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Roberto Gualdi

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## THÈSE EN COTUTELLE

présentée par

**Roberto GUALDI**

pour obtenir le grade de

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École Doctorale de Mathématiques et d'Informatique

et Escola de Doctorat de l'Universitat de Barcelona

Spécialité Mathématiques Pures

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# HEIGHT OF CYCLES IN TORIC VARIETIES

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dirigée par Alain YGER et Martín SOMBRA

soutenue le 20 septembre 2018

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*Alla bellezza, all'amore  
e alle sue messaggere  
presso di me.*



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# HEIGHT OF CYCLES IN TORIC VARIETIES

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*Abstract.* — We investigate in this work the relation between suitable Arakelov heights of a cycle in a toric variety and the arithmetic features of its defining Laurent polynomials. To this purpose, we associate to a Laurent polynomial certain concave functions which we call Ronkin functions and upper functions. We give upper bounds for the height of a complete intersection in terms of the associated upper functions. For a hypersurfaces, we prove a formula relating its height to the Ronkin function of the associated Laurent polynomial. We conjecture an analogous equality for a suitable average height in higher codimensions and indicate a strategy for the proof of a particular case. In all the treatment, we deal with convex geometrical objects such as polytopes, real Monge-Ampère measures and Legendre-Fenchel duality of concave functions. We suggest an algebraic framework for such a study and deepen the understanding of mixed integrals.

**Key words:** heights, toric varieties, Arakelov geometry, Ronkin functions, mixed integral.

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# HAUTEUR DE CYCLES DE VARIÉTÉS TORIQUES

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*Résumé.* — Nous étudions dans cette thèse la relation entre certaines hauteurs d'Arakelov de cycles de variétés toriques et les caractéristiques arithmétiques des polynômes de Laurent qui les définissent. Pour cela, nous associons à un polynôme de Laurent des fonctions concaves que nous appelons fonctions de Ronkin et fonctions supérieures. Nous donnons des bornes supérieures pour la hauteur d'une intersection complète faisant intervenir les fonctions supérieures associées. Dans le cas d'une hypersurface, nous montrons une formule liant sa hauteur à la fonction de Ronkin de son polynôme de Laurent. Nous proposons une égalité analogue pour des hauteurs moyennes appropriées en codimension supérieure et nous indiquons une stratégie pour la preuve d'un cas particulier. Dans ces travaux, nous utilisons des notions de géométrie convexe telles que les polytopes, les mesures de Monge-Ampère réelles et la dualité de Legendre-Fenchel de fonctions concaves. Nous les présentons dans un cadre algébrique adapté et nous développons l'étude des intégrales mixtes.

**Mots-clés:** hauteurs, variétés toriques, géométrie d'Arakelov, fonctions de Ronkin, intégrale mixte.

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# ALTURA DE CICLOS DE VARIEDADES TÓRICAS

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**Resumen.** — Estudiamos en esta tesis la relación entre algunas alturas de Arakelov de ciclos de variedades tóricas y las propiedades aritméticas de los polinomios de Laurent que los definen. Más precisamente, asociamos a un polinomio de Laurent ciertas funciones cóncavas que llamamos funciones de Ronkin y funciones superiores. Demostramos una cota superior de la altura de una intersección completa en términos de las funciones superiores asociadas. Para una hipersuperficie, damos una fórmula exacta que relaciona su altura a la función de Ronkin de su polinomio de Laurent. Sugerimos una igualdad análoga para apropiadas alturas esperadas en codimensión superior y indicamos una estrategia para la demostración de un caso particular. En todo este trabajo, utilizamos nociones de geometría convexa como politopos, medidas de Monge-Ampère reales y la dualidad de Legendre-Fenchel de funciones cóncavas. Presentamos estas herramientas en un cuadro algebraico oportuno y profundizamos en el estudio de las integrales mixtas.

**Palabras clave:** alturas, variedades tóricas, geometría de Arakelov, funciones de Ronkin, integral mixta.



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# Introduction

Finding the common solutions of a system of polynomial equations in several variables is a fundamental problem in mathematics. A beautiful result by Bernstein and Kouchnirenko, see [Ber75] and [Kou76], also known as the *BKK theorem* because of the contributions successively made by Khovanskii, allows to predict the number of solutions of such a system in terms of the combinatorial features of the involved polynomials. More precisely, let  $K$  be an algebraically closed field and  $f_1, \dots, f_n$  a  $n$ -tuple of  $n$  Laurent polynomials in  $n$  variables with coefficients in  $K$ . The BKK theorem asserts that the numbers of isolated solutions (counted with multiplicities) of the system  $f_1 = \dots = f_n = 0$  in  $(K^\times)^n$  is upper bounded by the mixed volume of the Newton polytopes of the polynomials. Moreover, such an estimate is exact for a generic choice of polynomials  $f_1, \dots, f_n$  with fixed Newton polytopes.

Though useful for the numerical solution of systems of polynomial equations, see [Stu02], [AG90] or [Li97], when  $K = \overline{\mathbb{Q}}$  the BKK theorem does not reveal anything about the arithmetic size of the isolated common zeros of the considered polynomials. Inspired by the analogy with the notion of degree, we make in this thesis some advances towards the understanding of the relation between the *height*, that is the arithmetical complexity, of the solutions of a system of polynomial equations and the arithmetic features of the involved polynomials.

The easiest example is the one of an irreducible polynomial  $f$  over  $\mathbb{Q}$  in one variable. In such a case and with respect to the canonical height, the relation between  $f$  and the complexity of any of its zeros  $\alpha$  is expressed by the classical equality<sup>1</sup>

$$\mathfrak{m}(f) = \deg(f) \cdot h_{\text{Weil}}(\alpha), \quad (1)$$

where  $\mathfrak{m}(f)$  denotes the logarithmic Mahler measure of  $f$ . A fertile point of view consists in considering the height as a function defined on the set of  $\overline{\mathbb{Q}}$ -points of an algebraic variety. With this geometric approach, an irreducible

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<sup>1</sup>See for instance [BG06, Proposition 1.6.6].

polynomial  $f$  over  $\mathbb{Q}$  defines a 0-cycle  $Z(f)$  on  $\mathbb{P}^1$  by considering the set of its zeros. Defining the height of a 0-cycle linearly in its irreducible components, one can hence rephrase (1) as

$$m(f) = h_{\text{Weil}}(Z(f)). \quad (2)$$

A first result we obtain in this text is a wide generalization of the relation (2). More precisely, given a proper toric variety  $X_\Sigma$  over an adelic field  $\mathbb{K}$  and an adelic semipositive toric metrized divisor  $\overline{D}$  on it, Arakelov geometry allows to define a height function  $h_{\overline{D}}$  on the set of cycles of  $X_\Sigma$ . Building on the recent results of [BPS14], we are able to express the height of the vanishing cycle of a Laurent polynomial  $f$  in  $X_\Sigma$  in terms of convex geometrical objects associated to  $\overline{D}$  and to  $f$ . To do this, we define a family of concave functions, the *v-adic Ronkin functions* of  $f$ , which can be viewed as the arithmetical analogue of the Newton polytope of  $f$  and contains enough arithmetical information to compute the height of its vanishing cycle. The formula we prove generalizes the well-known case of the canonical height, see [DP99] or [Mai00], and adapts to other relevant situations in arithmetic geometry, such as the one of the Fubini-Study height in projective spaces. Though not effective, it can be used to deduce lower and upper bounds for the height of toric hypersurfaces involving the arithmetics of  $\overline{D}$  and the Mahler measure of the defining polynomial of the hypersurface.

We then start exploring the case of higher codimensional cycles in a toric variety  $X_\Sigma$ . In particular, we focus on the computation of the height of the cycle defined by the common vanishing of a family of Laurent polynomials  $f_1, \dots, f_k$  over  $\mathbb{K}$ . Such a situation reveals to be wilder than the 1-codimensional case and we show that an analogous deterministic formula as the one for hypersurfaces can not be hoped. After giving a slight improvement of [MS16] concerning the upper bounds of the height of such an intersection, we adopt a probabilistic point of view and define a certain “expected value” for the height of the cycle defined by  $f_1, \dots, f_k$ . This approach reveals to be more promising than the deterministic one: we give a combinatorial equality for the degree of such a cycle and, inspired by a probabilistic reformulation of this geometrical result, we propose a formula for its expected height involving the Ronkin functions of  $f_1, \dots, f_k$ . In particular, the conjectured statement gives a guess for the height of the isolated solutions of the system  $f_1 = \dots = f_k = 0$ , representing an arithmetic version of the BKK theorem. We present ideas for the proof of the conjecture in a class of examples with  $k = 2$ .

We finally hope that our results could motivate a program relating the family of Ronkin functions of a cycle in a toric variety to its arithmetic properties; it would be interesting, for example, to know whether the essential minimum

or the distribution of small points on a hypersurface could be understood via the language developed here, in the philosophy of [BPS15] and [BPRS15].

We describe in the rest of the introduction the context and the main results of our study, leaving the precise definitions and statements to the body of the text.

Heights are a powerful tool in arithmetic geometry having played a prominent role in spectacular results during the twentieth century, such as the proof of Mordell conjecture by Faltings in [Fal83]. They now appear in many leading conjectures in number theory, such as the *abc* conjecture, Vojta’s conjecture and the Birch and Swinnerton-Dyer conjecture. Their systematic use was initiated by Northcott and Weil in [Nor50] and [Wei51] as a way to measure the “arithmetic complexity” of rational points on an algebraic variety defined over a number field. Subsequent works by Arakelov [Ara74], Gillet and Soulé [GS90] and Faltings [Fal91] established an arithmetic intersection theory where the height of an arbitrary dimensional cycle in an arithmetic variety can be defined as an arithmetic intersection number, in analogy with its degree. Independently, an equivalent definition of the height of a projective variety in terms of its Chow form was given by Nesterenko in [Nes77] [Nes84] and Philippon in [Phi91]; the comparison between the two approaches was made in [Sou91] and [Phi91].

The point of view we adopt in this thesis has a more analytic flavour and flourished in the second half of the nineties as a realization of the following aspiration of Soulé, see [Sou92, §1.5]:

A more dynamic approach would be an *adelic* variant of Arakelov geometry. The main object of study in this theory would be a smooth variety  $V$  over  $\mathbb{Q}$ , and vector bundles on  $V$  equipped with metrics at archimedean places, and  $p$ -adic analogs of these at finite places.

The foundations of such an *adelic Arakelov geometry* was laid by Zhang in [Zha95b] and successively developed by the works of Gubler [Gub98] and Chambert-Loir [Cha06]. Even if still incompatible with the existing definition of an arithmetic Chow ring, their constructions are enough to define heights, for which they moreover allow a greater flexibility than the original theory. Another advantage of the adelic approach is the similar treatment reserved to archimedean and non-archimedean places, especially thanks to the recent progress in the theory of forms and currents over Berkovich analytic spaces inaugurated by [CD12] and [GK17].

We review the outline of adelic Arakelov theory in Chapter 2, starting from basic notions on places over a field to fix the notations. The first ingredient one needs to consider is an *adelic field*, that is a field coming with a collection of inequivalent absolute values; arithmetically rich fields include, but are not restricted to, number fields and function fields of smooth projective curves. The constructions we present are philosophically the same unregardingly of the chosen base adelic field, accordingly to the intuition first due to Weil [Wei39] about the striking analogies between this kind of fields. To keep the notation and the exposition simpler in this introduction, we restrict to the case of the adelic field  $\mathbb{Q}$  and its set of places  $\mathfrak{M}$ , consisting of  $p$ -adic places and the unique archimedean one. The geometric input in the theory is given by the choice of a proper variety  $X$  over  $\mathbb{Q}$  and a divisor  $D$  on  $X$ . The field  $\mathbb{Q}$  coming with many different absolute values, that are distinct ways of measuring sizes, it is necessary to consider all of them “simultaneously” to compute the complexity of a  $\overline{\mathbb{Q}}$ -point of  $X$ . To do it one has to consider a suitable  *$v$ -adic analytification* of  $X$  for each place  $v$  of  $\mathbb{Q}$ . The use of Berkovich geometry, as introduced in [Ber90], has two main advantages. The first is that the topological space underlying the Berkovich analytification of a variety is more concrete than, for instance, the Grothendieck topology one needs to consider for Tate’s rigid analytic spaces. Secondly, Berkovich’s construction also applies for archimedean places, in which case it yields the usual complexification of a variety over  $\text{Spec } \mathbb{C}$ ; this allows to push the formal analogy between archimedean and non-archimedean places of  $\mathbb{Q}$  even further. For every  $v \in \mathfrak{M}$ , consider hence the analytic space  $X_v^{\text{an}}$  and the analytic line bundle  $\mathcal{O}(D)_v^{\text{an}}$ . There is a notion of  *$v$ -adic semipositive metric* one can put on such a line bundle and, for every  $d$ -dimensional algebraic cycle  $Z$  on  $X$ , of a measure  $c_1(\overline{D})^{\wedge d} \wedge \delta_{Z_v^{\text{an}}}$ . Such a measure, which is the protagonist of the elegant results on the equidistribution of Galois orbits of points of small height proved in [SUZ97] [Bil97] [Yua08] [Aut06], appears in an arithmetical Bézout formula over  $X_v^{\text{an}}$ , allowing to define the  *$v$ -adic local height* of  $Z$  recursively on the dimension of the cycle. By combing the local information coming from each place and under some integrability hypotheses on the choice of the collection of the  $v$ -adic metrics, this gives a notion of *global height*  $h_{\overline{D}}(Z)$  of the cycle  $Z$  with respect to the choice of the semipositively metrized divisor  $\overline{D}$ .

The height of a variety contains deep information about its arithmetic properties. Together with its degree, for instance, it can be used to give lower and upper bounds for the *essential minimum* of the variety, controlling the density of its algebraic points of small height, see [Zha95a].

Because of the tremendous freedom in the modern definition of heights,

the known values of such quantities are very few compared to the extent of the possibilities. The seek for examples, as often in algebraic geometry, has led researchers to the realm of toric varieties, whose combinatorial interpretation has proved to be a “remarkably fertile testing ground for general theories”. The dawn of toric geometry can be tracked in the seventies, with the works of Demazure [Dem70], Kempf, Knudsen, Mumford and Saint-Donat [KKMS73] and Miyake and Oda [OM75]. It was anyway during the nineties, with the expository texts by Fulton [Ful93] and Oda [Oda88] that toric geometry knew an explosion. These two books, together with the recent and almost omniscient [CLS11] are nowadays the standard references for the subject.

A *toric variety* over a field  $K$  is a normal algebraic variety equipped with the action of a multiplicative split torus  $\mathbb{T} = \mathbb{G}_{m,K}^n$  acting with an open dense orbit on it. It is customary to denote by  $M$  the character lattice of  $\mathbb{T}$  and by  $N$  its dual; they induce real vector spaces  $M_{\mathbb{R}}$  and  $N_{\mathbb{R}}$  respectively by tensoring with  $\mathbb{R}$ . The main interests in studying toric varieties is that their algebro-geometric properties can be read in terms of combinatorial objects defined on  $M_{\mathbb{R}}$  or  $N_{\mathbb{R}}$ , providing the subject with an authentic *toric dictionary*. For example, there exists a 1-1 correspondance between proper toric varieties with torus  $\mathbb{T}$  and *complete fans* in  $N_{\mathbb{R}}$ . Because of this, a toric variety will be denoted by  $X_{\Sigma}$  to stress it is associated to the fan  $\Sigma$ . Toric Cartier divisors on  $X_{\Sigma}$  are encoded by certain piecewise affine functions on  $N_{\mathbb{R}}$ , which allows to associate to them a convex polytope in  $M_{\mathbb{R}}$ . The polytope  $\Delta_D$  defined by a toric Cartier divisor  $D$  carries a lot of information about it. For instance, it describes the space of global sections of  $\mathcal{O}(D)$  and is enough to compute the degree of  $X_{\Sigma}$  with respect to  $D$ , namely by the equality

$$\deg_D(X_{\Sigma}) = n! \operatorname{vol}_M(\Delta_D), \quad (3)$$

where  $\operatorname{vol}_M$  is a suitably normalized Haar measure on  $M_{\mathbb{R}}$ .

It is natural to ask whether similar combinatorial formulas could hold for heights. The first arithmetical studies on toric varieties are due to Maillot, who described in [Mai00] the arithmetic intersection theory of Gillet and Soulé on a toric variety with line bundles equipped with their canonical metric. The systematic extension of the toric dictionary to adelic Arakelov theory, however, was inaugurated by Burgos Gil, Philippon and Sombra in the foundational work [BPS14], which we resume, together with some geometrical tools in toric geometry, in chapter 3. It turns out from their study that, for a toric divisor  $D$  and for any place  $v \in \mathfrak{M}$ , *toric semipositive metrics* on  $\mathcal{O}(D)_v^{\text{an}}$  are in bijection with continuous concave functions on the polytope  $\Delta_D$ . Encoding the datum of the semipositively toric metrized divisor  $\bar{D}$  in terms of the collection of its

$v$ -adic roof functions  $\vartheta_v$  (and under some global finiteness conditions), the global height of the cycle  $X_\Sigma$  can be expressed combinatorially as

$$h_{\overline{D}}(X_\Sigma) = (n+1)! \sum_{v \in \mathfrak{M}} \int_{\Delta_D} \vartheta_v \, d\text{vol}_M, \quad (4)$$

which is the arithmetic analogue of (3). The language developed in [BPS14] was exploited by the same authors in [BPS15] to compute the essential minimum of toric heights, with Moriwaki in [BMPS16] to study the positivity properties of toric metrized divisors and, together with Rivera-Letelier in [BPRS15], to explore more general versions of the equidistribution of small points in toric varieties. These works are contributing to the understanding of adelic Arakelov geometry by the comprehension of illustrative and computable examples where the studied phenomena can be visualized in terms of convex geometry. Once again, toric geometry has proven to be an oil lamp in the vast darkness of general theories.

A remarkable feature of the theory developed in [BPS14] is that it allows to treat a large spectrum of height functions on a toric variety, including the canonical heights studied by Maillot and the Fubini-Study height in projective space. On the other hand, their techniques rely strongly on the toric structure; as such, they only apply to the computation of the height of subvarieties which are themselves toric.

In the canonical case, a relation between the height of a general hypersurface, that is not necessarily toric, in a smooth projective toric variety and the Mahler measure of the corresponding polynomial is well-known, see for instance [Mai00]. Other computations have been performed by Cassaigne and Maillot in [CM00] for the Fubini-Study height of projective hypersurfaces. We here give a combinatorial formula for the height of Weil divisors in a toric variety holding for a much more general choice of metrics and using the dictionary of [BPS14].

Keeping the notation from toric geometry presented above, let  $Z$  be a Weil divisor on a toric variety  $X_\Sigma$  over  $\mathbb{Q}$ . To compute its height, and because of the results in [BPS14] for toric divisors, we can restrict to the case of  $Z$  being an irreducible hypersurface in  $X_\Sigma$  intersecting its dense open orbit. Under this assumption,  $Z$  is described by an irreducible Laurent polynomial  $f = \sum_m c_m \chi^m$  with rational coefficients. Its Newton polytope

$$\text{NP}(f) = \text{conv}\{m \in M : c_m \neq 0\}$$

is a closed polytope in  $M_{\mathbb{R}}$  capturing enough information for the intersection theoretical properties of  $Z$ . For instance, its degree with respect to a toric

divisor  $D$  on  $X_\Sigma$  is given by

$$\deg_D(Z) = \text{MV}_M(\Delta_D, \dots, \Delta_D, \text{NP}(f)), \quad (5)$$

where  $\text{MV}_M$  denotes the mixed volume of convex bodies in  $M_{\mathbb{R}}$  with respect to a suitably normalized Haar measure (refer to Proposition 4.2.1). The equality (5) is the 1-codimensional analogue of (3) and serves as an inspiration to the extension of (4) to irreducible hypersurfaces in  $X_\Sigma$ .

The first ingredient we need to do this is a correspondent of the mixed volume for concave functions. As suggested by looking at (4), a good candidate for this role is the notion of *mixed integral*: introduced by Philippon and Sombra in [PS08a], it is a polarized version of the integration of a concave function on a convex body. The mixed integral is only one of the many tools from convex geometry that we will need in this thesis; for this reason, we consacrate Chapter 1 to their presentation, mainly referring to the sources [Roc70] and [BPS14]. In doing this, we suggest an abstract algebraic approach for the constructions we present and treat convex geometry as “the study of semimodules over the semiring  $\mathbb{R}_{\geq 0}$ ”. We deepen the study of mixed integrals, clarifying their relation with mixed Monge-Ampère measures and tropical geometry (in particular tropical intersection theory), proving equalities for some particular cases and giving upper and lower bounds.

The second needed ingredient for a formula for the height of an irreducible hypersurface in a toric variety is an arithmetic analogue of the Newton polytope of  $f$ . For any place  $v \in \mathfrak{M}$ , there exists a certain tropicalization map  $\text{trop}_v$  from the  $v$ -adic analytic torus  $\mathbb{T}_v^{\text{an}}$  to the real vector space  $N_{\mathbb{R}}$ , whose fiber over any  $u \in N_{\mathbb{R}}$  has a distinguished subset  $\mathcal{B}_v(u)$ , namely its *Shilov boundary*, homeomorphic to a compact group. We define the  *$v$ -adic Ronkin function* of a Laurent polynomial  $f$  to be

$$\rho_{f,v} : N_{\mathbb{R}} \rightarrow \mathbb{R}, \quad u \mapsto \int_{\text{trop}_v^{-1}(u)} -\log |f(x)| d\text{Haar}_{\mathcal{B}_v(u)}, \quad (6)$$

where  $\text{Haar}_{\mathcal{B}_v(u)}$  is the Haar measure on  $\mathcal{B}_v(u)$  normalized to have total mass 1. When  $v$  is archimedean, such a function is related to the one studied by Passare and Rullgård in [PR04]. For non-archimedean places  $v \in \mathfrak{M}$ , the  $v$ -adic Ronkin function coincides instead with  $v$ -adic tropicalization of the polynomial  $f$ , see for instance [MS15]. In both cases  $\rho_{f,v}$  is a concave function on  $N_{\mathbb{R}}$  and its Legendre-Fenchel dual  $\rho_{f,v}^\vee$  is a concave function on the Newton polytope of  $f$ . It turns out that the family of  $v$ -adic Ronkin functions of  $f$  contains enough arithmetical information to compute the height of the corresponding hypersurface, and in fact we can prove the following theorem.

**Theorem 1.** *The height of  $Z$  with respect to an adelic semipositive toric metrized divisor  $\overline{D}$  is given by*

$$h_{\overline{D}}(Z) = \sum_{v \in \mathfrak{M}} \text{MI}_M(\vartheta_v, \dots, \vartheta_v, \rho_{f,v}^{\vee}),$$

with  $\vartheta_v$  being the  $v$ -adic roof function of  $\overline{D}$ .

Such a formula is compatible with the one deduced from [BPS14] for the case of hypersurfaces in  $X_{\Sigma}$  which are themselves toric and restricts to the classical relation for canonical heights. In particular, when  $X_{\Sigma}$  is the projective line and  $\overline{D}$  is the canonically metrized universal bundle, one recovers, as promised, the relation (2).

It seems arduous to use Theorem 1 to compute numerically the height of hypersurfaces, since the proved formula involves difficult-to-evaluate entities such as Ronkin functions and mixed integrals. For instance, it follows from (6) that the values of the archimedean Ronkin function are Mahler measure of perturbations of the Laurent polynomial  $f$ . Despite this, the stated equality extends the bridge between the arithmetic and the convex lands to non-toric cycles in  $X_{\Sigma}$  and clarifies the relation between the defining polynomial of an irreducible hypersurface and its height with respect to an adelic semipositive toric metrized line bundle. Importantly, this also shows that the collection of the  $v$ -adic Ronkin functions associated to a hypersurface contains enough arithmetical information about it; we may wonder whether other arithmetical properties of  $Z$  might be read in terms of such functions.

The construction of  $v$ -adic Ronkin functions and the proof of Theorem 1 are presented in chapter 4, in which we make efforts to treat the archimedean and non-archimedean case homogeneously, in the spirit of adelic Arakelov geometry. The exposition follows the one of [Gua17], where the results were first presented, but enriches it with the notion of  $v$ -adic Mahler measure and with some estimations.

A natural question concerns the extension of Theorem 1 to higher codimensions. We start exploring this direction in chapter 5, where we first remark that an analogous equality as the one for hypersurfaces can not be hoped. More precisely, we show that there exist polynomials  $f, g, g'$  sharing the same Ronkin functions for every place  $v \in \mathfrak{M}$  but for which the isolated solutions of  $f = g = 0$  and of  $f = g' = 0$  have different canonical height; as a result, the height of a complete intersection can not only depend on the Ronkin functions of the defining polynomials. The task of computing the height of the intersection of two polynomials by taking into account the way in which they interact

seems for the moment out of reach. A more down to earth problem consists in giving upper bounds for the height of the intersection cycle of a family of Laurent polynomials, as in [Som05], [Mai00] and [MS16]. We refine here the argument of [MS16] by defining for a Laurent polynomial  $f$  the  $v$ -adic upper function

$$\mu_{f,v} : N_{\mathbb{R}} \rightarrow \mathbb{R}, \quad u \mapsto - \max_{x \in B_v(u)} \log |f(x)|, \quad (7)$$

where we keep the same notation as the one used to define the  $v$ -adic Ronkin functions of  $f$  in (6). These functions play an important role in giving upper bounds of complete intersections in arbitrary codimension. More precisely, let  $k \in \{0, \dots, n\}$  and consider the family of Laurent polynomials  $f_1, \dots, f_k$  with coefficients in  $\mathbb{Q}$ . Each of them defines, by Zariski closure of its zero set in the torus, a 1-codimensional cycle in a toric variety  $X_{\Sigma}$ . Assuming that their intersection  $Z(f_1, \dots, f_k)$  in  $X_{\Sigma}$  has pure codimension  $k$ , we can prove the following.

**Theorem 2.** *For an adelic semipositive toric metrized divisor  $\overline{D}$  on  $X_{\Sigma}$ ,*

$$h_{\overline{D}}(Z(f_1, \dots, f_k)) \leq \sum_{v \in \mathfrak{M}} \text{MI}_M(\vartheta_v, \dots, \vartheta_v, \mu_{f_1,v}^{\vee}, \dots, \mu_{f_k,v}^{\vee}),$$

with  $\vartheta_v$  being the  $v$ -adic roof function of  $\overline{D}$ .

Since the authors of [MS16] were interested in a bound for the common solutions of a system of polynomial equations inside the torus, their result requires  $\overline{D}$  to be arithmetically nef; in our study of cycles in  $X_{\Sigma}$ , which can have components outside the dense open orbit, we can drop this assumption. Moreover, the functions we use give a slightly better upper bound than the one obtained by [MS16].

In chapter 6 we change point of view and propose to study a suitable expected value of the height of the complete intersection cycle  $Z(f_1, \dots, f_k)$ . To do this, we consider the family of cycles

$$Z(\zeta_1^* f_1, \dots, \zeta_k^* f_k)$$

obtained intersecting suitable *twisting* of the original polynomials by torsion points  $\zeta_1, \dots, \zeta_k$  of the algebraic torus of dimension  $n$ . We then define the *sup-expected height* of the intersection of the Laurent polynomials  $f_1, \dots, f_k$  with respect to a metrized divisor  $\overline{D}$  as a convenient average of  $h_{\overline{D}}(Z(\zeta_1^* f_1, \dots, \zeta_k^* f_k))$  over the set of  $k$ -tuples of torsion points and denote it by

$$\mathbb{E}[h_{\overline{D}}(f_1, \dots, f_k)].$$

As in the case of the toric variety and of its Weil divisors, the geometric situation serves as a guide. The degree with respect to a toric divisor  $D$  of the cycle  $Z(f_1, \dots, f_k)$  is seen to be given by the combinatorial expression

$$\deg_D(Z(f_1, \dots, f_k)) = \text{MV}_M(\Delta_D, \dots, \Delta_D, \text{NP}(f_1), \dots, \text{NP}(f_k)),$$

which is the higher codimensional analogue of (3) and (5). Inspired by such an equality we formulate the following guess.

**Conjecture 1.** *Let  $\overline{D}$  be an adelic semipositive toric metrized divisor on  $X_\Sigma$ . Then, the sup-expected height of the intersection of the family  $f_1, \dots, f_k$  with respect to  $\overline{D}$  is given by*

$$\mathbb{E}[h_{\overline{D}}(f_1, \dots, f_k)] = \sum_{v \in \mathfrak{M}} \text{MI}_M(\vartheta_v, \dots, \vartheta_v, \rho_{f_1, v}^\vee, \dots, \rho_{f_k, v}^\vee), \quad (8)$$

with  $\vartheta_v$  being the  $v$ -adic roof function of  $\overline{D}$ .

We finally show that Conjecture 1 is reasonable by giving a possible strategy for its proof in the case of two Laurent polynomials satisfying additional hypotheses.

# Résumé détaillé

La résolution d'un système d'équations polynomiales en plusieurs variables est un problème fondamental et transversal en Mathématiques. Un admirable résultat dû à Bernstein [Ber75] et Kouchnirenko [Kou76], connu comme le *théorème BKK* en raison des importantes contributions de Khovanskii, permet de prédire le nombre de solutions d'un tel système à partir de la combinatoire des polynômes qui y apparaissent. Plus précisément, soit  $K$  un corps algébriquement clos et  $f_1, \dots, f_n$  un  $n$ -uplet de polynômes de Laurent à  $n$  variables et à coefficients dans  $K$ . Le théorème BKK affirme que le nombre de solutions (isolées et comptées avec multiplicité) du système  $f_1 = \dots = f_n = 0$  dans le tore  $(K^\times)^n$  est borné supérieurement par le volume mixte des polytopes de Newton des polynômes. En outre, cette estimation est exacte pour une choix générique des polynômes  $f_1, \dots, f_n$  avec polytopes de Newton fixés. Bien qu'utile pour la résolution numérique des systèmes polynomiaux, voir par exemple [Stu02], [AG90] et [Li97], dans le cas  $K = \overline{\mathbb{Q}}$  le théorème BKK ne dévoile aucune information précise sur la taille arithmétique des zéros communs des polynômes. Inspirés par l'analogie avec la notion de degré, nous obtenons dans cette thèse des résultats liant la *hauteur*, c'est à dire la complexité arithmétique, des solutions d'un système d'équations polynomiales et les caractéristiques arithmétiques des polynômes.

Dans l'exemple le plus simple d'un polynôme irréductible  $f$  à une seule variable et à coefficients rationnels, la relation entre  $f$  et la complexité (canonique) de n'importe lequel de ses zéros  $\alpha$  est donnée par

$$m(f) = \deg(f) \cdot h_{\text{Weil}}(\alpha), \quad (9)$$

où  $m(f)$  désigne la mesure de Mahler logarithmique de  $f$ . Avec un point de vue plus géométrique, consistant à considérer la hauteur comme une fonction définie sur l'ensemble des  $\overline{\mathbb{Q}}$ -points d'une variété algébrique, linéairement en ses composantes irréductibles, le polynôme  $f$  définit le 0-cycle  $Z(f)$  de ses zéros sur  $\mathbb{P}^1$  et la relation (9) peut se réécrire comme

$$m(f) = h_{\text{Weil}}(Z(f)). \quad (10)$$

Un premier résultat que nous prouvons dans ce texte est une vaste généralisation de l'égalité (10). Plus précisément, pour une variété torique propre  $X_\Sigma$  définie sur un corps adélique  $\mathbb{K}$  et pour un diviseur torique équipé d'une métrique adélique, torique et semipositive  $\overline{D}$ , la géométrie d'Arakelov permet de définir une fonction hauteur  $h_{\overline{D}}$  sur l'ensemble des cycles de  $X_\Sigma$ . En utilisant les résultats récents de [BPS14], nous pouvons ici exprimer la hauteur du cycle donné par l'annulation d'un polynôme de Laurent  $f$  en  $X_\Sigma$  au moyen de certains objets de nature convexe associés à  $\overline{D}$  et à  $f$ . Pour cela, nous définissons une famille de fonctions concaves, les *fonctions de Ronkin  $v$ -adiques* de  $f$ , qui peuvent s'interpréter comme des analogues arithmétiques du polytope de Newton de  $f$  et qui contiennent suffisamment d'information arithmétique pour calculer la hauteur du cycle défini par  $f$ . La formule que nous montrons généralise le cas de la hauteur canonique qui apparaît dans [DP99] et [Mai00], et s'adapte aussi à d'autres situations remarquables en géométrie arithmétique comme le cas de la hauteur de Fubini-Study dans les espaces projectifs. Bien que non effective, cette égalité peut être utilisée pour produire des bornes inférieures et supérieures pour la hauteur d'une hypersurface d'une variété torique mettant en jeu certaines fonctions concaves associées à  $\overline{D}$  ainsi que la mesure de Mahler du polynôme de Laurent définissant l'hypersurface.

Nous explorons en suite le cas des cycles de plus haute codimension dans la variété torique  $X_\Sigma$ . En particulier, nous nous focalisons sur le calcul de la hauteur d'un cycle défini par l'annulation d'une famille de polynômes de Laurent  $f_1, \dots, f_k$  à coefficients dans  $\mathbb{K}$ . Cette situation se révèle plus délicate que le cas de codimension 1 et nous prouvons que l'on ne peut pas s'attendre à une formule déterministe analogue à celle démontrée pour une hypersurface. Nous améliorons légèrement un résultat de [MS16] et nous l'adaptons à notre situation pour donner des bornes supérieures pour la hauteur d'une telle intersection. Ensuite, nous prenons un point de vue probabiliste et nous définissons une certaine "valeur attendue" pour la hauteur du cycle défini par  $f_1, \dots, f_k$ . Cette approche paraît plus encourageant : nous fournissons une égalité combinatoire pour le degré d'un tel cycle et, inspirés par une reformulation probabiliste de ce résultat géométrique, nous proposons une formule pour son hauteur attendue mettant en jeu les fonctions de Ronkin de  $f_1, \dots, f_k$ . En particulier, l'égalité conjecturée donne un candidat pour la hauteur des solutions isolées du système  $f_1 = \dots = f_k = 0$  et représente une version arithmétique du théorème BKK. Nous présentons enfin des idées pour la preuve de la conjecture pour une classe d'exemples avec  $k = 2$ .

Nous espérons que nos résultats puissent motiver une étude plus profonde de la relation entre la famille de fonctions de Ronkin d'un cycle d'une variété

torique et ses propriétés arithmétiques. Il serait par exemple intéressant de savoir si le minimum essentiel ou la distribution des orbites de Galois des points de petite hauteur sur une hypersurface peuvent être compris *via* le langage développé ici, dans la philosophie de [BPS15] et [BPRS15].

Nous décrivons maintenant plus en détails les résultats de notre étude, faisant référence au contenu des chapitres pour des définitions et énoncés précis. De plus, nous nous limitons dans ce résumé au cas des variétés définies sur le corps de base  $\mathbb{Q}$ , laissant la situation d'un corps adélique général satisfaisant la formule du produit au corps du texte. Nous désignons par  $\mathfrak{M}$  l'ensemble des places de  $\mathbb{Q}$ , constitué des places  $p$ -adiques et de l'unique place archimédienne.

La notion de hauteur, qui a joué et joue encore un rôle fondamental en géométrie arithmétique, a été introduite par Northcott et Weil dans [Nor50] et [Wei51] comme une mesure de la “complexité arithmétique” des points rationnels sur une variété algébrique définie sur un corps de nombres. Les travaux successifs d'Arakelov [Ara74], Gillet et Soulé [GS90] et Faltings [Fal91] ont édifié une théorie d'intersection arithmétique où la hauteur d'un cycle de codimension arbitraire dans une variété arithmétique peut se définir en analogie avec son degré. Une définition équivalente de hauteur d'une variété projective a été donnée par Nesterenko [Nes77] [Nes84] et Philippon [Phi91].

L'approche que nous suivons dans cette thèse est appelée *géométrie d'Arakelov adélique* et a été développée entre autres par Zhang [Zha95b] Gubler [Gub98] et Chambert-Loir [Cha06]. Nous rappelons les constructions de base de cette théorie dans le chapitre 2. Pour résumer brièvement, fixons une variété propre  $X$  sur  $\mathbb{Q}$  et un diviseur de Cartier  $D$  sur  $X$ ; pour tout  $v \in \mathfrak{M}$  on peut considérer l'analytification  $v$ -adique de  $X$  et du fibré  $\mathcal{O}(D)$ , au sens de Berkovich. Une notion adéquate de *métrique  $v$ -adique semipositive* sur l'analytifié de  $\mathcal{O}(D)$  permet de définir la hauteur locale  $v$ -adique d'un cycle algébrique  $Z$  de  $X$ , au moyen d'une formule récursive de type Bézout. En combinant les informations locales obtenues aux différentes places de  $\mathbb{Q}$ , et sous certaines hypothèses d'intégrabilité, il est possible de définir la *hauteur globale*  $h_{\overline{D}}(Z)$  du cycle  $Z$  relative au choix  $\overline{D}$  du diviseur  $D$  et de la collection des métriques  $v$ -adiques semipositives sur les analytifiés de  $\mathcal{O}(D)$ .

Un des avantages de cette approche est le traitement similaire réservé aux cas archimédiens et non-archimédiens, grâce aussi aux développements récents de la théorie des formes différentielles et des courants sur les espaces de Berkovich, voir [CD12] et [GK17].

Du fait de la grande flexibilité de cette définition par rapport au con-

texte classique, rares sont les valeurs de hauteurs connues dans le cadre de la géométrie d'Arakelov. Comme souvent en géométrie algébrique, la recherche d'exemples conduit au domaine des variétés toriques, pour lesquelles les propriétés algébro-géométriques peuvent s'interpréter en termes des notions de géométrie convexe (*voir* les références classiques [Ful93], [Oda88] et [CLS11]).

L'extension du dictionnaire torique au cadre de la théorie d'Arakelov adélique a été inaugurée par l'œuvre fondamentale de Burgos Gil, Philippon et Sombra [BPS14], que nous décrivons brièvement, avec quelques résultats de géométrie torique, dans le chapitre 3. Soit  $X_\Sigma$  une variété torique propre sur  $\mathbb{Q}$ ; notons par  $M$  le réseau des caractères de son tore et par  $N$  le dual de  $M$ . La variété torique est associée à un éventail complet  $\Sigma$  dans l'espace vectoriel  $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ . À un diviseur  $D$  sur  $X_\Sigma$ , invariant sous l'action du tore, on peut associer un polytope convexe  $\Delta_D$  dans  $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ . Pour tout  $v \in \mathfrak{M}$ , les métriques  $v$ -adiques *toriques* semipositives sur l'analytifié  $v$ -adique de  $\mathcal{O}(D)$  sont en bijection avec les fonctions continues et concaves sur  $\Delta_D$ , de telle sorte qu'un diviseur torique équipé d'une métrique adélique torique et semipositive est décrit par la collection de ses *fonctions toiture  $v$ -adiques*  $\vartheta_v$ , avec  $\vartheta_v \equiv 0$  pour presque tous  $v \in \mathfrak{M}$ . La hauteur globale de  $X_\Sigma$  relative à un tel choix  $\overline{D}$  peut alors s'écrire de façon combinatoire comme

$$h_{\overline{D}}(X_\Sigma) = (n+1)! \sum_{v \in \mathfrak{M}} \int_{\Delta_D} \vartheta_v d\text{vol}_M, \quad (11)$$

qui est l'analogie arithmétique de la relation

$$\deg_D(X_\Sigma) = n! \text{vol}_M(\Delta_D) \quad (12)$$

donnant le degré de  $X_\Sigma$ . Dans les égalités précédentes,  $\text{vol}_M$  est la mesure de Haar sur  $M_{\mathbb{R}}$  normalisée pour que le volume d'un domaine fondamental de  $M$  soit 1. Le langage introduit dans [BPS14] a été employé successivement dans les articles [BPS15], [BMPS16] et [BPRS15] afin d'élargir la compréhension de la géométrie d'Arakelov adélique *via* l'étude d'exemples concrets, et plus précisément de différents aspects arithmétiques des variétés toriques.

La théorie développée par Burgos Gil, Philippon et Sombra s'applique à un grand nombre de fonctions hauteur sur une variété torique, y compris la hauteur canonique étudiée par exemple dans [DP99] et [Mai00] et la hauteur de Fubini-Study dans les espaces projectifs. Cependant, elle dépend fortement de la structure torique de la variété et elle ne fournit une formule que pour la hauteur des cycles qui sont eux-mêmes toriques. Nous nous intéressons

premièrement dans ce texte au calcul de la hauteur des hypersurfaces quelconques d’une variété torique. Le cas de la hauteur canonique, dans lequel la hauteur d’une hypersurface est liée à la mesure de Mahler du polynôme de Laurent qui la définit, est bien connu, *voir* par exemple [Mai00]. Des calculs ont été effectués par Cassaigne et Maillot dans [CM00] pour la hauteur de Fubini-Study d’hypersurfaces projectives. Nous prouvons ici une formule pour un choix plus général des métriques, en utilisant le dictionnaire établi en [BPS14].

Soit donc  $X_\Sigma$  une variété torique de dimension  $n$  fixée pour le reste de ce résumé et  $Z$  un cycle de  $X_\Sigma$  de codimension 1; on peut supposer que  $Z$  est irréductible et que son point générique appartient à l’orbite torique ouverte de  $X_\Sigma$ . Dans une telle situation,  $Z$  est décrit par un polynôme de Laurent  $f = \sum_m c_m \chi^m$  à coefficients dans  $\mathbb{Q}$ . Son polytope de Newton

$$\text{NP}(f) = \text{conv}\{m \in M : c_m \neq 0\}$$

permet de donner une expression combinatoire du degré de  $Z$  relatif à un diviseur torique  $D$  de  $X_\Sigma$ , notamment

$$\text{deg}_D(Z) = \text{MV}_M(\Delta_D, \dots, \Delta_D, \text{NP}(f)), \quad (13)$$

où  $\text{MV}_M$  est l’opérateur volume mixte des corps convexes dans  $M_{\mathbb{R}}$ . Cette égalité est l’analogie de (12) et constitue la source d’inspiration pour l’extension de (4) aux hypersurfaces irréductibles de  $X_\Sigma$ .

Le premier ingrédient dont nous avons besoin pour cette extension est celui d’*intégrale mixte*: introduit par Philippon et Sombra dans [PS08a], il représente la version polarisée de l’intégrale d’une fonction concave sur un corps convexe. L’intégrale mixte est seulement un des outils de la géométrie convexe que nous utilisons dans ce texte. Nous dédions donc le chapitre 1 à leur étude, en suivant en partie les sources [Roc70] et [BPS14]. Ce faisant, nous proposons une approche plus abstraite de la géométrie convexe, consistant à la considérer comme “l’étude des semi-modules sur le semi-anneau  $\mathbb{R}_{\geq 0}$ ”. Nous nous focalisons sur la notion d’intégrale mixte, en clarifiant ses relations avec la mesure de Monge-Ampère mixte et la géométrie tropicale (en particulier la théorie d’intersection tropicale), en prouvant des formules pour des situations spécifiques et en donnant des bornes supérieures et inférieures.

Le deuxième ingrédient est un analogue arithmétique du polytope de Newton de  $f$ . Pour chaque  $v \in \mathfrak{M}$ , il existe une application de tropicalisation  $\text{trop}_v$  définie sur le tore analytique  $v$ -adique à valeurs dans  $N_{\mathbb{R}}$ ; la fibre sur  $u \in N_{\mathbb{R}}$  de cette application contient un sous-ensemble distingué  $\mathcal{B}_v(u)$ , sa *frontière de Shilov*, qui est homéomorphe à un groupe compact. La *fonction de Ronkin*

$v$ -adique du polynôme de Laurent  $f$  est alors définie comme

$$\rho_{f,v} : N_{\mathbb{R}} \rightarrow \mathbb{R}, \quad u \mapsto \int_{\text{trop}_v^{-1}(u)} -\log |f(x)| d\text{Haar}_{\mathcal{B}_v(u)}, \quad (14)$$

où  $\text{Haar}_{\mathcal{B}_v(u)}$  est la mesure de Haar sur  $\mathcal{B}_v(u)$  normalisée pour que sa masse totale soit 1. Quand  $v$  est archimédien, cette fonction est liée à celle étudiée par Passare et Rullgård dans [PR04], tandis que la fonction de Ronkin de  $f$  à une place non-archimédienne coïncide avec la tropicalisation du polynôme au sens de [MS15]. Dans les deux cas,  $\rho_{f,v}$  est une fonction concave sur  $N_{\mathbb{R}}$  et son dual de Legendre-Fenchel  $\rho_{f,v}^{\vee}$  est une fonction concave définie sur le polytope de Newton de  $f$ . La famille des fonctions de Ronkin  $v$ -adiques de  $f$  contient assez d'informations pour le calcul de la hauteur de l'hypersurface correspondante.

**Théorème 1.** *Soit  $f$  le polynôme de Laurent définissant le cycle  $Z$  et  $\overline{D}$  un diviseur torique équipé d'une métrique adélique, torique et semipositive sur  $X_{\Sigma}$ . La hauteur de  $Z$  relative à  $\overline{D}$  est donnée par*

$$h_{\overline{D}}(Z) = \sum_{v \in \mathfrak{M}} \text{MI}_M(\vartheta_v, \dots, \vartheta_v, \rho_{f,v}^{\vee}),$$

où  $\vartheta_v$  est la fonction toiture  $v$ -adique de  $\overline{D}$ .

Quand  $X_{\Sigma}$  est la droite projective et  $\overline{D}$  le diviseur à l'infini muni de sa métrique canonique, l'énoncé précédent permet de retrouver la relation (10). Bien que difficile à appliquer numériquement, l'égalité du Théorème 1 étend le pont entre les domaines de la géométrie torique arithmétique et de la géométrie convexe à des cycles de  $X_{\Sigma}$  qui ne sont pas nécessairement toriques et clarifie la relation entre le polynôme de Laurent définissant une hypersurface et sa hauteur.

La définition des fonction de Ronkin  $v$ -adiques et la preuve du Théorème 1 (ainsi que celles de résultats similaires) sont présentées dans le chapitre 4, où les cas archimédiens et non-archimédien sont traités de manière analogue. L'exposition suit celle de [Gua17], en l'enrichissant de la notion de mesures de Mahler  $v$ -adiques et de quelques estimations.

Nous commençons à explorer le cas de codimension supérieure dans le chapitre 5, où nous remarquons d'abord qu'un énoncé analogue à celui du Théorème 1 ne peut pas être espéré. En effet, nous montrons avec un exemple en codimension 2 que la hauteur d'une intersection complète ne peut pas dépendre que des fonctions de Ronkin des polynômes de Laurent qui la

définissent. Nous nous concentrons alors sur la détermination d'une borne supérieure pour la hauteur d'un cycle obtenu comme zéros communs d'une famille de polynômes de Laurent, comme dans [Som05], [Mai00] et [MS16]. En pratique, nous raffinons l'argument de [MS16] en définissant, pour un polynôme de Laurent  $f$  à coefficients dans  $\mathbb{Q}$ , sa *fonction supérieure  $v$ -adique*

$$\mu_{f,v} : N_{\mathbb{R}} \rightarrow \mathbb{R}, \quad u \mapsto - \max_{x \in B_v(u)} \log |f(x)|, \quad (15)$$

où nous gardons les mêmes notations utilisées dans (14). Soit maintenant  $k \in \{0, \dots, n\}$  et considérons une famille  $f_1, \dots, f_k$  de polynômes de Laurent à coefficients rationnels. Supposons que les adhérences dans  $X_{\Sigma}$  de leurs ensembles de zéros s'intersectent proprement dans  $X_{\Sigma}$  et notons par  $Z(f_1, \dots, f_k)$  le cycle d'intersection de codimension  $k$  ainsi obtenu.

**Théorème 2.** *Soit  $\bar{D}$  un diviseur torique équipé d'une métrique adélique, torique et semipositive sur  $X_{\Sigma}$ . Alors,*

$$h_{\bar{D}}(Z(f_1, \dots, f_k)) \leq \sum_{v \in \mathfrak{M}} \text{MI}_M(\vartheta_v, \dots, \vartheta_v, \mu_{f_1,v}^{\vee}, \dots, \mu_{f_k,v}^{\vee}),$$

où  $\vartheta_v$  est la fonction toiture  $v$ -adique de  $\bar{D}$ .

La borne supérieure donnée par le Théorème 2 est légèrement plus fine de celle prouvée par [MS16], mais surtout est adaptée à l'étude des cycles de  $X_{\Sigma}$ .

Dans le chapitre 6, nous prenons un point de vue plus probabiliste et nous nous proposons d'étudier une valeur attendue appropriée pour la hauteur du cycle  $Z(f_1, \dots, f_k)$ . Pour ce faire, nous considérons la famille des cycles

$$Z(\zeta_1^* f_1, \dots, \zeta_k^* f_k)$$

obtenus comme intersections de certains *twisting* des polynômes originaux par des points de torsion  $\zeta_1, \dots, \zeta_k$  du tore algébrique de dimension  $n$ . La *hauteur sup-esperée* du cycle défini par les polynômes de Laurent  $f_1, \dots, f_k$  relative au diviseur metrisé  $\bar{D}$  est une moyenne convenable de  $h_{\bar{D}}(Z(\zeta_1^* f_1, \dots, \zeta_k^* f_k))$  sur l'ensemble des  $k$ -uplets de points de torsion mentionnés et nous la notons par

$$\mathbb{E}[h_{\bar{D}}(f_1, \dots, f_k)].$$

Le degré de  $Z(f_1, \dots, f_k)$  relatif à un diviseur torique  $D$  sur  $X_{\Sigma}$  est donné par l'expression combinatoire

$$\deg_D(Z(f_1, \dots, f_k)) = \text{MV}_M(\Delta_D, \dots, \Delta_D, \text{NP}(f_1), \dots, \text{NP}(f_k)),$$

qui est la généralisation de (12) et de (13). Inspirés par cette situation géométrique, nous proposons la conjecture suivante.

**Conjecture 1.** *Soit  $\bar{D}$  un diviseur torique équipé d'une métrique adélique, torique et semipositive sur  $X_\Sigma$ . Alors,*

$$\mathbb{E}[h_{\bar{D}}(f_1, \dots, f_k)] = \sum_{v \in \mathfrak{M}} \text{MI}_M(\vartheta_v, \dots, \vartheta_v, \rho_{f_1, v}^\vee, \dots, \rho_{f_k, v}^\vee), \quad (16)$$

où  $\vartheta_v$  est la fonction toiture  $v$ -adique de  $\bar{D}$ .

Nous concluons le chapitre en donnant quelques intuitions de la Conjecture 1 et notamment une stratégie possible pour la preuve du cas de deux polynômes de Laurent satisfaisants certaines hypothèses supplémentaires.

### Terminology and notations

A *variety*  $X$  is assumed to be a reduced and irreducible separated scheme of finite type over a field  $K$ . By an *irreducible hypersurface* in it we mean a closed integral subscheme of codimension 1 in  $X$ . For a field  $F$ ,  $X(F)$  stands for the set of  $F$ -points of  $X$ , that is the set of morphism  $\text{Spec } F \rightarrow X$  in the category of  $K$ -schemes. For every  $x \in X$ ,  $\kappa(x)$  denotes the residue field of  $X$  at  $x$ . A *divisor* on  $X$  is a Cartier divisor, unless otherwise stated.

The term *measure* on a topological space stands for a signed Borel measure on it; in particular, measures admit a well-defined push-forward via continuous mappings. A measure only taking non-negative real values on Borel subsets is called a *positive measure*.

The *power set* of a set  $E$  is the collection of all its subsets and is denoted by  $\mathcal{P}(E)$ . Finally, for any finite set  $E$ ,  $\#E$  denotes its cardinality.

# Convex geometry

We interpret here convex geometry as the study of semimodules of the semiring of nonnegative real numbers with the usual addition and multiplication. Using this algebraic point of view, we present convex bodies, concave functions and the related notion of Legendre-Fenchel duality. Moreover, the development of a general approach to polarization and mixed maps allows us to give a uniform study of mixed volumes, mixed real Monge-Ampère measures and mixed integrals. We especially focus on this last notion, proving several equalities and special cases and giving lower and upper bounds for them in terms of extremal values of the involved functions.

## 1.1

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### THE FORMALISM OF MIXED MAPS

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Mixed maps play a fundamental role in convex geometry, as we shall see later. In the next chapters we will frequently use the notions of mixed volumes, mixed Monge-Ampère measure and mixed integral. In this section we present a more algebraic and unified approach to their construction, based on semirings and semimodules.

**SEMIRINGS AND SEMIMODULES.** The definition of a commutative monoid is the same as the one of an abelian group, except that one does not require the existence of inverses.

**Definition 1.1.1.** A (*commutative*) *monoid* is a couple  $(M, +)$  of a nonempty set  $M$  and a binary operation  $+$  on  $M$ , which is assumed to be commutative, associative and admitting a neutral element  $0_M$ .

As for groups, the algebraic structure of monoids can be enriched by the definition of an additional binary operation.

**Definition 1.1.2.** A (*commutative*) *semiring* is a triple  $(S, +, \cdot)$  of a nonempty set  $S$  and two binary operation  $+$  and  $\cdot$  on  $S$ , called the addition and the multiplication, which are assumed to be commutative, associative and admitting neutral elements  $0_S$  and  $1_S$  respectively, with the multiplication distributing over the addition and the equality  $0_S \cdot s = 0_S$  holding for every  $s \in S$ .

**Example 1.1.3.** For any set  $E$ , its power set  $\mathcal{P}(E)$  has two semiring structures with addition and multiplication given by union and intersection, or conversely. Also, the set of positive integer numbers  $\mathbb{N}$  with the usual operation of addition and multiplication is a semiring. A relevant semiring in real convex geometry is the *tropical semifield*  $(\mathbb{R} \cup \{+\infty\}, \min, +)$ .

A map  $\varphi : S \rightarrow T$  between two semirings  $(S, +, \cdot)$  and  $(T, +, \cdot)$  is called a *semiring homomorphism* if for every  $s_1, s_2 \in S$ ,  $\varphi(s_1 + s_2) = \varphi(s_1) + \varphi(s_2)$ ,  $\varphi(s_1 \cdot s_2) = \varphi(s_1) \cdot \varphi(s_2)$ ,  $\varphi(0_S) = 0_T$  and  $\varphi(1_S) = 1_T$ .

**Example 1.1.4.** For any set  $E$ , the map sending each subset of  $E$  to its complement in  $E$  realizes an isomorphism of semirings between  $(\mathcal{P}(E), \cup, \cap)$  and  $(\mathcal{P}(E), \cap, \cup)$ .

As in classical commutative algebra, it is relevant to study the action of a semiring on an algebraic structure.

**Definition 1.1.5.** Let  $(S, +, \cdot)$  be a commutative semiring. A *S-semimodule* is a nonempty set  $M$  endowed with a binary operation  $+$  and a scalar multiplication  $\cdot : S \times M \rightarrow M$ , satisfying the following axioms for every  $s, s_1, s_2 \in S$  and  $m, m_1, m_2 \in M$ :

- (i)  $+$  is commutative, associative and admits a neutral element  $0_M$
- (ii)  $s \cdot (m_1 + m_2) = s \cdot m_1 + s \cdot m_2$
- (iii)  $(s_1 + s_2) \cdot m = s_1 \cdot m + s_2 \cdot m$
- (iv)  $(s_1 \cdot s_2) \cdot m = s_1 \cdot (s_2 \cdot m)$
- (v)  $s \cdot 0_M = 0_M$

$$(vi) \ 0_S.m = 0_M$$

$$(vii) \ 1_S.m = m.$$

In other words, a  $S$ -semimodule is a commutative monoid  $(M, +)$  together with a morphism of semirings

$$S \rightarrow \text{Hom}(M, M),$$

where  $\text{Hom}(M, M)$  is the noncommutative semiring of monoid endomorphisms on  $M$  with multiplication given by composition.

**Example 1.1.6.** Every commutative monoid  $(M, +)$  can be seen as a  $\mathbb{N}$ -semimodule with action given by

$$n.m := \underbrace{m + \cdots + m}_{n \text{ times}}$$

for every  $n \in \mathbb{N}$  and  $m \in M$ .

**Example 1.1.7.** For every semiring  $S$ , the *trivial  $S$ -semimodule* is the monoid  $(S, +)$  with action given by the multiplication in  $S$ .

A map  $\varphi : M \rightarrow N$  between two  $S$ -semimodules  $M$  and  $N$  is said to be a *morphism of  $S$ -semimodules* if it is a morphism of monoids respecting the action of  $S$ , that is  $\varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$  for all  $m_1, m_2 \in M$  and  $\varphi(s.m) = s.\varphi(m)$  for all  $s \in S$  and  $m \in M$ . Notice in particular that if  $\varphi : M \rightarrow N$  is a morphism of  $S$ -semimodules, then  $\varphi(0_M) = 0_N$ .

**GROTHENDIECK COMPLETION.** By forgetting the existence of additive inverses, any commutative ring has a semiring structure and, for a ring  $R$ , any  $R$ -module can be seen as a semimodule over the semiring associated to  $R$ . Conversely, the *Grothendieck completion* allows to construct rings and modules from semirings and semimodules, as follows.

**Definition 1.1.8.** The *Grothendieck group* of an abelian monoid  $(M, +)$  is the quotient  $\mathcal{G}(M)$  of the set  $M \times M$  by the equivalence relation  $\sim$  defined as

$$(a, b) \sim (c, d) \iff \exists m \in M \text{ such that } a + d + m = b + c + m.$$

It is given a binary operation by componentwise addition.

It is easy to show that the Grothendieck group of  $(M, +)$  is a well-defined abelian group, the neutral element being the class of  $(0_M, 0_M)$  and the inverse of  $[(a, b)]$  being  $[(b, a)]$ . It can be seen as the group of formal differences between elements of  $M$  and it comes with the morphism of monoids

$$\begin{aligned} M &\rightarrow \mathcal{G}(M) \\ m &\mapsto [(m, 0_M)]. \end{aligned} \tag{1.1}$$

Let now  $(S, +, \cdot)$  be a semiring. The Grothendieck group of the abelian monoid  $(S, +)$  can be given a multiplication by defining

$$(a, b) \cdot (c, d) := (ac + bd, ad + bc)$$

on representants. Such operation turns out to be well-defined on  $\mathcal{G}(S)$  and endows it with the structure of a commutative ring with unity  $[(1_S, 0_S)]$ .

Similarly, if  $S$  is a commutative semiring and  $M$  is a  $S$ -semimodule, the action of  $\mathcal{G}(S)$  on  $\mathcal{G}(M)$  induced by

$$(r, s) \cdot (m, n) := (r \cdot m + s \cdot n, r \cdot n + s \cdot m)$$

for every  $[(r, s)] \in \mathcal{G}(S)$  and  $[(m, n)] \in \mathcal{G}(M)$  gives  $\mathcal{G}(M)$  the structure of a  $\mathcal{G}(S)$ -module.

With the obvious transformation on morphisms, the Grothendieck completion defines a functor from the category of abelian monoids to the category of abelian groups, from the category of commutative semirings to the category of commutative rings with unity and from the category of  $S$ -semimodules to the category of  $\mathcal{G}(S)$ -modules.

**Example 1.1.9.** The Grothendieck completion of the semiring  $\mathbb{N}$  is  $\mathbb{Z}$ , while the one of the tropical semifield is  $\{0\}$ . For any semiring  $S$ , the Grothendieck completion of the trivial  $S$ -semimodule is the trivial  $\mathcal{G}(S)$ -module.

As emerges from the previous example, the semiring structure on  $\mathbb{N}$  is friendlier than the one on the tropical semifield. This difference is related to an algebraic property of the subjacent monoids.

**Definition 1.1.10.** An abelian monoid  $M$  is said to satisfy the *cancellation law* if for every  $m_1, m_2 \in M$  the equality  $m_1 + n = m_2 + n$  for some  $n \in M$  implies  $m_1 = m_2$ . A commutative semiring or a semimodule is said to satisfy the cancellation law if the subjacent additive monoid does.

A monoid satisfying the cancellation law is “not very far from being an abelian group”, in the sense of the following lemma.

**Lemma 1.1.11.** *The morphism of monoids  $M \rightarrow \mathcal{G}(M)$  in (1.1) is injective if and only if  $M$  satisfies the cancellation law.*

*Proof.* Suppose first the morphism is injective. Then, for every  $m_1, m_2 \in M$  satisfying  $m_1 + n = m_2 + n$  for some  $n \in M$ , one can consider the image of the two sides of the equality in  $\mathcal{G}(M)$ ; the group structure on  $\mathcal{G}(M)$  and the injectivity of the map imply that  $m_1 = m_2$ .

