak{m}} : H_* \left(\prod_i \underline{\mathcal{C}oh}_{0,\nu_i} \right) \rightarrow H_* \left(\prod_i \underline{\mathcal{C}oh}_{0,\nu_i} \right)_{loc,\mathfrak{m}}$$

for any $\nu \vdash d$, where we localize at $\mathfrak{m} \subset H_G^*(pt)$ — maximal homogeneous ideal with respect to homological grading. Since $H_*(\prod_i \underline{\mathcal{C}oh}_{0,\nu_i})$ is evidently a graded $H_G^*(pt)$ -module, the annihilator of c is a graded ideal for any $c \in H_*(\prod_i \underline{\mathcal{C}oh}_{0,\nu_i})$, therefore fully contained in \mathfrak{m} . This proves injectivity of $l_{\mathfrak{m}}$ and concludes the proof of the theorem. \square

Corollary 2.8.9. *The identity (2.10) holds in $H\mathbf{Ha}_C^{0,T}$.*

Proof. Note that we have the following chain of isomorphisms :

$$\begin{aligned} H_{G_d \times T}^*(T^* \mathcal{Q}uot_{0,d}^\circ) &\simeq H_{G_d \times T}^*(\mathcal{Q}uot_{0,d}^\circ) \simeq H^*(\underline{\mathcal{C}oh}_{0,d}) \otimes H_T^*(pt) \\ &\simeq S^d(H^*(\underline{\mathcal{C}oh}_{0,1})) [t] \simeq (H^*(C^d)[t; t_1, \dots, t_d])^{\mathfrak{S}_d}. \end{aligned}$$

Therefore, the closed embedding $\mathcal{C}_d \hookrightarrow T^* \mathcal{Q}uot_{0,d}^\circ$ defines a $(H^*(C^d)[t; t_1, \dots, t_d])^{\mathfrak{S}_d}$ -module structure on $H\mathbf{Ha}_C^{0,T}$, compatible with ρ . In particular, the identity (2.10) is well-defined in $H\mathbf{Ha}_C^{0,T}$.

Combining Corollary 2.8.3 and Theorem 2.8.4, we see that the natural morphism

$$H_*^T(\underline{\mathcal{Higgs}}_{0,d}^{nilp}) \rightarrow H_*^T(\underline{\mathcal{Higgs}}_{0,d})$$

is injective. Furthermore, it easily follows from the construction in Section 2.4 that the space $\bigoplus_d H_*^T(\underline{\mathcal{Higgs}}_{0,d}^{nilp})$ is a subalgebra in $H\mathbf{Ha}_C^{0,T}$. Since $H_*^T(\underline{\mathcal{Higgs}}_{0,1}^{nilp}) = H_*^T(\underline{\mathcal{Coh}}_{0,1}) \simeq H_*^T(\underline{\mathcal{Higgs}}_{0,1})$, all operators intervening in identity (2.10) belong to $\bigoplus_d H_*^T(\underline{\mathcal{Higgs}}_{0,d}^{nilp})$, and we conclude by Corollary 2.8.5 and Proposition 2.5.12. \square

Remark 2.8.10. Note that for $A = H$ the function g_C^{norm} takes the following form :

$$g_C^{norm}(z) = \frac{e(z\mathcal{O}(-\Delta))e(tz\mathcal{O}(\Delta))}{e(z)e(tz)} = \frac{(z-\Delta)(z+t+\Delta)}{z(z+t)} = 1 - \frac{\Delta(t+\Delta)}{z(t+z)},$$

where $\Delta \in H^2(C \times C)$ is the class of diagonal. Using this explicit expression, we can rewrite the identity (2.10) as a set of relations. In particular, for $\mathcal{L}_1 = t_1^{-1}$, $\mathcal{L}_2 = t_2^{-1}$, we get

$$[e_i, e_j]_3 - (t^2 + \Delta(t + \Delta))[e_i, e_j]_1 + t\Delta(t + \Delta)(e_i e_j + e_j e_i) = 0$$

for any $i, j \in \mathbb{Z}$, where $e_i = t_1^i$, and

$$[e_i, e_j]_n := \sum_{k=0}^n (-1)^k \binom{n}{k} (e_{i+k} e_{j+n-k} - e_{j+n-k} e_{i+k}).$$

For general \mathcal{L}_1 and \mathcal{L}_2 the relations become more complicated.

