# Méthodes de pilotage des flux avec prise : en compte des incertitudes prévisionnelles

Maxime Claisse

NNT : 2018SACLC022

# Thèse de Doctorat

de

L'Université Paris-Saclay

Préparée à

CentraleSupélec

Ecole Doctorale n°573
Interface

Sciences et technologies industrielles

par

**M. CLAISSE Maxime**

# Production Planning under Uncertainties and Forecasts Update

Thèse présentée et soutenue à Gif-sur-Yvette le 12/02/2018:

| | |
|---|---|
| M., CHU, Chengbin, | Professeur, CentraleSupélec<br>Directeur de thèse |
| M., JEMAI, Zied, | Maître de Conférences, Université de Tunis el Manar<br>Co-Directeur de thèse |
| M., KACEM, Imed, | Professeur, Université de Lorraine<br>Président du jury |
| M., BABAI, Mohamed-Zied, | Professeur, Kedge Business School<br>Rapporteur |
| Mme, GHAFFARI, Asma, | Maître de Conférences, CentraleSupélec<br>Examinateur |
| M., ARTIBA, Abdelhakim, | Professeur, Université de Valenciennes<br>Examinateur |
| M., VAN DELFT, Christian, | Professeur associé, HEC Paris<br>Examinateur |

# Remerciements

Je tiens tout d'abord à remercier vivement les professeurs Chengbin Chu et Zied Jemai qui ont encadré cette thèse. Leurs connaissances, culture, et vision scientifique et industrielle m'ont éclairé dans l'avancée de ces travaux, et m'ont également beaucoup apporté dans un contexte plus large sur l'appréhension de situations à double caractère technique et applicatif.

J'exprime ensuite mes remerciements les plus sincères à Monsieur Mohamed-Zied Babai et Monsieur Imed Kacem, qui, en tant que rapporteurs, ont accepté d'évaluer ce travail. Leurs pertinents conseils et remarques ont apporté une aide précieuse pour améliorer et mettre en relief ce travail.

Je remercie également Madame Asma Ghaffari, Monsieur Christian Van Delft, et Monsieur Abdelhakim Artiba, d'avoir accepté d'être membres du jury, pour l'intérêt qu'ils ont manifesté pour ces travaux, et pour leurs échanges constructifs.

Je tiens ensuite à remercier Messieurs Jean-Claude Bocquet, Bernard Yannou, et Vincent Mousseau, qui dirigent ou ont dirigé le laboratoire et l'équipe de recherche dans lesquels cette thèse a été effectuée, et grâce à qui mon environnement de travail était propice à un épanouissement professionnel et personnel.

Je remercie chaleureusement Delphine, Sylvie et Corinne, qui ont tout d'abord toujours été disponibles, à l'écoute, et dont l'aide m'a été particulièrement créatrice de valeur; et qui, également, m'ont offert des jours au labo agréables par leur bonne humeur, leur implication, et leur partage.

Cette thèse a été pour moi bien plus qu'une expérience professionnelle, mais également une période personnelle riche en rencontres. Un immense merci à ma famille du labo Ronay, Laurie, Toufic, Karim, Hakim, Hadi, Mehdi, ces années de thèse resteront nos moments de rencontre, remplis d'excellents souvenirs et de vives émotions.

Je ne remercierai jamais assez mes parents pour leur soutien inconditionnel, leur immense aide, et leur implication. "Quitte à faire des études, autant aller jusqu'au bout. Et puis, si tu ne le fais pas maintenant, tu ne le feras jamais"

Enfin, je remercie infiniment ma femme Denis, qui pendant ces années, a joué un rôle absolu. Elle a toujours été là pour me soutenir, m'aider, m'écouter, me faire taire quand il le fallait, me faire avancer, toujours dans le bon sens. Elle a su quoi dire, quoi faire, quoi penser pour que cette aventure aboutisse. Elle m'a supporté, poussé vers l'avant, elle m'a fait atteindre cette accomplissement;

elle m'a fait grandir.

Cette thèse s'est faite grâce à elle. Merci du fond du coeur mi amor.

# Abstract

The current environment of the international market of industrial goods, and the way it evolves through time, has created great challenges in recent years for organizations involved. Industrial companies have actually to face more and more volatility, respond to increasing service level, and be able to adapt themselves more rapidly, whereas costs need on the other way to be controlled and kept in a rational budget. Actors need consequently to implement adapted management processes in their organization, and one of the important points needed to be managed is the uncertain characteristic of the parameters and variables used in their decisional process. It may have indeed large consequences on the resulting state of health of the company, as it influences the methods of resources allocation that may finally not be coherent with the requirements of the market.

Among these methods, Production Planning, as part of tactical operations integrated into the Supply Chain process, is a key procedure allowing decisioners to balance demand and production resources. When this specific decisional process is considered, one of the most critical uncertainties to handle is the ones coming from the Forecasted Demand. In this context, in order to manage indicators at stake, such as service level and costs, best practices and management techniques have appeared, including notably organizational and decisional structures creating flexibility in the process. The Rolling-Plan Framework is a common practice of this family. However, it also creates instability since the updates procedures of the Rolling-Plan framework make the data set on which the decisions are taken change constantly. Consequently, although the gain in terms of flexibility is non-negligible for the uncertainties management, it generates on the other hand dynamics complexity.

We study in this work how to deal this dynamics complexity generated by Updates of the Forecasted Demand made in a Rolling-Plan Framework of a Production Planning Process. In particular, the question to which we answer is how to optimize the Production Plan in such a context.

This issue is tackled considering a single item single level production system. We first examine the dynamics and stochastic characteristics of a forecasted demand in the environment defined, in order to understand how uncertainties and dynamics of the variables involved impact the system. We hence manage to build a general mathematical model to represent it in the context of our study that can be exploitable

for analytical optimization. More precisely, we analyse the sources of uncertainties, and developped a double-level probabilistic structure to represent the dynamics and the uncertainties of the forecasted variables. We study analytically the behavior of such model, and how this modelling approach impacts the planning system in a rolling framework, in particular how it can improve the performances in terms of uncertainties management.

Then, the detailed study of special considerations allows us to design a theoretical optimization model of Production Planning taking into account the corresponding forecasted demand model. In particular, we design a specific and adapted stochastic dynamic programming approach, allowing then us to build a particular solutions computation framework. We proove that the method developed allows us to calculate the optimality of our problem in a general manner.

Besides, we apply the model and solution computation process in some study cases. By assuming some key characteristics for the forecasted demand, as for instance the use of the linear regression as forecasting technique, and for industrial parameters such as the production lead-time, we manage to get some valuable analytical results thanks to a dynamic computation process. In particular, we prove the optimality of solutions through an inductive reasoning based on expectations computation. Depending on the assumptions and configurations in terms of industrial parameters and variables, several prooves are realized to get the optimality. Then, analyses and studies are performed leading us to adapative solutions and optimal behaviors that have been proposed.

Solutions are also implemented and calculated with simulations in some particular numerical examples. Analyses and sensitivity studies are performed, highlighting the performances of our optimization method, compared to others. Moreover, the gain is quantified depending on the industrial environment faced by the system.

# Contents

# List of Figures

# List of Tables

# Chapter 0

# General Introduction

In this preliminary chapter, we briefly introduce the thesis so that the reader can understand the context of our work, the challenges it faces, and how we contribute to respond to them.

To do so, we first describe the background and the objectives of the thesis, before going through the approaches adopted. Finally, we present the outline of this manuscript.

# 1 | A Challenging Background

In manufacturing companies, value is created by transforming resources into goods that consumers and businesses are ready to buy. The supplying process allowing this value creation begins from the early stage of raw material supply and leads to final products. It needs to be managed efficiently to get as much value as possible. Specifically, this process needs to be designed, planned, executed and controlled to reach objectives under company's resources constraints. These are the aim of the Supply Chain Management.

One of the most important issues in that field is due to the general fact that an ideal management of the Supply Chain requires precisely knowing the whole functioning of the considered process, its environment, and its evolution. However, the complexity of companies networks, combined with today's markets diversity and variability, makes this impossible. A lack of knowledge is systematically encountered in any Supply Chain, creating uncertainties within the considered process. These uncertainties, especially the customers-related ones, have great negative impacts on the Supply Chain performances.

The future markets demand are indeed a key information that has big influence on the flows management efficiency. For instance, [Gruen, 2007] analyzed that 8.3% of the products worldwide are facing inventory shortages in the retail industry, among those 47% are due to poor information concerning the sales and the customers needs. Symetrically, 40% of the inventory surplus that become unsaleable are engendered by a blurred vision of the future requirements of the market in mass goods distribution. These lost sales cost more than 1% of the total turnover of the supermarket industry [Lightburn, 2003].

Uncertainties, and more specifically the ones related to the future customer demand, represent consequently an important challenge in the light of what's at stake.

To better understand what would be required tomorrow by the markets, it is a common practice to deal with forecasted customer demand. The demand uncertainties are in that case linked with the accuracy of the forecasts made to predict the future requirements: the more accurate the forecasts are, the less uncertainties there are about the future demand. Consequently, one of the methods applied in practice to cope with this uncertainties issue is to work on the accuracy of the forecasts. Dif-

ferent techniques exist, among those one consists of analyzing the historical data, revealing the way the supply operations have been executed and evolved, in order to understand how we could project this information into the future so that predictions become more accurate. The study of such methods was the subject of the thesis of [Hubert, 2013].

However, whatever the level of accuracy reached by these forecasting models, they will remain wrong, by definition. Thus, even if we succeed in reducing the level of uncertainties, in most cases they will stay non negligible. That is why managing supply operations under uncertainties remains a central issue in Supply Chain Management. In the specific case of Production Planning, [Mula et al., 2006] showed the importance of uncertainties management models, and [Sahin et al., 2013] qualified the consideration of uncertainties in the planning process as a major determinant of the market success of a company. Still, in spite of its importance, [Graves, 2011] pointed out that the academic literature involving production planning under uncertainties is relatively poor compared with deterministic counterparts.

In this thesis, our objective is to attempt fulfilling this gap. We are then particularly interested in studying how Production Planning under uncertainties can be managed.

One of the techniques the most commonly used in industry to handle uncertainties in Production Planning is to consider the problem through a Rolling Horizon framework [Sahin et al., 2013]. The main principle is to consider that, inasmuch as future customer demand cannot be known for sure, the process needs to be adaptive. Thus, in this environment, data is regularly updated and re-treated so that decisions taken within the supplying process are based on the most up-to-date sets of information. Uncertainties are consequently minimal, and operations are more appropriately executed to the current situation. A better reactivity is therefore observed, as well as diminution of mismanagement, leading finally to better performances [Penlesky et al., 1989].

However, although decisions are more consistent within the actual processing of the production operations and customer demand, it raises an important modeling issue. As a matter of fact, such environment creates instability due to frequent disturbances into the decision system. In particular, the models used to represent the future customer demand become dynamic, since demand forecasts are updated as well. This engenders arduous problem structure, especially in terms of cost mini-

mization. As a consequence, a lack of attention from academics is observed, whereas, notwithstanding the increase of the modeling complexity, the impact of taking into account the forecasts updates into the Rolling Horizon Planning framework would be widely positive [Sahin et al., 2013].

Our work aims at bringing material to contribute to this area.

# 2 | Purpose of the Thesis

It is therefore shown that uncertainties are a major issue in production planning. Besides, by improving the flexibility of the decisional process, the Rolling Horizon framework, commonly used in industry, allows a better management of the unknown evolution of the demand but increases instability and creates optimization modeling problems due to forecasts updates procedure.

As we will see in the following section, the existing literature suffers from a lack of models in such context, whereas it corresponds not only to the real proceedings of production planning and flows management in any manufacturing environment, but also to a technique proved to be efficient in practice.

The object of this thesis is consequently to reduce this gap. Hence, we consider in this work Production Planning under uncertainties in a Rolling Horizon framework. In particular, we focus our study on production planning while taking into account the dynamics of the demand forecasts updates.

In order to attempt responding to this challenge, the remainder of the manuscript will investigate the following questions:

- How do Rolling Horizon Framework and forecasts updates impact the Demand Forecasts Modeling?
- How do such models influence Supply Chain Performances?
- How can the forecasts update mechanism be modeled to be integrated into an optimization model?
- What optimization model could be used in such cases?
- How can optimal solutions be computed?
- Are these solutions more efficient than classical production planning methods, and to what degree do resulting production plans provide better performances?
- In which context can such solutions be implemented?

We present in the following section the approaches pursued to answer these questions.

# 3 | Approaches

## 3.1 | Research Methodology

First and foremost, it is important to precise that the previous questions this thesis attempts to answer have been examined in particular contexts. We indeed defined the perimeter of our work to the single-product single-level production planning problem, so that the central issues of our research questions, namely the uncertain and non-stationary characteristics of the forecasted demand, can be analyzed in depth without adding any external complexity to the problem. For the same reasons, we also make other assumptions, especially concerning the industrial environment. The specific hypotheses taken in this work will be defined more precisely in the next chapter.

In this context, we have tackled the problem through different steps of investigation:

We first developed a general framework of study for the considered issue. It enabled us to define in details the decisional system on which the modelings of the problem have been based. This framework has been used as a pattern to examine existing planning methods and understand the problem in a quantitative way by defining variables, parameters, and indicators revealing performances that are at stake. In particular, the demand-forecasting model has been deeply studied in order to represent the dynamics of the rolling horizon with updates.

Second, based on this framework, we develop a general optimization model. In order to catch both dynamic and stochastic features of the problem, a stochastic dynamic programming approach is developed. In particular, the forecast-updating mechanism is explicitly expressed in the stochastic modeling, and is integrated into the optimization process.

Then, analytical studies have been performed in which we were able to develop computational methods to get theoretical results of the corresponding optimization problem. It has been especially the case when linear regressions have been used as forecasting method in a fast delivery environment (zero- or one-period delivery lead-times).

The optimal solutions have been then tested through numerical simulations and compared with other planning methods so that performances enhancement can be evaluated.

## 3.2 | Contributions

The main value created by this thesis is twofold.

First, from an academic point of view, our work led us to developing theoretical optimization models and their associated optimization methods in a specific but not-yet-studied non-stationary stochastic context. The models and their corresponding solutions integrate indeed not only the uncertainties of the customer demand but also its complex dynamics due to forecast updates. Moreover, although these issues forced us to make relatively restrictive assumptions in terms of parameters to be taken into consideration, the results found bring important value. The techniques presented in this work are indeed purely analytic, and solutions are explicitly expressed. They also can be extended, allowing consequently to open new interesting perspectives, that will be discussed at the end of this manuscript.

Second, the problem studied here has a real-life industrial background that practitioners are facing regularly. Uncertainties and dynamic characteristics of the regularly updated customer demand are indeed the key factors influencing production quantities that are still nowadays poorly understood. Thus, our work, bringing theoretical answers to this specific field, has practical industrial implications by contributing to real-life complex problems on a daily basis. It contributes in particular to the current high stakes field of flexible and agile Supply Chain, managing uncertainties being nowadays at the heart of flow management.

# 4 | Thesis Structure

This dissertation is organized as follows.

In the first chapter, we will define precisely the problem considered in the thesis. To do so, we will begin by introducing and studying the production planning environment, before analyzing the currently used methods. Our objective is here to position clearly the thesis in its environment of study, and highlight its originality in the set of current modelings, computation methods, or analyses.

The second chapter is dedicated to the definition of the quantitative framework of this study. In particular, the industrial system considered and the decision system in which our works are inserted will be described in details. The impact of our positioning on the modeling approaches for production planning will be studied, and general methods will be assessed accordingly to understand quantitatively the consequences of the non-stationarity of demand to forecast.

In the third chapter, the optimization model developed in this thesis will be presented. To do so, we first describe our modeling in a standard case of study, and the solution methods that can be implemented to take into account the forecast updates in the optimization problem.

Then, we will study the specific case of simple linear regression method as forecasting method and fast delivery mode with no delivery lead-time. Explicit analytical expression of an optimal solution will be presented, and numerical study will follow to prove its efficiency.

Chapter four deals with a first extension of our model. The aim of this chapter is indeed to open the optimization models developed and the solution method implemented for more complex situations by relaxing some hypotheses. Specifically, we will present analytical results concerning a fast but non-null delivery lead time of one period. Simple linear regression will still be used for demand forecasting, and an extension to multiple linear regression will also be studied. In the same way as previously, simulations will also be performed and the outcome will be analyzed to understand the solutions in different situations. The goal is to study to what extent our model can be used.

We finally conclude this thesis with highlights of our work, limits, and perspectives for future researches.

# Chapter 1

# Problem Statement

In this chapter, we define precisely our positioning.

To do so, we first describe the context of Supply Chain Management and specifically the Production Planning field, to which our work contributes. Our objective is here to specify concepts that will be used in the sequel of the manuscript.

The second part of the chapter presents existing methods and academical works related to this thesis. We show here how current academic situation suffers from a lack of consideration for our issue.

Finally, this litterature review, along with the definition of the context of our study, leads us to the precise positioning of this manuscript and highlights its relevance.

# 1 | Context

## 1.1 | Supply Chain Management and Production Planning[1]

### 1.1.1 | Supply-Demand Relationship

Current economic and societal situations, and constant changes that take place in our societies make business environment increasingly challenging for any people and companies involved. Competitiveness and adaptability are thus nowadays among the main concerns of all companies willing to prosper in such a globalized and dynamic market. To succeed in developing such characteristics, companies must focus on high quality products and services, delivered with a good service level through flexible processes, while spending the least, in terms of money but also any of other strategic resources. Supply Chain Management is the field whose goal is to bring answers to such issues. Its purpose is indeed designing methods allowing efficient supply operations management. Then, two dimensions need to be made clear in order to define Supply Chain Management: what operations are concerned, and how to manage them.

The first dimension is directly related to the process of supplying goods, and all the necessary phases of the material flow to reach the final delivery to the customer. In a very basic scheme shown in figure 1.1 , we can summarize these steps into four chronological groups of operations: procurement, production, distribution, and sales. The first one aims at getting all products, components, raw materials, etc. needed to start the manufacturing process. It involves consequently multiple actors such as suppliers, and upstream logistics partners. The second one concerns the production of the goods, leading to products that company is ready to sell. Several production stages can appear, and various people and manufacturing companies may also be involved. Next, distribution is about the delivery process, but may also include final product inventory control and warehousing management. Finally, the sales take care of the interface between Supply Chain and the final customers, in terms of product deliveries, but also financial and information flows.

The figure 1.1 represents in a very simple manner the supply/demand relationships that exist in the Supply Chain, but it does not explicit the very large combi-

[1]This section has been written based on the works of: [Arnold et al., 2008], [Chopra and Meindl, 2013], [Cox et al., 1992], [Maimon et al., 1998], [Muckstadt and Sapra, 2010], [Peterson and Silver, 1979], [Pochet and Wolsey, 2006], [Rohde et al., 2000].

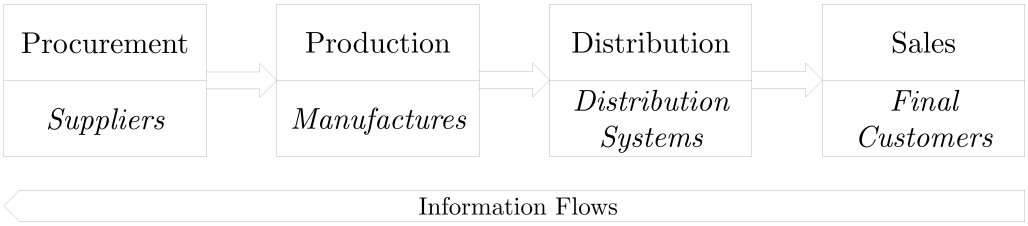



Figure 1.1: Basic steps of material flows and orientation of information flow in SC

natorial complexity that may appear in a supply and delivery network.

As long as companies belong to the same chain linked by supply and demand relationships, they belong to the same Supply Chain. Thus, there may be any number of companies involved in the same Supply Chain, where a customer is also the supplier of its downstream customers. Although a Supply Chain organization may consequently be very different from one industry to another, the framework of the types of operations making the process remains similar.

The second dimension refers to the type of decisions made to manage the operations of the stages of the supply process briefly described above. They can actually be classified according to the time horizon in which the companies, the people or the systems involved will be affected by the decision. The decisions are thus usually split into 4 levels: long-term, mid-term, short-term, or very-short-term ones. The figure 1.2 shows examples of decisions for each decision level, split by the operations types.

| | Procurement | Production | Distribution | Sales |
|---|---|---|---|---|
| Long Term | • Suppliers Selection<br>• Sourcing contract | • Localization & sizing of the production facilities | • Definition of the distribution structure | • Sales Strategy definition |
| Mid Term | • Capacity booking with Suppliers | • Capacity planning<br>• Master Production Plan | • Distribution capacity definition | • Mid-term Sales Planning |
| ShortTerm | • Sourcing Operations Management<br>• Personnal Plan | • Detailled Production plan<br>• Lot sizing | • Distribution plan<br>• Warehouse replenishment | • Short term Demand forecasts |
| Very Short Term | • Receipts and orders management | • Production Scheduling<br>• Shop Floor Control | • Deliveries and customers orders management | • Demand follow-up and monitoring |

Figure 1.2: Examples of operations arranged in the matrix "decision levels" and "steps of the supply process"

### 1.1.2 | Planning Procedures

Among all of the decisions needed to be made to manage Supply Chain operations, we can cluster some of them around an identical goal, namely: making plans. Planning decisions are key to Supply Chain Management, inasmuch as the ability of a company to respond to customer demands depends directly on what was planned by this company. One of the reasons is that supplying any good to a customer requests time. In fact, depending on the industry, the company's policies, or the product considered, multiple configurations may exist to deliver the product. These configurations may vary according to the moment of the supply process from which the operations are driven by the firm's customer orders, called the order penetration point. However, there will always be a delay corresponding to the time needed to fulfill the customer order. The figure 1.3 gives us 3 examples of these different supply chain configurations, engendering different delivery lead time, whether the company is organized as "engineering-to-order" (in which the design of the product is launched as soon as an order arrives), "make-to-order" (in which the production of the product is launched as soon as an order arrives), or "make-to-stock" (in which the delivery of the product is launched as soon as an order arrives).

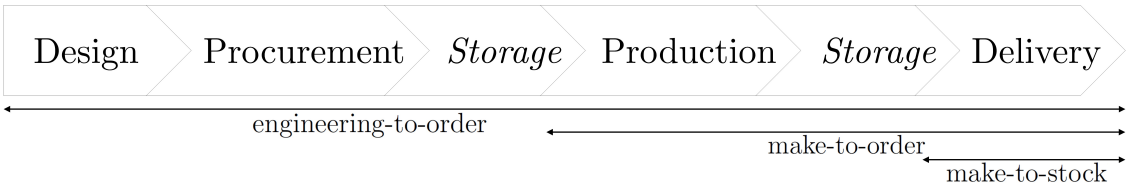



Figure 1.3: Examples of configurations according to the organization of the order penetration point

Moreover, in general, capacities -production, financial, or human resources- are limited. Therefore, in such a context in which a processing time is required and practical limitations have to be respected, any industrial company needs to plan its next operations to respond to the forthcoming customer demand. Planning deals consequently with the allocation of the capacities of the organizations involved over a future time horizon, thought and decided to provide requested goods or services while controlling generated costs.

Planning exists in all steps of the Supply Chain, from the procurement to the distribution, and consists more precisely of defining for the corresponding operations what is needed to be executed, how much, when, and where so that the supply process can be operated as "good" as possible, meaning the best service level possible

for customers under the finite resource and capacity of the company and the cost constraints to be respected.

When we consider specifically Production Planning, depending on the time horizon considered, we are respectively dealing with strategic business plan, sales and operations plan (S&OP), master production schedule (MPS), requirements plan or production plan, and production schedule or purchase control. We generally speak about top-down production planning hierarchy. Besides time horizon, the granularity of the details considered in the planning, the purpose of the plan, and the update frequency are key features that characterize these plans and distinguish them one from another.

The strategic plan aims at defining from a business point of view the general objectives and expectations of the company over the next two to ten years. The information used in this plan are very aggregated, dealing with major product groups, global turnover, etc., and the update frequency is very low, about several months. Then, the S&OP's goal is to define the production capacity and resources -materials and human- needed to respond to the market requirements represented by the strategic business plan. The information taken into account here are family of products, and the update frequency is greater, generally one per month.

As far as the MPS is concerned, the objective is to define precisely for the next months the production needs of each finished goods so that service level can be as high as possible -or so that the service level targeted can be reached-, while productions rates compared to capacity and other production constraints are respected. It is based on the forecasted demand of each final product, the production capacity, and the current workloads. Following the MPS, the requirements plan allows companies to get the needs of purchased goods or components, or produced sub-products, in order to respond to the demand of finished goods generated by the MPS. The granularity of the information used is consequently thin, inasmuch as details concerning all the production or procurement of any product are needed. The planning horizon of both MPS and the requirements plan corresponds to several months, and has to be greater than the requested delivery lead times.

Finally, the production schedule or purchase control is the short-term plan based on which the operations are executed. It is the tangible implementation of the previous ordered plans, managing the flows into the plant or in the purchasing operations with current production workforce, machines, or resources available. As it is day-to-day production or procurement operations management, the reviewing

frequency is generally daily, or every several days.

The following figure 1.4 sums up the level of granularity of the details used in the different plans presented above according to the time horizon of the corresponding plan.

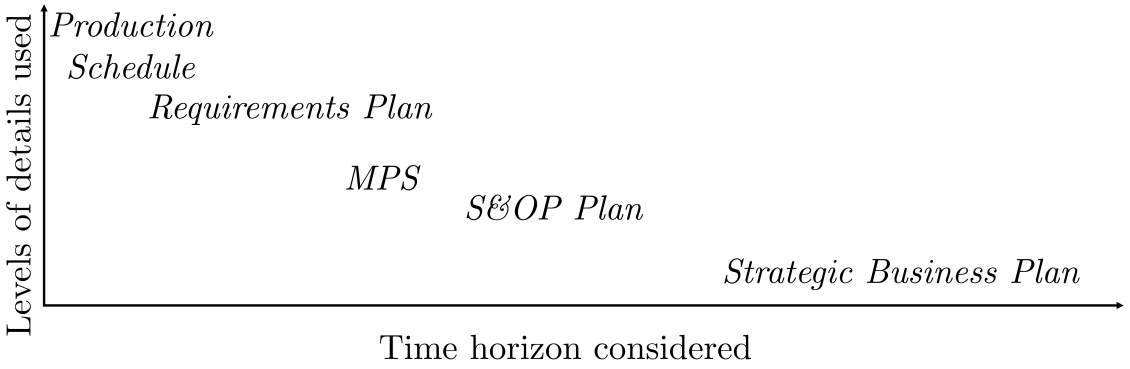



Figure 1.4: Granularity *versus* time horizon in different production planning operations

In this thesis, we particularly focus our work on studying Master Production Scheduling operations, i.e. decisions leading to the answers of **when and how many finished goods need to be produced** to fulfill the demand under potential costs, resources, and capacity constraints. Also, we will see that the studies presented in this manuscript have been realized under some specific hypotheses, described more precisely in the next sections, that make the MPS similar to the requirements plan. It is indeed particularly the case for the one stage production configuration. Consequently, in the remainder of this thesis, we will use indifferently the terms "master production schedule", "requirements plan", "production planning", or simply "planning" to refer to the master production schedule as defined previously.

The main issue of production planning that makes the exercise difficult is that planning is all about finding the best compromise. In a general manner, different departments have its own concerns or interests leading to different stakes, and consequently may take conflicting positions. In fact, from the perspective of the sales or marketing department, the main objective is to increase company's turnover. They are thus particularly concerned about customer service. As a consequence, they would tend for example to have high level of finished goods inventories and flexible production. As for the production department, its main goal is to maintain low production costs, and to avoid falling into over-production situations. These would

imply reducing the number of change-overs, maintain high level of raw material inventory to ensure production would not stop, and limit total production level to sidestep overflows. Finally, the finance department's attention is concentrated to the fixed assets costs and return on investment. They would consequently prefer keeping the inventories at a low level and decreasing the production assets and capacities, which finally would impact the service level in a negative way.

Therefore, conflicts exist between actors in the decision process leading to the production plan. In order to solve this situation, production planning is usually treated as an optimization problem, leading to the best trade-off between financial and market-oriented criteria, such as minimum costs or maximum profit, and service level respectively.

This optimization problem and its characteristics depend naturally on the environment in which the system evolves. That's why various mathematical models exist according to the industrial system studied.

In the following section, we go into details of the variety of configurations existing in production planning. We describe in particular the factors influencing the modeling of production planning as an optimization problem.

## 1.2 | The Production Planning and its study environment[2]

When a production planning problem is considered, the modeling process leading to decision support begins by defining the distinctive features of the system that need to be made clear to delineate the problem.

To do so, several considerations need to be taken into account. First of all, the industrial system defined by the elements representing the environment of the production operations has to be clarified. Some of these main elements to consider are listed in the following paragraph.

Second, the decision system, i.e. the variables, parameters and indicators wished to be integrated into the modeling has to be specified as well. In particular, the hypotheses concerning the customer demand modeling on the one hand, and the dynamics of the decision-making framework for which the optimization is needed on the other hand, are key factors influencing the modeling of production planning problems. These are precised in the second paragraph of this section.

[2]This section has been written based on the works of: [Arnold et al., 2008], [Cao, 2015], [Cheaitou, 2008], [Chopra and Meindl, 2013], [Graves et al., 1993], [Graves, 2011], [Hubert, 2013], [Muckstadt and Sapra, 2010], [Peterson and Silver, 1979], [Toomey, 2000].

### 1.2.1 | Industrial context

The environment in which the production planning decisions have impacts can be defined through several aspects. Here are the main features of the industrial context, allowing to understand more precisely the production planning problems that have to be considered.

#### i. Products

As described previously, Production Planning is about determining how many products need to be launched in production, and when. Before getting into these questions, it is essential to know the information concerning the kind of products that need to be planned so that the perimeter of the planning can be defined.

The number of types of products impacts indeed the modeling of the problem. Besides, if multiple items are considered, their production may not be independent of one another, for instance when they share similar components, or the same production process. Production plan will consequently be affected accordingly. Specific characteristics of some products may also influence the production plan, for example when the products considered are perishable, consumable or repairable, seasonal, etc.

#### ii. Supply network organization

Depending on which industries or companies are concerned, the structure of the supply process may be quite different. This network defines how materials, information, and financial flows are managed between supply chain protagonists. It can consist of many levels involving several partners, in different locations, or several internal production facilities. Besides, they may be organized as a serial production system, a parallel one, or a complex mix of them.

In our case study of the production planning, this structure influences the way one can model it, inasmuch as several production stages may exist. This engenders different requirements for each stage, not only in terms of procured materials -components, raw material, etc.-, but also in terms of information. It is indeed necessary to have specific plans. The organization of the production process is consequently a key industrial parameter in the modeling process of production planning.

**iii. Production lead-time**

By nature, any operation of a supply chain depends on upstream material flows and its procurement, and is supposed to fulfill the downstream requirements. Therefore, the time required to complete these tasks will impact the way operations are organized in any production level. In particular, it will impact the timing of when they have to be executed. The lead-times information is thus an important parameter to which plans are sensitive. This information may be known and defined clearly, when, for example, it is determined in advance with the corresponding suppliers, or considered as random, in the case of external or internal disruptions appearing during the execution of the operations and impacting their processing time. They are naturally dependent on the product considered, on the process executed to produce it, and the supplier executing it.

**iv. Production constraints**

Besides production lead-times, other aspects of the production context that affect directly the planning need to be integrated into the model, such as production capacity constraint, minimum of production quantity during a period, or productions of fixed lot sizes, etc.

**v. Cost structure**

One of the main factors influencing the decision-making in production planning is the costs induced by a plan. As described, production planning is actually a trade-off. In this context, the costs are here key to the optimization process: it is usually either the objective function to minimize, or one of the main constraints to respect so that final decision could satisfy financial stakes of the company.

Various types of costs may be considered in the planning models:

i) The first one represents the cost directly linked to the production orders. Depending on the production framework, two sources of expenses can exist:
   - ➢ The setup costs, or the fixed production costs, which occur when a production is placed and that do not depend on order quantity. It corresponds for instance to the costs generated by operations required to start up a production line or a machine. The existence of such costs is a major incentive to launch mass production in order to benefit from the economy of scale.
   - ➢ The variable costs, or unit costs, depends conversely on the volume ordered. These concern all quantity-dependent direct or indirect expenses such as com-

ponents purchasings, supplying equipment requested to produce one product, etc.

ii) The second type of costs that occur in a production system to be taken into account in the planning process is related to the inventory of the produced goods. It can be either the inventory holding costs, or the shortage costs:

- When the quantity produced is bigger than the customer demand, the company has an excess of products that must be kept. This surplus proves that too many products were produced, which means too much money invested. The surplus of production was indeed a source of expenditures. This capital is held until the products are finally sold, which generates an opportunity cost if the capital was used differently. Besides, carrying products in storage facilities requires fixed and variable costs for taking care of it, such as the equipment, the financial and human resources needed to manage the warehouses, insurance, taxes, damage, etc.