Conversely, if two elements  $m_1$  and  $m_2$  of  $M$  have the same image in  $\mathcal{G}(M)$ , they must satisfy  $(m_1, 0_M) \sim (m_2, 0_M)$ . The cancellation law on  $M$  implies that  $m_1 = m_2$ .  $\square$

**MIXED MAPS.** The semiring  $\mathbb{N}$  being an initial object in the category of commutative semirings, every natural number can be unambiguously thought as an element of a commutative semiring  $(S, +, \cdot)$ . With abuse of notation, for every  $k \in \mathbb{N}$  and  $m$  in a  $S$ -semimodule  $M$ , one will hence write  $k \cdot m$  to denote the action of the image of  $k$  in  $S$  on  $m$ ; by definition, this is nothing else than the sum of  $k$  copies of  $m$  in  $M$ .

**Definition 1.1.12.** Let  $S$  be a semiring,  $\varphi : M \rightarrow N$  a map between two  $S$ -semimodules and  $k$  a positive integer. A  $k$ -mixed map for  $\varphi$  is an application  $F : M^k \rightarrow N$  being symmetric, multilinear with respect to the  $S$ -semimodule structure on  $M$  and  $N$  and satisfying  $F(m, \dots, m) = k! \cdot \varphi(m)$  for each  $m \in M$ .

The following easy property of mixed maps follows directly from the definition and will be useful later on.

**Proposition 1.1.13.** *Let  $S$  be a semiring,  $\varphi : M \rightarrow N$  a map between two  $S$ -semimodules and  $k$  a positive integer. Assume also that  $N$  satisfies the cancellation law. If  $F$  is a  $k$ -mixed map for  $\varphi$ , then*

$$F(m_1, \dots, m_{k-1}, 0_M) = 0_N$$

for all  $m_1, \dots, m_{k-1} \in M$ .

*Proof.* By multilinearity and by the equality  $0_M + 0_M = 0_M$  one has

$$F(m_1, \dots, m_{k-1}, 0_M) = F(m_1, \dots, m_{k-1}, 0_M) + F(m_1, \dots, m_{k-1}, 0_M).$$

The claim follows then from the cancellation law in  $N$ .  $\square$

Under some hypotheses on the target semimodule one can show that a mixed map, if it exists, is unique.

**Proposition 1.1.14.** *In the same hypotheses of Definition 1.1.12, assume moreover that  $N$  is a group without nontrivial elements of  $k!$  torsion. Then there exists at most one  $k$ -mixed map  $M\varphi$  for  $\varphi$  and in such a case it is of the form*

$$M\varphi(m_1, \dots, m_k) = \sum_{j=1}^k (-1)^{k-j} \sum_{1 \leq i_1 < \dots < i_j \leq k} \varphi(m_{i_1} + \dots + m_{i_j}), \quad (1.2)$$

where for each  $n \in N$ ,  $-1.n$  is the additive inverse of  $n$ .

*Proof.* Let  $F$  be a mixed map for  $\varphi$ . For every positive integer  $r$  it must satisfy by definition

$$k!. \varphi(m_1 + \dots + m_r) = F(m_1 + \dots + m_r, \dots, m_1 + \dots + m_r)$$

for any choice of  $m_1, \dots, m_r \in M$ . The multilinearity of  $F$  yields

$$k!. \varphi(m_1 + \dots + m_r) = \sum_{i_1=1}^r \dots \sum_{i_k=1}^r F(m_{i_1}, \dots, m_{i_k}).$$

For every  $\ell_1, \dots, \ell_r \in \{0, \dots, k\}$  with  $\ell_1 + \dots + \ell_r = k$ , the number of summands for which each  $m_j$  appears exactly  $\ell_j$  times in the list  $m_{i_1}, \dots, m_{i_k}$  equals

$$N_{\ell_1, \dots, \ell_r}^k := \binom{k}{\ell_1} \cdot \binom{k - \ell_1}{\ell_2} \dots \binom{k - \ell_1 - \dots - \ell_{r-2}}{\ell_{r-1}}.$$

The symmetry of  $F$  implies hence that

$$k!. \varphi(m_1 + \dots + m_r) = \sum_{\substack{\ell_1, \dots, \ell_r \in \{0, \dots, k\} \\ \ell_1 + \dots + \ell_r = k}} N_{\ell_1, \dots, \ell_r}^k \cdot F_{\ell_1, \dots, \ell_r}(m_1, \dots, m_r), \quad (1.3)$$

where

$$F_{\ell_1, \dots, \ell_r}(m_1, \dots, m_r) := F(\underbrace{m_1, \dots, m_1}_{\ell_1}, \dots, \underbrace{m_r, \dots, m_r}_{\ell_r}).$$

The statement follows from the claim that the alternated sum

$$k!. \sum_{j=1}^k (-1)^{k-j} \sum_{1 \leq i_1 < \dots < i_j \leq k} \varphi(m_{i_1} + \dots + m_{i_j}) \quad (1.4)$$

coincides with  $k!. F(m_1, \dots, m_k)$ . Indeed, the  $k!$  multiples of two elements in the group  $N$  are the same if and only if their difference in  $N$  is of  $k!$  torsion.

By hypotheses on  $N$ , they must coincide.

To prove the claimed equality, the expression in (1.4) can be rewritten as

$$\sum_{j=1}^k (-1)^{k-j} \cdot \sum_{1 \leq i_1 < \dots < i_j \leq k} k! \cdot \varphi(m_{i_1} + \dots + m_{i_j}).$$

Developing each term of the sum as in (1.3), for any choice of  $\ell_1, \dots, \ell_k \in \{0, \dots, k\}$  with  $\ell_1 + \dots + \ell_k = k$ , the term  $F_{\ell_1, \dots, \ell_k}(m_1, \dots, m_k)$  appears in (1.4) with coefficient equal to  $(-1)^{k-j} N_{\ell_1, \dots, \ell_k}^k$  for each set  $\mathcal{S}$  of indices  $\{i_1, \dots, i_j\}$  containing the set  $\mathcal{L} = \{i : \ell_i \neq 0\}$ . Hence, its coefficient in (1.4) is given by

$$\sum_{\mathcal{S} \supseteq \mathcal{L}} (-1)^{k-|\mathcal{S}|} N_{\ell_1, \dots, \ell_k}^k = N_{\ell_1, \dots, \ell_k}^k \sum_{i=|\mathcal{L}|}^k (-1)^{k-i} \binom{k-|\mathcal{L}|}{i-|\mathcal{L}|}.$$

If  $|\mathcal{L}| \neq k$ , the Newton development of the binomial proves that this coefficient equals

$$N_{\ell_1, \dots, \ell_k}^k \sum_{i=0}^{k-|\mathcal{L}|} (-1)^{k-|\mathcal{L}|-i} \binom{k-|\mathcal{L}|}{i} = 0,$$

while for  $|\mathcal{L}| = k$  one must have  $\ell_1 = \dots = \ell_k = 1$  and hence the coefficient equals  $N_{1, \dots, 1}^k = k!$ .  $\square$

The uniqueness statement can be generalized to the case of target monoids satisfying the cancellation property. Recall that for a  $S$ -semimodule  $M$  an element  $s \in S$  is said to *act injectively* on  $M$  if for every  $m_1, m_2 \in M$  with  $m_1 \neq m_2$  one has  $s.m_1 \neq s.m_2$ . Clearly, an element which is multiplicatively invertible in  $S$  acts injectively on any  $S$ -semimodule.

**Corollary 1.1.15.** *Let  $S$  be a semiring,  $\varphi : M \rightarrow N$  a map between two  $S$ -semimodules and  $k$  a positive integer. Assume moreover that  $N$  satisfies the cancellation law and that  $k!$  acts injectively on  $N$ . Then there exists at most one  $k$ -mixed map for  $\varphi$ .*

*Proof.* The Grothendieck completion  $\mathcal{G}(N)$  of  $N$  has no nontrivial elements of  $k!$  torsion. Indeed,  $[(a, b)]$  is such if and only if, because of the cancellation law,  $k!.a = k!.b$ . By hypothesis this implies  $a = b$  and so the class of  $(a, b)$  is the neutral element of  $\mathcal{G}(N)$ .

Let  $\varepsilon : N \rightarrow \mathcal{G}(N)$  denote the monoid morphism defined in (1.1). It follows from the definition that for a  $k$ -mixed map  $F$  for  $\varphi$ , the composition  $\varepsilon \circ F$  is a  $k$ -mixed map for  $\varepsilon \circ \varphi$ . Hence, if  $F_1$  and  $F_2$  are two  $k$ -mixed maps for  $\varphi$ , Proposition 1.1.14 implies that  $\varepsilon \circ F_1 = \varepsilon \circ F_2$ . The equality  $F_1 = F_2$  follows from the injectivity of  $\varepsilon$  proved in Lemma 1.1.11.  $\square$

One can give a necessary condition for a map of semimodules to admit a mixed counterpart. To state it, one needs the following definition.

**Definition 1.1.16.** Let  $S$  be a semiring,  $M$  and  $N$  two  $S$ -semimodules and  $k$  a positive integer. A map  $\varphi : M \rightarrow N$  is called a *k-homogeneous map* of  $S$ -semimodules if it satisfies

$$\varphi(s.m) = s^k.\varphi(m)$$

for each  $s \in S$  and  $m \in M$ .

A map  $\varphi : M \rightarrow N$  is said to be a *homogeneous map* of  $S$ -semimodules if it is  $k$ -homogeneous for some positive integer  $k$ .

It follows directly from the definition that if  $\varphi : M \rightarrow N$  is a homogeneous map of  $S$ -semimodules then  $\varphi(0_M) = 0_N$ .

**Proposition 1.1.17.** Let  $S$  be a semiring,  $\varphi : M \rightarrow N$  a map between two  $S$ -semimodules and  $k$  a positive integer with  $k!$  acting injectively on  $N$ . If a  $k$ -mixed map for  $\varphi$  exists, then  $\varphi$  is  $k$ -homogeneous.

*Proof.* Suppose that a  $k$ -mixed map  $F$  for  $\varphi$  exists. The  $S$ -multilinearity of  $F$  implies that

$$k!.\varphi(s.m) = F(s.m, \dots, s.m) = s^k.F(m, \dots, m) = k!.(s^k.\varphi(m))$$

for each  $s \in S$  and  $m \in M$ . The fact that  $k!$  acts injectively on  $N$  implies then that  $\varphi(s.m) = s^k.\varphi(m)$ , as desired.  $\square$

The typical situation one will deal with is the one of  $S$  containing a copy of the positive rational numbers. In such a case, a  $k$ -homogeneous map  $\varphi$  of  $S$ -semimodules with target monoid satisfying the cancellation law admits no  $\ell$ -mixed map if  $\ell \neq k$  and at most one  $k$ -mixed map. Then, without ambiguity, one will simply speak about *the* mixed map of a homogeneous map  $\varphi$  of  $S$ -semimodules, without specifying its degree, and denote it by  $M\varphi$ .

**Remark 1.1.18.** A necessary and sufficient condition for a map  $\varphi$  of  $S$ -semimodules to admit a  $k$ -mixed map could be obtained using the notion of *combinatorial degree*, see [War79, Definition 2.7], which is the minimal positive integer  $d$  (if it exists) for which the equality

$$\sum_{\substack{1 \leq i_1 < \dots < i_\ell \leq d+1 \\ \ell \in \{1, \dots, d+1\}, \ell \text{ even}}} \varphi(m_{i_1} + \dots + m_{i_\ell}) = \sum_{\substack{1 \leq i_1 < \dots < i_\ell \leq d+1 \\ \ell \in \{1, \dots, d+1\}, \ell \text{ odd}}} \varphi(m_{i_1} + \dots + m_{i_\ell})$$

holds for any choice of  $m_1, \dots, m_{d+1} \in M$ . More information about this notion can be found in [DV09].

## 1.2

## CONVEX BODIES

In this section we present a first example of semimodule over the semiring of nonnegative real numbers, that is the one of convex bodies in a real vector space. Moreover, mixed volumes are introduced using the language developed in the previous section.

We fix throughout all the section a pair of reciprocally dual lattices  $N$  and  $M$  of rank  $n$  and we denote by  $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$  and  $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$  the associated  $n$ -dimensional real vector spaces. The duality between  $N_{\mathbb{R}}$  and  $M_{\mathbb{R}}$  is denoted by  $\langle \cdot, \cdot \rangle$ . Finally, if not otherwise stated,  $\mathbb{R}_{\geq 0}$  will stand for the semiring of nonnegative real numbers endowed with the usual addition and multiplication.

**DEFINITIONS.** Recall that a nonempty subset  $C$  of  $M_{\mathbb{R}}$  is said to be *convex* if it is closed under convex combinations, that is for every  $x, y \in C$  and  $\gamma \in [0, 1]$ , the point  $\gamma x + (1 - \gamma)y$  still belongs to  $C$ . For a convex subset  $C$  of  $M_{\mathbb{R}}$ , the *dimension* of  $C$  is the dimension of the minimal affine space containing it. The interior of  $C$  in this affine subspace is called the *relative interior* of  $C$ .

**Example 1.2.1.** For a collection of finitely many points  $x_1, \dots, x_r \in M_{\mathbb{R}}$ , the *convex hull* of  $x_1, \dots, x_r$ , that is

$$\text{conv}(x_1, \dots, x_r) := \left\{ \sum_{i=1}^r \lambda_i x_i : \lambda_i \geq 0, \sum_{i=1}^r \lambda_i = 1 \right\}$$

and the *cone* spanned by  $x_1, \dots, x_r$ , that is

$$\text{cone}(x_1, \dots, x_r) := \left\{ \sum_{i=1}^r \lambda_i x_i : \lambda_i \geq 0 \right\}$$

are convex subsets of  $M_{\mathbb{R}}$ .

The following notion reveals to be crucial when dealing with convex sets.

**Definition 1.2.2.** Let  $C$  be a closed convex subset of  $M_{\mathbb{R}}$ . The *support function* of  $C$  is the function  $\Psi_C : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}$  defined as

$$\Psi_C(u) := \inf_{x \in C} \langle x, u \rangle$$

for every  $u \in N_{\mathbb{R}}$ .

For a closed convex subset  $C$  of  $M_{\mathbb{R}}$  and for  $u \in N_{\mathbb{R}}$ , one can consider the set

$$C^u := \{x \in C : \langle x, u \rangle = \Psi_C(u)\}.$$

It is a closed convex subset of  $C$ , possibly empty. Clearly,  $C^0 = C$  while, for  $u \neq 0$  and  $\Psi_C(u) \in \mathbb{R}$ ,  $C^u$  is contained in an affine hyperplane of  $M_{\mathbb{R}}$  parallel to  $u^\perp := \{x \in M_{\mathbb{R}} : \langle x, u \rangle = 0\}$ .

**Definition 1.2.3.** Let  $C$  be a closed convex subset of  $M_{\mathbb{R}}$ . An *exposed face* of  $C$  is a nonempty subset of  $C$  coinciding with  $C^u$  for some  $u \in N_{\mathbb{R}}$ .

For any closed convex set  $C$ ,  $C = C^0$  is an exposed face of itself. Any exposed face of lower dimension can be written as the intersection of  $C$  with an affine hyperplane leaving  $C$  in one halfspace.

The *Minkowski sum* of two subsets  $A$  and  $B$  of  $M_{\mathbb{R}}$  is the set

$$A + B := \{a + b : a \in A, b \in B\} \subseteq M_{\mathbb{R}}.$$

It follows immediately from the definition that the Minkowski sum is commutative, associative and admits  $\{0\}$  as neutral element. The sum of two bounded sets is bounded and the sum of two convex subsets of  $M_{\mathbb{R}}$  is still convex. For every  $u \in N_{\mathbb{R}}$  and closed convex subsets  $C_1, C_2$  of  $M_{\mathbb{R}}$ , moreover, it is immediate to show that  $(C_1 + C_2)^u = C_1^u + C_2^u$ .

Also, for every subset  $A$  of  $M_{\mathbb{R}}$  and nonnegative real number  $\lambda$ , the *scaling* of  $A$  by  $\lambda$  is the set  $\lambda A := \{\lambda a : a \in A\}$ . As for the sum, the scaling preserves boundedness and convexity of sets. For every  $u \in N_{\mathbb{R}}$ , closed convex subset  $C$  of  $M_{\mathbb{R}}$  and  $\lambda \in \mathbb{R}_{\geq 0}$ , one has  $(\lambda C)^u = \lambda C^u$ .

**Definition 1.2.4.** A *convex body* in  $M_{\mathbb{R}}$  is a nonempty compact convex subset of  $M_{\mathbb{R}}$ . The set of convex bodies in  $M_{\mathbb{R}}$  is a  $\mathbb{R}_{\geq 0}$ -semimodule with the Minkowski sum and the scaling defined above. It is called the *semimodule of convex bodies* and denoted by  $\mathcal{B}(M_{\mathbb{R}})$ .

The semimodule  $\mathcal{B}(M_{\mathbb{R}})$  satisfies the cancellation law, see for instance [Sch14, Remark 1.7.6]; a different proof will be given in Corollary 1.3.15.

The properties of the Minkowski sum and of the scaling imply that for every  $u \in N_{\mathbb{R}}$  the map  $Q \mapsto Q^u$  for every  $Q \in \mathcal{B}(M_{\mathbb{R}})$  is a morphism of  $\mathbb{R}_{\geq 0}$ -semimodules from  $\mathcal{B}(M_{\mathbb{R}})$  to itself.

**Remark 1.2.5.** For every  $u \in N_{\mathbb{R}}$ , the linear function  $\langle \cdot, u \rangle$  on  $M_{\mathbb{R}}$  is continuous with respect to the euclidean topology on  $M_{\mathbb{R}}$  and then has a minimum

over any compact set. It follows that if  $Q$  is a convex body in  $M_{\mathbb{R}}$  the support function of  $Q$  only takes finite values on  $N_{\mathbb{R}}$  and

$$\Psi_Q(u) = \min_{x \in Q} \langle x, u \rangle$$

for every  $u \in N_{\mathbb{R}}$ . Moreover, for every  $u \in N_{\mathbb{R}}$  and  $Q \in \mathcal{B}(M_{\mathbb{R}})$ ,  $Q^u$  is nonempty.

**POLYTOPES.** By a *polyhedron* in  $M_{\mathbb{R}}$  one means a nonempty subset of  $M_{\mathbb{R}}$  obtained as the intersection of finitely many closed halfspaces; it is in particular closed and convex. By definition, a polyhedron  $P$  can be written as

$$P = \bigcap_{j=1}^r \{x \in M_{\mathbb{R}} : \langle a_j, x \rangle + \alpha_j \geq 0\}, \quad (1.5)$$

with  $a_1, \dots, a_r \in N_{\mathbb{R}}$  and  $\alpha_1, \dots, \alpha_r \in \mathbb{R}$ . By [Roc70, Theorem 19.1], any polyhedron  $P$  also has a representation in terms of vertices and directions, that is, in the notation of Example 1.2.1,

$$P = \text{conv}(b_1, \dots, b_s) + \text{cone}(b_{s+1}, \dots, b_t), \quad (1.6)$$

with  $b_1, \dots, b_t \in M_{\mathbb{R}}$ . The two previous equalities are respectively called the *H-representation* and the *V-representation* of the polytope  $P$  because of its expression in terms of hyperplanes or vertices.

**Definition 1.2.6.** A *polytope* in  $M_{\mathbb{R}}$  is a bounded polyhedron. A polytope is said to be *rational* if the slopes  $a_1, \dots, a_r$  in (1.5) can be chosen in  $N$ , to be *lattice* if the vertices  $b_1, \dots, b_t$  in (1.6) can be chosen in  $M$ .

An exposed face of a polytope will simply be called a *face*; this is coherent with the current terminology in convex geometry because of [Sch14, Corollary 2.4.2]. For a polytope of dimension  $d$  in  $M_{\mathbb{R}}$ , its 0-dimensional faces are called its *vertices*, its 1-dimensional faces are called its *edges* and its  $(d - 1)$ -dimensional faces are called its *facets*.

**Remark 1.2.7.** By the boundedness condition, the V-representation of  $P$  can be reduced to  $P = \text{conv}(b_1, \dots, b_s)$  for some  $b_1, \dots, b_s \in M_{\mathbb{R}}$ . In fact, a subset of  $M_{\mathbb{R}}$  is a polytope according to Definition 1.2.6 if and only if it is the convex hull of finitely many points, see [Sch14, Theorem 2.4.3 and Theorem 2.4.6].

Denoting by  $\mathcal{P}(M_{\mathbb{R}})$  and  $\mathcal{P}_r(M_{\mathbb{R}})$  respectively the sets of polytopes and rational polytopes in  $M_{\mathbb{R}}$ , one has that

$$\mathcal{P}_r(M_{\mathbb{R}}) \subseteq \mathcal{P}(M_{\mathbb{R}}) \subseteq \mathcal{B}(M_{\mathbb{R}}).$$

Since  $\{0\}$  is a rational polytope and  $\mathcal{P}(M_{\mathbb{R}})$  and  $\mathcal{P}_r(M_{\mathbb{R}})$  are closed under Minkowski sum and scaling they inherit from  $\mathcal{B}(M_{\mathbb{R}})$  the structure of  $\mathbb{R}_{\geq 0}$ -semimodules.

**Remark 1.2.8.** Any lattice polytope is rational. Anyway, the set of lattice polytopes is not closed under arbitrary scaling and hence fails to be a sub  $\mathbb{R}_{\geq 0}$ -semimodule of  $\mathcal{P}_r(M_{\mathbb{R}})$ .

Let  $P$  be a rational polytope of full dimension  $n$  in  $M_{\mathbb{R}}$  and  $F$  a facet of  $P$ . By definition,  $F$  is contained in a unique minimal affine subspace of  $M_{\mathbb{R}}$  of dimension  $n - 1$ , whose linear part has a one dimensional orthogonal subspace in  $N_{\mathbb{R}}$ . The rationality of  $P$  implies that this line in  $N_{\mathbb{R}}$  contain a rank one sublattice of  $N$ . The unique vector  $v_F \in N$  of minimal length satisfying  $P^{v_F} = F$  is called the *minimal inner integral vector* of  $F$ .

**MIXED VOLUMES.** Let  $\mathbb{R}$  denote the  $\mathbb{R}_{\geq 0}$ -semimodule of real numbers with the obvious action given by multiplication. For any Haar measure  $\mu$  on  $M_{\mathbb{R}}$ , any convex set is  $\mu$ -measurable, see for example [Lan86], and has finite measure if it is bounded. The application

$$\text{vol}_{\mu} : \mathcal{B}(M_{\mathbb{R}}) \rightarrow \mathbb{R} \tag{1.7}$$

associating to each convex body in  $M_{\mathbb{R}}$  its  $\mu$ -volume is hence a well-defined  $n$ -homogeneous map of  $\mathbb{R}_{\geq 0}$ -semimodules.

**Theorem 1.2.9.** *There exists a unique mixed map*

$$\text{MV}_{\mu} : \mathcal{B}(M_{\mathbb{R}})^n \rightarrow \mathbb{R}$$

for the map  $\text{vol}_{\mu}$  in (1.7).

*Proof.* The existence is proved for instance in [Ewa96, Chapter IV.3], up to a multiplicative constant. The uniqueness is a consequence of Theorem 1.1.14 since  $\mathbb{R}$  is a group and  $n!$  is invertible in  $\mathbb{R}_{\geq 0}$ .  $\square$

As a consequence of the general theory developed in section 1.1, one has an explicit form of such a mixed map.

**Definition 1.2.10.** The *mixed volume* with respect to  $\mu$  of the convex bodies  $Q_1, \dots, Q_n$  in  $M_{\mathbb{R}}$  is

$$\text{MV}_{\mu}(Q_1, \dots, Q_n) = \sum_{k=1}^n (-1)^{n-k} \sum_{1 \leq i_1 < \dots < i_k \leq n} \text{vol}_{\mu}(Q_{i_1} + \dots + Q_{i_k}).$$

By definition, the mixed volume operator is symmetric and multilinear with respect to Minkowski sum and scaling and it satisfies

$$\text{MV}_{\mu}(Q, \dots, Q) = n! \text{vol}_{\mu}(Q)$$

for every  $Q \in \mathcal{B}(M_{\mathbb{R}})$ .

**Notation 1.2.11.** The unique Haar measure on  $M_{\mathbb{R}}$  for which the volume of any fundamental domain of  $M$  is 1 is denoted by  $\mu_M$  and also called the *normalized Haar measure* on  $M_{\mathbb{R}}$ . In this case, the volume map in (1.7) and the mixed volume in Definition 1.2.10 are denoted by  $\text{vol}_M$  and  $\text{MV}_M$  respectively.

The mixed volume turns out to be monotone with respect to inclusion of convex bodies.

**Proposition 1.2.12.** For  $i = 1, \dots, n$ , let  $Q_i$  and  $R_i$  be convex bodies in  $M_{\mathbb{R}}$  with  $R_i \subseteq Q_i$ . Then,

$$\text{MV}_{\mu}(R_1, \dots, R_n) \leq \text{MV}_{\mu}(Q_1, \dots, Q_n).$$

*Proof.* See [Sch14, formula (5.25)]. □

For a subset  $A$  of  $M_{\mathbb{R}}$  and a vector  $x \in M_{\mathbb{R}}$ , the *translation* of  $A$  by  $x$  is defined as  $A + x := \{a + x : a \in A\}$  and can be seen as the Minkowski sum between  $A$  and the set  $\{x\}$ . With this terminology, the mixed volume is invariant under translation of its entries.

**Proposition 1.2.13.** For every family of convex bodies  $Q_1, \dots, Q_n$  in  $M_{\mathbb{R}}$  and of vectors  $x_1, \dots, x_n \in M_{\mathbb{R}}$ ,

$$\text{MV}_{\mu}(Q_1 + x_1, \dots, Q_n + x_n) = \text{MV}_{\mu}(Q_1, \dots, Q_n).$$

*Proof.* For  $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ , one has that

$$(Q_{i_1} + x_{i_1}) + \dots + (Q_{i_k} + x_{i_k}) = Q_{i_1} + \dots + Q_{i_k} + (x_{i_1} + \dots + x_{i_k}).$$

Since the volume with respect to a Haar measure  $\mu$  is invariant under translation in  $M_{\mathbb{R}}$ , the sums appearing in Definition 1.2.10 for  $\text{MV}_{\mu}(Q_1 + x_1, \dots, Q_n + x_n)$  and  $\text{MV}_{\mu}(Q_1, \dots, Q_n)$  coincide term by term, implying the claim. □

**Corollary 1.2.14.** *Let  $Q_2, \dots, Q_n$  be convex bodies in  $M_{\mathbb{R}}$  and  $x \in M_{\mathbb{R}}$ . Then,*

$$\text{MV}_{\mu}(\{x\}, Q_2, \dots, Q_n) = 0.$$

*Proof.* By writing  $\{x\} = \{0\} + x$ , the claim follows from Proposition 1.2.13 and Proposition 1.1.13, the convex body  $\{0\}$  being the neutral element for the Minkowski sum in  $\mathcal{B}(M_{\mathbb{R}})$ .  $\square$

### 1.3

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#### CONCAVE FUNCTIONS

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After introducing concave functions on real vector spaces, we review in this section the notion of Legendre-Fenchel duality. A more detailed and complete treatment of the subject can be found in [Roc70] or in [BPS14, Chapter 2]; as in the second reference, we will deal with concave functions instead of convex ones, even if the two cases lead to analogous results. Finally, we endow the set of concave functions with two different  $\mathbb{R}_{\geq 0}$ -semimodule structures, which turns out to be isomorphic via Legendre-Fenchel duality.

As in the previous section, let  $N$  and  $M$  be a pair of reciprocally dual lattices of rank  $n$  and  $N_{\mathbb{R}}$  and  $M_{\mathbb{R}}$  the associated real vector spaces.

**DEFINITIONS.** The *hypograph* of a function  $f : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}$  is defined as the subset

$$\Gamma(f) := \{(u, t) \in N_{\mathbb{R}} \times \mathbb{R} : u \in N_{\mathbb{R}}, t \leq f(u)\}$$

of  $N_{\mathbb{R}} \times \mathbb{R}$ . Denoting by  $\pi : N_{\mathbb{R}} \times \mathbb{R} \rightarrow N_{\mathbb{R}}$  the projection onto the first factor, it follows that  $\pi(\Gamma(f))$  is the subset of  $N_{\mathbb{R}}$  on which  $f$  takes finite values. In particular, it is nonempty if and only if  $f$  is not identically  $-\infty$ .

**Definition 1.3.1.** A function  $f : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}$  is said to be *concave* if its hypograph is a nonempty convex subset of  $N_{\mathbb{R}} \times \mathbb{R}$ . If moreover  $\Gamma(f)$  is closed,  $f$  is called a *closed concave function*.

Equivalently,  $f : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}$  is concave if and only if it is non identically  $-\infty$  and for every  $u_1, u_2 \in N_{\mathbb{R}}$  and  $\gamma \in [0, 1]$  the inequality

$$\gamma f(u_1) + (1 - \gamma)f(u_2) \leq f(\gamma u_1 + (1 - \gamma)u_2) \quad (1.8)$$

holds. A twice continuously differentiable function on an open set is concave if and only if its Hessian matrix is negative semi-definite at each point of the open set, see for example [Roc70, Theorem 4.5]. The *effective domain* of

a concave function  $f$  is the set  $\text{dom}(f)$  on which the function takes values different from  $-\infty$ ; as remarked above, it coincides with the projection to  $N_{\mathbb{R}}$  of the hypograph of  $f$ . Since the image of a convex set under a linear map is convex, it is a nonempty convex subset of  $N_{\mathbb{R}}$ . Any concave function is continuous in the relative interior of its domain by [Roc70, Theorem 10.1]. A concave function is closed if and only if it is upper semicontinuous. In particular, any continuous concave function whose effective domain is a closed convex set is closed.

**Example 1.3.2.** It is anyway not true in general that the effective domain of a closed concave function is closed. For instance, the function  $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$  defined as

$$f : u \mapsto \begin{cases} \log(u) & \text{if } u > 0 \\ -\infty & \text{otherwise} \end{cases}$$

is closed concave, but its effective domain is not closed.

Concave functions whose hypographs are simple convex subsets of  $N_{\mathbb{R}} \times \mathbb{R}$  are particularly easy to deal with and deserve a name.

**Definition 1.3.3.** A concave function is said to be *piecewise affine* if its hypograph is a polyhedron.

A function  $f : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}$  is piecewise affine if and only if its effective domain is a polyhedron in  $N_{\mathbb{R}}$  and there exist  $a_1, \dots, a_r \in M_{\mathbb{R}}$  and  $\alpha_1, \dots, \alpha_r \in \mathbb{R}$  such that

$$f(u) = \min_{i=1, \dots, r} (\langle a_i, u \rangle + \alpha_i) \quad (1.9)$$

for every  $u \in \text{dom}(f)$ , see [Roc70, page 172]. By the definition of a polyhedron, any piecewise affine concave function is closed.

Finally, the following notion will play a role in subsequent chapters. Even if a more general definition can be given for arbitrary concave functions, one restricts here to the case of closed concave functions, which is the one that will be relevant later on. A function  $f$  satisfying  $f(\lambda u) = \lambda f(u)$  for every  $\lambda \geq 0$  and  $u \in N_{\mathbb{R}}$  is called *conical*.

**Definition 1.3.4.** Let  $f : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}$  be a closed concave function and  $v_0 \in \text{dom}(f)$ . The *recession function* of  $f$  is the function  $\text{rec}(f) : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}$  defined as

$$\text{rec}(f)(u) := \lim_{\lambda \rightarrow \infty} \frac{f(v_0 + \lambda u)}{\lambda}$$

for every  $u \in N_{\mathbb{R}}$ .

The function  $\text{rec}(f)$  does not depend of the choice of  $v_0 \in \text{dom}(f)$  and is a closed conical concave function, see [Roc70, Theorem 8.5].

**LEGENDRE-FENCHEL DUALITY.** The Legendre-Fenchel dual of a concave function can be interpreted as a generalization of the notion of the support function of a closed convex subset introduced in Definition 1.2.2.

**Definition 1.3.5.** Let  $f : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}$  be a concave function. The *Legendre-Fenchel dual* of  $f$  is the function  $f^{\vee} : M_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}$  defined as

$$f^{\vee}(x) := \inf_{u \in N_{\mathbb{R}}} (\langle x, u \rangle - f(u))$$

for every  $x \in M_{\mathbb{R}}$ .

The Legendre-Fenchel dual of  $f$  is a closed concave function on  $M_{\mathbb{R}}$  satisfying, if  $f$  is closed, the equality  $(f^{\vee})^{\vee} = f$ , see [Roc70, Theorem 12.2]. The effective domain of  $f^{\vee}$  is hence a convex subset of  $M_{\mathbb{R}}$ , which one calls the *stability set* of  $f$  and denotes by  $\text{stab}(f)$ . Such a set can be interpreted as a control of the limit behaviour of  $f$ , in the sense of the following proposition.

**Proposition 1.3.6.** *Let  $f : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}$  be a closed concave function. Then, the recession function  $\text{rec}(f)$  is the support function of  $\text{stab}(f)$ .*

*Proof.* This is [Roc70, Theorem 13.3]. □

In particular, the stability set of a concave function with domain the whole  $N_{\mathbb{R}}$  does not need to be bounded: for instance, the stability set of  $-\exp : \mathbb{R} \rightarrow \mathbb{R}$  is the unbounded interval  $(-\infty, 0]$ .

The following statement about a special value of the Legendre-Fenchel dual follows immediately from its definition.

**Proposition 1.3.7.** *For every concave function  $f : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}$ ,*

$$f^{\vee}(0) = - \sup_{u \in N_{\mathbb{R}}} f(u).$$

The next example is fundamental when dealing with Legendre-Fenchel duality.

**Example 1.3.8.** For a closed convex subset  $C$  of  $M_{\mathbb{R}}$ , let  $\iota_C$  be the *indicator function* of  $C$ , that is the function on  $M_{\mathbb{R}}$  defined as

$$\iota_C(x) := \begin{cases} 0 & \text{if } x \in C \\ -\infty & \text{otherwise} \end{cases}.$$

It follows immediately from the definition that  $\iota_C^\vee = \Psi_C$ , see Definition 1.2.2. Conversely, the fact that  $C$  is closed implies that  $\iota_C$  is a closed concave function and the duality then yields  $\Psi_C^\vee = \iota_C$ . The indicator function and the support function of  $C$  are hence reciprocally dual concave functions.

The Legendre-Fenchel dual of a concave function is typically not as simple as in the previous example. One disposes of explicit expressions for it only in some remarkable cases: for instance, for smooth functions of Legendre type one can refer to [Roc70, chapter 26], while the proof of [Roc70, Theorem 19.2] provides the description of the Legendre-Fenchel dual of piecewise affine concave functions. Here is a particular case.

**Proposition 1.3.9.** *Let  $f$  be a piecewise affine concave function with  $\text{dom}(f) = N_{\mathbb{R}}$ , having a representation as in (1.9). Then,  $\text{stab}(f) = \text{conv}(a_1, \dots, a_r)$  and*

$$f^\vee(x) = \sup \left\{ \sum_{i=1}^r -\lambda_i \alpha_i : x = \sum_{i=1}^r \lambda_i a_i, \lambda_i \in [0, 1], \sum_{i=1}^r \lambda_i = 1 \right\}$$

for every  $x \in \text{stab}(f)$ .

*Proof.* This follows immediately from [BPS14, Proposition 2.5.5].  $\square$

In other words, for a concave function  $f$  as in the statement of Proposition 1.3.9, the Legendre-Fenchel dual  $f^\vee$  is the function parametrizing the “roof” of the closed convex set

$$\text{conv}((a_1, -\alpha_1), \dots, (a_r, -\alpha_r))$$

in  $M_{\mathbb{R}} \times \mathbb{R}$ . In particular, it is again a piecewise affine concave function.

Finally, one can generalize the notion of exposed face of a closed convex subset using the Legendre-Fenchel duality. Indeed, for a closed concave function  $f : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}$ , define a pairing  $P_f : M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}$  by setting

$$P_f(x, u) := f^\vee(x) + f(u) - \langle x, u \rangle.$$

It follows from the definition that  $P_f(x, u) \leq 0$  for every  $x \in M_{\mathbb{R}}$  and  $u \in N_{\mathbb{R}}$ . Moreover, the concavity of  $f$  assures that

$$P_f(x, \gamma u_1 + (1 - \gamma)u_2) \geq \gamma P_f(x, u_1) + (1 - \gamma)P_f(x, u_2) \quad (1.10)$$

for each  $x \in M_{\mathbb{R}}$ ,  $u_1, u_2 \in N_{\mathbb{R}}$  and  $\gamma \in [0, 1]$ . One can associate to every  $x \in M_{\mathbb{R}}$  a subset of  $N_{\mathbb{R}}$  by defining

$$x^* := \{u \in N_{\mathbb{R}} : P_f(x, u) = 0\}. \quad (1.11)$$

Such a set is clearly included in  $\text{dom}(f)$ . Since it is the preimage of the closed set  $[0, +\infty)$  under an upper semicontinuous function, it is closed. Moreover, it is a convex set because of (1.10). Analogously, for every  $u \in N_{\mathbb{R}}$  one can consider the closed convex subset  $u^*$  of  $\text{stab}(f)$ .

**Remark 1.3.10.** Following [BPS14, pages 47-48], the collections of sets  $x^*$  and  $u^*$  induce dual convex decompositions of certain subsets of  $N_{\mathbb{R}}$  and  $M_{\mathbb{R}}$  which almost coincide with  $\text{dom}(f)$  and  $\text{stab}(f)$ , see the cited reference for more details.

The set  $u^*$  is a generalization of the notion of the exposed face of a closed convex set relative to  $u$ , as the following example shows.

**Example 1.3.11.** Let  $C$  be a nonempty closed convex subset of  $M_{\mathbb{R}}$  and  $\Psi_C$  its support function. It follows from Example 1.3.8 that in this case

$$u^* = \{x \in M_{\mathbb{R}} : \iota_C(x) + \Psi_C(u) = \langle x, u \rangle\} = C^u$$

for any  $u \in N_{\mathbb{R}}$ .

The closed convex set  $u^*$  can be interpreted as the projection of certain exposed faces of the hypograph of  $f$ . To state this relation precisely, consider the lattice  $N \oplus \mathbb{Z}$  and its dual lattice  $M \oplus \mathbb{Z}$ , with associated vector spaces  $N_{\mathbb{R}} \times \mathbb{R}$  and  $M_{\mathbb{R}} \times \mathbb{R}$ , respectively. The duality between these two vector spaces is given by

$$\langle (x, t), (u, s) \rangle = \langle x, u \rangle + ts$$

for every  $(u, s) \in N_{\mathbb{R}} \times \mathbb{R}$  and  $(x, t) \in M_{\mathbb{R}} \times \mathbb{R}$ . Finally, denote by  $\pi : N_{\mathbb{R}} \times \mathbb{R} \rightarrow N_{\mathbb{R}}$  the projection onto the first factor.

**Proposition 1.3.12.** *Let  $f : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}$  be a closed concave function. Then, for every  $x \in M_{\mathbb{R}}$ ,*

$$x^* = \pi\left(\Gamma(f)^{(x, -1)}\right).$$

*Proof.* For every  $u \in N_{\mathbb{R}}$ ,  $(u, t) \in \Gamma(f)$  if and only if  $t \leq f(u)$ , because of the definition of the hypograph of  $f$ ; hence  $\langle x, u \rangle - t \geq \langle x, u \rangle - f(u)$  for all  $(u, t) \in \Gamma(f)$ . It follows then from the definition of the support function of a closed convex subset that

$$\Psi_{\Gamma(f)}(x, -1) = \inf_{(u, t) \in \Gamma(f)} (\langle x, u \rangle - t) = \inf_{u \in \text{dom}(f)} (\langle x, u \rangle - f(u)).$$

The fact that  $\langle x, u \rangle - f(u)$  is  $+\infty$  if  $u \notin \text{dom}(f)$  implies that  $\Psi_{\Gamma(f)}(x, -1) = f^\vee(x)$ . This yields

$$\begin{aligned} x^* &= \{u \in N_{\mathbb{R}} : \langle x, u \rangle - f(u) = \Psi_{\Gamma(f)}(x, -1)\} \\ &= \left\{u \in N_{\mathbb{R}} : (u, f(u)) \in \Gamma(f)^{(x, -1)}\right\} = \pi\left(\Gamma(f)^{(x, -1)}\right), \end{aligned}$$

concluding the proof.  $\square$

**OPERATIONS ON CONCAVE FUNCTIONS.** With the usual algebraic conventions on  $-\infty$  (in particular  $0 \cdot -\infty := -\infty$ ), the *pointwise sum* of two functions  $f, g$  on  $N_{\mathbb{R}}$  is the function

$$f + g : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}, \quad u \mapsto f(u) + g(u)$$

and the *scalar multiplication* of  $f$  by a nonnegative real number  $\lambda$  is the function

$$\lambda f : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}, \quad u \mapsto \lambda f(u).$$

**Definition 1.3.13.** Let  $C$  be a nonempty closed convex subset of  $N_{\mathbb{R}}$ . The set  $\mathcal{C}_{\text{dom}}(C)$  of closed concave functions on  $N_{\mathbb{R}}$  with effective domain coinciding with  $C$  is a  $\mathbb{R}_{\geq 0}$ -semimodule with the operation of pointwise sum and scalar multiplication. It is called the *semimodule of closed concave functions with effective domain  $C$* .

The claim in the previous definition is evident from (1.8) and the fact that the pointwise sum of two functions  $f$  and  $g$  takes finite values only when  $f$  and  $g$  simultaneously do. The semimodule  $\mathcal{C}_{\text{dom}}(C)$  is easily seen to satisfy the cancellation law and to have the indicator function of  $C$  as neutral element; moreover, its only invertible elements are constant functions on  $C$ . The properties of  $\mathcal{C}_{\text{dom}}(C)$  allow to infer less evident features of the semimodule of convex bodies, as follows.

**Proposition 1.3.14.** *The map  $\Psi : \mathcal{B}(M_{\mathbb{R}}) \rightarrow \mathcal{C}_{\text{dom}}(N_{\mathbb{R}})$ , associating to each convex body  $Q$  its support function  $\Psi_Q$  is an injective morphism of  $\mathbb{R}_{\geq 0}$ -semimodules.*

*Proof.* The support function of a convex body in  $M_{\mathbb{R}}$  is seen to be concave from (1.8). As proven in Remark 1.2.5, it has effective domain the whole  $N_{\mathbb{R}}$  and in particular it is closed. It is also immediately verified from the definition that  $\Psi$  is a morphism of  $\mathbb{R}_{\geq 0}$ -semimodules.

For what concerns injectivity, let  $Q_1$  and  $Q_2$  be convex bodies in  $M_{\mathbb{R}}$  with  $\Psi_{Q_1} = \Psi_{Q_2}$ . Then, the Legendre-Fenchel dual of these functions must coincide, that is, by Example 1.3.8,  $\iota_{Q_1} = \iota_{Q_2}$ ; in particular,  $Q_1 = Q_2$ .  $\square$

**Corollary 1.3.15.** *The semimodule  $\mathcal{B}(M_{\mathbb{R}})$  satisfies the cancellation law.*

*Proof.* Let  $Q_1, Q_2, P \in \mathcal{B}(M_{\mathbb{R}})$ , with  $Q_1 + P = Q_2 + P$ . Applying the map  $\Psi$  of the statement of Proposition 1.3.14 to both sides of the equality gives  $\Psi_{Q_1} + \Psi_P = \Psi_{Q_2} + \Psi_P$ . The cancellation property of  $\mathcal{C}_{\text{dom}}(N_{\mathbb{R}})$  and the injectivity of  $\Psi$  conclude the proof.  $\square$

One can consider other operations on concave functions; as they will appear in the following only for functions defined on  $M_{\mathbb{R}}$ , we prefer to give their definitions in this dual setting. The *sup-convolution* of two concave functions  $f, g$  on  $M_{\mathbb{R}}$  whose stability sets are not disjoint is the function

$$f \boxplus g : M_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}, \quad x \mapsto \sup_{y_1+y_2=x} (f(y_1) + g(y_2))$$

and the *right scalar multiplication* of  $f$  by a positive real number  $\lambda$  is the function

$$f\lambda : M_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}, \quad x \mapsto \lambda f(x/\lambda).$$

By convention, one sets  $f0 := \Psi_{\text{stab}(f)}$  for every concave function  $f$  on  $M_{\mathbb{R}}$ . In this occasion, the properties of convex subsets allow to state nontrivial features of such operations, thanks to the following remark on hypographs.

**Proposition 1.3.16.** *For every closed concave functions  $f, g$  on  $M_{\mathbb{R}}$  whose stability sets are not disjoint and for every positive real number  $\lambda$  one has that*

$$\Gamma(f \boxplus g) = \Gamma(f) + \Gamma(g)$$

and

$$\Gamma(f\lambda) = \lambda\Gamma(f).$$

*Proof.* The first equality follows from [AW89, equality (2.3)] and the fact that the sum of the hypographs of two closed concave function is closed. The second one is [AW89, equality (2.5)].  $\square$

It follows from the previous proposition that, whenever defined, the sup-convolution of two closed concave functions is a closed concave function and that its effective domain is the Minkowski sum of the effective domains of the summands. Similarly, the right scalar multiplication of a closed concave function by a positive real number is closed and concave with effective domain the scaling of the effective domain of the function by the same real number. Finally, [Roc70, Theorem 16.4] implies that  $\text{stab}(f \boxplus g) = \text{stab}(f) \cap \text{stab}(g)$  and  $\text{stab}(f\lambda) = \text{stab}(f)$  for every concave functions  $f, g$  on  $M_{\mathbb{R}}$  and  $\lambda \in \mathbb{R}_{\geq 0}$ .

**Definition 1.3.17.** Let  $C$  be a nonempty closed convex subset of  $N_{\mathbb{R}}$ . The set  $\mathcal{C}_{\text{stab}}(C)$  of closed concave functions on  $M_{\mathbb{R}}$  with stability set coinciding with  $C$  is a  $\mathbb{R}_{\geq 0}$ -semimodule with the operation of sup-convolution and right scalar multiplication. It is called the *semimodule of closed concave functions with stability set  $C$* .

The claim in the previous definition is a consequence of the properties cited above, of [AW89, Theorem 2.2] and of the fact that for every closed concave function  $f$  on  $M_{\mathbb{R}}$  with stability set  $C$  and  $\lambda \in \mathbb{R}_{\geq 0}$  one has

$$f \boxplus \Psi_C = f \quad \text{and} \quad \Psi_C \lambda = \Psi_C,$$

because of [Roc70, Theorem 16.4] and the definition of the support function. The two  $\mathbb{R}_{\geq 0}$ -semimodules defined above fulfill the following relation.

**Proposition 1.3.18.** *Let  $C$  be a nonempty closed convex subset of  $N_{\mathbb{R}}$ . Then, the map*

$$\mathcal{C}_{\text{dom}}(C) \rightarrow \mathcal{C}_{\text{stab}}(C), \quad f \mapsto f^{\vee}$$

*is an isomorphism of  $\mathbb{R}_{\geq 0}$ -semimodules.*

*Proof.* The Legendre-Fenchel dual of a closed concave function on  $N_{\mathbb{R}}$  with effective domain  $C$  is a closed concave function on  $M_{\mathbb{R}}$  with stability set  $C$ , by definition. The fact that  $(f^{\vee})^{\vee} = f$  for every closed function  $f$  implies that the map in the statement has an inverse given again by Legendre-Fenchel duality. Finally, these two maps are morphisms of  $\mathbb{R}_{\geq 0}$ -semimodules because of [Roc70, Theorem 16.1] and [Roc70, Theorem 16.4].  $\square$

A particular case of sup-convolution is given by the *translate* of a closed concave function  $f$  on  $M_{\mathbb{R}}$  by a point  $x_0 \in M_{\mathbb{R}}$ , which is defined to be  $\tau_{x_0} f := f \boxplus \iota_{\{x_0\}}$ ; more explicitly, it is the function

$$\tau_{x_0} f : M_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{-\infty\}, \quad x \mapsto f(x - x_0).$$

It follows from Proposition 1.3.16 that  $\Gamma(\tau_{x_0} f) = \Gamma(f) + (x_0, 0)$  and from the general results about sup-convolution that  $\tau_{x_0} f$  is a closed concave function satisfying

$$(\tau_{x_0} f)^{\vee} = f^{\vee} + x_0. \tag{1.12}$$

**Remark 1.3.19.** The theory presented above could be developed for convex functions instead of concave, with analogous results. Indeed, the map sending any concave function  $f$  to its opposite  $-f$  realizes an isomorphism between the  $\mathbb{R}_{\geq 0}$ -semimodules of concave and convex functions endowed with

pointwise sum and scalar multiplication. The same map induces an isomorphism between the  $\mathbb{R}_{\geq 0}$ -semimodule of concave functions endowed with sup-convolution and right scalar multiplication and the  $\mathbb{R}_{\geq 0}$ -semimodule of convex functions endowed with inf-convolution (see [Roc70, page 34]) and right scalar multiplication.

## 1.4

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### MONGE-AMPÈRE MEASURES

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Any concave function defined on a real vector space induces a positive measure on it by considering its supdifferential. We present here the definition of such a measure, called the real Monge-Ampère measure associated to the concave function, and of its mixed counterpart. We then focus on two peculiar cases, namely the piecewise affine and the smooth one.

As above, let  $N$  and  $M$  be reciprocally dual lattices of rank  $n$  and  $N_{\mathbb{R}}$  and  $M_{\mathbb{R}}$  the associated real vector spaces.

**MEASURES.** By a *measure* on a topological space  $X$  one means a signed Borel measure, that is a  $\sigma$ -additive function from the Borel subsets of  $X$  to  $\mathbb{R} \cup \{-\infty, +\infty\}$ , valued zero on the empty set. A measure taking nonnegative real values on any Borel subset of  $X$  is called a *positive* measure. The *support* of a measure  $\mu$  on  $X$  is the inclusion minimal closed subset  $\text{supp}(\mu)$  of  $X$  for which the map  $\mu$  is zero on each Borel subset of  $X$  disjoint from  $\text{supp}(\mu)$ .

**Example 1.4.1.** For every topological space  $X$ , the function mapping each Borel subset of  $X$  to 0 is a measure on  $X$ , called the *zero measure* on  $X$ . Its support is the empty set.

For all measures  $\mu, \mu_1, \mu_2$  on  $X$  and nonnegative real number  $\lambda$ , consider the measures  $\mu_1 + \mu_2$  and  $\lambda\mu$  on  $X$  defined as

$$(\mu_1 + \mu_2)(E) := \mu_1(E) + \mu_2(E)$$

and

$$(\lambda\mu)(E) := \lambda\mu(E)$$

for any Borel subset  $E$  of  $X$ . It follows easily from the definition that  $\text{supp}(\mu_1 + \mu_2) \subseteq \text{supp}(\mu_1) \cup \text{supp}(\mu_2)$  and, if  $\lambda \neq 0$ ,  $\text{supp}(\lambda\mu) = \text{supp}(\mu)$ .

**Definition 1.4.2.** Let  $C$  be a Borel subset of  $X$ . The set  $\mathcal{M}(C)$  of measures on  $X$  with support contained in  $C$  is a  $\mathbb{R}_{\geq 0}$ -semimodule with the operations introduced above. It is called the *semimodule of measures supported on  $C$* .

The claim in the previous definition is easily verified; the semimodule  $\mathcal{M}(C)$  has the zero measure as neutral element and it satisfies the cancellation law as any of its element admits an inverse.

**REAL MONGE-AMPÈRE MEASURES.** For a closed concave function  $f$  on  $N_{\mathbb{R}}$ , the *supdifferential* of  $f$  is a multivalued function from  $N_{\mathbb{R}}$  to  $M_{\mathbb{R}}$  defined as

$$\partial f : N_{\mathbb{R}} \rightarrow \mathcal{P}(M_{\mathbb{R}}), \quad u \mapsto u^*,$$

with  $u^*$  being the dual of  $u$  with respect to  $f$  as in (1.11); see [Roc70, Theorem 23.5] for equivalent formulations of such a notion. It is clear that the supdifferential is empty valued outside the effective domain of  $f$ . By [Roc70, Theorem 23.4], it is nonempty valued in the relative interior of  $\text{dom}(f)$ , but it can happen that  $\partial f(u) = \emptyset$  even if  $u \in \text{dom}(f)$ , as the following example shows.

**Example 1.4.3.** Consider the closed concave function  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  given by  $f(u) := -u \log u + u$ , with  $f(0) = 0$ . Its Legendre-Fenchel dual is given by  $x \mapsto -\exp(-x)$  because of [Roc70, Theorem 26.5]. One has

$$0^* = \pi\left(\Gamma(f^{\vee})^{(0,-1)}\right) = \emptyset$$

because of Proposition 1.3.12.

For any subset  $E$  of  $N_{\mathbb{R}}$ , one denotes

$$\partial f(E) := \bigcup_{u \in E} \partial f(u).$$

Let also  $\mu$  be a fixed Haar measure on  $M_{\mathbb{R}}$ .

**Definition 1.4.4.** Let  $f$  be a closed concave function on  $N_{\mathbb{R}}$ . The *real Monge-Ampère measure* of  $f$  with respect to  $\mu$  is the measure  $\mathcal{M}_{\mu}(f)$  on  $N_{\mathbb{R}}$  defined as

$$\mathcal{M}_{\mu}(f)(E) := \mu(\partial f(E))$$

for each Borel subset  $E$  of  $N_{\mathbb{R}}$ .

Since  $\mu$  is a positive measure, the real Monge-Ampère measure of any closed concave function  $f$  is also so. Moreover, because of the properties of the supdifferential stated above, the support of  $\mathcal{M}_{\mu}(f)$  is contained in  $\text{dom}(f)$ . Let  $C$  be a nonempty closed convex subset of  $N_{\mathbb{R}}$  containing an affine line. For

all strictly positive real number  $\lambda$  and closed concave function  $f$  with effective domain  $C$  one has that

$$\mathcal{M}_\mu(\lambda f)(E) = \mu(\partial(\lambda f)(E)) = \mu(\lambda \partial f(E)) = \lambda^n \mathcal{M}_\mu(f)(E)$$

for each Borel subset  $E$  of  $N_\mathbb{R}$ , see [Roc70, end of page 222]. Using the fact that  $\partial \iota_C(u) \subseteq \text{stab}(\iota_C) = \text{dom}(\Psi_C)$  for every  $u \in N_\mathbb{R}$ , one verifies that the set  $\partial \iota_C(N_\mathbb{R})$  lies in the linear subspace of  $M_\mathbb{R}$  orthogonal to the line contained in  $C$ ; in particular it is of dimension strictly smaller than  $n$  and then  $\mathcal{M}_\mu(\iota_C)$  is the zero measure on  $N_\mathbb{R}$ .

It follows that, with the notation introduced in Definition 1.4.2, the application

$$\mathcal{M}_\mu : \mathcal{C}_{\text{dom}}(C) \rightarrow \mathcal{M}(C) \tag{1.13}$$

associating to each closed concave function with effective domain  $C$  its real Monge-Ampère measure is a  $n$ -homogeneous map of  $\mathbb{R}_{\geq 0}$ -semimodules.

**Theorem 1.4.5.** *Let  $C$  be a nonempty closed convex subset of  $N_\mathbb{R}$  containing an affine line. There exists a unique mixed map*

$$\text{MM}_\mu : \mathcal{C}_{\text{dom}}(C)^n \rightarrow \mathcal{M}(C)$$

for the map in (1.13).

*Proof.* The existence is proved in [PR04, §5], up to a multiplicative constant. The uniqueness is a consequence of Theorem 1.1.14 since  $\mathcal{M}(C)$  is a group and  $n!$  is invertible in  $\mathbb{R}_{\geq 0}$ .  $\square$

As a consequence of the general theory developed in section 1.1, one has an explicit form for such a mixed map.

**Definition 1.4.6.** The *mixed real Monge-Ampère measure* with respect to  $\mu$  of the closed concave functions  $f_1, \dots, f_n$  on  $N_\mathbb{R}$  is

$$\text{MM}_\mu(f_1, \dots, f_n) = \sum_{k=1}^n (-1)^{n-k} \sum_{1 \leq i_1 < \dots < i_k \leq n} \mathcal{M}_\mu(f_{i_1} + \dots + f_{i_k}).$$

By definition, the mixed real Monge-Ampère operator is symmetric and multilinear with respect to pointwise sum and scalar multiplication and satisfies

$$\text{MM}_\mu(f, \dots, f) = n! \mathcal{M}_\mu(f)$$

for every  $f \in \mathcal{C}_{\text{dom}}(C)$ . It is in fact a positive measure, as shown in [PR04, §5], with total volume

$$\text{MM}_\mu(f_1, \dots, f_n)(N_{\mathbb{R}}) = \text{MV}_\mu(\text{stab}(f_1), \dots, \text{stab}(f_n)),$$

see [BPS14, Proposition 2.7.15]. In particular, it is a finite measure if the functions  $f_1, \dots, f_n$  have bounded stability sets.

**Notation 1.4.7.** As in section 1.2, when  $\mu$  is the normalized Haar measure on  $M_{\mathbb{R}}$  the mixed real Monge-Ampère operator in Definition 1.4.6 is denoted by  $\text{MM}_M$ .

**THE PIECEWISE AFFINE CASE.** When the involved concave functions are piecewise affine, the real Monge-Ampère operator admits a more explicit form, as follows. Denote by  $\pi : M_{\mathbb{R}} \times \mathbb{R} \rightarrow M_{\mathbb{R}}$  the projection onto the first factor and by  $\delta_v$  the *Dirac delta* at  $v$ , that is the measure on  $N_{\mathbb{R}}$  defined as

$$\delta_v(E) := \begin{cases} 1 & \text{if } v \in E \\ 0 & \text{otherwise} \end{cases}$$

for every Borel subset  $E$  of  $N_{\mathbb{R}}$ . Finally, let  $\mu$  be a Haar measure on  $M_{\mathbb{R}}$ .

**Proposition 1.4.8.** *For  $i = 1, \dots, n$ , let  $f_i$  be a piecewise affine concave function with effective domain  $N_{\mathbb{R}}$  and Legendre-Fenchel dual  $f_i^\vee$ . Then,*

$$\text{MM}_\mu(f_1, \dots, f_n) = \sum_{v \in N_{\mathbb{R}}} \text{MV}_\mu \left( \pi \left( \Gamma(f_1^\vee)^{(v, -1)} \right), \dots, \pi \left( \Gamma(f_n^\vee)^{(v, -1)} \right) \right) \delta_v,$$

and the sum is finite.

*Proof.* For a piecewise affine concave function  $f$ , [BPS14, Proposition 2.7.4] and Proposition 1.3.12 yield

$$\mathcal{M}_\mu(f) = \sum_{v \in N_{\mathbb{R}}} \mu(v^*) \delta_v = \sum_{v \in N_{\mathbb{R}}} \mu \left( \pi \left( \Gamma(f^\vee)^{(v, -1)} \right) \right) \delta_v.$$

The sum is moreover supported on finitely many  $v \in N_{\mathbb{R}}$ , corresponding to the directions of the finitely many exposed faces of  $\Gamma(f^\vee)$  of dimension  $n$ .

As a consequence of Proposition 1.3.16, Proposition 1.3.18 and the linearity of  $\pi$ , for every subset  $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$  one has

$$\begin{aligned} \mathcal{M}_\mu(f_{i_1} + \dots + f_{i_k}) &= \sum_{v \in N_{\mathbb{R}}} \mu \left( \pi \left( \Gamma(f_{i_1}^\vee \boxplus \dots \boxplus f_{i_k}^\vee)^{(v, -1)} \right) \right) \delta_v \\ &= \sum_{v \in N_{\mathbb{R}}} \mu \left( \pi \left( \Gamma(f_{i_1}^\vee)^{(v, -1)} \right) + \dots + \pi \left( \Gamma(f_{i_k}^\vee)^{(v, -1)} \right) \right) \delta_v, \end{aligned}$$

and the sum is finite. The statement follows then from the definition of the mixed real Monge-Ampère measure and of the mixed volume, rearranging the terms.  $\square$

**Example 1.4.9.** Let  $P_1, \dots, P_n$  be polytopes in  $M_{\mathbb{R}}$ . As a consequence of Proposition 1.4.8 one has

$$\text{MM}_{\mu}(\Psi_{P_1}, \dots, \Psi_{P_n}) = \text{MV}_{\mu}(P_1, \dots, P_n)\delta_0,$$

where  $\Psi_{P_i}$  is the support function of  $P_i$  for every  $i = 1, \dots, n$ .