Conjecture 2.8.11. *For any oriented Borel-Moore theory A , the morphism $\rho : A\mathbf{Ha}_C^{0,T} \rightarrow A\mathbf{Sh}_C$ of Theorem 2.5.5 is injective.*

We expect to prove Conjecture 2.8.11 in subsequent work [48] by analyzing the action of $A\mathbf{Ha}_C^{0,T}$ on $A\mathcal{M}_n^T$ for varying n .

2.9 Torsion-free sheaves on $\mathbb{P}(T^*C)$

In this section we provide an alternative description of $\mathcal{B}(d, n)$ as a moduli of sheaves on a surface.

Consider the surface $S := \mathbb{P}_C(\omega \oplus \mathcal{O})$, which compactifies the cotangent bundle T^*C . Denote by $D = D_\infty$ the complement of T^*C in S . Let also $i : D \hookrightarrow S$ be the natural embedding, and $\pi : S \rightarrow C$ the natural projection.

Let us recall some properties of sheaves on S ; we will closely follow the exposition in [49, Section 2]. The Neron-Severi group of S is given by $\text{NS}(S) = H^2(S, \mathbb{Z}) = \mathbb{Z}D \oplus \mathbb{Z}f$, where f is the class of a fiber of π . Thus, for any coherent sheaf E on S we will write $c_1(E) = c_{1,D}(E)D + c_{1,f}(E)f$. The product in $\text{NS}(S)$ is determined by the following formulas :

$$f^2 = 0, \quad Df = 1, \quad D^2 = 2 - 2g,$$

where the last equality follows from the fact that $\mathcal{O}_S(D)|_D \simeq \omega^{-1}$. Moreover, the canonical divisor of S is $K_S = -2D$. We will write elements of $H^{even}(S, \mathbb{Z})$ as triples $(a, b, c) \in \mathbb{Z} \oplus \text{NS}(S) \oplus \mathbb{Z}$; the same applies to $H^{even}(C, \mathbb{Z})$. In this fashion, Todd classes of S and C are respectively given by

$$\text{td } S = (1, D, 1 - g), \quad \text{td } C = (1, 1 - g),$$

and the push-forward along π in cohomology is given by

$$\pi_*(a, b_D D + b_f f, c) = (b_D, c).$$

With that in mind, given a sheaf $E \in \text{Coh } S$, the Chern character of its push-forward $R\pi_*E$ can be computed using Grothendieck-Riemann-Roch theorem. Namely, let $a = c_{1,D}(E)$, $b = c_{1,f}(E)$, $r = \text{rk } E$, and recall that

$$\text{ch}(E) = \left(r, c_1(E), \frac{c_1(E)^2 - 2c_2(E)}{2} \right) = (r, aD + bf, a^2(1-g) + ab - c_2(E)).$$

We have :

$$\begin{aligned} (\text{rk}(R\pi_*E), c_1(R\pi_*E) + (1-g)\text{rk}(R\pi_*E)) &= \text{ch}(R\pi_*E) \text{td } C = \pi_*(\text{ch } E \text{td } S) \\ &= \pi_*((r, aD + bf, a^2(1-g) + ab - c_2(E))(1, D, 1-g)) \\ &= (a+r, (r+a^2+2a)(1-g) + (a+1)b - c_2(E)). \end{aligned}$$

Therefore $\text{rk}(R\pi_*E) = a+r$, and $c_1(R\pi_*E) = (a+1)(a(1-g)+b) - c_2(E)$. In particular, suppose that $R^0\pi_*E = 0$, and $R^1\pi_*E$ is a torsion sheaf of degree d . Then $\text{rk}(R\pi_*E) = 0$, $c_1(R\pi_*E) = -d$, and we have

$$a = -r, \quad c_2(E) = d + (r-1)(r(1-g) - b).$$

For any nef divisor H on S , we can define a notion of *slope H -semistability* for sheaves on S . One example of nef divisor is given by f . Instead of giving general definitions, we will use the following characterization of f -semistable sheaves :