  These two sources of expenses represent the holding costs. Different ways of calculation may be used to evaluate them. Usually, the holding costs per product are approximated as proportional to the value of the products carried in stock. In the cases of temporal subdivisions through periods, holding costs are assumed to be dependent on the number of units held and to the number of periods during which they are kept.
- Contrary to circumstances engendering holding costs, there exist those in which the production level cannot meet all the customer requirements when they arrive. In that case, shortage costs arise. Different situations may be taken into consideration, among them, the two extremes being:
  - Customers may decide to be supplied with another company. In this case, the sale is considered as lost. This context represents a cost for the company, in which there is at least the lost profit that could have been earned. Next sales may also be affected, at least because of the degradation of the company image through non-satisfied customers point of view. Moreover, shortages of inventory generate indirect costs due to the fact it has to be managed. Operational organization must indeed be adapted: faster delivery modes, and consequently more costly, may be chosen for example, as a reaction of the lost sales to counterbalance them. This "lost sales" situation is consequently costly. In the usual approximation, this cost is considered as proportional to the number of

sales lost.

- On the contrary, customers may be ready to wait until their demand can be satisfied. In other words, their requirements not yet fulfilled because of shortages of stock are delayed, and cumulated with the subsequent ones, till the inventory reaches a level that could enable the requested deliveries and finally meet the demand. This delay causes additional costs due to "new" ordering management and, in a similar way as the lost sales situation, due to the degradation of the quality of the delivery perceived by customers. We usually refer to the "backlog" situations when we consider this kind of behavior when shortages appear. As in the lost sales case, the backlog costs are proportional to the number of products backlogged. Besides, it also depends on the number of periods during which the corresponding requirements have been kept.

As far as this thesis is concerned, we will consider in the remainder of the manuscript the uncapacitated single-product single-level production planning problem. No specific characteristics is kept concerning the product, and the different production lead-times studied are equal to zero, one or two periods. Finally, the costs chosen in our studies correspond to classical ones, the inventory holding costs, the unit production costs, and the backlog costs.

### 1.2.2 | Decisional system considered

In order to complete the positioning in the field of production planning, we need to define the key features or dimensions of the framework the decisions resulting from the planning procedure are inserted in. Among them, we first define hereafter the possible ways to model the customer demand, main input data of planning models, and then the dynamics of the system for which optimization is needed.

#### i. Customer Demand Modelings

Once the industrial context has been defined, we know more precisely the internal and external environment of the production system that can respond to a request. Still, it is necessary to determine thereafter what kind of request the system is supposed to fulfill.

Since production planning decisions deal with the future, customer demand generating production requirements is composed of two parts: the demand already

received (frim orders) and the demand expected to be received later. The latter part has to be estimated. Besides, many characteristics may change from a market to another one, making the environment influencing the customer demand very changeable. Thus, depending on the context, the demand modeling will be different. Several hypotheses or specificities have to be taken into consideration consequently. We give here some examples of characteristics of the customer demand that impact the planning problem, its modeling process, and solution methods.

**Deterministic versus Random Variables:** in the case of a "make-to-order" environment, or when the market characteristics allow the company to know the future customer requests relatively precisely, demand is modeled through deterministic variables. It is especially the case where few variations of the demand patterns are observed in a large historical data-base, and are expected to stay unchanged. Therefore, the demand can be predicted accurately enough to consider it as determined. In this case, products considered are generally well known, from the customers point of view, as well as for suppliers and manufacturers.

When, on the contrary, the situation does not permit such hypotheses, forecasts that are made to model future demand are less precise, and the risk of forecasts errors exists. That's the reason why, the uncertainties of the demand need to be caught by the models representing it. In that sense, random variables are usually used, and the corresponding problem is then described as stochastic production planning.

Depending on the environment of study, there are two classes of models for random demand that can be considered to capture the demand uncertainties [Lee and Nahmias, 1993]. In the first one, the demand is usually characterized as "uncertain", and is defined through a known probabilistic density function (pdf). In the second one, the context and the available data do not enable to compute accurately any demand pdf. The demand is thus considered as "unknown". Different models can be used in that context. Some of them are for example to use conditional pdf, or several levels of stochastic modeling such as defining a pdf for the parameters of the demand pdf. In this case, it will be regularly updated as soon as new information becomes available.

**Forecasts and Uncertainties Computation techniques:** In a general manner, demand forecasts, and, if considered, their corresponding uncertainties, are cal-

culated based on various sources of information. Generally, they are determined thanks to both internal and external data. Historical data as previous sales, orders, or states of the system -inventory levels, resources allocated, etc.- are for instance internal one commonly used. Other factors, such as the potential evolution of the market, expert judgments on trends in any of the domains influencing the company environment -financial, legal, social, ..., are examples of external information that can be treated to generate forecasts.

Thus, depending on the availability of the historical data, its patterns, and the decision-makers' objectives concerning the forecasts features, different methods of computation can be used. When there is few historical data or when it is unreliable, qualitative methods are used. They are based on experts opinions, market surveys, or any non-quantifiable information. Conversely, if enough data is available, quantitative statistical methods are implemented. In this case, two categories of method exist: the intrinsic or extrinsic ones. The techniques belonging to the first category are based on historical data. The others use moreover external influencing factors and draw causal links between them and the forecasted variable.

As far as the uncertainties are concerned, they can be directly calculated through the statistical method used to compute the forecasted demand as the associated statistical estimation of the variance of the forecasts. This can be seen as an "*a priori*" uncertainties calculation. Other methods model the forecasts errors made during the previous periods of study and project them to the future periods of the planning horizon as their associated empirical variance. This projection can depend on the value of the forecasted demand -relative model-, can be independent -absolute model-, or both -mix model- [Hubert, 2013], and is generally growing according to the time interval considered between the forecast date and the period considered. They can be qualified as "*a posteriori*" uncertainties calculation methods. In any case, these calculations lead to a definition of the variance of the forecasts.

The selection of the best method in terms of accuracy is one of the major challenges in forecasting. This concerns not only the forecasting technique to be chosen appropriately, but also the optimization of its parameters according to the objectives targeted and the working environment [Chopra and Meindl, 2013]. As seen in chapter 0, uncertainties are indeed a major source of complexity within the production planning modeling and its solution computation, impacting the supply chain performances in a negative way. That is the reason why being able to get more

accurate forecasts or, in other words, decreasing their uncertainties, is an objective shared by many industrial companies.

**Static versus Dynamic Demand:** Besides forecasts and uncertainties computation methods, another important element of the demand modeling influences the supply chain performances: its temporal dependency. The customer demand defined in the previous paragraphs may indeed evolve over time. Two cases may consequently be considered here: whether this evolution is taken into account in the demand modeling or not.

The first case -static demand- considers the demand as being modelled with constant parameters over time. The customer needs are then stationary, and the demand rate per time unit is the same for all future period considered. These strong hypotheses make such models theoretical, and thus they may be far from realistic situations. However, they allow to calculate robust solutions with respect to costs parameters and demand rate quantification. Besides, they usually represent interesting solutions to begin with in more complex situations [Lee and Nahmias, 1993].

When, on the contrary, we speak about dynamic demand, it is considered as time-varying, i.e. changing over time. It is usually represented in this context as a series of as many elements as periods in the planning horizon, whose elements vary from each other, modeling the fluctuations of the demand. These variations may be independent, or determined by an explicit time-dependent function.

### ii. Modeling framework of the dynamics of the decision system within production planning

In the same way as the customer demand, other features of the system may also be changing over time, making the decision environment evolve. Thus, when we deal with production planning, another important point is to set to what degree of consideration the dynamics of the problem is integrated into the optimization model.

According to this, various frameworks of study exist, starting from the simplest model with strong hypotheses to the most realistic ones, leading to different solutions whose efficiency may vary in a non negligible manner: the more accurate the model is in terms of evolution of the time-related variables, the more flexible the decision system is, being thus able to respond more adequately to the uncertainties of the

environment, especially the customer demand.

These frameworks depend on the temporal structure of the decision process resulting from the planning problem considered. In particular, whether the evolution of the decisions is considered or not into the modeling of the planning problem, which is represented through the concept of fixed or rolling plan framework.

**Fixed versus Rolling Plan framework:** The key concept differentiating these two environments is the date during which the planning process is executed, and whether the decision-maker would consider it as a new dimension of the problem. In the remainder, we call this date the forecasts date, inasmuch as it represents a time instant where forecasted demands are calculated.

i) Fixed Plan framework:

On the one hand, when the fixed plan framework is considered, the plan is calculated at the beginning of a specific forecasts date, that we note $t$, for future periods of the planning horizon, whose length is noted $N$. Consequently, as shown in figure 1.5, one unique plan is therefore computed at this precise instant, the beginning of period $t$, for the future periods, from $t+L$ to $t+L+N-1$, $L$ being the production lead-time. Models of such problems are usually optimization problems minimizing the costs or maximizing the service level for this unique plan specifically.

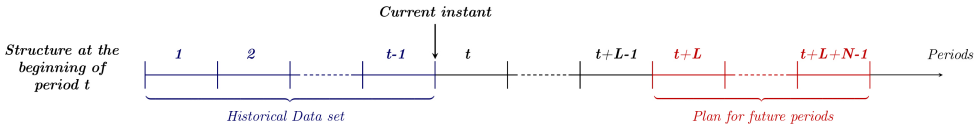



Figure 1.5: A fixed plan framework example

The simplest models consider a one-period problem ($N = 1$) in this context of fixed plan. Others take into account several periods, with time-varying demand or not.

The outcome of this planning procedure is a serie of production -or purchase- orders to be released during the corresponding periods, minus the lead-time. Thus, those for which the lead-time requires to be executed in the current forecasts date $t$ are consequently launched actually in the production process.

ii) Rolling Plan Framework:

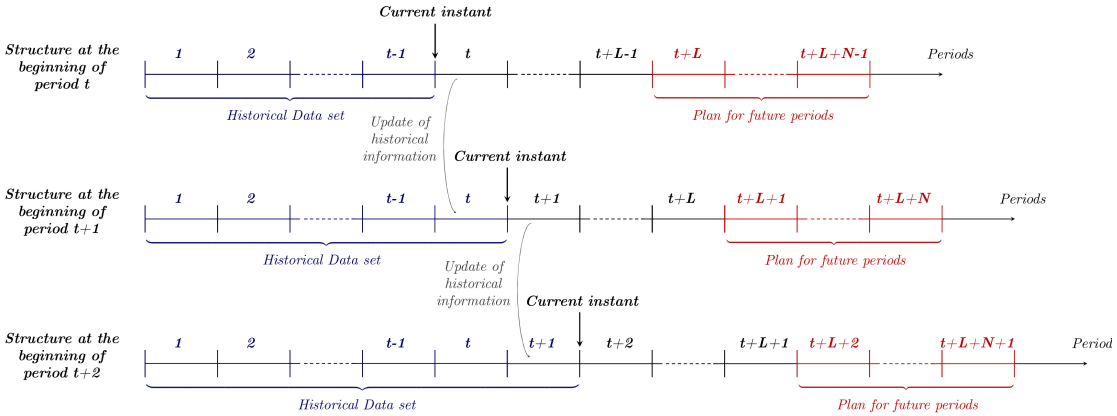
In practice though, one period later, a new similar procedure of planning begins. All the previous calculations of the fixed plan structure start again, taking into account the latest state of the system, which is precisely what happened until the end of period $t$. The latter results calculated at the beginning of $t$ may consequently not be suitable anymore for the new forecast data.
That is why other models consider this new dimension -the date of forecasts- and deal with rolling plan framework.

In this case, we take into account the evolution of the state of the system from one date of forecasts to the next one, leading to evolving or updated decisions. Once a first plan has been computed at the specific date of forecasts $t$, the historical data is indeed updated as soon as the period $t$ has passed, based on the latest information available. This new set of information is used to compute another plan for the next periods at the beginning of period $t+1$. In the same way, a new plan is next calculated at the beginning of period $t+2$ by taking into account the newly available information that appear during period $t+1$, etc. This dynamics is taken into account in the rolling plan framework.

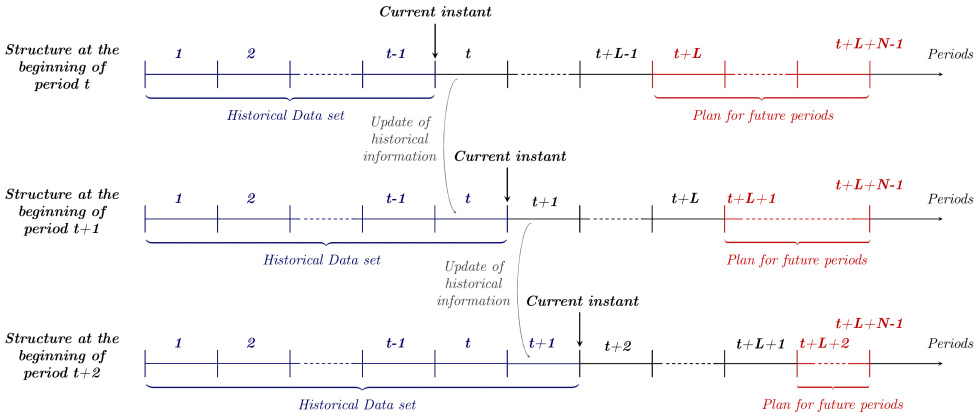
In this situation, two different cases may arise, whether the length of the planning horizon $N$ is fixed for all date of forecasts, or if the last period of the planning is defined at the first date of forecasts, and remain unchanged when time goes on. We call them respectively "rolling horizon framework" and "rolling decisions framework" as in [Cheaitou, 2008].

- When the length of the planning horizon $N$ does not change, it means the last period considered for each plan moves forward according to the forecasts date. The whole $N$ periods plan is sliding while time evolves. An example of a rolling horizon framework is presented through figure 1.6a
- On the contrary, when the last period of any plan is fixed, the length of the planning horizon decreases with respect to the increase of the date of forecasts. The last period is always the same in any planning considered. This case is generally observed when seasonal products with sales deadlines are considered, or when a production outsourcing contract has a end date. Figure 1.6b shows an example of a rolling decisions configuration.

The resulting production plan of such rolling plan framework planning models is consequently the series of actual orders released period after period. Each of




(a) A Rolling Horizon Framework




(b) A Rolling Decision Framework

Figure 1.6: Examples of Rolling Horizon and Decision frameworks

them are calculated with different sets of information, as it has been updated progressively. The final production orders cannot consequently be completely adapted with respect to the overall final information, as it has be built gradually at each step of the process, creating deviation compared with the optimal a posteriori plan. However, each decision is based on the most accurate set of information on hand on the date of forecasts.

The decision to consider a rolling environment (either rolling decisions plan or rolling horizon plans) or not in the planning problem modelled depends on contextual factors such as the ability to update the information database, the accuracy of the forecasts, the nature of the products considered, etc.

However, in general terms, both kinds of Rolling Plan framework give better performances, as it allows adjustments on a continuous-flow basis with respect to new information available for the system state as well as for future periods [Penlesky et al., 1989], [Penlesky et al., 1991]. Production orders actually released become thus

more accurate to the latest information available in the supply chain. By taking into account the evolution of the state of the system into the dynamics caught by the decision process of production planning. Rolling Plan frameworks allow better flexibility of the process, and thus present great benefits as far as the flows management is concerned.

**Non-stationary versus Stationary Demand Forecasts:** in Rolling Plan frameworks, as seen previously, new information arrives between successive dates of forecasts: the system state is updated as well as historical data to calculate a new plan for the next periods. Different uses of this new information may be considered, especially concerning the forecasted demand for the next periods.

If at each iteration, the forecasted demand for the following periods are indeed computed again based on the newly up-to-date historical data, the forecasts are updated, and evolve from one date of forecasts to the next one. Then, not only the state of the system -inventory levels, production capacity status, etc.- is used to determine the new production plan, but also newly adjusted demand forecasts. For a fixed considered period in the planning horizon, a forecasted demand depends consequently not only on the period considered -the index of the demand series- but also on the date of forecasts -the instant of calculation. Whether the model of production planning includes this process of update of forecasts with respect to forecasts date or not, the demand variable may be represented as a non-stationary one with respect to the date of forecasts.

### 1.2.3 | Conclusion

As a conclusion of this section, for a decision maker in production planning, the first step consists of modeling and defining the characteristics of the problem. He needs to analyze the external environment of the production system, the internal organization, the constraints, and the capabilities and the capacity of the resources. He must define properly the objectives of the company and make some assumptions for the resulting model can be handled from a computational point of view.

This would indeed lead to the structure of the planning problem to study through the formalization of the key features to consider among the different dimensions and hypotheses presented above: first of all, the elements representing the industrial context, then the hypotheses characterizing the decisional structure of the process,

including the customer-demand modeling, and the time-related hypotheses defining the dynamics of the decision system considered.

Once delineated, the planning problem is solved either through computation methods, exact algorithms or efficient heuristics with respect to the assumptions adopted. A literature review of some of them will be presented in the next section.

# 2 | Litterature Review and Positioning

In this section, we describe more in details the positioning of this thesis into this large production planning area by emphasising the problem considered in this work combined with a literature review of existing planning methods.

We first examine the current academic works concerning various planning problems through different configurations that may exist in terms of hypotheses about the decision system considered. Two types of classification features are then used here, whether the demand is considered as deterministic or random and static or dynamic on the one hand, and the degree of consideration of the evolution of the decision system on the other hand: fixed or rolling plan framework, combined or not with non-stationarity of the forecasted demand. In particular, as the closest to our research, the context of rolling plan framework is particularly studied.

We finally conclude this chapter by making into perspective the issues described in the general introduction through planning environment described previously, the analysis of the state-of-the-art literature, and the challenges presented in chapter 0.

## 2.1 | Methods of Production Planning in a Fixed Plan Context

As seen in chapter one, production planning is a vast field of analyses and techniques that depends on a large range of hypotheses. Based on the context of study, the industrial framework of the company defining its supply chain, the market and customer demand considerations, and the time-related context defined by decision-makers, various modelling approaches can be used, from optimization models to business representation procedures. To solve this problem, many solution techniques may be implemented, such as for instance procedures of calculations, simulations, or analytical studies.

In this section, we present some classical general models and methods for production planning or inventory control for some usual "push" systems, chosen among the different dimensions of the problem described in the previous chapter.

In order to highlight the influence of the dynamic features, as well as the importance of the customer demand hypotheses, we particularly focus the following non-exhaustive description, summarized in table 1.1, on these dimensions. Thus, despite the potential dependency on the industrial environment -number or nature of the products considered, number of production levels in the supply chain network, production constraints, nature of costs, etc. -, the methods are classified according to the modeling framework of the customer demand defined. The first paragraph is dedicated to the classical methods used in simple cases, the second one describes in more details models adapted to more complex situations, in which both uncertainties and dynamic demand are taken into consideration.

### 2.1.1 | Basic Planning methods

We present briefly here standard methods for production planning in basic frameworks: random static demand with periodic review, or deterministic and dynamic demand.

When a planning frequency is defined and uncertain demand is considered, the static inventory control policies -re-order point and order-up-to-level policies- are adapted methods for production planning. They allow to get adjusted required production releases at the right timing [Peterson and Silver, 1979] [Zipkin, 2000].

In a different context, when we consider the case of a dynamic and deterministic demand, the widespread method of production planning in industry is material resources planning (MRP). Based on the bill of materials of the products considered and the production lead times of any of its components, it allows the planners to compute the net requirements of all sub-products and components involved in the production process, meaning how many have to be available on stock, and when, so that the final customer demand can be fulfilled. [Orlicky, 1974]. An improved version (MRPII) is often used when capacity is considered [Vollmann et al., 1997]. This calculation procedure leads to a requirements plan that can be used as a production plan.

However, this plan may sometimes have some drawbacks in terms of costs. In fact, the subdivision into simple sub-plans for any component following the product

structure may give suboptimal solutions compared with the global minimum cost. It is especially the case for some environments including for instance non-negative set-up costs, multi-products, or capacity constraints [Pochet and Wolsey, 2006]. In these situations, we generally use as a consequence some lot-sizing rules, resulting from optimization models of the production orders taking into account the specificities of the situation. Many models exist depending on the context of the application. Some of them are for intance classified in the reviews of [Jans and Degraeve, 2008] and [Karimi et al., 2003], and examples of optimisation algorithms are given in [Jans and Degraeve, 2007].

### 2.1.2 | Dynamic Planning methods under uncertainties

Although the previous methods are suitable and efficient in many situations, their main drawback is that they do not consider uncertainties. In particular, we are interested in this thesis in integrating uncertain parameters into the modeling of the demand to respond to.

In this context, similar techniques can be implemented in which the stochastic aspects are taken into account. For instance, the MRP production systems under uncertainties has been studied. In this context, safety stocks may be relevant to cope with the randomness of the demand [Graves, 2011], [Caridi and Cigolini, 2000].

As in the deterministic approach though, the standard MRP procedure does not provide the minimum cost planning in general -the minimum expected cost in this stochastic context. That's the reason why mathematical models for lots sizes optimization have also been developed for stochastic environments [Bitran and Tirupati, 1993] [Karlin, 1960]. Various methods have been studied; such as the scenario-based approach [Guan et al., 2006] [Pochet and Wolsey, 2006], the expectated cost optimization [Helber et al., 2013], [Tempelmeier, 2011], a service level constraint [Tarim et al., 2011], or dynamic programming approach [Gupta and Maranas, 2003], [Gupta and Maranas, 2000]. An analytical computation of an optimal solution is also described in [Cao, 2015].

## 2.2 | Methods of Production Planning in a Rolling Plan Context

### 2.2.1 | Deterministic Demand

In the case of dynamic demands with respect to the periods considered combined with a rolling plan with respect to the forecasts dates, we are dealing with a complex modeling and problem-solving environment, inasmuch as the data to take into consideration are regularly updated. It adds consequently instability and uncertainties in the system [Tempelmeier, 2011], [Blackburn et al., 1986], creating a highly variable environment [Sridharan et al., 1988].

However, the implementation of this procedure in real cases has been proved to be efficient [Sahin et al., 2013]. Besides, [Baker, 1977] experimented in details how operations are managed in rolling plans, and concluded that such framework allows the finally obtained production plan to be systematically a low cost solution: compared with the optimal solution calculated based on all a posteriori information, the rolling plan deviation never exceeds 10% in terms of costs.

It has consequently been the object of several studies, so that the benefits of such model can be exploited, while trying to cope with the modeling difficulties. In a two stages assembly line context, [Bitran et al., 1982] uses a hierarchical method based on multi-steps linear programming models, and proves its efficient in this rolling plan context. Other techniques may be used, such as a frozen schedule for instance [Sridharan and Berry, 1990a]. It corresponds to an interval of time in which the production plan does not change fromthe date of forecasts. For instance, if, at the forecasts date $t$, the plan of $N$ periods is computed in which we define a frozen interval of $l$ periods, the next plan calculated at the forecasts date $t+1$ remains unchanged for periods $t+1$ to $t+l-1$ compared with the previous one. The length $l$ can depend on the date of forecasts, or be determined and kept constant for all of them. In both situations, the calculation of its length has a direct impact on the variability and uncertainties of the system [Sridharan and LaForge, 1994b]: the shorter it is, the more up-to-date the orders are, and also, consequently, the more accurate and coherent they are regarding the demand to which they fulfill. Other parameters have a direct influence on the performances of the plan: the re-planning frequency or time between orders (TBO), the planning length, the choice of the lot sizing rules, the method of safety stocks quantification, and the a posteriori forecasts errors [Yeung et al., 1998]. The objective is then to calculate an adapted

set of variables allowing to compute an efficient production plan with respect to cost and stability.

To that end, [Lundin and Morton, 1975] and [Schwarz, 1977] analyzed the behavior of a rolling horizon systems according to the frequency of re-planning, and concluded that situations in which the TBO is proportional to the length of the horizon provide good solution in terms of costs. In particular, the configuration with a planning horizon equal to 5 times the TBO gave only a 1% deviation from the a posteriori optimal cost in their settings. [Bookbinder and H'ng, 1986] confirmed this property of proportionality testing the classical lot sizing model of [Wagner and Whitin, 1958] in a rolling horizon framework. They found indeed a proportionality factor equal to 4 to ensure a 1% deviation from the optimality. However, the planning horizon length must exceed a certain limit: [Federgruen and Tzur, 1994] has indeed shown a "truncated horizon effect" which has big negative impacts on final costs when the planning horizon is too short.

As far as the frozen interval length is concerned, [Sridharan et al., 1988] and [Sridharan et al., 1987] proved that when it is kept under 50% of the planning horizon, it has low effect on costs. However, the higher the re-planning frequency is, the better in terms of costs [Sridharan and Berry, 1990b].

An important issue which has been the object of various investigations is the choice of the lot sizing rules. Among them, several authors tested various lot sizing heuristics and exact algorithms for a fixed plan, but implemented in a rolling horizon environment. The results indicate that in this rolling context, optimal sub plans do not provide optimality, and are often overtaken by simpler heuristics. By comparing the Silver-Meal heuristics [Silver and Meal, 1973], a modified Silver-Meal heuristics [Silver and Miltenburg, 1984], and the Wagner-Within algorithm, [Bookbinder and H'ng, 1986] [Blackburn and Millen, 1980] proved that, even if the plan coming from the Wagner-Within lot sizes rule is contained within 1% of the optimality in terms of costs, the modified Silver-Meal heuristic gave better solutions in rolling schedules. Actually, [Ristroph, 1990] described the relation between the planning horizon length and the performances of lot sizing rules, adding to the comparison the classical least-unit-cost (LUC) and part-period rule (PPR) heuristics: the performance of the modified Silver-Meal heuristic is inversely proportional to the planning horizon length. Thus, when long planning horizons are considered, the Wagner-Within rule provides finally a better plan. Besides, when the demand forecasts accuracy is low, the modified Silver-Meal heuristics is also preferable [Russell and Urban, 1993]. In

order to compensate the short planning horizon obstacle that makes the Wagner-Within algorithm less insteresting, [Stadtler, 2000] proposed some modifications to improve its performance.

Concerning the safety stocks quantification under a rolling-horizon framework, several procedures of quantification are discussed in [Sridharan et al., 1988].

### 2.2.2 | Random Demand

The above-mentioned studies were conducted considering a deterministic customer demand. In a stochastic context in which the demand uncertainties are taken into consideration, the model may be difficult to handle. [Wemmerlöv, 1986] cited for instance the excess of inventory and the decrease of the service level engendered in such situations. Various configurations and techniques may be implemented though.

First of all, in terms of inventory control policies, the dynamic ones allow the planner to quantify production orders in a forecast-based environment, with a specific re-planning frequency both calculated under service level constraint [Babai and Dallery, 2009]. [Federgruen, 1993] also developed an adaptive base-stock policy allowing optimization of production orders with information updating and regular re-planning.

In production planning, the same parameters and influencing variables are studied and quantified as in the deterministic approach detailed in the previous paragraph, although the service level concept must be added as it has an important role in the planning process [Sridharan and LaForge, 1994a].

The MRP procedure has been studied under rolling horizon and random demand by [Whybark and Williams, 1976] to compare the effectiveness of the safety stocks and the inflated lead times. He concluded that safety stocks allow better management of the uncertainties in such contexts.

Safety stocks are thus investigated by [Tang and Grubbström, 2002] that explained that as soon as they are optimized according to the forecasts errors, there are less constraints in the choice of the TBO and the frozen interval length inasmuch as they do not impact much the final service level and costs. The calculation of safety stocks are especially discussed in [Whybark and Williams, 1976], [Sridharan and LaForge, 1994a], and [Campbell, 1992] who detailed the influence of forecasts accuracy in the computation process. [Vargas and Metters, 2011] proposed an optimization algorithm for production planning in rolling horizon framework when demand is stationary and proportional unit costs.

In some industries where supply lead-times are considerable, uncertainties have big influence on the performances of the planning -cost, service level, ... -, and specific modeling approahces need to be implemented. [Cao, 2015] studies for example the case in the automotive industry, whereas and [Fisher and Raman, 1996] focuses particularly on the fashion industry, for which the seasonal effect -new products launches in particular- has great consequences on the planning management. A fast reactive decision environment is indeed required.

In a similar way as in the deterministic case, the lot sizing rules have also been evaluated in a stochastic environment. In particular, [Wemmerlöv and Whybark, 1984] tested deterministic lot size rules with a safety stock quantified to reach a fixed service level, and concluded that in an uncertain context, the results are very different from the deterministic settings. They especially pointed out the efficiency of the PPR rule. Moreover, the more uncertain the demand is, the less impact the lead times and the TBO have on heuristics efficiency, although the global performance decreases. [Bookbinder and H'ng, 1986] proposed a heuristic for lot sizes when a service level constraint is given. Finally, [Sridharan and Berry, 1990a] noted that with demand uncertainties, the costs and service level are more sensitive to the lot sizing rule than the re-planning frequency or the length of the planning horizon.

### 2.2.3 | Synthesis

According to the hypotheses defining the decision system in which the production planning is considered, different methods exist in order to get as much value as possible from the plan resulting from the computation process.

Thus, depending on the choice of taking into account the randomness of the customer demand, its dynamic characteristic, or the evolution of the decisions from one date to another, different computation techniques may be implemented, whether the decision maker would rather target cost optimization, service level improvement, stability, or flexibility into his Supply Chain. We present in Table 1.1 a brief overview of methodologies that can be used to manage production plan in different possible configurations.

Table 1.1: Methods of Production Planning/Inventory Control

| Demand time axis | Plan framework | Deterministic Demand | Stochastic Demand |
|---|---|---|---|
| Static | Rolling | | – Order Up-to-level policy<br>– Reorder Point policy |
| Dynamic | Fixed | – MRP procedure<br>– Lot Sizing problem | – MRP procedure under uncertainties<br>– Stochastic Lot Sizing problem<br>– Safety Stocks Optimization |
| | Rolling | – Re-planning adjustments<br>– Adapted heuristics<br>– Safety Stocks quantification<br>– Frozen Schedules | – Dynamic Inventory Control policies<br>– Re-planning adjustments<br>– Adapted heuristics<br>– Safety Stocks quantification<br>– Frozen Schedules |

## 2.3 | Production Planning in a Rolling Plan Context under uncertainties and Forecasts Update

Contrary to the previous techniques in which the customer demand is dynamic but not updated at each step of the rolling process, in this paragraph, we present existing works under a Rolling Plan framework, combined with a random and non-stationary demand. The forecasts update procedure is indeed considered here into the proceedings of the computations of the production plan at each iteration. Our contributions presented in the remainder of this thesis is embedded into this configuration of study.

Literature dealing with such contexts can be divided into four categories. The first one corresponds to a set of theoretical models, in which possible optimization modelings are studied in a general manner. Then, other works consider the error bounds of the forecasts in order to manage the orders quantification. Some authors focus then their study on some specific products, allowing to get results for production plan in a particular environment. Finally, the forecasts update, creating

forecasts evolution at each step of the rolling process, is modelled by some authors as a stochastic process, more specifically as a martingale.

### 2.3.1 | Theoretical Modelings

As far as the modeling process of the non-stationary rolling horizon planning is concerned, some works exist in which a theoretical framework of study is built. These models allow the authors to develop an optimization model for the considered problem. Two different optimization techniques can be found: dynamic programming on the one hand, and online optimization on the other hand. We present these techniques in the following paragraphs.

#### i. Dynamic Programming Approach

[Michalska and Mayne, 1995] and [Morton, 1979] developed models where the information is considered as non-stationary with respect to the date of each step of the rolling plan framework. They modelled the problem with dynamic programming, and present some theoretical ways to consider the mathematical problem, without focusing specifically on production planning. As for [Sethi and Sorger, 1991], they have been ones of the first to develop a theoretical stochastic dynamic programming model in the specific case of production planning with forecasts update. In order to represent this dynamics into an optimization model, they describe precisely the variables at stake, and how they have to be defined and treated so that the minimum of the expected cost could be reached through a recursive calculation in which several minimization computations have to be executed step by step, based on a growing set of information. This work is about how one can build an optimization model in a theoretical way, but does not aim at developing solution methods. In [Sethi et al., 2001] and [Sethi et al., 2006], the authors applied the previous modeling to some specific applications. However, results remain purely theoretical, and no computational results are reported.