The real Monge-Ampère measure of piecewise affine concave functions is strictly related to tropical intersection theory, as introduced in [Mik06, §4] or, equivalently because of [Kat12, §5], in [AR10]. We recall here briefly the main ingredients needed to state such a relation, referring to [MS15] for a more exhaustive introduction to the realm of tropical geometry.

For every  $k \in \{0, \dots, n\}$ , a *tropical  $k$ -cycle* in  $N_{\mathbb{R}}$  is a rational polyhedral complex in  $N_{\mathbb{R}}$  of pure dimension  $k$ , endowed with integer weights on top dimensional cells satisfying a certain *balancing condition* at  $(k-1)$ -dimensional faces, see [Kat12, Definition 5.1]. In particular, to each piecewise affine concave function  $f$  with effective domain  $N_{\mathbb{R}}$  and rational slopes one can associate a tropical  $(n-1)$ -cycle on  $N_{\mathbb{R}}$  whose underlying polyhedral complex is the corner set of  $f$ , that is the set of points where the minimum in (1.9) is attained at least twice. It is balanced when equipped with weights defined locally as the (opposite of the) weights in [AR10, Definition 3.4], see [AR10, Proposition 3.7 (a)]. Such a tropical cycle is called the *tropical Weil divisor* associated to the function  $f$ .

For  $i = 1, \dots, n$  let  $f_i$  be a piecewise affine concave functions with rational slopes and  $\text{dom}(f_i) = N_{\mathbb{R}}$ . Following [Mik06, §4] and [AR10] one can construct a tropical 0-cycle  $f_1 \cdots f_n \cdot N_{\mathbb{R}}$  as the tropical intersection product of the tropical Weil divisors associated to  $f_1, \dots, f_n$ . It is a collection of finitely many points  $v_1, \dots, v_r \in N_{\mathbb{R}}$  with weights  $\omega_1, \dots, \omega_r \in \mathbb{Z}$  respectively, to which one can associate the discrete measure

$$\delta_{f_1 \cdots f_n \cdot N_{\mathbb{R}}} := \sum_{j=1}^r \omega_j \delta_{v_j}$$

on  $N_{\mathbb{R}}$ .

**Proposition 1.4.10.** *For  $i = 1, \dots, n$ , let  $f_i$  be a piecewise affine concave function with effective domain  $N_{\mathbb{R}}$  and rational slopes. Then,*

$$\text{MM}_M(f_1, \dots, f_n) = \delta_{f_1 \cdots f_n \cdot N_{\mathbb{R}}}.$$

*Proof.* Let  $f$  be a piecewise affine concave function with effective domain  $N_{\mathbb{R}}$  and rational slopes. It is well-known to tropical geometers, see for instance [BB13, first bullet of Example 5.6], that the weight of a vertex  $v$  in the top tropical self intersection product of  $f \cdot N_{\mathbb{R}}$  coincides with the normalized volume of the cell corresponding to  $v$  in the dual convex decomposition of the stability set of  $f$ . This can also be verified directly using [AR10, Definition 3.4] recursively. Then,

$$\delta_{f \dots f \cdot N_{\mathbb{R}}} = \mathcal{M}_M(f)$$

because of [BPS14, Proposition 2.7.4]. Moreover, the map associating to a  $n$ -tuple  $(f_1, \dots, f_n)$  of piecewise affine concave functions with effective domain  $N_{\mathbb{R}}$  and rational slopes the discrete measure  $\delta_{f_1 \dots f_n \cdot N_{\mathbb{R}}}$  is symmetric and multilinear with respect to pointwise sum because of the corresponding properties of the tropical intersection product, see [AR10, Remark 3.6 and Theorem 9.10]. The claim follows then from Theorem 1.1.14 since  $\mathcal{M}(N_{\mathbb{R}})$  is a group and  $n!$  is invertible in  $\mathbb{R}_{\geq 0}$ .  $\square$

**THE SMOOTH CASE.** We suppose fixed, throughout the whole subsection, the choice of a basis of  $M$  and of its dual basis of  $N$ , which determines an isomorphism  $N_{\mathbb{R}} \simeq \mathbb{R}^n$ . When  $f$  is a smooth concave function on  $\mathbb{R}^n$ , its real Monge-Ampère measure can be expressed in terms of its Hessian matrix, which measures how concave the function is in the neighbourhood of a point.

**Proposition 1.4.11.** *Let  $f$  be a smooth concave function with effective domain  $\mathbb{R}^n$ . Then,*

$$\mathcal{M}_M(f) = (-1)^n \det(\text{Hess}(f)) \lambda,$$

where  $\lambda$  denotes the ordinary Lebesgue measure on  $\mathbb{R}^n$ .

*Proof.* See [RT77, Proposition 3.4].  $\square$

The Hessian matrix of a smooth concave function on  $\mathbb{R}^n$  is semi-negative definite at each point of the domain by [Roc70, Theorem 4.5]. It follows that  $(-1)^n \det(\text{Hess}(f))$  is nonnegative at each point, hence its product with the Lebesgue measure is a positive measure on  $\mathbb{R}^n$ .

**Remark 1.4.12.** The previous proposition still holds when  $f$  is a twice continuously differentiable function on a convex open subset of  $\mathbb{R}^n$ .

The mixed real Monge-Ampère measure of a  $n$ -tuple of smooth concave functions with effective domain  $\mathbb{R}^n$  can be constructed as a shadow of its complex counterpart as explained in [Ras01] and [PR04, §5], see [Yge15, §2]

for an expository treatment. To do this, consider the complex torus  $(\mathbb{C}^*)^n$  and the continuous application

$$-\text{Log} : (\mathbb{C}^*)^n \rightarrow \mathbb{R}^n, \quad (z_1, \dots, z_n) \mapsto (-\log |z_1|, \dots, -\log |z_n|). \quad (1.14)$$

For every  $i = 1, \dots, n$ , if  $f_i$  is a smooth concave function with effective domain  $\mathbb{R}^n$  the function  $F_i := -f_i \circ (-\text{Log})$  is a plurisubharmonic function on  $(\mathbb{C}^*)^n$ . Then, denoting

$$dd^c := \frac{i}{\pi} \partial \bar{\partial},$$

$dd^c F_i$  is a closed positive current of bidegree  $(1, 1)$  on the complex torus. Using the technique developed by Bedford and Taylor in [BT76] and [BT82], one can define the wedge product

$$\text{MA}_{\mathbb{C}}(F_1, \dots, F_n) := (dd^c F_1) \wedge \dots \wedge (dd^c F_n),$$

which is called the *complex mixed Monge-Ampère measure* of  $F_1, \dots, F_n$ . It is a closed and positive current of bidegree  $(n, n)$ , hence defining a positive measure on  $(\mathbb{C}^*)^n$ . It follows from [PR04, equality (15) at page 498] that

$$(-\text{Log})_* \text{MA}_{\mathbb{C}}(F_1, \dots, F_n) = \text{MM}_M(f_1, \dots, f_n),$$

expressing the mixed real Monge-Ampère measure as the push-forward of a complex current.

## 1.5

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### MIXED INTEGRALS

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The mixed integral of a family of concave functions defined on compact domains was introduced by Philippon and Sombra in [PS08a] and further studied by the same authors in [PS08b]. We review here this notion and collect some properties which will be useful in the following. In particular, we interpret the recursive formula in [PS08b, §8] in terms of Monge-Ampère measures, study the special case of mixed integrals involving the indicator function of a line segment and give lower and upper bounds for such objects.

As usual, we fix a pair of reciprocally dual lattices  $N$  and  $M$ , of rank  $n$  and we denote by  $N_{\mathbb{R}}$  and  $M_{\mathbb{R}}$  the associated  $n$ -dimensional real vector spaces.

**DEFINITION.** The set  $\mathcal{C}_c(M_{\mathbb{R}})$  of closed concave functions on  $M_{\mathbb{R}}$  with compact effective domain can be given the structure of an  $\mathbb{R}_{\geq 0}$ -semimodule with sup-convolution and right multiplication. Indeed, it is easily verified that any

closed concave function with compact effective domain has stability set the whole  $N_{\mathbb{R}}$ , which assures that the sup-convolution is well-defined in  $\mathcal{C}_c(M_{\mathbb{R}})$ . Unless otherwise stated,  $\mathcal{C}_c(M_{\mathbb{R}})$  will always be endowed with such a  $\mathbb{R}_{\geq 0}$ -semimodule structure.

For any Haar measure  $\mu$  on  $M_{\mathbb{R}}$ , consider the application

$$I_{\mu} : \mathcal{C}_c(M_{\mathbb{R}}) \rightarrow \mathbb{R}$$

$$g \mapsto \int_{\text{dom}(g)} g \, d\mu, \quad (1.15)$$

which takes finite values since the domain of  $g$  is compact. One observes that  $I_{\mu}$  is homogeneous of degree  $(n + 1)$ . Indeed, for all  $g \in \mathcal{C}_c(M_{\mathbb{R}})$  with  $\dim \text{dom}(g) = n$  and  $\lambda \in \mathbb{R}_{>0}$ ,

$$I_{\mu}(\lambda \cdot g) = \int_{\text{dom}(g\lambda)} (g\lambda) \, d\mu = \int_{\lambda \text{dom}(g)} \lambda g(u/\lambda) \, d\mu(u) = \lambda^{n+1} I_{\mu}(g)$$

by a change of variables. When  $\lambda = 0$  or the dimension of  $\text{dom } g$  is not maximal, the same equality is obvious from the definition.

**Theorem 1.5.1.** *There exists a unique mixed map*

$$\text{MI}_{\mu} : \mathcal{C}_c(M_{\mathbb{R}})^{n+1} \rightarrow \mathbb{R}$$

for the map  $I_{\mu}$  in (1.15).

*Proof.* Existence is proved in [PS08a, Proposition IV.5 (a) and (b)]. Uniqueness is a consequence of Theorem 1.1.14 since  $\mathbb{R}$  is a group without nontrivial elements of  $n!$  torsion.  $\square$

As a consequence of the general theory developed in section 1.1, one can give the explicit form of  $\text{MI}_{\mu}$ : for  $g_0, \dots, g_n \in \mathcal{C}_c(M_{\mathbb{R}})$ ,

$$\text{MI}_{\mu}(g_0, \dots, g_n) = \sum_{k=0}^n (-1)^{n-k} \sum_{0 \leq i_0 < \dots < i_k \leq n} I_{\mu}(g_{i_0} \boxplus \dots \boxplus g_{i_k}).$$

The remark that the effective domain of the sup-convolution of two concave functions is the Minkowski sum of their effective domains allows to recover the equality of [PS08a, Définition IV.4].

**Definition 1.5.2.** Let  $g_0, \dots, g_n$  be closed concave functions on  $M_{\mathbb{R}}$  with effective domains the convex bodies  $Q_0, \dots, Q_n$  respectively. The *mixed integral* of  $g_0, \dots, g_n$  with respect to  $\mu$  is

$$\text{MI}_{\mu}(g_0, \dots, g_n) := \sum_{k=0}^n (-1)^{n-k} \sum_{0 \leq i_0 < \dots < i_k \leq n} \int_{Q_{i_0} + \dots + Q_{i_k}} (g_{i_0} \boxplus \dots \boxplus g_{i_k}) \, d\mu.$$

By definition, the mixed integral operator is symmetric and multilinear with respect to sup-convolution and it satisfies

$$\text{MI}_\mu(g, \dots, g) = (n+1)! \int_Q g \, d\mu$$

for every closed concave function  $g$  on  $M_{\mathbb{R}}$  with compact support  $Q$ . As for the mixed volume and the mixed Monge-Ampère operator, when  $\mu$  is the normalized Haar measure  $\text{vol}_M$  on  $M_{\mathbb{R}}$  one will simply write  $\text{MI}_M$ .

**RELATIONS WITH MIXED VOLUMES.** The notion of mixed integral is closely related to the one of mixed volume, from which it inherits many of its properties. Generally speaking, the mixed integral of concave functions over  $M_{\mathbb{R}}$  can be written in terms of the mixed volume in  $M_{\mathbb{R}} \times \mathbb{R}$  of their hypographs. To state the precise relation, one needs to introduce some notation. For a closed concave function  $g$  with effective domain a convex body  $Q$  and a real number  $\gamma \leq \min_Q g$ , set

$$\Gamma_\gamma(g) := \{(x, t) : x \in Q, t \in [\gamma, g(x)]\} \subset M_{\mathbb{R}} \times \mathbb{R}.$$

It is a convex body contained in the hypograph of  $g$ . Also, for a collection of  $n+1$  convex bodies  $Q_0, \dots, Q_n$  in  $M_{\mathbb{R}}$  and for every  $i \in \{0, \dots, n\}$ , let

$$\text{MV}_M(\hat{Q}_i) := \text{MV}_M(Q_0, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_n)$$

denote the mixed volume of all the bodies of the family except  $Q_i$ . Finally, for a Haar measure  $\mu$  on  $M_{\mathbb{R}}$ , denote by  $\tilde{\mu}$  the Haar measure on  $M_{\mathbb{R}} \times \mathbb{R}$  obtained as the product of  $\mu$  with the Lebesgue measure on  $\mathbb{R}$ .

**Proposition 1.5.3.** *For  $i = 0, \dots, n$ , let  $g_i$  be a closed concave function defined on a convex body  $Q_i$  and  $\gamma_i$  a real number, with  $\gamma_i \leq \min(\min_{Q_i} g_i, 0)$ . Then*

$$\text{MI}_\mu(g_0, \dots, g_n) = \text{MV}_{\tilde{\mu}}(\Gamma_{\gamma_0}(g_0), \dots, \Gamma_{\gamma_n}(g_n)) + \sum_{i=0}^n \gamma_i \text{MV}_\mu(\hat{Q}_i).$$

*Proof.* This is [PS08a, Proposition IV.5 (d)]. □

A useful consequence of the equality in the previous statement is the monotonicity of the mixed integral operator.

**Proposition 1.5.4.** *For  $i = 0, \dots, n$ , let  $g_i$  and  $h_i$  be closed concave functions defined on the same convex body  $Q_i$ , with  $h_i \leq g_i$ . Then,*

$$\text{MI}_\mu(h_0, \dots, h_n) \leq \text{MI}_\mu(g_0, \dots, g_n).$$

*Proof.* This is [PS08b, Proposition 8.1] and it follows immediately from Proposition 1.5.3 and the monotonicity of mixed volumes.  $\square$

Conversely, one can recover the mixed volume of the domains of a family of  $n$  concave functions via mixed integrals. Recall that  $\delta_0$  denotes the function which is valued 1 on  $\{0\}$ ,  $-\infty$  otherwise.

**Proposition 1.5.5.** *For  $i = 1, \dots, n$ , let  $g_i$  be a closed concave function defined on a convex body  $Q_i$  in  $M_{\mathbb{R}}$ . Then*

$$\text{MI}_{\mu}(\delta_0, g_1, \dots, g_n) = \text{MV}_{\mu}(Q_1, \dots, Q_n).$$

*Proof.* For any  $i = 1, \dots, n$  choose  $\gamma_i \leq \min(\min_{Q_i} g_i, 0)$  and let  $\gamma_0 := -1$ . With the notation  $Q_0 = \{0\}$ , Proposition 1.5.3 implies that

$$\begin{aligned} \text{MI}_{\mu}(-\delta_0, g_1, \dots, g_n) &= \text{MV}_{\bar{\mu}}(Q_0 \times \{-1\}, \Gamma_{\gamma_1}(g_1), \dots, \Gamma_{\gamma_n}(g_n)) \\ &\quad + \sum_{i=0}^n \gamma_i \text{MV}_{\mu}(\hat{Q}_i). \end{aligned}$$

Applying Corollary 1.2.14 one obtains

$$\text{MI}_{\mu}(-\delta_0, g_1, \dots, g_n) = -\text{MV}_{\mu}(Q_1, \dots, Q_n). \quad (1.16)$$

Since  $\delta_0 \boxplus (-\delta_0) = \iota_0$ , which is the neutral element for the sup-convolution, multilinearity and Proposition 1.1.13 yield

$$\text{MI}_{\mu}(\delta_0, g_1, \dots, g_n) + \text{MI}_{\mu}(-\delta_0, g_1, \dots, g_n) = 0,$$

hence, together with (1.16), the claim.  $\square$

**Corollary 1.5.6.** *For  $i = 0, \dots, n$ , let  $g_i$  be a closed concave function defined on a convex body  $Q_i$  in  $M_{\mathbb{R}}$ . Then, for every  $c \in \mathbb{R}$ ,*

$$\text{MI}_M(g_0, \dots, g_{n-1}, g_n + c) = \text{MI}_M(g_0, \dots, g_n) + c \cdot \text{MV}_M(Q_0, \dots, Q_{n-1}).$$

*Proof.* By writing  $g_n + c = g_n \boxplus (c \cdot \delta_0)$  one has that

$$\begin{aligned} \text{MI}_M(g_0, \dots, g_{n-1}, g_n + c) &= \text{MI}_M(g_0, \dots, g_{n-1}, g_n) \\ &\quad + c \cdot \text{MI}_M(g_0, \dots, g_{n-1}, \delta_0) \end{aligned}$$

because of the multilinearity of the mixed integral and the fact that  $c \cdot \delta_0 = \delta_0 \cdot c$ . The claim follows then by symmetry from Proposition 1.5.5.  $\square$

Finally, analogously to Proposition 1.2.13, the mixed integral is invariant with respect to translation of the entries.

**Proposition 1.5.7.** *For  $i = 0, \dots, n$ , consider  $x_i \in M_{\mathbb{R}}$  and  $g_i$  a closed concave function defined on a convex body  $Q_i$ . Then,*

$$\text{MI}_{\mu}(\tau_{x_0}g_0, \dots, \tau_{x_n}g_n) = \text{MI}_{\mu}(g_0, \dots, g_n).$$

*Proof.* For every  $i = 0, \dots, n$ , choose  $\gamma_i \in \mathbb{R}$  with  $\gamma_i \leq \min(\min_{Q_i} g_i, 0)$ . The definition of the translate yields that  $\tau_{x_i}g_i$  is defined on  $Q_i + x_i$  and that  $\Gamma_{\gamma_i}(\tau_{x_i}g_i) = \Gamma_{\gamma_i}(g_i) + (x_i, 0)$ . The claim then follows from Proposition 1.5.3 and Proposition 1.2.13.  $\square$

**A RECURSIVE FORMULA.** Mixed integrals satisfy a useful recursive relation, which was proved in [PS08b, Proposition 8.5]. When the effective domains of the involved functions are rational polytopes, one can interpret such a formula in terms of mixed Monge-Ampère measures via Legendre-Fenchel duality.

Recall first that if  $u \in N_{\mathbb{Q}} \setminus \{0\}$ , the intersection  $M(u) := M \cap u^{\perp}$  is a lattice of rank  $n - 1$  spanning the linear space  $u^{\perp}$  and hence it induces a normalized Haar measure  $\text{vol}_{M(u)}$  on  $u^{\perp}$ . Moreover, for every convex body  $Q$  in  $M_{\mathbb{R}}$  the subset  $Q^u$  is contained in an affine subspace of  $M_{\mathbb{R}}$  of codimension 1 and parallel to  $u^{\perp}$ . Then, if  $g_i$  is a concave function defined on a convex body  $Q_i$  in  $M_{\mathbb{R}}$  for each  $i = 0, \dots, n$ , Proposition 1.5.7 allows to consider the mixed integral with respect to  $\text{vol}_{M(u)}$  of  $g_0|_{Q_0^u}, \dots, g_n|_{Q_n^u}$ , as functions defined on convex subsets of  $u^{\perp}$ .

Finally, a vector  $u \in N$  is said to be *primitive* if it is nonzero and there is no other element  $u' \in N$  such that  $ku' = u$  for some positive integer  $k$ .

**Theorem 1.5.8.** *For  $i = 0, \dots, n$ , let  $g_i$  be a continuous concave function on a rational polytope  $Q_i$  in  $M_{\mathbb{R}}$ . Then*

$$\begin{aligned} \text{MI}_M(g_0, \dots, g_n) = & - \sum_{\substack{u \in N \\ \text{primitive}}} \Psi_{Q_0}(u) \text{MI}_{M(u)}(g_1|_{Q_1^u}, \dots, g_n|_{Q_n^u}) \\ & - \int_{N_{\mathbb{R}}} g_0^{\vee} d\text{MM}_M(g_1^{\vee}, \dots, g_n^{\vee}), \end{aligned}$$

*the first sum being finite.*

*In particular, if  $g_i$  is a piecewise affine concave function on  $Q_i$  with hypograph  $\Gamma_i$  for any  $i = 0, \dots, n$ , denoting by  $\pi : M_{\mathbb{R}} \times \mathbb{R} \rightarrow M_{\mathbb{R}}$  the projection onto the*

first factor, one has

$$\begin{aligned} \text{MI}_M(g_0, \dots, g_n) = & - \sum_{\substack{u \in N \\ \text{primitive}}} \Psi_{Q_0}(u) \text{MI}_{M(u)}(g_1|_{Q_1^u}, \dots, g_n|_{Q_n^u}) \\ & - \sum_{v \in N_{\mathbb{R}}} g_0^{\vee}(v) \text{MV}_M \left( \pi \left( \Gamma_1^{(v, -1)} \right), \dots, \pi \left( \Gamma_n^{(v, -1)} \right) \right). \end{aligned}$$

*Proof.* By [BPS14, Proposition 2.5.23 (1)], any continuous concave function on a polytope can be approximated, with respect to uniform convergence, by a sequence of piecewise affine concave functions on the polytope itself. On the other hand, the Legendre-Fenchel duality and the real Monge-Ampère operator are continuous with respect to uniform limits of concave functions, see [BPS14, Proposition 2.2.3] and [RT77, §3], respectively. It is not difficult to show that the same holds for mixed integrals. Thanks to Proposition 1.4.8, it is hence enough to prove the formula in the particular case of  $g_0, \dots, g_n$  being piecewise affine concave functions.

Let hence  $g_i$  be a concave piecewise affine function on the rational polytope  $Q_i$  in  $M_{\mathbb{R}}$ ,  $\Gamma_i$  its hypograph, for  $i = 0, \dots, n$ . The choice of a basis of  $N$  (and of the dual basis of  $M$ ) endows  $N_{\mathbb{R}}$  and  $M_{\mathbb{R}}$  with an euclidean structure, allowing to consider the sets

$$\mathbb{S}^{n-1} := \{w \in N_{\mathbb{R}} : \|w\| = 1\} \subseteq N_{\mathbb{R}}$$

and

$$\mathbb{S}_-^n := \{(v, t) \in N_{\mathbb{R}} \times \mathbb{R} : \|(v, t)\| = 1, t < 0\} \subseteq N_{\mathbb{R}} \times \mathbb{R}.$$

After a change of sign due to the use of a different notation, [PS08b, Proposition 8.5] affirms that

$$\begin{aligned} \text{MI}_M(g_0, \dots, g_n) = & - \sum_{w \in \mathbb{S}^{n-1}} \Psi_{Q_0}(w) \text{MI}_{n-1}(g_1|_{Q_1^w}, \dots, g_n|_{Q_n^w}) \\ & - \sum_{r \in \mathbb{S}_-^n} \Psi_{\Gamma_0}(r) \text{MV}_n(\Gamma_1^r, \dots, \Gamma_n^r), \quad (1.17) \end{aligned}$$

where, on the right hand side, one refers to the mixed integral with respect to the measure obtained restricting  $\text{vol}_M$  to  $w^\perp$  and to the mixed volume with respect to the restriction of  $\text{vol}_{M \oplus \mathbb{Z}}$  to  $r^\perp$ .

Concerning the first sum on the right hand side of (1.17), if a term in the sum is different from zero, then there exists a subset  $I \subset \{1, \dots, n\}$  such that the Minkowski sum of  $Q_i^w$ , with  $i \in I$ , is of dimension  $n-1$ ; in particular, denoting

$Q := Q_1 + \cdots + Q_n$ ,  $Q^w = Q_1^w + \cdots + Q_n^w$  needs to be of dimension  $n - 1$ . As a consequence, one can restrict the sum to the set of vectors  $w \in \mathbb{S}^{n-1}$  for which  $Q^w$  is a  $(n - 1)$ -dimensional face of  $Q$ . This set is included in the set of vectors of unitary length which are perpendicular to a  $(n - 1)$ -dimensional face of  $Q$ , hence it is finite since  $Q$  is a polytope. Moreover, since  $Q$  is rational, the ray spanned by such a vector  $w$  contains a unique primitive vector  $u \in N$ . The linearity of  $\Psi_{Q_0}$  yields hence the equality

$$\begin{aligned} \sum_{w \in \mathbb{S}^{n-1}} \Psi_{Q_0}(w) \text{MI}_{n-1}(g_1|_{Q_1^w}, \dots, g_n|_{Q_n^w}) \\ = \sum_{\substack{u \in N \\ \text{primitive}}} \frac{\Psi_{Q_0}(u)}{\|u\|} \text{MI}_{n-1}(g_1|_{Q_1^u}, \dots, g_n|_{Q_n^u}). \end{aligned}$$

The fact that the restriction of  $\text{vol}_M$  to  $u^\perp$  is equal to the measure  $\text{vol}_{M(u)}$  multiplied by  $\|u\|$ , see [BPS14, proof of Corollary 2.7.10], allows to conclude that the first sum in (1.17) coincides with the desired one.

Regarding the second sum in (1.17), there exists an obvious bijection between  $\mathbb{S}_-^n$  and  $N_{\mathbb{R}}$  given by associating to each  $r \in \mathbb{S}_-^n$  the only vector  $v \in N_{\mathbb{R}}$  such that  $(v, -1)$  lies on the line spanned by  $r$ . Hence,

$$\sum_{r \in \mathbb{S}_-^n} \Psi_{\Gamma_0}(r) \text{MV}_n(\Gamma_1^r, \dots, \Gamma_n^r) = \sum_{v \in N_{\mathbb{R}}} \frac{\Psi_{\Gamma_0}(v, -1)}{\|(v, -1)\|} \text{MV}_n(\Gamma_1^{(v, -1)}, \dots, \Gamma_n^{(v, -1)}).$$

Directly by the definition of the Legendre-Fenchel duality, one has the equality  $\Psi_{\Gamma_0}(v, -1) = g_0^\vee(v)$ . The statement follows then from the fact that for every Borel set  $E$  in  $(v, -1)^\perp$ , the measure of  $E$  with respect to the restriction of  $\text{vol}_{M \oplus \mathbb{Z}}$  to  $(v, -1)^\perp$  equals  $\|(v, -1)\| \cdot \text{vol}_M(\pi(E))$ , again by [BPS14, proof of Corollary 2.7.10].  $\square$

**Remark 1.5.9.** For a rational polytope  $P$  of full dimension  $n$  in  $M_{\mathbb{R}}$ , every facet  $F$  of  $P$ , that is a face of dimension  $n - 1$ , admits a distinguished orthogonal vector: it is the unique primitive vector  $v_F \in N$  which satisfies  $P^{v_F} = F$ . Under the additional assumption that the Minkowski sum  $Q := Q_1 + \cdots + Q_n$  is of dimension  $n$  in  $M_{\mathbb{R}}$ , the formula in Theorem 1.5.8 can be written as

$$\begin{aligned} \text{MI}_M(g_0, \dots, g_n) = - \sum_F \Psi_{Q_0}(v_F) \text{MI}_{M(v_F)}(g_1|_{Q_1^{v_F}}, \dots, g_n|_{Q_n^{v_F}}) \\ - \int_{N_{\mathbb{R}}} g_0^\vee d\text{MM}_M(g_1^\vee, \dots, g_n^\vee), \end{aligned}$$

the first sum being over the finite set of facets of the polytope  $Q$ . Indeed, in such a situation the application  $F \mapsto v_F$  realizes a bijection between the set of facets of  $Q$  and the set of primitive vectors  $u \in N$  for which  $Q^u$  is a  $(n-1)$ -dimensional face of  $Q$ , which are the only vectors for which the term of the sum in the statement of the theorem does not vanish.

**Remark 1.5.10.** The statement of Theorem 1.5.8 can be reformulated in terms of Legendre-Fenchel duality. For  $i = 0, \dots, n$ , let  $f_i$  be a concave function on  $N_{\mathbb{R}}$  with stability set a rational polytope  $Q_i$  in  $M_{\mathbb{R}}$ . Under the assumption that  $Q_1 + \dots + Q_n$  is of dimension  $n$  in  $M_{\mathbb{R}}$ , Remark 1.5.9 yields

$$\begin{aligned} \text{MI}_M(f_0^\vee, \dots, f_n^\vee) &= - \sum_F \Psi_{Q_0}(v_F) \text{MI}_{M(v_F)}(f_1^\vee|_{Q_1^{v_F}}, \dots, f_n^\vee|_{Q_n^{v_F}}) \\ &\quad - \int_{N_{\mathbb{R}}} f_0 \, d\text{MM}_M(f_1, \dots, f_n). \end{aligned} \quad (1.18)$$

Indeed, it is sufficient to readily apply the previous theorem to the functions  $f_0^\vee, \dots, f_n^\vee$ , which are continuous on their domain and satisfy the equality  $(f_i^\vee)^\vee = f_i$  for each  $i = 0, \dots, n$  by concavity and closedness. It is easy to verify that the choice  $f_0 = \dots = f_n = f$  in (1.18) yields the formula in [BPS14, Corollary 2.7.10].

An application of the recursive formula proved above is the computation of the mixed integral when all except one entry are indicator functions in the sense of Example 1.3.8.

**Corollary 1.5.11.** *Let  $Q_1, \dots, Q_n$  be rational polytopes in  $M_{\mathbb{R}}$  and  $f$  a concave function on  $N_{\mathbb{R}}$  with stability set a rational polytope. Then*

$$\text{MI}_M(\iota_{Q_1}, \dots, \iota_{Q_n}, f^\vee) = -\text{MV}_M(Q_1, \dots, Q_n) \cdot f(0).$$

*Proof.* By symmetry, one can develop the recursive formula in Remark 1.5.10 with respect to  $f^\vee$  to obtain

$$\text{MI}_M(\iota_{Q_1}, \dots, \iota_{Q_n}, f^\vee) = - \int_{N_{\mathbb{R}}} f \, d\text{MM}_M(\iota_{Q_1}^\vee, \dots, \iota_{Q_n}^\vee),$$

the indicator functions  $\iota_{Q_1}, \dots, \iota_{Q_n}$  being zero where defined. The duality in Example 1.3.8 and the fact that

$$\text{MM}_M(\Psi_{Q_1}, \dots, \Psi_{Q_n}) = \text{MV}_M(Q_1, \dots, Q_n) \delta_0$$

because of Example 1.4.9 conclude the proof.  $\square$

**A SPECIAL CASE.** It is interesting to give a formula expressing the mixed integral of a  $(n + 1)$ -tuple of concave functions on  $M_{\mathbb{R}}$  where one of them is the indicator function of a line segment.

Let  $m$  be a primitive vector of  $M$  and consider the quotient  $P := M/\mathbb{Z}m$ . Since  $m$  is primitive,  $P$  is a lattice of rank  $n - 1$ . By abuse of notation, let  $\pi$  denote both the projection from  $M$  to  $P$  and the induced linear map from  $M_{\mathbb{R}}$  to  $P_{\mathbb{R}}$ . For each closed concave function  $g$  defined on a compact subset  $B$  of  $M_{\mathbb{R}}$ , let

$$\pi_*g : \pi(B) \rightarrow \mathbb{R}, \quad x \mapsto \max_{y \in \pi^{-1}(x)} g(y) \quad (1.19)$$

be the *direct image* of  $g$  by  $\pi$ . It is a well defined closed concave function with domain a bounded subset of  $P_{\mathbb{R}}$ , see [Roc70, Theorem 5.7 and Theorem 9.2]. Finally, for  $x_1, x_2 \in M_{\mathbb{R}}$ , denote by  $\overline{x_1x_2}$  the line segment in  $M_{\mathbb{R}}$  with extremal points  $x_1$  and  $x_2$ . The following lemma is a generalization of [Ewa96, exercise 3 at page 128].

**Lemma 1.5.12.** *In the above hypotheses and notations and for  $n \geq 2$ , let  $Q_1, \dots, Q_{n-1}$  be polytopes in  $M_{\mathbb{R}}$ . Then,*

$$\text{MV}_M(\overline{0m}, Q_1, \dots, Q_{n-1}) = \text{MV}_P(\pi(Q_1), \dots, \pi(Q_{n-1})).$$

*Proof.* The vector  $m$  being primitive, it can be extended to a basis of the lattice  $M$ , see for instance [Lek69, Theorem 5 at page 21]. We suppose fixed throughout the proof such a basis  $(m_1, \dots, m_{n-1}, m)$  of  $M$  and the induced isomorphism  $M_{\mathbb{R}} \simeq \mathbb{R}^n$ ; under this identification, the normalized volume  $\text{vol}_M$  corresponds to the Lebesgue measure  $\text{vol}_n$  on  $\mathbb{R}^n$ . Since  $(\pi(m_1), \dots, \pi(m_{n-1}))$  is a basis of  $P$ , such a lattice is isomorphic to the span of  $m_1, \dots, m_{n-1}$  in  $M$  and hence it is identified with the linear subspace  $\mathbb{R}^{n-1} \times \{0\}$  of  $\mathbb{R}^n$ . Moreover,  $\text{vol}_P$  corresponds to the  $(n - 1)$ -dimensional Lebesgue measure  $\text{vol}_{n-1}$  on  $\mathbb{R}^{n-1} \times \{0\}$  and the map  $\pi$  to the vertical projection.

The claim reduces then to the particular case of a family of polytopes  $Q_1, \dots, Q_{n-1}$  in  $\mathbb{R}^n$ ,  $m = (0, \dots, 0, 1)$  and  $\pi$  the vertical projection. Denoting by  $S$  the vertical segment of unitary length and rearranging the terms in the definition of mixed volume one obtains

$$\begin{aligned} \text{MV}_n(S, Q_1, \dots, Q_{n-1}) = \\ \sum_{k=1}^{n-1} (-1)^{n-1-k} \sum_{1 \leq i_1 < \dots < i_k \leq n-1} (\text{vol}_n(S + Q_{i_1} + \dots + Q_{i_k}) - \text{vol}_n(Q_{i_1} + \dots + Q_{i_k})) \end{aligned}$$

since the  $n$ -dimensional volume of a line segment vanishes for  $n \geq 2$ . To prove the claim it is hence enough to show that for each polytope  $Q$  in  $\mathbb{R}^n$  the

equality

$$\text{vol}_n(S + Q) - \text{vol}_n(Q) = \text{vol}_{n-1}(\pi(Q))$$

holds. But  $Q \subset S + Q$  and the difference of their volumes coincides with the integral over  $\pi(Q) = \pi(S + Q)$  of the difference between the concave functions parametrizing the roof of the polytope  $S + Q$  and  $Q$  respectively. Such a difference being constantly equal to 1 on  $\pi(Q)$ , the claim follows from the definition of the Lebesgue integral, concluding the proof.  $\square$

**Proposition 1.5.13.** *In the above hypotheses and notations, let  $g_i$  be a continuous concave function defined on a polytope  $Q_i$  in  $M_{\mathbb{R}}$ , for  $i = 1, \dots, n$ . Then,*

$$\text{MI}_M(\iota_{\overline{0m}}, g_1, \dots, g_n) = \text{MI}_P(\pi_*g_1, \dots, \pi_*g_n).$$

*Proof.* For  $n = 1$ , the claim follows from Corollary 1.5.11. Assume hence  $n \geq 2$ . Choosing  $\gamma_0 = 0$  and for each  $i = 1, \dots, n$  a nonpositive real number  $\gamma_i$  such that  $\gamma_i \leq \min_{x \in Q_i} g_i(x)$ , Proposition 1.5.3 implies that

$$\begin{aligned} \text{MI}_M(\iota_{\overline{0m}}, g_1, \dots, g_n) &= \text{MV}_{M \oplus \mathbb{Z}}(\overline{(0,0)(m,0)}, \Gamma_{\gamma_1}(g_1), \dots, \Gamma_{\gamma_n}(g_n)) \\ &\quad + \sum_{i=1}^n \gamma_i \text{MV}_M(\overline{0m}, Q_1, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_n). \end{aligned}$$

By the hypotheses on  $m$ ,  $(m, 0)$  is a nonzero primitive vector of the lattice  $M \oplus \mathbb{Z}$ . The map  $\pi' := \pi \times \text{id}_{\mathbb{Z}} : M \oplus \mathbb{Z} \rightarrow P \oplus \mathbb{Z}$  is a surjective group homomorphism, giving  $(M \oplus \mathbb{Z})/\mathbb{Z}(m, 0) \simeq P \oplus \mathbb{Z}$ . Then, by Lemma 1.5.12,

$$\begin{aligned} \text{MI}_M(\iota_{\overline{0m}}, g_1, \dots, g_n) &= \text{MV}_{P \oplus \mathbb{Z}}(\pi'(\Gamma_{\gamma_1}(g_1)), \dots, \pi'(\Gamma_{\gamma_n}(g_n))) \\ &\quad + \sum_{i=1}^n \gamma_i \text{MV}_P(\pi(Q_1), \dots, \pi(Q_{i-1}), \pi(Q_{i+1}), \dots, \pi(Q_n)). \end{aligned}$$

The statement follows hence from Proposition 1.5.3 applied to the concave functions  $\pi_*g_1, \dots, \pi_*g_n$ , the direct image of  $g_i$  by  $\pi$  being a concave function defined on  $\pi(Q_i)$  and satisfying

$$\begin{aligned} \pi'(\Gamma_{\gamma_i}(g_i)) &= \{(\pi(y), t) \in \pi(Q_i) \times \mathbb{R} : \gamma_i \leq t \leq g_i(y)\} \\ &= \{(x, t) \in \pi(Q_i) \times \mathbb{R} : \gamma_i \leq t \leq (\pi_*g_i)(x)\} \\ &= \Gamma_{\gamma_i}(\pi_*g_i) \end{aligned}$$

for every  $i = 1, \dots, n$ .  $\square$

**INEQUALITIES FOR MIXED INTEGRALS.** Mixed integrals are typically difficult to compute. It is therefore useful to dispose of computable lower and upper bounds for these quantities.

**Theorem 1.5.14.** *For  $i = 0, \dots, n$ , let  $g_i$  be a continuous concave function on a rational polytope  $Q_i$  in  $M_{\mathbb{R}}$ . Then,*

$$\text{MI}_M(g_0, \dots, g_n) \geq \max_i \left( -g_i^\vee(0) \cdot \text{MV}_M(\hat{Q}_i) + \sum_{j \neq i} \min_{Q_j}(g_j) \cdot \text{MV}_M(\hat{Q}_j) \right)$$

and

$$\text{MI}_M(g_0, \dots, g_n) \leq - \sum_{i=0}^n g_i^\vee(0) \cdot \text{MV}_M(\hat{Q}_i).$$

*Proof.* For the first inequality, consider for each  $j = 0, \dots, n$  the constant function  $f_j$  on  $Q_j$  with value  $\min_{Q_j}(g_j)$ , that is

$$f_j = \iota_{Q_j} + \min_{Q_j}(g_j).$$

It is obvious from the definition that  $f_j \leq g_j$  for each  $j = 0, \dots, n$ . Hence, using Proposition 1.5.4 and repeatedly applying Lemma 1.5.6, one has, for each  $i = 0, \dots, n$ ,

$$\begin{aligned} \text{MI}_M(g_0, \dots, g_n) &\geq \text{MI}_M(f_0, \dots, f_{i-1}, g_i, f_{i+1}, \dots, f_n) \\ &= \text{MI}_M(\iota_{Q_0}, \dots, \iota_{Q_{i-1}}, g_i, \iota_{Q_{i+1}}, \dots, \iota_{Q_n}) \\ &\quad + \sum_{j \neq i} \min_{Q_j}(g_j) \cdot \text{MV}_M(\hat{Q}_j). \end{aligned}$$

Corollary 1.5.11 and the arbitrariness of  $i \in \{0, \dots, n\}$  hence imply the first inequality.

Similarly, for the second one, consider for each  $i = 0, \dots, n$  the constant function  $h_i$  on  $Q_i$  with value  $\max_{Q_i}(g_i)$ , that is

$$h_i = \iota_{Q_i} + \max_{Q_i}(g_i) = \iota_{Q_i} - g_i^\vee(0)$$

because of Proposition 1.3.7. The relation  $h_i \geq g_i$  for each  $i = 0, \dots, n$ , Proposition 1.5.4 and Lemma 1.5.6 give

$$\begin{aligned} \text{MI}_M(g_0, \dots, g_n) &\leq \text{MI}_M(h_0, \dots, h_n) \\ &= \text{MI}_M(\iota_{Q_0}, \dots, \iota_{Q_n}) - \sum_{i=0}^n g_i^\vee(0) \cdot \text{MV}_M(\hat{Q}_i). \end{aligned}$$

The obvious remark that the first summand of the last term is zero concludes the proof.  $\square$

# Adelic Arakelov theory

In this chapter we picture the general framework in which we will be set in the following. Before giving the definition of height and state some of its properties, we briefly recall facts about adelic fields, giving all the results that will be needed in the subsequent chapters. Then, we move to a succinct description of the basic constructions in Berkovich theory, which is the key to do analysis on algebraic varieties; we insist on the common viewpoint and language that can be adopted in the archimedean and non-archimedean case.

## 2.1

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### ABSOLUTE VALUES

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The arithmetic features of a field can be thought as the understanding of the absolute values over it, that is of the ways of measuring the size of its elements. We review here classical notions related to absolute values, fixing the notations and terminology for the subsequent sections.

**DEFINITION AND FIRST EXAMPLES.** Recall that  $\mathbb{R}_{\geq 0}$  denotes the set of non-negative real numbers.

**Definition 2.1.1.** An *absolute value* on a field  $K$  is a map

$$|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$$

satisfying the following axioms for all  $x, y \in K$ :

- (i)  $|x| = 0$  if and only if  $x = 0_K$
- (ii) (*multiplicativity*)  $|x \cdot y| = |x| \cdot |y|$
- (iii) (*triangle inequality*)  $|x + y| \leq |x| + |y|$ .

**Remark 2.1.2.** Axioms (i) and (ii) immediately imply that  $|1_K| = 1$  and that  $|-1_K| = 1$ . As a consequence,  $|x^{-1}| = |x|^{-1}$  for every  $x \in K^*$  and  $|-x| = |x|$  for every  $x \in K$ .

**Example 2.1.3.** For any field  $K$ , the function  $|\cdot|_{\text{tr}}$  defined as

$$|x|_{\text{tr}} := \begin{cases} 0 & \text{if } x = 0_K \\ 1 & \text{otherwise} \end{cases}$$

is an absolute value, which is called the *trivial absolute value* over  $K$ .

**Example 2.1.4.** Over the field  $\mathbb{Q}$  of rational numbers, the usual (euclidean) absolute value is an absolute value, which will be denoted by  $|\cdot|_{\infty}$ . Also, for every prime number  $p$ , the function  $|\cdot|_p$  defined by

$$|x|_p := p^{-\ell},$$

where  $x = p^{\ell} \cdot \frac{a}{b}$  with  $a$  and  $b$  both coprime with  $p$  and  $\ell \in \mathbb{Z}$ , is an absolute value over  $\mathbb{Q}$ , called the *p-adic absolute value*.

The nature of an absolute value over a field is strongly determined by its behaviour with respect to the sum and more precisely by meeting or not the archimedean property.

**Definition 2.1.5.** If an absolute value over  $K$  satisfies, instead of axiom (iii) in Definition 2.1.1, the stronger inequality

$$|x + y| \leq \max\{|x|, |y|\} \tag{2.1}$$

for all  $x, y \in K$ , then it is called a *non-archimedean absolute value*. Otherwise, it is called an *archimedean absolute value*.

The inequality (2.1) appearing in the previous definition is often referred to as the *ultrametric inequality*.

One can characterize non-archimedean absolute values in terms of the values they take on a distinguished subring of  $K$ .

**Proposition 2.1.6.** *Let  $K$  be a field and, with abuse of notation, denote by  $\mathbb{Z}$  the image of the ring  $\mathbb{Z}$  in  $K$ . For an absolute value  $|\cdot|$  on  $K$ , the following statements are equivalent:*

1.  $|\cdot|$  is non-archimedean
2.  $|m| \leq 1$  for every  $m \in \mathbb{Z}$
3.  $|\cdot|$  is bounded on  $\mathbb{Z}$ .

*Proof.* The implications (1)  $\Rightarrow$  (2) and (2)  $\Rightarrow$  (3) are obvious, since  $|1_K| = |-1_K| = 1$ . To prove (3)  $\Rightarrow$  (1), suppose  $B \geq 1$  is an upper bound for  $|\cdot|$  on  $\mathbb{Z}$ . For any  $x \in K$  with  $|x| \geq 1$  and for any positive integer  $n$ , the binomial theorem and the definition of absolute value imply that

$$|x+1|^n = |(x+1)^n| = \left| \sum_{k=0}^n \binom{n}{k} x^k \right| \leq \sum_{k=0}^n \binom{n}{k} |x|^k \leq (n+1)B|x|^n,$$

hence

$$|x+1| \leq \sqrt[n]{(n+1)B} \cdot |x|.$$

Since the previous inequality must be satisfied for arbitrarily big  $n$ , one must have that if  $|x| \geq 1$ , then  $|x+1| \leq |x|$ .

Choose now  $x, y \in K$ . Whenever one of the two elements is  $0_K$ , the ultrametric inequality is trivially satisfied. Assume then that  $x, y \in K^\times$  and, without loss of generality, that  $|x| = \max\{|x|, |y|\}$ . Then, applying the previous inequality to  $x/y \in K$ , which has absolute value greater than 1, one can deduce that

$$|x+y| = |y| \left| \frac{x}{y} + 1 \right| \leq |y| \frac{|x|}{|y|} = |x| = \max\{|x|, |y|\}.$$

The absolute value  $|\cdot|$  is then non-archimedean.  $\square$

**Corollary 2.1.7.** *A field of positive characteristic only admits non-archimedean absolute values.*

*Proof.* This follows from the previous proposition and the fact that the image of  $\mathbb{Z}$  in a field of positive characteristic is finite.  $\square$

**Example 2.1.8.** Over any field  $K$ , the trivial absolute value  $|\cdot|_{\text{tr}}$  is a non-archimedean absolute value. As a direct application of Proposition 2.1.6, the  $p$ -adic absolute values over  $\mathbb{Q}$  are non-archimedean, while the usual absolute value is archimedean.

**ABSOLUTE VALUES FROM GEOMETRY.** Algebraic geometry provides a wide source of examples for non-archimedean absolute values, via the following construction. Let  $S$  be an integral noetherian normal separated scheme and let  $K := K(S)$  be its function field. For any prime divisor  $T$  of  $S$ , which is a closed integral subscheme of  $S$  of codimension 1, the local ring  $\mathcal{O}_{T,S}$  at the generic point of  $T$  is an integrally closed noetherian local ring of Krull dimension 1, hence a discrete valuation ring. Denote by  $\text{ord}_T$  the associated valuation on  $\mathcal{O}_{T,S}$ , which extends to a valuation on its fraction field, coinciding with  $K$ . For any  $c > 1$ , the map  $c^{-\text{ord}_T(\cdot)}$  is easily seen to be a non-archimedean absolute value on  $K$ . Here are two particular cases, together with a convenient choice of the constant  $c$ .

**Example 2.1.9.** Suppose that the scheme  $S$  is of dimension 1. Then, for every  $f \in K$ , set

$$|f|_T := c_T^{-\text{ord}_T(f)}$$

with

$$c_T = \begin{cases} e & \text{if } \kappa(T) \text{ is infinite} \\ \#\kappa(T) & \text{otherwise} \end{cases},$$

where  $\kappa(T)$  denotes the residue field of  $S$  at  $T$ .

As a particular case, let  $K$  be a number field and denote by  $\mathcal{O}_K$  its ring of integers. The affine scheme  $S = \text{Spec } \mathcal{O}_K$  is an integral noetherian normal separated scheme with function field  $K$ . A prime divisor of  $S$  is a closed point of  $S$  and it is then given by a maximal ideal  $\mathfrak{p}$  of  $\mathcal{O}_K$  with residue field  $\kappa(\mathfrak{p}) \simeq \mathcal{O}_K/\mathfrak{p}$ , so that  $c_{\mathfrak{p}} = \#\kappa(\mathfrak{p}) = N(\mathfrak{p})$ , the absolute norm of  $\mathfrak{p}$ . For every  $f \in K$  one has then that

$$|f|_{\mathfrak{p}} = N(\mathfrak{p})^{-\text{ord}_{\mathfrak{p}}(f)},$$

where  $\text{ord}_{\mathfrak{p}}(f)$  coincides with the exponent of the ideal  $\mathfrak{p}$  in the factorization of the principal fractional ideal  $(f)$ . The absolute value  $|\cdot|_{\mathfrak{p}}$  is called the  *$\mathfrak{p}$ -adic absolute value* on  $K$ , compatibly with the fact that when  $K = \mathbb{Q}$  the function  $|\cdot|_{(p)}$  coincides with the  $p$ -adic absolute value introduced in Example 2.1.4.

**Example 2.1.10.** Suppose that  $S$  is a normal projective variety of dimension  $n$  over a field  $k$  and let  $E$  be an ample divisor over  $X$ . Then, for each  $f \in K$  let

$$|f|_T := c_k^{-\deg_E(T) \text{ord}_T(f)}$$

with

$$c_k = \begin{cases} e & \text{if } k \text{ is infinite} \\ \#k & \text{otherwise} \end{cases}$$

and  $\deg_E(T)$  denoting the degree of  $T$  with respect to  $E$  as in [Ful98], see also the beginning of section 2.4. In particular, the fact that  $c_k^{\deg_E(T)} > 1$  follows from the Nakai-Moishezon criterion, see [Laz04, Theorem 1.2.23].

The definitions of the two previous examples agree in the case of a regular projective curve  $S$  over a field  $k$ . In such a case, a prime divisor  $T$  is a closed point of  $S$ , hence its residue field is a finite extension of  $k$ , see [Gro60, I, ch.1, Proposition 6.4.2]. Hence, one has that  $c_T = e$  if  $k$  is infinite, while  $c_T = \#k^{[\kappa(T):k]} = \#k^{\deg(T)}$  if  $k$  is finite. Notice that this choice of the constant differs from the one made in [BPS14, Example 1.5.4].

**PLACES.** Every absolute value  $|\cdot|$  over a field  $K$  induces a distance by setting  $d(x, y) := |x - y|$  for every  $x, y \in K$ , and hence a topology on  $K$ .

**Definition 2.1.11.** Two absolute values on a field  $K$  are called *equivalent* if they induce the same topology on  $K$ . A class of equivalent absolute values is called a *place* of  $K$ .

For a place  $v$  over a field, we will often denote by  $|\cdot|_v$  an absolute value in the class  $v$ . It is possible to characterize equivalent absolute values over a field.

**Proposition 2.1.12.** *Two absolute values  $|\cdot|_1$  and  $|\cdot|_2$  on a field  $K$  are equivalent if and only if there exists  $\lambda \in \mathbb{R}_{>0}$  such that*

$$|x|_1 = |x|_2^\lambda$$

*for every  $x \in K$ . In particular, if  $|\cdot|_1$  and  $|\cdot|_2$  are equivalent absolute values on  $K$  and  $x \in K$ ,  $|x|_1 = 1$  if and only if  $|x|_2 = 1$ .*

*Proof.* This is [Lan02, Proposition XII.1.1]. □

**Remark 2.1.13.** Any positive power of a non-archimedean absolute value is again an absolute value. This is not the case for an archimedean absolute value  $|\cdot|$ . In fact,  $|\cdot|^\lambda$  is an absolute value for any  $\lambda \in (0, 1]$ , but the triangular inequality can fail for  $\lambda > 1$  (for instance, the function  $|\cdot|_\infty^2$  on  $\mathbb{Q}$  is not an absolute value).

It follows immediately from Proposition 2.1.12 that the trivial absolute value is only equivalent to itself. Another consequence of the proposition is given in the following remark.

**Remark 2.1.14.** In the setting of the previous subsection, suppose moreover that  $S$  is a *finite affine scheme*, meaning that any finite subset of  $S$  is contained in an affine open subset of  $S$ ; this is true for instance if  $S$  is a quasi-projective scheme over an affine scheme, see [Liu02, Proposition 3.3.36]. In such a case, if  $T_1$  and  $T_2$  are two distinct prime divisors of  $S$ , the associated absolute values  $|\cdot|_{T_1}$  and  $|\cdot|_{T_2}$  are nontrivial and inequivalent. Indeed, let  $\text{Spec } A$  be an affine open subscheme of  $S$  containing the generic points of  $T_1$  and  $T_2$ , corresponding to two different prime ideals  $\mathfrak{p}_1$  and  $\mathfrak{p}_2$  of  $A$  of height 1. Since  $\mathfrak{p}_1 \neq \mathfrak{p}_2$  there exists an element  $f \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$ . Then,  $\text{ord}_{T_1}(f) \geq 1$ , while  $\text{ord}_{T_2}(f) = 0$ , hence the inequivalence of  $|\cdot|_{T_1}$  and  $|\cdot|_{T_2}$  follows from Proposition 2.1.12.

As an easy consequence of Proposition 2.1.12 and Proposition 2.1.6, the property of being an archimedean or a non-archimedean absolute value is stable under equivalence of absolute values and it is then meaningful to speak about *archimedean places* and *non-archimedean places* over a field. The study of the set of places over a field plays a central role in adelic Arakelov geometry. As a basic example, the places over  $\mathbb{Q}$  have been exhaustively described by Ostrowski.

**Theorem 2.1.15** (Ostrowski). *Every non-trivial absolute value over  $\mathbb{Q}$  is equivalent either to a  $p$ -adic absolute value or to the usual absolute value.*

*Proof.* This was proved in [Ost16], see also [Kob84, Theorem I.2.1].  $\square$

**Remark 2.1.16.** The equivalent statement of Theorem 2.1.15 for a number field  $K$  asserts that any non-trivial absolute value on  $K$  is equivalent either to a  $\mathfrak{p}$ -adic absolute value for a maximal ideal  $\mathfrak{p}$  in  $\mathcal{O}_K$  as in Example 2.1.9 or to the absolute value

$$|\cdot|_{\sigma} := |\sigma(\cdot)|_{\infty}$$

with  $\sigma : K \hookrightarrow \mathbb{C}$  an embedding and  $|\cdot|_{\infty}$  the euclidean absolute value on  $\mathbb{C}$ , see [BS66, Theorem 1 at page 280] for a proof.

**ALGEBRAICALLY CLOSED COMPLETE FIELDS.** Fields equipped with absolute values play a central role in the remaining of the chapter and hence deserve a name.

**Definition 2.1.17.** A *valued field* is the datum of a field  $K$  together with an absolute value  $|\cdot|$  over it. An *isometry* between two valued fields  $(K_1, |\cdot|_1)$  and  $(K_2, |\cdot|_2)$  is an isomorphism of fields  $\varphi : K_1 \rightarrow K_2$  such that

$$|x|_1 = |\varphi(x)|_2$$

for every  $x \in K_1$ .

A valued field  $(K, |\cdot|)$  is said to be archimedean or non-archimedean depending on the nature of the absolute value  $|\cdot|$ . It is clear from the definition that if  $(K_1, |\cdot|_1)$  and  $(K_2, |\cdot|_2)$  are isometrical valued fields, then  $(K_1, |\cdot|_1)$  is archimedean (respectively, non-archimedean) if and only if  $(K_2, |\cdot|_2)$  is so. In the adelic definition of height one will present later on, one needs to deal with valued fields with friendly properties.

**Definition 2.1.18.** An *algebraically closed complete field* is a valued field  $(K, |\cdot|)$  with  $K$  an algebraically closed field which is complete for the topology induced by  $|\cdot|$ .

It is easy to construct an algebraically closed complete field from the datum of a valued field  $(F, |\cdot|_F)$ . As a first step, the completion  $\hat{F}$  of  $F$  with respect to the topology given by  $|\cdot|_F$  is naturally equipped with an absolute value  $|\cdot|$  extending  $|\cdot|_F$ . From [BG06, Remark 1.2.8],  $|\cdot|$  extends uniquely to the algebraic closure of  $\hat{F}$ . Finally, the completion  $K$  of the algebraic closure of  $\hat{F}$  with respect to the corresponding topology is equipped with the extension  $|\cdot|_K$  of the absolute value  $|\cdot|$ . It follows for instance from [Bru63, Proposition 5, ch.2] and [BG06, Theorem 1.2.6] that  $K$  is algebraically closed, hence the pair  $(K, |\cdot|_K)$  is an algebraically closed complete field.

Notice that, if instead of  $|\cdot|$  one considers an equivalent absolute value  $|\cdot|^\lambda$  on  $F$ , with  $\lambda \in \mathbb{R}_{>0}$ , the previous construction gives the same field  $K$  (since the completion operation is purely topological) and the absolute value  $|\cdot|_K^\lambda$  on  $K$ . Otherwise said, any place over  $F$  determines an algebraically closed complete field  $(K, |\cdot|_K)$  uniquely up to equivalence of the absolute value.

**Example 2.1.19.** By Ostrowski theorem, the algebraically closed complete fields that can be obtained from  $\mathbb{Q}$  via the construction described above are  $(\overline{\mathbb{Q}}, |\cdot|_{\text{tr}})$  from the trivial absolute value,  $(\mathbb{C}_p, |\cdot|_p)$  for each rational prime number  $p$  and  $(\mathbb{C}, |\cdot|_\infty)$  from the unique archimedean absolute value.

The following well-known result characterizes algebraically closed complete archimedean fields.

**Theorem 2.1.20.** *Let  $(K, |\cdot|)$  be an algebraically closed complete archimedean field. Then, there exists an isometry between  $(K, |\cdot|)$  and  $\mathbb{C}$  with a power of the usual absolute value. Moreover, this isometry is unique up to complex conjugation.*

*Proof.* Suppose that  $(K, |\cdot|)$  is an algebraically closed complete archimedean field. By Corollary 2.1.7,  $K$  has characteristic 0 and then it contains a copy of  $\mathbb{Q}$ , which we denote again by  $\mathbb{Q}$  with abuse of notation. The restriction of  $|\cdot|$

to  $\mathbb{Q}$  is archimedean by Proposition 2.1.6 and then it is a power of the usual absolute value on  $\mathbb{Q}$  by Ostrowski theorem. Since  $K$  is complete with respect to  $|\cdot|$  it must then contain a copy of  $\mathbb{R}$ . The condition of being algebraically closed implies that  $K$  contains a copy of  $\mathbb{C}$ . Then,  $(K, |\cdot|)$  is a Banach algebra over  $\mathbb{C}$  and also a field; Gelfand-Mazur theorem, see [Rud91, Theorem 10.14], asserts that  $K$  must be isomorphic to  $\mathbb{C}$ . The isomorphism  $\varphi : \mathbb{C} \rightarrow K$  induces an absolute value  $|\varphi(\cdot)|$  on  $\mathbb{C}$ , which restricts on  $\mathbb{Q}$  to a real positive power  $|\cdot|_\infty^\lambda$  of the usual absolute value. The construction presented before Example 2.1.19 proves that there exists a unique extension of  $|\cdot|_\infty^\lambda$  from  $\mathbb{Q}$  to  $\mathbb{C}$ , that is the same power of the euclidean absolute value on  $\mathbb{C}$ . It follows that  $\varphi$  is an isometry between  $(K, |\cdot|)$  and  $(\mathbb{C}, |\cdot|_\infty^\lambda)$ .

Suppose now that  $\varphi_1$  and  $\varphi_2$  are two such isometries. Then  $\varphi_1 \circ \varphi_2^{-1}$  is an isometry of  $(\mathbb{C}, |\cdot|_\infty^\lambda)$ , in particular a continuous automorphism of  $\mathbb{C}$ . As such, it fixes  $\mathbb{Q}$  and, by continuity, also  $\mathbb{R}$ . The equality  $\mathbb{C} = \mathbb{R} + i\mathbb{R}$  and the fact that  $i$  must be mapped to a square root of  $-1$  imply that  $\varphi_1 \circ \varphi_2^{-1}$  is the identity map or the complex conjugation, proving the last statement.  $\square$

A consequence of this result is a characterization of the archimedean places over a field, extending part of the statements of Remark 2.1.16.