Lemma 2.9.1 ([49, Lemma 4.3]). *A torsion-free sheaf E on S is f -semistable if and only if its generic fiber over C is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(l)^{\oplus m}$ for some $l \in \mathbb{Z}$, $m \in \mathbb{N}$.*

Theorem 2.9.2. *Stable Higgs triples $(\mathcal{E}, \alpha, \theta) \in \mathcal{B}(d, n)$ are in one to one correspondence with f -semistable torsion-free sheaves E on S equipped with a trivialization $\varphi : E|_D \xrightarrow{\sim} \mathcal{O}_C^{\oplus n}$, satisfying the following numerical conditions :*

$$\text{rk } E = n, \quad c_{1,f}(E) = n(2-2g), \quad c_{1,D}(E) = -n, \quad c_2(E) = d + n(n-1)(g-1). \quad (2.28)$$

We will need the following technical result :

Lemma 2.9.3 (Derived see-saw principle). *Let $E^\bullet \in \mathcal{D}^b(\text{Coh } S)$ be a complex of sheaves, such that $R\pi_*(E^\bullet) = 0$. Then we have an isomorphism $E^\bullet \simeq \pi^*(N^\bullet) \otimes \mathcal{O}_S(-D)$, where $N^\bullet = R\pi_*(E^\bullet \otimes \mathcal{O}_S(D))$.*

Proof. If E is a coherent sheaf, then the condition $R\pi_*E = 0$ means that at each point of the base $c \in C$ the fiber E_c is isomorphic to a direct sum of several copies of $\mathcal{O}_{\mathbb{P}^1}(-1)$ [51, Corollary 5.4]. In particular, this implies that $E(D)_c$ is trivial at each point c , and thus the natural map $\pi^*\pi_*(E(D)) \rightarrow E(D)$ is an isomorphism. We can therefore conclude by taking N to be $\pi_*(E(D))$.

In the general case, we proceed by induction on the cohomological length of E^\bullet . Let E^\bullet be a complex of length $k+1$ starting in degree 0, and consider the following exact triangle :

$$\mathcal{H}^0(E^\bullet) \rightarrow E^\bullet \rightarrow E_{\geq 1}^\bullet \xrightarrow{+1}$$

Applying the functor $R\pi_*$ to the triangle above yields an isomorphism $R\pi_*(E_{\geq 1}^\bullet)[-1] \xrightarrow{\sim} R\pi_*\mathcal{H}^0(E^\bullet)$. However, cohomology of the complex on the left is concentrated in degrees strictly bigger than 1, while $R\pi_*\mathcal{H}^0(E^\bullet)$ lives in degrees 0 and 1. Therefore both complexes must be null-homotopic, and by induction

$$\begin{aligned} \mathcal{H}^0(E^\bullet) &\simeq \pi^*(R\pi_*\mathcal{H}^0(E^\bullet(D))) \otimes \mathcal{O}_S(-D), \\ E_{\geq 1}^\bullet &\simeq \pi^*(R\pi_*(E_{\geq 1}^\bullet(D))) \otimes \mathcal{O}_S(-D). \end{aligned}$$

Since both isomorphisms above are obtained by applying the counit map $\pi^*R\pi_* \Rightarrow \text{Id}_S$, it also induces an isomorphism $\pi^*R\pi_*(E^\bullet(D)) \xrightarrow{\sim} E^\bullet(D)$. \square

Recall [33, Section 3.4] that $R\pi_*$ admits a right adjoint $\pi^!$, given by

$$\pi^! : \mathcal{E}^\bullet \mapsto \pi^*(\mathcal{E}^\bullet) \otimes \omega_\pi[1],$$

where ω_π is the relative canonical bundle. The latter is given by

$$\omega_\pi = \omega_S \otimes \pi^*\omega^{-1} = \mathcal{O}_S(-2D) \otimes \pi^*\omega^{-1}.$$

Since $\mathcal{O}_S(D)|_D \simeq \omega^{-1}$, we deduce $i^*\omega_\pi \simeq \omega$.