#### ii. Online Optimization Approach

In another context, [Dangelmaier and Degener, 2011] developed a methodology for production planning in a specific rolling plan framework in which the only information available at the time of the decision is the one whose due date does not allow any time buffer, without any demand information for the future periods whose production orders may be delayed. At each iteration then, the only required demand information is released progressively. In this context, they built an online optimiza-

tion procedure to minimize the risk of shortages. Theoretical mathematical model is then described to represent such a configuration.

### 2.3.2 | Performance bounds

Modeling the problem considered here being relatively complex, an approach used in the literature to end up with practical results is to work with bounds, especially error bounds. Some authors were indeed able to model the interval in which the difference between the a posteriori optimal cost for the rolling plan considered and the generated cost of a sequential optimization of multiple fixed-plan problems is included.

These models used a general study environment of the control theory to model the rolling-horizon planning problem.

A review of such works is given by [Chand et al., 2002].

#### i. Markov Chain Models

To do so, [Alden and Smith, 1992] and [Hopp, 1989] used a Markov chain modeling approach to make this error bounds explicit. Analyses were performed to put into perspectives the performances of different planning methods, especially the influence of the length of the horizon in the sequential optimization of multiple fixed plan framework problems, or the impact of the non-stationarity modeling on the costs deviation [Hernandez-Lerma and Lasserre, 1990].

#### ii. Dynamic Optimization

In the same environment, [Morton and Pentico, 1995] developped heuristics in order to minimize the cost deviation between the a posteriori minimal cost and the cost generated by the sequence of different fixed-plan problems. In this approach then, cost performances are at stake, and the objective is to understand how to manage the inventory in a theoretical way for flow management modeling.

### 2.3.3 | Specific Products

Rolling horizon planning with forecasts updates has also been studied in some particular cases where hypotheses allow to precise more specifically the optimization model that can be applicable to face the associated challenges. As examples, we present here some works concerning products whose production horizon have a length limited to two periods, and fashion products whose sourcing contract is

flexible over a rolling decision framework made up with two orders.

**i. Two-Period Dynamic Programming**

When short-life products are considered, the length of the horizon to be taken into account do not exceed several periods. In that case, models using dynamic programming may be explicitely built, and specific solutions can be found to optimize the total expected cost of a rolling horizon planning problem under uncertainties. We cite here the works of [Fisher et al., 2001] and [Bradford and Sugrue, 1990], as well as the thesis of [Cheaitou, 2008] who presents a rich variety of modelings and their corresponding solution methods.

As for [Gurnani and Tang, 1999], the authors developped a model considering one period in which two production releases are authorized. A similar framework is also adopted by [Choi et al., 2003] for the seasonal products.

**ii. Flexible Contracts**

In the fashion industry, in which products have a short-life cycle, some authors also analyse how the supply contracts may improve the service level thanks to coordination and sharing information in a rolling plan framework. This is particularly the work of [Donohue, 2000] for example, that shows the benefices of flexible contracts in such a dynamic environment.

Although there may be some similar hypotheses in terms of context of study, this large field of research is larger than only quantitative optimization of production planning. Thus, we won't go into this area much deeper.

### 2.3.4 | Forecast Evolution Modeling

Another way that allows modeling of the planning problem in an uncertain and non-stationary context is to model the potential evolution of the forecasts that may appear from one date of forecast to the next one.

With this aim, [Graves, 1999] modeled the non-stationary demand as a autoregressive integrated moving average process, and was able to compute an efficient safety stock level for a single item inventory system. [Heath and Jackson, 1994] proposed to represent the dynamics of the demand at each iteration of the rolling plan as a stochastic process, and more specifically a martingale. They consider indeed that the expectation of the forecasts of a period $j > t$ based upon the information up to $t+1$ should be the forecast of this same period based upon the information up to the current date of forecast $t$. Using this relation, they performed simulations to

study the efficiency of some planning methods in a basic scheme of production. By implementing this forecasts evolution modeling, [Albey et al., 2015] study a similar case in the specific semi-conductors industry.

## 2.4 | Our Positioning

The high level of complexity of the planning problems under uncertainties and rolling plan framework taking into consideration forecasts update makes the optimal solution computation very difficult.

That's why the literature described above deals first of all with mathematical models that remains purely theoretical, and, in this case, do not give explicit solution to the problem. These models are built either through the construction of dynamic programming or online optimization models.

Then, other authors studied the solutions of the problem, but, to do so, focused on some specific kind of products, as for short-life cycle products so that the exponential complexity of the problem may be controlled for example.

Finally, the difficulties of the modeling process has been overcome in other works by studying either bounds of cost deviation in a sequential optimization process, or by modeling precisely the evolution of the forecasted demand from one forecast date to another one, which restrains the solution space.

Consequently, current academic works dealing with such decisional environment are, to the best of our knowledge, are restricted to special cases, under strong modeling hypotheses, or only theoretical mathematical models. This conclusion is also supported by reviews conducted by [Jans and Degraeve, 2008] who ends his analysis declaring that results concerning non-stationary production planning models are too few, and by [Sahin et al., 2013] who affirms that although considering forecast updates in rolling plans may increase the modelling complexity, it may reversely generate great benefits in terms of cost and/or stability for the solution computed.

From the analysis above, we can see the importance of uncertainties management and therefore the great interest of the Rolling Plan framework enabling decision makers to work in a flexible decision system for flow management. Current literature, however, lacks of knowledge about such configurations. This situation makes the topic of our study relevant, since it consists of developing optimization models for production planning under uncertainties and forecast updates.

We therefore have to investigate the forecast update mechanism and integrate it into a stochastic optimization model for production planning.

As for the industrial context for our study, as stated in the general introduction, we will only consider the single-item single-level production environment in order to focus our attention to the goal described right above.

# Chapter 2

# Problem Description: Forecasts Update and Production Planning

In this chapter, we explain the mathematical modeling choices we made in terms of modeling and decision making to solve the problem described previously.

The quantitative framework that we will follow all along the remainder of this manuscript is then described here in details. We first define the modeling of the customer demand taking into account our positioning. Second, we present the specific configuration chosen for our work. In particular, we emphasize the challenges we have to face by studying some standard key performance indicators and the behavior of our modeling choice in such a framework in both analytical and numerical ways.

# 1 | Forecast Update and Demand Modeling

First of all, we discuss the choice of the customer demand modeling. As described in the positioning of our study, the main characteristics to take into consideration here are the uncertainties and the dynamics of the forecasts updates in the Rolling Plan framework.

## 1.1 | Uncertain and Dynamic Demand Modeling

### 1.1.1 | Forecasts and Uncertainties

When we talk about uncertainties modeling, several challenges appear, whatever the system considered. It has been indeed acknowledged that the quantitative issues, related to the representation techniques and the assessment of uncertainties, are complex.

The question of their assessment in production planning has been studied by [Hubert, 2013] and other academical works presented in the literature review. In addition, in this section, we study specifically the representation techniques. We indeed ask ourselves what ways can be used as modeling practice to embody the uncertainties in the context of the problem on which we focus, namely the non-stationary demand of the rolling plan framework. The objective is to be able to integrate them in the optimization process of production planning.

As described in the first chapter, the key demand information on which the decision system leading to production plan is based on the future customer requirements. That is the reason why it is necessary to work with forecasted requirements. The uncertainties considered here are consequently due to the forecasting process.

In such a context, it is commonly conceded that the mathematical language of uncertainty is probability [De Rocquigny, 2012]. For our problem, we consider consequently customer demand as random variable managed by probability theory. Thus, for a specific period $i$, we choose to represent the customer demand by the random variable $D_i$. We also consider that this random variable will be represented by its associated probabilistic density function (pdf) noted $\phi_i$.

When we speak about uncertainties in systems modeling, it is important to note there exist two different sources from which they can appear. In fact, both

randomness and epistemic uncertainties are included into the uncertainties modeling process [Riesch, 2012] [Helton, 1994].

The randomness is the natural aleatory characteristics of any event. Specifically, in terms of customer demand, it represents the human behavior of the protagonists of the supply chain that cannot be predictable. On the other hand, the epistemic uncertainties are relative to lack of knowledge. The incomplete characteristic of the data used into the modeling process creates this particular epistemic uncertainties. This statistical uncertainties are quantifiable. In our problem, dealing with uncertainties means considering these two sources into the customer demand modeling.

The rolling plan framework, as described in the previous chapter, ensures us to work with as up-to-date data as possible. In this two-dimensional representation of uncertainties, it actually allows us to minimize the epistemic uncertainties faced by the system at the moment of the decision making. In any case, the randomness of the forecasted demand cannot be neither downplayed nor neglected.

The value of the rolling plan framework takes place consequently in the flexibility gain for the uncertainties management.

However, it requires more complex time-related features that will be discussed in the following, especially when we consider the forecast update procedure at each step of the rolling plan.

### 1.1.2 | Forecast updates and non-stationarity

This forecasted demand update mechanism makes the learning data set on which the $\phi_i$ has been calculated to represent quantitatively $D_i$ evolve with respect to the date of forecasts considered.

We need therefore to consider this temporal dimension into our demand modeling: we take into account the index of the date of forecast $j$ for the demand in period $i \geqslant j$, which can consequently legitimately be noted $D_{i,j}$. In the same manner, we will note $\phi_{i,j}$ its corresponding pdf. We note $\Delta_{j-1}$ the set of all information on hand at the beginning of the date of forecasts $j$, used to calculate $\phi_{i,j}$ for any $i \geqslant j$.

Then, the objective of minimizing the epistemic uncertainties described previously corresponds to establishing a production plan under the use of the $D_{i,j}$ for which we have $i - j$ periods optimized. The following figures illustrate the learning

set $\Delta_{t-1}$ to calculate the pdf of random variables $D_{i,t}$, $i \geqslant t$ (figure 2.1) and its evolution from the date of forecast $t$ to $t+1$, engendering the evolution of $\phi_i$ from $\phi_{i,t}$ to $\phi_{i,t+1}$ (figure 2.2), where $L$ is the lead time and $N$ the planning horizon.

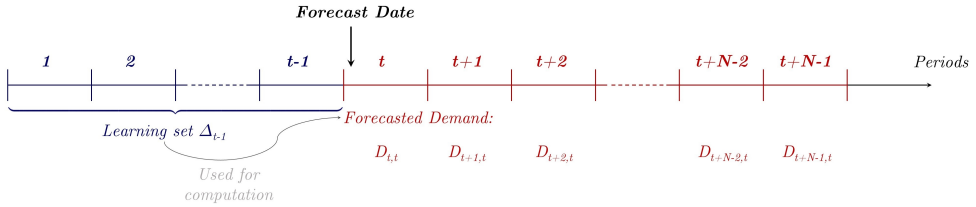



Figure 2.1: The learning set $\Delta_{t-1}$

The update of $\Delta_{t-1}$ generating $\Delta_t$ creates the changes from the pdf representing $D_{i,t}$ to $D_{i,t+1}$.

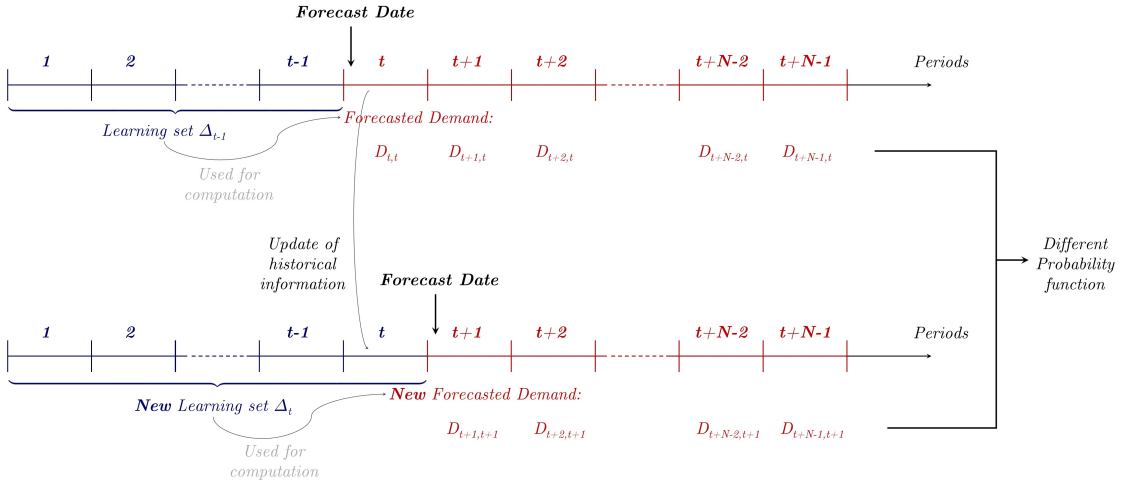



Figure 2.2: From $\Delta_{t-1}$ to $\Delta_t$

The new forecasted demands $D_{t+1,t+1}, ..., D_{t+N-1,t+1}$ together with their corresponding pdf $\phi_{t+1,t+1}, ..., \phi_{t+N-1,t+1}$ have been calculated based on $\Delta_t$, taking into account more information than what existed at the previous iteration, $\Delta_{t-1}$. As a consequence, for a given $i \geqslant t+1$, $\phi_{i,t+1}$ has evolved from $\phi_{i,t}$.

In such a rolling plan framework with forecast update, the forecasted demand needs consequently to be considered as non-stationary with respect to the date of forecasts.

## 1.2 | PDF Computation process

In order to model these characteristics, the challenge is to understand how, for any $i \geqslant t+1$, $\phi_{i,t}$ evolves to $\phi_{i,t+1}$.

These changes are due to the new information that make us pass from $\Delta_{t-1}$ to $\Delta_t$. The key information arriving from the sequence of events taking place during period $t$ that allows this update is the observation of the real customer demand that occured during period $t$. Therefore, we need to precise how we could quantify this observation so that the evolution that we want to model can be made explicit.

The a-posteriori real customer demand faced by the production planning system during one period can be considered as an observation, or a random sample, of the unknown -theoretical- random variable representing the demand. Its pdf and parameters are unknown [Graves et al., 1993]. In the perimeter previously described, its corresponding uncertainties are only aleatory. This random variable could have been calculable if an infinite set of data would have been accessible to quantify it.

However, in our study, such information is unavailable: the theoretical pdf that defines the real demand variable is unknown.

Then, to be able to characterize the real a-posteriori observed demand faced by the system, we consider that it will be an observation, or a random sample, of the random variable whose epistemic uncertainties is minimal. In other words, a sample of the forecasted demand is calculated based on the information of the latest date of forecast. This allows us to deal with the most accurate form available to represent the unknown theoretical demand, and then to use data as close to reality as possible. Using the notation defined in the previous paragraph, we consider consequently that the real demand in any period $i$ is a random sample of the random variable $D_{i,i}$, based on the information available up to $i-1 : \Delta_{i-1}$.

This assumption allows us to define our demand modeling pattern. It helps us indeed to understand that, for any $i \geqslant t+1$, $\phi_{i,t}$ evolves to $\phi_{i,t+1}$ by adding the information of a random sample of $D_{t,t}$.

### 1.2.1 | Conditional Probabilities

In the study of production planning considered here, the particular periods $i$ that interest us at the beginning of period $t$ are $i = t+L, .., t+L+N-1$, as seen in 1.6a. Specifically, the production decision taken during period $t$ will impact the period $t+L$. Thus, at the beginning of period $t$, one of the most important challenge faced by planners is the modeling of the demand in period $t+L$.

Following the reasoning made above, the best information that we could have to model it is $D_{t+L,t+L}$, whose corresponding pdf $\phi_{t+L,t+L}$ would be calculated based

on $\Delta_{t+L-1}$. However, at the beginning of period $t$, we only have $\Delta_{t-1}$ as available historical data.

However, we know that $\phi_{t+1,t}$ evolves to $\phi_{t+1,t+1}$ by adding the information of a random sample of $D_{t,t}$ as explained previously. But inasmuch as $\Delta_{t-1}$ is completely known at the beginning of period $t$, $\phi_{t,t}$ is calculable, and consequently $D_{t,t}$ can be entirely characterized.

By combining $\Delta_{t-1}$ and this new information about $D_{t,t}$ and its corresponding pdf, it allows us to get clearer characteristics of $\Delta_t$: we can describe $\Delta_t$ conditional to $D_{t,t}$.

Finally, the direct corollary of this conditional description of $\Delta_t$ is that we can consequently be able to compute conditionally $\phi_{t+1,t+1}$. Once retrieved, it gives us, in the same manner, a conditional modeling of $D_{t+1,t+1}$.

Iteratively, using the same procedure, we can calculate $\forall i > t$, $\phi_{t+i,t+i}$ conditionally to variables $D_{k,k}$, $t \leqslant k < i$.

This particular way to represent the demand ensures us to catch the dynamics of the update of any $\phi_{i,t}$ to $\phi_{i,t+L}$, $i \geqslant t + L$. This update, faced by any planning system in a rolling framework, interests us particularly, inasmuch as it impacts the performance of the order made at $t$, received at $t + L$.

This method requires iterative computation of conditional pdf $\phi_{k,k}$ for any $k$ such that $t \leqslant k \leqslant t+L$, each of them being calculated given the previous random variables $D_{j,j}$, $t \leqslant j \leqslant k$. At each step of the computation, the update of the data based on information that is non available at the beginning of period $t$ is modelled through conditional probability, using the most up-to-date random variables to represent this information.

In this context, we can use the appropriate following notation:

- For the specific period $t$:
  - $D_t$: the random variable representing the demand in period $t$, calculated based on $\Delta_{t-1}$ at the beginning of period $t$.
  - $\phi_t$ its corresponding pdf.
- For the following periods $i > t$:
  - $\Delta_{i-1|(D_k)_{t\leqslant k<i}}$: the union set of $\Delta_{i-1}$, and $(D_k)_{t\leqslant k<i}$, which are calculated iteratively. As described, it represents the information on which the forecasts made at the beginning of period $i$ would be based, but because of a lack

of knowledge at te beginning of period $t$, the set is defined conditionally to $(D_k)_{t\leqslant k<i}$.

- $D_{i|(D_k)_{t\leqslant k<i}}$: the random variable representing the demand in period $i$, calculated based on $\Delta_{i-1|(D_k)_{t\leqslant k<i}}$. These are computed iteratively conditionally to random variables $D_k$ for $t < k < i$.
- $\phi_{i|(D_k)_{t\leqslant k<i}}$ Their corresponding pdf.
- $\mathbb{E}[\ .\ |(D_k)_{t\leqslant k<i}]$: the random variable of the corresponding conditional expectation of variable $D_{i|(D_k)_{t\leqslant k<i}}$. This represents mathematically the conditional expectation of $D_{i|(D_k)_{t\leqslant k<i}}$ with respect to the $\sigma$-algebra generated by the collection of random variables $(D_k)_{t\leqslant k<i}$.

An illustration of this modeling procedure is presented in 2.3. For ease of reading, we will note $\forall i > t$:

- $\Delta_{i-1|t} = \Delta_{i-1|(D_k)_{t\leqslant k<i}}$
- $D_{i|t} = D_{i|(D_k)_{t<k<i}}$
- $\phi_{i|t} = \phi_{i|(D_k)_{t<k<i}}$
- $\mathbb{E}_{i-1|t}[\ .\ ] = \mathbb{E}[\ .\ |(D_k)_{t\leqslant k<i}]$

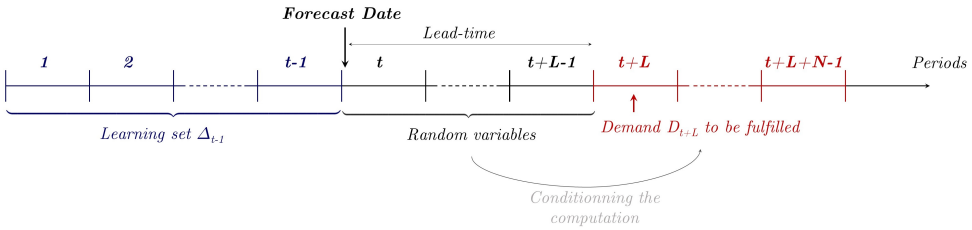



Figure 2.3: Method of conditional computation of the demand

### 1.2.2 | A Dynamic Multiple-level Probabilistic Structure

As set above, the random variables modelling the demand in periods $i > t$ catch the dynamics of the forecast update that the system will face during the evolution of the rolling plan from $t$ to $t + L$. This probabilistic conditional definition is actually based on the fact that at the beginning of period $t$, planners suffer from a lack of knowledge for future periods from $t$ to $t + L$. This deficit of information is indeed represented through the previous probabilistic structure, as described in [Helton, 1994].

This particular form of demand modeling can actually be interpreted as the following model $\forall i > t$,:

$$\begin{cases} D_{i|t} & \sim \phi_{i|t}(\theta_{i,t}) \\ \theta_{i,t} & = \mathcal{G}_{i,t}\left((D_k)_{t\leqslant k<i}\right) \end{cases}$$

In this structure, $\theta_{i,t}$ denotes the vector of parameters of the pdf of the random variable $D_{i|t}$. It may for example be the mean and the standard deviation, in the specific case of a normal distribution. Being computed conditionally to random variables $(D_k)_{t\leqslant k<i}$ implies that this specific vector $\theta$ depends directly on them. More precisely, we define by $\mathcal{G}_{i,t}$ the explicit relation between these parameters $\theta_{i,t}$ and $(D_k)_{t\leqslant k<i}$.

This $\mathcal{G}_{i,t}$ allows us to characterize $\phi_{i|t}$ with respect to $(D_k)_{t\leqslant k<i}$ through $\theta_{i,t}$. On the other hand, we can write a similar system for any $D_{k|t}$, $t \leqslant k < i$. We have for example $D_{i-1|t}$ whose parameter vectors $\theta_{i-1|t}$ of its pdf $\phi_{i-1|t}$ can be expressed with respect to $(D_k)_{t\leqslant k<i-1}$).

$$\begin{cases} D_{i|t} & \sim \phi_{i|t}(\theta_{i,t}) \\ \theta_{i,t} & = \mathcal{G}_{i,t}\left((D_k)_{t\leqslant k<i}\right) \end{cases} \quad (2.1) \qquad \text{and} \qquad \begin{cases} D_{i-1|t} & \sim \phi_{i-1|t}(\theta_{i-1,t}) \\ \theta_{i-1,t} & = \mathcal{G}_{i-1,t}\left((D_k)_{t\leqslant k<i-1}\right) \end{cases} \quad (2.2)$$

Equations 2.2 allows us to write $D_{i-1|t}$ as a random variable whose pdf depends directly on $((D_k)_{t\leqslant k<i-1})$: $D_{i-1|t} \sim \phi_{i-1|t}\left(\mathcal{G}_{i-1,t}\left((D_k)_{t\leqslant k<i-1}\right)\right)$. Knowing that $\theta_{i,t} = \mathcal{G}_{i,t}\left((D_k)_{t\leqslant k<i}\right)$, we can specify $\theta_{i,t}$ as a random variable defined with respect to $((D_k)_{t\leqslant k<i-1})$ on the one hand, and $D_{i-1|t}$ already determined with respect to $((D_k)_{t\leqslant k<i-1})$ on the other hand. It is consequently a random variable depending on $(D_{i-1|t}, (D_k)_{t\leqslant k<i-1})$; we note $\psi_{i,t}$ its corresponding pdf. This leads us to the double level probabilistic structure:

$$\begin{cases} D_{i|t} & \sim \phi_{i|t}(\theta_{i,t}) \\ \theta_{i,t} & \sim \psi_{i|t}\left(\left(D_{i-1|t}, (D_k)_{t\leqslant k<i-1}\right)\right) \end{cases} \quad (2.3)$$

This particular structure is showing theoretically the conditional link developped in the previous paragraph that exists between $D_{i|t}$ and $D_{i-1|t}$ though $\psi_{i|t}$. We have indeed an explicit recursive link between $D_{i|t}$ and $D_{i-1|t}$, all along with function $\mathcal{G}_{i,t}$ expressed with respect to $\left(D_{i-1|t}, (D_k)_{t\leqslant k<i-1}\right)$.

In a similar way, we can build a multiple-level probabilistic structure in order to get expressed the iterative conditional relations between $D_{i|t}$ and $(D_k)_{t\leqslant k<i}$, getting retrieved at each step of the computation of the $D_{i|t}$, $i \geqslant t$, as presented in the previous section.

Relation 2.3 has a double interest, as, first of all, it gives a -stochastic- representation of the random variable $D_{i,i}$ minimizing the epistemic uncertainties, and, second, it includes the dynamics of the forecast update facing by $D_{i,t}$ until it is updated to $D_{i,i}$. It takes finally into account the non-stationarity relative to the update mechanism required to the uncertainties management through flexibility given by the rolling plan framework.

### 1.3 | Conclusion

We defined in this section a framework of modeling of the forecasted demand of any period $i > t$ at the precise forecasts date $t$, taking into account the forecast update that will take place during periods between $t$ and $i$. Both the non-stationarity and uncertainties generated by the update mechanism of the rolling plan are consequently represented into our model of the forecasted demand.

This modeling creates first of all a complex dynamics of computation inasmuch as an iterative way of conditional computation is required to represent the demand of the objective period, and second a double-level probabilistic structure which gives us finally a stochastic representation of the demand, whose distribution parameters are random and recursively defined period after period.

In the next section, we will specify concretely the mathematical choice made to get this model explicit, and study this representation to better understand its benefits.

## 2 | Specifications of our Study

In this section, we go into details of the hypotheses in our work so that the study can be analyzed in depth, and so that we can go through explicit calculations to answer our questions expressed in previous paragraph 2.

### 2.1 | Demand Forecasts Hypotheses

As seen in section 1, the modeling of the demand as a non-stationary random variable, that takes into account the update of the forecasts, is source of non-negligible complexity. That is the reason why we intentionally chose in this study to work with

relative simple forecasting representations and methods of computation so that techniques presented previously can be explicitly calculable and implementable. This will allow us to attain one particular objective presented in 2, and whose relevance is justified at the end of the literature review, which is to be able to compute solutions expressly.

### 2.1.1 | Mathematical Models choice

The details of the variables and the modeling definition that we will follow in our thesis are presented here.

#### i. General form of the PDF

As set in the previous section, the forecasted demand in period $i$ calculated at the beginning of period $t$ is a random variable defined through its PDF $\phi_{i|t}$.

We will see in the next chapters that our production planning problem implies various integration issues. To that matter, we chose to work with a triangular distribution. It is indeed a classical linear distribution allowing integration properties since it is linear, with good approximation properties though. We assume consequently that $\forall i, t \leqslant i \leqslant t+N-1$ , $\phi_{i|t}$ is the pdf of a centered triangular law, whose mean is noted $m_i$ and its standard deviation $\sigma_i$. In that context, the vector of the parameters of $\phi_{i|t}$, namely $\theta_{i|t}$, is here of length 2 and made up of $m_i$ and $\sigma_i$. We note arbitrarily $m_i - 2s_i$ and $m_i + 2s_i$ the boundaries of this triangular PDF. We have then $s_i^2 = 3/2 \times \sigma_i^2$, $(\sigma_i \neq 0)$. As defined, we can write $\forall z \in \mathbb{R}$:

$$\phi_{i|t}(z) = \begin{cases} \phi_1 = \frac{1}{4s_i^2}(z - m_i + 2s_i) & \text{if } m_i - 2s_i \leqslant z \leqslant m_i \\ \phi_2 = \frac{1}{4s_i^2}(m_i + 2s_i - z) & \text{if } \quad m_i \leqslant z \leqslant m_i + 2s_i \\ 0 & \text{otherwise} \end{cases}$$

#### ii. Forecasting method for the parameters $\theta_{i|t}$

In order to compute this PDF, and especially its parameters $\theta_{i|t}$ so that $D_{i|t}$ can be represented analytically and numerically, a forecasting technique will be used.

In the large field of forecasting techniques, well presented in a comprehensive manner in [Hubert, 2013], it is noteworthy that one of them was particularly interesting for our study, namely the linear regression.

It is indeed a statistical method that drew our attention for a number of reasons. First of all, the linear regression technique is widely used, known, and easy to implement in industrial environment where various information systems allow companies to have large databases. Second, well calibrated, it can provide good results in terms of forecasting performances. Then, as quantitative method, the process of forecasting can integrate any data that can be exploited, allowing a large flexibility of use. Also, qualitative judgments can be integrated through qualitative predictors vectors if necessary. Finally, the mathematical procedure and computations are well modelled in a well known framework of study.

### 2.1.2 | Linear Regression parameters Computation

We define then the following variables to precise our study with linear regression as forecasting technique.

Let $p \geqslant 2$ be the number of known predictor variables $(\Lambda_k)_{1\leqslant k\leqslant p}$ of length $t+N-1$. We note $\Lambda$ the corresponding $t+N-1 \times p$ known matrix, and $(\lambda_{ik})_{\substack{1\leqslant k\leqslant p \\ 1\leqslant i\leqslant t+N-1}} = (\Lambda_k)_{1\leqslant k\leqslant p}$.

The first vector $\Lambda_1$ is considered as the constant vector equal to 1, and the second $\Lambda_2$ equal to the "time vector" representing the trend of the data: $\lambda_{i2} = i$, $\forall i, 1 \leqslant i \leqslant t+N-1$.

Thus, as well known [Dodge and Rousson, 2004], at the beginning of $t$, when a forecasting procedure is executed to get the forecasted demand from $t$ to $t+N-1$, the $t-1$ first observations of $\Lambda$ are used as learning set to calculate the $p$ coefficients of the predictive linear function. The following $N$ observations $(\lambda_{ik})_{\substack{1\leqslant k\leqslant p \\ t\leqslant i\leqslant t+N-1}}$ are used in the predictive set to get the forecasts. We note consequently $\Lambda_l(t) = (\lambda_{ik})_{\substack{1\leqslant k\leqslant p \\ 1\leqslant i<t}}$ the subset of $\Lambda$ used in the learning set for the regression calibration at the beginning of period $t$, and $\Lambda_p(t) = (\lambda_{ik})_{\substack{1\leqslant k\leqslant p \\ 1\leqslant i<t}}$ the projection set used for predictions for $i$ in $[\![t, t+N-1]\!]$.

We note $A(t)$ the vector of the coefficients of the linear regression calculated based on $\Lambda_l(t)$: $A(t) = (a_j(t))_{1\leqslant j\leqslant p}$. $\forall t$, $A(t)$ is then made up of the $p$ coefficients of the predictive linear function. These $a_j(t)$ are supposed to be calculated with the least square method. Then, $\forall i$, $t \leqslant i \leqslant t+N-1$ , the predicted vector $\hat{Y}_i(t) = \sum_{j=1}^{p} a_j(t).\lambda_{ij}$ represents the output of the linear regression.

In the context of the customer demand as modelled previously, the dependent variable of the regression executed at the beginning of period $t$ is the observed demand. We note it $d_i$, $\forall i,\ 1 \leqslant i < t$, and $\overline{d(t)}$ the corresponding vector of length $t-1$. Consequently, the predicted vector corresponding to the linear regression represents the mean of the forecasted demand in any period $i$: $m_i$, with $t \leqslant i \leqslant t+N-1$. It becomes then equal to $\hat{Y}_i = \sum_{j=1}^{p} a_j(t).\lambda_{ij}$.

However, the iterative conditional computation of the pdf $\phi_{i|t}$ for any $i > t$ described in the previous section is concretely interpreted here as the fact that $m_i$ will systematically be actually calculated based on all information up to $i-1$. This means that at the beginning of period $t$, we only keep $m_t = \hat{Y}_t = \sum_{j=1}^{p} a_j(t).\lambda_{tj}$ as output of the regression made based on $\Lambda_l(t)$. As for the following periods $i > t$, the value kept as mean of the random variable $D_{i|t}$ will be consequently $m_i = \sum_{j=1}^{p} a_j(i).\lambda_{ij}$ whose corresponding coefficients $(a_j(i))_{1 \leqslant j \leqslant p}$ are re-computed from another regression based on $\Lambda_l(i)$.

We detail in the following relations how these coefficients $A(i)$ are computed using the least square method, $\forall i \in [\![t, t+N-1]\!]$.