**Proposition 2.1.21.** *The archimedean places over a field  $K$  are in bijection with the embeddings of  $K$  in  $\mathbb{C}$  up to complex conjugation.*

*Proof.* Let  $\sigma$  be an embedding of  $K$  into  $\mathbb{C}$ . Then, the map  $|\cdot|_\sigma : K \rightarrow \mathbb{R}$ ,  $x \mapsto |\sigma(x)|_\infty$  is an absolute value over  $K$ . Moreover, as it restricts to the usual euclidean one on the image of  $\mathbb{Z}$  in  $K$ , it is archimedean. We claim that the function which associates to  $\sigma$  the class of the absolute value  $|\cdot|_\sigma$  is a bijection between the embedding of  $K$  in  $\mathbb{C}$  up to complex conjugation and the archimedean places of  $K$ .

One first shows that the map is injective. Suppose that  $\sigma_1$  and  $\sigma_2$  have the same image, that is  $|\cdot|_{\sigma_1}$  and  $|\cdot|_{\sigma_2}$  are equivalent absolute values; since each place of  $K$  has a unique representative restricting to the usual euclidean absolute value on the image of  $\mathbb{Z}$  in  $K$ , one has  $|\cdot|_{\sigma_1} = |\cdot|_{\sigma_2}$ . Let  $K_v$  be the completion of the algebraic closure of the completion of  $K$  with respect to such an absolute value. For  $i = 1, 2$ , there exists an extension  $\bar{\sigma}_i$  of  $\sigma_i$  to an isomorphism between  $K_v$  and  $\mathbb{C}$ . Moreover, the map  $|\cdot|_{\bar{\sigma}_i} : K_v \rightarrow \mathbb{R}$ ,  $x \mapsto |\bar{\sigma}_i(x)|_\infty$  is an absolute value on  $K_v$  extending  $|\cdot|_{\sigma_i}$ ; it follows that  $\bar{\sigma}_i$  is an isometry between  $K_v$  and  $(\mathbb{C}, |\cdot|_\infty)$ . By the uniqueness statement in Theorem 2.1.20,  $\bar{\sigma}_1$  and  $\bar{\sigma}_2$  must coincide up to complex conjugation and the same holds hence for  $\sigma_1$  and  $\sigma_2$ .

For the surjectivity, let  $v$  be an archimedean place of  $K$  and denote by  $|\cdot|_v$

the representative of  $v$  restricting to the usual euclidean absolute value on the image of  $\mathbb{Z}$  in  $K$ . Denote again by  $|\cdot|_v$  the extension of the absolute value to the algebraically closed complete field  $K_v$ . Let  $\iota$  be the embedding of  $K$  in  $K_v$  and  $\varphi$  the unique isometry between  $K_v$  and  $(\mathbb{C}, |\cdot|_\infty)$  given by Theorem 2.1.20, up to complex conjugation. The embedding  $\varphi \circ \iota$  of  $K$  into  $\mathbb{C}$  is a preimage of  $v$  since  $|x|_{\varphi \circ \iota} = |\varphi(\iota(x))|_\infty = |\iota(x)|_v = |x|_v$  for every  $x \in K$  by definition of isometry.  $\square$

**Remark 2.1.22.** The previous proposition immediately implies that there is no archimedean absolute value on a field of positive characteristic, as already proved, with a different argument, in Corollary 2.1.7.

## 2.2

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### BERKOVICH ANALYTIFICATION

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We recall here the basis of Berkovich's theory, focusing as in [BPS14, §1.2] on the description of the analytification functor for algebraic varieties; we refer instead to [Ber90] for an exhaustive treatment of the subject. We underline how Berkovich's construction is compatible with the classical complex analytification and also briefly describe the Galois action on such analytic spaces. We suppose fixed for this section a valued field  $(K, |\cdot|)$ , with  $K$  complete with respect to the topology induced by  $|\cdot|$ .

**ANALYTIFICATION OF A VARIETY.** For an affine variety  $X = \text{Spec } A$  over  $\text{Spec } K$ , the structure morphism  $\iota : K \hookrightarrow A$  gives  $A$  the structure of a  $K$ -algebra. By a *multiplicative seminorm* on  $A$  one means a map

$$\|\cdot\| : A \rightarrow \mathbb{R}_{\geq 0}$$

which is multiplicative, non identically zero and satisfying the triangle inequality for every pair of elements of  $A$ .

**Definition 2.2.1.** The *Berkovich analytification* of the affine variety  $X = \text{Spec } A$  is the set

$$X^{\text{an}} := \{\|\cdot\|_x, \text{ multiplicative seminorms on } A : \|\iota(k)\|_x = |k| \text{ for all } k \in K\}$$

endowed with the coarsest topology making the map

$$|a(\cdot)| : X^{\text{an}} \rightarrow \mathbb{R}, \quad \|\cdot\|_x \mapsto \|a\|_x \tag{2.2}$$

continuous for each  $a \in A$ .

A point of  $X^{\text{an}}$  will be denoted alternatively by  $x$  or by  $\|\cdot\|_x$ . For every  $\|\cdot\|_x \in X^{\text{an}}$ , the set  $\ker \|\cdot\|_x := \{a \in A : \|a\|_x = 0\}$  is a prime ideal in  $A$ , hence one deduces a map

$$\begin{aligned} \pi_X : X^{\text{an}} &\rightarrow X \\ \|\cdot\|_x &\mapsto \ker \|\cdot\|_x. \end{aligned} \tag{2.3}$$

The map  $\pi_X$  is continuous; indeed, for every principal open subset  $D(a)$  of  $X$  with  $a \in A$ , one has

$$\pi_X^{-1}(D(a)) = \{\|\cdot\|_x \in X^{\text{an}} : \|a\| \neq 0\},$$

which is open in  $X^{\text{an}}$  because of the continuity of the map (2.2).

In order to define a sheaf on  $X^{\text{an}}$  as in [Ber90, Definition 1.5.3], one needs an analytic analogue of the residue field at a point. To do this, remark that for  $x \in X^{\text{an}}$  the field  $\kappa(\pi_X(x)) = \text{Frac}(A/\ker \|\cdot\|_x)$  is endowed with a well-defined norm  $\|\cdot\|_x$  extending the absolute value on  $K$ ; the corresponding completion is denoted by  $\mathcal{H}(x)$  and is called the *complete residue field at  $x$* . For any  $a \in A$ , denoting by  $a(x)$  the image of  $a$  in  $\mathcal{H}(x)$  and by  $|\cdot|$  the absolute value on  $\mathcal{H}(x)$ , the equality  $|a(x)| = \|a\|_x$  justifies the notation used in (2.2). Analytic functions can now be defined as functions to the complete residue field which are locally approximated by algebraic rational functions.

**Definition 2.2.2.** An *analytic function* on an open subset  $U$  of  $X^{\text{an}}$  is a function

$$f : U \rightarrow \prod_{x \in U} \mathcal{H}(x)$$

such that:

1.  $f(x) \in \mathcal{H}(x)$  for every  $x \in U$
2. for every  $x \in U$  there exists an open neighbourhood  $V$  of  $x$ ,  $V \subseteq U$ , such that for every  $\varepsilon > 0$  there exist  $a, b \in A$  with

$$\left| f(y) - \frac{a(y)}{b(y)} \right| < \varepsilon$$

for all  $y \in V$  (one requires that  $b \notin \ker \|\cdot\|_y$  for all  $y \in V$ ).

The set of analytic functions on  $U$  is a ring denoted by  $\mathcal{O}_{X^{\text{an}}}(U)$ . By [Ber90, §1.5],  $\mathcal{O}_{X^{\text{an}}}$  is a sheaf of ring turning  $(X^{\text{an}}, \mathcal{O}_{X^{\text{an}}})$  into a locally ringed space.

**Remark 2.2.3.** It follows from the definition that every regular function on an algebraic open subset  $U$  of  $X$  induces by composition an analytic function on  $U^{\text{an}} := \pi_X^{-1}(U)$ . This turns the map  $\pi_X$  of (2.3) into a morphism of ringed spaces.

When the variety  $X$  is not necessarily affine, the analytic space  $(X^{\text{an}}, \mathcal{O}_{X^{\text{an}}})$  is obtained by a gluing process, see [Ber90, Remark 3.4.2] for the details. It is a locally ringed space called the *Berkovich analytification* of  $X$ . Its underlying topological space enjoys friendly properties, as follows.

**Theorem 2.2.4.** *For every variety  $X$  over  $K$ , the topological space  $X^{\text{an}}$  is nonempty, locally compact, Hausdorff and arcwise connected.*

*Proof.* See [Ber90, Theorem 1.2.1, Theorem 3.4.8 and Theorem 3.5.3].  $\square$

Finally,  $\mathcal{O}_X$ -modules on  $X$  can be analytified via the classical inverse image construction described in [Har77, II.5].

**Definition 2.2.5.** Let  $\mathcal{F}$  be a sheaf of  $\mathcal{O}_X$ -modules on  $X$ . Then, the *analytification* of  $\mathcal{F}$  is the sheaf of  $\mathcal{O}_{X^{\text{an}}}$ -modules defined as  $\mathcal{F}^{\text{an}} := \pi_X^* \mathcal{F}$ .

**THE ARCHIMEDEAN CASE.** For an affine scheme  $X = \text{Spec } A$  over  $K$ , a  $K$ -point of  $X$  corresponds to a morphism  $A \rightarrow K$  extending the identity on  $K$ . The composition of such a morphism with the absolute value  $|\cdot|$  of  $K$  is verified to be a multiplicative seminorm on  $A$  extending  $|\cdot|$ , hence a point of the analytic variety  $X^{\text{an}}$ . There exists then a map

$$X(K) \rightarrow X^{\text{an}} \tag{2.4}$$

whose images are called the *rigid points* of  $X^{\text{an}}$ . The map in (2.4) is injective; indeed, two different  $K$ -points of  $X$  need to have different images, then the kernels of the corresponding maps  $A \rightarrow K$  are different, implying that the kernel of the associated seminorms are different.

**Proposition 2.2.6.** *Let  $(K, |\cdot|)$  be an algebraically closed complete archimedean field and  $X$  a variety over  $K$ . Then, the map in (2.4) realizes a bijection between the set of  $K$ -points of  $X$  and the Berkovich analytification of  $X$ .*

*Proof.* The statement being checked locally, one can assume that  $X$  is an affine variety over  $\text{Spec } K$ , let  $X = \text{Spec } A$ .

The injectivity of the map in (2.4) is valid without assumptions on  $K$  and has been proved before the statement of the proposition.

To check the surjectivity, let  $x \in X^{\text{an}}$  and consider its complete residue field

$\mathcal{H}(x)$ . It is a field extension of  $K$ , endowed with an absolute value  $|\cdot|_{\mathcal{H}(x)}$  satisfying  $|a(x)|_{\mathcal{H}(x)} = \|a\|_x$  for every  $a \in A$ , see the discussion of the previous subsection. In particular,  $(\mathcal{H}(x), |\cdot|_{\mathcal{H}(x)})$  is an archimedean complete field extending  $K$ , then it embeds isometrically in  $(K, |\cdot|)$  because of Theorem 2.1.20. Fix such an isometrical embedding  $\varphi$ . The association  $a \mapsto a(x)$  gives a morphism  $A \rightarrow \mathcal{H}(x)$  and hence composing with  $\varphi$  one gets a  $K$ -point of  $X$ . This is a preimage of  $x$  by (2.4) since

$$|\varphi(a(x))| = |a(x)|_{\mathcal{H}(x)} = \|a\|_x$$

for all  $a \in A$ . □

**Remark 2.2.7.** More generally, when  $(K, |\cdot|)$  is an algebraically closed complete archimedean field and  $X$  is a variety over  $K$ , the analytic space  $(X^{\text{an}}, \mathcal{O}_{X^{\text{an}}})$  described in the previous subsection coincides with the usual complex analytification of  $X$  treated in [Ser56], see [Ber90, Example 1.5.4 (i)] for the case of the affine space.

**A GAGA PROPERTY.** The features of the analytic space  $X^{\text{an}}$  are strictly related to the ones of the original algebraic variety  $X$ . The following theorem exhibits an instance of this correspondence that will be used later on.

**Theorem 2.2.8.** *For a variety  $X$  over  $K$ ,  $X$  is proper if and only if  $X^{\text{an}}$  is compact.*

*Proof.* When  $K$  is non-archimedean this is [Ber90, Theorem 3.4.8 (ii) and Theorem 3.5.3 (ii)], otherwise this was proved in [Ser56, Proposition 6]. □

The interplay between the properties of an algebraic variety and the ones of its analytification can be extended to the sheaf-theoretical level, see [Ser56, §3] for the archimedean case and [Ber90, §3.4] for the non-archimedean nontrivial one.

**GALOIS ACTION.** For an affine variety  $X = \text{Spec } A$  over  $K$ , Berkovich's construction gives an analytic space which one here denotes by  $X_K^{\text{an}}$  to stress its dependance from the base field. The completion  $\mathbb{C}_K$  of an algebraic closure of  $K$  is an algebraically closed field coming endowed with the unique extension of the absolute value of  $K$ , which one denotes again by  $|\cdot|$  with abuse of notation. The base change

$$X_{\mathbb{C}_K} = \text{Spec}(A \otimes_K \mathbb{C}_K)$$

is hence an affine variety over the algebraically closed complete field  $\mathbb{C}_K$ . In particular, one can consider the Berkovich analytification of such a variety and denote it by  $X_{\mathbb{C}_K}^{\text{an}}$ . The *absolute Galois group*  $\text{Gal}(\overline{K}^{\text{sep}}/K)$  of  $K$  acts on  $X_{\mathbb{C}_K}^{\text{an}}$ . Indeed, for an element  $\sigma$  of such a group and a seminorm  $\|\cdot\|_x$  on  $A \otimes_K \mathbb{C}_K$ , let  $\|\cdot\|_{\sigma(x)}$  be the map defined by

$$\left\| \sum_i a_i \otimes \alpha_i \right\|_{\sigma(x)} := \left\| \sum_i a_i \otimes \sigma(\alpha_i) \right\|_x$$

for every  $\sum_i a_i \otimes \alpha_i \in A \otimes_K \mathbb{C}_K$ . It is easily checked that the map  $\|\cdot\|_{\sigma(x)}$  is a multiplicative seminorm on  $A \otimes_K \mathbb{C}_K$ . Moreover, it extends the absolute value on  $K$  since  $|\sigma(\cdot)|$  is an absolute value on  $K$  and then it must coincide with  $|\cdot|$  because of [BG06, Remark 1.2.8]; hence  $\|\cdot\|_{\sigma(x)}$  is a point of  $X_{\mathbb{C}_K}^{\text{an}}$ . The previous construction extends to the case of a general, not necessarily affine, variety over  $K$ .

**Proposition 2.2.9.** *For a variety  $X$  over  $K$ ,*

$$X_K^{\text{an}} \simeq X_{\mathbb{C}_K}^{\text{an}} / \text{Gal}(\overline{K}^{\text{sep}}/K).$$

*Proof.* This follows from [Ber90, Corollary 1.3.6]. □

**Remark 2.2.10.** Thanks to Proposition 2.2.9, the study of the analytic space  $X_K^{\text{an}}$  can be reduced to the one of  $X_{\mathbb{C}_K}^{\text{an}}$ . In particular, it will be enough to deal with varieties defined over algebraically closed complete fields and to define objects which are compatible with the action of the absolute Galois group. This agrees with the philosophy adopted in [BPS14, Remark 1.1.5] when  $K = \mathbb{R}$ , in which case the absolute Galois group consists of the identity and of the complex conjugation.

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## 2.3 ADELIC FIELDS

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An adelic field is the fundamental basis over which one can develop a theory of heights and can be intended as a field together with a rich arithmetic structure. We present here the definition and basic examples of adelic fields (including global fields) and introduce a notion of equivalence of adelic fields.

**DEFINITIONS AND EXAMPLES.** Loosely speaking, an adelic field is a field enriched with a weighted collection of inequivalent absolute values over it.

**Definition 2.3.1.** An *adelic field* is the datum  $(\mathbb{K}, (|\cdot|_v, n_v)_{v \in \mathfrak{M}})$  of a field  $\mathbb{K}$ , a set of places  $\mathfrak{M}$  over  $\mathbb{K}$  and a choice, for each  $v \in \mathfrak{M}$ , of a real positive number  $n_v$  and a representant  $|\cdot|_v$  of  $v$  satisfying:

- (i) if  $|\cdot|_v$  is non-archimedean,  $\log |\cdot|_v$  is a discrete valuation on  $\mathbb{K}$  up to a multiplicative constant
- (ii) for each  $x \in \mathbb{K}^*$ , the set of  $v \in \mathfrak{M}$  for which  $|x|_v \neq 1$  is finite.

It follows from Proposition 2.1.12 that the two conditions in the previous definition do not depend on the choice of the representant of the place  $v$ .

**Notation 2.3.2.** To lighten the notation and when there is no ambiguity, one will simply denote an adelic field by  $\mathbb{K}$ . One will prefer the bold notation when dealing with adelic fields, to underline the extra arithmetic structure they enjoy.

Any valued field  $(K, |\cdot|)$  has a trivial adelic structure, assuming that  $|\cdot|$  is associated with a discrete valuation if non-archimedean; it is indeed enough to consider  $(K, |\cdot|, 1)$ . In particular, any field can be seen as an adelic field by equipping it with its trivial absolute value.

**Example 2.3.3.** The archetypical example of an adelic field is the field  $\mathbb{Q}$  of rational numbers, with  $\mathfrak{M}$  the set of all its places as described in Ostrowski theorem. Unless otherwise stated,  $\mathbb{Q}$  is assumed to be equipped with the adelic structure consisting of such a collection of places, representants chosen as in Example 2.1.4 and weights equal to 1.

**Example 2.3.4.** Let  $S$  be a normal projective variety over a field  $k$  and  $E$  an ample divisor over  $X$ . Let  $\mathbb{K} = K(S)$  be the function field of  $S$ ,  $\mathfrak{M}$  be the set of prime divisors of  $S$ . With the collection of absolute values associated to prime divisors as in Example 2.1.10 (which are inequivalent because of Remark 2.1.14) and weights equal to 1, one obtains an adelic structure on  $\mathbb{K}$ .

If  $(\mathbb{K}, (|\cdot|_v, n_v)_{v \in \mathfrak{M}})$  is an adelic field, one denotes by  $\mathfrak{M}_\infty$  the subset of  $\mathfrak{M}$  consisting of archimedean places. The following property is an easy but fundamental consequence of the definition.

**Proposition 2.3.5.** *An adelic field  $\mathbb{K}$  only admits finitely many archimedean places.*

*Proof.* By Corollary 2.1.7, one can reduce to the case of a field  $\mathbb{K}$  of characteristic zero. In such a situation,  $\mathbb{K}$  contains a copy of  $\mathbb{Q}$  and any archimedean

absolute value  $|\cdot|_v$  on  $\mathbb{K}$  restricts to an archimedean absolute value on  $\mathbb{Q}$ . By Ostrowski theorem, one has hence  $|2|_v > 1$ . The second axiom in Definition 2.3.1 concludes the proof.  $\square$

**THE PRODUCT FORMULA.** For a nonzero element  $x$  of an adelic field  $\mathbb{K}$ , the *defect* of  $x$  is defined as

$$\text{def}(x) := \sum_{v \in \mathfrak{M}} n_v \log |x|_v.$$

Notice that it is a well-defined real number because of the second axiom in Definition 2.3.1. Moreover, the map  $\text{def} : \mathbb{K}^* \rightarrow \mathbb{R}$  is a group homomorphism; its image is hence a subgroup of  $\mathbb{R}$  denoted by  $\text{def}(\mathbb{K})$  and called the *defect* of  $\mathbb{K}$ . For instance, if  $(K, |\cdot|)$  is a valued field and  $|\cdot| = a^{-\nu(\cdot)}$  for some  $a \in \mathbb{R}_{>0}$  and a discrete valuation  $\nu$ , the defect of the adelic field  $(K, |\cdot|, 1)$  is the value group of  $K$  multiplied by  $\log a$ .

**Definition 2.3.6.** An adelic field  $(\mathbb{K}, (|\cdot|_v, n_v)_{v \in \mathfrak{M}})$  is said to satisfy the *product formula* if  $\text{def}(\mathbb{K}) = \{0\}$ .

The adelic fields introduced above rejoice this fundamental property.

**Example 2.3.7.** The field  $\mathbb{Q}$  with the adelic structure introduced in Example 2.3.3 satisfies the product formula because of [BG06, Proposition 1.4.4].

**Example 2.3.8.** For a normal projective variety  $S$  over a field  $k$  and an ample divisor  $E$ , the function field  $\mathbb{K} = K(S)$  with the adelic structure given in Example 2.3.4 satisfies the product formula. Indeed, for every  $f \in \mathbb{K}$  one has

$$\sum_{T \in \mathfrak{M}} \log |f|_T = -\log(c_k) \sum_{T \in \mathfrak{M}} \deg_E(T) \text{ord}_T(f) = -\log(c_k) \deg_E(\text{div}(f)) = 0$$

because the degree of a principal divisor is zero, see also [BG06, Proposition 1.4.7].

**Remark 2.3.9.** The definition of adelic fields can be generalized to the one of  $M$ -fields, see [Gub97, §2] for a precise treatment of this notion. In such a setting, the set of places  $\mathfrak{M}$  is considered as a measure space  $(\mathfrak{M}, \mu)$  and the non-archimedean absolute values are not required to be associated to discrete valuations.

**EQUIVALENCE OF ADELIC FIELDS.** When  $\mathbb{K}$  is an adelic field, a different choice of representant for a certain place  $v$  can be compensated by a modification on the corresponding weight, without changing the arithmetical nature of  $\mathbb{K}$ . One can make this formal thanks to the following.

**Definition 2.3.10.** Two adelic fields  $(\mathbb{K}_1, (|\cdot|_v, n_v)_{v \in \mathfrak{M}_1})$  and  $(\mathbb{K}_2, (|\cdot|_w, \ell_w)_{w \in \mathfrak{M}_2})$  are said to be *equivalent* if there exist an isomorphism  $\varphi : \mathbb{K}_1 \rightarrow \mathbb{K}_2$  and a bijection  $f : \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$  satisfying

$$n_v \log |x|_v = \ell_{f(v)} \log |\varphi(x)|_{f(v)}$$

for all  $x \in \mathbb{K}_1^*$  and  $v \in \mathfrak{M}_1$ .

The fundamental example of equivalent adelic fields is given by “rescaling”: let  $(\mathbb{K}, (|\cdot|_v, n_v)_{v \in \mathfrak{M}})$  be an adelic field and choose  $\lambda_v \in \mathbb{R}_{>0}$  for any place  $v \in \mathfrak{M}$ , with  $\lambda_v \in (0, 1]$  if  $v$  is archimedean. Then  $(\mathbb{K}, (|\cdot|_v^{\lambda_v}, n_v/\lambda_v)_{v \in \mathfrak{M}})$  is an adelic field equivalent to  $(\mathbb{K}, (|\cdot|_v, n_v)_{v \in \mathfrak{M}})$ .

**Proposition 2.3.11.** *If  $\mathbb{K}_1$  and  $\mathbb{K}_2$  are two equivalent adelic fields,  $\text{def}(\mathbb{K}_1) = \text{def}(\mathbb{K}_2)$ . In particular,  $\mathbb{K}_1$  satisfies the product formula if and only if  $\mathbb{K}_2$  does.*

*Proof.* Denote by  $\varphi$  the isomorphism between  $\mathbb{K}_1$  and  $\mathbb{K}_2$  given by Definition 2.3.10. It follows immediately from the definition that  $\text{def}(x) = \text{def}(\varphi(x))$  for every  $x \in \mathbb{K}_1^*$ , which implies the claim.  $\square$

**EXTENSIONS OF ADELIC FIELDS.** Let  $\mathbb{K}$  be an adelic field and  $\mathbb{F}$  a finite field extension of  $\mathbb{K}$ . For a place  $v$  on  $\mathbb{K}$  with representant  $|\cdot|_v$ , the algebra  $\mathbb{F} \otimes_{\mathbb{K}} \mathbb{K}_v$  can be decomposed as a direct sum of finitely many local Artinian  $\mathbb{K}_v$ -algebras  $E_w$ , one for each extension  $w$  of  $v$  to  $\mathbb{F}$ , see [MS16, Lemma 3.4]. Set

$$n_w := \frac{\dim_{\mathbb{K}_v}(E_w)}{[\mathbb{F} : \mathbb{K}]} n_v.$$

If  $\mathfrak{N}$  denotes the set of places restricting to a place in  $\mathfrak{M}$  and  $|\cdot|_w$  the unique absolute value in  $w \in \mathfrak{N}$  restricting to the representant of a place in  $\mathfrak{M}$ ,  $\mathbb{F}$  can be given the structure  $(\mathbb{F}, (|\cdot|_w, n_w)_{w \in \mathfrak{N}})$ .

**Proposition 2.3.12.** *Let  $\mathbb{K}$  be an adelic field and let  $\mathbb{F}$  be a finite field extension of  $\mathbb{K}$ . With the structure described above,  $\mathbb{F}$  is an adelic field. Moreover  $\mathbb{F}$  satisfies the product formula if  $\mathbb{K}$  does.*

*Proof.* Any absolute value on  $\mathbb{F}$  restricting to a non-archimedean absolute value on  $\mathbb{K}$  associated to a discrete valuation is still non-archimedean and associated to a discrete valuation, see for instance [Lan02, Proposition XII.4.2]. To prove the second property in Definition 2.3.1, one can restrict to the non-archimedean places in  $\mathfrak{N}$  because of the adelicity of  $\mathbb{K}$ , Proposition 2.3.5 and the fact that there are only finitely many places on  $\mathbb{F}$  restricting to a fixed one in  $\mathfrak{M}$ . Let hence  $x \in \mathbb{F}^*$  and denote by  $f$  its minimal polynomial over  $\mathbb{K}$ . For almost all  $v \in \mathfrak{M}$  all the coefficients of  $f$  have absolute value 1; since there are finitely many extensions of a fixed place of  $\mathbb{K}$  to  $\mathbb{F}$ , for almost all  $w \in \mathfrak{N}$  all the coefficients of  $f$  have  $w$ -adic absolute value equal to 1. For such a  $w$  one must have  $|x|_w = 1$ ; indeed, if by contradiction  $|x|_w \neq 1$  one would have that

$$0 = |0|_w = |f(x)|_w = \max\{|x|_w^d, 1\},$$

with  $d = [\mathbb{F} : \mathbb{K}]$ , which is absurd.

The last statement is [MS16, Proposition 3.7 (2)], see also [Gub97, Remark 2.5] for the case of arbitrary  $M$ -fields.  $\square$

Unless otherwise stated, any finite extension of an adelic field  $\mathbb{K}$  will be considered endowed with the adelic structure described above. In particular, finite extensions of the adelic fields introduced in Example 2.3.3 and Example 2.3.4 are adelic fields satisfying the product formula.

**Remark 2.3.13.** It was proven in [AW45, Theorem 3] that if an adelic field  $\mathbb{K}$  satisfies the product formula and its non-archimedean places have residue field of finite order, then  $\mathbb{K}$  must be a number field or a finite extension of the function field of a regular projective curve. Such fields, which are so axiomatically characterized, are also known as *global fields*.

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## 2.4. DEGREES, LOCAL AND GLOBAL HEIGHTS

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Let  $(\mathbb{K}, (|\cdot|_v, n_v)_{v \in \mathfrak{N}})$  be an adelic field satisfying the product formula and consider a fixed normal proper variety  $X$  of dimension  $n$  over  $\mathbb{K}$  for the entire section. Using the recent developments of the notions of differential forms and currents over Berkovich spaces, we define local heights of cycles on  $X$  in an analogous way at all places. We finally combine local information to introduce global heights.

**ALGEBRAIC CYCLES.** Arakelov geometry allows to define heights as real-valued functions on the set of subvarieties of  $X$  of arbitrary dimension. A good environment for this purpose is provided by the notion of cycles on  $X$ .

**Definition 2.4.1.** For  $d \in \mathbb{N}$ , a  $d$ -cycle on  $X$  is a finite formal sum with integer coefficients of irreducible subvarieties of  $X$  of dimension  $d$ .

With the obvious sum operation, the set of  $d$ -cycles on  $X$  is a group and it is denoted by  $Z_d(X)$ . It is the free abelian group

$$Z_d(X) = \bigoplus_V \mathbb{Z} \cdot [V]$$

generated by the  $d$ -dimensional irreducible subvarieties  $V$  of  $X$ . Whenever  $d > n$ , the group  $Z_d(X)$  is trivial,  $Z_n(X) \simeq \mathbb{Z}$  since  $X$  is the only  $n$ -dimensional irreducible subvariety of  $X$  while the group  $Z_{n-1}(X)$  is called the *group of Weil divisors* of  $X$ . For a cycle  $Z$  on  $X$ , the topological union of the subvarieties of  $X$  appearing with nonzero coefficients in  $Z$  is called the *support* of  $Z$  and is denoted by  $|Z|$ . One says that a cycle  $Z$  of  $X$  is *effective* if all the coefficients of  $Z$  are nonnegative; one write in this case  $Z \geq 0$ .

From a nonzero rational function  $f \in K(X)^*$  one can define the Weil divisor  $[\text{div}(f)]$  of  $X$  as in [Har77, Definition at page 131]. This allows to associate to a Cartier divisor  $D = \{(U_\alpha, f_\alpha)\}_\alpha$  on  $X$  both a Weil divisor  $[D]$  and a line bundle  $\mathcal{O}(D)$  by setting

$$\mathcal{O}(D)(U) := \{f \in K(X)^* : ([\text{div}(f)] + D)|_U \geq 0\}$$

for each open subset  $U$  of  $X$ . It is the sub  $\mathcal{O}_X$ -module of the constant sheaf  $\mathcal{K}_X$  of rational functions generated by  $f_\alpha^{-1}$  on  $U_\alpha$ . The section 1 of  $\mathcal{K}_X$  produces a distinguished rational section  $s_D$  of  $\mathcal{O}(D)$  satisfying  $\text{div}(s_D) = D$ , where  $\text{div}(s)$  denotes the Cartier divisor corresponding to the rational section  $s$ . By abuse of notation, the support  $|D|$  of a Cartier divisor is meant to be the support of the associated Weil divisor  $[D]$ .

**Definition 2.4.2.** A family of cycles  $Z_1, \dots, Z_r$  on  $X$  is said to *intersect properly* if for every  $I \subseteq \{1, \dots, r\}$ , each irreducible component of  $\bigcap_{i \in I} |Z_i|$  has codimension  $\sum_{i \in I} \text{codim}(Z_i)$ .

Similarly, let  $Z$  be a  $d$ -dimensional cycle on  $X$  and let  $s_i$  be a rational section of a line bundle  $\mathcal{L}_i$  on  $X$  for  $i = 1, \dots, r$ . Then  $s_1, \dots, s_r$  are said to *meet  $Z$  properly* if for every  $I \subseteq \{1, \dots, r\}$ , each irreducible component of  $|Z| \cap \bigcap_{i \in I} |\text{div}(s_i)|$  has dimension  $d - \#I$ .

Let  $\varphi : Y \rightarrow X$  be a proper morphism between two varieties over  $\mathbb{K}$ ; this is for example the case when  $\varphi$  is a closed immersion or when it is a morphism between two proper varieties. Then,  $\varphi$  induces a group homomorphism  $\varphi_* : Z_d(Y) \rightarrow Z_d(X)$  for each  $d \in \mathbb{N}$  by setting

$$f_* V := \begin{cases} [K(V) : K(f(V))]f(V) & \text{if } \dim(f(V)) = \dim(V), \\ 0 & \text{otherwise} \end{cases}$$

for any irreducible subvariety  $V$  of  $Y$  of dimension  $d$ . For  $Z \in Z_d(Y)$ , the cycle  $\varphi_* Z \in Z_d(X)$  is called the *push-forward* of  $Z$  via  $\varphi$ , see [Ful98, §1.4] for more details.

The first steps of the intersection theory developed in [Ful98] allow to define the intersection between a Cartier divisor and a cycle on  $X$ , provided they intersect properly.

**Definition 2.4.3.** Let  $V$  be an irreducible subvariety of  $X$  of dimension  $d$ , with  $\iota : V \hookrightarrow X$  the corresponding closed embedding. Let  $\mathcal{L}$  be a line bundle on  $X$  and  $s$  a rational section of  $\mathcal{L}$  meeting  $V$  properly. The *intersection cycle* between  $s$  and  $V$  is defined as

$$\operatorname{div}(s) \cdot V := \iota_*[\operatorname{div}(\iota^* s)]$$

where  $\operatorname{div}(\iota^* s)$  denotes the Cartier divisor on  $V$  associated to the section  $\iota^* s$  of the restricted line bundle  $\iota^* \mathcal{L}$ .

Extending by linearity, one defines the *intersection cycle*  $\operatorname{div}(s) \cdot Z$  between a  $d$ -cycle  $Z$  of  $X$  and a section  $s$  of  $\mathcal{L}$  meeting  $Z$  properly.

It follows from the definition that the intersection cycle  $\operatorname{div}(s) \cdot Z$ , whenever defined, is a  $(d - 1)$ -dimensional cycle on  $X$ . An extension of the notion presented in the previous definition and the study of its properties can be found in [Ful98, §2.3].

**Remark 2.4.4.** One can study the effect of base change on the group of algebraic cycles of  $X$ . Let  $V$  be an irreducible subvariety of  $X$  of pure dimension  $d$  and  $\mathbb{F}$  a finite extension of  $\mathbb{K}$ . The base change  $V_{\mathbb{F}}$  of  $V$  is a separated scheme of finite type over  $\mathbb{F}$ , with a closed embedding in  $X_{\mathbb{F}}$ ; however, it does not need to be reduced and irreducible. The definition in [Ful98, §1.5] allows to associated to it a  $d$ -cycle  $[V_{\mathbb{F}}]$  of  $X_{\mathbb{F}}$ . Extending by linearity, one obtains a group homomorphism

$$\operatorname{ext}_{\mathbb{F}/\mathbb{K}} : Z_d(X) \rightarrow Z_d(X_{\mathbb{F}}). \quad (2.5)$$

The set of all the algebraic extensions of  $\mathbb{K}$  in a fixed algebraic closure  $\overline{\mathbb{K}}$  are a directed set and the groups of  $d$ -cycles on  $X_{\mathbb{F}}$  with the corresponding extension morphisms form a direct system. Its direct limit is canonically isomorphic to the group  $Z_d(X_{\overline{\mathbb{K}}})$ .

**DEGREES.** The notion of height of a cycle in Arakelov geometry arises as an arithmetic analogue of the one of degree in classical intersection theory. For this reason, one recalls in this subsection such a geometric construction. One assumes here that the variety  $X$  is smooth and projective.

**Definition 2.4.5.** Let  $Z$  be a  $d$ -cycle on  $X$  and  $D_1, \dots, D_d$  a family of Cartier divisors over  $X$ . The *degree of  $Z$  with respect to  $D_1, \dots, D_d$*  is defined by the recursive formula

$$\deg_{D_1, \dots, D_d}(Z) := \deg_{D_1, \dots, D_{d-1}}(\operatorname{div}(s_d) \cdot Z)$$

for any choice of a rational section  $s_d$  of  $\mathcal{O}(D_d)$  meeting  $Z$  properly, and by setting

$$\deg\left(\sum_p m_p p\right) := \sum_p m_p [\kappa(p) : \mathbb{K}]$$

for any 0-cycle  $\sum_p m_p p$  on  $X$ .

The previous definition is well-posed. First, the assumptions on  $X$  assure that there always exists a section  $s_d$  of  $\mathcal{O}(D_d)$  meeting  $Z$  properly, see [Liu02, Proposition 9.1.11]. Secondly, the degree of  $Z$  with respect to  $D_1, \dots, D_d$  does not depend of the choice of the sections  $s_i$  of  $\mathcal{O}(D_i)$  for any  $i = 1, \dots, d$  because of the discussion in [Ful98, §2.5]. One has then a map

$$\deg_{D_1, \dots, D_d} : Z_d(X) \rightarrow \mathbb{Z}$$

satisfying the following properties.

**Theorem 2.4.6.** *Let  $X$  be a smooth and projective variety over  $\mathbb{K}$  and  $D_1, \dots, D_d$  a family of divisors over  $X$ . Then*

1. *the degree of  $d$ -dimensional cycles on  $X$  is multilinear and symmetric in the choice of the divisors and invariant under linear equivalence*
2. (Projection formula) *if  $\varphi : Y \rightarrow X$  is a dominant morphism between two smooth and projective varieties over  $\mathbb{K}$ ,*

$$\deg_{\varphi^* D_1, \dots, \varphi^* D_d}(Z) = \deg_{D_1, \dots, D_d}(\varphi_* Z)$$

*for any  $d$ -cycle  $Z$  of  $Y$*

3. if  $Z \in Z_d(X)$  and  $\mathbb{F}$  is a finite field extension of  $\mathbb{K}$ ,

$$\deg_{D_1, \dots, D_d}(Z) = \deg_{D_1, \mathbb{F}, \dots, D_d, \mathbb{F}}(\text{ext}_{\mathbb{F}/\mathbb{K}}(Z))$$

with  $\text{ext}_{\mathbb{F}/\mathbb{K}}$  as in (2.5) and  $D_{i, \mathbb{F}}$  standing for the extension of  $D_i$  to  $X_{\mathbb{F}}$  for any  $i = 1, \dots, d$ .

*Proof.* The first property follows from [Ful98, Proposition 2.3 (b) and Theorem 2.4] and directly from the definition. The projection formula is a consequence of [Ful98, Proposition 2.3 (c)]. The last property is proved in [Ful98, Example 6.2.9].  $\square$

**Remark 2.4.7.** For a morphism  $\varphi : Y \rightarrow X$  and a Cartier divisor  $D$  on  $X$ , the pull-back of  $D$  by  $\varphi$  is only defined when the image of  $\varphi$  is not contained in the support of  $D$ . In particular, the dominance hypothesis in Theorem 2.4.6 (2) assures that the left hand side term of the equality therein is well-defined. A more general version of the projection formula is given in [Ful98, Proposition 2.3 (c)] for arbitrary proper morphisms and pseudo-divisors in the sense of [Ful98, Definition 2.2.1].

**Remark 2.4.8.** Since birational morphisms are dominant, the projection formula in Theorem 2.4.6 allows to extend the notion of degree to proper ambient varieties which only are birational to smooth and projective varieties, without necessarily being such.

**Remark 2.4.9.** The notion of degree arises from the deeper definition of an intersection theory on  $X$  and can be introduced in terms of a top intersection product of the class of a cycle in the Chow ring of  $X$  with the Chern class of a line bundle, see [Ful98] for a detailed treatment or also [Har77, Appendix A] for a more concise introduction.

**Notation 2.4.10.** If  $d \in \mathbb{N}$  and  $D$  is a Cartier divisor over a proper variety  $X$  which is birational to a smooth and projective variety, it is customary to denote by  $\deg_D$  the application  $\deg_{D, \dots, D}$  on  $Z_d(X)$ .

**METRIZED DIVISORS.** To define heights in the setting of adelic Arakelov geometry, one needs to consider an additional datum of analytic nature. For every  $v \in \mathfrak{M}$ , denote by  $\mathbb{C}_v$  the completion of the algebraic closure of the completion of  $\mathbb{K}$  with respect to  $v$ . It comes equipped with an absolute value that one denotes again, with abuse of notation, by  $|\cdot|_v$ . The symbol  $X_v^{\text{an}}$  stands for the Berkovich analytification, as in section 2.2, of the base change of  $X$  to the field  $\mathbb{C}_v$ . Because of [Gro61, Proposition 5.4.2 (iii)] and Theorem 2.2.8,

the topological space underlying such an analytic space is compact. Also, a base change and Definition 2.2.5 allow to consider the  $v$ -adic analytification  $\mathcal{L}_v^{\text{an}}$  of a line bundle  $\mathcal{L}$  on  $X$ ; it is an analytic line bundle over  $X_v^{\text{an}}$ .

**Definition 2.4.11.** Let  $D$  be a Cartier divisor on  $X$  and  $v$  a place of  $\mathbb{K}$ . A  $v$ -adic metric on  $D$  is an assignment associating to every open subset  $U$  of  $X_v^{\text{an}}$  and analytic local section  $s \in \mathcal{O}(D)_v^{\text{an}}(U)$  a continuous function

$$\|s(\cdot)\| : U \rightarrow \mathbb{R}_{\geq 0} \quad (2.6)$$

such that

- (i) it is compatible with restrictions
- (ii) it is invariant under the action of  $\text{Gal}(\overline{\mathbb{K}}_v^{\text{sep}}/\mathbb{K}_v)$
- (iii)  $\|s(p)\| = 0$  if and only if  $s(p) = 0$
- (iv)  $\|(fs)(p)\| = |f(p)|_v \cdot \|s(p)\|$  for all  $f \in \mathcal{O}_{X_v^{\text{an}}}(U)$  and  $p \in U$ .

A Cartier divisor together with a  $v$ -adic metric is also called a  $v$ -adic metrized divisor and denoted by  $\overline{D}_v$  or also by  $(D, \|\cdot\|)$ .

To give a definition of height for a large spectrum of metrized divisors, it is particularly useful to identify a class of relevant choices of metrics. A  $v$ -adic metric on a Cartier divisor  $D$  is said to be *elementary* if

- when  $v$  is archimedean, the function  $\|s(\cdot)\|$  in (2.6) is smooth for every open  $U$  and local section  $s$  on  $U$ , see also [Cha11, §1.2.1]
- when  $v$  is non-archimedean, the function  $\|s(\cdot)\|$  is induced by an algebraic model of  $X$  and  $D$  over  $\mathbb{C}_v^\circ := \{\alpha \in \mathbb{C}_v : |\alpha|_v \leq 1\}$ , see [GK17, §8.12].

**Remark 2.4.12.** For a non-archimedean place  $v$  of  $\mathbb{K}$ , it is equivalent for a  $v$ -adic metric on  $D$  to be induced by an algebraic or by a formal model of  $D$ , see [GK17, §8.8 and Proposition 8.13].

To a Cartier divisor  $D$  equipped with an elementary  $v$ -adic metric one can associate a  $(\delta)$ -form of bidegree  $(1, 1)$ , called the *first Chern form of  $\overline{D}_v$*  and denoted by

$$c_1(\overline{D}_v),$$

see [Cha11, §1.2.2] when  $v$  is archimedean, [GK17, Definition 9.12 and Remark 9.16] otherwise. Also, an elementary  $v$ -adic metric is said to be *semi-positive* if  $c_1(\overline{D}_1)$  is a non-negative form on  $X_v^{\text{an}}$ , see [Cha11, §1.2.7] when  $v$  is archimedean, [GK17, Definition 2.6] otherwise for more precise definitions.

**Remark 2.4.13.** In the non-archimedean case, an elementary metric is semi-positive if and only if the associated algebraic model  $\mathcal{D}$  is vertically nef, see [GK15, Theorem 6.10 (1) and (4)]. This was indeed the original definition given by Zhang in [Zha95a, §1.2] and coincides with the one in [BPS14, Definition 1.3.12].

A  $v$ -adic metrized divisor  $\overline{D}_v$  is said to be *semipositive* if the corresponding metric can be approximated, in the sense of [BPS14, §1.4], by semipositive elementary metrics.

Sums and pull-backs of  $v$ -adic metrized divisors can be defined as in [Cha11, §1.2]; in particular, sums and pull-backs of  $v$ -adic semipositively metrized divisor are still semipositively metrized.

**LOCAL HEIGHTS.** Fix a place  $v \in \mathfrak{M}$ . For any  $d$ -dimensional subvariety  $Y$  of  $X$  and for any  $d$ -tuple of  $v$ -adic semipositive metrized divisors  $\overline{D}_{0,v}, \dots, \overline{D}_{d-1,v}$  on  $X$ , there exists a positive measure

$$c_1(\overline{D}_{0,v}) \wedge \cdots \wedge c_1(\overline{D}_{d-1,v}) \wedge \delta_{Y^{\text{an}}} \quad (2.7)$$

on  $X_v^{\text{an}}$ , which was first introduced in [Cha06, Définition 2.4 and Proposition 2.7 b)] in the non-archimedean setting and extended in [Gub07, §3.8] under weaker assumptions. The suggestive notation for the measure in (2.7) is compatible, in the elementary case, with the wedge product of the first Chern forms, see [CD12, §6.9] and [GK17, Theorem 10.5] for the non-archimedean setting. In the general case, it is obtained by a limit argument.

**Definition 2.4.14.** Let  $Z$  be a  $d$ -dimensional cycle in  $X$  and  $(\overline{D}_{0,v}, s_0), \dots, (\overline{D}_{d,v}, s_d)$  a collection of  $v$ -adic semipositive metrized divisors on  $X$  with rational sections of the corresponding line bundles, with  $s_0, \dots, s_d$  meeting  $Z$  properly. The  $v$ -adic local height of  $Z$  in  $X$  with respect to  $(\overline{D}_{i,v}, s_i)$  for  $i = 0, \dots, d$  is defined, linearly in its irreducible components, by the recursive formula

$$\begin{aligned} h_{\overline{D}_{0,v}, \dots, \overline{D}_{d,v}}(Z; s_0, \dots, s_d) &:= h_{\overline{D}_{0,v}, \dots, \overline{D}_{d-1,v}}(Z \cdot \text{div}(s_d); s_0, \dots, s_{d-1}) \\ &\quad - \int_{X_v^{\text{an}}} \log \|s_d\|_{d,v} c_1(\overline{D}_{0,v}) \wedge \cdots \wedge c_1(\overline{D}_{d-1,v}) \wedge \delta_{Z_v^{\text{an}}}, \end{aligned}$$

where  $\|\cdot\|_{d,v}$  denotes the metric of  $\overline{D}_{d,v}$  and one sets the height of the zero cycle to be zero.

The integrals appearing in the previous definition are well-defined, as shown in [CT09, Théorème 4.1] in both the archimedean and non-archimedean setting, and [GH17, Theorem 1.4.3] for the case of non-archimedean valuations which are not necessarily discrete. The  $v$ -adic local height function is moreover symmetric and multilinear with respect to sums of metrized divisors with rational sections of the associated line bundles, see [Gub03, Proposition 3.4 and Remark 9.3].

**GLOBAL HEIGHTS.** The adelic structure on the field  $\mathbb{K}$  allows to define a *semipositive metrized divisor*  $\bar{D}$  on  $X$  by the choice, for every place  $v \in \mathfrak{M}$ , of a continuous semipositive  $v$ -adic metric on  $\mathcal{O}(D)_v^{\text{an}}$ . This global definition induces a notion of a  $v$ -adic local height function at each place of  $\mathbb{K}$ . However, some care has to be taken when defining global heights as sums of such  $v$ -adic local heights, since they do not need to be well-defined in general.

**Definition 2.4.15.** A  $d$ -dimensional irreducible subvariety  $Y$  of  $X$  is said to be *integrable* with respect to the choice of  $d + 1$  semipositive metrized divisors  $\bar{D}_0, \dots, \bar{D}_d$  if there exists a birational proper map  $\varphi : Y' \rightarrow Y$ , with  $Y'$  projective, and sections  $s_i$  of  $\varphi^* \mathcal{O}(D_i)$  for each  $i = 0, \dots, d$ , meeting  $Y'$  properly, such that the  $v$ -adic local height

$$h_{\varphi^* \bar{D}_0, v, \dots, \varphi^* \bar{D}_d, v}(Y'; s_0, \dots, s_d)$$

is zero for all but finitely many places  $v \in \mathfrak{M}$ . A  $d$ -dimensional cycle is said to be *integrable* if each of its irreducible components is. If  $Y$  is an integrable  $d$ -dimensional irreducible subvariety, the *global height of  $Y$  in  $X$  with respect to  $\bar{D}_0, \dots, \bar{D}_d$*  is defined as

$$h_{\bar{D}_0, \dots, \bar{D}_d}(Y) := \sum_{v \in \mathfrak{M}} n_v h_{\varphi^* \bar{D}_0, v, \dots, \varphi^* \bar{D}_d, v}(Y'; s_0, \dots, s_d).$$

The *global height* of integrable cycles is defined by linearity.

The previous definition does not depend on the choice of the projective resolution  $Y'$  of  $Y$  nor of the sections  $s_0, \dots, s_d$ , as a consequence of [Gub03, Proposition 3.6 and Remark 9.3], [BPS14, Theorem 1.4.17 (3)] and the product formula on  $K$ .

The following statement, which resumes the properties of the global height, is the analog of Theorem 2.4.6.

**Theorem 2.4.16.** *Let  $X$  be a variety over  $\mathbb{K}$  and  $\bar{D}_0, \dots, \bar{D}_d$  a family of semipositive metrized divisors over  $X$ . Then*

1. the global height of  $d$ -dimensional cycles on  $X$  is multilinear and symmetric in the choice of the semipositive metrized divisors
2. (Arithmetic projection formula) if  $\varphi : Y \rightarrow X$  is a dominant morphism between two proper varieties over  $\mathbb{K}$ s and  $Z$  is a  $d$ -cycle in  $Y$ , the cycle  $\varphi_*Z$  is integrable with respect to  $\overline{D}_0, \dots, \overline{D}_d$  if and only if  $Z$  is integrable with respect to  $\varphi^*\overline{D}_0, \dots, \varphi^*\overline{D}_d$  and in this case

$$h_{\varphi^*\overline{D}_1, \dots, \varphi^*\overline{D}_d}(Z) = h_{\overline{D}_1, \dots, \overline{D}_d}(\varphi_*Z)$$

3. if  $Z \in Z_d(X)$  and  $\mathbb{F}$  is a finite field extension of  $\mathbb{K}$  with the adelic structure described in Proposition 2.3.12,

$$h_{\overline{D}_1, \dots, \overline{D}_d}(Z) = h_{\overline{D}_1, \mathbb{F}, \dots, \overline{D}_d, \mathbb{F}}(\text{ext}_{\mathbb{F}/\mathbb{K}}(Z))$$

with  $\text{ext}_{\mathbb{F}/\mathbb{K}}$  as in (2.5) and  $\overline{D}_{i, \mathbb{F}}$  standing for the extension of  $\overline{D}_i$  to  $X_{\mathbb{F}}$  for any  $i = 1, \dots, d$ .

*Proof.* The first statement is proved in [Gub03, Proposition 3.4 and Remark 9.3]. For (2), the statement about integrability is [BPS14, Proposition 1.5.8 (2)], while the equality of the global heights follows from the same property on local heights, as proved in [Gub03, Proposition 3.6 and Remark 9.3] in the more general context of pseudo-divisors. Last, property (3) is proved as in [BPS14, Proposition 1.5.10].  $\square$

**Remark 2.4.17.** It follows from the invariance of the height under extension of the base field proved in Theorem 2.4.16 (3) that one can speak about the global height of a cycle  $Z$  defined over  $\overline{\mathbb{K}}$ . Indeed, it is sufficient to compute the height of  $Z$  in any finite adelic field extension  $\mathbb{F}$  of  $\mathbb{K}$  over which  $Z$  is defined.

**Notation 2.4.18.** When  $d \in \mathbb{N}$  and  $\overline{D}$  is a semipositive metrized divisor on  $X$ , one says that  $Z \in Z_d(X)$  is  $\overline{D}$ -integrable if it is integrable with respect to  $\overline{D}, \dots, \overline{D}$ . Moreover, as in Notation 2.4.10 for the degree, one denotes by  $h_{\overline{D}}$  the application  $h_{\overline{D}, \dots, \overline{D}}$  defined on the set of  $d$ -dimensional  $\overline{D}$ -integrable cycles of  $X$ .



# Toric varieties

Toric geometry lies in a beautiful crossroad between algebraic geometry and convex geometry and represents the universe we will be set in for the rest of the study. After recalling the first definitions and characterizations of toric varieties and toric divisors, we present some geometrical result as the combinatorial description of the Weil divisor of a Laurent polynomial and of its intersection with toric orbits. Then, we move to the arithmetic setting by resuming the main constructions of [BPS14]. To keep the treatment of archimedean and non-archimedean places on equal footing, we rephrase their description of the Chambert-Loir measure of semipositive toric metrized divisors in terms of minimal boundaries of tropical fibers. We finally extend their integrability result to arbitrary base adelic fields.

## 3.1

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### TORIC GEOMETRY

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We recall briefly in this section the basic constructions and results in toric geometry, including the combinatorial characterization and description of toric varieties and toric Cartier divisors. We then give a useful result on suitable toric resolutions.

A field  $K$  is fixed for the whole section.

**TORIC VARIETIES.** Recall that by a *split torus* over  $K$  one means a group scheme over  $K$  isomorphic to

$$\mathbb{G}_{m,K}^n := \operatorname{Spec} K[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$$

for some  $n \in \mathbb{N}$ . The lattice of one-parameter subgroups of a split torus  $\mathbb{T}$  is the free group of rank  $n$

$$N := \operatorname{Hom}(\mathbb{G}_m, \mathbb{T}).$$

Its dual lattice

$$M := N^\vee = \operatorname{Hom}(N, \mathbb{Z})$$

is canonically isomorphic to the lattice  $\operatorname{Hom}(\mathbb{T}, \mathbb{G}_m)$  of characters of  $\mathbb{T}$ . A basis of the  $K$ -algebra  $K[M]$  will be denoted, accordingly to [Ful93, beginning of §1.3], by  $(\chi^m)_{m \in M}$  and the element of  $K[M]$  called *Laurent polynomial over  $K$* .

**Definition 3.1.1.** A *toric variety* with torus  $\mathbb{T}$  is a normal variety  $X$  over  $K$  equipped with a dense open embedding  $\mathbb{T} \hookrightarrow X$  and an action of  $\mathbb{T}$  on  $X$  extending the multiplication of  $\mathbb{T}$ .

If  $X$  is a toric variety with torus  $\mathbb{T}$  one has hence the following commutative diagram

$$\begin{array}{ccc} \mathbb{T} \times_K X & \longrightarrow & X \\ \uparrow & & \uparrow \\ \mathbb{T} \times_K \mathbb{T} & \longrightarrow & \mathbb{T} \end{array}$$

of schemes over  $K$ .

Toric varieties with torus  $\mathbb{T}$  admit a nice combinatorial description in terms of convex geometrical objects in the real vector space  $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ . By a *strongly convex polyhedral rational cone* in  $N_{\mathbb{R}}$  one here means a convex set of the form

$$\sigma = \operatorname{cone}(u_1, \dots, u_r)$$

with  $r \in \mathbb{N}$ ,  $u_1, \dots, u_r \in N$ , notation as in Example 1.2.1 and having  $\{0\}$  as a face (see other equivalent definitions in [CLS11, Proposition 1.2.12]). The vector  $u_1, \dots, u_r$  are called the *generators* of  $\sigma$ .

**Notation 3.1.2.** From now on, whenever dealing with toric varieties, by a *cone* in  $N_{\mathbb{R}}$  one will always mean a strongly convex polyhedral rational cone.

A *fan* in  $N_{\mathbb{R}}$  is a finite collection of cones in  $N_{\mathbb{R}}$  such that any face of  $\sigma \in \Sigma$  is still in  $\Sigma$  and the intersection of two cones in  $\Sigma$  is a face of both. The collection of cones of dimension  $k$  in the fan  $\Sigma$  will be denoted by  $\Sigma^{(k)}$ . The *support* of a fan  $\Sigma$  in  $N_{\mathbb{R}}$  is the set-theoretical union of its cones, that is

$$|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma.$$

From any fan  $\Sigma$  in  $N_{\mathbb{R}}$  one can construct a toric variety  $X_{\Sigma}$  as follows. For any cone  $\sigma$  in  $\Sigma$ , consider its dual

$$\sigma^{\vee} := \{x \in M_{\mathbb{R}} : \langle x, u \rangle \geq 0 \text{ for all } u \in \sigma\} \subseteq M_{\mathbb{R}}.$$

It is a cone in  $M_{\mathbb{R}}$  of dimension  $n$ , see [CLS11, Proposition 1.2.12]. The spectrum of the corresponding monoid algebra

$$X_{\sigma} := \text{Spec } K[M \cap \sigma^{\vee}]$$

is an affine normal variety of dimension  $n$ . Gluing such varieties accordingly to the intersection of the corresponding cones as in [Ful93, §1.4] gives a normal variety  $X_{\Sigma}$  which is indeed toric.

**Example 3.1.3.** By definition, a fan  $\Sigma$  always contains the trivial cone  $\{0\}$ . The affine variety associated to it is

$$X_{\{0\}} = \text{Spec } K[M \cap \{0\}^{\vee}] = \text{Spec } K[M]$$

which is isomorphic to the torus  $\mathbb{T}$ . Such an open subset of  $X_{\Sigma}$  is called the *dense open orbit* of the toric variety  $X_{\Sigma}$ .

One main feature of toric geometry is that any toric variety with torus  $\mathbb{T}$  arises from a fan.

**Theorem 3.1.4.** *Let  $X$  be a toric variety with torus  $\mathbb{T}$ . Then, there exists a fan  $\Sigma$  in  $N_{\mathbb{R}}$  such that  $X$  is isomorphic to  $X_{\Sigma}$ .*

*Proof.* This was proved in [KKMS73, §I.2, Theorem 6 (i)], see also [CLS11, Corollary 3.1.8].  $\square$

The geometric properties of the toric variety  $X_{\Sigma}$  can be read on the corresponding fan. Here are two examples of this dictionary.

**Proposition 3.1.5.** *Let  $X_{\Sigma}$  be a toric variety over  $K$ . Then*

1.  $X_\Sigma$  is proper if and only if the fan  $\Sigma$  is complete, that is  $|\Sigma| = N_\mathbb{R}$
2.  $X_\Sigma$  is smooth if and only if each cone  $\sigma \in \Sigma$  is generated by a part of a basis of  $N$ .

*Proof.* The first property is [CLS11, Theorem 3.4.6], the second is [CLS11, Theorem 3.1.19 (a)].  $\square$

Morphisms between toric varieties preserving the toric structure also admit an interpretation in terms of convex geometry. Let  $\mathbb{T}_1$  and  $\mathbb{T}_2$  be two split tori over  $K$  with lattice of one-parameter subgroups  $N_1$  and  $N_2$  respectively. Given two toric varieties  $X_1$  and  $X_2$  with torus  $\mathbb{T}_1$  and  $\mathbb{T}_2$  respectively, a *toric morphism* between  $X_1$  and  $X_2$  is a morphism  $\varphi : X_1 \rightarrow X_2$  mapping the dense open orbit  $X_{1,0} \simeq \mathbb{T}_1$  of  $X_1$  into the dense open orbit  $X_{2,0} \simeq \mathbb{T}_2$  of  $X_2$  (see Example 3.1.3) and such that the restriction  $\varphi|_{X_{1,0}} : X_{1,0} \rightarrow X_{2,0}$  is a group scheme homomorphism. Toric morphisms are equivariant in the sense that they are compatible with the torus action, see [CLS11, page 126]. In the following statement, one denotes by  $\ell_\mathbb{R}$  the unique extension of a linear map  $\ell : N_1 \rightarrow N_2$  to a  $\mathbb{R}$ -linear map  $(N_1)_\mathbb{R} \rightarrow (N_2)_\mathbb{R}$ .

**Theorem 3.1.6.** *In the above notations, let also  $\Sigma_1$  and  $\Sigma_2$  be the fans of  $X_1$  and  $X_2$  respectively in  $(N_1)_\mathbb{R}$  and  $(N_2)_\mathbb{R}$ . There is a bijection between the set of toric morphisms from  $X_1$  to  $X_2$  and the set of linear maps  $\ell : N_1 \rightarrow N_2$  for which for any cone  $\sigma \in \Sigma_1$  the set  $\ell_\mathbb{R}(\sigma)$  is contained in a cone of  $\Sigma_2$ . Moreover, the toric morphism  $\varphi : X_1 \rightarrow X_2$  corresponding to  $\ell$  is proper if and only if  $\ell_\mathbb{R}^{-1}(|\Sigma_2|) = |\Sigma_1|$ .*

*Proof.* The first part of the statement was proved in [Oda88, Theorem 1.13], the second one is [Oda88, Theorem 1.15].  $\square$

**TORIC ORBITS.** For a toric variety  $X$  over  $K$  with torus  $\mathbb{T}$ , the torus action determines a decomposition of  $X$  into orbits, see [Ful93, §3.1] for a more detailed treatment. To define them, consider for any cone  $\tau$  of dimension  $k$  in  $\Sigma$  the variety

$$O_\tau = \text{Spec } K[M \cap \tau^\perp], \quad (3.1)$$

where  $\tau^\perp = \{x \in M_\mathbb{R} : \langle x, u \rangle = 0 \text{ for all } u \in \tau\}$ . This is a split torus of dimension  $n - k$  over  $K$  with lattice of one-parameter subgroups

$$N(\tau) := N/(N \cap \tau)$$

and it is called the *orbit* associated to  $\tau$ . As an example, when  $\tau = \{0\}$  one obtains the dense open orbit of Example 3.1.3. Such an orbit admits a closed embedding in  $X_\tau$  defined by the morphism of  $K$ -algebras

$$K[M \cap \tau^\vee] \rightarrow K[M \cap \tau^\perp], \quad \chi^m \mapsto \begin{cases} \chi^m & \text{if } m \in \tau^\perp \\ 0 & \text{otherwise} \end{cases}. \quad (3.2)$$

The Zariski closure  $V(\tau)$  of  $O_\tau$  in the full toric variety  $X_\Sigma$  is called the *orbit closure* associated to  $\tau$ . It is a closed subvariety of  $X_\Sigma$  of dimension  $n - k$  which is isomorphic to a toric variety with torus  $O_\tau$ . Its corresponding fan is the *star* of  $\Sigma$  at  $\tau$ , denoted by  $\Sigma(\tau)$ , that is the fan in  $N(\tau)_\mathbb{R}$  consisting of the projections  $\sigma(\tau)$  of the cones  $\sigma$  in  $\Sigma$  containing  $\tau$ . The isomorphism between  $X_{\Sigma(\tau)}$  and  $V(\tau)$  can be visualized explicitly by gluing the morphisms  $X_{\sigma(\tau)} \rightarrow X_\sigma$  defined by

$$K[M \cap \sigma^\vee] \rightarrow K[M \cap \sigma^\vee \cap \tau^\perp], \quad \chi^m \mapsto \begin{cases} \chi^m & \text{if } m \in \tau^\perp \\ 0 & \text{otherwise} \end{cases} \quad (3.3)$$

for every  $\sigma \in \Sigma$  containing  $\tau$ .