Proof of Theorem 2.9.2. (1) Let us begin with an f -semistable torsion-free sheaf $E \in \text{Coh } S$, equipped with a trivialization $\varphi : E|_D \xrightarrow{\sim} \mathcal{O}_C^{\oplus n}$ and satisfying the numerical conditions (2.28). Lemma 2.9.1 implies that the generic fiber of E over C is isomorphic to $\mathcal{O}(-1)^{\oplus n}$. In particular, both π_*E and $R^1\pi_*E$ have finite support. Since E is torsion-free, π_*E is as well, so that $\pi_*E = 0$. Moreover, by computation at the beginning of the section $R^1\pi_*E$ is a torsion sheaf of degree d . All in all, we have $R\pi_*E[1] \simeq R^1\pi_*E =: \tilde{\mathcal{E}} \in \text{Coh}_{0,d} C$.

Adjunction $R\pi_* \dashv \pi^!$ provides us with the unit map $u : E \rightarrow \pi^!R\pi_*E$. It can be completed to a triangle with mapping cone $K := \text{Cone}(u)$:

$$E \xrightarrow{u} \pi^!R\pi_*E \rightarrow K \xrightarrow{+1} \quad (2.29)$$

If we apply $R\pi_*$ to this triangle, the map u gets sent to the identity $R\pi_*E \rightarrow R\pi_*E$. Therefore $R\pi_*K = 0$, and by Lemma 2.9.3 we have

$$K \simeq \pi^*(R\pi_*(K(D))) \otimes \mathcal{O}_S(-D) = \pi^!(R\pi_*(K(D)) \otimes \omega) \otimes \mathcal{O}_S(D)[-1] = \pi^!(N) \otimes \mathcal{O}_S(D),$$

where $N := R\pi_*(K(D)) \otimes \omega[-1]$.

Twisting the triangle (2.29) by $\mathcal{O}_S(-D)$ before pushing it down to C , we get :

$$R\pi_*(E(-D)) \rightarrow R\pi_*E \otimes (\mathcal{O} \oplus \omega) \rightarrow N \xrightarrow{+1}$$

where the middle term is computed using projection formula :

$$R\pi_*(\pi^!R\pi_*E(-D)) = R\pi_*E \otimes \omega^{-1} \otimes R\pi_*(\mathcal{O}(-3D)[1]) = R\pi_*E \otimes (\mathcal{O} \oplus \omega).$$

Let us denote $\mathcal{E} := R^1\pi_*E \otimes \omega = R\pi_*E \otimes \omega[1]$, and $W = N[1]$. We can then rewrite the triangle above as follows :

$$R\pi_*(E(-D))[1] \rightarrow (\mathcal{E} \otimes \omega^{-1}) \oplus \mathcal{E} \rightarrow W \xrightarrow{+1}$$

Let us denote the second map by $(\theta \otimes \omega^{-1}, \iota)$, where $\theta \in \mathbb{H}\text{om}(\mathcal{E}, W \otimes \omega)$ and $\iota \in \mathbb{H}\text{om}(\mathcal{E}, W)$. Restricting the triangle (2.29) to the divisor at infinity via the map i^* , we get :

$$\begin{aligned} i^*E \rightarrow R\pi_*E \otimes i^*(\omega_\pi[1]) \rightarrow N \otimes i^*(\omega_\pi(D)[1]) \xrightarrow{+1} \\ \Leftrightarrow \mathcal{O}^{\oplus n} \rightarrow R\pi_*E \otimes \omega[1] \rightarrow N[1] \xrightarrow{+1} \Leftrightarrow \mathcal{O}^{\oplus n} \rightarrow \mathcal{E} \rightarrow W \xrightarrow{+1} \end{aligned}$$

Let us denote the map $\mathcal{O}^{\oplus n} \rightarrow \mathcal{E}$ by α . We have thus obtained an isomorphism $W \simeq \text{Cone}(\alpha)$. In particular, θ gets identified with an element in $\mathbb{E}\text{xt}^1(\mathcal{E}, \mathcal{O}^{\oplus n} \xrightarrow{\alpha} \mathcal{E})$. Putting everything together, we obtain a Higgs triple $(\mathcal{E}, \alpha, \theta)$. We'll prove its stability later on.