➢ For the specific period $i = t$, we have from the least square method:

$$A(t) = \left(\Lambda_l(t)^T \Lambda_l(t)\right)^{-1} \Lambda_l(t)^T \times \overline{d(t)}$$

where $M^T$ is the transpose of any matrix $M$. This leads us to

$$\forall j \in [\![1, p]\!],\ a_j(t) = \sum_{k=1}^{t-1} \left(\left(\Lambda_l(t)^T \Lambda_l(t)\right)^{-1} \Lambda_l(t)^T\right)_{jk}.\ d_k$$

In order to lighten the notation, we define $\forall i \in [\![t, t+N-1]\!]$, $\forall k \in [\![1, t-1]\!]$, $\forall j \in [\![1, p]\!]$:

$$\begin{aligned}
\alpha_{jk}(i) &= \left(\left(\Lambda_l(i)^T \Lambda_l(i)\right)^{-1} \Lambda_l(i)^T\right)_{jk} \\
&= \sum_{m=1}^{p} \left(\Lambda_l(i)^T \Lambda_l(l)\right)^{-1}_{jm}.\ \left(\Lambda_l(i)^T\right)_{jk} \\
&= \sum_{m=1}^{p} \left(\Lambda_l(i)^T \Lambda_l(i)\right)^{-1}_{jm}.\ \lambda_{kj}
\end{aligned}$$

The elements of $A(t)$ are then re-written as simply

$$a_j(t) = \sum_{k=1}^{t-1} \alpha_{jk}(t).\ d_k$$

This being known, we can compute the sought $m_t$ thanks to the fact that the output vector $(m_i)_{t \leqslant i \leqslant t+N-1}$ is equal to $A(t)^T \times \Lambda_p(t)$:

$$m_t = \sum_{j=1}^{p} a_j(t).\lambda_{tj} = \sum_{j=1}^{p} \left( \sum_{k=1}^{t-1} \alpha_{jk}(t).\ d_k \right) \lambda_{tj}$$

which gives us finally

$$m_t = \sum_{k=1}^{t-1} \left( \sum_{j=1}^{p} \alpha_{jk}(t).\ \lambda_{tj} \right) d_k \tag{2.4}$$

➢ For any period $i > t$, as explained in 1, the iterative computation of conditional pdf of random variables $\left(D_{i|t}\right)_{t<i\leqslant t+N-1}$ are integrated into the learning set of the calculation of its parameters $\theta_{i|t}$. In other words, we consider $\Lambda_l(i)$ and the concatenation of $\overline{d(t)}$ and $\left(D_{l|t}\right)_{t\leqslant l<i}$ in the computation of $A(i)$ that becomes random:

$$\forall j \in [\![1, p]\!], a_j(i) = \sum_{k=1}^{t-1} \alpha_{jk}(i).\ d_k + \sum_{l=t}^{i-1} \alpha_{jl}(i).\ D_{l|t}$$

The mean $m_i$ of the considered random variable $D_{i|t}$ is consequently, $\forall i > t$:

$$\begin{aligned} m_i &= \sum_{j=1}^{p} a_j(i).\lambda_{ij} \\ &= \sum_{j=1}^{p} \left( \sum_{k=1}^{t-1} \alpha_{jk}(i).\ d_k + \sum_{l=t}^{i-1} \alpha_{jl}(i).\ D_{l|t} \right) \lambda_{ij} \\ &= \sum_{k=1}^{t-1} \left( \sum_{j=1}^{p} \alpha_{jk}(i).\ \lambda_{ij} \right) d_k + \sum_{l=t}^{i-1} \left( \sum_{j=1}^{p} \alpha_{jl}(i).\ \lambda_{ij} \right) D_{l|t} \end{aligned}$$

We note $\forall i \in [\![t, t+N-1]\!], \forall k \in [\![1, i-1]\!]$:

$$\beta_{m_i}^k = \sum_{j=1}^{p} \alpha_{jk}(i).\ \lambda_{ij} = \sum_{j=1}^{p} \left( \sum_{m=1}^{p} \left(\Lambda_l(i)^T \Lambda_l(i)\right)_{jm}^{-1}.\ \lambda_{kj} \right).\ \lambda_{ij}$$

Then, we finally have

$$m_i = \sum_{k=1}^{t-1} \beta_{m_i}^k d_k + \sum_{k=t}^{i-1} \beta_{m_i}^k D_{k|t} \tag{2.5}$$

The other parameter of the pdf $\phi_{i|t}$ defined in paragraph i that is required to express literally the variables modelled is the standard deviation $\sigma_i$. This parameter represents the uncertainties linked with the random variable $D_{i|t}$, and, as previously said, can be calculated (or estimated) with many methods. We can basically classify them into two different categories: the *a posteriori* and *a priori* methods. The first ones consist of considering the errors made during the previous forecasting procedures, analyzing them, and projecting the trends for forecasted periods. We can cite as illustrative examples of such techniques the works of [Hubert, 2013] and [Babai, 2005]. The second methods consider conversely the forecasting techniques used and its associated forecasting uncertainties represented by the error from the obtained model and the error associated with future observations, calculated based on the theoretical statistical variance of the estimated vector $A(t)$ [Friedman et al., 2001]. We choose for modelling and computational issues the second type of uncertainties: the statistical *a priori* ones.

In our case of study in which we consider the linear regression, the corresponding uncertainties of prediction are expressed with the following relations:

Let $\sigma_{\varepsilon_i}$ be the non-biased estimator of the standard deviation of the errors of the regression $\varepsilon_i$, for $i \in [\![1, t-1]\!]$:

$$\sigma_{\varepsilon_i}^2 = \frac{1}{i-p-2} \left( \sum_{k=1}^{t-1} \left( d_k - A(i)^T \Lambda_l(i)_k \right)^2 + \sum_{k=t}^{i-1} \left( D_{k|t} - A(i)^T \Lambda_l(i)_k \right)^2 \right)$$

with $\Lambda_l(i)_k = (\lambda_{km})_{1 \leqslant m \leqslant p}$ being the $k^{th}$ row of the learning matrix $\Lambda_l(i)$. In other words, the product $A(i)^T \Lambda_l(i)_k$ is the output of the regression made at the date $i$ for the already observed data for period $k < i$.

Then, we can write that, for any $j$ in $[\![1, p]\!]$ the variance of the $j^{th}$ element of $A(i)$ is $\sigma_{\varepsilon_i}^2 \left( \Lambda_l(i)^T \Lambda_l(i) \right)^{-1}$. We note $c_i = \left( \Lambda_l(i)^T \Lambda_l(i) \right)^{-1}$. We have consequently $\forall i \in [\![t, t+N-1]\!]$:

$$\sigma_i = \sigma_{\varepsilon_i} \sqrt{1 + \Lambda_p(i)_i \times c_i \times \Lambda_p(i)_i^T} \tag{2.6}$$

where $\Lambda_p(i)_i = (\lambda_{ij})_{1 \leqslant j \leqslant p}$ is the $i^{th}$ row of the projection matrix $\Lambda_p(i)$.

To end this section, we see that the specific case of the linear regression used as forecasting method presented here finally enables us to compute and express litterally the pdf for any demand forecasts $D_{i|t}$, $t \leqslant i \leqslant t + N - 1$ defined in 1. The relations 2.5, 2.4, and 2.6 clarify indeed $\theta_{i|t}$ such that the iterative computation method explained in section 1 can be implemented. Compared to equation 2.3, we have indeed expressed the $\mathcal{G}_{i,t}$ function.

It allows to take into consideration the forecast update mechanism into the modeling of the forecasted demand concretly in the model of the pdf. We have consequently here explicit equations defining the forecast update into the demand modeling through $D_{i|t}$.

## 2.2 | Studies of the Demand Modeling Choice

Now that the details of the mathematical choice of the demand forecasts have been defined, they can lead us to interesting analyses in order to understand the benefits of such dynamic and double level probabilistic modelling.

Litteral calculations may indeed be realized to highlight the characteristics of such a modelling approach, especially concerning one of the main challenges cited in section 1, which is the uncertainties estimation and management. Analytical studies is then presented here to this end.

Then, numerical simulations will be performed to illustrate these characteristics.

### 2.2.1 | Analytical Study

We present here in details an analysis of the *a priori* uncertainties of the demand forecasts that is taken into consideration in the decision process leading to the production plan at the beginning of any period $t$ in the specific case chosen above of the linear regression. To do so, we will go in depth into its computation our double-level probability structure modeling.

As described previously, the production decision taken during period $t$ will impact the period $t + L$, where $L$ is the production lead-time. We have seen in the previous section that in our modelling of the corresponding demand forecast $D_{t+L|t}$, the dynamics of the forecast updates from $t$ to $t + L - 1$ will be integrated into the representation of $D_{t+L|t}$ through a conditional iterative computation of its pdf whose parameters are consequently stochastic.

The aim of this study is to understand how these parameters are modelled. In other words, we are studying the $\psi_{t+L|t}$ defined in equation 2.3. In order to better understand the computations, we will take as example the simple linear regression.

In the case of simple linear regression, we have $p = 2$, meaning the matrix $\Lambda$ is only made up of the constant vector equal to 1, $\Lambda_{i1} = 1$, $\forall i \in [\![1, t+N-1]\!]$, and the "trend" vector, such as $\Lambda_{i2} = i$, $\forall i \in [\![1, t+N-1]\!]$. In such context, the $\beta^k_{m_i}$ defined in equation 2.5, and the $c_i$ of the equation 2.6 can be explicitely calculated. We basically have indeed: $\forall i \in [\![t, t+N-1]\!]$

$$m_i = a_1(i) + a_2(i).i \quad \text{and} \quad \sigma_i^2 = \sigma^2_{\varepsilon_i}.\left(\frac{1}{i} + \frac{(i/2)^2}{\sum_{j=1}^{i-1}(j-i/2)^2}\right) \tag{2.7}$$

where $\sigma^2_{\varepsilon_i} = \frac{1}{i-4}\left(\sum_{k=1}^{t-1}\left(d_k - a_1(i) - a_2(i)k\right)^2 + \sum_{k=t}^{i-1}\left(D_{k|t} - a_1(i) - a_2(i)k\right)^2\right)$.

From here, the coefficients $a_1(i)$ and $a_2(i)$ of the regression based on the learning set $\Lambda_l(i)$ are only noted $a_1$ and $a_2$ for more ease of reading, $\forall t < i \leqslant t+N-1$, the particular form of $\Lambda$ in the specific case of the simple linear regression gives us thanks to the least square method:

$$\begin{cases} a_2 = \alpha_1^i\left(\sum_{k=1}^{t-1} d_k + \sum_{k=t}^{i-1} D_{k|t}\right) + \alpha_2^i\left(\sum_{k=1}^{t-1} k d_k + \sum_{k=t}^{i-1} k D_k\right) \\ a_1 = \dfrac{1}{i-1}\left(\sum_{k=1}^{t-1} d_k + \sum_{k=t}^{i-1} D_k\right) - a_2\dfrac{i}{2} \end{cases} \quad \text{where} \quad \begin{cases} \alpha_1^i = \dfrac{-6}{(i-1)(i-2)} \\ \alpha_2^i = \dfrac{12}{i(i-1)(i-2)} \end{cases}$$

which leads us finally to the literal expression of the function $\mathcal{G}_{i,t}$ defined in 2.1:

$$m_i = \sum_{k=1}^{t-1} \frac{2(3k-i-1)}{(i-1)(i-2)} d_k + \sum_{k=t}^{i-1} \frac{2(3k-i-1)}{(i-1)(i-2)} D_{k|t} \tag{2.10}$$

For the specific period $t+L$ that interests us, we can consequently write that $\beta^k_{m_{t+L}} = \frac{2(3k-t-L-1)}{(t+L-1)(t+L-2)}$. This coefficient represents the dynamic links between $m_{t+L}$ and the previous random variables $D_{k|t}$ for $t \leqslant k \leqslant t+L$ that are integrated into its computation. As explained ,this consideration allows to take into account the forecast update that will take place during the rolling horizon, but adds uncertainties into the computation by creating a double level probability structure. This framework forces us to compute $m_{t+L}$ iteratively by calculating successively the $\phi_{k|t}$ from $k = t$ to $k = t+L-1$.

The question now is: "How does this iterative calculation impact on $\theta_{t+L,t}$?" Or, in other words, what can we know about $\psi_{t+L|t}$ from 2.3?

To answer this question, we will first of all study the mean and the standard deviation of the random variable $m_i$ defined in 2.10. Then, the same study will be executed for $\sigma_i$. Results will give us relevant information about the parameters of $\phi_{t+L|t}$ in our context of forecast update.

**i. Probabilistic Properties**

Before any calculation, an important remark is necessary in order to justify the probabilistic environment in which the following results are computed. In particular, it is important to understand the probability sets on which our variables are defined to be able to compute any stochastic features such as the expectation of our random variables.

Actually, in our modeling, as far as probabilistic spaces and measures are concerned, the unknown random variables defining the real customer demand $D_i$ for $i$ from 0 to $t+N-1$ define a stochastic process taking real values, meaning basically in the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The corresponding -unknown- density function $\phi_i$ then defines a measure on the $\sigma$-algebra generated by $D_i$ included into $\mathcal{B}(\mathbb{R})$, noted $\mathcal{F}_i$. At the beginning of a period $t$, the observations $d_i$ for $i \in [\![1, t-1]\!]$ represent one -among multiple possibilities- data trajectory included in $\mathcal{F}_{t-1}$. For next periods $j \geqslant t$, the unknown $D_j$ define anonther $\sigma$-algebra $\mathcal{F}_j$ such as $\mathcal{F}_{t-1} \subset \mathcal{F}_j$ inasmuch as the potential trajectories $(d_k)_{0 \leqslant k \leqslant t-1}$ are strictly included in the set of possible observations $(d_k)_{0 \leqslant k \leqslant j}$ for $j > t-1$.

By characterizing recursively $D_{i+1|t}$ with respect to $D_{i|t}$ through the conditional density function $\phi_{i+1|t}$, its corresponding measure so built is defined on $\mathcal{F}_{i+1}$, in which $\mathcal{F}_i$ is included. Consequently, $D_{i|t}$ and $D_{i+1|t}$ are random variables considered in two different probability spaces, as their sets of events are not similar. Therefore, although the definition of $\mathcal{F}_{i+1}$ depends on $D_{i|t}$, their sampling are independent.

In terms of expectation computation, this independence makes any joint density $\phi_{D_{i|t}, D_{k|t}}$ of any $k$, $i$ such that $t \leqslant k < i$ be equal to the product of their marginal density function. In other words, given any vector $X \in \mathbb{R}^{i-t}$ such that $(D_{k|t})_{t \leq k < i} =$

$X$, we have $\forall z \in \mathbb{R}$:

$$\begin{aligned}\phi_{i|t}(\theta_{i,t}, z) &= \phi_{i|(D_{k|t})_{t\leq k<i}=X}(\theta_{i,t}, z)\\ &= \phi_{i|t}(\mathcal{G}_{i,t}(X), z)\end{aligned}$$

Therefore, for any measurable function $g$ in $(\mathbb{R}, \mathcal{F}_i)$, we have, $\forall X \in \mathbb{R}^{i-t}$

$$\begin{aligned}\mathbb{E}_{i-1}\left[g(D_{i|t})\right] &= \mathbb{E}\left[g(D_{i|t}) \mid (D_{k|t})_{t\leq k<i} = X\right]\\ &= \int_{\mathbb{R}} g(z)\phi_{i|t}(\mathcal{G}_{i,t}(X), z)dz\end{aligned}$$

This result allows us to ensure that our calculation of $\phi_{i|t}(\theta_{i,t}, z)$ is relevant with respect to the variables $(D_{k|t})_{t\leq k<i}$, for any $i$ between $t$ and $t+N-1$.

**ii. Characteristics of $\psi_{t+L|t}$**

We aim here at defining the stochastic features of $m_i$ and $\sigma_i$ defined in 2.7 so that the double-level probabilistic structure can be deeply understood in the context of the simple linear regression.

**Mean $m_{t+L}$:** we first study the mean of the random variable $m_{t+L}$ conditional to $\Delta_{t+L-1|t}$.

**Property 2.2.1-1.** *The variable $\mu_{t+L}$ defined as follows is an unbiased estimator of $\mathbb{E}_{t+L-1}[m_{t+L}]$:*

$$\mu_{t+L} = \sum_{k=1}^{t-1} \beta^k_{m_{t+L}} d_k + \sum_{k=t}^{t+L-1} \beta^k_{m_{t+L}} m_k$$

*The corresponding variance $\mathbb{V}ar_{t+L-1}[m_{t+L}]$ is equal to*

$$\mathbb{V}ar_{t+L-1}[m_{t+L}] = \sum_{k=t}^{t+L-1} \left(\beta^k_{m_{t+L}}\right)^2 \sigma_k^2$$

*Proof.* The equation 2.10 gives us basically that the variable $m_{t+L}$ is linear with respect to $\left(D_{k|t}\right)_{t\leq k<t+L}$. Moreover, the previous paragraph ensures us that the covariance between any $D_{i|t}$ and $D_{j|t}$, $\forall i,\ j$, is equal to zero. □

**Standard deviation $\sigma_{t+L}$:** in the same manner, we study the random variable $\sigma_{t+L}$ by computing its mean and standard deviation.

**Lemma 2.2.1-2.** *The variable $\mathbb{E}_{t+L-1}\left[\sigma^2_{t+L}\right]$ is calculable with respect to $(d_k)_{1\leqslant k<t}$, and $(m_k, \sigma_k)_{t\leqslant k<t+L}$.*

*Proof.* We note $\alpha_3 = \frac{2(2i-1)}{(i-1)(i-2)}$ and $\alpha_4 = \frac{-2(i+1)}{(i-1)(i-2)}$. We have $\forall k \in [\![t, t+L-1]\!]$:

$$\mathbb{E}_{t+L-1}\left[D_{k|t}\right] = \alpha_4 \left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{k-1} m_j\right) - \alpha_1 \left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{k-1} j m_j\right)$$
$$\mathbb{V}ar_{t+L-1}\left[D_{k|t}\right] = {\alpha_1}^2 \sum_{j=t}^{k-1} j^2\sigma_j^2 - 2\alpha_1\alpha_4 \sum_{j=t}^{k-1} j\sigma_j^2 + \alpha_4^2 \sum_{j=t}^{k-1} \sigma_j^2$$

Moreover, we can compute

$$\mathbb{E}_{t+L-1}\left[a_1\right] = \alpha_3 \left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{t+L-1} m_j\right) + \alpha_1 \left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{t+L-1} j m_j\right)$$
$$\mathbb{V}ar_{t+L-1}\left[a_1\right] = {\alpha_3}^2 \sum_{j=t}^{t+L-1} \sigma_j^2 + \alpha_1^2 \sum_{j=t}^{t+L-1} j^2\sigma_j^2 + 2\alpha_1\alpha_3 \sum_{j=t}^{t+L-1} j\sigma_j^2$$
$$\mathbb{E}_{t+L-1}\left[a_2\right] = \alpha_1 \left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{t+L-1} m_j\right) + \alpha_2 \left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{t+L-1} j m_j\right)$$
$$\mathbb{V}ar_{t+L-1}\left[a_2\right] = {\alpha_1}^2 \sum_{j=t}^{t+L-1} \sigma_j^2 + \alpha_2^2 \sum_{j=t}^{t+L-1} j^2\sigma_j^2 + 2\alpha_1\alpha_2 \sum_{j=t}^{t+L-1} j\sigma_j^2$$

The independence gives us $\forall k \in [\![t, t+L-1]\!]$:

$$Cov(a_1, a_2) = \alpha_1\alpha_3 \sum_{j=t}^{t+L-1} \sigma_j^2 + (\alpha_1^2 + \alpha_2\alpha_3) \sum_{j=t}^{t+L-1} j\sigma_j^2 + \alpha_1\alpha_2 \sum_{j=t}^{t+L-1} j^2\sigma_j^2$$
$$Cov(D_{k|t}, a_1) = \alpha_1 k\sigma_k^2 + \alpha_3\sigma_k^2$$
$$Cov(D_{k|t}, a_2) = \alpha_2 k\sigma_k^2 + \alpha_1\sigma_k^2$$

The equalities expressed above help us get the explicit expressions for $\mathbb{E}_{t+L-1}\left[D_{k|t}^2\right]$, $\mathbb{E}_{t+L-1}\left[D_{k|t}.a_1\right]$, $\mathbb{E}_{t+L-1}\left[D_{k|t}.a_2\right]$ for any $k$ in $[\![t, t+L-1]\!]$; and $\mathbb{E}_{t+L-1}\left[a_1^2\right]$, $\mathbb{E}_{t+L-1}\left[a_2^2\right]$, and $\mathbb{E}_{t+L-1}\left[a_1a_2\right]$.

On the other hand, based on 2.7, we have $\mathbb{E}_{t+L-1}\left[\sigma_{t+L}^2\right] = \gamma_i \ . \ \mathbb{E}_{t+L-1}\left[\sigma_{\varepsilon_i}^2\right]$, where $\gamma_i$ is equal to $\left(\frac{1}{i} + \frac{(i/2)^2}{\sum_{j=1}^{i-1}(j-i/2)^2}\right)$ and $\sigma_{\varepsilon_i}^2 = \frac{1}{i-4}\left[\sum_{k=1}^{t-1}(d_k - a_1 - a_2k)^2 + \sum_{k=t}^{t+L-1}(D_{k|t} - a_1 - a_2k)^2\right]$.

Thus, $\sigma_{\varepsilon_i}^2$ is proportional to the sum of the *a posteriori* squarred errors equals to $(d_k - a_1 - a_2k)^2$ for $k < t$ and the *a priori* squarred errors equals to $(D_{k|t} - a_1 - a_2k)^2$ for $k \in [\![t, t+L-1]\!]$. We can consequently easily express $\mathbb{E}_{t+L-1}\left[\sigma_{\varepsilon_i}^2\right]$ with respect to the variables calculated above inasmuch as both operators $\mathbb{E}[.]$ and $Cov(.,.)$ are linear ones. □

*Remark*: complete results are available in appendix.

**Lemma 2.2.1 - 3.** *The variable* $\mathbb{V}ar_{t+L-1}\left[\sigma^2_{t+L}\right]$ *is calculable with respect to* $(d_k)_{1\leqslant k<t}$, *and* $(m_k, \sigma_k)_{t\leqslant k<t+L}$.

*Proof.* As previously noted, $\sigma^2_{t+L} = \gamma_i \,.\, \sigma^2_{\varepsilon_i}$. Being able to express $\mathbb{V}ar_{t+L-1}\left[\sigma^2_{t+L}\right]$ is consequently equivalent to being able to describe $\mathbb{V}ar_{t+L-1}\left[\sigma^2_{\varepsilon_i}\right]$ , and then even $\mathbb{V}ar_{t+L-1}\left[\sum_{k=1}^{t-1}(d_k - a_1 - a_2k)^2 + \sum_{k=t}^{t+L-1}(D_{k|t} - a_1 - a_2k)^2\right]$.

Let $e_k$ be the quantity equal to $d_k - a_1 - a_2k$ for $k < t$ and to $D_{k|t} - a_1 - a_2k$ for $k \geqslant t$. By definition, $\mathbb{V}ar_{t+L-1}\left[\sum_{k=1}^{t+L-1} e_k^2\right]$ is equal to the sum of $\sum_{k=1}^{t+L-1}\mathbb{V}ar_{t+L-1}(e_k^2)$ and of $2\sum_{1\leqslant i<j\leqslant t+L-1} cov(e_i^2, e_j^2)$.

➢ We first study the computation of $\sum_{k=1}^{t+L-1}\mathbb{V}ar_{t+L-1}(e_k^2)$.

By definition of the least square method, the random variable $e$ is a centered variable. Considering moreover that it follows a Gaussian distribution, we can write its $2^{nd}$ and $4^{th}$ moments $\forall k \in [\![1, t+L-1]\!]$:

$$\begin{aligned}\mathbb{E}_{t+L-1}\left[e_k^2\right] &= \mathbb{V}ar_{t+L-1}\left[e_k\right]\\ \mathbb{E}_{t+L-1}\left[e_k^4\right] &= 3\mathbb{V}ar_{t+L-1}\left[e_k\right]^2\end{aligned}$$

which gives us consequently $\mathbb{E}_{t+L-1}\left[e_k^4\right] = 3\mathbb{E}_{t+L-1}\left[e_k^2\right]^2$.

On the other hand, we know, by definition, that $\mathbb{V}ar_{t+L-1}\left[e_k^2\right] = \mathbb{E}_{t+L-1}\left[e_k^4\right] - \mathbb{E}_{t+L-1}\left[e_k^2\right]^2$.

Finally, we have then $\mathbb{V}ar_{t+L-1}\left[e_k^2\right] = 2\mathbb{E}_{t+L-1}\left[e_k^2\right]^2$, which is known with respect to $(d_k)_{1\leqslant k<t}$, and $(m_k, \sigma_k)_{t\leqslant k<t+L}$ based on the proof of 2.2.1 - 2.

➢ We examine then the term $\sum_{1\leqslant i<j\leqslant t+L-1} cov(e_i^2, e_j^2)$.

$\forall i, j$ s.t. $1 \leqslant i < j \leqslant t+L-1$, $cov(e_i^2, e_j^2) = \mathbb{E}_{t+L-1}\left[e_i^2 e_j^2\right] - \mathbb{E}_{t+L-1}\left[e_i^2\right]\mathbb{E}_{t+L-1}\left[e_j^2\right]$

As seen right above, the terms $\mathbb{E}_{t+L-1}\left[e_i^2\right]$ are known with respect to $(d_k)_{1\leqslant k<i}$, and $(m_k, \sigma_k)_{t\leqslant k<i}$. Moreover, in the same manner, we can easily confirm that $\mathbb{V}ar_{t+L-1}\left[e_i\right]$ is also known $\forall i$ s.t. $1 \leqslant i \leqslant t+L-1$, as well as $cov(e_i, e_j)$, $\forall i, j$ s.t. $1 \leqslant i < j \leqslant t+L-1$.

On the other hand, admitting the $(e_i)_{1\leqslant i\leqslant t+L-1}$ is a Gaussian vector, we can write $\mathbb{E}_{t+L-1}\left[e_i^2 e_j^2\right] = \mathbb{V}ar_{t+L-1}\left[e_i\right]\mathbb{V}ar_{t+L-1}\left[e_j\right] + 2cov(e_i, e_j)^2$.

This completes the proof as each term can be expressed with respect to the wished variable. □

**Property 2.2.1 - 4.** *Assuming the random variable $\sigma_{t+L}$ follows a Gaussian distribution, it can be represented through its conditional expectation and variance, expressed explicitly with respect to previous states $(d_k)_{1\leqslant k<t}$, and $(m_k, \sigma_k)_{t\leqslant k<t+L}$.*

*Proof.* Being Gaussian, we can write the $2^{nd}$ and $4^{th}$ corresponding moments:

$$\begin{aligned}\mathbb{E}_{t+L-1}\left[\sigma_{t+L}^2\right] &= \mathbb{V}ar_{t+L-1}\left[\sigma_{t+L}\right] + \mathbb{E}_{t+L-1}\left[\sigma_{t+L}\right]^2 \\ \mathbb{E}_{t+L-1}\left[\sigma_{t+L}^4\right] &= 3\mathbb{V}ar_{t+L-1}\left[\sigma_{t+L}\right]^2 + 6\mathbb{V}ar_{t+L-1}\left[\sigma_{t+L}\right]\mathbb{E}_{t+L-1}\left[\sigma_{t+L}\right]^2 + \mathbb{E}_{t+L-1}\left[\sigma_{t+L}\right]^4\end{aligned}$$

which lead us to:

$$\mathbb{E}_{t+L-1}\left[\sigma_{t+L}\right]^4 = \frac{1}{2}\left(3\mathbb{E}_{t+L-1}\left[\sigma_{t+L}^2\right]^2 - \mathbb{E}_{t+L-1}\left[\sigma_{t+L}^4\right]\right)$$

On the other hand, $\mathbb{V}ar_{t+L-1}\left[\sigma_{t+L}^2\right] = \mathbb{E}_{t+L-1}\left[\sigma_{t+L}^4\right] - \mathbb{E}_{t+L-1}\left[\sigma_{t+L}^2\right]^2$ and $\mathbb{V}ar_{t+L-1}\left[\sigma_{t+L}\right] = \mathbb{E}_{t+L-1}\left[\sigma_{t+L}^2\right] - \mathbb{E}_{t+L-1}\left[\sigma_{t+L}\right]^2$, so we get finally:

$$\begin{aligned}\mathbb{E}_{t+L-1}\left[\sigma_{t+L}\right]^4 &= \mathbb{E}_{t+L-1}\left[\sigma_{t+L}^2\right]^2 - \tfrac{1}{2}\mathbb{V}ar_{t+L-1}\left[\sigma_{t+L}^2\right] \\ \mathbb{V}ar_{t+L-1}\left[\sigma_{t+L}\right] &= \mathbb{E}_{t+L-1}\left[\sigma_{t+L}^2\right] - \sqrt{\mathbb{E}_{t+L-1}\left[\sigma_{t+L}^2\right]^2 - \tfrac{1}{2}\mathbb{V}ar_{t+L-1}\left[\sigma_{t+L}^2\right]}\end{aligned}$$

The lemmas 2.2.1 - 2 and 2.2.1 - 3 have shown that $\mathbb{E}_{t+L-1}\left[\sigma_{t+L}^2\right]$ and $\mathbb{V}ar_{t+L-1}\left[\sigma_{t+L}^2\right]$ can be expressed litterally with respect to the previous state variables $(d_k)_{1\leqslant k<t}$, and $(m_k, \sigma_k)_{t\leqslant k<t+L}$. Consequently, it is also the case for the expectation and the variance of the variable $\sigma_{t+L}$.

□

### iii. Conclusion

Our choice in terms of demand modeling has been determined so that it can include the dynamics of the forecast update through the rolling horizon. This motivation leads us to the mathematical modeling presented in this chapter, and we thus came up with an iteratively defined forecasted demand, by basing the computation on random variables defined for future periods.

In order to represent this demand, and exploit it, we work with conditional definitions, based on future stochastic features calculated at the date of forecasts $t$. This creates a double-level probabilistic structure for any demand in period $i < t$.

In the current section, we analyzed theoretically the impact of such modeling on the stochastic features of the demand modelled in a concrete example. We prove

that, using linear regression as forecasting technique, the stochastic characteristics of the distribution defining the demand are random variables. In particular, we were able to express their features literally with respect to variables defining previous periods.

The relations explicitally written here bring value in terms of modeling approach as they highlight the iterative link between a considered future period that we want to represent on the one hand, and the future demands that will evolve in the next forecasts dates through forecast update procedure and that impacts on the quantification of the considered forecast demand on the other hand.

In other words, we quantified theoretically the impact of the forecast updates from $t$ to $i$ on any forecasted demand model $D_{i|t}$.

In particular, our results show that the uncertainties are greater due to the generated variance of the expectation of the demand, and also of the corresponding standard deviation.

As shown previously in this paragraph, the variance of the parameters $\theta_{i|t}$ through $\psi_{i|t}$ create indeed variability in the computation of $D_{i|t}$ due to different scenarios $(D_{k|t})_{t\leqslant k<i}$. The following illustrations give us a representation of this double-level probability structure in the specific case of Gaussian variable. It shows how the variance of both the mean and the standard deviation of $\phi_{i|t}$ may influence the distribution. In the first case, 2.4, the distribution may be translated from one point to another, conditional to $(D_{k|t})_{t\leqslant k<i}$. In the second case 2.5, the width of the distribution may be influenced by these random samplings.

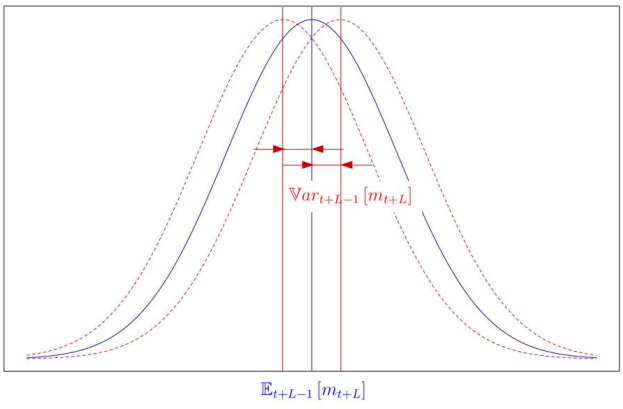



Figure 2.4: The influence of the variance of $m_{t+L}$

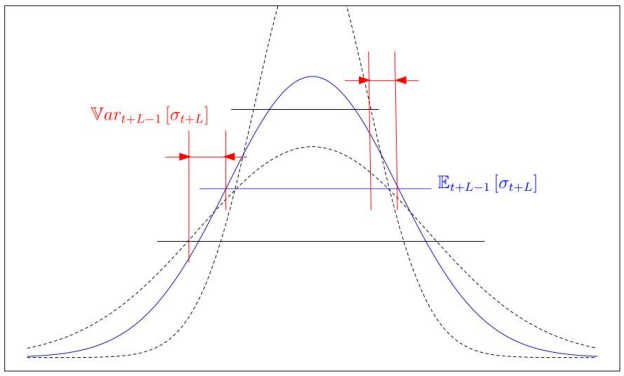



Figure 2.5: The influence of the variance of $\sigma_{t+L}$

It is important to note that necessarily, the key probability $\mathbb{P}(D_{t+L|t} > X)$ influencing the production plan will be impacted. Consequently, it will be crucial to take into account the link between $(D_{k|t})_{t\leqslant k<i}$ and $D_{t+L|t}$ described in details here so that we can quantify the induced variability and finally exploit it. This information combined with $\mathbb{P}(D_{t+L|t} > X|(D_{k|t})_{t\leqslant k<i})$ enable us to get clearer information on $\mathbb{P}(D_{t+L|t} > X)$.