**Remark 3.1.7.** For every  $\tau \in \Sigma^{(1)}$ , the orbit closure  $V(\tau)$  is an irreducible subvariety of  $X_\Sigma$  of codimension 1, hence a prime Weil divisor on  $X_\Sigma$  not intersecting the dense open orbit  $X_0$ .

**DIVISORS ON TORIC VARIETIES.** Given a toric variety  $X$  over  $K$  with torus  $\mathbb{T}$ , the Cartier divisors on  $X$  admitting a combinatorial description are the ones behaving well under the torus action.

**Definition 3.1.8.** A *toric Cartier divisor* on the toric variety  $X$  is a Cartier divisor  $D$  on  $X$  satisfying  $\pi_2^* D = \alpha^* D$ , with  $\alpha : \mathbb{T} \times_K X \rightarrow X$  the torus action given by Definition 3.1.1 and  $\pi_2 : \mathbb{T} \times_K X \rightarrow X$  the projection onto the second factor.

A *virtual support function* on a fan  $\Sigma$  in  $N_\mathbb{R}$  is a function  $\Psi : |\Sigma| \rightarrow \mathbb{R}$  whose restriction to any cone of  $\Sigma$  is a linear integral functional; otherwise said, for every cone  $\sigma \in \Sigma$  there exists a vector  $m_\sigma \in M$  for which  $\Psi(u) = \langle m_\sigma, u \rangle$  for every  $u \in \sigma$ . If  $X_\Sigma$  is the toric variety associated to the fan  $\Sigma$ , any virtual support function on  $\Sigma$  gives a well-defined toric Cartier divisor by considering

$$D_\Psi := \{(X_\sigma, \chi^{-m_\sigma})\}_{\sigma \in \Sigma},$$

see for instance [Ful93, §3.3]. All toric Cartier divisors arise in this way.

**Theorem 3.1.9.** *Let  $X_\Sigma$  be the toric variety associated to the fan  $\Sigma$  and let  $D$  be a toric Cartier divisor on  $X_\Sigma$ . Then, there exists a virtual support function  $\Psi$  on  $\Sigma$  such that  $D = D_\Psi$ .*

*Proof.* This is [KKMS73, §I.2, Theorem 9].  $\square$

The geometric properties of  $D_\Psi$  corresponds to the ones of the function  $\Psi$ , as follows.

**Proposition 3.1.10.** *The divisor  $D_\Psi$  on the toric variety  $X_\Sigma$  is generated by global sections if and only if  $\Psi$  is concave. Moreover, it is ample if and only if  $\Psi$  is strictly concave on  $\Sigma$ , that is the domain of linearity of  $\Psi$  coincide with the cones of  $\Sigma$ .*

*Proof.* This is proved in [Ful93, §3.4].  $\square$

To a toric Cartier divisor  $D_\Psi$  one can associate a line bundle  $\mathcal{O}(D_\Psi)$  and a distinguished rational section  $s_\Psi$  of  $\mathcal{O}(D_\Psi)$ , also called the *distinguished toric section*, in such a way that  $\text{div}(s_\Psi) = D_\Psi$ . The associated Weil divisor is

$$[\text{div}(s_\Psi)] = [D_\Psi] = \sum_{\tau \in \Sigma^{(1)}} -\Psi(v_\tau)V(\tau), \quad (3.4)$$

where  $v_\tau$  denotes the minimal nonzero integral vector of  $\tau$  for any  $\tau \in \Sigma^{(1)}$ , see [Ful93, page 66]. In particular,  $s_\Psi$  is regular and nowhere vanishing on  $X_0$ . A toric divisor  $D_\Psi$  also determines a polyhedron

$$\Delta_\Psi := \{x \in M_{\mathbb{R}} : \langle x, u \rangle \geq \Psi(u) \text{ for all } u \in |\Sigma|\} \quad (3.5)$$

in  $M_{\mathbb{R}}$ , which is bounded whenever  $X_\Sigma$  is proper, see [Ful93, Proposition at page 67]. If  $D_\Psi$  is generated by global sections,  $\Delta_\Psi$  coincides with the stability set of  $\Psi$  and  $\Psi$  is the support function of  $\Delta_\Psi$  in the sense of Remark 1.2.5.

**TORIC RESOLUTIONS.** In the study of the arithmetic of cycles in a toric variety the following notion will be relevant. Recall that a polytope  $P$  in  $M_{\mathbb{R}}$  determines a concave function on  $N_{\mathbb{R}}$  as in Remark 1.2.5.

**Definition 3.1.11.** A complete fan  $\Sigma$  in  $N_{\mathbb{R}}$  and a polytope  $P \subset M_{\mathbb{R}}$  are said to be *compatible* if the support function  $\Psi_P$  is linear on each cone of  $\Sigma$ .

The following useful results shows that, up to a birational transformation, a toric variety can be always supposed to enjoy friendly properties and have fan compatible with any finite number of given polytopes.

**Proposition 3.1.12.** *Let  $X_\Sigma$  be a proper toric variety over  $K$  and  $P_1, \dots, P_r$  a finite family of polytopes in  $M_{\mathbb{R}}$ . Then, there exist a smooth projective toric variety  $X_{\Sigma'}$  with fan  $\Sigma'$  in  $N_{\mathbb{R}}$  and a proper toric morphism  $\pi : X_{\Sigma'} \rightarrow X_\Sigma$  satisfying:*

1.  $\pi$  restricts to the identity on the dense open orbit of  $X_{\Sigma'}$  and  $X_\Sigma$
2.  $P_1, \dots, P_r$  are compatible with  $\Sigma'$ .

*Proof.* One can always refine the complete fan  $\Sigma$  to a fan  $\Sigma'$  in such a way that  $\Psi_{P_1}, \dots, \Psi_{P_r}$  are linear on each cone of  $\Sigma'$ . After possibly refining again, one can suppose that  $\Sigma'$  is the fan of a projective toric variety (because of the toric Chow lemma, see [CLS11, Theorem 6.1.18]) and that each of its cones is generated by a part of a basis of  $N$  (see [Ful93, §2.6]). The associated toric variety  $X_{\Sigma'}$  is smooth, projective and it satisfies (2). Finally, since  $\Sigma'$  is a refinement of  $\Sigma$ , the toric morphism  $\pi$  corresponding to the identity on  $N$  by Theorem 3.1.6 is proper and restricts to the identity on the dense open orbit of  $X_{\Sigma'}$ .  $\square$

## 3.2 DIVISORS OF LAURENT POLYNOMIALS

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The present section focuses on the combinatorial description of the Weil divisor on a toric variety of the rational function coming from a Laurent polynomial. This result will be used in the proof of the main theorems in the next chapter. We then also study its intersection with toric orbits.

**THE SETTING.** Fix for all the section the choice of a field  $K$  and of a split torus  $\mathbb{T}$  of dimension  $n$  over  $K$ . Let  $N$  be its lattice of one-parameter subgroups,  $M$  its dual and  $N_{\mathbb{R}}$  and  $M_{\mathbb{R}}$  the associated vector spaces. Choose a smooth proper toric variety  $X_\Sigma$  with torus  $\mathbb{T}$ . The associated fan is then complete by Proposition 3.1.5 and all of its cones are generated by part of a basis of  $N$ . The dense open orbit of  $X_\Sigma$  described in Example 3.1.3 is isomorphic to  $\mathbb{T}$ ; the function field of  $X_\Sigma$  coincides hence with  $K(M)$ . In particular, any Laurent polynomial  $f = \sum c_m \chi^m$  is a regular function on  $X_0$  and corresponds hence to a rational function on  $X_\Sigma$ , which one denotes again, with abuse of notation, by  $f$ . The following notion, which can be introduced in a more general setting, is playing a role in all the following description.

**Definition 3.2.1.** Let  $R$  be a ring and  $f \in R[M]$  a Laurent polynomial with coefficients in  $R$ ,  $f = \sum_m c_m \chi^m$ . The *Newton polytope*  $\text{NP}(f)$  of  $f$  is the convex hull in  $M_{\mathbb{R}}$  of the finite set  $\{m \in M : c_m \neq 0_R\}$ .

The Newton polytope of  $f$  is a polytope in  $M_{\mathbb{R}}$  according to Definition 1.2.6 because of Remark 1.2.7.

**THE DIVISOR.** For each cone  $\tau$  in  $\Sigma$  of dimension 1, denote by  $v_\tau$  its minimal nonzero integral vector, which generates  $\tau \cap N$  as a monoid. If  $\sigma$  is a *smooth cone*, that is generated by part of a basis of  $N$ , of dimension  $n$  in  $N_{\mathbb{R}}$ , the collection  $(v_\tau)_\tau$ , with  $\tau$  ranging in the set of one dimensional faces of  $\sigma$ , is a basis of  $N$  and hence gives a dual basis  $(v_\tau^\vee)_\tau$  of the lattice  $M$ .

**Lemma 3.2.2.** *Let  $\sigma$  be a strongly convex polyhedral rational cone in  $N_{\mathbb{R}}$ . For every face  $\tau$  of  $\sigma$  of dimension 1, the orbit closure  $V(\tau)$  in the affine toric variety  $X_\sigma$  is the subvariety corresponding to the prime ideal*

$$\mathfrak{p} = (\chi^m : m \in \sigma^\vee \cap M, m \notin \tau^\perp)$$

of  $\mathcal{O}(X_\sigma) = K[M \cap \sigma^\vee]$ . Moreover, if  $\sigma$  is smooth and of maximal dimension in  $N_{\mathbb{R}}$ ,  $\mathfrak{p}$  is principal and generated by  $\chi^{v_\tau^\vee}$ .

*Proof.* Recall that the orbit closure  $V(\tau)$  is the toric variety  $\text{Spec } K[M \cap \sigma^\vee \cap \tau^\perp]$  and can be embedded in  $X_\sigma = \text{Spec } K[M \cap \sigma^\vee]$  via the surjection in (3.3). Then  $V(\tau)$  is seen as the subvariety of  $X_\sigma$  corresponding to the kernel of such homomorphism, that is

$$\mathfrak{p} = \bigoplus_{\substack{m \in \sigma^\vee \cap M \\ m \notin \tau^\perp}} K\chi^m = (\chi^m : m \in \sigma^\vee \cap M, m \notin \tau^\perp),$$

proving the first statement.

Suppose now that  $\sigma$  is a smooth cone of dimension  $n$  in  $N_{\mathbb{R}}$ ; denote by  $v_1, v_2, \dots, v_n$  the basis of  $N$  given by the minimal integral vectors of the rays of  $\sigma$ , with the assumption that  $v_1 = v_\tau$ . By definition,

$$\sigma = \mathbb{R}_{\geq 0}v_1 + \dots + \mathbb{R}_{\geq 0}v_n.$$

As a result, denoting by  $(v_i^\vee)_{i=1, \dots, n}$  the basis of  $M$  dual to  $(v_i)_{i=1, \dots, n}$ , one has that

$$\langle v_i^\vee, u \rangle = \lambda_i \geq 0$$

for every  $i \in \{1, \dots, n\}$  and for every  $u = \sum_i \lambda_i v_i \in \sigma$ . In particular,  $v_\tau^\vee \in \sigma^\vee$ . It is easy to check that  $v_\tau^\vee$  is integrally valued on each element of  $N$  and hence it belongs to  $M$ . It follows then from  $\langle v_\tau^\vee, v_\tau \rangle = 1$  that

$$\left(\chi^{v_\tau^\vee}\right) \subseteq \mathfrak{p}.$$

For the reverse inclusion, consider  $m \in \sigma^\vee \cap M$  with  $m \notin \tau^\perp$ . By assumption,  $\langle m, v_\tau \rangle \in \mathbb{Z}$  and  $\langle m, v_\tau \rangle \geq 0$ ; moreover, since  $m \notin \tau^\perp$ , one has  $\langle m, v_\tau \rangle \geq 1$ . For each  $u = \sum_i \lambda_i v_i \in \sigma$  one has

$$\langle m - v_\tau^\vee, u \rangle = \lambda_1 \langle m - v_\tau^\vee, v_\tau \rangle + \sum_{i \geq 2} \lambda_i \langle m, v_i \rangle \geq \lambda_1 (\langle m, v_\tau \rangle - 1) \geq 0.$$

As a result,  $m - v_\tau^\vee \in \sigma^\vee \cap M$  and hence  $\chi^m = \chi^{v_\tau^\vee} \cdot \chi^{m - v_\tau^\vee} \in (\chi^{v_\tau^\vee})$ , completing the proof.  $\square$

**Remark 3.2.3.** The last statement of the previous lemma is not true for a general strongly convex polyhedral rational cone  $\sigma$  of maximal dimension in  $N$ . For example, if  $\sigma$  has more than  $n$  faces of dimension 1, the divisor class group of  $X_\sigma$ , which is generated by the classes of the orbit closures associated to the rays, turns out to be nontrivial, as a consequence of [Ful93, Proposition at page 63].

For a nonzero Laurent polynomial  $f \in K[M]$ , the subset  $V(f)$  of zeros of  $f$  in  $X_0$  is a closed subscheme of the dense open orbit. Its closure in  $X_\Sigma$  is a closed subscheme of  $X_\Sigma$ , denoted by  $\overline{V(f)}$ . Taking into account multiplicities, one can consider the associated Weil divisor  $[\overline{V(f)}]$ . It is the zero cycle when  $f$  is a monomial.

**Theorem 3.2.4.** *For a nonzero Laurent polynomial  $f$ ,*

$$[\operatorname{div}(f)] = [\overline{V(f)}] + \sum_{\tau \in \Sigma(1)} \Psi_{\operatorname{NP}(f)}(v_\tau) V(\tau),$$

where  $\Psi_{\operatorname{NP}(f)}$  denotes the support function of the Newton polytope of  $f$ . In particular,  $[\overline{V(f)}]$  is rationally equivalent to the cycle  $-\sum_{\tau \in \Sigma(1)} \Psi_{\operatorname{NP}(f)}(v_\tau) V(\tau)$  on  $X_\Sigma$ .

*Proof.* By [Ful93, formula at page 55], the irreducible components of  $X_\Sigma \setminus X_0$  are exactly the orbit closures  $V(\tau)$ , with  $\tau$  ranging in the set of 1 dimensional

cones of  $\Sigma$ . Since moreover  $f$  is a regular function on  $X_0$ , it follows from the classical theory of divisors that

$$[\operatorname{div}(f)] = \left[ \overline{V(f)} \right] + \sum_{\tau} \nu_{\tau}(f) V(\tau),$$

where  $\nu_{\tau}(f) \in \mathbb{Z}$  is the order of vanishing of  $f$  along  $V(\tau)$ . The statement of the theorem then follows from the fact that, for every  $\tau \in \Sigma^{(1)}$ , such an order equals  $\Psi_{\text{NP}(f)}(v_{\tau})$ .

This claim can be proved locally; fix a ray  $\tau \in \Sigma^{(1)}$  and let  $\sigma$  be any maximal dimensional cone of  $\Sigma$  containing  $\tau$ . The fan being complete and consisting of smooth cones, such a  $\sigma$  exists and the minimal integral vectors  $v_1, \dots, v_n$  of its rays are a basis of  $N$ . Assume moreover that  $v_1 = v_{\tau}$  and, for simplicity, denote by  $R := K[M \cap \sigma^{\vee}]$  the ring of regular functions over  $X_{\sigma}$ . The order of vanishing of  $f$  along  $V(\tau)$  is computed as the valuation of  $f$  determined by the valuation ring  $R_{\mathfrak{p}}$ , the localization of  $R$  at the prime ideal  $\mathfrak{p}$  corresponding to the subvariety  $V(\tau)$  in  $X_{\sigma}$ . By Lemma 3.2.2, the cone  $\sigma$  being smooth and maximal dimensional, one has that  $\mathfrak{p} = (\chi^{v_{\tau}^{\vee}})$ . The maximal ideal  $\mathfrak{p}R_{\mathfrak{p}}$  of  $R_{\mathfrak{p}}$  is hence the principal ideal generated by  $\chi^{v_{\tau}^{\vee}}$ .

Suppose first that  $f = \sum_m c_m \chi^m$  lies in  $R$ , that is every  $m$  appearing in  $f$  belongs to  $M \cap \sigma^{\vee}$ . By definition of the valuation in  $R_{\mathfrak{p}}$ ,

$$\begin{aligned} \nu_{\tau}(f) &= \max \left\{ l \in \mathbb{N} : f \in (\mathfrak{p}R_{\mathfrak{p}})^l \right\} = \max \left\{ l \in \mathbb{N} : f \in (\chi^{lv_{\tau}^{\vee}}) \right\} \\ &= \max \left\{ l \in \mathbb{N} : \chi^{m-lv_{\tau}^{\vee}} \in R_{\mathfrak{p}} \text{ for all } m \text{ with } c_m \neq 0 \right\}. \end{aligned}$$

The condition  $\chi^{m-lv_{\tau}^{\vee}} \in R_{\mathfrak{p}}$  is equivalent to the fact that  $\langle m, v_{\tau} \rangle \geq l$ . Indeed, if the first is true, then

$$\langle m, v_{\tau} \rangle - l = \langle m - lv_{\tau}^{\vee}, v_{\tau} \rangle \geq 0.$$

Conversely, for each  $u = \sum_i \lambda_i v_i \in \sigma$  one has

$$\langle m - lv_{\tau}^{\vee}, u \rangle = \lambda_1 (\langle m, v_{\tau} \rangle - l) + \sum_{i \geq 2} \lambda_i \langle m, v_i \rangle \geq 0,$$

and so  $m - lv_{\tau}^{\vee} \in \sigma^{\vee} \cap M$ . As a consequence,

$$\begin{aligned} \nu_{\tau}(f) &= \max \left\{ l \in \mathbb{N} : \langle m, v_{\tau} \rangle \geq l \text{ for all } m \text{ with } c_m \neq 0 \right\} \\ &= \min \left\{ \langle m, v_{\tau} \rangle : m \text{ with } c_m \neq 0 \right\} = \Psi_{\text{NP}(f)}(v_{\tau}). \end{aligned}$$

For a general  $f = \sum_m c_m \chi^m$ , the fact that  $\sigma^\vee$  has dimension  $n$  in  $M_{\mathbb{R}}$  ( $\sigma$  is indeed strongly convex) assures that there exists a big enough vector  $m_0 \in M \cap \sigma^\vee$  for which  $m + m_0 \in M \cap \sigma^\vee$  for each  $m$  such that  $c_m \neq 0$ . Hence

$$f = \frac{\sum_m c_m \chi^{m+m_0}}{\chi^{m_0}},$$

with both the numerator and the denominator belonging to  $R$ . Applying the result for such elements one deduces

$$\begin{aligned} \nu_\tau(f) &= \nu_\tau \left( \sum_m c_m \chi^{m+m_0} \right) - \nu_\tau(\chi^{m_0}) = \Psi_{\text{NP}(f)+m_0}(v_\tau) - \langle m_0, v_\tau \rangle \\ &= \Psi_{\text{NP}(f)}(v_\tau), \end{aligned}$$

concluding the proof.  $\square$

**Remark 3.2.5.** Given a toric Cartier divisor  $D_\Psi$  constructed from a virtual support function  $\Psi$  on  $N_{\mathbb{R}}$ , it can be deduced from (3.4) and Theorem 3.2.4 that  $f$  is a regular global section of  $\mathcal{O}(D_\Psi)$  if and only if

$$\Psi_{\text{NP}(f)}(v_\tau) \geq \Psi(v_\tau)$$

for every  $\tau \in \Sigma^{(1)}$ . Since the rays of  $\Sigma$  span the whole  $N_{\mathbb{R}}$ , this is equivalent to the fact that  $\text{NP}(f) \subseteq \Delta_\Psi$ , in the notation of (3.5). This result agrees with [Ful93, Lemma at page 66].

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### 3.3 OUTSIDE THE DENSE OPEN TORUS

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Let  $f$  be a nonzero Laurent polynomial with coefficients in  $\mathbb{K}$  with Newton polytope compatible with a complete fan  $\Sigma$ . The zero set of  $f$  in  $X_0$  can be closed to the proper toric variety  $X_\Sigma$ . We here study, via initial forms of polynomials, the intersection of this hypersurface with the toric orbits of  $X_\Sigma$ .

**FACES ASSOCIATED TO CONES.** For a polytope  $P$  and a compatible fan  $\Sigma$ , it is interesting to associated to each cone of the fan a face of the polytope.

**Definition 3.3.1.** Let  $P$  be a polytope in  $M_{\mathbb{R}}$  and  $\sigma$  a strongly convex rational polyhedral cone in  $N_{\mathbb{R}}$  on which the support function  $\Psi_P$  is linear. The  $\sigma$ -face of  $P$  is the subset

$$F(\sigma, P) := P^u$$

of  $P$ , with  $u \in \text{ri}(\sigma)$ .

The fact that the definition of this set does not depend on the choice of  $u$  and a more explicit description of it are given in the following.

**Proposition 3.3.2.** *In the hypotheses of Definition 3.3.1, the set  $F(\sigma, P)$  does not depend on the choice of  $u \in \text{ri}(\sigma)$ . Moreover, for any finite family  $w_1, \dots, w_r$  of generators of  $\sigma$ ,*

$$F(\sigma, P) = P^{w_1} \cap \dots \cap P^{w_r}.$$

*Proof.* The cone  $\sigma$  being strictly convex, one can first assume that the vectors  $w_1, \dots, w_r$  are the generators of the rays of  $\sigma$ . Hence, any  $u \in \text{ri}(\sigma)$  can be written as

$$u = \sum_{i=1}^r \lambda_i w_i,$$

with  $\lambda_1, \dots, \lambda_r \in \mathbb{R}_{>0}$ . For any  $x \in P$ , moreover, one has that

$$\langle x, u \rangle - \Psi_P(u) = \sum_{i=1}^r \lambda_i (\langle x, w_i \rangle - \Psi_P(w_i))$$

since  $\Psi_P$  is linear on  $\sigma$ . By definition of the support function of a polytope,  $\langle \cdot, v \rangle - \Psi_P(v)$  is nonnegative on  $P$  for every  $v \in N_{\mathbb{R}}$ . Hence, the left hand side of the previous equality is zero if and only if each summand on the right hand side is, which implies

$$P^u = \bigcap_{i=1}^r \{x \in P : \langle x, w_i \rangle - \Psi_P(w_i) = 0\} = \bigcap_{i=1}^r P^{w_i}.$$

In particular, this does not depend on the choice of  $u \in \text{ri}(\sigma)$ . An analogous argument shows that for every  $w \in \sigma$  one has  $P^w \supseteq \bigcap_{i=1}^r P^{w_i}$ . This inclusion, together with the fact that any family of generators of  $\sigma$  contains a generator for each ray, imply the general statement.  $\square$

**Remark 3.3.3.** The argument of the proof also shows that for every  $u \in \sigma$  not necessarily in the relative interior of  $\sigma$  one has  $F(\sigma, P) \subseteq P^u$ .

Recall that for a cone  $\sigma$  in  $N_{\mathbb{R}}$ , its *orthogonal* is the linear subspace of  $M_{\mathbb{R}}$  defined as  $\sigma^{\perp} = \{x \in M_{\mathbb{R}} : \langle x, u \rangle = 0 \text{ for all } u \in \sigma\}$ . Its dimension coincides with  $\text{codim}(\sigma)$ , which is the codimension of the minimal linear space of  $N_{\mathbb{R}}$  containing  $\sigma$ .

**Remark 3.3.4.** An argument analogous to the one of the proof of Proposition 3.3.2 shows that

$$\sigma^\perp = \bigcap_{u \in \sigma} u^\perp = w_1^\perp \cap \cdots \cap w_r^\perp$$

for any finite family  $w_1, \dots, w_r$  of generators of  $\sigma$ .

**Corollary 3.3.5.** *The  $\sigma$ -face of  $P$  is a nonempty face of  $P$  described as  $F(\sigma, P) = (a + \sigma^\perp) \cap P$  for any  $a \in F(\sigma, P)$ . In particular, it has dimension at most  $\text{codim}(\sigma)$ .*

*Proof.* The fact that  $F(\sigma, P)$  is a nonempty face of  $P$  is immediate from its definition. For every  $u \in \sigma$ , Remark 3.3.3 implies  $a \in P^u$ , hence the equality  $P^u = (a + u^\perp) \cap P$ . The claim  $F(\sigma, P) = (a + \sigma^\perp) \cap P$  follows then from Proposition 3.3.2 and Remark 3.3.4.  $\square$

However, it can happen that the dimension of  $F(\sigma, P)$  is strictly smaller than  $\text{codim}(\sigma)$ , as the following example shows.

**Example 3.3.6.** Let  $\Delta$  be the unit simplex in  $\mathbb{R}^2$  and  $\sigma$  the one dimensional cone in  $\mathbb{R}^2$  generated by the vector  $(1, 1)$ . The support function of  $\Delta$  is conic, hence it is linear on  $\sigma$ . By definition,

$$F(\sigma, \Delta) = \Delta^{(1,1)} = \{(x_1, x_2) \in \Delta : x_1 + x_2 = 0\} = \{(0, 0)\},$$

while  $\text{codim}(\sigma) = 1$ .

**INTERSECTIONS WITH TORIC ORBITS.** Let  $f \in \mathbb{K}[M]$  be a nonzero Laurent polynomial with Newton polytope  $\text{NP}(f) \subset M_{\mathbb{R}}$ ,

$$f = \sum_{m \in \text{NP}(f) \cap M} c_m \chi^m \tag{3.6}$$

with  $c_m \in \mathbb{K}$ . The following definition will be crucial for the rest of the subsection.

**Definition 3.3.7.** For a face  $F$  of  $\text{NP}(f)$ , the  $F$ -initial form of  $f$  is the polynomial

$$\text{in}_F(f) := \sum_{m \in F \cap M} c_m \chi^m.$$

**Example 3.3.8.** In particular  $\text{in}_{\text{NP}(f)}(f) = f$ , while if  $m$  is a vertex of  $\text{NP}(f)$ , the initial form  $\text{in}_{\{m\}}(f)$  coincides with the monomial of order  $m$  of  $f$ .

Fix now the choice of a fan  $\Sigma$  compatible with  $\text{NP}(f)$  and denote by  $\overline{V(f)}$  the closure of the zero set of  $f$  in the torus to the toric variety  $X_\Sigma$ . One is interested in describing the intersection of  $\overline{V(f)}$  with the toric orbits of  $X_\Sigma$ . Recall that the toric orbits of  $X_\Sigma$  correspond to the cones of the fan  $\Sigma$  and that one denotes by  $O_\sigma$  the orbit associated to the cone  $\sigma \in \Sigma$ ; it is of dimension  $\dim(\sigma)$  and can be abstractly described as  $\text{Spec } \mathbb{K}[M \cap \sigma^\perp]$ . Denote also by  $F(\sigma)$  the  $\sigma$ -face  $F(\sigma, \text{NP}(f))$  as in Definition 3.3.1. If  $a \in F(\sigma) \cap M$ , then  $F(\sigma) \subset a + \sigma^\perp$  because of Corollary 3.3.5 and then

$$\frac{\text{in}_{F(\sigma)}(f)}{\chi^a} = \sum_{m \in F(\sigma) \cap M} c_m \chi^{m-a}$$

is a Laurent polynomial in  $\mathbb{K}[M \cap \sigma^\perp]$ .

**Theorem 3.3.9.** *In the above hypotheses and notations, for every cone  $\sigma$  of  $\Sigma$ , the intersection of  $\overline{V(f)}$  with the torus orbit  $O_\sigma$  is the zero set of*

$$\frac{\text{in}_{F(\sigma)}(f)}{\chi^a}$$

in  $O_\sigma$ , for any choice of  $a \in F(\sigma) \cap M$ .

*Proof.* Consider the affine toric subvariety  $X_\sigma = \text{Spec } \mathbb{K}[M \cap \sigma^\vee]$  of  $X_\Sigma$ . The toric orbit  $O_\sigma$  is a closed subscheme of  $X_\sigma$ , with immersion described in (3.2). Let  $a \in F(\sigma) \cap M$ . Because of Remark 3.3.3,  $a \in \text{NP}(f)^u$  for every  $u \in \sigma$ , hence  $m - a \in \sigma^\vee$  for all  $m \in \text{NP}(f) \cap M$ ; it follows that  $f \cdot \chi^{-a} \in \mathbb{K}[\sigma^\vee \cap M]$ . From Theorem 3.2.4 and [Ful93, Lemma at page 61],

$$[\text{div}(f \cdot \chi^{-a})]$$

is the cycle associated to the closure of  $V(f)$  to  $X_\sigma$ . Then, the closure of the zero set of  $f$  to  $X_\sigma$  is the closed set in  $X_\sigma$  defined by the ideal of  $\mathbb{K}[M \cap \sigma^\vee]$  generated by  $f \cdot \chi^{-a}$ . Its intersection with the closed subvariety  $O_\sigma$  of  $X_\sigma$  is then given by the zero set in  $O_\sigma$  of the image of  $f \cdot \chi^{-a}$  by (3.2). With the notation of (3.6) and because of Corollary 3.3.5, this coincides with

$$\sum_{m-a \in \sigma^\perp \cap M} c_m \chi^{m-a} = \sum_{\substack{m \in a + (\sigma^\perp \cap M) \\ m \in \text{NP}(f)}} c_m \chi^{m-a} = \sum_{m \in F(\sigma) \cap M} c_m \chi^{m-a},$$

implying the claim.  $\square$

**Remark 3.3.10.** When  $\sigma = \{0\}$ , one has that  $F(\sigma) = \text{NP}(f)$  and hence that  $\text{in}_{F(\sigma)}(f) = f$ , as in Example 3.3.8. The claim of Theorem 3.3.9 is obvious in such a case,  $O_\sigma$  being the dense open orbit of  $X_\Sigma$ .

## 3.4

## HEIGHTS OF TORIC VARIETIES

We recall here briefly the main results of [BPS14] concerning the adelic Arakelov geometry of toric varieties. The choices of a base adelic field  $\mathbb{K}$  satisfying the product formula and of a proper toric variety  $X_\Sigma$  of dimension  $n$  over  $\mathbb{K}$  are supposed to be fixed for the whole section.

**TROPICALIZATIONS.** Let  $\mathbb{T}$  be a split torus of dimension  $n$  over  $\mathbb{K}$ . As before, denote by  $N$  and  $M$  the character and cocharacter groups of  $\mathbb{T}$  and by  $N_{\mathbb{R}}$  and  $M_{\mathbb{R}}$  the associated (reciprocally dual) real vector spaces. For every place  $v \in \mathfrak{M}$ , denote by  $\mathbb{C}_v$  the completion of an algebraic closure of the completion of  $\mathbb{K}$  with respect to  $v$ . The  $v$ -adic analytic torus is then by definition

$$\mathbb{T}_v^{\text{an}} = \{ \|\cdot\|_x \text{ multiplicative seminorm on } \mathbb{C}_v[M] : \|k\|_x = |k|_v \text{ for all } k \in \mathbb{C}_v \},$$

see section 2.2, and it is equivalently denoted by  $\mathbb{T}_v^{\text{an}}$  or  $\mathbb{T}_{\mathbb{C}_v}^{\text{an}}$  to stress the field of definition.

**Remark 3.4.1.** Suppose that  $v$  is an archimedean place of  $\mathbb{K}$ . Because of Proposition 2.2.6 the set  $\mathbb{T}_v^{\text{an}}$  is in bijection with the set of  $\mathbb{C}_v$ -points of  $\mathbb{T}_{\mathbb{C}_v}$  by setting, for every  $\mathbb{C}_v$ -point  $p$ ,

$$\|f\|_p := |f(p)|_v$$

for all  $f \in \mathbb{C}_v[M]$ . Proposition 2.1.20 asserts now that there exists, up to complex conjugation, a unique isometry  $\sigma$  between  $\mathbb{C}_v$  and  $\mathbb{C}$ , hence

$$\|f\|_p = |f(p)|_v = |\sigma(f(p))|_\infty = |f^\sigma(\sigma(p))|_\infty$$

where  $f^\sigma = \sum_m \sigma(c_m)\chi^m$  if  $f = \sum_m c_m\chi^m$ . The remark that the  $\mathbb{C}_v$ -points of  $\mathbb{T}_{\mathbb{C}_v}$  are in bijection with the  $\mathbb{C}$ -points of  $\mathbb{T}_{\mathbb{C}}$  since the two fields are isomorphic implies that the set  $\mathbb{T}_v^{\text{an}}$  consists of the points  $\|\cdot\|_z$  with  $z \in \mathbb{T}_{\mathbb{C}}(\mathbb{C})$ , where

$$\|f\|_z := |f^\sigma(z)|_\infty$$

for every  $f \in \mathbb{C}_v[M]$ . For an archimedean place  $v$  of  $\mathbb{K}$  one will always identify  $\mathbb{T}_v^{\text{an}}$  with  $\mathbb{T}_{\mathbb{C}}(\mathbb{C})$  unless otherwise mentioned.

The following map plays a crucial role in all the following treatment.

**Definition 3.4.2.** Let  $v \in \mathfrak{M}$ . The  $v$ -adic tropicalization map is the application  $\text{trop}_v : \mathbb{T}_v^{\text{an}} \rightarrow N_{\mathbb{R}} = \text{Hom}(M, \mathbb{R})$  defined by

$$(\text{trop}_v(\|\cdot\|_x))(m) := -\log \|\chi^m\|_x$$

for every  $\|\cdot\|_x \in \mathbb{T}_v^{\text{an}}$ .

Thanks to the formalism of Berkovich construction, such a map turns out to be a continuous application.

**Remark 3.4.3.** The  $v$ -adic tropicalization map coincides (up to a field extension) with the valuation map  $\text{val}$  used by Burgos Gil, Philippon and Sombra, see [BPS14, (4.1.2) and Remark 4.1.3].

**Remark 3.4.4.** In the archimedean case, the choice of a basis for  $M$  allows to write the tropicalization map in the more familiar form

$$\text{trop}((z_1, \dots, z_n)) = (-\log |z_1|, \dots, -\log |z_n|),$$

which coincides with the one considered in (1.14).

**Remark 3.4.5.** When  $v$  is non-archimedean, one can construct a suitable section of  $\text{trop}_v$ . Indeed, for each  $u \in N_{\mathbb{R}}$ , one can consider the map associating to a Laurent polynomial  $f = \sum c_m \chi^m$  over  $\mathbb{C}_v$  the real value

$$\|f\|_{\kappa_v(u)} := \max_m |c_m|_v e^{-\langle m, u \rangle}. \quad (3.7)$$

It is easy to verify that  $\|\cdot\|_{\kappa_v(u)}$  is a multiplicative seminorm on  $\mathbb{C}_v[M]$  extending the absolute value  $|\cdot|_v$  of  $\mathbb{C}_v$ ; hence it corresponds to a point  $\kappa_v(u) \in \mathbb{T}_v^{\text{an}}$ , called the *Gauss point* over  $u$ . The application  $\kappa_v : N_{\mathbb{R}} \rightarrow \mathbb{T}_v^{\text{an}}$  defined by

$$\kappa_v : u \rightarrow \|\cdot\|_{\kappa_v(u)}$$

is proved to be a continuous section of  $\text{trop}_v$ , see for instance [BPS14, Proposition-Definition 4.2.12],  $\kappa_v$  coinciding with  $\theta_0 \circ \mathbf{e}$  in the cited reference.

The  $v$ -adic tropicalization map allows to consider the following fundamental construction. Set

$$\mathcal{B}_v := \begin{cases} (\mathbb{S}^1)^n & \text{if } v \text{ is archimedean} \\ \{1\} & \text{otherwise} \end{cases}. \quad (3.8)$$

It is a compact group. There exists an embedding  $\iota_v : N_{\mathbb{R}} \times \mathcal{B}_v \rightarrow \mathbb{T}_v^{\text{an}}$ , fitting in the commutative diagram

$$\begin{array}{ccc} N_{\mathbb{R}} \times \mathcal{B}_v & \xrightarrow{\iota_v} & \mathbb{T}_v^{\text{an}} \\ \downarrow & \swarrow \text{trop}_v & \\ N_{\mathbb{R}} & & \end{array} \quad (3.9)$$

with the vertical arrow being the projection onto the first factor. In the archimedean case, it is determined by the choice of a homeomorphism  $(\mathbb{C}^*)^n \simeq N_{\mathbb{R}} \times (\mathbb{S}^1)^n$ , while in the non-archimedean case it coincides with the map  $(u, 1) \mapsto \kappa_v(u)$ . The image of  $\iota_v$  is homeomorphic to  $N_{\mathbb{R}} \times \mathcal{B}_v$  and it is a deformation retract of the analytic torus, coinciding with it if  $v$  is archimedean.

**AMOEBAS.** It can be seen that for any closed subvariety  $Y$  of  $\mathbb{T}$  and for every  $v \in \mathfrak{M}$ , there is a natural analytic closed embedding corresponding to an inclusion of sets  $Y_v^{\text{an}} \subseteq \mathbb{T}_v^{\text{an}}$ , making the following definition meaningful.

**Definition 3.4.6.** Let  $f$  be a non zero Laurent polynomial in  $\mathbb{K}[M]$  and  $V(f)$  the associated closed subvariety of  $\mathbb{T}$ . For  $v \in \mathfrak{M}$ , the subset

$$\mathcal{A}_{f,v} := \text{trop}_v(V(f)_v^{\text{an}}) \subseteq N_{\mathbb{R}}$$

is called the  $v$ -adic amoeba of  $f$ .

The so-defined set has been widely studied in the literature. When  $v$  is archimedean, it coincides (up to a change of sign) with the notion of amoeba studied by Gelfand, Kapranov and Zelevinsky in [GKZ08] and by Passare and Rullgård in [PR04].

In the non-archimedean case, the amoeba of a non zero Laurent polynomial  $f$  coincides with the corner locus of the associated tropical polynomial (see [EKL06, Theorem 2.1.1]), and hence can be weighted to be a balanced polyhedral complex of pure dimension  $n - 1$ . It exhibits moreover a duality with the subdivision of the Newton polytope of  $f$  coming from the valuation of its coefficients, as shown in [EKL06, Corollary 2.1.2].

**ARAKELOV TORIC GEOMETRY.** Let now consider the proper toric variety  $X_{\Sigma}$  and let  $\mathbb{T}$  be its torus, with lattice of one-parameter subgroups  $N$ . As usual, let  $M$  be the dual of  $N$  and  $N_{\mathbb{R}}$  and  $M_{\mathbb{R}}$  the associated real vector spaces. Let  $D_{\Psi}$  be the toric divisor on  $X_{\Sigma}$  associated virtual support function  $\Psi$  on  $\Sigma$  as in Theorem 3.1.9. The metrics on  $\mathcal{O}(D_{\Psi})$  admitting a combinatorial description

are the ones which are invariant under the action of a certain compact torus, see [BPS14, §4.2] for more details about this notion. In concrete terms, a  $v$ -adic metric  $\|\cdot\|_v$  on  $\mathcal{O}(D_\Psi)_v^{\text{an}}$  is called a  $v$ -adic toric metric if the map

$$(X_0)_v^{\text{an}} \rightarrow \mathbb{R}, \quad p \mapsto \|s_\Psi(p)\|_v$$

is constant along the fibers of the  $v$ -adic tropicalization map  $\text{trop}_v : (X_0)_v^{\text{an}} \rightarrow N_{\mathbb{R}}$  introduced in Definition 3.4.2. A toric divisor  $D$  together with a  $v$ -adic toric metric on  $\mathcal{O}(D)$  is called a  $v$ -adic toric metrized divisor. To a  $v$ -adic toric metrized divisor  $\overline{D}_v$  one can associate the real-valued map  $\psi_{\overline{D}_v}$  on  $N_{\mathbb{R}}$  satisfying the equality

$$\psi_{\overline{D}_v} \circ \text{trop}_v = \log \|s_D\|_v \tag{3.10}$$

on the analytic torus  $(X_0)_v^{\text{an}}$ ,  $s_D$  being the distinguished rational section of  $\mathcal{O}(D)$ .

The map  $\psi_{\overline{D}_v}$ , which will be referred to as the *metric function* of  $\overline{D}_v$ , has been introduced by Burgos Gil, Philippon and Sombra in their study of Arakelov geometry of toric varieties to encode many arithmetic properties of  $\overline{D}_v$ , see [BPS14, Chapter 4]. For instance, it is smooth in the archimedean case if the metric is smooth, while in the non-archimedean setting it is rational piecewise affine if the metric is algebraic, see [BPS14, Theorem 4.5.10 (1)] and [GH17, Proposition 2.5.5]. Also, the semipositivity of  $\overline{D}_v$  is translated into the concavity of its corresponding metric function.

**Theorem 3.4.7.** *Let  $D$  be the toric divisor associated to the virtual support function  $\Psi$ . The assignment  $\|\cdot\|_v \mapsto \psi_{\overline{D}_v}$  is a bijection between the space of  $v$ -adic semipositive toric metrics on  $\mathcal{O}(D)_v^{\text{an}}$  and the space of concave functions  $\psi$  on  $N_{\mathbb{R}}$  such that  $|\psi - \Psi|$  is bounded.*

*Proof.* This is [BPS14, Theorem 4.8.1 (1)]. The extension to the general non-archimedean case is [GH17, Theorem 2.5.8].  $\square$

If  $\overline{D}_v$  is a  $v$ -adic semipositive toric metrized divisor, the Legendre-Fenchel dual of the metric function of  $\overline{D}_v$  is called the *roof function* of  $\overline{D}_v$  and denoted by  $\vartheta_{\overline{D}_v}$ : it is a concave function on  $M_{\mathbb{R}}$  with effective domain the polytope  $\Delta_\Psi$ . The correspondance between  $v$ -adic semipositive toric metrics and concave functions is moreover well-behave with respect to sums, as follows.

**Proposition 3.4.8.** *Let  $v \in \mathfrak{M}$  and  $\overline{D}_1, \overline{D}_2$  two  $v$ -adic semipositive toric metrized divisors, with metric functions  $\psi_1, \psi_2$  respectively and roof functions  $\vartheta_1, \vartheta_2$  respectively. Then,  $\overline{D}_1 + \overline{D}_2$  is a semipositive toric metrized divisor with metric function  $\psi_1 + \psi_2$  and roof function  $\vartheta_1 \boxplus \vartheta_2$ .*

*Proof.* The first statement is [BPS14, Proposition 4.3.14 (1)]. The second one follows from it by Proposition 1.3.18.  $\square$

A toric divisor admits a  $v$ -adic semipositive metric if and only if it is generated by global sections, as proved in [BPS14, Corollary 4.8.5]. For such divisors, moreover, there exists a distinguished choice of a  $v$ -adic semipositive metric.

**Definition 3.4.9.** Let  $D$  be a toric divisor generated by global sections,  $\Psi$  its associated virtual support function. The  *$v$ -adic canonical metric* on  $D$  is the semipositive toric metric on  $\mathcal{O}(D)_v^{\text{an}}$  corresponding to  $\Psi$  in the bijection of Theorem 3.4.7.

In the non-archimedean case, the canonical metric on  $D$  coincides with the algebraic metric induced by the canonical model of  $X_\Sigma$  and  $D$ , see [BPS14, Example 4.5.4].

For semipositive  $v$ -adic toric metrized divisors, the measure in (2.7) can be expressed in terms of the associated metric functions. To do this, recall that there exists an embedding

$$\iota_v : N_{\mathbb{R}} \times \mathcal{B}_v \rightarrow X_{0,v}^{\text{an}}$$

which fits into the commutative diagram (3.9), and denote by  $\text{Haar}_{\mathcal{B}_v}$  the Haar measure on  $\mathcal{B}_v$  normalized to have total mass 1.

**Theorem 3.4.10.** For  $i = 0, \dots, n-1$ , let  $\overline{D}_{i,v}$  be a semipositive  $v$ -adic toric metrized divisor on  $X_\Sigma$ ,  $\Psi_i$  the virtual support function associated to  $D_i$  and  $\psi_{i,v}$  the metric function of  $\overline{D}_{i,v}$ . Then, the positive measure

$$c_1(\overline{D}_{0,v}) \wedge \cdots \wedge c_1(\overline{D}_{n-1,v}) \wedge \delta_{X_{\Sigma,v}^{\text{an}}}$$

is zero outside  $X_{0,v}^{\text{an}}$  and

$$\begin{aligned} c_1(\overline{D}_{0,v}) \wedge \cdots \wedge c_1(\overline{D}_{n-1,v}) \wedge \delta_{X_{\Sigma,v}^{\text{an}}} |_{X_{0,v}^{\text{an}}} \\ = (\iota_v)_* (\text{MM}_M(\psi_{0,v}, \dots, \psi_{n-1,v}) \times \text{Haar}_{\mathcal{B}_v}). \end{aligned}$$

In particular,

$$\begin{aligned} (\text{trop}_v)_* (c_1(\overline{D}_{0,v}) \wedge \cdots \wedge c_1(\overline{D}_{n-1,v}) \wedge \delta_{X_{\Sigma,v}^{\text{an}}} |_{X_{0,v}^{\text{an}}}) \\ = \text{MM}_M(\psi_{0,v}, \dots, \psi_{n-1,v}) \end{aligned}$$

as measures on  $N_{\mathbb{R}}$ .

*Proof.* The first statement follows from [BPS14, Theorem 1.4.10 (1)] and [GH17, Corollary 1.4.5]. The expression for the measure in the archimedean and the discrete non-archimedean case is obtained from [BPS14, Theorem 4.8.11] and multilinearity; the general non-archimedean case is deduced from [GH17, Theorem 2.5.10]. The last assertion is an easy consequence of the commutativity of the diagram (3.9).  $\square$

Moving to the global case, a *semipositive toric metric* on a toric divisor  $D$  is a choice, for each place  $v \in \mathfrak{M}$ , of a semipositive  $v$ -adic toric metric on the line bundle  $\mathcal{O}(D)$ . The toric divisor  $D$  together with a semipositive toric metric is called a *semipositive toric metrized divisor* and it is denoted by  $\overline{D}$ . From the point of view of convex geometry, the semipositive toric metrized divisor  $\overline{D}$  is completely described by the collection  $(\psi_v)_{v \in \mathfrak{M}}$  of its metric functions or, equivalently, by the collection  $(\vartheta_v)_{v \in \mathfrak{M}}$  of its roof functions. A notion of well-behaving toric metrics was defined in [BPS14, Definition 4.9.1].

**Definition 3.4.11.** A toric metric  $(\|\cdot\|_v)_{v \in \mathfrak{M}}$  on a toric divisor is said to be *adelic* if for all but finitely many  $v \in \mathfrak{M}$  the  $v$ -adic toric metric  $\|\cdot\|_v$  is the canonical one, in the sense of Definition 3.4.9.

In convex terms, a toric metric on the toric divisor  $D$  associated to the virtual support function  $\Psi$  is adelic if and only if the family  $(\psi_v)_v$  of its metric functions satisfies  $\psi_v = \Psi$  for all but finitely many  $v \in \mathfrak{M}$ . It follows from [BPS14, Theorem 5.2.4] that any toric subvariety of  $X_\Sigma$  is integrable with respect to the choice of adelic semipositive toric metrized divisors. In particular, one can compute the global height of the  $n$ -dimensional cycle  $X_\Sigma$  with respect to such choices.

**Theorem 3.4.12.** *Let  $\overline{D}_0, \dots, \overline{D}_n$  be toric divisors over  $X_\Sigma$ , equipped with adelic semipositive toric metrics. Then*

$$h_{\overline{D}_0, \dots, \overline{D}_n}(X_\Sigma) = \sum_{v \in \mathfrak{M}} n_v \text{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n,v}),$$

where  $\vartheta_{i,v}$  is the roof function of  $\overline{D}_{i,v}$ , for every  $i = 0, \dots, n$  and  $v \in \mathfrak{M}$ .

*Proof.* This is [BPS14, Theorem 5.2.5].  $\square$

**TORIC LOCAL HEIGHTS.** Recall from Definition 3.4.9 that any toric divisor generated by global sections admits a distinguished  $v$ -adic semipositive toric metric, the canonical metric. This allows to define a local height with respect

to toric divisors that is independent of the choice of the sections. Such a notion was introduced in [BPS14, §5.1] as a key step in the proof of the formula for the global height of a toric variety.

**Definition 3.4.13.** For a place  $v$  of  $\mathbb{K}$ , let  $\bar{D}_0, \dots, \bar{D}_d$  be toric divisors on  $X_\Sigma$ , endowed with  $v$ -adic semipositive toric metrics. Denote by  $\bar{D}_0^{\text{can}}, \dots, \bar{D}_d^{\text{can}}$  the same divisors equipped with their  $v$ -adic canonical metric. Let  $Y$  be an irreducible  $d$ -dimensional subvariety of  $X_\Sigma$  and  $\varphi : Y' \rightarrow Y$  a birational morphism, with  $Y'$  projective. The  $v$ -adic toric local height of  $Y$  with respect to  $\bar{D}_0, \dots, \bar{D}_d$  is defined as

$$h_{\bar{D}_0, \dots, \bar{D}_d}^{\text{tor}}(Y) := h_{\varphi^* \bar{D}_0, \dots, \varphi^* \bar{D}_d}(Y'; s_0, \dots, s_d) - h_{\varphi^* \bar{D}_0^{\text{can}}, \dots, \varphi^* \bar{D}_d^{\text{can}}}(Y'; s_0, \dots, s_d),$$

where  $s_i$  is a rational section of  $\varphi^* \mathcal{O}(D_i)$ , for every  $i = 0, \dots, d$  and  $s_0, \dots, s_d$  meet  $Y'$  properly. The definition extends by linearity to any cycle of dimension  $d$ .

The toric local height of a cycle does neither depend on the choice of the sections  $s_0, \dots, s_d$ , nor on the birational model  $Y'$  of  $Y$  because of [BPS14, Theorem 1.4.17 (2) and (3)]. Moreover, the definition is nonempty: Chow's lemma provides  $Y$  with a projective birational model, while the moving lemma assures the existence of rational sections meeting  $Y'$  properly.

### 3.5 INTEGRABILITY

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Keeping the notations of the previous section, we extend the integrability statement of toric cycles in a toric variety in [BPS14, Proposition 5.2.4] with respect to adelic semipositive toric metrized divisor to all cycles in  $X_\Sigma$ .

**THE MULTIPROJECTIVE CASE.** The first step of the extension consists in proving the integrability of cycles in a very particular case. Let  $d \in \mathbb{N}$  and let  $n_0, \dots, n_d \in \mathbb{N}_{\geq 1}$ . Consider the toric variety

$$\mathbb{P} := \mathbb{P}_{\mathbb{K}}^{n_0} \times \dots \times \mathbb{P}_{\mathbb{K}}^{n_d}.$$

For every  $i = 0, \dots, d$  denote by  $\bar{H}_i^{\text{can}}$  the divisor at infinity of  $\mathbb{P}_{\mathbb{K}}^{n_i}$  endowed with the canonical metric at each place. It gives, by pulling-back via the projection  $\pi_i : \mathbb{P} \rightarrow \mathbb{P}_{\mathbb{K}}^{n_i}$ , a metrized divisor  $\bar{G}_i$  on  $\mathbb{P}$ .

**Proposition 3.5.1.** *In the above hypotheses and notations, every  $d$ -dimensional cycle of  $\mathbb{P}$  is integrable with respect to  $\bar{G}_0, \dots, \bar{G}_d$ .*

*Proof.* Let  $Z$  be a  $d$ -dimensional cycle in  $\mathbb{P}$ . By linearity, we can assume that  $Z$  is a prime cycle, that is an irreducible subvariety. By the moving lemma, one can choose  $s_0, \dots, s_d$ , sections of  $G_0, \dots, G_d$  respectively, intersecting  $Z$  properly. The section  $s_i$  can be thought as an hyperplane in  $\mathbb{P}^{n_i}$ ; the corresponding dual point is  $\mathbf{s}_i := [s_{0,i} : \dots : s_{n_i,i}]$ .

Because of [Gub97, Example 1.3], for a non-archimedean  $v \in \mathfrak{M}$ , the  $v$ -adic local height of  $Z$  with can be written as

$$h_{\overline{G_0, v}, \dots, \overline{G_d, v}}(Z; s_0, \dots, s_d) = \log |\mathrm{Ch}_Z|_v - \log |\mathrm{Ch}_Z(\mathbf{s}_0, \dots, \mathbf{s}_d)|_v$$

where  $\mathrm{Ch}_Z$  is the Chow form associated to the irreducible subvariety  $Z$  and  $|\mathrm{Ch}_Z|_v$  is the  $v$ -adic Gauss norm of the multihomogeneous polynomial  $\mathrm{Ch}_Z$ . The definition of adelic field and Proposition 2.3.5 assures now that for almost all  $v$ , all coefficients of  $\mathrm{Ch}_Z$  and the element  $\mathrm{Ch}_Z(\mathbf{s}_0, \dots, \mathbf{s}_d)$  have  $v$ -adic absolute value equal to 1 and hence  $h_{\overline{G_0, v}, \dots, \overline{G_d, v}}(Z; s_0, \dots, s_d) = 0$ .  $\square$

**THE TORIC CASE.** The result of the previous subsection reveals in fact to be enough to the treat the general toric case. One starts by considering the situation in which all the divisors are equipped with their canonical metrics in the sense of Definition 3.4.9.

**Proposition 3.5.2.** *Let  $X_\Sigma$  be a proper toric variety over  $\mathbb{K}$  and let  $D_0, \dots, D_d$  be toric divisors on  $X$  generated by their global sections. Then, every  $d$ -dimensional cycle in  $X$  is integrable with respect to  $\overline{D_0}^{\mathrm{can}}, \dots, \overline{D_d}^{\mathrm{can}}$ .*

*Proof.* Since the divisors are toric divisor generated by their global sections, there exist toric morphisms  $\varphi_0, \dots, \varphi_d$ , with  $\varphi_i : X \rightarrow \mathbb{P}_{\mathbb{K}}^{n_i}$  such that  $\mathcal{O}(D_i) \simeq \varphi_i^*(\mathcal{O}(1))$  for each  $i = 0, \dots, d$  and certain  $n_i \in \mathbb{N}$ . On the product variety

$$\mathbb{P} := \mathbb{P}_{\mathbb{K}}^{n_0} \times \dots \times \mathbb{P}_{\mathbb{K}}^{n_d}$$

consider for each  $i = 0, \dots, d$  the line bundle  $\mathcal{O}(e_i) := \pi_i^*(\mathcal{O}(1))$ , where  $\pi_i : \mathbb{P} \rightarrow \mathbb{P}_{\mathbb{K}}^{n_i}$  is the projection on the  $i$ -th factor. The line bundle  $\mathcal{O}(e_i)$  can be endowed with the pull-back metric of the canonical metric for each  $i$ . The morphism

$$\varphi := (\varphi_0, \dots, \varphi_d) : X \rightarrow \mathbb{P}$$

is such that

$$\varphi^* \overline{\mathcal{O}(e_i)} = (\pi_i \circ \varphi)^* \overline{\mathcal{O}(1)} = \varphi_i^* \overline{\mathcal{O}(1)} \simeq \overline{\mathcal{O}(D_i)}^{\mathrm{can}}$$

since the morphism  $\varphi_i$  is toric and the pull back of the canonical metric is the canonical metric. Let  $Z$  be a cycle of  $X$  of dimension  $d$ . By [BPS14,

Proposition 1.5.8 (2)] and Proposition 3.5.1 one has that  $Z$  is integrable with respect to  $\overline{D}_0^{\text{can}}, \dots, \overline{D}_d^{\text{can}}$ .  $\square$

Combined with the following easy lemma, the previous proposition yields the main result of the section.

**Lemma 3.5.3.** *Let  $X$  be a proper variety over  $\mathbb{K}$  and let  $D_0, \dots, D_d$  be divisors on  $X$ . For all  $i = 0, \dots, d$  let  $\overline{D}_i = (D_i, (\|\cdot\|_{i,v})_{v \in \mathfrak{M}})$  and  $\overline{D}'_i = (D_i, (\|\cdot\|'_{i,v})_{v \in \mathfrak{M}})$  be two semipositive metrized divisors. Assume that for all but finitely many  $v \in \mathfrak{M}$  one has  $\|\cdot\|_{i,v} = \|\cdot\|'_{i,v}$  for every  $i = 0, \dots, d$ . Then, a cycle  $Z$  of dimension  $d$  in  $X$  is integrable with respect to  $\overline{D}_0, \dots, \overline{D}_d$  if and only if it is integrable with respect to  $\overline{D}'_0, \dots, \overline{D}'_d$ .*

*Proof.* The statement follows immediately from the definition of integrable cycle and the fact that the  $v$ -adic local heights with respect to  $\overline{D}_0, \dots, \overline{D}_d$  and to  $\overline{D}'_0, \dots, \overline{D}'_d$  coincide for almost every  $v \in \mathfrak{M}$  by hypothesis.  $\square$

**Theorem 3.5.4.** *Let  $X_\Sigma$  be a proper toric variety over  $\mathbb{K}$  and let  $\overline{D}_0, \dots, \overline{D}_d$  be adelic semipositive toric metrized divisors on  $X$ . Then, every  $d$ -dimensional cycle in  $X$  is integrable with respect to  $\overline{D}_0, \dots, \overline{D}_d$ .*

*Proof.* By the characterization of semipositive toric metrics given in Theorem 3.4.7, the divisors  $D_0, \dots, D_d$  are generated by their global sections. Moreover, the adelicity assumption is equivalent to the fact that the local metrics differ from the canonical ones at finitely many places. The statement follows then from Lemma 3.5.3 and Proposition 3.5.2.  $\square$

**Remark 3.5.5.** Theorem 3.5.4 extends both [BPS14, Proposition 5.2.4] (where the integrability was proved for toric orbits) and the application of [BPS14, Proposition 1.5.14] (which only holds for global fields) to the toric case. Moreover, it answers positively [MS16, Question 4.16] in the case of adelic semipositive toric metrized divisors on a toric variety.



# Heights of hypersurfaces

We prove in this chapter the announced formula for the global height of a cycle of codimension 1 in a toric variety with respect to a choice of adelic semipositively metrized toric divisors. To do it, we first define and study local Ronkin functions associated to a Laurent polynomial. Their value at 0 play the role of local Mahler measures of the polynomial and appear in the expression of lower and upper bounds for the height. Finally, we present examples in which the formula can be applied.

We supposed fixed for the whole chapter an adelic field  $(\mathbb{K}, (|\cdot|_v, n_v)_{v \in \mathfrak{M}})$  satisfying the product formula and a proper toric variety  $X_\Sigma$  over  $\mathbb{K}$  of dimension  $n$ , with dense open orbit  $X_0$ . We denote by  $\mathbb{T}$  the corresponding torus, by  $M$  and  $N$  respectively its character and cocharacter lattice and by  $M_{\mathbb{R}}$  and  $N_{\mathbb{R}}$  the associated real vector spaces.

## 4.1

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### RONKIN FUNCTIONS

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We here associate to any Laurent polynomial in  $\mathbb{K}[M]$  and place of  $\mathbb{K}$  a concave function, called the Ronkin function of the polynomial, which will be a key ingredient in the formula for the height of an hypersurface in a toric variety. We then present a number of its properties and express the maximum and minimum of its Legendre-Fenchel dual.