(2) In the opposite direction, let us start with a stable triple $(\mathcal{E}, \alpha, \theta) \in \mathcal{B}(d, n)$. The natural map $\iota : \mathcal{E} \rightarrow \alpha[1]$ coupled with the Higgs field θ gives rise to a morphism $\mathcal{E} \otimes \omega^{-1} \otimes (\mathcal{O} \oplus \omega) \rightarrow \alpha[1]$. Since

$$\mathcal{E} \otimes \omega^{-1} \otimes (\mathcal{O} \oplus \omega) = R\pi_*(\pi^!(\mathcal{E} \otimes \omega^{-1}) \otimes \mathcal{O}_S(-D)),$$

by adjunction we obtain a map $\Psi : \pi^!(\mathcal{E} \otimes \omega^{-1}) \rightarrow \pi^!\alpha \otimes \mathcal{O}_S(D)[1]$. Define $E := \text{Cone}(\Psi)[-2]$. Since $\pi^!$ is obtained from π^* by applying an invertible map $(\pi^*(\omega^{-1}) \otimes -)[1]$, it suffices to study the map $\Psi' : \pi^*\mathcal{E} \rightarrow \pi^*(\alpha \otimes \omega) \otimes \mathcal{O}_S(D)[1]$.

Recall (see section 2.6) that as a complex α is quasi-isomorphic to $K \oplus J[-1]$, where $K = \text{Ker } \alpha$, $J = \text{Coker } \alpha$. Thus, we can express Ψ' as a sum of $\psi_e : \pi^* \mathcal{E} \rightarrow (\pi^* K \otimes \omega) \otimes \mathcal{O}_S(D)[1]$ and $\psi_h : \pi^* \mathcal{E} \rightarrow \pi^*(J \otimes \omega) \otimes \mathcal{O}_S(D)$. Let $M \in \text{Ext}^1(\pi^* \mathcal{E}, \pi^* K(D))$ be the extension given by ψ_e . Then the two-step complex given by Ψ' is quasi-isomorphic to $M \rightarrow \pi^* J \otimes \mathcal{O}_S(D)$, where the arrow is given by composing ψ_h with $M \rightarrow \pi^* \mathcal{E}$. Therefore the cone of Ψ' has length 1 if and only if ψ_h is surjective.

Lemma 2.9.4. *The map ψ_h is surjective if and only if the triple $(\mathcal{E}, \alpha, \theta)$ is stable.*

Proof. In order to simplify the notation, we will omit the subscript and write $\psi = \psi_h$ throughout the proof. It's easy to see that ψ is adjoint to

$$(\theta, \iota) : \mathcal{E} \rightarrow J \otimes (\omega \oplus \mathcal{O}),$$

where $\theta = \theta_h$, and $\iota : \mathcal{E} \rightarrow J$ is the natural projection. Let $c \in C$, and $s = \pi^{-1}(c) = \mathbb{P}(\omega_c \oplus \mathcal{O}_c)$. If we choose an identification $\omega_c \simeq \mathcal{O}_c$, the fiber ψ_s at the point s is given by a linear combination $a\theta_c + b\iota_c$ for some $a, b \in \mathbb{k}$. In particular, ψ is surjective if and only if it is surjective at each point $s \in S$, that is for every $c \in C$ and $[a : b] \in \mathbb{P}^1$ the map $a\theta_c + b\iota_c$ is surjective.

Suppose that ψ is not surjective. Then there exists a point $s \in \pi^{-1}(c)$, $c \in C$ where surjectivity fails. This means that $J'_c := \text{Im}(a\theta_c + b\iota_c)$ is a proper subsheaf of J_c for some $a \neq 0$, b . Denote $\mathcal{E}'_c = \iota^{-1}(J'_c)$; then $\theta_c(\mathcal{E}'_c) \subset J'_c$. Further, let

$$\mathcal{E}' = \mathcal{E}'_c \oplus \bigoplus_{p \in C \setminus \{c\}} \mathcal{E}_p, \quad J' = J'_c \oplus \bigoplus_{p \in C \setminus \{c\}} J_p.$$

Then $\text{Im } \alpha \subset \mathcal{E}'$ and $\theta(\mathcal{E}') \subset J'$, which precludes the triple $T = (\mathcal{E}, \alpha, \theta)$ from being stable.