In the next section, we present numerically some results showing the benefits of taking into account this epistemic uncertainties of the parameters of the distribution.

### 2.2.2 | Simulation results

In this section, we present the simulation results in which we calculated the stock level of a classical stochastic MRP approach, based on three different ways of modeling the demand as input of the MRP approach. Specifically, we compare results based on a fixed horizon framework, a classical rolling plan framework, and our dynamic and double-level probabilistic rolling approach.

The aim is to show the benefits of considering the impact of the forecast update into the modeling of the demand in terms of uncertainties. To do so, we particularly focus the analysis on the average observed service level.

We follow the following procedure. We defined the real demand as a series of 90 random variables, whose mean and standard deviation are defined as a fixed chronological expanding number and a constant fraction of it respectively. We considered that our analysis began at the beginning of the $50^{th}$ period. Thus, we generated a

random sampling of the first 49 observations of the demand to represent our learning base as historical observed data. On the other hand, 10 000 samplings of the series of the demand from period 50 to period 90 have been drawn.

For each of them, three planning processes have been executed.

➢ First, in the fixed horizon context, from the learning base made up with 49 observations of the real demand, we calculated the forecasted demand for the 41 next periods. These forecasts are, as presented in the previous section, defined through their triangular pdf, and particularly their mean and standard deviation. We calculated them using the formulas presented previously based on the simple linear regression.

Once the forecasted demand computed, we calculated the production plan using standard net requirements calculation, in which a targeted service level $\alpha$ is defined to calculate a safety stock $\sigma_i \times \Phi_{st}^{-1}(\alpha)$ in order to handle the uncertainties $\sigma_i$, $\phi_{st}^{-1}(\alpha)$ being the standard centered triangular law. The production lead-time is defined as equal to 4 periods. The targeted service level $\alpha$ is fixed at 95%.

Once the forecast demand is known, all this plan calculated at beginning of period 50, and then based on all information up to 49 is kept, and compared with all the 10 000 scenarios generated in the first place. We calculated thus the real a posteriori service level faced by the system as the ratio of quantities served for each of scenario, and compute the observed expectation of these ratios.

➢ Second, we place ourselves in a context of a classical rolling decision framework. As in the first case, from the learning base made up of 49 observations of the real demand, we calculated the forecasted demand for the 41 next periods. Then, we calculated the production plan using the same standard net requirements calculation as in the first case, with also a production lead-time of 4 periods and a targeted service level $\alpha$ fixed at 95%.

Once calculated, the first decision of production to be launched during the first period is actually executed. Then, for each scenario, the real observed sample of the demand of the following period is revealed. The process restarts again, but from period 51 with a new learning base of 50 periods.

Finally, for each scenario, we have a series of production launches at each step of the rolling plan. The a posteriori service level is finally calculated for each scenario, and we compute their observed expectation.

➢ Third, we use a rolling decision framework as defined in our approach presented in the previous sections.

In the same manner, from the learning base made up with 49 observations of the real demand, we calculated the forecasted demand for the 41 next periods. But here, we take into account the forecast updates that the system will face from the current forecast up to the delivery of the first production (here, 4 periods later). We consider indeed the use of the double-level probabilistic structure presented above allowing to take into account the epistemic uncertainties added by the forecast update mechanism.
We calculated then production plan using the standard net requirements calculation, with also a production lead-time of 4 periods and a targeted service level $\alpha$ fixed at 95% for the first level of uncertainties, and 75% for the second level -epistemic-uncertainties. This second-level service level target corresponds to the idea to be able to face 75% of the cases for the computation of the values of the parameters of the pdf of any random variable representing the forecasted demand of period $i$.
As previously, once calculated, the first decision of production to be launched during the first period is actually executed, and, for each scenario, the real observed sample of the demand of the following period is revealed. The process restarts again, but from period 51 with a new learning base of 50 periods.
Finally, for each scenario, we have a time series of production launches at each step of the rolling plan. The a posteriori service level is finally calculated for each of them, and we compute their observed expectation.

The following figures show the results obtained for the observed mean of the service levels depending on the demand model chosen in the case of a specific learning base chosen arbitrarily from period 1 to period 49.

Figure 2.6 illustrates that the forecasted demand based only on the learning set up to period 49 is not sufficient to get the targeted service level for the whole plan: the service level is divergent. In fact, here, our sampling of the first learning base $\Delta_{t-1}$ containing information up to $49^{th}$ period makes the service level diverge to zero, but the case of a divergence to infinity could have also been observed. The lack of up-to-date information all along the proceedings of the production plan make the errors accumulate themselves period after period. Thus, based on a fixed set of information, the service level diverges on average.

In the case of a rolling plan framework, the divergence is avoided thanks to the adaptation of the system over time to the evolution of the real observed demand. But, inasmuch as the risk of error from the period of the date of forecast to the period of the reception of the required quantities does not take into account the epistemic

uncertainties coming from the incomplete learning set of the forecast procedure, this does not consider the forecast update that the system will face necessarily during this period. Consequently, by taking into consideration this forecast update and its corresponding uncertainties, we can see in figure 2.6 that our rolling plan approach integrating the forecast update is more efficient with respect to the targeted service level compared with the classical rolling plan framework.

We illustrate consequently here the benefits of using a double-level probabilistic system to model the customer demand: the forecast update procedure creates epistemic uncertainties which are not taken into consideration in the classical rolling-plan framework. The targeted service level cannot therefore be reached -in average- because these uncertainties engender a spread of the value of the parameters of the pdf of the demand appearing during the lead time. In the double-level probabilistic framework on the contrary, the forecast update creating uncertainties are taken into account in the demand modeling, and consequently into the production plan. The service level targeted is reached, inasmuch as the lack of knowledge coming from the next update is here integrated into the production plan.

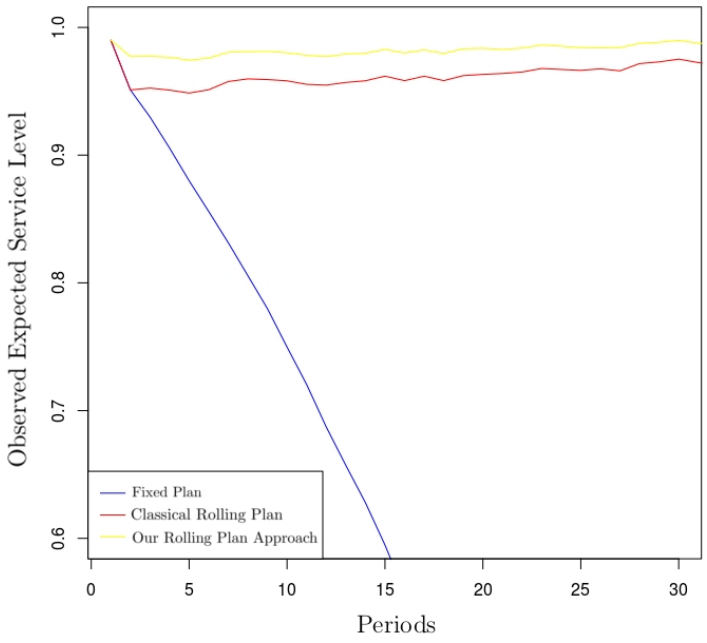



Figure 2.6: The expected service level through the planning horizon

Besides, in order to examine more in details the benefits from our framework that takes into consideration forecast updates compared with a classical rolling plan framework, we calculated the improvement realized in our modeling compared with

a classical framework with respect to the lead time. This improvement is quantified as the difference of the percentage points of the service level of the two modeling approaches.

Actually, the double-level probabilistic structure that allows us to integrate the dynamics and the uncertainties generated by the forecast update is more efficient so as the lead time. Actually, the bigger the production lead-time is, the higher the epistemic uncertainties during the lead time. This lead-time-dependent benefit is presented in figure 2.7: for longer lead-time, the benefits of our modeling on the average service level are higher than classical approaches.

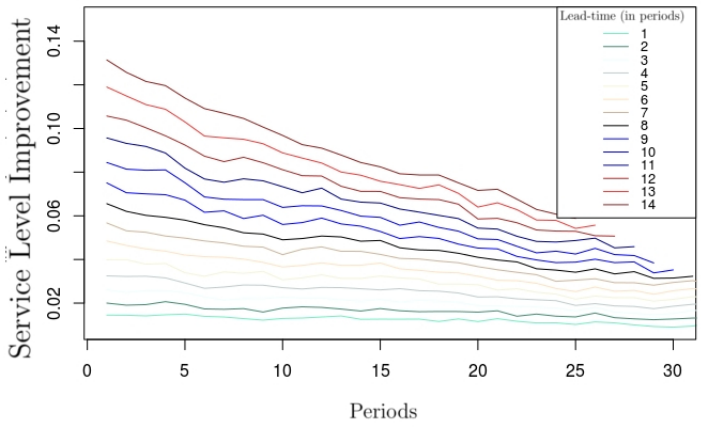



Figure 2.7: The improvements of the service level between a classical and our approach of the Rolling Plan Framework

## 2.3 | Conclusion and Impact on the Production Planning Optimization Problem

In this section, we have presented the forecasting method that we will use in the remaining of our manuscript. We described quantitatively the detailed consequences of choosing the double-level probabilistic modeling to represent the demand in our rolling horizon with forecast update context.

Then, once defined, we were able to compute the stochastic features of such a model in an analytical way. This has allowed us to understand the impact of the forecast update procedure on the modeling parameters of the forecasted demand.

Finally, we simulated demand scenarios and rolling plan frameworks to calculate production order in different contexts of study to illustrate the benefits of our

forecast modeling on a planning indicators, namely the a posteriori expected service level.

Such settings of study imply both dynamic and stochastic characteristics of the planning problem, corresponding to these demand forecasts features.

The following question raises consequently and naturally: what type of optimization method can be applied to solve such a problem. In the field of optimization involving uncertainties, both stochastic programming or robust optimization could be used. The second one, however, fits more when the distribution of the random parameters or variables are unknown. Moreover, the first one can be implemented dynamically with a multi-stage context, allowing consequently to treat the non-stationarity of our problem.

As a consequence, stochastic dynamic programming seems to be an appropriate optimization method to be applied.

We will present in the following chapter how we can model the production planning problem in a rolling plan framework taking into account the forecast update mechanism with stochastic dynamic programming when linear regression is used for demand forecasting.

# Chapter 3

# Optimization Model for the Production Planning System with Forecasts Update

In this chapter, we develop an optimization model applied to answer our issue, namely production planning taking into account forecast updates.

To do so, we integrate the forecasted demand modeling presented in the previous chapter into an optimization framework. This framework takes into account both the uncertain and non-stationary characteristics of the demand forecasts. We analytically develop a method to solve this model.

More specifically, the first part of the chapter is dedicated to the detailed description of the functions, variables, and parameters of the optimization problem, and presents a solution method based on theoretical analysis.

In the second part of the chapter, we apply this model for a specific and simple case study as an illustrating example. Solutions are then computed, with the method explained in the previous chapter, leading us to analytical results. Finally, numerical analyses are presented to illustrate the results of our optimization model, and to better understand how the parameters influence the performances.

# 1 | Problem Modeling

## 1.1 | System at stake, and objective

### 1.1.1 | General Variables

As described in details in the previous chapters, we consider here an uncapacitated single-item single-level production planning problem with backlog allowed, under demand forecasts uncertainties, and taking into account the update of the forecasts occuring at each step of the rolling horizon framework.

We treat the problem as a cost minimization problem, in which the costs considered are the unit production costs, the inventory holding costs, and the backlog costs. The objective is to define, at the beginning of period $t$, the production quantities for the next $N$ periods, using historical data $\Delta_{t-1}$ from periods 1 to $t-1$, that minimize these costs, taking into account the forecast update mechanism.

We define the system parameters as described in the following 3.1:

Table 3.1: Parameters Definition

| | | |
|---|---|---|
| Cost Parameters: | $H$ | The inventory holding cost per unit per period |
| | $B$ | The backlog cost per unit per period |
| | $p$ | The per-unit production cost |
| Industrial Parameter: | $L$ | The production Lead-time |

We note moreover the variables defining the industrial system state at each period $i$ as detailed in table 3.2:

Table 3.2: State Variables Definition

| | | |
|---|---|---|
| $X_i$ | The random variable representing the available inventory at the beginning of period $i$ | $i = t..N$ |
| $x_i$ | The real available inventory at the beginning of period $i$ | $i = 1..t-1$ |
| $c_i$ | The production order launched during period $i$, received at $i+L$ | $i = 1..t+N-1-L$ |
| $K_i$ | The random variable representing the production cost for period $i$ | $i = t..N$ |
| $W_j$ | The expected total production cost for a production plan from $j$ to $t+N-1$ | $j = t..N$ |

We consider specifically that $\forall i \in [\![t, t+N-1-L]\!]$, $K_i$ is made up of the inventory cost and/or backlog costs, as well as the production costs:

$$K_i = \begin{cases} -B.\left(X_i + c_{i-L} - D_{i|t}\right) + p.c_i & \text{when } X_i + c_{i-L} - D_i < 0 \\ H.\left(X_i + c_{i-L} - D_{i|t}\right) + p.c_i & \text{otherwise} \end{cases}$$

For the last $L$ periods, we have for $i \in [\![t+N-1-L, t+N-1]\!]$, $K_i = -B.\left(X_i + c_{i-L} - D_{i|t}\right)$ when $X_i + c_{i-L} - D_{i|t} < 0$, and $H.\left(X_i + c_{i-L} - D_{i|t}\right)$ otherwise. $K_i$ is indeed in these cases made up only of the holding/backlog cost as $c_i$ is not defined.

Inasmuch as the planning problem from period $t$ to the planning horizon $t+N-1$ is considered, we assume all variables $d_i$ and $x_i$, $i < t$ are known as historical data, included into the learning set $\Delta_{t-1}$.

### 1.1.2 | States of the System

At each period $i$ of the planning that interests us, *ie*, $i \geqslant t$, the state of the system is described entirely through specific variables. Precise values of these variables create the information set needed to understand the instantaneous picture of the considered environment at this instant $i$, defined at the beginning of the planning process, *ie* at the beginning of period $t$.

**i. Basic State of any period $i$:**

Considering the variables presented in 3.2, the system state at the beginning of a period $i$ is made up of the followings: the inventory level at the beginning of the period $X_i$, and the previous order launched $(c_j)_{i-L \leqslant j < i}$. Considering the learning set presented in chapter 2, the information also included into the state of the system in period $i$ is all the historical data -including the historical demand- previously noted $\Delta_{t-1}$. In the specific case of the linear regression that we will use in the remaining of this thesis, and that is described in section 2 of the previous chapter, it corresponds for example to the learning matrix of size $(i-1) \times p$, and the $t-1$ historical demand data $(d_i)_{1 \leqslant i < t}$.

We define this state as the basic state of the system. All the data considered here are indeed raw data, not yet analyzed nor used by the planner into any calculation process.

**ii. Advanced State of any period $i$:**

The basic state of the system allows the planner to identify the details of the context in which the production decisions have to be taken, but is not sufficient to make them up. The decisional system cannot indeed be described entirely without another level of information, in particular the demand forecasts information.

In fact, as previously defined in chapter 2, in order to take into account the forecast update procedure of the rolling plan into the modelling of our forecasts, they are calculated recursively and conditionnaly from period $t$ to the considered period $i$. Thus, to be able to understand the precise situation of the decision environment of period $i$, demand forecasts $D_{k|t}$ for $k \in [\![t, i-1]\!]$ have to be included in the system state. These are indeed variables conditional to which the computation of the future forecasts of periods $j \geqslant i$ are based, and which are therefore required to take any decision at the beginning of period $i$.

These variables are completely defined by their random parameters $\theta_{k|t}$, computed recursively from the current forecast date $t$.

As a consequence, we define these outputs of the forecasting process -*ie* the vector of parameters $\theta_{k|t}$, for $t \leqslant k \leqslant i$ - as an augmented state for the system in period $i$, that complete the set of information from which the decision can be taken. We define it as the advanced state of a period $i$. It is consequently made up of the recursive demand forecasts information from $t$ to $i-1$, namely $D_{k|t}$ for $t \leqslant k < i$, and the forecast information for the parameters of period $i$. Each of these forecasted demand is entirely defined through the conditional double-level definition of their parameters set $\theta_{k|t}$, which are computed recursively and conditional to the previous states so that the forecast update mechanism can be included in the information defining this advanced state of any period $i$.

### 1.1.3 | Objective

The problem formulated here is a cost minimization problem. We are seeking for the production plan over the $N$ following periods of the planning horizon from the beginning of the first date of forecasts noted $t$. We consider in particular the minimization of the expected total cost $W_t$ with respect to the stochastic process $\left(D_{i|t}\right)_{t \leqslant i \leqslant t+N-1}$ defined in the previous chapter as the dynamic modeling of the demand that take into account the forecast updates of the rolling plan framework.

The expected cost $W_t$ is by definition the expectation conditional to $\Delta_{t-1}$ of the

cumulated cost of all periods from $t$ to the planning horizon $t+N-1$. It depends consequently naturally on the state of the system in period $t$, and on the decision variables $(c_i)_{t\leqslant i\leqslant t+N-1-L}$ whose optimal point is sought:
$W_t\left(x_t, \Delta_{t-1}, (c_i)_{t-L\leqslant i<t}, (c_i)_{t\leqslant i\leqslant t+N-1-L}, \left(D_{i|t}\right)_{t\leqslant i\leqslant t+N-1}\right) =$

$$\mathbb{E}_{t-1}\left[\sum_{i=t}^{t+N-1-L} K_i\left(X_i, D_{i|t}, c_{i-L}, c_i\right) + \sum_{i=t+N-L}^{t+N-1} K_i\left(X_i, D_{i|t}, c_{i-L}\right)\right] \tag{3.1}$$

## 1.2 | Computational framework: Stochastic Dynamic Programming

As explained, in order to take into account the dynamics of the forecast update procedure into the optimization problem, we model the demand as a stochastic historical-dependent process. Each forecasted demand $D_{i|t}$ at any period $i > t$ is defined conditional to the previous stochastic advanced states $t \leqslant k < i$, making its pdf dependent on the random parameters of previous states (see chapter 2).

Despite its interesting characteristics that allow to depict stochastic scenarios occurring during the rolling framework, this iterative modeling creates a computational issue in terms of cost function calculation. Actually, the expectation conditional to the state of $t-1$ used to compute the cost function of the optimization problem at the beginning of period $t$ does not permit us to develop the expression of the cost function in 3.1. It depends indeed on variables $D_{i|t}$, $i > t$ that are defined conditional to states $k$, for $t \leqslant k < i$. We are consequently not able to calculate an expectation of any measurable function $g$ such as $\mathbb{E}_{t-1}\left[g\left(D_{i|t}\right)\right]$ for any $i > t$ inasmuch as it is not defined on the right $\sigma$-algebra.

Consequently, in order to be able to integrate this dynamic feature into our optimization problem, we will model the problem as a multi-stage stochastic problem: each period represents a step of the optimization problem whose optimal point depends on the previous ones.

We describe in details the relations between the stages of the problem in the following subsections. First of all, we make explicit the iterative link between the cost functions of the stages considered, then we detail the dynamics of the states of the system from one period to another, and finally we explain how these help us to get the optimal solution of our problem.

### 1.2.1 | Recursiveness of the Cost Function Computation

Assuming we are able to compute the conditional expected cost $W$ for a specific period $j$ such that $t < j \leqslant t + N - L$:

$$W_j\left(X_j, \Delta_{j-1|t}, (c_i)_{j-L\leqslant i<j}, (c_i)_{j\leqslant i\leqslant t+N-1-L}, \left(D_{i|t}\right)_{j\leqslant i\leqslant t+N-1}\right)$$

It implicitly supposes that the system state at the beginning of $j$ is entirely known, *ie* we can calculate any function conditional to $\Delta_{j-1|t}$.

Moreover, the advanced information on $\left(D_{i|t}\right)_{j\leqslant i\leqslant t+N-1}$ are, as previously defined, calculated recursively based on $\Delta_{j-1|t}$. For an easier reading, it is not necessary to let it explicit in the variables of $W_j$.

With such hypothesis, the minimum of this function with respect to $(c_i)_{j\leqslant i\leqslant t+N-1-L}$ would consequently be calculable. We note $V_j$ this value, and call it Value Function of period $j$, and $(c_i^*)_{j\leqslant i\leqslant t+N-1-L}$ its argument of the minimum:

**Definition 1.2.1-1.**
$V_j\left(X_j, \Delta_{j-1|t}, (c_i)_{j-L\leqslant i<j}\right)$

$$\begin{aligned}
&= W_j\left(X_j, \Delta_{j-1|t}, (c_i)_{j-L\leqslant i<j}, (c_i^*)_{j\leqslant i\leqslant t+N-1-L}\right)\\
&= \min_{(c_i)_{j\leqslant i\leqslant t+N-1-L}} W_j\left(X_j, \Delta_{j-1|t}, (c_i)_{j-L\leqslant i<j}, (c_i)_{j\leqslant i\leqslant t+N-1-L}\right)\\
&= \min_{(c_i)_{j\leqslant i\leqslant t+N-1-L}} \mathbb{E}_{j-1}\left[\sum_{i=j}^{t+N-1-L} K_i\left(X_i, D_{i|t}, c_{i-L}, c_i\right) + \sum_{i=t+N-L}^{t+N-1} K_i\left(X_i, D_{i|t}, c_{i-L}\right)\right]
\end{aligned}$$

The Value Function of a period $j$, as defined here, follows an important property: we have indeed the following recursive equation:

**Property 1.2.1-2.** *$\forall t < j < t + N - 1 - L$, the Value Function $V_j$ respects the following recursive equation:*

$$V_j\left(X_j, \Delta_{j-1|t}, (c_i)_{j-L\leqslant i<j}\right) = \min_{c_j}\left\{\mathbb{E}_{j-1}\left[K_j + V_{j+1}\left(X_{j+1}, \Delta_{j|t}, (c_i)_{j-L<i\leqslant j}\right)\right]\right\} \quad (3.2)$$

*Proof.* At stage $j$, the expected total cost $W_j$ is equal to:

$$W_j\left(X_j, \Delta_{j-1|t}, (c_i)_{j-L\leqslant i<j}, (c_i)_{j\leqslant i\leqslant t+N-1-L}\right)$$

$$\begin{aligned}
&=\mathbb{E}_{j-1}\left[\sum_{i=j}^{t+N-1-L} K_i\left(X_i, D_{i|t}, c_{i-L}, c_i\right) + \sum_{i=t+N-L}^{t+N-1} K_i\left(X_i, D_{i|t}, c_{i-L}\right)\right]\\
&=\mathbb{E}_{j-1}\left[K_j\left(X_j, D_{j|t}, c_{j-L}, c_j\right)\right]\\
&\quad+\mathbb{E}_{j-1}\left[\sum_{i=j+1}^{t+N-1-L} K_i\left(X_i, D_{i|t}, c_{i-L}, c_i\right) + \sum_{i=t+N-L}^{t+N-1} K_i\left(X_i, D_{i|t}, c_{i-L}\right)\right]
\end{aligned}$$

Given the definition of the probability spaces used here, defined in the previous chapter, we can use the following law of iterated expectations $\forall j > t$:

$$\begin{aligned}
&\mathbb{E}_{j-1}\left[\sum_{i=j+1}^{t+N-1-L} K_i\left(X_i, D_{i|t}, c_{i-L}, c_i\right) + \sum_{i=t+N-L}^{t+N-1} K_i\left(X_i, D_{i|t}, c_{i-L}\right)\right]\\
=&\mathbb{E}_{j-1}\left[\mathbb{E}_j\left[\sum_{i=j+1}^{t+N-1-L} K_i\left(X_i, D_{i|t}, c_{i-L}, c_i\right) + \sum_{i=t+N-L}^{t+N-1} K_i\left(X_i, D_{i|t}, c_{i-L}\right)\right]\right]
\end{aligned}$$

As a consequence, we can rewrite $W_j$ as:

$$W_j\left(X_j, \Delta_{j-1|t}, (c_i)_{j-L\leqslant i<j}, (c_i)_{j\leqslant i\leqslant t+N-1-L}\right) =$$

$$\mathbb{E}_{j-1}\left[K_j\left(X_j, D_{j|t}, c_{j-L}, c_j\right) + W_{j+1}\left(X_{j+1}, \Delta_{j|t}, (c_i)_{j-L<i\leqslant j}, (c_i)_{j<i\leqslant t+N-L}\right)\right]$$

which gives us for the value function:

$$\begin{aligned}
&V_j\left(X_j, \Delta_{j-1|t}, (c_i)_{j-L\leqslant i<j}\right)\\
&= \min_{(c_i)_{j\leqslant i\leqslant t+N-1-L}} \left\{\mathbb{E}_{j-1}\left[K_j\left(X_j, D_{j|t}, c_{j-L}, c_j\right)\right.\right.\\
&\qquad\qquad\left.\left. + W_{j+1}\left(X_{j+1}, \Delta_{j|t}, (c_i)_{j-L<i\leqslant j}, (c_i)_{j<i\leqslant t+N-L}\right)\right]\right\}
\end{aligned}$$

Moreover, inasmuch as $K_j$ does not depend on $(c_i)_{i\geqslant j+1}$, it is also the case for $\mathbb{E}_{j-1}\left[K_j\left(X_j, D_{j|t}, c_{j-L}, c_j\right)\right]$. Thus, we can separate the min over $(c_i)_{j\leqslant i\leqslant t+N-1-L}$ into two minimum functions over two seperate sets, and distribute the second one over the last part of the inner sum:

$$
\begin{aligned}
&V_j\left(X_j, \Delta_{j-1|t}, (c_i)_{j-L\leqslant i<j}\right) \\
&= \min_{c_j}\left\{ \min_{(c_i)_{j<i\leqslant t+N-1-L}} \left\{ \mathbb{E}_{j-1}\Big[K_j\left(X_j, D_{j|t}, c_{j-L}, c_j\right)\right.\right. \\
&\qquad\qquad \left.\left. + W_{j+1}\left(X_{j+1}, \Delta_{j|t}, (c_i)_{j-L<i\leqslant j}, (c_i)_{j<i\leqslant t+N-L}\right)\Big]\right\}\right\} \\
&= \min_{c_j}\left\{\mathbb{E}_{j-1}\Big[\ K_j\left(X_j, D_{j|t}, c_{j-L}, c_j\right)\right. \\
&\qquad\qquad \left. + \min_{(c_i)_{j<i\leqslant t+N-1}}\left\{W_{j+1}\left(X_{j+1}, \Delta_{j|t}, (c_i)_{j-L<i\leqslant j}, (c_i)_{j<i\leqslant t+N-L}\right)\Big]\right\}\right\}
\end{aligned}
$$

Finally, with the previous definition of the value function, we have consequently:

$$
V_j\left(X_j, \Delta_{j-1|t}, (c_i)_{j-L\leqslant i<j}\right) = \min_{c_j}\left\{\mathbb{E}_{j-1}\left[K_j + V_{j+1}\left(X_{j+1}, \Delta_{j|t}, (c_i)_{j-L<i\leqslant j}\right)\right]\right\}
$$

□

### 1.2.2 | Dynamic of the States

As defined previously, the states correspond directly to the picture of the planning system at the beginning of the considered period -what we defined as the basic state of the system- on the one hand, and the analytical data resulting from the forecasting process executed at the beginning of this period, giving us on the other hand, at this precise forecast date, an advanced state of the system on which the planning decision will be based.

These states move from one period to the next one, following the evolutive flows of the system in its environment. In the case of the basic state, these flows include in particular the update of the inventory level, and the real demand record. As far as the advanced state is concerned, it includes the update of the forecasts, heart of our issue.

We define this evolution as the law of motion of the system, or the dynamics of states from one period to another one. We show here that we can define explicit functions to set this dynamics of the states.

#### i. Basic State

When we considering the data defining the basic state of the system, we have, on the one hand, the inventory level which evolves from one period $j$ to the following

one $j+1$ as the classic inventory update equation with backlog: $\forall j,\ j = t..t+N-1,\ X_{j+1} = X_j + c_{j-L} - D_j$.

Modeling this update at the beginning of period $t$, we have consequently, with the previous definitions of our variables at stake: $\forall j,\ j = t..t+N-1,\ X_{j+1} = X_j + c_{j-L} - D_{j|t}$. We note it $\nu$: $X_{j+1} = \nu\left(X_j, c_{j-L}, D_{j|t}\right)$

As far as the historical information is concerned, two cases arise depending on the data considered.

The first one is basically the case of the demand information corresponding to periods $j < t$. Here, recorded historical database is considered. Thus, when the system evolves from state $j$ to the next one, there is no change for this set of information.

The second one corresponds more specifically to the static set of information used as learning set for the whole planning horizon without any dependency with respect to the date of forecast. In this case, no update occurs during the rolling plan proceedings, and there is here no law of motion to be modelled. It is the case for example when we consider a linear regression, where the matrix $\Lambda(t)$ used for both learning and projection is known at the beginning of $t$. The only change is consequently the split limit between learning set and test set.

**ii. Advanced State**

The advanced state of the system in any period $j \geqslant t$ is made up with the forecasts information from $t$ to $j-1$. In the same manner, the advanced state of the system in the next period $j+1$ is therefore made up of the forecasts information from $t$ to $j$. The only change from the advanced state in $j$ and the one in $j+1$ is the new information of $D_{j|t}$ which has to be calculated, and finally integrated into the set of information of the advanced state. The law of motion that interests us here is consequently how to get the forecasts information at the beginning of period $j+1$ from the state in $j$, or, in other words, how to define and characterize $D_{j|t}$ from demand information up to $j-1$.

The forecasted demand in $j$ corresponds to the random variable $D_{j|t}$ computed dynamically from $k = t$ to $j$. It is entirely determined through the definition of the parameters $\theta_{j|t}$ of its pdf with respect to the previous states data.

Depending on the forecasting technique used, the computation method differs. However, the forecasting procedure will systematically make us able to define the function $\mathcal{G}_{j|t}$ based on the state in $j$, $\Delta_{j-1|t}$, that allows to express litterally the

parameters of the pdf of the new forecasts sought $D_{j|t}$, and consequently to define it precisely (see chapter 2, equation 2.3):

$$\theta_{j|t} = \mathcal{G}_{j|t}\left(\Delta_{j-1|t}\right)$$

$D_{j|t}$ can consequently be well determined with respect to the state in $j$, and then, by including it into the advanced state of period $j$, the information of the advanced state in period $j+1$ is complete: $\Delta_{j|t} = \Delta_{j-1|t} \cup \{D_{j|t}\}$

**iii. Dynamics of the states: laws of motion**

Finally, when a period $j$ is completely known through both its basic state information and its advanced state of information, we are able to define precisely the future period states. They can indeed be explicitely computed with respect to the previous state, thanks to the following laws of motion:

$$\begin{cases} \theta_{j|t} & = \mathcal{G}_{j|t}(\Delta_{j-1|t}) \\ X_{j+1} & \nu\left(X_j, c_{j-L}, D_{j|t}\right) \end{cases} \tag{3.3}$$

We will see in the next section how this evolution will be used in the solution process.

### 1.2.3 | Dynamic Equation of the Problem

Let $\mathcal{P}$ be the planning problem consisting of the minimization of the function $W_t$ defined in 3.1 with respect to the decision variables $(c_i)_{t\leqslant i\leqslant t+N-1-L}$ taking into account the forecast update mechanism by using the dynamically defined conditional stochastic process $\left(D_{i|t}\right)_{t\leqslant i\leqslant t+N-1}$.

We show here in details the computational framework of the solutions of $\mathcal{P}$. This method will allow us to calculate all the value functions and their corresponding argument of the minimum for any $i \geqslant t$, or, in other words, the solutions of our problem.

**Lemma 1.2.3 - 3.** *$\forall j,\ t \leqslant j < t+N-L$, if $\wp$ is a measurable function with respect to the $\sigma$-algebra generated by $D_{j|t}$, then its expectation conditional to the state in $j$ can be calculated and expressed literally with respect to the information of this state.*

*Proof.* Let $j$ be a period index in $[\![t, t+N-1]\!]$. Let $\wp$ be a measurable function with respect to the random variable $D_{j|t}$ and its corresponding measure $\phi_{j|t}$.

As defined previsouly, $\phi_{j|t}$ depends on its parameters $\theta_{j|t}$. Based on 3.3, these are calculable through $\theta_{j|t} = \mathcal{G}_{j|t}(\Delta_{j-1|t})$.