**SHILOV BOUNDARIES.** To keep the treatment of the archimedean and non-archimedean settings uniform, the following functional analytic notion results

to be crucial.

**Definition 4.1.1.** Let  $K$  be an algebraically closed complete field as in Definition 2.1.18 and  $A$  a  $K$ -algebra. For any set  $\mathcal{S}$  of multiplicative seminorms on  $A$  extending the absolute value of  $K$ , a *boundary* of  $\mathcal{S}$  is a subset  $\mathcal{B} \subseteq \mathcal{S}$  such that

$$\max_{x \in \mathcal{S}} \|a\|_x = \max_{x \in \mathcal{B}} \|a\|_x$$

for every  $a \in A$ .

In other words, any boundary of  $\mathcal{S}$  contains the whole information about the maximal values that the seminorms in  $\mathcal{S}$  can attain. As a trivial example,  $\mathcal{S}$  is a boundary of  $\mathcal{S}$ .

**Remark 4.1.2.** Differently from the classical notion of boundary of a Banach algebra, one here allows to define the boundary of an arbitrary subset of the whole structure space.

For any place  $v \in \mathfrak{M}$ , one is here especially interested in the boundary of the fibers of the  $v$ -adic tropicalization map

$$\text{trop}_v : \mathbb{T}_v^{\text{an}} \rightarrow N_{\mathbb{R}}$$

introduced in Definition 3.4.2. It follows immediately from the definitions that for every  $u \in N_{\mathbb{R}}$

$$\text{trop}_v^{-1}(u) = \left\{ \|\cdot\|_x \in \mathbb{T}_v^{\text{an}} : \|\chi^m\|_x = e^{-\langle m, u \rangle} \text{ for all } m \in M \right\}.$$

**Remark 4.1.3.** If  $v$  is an archimedean place of  $\mathbb{K}$ , it corresponds to an embedding  $\sigma : \mathbb{K} \hookrightarrow \mathbb{C}$  up to complex conjugation because of Proposition 2.1.21, then the analytic torus  $\mathbb{T}_v^{\text{an}}$  can be identified with the  $\mathbb{C}$ -points of  $\mathbb{T}_{\mathbb{C}}$  as in Remark 3.4.1. The choice of a basis of  $M$  and of the dual basis of  $N$  identifies hence  $\mathbb{T}_v^{\text{an}}$  with  $(\mathbb{C}^*)^n$  and  $N_{\mathbb{R}}$  with  $\mathbb{R}^n$ . Under such identifications, it follows directly from the definition that

$$\begin{aligned} \text{trop}_v^{-1}(u) &= \{(z_1, \dots, z_n) \in (\mathbb{C}^*)^n : |z_i|_{\infty} = e^{-u_i} \text{ for } i = 1, \dots, n\} \\ &= \{(e^{-u_1 + i\theta_1}, \dots, e^{-u_n + i\theta_n}) : \theta_1, \dots, \theta_n \in [0, 2\pi)\} \end{aligned}$$

for every  $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ .

In the complex case, the existence of a unique minimal boundary for an algebra of functions on a compact space has been proved by Shilov. The following result, which equally holds in the non-archimedean case, is well-known by experts.

**Proposition 4.1.4.** *For every  $u \in N_{\mathbb{R}}$ , the set  $\text{trop}_v^{-1}(u)$  has a unique minimal boundary. In the archimedean case, it coincides with the whole fiber, while in the non-archimedean case it consists of the Gauss point over  $u$ , as in (3.7).*

*Proof.* Suppose first that  $v$  is an archimedean place of  $\mathbb{K}$  (corresponding to an embedding  $\sigma$  of  $\mathbb{K}$  in  $\mathbb{C}$ ) and fix the choice of a basis of  $M$ . One can identify  $\mathbb{T}_v^{\text{an}}$  with  $(\mathbb{C}^*)^n$  by considering for every  $z = (z_1, \dots, z_n) \in (\mathbb{C}^*)^n$  the seminorm  $\|\cdot\|_z$  defined as

$$\|f\|_z = |f^\sigma(z)|_\infty$$

for every  $f \in \mathbb{C}_v[T_1, \dots, T_n]$ , as in Remark 3.4.1. Assuming that the coordinates of  $u$  in the basis dual to the chosen one are  $(u_1, \dots, u_n)$ , a point  $\tilde{z} \in \text{trop}_v^{-1}(u)$  is of the form

$$\tilde{z} := (e^{-u_1+i\theta_1}, \dots, e^{-u_n+i\theta_n}),$$

with  $\theta_i \in [0, 2\pi)$  for every  $i = 1, \dots, n$ , because of Remark 4.1.3. It is easy to check that such a point is the only one in  $\text{trop}_v^{-1}(u)$  for which

$$\left\| \left( T_1 + \sigma^{-1}(e^{-u_1+i\theta_1}) \right) \dots \left( T_n + \sigma^{-1}(e^{-u_n+i\theta_n}) \right) \right\|_z$$

is maximal. As a consequence,  $\tilde{z}$  must belong to any boundary of  $\text{trop}_v^{-1}(u)$ . This proves that the only boundary of  $\text{trop}_v^{-1}(u)$  is the set itself.

Suppose now that  $v$  is a non-archimedean place of  $\mathbb{K}$ . Then, for every  $\|\cdot\|_x \in \text{trop}_v^{-1}(u)$  and for every  $f = \sum c_m \chi^m \in \mathbb{C}_v[M]$  one has by the triangular inequality

$$\|f\|_x \leq \max_m |c_m|_v \|\chi^m\|_x = \max_m |c_m|_v e^{-\langle m, u \rangle}.$$

This trivially implies that the Gauss norm  $\|f\|_{\kappa_v(u)} = \max_m |c_m|_v e^{-\langle m, u \rangle}$  is a boundary for  $\text{trop}_v^{-1}(u)$ .  $\square$

For  $u \in N_{\mathbb{R}}$ , one will denote the unique minimal boundary of  $\text{trop}_v^{-1}(u)$  described in the previous proposition by  $\mathcal{B}_v(u)$ . If not otherwise mentioned, such a set is considered to be endowed with the topology induced from  $\mathbb{T}_v^{\text{an}}$ . Recalling the definition of the compact group  $\mathcal{B}_v$  in (3.8), the boundary  $\mathcal{B}_v(u)$  is hence homeomorphic to  $\mathcal{B}_v$  for every  $u \in N_{\mathbb{R}}$ . This allows to define the measure

$$\sigma_{u,v} := \text{Haar}_{\mathcal{B}_v(u)} \tag{4.1}$$

on  $\text{trop}_v^{-1}(u)$ , which is the Haar measure on the compact group  $\mathcal{B}_v(u)$  normalized to have total mass 1; it is a finite measure on  $\text{trop}_v^{-1}(u)$ , supported on  $\mathcal{B}_v(u)$  and distributing homogeneously on this set. In the non-archimedean case it coincides with the Dirac delta at the Gauss point over  $u$ .

**DEFINITIONS AND FIRST PROPERTIES.** The terminology introduced in the previous subsection allows to give a definition of the  $v$ -adic Ronkin function of a Laurent polynomial independently of the nature of the place  $v$ .

**Definition 4.1.5.** Let  $f$  be a nonzero Laurent polynomial over  $\mathbb{K}$ . For a place  $v \in \mathfrak{M}$ , the  $v$ -adic Ronkin function of  $f$  is the map  $\rho_{f,v} : N_{\mathbb{R}} \rightarrow \mathbb{R}$  defined as

$$\rho_{f,v}(u) := \int_{\text{trop}_v^{-1}(u)} -\log \|f\|_x d\sigma_{u,v}(x)$$

for every  $u \in N_{\mathbb{R}}$ , with  $\sigma_{u,v}$  as in (4.1).

The integral in the previous definition is finite. Indeed, logarithmic singularities are integrable in the archimedean case, and the Gauss norm of a nonzero Laurent polynomial is positive in the non-archimedean case. The following two remarks give a more explicit expression for the  $v$ -adic Ronkin function of a Laurent polynomial  $f = \sum_m c_m \chi^m \in \mathbb{K}[M]$ .

**Remark 4.1.6.** Assume that  $v$  is an archimedean place of  $\mathbb{K}$  and let  $\sigma$  be the corresponding embedding of  $\mathbb{K}$  in  $\mathbb{C}$ , up to complex conjugation. Fixing the choice of a basis of  $M$  and of the dual basis of  $N$ , one identifies  $\mathbb{T}_v^{\text{an}}$  with  $(\mathbb{C}^*)^n$  as in Remark 3.4.1. Then, for every  $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ ,

$$\begin{aligned} \rho_{f,v}(u) &= \int_{\text{trop}_v^{-1}(u)} -\log \|f\|_z d\sigma_{v,u}(z) \\ &= \int_{\text{trop}_v^{-1}(u)} -\log |f^\sigma(z)|_\infty d\text{Haar}_{\text{trop}_v^{-1}(u)}(z) \\ &= -\frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \log \left| f^\sigma(e^{-u_1+i\theta_1}, \dots, e^{-u_n+i\theta_n}) \right|_\infty d\theta_1 \dots d\theta_n, \end{aligned}$$

where  $f^\sigma = \sum_m \sigma(c_m) \chi^m \in \mathbb{C}[M]$ . In particular, one has that the equality  $\rho_{f,v}(u) = -N_{f^\sigma}(-u)$  for every  $u \in N_{\mathbb{R}}$ , where  $N_f$  is the classical Ronkin function associated to a Laurent polynomial with complex coefficients, see for instance [PR04, §2].

**Remark 4.1.7.** If  $v$  is a non-archimedean place of  $\mathbb{K}$ , it is easily checked that

$$\rho_{f,v}(u) = \min_m (\langle m, u \rangle - \log |c_m|_v)$$

for every  $u \in N_{\mathbb{R}}$ . Such a function on  $N_{\mathbb{R}}$  is also known as the  $v$ -adic tropicalization of  $f$  and denoted by  $f^{\text{trop},v}$ , see for instance [MS15, equality (2.4.1)].

The following property of the Ronkin function follows immediately from its definition.

**Proposition 4.1.8.** *For every nonzero Laurent polynomials  $f, g \in \mathbb{K}[M]$ ,  $v \in \mathfrak{M}$  and  $c \in \mathbb{K}^*$ , one has  $\rho_{f \cdot g, v} = \rho_{f, v} + \rho_{g, v}$  and  $\rho_{cf, v} = -\log |c|_v + \rho_{f, v}$ .*

Recall that from Definition 3.2.1 one can associate to a Laurent polynomial  $f$  its Newton polytope  $\text{NP}(f)$ . It reveals to be useful to consider the support function  $\Psi_{\text{NP}(f)}$  of  $\text{NP}(f)$ , as in Remark 1.2.5; it is a concave function on  $N_{\mathbb{R}}$ , with domain  $N_{\mathbb{R}}$ , stability set  $\text{NP}(f)$  and Legendre-Fenchel dual the indicator function of  $\text{NP}(f)$  because of Example 1.3.8.

**Proposition 4.1.9.** *Let  $f$  be a nonzero Laurent polynomial over  $\mathbb{K}$  and  $v \in \mathfrak{M}$ . Then:*

1.  $\rho_{f, v}$  is a continuous concave function on  $N_{\mathbb{R}}$  (in particular it is closed) and it is affine on each connected component of the complement of the  $v$ -adic amoeba of  $f$  (see Definition 3.4.6)
2.  $|\rho_{f, v} - \Psi_{\text{NP}(f)}|$  is bounded on  $N_{\mathbb{R}}$
3. the stability set of  $\rho_{f, v}$  coincides with  $\text{NP}(f)$  and  $\text{rec}(\rho_{f, v}) = \Psi_{\text{NP}(f)}$ .

*Proof.* Thanks to Remark 4.1.7, the statements in (1) are trivial if  $v$  is a non-archimedean place of  $\mathbb{K}$ . Otherwise, the concavity of  $\rho_{f, v}$  and its affinity outside the  $v$ -adic amoeba follow from Remark 4.1.6 and [PR04, Theorem 1]. As a consequence of the concavity,  $\rho_{f, v}$  is continuous on  $N_{\mathbb{R}}$ , hence closed. To prove (2), suppose that  $f = \sum_m c_m \chi^m$  and let

$$\gamma_v(f) := \begin{cases} \#\{m \in M : c_m \neq 0\} & \text{if } v \text{ is archimedean} \\ 1 & \text{otherwise} \end{cases}.$$

For any  $u \in N_{\mathbb{R}}$  and  $x \in \text{trop}_v^{-1}(u)$ , the inequality

$$\|f\|_x \leq \gamma_v(f) \cdot \max_m (|c_m|_v \|\chi^m\|_x)$$

implies that

$$\begin{aligned} -\log \|f\|_x &\geq \min_m (\langle m, u \rangle - \log |c_m|_v) - \log \gamma_v(f) \\ &\geq \Psi_{\text{NP}(f)}(u) - \max_m \log |c_m|_v - \log \gamma_v(f) \end{aligned}$$

and hence

$$\rho_{f, v}(u) \geq \Psi_{\text{NP}(f)}(u) - \log \left( \gamma_v(f) \max_m |c_m|_v \right)$$

for every  $u \in N_{\mathbb{R}}$ . For a reverse inequality, denote by  $\mathcal{V}(f)$  the set of vertices of  $\text{NP}(f)$ . Then, for every  $m \in \mathcal{V}(f)$  the Ronkin function of  $f$  coincides with  $\langle m, u \rangle - \log |c_m|_v$  in a nonempty open subset of  $N_{\mathbb{R}}$  (this follows from [PR04, Proposition 2] for archimedean places and from [EKL06, Corollary 2.1.2] for non-archimedean ones). By the concavity of  $\rho_{f,v}$ , one deduces hence that

$$\rho_{f,v}(u) \leq \Psi_{\text{NP}(f)}(u) - \log \min_{m \in \mathcal{V}(f)} |c_m|_v,$$

for every  $u \in N_{\mathbb{R}}$ , concluding the proof of (2).

The statements in (3) follows directly from (2). Indeed, since  $|\rho_{f,v} - \Psi_{\text{NP}(f)}|$  is bounded,  $\text{stab}(\rho_{f,v}) = \text{stab}(\Psi_{\text{NP}(f)}) = \text{NP}(f)$ . The last equality follows then from Proposition 1.3.6.  $\square$

The calculation of the  $v$ -adic Ronkin function of a nonzero Laurent polynomial is typically very difficult. Anyway, an explicit expression for it is available in the following two simple situations.

**Example 4.1.10.** It follows from the definition that, for every  $v \in \mathfrak{M}$ , the  $v$ -adic Ronkin function of the monomial  $\chi^m$  coincides with the linear function  $m$  on  $N_{\mathbb{R}}$  for every  $m \in M$ .

**Example 4.1.11.** For any  $m, m' \in M$  with  $m \neq m'$  and for every  $v \in \mathfrak{M}$ , the  $v$ -adic Ronkin function of the binomial  $f = \chi^m - \chi^{m'}$  coincides with the support function of the segment  $\overline{mm'}$ , that is

$$\rho_{f,v}(u) = \min(\langle m, u \rangle, \langle m', u \rangle)$$

for every  $u \in N_{\mathbb{R}}$ . To prove this, remark first that one can restrict to the case of the binomial  $f = \chi^m - 1$  with  $m \neq 0$  because of Proposition 4.1.8, Example 4.1.10 and by factoring with a monomial. When  $v$  is non-archimedean, the statement follows immediately from Remark 4.1.7. Otherwise, after the choice of a basis of  $M$ , Remark 4.1.6 allows to write

$$\rho_{f,v}(u) = -\frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \log |e^{-m_1 u_1 + i m_1 \theta_1} \dots e^{-m_n u_n + i m_n \theta_n} - 1| d\theta_1 \dots d\theta_n,$$

with  $m_1, \dots, m_n$  being the coordinates of  $m$  in such a basis,  $u_1, \dots, u_n$  the coordinates of  $u$  in the dual one and  $|\cdot|$  denoting the euclidean absolute value on  $\mathbb{C}$ . Assuming that  $m_1 > 0$ , which is always possible since  $m \neq 0$ , Jensen's formula yields, for every  $\theta_2, \dots, \theta_n$ ,

$$\int_{\theta_1 \in [0,2\pi]} \log |e^{-m_1 u_1 + i m_1 \theta_1} \dots e^{-m_n u_n + i m_n \theta_n} - 1| d\theta_1 = -2\pi \sum_{j=1}^k \log \frac{|\alpha_j|}{e^{-m_1 u_1}} \quad (4.2)$$

with  $\alpha_1, \dots, \alpha_k$  being the zeros of the univariate polynomial

$$(e^{-m_2 u_2 + i m_2 \theta_2} \dots e^{-m_n u_n + i m_n \theta_n}) T - 1$$

lying inside the closed disk of radius  $e^{-m_1 u_1}$ , repeated according to multiplicity. The only complex zero of the above polynomial has modulus  $e^{m_2 u_2 + \dots + m_n u_n}$ ; the integral in (4.2) is then zero if  $m_1 u_1 + \dots + m_n u_n > 0$ , otherwise it equals  $-2\pi(m_1 u_1 + \dots + m_n u_n)$ . It follows that

$$\rho_{f,v}(u) = \min(m_1 u_1 + \dots + m_n u_n, 0),$$

hence the claim.

**LOCAL MAHLER MEASURES.** For a Laurent polynomial  $f$  in  $n$  variables and complex coefficients, one defines its (*logarithmic*) *Mahler measure* as

$$m(f) := \frac{1}{(2\pi)^n} \int_{\theta_1, \dots, \theta_n \in [0, 2\pi]} \log \left| f(e^{i\theta_1}, \dots, e^{i\theta_n}) \right| d\theta_1 \dots d\theta_n, \quad (4.3)$$

with  $|\cdot|$  denoting the usual absolute value on  $\mathbb{C}$ . Such a quantity is notoriously difficult to compute and is sometimes related to special values of  $L$ -functions, see [Smy81], [Den97], [Boy98] and [Lal08].

**Remark 4.1.12.** Let  $f = \sum_m c_m \chi^m$  be a nonzero Laurent polynomial in  $\mathbb{K}[M]$  and  $v$  an archimedean place of  $\mathbb{K}$ , associated to the embedding  $\sigma : \mathbb{K} \hookrightarrow \mathbb{C}$ . After the choice of a basis of  $M$ , Remark 4.1.6 asserts that

$$\rho_{f,v}(u) = -m \left( \sum_m \sigma(c_m) e^{-m_1 u_1 - \dots - m_n u_n} T_1^{m_1} \dots T_n^{m_n} \right)$$

for every  $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ .

The previous remark suggests to introduce the following local correspondent for the Mahler measure of a Laurent polynomial.

**Definition 4.1.13.** Let  $f$  be a nonzero Laurent polynomial in  $\mathbb{K}[M]$ . For every  $v \in \mathfrak{M}$ , the real number  $m_v(f) := -\rho_{f,v}(0)$  is called the *v-adic Mahler measure* of  $f$ .

The name given in Definition 4.1.13 is justified by the following remark.

**Remark 4.1.14.** Let  $f = \sum_m c_m \chi^m$  be a nonzero Laurent polynomial in  $\mathbb{K}[M]$ . If  $v$  is an archimedean place of  $\mathbb{K}$  corresponding to the embedding  $\sigma : K \hookrightarrow \mathbb{C}$ , it follows immediately from Remark 4.1.12 that

$$m_v(f) = \mathfrak{m}(f^\sigma),$$

where  $f^\sigma := \sum_m \sigma(c_m) \chi^m$ . If instead  $v$  is non-archimedean, Remark 4.1.7 yields the equality

$$m_v(f) = \log \max_m |c_m|_v.$$

In both cases, the  $v$ -adic Mahler measure of  $f$  introduced in Definition 4.1.13 is the logarithm of the ‘local measure’  $M_v(f)$  of  $f$  at  $v$  defined in [Phi86, page 19].

**Example 4.1.15.** Let  $\mathbb{K} = \mathbb{Q}$  and  $f$  a nonzero Laurent polynomial in  $\mathbb{Z}[M]$  which is *primitive*, that is the greatest common divisor of all its coefficients is 1. Then, Remark 4.1.14 implies that  $m_\infty(f) = \mathfrak{m}(f)$ , while  $m_v(f) = 0$  for all non-archimedean places  $v$ .

The notion of  $v$ -adic Mahler measure allows to express the extrema of the Legendre-Fenchel dual of the  $v$ -adic Ronkin function of  $f$  in a compact way.

**Proposition 4.1.16.** *Let  $f$  be a nonzero Laurent polynomial with coefficients in  $\mathbb{K}$ ,  $f = \sum_m c_m \chi^m$  and  $v \in \mathfrak{M}$ . Then,*

$$\max \rho_{f,v}^\vee = m_v(f)$$

and

$$\min \rho_{f,v}^\vee = \min_{m \in \mathcal{V}(f)} \log |c_m|_v$$

where  $\mathcal{V}(f)$  denotes the set of vertices of the Newton polytope of  $f$ .

*Proof.* The function  $\rho_{f,v}^\vee$  is a closed concave function with effective domain  $\text{NP}(f)$  because of Proposition 4.1.9 (3). Since it is closed, it is continuous on  $\text{NP}(f)$ . It follows hence from Proposition 1.3.7 and Definition 4.1.13 that

$$\max \rho_{f,v}^\vee = -\rho_{f,v}(0) = m_v(f).$$

Concerning the minimum, remark first that this has to be attained at a vertex of the Newton polytope of  $f$ . Indeed, if  $x \in \text{NP}(f)$  it can be written by Remark 1.2.7 as a convex combination  $x = \sum_{m \in \mathcal{V}(f)} \lambda_m m$  of the vertices of  $\text{NP}(f)$ ,

with  $\lambda_m \in [0, 1]$  for every  $m \in \mathcal{V}(f)$  and  $\sum_m \lambda_m = 1$ . The concavity of  $\rho_{f,v}^\vee$  imposes then that

$$\rho_{f,v}^\vee(x) \geq \sum_{m \in \mathcal{V}(f)} \lambda_m \rho_{f,v}^\vee(m) \geq \min_{m \in \mathcal{V}(f)} \rho_{f,v}^\vee(m) \sum_{m \in \mathcal{V}(f)} \lambda_m = \min_{m \in \mathcal{V}(f)} \rho_{f,v}^\vee(m).$$

The statement follows then from the fact that the value of the function  $\rho_{f,v}^\vee$  at a vertex  $m$  of the Newton polytope of  $f$  coincides with  $\log |c_m|_v$ . But, when  $v \in \mathfrak{M}$  is archimedean, this is a consequence of Remark 4.1.6 and [PR04, Theorem 2], otherwise it follows from Remark 4.1.7 and Proposition 1.3.9.  $\square$

## 4.2

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### LOCAL AND GLOBAL HEIGHTS OF HYPERSURFACES

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We present in this section combinatorial formulas for the degree, the local and global height of an effective cycle in  $X_\Sigma$  of pure codimension 1. When all the components of the cycle cut the open dense orbit of  $X_\Sigma$ , such formulas are given in terms of the Newton polytope and the Ronkin functions of its defining polynomial.

**THE SETTING.** One fixes here the terminology and notation adopted throughout the whole section. For an effective cycle  $Z$  of pure codimension 1 whose prime components intersect the dense open orbit  $X_0$  of  $X_\Sigma$  one has

$$Z = \sum_{i=1}^r \ell_i Y_i$$

for positive integers  $\ell_1, \dots, \ell_r$  and prime divisors  $Y_1, \dots, Y_r$  intersecting  $X_0$ . For every  $i = 1, \dots, r$ , the closed irreducible subvariety of  $X_0$  obtained as the intersection between  $Y_i$  and  $X_0$  is associated to a prime ideal of height one in  $\mathbb{K}[M]$ , which is principal since  $\mathbb{K}[M]$  is a unique factorization domain; denote by  $f_i$  an irreducible Laurent polynomial generating such an ideal. The Laurent polynomial  $f = f_1^{\ell_1} \cdots f_r^{\ell_r}$  is called a *defining polynomial* for the cycle  $Z$  and is uniquely defined up to multiplication by an invertible element of  $\mathbb{K}[M]$ , that means by a monomial. Moreover,

$$\left[ \overline{V(f)} \right] = Z, \tag{4.4}$$

that is the cycle associated to the closure of the subscheme  $V(f)$  in  $X_\Sigma$  agrees with  $Z$ , see [Ful98, §1.5]. Let

$$\Psi_f := \Psi_{\text{NP}(f)}$$

be the support function, in the sense of Remark 1.2.5, of the Newton polytope  $\text{NP}(f)$  of  $f$ ; it is a piecewise linear function with effective domain  $N_{\mathbb{R}}$ . It is not necessarily a virtual support function on the fan  $\Sigma$ ; when it is, it defines a toric divisor  $D_f$  on  $X_{\Sigma}$  by Theorem 3.1.9. Such a divisor  $D_f$  is generated by global sections because of Proposition 3.1.10 and it comes with a distinguished toric section  $s_f$  of  $\mathcal{O}(D_f)$ . The Laurent polynomial  $f$  can be seen as a rational function on  $X_{\Sigma}$ , inducing hence a rational section  $fs_f$  of  $\mathcal{O}(D_f)$ , whose associated Weil divisor is

$$[\text{div}(fs_f)] = [\text{div}(f)] + [\text{div}(s_f)] = Z \quad (4.5)$$

because of Theorem 3.2.4, (3.4) and (4.4).

**DEGREES.** The invariance of the degree under birational transformations allows to prove a combinatorial formula for the degree of the cycle  $Z$  in terms of its defining polynomial  $f$ .

**Proposition 4.2.1.** *Let  $D_{\Psi_1}, \dots, D_{\Psi_{n-1}}$  be toric divisors on  $X_{\Sigma}$  generated by global sections,  $Z$  an effective cycle on  $X_{\Sigma}$  of pure codimension 1 and prime components intersecting  $X_0$ , with defining polynomial  $f$ . Then*

$$\deg_{D_{\Psi_1}, \dots, D_{\Psi_{n-1}}}(Z) = \text{MV}_M(\Delta_{\Psi_1}, \dots, \Delta_{\Psi_{n-1}}, \text{NP}(f)),$$

where  $\text{MV}_M$  denotes the mixed volume function associated to the measure  $\text{vol}_M$  (see Notation 1.2.11) and  $\Delta_{\Psi_i}$  the polytope associated to the toric divisor  $D_{\Psi_i}$ , for each  $i = 1, \dots, n-1$ .

*Proof.* Consider the smooth projective toric variety  $X_{\Sigma'}$  and the proper toric morphism  $\pi : X_{\Sigma'} \rightarrow X_{\Sigma}$  given by Lemma 3.1.12. Since the support function  $\Psi_f$  is a virtual support function on  $\Sigma'$ , one can consider the corresponding toric divisor  $D_f$  on  $X_{\Sigma'}$  and the associated distinguished rational section  $s_f$  of  $\mathcal{O}(D_f)$ . The definition of  $\pi$ , together with (4.5), assures that

$$\pi_*([\text{div}(fs_f) \cdot X_{\Sigma'}]) = \pi_*([\text{div}(fs_f)]) = Z.$$

Definition 2.4.5 and the projection formula in Theorem 2.4.6 yield hence

$$\deg_{D_{\Psi_1}, \dots, D_{\Psi_{n-1}}}(Z) = \deg_{\pi^*D_{\Psi_1}, \dots, \pi^*D_{\Psi_{n-1}}, D_f}(X_{\Sigma'}).$$

The function  $\Psi_f$  being concave,  $D_f$  is generated by global sections. Moreover, the virtual support functions associated to the toric divisor  $\pi^*D_{\Psi_i}$  on  $X_{\Sigma'}$  agrees with  $\Psi_i$ , for every  $i = 1, \dots, n-1$ . The combinatorial description in [Oda88, Proposition 2.10] of the degree of a toric variety with respect to toric divisor generated by global sections concludes then the proof.  $\square$

**Remark 4.2.2.** By [Ful93, formula at page 55], the irreducible components of  $X_\Sigma \setminus X_0$  are the orbit closures  $V(\tau)$ , with  $\tau$  ranging in the set of 1 dimensional cones of  $\Sigma$ . It follows that if  $Z$  is a prime divisor of  $X_\Sigma$  not intersecting  $X_0$ , it coincides with  $V(\tau)$  for some  $\tau \in \Sigma^{(1)}$ . In such a case, the degree of  $Z$  with respect to a collection  $D_{\Psi_1}, \dots, D_{\Psi_{n-1}}$  of toric divisors on  $X_\Sigma$  generated by global sections is given by

$$\deg_{D_{\Psi_1}, \dots, D_{\Psi_{n-1}}}(V(\tau)) = MV_{M(v_\tau)}(\Delta_{\Psi_1}^{v_\tau}, \dots, \Delta_{\Psi_{n-1}}^{v_\tau}),$$

where  $v_\tau$  is the minimal nonzero integral vector of  $\tau$ , see [BPS14, formulæ (3.4.1) and (3.4.4)].

**Remark 4.2.3.** The reduction to the case of a smooth projective toric variety employed in the proof of Proposition 4.2.1 equally works when computing the local height of the cycle  $Z$  with respect to a family of  $v$ -adic semipositive toric metrized divisors  $\overline{D}_{0,v}, \dots, \overline{D}_{n-1,v}$ . Indeed, let  $f$  be a defining polynomial for  $Z$ ,  $X_{\Sigma'}$  and  $\pi$  as in the statement of Lemma 3.1.12. For every family of rational sections  $s_0, \dots, s_{n-1}$  of  $\mathcal{O}(D_0), \dots, \mathcal{O}(D_{n-1})$  respectively for which the following local heights are well-defined, the local arithmetic projection formula in [BPS14, Theorem 1.4.17 (2)] asserts that

$$h_{\overline{D}_{0,v}, \dots, \overline{D}_{n-1,v}}(Z; s_0, \dots, s_{n-1}) = h_{\pi^* \overline{D}_{0,v}, \dots, \pi^* \overline{D}_{n-1,v}}(Z'; \pi^* s_0, \dots, \pi^* s_{n-1}),$$

where  $Z'$  is the cycle in  $X_{\Sigma'}$  associated to the subscheme obtained as the closure of  $V(f)$  and has hence  $f$  as a defining polynomial. Because of [BPS14, Proposition 4.3.19], the pull-back of  $\overline{D}_{i,v}$  via  $\pi$  is a  $v$ -adic semipositive toric metrized divisor on  $X_{\Sigma'}$  whose metric function coincides with the one of  $\overline{D}_{i,v}$ , for every  $i = 0, \dots, n-1$  and  $v \in \mathfrak{M}$ . It follows that any combinatorial formula for the local height of  $Z'$  in  $X_{\Sigma'}$  with respect to  $\pi^* \overline{D}_{0,v}, \dots, \pi^* \overline{D}_{n-1,v}$  only involving the defining polynomial of  $Z$  and the metric functions of the metrized divisors equally holds for the local height of  $Z$  in  $X_\Sigma$  with respect to  $\overline{D}_{0,v}, \dots, \overline{D}_{n-1,v}$ . Similarly, the reduction step can be adopted when dealing with the integrability and the global height of  $Z$ , because of Theorem 2.4.16.

**LOCAL HEIGHTS.** The relation between the Ronkin function of a Laurent polynomial and the height of the corresponding toric hypersurface appears thanks to the following notion.

**Definition 4.2.4.** Let  $f$  be a nonzero Laurent polynomial with coefficients in  $\mathbb{K}$  and  $D_f$  the divisor associated to its Newton polytope. For a place  $v \in \mathfrak{M}$ , the  $v$ -adic Ronkin metric on  $D_f$  is the  $v$ -adic semipositive toric metric on  $\mathcal{O}(D_f)_v^{\text{an}}$  corresponding to the  $v$ -adic Ronkin function  $\rho_{f,v}$  via Theorem 3.4.7.

The previous definition makes sense since, for every  $v \in \mathfrak{M}$ , the  $v$ -adic Ronkin function of  $f$  is concave on  $N_{\mathbb{R}}$  and has bounded difference from  $\Psi_f$  because of Proposition 4.1.9. If not otherwise specified,  $\overline{D}_{f,v}$  will denote the divisor  $D_f$  equipped with the  $v$ -adic Ronkin metric  $\|\cdot\|_{f,v}$  defined above. By definition,

$$\log \|s_f\|_{f,v} = \rho_{f,v} \circ \text{trop}_v \quad (4.6)$$

on  $X_{0,v}^{\text{an}}$ . To lighten the notation, we will drop the subscript  $v$  whenever the choice of the place is clear from the context.

**Proposition 4.2.5.** *Let  $f$  and  $g$  be two nonzero Laurent polynomials and assume that  $\Psi_f$  and  $\Psi_g$  are virtual support functions on the fan of  $X_{\Sigma}$ . Then,  $\overline{D}_f + \overline{D}_g = \overline{D}_{f \cdot g}$ .*

*Proof.* The equality  $\text{NP}(f \cdot g) = \text{NP}(f) + \text{NP}(g)$  implies that  $\Psi_{f \cdot g} = \Psi_f + \Psi_g$ . In particular,  $\Psi_{f \cdot g}$  is a virtual support function on the fan  $\Sigma$  and then defines a toric divisor  $D_{f \cdot g}$  on  $X_{\Sigma}$  which satisfies  $D_{f \cdot g} = D_f + D_g$  because of [Ful93, §3.4]. The statement follows now from Proposition 3.4.8 and Proposition 4.1.8.  $\square$

The key property of the Ronkin metric is given in the following proposition.

**Proposition 4.2.6.** *Let  $X_{\Sigma}$  be a smooth projective toric variety,  $Z$  an effective cycle on  $X_{\Sigma}$  of pure codimension 1 and prime components intersecting  $X_0$ . Let  $f$  be a defining polynomial for  $Z$  and assume that  $\Psi_f$  is a virtual support function on the fan  $\Sigma$ . For a fixed place  $v \in \mathfrak{M}$ , let  $\overline{D}_0, \dots, \overline{D}_{n-1}$  be toric divisors on  $X_{\Sigma}$ , equipped with  $v$ -adic semipositive toric metrics. Then*

$$h_{\overline{D}_0, \dots, \overline{D}_{n-1}}(Z; s_0, \dots, s_{n-1}) = h_{\overline{D}_0, \dots, \overline{D}_{n-1}, \overline{D}_f}(X_{\Sigma}; s_0, \dots, s_{n-1}, f s_f), \quad (4.7)$$

for every choice of rational sections  $s_0, \dots, s_{n-1}$  of  $\mathcal{O}(D_0), \dots, \mathcal{O}(D_{n-1})$  respectively with  $[\text{div}(s_0)], \dots, [\text{div}(s_{n-1})], Z$  intersecting properly.

*Proof.* The product  $f s_f$  is a rational section of  $\mathcal{O}(D_f)$  on  $X_{\Sigma}$  with associated Weil divisor

$$[\text{div}(f s_f)] = [\text{div}(f)] + [\text{div}(s_f)] = Z$$

by Theorem 3.2.4, (3.4) and (4.4). Hence, the sections  $s_0, \dots, s_{n-1}, f s_f$  meet  $X_{\Sigma}$  properly and the right hand side term in (4.7) is well defined.

Definition 2.4.14 stating that

$$\begin{aligned} h_{\overline{D}_0, \dots, \overline{D}_{n-1}}(Z; s_0, \dots, s_{n-1}) &= h_{\overline{D}_0, \dots, \overline{D}_{n-1}, \overline{D}_f}(X_{\Sigma}; s_0, \dots, s_{n-1}, f s_f) \\ &\quad + \int_{X_{\Sigma}^{\text{an}}} \log \|f s_f\|_{f,v} c_1(\overline{D}_0) \wedge \dots \wedge c_1(\overline{D}_{n-1}), \end{aligned}$$

the proposition follows from the vanishing of the integral on the right hand side. Indeed, thanks to Theorem 3.4.10, such an integral is supported on the analytification of the dense open orbit of  $X_\Sigma$ , where the rational function  $f$  is regular. Together with the definition of the Ronkin metric in (4.6), this yields

$$\begin{aligned} \int_{X_\Sigma^{\text{an}}} \log \|f s_f\|_{f,v} c_1(\overline{D}_0) \wedge \cdots \wedge c_1(\overline{D}_{n-1}) \\ = \int_{X_0^{\text{an}}} \log |f|_v c_1(\overline{D}_0) \wedge \cdots \wedge c_1(\overline{D}_{n-1}) \\ + \int_{X_0^{\text{an}}} (\rho_{f,v} \circ \text{trop}_v) c_1(\overline{D}_0) \wedge \cdots \wedge c_1(\overline{D}_{n-1}). \end{aligned}$$

For every  $i = 0, \dots, n-1$ , denote by  $\psi_i$  the metric function of  $\overline{D}_i$ . The tropicalization map being continuous, the change of variables formula and Theorem 3.4.10 imply on the one hand that

$$\int_{X_0^{\text{an}}} (\rho_{f,v} \circ \text{trop}_v) c_1(\overline{D}_0) \wedge \cdots \wedge c_1(\overline{D}_{n-1}) = \int_{N_{\mathbb{R}}} \rho_{f,v} d\text{MM}_M(\psi_0, \dots, \psi_{n-1}).$$

On the other hand, Theorem 3.4.10, together with the change of variables formula and Fubini's theorem, gives

$$\begin{aligned} \int_{X_0^{\text{an}}} \log |f|_v c_1(\overline{D}_0) \wedge \cdots \wedge c_1(\overline{D}_{n-1}) = \\ \int_{N_{\mathbb{R}}} \left( \int_{\mathcal{B}_v} (\log |f|_v \circ \iota_v) d\text{Haar}_{\mathcal{B}_v} \right) d\text{MM}_M(\psi_0, \dots, \psi_{n-1}). \end{aligned}$$

The definition of the maps  $\iota_v$  and  $\rho_{f,v}$  assures that the inner integral coincides with the opposite of the  $v$ -adic Ronkin function of  $f$ , concluding the proof.  $\square$

**TORIC LOCAL HEIGHTS.** Recall that for any place  $v \in \mathfrak{M}$ , Definition 3.4.13 gives a definition of a local height on a toric variety which is independent of sections. The following theorem gives a formula for the toric local height of the cycle  $Z$  in terms of its defining polynomial.

**Theorem 4.2.7.** *Let  $X_\Sigma$  be a proper toric variety,  $Z$  an effective cycle on  $X_\Sigma$  of pure codimension 1 and prime components intersecting  $X_0$ . For a place  $v$  of  $\mathbb{K}$ , let  $\overline{D}_0, \dots, \overline{D}_{n-1}$  be toric divisors on  $X_\Sigma$ , equipped with  $v$ -adic semipositive toric metrics. Then*

$$h_{\overline{D}_0, \dots, \overline{D}_{n-1}}^{\text{tor}}(Z) = \text{MI}_M(\vartheta_0, \dots, \vartheta_{n-1}, \rho_{f,v}^\vee) - \deg_{D_0, \dots, D_{n-1}}(X_\Sigma) \cdot m_v(f),$$

where  $f$  is a defining polynomial for  $Z$ ,  $m_v(f)$  its  $v$ -adic Mahler measure and  $\vartheta_i$  is the roof function of  $\bar{D}_i$ , for  $i = 0, \dots, n-1$ .

*Proof.* Because of Remark 4.2.3, one can assume that  $X_\Sigma$  is a smooth projective toric variety on whose fan  $\Psi_f$  is a virtual support function. Thanks to the moving lemma, one can choose rational sections  $s_0, \dots, s_{n-1}$  of  $\mathcal{O}(D_0), \dots, \mathcal{O}(D_{n-1})$  respectively such that  $[\text{div}(s_0)], \dots, [\text{div}(s_{n-1})], Z$  intersect properly. Proposition 4.2.6 implies then that

$$h_{\bar{D}_0, \dots, \bar{D}_{n-1}}^{\text{tor}}(Z) = h_{\bar{D}_0, \dots, \bar{D}_{n-1}, \bar{D}_f}(X_\Sigma; s_0, \dots, s_{n-1}, f s_f) - h_{\bar{D}_0^{\text{can}}, \dots, \bar{D}_{n-1}^{\text{can}}, \bar{D}_f}(X_\Sigma; s_0, \dots, s_{n-1}, f s_f).$$

By adding and subtracting the quantity

$$h_{\bar{D}_0^{\text{can}}, \dots, \bar{D}_{n-1}^{\text{can}}, \bar{D}_f^{\text{can}}}(X_\Sigma; s_0, \dots, s_{n-1}, f s_f)$$

on the right hand side, one obtains that

$$h_{\bar{D}_0, \dots, \bar{D}_{n-1}}^{\text{tor}}(Z) = h_{\bar{D}_0, \dots, \bar{D}_{n-1}, \bar{D}_f}^{\text{tor}}(X_\Sigma) - h_{\bar{D}_0^{\text{can}}, \dots, \bar{D}_{n-1}^{\text{can}}, \bar{D}_f}^{\text{tor}}(X_\Sigma).$$

Denote by  $\Psi_i$  the virtual support function on  $\Sigma$  associated to the toric divisor  $D_i$ , for every  $i = 0, \dots, n-1$ . Thanks to [BPS14, Corollary 5.1.9], the previous equality yields

$$h_{\bar{D}_0, \dots, \bar{D}_{n-1}}^{\text{tor}}(Z) = \text{MI}_M(\vartheta_0, \dots, \vartheta_{n-1}, \rho_{f,v}^\vee) - \text{MI}_M(\Psi_0^\vee, \dots, \Psi_{n-1}^\vee, \rho_{f,v}^\vee).$$

Since they admit by hypothesis a semipositive toric metric, the toric divisors  $D_0, \dots, D_{n-1}$  are generated by global sections. For every  $i = 0, \dots, n-1$ , the function  $\Psi_i$  is hence concave and conic and so it is the support function of the polytope  $\Delta_i := \text{stab}(\Psi_i) \subseteq M_{\mathbb{R}}$ . The statement of the theorem follows from a combination of Example 1.3.8, Corollary 1.5.11, Definition 4.1.13 and the combinatorial expression for the degree of a toric variety with respect to toric divisors generated by their global sections, see for example [Oda88, Proposition 2.10].  $\square$

**GLOBAL HEIGHTS.** The main ingredient in the proof of the result concerning the global case is the following.

**Definition 4.2.8.** Let  $f$  be a nonzero Laurent polynomial over  $\mathbb{K}$  and  $D_f$  the divisor associated to its Newton polytope. The *Ronkin metric* on  $D_f$  is the choice, for every place  $v \in \mathfrak{M}$ , of the  $v$ -adic Ronkin metric on  $D_f$  defined in Definition 4.2.4.

Unless otherwise stated,  $\overline{D}_f$  will denote the toric divisor  $D_f$  equipped with its Ronkin metric. By definition, it is a semipositive toric metrized divisor.

**Lemma 4.2.9.** *The Ronkin metric on  $D_f$  is adelic.*

*Proof.* For a non-archimedean place  $v \in \mathfrak{M}$ , the function  $\rho_{f,v}$  coincides with the tropicalization of the Laurent polynomial  $f$ , as claimed in Remark 4.1.7. The fact that  $f$  has finitely many nonzero coefficients and the second axiom in Definition 2.3.1 imply that  $\rho_{f,v} = \Psi_f$  for all but finitely many non-archimedean places. The statement follows then from Lemma 2.3.5.  $\square$

The definition of such a toric metrized divisor and the study of the local height of  $Z$  in Proposition 4.2.6 allow to give a formula for its global height, implying Theorem 1 in the introduction.

**Theorem 4.2.10.** *Let  $Z$  be an effective cycle on  $X_\Sigma$  of pure codimension 1 and prime components intersecting  $X_0$ . Let  $\overline{D}_0, \dots, \overline{D}_{n-1}$  be toric divisors on  $X_\Sigma$ , equipped with adelic semipositive toric metrics. Then,*

$$h_{\overline{D}_0, \dots, \overline{D}_{n-1}}(Z) = \sum_{v \in \mathfrak{M}} n_v \text{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n-1,v}, \rho_{f,v}^\vee),$$

where  $f$  is a defining polynomial for  $Z$  and  $\vartheta_{i,v}$  is the roof function of  $\overline{D}_{i,v}$ , for every  $i = 0, \dots, n-1$  and  $v \in \mathfrak{M}$ .

*Proof.* Because of Remark 4.2.3, one can assume that  $X_\Sigma$  is a smooth projective toric variety on whose fan  $\Psi_f$  is a virtual support function. Let hence  $s_0, \dots, s_{n-1}$  be rational sections of  $\mathcal{O}(D_0), \dots, \mathcal{O}(D_{n-1})$  respectively such that  $[\text{div}(s_0)], \dots, [\text{div}(s_{n-1})], Z$  intersect properly. Because of Proposition 4.2.6, the  $v$ -adic local height of  $Z$  with respect to the above choice of sections is given by

$$h_{\overline{D}_0, \dots, \overline{D}_{n-1}, v}(Z; s_0, \dots, s_{n-1}) = h_{\overline{D}_0, \dots, \overline{D}_{n-1}, v, \overline{D}_f}(X_\Sigma; s_0, \dots, s_{n-1}, f s_f). \quad (4.8)$$

Summing with the associated local weights over the places of  $\mathbb{K}$ , the global height of  $Z$  is seen to satisfy

$$h_{\overline{D}_0, \dots, \overline{D}_{n-1}}(Z) = h_{\overline{D}_0, \dots, \overline{D}_{n-1}, \overline{D}_f}(X_\Sigma).$$

Theorem 3.4.12 concludes hence the proof.  $\square$

Many comments can be made regarding the result of Theorem 4.2.10 and its proof.

**Remark 4.2.11.** As in Remark 4.2.2, if  $Z$  is an irreducible hypersurface on  $X_\Sigma$  not intersecting  $X_0$  it coincides with  $V(\tau)$  for a 1 dimensional cone  $\tau$  of the fan  $\Sigma$ . In such a case,  $Z$  is integrable with respect to a family of adelic semipositive toric metrized divisors  $\overline{D}_0, \dots, \overline{D}_{n-1}$  on  $X_\Sigma$  and its global height is given by

$$h_{\overline{D}_0, \dots, \overline{D}_{n-1}}(V(\tau)) = \sum_{v \in \mathfrak{M}} n_v \text{MI}_{M(v_\tau)}(\vartheta_{0,v} |_{\Delta_0^{v_\tau}}, \dots, \vartheta_{n-1,v} |_{\Delta_{n-1}^{v_\tau}}),$$

see [BPS14, Proposition 5.1.11 and Proposition 5.2.4]. In the previous formula,  $\Delta_i$  is the polytope associated to the divisor  $D_i$  and  $\vartheta_{i,v}$  is the roof function of  $\overline{D}_{i,v}$  for every  $i = 0, \dots, n-1$  and  $v \in \mathfrak{M}$ , while  $v_\tau$  is the minimal nonzero integral vector of  $\tau$ .

**Remark 4.2.12.** Theorem 4.2.10 and Remark 4.2.11 allow to give a combinatorial formula for the height of *any* Weil divisor  $Z$  in  $X_\Sigma$  with respect to a choice of adelic semipositive toric metrized divisors. It is indeed sufficient to use the linearity of the height function and to write

$$Z = \sum_{i=1}^r \ell_i Y_i - \sum_{i=r+1}^s \ell_i Y_i + \sum_{\tau \in \Sigma^{(1)}} \ell_\tau V(\tau)$$

with  $Y_1, \dots, Y_s$  prime divisors intersecting  $X_0$  and  $\ell_1, \dots, \ell_s$  positive integers.

**Remark 4.2.13.** The fact that the hand side of the formula in Theorem 4.2.10 is independent of the choice of the defining polynomial  $f$  for  $Z$  can also be checked directly. Indeed, if  $f'$  is another such polynomial, it must be  $f' = c\chi^m \cdot f$  with  $c \in \mathbb{K}^\times$  and  $m \in M$ . It follows from Proposition 4.1.8 and Example 4.1.10 that for every  $v \in \mathfrak{M}$  one has  $\rho_{f',v} = \rho_{f,v} + m_0 - \log |c|_v$ . The definition of the Legendre-Fenchel duality together with (1.12) yields then

$$\rho_{f',v}^\vee = \tau_{m_0}(\rho_{f,v}^\vee + \log |c|_v).$$

Using Proposition 1.5.7 and Corollary 1.5.6,

$$\begin{aligned} \text{MI}_M(\vartheta_0, \dots, \vartheta_{n-1}, \rho_{f',v}^\vee) &= \text{MI}_M(\vartheta_0, \dots, \vartheta_{n-1}, \rho_{f,v}^\vee) \\ &\quad + \log |c|_v \cdot \text{MV}_M(\Delta_0, \dots, \Delta_{n-1}), \end{aligned}$$

with  $\Delta_i$  being the polytope associated to  $D_i$  for every  $i = 0, \dots, n-1$ . The claimed independance is then a consequence of the product formula on  $\mathbb{K}$ .

**Remark 4.2.14.** Local and global heights of cycles are symmetric and multilinear with respect to sums of semipositive metrized divisors, provided that all terms are defined. The formulas obtained for 1-codimensional cycles in toric varieties are consistent with these properties, the sum of semipositive toric metrized divisors corresponding to the sup-convolution of the associated roof functions, see Proposition 3.4.8.

**Remark 4.2.15.** The proof of Theorem 4.2.10 also shows directly that the cycle  $Z$  is integrable with respect to  $\overline{D}_0, \dots, \overline{D}_{n-1}$ . Indeed, because of Lemma 4.2.9, each member of the family  $\overline{D}_0, \dots, \overline{D}_{n-1}, \overline{D}_f$  is an adelic semipositive toric metrized divisor on  $X_\Sigma$ . As a consequence of the first assertion in [BPS14, Proposition 5.2.4],  $X_\Sigma$  is integrable with respect to such a choice of metrized divisors and hence [BPS14, Proposition 1.5.8 (1)] allows to conclude that

$$h_{\overline{D}_0, v, \dots, \overline{D}_{n-1}, v, \overline{D}_f, v}(X_\Sigma; s_0, \dots, s_{n-1}, f s_f) = 0$$

for all but finitely many places  $v \in \mathfrak{M}$ . Comparing with (4.8), one deduces the integrability of  $Z$  with respect to  $\overline{D}_0, \dots, \overline{D}_{n-1}$ .

**Remark 4.2.16.** The results obtained in the section can be easily extended to the case of adelic DSP toric metrized divisors by using arguments analogous to the ones of [BPS14, Remark 5.1.10].

Finally, one can combine the formula just proved with the bounds on mixed integrals studied in Chapter 1 to get a window for the values of the global height of an effective cycle of codimension 1. For the sake of simplicity, one only considers here the case in which  $\overline{D}_0 = \dots = \overline{D}_{n-1}$ .

**Proposition 4.2.17.** *Let  $Z$  be an effective cycle on  $X_\Sigma$  of pure codimension 1 and prime components intersecting  $X_0$ ,  $\overline{D}$  a toric divisors on  $X_\Sigma$  equipped with adelic semipositive toric metrics. Then,*

$$h_{\overline{D}}(Z) \geq \sum_v n_v \left( n! \operatorname{vol}_M(\Delta) m_v(f) + n \min(\vartheta_v) \operatorname{MV}_M(\Delta, \dots, \Delta, \operatorname{NP}(f)) \right)$$

and

$$h_{\overline{D}}(Z) \leq \sum_v n_v \left( n! \operatorname{vol}_M(\Delta) m_v(f) - n \psi_v(0) \operatorname{MV}_M(\Delta, \dots, \Delta, \operatorname{NP}(f)) \right),$$

where  $f$  is a defining polynomial for  $Z$ ,  $\Delta$  is the polytope associate to  $D$  and  $\psi_v$  and  $\vartheta_v$  are respectively the metric and the roof function of  $\overline{D}_v$ , for every  $v \in \mathfrak{M}$ .

*Proof.* Both the bounds follow from Theorem 1.5.14 and Theorem 4.2.10, together with Proposition 4.1.16 and the definition of the  $v$ -adic Mahler measure of a Laurent polynomial.  $\square$

The adelicity of  $\overline{D}$  and the expression for the non-archimedean Mahler measures of  $f$  in Remark 4.1.14 immediately imply that the infinite sums appearing in Proposition 4.2.17 have in fact finitely many nonzero terms.

### 4.3 EXAMPLES

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We apply in this section the formula in Theorem 4.2.10 to four particular cases. In the first one, we focus on specific hypersurfaces of  $X_\Sigma$ , while in the following three we made relevant choices of the metrized divisors.

**BINOMIAL HYPERSURFACES.** For a primitive vector  $m$  in  $M$  one can consider the Laurent binomial  $f = \chi^m - 1$ ; it is irreducible in  $\mathbb{K}[M]$  as can be verified by considering its Newton polytope. Hence the closure  $Z$  in  $X_\Sigma$  of the subvariety  $V(f)$  of the torus  $\text{Spec } \mathbb{K}[M]$  is an irreducible hypersurface of  $X_\Sigma$  with defining polynomial  $f$ .

Let  $\overline{D}_0, \dots, \overline{D}_{n-1}$  be toric divisors on  $X_\Sigma$ , equipped with adelic semipositive toric metrics, with  $\vartheta_{i,v}$  the roof function of  $\overline{D}_{i,v}$ , for every  $i = 0, \dots, n-1$  and  $v \in \mathfrak{M}$ . By Example 4.1.11,  $\rho_{f,v}$  coincides for every  $v \in \mathfrak{M}$  with the support function of the segment  $\overline{0m}$  in  $M_{\mathbb{R}}$ . The formula in Theorem 4.2.10 implies then that

$$h_{\overline{D}_0, \dots, \overline{D}_{n-1}}(Z) = \sum_{v \in \mathfrak{M}} n_v \text{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n-1,v}, \iota_{\overline{0m}}),$$

because of Example 1.3.8. Considering the quotient lattice  $P := M/\mathbb{Z}m$  and the associated projection  $\pi : M \rightarrow P$ , Proposition 1.5.13 allows to deduce

$$h_{\overline{D}_0, \dots, \overline{D}_{n-1}}(Z) = \sum_{v \in \mathfrak{M}} n_v \text{MI}_P(\pi_*\vartheta_{0,v}, \dots, \pi_*\vartheta_{n-1,v}), \quad (4.9)$$

with  $\pi_*\vartheta_{i,v}$  denoting the direct image of  $\vartheta_{i,v}$  by  $\pi$  for every  $i = 0, \dots, n-1$  and  $v \in \mathfrak{M}$ , see (1.19).

**Remark 4.3.1.** Let  $Q$  be the dual lattice of  $P = M/\mathbb{Z}m$ . The projection  $\pi : M \rightarrow P$  induces an injective dual map  $Q \rightarrow N$ , with image  $m^\perp \cap N$ . By identifying  $Q$  with such an image, which is a saturated sublattice of  $N$ , one

can consider the restriction of the fan  $\Sigma$  to  $Q_{\mathbb{R}}$ ; its corresponding toric variety  $X_{\Sigma_Q}$  is proper and has torus  $\text{Spec } \mathbb{K}[P]$ . It also comes with a toric morphism  $\varphi : X_{\Sigma_Q} \rightarrow X_{\Sigma}$ , whose restriction to the dense open orbit coincides with the closed immersion of split tori  $\text{Spec } \mathbb{K}[P] \rightarrow \text{Spec } \mathbb{K}[M]$  given by the surjection  $\pi : M \rightarrow P$ , see [BPS14, pages 81-83]. Finally, the push-forward of the cycle  $X_{\Sigma_Q}$  by  $\varphi$  is the cycle  $Z$  associated to the hypersurface defined by  $\chi^m - 1$ . Indeed, the image of  $\varphi$  coincides by properness with the closure in  $X_{\Sigma}$  of the image of  $\text{Spec } \mathbb{K}[P] \rightarrow \text{Spec } \mathbb{K}[M]$ , which is an irreducible  $(n-1)$ -dimensional subscheme of  $\text{Spec } \mathbb{K}[M]$  contained in  $V(\chi^m - 1)$  as  $\chi^{\pi(m)} - 1 = 0$ .

Hence, equality (4.9) can also be obtained from Theorem 3.4.12, the arithmetic projection formula stated in Theorem 2.4.16 and the fact that  $\pi_* \vartheta_{i,v}$  is the roof function of the pull-back of  $\overline{D}_{i,v}$  via  $\pi$ , for every  $i = 0, \dots, n-1$  and  $v \in \mathfrak{M}$ , because of [BPS14, Proposition 4.3.19 and Proposition 2.3.8 (3)].

**THE CANONICAL HEIGHT.** A toric divisor  $D$  on  $X_{\Sigma}$  generated by global sections admits by Definition 3.4.9 a distinguished semipositive toric metric at any place. The metrized divisor obtained by the choice of such a family of  $v$ -adic canonical metrics is denoted by  $\overline{D}^{\text{can}}$ ; it is an adelic semipositive toric metrized divisor.

For a cycle  $Z$  of dimension  $d$  in  $X_{\Sigma}$ , the *canonical global height* of  $Z$  with respect to a family  $D_0, \dots, D_d$  of toric divisors on  $X_{\Sigma}$  generated by global sections is defined to be its global height with respect to  $\overline{D}_0^{\text{can}}, \dots, \overline{D}_d^{\text{can}}$  and it is also denoted by  $h_{D_0, \dots, D_d}^{\text{can}}(Z)$ . The machinery developed in the previous sections allows to express the canonical global height of an effective cycle on  $X_{\Sigma}$  of pure codimension 1 via convex geometry. Recall that by  $m_v(f)$  one means the  $v$ -adic Mahler measure of the Laurent polynomial  $f$ .

**Proposition 4.3.2.** *Let  $Z$  be an effective cycle on  $X_{\Sigma}$  of pure codimension 1 and prime components intersecting  $X_0$  and  $D_0, \dots, D_{n-1}$  a family of toric divisors on  $X_{\Sigma}$  generated by global sections. The canonical global height of  $Z$  with respect to  $D_0, \dots, D_{n-1}$  is given by*

$$h_{D_0, \dots, D_{n-1}}^{\text{can}}(Z) = \deg_{D_0, \dots, D_{n-1}}(X_{\Sigma}) \cdot \sum_{v \in \mathfrak{M}} n_v m_v(f)$$

for any choice of a defining polynomial  $f$  for  $Z$ .

*Proof.* Denoting for any  $i = 0, \dots, n-1$  by  $\Psi_i$  the function associated to  $D_i$ , the property of being globally generated implies that  $\Psi_i$  is the support function of the lattice polytope  $\Delta_i := \text{stab}(\Psi_i) \subseteq M_{\mathbb{R}}$ . The roof function of

$\overline{D}_{i,v}^{\text{can}}$  is hence  $\iota_{\Delta_i}$  for every  $i = 0, \dots, n-1$  and for every  $v \in \mathfrak{M}$ , because of Example 1.3.8. It follows then from Theorem 4.2.10 and Corollary 1.5.11 that

$$h_{D_0, \dots, D_{n-1}}^{\text{can}}(Z) = -\text{MV}_M(\Delta_0, \dots, \Delta_{n-1}) \cdot \sum_{v \in \mathfrak{M}} n_v \rho_{f,v}(0),$$

with  $f$  any defining polynomial for  $Z$ . To conclude, recall that the degree of  $X_\Sigma$  with respect to  $D_0, \dots, D_{n-1}$  is given by the mixed volume of the associated polytopes, as proved in [Oda88, Proposition 2.10], and that  $m_v(f) = -\rho_{f,v}(0)$  by Definition 4.1.13.  $\square$

The case of the base field  $\mathbb{Q}$  with the adelic structure described in Example 2.3.3 is particularly interesting for arithmetic purposes. In [Mai00, Proposition 7.2.1], Maillot expressed the canonical height of a hypersurface in a toric variety over  $\mathbb{Q}$  in terms of the Mahler measure of the associated section. While its proof relies on the study of the arithmetic Chow ring of the ambient toric variety, one here deduces his result from Proposition 4.3.2.