Conversely, suppose that $T = (\mathcal{E}, \alpha, \theta)$ is destabilized by the subsheaf $\mathcal{E}' \subset \mathcal{E}$. Denote $J' = \iota(\mathcal{E}')$. Let us choose a point $c \in C$, such that $\mathcal{E}'_c \subset \mathcal{E}_c$ is a proper subsheaf. For a point $s \in \pi^{-1}(c)$, consider the degree of the sheaf $\text{Im } \psi_s$. By assumption $\psi(\mathcal{E}'_c) \subset J'_c$, therefore

$$\begin{aligned} \deg \text{Im } \psi_s &= \deg \mathcal{E}_c - \deg \text{Ker } \psi_s = \deg \mathcal{E}'_c + \deg \mathcal{E}_c / \mathcal{E}'_c - \deg(\text{Ker } \psi_s \cap \mathcal{E}'_c) - x \\ &= \deg J'_c + \deg \mathcal{E}_c / \mathcal{E}'_c - x = \deg J_c - x, \end{aligned}$$

where $x = \deg(\text{Ker } \psi_s / \text{Ker } \psi_s \cap \mathcal{E}'_c)$. We deduce that $\text{Ker } \psi_s \subset \mathcal{E}'_c$. Since this is true for every c and s , the map ψ induces an automorphism of $\mathcal{E} / \mathcal{E}'$. In particular, the map $a \text{Id} + b\theta_c$ is an automorphism of $\mathcal{E} / \mathcal{E}'$ for each $[a : b] \in \mathbb{P}^1$. However, if we regard $\mathcal{E} / \mathcal{E}'$ as a \mathbb{k} -module, θ must possess an eigenvalue λ , so that $\theta - \lambda \text{Id}$ cannot be invertible. Thus ψ is not surjective. \square

Due to the Lemma above, E is isomorphic to $\text{Ker}(\Psi)$. Moreover, E sits inside a torsion-free sheaf M , which in its turn is obtained as an extension of a torsion-free sheaf $\pi^* K(D)$ by $\pi^* \mathcal{E}$. As a consequence, $\text{Tor } E$ is contained in $\pi^* \mathcal{E}$. However, restricting Ψ to D , we can check that $E|_D \simeq \mathcal{O}^{\oplus n}$. Hence the support of $\text{Tor } E$ must be disjoint from D . Since for every subsheaf of $\pi^* \mathcal{E}$ its support intersects D , the sheaf E is torsion-free. Furthermore, outside of the support of \mathcal{E} the complex Ψ is isomorphic to $\mathcal{O}(-D)^{\oplus n}$. By Lemma 2.9.1, it implies that E is f -semistable.

Let us compute the Chern character of E . We have :

$$\begin{aligned} \text{ch}(E) &= \text{ch}(\pi^* \mathcal{E}) - \text{ch}(\pi^* \alpha \otimes \mathcal{O}_S(D)[1]) \\ &= \text{ch}(\pi^* \omega^{-1} \otimes \mathcal{O}(-2D)[1]) (\text{ch}(\pi^* \mathcal{E} \otimes \omega^{-1}) - \text{ch}(\pi^* \alpha \otimes \mathcal{O}_S(D)[1])) \\ &= (1, -2D + (2 - 2g)f, 0) ((0, df, 0) - (-n, df, 0)(1, D, 1 - g)) \\ &= (n, -nD + n(2 - 2g)f, -n(1 - g) - d). \end{aligned}$$

Therefore $\text{rk } E = n$, $c_{1,f}(E) = n(2 - 2g)$, $c_{1,D}(E) = -n$, and

$$c_2(E) = c_1(E)^2 / 2 - \text{ch}_2(E) = d + n(n - 1)(g - 1).$$

Finally, it is easy to check that the procedure (2) is inverse to (1). Suppose that the triple we've obtained from (E, φ) is not stable. Then E cannot be a sheaf, but rather a complex of sheaves, by Lemma 2.9.4. Therefore the resulting triple is necessarily stable, and we may conclude. \square

Let us denote by $M_f(r, c_1, c_2)$ the moduli space of f -semistable torsion-free sheaves E on S with trivialization at D , such that $\text{rk } E = r$, $c_1(E) = c_1$, $c_2(E) = c_2$. It is a smooth quasi-projective variety, see [35, Chapter 4].