We have consequently:

$$\begin{aligned}\mathbb{E}_{j-1}\left[\wp\left(D_{j|t}\right)\right] &= \int_{\mathbb{R}} \wp(z)\phi_{j|t}(z,\theta_{j|t})dz \\ &= \int_{\mathbb{R}} \wp(z)\phi_{j|t}(z,\mathcal{G}_{j|t}(\Delta_{j-1|t}))dz\end{aligned}$$

Since $\mathbb{E}_{j-1}$ is the expectation operator conditional to the state in $j$, the integrability is assessed knowing $(D_k)_{t\leqslant k<j}$. In other words, $\Delta_{j-1|t}$ is completely determined at this level of the calculation.

By cascading definitions, we are consequently able to express precisely $\phi_{i|t}$, and therefore $\mathbb{E}_{j-1}\left[\wp\left(D_{j|t}\right)\right]$, thanks to the measurability of $\wp$ taken as assumption. □

**Lemma 1.2.3-4.** *For periods $j \geqslant t+N-L$, the value of $W_j$ can be calculated literally with respect to the state at the beginning of period $j-1$ and $D_{j-1|t}$.*

*Proof.* When periods $j \geqslant t+N-L$ are considered, no decision variables exist. Consequently, the total cost function $W_j$ can be directly written as

$$W_j = \mathbb{E}_{j-1}\left[\sum_{i=j}^{t+N-1} K_i\left(X_i, D_{i|t}, c_{i-L}\right)\right]$$

1. Specifically, for the last period $t+N-1$, we have:

$$W_{t+N-1} = \mathbb{E}_{t+N-2}\Big[K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-1-L}\right)\Big]$$

The variables $X_{t+N-1}$ and $c_{t+N-1-L}$ being part of the basic state of period $t+N-1$ conditional to which the computation of $W_{t+N-1}$ is made, we can easily confirm we are in the scope of the lemma 1.2.3-3. Consequently, $W_{t+N-1}$ can be known and expressed literally with respect to the state in $t+N-1$. Then, we note it $W_{t+N-1}\left(X_{t+N-1}, \Delta_{t+N-2|t}, c_{t+N-1-L}\right)$.

Besides, the law of motion 3.3 allows us to write $X_{t+N-1}$ as a variable with respect to the state in $t+N-2$: $X_{t+N-1} = \nu\left(X_{t+N-2}, c_{t+N-2-L}, D_{t+N-2|t}\right)$, and $\Delta_{t+N-2|t}$ as a union set of $\Delta_{t+N-3|t}$ and $D_{t+N-2|t}$. $W_{t+N-1}$ can be therefore finally entirely defined with respect to the state in $t+N-2$ and $D_{t+N-2|t}$. We note it then finally $W_{t+N-1}\left(X_{t+N-2}, \Delta_{t+N-3|t}, c_{t+N-1-L}, c_{t+N-2-L}, D_{t+N-2|t}\right)$

2. Now, considering the next stage $W_{t+N-2}$, we have

$$W_{t+N-2} = \mathbb{E}_{t+N-3}\Big[K_{t+N-2}\left(X_{t+N-2}, D_{t+N-2|t}, c_{t+N-2-L}\right) + K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-1-L}\right)\Big]$$

which can be transformed thanks to the law of iterated expectations as

$$W_{t+N-2} = \mathbb{E}_{t+N-3}\Big[K_{t+N-2}\left(X_{t+N-2}, D_{t+N-2|t}, c_{t+N-2-L}\right)\Big] + \mathbb{E}_{t+N-3}\Big[\mathbb{E}_{t+N-2}\Big[K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-1-L}\right)\Big]\Big]$$

which finally leads us to the following relation defining $W_{t+N-2}$:

$$W_{t+N-2} = \mathbb{E}_{t+N-3}\Big[\, K_{t+N-2}\left(X_{t+N-2}, D_{t+N-2|t}, c_{t+N-2-L}\right) + W_{t+N-1}\left(X_{t+N-2}, \Delta_{t+N-3|t}, c_{t+N-1-L}, c_{t+N-2-L}, D_{t+N-2|t}\right)\Big]$$

In the same manner, the variables $X_{t+N-2}$, $\Delta_{t+N-3|t}$, $c_{t+N-1-L}$, and $c_{t+N-2-L}$ are part of the state of period $t+N-2$. $W_{t+N-2}$ being computed conditional to this period, we are again in the perimeter of the lemma 1.2.3 - 3.

The steps one and two can be then executed for $W_{t+N-2}$ and $W_{t+N-3}$, and, similarly, for any $j > t+N-L$ in an iterative manner.

This procedure allows us to finally get the fact that for any $j \geqslant t+N-L$, $W_j$ can be entirely defined with respect to the state in $j-1$ and $D_{j-1|t}$.

□

In particular, the lemma 1.2.3 - 4 is true for $j = t+N-L$, meaning that $W_{t+N-L}$ can be expressed literally depending on the state in $t+N-L-1$ and $D_{t+N-L-1|t}$. We note it then finally

$$W_{t+N-L}\left(X_{t+N-L-1}, \Delta_{t+N-L-2|t}, (c_i)_{t+N-2L-1 \leqslant i < t+N-L-1}, D_{t+N-L-1|t}\right)$$

**Lemma 1.2.3 - 5.** *The value function $V_{t+N-L-1}$ can be computed and expressed literally with respect to the information of the state in $t+N-L-1$. This calculation allows us to get the last decision variable optimal value $c^*_{t+N-L-1}$*

*Proof.* First of all, based on the definition 1.2.1 - 1 and the lemma 1.2.3 - 4, we known $V_{t+N-L-1}$ exists, and can be written as

$$V_{t+N-L-1}\left(X_{t+N-L-1}, \Delta_{t+N-L-2|t}, (c_i)_{t+N-2L-1\leqslant i<t+N-L-1}\right)$$

$$\begin{aligned}
&= W_{t+N-L-1}\left(X_{t+N-L-1}, \Delta_{t+N-L-2|t}, (c_i)_{t+N-2L-1\leqslant i<t+N-L-1}, c^*_{t+N-L-1}\right)\\
&= \min_{c_{t+N-L-1}} W_{t+N-L-1}\left(X_{t+N-L-1}, \Delta_{t+N-L-2|t}, (c_i)_{t+N-2L-1\leqslant i<t+N-L-1}\right)\\
&= \min_{c_{t+N-L-1}} \left\{ \mathbb{E}_{t+N-L-2}\left[ K_{t+N-L-1}\left(X_{t+N-L-1}, D_{t+N-L-1|t}, c_{t+N-2L-1}, c_{t+N-L-1}\right)\right.\right.\\
&\qquad \left.\left. + \sum_{i=t+N-L}^{t+N-1} K_i\left(X_i, D_{i|t}, c_{i-L}\right)\right]\right\}
\end{aligned}$$

Using the law of iterated expectations, we get:

$$V_{t+N-L-1}\left(X_{t+N-L-1}, \Delta_{t+N-L-2|t}, (c_i)_{t+N-2L-1\leqslant i<t+N-L-1}\right)$$

$$\begin{aligned}
&= \min_{c_{t+N-L-1}} \left\{ \mathbb{E}_{t+N-L-2}\left[ K_{t+N-L-1}\left(X_{t+N-L-1}, D_{t+N-L-1|t}, c_{t+N-2L-1}, c_{t+N-L-1}\right)\right.\right.\\
&\qquad \left.\left. + W_{t+N-L}\right]\right\}
\end{aligned}$$

As described previsouly, the lemma 1.2.3-4 gives us the dependency of $W_{t+N-L}$ with respect to the state in $t+N-L-1$ and $D_{t+N-L-1|t}$. Moreover, by definition, the variables $X_{t+N-L-1}$ and $c_{t+N-2L-1}$ are part of the state of $t+N-L-1$. Therefore, we can apply the lemma 1.2.3-3 to be able to express

$\mathbb{E}_{t+N-L-2}\left[K_{t+N-L-1} + W_{t+N-L}\right]$ with respect to $c_{t+N-L-1}$.

In the context of production planning in which our problem is defined, we can assume the set of possible solutions for $c_{t+N-L-1}$ and the corresponding values of $V_{t+N-L-1}\left(X_{t+N-L-1}, \Delta_{t+N-L-2|t}, (c_i)_{t+N-2L-1\leqslant i<t+N-L-1}\right)$ are necessarily bounded. We then assume the fact that the min exists, and, as soon as

$\mathbb{E}_{t+N-L-2}\left[K_{t+N-L-1} + W_{t+N-L}\right]$ is known, can be determined.

□

**Theorem 1.2.3-6.** *$\mathcal{P}$ is solvable dynamically based on information on hand at the beginning of the forecast date $t$, namely the inventory level $x_t$, the historical information $\Delta_{t-1}$ -including the real demand observations up to period $t-1$-, and the already launched but not yet received productions $(c_i)_{t-L\leqslant i<t}$.*

*The dynamic equation on which the solutions computation is based is the following recursive system:*

$\forall t \leqslant j < t+N-L-1,$

$$\begin{cases} V_j\left(X_j, \Delta_{j-1|t}, (c_i)_{j-L\leqslant i<j}\right) = \\ \qquad \min\limits_{c_j}\left\{\mathbb{E}_{j-1}\Big[K_j\left(X_j, D_{j|t}, c_{j-L}, c_j\right) \right. \\ \qquad\qquad \left. + V_{j+1}\left(\nu\left(X_j, c_{j-L}, D_{j|t}\right), \Delta_{j-1|t}, D_{j|t}, (c_i)_{j-L<i\leqslant j}\right)\Big]\right\} \\ \\ V_{t+N-L-1} \quad \textit{known with respect with } X_{t+N-L-2},\ D_{t+N-L-2|t},\ \Delta_{t+N-L-3|t}, \\ \qquad\qquad \textit{and } (c_i)_{t+N-2L-2<i\leqslant t+N-L-2} \end{cases}$$

*Proof.* The property of the value function 1.2.1-2 makes explicit the recursiveness it respects for $t \leqslant j \leqslant t+N-L-2$ by linking value functions of two following states one to another. We showed that

$$V_j\left(X_j, \Delta_{j-1|t}, (c_i)_{j-L\leqslant i<j}\right) = \min_{c_j}\left\{\mathbb{E}_{j-1}\Big[K_j + V_{j+1}\left(X_{j+1}, \Delta_{j|t}, (c_i)_{j-L<i\leqslant j}\right)\Big]\right\}$$

As already applied in the previous proofs, we can easily make $V_{j+1}$ dependent on the state in $j$ and $D_{j|t}$ by using the laws of motion defined in 3.3: we have indeed $X_{j+1} = \nu\left(X_j, c_{j-L}, D_{j|t}\right)$ and $\Delta_{j|t}$ be a union set of $\Delta_{j-1|t}$ and $D_{j|t}$, defined $\theta_{j|t} = \mathcal{G}_{j|t}\left(\Delta_{j-1|t}\right)$.

Consequently, the laws of motion allow us to confirm that the quantity $K_j + V_{j+1}$ is a function depending on the states in $j$ and $D_{j|t}$. In this case, we know from Lemma 1.2.3-3 that $\mathbb{E}_{j-1}\left[K_j + V_{j+1}\right]$ exists and can be computed by integration.

We can confirm consequently that the dynamics of the value function not only exists, but also allows us to compute it from one state to another dynamically by solving at each stage of the recursive procedure a minimization problem.

Finally, we know from 1.2.3-5 that $V_{t+N-L-1}$ can be known with respect with the states of period $t+N-L-1$. By using the same reasoning based on laws of motion, the final value function of the dynamic equation $V_{t+N-L-1}$ can be expressed literrally with respect to $X_{t+N-L-2}$, $D_{t+N-L-2|t}$, $\Delta_{t+N-L-3|t}$, and $(c_i)_{t+N-2L-2<i\leqslant t+N-L-2}$.

Being set, this first statement gives us the initialization of the induction reasoning, allowing to roll out the calculations of the dynamic system in order to compute the solution $(c_i^*)_{t\leqslant i\leqslant t+N-L-1}$. □

### 1.2.4 | Conclusion on the solutions computation procedure

The integration of the forecast update mechanism through the definition of the demand forecasts as a conditional recursive stochastic process creates a complex multi-dimensional and dynamic environment for a solutions computation system.

However, by characterizing precisely the states of the system in each period and the forecast update through their dynamic link from one period to the next on the one hand, and by using a multi-stage stochastic optimization framework modelled through its dynamic nested form, we are able to highlight properties respected by our system.

In particular, specific declarations and properties enable us to calculate step by step the value function from the planning horizon $t + N - 1$ to our first period of interest $t$ thanks to conditional expectation computation and minimization problem.

This inductive reasoning based on the dynamic equation that rules the system finally gives us the solution of our problem $(c_i^*)_{t \leqslant i \leqslant t+N-L-1}$. To be able to achieve such computations, some conditions need to be met. First, the forecasting method used has to be defined such that the law of motion characterizing the stochastic process of the demand $\mathcal{G}_{i|t}$ can be explicitly written with respect to the previous states information. This relation is entirely determined by the forecast technique used by the planner. Second, the sets of possible solutions for minimization problems have to be bounded so that an optimal point exists.

## 2 | A First Case Study

In this section, we present a first illustrative example of the calculation process and solution computation presented above in a simple case study.

The first part describes the industrial parameters and the assumptions defining the decisional system in which our first example evolves. The second part is dedicated to the implementation of the solutions calculation procedure, which ends with the literal expressions of the solutions of our problem. Finally, we end this section by presenting a numerical study illustrating concretely the benefits of our method compared with other classical techniques.

## 2.1 | Environment Description

### 2.1.1 | Industrial Parameters

In the example developed here, we deliberately chose a simple case. We aim indeed to be able to make the computational details explicit so that the mechanism of the solution-search procedure is clear.

We consequently chose to work with a production lead-time equal to zero period in this illustrative example ($L = 0$). The order launched at any period $i$ is then assumed to be received before the end of this same period.

The forecast update procedure is consequently taken into account into the inventory projection for the next periods after ordering/receiving the orders.

Moreover, as defined in the previous section, we are working with a backlog-allowed system, in which the costs that interest us are the inventory-holding and/or backlog cost and the unit production cost per period.

The production does not face any constraint -capacity and/or quantitative restrictions, and is supposed to be a one-level production.

### 2.1.2 | Decision System

#### i. Demand

The demand modelling chosen here follows the definition given in the previous chapter. Its pdf $\phi_{i|t}$ for any $i \geqslant t$ is indeed supposed to be a centered triangular one, and the corresponding parameters $\theta_{i|t}$ are defined as the mean $m_i$ and an equivalent of its standard deviation $s_i$, as defined in paragraph i.

The forecasting method that will be used in this section is the simple linear regression. Compared with the demand modelling described in chapter 2 section 2, the learning matrix $\Lambda$ is consequently of width 2, with $\Lambda_1$ being the constant vector equal to 1, and $\Lambda_2$ the "temporal" vector as known as "trend", equal to $i$ in row $i$.

The parameters of any pdf $\phi_{i|t}$ can consequently be calculated explicitly.

#### ii. States

In the previous section, we defined the basic and the advanced states as the information required to describe entirely the system in a specific period $i > t$. They correspond respectiveley to the inventory information and the static historical data on the one hand, and the demand forecasts information up to $i$ on the other hand, in particular the parameters $\theta_{k|t}$ for $k \in [\![t, i]\!]$.

In our example, we use the simple linear regression as forecasting method. Therefore, we place ourselves in the case of equations 2.7 as definition for $\theta_{i|t}$:

$$m_i = a_1(i) + a_2(i).i \quad \text{and} \quad \sigma_i^2 = \sigma_{\varepsilon_i}^2.\left(\frac{1}{i} + \frac{(i/2)^2}{\sum_{j=1}^{i-1}(j - i/2)^2}\right)$$

where $a_1(i)$ and $a_2(i)$ are the multiplying factors of the regression corresponding respectively to the coefficients for the constant vector equal to 1, and the trend vector, both of size $i$. We omit the $''(i)''$ for the following equations, and use only as notation $a_1$ and $a_2$ to make the reading easier.
As for $\sigma_{\varepsilon_i}$, it is equal to $\sigma_{\varepsilon_i}^2 = \frac{1}{i-4}\left(\sum_{k=1}^{t-1}(d_k - a_1 - a_2k)^2 + \sum_{k=t}^{i-1}\left(D_{k|t} - a_1 - a_2k\right)^2\right)$.
The least square method give us the following relations allowing us to define $\theta_{i|t}$ in details:

$$\begin{cases} a_2 = \alpha_1^i\left(\sum_{k=1}^{t-1} d_k + \sum_{k=t}^{i-1} D_{k|t}\right) + \alpha_2^i\left(\sum_{k=1}^{t-1} kd_k + \sum_{k=t}^{i-1} kD_k\right) \\ a_1 = \frac{1}{i-1}\left(\sum_{k=1}^{t-1} d_k + \sum_{k=t}^{i-1} D_k\right) - a_2\frac{i}{2} \end{cases} \quad \text{where} \quad \begin{cases} \alpha_1^i = \frac{-6}{(i-1)(i-2)} \\ \alpha_2^i = \frac{12}{i(i-1)(i-2)} \end{cases}$$

Advanced states of period $i$ is consequently defined with the detailed description of any $m_k$, $k \in [\![t, i]\!]$, as previously right above. The historical data $(d_i)_{1 \leqslant i < t}$ and the inventory level $X_k$, being the basic state, complete the whole definition of any state.

### 2.1.3 | Law of Motion

The law of motion or dynamic of the states of the system is the dynamic equation making the link between the state of the system in any period $i$ and the state of the following period $i+1$ explicit.

As defined in the previous section, we are consequently interested here in defining expressly the equation 3.3, and more specifically the function $\mathcal{G}_{i+1|t}(\Delta_{i|t})$. It enables us to characterize the advanced state in $i+1$ based on the information of period $i$, *ie* up to $i-1$. This makes the forecast update mechanism explicit inasmuch as we determine here the dynamic relation between two forecasts calculated at two following forecast dates in the rolling process.

As proved in chapter 2, for the first element of $\theta_{i+1|t}$, $m_{i+1}$, we have:

$$m_{i+1} = \sum_{k=1}^{t-1} \frac{2(3k - i - 2)}{i(i-1)} d_k + \sum_{k=t}^{i} \frac{2(3k - i - 2)}{i(i-1)} D_{k|t} \tag{3.7}$$

or, with the notation defined in chapter 2: $\beta^k_{m_{i+1}} = \frac{2(3k-i-2)}{i(i-1)}$.

Thanks to this explicit definition of the coefficient $\beta^k_{m_{i+1}}$, we can express the relation between variables conditional to which $\theta_{i+1|t}$ is calculated, which are the random variables $D_{k|t}$ for $t \leqslant k \leqslant i$. Up to period $i-1$, they are included in the advanced state $\Delta_{i|t}$. The law of motion for $m_{i+1}$ is therefore completely known at this stage.

As far as the uncertainty parameter $\sigma_i$, it is considered in this example as a static parameter. It means the series $(\sigma_i)_{t+1\leqslant i\leqslant t+N-1}$ can be defined entirely at the beginning of period $t$, without any dynamic computation conditionally to variables $D_{k|t}$, $t \leqslant k < i$. It can be interpreted as the fact that the uncertainties are not considered in this example to be dependent on the forecast updates that are proceeded during the rolling horizon framework.

In this context, the definition of the coefficients $\beta^k_{m_i}$ are therefore sufficient to know entirely the dynamics of the forecast update, and consequently to be able to explicitly describe the law of motion of the system.

As a conclusion, the dynamic link between states from $i$ to $i+1$ is entirely defined thanks to the classical inventory update equation 3.3, and the forecasts parameter evolution 3.7

## 2.2 | Optimal Solutions Computation

### 2.2.1 | Analytical Calculations

**Theorem 2.2.1-1.** *Under the assumptions summed-up in section 1, the optimal solutions of our problem are:*
*when* $B \geqslant H + 2p$*:*

$$\begin{cases} \forall i \mid t \leqslant i < t+N-1,\ c^*_i = 2s_i\left(1-\sqrt{\dfrac{2H}{H+B}}\right) + m_i - X_i \\ c^*_{t+N-1} = 2s_{t+N-1}\left(1-\sqrt{\dfrac{2(H+p)}{H+B}}\right) + m_{t+N-1} - X_{t+N-1} \end{cases}$$

*Otherwise:*

$$\begin{cases} \forall i \mid t \leqslant i < t+N-1,\ c_i^* = 2s_i\left(1-\sqrt{\dfrac{2H}{H+B}}\right) + m_i - X_i \\ \qquad\qquad c_{t+N-1}^* = 2s_{t+N-1}\left(\sqrt{\dfrac{2(B-p)}{H+B}}-1\right) + m_{t+N-1} - X_{t+N-1} \end{cases}$$

The sign of $B-H-2p$ only impacts the last production order as side effect. When the backlog costs are below $H+2p$, it is more valuable to produce less than expected net requirements $m_{t+N-1} - X_{t+N-1}$, whereas high backlog costs force the system to make the last period excess stock in expectation: $c_{t+N-1}^*$ is indeed greater than expected net requirements in that case.

*Proof.* We only present here the proof for the first case in which $B-2H-p \geqslant 0$. The proof corresponding to the other case follows a similar reasoning and computational framework.

**i. Proof Principle**

We use here a proof by induction using the model developped in the previous section 2. In fact, we show recursively that the value functions $V_i$, $\forall i,\ t \leqslant i \leqslant t+N-1$ follows a general form with respect to the states in period $i$ and $D_{i|t}$ thanks to the dynamic equation 3. This specific shape allows us to compute its corresponding argument of the minimum $c_i^*$, solutions of our problem.

To do so, we first demonstrate the inductive step, by illustrating how the value function respects a special property from $i$ to $i-1$ thanks to the integration property of section 3, the dynamic equation of the system, and the laws of motion defined previously. Second, we show the last period follows the same property, which defines the base case of our demonstration.

**ii. Expectation and Minimum Computation: the Dynamic Equation of the Value Function for any $i$**

Based on theorem 1.2.3 - 6, we prove here that if, for any $t < i \leqslant t+N-1$, $V_i$ respects a special form, $V_{i-1}$ also follows this particular property. The calculation to which this result comes from is a two-step calculation process. It consists of a minimization problem that allows us to compute $c_i^*$ in this specific case. Then, the laws of motion are applied, which makes the relations be explicitly expressed through $(i-1)^{th}$ period.

In particular, we assume $V_i$ is a linear function with respect to the variables defining the state in the $i-1^{th}$ period, to $D_{i-1|t}$ and to $c_{i-1}$ as follows:

$$V_i\left(X_{i-1}, c_{i-1}, D_{i-1|t}, \Delta_{i-2|t}\right) = \gamma^i_{D_{i-1}} D_{i-1|t} - (c_{i-1} + X_{i-1})p + \sum_{k=t}^{i-2} \gamma^i_{D_k} D_{k|t} + \Gamma_i$$

where $\gamma^i_Y$ are fixed coefficients in $\mathbb{R}$, corresponding to the multiplying factor for any variable $Y$ in the definition of $V_i$, and $\Gamma_i$ a real corresponding to the intercept for $V_i$.

In order to calculate $V_{i-1}$ in this context, we use theorem 1.2.3 - 6: first, we integrate $V_i$ with respect to $D_{i-1|t}$ conditionally to the $(i-2)^{th}$ state, we then add the expected cost of period $i-1$ with respect to the $(i-2)^{th}$ state, and finally we compute its argument of the minimum $c^*_{i-1}$.

Firstofall, the integration of $V_i$ gives us:

$$\mathbb{E}_{i-2}\left[V_i\left(X_{i-1}, c_{i-1}, D_{i-1|t}, \Delta_{i-2|t}\right)\right] = \gamma^i_{D_{i-1}} m_{i-1} - (c_{i-1} + X_{i-1})p + \sum_{k=t}^{i-2} \gamma^i_{D_k} D_{k|t} + \Gamma_i$$

Second, we need to add $\mathbb{E}_{i-2}\left[K_{i-1}\left(X_{i-1}, D_{i-1|t}, c_{i-1}\right)\right]$. By definition of $K_{i-1}\left(X_{i-1}, D_{i-1|t}, c_{i-1}\right)$, we can compute its expected value with respect to random variable $D_{i-1|t}$ conditionally to the $i-2^{th}$ state. Thus, by integrating it with respect to its density function $\phi_{i-1|t}$, we find $\mathbb{E}_{i-2}\left[K_{i-1}\left(X_{i-1}, D_{i-1|t}, c_{i-1}\right)\right]$ as a function of $X_{i-1}$, $c_{i-1}$, and the advanced information $\theta_{i-1|t}$ of the state of period $i-1$ -$m_{i-1}$ and $s_{i-1}$:

$$\mathbb{E}_{i-2}\left[K_{i-1}\left(X_{i-1}, D_{i-1|t}, c_{i-1}\right)\right] =$$
$$\begin{cases} -B(X_{i-1} + c_{i-1} - m_{i-1}) + pc_{i-1} & if\ X_{i-1} + c_{i-1} < m_{i-1} - 2s_{i-1} \\ -B(X_{i-1} + c_{i-1} - m_{i-1}) + pc_{i-1} & \\ \quad + \frac{H+B}{24s^2_{i-1}}(X_{i-1} + c_{i-1} - m_{i-1} + 2s_{i-1})^3 & if\ m_{i-1} - 2s_{i-1} \le X_{i-1} + c_{i-1} < m_{i-1} \\ -B(X_{i-1} + c_{i-1} - m_{i-1}) + pc_{i-1} & \\ \quad + \frac{H+B}{24s^2_{i-1}}(X_{i-1} + c_{i-1} - m_{i-1} + 2s_{i-1})^3 & if\ m_{i-1} \le X_{i-1} + c_{i-1} < m_{i-1} + 2s_{i-1} \\ H(X_{i-1} + c_{i-1} - m_{i-1}) + pc_{i-1} & if\ 2s_{i-1} + m_{i-1} \le X_{i-1} + c_{i-1} \end{cases}$$

Finally, we need to solve the minimization problem
$\min_{c_{i-1}} \mathbb{E}_{i-2}\left[K_{i-1}\left(X_{i-1}, D_{i-1|t}, c_{i-1}\right) + V_i\left(X_{i-1}, c_{i-1}, D_{i-1|t}, \Delta_{i-2|t}\right)\right]$:
The function $c_{i-1} \to \mathbb{E}_{i-2}\left[K_{i-1}\left(X_{i-1}, D_{i-1|t}, c_{i-1}\right)\right]$ is easily identifiable as double differentiable for all $c_{i-1} \in \mathbb{R}$. Moreover, as shown in the following equation 3.8,

we have $\forall c_{i-1} \in \mathbb{R},\ \frac{\partial^2}{\partial c_{i-1}^2}\mathbb{E}_{i-2}\left[K_{i-1}\left(X_{i-1}, D_{i-1|t}, c_{i-1}\right)\right] \geq 0$. Knowing its first-order partial derivative is equal to $-B+p$ when $c_{i-1} = m_{i-1} - 2s_{i-1} + X_{i-1}$ and to $H+p$ when $c_{i-1} = m_{i-1} + 2s_{i-1} + X_{i-1}$, we can establish the convexity of the function $\mathbb{E}_{i-2}\left[K_{i-1}\left(X_{i-1}, D_{i-1|t}, c_{i-1}\right)\right]$ with respect to $c_{i-1}$ inasmuch as $B > p$.

$$\frac{\partial^2}{\partial c_{i-1}^2}\mathbb{E}_{N-1}\left[K_{i-1}\left(X_{i-1}, D_{i-1|t}, c_{i-1}\right)\right] =$$
$$\begin{cases} 0 & if\ X_{i-1}+c_{i-1} < m_{i-1} - 2s_{i-1} \\ \frac{H+B}{4s_{i-1}^2}(X_{i-1}+c_{i-1}-m_{i-1}+2s_{i-1}) & if\ m_{i-1}-2s_{i-1} \leq X_{i-1}+c_{i-1} < m_{i-1} \\ -\frac{H+B}{4s_{i-1}^2}(X_{i-1}+c_{i-1}-m_{i-1}-2s_{i-1}) & if\ m_{i-1} \leq X_{i-1}+c_{i-1} < m_{i-1}+2s_{i-1} \\ 0 & if\ 2s_{i-1}+m_{i-1} \leq X_{i-1}+c_{i-1} \end{cases} \tag{3.8}$$

Moreover, we know $c_{i-1} \to \mathbb{E}_{i-2}\left[V_i\left(X_{i-1}, c_{i-1}, D_{i-1|t}, \Delta_{i-2|t}\right)\right]$ is linear. Therefore, the minimum of the function
$c_{i-1} \to \mathbb{E}_{i-2}\left[K_{i-1}\left(X_{i-1}, D_{i-1|t}, c_{i-1}\right) + V_i\left(X_{i-1}, c_{i-1}, D_{i-1|t}, \Delta_{i-2|t}\right)\right]$ exists. We compute its argument of the minimum using derivative computation with respect to our variable of interest: the equation 3.9
$\frac{\partial}{\partial c_{i-1}}\mathbb{E}_{i-2}\left[K_{i-1}\left(X_{i-1}, D_{i-1|t}, c_{i-1}\right) + V_i\left(X_{i-1}, c_{i-1}, D_{i-1|t}, \Delta_{i-2|t}\right)\right] = 0$ has a solution in $c_{i-1}$:

$$\frac{\partial}{\partial c_{i-1}}\mathbb{E}_{i-2}\left[K_{i-1}\left(X_{i-1}, D_{i-1|t}, c_{i-1}\right) + V_i\left(X_{i-1}, c_{i-1}, D_{i-1|t}, \Delta_{i-2|t}\right)\right] = 0$$
$$\Leftrightarrow \begin{cases} -B & if\ X_{i-1}+c_{i-1} < m_{i-1} - 2s_{i-1} \\ \frac{1}{2}(H-B) + \frac{H+B}{2s_{i-1}}(X_{i-1}+c_{i-1}-m_{i-1}) + \frac{H+B}{8s_{i-1}^2}(X_{i-1}+c_{i-1}-m_{i-1})^2 & if\ m_{i-1}-2s_{i-1} \leq X_{i-1}+c_{i-1} < m_{i-1} \\ \frac{1}{2}(H-B) + \frac{H+B}{2s_{i-1}}(X_{i-1}+c_{i-1}-m_{i-1}) - \frac{H+B}{8s_{i-1}^2}(X_{i-1}+c_{i-1}-m_{i-1})^2 & if\ m_{i-1} \leq X_{i-1}+c_{i-1} < m_{i-1}+2s_{i-1} \\ H & if\ 2s_{i-1}+m_{i-1} \leq X_{i-1}+c_{i-1} \end{cases} = 0 \tag{3.9}$$

According to the sign of the quantity $B-H$, the solution of 3.9 belongs either to $[m_{i-1} - X_{i-1} - 2s_{i-1}, m_{i-1} - X_{i-1}]$ or $[m_{i-1} - X_{i-1},\ \ m_{i-1} - X_{i-1} + 2s_{i-1}]$. When

specifically $B - H \geq 0$, the solution appears in $[m_{i-1} - X_{i-1}, m_{i-1} - X_{i-1} + 2s_{i-1}]$, and is consequently the solution of the quadratic equation in $c_{i-1}$: $\frac{1}{2}(H - B) + \frac{H+B}{2s_{i-1}}(X_{i-1} + c_{i-1} - m_{i-1}) - \frac{H+B}{8s_{i-1}^2}(X_{i-1} + c_{i-1} - m_{i-1})^2 = 0$. It appears to be equal to

$$c_{i-1}^* = m_{i-1} - X_{i-1} + 2s_{i-1}\left(1 - \sqrt{\frac{2H}{H+B}}\right)$$

.