**Corollary 4.3.3** (Maillot). *In the hypotheses and notations of Proposition 4.3.2, assume moreover that the base adelic field is  $\mathbb{Q}$  with its usual adelic structure. Let  $f$  be a defining polynomial for  $Z$  having as coefficients integer numbers with greatest common divisor 1. Then,*

$$h_{D_0, \dots, D_{n-1}}^{\text{can}}(Z) = \deg_{D_0, \dots, D_{n-1}}(X_\Sigma) \cdot \mathfrak{m}(f).$$

*Proof.* The statement follows directly from Proposition 4.3.2 and Example 4.1.15.  $\square$

**THE  $\rho$ -HEIGHT.** The strategy adopted to prove the formula for the height of an hypersurface suggests the introduction of a distinguished height function. Let  $Z$  be an effective cycle on  $X_\Sigma$  of pure codimension 1 and prime components intersecting  $X_0$  and assume that the support function of the Newton polytope of a defining polynomial for  $Z$  is a virtual support function on the fan  $\Sigma$ . By Lemma 3.1.12, this is always the case up to a birational toric transformation. In this setting, the choice of a defining polynomial  $f$  for  $Z$  determines a toric divisor  $D_f$  on  $X_\Sigma$  and a distinguished toric metric on it, the Ronkin metric, as introduced in Definition 4.2.8. The so-obtained metrized divisor, which is denoted by  $\overline{D}_f$ , is an adelic semipositive toric metrized divisor by Lemma 4.2.9.

**Definition 4.3.4.** In the above hypotheses and notations, the  $\rho$ -height of  $Z$ , denoted by  $h_\rho(Z)$ , is defined as its global height with respect to  $\overline{D}_f, \dots, \overline{D}_f$ , for a choice of a defining polynomial  $f$  for  $Z$ .

As shown below, the  $\rho$ -height of  $Z$  is independent of the choice of the defining polynomial  $f$ . Even if it is not clear whether such a height has a significant geometrical interpretation or arithmetical application, its combinatorial expression is simpler than the general case.

**Proposition 4.3.5.** *In the above hypotheses and notations, the  $\rho$ -height of  $Z$  is given by*

$$h_\rho(Z) = (n+1)! \sum_{v \in \mathfrak{M}} n_v \int_{\text{NP}(f)} \rho_{f,v}^\vee d\text{vol}_M,$$

where  $f$  is a defining polynomial for  $Z$  and  $\text{NP}(f)$  is its Newton polytope.

*Proof.* The statement follows trivially from Theorem 4.2.10, Proposition 4.1.9 (3) and the properties of mixed integrals.  $\square$

**Remark 4.3.6.** The equality in Proposition 4.3.5 shows that the  $\rho$ -height of  $Z$  does not depend on the choice of a defining polynomial for it. Indeed, if  $f'$  is another such polynomial, it must satisfy  $f' = c\chi^m \cdot f$  for some nonzero monomial  $c\chi^m \in \mathbb{K}[M]$ . An argument analogous to the one of Remark 4.2.13 proves hence the claimed independence.

It is significant to stress that the formula in Proposition 4.3.5, though compact, is difficult to evaluate because of the complexity of the archimedean Ronkin function.

**THE FUBINI-STUDY HEIGHT.** As a last example, let the ambient toric variety  $X_\Sigma$  be the  $n$ -dimensional projective space over  $\mathbb{K}$ . Denote by  $D_\infty$  the toric divisor on  $\mathbb{P}_{\mathbb{K}}^n$  whose associated Weil divisor is the hyperplane at infinity; the corresponding sheaf is the universal line bundle  $\mathcal{O}(1)$  on  $\mathbb{P}_{\mathbb{K}}^n$ . If not otherwise specified, the notation  $\overline{D}_\infty$  will refer to  $D_\infty$  equipped with the Fubini-Study metric at archimedean places, see [BPS14, Example 1.1.2], and the canonical one at non-archimedean places, in the sense of Definition 3.4.9. It turns out that  $\overline{D}_\infty$  is an adelic semipositive toric metrized divisor. For an effective cycle  $Z$  on  $\mathbb{P}_{\mathbb{K}}^n$  of pure codimension 1, the global height

$$h_{\text{FS}}(Z) := h_{\overline{D}_\infty, \dots, \overline{D}_\infty}(Z)$$

is called the *Fubini-Study height* of  $Z$ .

**Remark 4.3.7.** The Fubini-Study height defined here coincides with the one introduced in [Fal91] and studied in [Phi95]. Examples of the computation of such height for projective hypersurfaces can be found in [CM00].

Specializing Theorem 4.2.10, one can write the Fubini-Study height of a projective hypersurface in terms of convex geometry. To do so, recall that  $\mathfrak{M}_\infty$  denotes the collection of archimedean places of  $\mathbb{K}$ , which is a finite set by Lemma 2.3.5. After fixing an isomorphism  $M \simeq \mathbb{Z}^n$ , consider the standard simplex

$$\Delta^n := \{(x_1, \dots, x_n) : x_1 + \dots + x_n \leq 1, x_i \geq 0 \text{ for all } i = 1, \dots, n\}$$

in  $M_{\mathbb{R}} \simeq \mathbb{R}^n$  and, agreeing that  $x_0 := 1 - \sum_{i=1}^n x_i$ , set the function  $\vartheta_{\text{FS}} : \Delta^n \rightarrow \mathbb{R}$  to be

$$\vartheta_{\text{FS}}(x) := -\frac{1}{2} \sum_{i=0}^n x_i \log x_i,$$

which is defined on the boundary of  $\Delta^n$  by continuity.

**Proposition 4.3.8.** *Let  $Z$  be an effective cycle on  $\mathbb{P}_{\mathbb{K}}^n$  of pure codimension 1 and prime components intersecting  $X_0$ . The Fubini-Study height of  $Z$  is given by*

$$h_{\text{FS}}(Z) = \sum_{v \in \mathfrak{M}_\infty} n_v \text{MI}_M(\vartheta_{\text{FS}}, \dots, \vartheta_{\text{FS}}, \rho_{f,v}^\vee) + \sum_{v \in \mathfrak{M} \setminus \mathfrak{M}_\infty} n_v m_v(f),$$

where  $f$  is a defining polynomial for  $Z$ .

*Proof.* The roof functions of the metrized divisor  $\overline{D}_\infty$  are given by the function  $\vartheta_{\text{FS}}$  at archimedean places, as remarked in [BPS14, Example 2.4.3 and Example 4.3.9 (2)] and by the indicator function of  $\Delta^n$  at non-archimedean places, by [BPS14, Example 4.3.9 (1)] and Example 1.3.8. The statement follows then from Theorem 4.2.10, Corollary 1.5.11 and Definition 4.1.13, together with the fact that  $\text{MV}_M(\Delta^n, \dots, \Delta^n) = 1$  because of the definition of mixed volume and Notation 1.2.11.  $\square$

Example 4.1.15 allows to deduce without difficulties the following special case.

**Corollary 4.3.9.** *Assume the base adelic field to be  $\mathbb{Q}$  with its usual adelic structure. The Fubini-Study height of an effective cycle  $Z$  on  $\mathbb{P}_{\mathbb{Q}}^n$  of pure codimension 1 and prime components intersecting  $X_0$  is given by*

$$h_{\text{FS}}(Z) = \text{MI}_M(\vartheta_{\text{FS}}, \dots, \vartheta_{\text{FS}}, \rho_{f,\infty}^\vee),$$

where  $f$  is a defining polynomial for  $Z$  whose coefficients are integer numbers with greatest common divisor 1.

Because of the presence of an archimedean Ronkin function, the formula in Corollary 4.3.9 appears arduous to evaluate. One can however give estimations for it in some special situations, as the following.

**Example 4.3.10.** Consider the zero set of the homogeneous polynomial  $T_0 + T_1 + T_2$  in  $\mathbb{P}_{\mathbb{Q}}^2$ . Its associated cycle  $Z$  has a unique irreducible component, of multiplicity 1, and with defining polynomial  $f = 1 + x + y$ . By taking advantage of the fact that  $\text{NP}(f)$  coincides with the standard simplex  $\Delta$  in  $\mathbb{R}^2$ , which is also the polytope associated to  $D_{\infty}$  and has normalized volume  $1/2$ , one can use Proposition 4.2.17 to give concise bounds for  $h_{\text{FS}}(Z)$ . For the upper bound,

$$h_{\text{FS}}(Z) \leq \mathfrak{m}(f) - 2\psi_{\infty}(0),$$

where  $\psi_{\infty}$  is the metric function associated to the Fubini-Study height. Using its explicit expression as in [BPS14, Example 2.4.3] one deduces that  $\psi_{\infty}(0) = -(1/2)\log 3$ . The lower bound is instead, using directly the more precise Theorem 1.5.14,

$$h_{\text{FS}}(Z) \geq \max\left(\mathfrak{m}(f) + 2\min_{\Delta} \vartheta_{\infty}, -\psi_{\infty}(0) + \min_{\Delta} \vartheta_{\infty} + \min_{\Delta} \rho_{f,\infty}^{\vee}\right).$$

By the concavity of  $\vartheta_{\infty}$  and the same argument in the proof of Proposition 4.1.16, such a minimum has to be attained at a vertex of  $\Delta$ . The symmetry under the transformations  $(x_1, x_2) \mapsto (1 - x_1 - x_2, x_2)$  and  $(x_1, x_2) \mapsto (x_1, 1 - x_1 - x_2)$  assures that the value of the function is the same at all the vertices of  $\Delta$ , then  $\min \vartheta_{\infty} = \vartheta_{\infty}(0, 0) = 0$ . It is instead straightforward from Proposition 4.1.16 that  $\min \rho_{f,\infty}^{\vee} = 0$ . One deduces that

$$\max\left(\mathfrak{m}(f), \frac{\log 3}{2}\right) \leq h_{\text{FS}}(Z) \leq \mathfrak{m}(f) + \log 3.$$

Using the fact that

$$\mathfrak{m}(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) = 0.3230659\dots$$

as proved by Smyth in the appendix of [Boy81], with  $L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{1}{n^s}$ ,

$$\frac{\log 3}{2} \leq h_{\text{FS}}(Z) \leq \mathfrak{m}(f) + \log 3,$$

which gives

$$0.549306 < h_{\text{FS}}(Z) < 1.421679.$$



# Upper bounds of heights

We show in this chapter that the formula for the height of an hypersurface proven in the previous chapter can not be generalized to higher codimensions. We then prove combinatorial upper bounds for the global height of a complete intersection in a toric variety in terms of the defining polynomials, slightly refining an argument by Martínez and Sombra. To do this, we introduce the notion of upper functions of polynomials using the same language adopted for the definition of Ronkin functions.

We suppose fixed for the whole chapter an adelic field  $(\mathbb{K}, (n_v, |\cdot|_v)_{v \in \mathfrak{M}})$  satisfying the product formula and a proper toric variety  $X_\Sigma$  over  $\mathbb{K}$  of dimension  $n$ , with dense open orbit  $X_0$ . As in the previous exposition, we denote by  $\mathbb{T}$  the corresponding torus, by  $M$  and  $N$  respectively its character and cocharacter lattice and by  $M_{\mathbb{R}}$  and  $N_{\mathbb{R}}$  the associated real vector spaces.

## 5.1

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### MOTIVATION

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We present here a counterexample for the straightforward extension of the formula for the height of an hypersurface in a toric variety to higher codimensions.

**A GUESS.** Recall that for a nonzero Laurent polynomial  $f$  with coefficients in  $\mathbb{K}$ , with Newton polytope compatible with the fan  $\Sigma$  in the sense of Definition 3.1.11, it is possible to consider its associated toric divisor  $D_f$  by Theorem 3.1.9; it comes with a distinguished toric section  $s_f$ . Let  $k \in \{0, \dots, n\}$  and

consider  $f_1, \dots, f_k$  a family of nonzero Laurent polynomials with coefficients in  $\mathbb{K}$  and Newton polytopes compatible with the fan  $\Sigma$ ; this last hypothesis can always be verified up to a birational transformation because of Proposition 3.1.12. Under the assumption that the sections  $f_i s_{f_i}$  meet  $X_\Sigma$  properly with  $i = 1, \dots, k$ , denote by

$$Z(f_1, \dots, f_k) := \operatorname{div}(f_k s_{f_k}) \cdots \operatorname{div}(f_1 s_{f_1}) \cdot X_\Sigma$$

the corresponding intersection cycle. To lighten the notation, one will also denote the family  $(f_1, \dots, f_k)$  by  $\underline{f}$  and its intersection cycle by  $Z(\underline{f})$ . Such a cycle is, whenever defined, of dimension  $n - k$  and its restriction to the dense open orbit of  $X_\Sigma$  is the cycle associated to the closed subscheme of the ideal  $(f_1, \dots, f_k)$  in  $\mathbb{K}[M]$ .

Choose  $n - k + 1$  semipositive adelic toric metrized divisors  $\bar{D}_0, \dots, \bar{D}_{n-k}$  on  $X_\Sigma$  and let  $\vartheta_{i,v}$  be the  $v$ -adic roof function of  $D_i$  for every  $i = 0, \dots, n - k$  and  $v \in \mathfrak{M}$ . One is interested in a combinatorial expression, in terms of convex geometry, of the global height

$$h_{\bar{D}_0, \dots, \bar{D}_{n-k}}(Z(\underline{f})).$$

For  $k = 0$ , Theorem 3.4.12 asserts that

$$h_{\bar{D}_0, \dots, \bar{D}_n}(Z(\underline{f})) = h_{\bar{D}_0, \dots, \bar{D}_n}(X_\Sigma) = \sum_{v \in \mathfrak{M}} n_v \operatorname{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n,v})$$

while for  $k = 1$ , it follows from Theorem 4.2.10 that

$$h_{\bar{D}_0, \dots, \bar{D}_{n-1}}(Z(\underline{f})) = h_{\bar{D}_0, \dots, \bar{D}_{n-1}}(\overline{V(f_1)}) = \sum_{v \in \mathfrak{M}} n_v \operatorname{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n-1,v}, \rho_{f_1,v}^\vee).$$

One can ask whether a similar formula could hold for higher values of  $k$ , namely whether the global height  $h_{\bar{D}_0, \dots, \bar{D}_{n-k}}(Z(f_1, \dots, f_k))$  could be given by

$$\sum_{v \in \mathfrak{M}} n_v \operatorname{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n-k,v}, \rho_{f_1,v}^\vee, \dots, \rho_{f_k,v}^\vee).$$

**A COUNTEREXAMPLE.** The extension of the formula proposed above can not hold for all  $k$ . Indeed, if it did, the height of  $Z(f_1, \dots, f_k)$  would only depend on the family of Ronkin functions of the involved polynomials, which turns out to be false even in codimension 2, as the next example shows.

**Example 5.1.1.** Let  $\zeta_3$  and  $\zeta_4$  denote respectively a primitive third and fourth root of unity and let  $\mathbb{K}$  be the number field  $\mathbb{Q}(\zeta_3, \zeta_4)$  endowed with the adelic

structure coming from  $\mathbb{Q}$  as in Proposition 2.3.12. After having chosen a basis of the lattice  $M$ , consider the polynomials

$$f = 1 + x + y, \quad g = 1 + \zeta_3^2 x + \zeta_3 y, \quad g' = 1 - x + \zeta_4 y$$

in  $\mathbb{K}[x^{\pm 1}, y^{\pm 1}]$ . It is easily verified that for each place  $v$  of  $\mathbb{K}$ , the  $v$ -adic Ronkin functions of  $f$ ,  $g$  and  $g'$  coincide. The canonical height on  $\mathbb{P}_{\mathbb{K}}^2$  with respect to the Cartier divisor with corresponding Weil divisor the line at infinity is also called the *Weil height* on  $\mathbb{P}_{\mathbb{K}}^2$  and denoted by  $h_{\text{Weil}}$ . It coincides with the height defined in [BG06, §1.5.1]. There are two possible embeddings of  $\mathbb{K}$  in  $\mathbb{C}$  up to complex conjugation: the first sends  $\zeta_3$  to  $e^{2\pi i/3}$  and  $\zeta_4$  to  $i$ , the second  $\zeta_3$  to  $e^{2\pi i/3}$  and  $\zeta_4$  to  $-i$ . Then, using Proposition 2.1.21,

$$\begin{aligned} h_{\text{Weil}}(Z(f, g)) &= h_{\text{Weil}}((\zeta_3 : 1 : -\zeta_3 - 1)) = \sum_{v \in \mathfrak{M}_{\mathbb{K}}} n_v \log^+ |\zeta_3 + 1|_v \\ &= 2 \cdot \frac{1}{2} \log^+ |e^{2\pi i/3} + 1|_{\infty} = 0 \end{aligned}$$

since  $\zeta_3 + 1$  is an algebraic integer and  $n_v = [\mathbb{C} : \mathbb{R}]/[\mathbb{K} : \mathbb{Q}] = 1/2$  for every archimedean place  $v$  of  $\mathbb{K}$ , see [MS16, Example 3.8]. Analogously,

$$\begin{aligned} h_{\text{Weil}}(Z(f, g')) &= h_{\text{Weil}}((\zeta_4 : 1 : -\zeta_4 - 1)) = \sum_{v \in \mathfrak{M}_{\mathbb{K}}} n_v \log^+ |\zeta_4 + 1|_v \\ &= \frac{1}{2} \log^+ |i + 1|_{\infty} + \frac{1}{2} \log^+ |-i + 1|_{\infty} = \frac{1}{2} \log 2. \end{aligned}$$

This shows that, in general, the height of a complete intersection can not only depend on the Ronkin functions of the defining polynomials.

## 5.2

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### UPPER FUNCTIONS OF POLYNOMIALS

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We introduce here the notions of upper function and gross upper function associated to a Laurent polynomial in  $\mathbb{K}[M]$  for each place of  $\mathbb{K}$ . We then study such functions and prove a number of properties they rejoice.

**DEFINITIONS AND EXAMPLES.** Recall that for any place  $v \in \mathfrak{M}$ , the analytic torus  $\mathbb{T}_v^{\text{an}}$  consists, set-theoretically, of the multiplicative seminorms on  $\mathbb{C}_v[M]$  extending  $|\cdot|_v$ . Moreover, there exists a tropicalization map  $\text{trop}_v : \mathbb{T}_v^{\text{an}} \rightarrow N_{\mathbb{R}}$  as in Definition 3.4.2.

**Definition 5.2.1.** Let  $f$  be a nonzero Laurent polynomial over  $\mathbb{K}$ . For a place  $v \in \mathfrak{M}$ , the  $v$ -adic upper function of  $f$  is the map  $\mu_{f,v} : N_{\mathbb{R}} \rightarrow \mathbb{R}$  defined as

$$\mu_{f,v}(u) := - \max_{x \in \text{trop}_v^{-1}(u)} \log \|f\|_x$$

for every  $u \in N_{\mathbb{R}}$ .

It is clear from Definition 4.1.1 that

$$\mu_{f,v}(u) = - \max_{x \in \mathcal{B}_v(u)} \log \|f\|_x, \quad (5.1)$$

where  $\mathcal{B}_v(u)$  denotes the boundary of the fiber of the tropicalization map over  $u$  as in the discussion after Proposition 4.1.4. The explicit description of such a boundary proved in the cited proposition allows to give a more explicit expression for the  $v$ -adic upper function and to check that it takes a real value at each  $u \in N_{\mathbb{R}}$ .

**Remark 5.2.2.** Assume that  $v$  is an archimedean place of  $\mathbb{K}$  and let  $\sigma$  be the corresponding embedding of  $\mathbb{K}$  in  $\mathbb{C}$ , up to complex conjugation. Fixing the choice of a basis of  $M$  and of the dual basis of  $N$ , one identifies  $\mathbb{T}_v^{\text{an}}$  with  $(\mathbb{C}^*)^n$  as in Remark 3.4.1. Then, for every  $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ ,

$$\mu_{f,v}(u) = - \max_{\theta_1, \dots, \theta_n \in [0, 2\pi]} \log \left| f^\sigma(e^{-u_1 + i\theta_1}, \dots, e^{-u_n + i\theta_n}) \right|_\infty,$$

where  $f^\sigma = \sum_m \sigma(c_m) \chi^m \in \mathbb{C}[M]$ .

**Remark 5.2.3.** If  $v$  is a non-archimedean place of  $\mathbb{K}$ , it follows immediately from (5.1) and Proposition 4.1.4 that  $\mu_{f,v} = f^{\text{trop},v}$  as in Remark 4.1.7.

A more explicit function associated to a nonzero Laurent polynomial is given in the following definition, which coincides with [MS16, (1.2)].

**Definition 5.2.4.** Let  $f = \sum_m c_m \chi^m$  be a nonzero Laurent polynomial over  $\mathbb{K}$ . For a place  $v \in \mathfrak{M}$ , the *gross  $v$ -adic upper function* of  $f$  is the map  $\tilde{\mu}_{f,v} : N_{\mathbb{R}} \rightarrow \mathbb{R}$  defined as

$$\tilde{\mu}_{f,v}(u) := \begin{cases} -\log \left( \sum_m |c_m|_v e^{-\langle m, u \rangle} \right) & \text{if } v \text{ is archimedean} \\ \mu_{f,v}(u) & \text{otherwise} \end{cases} \quad (5.2)$$

for every  $u \in N_{\mathbb{R}}$ .

When  $v$  is non-archimedean, it follows immediately from Remark 5.2.3 and Definition 5.2.4 that

$$\tilde{\mu}_{f,v} = \mu_{f,v} = \rho_{f,v}. \quad (5.3)$$

The relation among the Ronkin function, the upper function and the gross upper function is instead less tame in the archimedean setting, as the next two examples show.

**Example 5.2.5.** Let  $\mathbb{K} = \mathbb{Q}$ ,  $M = \mathbb{Z}$  and  $f = x^2 + x + 1$ . For the unique archimedean absolute value  $v$  of  $\mathbb{K}$  one has by Remark 5.2.2 that

$$\mu_{f,v}(u) = - \max_{\theta \in [0, 2\pi]} \log |f(e^{-u+i\theta})| = - \log(e^{-2u} + e^{-u} + 1) = \tilde{\mu}_{f,v}(u)$$

for every  $u \in N_{\mathbb{R}}$ .

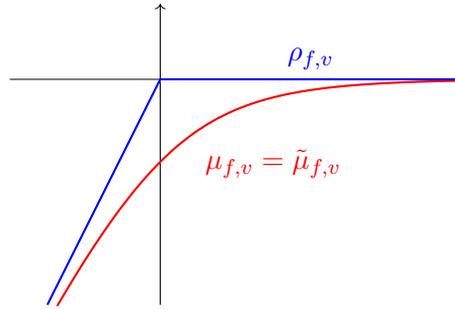


Figure 5.1: The case of Example 5.2.5.

**Example 5.2.6.** Let again  $\mathbb{K} = \mathbb{Q}$ ,  $M = \mathbb{Z}$  and  $v$  be the unique archimedean absolute value of  $\mathbb{K}$ . For the Laurent polynomial  $f = x^2 + x - 1$  one has

$$|f(e^{-u+i\theta})|^2 = e^{-4u} + 3e^{-2u} + 1 + 2e^{-u}(e^{-2u} - 1) \cos \theta - 4e^{-2u} \cos^2 \theta.$$

This function is the restriction to the closed interval  $[-1, 1]$  of a concave parabola in  $\cos \theta$ . Its maximum is then

$$\max_{\theta \in [0, 2\pi]} |f(e^{-u+i\theta})|^2 = \begin{cases} e^{-4u} + 2e^{-3u} - e^{-2u} - 2e^{-u} + 1 & \text{if } \sinh(u) \leq -2 \\ \frac{5}{4}(e^{-4u} + 2e^{-2u} + 1) & \text{if } \sinh(u) \in [-2, 2], \\ e^{-4u} - 2e^{-3u} - e^{-2u} + 2e^{-u} + 1 & \text{if } \sinh(u) \geq 2 \end{cases}$$

implying that

$$\mu_{f,v}(u) = \begin{cases} -\frac{1}{2} \log(e^{-4u} + 2e^{-3u} - e^{-2u} - 2e^{-u} + 1) & \text{if } u \leq \log(\sqrt{5} - 2) \\ -\frac{1}{2} \log\left(\frac{5}{4}(e^{-4u} + 2e^{-2u} + 1)\right) & \text{otherwise} \\ -\frac{1}{2} \log(e^{-4u} - 2e^{-3u} - e^{-2u} + 2e^{-u} + 1) & \text{if } u \geq \log(\sqrt{5} + 2) \end{cases}.$$

The gross  $v$ -adic upper function of  $f$  is instead, as in Example 5.2.5

$$\tilde{\mu}_{f,v}(u) = -\log(e^{-2u} + e^{-u} + 1).$$

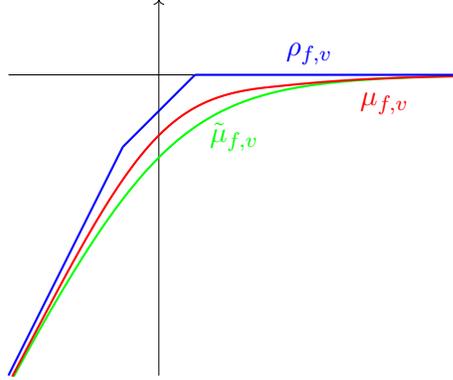


Figure 5.2: The case of Example 5.2.6.

Despite their wild behaviour in the archimedean setting, one can give a sequence of inequalities relating the Ronkin function, the upper function and the gross upper function of a nonzero Laurent polynomial at any place  $v$ .

**Proposition 5.2.7.** *For a nonzero Laurent polynomial  $f \in \mathbb{K}[M]$  and a place  $v \in \mathfrak{M}$ ,*

$$\tilde{\mu}_{f,v} \leq \mu_{f,v} \leq \rho_{f,v}.$$

*Proof.* The statement is trivial for non-archimedean places because of (5.3). Assume hence that  $v$  is archimedean; for every  $u \in N_{\mathbb{R}}$ , each  $x \in \text{trop}_v^{-1}(u)$  is an archimedean absolute value over  $\mathbb{K}[M]$  satisfying  $\|\chi^m\|_x = e^{-\langle m, u \rangle}$  by the definition of the map  $\text{trop}_v$ , see Definition 3.4.2. Then, if  $f = \sum_m c_m \chi^m$ , the triangular inequality yields

$$\|f\|_x \leq \sum_m |c_m|_v e^{-\langle m, u \rangle}$$

for every  $x \in \text{trop}_v^{-1}(u)$ . So, for every  $u \in N_{\mathbb{R}}$ ,

$$\mu_{f,v}(u) = \min_{x \in \text{trop}_v^{-1}(u)} -\log \|f\|_x \geq -\log \left( \sum_m |c_m|_v e^{-\langle m, u \rangle} \right) = \tilde{\mu}_{f,v}(u),$$

proving the first inequality. Regarding the second one, Definition 5.2.1 gives

$$-\log \|f\|_x \geq \mu_{f,v}(u)$$

for all  $x \in \text{trop}_v^{-1}(u)$ . The monotonicity of the integral on  $\text{trop}_v^{-1}(u)$  and the fact that the total mass of the measure in (4.1) is 1, together with the definition of the Ronkin function in Definition 4.1.5 imply that  $\rho_{f,v}(u) \geq \mu_{f,v}(u)$  for all  $u \in N_{\mathbb{R}}$ .  $\square$

**PROPERTIES.** To study the properties of upper functions and gross upper functions, one can start by considering their behaviour with respect to products and scalar multiplications, as done in Proposition 4.1.8 for Ronkin functions.

**Proposition 5.2.8.** *For every nonzero Laurent polynomials  $f, g \in \mathbb{K}[M]$ ,  $v \in \mathfrak{M}$  and  $c \in \mathbb{K}^*$ , one has*

1.  $\mu_{f \cdot g, v} \geq \mu_{f, v} + \mu_{g, v}$
2.  $\tilde{\mu}_{f \cdot g, v} \geq \tilde{\mu}_{f, v} + \tilde{\mu}_{g, v}$
3.  $\mu_{cf, v} = -\log |c|_v + \mu_{f, v}$
4.  $\tilde{\mu}_{cf, v} = -\log |c|_v + \tilde{\mu}_{f, v}$ .

*Proof.* All the claims are obvious in the non-archimedean setting because of (5.3) and Proposition 4.1.8. Assume then that  $v$  is an archimedean place of  $\mathbb{K}$  and let  $f = \sum_m a_m \chi^m$  and  $g = \sum_l b_l \chi^l$ . To prove 2,

$$\begin{aligned} \tilde{\mu}_{f \cdot g, v}(u) &= -\log \left( \sum_k \left| \sum_{m+l=k} a_m b_l \right|_v e^{-\langle k, u \rangle} \right) \\ &\geq -\log \left( \sum_k \sum_{m+l=k} |a_m|_v |b_l|_v e^{-\langle k, u \rangle} \right) \\ &= -\log \left( \sum_{m, l} |a_m|_v |b_l|_v e^{-\langle m, u \rangle} e^{-\langle l, u \rangle} \right) \\ &= \tilde{\mu}_{f, v} + \tilde{\mu}_{g, v}. \end{aligned}$$

The other claims follow easily from the definitions.  $\square$

The following statement is instead the equivalent of Proposition 4.1.9. Recall that for a nonzero Laurent polynomial  $f \in \mathbb{K}[M]$ , the map  $\Psi_{\text{NP}(f)}$  denotes the support function of the Newton polytope of  $f$ , see Definition 3.2.1 and Remark 1.2.5.

**Proposition 5.2.9.** *Let  $f$  be a nonzero Laurent polynomial over  $\mathbb{K}$  and  $v \in \mathfrak{M}$ . Then,*

1.  $\mu_{f,v}$  and  $\tilde{\mu}_{f,v}$  are continuous concave functions on  $N_{\mathbb{R}}$  (in particular they are closed)
2.  $|\mu_{f,v} - \Psi_{\text{NP}(f)}|$  and  $|\tilde{\mu}_{f,v} - \Psi_{\text{NP}(f)}|$  are bounded on  $N_{\mathbb{R}}$
3. the stability sets of  $\mu_{f,v}$  and  $\tilde{\mu}_{f,v}$  coincide with  $\text{NP}(f)$  and  $\text{rec}(\mu_{f,v}) = \text{rec}(\tilde{\mu}_{f,v}) = \Psi_{\text{NP}(f)}$ .

*Proof.* For a non-archimedean place  $v$  of  $\mathbb{K}$  the three claims follow from (5.3) and the analogous properties of the Ronkin function proved in Proposition 4.1.9.

Assume hence that  $v$  is an archimedean place of  $\mathbb{K}$  and let  $f = \sum_m c_m \chi^m$ . As in the proof of Proposition 4.1.9, denote by  $\gamma(f)$  the number of nonzero coefficients of  $f$ . Then, for every  $u \in N_{\mathbb{R}}$

$$\begin{aligned} \tilde{\mu}_{f,v}(u) &= -\log \left( \sum_m |c_m|_v e^{-\langle m, u \rangle} \right) \geq -\log \left( \gamma(f) \max_m |c_m|_v \cdot \max_m e^{-\langle m, u \rangle} \right) \\ &= \min_m \langle m, u \rangle - \log(\gamma(f) \max_m |c_m|_v) = \Psi_{\text{NP}(f)}(u) - \log(\gamma(f) \max_m |c_m|_v). \end{aligned}$$

For a reverse inequality, Proposition 5.2.7 and Proposition 4.1.9 (2) imply that there exists a real constant  $c$  for which

$$\tilde{\mu}_{f,v}(u) \leq \Psi_{\text{NP}(f)}(u) + c$$

for every  $u \in N_{\mathbb{R}}$ . This proves (2) for the function  $\tilde{\mu}_{f,v}$ ; the same claim for  $\mu_{f,v}$  follows hence easily from Proposition 5.2.7 and Proposition 4.1.9 (2).

As a result, one can treat  $\tilde{\mu}_{f,v}$  as the metric function corresponding to a toric metric on the toric Cartier divisor associated to  $\text{NP}(f)$  on a certain toric variety; the concavity of  $\tilde{\mu}_{f,v}$  follows then from the geometric approach of [MS16, Proposition 6.2] and the characterization of semipositive toric metrics in Theorem 3.4.7.

To prove (1) for  $\mu_{f,v}$ , fix the choice of a basis of  $M$  and of its dual basis of  $N$  and let  $\sigma$  be the embedding of  $\mathbb{K}$  in  $\mathbb{C}$ , up to complex conjugation,

corresponding to the place  $v$ . By Remark 5.2.2, the  $v$ -adic upper function of  $f$  can then be written as

$$\mu_{f,v}(u) = \min_{\theta_1, \dots, \theta_n \in [0, 2\pi]} -\log \left| f^\sigma(e^{-u_1+i\theta_1}, \dots, e^{-u_n+i\theta_n}) \right|_\infty.$$

Consider for any  $(\theta_1, \dots, \theta_n) \in [0, 2\pi]^n$  the function  $F_{\theta_1, \dots, \theta_n} : N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{\infty\}$  defined by

$$F_{\theta_1, \dots, \theta_n} : u \mapsto -\log \left| f^\sigma(e^{-u_1+i\theta_1}, \dots, e^{-u_n+i\theta_n}) \right|_\infty.$$

Setting

$$f_{\theta_1, \dots, \theta_n} = \sum_m \sigma(c_m) e^{i(m_1\theta_1 + \dots + m_n\theta_n)} \chi^m \in \mathbb{C}[M],$$

one sees that the function

$$F_{\theta_1, \dots, \theta_n}(u) = -\log |f_{\theta_1, \dots, \theta_n}(e^{-u_1}, \dots, e^{-u_n})|_\infty$$

is concave because of [Yge15, Proposition 2.4] and the fact that  $\log |f_{\theta_1, \dots, \theta_n}|_\infty$  is a plurisubharmonic function on  $(\mathbb{C}^*)^n$ . The pointwise infimum of any family of concave functions being again concave because of [Roc70, Theorem 5.5], one deduces that  $\mu_{f,v}$  is so.

Being concave on  $N_{\mathbb{R}}$ ,  $\mu_{f,v}$  and  $\tilde{\mu}_{f,v}$  are continuous on  $N_{\mathbb{R}}$ , then closed. Property (3) is a direct consequence of (2) as in the proof of Proposition 4.1.9.  $\square$

### 5.3

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#### UPPER BOUNDS

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Using an analogous idea to the one of [MS16], we prove here combinatorial expressions giving upper bounds for the height of cycles in toric varieties in terms of the defining polynomials. We first give some general definitions and properties and then describe the toric case.

**SMALL SECTIONS.** The existence of a metric on line bundles, which measures sections, allows to introduce the following fundamental notion, see also [Sou92, §3.6].

**Definition 5.3.1.** Let  $\overline{D} = (D, (\|\cdot\|_v)_{v \in \mathfrak{M}})$  be a metrized divisor on a proper variety  $X$  over  $\mathbb{K}$ . For a place  $v \in \mathfrak{M}$ , a global section  $s$  of  $\mathcal{O}(D)$  is said to be  $\overline{D}_v$ -small if

$$\log \|s(p)\|_v \leq 0$$

for every point  $p \in X_v^{\text{an}}$ ,  $p \notin |\text{div}(s)|$ . A section  $s$  of  $\mathcal{O}(D)$  is said to be  $\overline{D}$ -small if it is  $\overline{D}_v$ -small for every  $v \in \mathfrak{M}$ .

Small sections are particularly useful to give recursive bounds for the height of algebraic cycles.

**Lemma 5.3.2.** *Let  $X$  be a proper variety over  $\mathbb{K}$  and  $\overline{D}_{0,v}, \dots, \overline{D}_{d,v}$  a family of  $v$ -adic semipositive metrized divisors on  $X$  for a place  $v \in \mathfrak{M}$ . Let  $Z$  be a  $d$ -dimensional effective cycle in  $X$  and  $s_0, \dots, s_d$  rational sections of  $\mathcal{O}(D_0), \dots, \mathcal{O}(D_d)$  respectively, meeting  $Z$  properly. If  $s_d$  is a  $\overline{D}_{d,v}$ -small section, then*

$$h_{\overline{D}_{0,v}, \dots, \overline{D}_{d-1,v}}(Z \cdot \text{div}(s_d); s_0, \dots, s_{d-1}) \leq h_{\overline{D}_{0,v}, \dots, \overline{D}_{d,v}}(Z; s_0, \dots, s_d).$$

*Proof.* The cycle  $Z$  being effective and the metrized divisors semipositive, the measure  $c_1(\overline{D}_{0,v}) \wedge \dots \wedge c_1(\overline{D}_{d-1,v}) \wedge \delta_{Z_v^{\text{an}}}$  is a positive measure on  $X_v^{\text{an}}$ . The definition of local height, the monotonicity of the integral and the hypothesis on  $s_d$  imply then that the difference between the  $v$ -adic local height of  $Z \cdot \text{div}(s_d)$  and the height of  $Z$  with respect to the chosen sections is

$$\int_{X_v^{\text{an}}} \log \|s_d\|_{d,v} c_1(\overline{D}_{0,v}) \wedge \dots \wedge c_1(\overline{D}_{d-1,v}) \wedge \delta_{Z_v^{\text{an}}} \leq 0,$$

yielding the desired inequality.  $\square$

**Proposition 5.3.3.** *Let  $X$  be a smooth projective variety over  $\mathbb{K}$ ,  $\overline{D}_0, \dots, \overline{D}_d$  a collection of semipositive metrized divisors on  $X$  and  $Z$  a  $d$ -dimensional effective cycle on  $X$ . If  $s_d$  is a  $\overline{D}_d$ -small section of  $\mathcal{O}(D_d)$  meeting  $Z$  properly,*

$$h_{\overline{D}_0, \dots, \overline{D}_{d-1}}(Z \cdot \text{div}(s)) \leq h_{\overline{D}_0, \dots, \overline{D}_d}(Z),$$

*provided both terms are defined.*

*Proof.* The assumptions on  $X$  assure the existence of sections  $s_0, \dots, s_{d-1}$  such that  $s_0, \dots, s_{d-1}, s$  meet  $Z$  properly. Whenever both cycles in the claim are integrable, the statement is immediate from Lemma 5.3.2.  $\square$

**THE TORIC CASE.** Fix now a proper toric variety  $X_\Sigma$  of dimension  $n$  over  $\mathbb{K}$ . For a nonzero Laurent polynomial  $f$  in  $\mathbb{K}[M]$ , whose Newton polytope is compatible with the fan  $\Sigma$  in the sense of Definition 3.1.11, one can consider the divisor  $D_f$  on  $X_\Sigma$  associated to the support function  $\Psi_{\text{NP}(f)}$  by Theorem 3.1.9. Thanks to Theorem 3.4.7 and Proposition 5.2.9, the collection of  $v$ -adic upper functions of  $f$  define a semipositive toric metric on  $D_f$ .

**Definition 5.3.4.** In the notation and hypotheses above, the  $\mu$ -metric on  $D_f$  is the semipositive toric metric on  $D_f$  corresponding to the family  $(\mu_{f,v})_{v \in \mathfrak{M}}$  via Theorem 3.4.7.

One will denote by  $\|\cdot\|_{\mu,v}$  the  $v$ -adic  $\mu$ -metric of  $D_f$  and by  $\overline{D}_f^\mu$  the divisor  $D_f$  endowed with its  $\mu$ -metric. A first easy property of  $\overline{D}_f^\mu$  is given by the following.

**Proposition 5.3.5.** *The  $\mu$ -metric on  $D_f$  is adelic.*

*Proof.* The statement is a direct consequence of Proposition 2.3.5, Lemma 4.2.9 and (5.3).  $\square$

The main interest of the  $\mu$ -metric on  $D_f$  is its relation with small sections and is stated in the next proposition.

**Proposition 5.3.6.** *The section  $f s_f$  of  $\mathcal{O}(D_f)$  is  $\overline{D}_f^\mu$ -small in the sense of Definition 5.3.1*

*Proof.* Let  $v \in \mathfrak{M}$ . For a point  $p \in X_{0,v}^{\text{an}}$ ,  $p \notin |\text{div}(f s_f)|$ , the definition of the metric and the one of the  $v$ -adic upper function of  $f$  yield

$$\log \|(f s_f)(p)\|_{\mu,v} = \log |f(p)| + \mu_{f,v}(\text{trop}_v(p)) \leq 0.$$

The continuity of the metric implies that the inequality equally holds, outside the divisor of  $f s_f$ , in  $X_{\Sigma,v}^{\text{an}} \setminus X_{0,v}^{\text{an}}$ , concluding the proof.  $\square$

The following statement is the main result of the chapter.

**Theorem 5.3.7.** *Let  $X_\Sigma$  be a smooth projective toric variety of dimension  $n$  over  $\mathbb{K}$ ,  $k \in \{0, \dots, n\}$  and  $f_1, \dots, f_k$  nonzero Laurent polynomials in  $\mathbb{K}[M]$  with Newton polytopes compatible with the fan  $\Sigma$ . Assume that the sections  $f_1 s_{f_1}, \dots, f_k s_{f_k}$  of  $\mathcal{O}(D_{f_1}), \dots, \mathcal{O}(D_{f_k})$  respectively meet  $X_\Sigma$  properly and let*

$$Z(f_1, \dots, f_k) := \text{div}(f_1 s_{f_1}) \cdots \text{div}(f_k s_{f_k}) \cdot X_\Sigma \quad (5.4)$$

*be the corresponding cycle in  $X_\Sigma$ , of dimension  $n - k$ . Let  $\overline{D}_0, \dots, \overline{D}_{n-k}$  be adelic semipositive toric metrized divisors on  $X_\Sigma$ . Then*

$$h_{\overline{D}_0, \dots, \overline{D}_{n-k}}(Z(f_1, \dots, f_k)) \leq \sum_{v \in \mathfrak{M}} \text{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n-k,v}, \mu_{f_1,v}^\vee, \dots, \mu_{f_k,v}^\vee),$$

*where  $\vartheta_{i,v}$  is the  $v$ -adic roof function of  $\overline{D}_i$  for each  $i = 0, \dots, n - k$  and  $v \in \mathfrak{M}$ .*

*Proof.* The proof is done by induction on  $k$ . The claim for  $k = 0$  is obvious from Theorem 3.4.12. Assume hence that the inequality is true for  $k - 1$ , with  $k \geq 1$ . Consider the Laurent polynomials  $f_1, \dots, f_k$  and the adelic semi-positive toric metrized divisors  $\overline{D}_0, \dots, \overline{D}_{n-k}$ . Since the family of sections  $f_1 s_{f_1}, \dots, f_k s_{f_k}$  meets  $X_\Sigma$  properly, so does the subfamily  $f_1 s_{f_1}, \dots, f_{k-1} s_{f_{k-1}}$  and one has by the commutativity of intersection products, see [Ful98, Theorem 2.4],

$$Z(f_1, \dots, f_k) = \text{div}(f_k s_{f_k}) \cdot Z(f_1, \dots, f_{k-1}).$$

Thanks to Theorem 3.5.4, the hypotheses of the theorem and Proposition 5.3.5, the cycle  $Z(f_1, \dots, f_k)$  is integrable with respect to  $\overline{D}_0, \dots, \overline{D}_{n-k}$  and  $Z(f_1, \dots, f_{k-1})$  is integrable with respect to  $\overline{D}_0, \dots, \overline{D}_{n-k}, \overline{D}_{f_k}^\mu$ . Moreover, the cycle  $Z(f_1, \dots, f_{k-1})$  is effective since it is obtained by intersecting effective Cartier divisors on  $X_\Sigma$ .

Hence, Proposition 5.3.3 and Proposition 5.3.6 imply that

$$h_{\overline{D}_0, \dots, \overline{D}_{n-k}}(Z(f_1, \dots, f_k)) \leq h_{\overline{D}_0, \dots, \overline{D}_{n-k}, \overline{D}_{f_k}^\mu}(Z(f_1, \dots, f_{k-1})).$$

The induction hypothesis and the symmetry of the mixed integral operator, together with the fact that the  $v$ -adic roof function of  $\overline{D}_{f_k}^\mu$  is  $\mu_{f_k, v}^\vee$  because of Definition 5.3.4, conclude the proof.  $\square$

The  $v$ -adic upper function of a Laurent polynomial being more complicated than the gross  $v$ -adic upper function, one can prove the following more computable, though weaker, upper bound.

**Corollary 5.3.8.** *In the same hypotheses and notation of Theorem 5.3.7,*

$$h_{\overline{D}_0, \dots, \overline{D}_{n-k}}(Z(f_1, \dots, f_k)) \leq \sum_{v \in \mathfrak{M}} \text{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n-k,v}, \tilde{\mu}_{f_1,v}^\vee, \dots, \tilde{\mu}_{f_k,v}^\vee).$$

*Proof.* The statement immediately follows from the inequality in Theorem 5.3.7, Proposition 5.2.7, [BPS14, Proposition 2.2.2] and Proposition 1.5.4.  $\square$

**Remark 5.3.9.** The Legendre-Fenchel dual of the gross  $v$ -adic upper function of a Laurent polynomial has an explicit expression, see for instance [MS16, Equality (6.7) in Proposition 6.2].

**Remark 5.3.10.** Under stronger assumptions on the metrized divisors  $\overline{D}_0, \dots, \overline{D}_{n-k}$ , one might prove upper bounds for the height of the restriction to the dense open torus of the intersection cycle of  $k$  Laurent polynomials. For instance, [MS16, Theorem 1.1] gives a bound for the height of the common solutions in the torus of a system of  $n$  Laurent polynomials with respect to a *nef* toric metrized divisor.

# Average heights

The computations in Example 5.1.1 show that the height of a complete intersection can not only depend on the Ronkin functions of the generating Laurent polynomials. In this chapter, we conjecture a formula for a certain average height of the cycle obtained as a complete intersection in a toric variety. We then focus on the strategy for the proof of a family of cases.

The choice of an adelic field  $(\mathbb{K}, (|\cdot|_v, n_v)_{v \in \mathcal{M}})$  is assumed throughout the whole section. We also fix the choice of an algebraic closure  $\overline{\mathbb{K}}$  of  $\mathbb{K}$ . By abuse of notation, we will identify the set of closed points of a variety  $X$  over  $\mathbb{K}$  with the set of its  $\overline{\mathbb{K}}$ -points, see [Gro60, I, ch.1, Proposition 6.4.2], and call them the *algebraic points* of  $X$ .

## 6.1

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### INVARIANT MEANS ON GROUPS

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The definition of an ‘equidistributed’ average on a group  $G$  requires some caution. Indeed, when the group is compact, there exists a unique invariant probability measure, namely the normalized Haar measure, on  $G$ . A well-known theorem of Weil, however, affirms that if a Polish topological group admits a left-invariant  $\sigma$ -additive measure, then it is locally compact, see [Wei40]. This implies then that not all topological groups have a left-invariant  $\sigma$ -additive measure, as is the case for discrete countable groups. In these situations, averages of random variables can not be defined to be expected value with respect to a measure as in the classical case.

This problem can be overtaken by considering coarser versions of left-invariant

integration on  $G$  with finite total mass, in particular weakening the requirement of  $\sigma$ -additivity. Recall that a *finitely additive probability measure* on a measurable space  $E$  is a positive measure  $\mu$  satisfying  $\mu(E) = 1$  and

$$\mu(A_1 \cup \cdots \cup A_k) = \mu(A_1) + \cdots + \mu(A_k)$$

for every finite collection of pairwise disjoint measurable sets  $A_1, \dots, A_k$ .

**Definition 6.1.1.** A discrete group  $G$  is *amenable* if there exists a finitely additive probability measure  $\mu$  on  $G$  which is left-invariant.

Amenable groups were introduced by von Neumann in [Neu29] to study obstructions to instances of the Banach-Tarski paradox. Contributions to the theory have been successively made by Følner, Day and Gromov among others. References can be found in the monograph [Pat88] or in the more recent and concise [CC10, chapter 4].

**Example 6.1.2.** Finite groups are amenable as they admit the normalized counting measure. Instead, the free group of rank 2 is non-amenable, see for instance [CC10, Theorem 4.4.7].

Let  $\mu$  be a finitely additive probability measure on a measurable space  $E$  and denote by  $\ell_\infty(E)$  the vector space of bounded real-valued functions on  $E$ . One can define the integral with respect to  $\mu$  of a function in  $\ell_\infty(E)$  by approximation via simple functions, as in the classical Lebesgue case, see [CC10, proof of Theorem 4.1.8] for a detailed construction. The so defined integral is a linear functional on  $\ell_\infty(E)$ , monotone and continuous with respect to the sup-norm of bounded functions.

**Definition 6.1.3.** Let  $G$  be an amenable group and  $\mu$  a left-invariant finitely additive probability measure on  $G$ . The *mean* of a bounded real-valued function  $f$  over  $G$  with respect to  $\mu$  is

$$\mathbb{E}_\mu[f] := \int_G f \, d\mu.$$

The mean with respect to  $\mu$  is a left-invariant linear functional on  $\ell_\infty(G)$  which is 1 for the indicator function of  $G$  and which takes non-negative values on non-negative valued functions. Conversely, any linear functional on  $\ell_\infty(G)$  satisfying the previous properties induce a left-invariant finitely additive probability measure on  $G$  by evaluating it at indicator functions.

**Remark 6.1.4.** It is shown in [CC10, Proposition 4.4.4] that the existence of a left-invariant finitely additive probability measure on a discrete group  $G$  is equivalent to the existence of a right-invariant one and also to the existence of an invariant one.

**Remark 6.1.5.** The name ‘amenable’ given in Definition 6.1.1 refers indeed to the fact that a meaningful mean can be defined.

Amenability is a property stable under considering subgroups, quotients, extensions and direct products, see [CC10, section 4.5]. The following result will also be particularly useful.

**Proposition 6.1.6.** *The direct limit of amenable groups is amenable.*

*Proof.* Let  $(I, \leq)$  be a directed set and let  $(G_i, f_{ij})$  be a directed system of groups over  $I$  and denote by  $G$  its direct limit. By hypothesis, for every  $i \in I$  there exists a left-invariant finitely additive probability measure  $\mu_i$  on  $G_i$ ; by pushing forward via the canonical function  $G_i \rightarrow G$ , one can consider  $\mu_i$  as a finitely additive measure probability measure on  $G$ . The mean  $\mathbb{E}_{\mu_i}$  on  $G$  with respect to  $\mu_i$  is a continuous linear functional on  $(\ell_\infty(G), \|\cdot\|_\infty)$ , hence it is an element of the Banach space  $(\ell_\infty(G))^*$ , with dual norm

$$\|\mathbb{E}_{\mu_i}\| = \sup\{|\mathbb{E}_{\mu_i}(f)| : f \in \ell_\infty(G), \|f\|_\infty \leq 1\} = \mathbb{E}_{\mu_i}(1) = 1$$

because of the monotonicity of the integral. It follows that the net  $(\mathbb{E}_{\mu_i})_{i \in I}$  is contained in the unit ball of  $(\ell_\infty(G))^*$ , which is compact with respect to the weak-\* topology because of the Banach-Alaoglu theorem, see for instance [Rud91, theorem 3.15]. There exists hence a subnet  $(\mathbb{E}_{\mu_j})_{j \in J}$  converging to a linear functional  $\mathbb{E}$  on  $\ell_\infty(G)$ . It follows by passage to the limit that  $\mathbb{E}(1) = 1$  and that  $\mathbb{E}$  takes non-negative values on non-negative functions. Moreover,  $\mathbb{E}$  is left-invariant on  $G$ . Indeed, for every  $g \in G$  and  $f \in \ell_\infty(G)$ ,

$$\mathbb{E}(g.f) = \lim \mathbb{E}_{\mu_j}(g.f) = \lim \mathbb{E}_{\mu_j}(f) = \mathbb{E}(f),$$

since there exists  $j_0 \in J$  such that  $g$  lies in the canonical image of  $G_{j_0}$  in  $G$  for every  $j \geq j_0$  and  $\mathbb{E}_{\mu_j}$  is  $G_{j_0}$ -invariant. Then, by setting  $\mu(A) := \mathbb{E}(1_A)$  for every  $A \subseteq G$ , one obtains a left-invariant finitely additive probability measure  $\mu$  on  $G$ .  $\square$

## 6.2

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### THE TWISTED FAMILY

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We introduce in this section the family of Laurent polynomials over which the average will be computed. Loosely speaking, it consists of a collection

of Laurent polynomials over  $\overline{\mathbb{K}}$  obtained by twisting the original polynomials by torsion points in the torus. Their zero sets are then translate of a fixed hypersurface by torsion points.

Throughout the whole section, let  $M$  be a lattice of dimension  $n$  and  $N_{\mathbb{R}}$  the real vector space associated to its dual lattice  $N$ .

**TRANSLATED SUBVARIETIES.** Fix for this subsection an arbitrary field  $K$ . Denote by  $\mathbb{T} := \text{Spec } K[M]$  the  $n$ -dimensional torus over  $K$  with character lattice  $M$  and by  $\mathbb{T}_{\overline{K}}$  its base change to a fixed algebraic closure  $\overline{K}$  of  $K$ . Any  $m \in M = \text{Hom}(\mathbb{T}, \mathbb{G}_m)$  induces hence a morphism

$$\mathbb{T}(K) \rightarrow \mathbb{G}_m(K) = K^*.$$

The image of a point  $x \in \mathbb{T}(K)$  by such a morphism is denoted by  $x^m$ .

**Definition 6.2.1.** Let  $f = \sum_m c_m \chi^m$  be a Laurent polynomial in  $K[M]$  and  $x \in \mathbb{T}(K)$ . The *twisted of  $f$  by  $x$*  is the Laurent polynomial

$$x^* f = \sum_m (c_m x^m) \chi^m$$

in  $K[M]$ .

**Remark 6.2.2.** The group scheme multiplication on  $\mathbb{T}$  induces a multiplication morphism on  $\mathbb{T}(K)$ . In particular, the multiplication by the element  $x \in \mathbb{T}(K)$  is a group homomorphism on  $\mathbb{T}(K)$ , denoted by  $\text{mult}_x$ . On the other hand, any polynomial  $f \in K[M]$  gives a mapping from  $\mathbb{T}(K)$  to  $K$ . The pull-back of this mapping by  $\text{mult}_x$  is the mapping associated with the polynomial  $x^* f$ , which justifies the notation adopted in Definition 6.2.1.

From the definition and the fact that  $\mathbb{T}(K)$  is an abelian group, one easily proves that for  $f \in K[M]$  and  $x, y \in \mathbb{T}(K)$

$$(xy)^* f = x^*(y^* f). \tag{6.1}$$

Definition 6.2.1 allows to introduce the notion of translated subvarieties in a torus from an algebraic point of view. For a subvariety  $V$  of  $\mathbb{T}$ , denote by  $I(V)$  the corresponding ideal in  $K[M]$ .

**Definition 6.2.3.** Let  $V$  be a subvariety of  $\mathbb{T}$  and  $x \in \mathbb{T}(K)$ , with inverse  $x^{-1}$ . The *translated of  $V$  by  $x$*  is the subvariety  $xV$  of  $\mathbb{T}$  corresponding to the ideal

$$(x^{-1})^* I(V) := \{(x^{-1})^* f : f \in I(V)\} \tag{6.2}$$

in  $K[M]$ .

**Remark 6.2.4.** Using (6.1), one easily checks that the translation by any  $x \in \mathbb{T}(K)$  is a bijection on the set of subvarieties of  $\mathbb{T}$ , with inverse given by the translation by  $x^{-1}$ . The operation in (6.2) preserving the primality of an ideal, the translation by  $x$  sends moreover irreducible subvarieties of  $\mathbb{T}$  into irreducible subvarieties of  $\mathbb{T}$ . In particular, it preserves the dimension.

There exists a more geometric interpretation of the notion of translated variety. To see it, consider that the group scheme structure on  $\mathbb{T}$  gives the homomorphism of abelian groups  $\mathbb{T}(\overline{K}) \times \mathbb{T}(\overline{K}) \rightarrow \mathbb{T}(\overline{K})$  at the level of algebraic points. This allows to consider, for each  $x \in \mathbb{T}(\overline{K})$  and  $A \subseteq \mathbb{T}(\overline{K})$ , the set  $x.A := \{xa : a \in A\} \subseteq \mathbb{T}(\overline{K})$ .

**Proposition 6.2.5.** *Let  $V$  be a subvariety of  $\mathbb{T}_{\overline{K}}$  and  $x \in \mathbb{T}(\overline{K})$ . The set of closed points of  $xV$  coincides with  $x.V(\overline{K})$ .*

*Proof.* Let  $I(V)$  denote the ideal of  $V$  in  $\overline{K}[M]$ . By definition, the set  $V(\overline{K})$  consists of the points of  $\mathbb{T}(\overline{K})$  which are zeros of  $f$  for every  $f \in I(V)$ . But a point  $y \in \mathbb{T}(\overline{K})$  is a zero of  $f$  if and only if  $xy$  is a zero of  $(x^{-1})^*f$ . The claim follows then from Definition 6.2.3.  $\square$

**TORSION POINTS.** Coming back to the case of an adelic base field  $\mathbb{K}$ , let  $\mathbb{T}$  be the split torus  $\text{Spec } \mathbb{K}[M]$ . The neutral element of the abelian group  $\mathbb{T}(\overline{\mathbb{K}})$  of  $\overline{\mathbb{K}}$ -points of  $\mathbb{T}$  is denoted by  $\mathbf{1}$ . The following definition introduces a distinguished subgroup of  $\mathbb{T}(\overline{\mathbb{K}})$ .

**Definition 6.2.6.** For any  $d \in \mathbb{N}$ , the *subgroup of  $d$ -torsion points of  $\mathbb{T}(\overline{\mathbb{K}})$*  is the subgroup

$$\tau(d) := \{\zeta \in \mathbb{T}(\overline{\mathbb{K}}) : \zeta^d = \mathbf{1}\},$$

that is the set of points in  $\mathbb{T}(\overline{\mathbb{K}})$  with order dividing  $d$ .

The choice of a system of coordinates on  $M$  specifies an isomorphism of  $\mathbb{T}$  with  $\mathbb{G}_m^n$  and hence a group isomorphism between  $\mathbb{T}(\overline{\mathbb{K}})$  and the set  $(\overline{\mathbb{K}}^*)^n$  endowed with componentwise multiplication and neutral element  $\mathbf{1} = (1, \dots, 1)$ .

**Proposition 6.2.7.** *The subgroup  $\tau(d)$  is finite. More precisely, if  $d$  is not divisible by the characteristic of  $\mathbb{K}$ , the cardinality of  $\tau(d)$  equals  $d^n$ , otherwise it equals  $a^n$ , with  $a$  being the highest factor of  $d$  not divisible by  $\text{char}(\mathbb{K})$ .*

*Proof.* After a choice of coordinates on  $M$ , an element  $\zeta \in (\overline{\mathbb{K}}^*)^n$  is in  $\tau(d)$  if and only if its components are  $d$ -roots of unity in  $\overline{\mathbb{K}}$ . If the characteristic of  $\mathbb{K}$

does not divide  $d$ , the algebraically closed field  $\overline{\mathbb{K}}$  has exactly  $d$  distinct  $d$ -roots of unity; the fact that no root appears with multiplicity two follows from the fact that the formal derivative of  $T^d - 1$  is  $dT^{d-1}$ , which has no nonzero roots. Otherwise, assume that  $p = \text{char } \mathbb{K}$  divides  $d$ ; writing  $d = p^\alpha a$  with  $a$  an integer coprime with  $p$ , one has that  $\alpha = \nu_p(d)$  is the  $p$ -adic valuation of  $d$ . It follows that a  $d$ -root of unity is a zero of the polynomial

$$T^d - 1 = (T^a - 1)^{p^\alpha},$$

hence of the polynomial  $T^a - 1$ . Since  $p$  does not divide  $a$ , it follows from the first part of the proof that this polynomial has exactly  $a = d/p^\alpha$  different roots in  $\overline{\mathbb{K}}$ .  $\square$

It will be useful to consider the *subgroup of torsion points*  $\tau$  of  $\mathbb{T}(\overline{\mathbb{K}})$ , which is

$$\tau := \bigcup_{d \in \mathbb{N}} \tau(d). \quad (6.3)$$

It is the subgroup of  $\mathbb{T}(\overline{\mathbb{K}})$  consisting of its elements of finite order. A choice of coordinates on  $M$  yields  $\tau = \mu_\infty^n$ , with  $\mu_\infty$  denoting the group of roots of unity in  $\overline{\mathbb{K}}$ . Moreover, as a countable union of finite sets,  $\tau$  is countable.

**Proposition 6.2.8.** *The group  $\tau$  is amenable.*

*Proof.* Let  $(\mathbb{N}, |)$  be the directed set of positive integers endowed with the divisibility relation. Whenever  $d_1, d_2 \in \mathbb{N}$  with  $d_1 | d_2$ , any  $d_1$ -torsion point is also a  $d_2$ -torsion point, hence there is a trivial injection  $\tau(d_1) \hookrightarrow \tau(d_2)$ . With these notations,

$$\varinjlim_d \tau(d) \simeq \tau.$$

Since each of the groups  $\tau(d)$  is amenable because of Proposition 6.2.7 and Example 6.1.2, the claim follows from Proposition 6.1.6.  $\square$

**Remark 6.2.9.** More directly, Proposition 6.2.8 follows from the fact that every abelian group is amenable, see [CC10, Theorem 4.6.1].