Corollary 2.9.5. *We have an isomorphism of moduli spaces $\mathcal{B}(d, n) \simeq M_f(n, 0, d)$. In particular, $\mathcal{B}(d, 1) \simeq \text{Hilb}_d T^*C$.*

Proof. Since constructions in Proposition 2.9.2 work in flat families, we obtain an isomorphism

$$\mathcal{B}(d, n) \simeq M_f(n, -nD + n(2 - 2g)f, d + n(n - 1)(g - 1)).$$

Furthermore, the map $E \mapsto E(D) \otimes \pi^*\omega$ establishes an isomorphism

$$M_f(n, -nD + n(2 - 2g)f, d + n(n - 1)(g - 1)) \simeq M_f(n, 0, d).$$

We prove the first statement by taking the composition of two maps above. For the second statement, let E_1 be a locally free sheaf on S of rank 1 and $c_1(E) = 0$. By see-saw principle, $E_1 \simeq \pi^*\pi_*E_1$. For instance, if $E_1|_D \simeq \mathcal{O}_C$, then $E_1 \simeq \mathcal{O}_S$. By consequence, the isomorphism

$$\begin{aligned} M_f(1, 0, d) &\xrightarrow{\sim} \text{Hilb}_d T^*C, \\ E &\mapsto (E|_{T^*C} \subset E^{\vee\vee}|_{T^*C} \simeq \mathcal{O}_{T^*C}) \end{aligned}$$

provides us with the desired description of $\mathcal{B}(d, n)$. \square

Corollary 2.9.6. *For any $n > 0$, the space $\bigoplus_d H_*^T(\mathcal{B}(d, n))$ admits a natural $H\mathbf{Ha}_C^{0,T}$ -module structure.* \square

It is instructive to compare this corollary with recent works of Neguț [60, 57]. For any smooth projective surface S and a nef divisor H , he considers the moduli space \mathcal{M} of H -stable sheaves on S with varying second Chern class, and for every $n \in \mathbb{Z}$ defines an operator $e_n : K(\mathcal{M}) \rightarrow K(\mathcal{M} \times S)$ by Hecke correspondences. These operators generate a subalgebra \mathcal{A} inside $\bigoplus_{k>0} \text{Hom}(K(\mathcal{M}) \rightarrow K(\mathcal{M} \times S^k))$, which can be then projected to a shuffle algebra \mathcal{V}_{sm} . The content of Conjecture 3.20 in [60] is that this projection is supposed to be an isomorphism. The conjecture is proved under rather restrictive assumptions; for instance, it is required that $K(S \times S) \simeq K(S) \otimes K(S)$.

Let us now take $S = T^*C$ together with a scaling action of $T \simeq \mathbb{G}_m$, and replace usual K -groups with their T -equivariant counterpart. In this case, the algebra \mathcal{V}_{sm} can be identified with the subalgebra of $K\mathbf{Sh}_C^{norm}$, generated by $K(\text{B}\mathbb{G}_m) \subset K\mathbf{Sh}_C^{norm}[1] \simeq K^T(C \times \text{B}\mathbb{G}_m)$. If we further replace K -groups by Borel-Moore homology, then by Corollary 2.8.5 homological version of \mathcal{V}_{sm} is realized as a subalgebra of $H\mathbf{Ha}_C^{0,T}$. Therefore, one can regard Corollary 2.9.6 as a ‘‘homological non-compact’’ version of Neguț’s conjecture for $S = T^*C$, $c_1 = 0$, and stability condition given by f . Another modest gain of our approach is that $A\mathbf{Ha}_C^{0,T}$ gets abstracted away from its natural representations, unlike \mathcal{A} .

In general, one expects that the moduli of framed sheaves on $\mathbb{P}(T^*C)$ with non-trivial first Chern class can be recovered from the moduli of stable Higgs triples of *positive* rank. Nevertheless, as stability condition for triples varies, Lemma 2.9.4 seems to suggest that the corresponding objects on S don’t have to lie in the usual heart of $\mathcal{D}(\text{Coh } S)$.

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