In this special case, we can consequently finally express the value function of period $i-1$:

$$V_{i-1}\left(X_{i-1}, \Delta_{i-2|t}\right) = \min_{c_{i-1}}\left\{\mathbb{E}_{i-2}\Big[\ K_{i-1}\left(X_{i-1}, D_{i-1|t}, c_{i-1}\right) + V_i\left(X_{i-1}, c_{i-1}, D_{i-1|t}, \Delta_{i-2|t}\right)\Big]\right\}$$

$$\Leftrightarrow \begin{cases} V_{i-1}\left(X_{i-1}, \Delta_{i-2|t}\right) = 2Hs_{i-1}\left(1 - \frac{2}{3}\sqrt{\frac{2H}{H+B}}\right) - pX_{i-1} + \gamma_{D_{i-1}}^i m_{i-1} \\ \qquad\qquad + \sum_{k=t}^{i-2}\gamma_{D_k}^i D_{k|t} + \Gamma_i \\ c_{i-1}^* = 2s_{i-1}\left(1 - \sqrt{\frac{2H}{H+B}}\right) + m_{i-1} - X_{i-1} \end{cases}$$

By finally applying the laws of motion of paragraph 3, we express $V_{i-1}$ with respect to the random variable $D_{i-2|t}$ and the $i-2^{th}$ state variables. The result is a linear function such that:

$$V_{i-1}\left(X_{i-2}, \Delta_{i-2|t}\right) = \gamma_{D_{i-2}}^{i-1} D_{i-2|t} - (c_{i-2} + X_{i-2})p + \sum_{k=t}^{i-3}\gamma_{D_k}^{i-1} D_{k|t} + \Gamma_{i-1}$$

where

$$\gamma_{D_{i-2}}^{i-1} = \left(\gamma_{D_{i-1}}^i \beta_{m_{i-1}}^{i-2} + \gamma_{D_{i-2}}^i + p\right)$$

$$\forall k, t < k < N,\ \gamma_{D_k}^{i-1} = \gamma_{D_k}^i + \frac{2(3k-i)}{(i-2)(i-3)}$$

$$\text{and } \Gamma_{i-1} = 2s_{i-1}H\left(1 - \frac{2}{3}\sqrt{\frac{2H}{H+B}}\right) + \Gamma_i + \frac{2\gamma_{D_{i-1}}^i}{(i-2)(i-3)}\left(\sum_{j=1}^{t-1}(3j-i)d_i\right)$$

We just proved the inductive step of our proof of theorem 2.2.1 - 1 $\forall i$ such as

$t < i \leqslant t + N - 1$:

**Property 2.2.1 - 2.** *If $V_i$ is a linear function with respect to the variables of the $i - 1^{th}$ state, to $D_{i-1|t}$ and to $c_{i-1}$, then $c^*_{i-1} = 2s_{i-1}\left(1 - \sqrt{\frac{2H}{H+B}}\right) + m_{i-1} - X_{i-1}$, and $V_{i-1}$ is a linear function with respect to the variables of the $i - 2^{th}$ state, to $D_{i-2|t}$ and to $c_{i-2}$*

### iii. Calculation of the Base Case

We demonstrate here that the last value function of our planning problem $V_{t+N-1}$ follows the property previously announced.

By proceeding in the same manner as previously, the proof described here follows 3 steps of calculation: first, the integration of $K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-1}\right)$ conditional to the state of period $t+N-2$ with respect to $\phi_{t+N-1|t}$. Then, the minimization problem of the function that interests us with respect to $c_{t+N-1}$. Finally, the transformation of the resulted value function into an expression with respect to the variables describing the state of $t + N - 3$, $c_{t+N-2}$ and $D_{t+N-2|t}$ thanks to the laws of motion.

We first express $\mathbb{E}_{t+N-2}\left[K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-1}\right)\right]$ as a function of $X_{t+N-1}$, $c_{t+N-1}$, and the advanced information $\theta_{t+N-1|t}$ of the state of period $t + N - 1$: $m_{t+N-1}$ and $s_{t+N-1}$. To do so, we use the integration with respect to the density function $\phi_{t+N-1|t}$:

$$\mathbb{E}_{t+N-2}\left[K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-1}\right)\right] =$$

$$\begin{cases} -B(X_{t+N-1} + c_{t+N-1} - m_{t+N-1}) + pc_{t+N-1} & \text{in } case\#1 \\ -B(X_{t+N-1} + c_{t+N-1} - m_{t+N-1}) + pc_{t+N-1} \\ \quad + \frac{H+B}{24s^2_{t+N-1}}(X_{t+N-1} + c_{t+N-1} - m_{t+N-1} + 2s_{t+N-1})^3 & \text{in } case\#2 \\ -B(X_{t+N-1} + c_{t+N-1} - m_{t+N-1}) + pc_{t+N-1} \\ \quad + \frac{H+B}{24s^2_{t+N-1}}(X_{t+N-1} + c_{t+N-1} - m_{t+N-1} + 2s_{t+N-1})^3 & \text{in } case\#3 \\ H(X_{t+N-1} + c_{t+N-1} - m_{t+N-1}) + pc_{t+N-1} & \text{in } case\#4 \end{cases}$$

$$\begin{aligned} case\#1 &\Leftrightarrow \quad if\ X_{t+N-1} + c_{t+N-1} < m_{t+N-1} - 2s_{t+N-1} \\ case\#2 &\Leftrightarrow \quad if\ m_{t+N-1} - 2s_{t+N-1} \leq X_{t+N-1} + c_{t+N-1} < m_{t+N-1} \\ case\#3 &\Leftrightarrow \quad if\ m_{t+N-1} \leq X_{t+N-1} + c_{t+N-1} < m_{t+N-1} + 2s_{t+N-1} \\ case\#4 &\Leftrightarrow \quad if\ 2s_{t+N-1} + m_{t+N-1} \leq X_{t+N-1} + c_{t+N-1} \end{aligned}$$

We then go to the second step of the computation process, which is the argument

of the minimum computation. Like the case we had $\forall i$, the function $c_{t+N-1} \rightarrow \mathbb{E}_{t+N-2}\left[K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-1}\right)\right]$ is double differentiable for all $c_{t+N-1} \in \mathbb{R}$, and $\frac{\partial^2}{\partial c_{t+N-1}^2}\mathbb{E}_{t+N-2}\left[K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-1}\right)\right] \geq 0$, $\forall c_{t+N-1} \in \mathbb{R}$. Knowing moreover its first-order partial derivative is equal to $-B+p$ when $c_{t+N-1} = m_{t+N-1} - 2s_{t+N-1} + X_{t+N-1}$ and to $H+p$ when $c_{t+N-1} = m_{t+N-1} + 2s_{t+N-1} + X_{t+N-1}$, $c_{t+N-1} \rightarrow \mathbb{E}_{t+N-2}\left[K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-1}\right)\right]$ is convex as long as $B > p$.

Then, the following equation 3.10: $\frac{\partial}{\partial c_{t+N-1}}\mathbb{E}_{t+N-2}\left[K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-1}\right)\right] = 0$ has one solution in $c_{t+N-1}$, which is the minimum we are looking for.

$$\frac{\partial}{\partial c_{t+N-1}}\mathbb{E}_{t+N-2}\left[K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-1}\right)\right] = 0$$

$$\Leftrightarrow \begin{cases} -B+p & \text{in } case\#1 \\ \frac{1}{2}(H-B+2p) & \\ +\frac{H+B}{2s_{t+N-1}}(X_{t+N-1}+c_{t+N-1}-m_{t+N-1}) & \text{in } case\#2 \\ +\frac{H+B}{8s_{t+N-1}^2}(X_{t+N-1}+c_{t+N-1}-m_{t+N-1})^2 & \\ \frac{1}{2}(H-B+2p) & \\ +\frac{H+B}{2s_{t+N-1}}(X_{t+N-1}+c_{t+N-1}-m_{t+N-1}) & \text{in } case\#3 \\ -\frac{H+B}{8s_{t+N-1}^2}(X_{t+N-1}+c_{t+N-1}-m_{t+N-1})^2 & \\ H+p & \text{in } case\#4 \end{cases} = 0 \qquad (3.10)$$

According to the sign of the quantity $B-(H+2p)$, the solution of 3.10 belongs either to the interval $[m_{t+N-1} - X_{t+N-1} - 2s_{t+N-1}, m_{t+N-1} - X_{t+N-1}]$ or $[m_{t+N-1} - X_{t+N-1}, m_{t+N-1} - X_{t+N-1} + 2s_{t+N-1}]$. When specifically, as admitted in this proof, $B-(H+2p) \geq 0$, the solution appears in $[m_{t+N-1} - X_{t+N-1}, m_{t+N-1} - X_{t+N-1} + 2s_{t+N-1}]$. The corresponding quadratic equation has one minimum on this interval, which is

$$c_{t+N-1}^* = 2s_{t+N-1}\left(1 - \sqrt{\frac{2(H+p)}{H+B}}\right) + m_{t+N-1} - X_{t+N-1}$$

.

We can therefore expressed the value function for $i = t+N-1$:

$$\begin{aligned} V_{t+N-1}(X_{t+N-1}, \Delta_{t+N-2|t}) =& p\left(m_{t+N-1} - X_{t+N-1}\right) \\ &+ 2s_{t+N-1}\left(H+p\right)\left(1 - \frac{2}{3}\sqrt{\frac{2(H+p)}{H+B}}\right) \end{aligned}$$

This expression of the value function is true for all values of $c_{t+N-1}^*$ with respect

to $X_{t+N-1}$. It includes especially values such that $c^*_{t+N-1} \leq 0$, that don't exist concretely inasmuch as a negative order has no sense in practice. Such cases arise when $X_{t+N-1} \geq 2s_{t+N-1}(1 - \sqrt{\frac{2(H+p)}{H+B}}) + m_{t+N-1}$. For the following, we make the assumption that we do not face such cases.

To lighten the notation, we note $\Gamma_{t+N-1}$ the part of $V_{t+N-1}$ that does not depend on the $t+N-2^{th}$ state: $\Gamma_{t+N-1} = 2s_{t+N-1}(H+p)(1 - \frac{2}{3}\sqrt{\frac{2(H+p)}{H+B}})$. This makes finally $V_{t+N-1}$ be equal to

$$V_{t+N-1}(X_{t+N-1}, \Delta_{t+N-2|t}) = \Gamma_{t+N-1} + p(m_{t+N-1} - X_{t+N-1})$$

.

The last step of our process of calculation of the value function is the expression of the function with respect to the $t+N-2^{th}$ state variables, and $D_{t+N-2|t}$. Using the laws of motion 3. We have indeed:

$$\begin{aligned} V_{t+N-1}\left(X_{t+N-2}, c_{t+N-2}, D_{t+N-2|t}, \Delta_{t+N-3|t}\right) = \\ \left(\left(\beta^{t+N-2}_{m_{t+N-1}} + 1\right) D_{t+N-2} - c_{t+N-2} - X_{t+N-2} + \sum_{k=t}^{t+N-3} \beta^k_{m_{t+N-1}} D_{k|t}\right) p \\ + \Gamma_{t+N-1} \end{aligned}$$

**iv. Conclusion**

By defining

$$\begin{aligned} \gamma^{t+N-1}_{D_{t+N-2}} &= \left(\beta^{t+N-2}_{m_{t+N-1}} + 1\right) p \\ \gamma^{t+N-1}_{D_k} &= \beta^k_{m_{t+N-1}} \end{aligned}$$

we can confirm the specific assumption we made at the beginning of the inductive step of the recursive proof is true for $i = t+N-1$.

Therefore, it is, by induction, true $\forall i$ such as $t \leqslant i \leqslant t+N-1$

□

### 2.2.2 | Interpretation

The optimal series of production quantities $(c^*_i)_{t \leqslant i \leqslant t+N-1}$ calculated here corresponds to the optimal production plan for the periods from $t$ to $t+N-1$ minimizing the backlog/inventory cost and unit production cost in the case of a production lead-time equal to zero, *ie* the production quantities are received during the period they

have been ordered. It takes into account the forecast update dynamics of the case of a simple linear regression forecasting procedure. It is indeed integrated into the dynamic calculation of the nested form of the value function the law of motion of the system, making the update of the forecasts explicit through the consideration of the stochastic demand up to the considered period during the dynamic programming computation process.

## 2.3 | Numerical Study

In this section, we show the benefits of our modeling and computing framework by illustrating numerically the results presented above and comparing them with other classical production planning methods in the context that interests us, namely the stochastic and rolling plan framework environment.

To do so, we run several simulations of a production system in a rolling horizon framework and analyse the absolute and relative costs of each method. We compare these numerical results in a first part, before analysing more precisely in a second part the evolution of the solution performances with respect to different industrial parameters.

### 2.3.1 | Numerical Assumptions

We assume here we have a planning horizon $N$ of length equal to 4 periods. The historical data are available for $t = 17$ periods. The production plan needed to be calculated is consequently $(c_i)$ for $i \in \{17, 18, 19, 20\}$. At first, We assumed arbitrarily $p = 1$, $B = 6$, $H = 2$, and the following initial state:
$(d_i)_{i=1..16} = (48, 52, 57, 59, 60, 55, 64, 72, 58, 71, 74, 82, 73, 80, 71, 78)$, $x_{16} = 20$, and $(s_i)_{i=17..20} = (5, 6, 7, 8)$. We chose arbitrarily the case $B \geq H + 2p$ as illustration. The behaviors of the results stay similar otherwise.

### 2.3.2 | Methods of computation

In order to compute the production orders, we use our method presented above and compare it to the following standard stochastic net requirements calculation procedures: at each period $j$, the corresponding production orders are equal to $m_j + \Phi_{tr}^{-1}(\tau).\sigma_j - x_{j-1}$. $\Phi_{tr}$ represents the standard cumulative density function of the triangular distribution and $\tau$ the service level targeted.

This quantity of production order represents basically the classical expected net requirements, to which we added a safety stock quantified according to the service level targeted $\tau$ and the uncertainties faced during the corresponding period $\sigma_j$.

The simulation is run such that, at each iteration $i$ of the rolling-horizon, we generate $M = 20$ different random samples $d_i$, which makes the total number of random scenarios $(d_{17}, d_{18}, d_{19}, d_{20})$ equal to $20^4 = 160000$. For each of them, we calculate the total costs of each method to assess, and the inventory level at the end of any period from $t = 17$ to $t + N - 1 = 20$. The averaged results are presented in the following table 3.3.

### 2.3.3 | Results

Table 3.3: Simulation Results: Averaged Costs and Inventory

| **Methods used** | **Average total cost** | **Average inventory end of period** | | | |
|---|---|---|---|---|---|
| | | 17 | 18 | 19 | 20 |
| Our method | **380.43** | 2.67 | 3.53 | 4.10 | 2.14 |
| Standard Orders Calculation with $\tau =$ | | | | | |
| 50% | 391.54 | -0.26 | 0.01 | 0.00 | 0.00 |
| 52% | 389.26 | 0.18 | 0.34 | 0.38 | 0.41 |
| 54% | 386.90 | 0.42 | 0.59 | 0.74 | 0.79 |
| 56% | 386.09 | 0.61 | 1.01 | 1.06 | 1.23 |
| 58% | 384.36 | 0.86 | 1.32 | 1.33 | 1.59 |
| 60% | 383.51 | 1.01 | 1.53 | 1.78 | 2.02 |
| 62% | 382.94 | 1.27 | 1.84 | 2.14 | 2.44 |
| 64% | 382.25 | 1.54 | 2.16 | 2.51 | 2.86 |
| 66% | 381.99 | 1.81 | 2.49 | 2.89 | 3.29 |
| 68% | 381.81 | 2.08 | 2.82 | 3.27 | 3.73 |
| 70% | 381.56 | 2.37 | 3.16 | 3.67 | 4.19 |
| 72% | 381.99 | 2.65 | 3.50 | 4.07 | 4.65 |
| 74% | 381.96 | 2.95 | 3.86 | 4.49 | 5.13 |
| 76% | 383.14 | 3.27 | 4.24 | 4.93 | 5.63 |
| 78% | 384.47 | 3.60 | 4.64 | 5.39 | 6.16 |
| 80% | 385.72 | 3.95 | 5.06 | 5.89 | 6.73 |
| 82% | 387.48 | 4.32 | 5.50 | 6.41 | 7.32 |
| 84% | 390.00 | 4.72 | 5.99 | 6.97 | 7.97 |
| 86% | 393.35 | 5.15 | 6.51 | 7.58 | 8.66 |
| 88% | 397.24 | 5.63 | 7.08 | 8.24 | 9.42 |
| 90% | 401.35 | 6.15 | 7.70 | 8.97 | 10.25 |

We note that the stochastic dynamic programming approach has in average a lower cost than any standard dynamic safety stock policies. By taking into consideration the potential evolution of the forecasts through the dynamic calculation of the value function, the production orders are indeed more accurate as the iterations go on. It is particularly visible in the last period: the average inventory at the end of the planning horizon is lower than the one corresponding to the stochastic MRP procedures with similar cost. Whereas safety stocks cumulate themselves periods after periods, the stochastic dynamic approach is more adapted.

As shown in figure 3.1, our method allows not only to get the optimal safety stock faced with uncertainties, but also to be more adaptative to the real proceeding of the physical flows when rolling plan moves forward. Taking indeed into consideration the forecast update into the calculation of the optimal quantity to be produced, we have a cost globally better in average that any procedure of optimization of safety stock. The figure 3.1 illustrates the costs of any safety stock quantification with respect with $\tau$ in blue line, and the cost of our method in black.

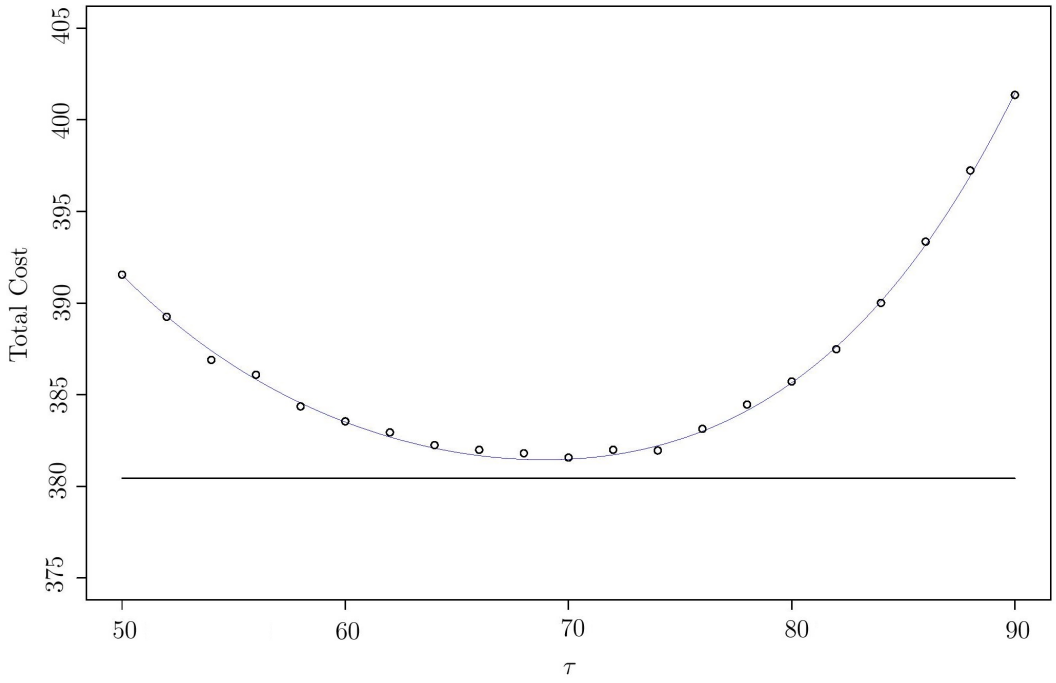



Figure 3.1: Costs Comparison with respect to $\tau$

These results have been calculated in a specific industrial and decisional environment, where potentially impacting parameters have been numerically set arbitrarily. We study this impact in the following section by examining how the results and performances evolve with respect to these parameters. In particular, we focus on the

analyse of the influence of the uncertainties and of the cost parameters.

### 2.3.4 | Sensitivity Analysis

We study then here the results of our dynamic approach in several cases where either the costs or the uncertainties are changing. We analyze specifically the sensitivity of the performance of our method compared with classical dynamic safety stock policies in production planning previously presented. We highlight how the benefits of using our method evolve according to the system parameters.

Our first study considers the variations of the uncertainties of the forecasted demand. In the second one, we analyse how the fluctuations of cost parameters impact the performance of the method.

We calculate specifically as indicator and output the gain between our method and the procedure of optimization of the safety stock corresponding to the $\tau$ the least costly of any of the policies. This benefit is expressed in percentage of cost reduction, probability of stock out reduction, and inventory level reduction. We study in both sections its evolution with respect to the corresponding variable of interest.

The initial state and the parameters of the system considered are assumed to be the same as the previous study: $N = 4$, $t = 17$, $x_{16} = 20$, and $(d_i)_{i=1..16} = (48, 52, 57, 59, 60, 55, 64, 72, 58, 71, 74, 82, 73, 80, 71, 78)$.

#### i. Uncertainties

As previously defined, the uncertainties of the system, modelled through the variable $s_i$, are supposed non-dynamic with respect to the rolling horizon plan, meaning that we know their value from the current forecast date $t$ to the planning horizon $t+N-1$ when planning decision are made at $t$, and that we do not consider them as conditional to the future forecasts during the rolling plan proceedings.

Here, to examine the sensibility of our method to the uncertainties, we made the vector $s = (s_{17}, s_{18}, s_{19}, s_{20})$ vary such that $s_{17} \in [\![1, 60]\!]$, and $s_{i+1} = s_i + 2$ for any $i = 17, 18$ or 19.

For each of the scenario of the study, we calculate two kinds of outputs. First, the total average costs and averaged inventory at the end of periods considered for our method. Then, in a relative way, we compare the cost, inventory level, and probability of stock out reduction between our method and the least costly of any

standard safety stock policies among those whose safety stocks vary from $\tau = 50\%$ to $\tau = 90\%$.

This simulation was made under the following assumptions of costs parameters: $H = 2$, $p = 20$. We chose here the case where $B$ is greater than $H + 2p$. We define it arbitrarily as $B = 1.5 \times (H + 2p) = 63$. While $H \leq B$, results have similar behavior in the other case $B \leq H + 2p$.

Absolute results are shown in figures 3.2 and 3.3. Costs is increasing with respect to uncertainty inasmuch as optimal safety stock represented by the quantity $2s_i \left(1 - \sqrt{\frac{2H}{H+B}}\right)$ is proportional to the uncertainty quantification $s_i$ of period $i$. This linearity property is clearly visible in figure 3.3.

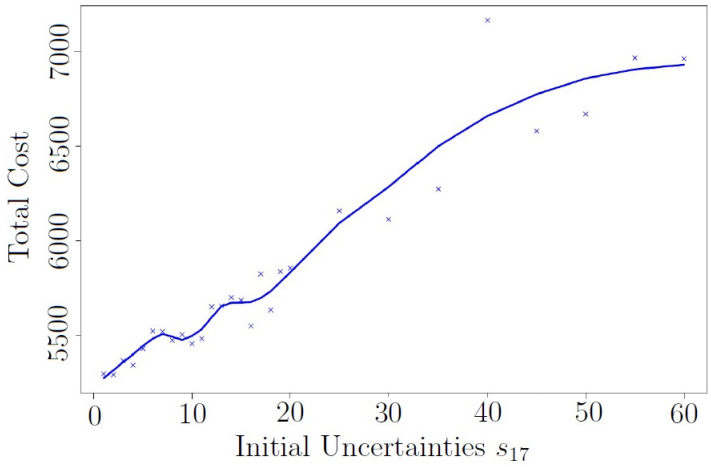



Figure 3.2: Evolution of the Cost of the Stochastic Dynamic Approach with respect to the initial uncertainty

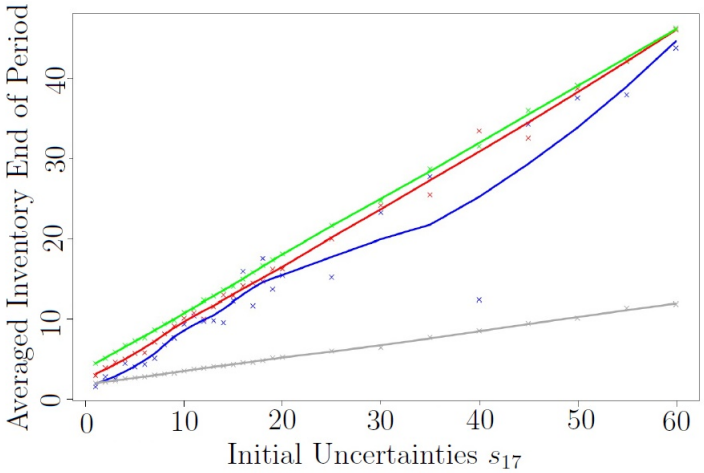



Figure 3.3: Evolution of the averaged inventory at the end of periods with respect to the initial uncertainty

As for the relative results, they are presented in the following figures 3.4, 3.5, and 3.6. Figure 3.4 shows detailled results of the evolution of the cost difference between our method and the best standard safety stock policy in terms of costs among those whose safety stocks vary from $\tau = 50\%$ to $\tau = 90\%$. It reveals the interest of using our method especially in an uncertain context: the benefits of our methods grows in absolute value as the initial uncertainty $s_{17}$.

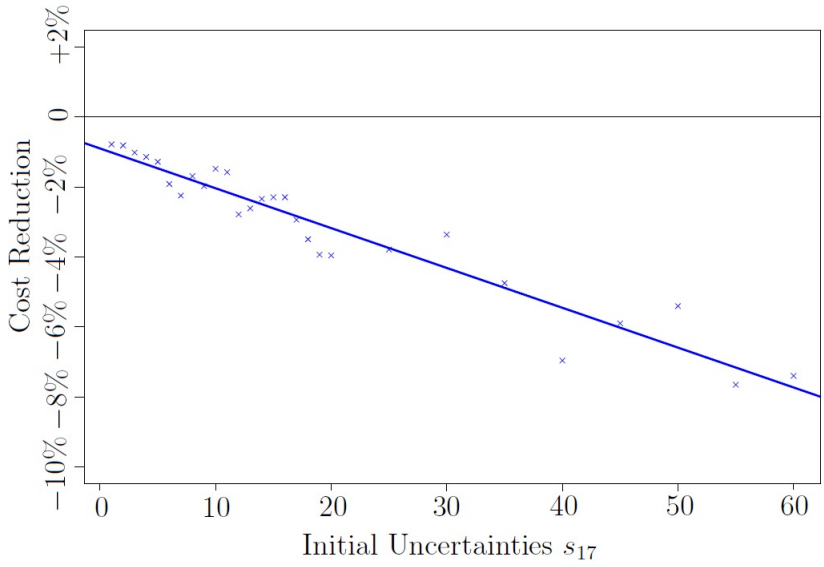



Figure 3.4: Evolution of the Cost reduction with respect to the initial uncertainty

Figures 3.5 and 3.6 show relative results concerning respectively the averaged inventory and the averaged percentage of inventory shortage at the end of periods $17, 18, 19$, and $20$.

The relative difference of inventory level between our method and the best standard safety stock policy is presented in 3.5. The percentage points of the difference of probability of shortages is shown in 3.6.

Both expose a better service level for our method compared with standard ones. We have indeed higher inventory level from periods 17 to 19 for any $s_{17}$ used. As for last period 20, inasmuch as it corresponds to the end of the planning horizon, the optimal solution imposes the production orders to be lower. Since it cannot be used as backlog any further, in order to reduce inventory costs, orders are consequently reduced .

It is also interesting to notice that, in general, the percentage of inventory difference does not evolve when uncertainty grows up. On the contrary, we note a slightly decreasing shape of the inventory level difference. The service level performance of our method is consequently better than any standard procedure, but not increasing.

The difference in percentage points of stock-out is globally better for our method when uncertainty grows. We can see indeed that when $s_{17}$ stays low, the case where there is no difference of service level between those methods is more frequent. This is explained by the fact that when $s_{17}$ is relatively small, both standard procedure and our method manage to get 100% service level, the difference between them being thus equal to zero. This explains the shape of the period 17 curve. However, the results of better service level illustrated by figure 3.5 are clearly confirmed here.

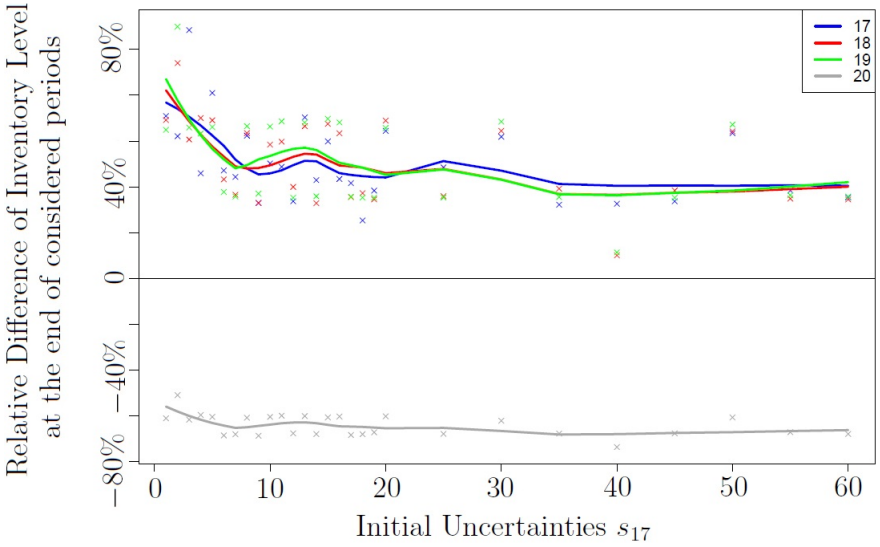



Figure 3.5: Evolution of the relative difference of the averaged inventory level with respect to the initial uncertainty

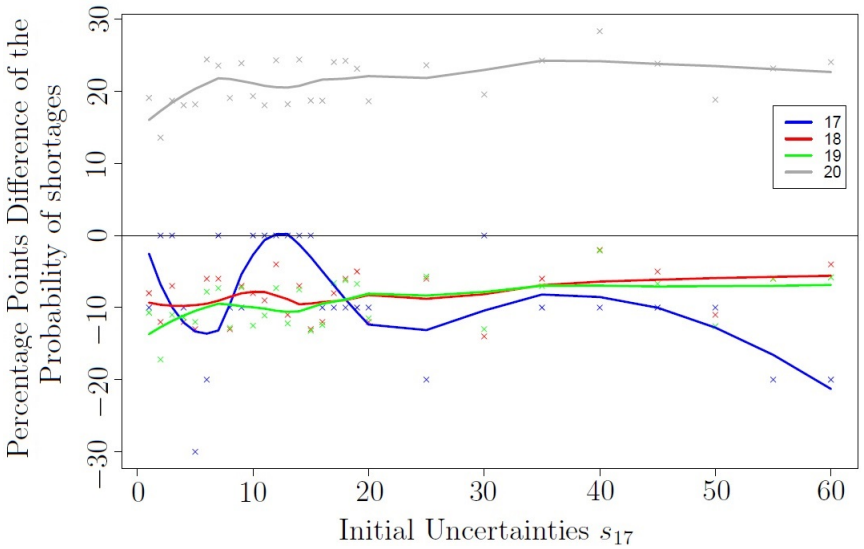



Figure 3.6: Evolution of the difference of percentage points of probability of shortage with respect to the initial uncertainty

To conclude this section, the stochastic dynamic approach that takes into account forecast update has lower averaged cost than the least costly standard policy. We

notice moreover that the more uncertain the system is, the more cost-efficient our method is.

Besides, as explained previously, it has also globally a higher service level for any period, except the last one of the planning horizon. This result is true for any level of random influence, represented by the stochastic parameters $s_{17}$. Little influence of the uncertainties is observed on service-level performances.

**ii. Costs Parameters**

In this section, we study the same outputs as in the previous analyses, with similar assumptions in terms of system definition. However, we define $s_{17} = 10$, and the variables of interest are here the cost parameters $H$, $p$ and $B$. The objective is to study in which industrial context the stochastic dynamic approach is fruitful to use according to the cost structure of the production system considered.

To do so, we analyse the sensitivity of the performance of our method with respect to evolution of ratio $\frac{H}{p}$ on the one hand, and to $B$ on the other hand.

The first case examines consequently the interests of using the stochastic dynamic programming approach according to the value of the product considered relatively to its holding costs.

The second one inspects how evolve the performance of our method when backlog costs are more consequent.

The results are, as previsouly, compared relatively to the least costly of the standard safety stock policy among those whose service level vary from $\tau = 50\%$ to $\tau = 90\%$.

**Ratio $\frac{H}{p}$ :**

$B$ is here set up to 200 in order to insure the backlog costs is always greater than the required threshold.

We make the ratio $\frac{H}{p}$ varies from 0.05 -high value product case- to 1- low value product case or high inventory holding costs case, and calculation for each of these values the total cost reduction, and the evolution of the improvement of service level, through the inventory level difference on the one hand, and the difference of the probability of stock out on the other hand.

Results are shown in figures 3.7, 3.8, and 3.9 with respectively the evolution of the cost reduction, the relative difference of inventory level at the end of the

periods, and the difference of percentage points of probability of shortage between our method and the best safety stock policy among those whose $\tau$ vary from $\tau = 50\%$ to $\tau = 90\%$.