Unfortunately, the proof of Proposition 6.1.6 does not provide an explicit finitely additive invariant measure on  $\tau$ . In the setting of Proposition 6.2.8 and denoting by  $\delta_x$  the Dirac delta at a point  $x$ , it states that there exists a subnet of

$$\left( \frac{1}{\#\tau(d)} \sum_{\zeta \in \tau(d)} \delta_\zeta \right)_{d \in (\mathbb{N}, |)} \quad (6.4)$$

whose corresponding means converge weakly in  $(\ell_\infty(\tau))^*$ . Anyway, the net (6.4) itself is in general not convergent (see the example below) and the existence of a converging subnet, which is not unique, relies on the highly ineffective Banach-Alaoglu theorem.

**Example 6.2.10.** Let  $\mathbb{T}$  denote the one dimensional torus over  $\overline{\mathbb{Q}}$ . Then, the net in (6.4) is not weak-\* convergent. Indeed, if it was, the net

$$\left( \frac{1}{d} \sum_{\zeta \in \tau(d)} f(\zeta) \right)_{d \in (\mathbb{N}, |)} \quad (6.5)$$

would be convergent in  $\mathbb{R}$  for any bounded function  $f$  on  $\tau$ . Consider the subset

$$A := \{ \zeta \in \tau : \nu_2(\text{ord}(\zeta)) \text{ is odd} \}$$

of  $\tau$ , where  $\nu_2(\text{ord}(\zeta))$  denotes the 2-adic valuation of the order of  $\zeta$ . Sorting the elements of  $A \cap \tau(d)$  by the 2-adic valuation of their orders and using the properties of the Euler totient function one shows that for every  $d \in \mathbb{N}$

$$\frac{\#(A \cap \tau(d))}{d} = \begin{cases} \frac{1}{3} \left( 1 - \frac{1}{2^{\nu_2(d)}} \right) & \text{if } \nu_2(d) \text{ is even} \\ \frac{1}{3} \left( 2 - \frac{1}{2^{\nu_2(d)}} \right) & \text{otherwise} \end{cases}.$$

Hence, the net (6.5) fails to be convergent for the indicator function of  $A$ , as it has  $1/3$  and  $2/3$  as cluster points.

One will give a formal construction of an *ad-hoc* measure in the particular case of interest for this chapter, see Remark 6.4.3.

**TWISTED FAMILIES.** For a torsion point  $\zeta \in \tau$ , the residue field of  $\mathbb{T}$  at  $\zeta$  is a finite field extension of  $\mathbb{K}$ , denoted by  $\mathbb{K}(\zeta)$ . Considering  $\zeta$  as a  $\overline{\mathbb{K}}$ -point of  $\mathbb{T}$ , any character  $m \in M = \text{Hom}(\mathbb{T}, \mathbb{G}_m)$  induces a morphism

$$\mathbb{T}(\overline{\mathbb{K}}) \rightarrow \mathbb{G}_m(\overline{\mathbb{K}}) = \overline{\mathbb{K}}^*.$$

Compatibly with the notation in the beginning of the section, the image of  $\zeta$  by such morphism is denoted by  $\zeta^m$  and it lies in  $\mathbb{K}(\zeta)$ . For a Laurent polynomial  $f = \sum_m c_m \chi^m \in \mathbb{K}[M]$ , the twist of  $f$  by  $\zeta$  as in Definition 6.2.1 is then the polynomial

$$\zeta^* f = \sum_m (c_m \zeta^m) \chi^m$$

in  $\mathbb{K}(\zeta)[M]$ . In particular, if  $d \in \mathbb{N}$  is such that  $\zeta \in \tau(d)$ , then  $\zeta^m$  is a  $d$ -root of unity in  $\overline{\mathbb{K}}$  for any  $m \in M$ , so  $\zeta^* f \in \mathbb{K}_d[M]$ , where  $\mathbb{K}_d$  is the cyclotomic field obtained by adding to  $\mathbb{K}$  all its  $d$ -roots of unity.

**Remark 6.2.11.** When  $M$  is given a fixed system of coordinates, one can write  $m = (m_1, \dots, m_n)$  and  $\zeta = (\zeta_1, \dots, \zeta_n)$ . Then,  $\mathbb{K}(\zeta)$  is the minimal field extension of  $\mathbb{K}$  containing  $\zeta_1, \dots, \zeta_n$  and  $\zeta^m = \zeta_1^{m_1} \cdots \zeta_n^{m_n}$ .

One can study the effect of twisting on the combinatorial information associated to a Laurent polynomial.

**Proposition 6.2.12.** *Let  $f \in \mathbb{K}[M]$  and  $\zeta \in \tau$ . Then,  $f$  and  $\zeta^* f$  share the same Newton polytope,  $v$ -adic Ronkin function,  $v$ -adic upper function and gross  $v$ -adic upper function for every place  $v$  of  $\mathbb{K}(\zeta)$ .*

*Proof.* For any  $m \in M$ ,  $\zeta^m$  is the image in  $\overline{\mathbb{K}}^*$  of an element of finite order by a group homomorphism, hence it is a root of unity in  $\overline{\mathbb{K}}$ .

Let  $f = \sum_m c_m \chi^m$ . By the above,  $c_m \zeta^m \neq 0$  whenever  $c_m \neq 0$ , hence  $\text{NP}(f) = \text{NP}(\zeta^* f)$ . Also, since a root of unity has absolute value equal to 1 at all places,

$$|c_m \zeta^m|_v = |c_m|_v \quad (6.6)$$

for any place  $v$  of  $\mathbb{K}(\zeta)$ .

Let  $v$  be a non-archimedean place of  $\mathbb{K}(\zeta)$ . In this case, the  $v$ -adic Ronkin function, the  $v$ -adic upper function and the gross  $v$ -adic upper function coincide, see Remark 5.3. Because of (6.6), moreover

$$\rho_{f,v} = \min_m (\langle m, \cdot \rangle - \log |c_m|_v) = \rho_{\zeta^* f, v}.$$

Assume now that  $v$  is archimedean: because of Proposition 2.1.21,  $v$  corresponds to an embedding  $\sigma : \mathbb{K}(\zeta) \rightarrow \mathbb{C}$  up to complex conjugation. In the notation of Remark 4.1.6, for  $u \in N_{\mathbb{R}}$ ,

$$\begin{aligned} \rho_{\zeta^* f, v}(u) &= \frac{1}{(2\pi)^n} \int_{[0, 2\pi]^n} -\log |f^\sigma(\sigma(\zeta_1)e^{-u_1+i\theta_1}, \dots, \sigma(\zeta_n)e^{-u_n+i\theta_n})| d\theta \\ &= \rho_{f, v}(u) \end{aligned}$$

by an affine change of variables in the variables  $\theta_1, \dots, \theta_n$ . Similarly,  $\mu_{f, v} = \mu_{\zeta^* f, v}$ . The equality  $\hat{\mu}_{f, v} = \hat{\mu}_{\zeta^* f, v}$  follows again from (6.6).  $\square$

Let  $\underline{f} = (f_1, \dots, f_k)$  be a  $k$ -tuple of Laurent polynomials with coefficients in  $\mathbb{K}$ . Operating a componentwise twisting, one gets a family of  $k$ -tuples of Laurent polynomials with coefficients in  $\overline{\mathbb{K}}$ .

**Definition 6.2.13.** Let  $k \in \mathbb{N}$  and  $f_1, \dots, f_k \in \mathbb{K}[M]$ . The *twisted family* of  $\underline{f} = (f_1, \dots, f_k)$  is the family

$$\mathcal{T}(\underline{f}) := \{(\zeta_1^* f_1, \dots, \zeta_k^* f_k) : \zeta_1, \dots, \zeta_k \in \tau\}.$$

Similarly, for any  $d \in \mathbb{N}$ , the  $d$ -twisted family of  $\underline{f} = (f_1, \dots, f_k)$  is the family

$$\mathcal{T}_d(\underline{f}) := \{(\zeta_1^* f_1, \dots, \zeta_k^* f_k) : \zeta_1, \dots, \zeta_k \in \tau(d)\}.$$

**Remark 6.2.14.** By Proposition 6.2.7,  $\mathcal{T}_d(\underline{f})$  is a finite set, of cardinality  $d^{nk}$  if  $d$  is not divisible by the characteristic of  $\mathbb{K}$ . Moreover, there exists a finite field extensions  $\mathbb{F}$  of  $\mathbb{K}$  such that each component of each element of the family  $\mathcal{T}_d(\underline{f})$  is a Laurent polynomial with coefficients in  $\mathbb{F}$ .

The analogous relation on the subgroups, see (6.3), implies that

$$\mathcal{T}(\underline{f}) = \bigcup_{d \in \mathbb{N}} \mathcal{T}_d(\underline{f}).$$

## 6.3

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### THE GEOMETRY OF TWISTED FAMILIES

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Our first goal is to understand the geometrical properties one can expect “on average” for the twisted family of  $\underline{f} = (f_1, \dots, f_k)$ , namely the intersection of the associated sections and the corresponding degree, when well-defined.

**THE SETTING.** One fixes in this subsection the terminology and objects that will be used in the following geometric study. First of all, thanks to Proposition 3.1.12, one can choose a smooth projective toric variety  $X_\Sigma$  over  $\overline{\mathbb{K}}$  whose fan  $\Sigma$  is compatible with the Newton polytopes  $\text{NP}(f_1), \dots, \text{NP}(f_k)$  of  $f_1, \dots, f_k$  respectively. For every  $i = 1, \dots, k$ , let moreover  $D_{f_i}$  be the toric Cartier divisor on  $X_\Sigma$  associated to  $f_i$  as in section 4.2; it is generated by global sections because of Proposition 3.1.10. Let also  $s_{f_i}$  be the corresponding distinguished toric section of  $\mathcal{O}(D_{f_i})$ . In particular, Proposition 6.2.12 assures that for any torsion point  $\zeta \in \tau$  one can consider the rational section  $\zeta^* f_i s_{f_i}$  of  $\mathcal{O}(D_{f_i})$ . The same proposition, together with Theorem 3.2.4 and Definition 6.2.3, implies that

$$\text{div}(\zeta^* f_i s_{f_i}) \cdot X_\Sigma = \left[ \overline{V(\zeta^* f_i)} \right] = \left[ \zeta^{-1} V(f_i) \right]. \quad (6.7)$$

By  $\mathcal{B}(\underline{f})$  one means the set of  $k$ -tuple of torsion points  $\underline{\zeta} = (\zeta_1, \dots, \zeta_k) \in \tau^k$  for which the cycles introduced in (6.7) does *not* intersect properly in  $X_\Sigma$ . In particular, whenever  $\underline{\zeta} \notin \mathcal{B}(\underline{f})$ , one can consider the intersection cycle

$$Z_{\underline{f}}(\underline{\zeta}) := \text{div}(\zeta_1^* f_1 s_{f_1}) \cdots \text{div}(\zeta_k^* f_k s_{f_k}) \cdot X_\Sigma \quad (6.8)$$

in the sense of Definition 2.4.3. If the irreducible components of the support of such an intersection lie in the dense open torus of  $X_\Sigma$  one has

$$Z_f(\underline{\zeta}) = \left[ \overline{V(\zeta_1^* f_1, \dots, \zeta_k^* f_k)} \right].$$

**PROPER INTERSECTION.** One studies here the cardinality of the set  $\mathcal{B}(f)$  when  $k = 2$ . To simplify the notations, let  $f, g$  be the two fixed nonzero Laurent polynomials with coefficients in  $\mathbb{K}$  and denote by  $D_f, D_g, s_f, s_g$  the corresponding toric Carter and distinguished rational sections introduced above. To study the conditions under which the divisors associated to the twisted polynomials intersect properly in  $X_\Sigma$ , consider, for two subvarieties  $V$  and  $W$  of  $\mathbb{T}_{\overline{\mathbb{K}}}$ , the subset

$$\mathcal{P}(V, W) := \{x \in \mathbb{T}(\overline{\mathbb{K}}) : xV \supseteq W\} \quad (6.9)$$

of  $\mathbb{T}(\overline{\mathbb{K}})$ . Because of the density of the closed points in a variety, see [Gro60, IV, Troisième partie, Corollaire 10.4.8], one also has

$$\mathcal{P}(V, W) = \{x \in \mathbb{T}(\overline{\mathbb{K}}) : x.V(\overline{\mathbb{K}}) \supseteq W(\overline{\mathbb{K}})\}. \quad (6.10)$$

The main interest of such a set is given by the following two properties, which clarify its relation with proper intersection.

**Lemma 6.3.1.** *Let  $V$  and  $W$  be two subvarieties of  $\mathbb{T}_{\overline{\mathbb{K}}}$ , with  $V$  an hypersurface and  $W$  irreducible. For a point  $x \in \mathbb{T}(\overline{\mathbb{K}})$ , one has that  $xV$  and  $W$  intersect properly if and only if  $x \notin \mathcal{P}(V, W)$ .*

*Proof.* It follows from Remark 6.2.4 that  $xV$  is a hypersurface in  $\mathbb{T}_{\overline{\mathbb{K}}}$ . The subvarieties  $xV$  and  $W$  fail to intersect properly if and only if the codimension of their intersection is strictly smaller than the sum of the codimensions. Since  $xV$  has codimension 1, this happens exactly when the dimension of  $xV \cap W$  equals the dimension of  $W$ . Since  $W$  is irreducible, this is equivalent to the condition  $W \subseteq xV$ , that is  $x \in \mathcal{P}(V, W)$  by definition.  $\square$

**Lemma 6.3.2.** *Let  $V$  and  $W$  be two subvarieties of  $\mathbb{T}_{\overline{\mathbb{K}}}$ , with  $V$  of positive codimension and  $W(\overline{\mathbb{K}})$  nonempty. Then  $\mathcal{P}(V, W)$  is contained in the set of  $\overline{\mathbb{K}}$ -points of a proper subvariety of  $\mathbb{T}_{\overline{\mathbb{K}}}$ .*

*Proof.* Let  $p$  be a closed point of  $W$ , which exists by hypothesis. It follows from (6.10) that

$$\begin{aligned} \mathcal{P}(V, W) &\subseteq \{x \in \mathbb{T}(\overline{\mathbb{K}}) : p \in x.V(\overline{\mathbb{K}})\} = \{x \in \mathbb{T}(\overline{\mathbb{K}}) : x^{-1}p \in V(\overline{\mathbb{K}})\} \\ &= \{x \in \mathbb{T}(\overline{\mathbb{K}}) : x^{-1} \in p^{-1}.V(\overline{\mathbb{K}})\} = \{x \in \mathbb{T}(\overline{\mathbb{K}}) : x^{-1} \in (p^{-1}V)(\overline{\mathbb{K}})\} \end{aligned}$$

by Proposition 6.2.5. Since  $V$  has positive codimension, the same property holds for  $p^{-1}V$ , so one can choose a nonzero Laurent polynomial  $g$  in the ideal of  $p^{-1}V$ , say  $g = \sum_m c_m \chi^m$ . An algebraic point of the torus is a zero of  $g$  if and only if its inverse is a zero of  $g' := \sum_m c_m \chi^{-m}$ , hence

$$\mathcal{P}(V, W) \subseteq \{x \in \mathbb{T}(\overline{\mathbb{K}}) : g(x^{-1}) = 0\} = V(g')(\overline{\mathbb{K}}),$$

from which the claim follows.  $\square$

The last ingredient needed to control the occurrence of “bad” intersections is the following theorem, which was conjectured by Lang in [Lan83, page 220] and proved independently in [Lau84, Théorème 2] and [SA94, Proposition 1.6]. By a *torsion subvariety* of  $\mathbb{T}_{\overline{\mathbb{K}}}$  one means a translated of an irreducible closed subtorus of  $\mathbb{T}_{\overline{\mathbb{K}}}$  by a torsion point in  $\mathbb{T}(\overline{\mathbb{K}})$ .

**Theorem 6.3.3** (Manin-Mumford). *Let  $\mathbb{K}$  be a field of characteristic 0 and  $Y$  an algebraic subvariety of  $\mathbb{T}_{\overline{\mathbb{K}}}$ . The set of torsion points of  $\mathbb{T}_{\overline{\mathbb{K}}}$  lying on  $Y$  is a finite union of the sets of torsion points of torsion subvarieties of  $\mathbb{T}_{\overline{\mathbb{K}}}$ .*

The following statement is a consequence of this powerful theorem.

**Proposition 6.3.4.** *Let  $\mathbb{K}$  be a field of characteristic 0,  $W$  an irreducible hypersurface in  $\mathbb{T}_{\overline{\mathbb{K}}}$  with  $W(\overline{\mathbb{K}})$  nonempty and  $f$  a Laurent polynomial in  $\mathbb{K}[M]$  not being a monomial. Then, there exists an absolute positive constant  $C$  such that for all  $d \in \mathbb{N}$  the number of torsion points  $\zeta \in \tau(d)$  such that  $V(\zeta^* f)$  does not meet  $W$  properly is at most  $C \cdot d^{n-1}$ .*

*Proof.* Assume that  $\zeta \in \tau(d)$  is such that  $V(\zeta^* f)$  does not meet  $W$  properly. Lemma 6.3.1 and the equality  $V(\zeta^* f) = \zeta^{-1}V(f)$  given by (6.2.3) imply then that  $\zeta^{-1} \in \mathcal{P}(V(f), W)$ . Denote by  $Y$  the proper algebraic subvariety of  $\mathbb{T}_{\overline{\mathbb{K}}}$  containing  $\mathcal{P}(V(f), W)$  by Lemma 6.3.2. It follows that  $\zeta^{-1}$  is a torsion point of  $\mathbb{T}_{\overline{\mathbb{K}}}$  lying on  $Y$ . Manin-Mumford theorem assures that

$$\tau \cap Y(\overline{\mathbb{K}}) = \bigcup_{i=1}^C (\tau \cap T_i(\overline{\mathbb{K}}))$$

for a fixed natural number  $C$ , with  $T_1, \dots, T_C$  being torsion subvarieties of  $\mathbb{T}_{\overline{\mathbb{K}}}$ . Since  $Y$  is proper, each  $T_i$  is of dimension at most  $n-1$ , then  $T_i(\overline{\mathbb{K}})$  contains at most  $d^{n-1}$   $d$ -torsion points for any  $i = 1, \dots, C$ , implying the statement.  $\square$

One can finally show that the set of pairs of torsion points for which the variety generated by the twisted  $f$  and  $g$  has uncorrect dimension is “tiny”.

**Theorem 6.3.5.** *Let  $\mathbb{K}$  be a field of characteristic 0,  $f$  and  $g$  two Laurent polynomials in  $\mathbb{K}[M]$  which are not monomials. Then, the number of  $\underline{\zeta} = (\zeta_1, \zeta_2) \in \tau(d)^2$  such that the sections  $\zeta_1^* f s_f$  and  $\zeta_2^* g s_g$  do not meet  $X_\Sigma$  properly is at most  $C \cdot d^{2n-1}$  for a constant  $C$  which does not depend on  $d$ .*

*Proof.* Recall first that the divisors of  $\zeta_1^* f s_f$  and  $\zeta_2^* g s_g$  have generic points inside the dense open orbit  $X_0$  of  $X_\Sigma$ . Hence, their supports do not contain any 1-codimensional closed toric orbit, which are the irreducible components of  $X_\Sigma \setminus X_0$ . As a consequence, the sections  $\zeta_1^* f s_f$  and  $\zeta_2^* g s_g$  meet  $X_\Sigma$  properly if and only if each irreducible component of

$$X_0 \cap |\operatorname{div}(\zeta_1^* f s_f)| \cap |\operatorname{div}(\zeta_2^* g s_g)| = V(\zeta_1^* f) \cap V(\zeta_2^* g)$$

is of codimension 2 in  $X_0$ , that is if and only if the variety  $V(\zeta_1^* f)$  meet  $V(\zeta_2^* g)$  properly in the torus.

Secondly, the invariance of proper intersection under translation assures that the varieties  $V(\zeta_1^* f) = \zeta_1^{-1} V(f)$  and  $V(\zeta_2^* g) = \zeta_2^{-1} V(g)$  meet properly in the torus if and only if  $\zeta_2 \zeta_1^{-1} V(f)$  and  $V(g)$  do. It follows that the number of pairs  $\underline{\zeta} = (\zeta_1, \zeta_2) \in \tau(d)^2$  for which the sections in the statement do not meet properly is equal to  $\#\tau(d)$  times the number of  $\zeta \in \tau(d)$  for which  $\zeta^{-1} V(f) = V(\zeta^* f)$  and  $V(g)$  do not meet properly. One can then reduce the study to the situation where  $V(g)$  is fixed and  $V(f)$  is translated by  $d$ -torsion points.

Since  $g$  is not a monomial,  $V(g)$  is of codimension 1 in  $X_0$  and has at least a  $\overline{\mathbb{K}}$ -point. Let  $W_1, \dots, W_r$  be the irreducible components of  $V(g)$  over  $\overline{\mathbb{K}}$ , with  $r \in \mathbb{N}$ . For each  $i = 1, \dots, r$ , Proposition 6.3.4 asserts that there exists  $C_i \in \mathbb{R}_{>0}$  independent of  $d$  such that the number of torsion points  $\zeta \in \tau(d)$  for which  $V(\zeta^* f)$  does not meet  $W_i$  properly is at most  $C_i \cdot d^{n-1}$ . Since proper intersection is equivalent to the proper intersection of the irreducible components, one deduces that the number of  $\zeta \in \tau(d)$  for which  $V(\zeta^* f)$  does not meet  $V(g)$  properly is upper bounded by

$$C_1 \cdot d^{n-1} + \dots + C_r \cdot d^{n-1} = C \cdot d^{n-1},$$

with  $C := C_1 + \dots + C_r$ .

Coming back to the general case, the above argument allows to infer that the number of  $\underline{\zeta} = (\zeta_1, \zeta_2) \in \tau(d)^2$  such that the sections  $\zeta_1^* f s_f$  and  $\zeta_2^* g s_g$  do not meet properly is at most

$$\#\tau(d) \cdot C \cdot d^{n-1} = C \cdot d^{2n-1}$$

because of Proposition 6.2.7, concluding the proof.  $\square$

It follows that under the hypotheses of the previous theorem, the portion of pairs  $(\zeta_1, \zeta_2)$  in  $\tau(d)^2$  for which the sections  $\zeta_1^* f s_f$  and  $\zeta_2^* g s_g$  do not meet  $X_\Sigma$  properly is negligible, that is

$$\frac{\#(\mathcal{B}(f, g) \cap \tau(d)^2)}{\#\tau(d)^2} \rightarrow 0 \quad (6.11)$$

as  $d \rightarrow \infty$ . Otherwise said, the twisted divisors of  $f$  and  $g$  meet  $X_\Sigma$  properly *generically* in  $\tau^2$ .

**Remark 6.3.6.** Using the description of the intersection of  $\overline{V(f)}$  with the toric orbits of  $X_\Sigma$  presented in Theorem 3.3.9, one might show that moreover the intersection of the varieties  $V(\zeta_1^* f)$  and  $V(\zeta_2^* g)$  has generic points inside the open dense torus of  $X_\Sigma$  for a generic choice of  $(\zeta_1, \zeta_2)$  in  $\tau^2$ .

**THE DEGREE.** Coming back to the general case of an arbitrary  $k$  and in the notation of the beginning of the section, the next result describes the geometric degree of the intersection cycle  $Z_{\underline{f}}(\underline{\zeta})$  defined in (6.8).

**Theorem 6.3.7.** *Let  $D_1, \dots, D_{n-k}$  be toric divisors on  $X_\Sigma$  generated by global sections,  $\underline{f} = (f_1, \dots, f_k)$  a family of Laurent polynomials in  $\mathbb{K}[M]$ . For every  $\underline{\zeta} \in \tau^k \setminus \mathcal{B}(\underline{f})$  one has*

$$\deg_{D_1, \dots, D_{n-k}}(Z_{\underline{f}}(\underline{\zeta})) = \text{MV}_M(\Delta_1, \dots, \Delta_{n-k}, \text{NP}(f_1), \dots, \text{NP}(f_k)),$$

where  $\text{MV}_M$  denotes the mixed volume operator associated to the measure  $\text{vol}_M$  and  $\Delta_i$  the polytope associated to the toric divisor  $D_i$  for each  $i = 1, \dots, n-k$ .

*Proof.* Applying repeatedly the definition, one obtains

$$\deg_{D_1, \dots, D_{n-k}}(Z_{\underline{f}}(\underline{\zeta})) = \deg_{D_1, \dots, D_{n-k}, D_{f_1}, \dots, D_{f_k}}(X_\Sigma).$$

Since all the divisor  $D_1, \dots, D_{n-k}, D_{f_1}, \dots, D_{f_k}$  are generated by global sections, the claim follows from [Oda88, Proposition 2.10].  $\square$

One can give a probabilistic consequence of Theorem 6.3.7. For any choice  $D_1, \dots, D_{n-k}$  of toric divisors on  $X_\Sigma$  generated by global sections, define the map

$$\text{Deg}_{D_1, \dots, D_{n-k}}(\underline{f}) : \tau^k \rightarrow \mathbb{Z}$$

by setting

$$\text{Deg}_{D_1, \dots, D_{n-k}}(\underline{f})(\underline{\zeta}) := \begin{cases} \deg_{D_1, \dots, D_{n-k}}(Z_{\underline{f}}(\underline{\zeta})) & \text{if } \underline{\zeta} \notin \mathcal{B}(\underline{f}) \\ 0 & \text{otherwise} \end{cases}. \quad (6.12)$$

Such a map is bounded on  $\tau^k$  because of Theorem 6.3.7. It is then meaningful to consider a suitable average of it on  $\tau^k$ .

**Definition 6.3.8.** Let  $D_1, \dots, D_{n-k}$  be toric divisors on  $X_\Sigma$  generated by global sections. The *sup-expected degree* of the family of Laurent polynomials  $\underline{f}$  with respect to  $D_1, \dots, D_{n-k}$  is defined as

$$\mathbb{E}[\text{Deg}_{D_1, \dots, D_{n-k}}(\underline{f})] := \limsup_{d \rightarrow \infty} \frac{1}{\#\tau(d)^k} \sum_{\underline{\zeta} \in \tau(d)^k} \text{Deg}_{D_1, \dots, D_{n-k}}(\underline{f})(\underline{\zeta}).$$

For  $k = 2$ , the following result is an immediate consequence of the combinatorial description of the degree of the intersection cycle  $Z_{\underline{f}}(\underline{\zeta})$  and the fact that the set  $\mathcal{B}(\underline{f})$  is small in  $\tau^2$ .

**Corollary 6.3.9.** Let  $\mathbb{K}$  be a field of characteristic 0,  $f$  and  $g$  two Laurent polynomials in  $\mathbb{K}[M]$  and  $D_1, \dots, D_{n-2}$  toric divisors on  $X_\Sigma$  generated by global sections. The sup-expected degree of  $\underline{f}$  with respect to  $D_1, \dots, D_{n-2}$  is in fact a limit and

$$\mathbb{E}[\text{Deg}_{D_1, \dots, D_{n-2}}(\underline{f})] = \text{MV}_M(\Delta_1, \dots, \Delta_{n-2}, \text{NP}(f), \text{NP}(g)),$$

where  $\Delta_i$  denotes the polytope associated to the toric divisor  $D_i$  for every  $i = 1, \dots, n-2$ .

*Proof.* The term in the limit of the definition of the sup-expected degree equals

$$\frac{\#\tau(d)^2 - \#(\mathcal{B}(f, g) \cap \tau(d)^2)}{\tau(d)^2} \cdot \text{MV}_M(\Delta_1, \dots, \Delta_{n-2}, \text{NP}(f), \text{NP}(g))$$

because of Theorem 6.3.7, which implies the desired statement as  $d \rightarrow \infty$  thanks to (6.11).  $\square$

## 6.4

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### THE ARITHMETICS OF TWISTED FAMILIES

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Inspired by the result in Corollary 6.3.9, we conjecture here an analogous claim for the average height of a complete intersection in a toric variety. We then give intuitions for it to hold under additional assumptions.

As in the previous section, let  $k \in \{0, \dots, n\}$  and  $\underline{f} = (f_1, \dots, f_k)$  a fixed family of  $k$  nonzero Laurent polynomials with coefficients in  $\mathbb{K}$ . Let  $\Sigma$  be a fan compatible with the Newton polytopes of the Laurent polynomials.

**THE SUP-EXPECTED HEIGHT.** Adopting the same notation of the previous section, when  $\underline{\zeta} = (\zeta_1, \dots, \zeta_k) \in \tau^k \setminus \mathcal{B}(f)$  the cycles

$$\left[ \overline{V(\zeta_i^* f_i)} \right] = \text{div}(\zeta_i^* f_i s_{f_i}) \cdot X_\Sigma$$

for  $i = 1, \dots, k$  meet properly in  $X_{\Sigma, \overline{\mathbb{K}}}$ . In such a case, as in (6.8), denote by

$$Z_{\underline{f}}(\underline{\zeta}) := \text{div}(\zeta_1^* f_1 s_{f_1}) \cdot \dots \cdot \text{div}(\zeta_k^* f_k s_{f_k}) \cdot X_\Sigma$$

the cycle associated to the intersection over  $\overline{\mathbb{K}}$  of the corresponding twistings. If  $d$  is such that  $\underline{\zeta} \in \tau(d)^k$ , such a cycle is defined over the cyclotomic field extension  $\mathbb{K}_d$  of  $\mathbb{K}$  obtained adding to  $\mathbb{K}$  all the  $d$ -roots of unity.

Let moreover  $\overline{D}_0, \dots, \overline{D}_{n-k}$  be toric divisors on  $X_\Sigma$ , equipped with adelic semipositive toric metrics and consider the map

$$H_{\overline{D}_0, \dots, \overline{D}_{n-k}}(\underline{f}) : \tau^k \rightarrow \mathbb{R}$$

defined as

$$H_{\overline{D}_0, \dots, \overline{D}_{n-k}}(\underline{f}) : \underline{\zeta} \mapsto \begin{cases} h_{\overline{D}_0, \dots, \overline{D}_{n-k}}(Z_{\underline{f}}(\underline{\zeta})) & \text{if } \underline{\zeta} \notin \mathcal{B}(f) \\ 0 & \text{otherwise} \end{cases}.$$

Such a notion is well-defined thanks to Theorem 3.5.4 and the definition of  $\mathcal{B}(f)$ . In contrast with the geometric degree, which is constant by Theorem 6.3.7 whenever the intersection has the expected dimension, Example 5.1.1 shows that the function  $H_{\overline{D}_0, \dots, \overline{D}_{n-k}}(\underline{f})$  may behave wildly, even for  $k = 2$ . The following proposition describes a first property of such a function.

**Proposition 6.4.1.** *The map  $H_{\overline{D}_0, \dots, \overline{D}_{n-k}}(\underline{f})$  is upper bounded on  $\tau^k$ .*

*Proof.* The claim follows from the definition, from Theorem 5.3.7 and Proposition 6.2.12.  $\square$

Recalling that  $\tau(d)$  denotes the subgroup of  $d$ -torsion points of  $\mathbb{T}(\overline{\mathbb{K}})$ , one can consider the net of real numbers

$$\left( \frac{1}{\#\tau(d)^k} \sum_{\underline{\zeta} \in \tau(d)^k} H_{\overline{D}_0, \dots, \overline{D}_{n-k}}(\underline{f})(\underline{\zeta}) \right)_{d \in (\mathbb{N}, |)}.$$

As showed in Example 6.2.10, there is no general argument to infer that such a net is convergent. Anyway, it is upper bounded because of Proposition 6.4.1, suggesting to consider its higher accumulation point, which is finite.

**Definition 6.4.2.** Let  $\bar{D}_0, \dots, \bar{D}_{n-k}$  be toric divisors on  $X_\Sigma$ , equipped with adelic semipositive toric metrics. The *sup-expected height* of the family of Laurent polynomials  $\underline{f} = (f_1, \dots, f_k)$  with respect to  $\bar{D}_0, \dots, \bar{D}_{n-k}$  is defined as

$$\mathbb{E}[H_{\bar{D}_0, \dots, \bar{D}_{n-k}}(\underline{f})] := \limsup_{d \rightarrow \infty} \frac{1}{\#\tau(d)^k} \sum_{\zeta \in \tau(d)^k} H_{\bar{D}_0, \dots, \bar{D}_{n-k}}(\underline{f})(\zeta).$$

**Remark 6.4.3.** The group  $\tau^k$  is amenable as direct product of amenable groups, see Proposition 6.2.8 and [CC10, Corollary 4.5.14]. The superior limit gives a choice of a subnet of real numbers obtained by integrating against a product of counting measures on  $\tau$ . By the Banach-Alaoglu theorem, the subnet of such measures has a subnet which converges to a finitely additive measure  $\mu$  on  $\tau^k$ . The sup-expected height coincides with the the mean of the bounded function  $H$  with respect to such a measure  $\mu$ , justifying the notation  $\mathbb{E}[H]$ .

The analogous claim of the geometric case in Corollary 6.3.9 is given by the following conjecture.

**Conjecture 6.4.4.** Let  $\bar{D}_0, \dots, \bar{D}_{n-k}$  be toric divisors on  $X_\Sigma$ , equipped with adelic semipositive toric metrics. The sup-expected height of the family  $\underline{f}$  with respect to  $\bar{D}_0, \dots, \bar{D}_{n-k}$  defined in Definition 6.4.2 is in fact a limit and

$$\mathbb{E}[H_{\bar{D}_0, \dots, \bar{D}_{n-k}}(\underline{f})] = \sum_{v \in \mathfrak{M}} n_v \text{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n-k,v}, \rho_{f_1,v}^\vee, \dots, \rho_{f_k,v}^\vee),$$

where  $\vartheta_{i,v}$  is the roof function of  $\bar{D}_{i,v}$  for every  $i = 0, \dots, n-k$  and  $v \in \mathfrak{M}$ .

**Remark 6.4.5.** It would be a consequence of the fact that  $\tau^k \cap \mathcal{B}(\underline{f})$  is tiny, of Theorem 5.3.7 and of Proposition 6.2.12 that

$$\mathbb{E}[H_{\bar{D}_0, \dots, \bar{D}_{n-k}}(\underline{f})] \leq \sum_{v \in \mathfrak{M}} n_v \text{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n-k,v}, \mu_{f_1,v}^\vee, \dots, \mu_{f_k,v}^\vee).$$

The equality in Conjecture 6.4.4 does not contradict such an upper bound because of Proposition 5.2.7, [BPS14, Proposition 2.2.2] and the monotonicity of mixed integrals.

To state a particular case of Conjecture 6.4.4 which seems easier to prove, one needs the following notion.

**Definition 6.4.6.** A lattice polytope in  $M_{\mathbb{R}}$  is said to be *irreducible* if whenever it is the Minkowski sum of two lattice polytopes in  $M_{\mathbb{R}}$  one of the two summand consists of a single point.

Otherwise said, a lattice polytope is irreducible if it can not be written as the Minkowski sum of two non-invertible elements in the monoid of lattice polytopes in  $M_{\mathbb{R}}$ . The main property of irreducible polytopes is that they gives sufficient conditions for the irreducibility of Laurent polynomials.

**Proposition 6.4.7.** *Let  $K$  be a field and  $f$  a Laurent polynomial in  $K[M]$ . If the Newton polytope  $\text{NP}(f)$  of  $f$  is irreducible in  $M_{\mathbb{R}}$ , then  $f$  is an irreducible Laurent polynomial in  $K[M]$ .*

*Proof.* Suppose that  $f = gh$  with  $g, h \in K[M]$ , which implies that  $\text{NP}(f) = \text{NP}(g) + \text{NP}(h)$ . The fact that the Newton polytope of  $f$  is irreducible implies that one between  $\text{NP}(g)$  and  $\text{NP}(h)$  consists of a point, hence that one between  $g$  and  $h$  is a monomial, hence a unit in  $K[M]$ .  $\square$

One can now state a relevant particular case of Conjecture 6.4.4.

**Claim 6.4.8.** *Let  $\mathbb{K}$  be a number field,  $f, g$  two Laurent polynomials in  $\mathbb{K}[M]$  and  $\overline{D}_0, \dots, \overline{D}_{n-2}$  semipositively adelic toric metrized divisors on  $X_{\Sigma}$ . Assume moreover that*

1. *the Newton polytopes  $\text{NP}(f)$  and  $\text{NP}(g)$  are irreducible and distinct*
2. *the coefficients of  $f$  and  $g$  lie in the ring of integers  $\mathcal{O}_{\mathbb{K}}$  of  $\mathbb{K}$  and the coefficients corresponding to the vertices of their Newton polytopes are roots of unity in  $\mathbb{K}$*
3. *if  $v$  is archimedean, the  $v$ -adic metric on  $D_i$  is smooth for all  $i = 0, \dots, n-2$*
4. *if  $v$  is non-archimedean, the  $v$ -adic metric on  $D_i$  is the canonical one for all  $i = 0, \dots, n-2$ .*

Then,

$$\begin{aligned} \lim_{d \rightarrow \infty} \frac{1}{\#\tau(d)^2} \sum_{\zeta \in \tau(d)^2} H_{\overline{D}_0, \dots, \overline{D}_{n-2}}(f)(\zeta) \\ = \sum_{v \in \mathfrak{M}} n_v \text{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n-2,v}, \rho_{f,v}^{\vee}, \rho_{g,v}^{\vee}). \end{aligned} \quad (6.13)$$

The remaining of the section is consacrated to the indication of a strategy for the proof of such a statement. It would follow from the verification of Claim 6.4.10 and Claim 6.4.12, which concern the archimedean and non-archimedean contributions to the average height separately.

**THE ARCHIMEDEAN PART.** Throughout the subsection, let  $|\cdot|$  stand for the usual euclidean absolute value on  $\mathbb{C}$  and denote by

$$\mathbb{S} := \{\mathbf{t} \in (\mathbb{C}^\times)^n : |t_1| = \cdots = |t_n| = 1\}$$

the compact torus in the complex torus  $(\mathbb{C}^\times)^n$ . For a Laurent polynomial  $f$  with coefficients in  $\mathbb{C}$ , denote by  $V(f)$  its zero set in  $(\mathbb{C}^\times)^n$  and by  $\mathbf{t}^*f$  the twisted of  $f$  by  $\mathbf{t} \in \mathbb{S}$  as in Definition 6.2.1.

**Claim 6.4.9.** *Let  $f, g \in \mathbb{C}[M]$  be two Laurent polynomials with irreducible and distinct Newton polytopes. Let moreover  $\overline{D}_i$  be a divisor together with a smooth metric on the complex analytification of  $\mathcal{O}(D_i)$  for all  $i = 0, \dots, n-2$ . Then, the map  $\mathbb{S} \rightarrow \mathbb{R}$ ,*

$$\mathbf{t} \mapsto \int_{V(f)} \log |\mathbf{t}^*g| c_1(\overline{D}_0) \wedge \cdots \wedge c_1(\overline{D}_{n-2})$$

is continuous.

*Idea.* First of all remark that the condition on the Newton polytopes of  $f$  and  $g$  implies that  $\overline{V(f)} \cap \overline{V(\mathbf{t}^*g)}$  is of dimension strictly smaller than  $n-1$  for any  $\mathbf{t}$  in the  $n$ -dimensional complex torus. Indeed, for what concerns the components outside the dense open orbit  $X_0$ , the claim is clear from the definition of the varieties. Otherwise, it follows from the fact that  $V(f)$  and  $V(\mathbf{t}^*g)$  are irreducible because of Proposition 6.2.12 and Proposition 6.4.7 and distinct since their Newton polytopes are different, so they can not share any irreducible component in  $X_0$ .

A statement such as the one of [Sto67, Theorem 4.9] is required for the proof of the claim; however, the quoted result can not be applied directly to the present situation because of some technical difficulties and needs to be refined.  $\diamond$

Given a positive integer  $d$ , denote by  $\mathbb{K}_d$  the cyclotomic field extension obtained by adding to  $\mathbb{K}$  all the  $d$ -roots of unity of  $\overline{\mathbb{K}}$ . With the structure defined in Proposition 2.3.12, it is an adelic field; write  $\mathfrak{M}_d$  for its set of places and use the notation  $w|v$  to mean that  $w$  is a place of  $\mathbb{K}_d$  extending the place  $v$  of  $\mathbb{K}$ .

**Claim 6.4.10.** *Assume that the hypotheses of Claim 6.4.8 are satisfied. Then, for every archimedean place  $v$  over  $\mathbb{K}$*

$$\frac{1}{\#\tau(d)} \sum_{w|v} n_w \sum_{\zeta \in \tau(d)} \int_{X_{0,w}^{\text{an}}} \log \|\zeta^*g\|_{g,w} c_1(\overline{D}_{0,w}) \wedge \cdots \wedge c_1(\overline{D}_{n-2,w}) \wedge \delta_{\overline{V(f)}_w^{\text{an}}} \rightarrow 0 \quad (6.14)$$

as  $d \rightarrow \infty$ , where  $\|\cdot\|_{g,w}$  denotes the  $w$ -adic Ronkin metric on  $D_g$ .

*Idea.* A place  $v$  of  $\mathbb{K}$  corresponds, because of Proposition 2.1.21, to an embedding  $\sigma : \mathbb{K} \hookrightarrow \mathbb{C}$  up to complex conjugation. Analogously, a place  $w$  of  $\mathbb{K}_d$  extending  $v$  corresponds to an embedding  $\tau_w : \mathbb{K}_d \hookrightarrow \mathbb{C}$  extending  $\sigma$ , up to complex conjugation.

By writing

$$\log \|\zeta^* g s_g\|_{g,w} = \log |\zeta^* g| + \rho_{g,v}$$

and by reducing the double sums (over the complex embedding and over the  $d$ -torsion points) to a single sum performed over the  $d$ -torsion points of  $\mathbb{K}$  one could obtain (at least for  $d$  prime)

$$\frac{1}{\#\tau(d)} \sum_{\zeta \in \tau(d)} \int_{X_{0,w}^{\text{an}}} \log |\zeta^* g|_w d\mu_w + \int_{X_{0,w}^{\text{an}}} \rho_{g,v} d\mu_w$$

with  $\mu_w := c_1(\overline{D}_{0,w}) \wedge \cdots \wedge c_1(\overline{D}_{n-2,w}) \wedge \delta_{\overline{V(f)}_w^{\text{an}}}$  for simplicity of the exposition. The first term in the previous sum is a Riemann sum of the continuous function in Claim 6.4.9, then it converges for  $d \rightarrow \infty$  to its integral over  $\mathbb{S}$ . Fubini's theorem and the definition of the Ronkin function would conclude the proof of the claim.  $\diamond$

**THE NON-ARCHIMEDEAN PART.** Fix in this subsection a torsion point  $\zeta \in \tau(d)$  for some  $d \in \mathbb{N}$  and let  $\mathbb{K}_d$  be the cyclotomic extension of  $\mathbb{K}$  containing the  $d$ -roots of unity. It is a finite field extension of  $\mathbb{K}$  and hence it is endowed with an extended adelic structure, see Proposition 2.3.12. For any place  $w$  of such an adelic field  $\mathbb{K}_d$ , the notation  $\overline{D}_{g,w}$  stands for the divisor  $D_g$  on  $X_\Sigma$  associated to  $g$  equipped with its  $w$ -adic Ronkin metric introduced in Definition 4.2.4.

**Lemma 6.4.11.** *Under the hypotheses of Claim 6.4.8 the  $w$ -adic Ronkin metric on  $D_g$  is the canonical metric for all non-archimedean places  $w$  of  $\mathbb{K}_d$ .*

*Proof.* Writing  $g = \sum c_m \chi^m$  and recalling the form of the non-archimedean Ronkin function given in Remark 4.1.7 one has

$$\rho_{g,w}(u) = \min(\langle m, u \rangle - \log |c_m|_w)$$

for every  $u \in N_{\mathbb{R}}$ . By Proposition 1.3.9, the Legendre-Fenchel dual of  $\rho_{g,w}$  is given by the function parametrizing the roof of the convex envelop of the points  $(m, \log |c_m|_w) \in M_{\mathbb{R}} \times \mathbb{R}$  with  $m$  ranging in the lattice points of  $\text{NP}(g)$ . The hypotheses on  $g$  assure that  $c_m \in \mathcal{O}_{\mathbb{K}(\zeta)}$ , hence that  $\log |c_m|_w \leq 0$  for

every  $m \in \text{NP}(g)$  and that  $\log |c_m|_w = 0$  when  $m$  is a vertex of  $\text{NP}(g)$ . It follows that  $\rho_{g,w}^\vee$  is the indicator function of  $\text{NP}(g)$ , hence that  $\rho_{g,w} = \Psi_{\text{NP}(g)}$  because of Example 1.3.8.  $\square$

The key property of the non-archimedean case that will be needed in the proof of Claim 6.4.8 is the vanishing of a certain integral.

**Claim 6.4.12.** *Assume that the hypotheses of Claim 6.4.8 are satisfied. For all non-archimedean places  $w$  of  $\mathbb{K}_d$  one has*

$$\int_{X_{\Sigma,w}^{\text{an}}} \log \|\zeta^* g s_g\|_{g,w} c_1(\overline{D}_{0,w}) \wedge \cdots \wedge c_1(\overline{D}_{n-2,w}) \wedge \delta_{\overline{V(f)}_w^{\text{an}}} = 0.$$

*Idea.* Let  $w$  be a non-archimedean place of  $\mathbb{K}_d$  that one supposes fixed. To simplify the notation, the index  $w$  will hence be dropped from the absolute values, uniformizers, analytifications, tropicalizations and Ronkin functions appearing in the following, whenever the context allows.

Let  $(\mathbb{K}_d)_w$  be the completion of  $\mathbb{K}_d$  with respect to  $w$  and  $(\mathbb{K}_d)_w^\circ$  its valuation ring. Since  $\mathbb{K}_d$  is an adelic field by Proposition 2.3.12,  $w$  is associated to a discrete valuation, hence the valuation ring is local and its maximal ideal  $(\mathbb{K}_d)_w^{\circ\circ}$  is principal, generated by a fixed uniformizer  $\varpi$ .

Because of Lemma 6.4.11 and of the hypotheses of Claim 6.4.8, all the  $w$ -adic metrics on  $D_0, \dots, D_{n-2}, D_g$  are the canonical one. In particular, they all are algebraic and come from their canonical model described in [BPS14, Definition 3.6.3], see [BPS14, Example 4.5.4]. Denote hence by  $\mathcal{X}_\Sigma$  the canonical model of  $X_\Sigma$  over  $\text{Spec}(\mathbb{K}_d)_w^\circ$  and by  $\mathcal{D}_0, \dots, \mathcal{D}_{n-2}, \mathcal{D}_g$  the canonical model of the divisors  $D_0, \dots, D_{n-2}, D_g$  respectively. Developing the definition of the Chambert-Loir measure as in [BPS14, Remark 1.4.14], one has then

$$\begin{aligned} \int_{X_{\Sigma,w}^{\text{an}}} \log \|\zeta^* g s_g\|_{g,w} c_1(\overline{D}_{0,w}) \wedge \cdots \wedge c_1(\overline{D}_{n-2,w}) \wedge \delta_{\overline{V(f)}_w^{\text{an}}} \\ = \log |\varpi| \sum_V \text{ord}_V(\zeta^* g s_g) \deg_{\mathcal{D}_0, \dots, \mathcal{D}_{n-2}}(V), \end{aligned}$$

where  $\zeta^* g s_g$  is considered as a rational section of  $\mathcal{O}(\mathcal{D}_g)$  and the sum on the right hand side is running over the irreducible components  $V$  of the special fiber of the normalization of the closure of  $\overline{V(f)}$  in  $\mathcal{X}_\Sigma$ . The hypotheses on the Laurent polynomials  $f$  and  $g$  could be used to prove that the section  $\zeta^* g s_g$  is regular on the model  $\mathcal{X}_\Sigma$  and that it does not vanish on any irreducible component of the special fiber of  $V(f)$ ; one could in particular exploit the fact that the special fibers of  $f$  and  $g$  are obtained as the zero sets of the reduction

of  $f$  and  $g$  modulo  $\varpi$ . Since the coefficients corresponding to the vertices of their Newton polytopes are roots of unity in  $\mathbb{K}$ , the reductions modulo  $\varpi$  have the same Newton polytope of the original polynomials. The hypotheses on  $\text{NP}(f)$  and  $\text{NP}(g)$  would allow to conclude then that  $\text{ord}_V(\zeta^* g s_g) = 0$  for all  $V$ .  $\diamond$

**THE MAIN STRATEGY.** This subsection explains how to deduce a proof of Claim 6.4.8 from Claim 6.4.10 and Claim 6.4.12. One will need the following technical lemma. Recall that  $\mathbf{1}$  denotes the neutral element of the group  $\tau$ ; it is a  $d$ -torsion point of  $\mathbb{T}(\overline{\mathbb{K}})$  for any  $d \in \mathbb{N}$ .

**Lemma 6.4.13.** *Let  $\overline{D}_0, \dots, \overline{D}_{n-2}$  be toric divisors on  $X_\Sigma$  equipped with adelic semipositive toric metrics and let  $\underline{f} = (f, g)$  be a pair of Laurent polynomials over  $\mathbb{K}$ . Then,*

$$\frac{1}{\#\tau(d)^2} \sum_{\zeta_1, \zeta_2 \in \tau(d)} H_{\overline{D}_0, \dots, \overline{D}_{n-2}}(\underline{f})(\zeta_1, \zeta_2) = \frac{1}{\#\tau(d)} \sum_{\zeta \in \tau(d)} H_{\overline{D}_0, \dots, \overline{D}_{n-2}}(\underline{f})(\mathbf{1}, \zeta)$$

for every  $d \in \mathbb{N}$ .

*Proof.* Consider the map  $\phi : \tau(d)^2 \rightarrow \tau(d)^2$  defined by  $(\zeta_1, \zeta_2) \mapsto (\mathbf{1}, \zeta_1^{-1} \zeta_2)$ . It is a group homomorphism with  $\text{Im}(\phi) = \{\mathbf{1}\} \times \tau(d)$  and  $\ker(\phi) = \{(\zeta, \zeta) : \zeta \in \tau(d)\}$ . One shows that

$$H_{\overline{D}_0, \dots, \overline{D}_{n-2}}(\underline{f}) \circ \phi = H_{\overline{D}_0, \dots, \overline{D}_{n-2}}(\underline{f}). \quad (6.15)$$

Indeed, the translated of the varieties  $\overline{V}(f)$  and  $\overline{V}(g)$  by  $\zeta_1$  and  $\zeta_2$  intersect properly if and only their translated by  $\mathbf{1}$  and  $\zeta_1^{-1} \zeta_2$  do, which proves (6.15) on  $\mathcal{B}(\underline{f})$ . Otherwise, the claim follows from the invariance of the height under twisting by a torsion point, the metrics on  $D_0, \dots, D_{n-2}$  being toric.

Then

$$\begin{aligned} \sum_{\zeta_1, \zeta_2 \in \tau(d)} H_{\overline{D}_0, \dots, \overline{D}_{n-2}}(\underline{f})(\zeta_1, \zeta_2) &= \sum_{\zeta_1, \zeta_2 \in \tau(d)} (H_{\overline{D}_0, \dots, \overline{D}_{n-2}}(\underline{f}) \circ \phi)(\zeta_1, \zeta_2) \\ &= \sum_{\zeta \in \tau(d)} \sum_{(\zeta_1, \zeta_2) \in \phi^{-1}(\mathbf{1}, \zeta)} H_{\overline{D}_0, \dots, \overline{D}_{n-2}}(\underline{f})(\mathbf{1}, \zeta) \\ &= \#\tau(d) \sum_{\zeta \in \tau(d)} H_{\overline{D}_0, \dots, \overline{D}_{n-2}}(\underline{f})(\mathbf{1}, \zeta), \end{aligned}$$

implying the claim.  $\square$

Assume from now on that the hypotheses of Claim 6.4.8 are satisfied. Because of Theorem 2.4.16 (2) and Proposition 3.1.12, one can moreover assume that  $X_\Sigma$  is a smooth projective toric variety with fan compatible with the Newton polytopes  $\text{NP}(f)$  and  $\text{NP}(g)$ .

*Step 1.* Fix for the moment a positive integer  $d$  and a torsion point  $\zeta \in \tau(d)$  for which  $\text{div}(fs_f)$  and  $\text{div}(\zeta^*gs_g)$  intersect properly in  $X_\Sigma$ . The sections  $fs_f$  and  $\zeta^*gs_g$  are defined over the base change of  $X_\Sigma$  to the cyclotomic field  $\mathbb{K}_d$  obtained by adding to  $\mathbb{K}$  all the  $d$ -roots of unity of  $\overline{\mathbb{K}}$ . Let  $(\mathbb{K}_d, (n_w, |\cdot|_w)_{w \in \mathfrak{M}_d})$  be the adelic structure on  $\mathbb{K}_d$  induced by the one of  $\mathbb{K}$  as in Proposition 2.3.12. Let also

$$Z_{(f,g)}(\mathbf{1}, \zeta) = \text{div}(\zeta^*gs_g) \cdot \text{div}(fs_f) \cdot X_\Sigma$$

as in (6.8). Using Theorem 3.2.4 and the irreducibility of  $f$  given by Proposition 6.4.7 one can write

$$Z_{(f,g)}(\mathbf{1}, \zeta) = \text{div}(\zeta^*gs_g) \cdot \overline{V(f)}. \quad (6.16)$$

Since  $X_\Sigma$  is projective and  $\text{div}(fs_f)$  and  $\text{div}(\zeta^*gs_g)$  are assumed to meet properly in  $X_\Sigma$ , the moving lemma assures the existence of sections  $s_0, \dots, s_{n-2}$  of  $\mathcal{O}(D_0), \dots, \mathcal{O}(D_{n-2})$  respectively with

$$\text{div}(s_0), \dots, \text{div}(s_{n-2}), \text{div}(fs_f), \text{div}(\zeta^*gs_g)$$

intersecting properly. In particular, the sections  $s_0, \dots, s_{n-2}, \zeta^*gs_g$  intersect  $\overline{V(f)}$  properly. The recursive definition of local heights, together with (6.16), asserts that for every place  $w \in \mathfrak{M}_d$

$$\begin{aligned} h_{\overline{D}_0, w, \dots, \overline{D}_{n-2}, w} (Z_{(f,g)}(\mathbf{1}, \zeta); s_0, \dots, s_{n-2}) = \\ h_{\overline{D}_0, w, \dots, \overline{D}_{n-2}, w, \overline{D}_g, w} (\overline{V(f)}; s_0, \dots, s_{n-2}, \zeta^*gs_g) \\ + \int_{X_{\Sigma, w}^{\text{an}}} \log \|\zeta^*gs_g\|_{g, w} c_1(\overline{D}_0, w) \wedge \dots \wedge c_1(\overline{D}_{n-2}, w) \wedge \delta_{\overline{V(f)}_w^{\text{an}}}, \end{aligned}$$

with  $\overline{D}_{g, w} = (D_g, \|\cdot\|_{g, w})$  denoting the divisor  $D_g$  with its  $w$ -adic Ronkin metric, as in Definition 4.2.4.

The integral term on the right hand side is zero for all non-archimedean places, thanks to Claim 6.4.12. Because of Proposition 2.3.12 and Lemma 2.3.5, such an integral term is nonzero only for finitely many places  $w$  of  $\mathbb{K}_d$ . Moreover, the local heights of  $Z_{(f,g)}(\mathbf{1}, \zeta)$  and  $\overline{V(f)}$  are almost all zeros by Theorem 3.5.4 and the fact that the integrability of a cycle does not depend of the choice of

the sections by [BPS14, Proposition 1.5.8 (1)]. Summing over all the places  $w \in \mathfrak{M}_d$  one obtains the relation between global heights

$$\begin{aligned} h_{\overline{D}_0, \dots, \overline{D}_{n-2}}(Z_{(f,g)}(\mathbf{1}, \zeta)) &= h_{\overline{D}_0, \dots, \overline{D}_{n-2}, \overline{D}_g}(\overline{V}(f)) \\ &+ \sum_{v \in \mathfrak{M}_{d, \infty}} n_w \int_{X_{\Sigma, w}^{\text{an}}} \log \|\zeta^* g s_g\|_{g, w} c_1(\overline{D}_{0, w}) \wedge \cdots \wedge c_1(\overline{D}_{n-2, w}) \wedge \delta_{\overline{V}(f)_w}^{\text{an}}. \end{aligned} \quad (6.17)$$

with  $\mathfrak{M}_{d, \infty}$  denoting the set of archimedean places in  $\mathfrak{M}_d$  and  $\overline{D}_g$  the divisor associated to  $g$  equipped with its Ronkin metric, see Definition 4.2.8.

*Step 2.* Recall that by  $\mathcal{B}(f, g)$  one means the subset of torsion points for which the corresponding twisted varieties of  $f$  and  $g$  do not meet properly. It follows from the hypotheses of Claim 6.4.8 that  $(\mathbf{1}, \zeta) \notin \mathcal{B}(f, g)$  for any choice of  $\zeta \in \tau$ . Indeed, if this was not the case, the divisors  $\text{div}(f s_f)$  and  $\text{div}(\zeta^* g s_g)$  would not meet properly in the dense open orbit  $X_0$  (since their support do not contain any closed toric 1-codimensional orbit); then, the Laurent polynomials  $f$  and  $\zeta^* g$  would have a common irreducible factor in  $\mathbb{K}[M]$ , which is absurd since they are irreducible by Proposition 6.4.7 and distinct by Proposition 6.2.12. Hence, for any  $d \in \mathbb{N}$ , averaging on the subgroup of  $d$ -torsion points of  $\mathbb{T}(\overline{\mathbb{K}})$  yields

$$\begin{aligned} \frac{1}{\#\tau(d)} \sum_{\zeta \in \tau(d)} H_{\overline{D}_0, \dots, \overline{D}_{n-2}}(f)(\mathbf{1}, \zeta) \\ &= \frac{1}{\#\tau(d)} \sum_{\zeta \in \tau(d)} h_{\overline{D}_0, \dots, \overline{D}_{n-2}, \overline{D}_g}(\overline{V}(f)) \\ &+ \frac{1}{\#\tau(d)} \sum_{\zeta \in \tau(d)} \sum_{w \in \mathfrak{M}_{d, \infty}} n_w \int_{X_{\Sigma, w}^{\text{an}}} \log \|\zeta^* g s_g\|_{g, w} d\mu_w \end{aligned} \quad (6.18)$$

where

$$\mu_w := c_1(\overline{D}_{0, w}) \wedge \cdots \wedge c_1(\overline{D}_{n-2, w}) \wedge \delta_{\overline{V}(f)_w}^{\text{an}}$$

for simplicity of notation.

The summands of the first term on the right hand side of (6.18) are constant in  $\zeta$ , so such a term coincides with

$$h_{\overline{D}_0, \dots, \overline{D}_{n-2}, \overline{D}_g}(\overline{V}(f)) = \sum_{v \in \mathfrak{M}} n_v \text{MI}_M(\vartheta_{0, v}, \dots, \vartheta_{n-2, v}, \rho_{f, v}^\vee, \rho_{g, v}^\vee)$$

because of Theorem 4.2.10, Theorem 2.4.16 (3) and the symmetry of mixed integral.

The second term on the right hand side of (6.18) can be rewritten as

$$\begin{aligned} & \frac{1}{\#\tau(d)} \sum_{w \in \mathfrak{M}_{d,\infty}} n_w \sum_{\zeta \in \tau(d)} \int_{X_{\Sigma,w}^{\text{an}}} \log \|\zeta^* g s_g\|_{g,w} d\mu_w \\ &= \sum_{v \in \mathfrak{M}_\infty} \frac{1}{\#\tau(d)} \sum_{w|v} n_w \sum_{\zeta \in \tau(p)} \int_{X_{\Sigma,w}^{\text{an}}} \log \|\zeta^* g s_g\|_{g,w} d\mu_w \end{aligned}$$

The fact that  $\mu_w$  is supported on  $\overline{V(f)}_w^{\text{an}}$ , which has no components outside  $X_{0,w}^{\text{an}}$ , and that the measure  $\mu_w$  does not charge closed subsets allow to restrict the integrals in the previous sum over  $X_{0,w}^{\text{an}}$ . Each of the finitely many terms of the sum over  $v \in \mathfrak{M}_\infty$  converges to 0 as  $d$  is going to  $\infty$  because of Claim 6.4.10.

Hence, the quantity in (6.18) converges to the limit

$$\sum_{v \in \mathfrak{M}} n_v \text{MI}_M(\vartheta_{0,v}, \dots, \vartheta_{n-2,v}, \rho_{f,v}^\vee, \rho_{g,v}^\vee)$$

as  $d \rightarrow \infty$ . Lemma 6.4.13 would then conclude the proof of Claim 6.4.8.

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