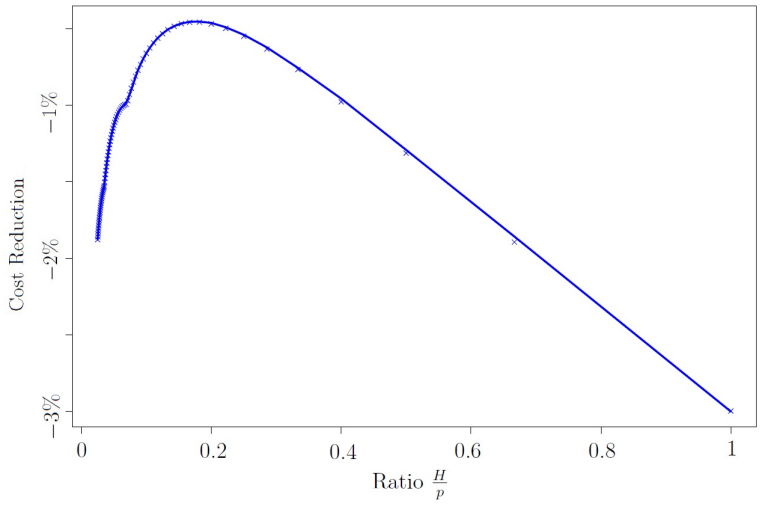



Figure 3.7: Evolution of the Cost reduction with respect to the ratio $\frac{H}{p}$

We observe here that the cost of our method is always lower than the best standard one in figure 3.7 for any value of $\frac{H}{p}$.

The interesting concave shape of the curve shows also that there is a minimum of absolute cost gain corresponding to the passing through the threshold from which inventory holding cost gets non negligible compared with unit production cost.

Beyond this point, in both decreasing or increasing $\frac{H}{p}$, relative stochastic dynamic programming cost improves. As a consequence, the benefits are always positive, and increase as we get further from $\frac{H}{p} = 0.2$.

The threshold point, which is the argument of the maximum of the cost reduction, and from which the value of the per-unit production cost becomes close to the same order of magnitude of the inventory cost, is also visible on the results of the inventory levels and the probability of stock out results.

From this point, the difference of service level between our method and the least costly standard one is indeed relatively small.

Moreover, the evolutions of the difference of inventory level and probability of shortages are expected. We saw indeed in 2.2.1 - 1 that the optimal production orders $c_k^*$ for $k < N$ are independent of $p$. Therefore, its corresponding average inventory level is not changing for periods 17, 18, and 19. The shape of the relative difference presented in figure 3.8 corresponds consequently only to changes of the stochastic standard procedure inventory level used as base of comparison.

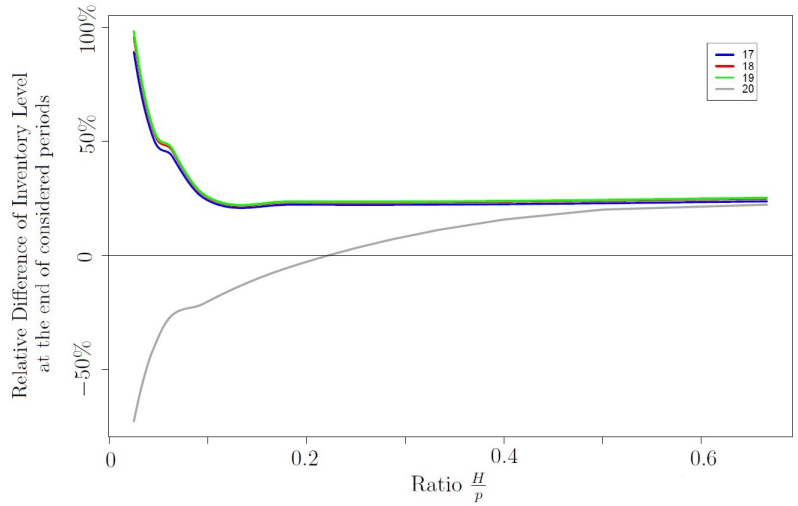



Figure 3.8: Evolution of the the relative difference of the averaged inventory level with respect to the ratio $\frac{H}{p}$

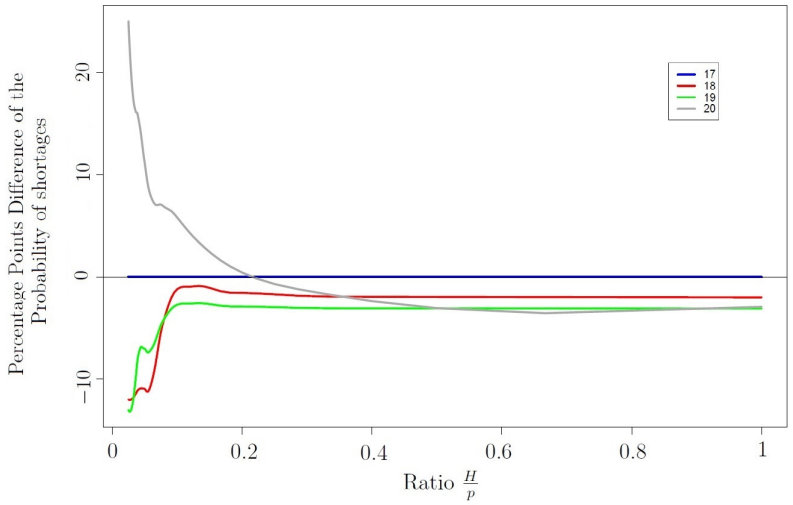



Figure 3.9: Evolution of the difference of percentage points of probability of shortage with respect to the ratio $\frac{H}{p}$

Besides, we note the threshold seen in figure 3.7 appears when inventory level of period 20 becomes greater than the one of the least costly of the standard procedures. As a consequence, from this point, the balance between $H$ and $p$ allows more stock with least cost.

**Ratio $\frac{B}{H}$** :

When high value products are considered, that is to say when $p$ takes great values, it is consistent to believe the backlog costs take also a greater importance inasmuch as delays for costly products has strong impact on the proceedings of further supply chain actions. In that context, we finally analyse a third case study where we make

$B$ increase, along with a proportional growth of $p$, while $H$ remains constant. These analyses are consequently presented with respect to ratio $\frac{B}{H}$, illustrating different areas of some delay policies: high $\frac{B}{H}$ imposes a strict environment where delays are costly, whereas a low ratio allows more flexibility as it is more convenient in terms of cost of delay.

Relative results compared with the least cost standard safety stock policy are presented in the following figures 3.10, 3.11, and 3.12.

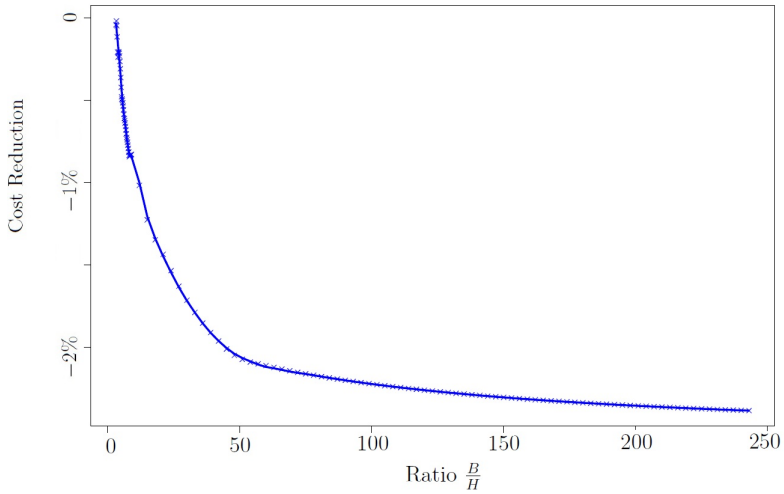



Figure 3.10: Evolution of the Cost reduction with respect to the unit product cost $p$, $B$ being adapted accordingly

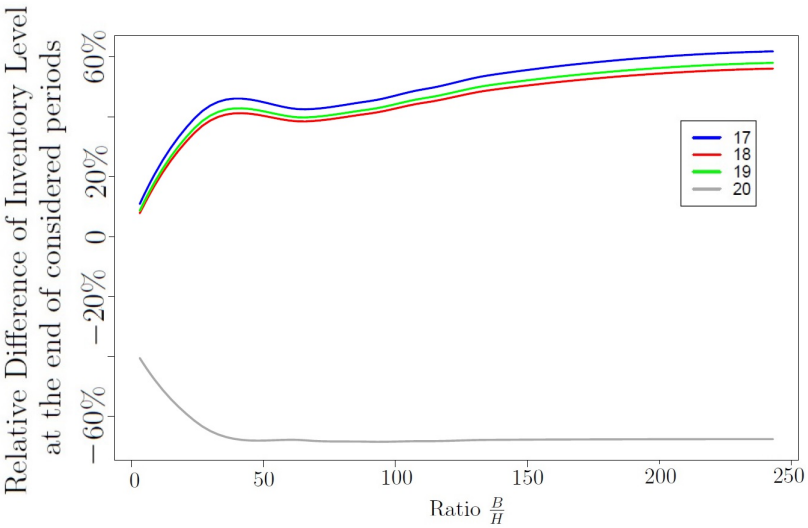



Figure 3.11: Evolution of the the relative difference of the average inventory level with respect to the unit product cost $p$, $B$ being adapted accordingly

Our method still have lower average total costs thanks to the forecast update.

As far as service level and inventory level are concerned, as previously, we get

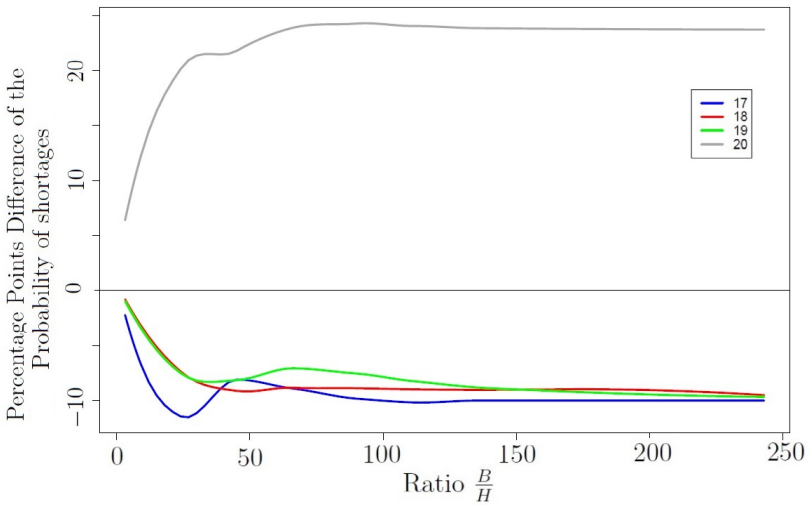



Figure 3.12: Evolution of the difference of percentage points of probability of shortage with respect to the unit product cost $p$, $B$ being adapted accordingly

better service level for any period except the last one since solution adapts itself to the end of the planning period.

However, the shapes of these graphs reveal a property. It is interesting to notice that an asymptotic behavior seems to appear with high values of $B$. That is to say, from a given environment of high backlog costs - $\frac{B}{H}$ being equal to approximatively 70, the increase of these costs does not improve the performance of our method relatively to stochastic MRP procedure: there is a limit of the enhancement of our method with respect to a growing $B$.

### iii. Concluding Remarks for the numerical study

We were able to run several simulations to first illustrate the benefits of our modeling framework - and its corresponding solutions- for the optimization of a stochastic production plan taking into account the forecast update mechanism that occurs during rolling plan scheme. Second, we challenged these benefits with classical planning procedures, and analyse them with respect to the system parameters to understand their influence.

This study highlights several interesting points. Firstofall, our method allows to get better results in terms of costs. Moreover, depending on the system environment in which the method is applied, it can bring more value.

In particular, the more uncertain the system is, the more interesting it is to use our method in terms of average costs and service level.

In terms of costs parameters, the average gain gets bigger when the holding costs increase relatively to production costs -low value products. As for backlog costs, solution performance increase in terms of relative difference compared with stochastic safety stock with respect to $B$, although an upper limit of improvement appears as long as $B$ grows. These results may particularly be useful to define which kind of industries may take advantages of such production planning method.

## 2.4 | Conclusion of our Simple Case Study

We presented in this section a first simple example of our method of production planning taking into account the forecast updates under a rolling plan framework.

To focus our study on the optimization procedure in such a complex situation, this example has been conducted under strong hypotheses. In particular, we chose to work here with a basic industrial case of instantaneous delivery -production and/or delivery lead-time equal to zero-, with a simple linear regression forecasting technique.

These hypotheses however gave us the opportunity to illustrate analytically three important axes of our method: the modelling of the problem through conditional and dynamic variables, the solution-computation procedure, through the nested form of a stochastic dynamic program, and the numerical performances of our method thanks to a numerical simulation.

The modelling part consists mostly in three steps. First, the definition of the states of the system for each period of the planning horizon, second, the dynamic laws that link the states between one another, including the forecast update procedure from one state to the next one, and finally the stochastic dynamic programming procedure allowing us to compute the solutions.

The solution-computation procedure is based on the nested form of the dynamic programming. Thanks to the dynamic equation respected by the value function of the system, the integration properties of the conditional dynamic variables, and the laws of motion, we were able to prove an induction reasoning allowing us to compute analytically an optimal production plan.

Finally, simulations have been conducted to illustrate the performance of the solution and to highlight the benefits of this method. Absolute results have been

presented, and show that taking into account the forecast update into the optimization procedure brings more value in terms of cost reduction than any standard planning policy. Sensitivity analyses have also been performed, showing in particular that the more uncertain the context is on the one hand, and the more high value the product is on the other hand, the more interesting our method is compared with the safety stock optimization policies. Thus, even in a zero lead-time system, integrating forecast update mechanism into production planning appears a value-adding practice for practitioners using rolling-horizon framework.

Two main hypotheses have an important impact on the planning: the short lead-time and the modeling of the forecasts. From this point of view,the results showing that we can express literally the production control quantity minimizing the expected total cost while using modeling of forecast update then open new perspectives. We will see in the next chapter how we can manage such practices in more complex situations: a strictly positive lead-time on the one hand, and a multiple regression on the other hand.

# Chapter 4

# Study cases of Fast Delivery and use of Linear Regression

In this chapter, we aim at applying our method in two different situations. Our goal is indeed here to study our production planning problem in the non-stationary and stochastic environment generated by the forecast update procedure under a rolling plan framework in an analytical way for more complex cases. Extensions of the model presented in the previous chapter will be then presented, as well as their solution methods.

The first section will be dedicated to the development of our stochastic dynamic programming method for a product whose production lead-time is equal to one period. As for the forecasting method used, it will be kept to be a simple linear regression.

As for the second part of the chapter, we will develop the method in an environment of use of the multiple linear regression as forecasting method, in order to generalize to various industrial cases. The model and the solutions computation will indeed be computed in an analytical way so that they could be applied in a large spectrum of potential contexts. Simulations will also be carried out and resulting outcome analyzed in order to understand the solutions encountered in different situations.

# 1 | Model with One-Period Lead Times

In this section, we generalize the problem modelling and solution method presented previously by relaxing one of the restrictive assumption: the instantaneous delivery (or zero lead time) assumption. We consider indeed here the lead-time to be equal to one period, although we still stay in a fast delivery mode.

We explain how the model evolves, and how the solution computation procedure changes in this new environment. Moreover, a new simulation is realized to illustrate the solutions found.

## 1.1 | Problem description and modelling

We still consider an $N$-period production problem in a rolling plan framework. We assume that the production lead-time is equal to 1 period. The objective is kept to be the search of the production plan from the current forecast date $t$ to the last period of the horizon $t+N-1$, including the forecast update procedure of the rolling horizon into the calculation of the optimal planning.

We use the same notation as previously. In this context, we have $\forall i = 1..t+N-1, K_i$ is equal to $-B.(X_i + c_{i-1} - D_i) + p.c_i$ when $X_i + c_{i-1} - D_i < 0$ and to $H.(X_i + c_{i-1} - D_i) + p.c_i$ otherwise. For the last period, $K_{t+N-1}$ is only made up of the holding/backlog cost as $c_{t+N-1}$ is not defined.

### 1.1.1 | Definition of the States of the System

**i. Basic state of the system**

For any period $i$, the basic state of the system is defined through the available inventory at the beginning of period $i$, $X_i$ -or $x_t$ for the specific period $t$-, and the historical data of the demand information, from period 1 to period $t-1$: $,(d_i)_{i=1..t-1}$.

**ii. Advanced state of the system**

As previously defined, random variables $D_{k|t}, t \leq k < i$ are conditionnally taken into the definition of the demand of period $i$. More specifically, the parameters of the pdf of $D_{i|t}$, namely the vector $\theta_{i|t}$, are computed with respect to all information before period $i$: $((d_i)_{1\leq i<t}, (D_{k|t})_{t\leq k<i})$. Consequently, in this specific context of study, this definition does not change compared with the situation previously examined. Thus,

the parameters $\theta_{k|t}$ for $t \leqslant k \leqslant i$ defining $(D_{k|t})_{t \leq k < i}$) are also needed to characterize the state of the system in period $i$, known as the advanced state of the system.

### 1.1.2 | Forecasted Demand Modeling

We use the same modeling framework as in the previous chapter: $m_i$ and $\sigma_i$ are respectively the mean and the standard deviation of the random variable $D_{i|t}$ following a centered triangular distribution between $m_i - 2s_i$ and $m_i + 2s_i$, where $s_i^2 = 3/2 \times \sigma_i^2$. $\theta_{i|t}$ is equal to the vector $(m_i, s_i)$. $m_i$ is supposed to be calculated with the simple linear regression method. Therefore, its $y$-intercept $a_1(i)$ and slope $a_2(i)$ are following the same relations as described previously.

### 1.1.3 | Dynamics of the States: laws of motion of the system

The law of motion, being defined as the dynamic equation which explicitly links the states of the system in any pair of successive periods $i$ and $i+1$, only depends on the demand forecast method used, and the type of industrial configuration in case of stock-out. Here, the situation is the same as in the previous chapter: simple linear regression is used for demand forecasting, and unmet demand is backlogged. We can consequently re-write directly the laws of motion 3:

$$m_{i+1} = \sum_{k=1}^{t-1} \frac{2(3k-i-2)}{i(i-1)} d_k + \sum_{k=t}^{i} \frac{2(3k-i-2)}{i(i-1)} D_{k|t} \tag{4.1}$$

or, with the notation defined in chapter 2: $\beta^k_{m_{i+1}} = \frac{2(3k-i-2)}{i(i-1)}$.

As in paragraph 3 of the second section of the previous chapter, used together with the classical inventory update equation, 4.1 defines consequently completely the laws of motion of the system, as it makes the link between the parameters of $\phi_{i+1|t}$ and $\Delta_{i|t}$ - the demand information up to $i$ - explicit:

$$\begin{cases} \theta_{j|t} & = \mathcal{G}_{j|t}(\Delta_{j-1|t}) \\ X_{j+1} & = \nu\left(X_j, c_{j-1}, D_{j|t}\right) \end{cases}$$

In order to simplify the notation and make the link 4.1 clearer between states, we will use the following notation $\forall i$, such as $t < i \leqslant t + N - 1$.

As defined in chapter 2 we have,

$$m_i = \sum_{k=1}^{t-1} \beta_{m_i}^k d_k + \sum_{k=t}^{i-1} \beta_{m_i}^k D_{k|t}$$

Equation 4.1 allows us to get the value of $\beta_{m_i}^k$ in our case study. We will note, from now on, $\forall j < i$ the variable $\beta_{m_i}\left(\Delta_{j|t}\right)$ defined as:

$$\beta_{m_i}\left(\Delta_{j|t}\right) = \sum_{k=1}^{t-1} \beta_{m_i}^k d_k + \sum_{k=t}^{j} \beta_{m_i}^k D_{k|t}$$

thanks to which we are able to re-write the law of motion as, $\forall j < i$:

$$m_i = \beta_{m_i}\left(\Delta_{j|t}\right) + \sum_{k=j+1}^{i-1} \beta_{m_i}^k D_{k|t}$$

For example, $m_i$ can consequently be simply noted $m_i = \beta_{m_i}\left(\Delta_{i-2|t}\right) + \beta_{m_i}^{i-1} D_{i-1|t}$.

### 1.1.4 | Objective Function and Value Function

In this new example, we study an industrial environment in which the production lead-time is equal to one period. Thus, the sought optimal production planning is here $(c_j^*)_{t \leq j < t+N-1}$. This time series minimizes the expected total cost from $t$ to $t+N-1$, $W_t$.

This total cost depends here specifically on the state of the system in $t$, the previous production orders already launched in $t-1$, and the production orders $(c_j)_{t \leq j < t+N-1}$ as decision variables :

$$W_t\left(x_t, \Delta_{t-1|t}, c_{t-1}, (c_j)_{t \leq j < t+N-1}\right)$$

$$= \mathbb{E}_{t-1}\left[\sum_{i=t}^{t+N-2} K_i\left(X_i, D_{i|t}, c_{i-1}, c_i\right) + K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-2}\right)\right]$$

We define as previously the value function $V_j$, $t \leq j < t+N-1$ as the minimum of the expected cost of the subhorizon from $j$ to $t+N-1$, $W_j$. We note $(c_k^*)_{j \leq k < t+N-1}$ its argument of the minimum. In this new example, we have $V_j$ depending on $X_j$, $\Delta_{j-1|t}$, and $c_{j-1}$ such as $\forall j \in [\![t, t+N-2]\!]$,

$V_j(X_j, \Delta_{j-1|t}, c_{j-1})$

$$
\begin{aligned}
&= W_j\left(X_j, D_{j-1|t}, c_{j-1},\ (c_k^*)_{j\le k<t+N-1}\right)\\
&= \min_{(c_k)_{j\le k<t+N-1}} \left\{ \mathbb{E}_{j-1}\left[ \sum_{i=j}^{t+N-2} K_i\left(X_i, D_{i|t}, c_{i-1}, c_i\right)\right.\right.\\
&\qquad\qquad \left.\left. + K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-2}\right)\right]\right\}
\end{aligned}
$$

### 1.1.5 | Dynamic Equation of the system

In the same manner as we described in chapter 3, the value function defined here can be re-written through a nested form (see paragraph 3), we have here an equivalent property:

**Property 1.1.5-1.** *The Value Function* $V_j\left(X_j, \Delta_{j-1|t}, c_{j-1}\right)$ *respects the dynamic equation* $\forall j \in [\![t, t+N-2]\!]$*:*

$V_j(X_j, \Delta_{j-1|t}, c_{j-1})$

$$
= \min_{c_j}\left\{ \mathbb{E}_{j-1}\left[ K_j\left(X_j, D_{j|t}, c_{j-1}, c_j\right) + V_{j+1}\left(\nu\left(X_j, c_{j-1}, D_{j|t}\right), \mathcal{G}_{j|t}(\Delta_{j-1|t}), c_j\right)\right]\right\}
$$

*Proof.* The proof follows the same reasoning as in chapter 3, theorem 1.2.3-6 □

## 1.2 | Analytical Results

**Theorem 1.2.0-1.** *Given the assumptions defined previously, the optimal production plan* $(c_j^*)_{t\le j<t+N-1}$*:*

$\forall j,\ t \le j < t+N-2,$

$$
\left\{
\begin{array}{rl}
c_j^* = & \beta_{m_{j+1}}\left(\Delta_{j-1|t}\right) - X_j - c_{j-1} + \frac{j+4}{j} m_j + \frac{3(B-H)}{2(H+B)} s_{j+1}\\
c_{t+N-2}^* = & \beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) - X_{t+N-2} - c_{t+N-3} + \frac{t+N+2}{t+N-2} m_{t+N-2}\\
& + \frac{3(B-H-2\,p)}{2(H+B)} s_{t+N-1}
\end{array}
\right. \tag{4.2}
$$

*Proof.* This theorem is proved through the arguments provided in the following paragraphs.

### 1.2.1 | Proof Principle

We use the same framework as in the previous example: our proof will be developed by induction using the model developed in the previous chapter 2. We show recursively that the value functions $V_i$, $\forall i,\ t \leqslant i < t+N-1$ follows a general form with respect to the states in period $i$ and $D_{i|t}$. This specific shape allows us to compute its corresponding optimum argument of $c_i^*$, solutions of our problem.

To do so, we first demonstrate the inductive step, by illustrating how the value function respects a special property from $i$ to $i-1$ thanks to the integration property of section 3, the dynamic equation of the system, and the laws of motion defined previously. We then show the last two periods follow the same property, which defines the base case of our demonstration.

For each of these two parts of the proof, we follow the same steps of computation:

- Integration of the total costs from the considered period so that the expectation can be expressed literally with respect to the corresponding state information;
- Minimization of this expected total cost with respect to the corresponding current production order:
- Approximation of this function through a continuous quadratic polynomial function with respect to the new current production order;
- Expression of this newly calculated value function with respect to the previous state information thanks to the laws of motion.

### 1.2.2 | Iterative Step of the Proof

Our goal is here to demonstrate a recursive form of $V_j$ and its corresponding optimum argument of $c_j^*$, for all $j$ such as $t \leqslant j < t+N-3$.

To do so, we define $V_j$ as a specific function with respect to the variables of state in $j-1$ and $D_{j-1|t}$, and we use the theorem 1.2.3 - 6 as follows.

Let $V_j$, for any $j$ such as $t \leqslant j < t+N-3$ be equal to the following function depending on the information of the state in $j-1$, and expressed with respect to $D_{j-1|t}$, specifically :

$$
\begin{aligned}
&V_j\left(X_{j-1}, \Delta_{j-2|t}, D_{j-1|t}, c_{j-1}, c_{j-2}\right) \\
&= \gamma_2^j \left(c_{j-1} + c_{j-2} + X_{j-1} - \tfrac{j+3}{j-1} D_{j-1|t} - \beta_{m_j}\left(\Delta_{j-2|t}\right)\right)^2 \\
&\quad + \gamma_{c1}^j\, c_{j-1} + \gamma_{c2}^j\, c_{j-2} + \gamma_x^j\, X_{j-1} + \gamma_d^j\, D_{j-1|t} \\
&\quad + \gamma_\beta^j \beta\left(\Delta_{j-2|t}\right) + \Gamma_j
\end{aligned}
$$

where, in this definition, $\gamma_2^j, \gamma_{c1}^j, \gamma_{c2}^j, \gamma_x^j, \gamma_d^j, \gamma_\beta^j$, and $\Gamma_j$ are real numbers that do not depend on any period $k$ such that $t \leq k < j$, and $\beta\left(\Delta_{j-2|t}\right)$ is a linear combination of the demand information $\Delta_{j-3|t}$. Moreover, we assume $\gamma_2^j > 0$ (Peut-on justifier?) since, as seen before, it represents generally a positive linear combination of industrial parameters.

We calculate here through the four steps previously described the next value function $V_{j-1}$ with respect to the state in $j-2$ and $D_{j-2|t}$, knowing that, thanks to the theorem 1.2.3 - 6, $V_{j-1}$ is equal to :

$$V_{j-1}\left(X_{j-1}, \Delta_{j-2|t}, c_{j-2}\right) = \min_{c_{j-1}} \Big\{ \mathbb{E}_{j-2}\Big[ K_{j-1}\left(X_{j-1}, D_{j-1|t}, c_{j-1}, c_{j-2}\right) + V_j\left(X_{j-1}, \Delta_{j-2|t}, D_{j-1|t}, c_{j-1}, c_{j-2}\right)\Big]\Big\}$$

**i. Integration of the total costs**

We first compute the expectation of the value function of $j$ defined with respect to the information of state $j-1$ and $D_{j-1|t}$ by hypothesis as the inductive assumption. To do so, we integrate its value with respect to $\phi_{j-1|t}$, conditionally to the information up to period $j-2$ (*ie* the state of period $j-1$):

$$\mathbb{E}_{j-2}\Big[ V_j\left(X_{j-1}, \Delta_{j-2|t}, D_{j-1|t}, c_{j-1}, c_{j-2}\right)\Big]$$

Then, the costs of current period $j$ are also integrated with respect to the current random variable $D_{j-1|t}$:

$$\mathbb{E}_{j-2}\Big[ K_{j-1}\left(X_{j-1}, D_{j-1|t}, c_{j-1}, c_{j-2}\right)\Big]$$

The detailed results are described in the appendix.

**ii. Minimization Problem**

The resulting function with respect to $c_{j-1}$

$$c_{j-1} \to \mathbb{E}_{j-2}\Big[ K_{j-1}\left(X_{j-1}, D_{j-1|t}, c_{j-1}, c_{j-2}\right) + V_j\left(X_{j-1}, \Delta_{j-2|t}, D_{j-1|t}, c_{j-1}, c_{j-2}\right)\Big]$$

is a continuous function, and double differentiable. Its second derivative is equal to $2\gamma_2^j$, which is strictly positive. It is consequently continous and convex with respect to $c_{j-1}$, we can find its argument of the minimum.

To do so, we look for the solution of the following equation

$$\frac{\partial}{\partial c_{j-1}}\mathbb{E}_{j-2}\Big[K_{j-1}\left(X_{j-1}, D_{j-1|t}, c_{j-1}, c_{j-2}\right) + V_j\left(X_{j-1}, \Delta_{j-2|t}, D_{j-1|t}, c_{j-1}, c_{j-2}\right)\Big] = 0 \tag{4.3}$$

This equation 4.3 has only one solution on our interval of interest, and is equal to:

$$c^*_{j-1} = \beta_{m_j}\left(\Delta_{j-2|t}\right) - X_{j-1} - c_{j-2} + \frac{j+3}{j-1}m_{j-1} - \frac{p+\gamma^j_{c1}}{2\gamma^j_2} \tag{4.4}$$

which gives us the value function of the next period, in other words $V_{j-1}$, with respect to the information state of period $j-1$, which is $X_{j-1}$, $\Delta_{j-2|t}$, and $c_{j-2}$. The detailed calculation is available in the appendix.

**Remark 1.2.2-2.** *Theorem 1.2.3-6 shows that, at each step of the dynamic programming approach, any value function $V_{i+1}$, $i \in [\![t, t+N-2]\!]$ needs to be integrated with respect to the demand $D_{i|t}$, and its corresponding pdf $\phi_{i|t}$, conditionally to the information of period $i$, ie $\Delta_{i|t}$. It is therefore also the case in our inductive reasoning. In other words, applied to our case study, $V_{j-1}\left(X_{j-1}, \Delta_{j-2|t}, c_{j-2}\right)$ needs to be integrated with respect to $D_{j-2|t}$.*

*However, two issues arise here.*

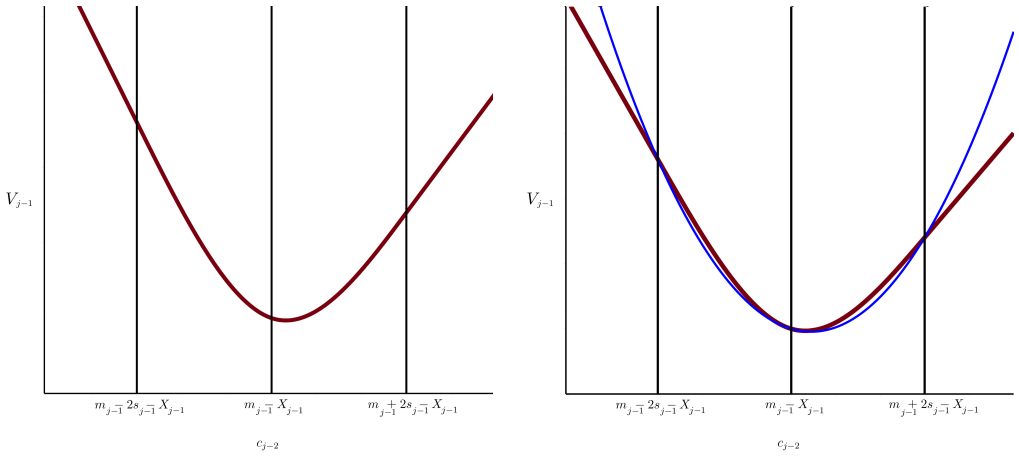
*First, $V_{j-1}\left(X_{j-1}, \Delta_{j-2|t}, c_{j-2}\right)$ is expressed with respect to $X_{j-1}$, $\Delta_{j-2|t}$, and $c_{j-2}$, which is the $j-1$ state information. The one from the state of $j-2$ needs to be explicit in order to compute this expectation though. That is why, as seen in the previous chapter and in 2, we will apply the laws of motion.*

*Second, we have, because the production lead-time is 1-period long, a historical order that has to be taken into consideration into the state of the system. In particular, as presented in the expression of $V_{j-1}\left(X_{j-1}, \Delta_{j-2|t}, c_{j-2}\right)$ in appendix, $c_{j-2}$ appears to be a variable on which the value function depends. Specifically, due to the triangular distribution of the demand, the shape of the value function $V_{j-1}$ changes depending on $c_{j-2}$, and in particular its form with respect to $m_{j-1}$. Consequently, the link between $V_{j-1}$ and previous state information $\Delta_{j-3|t}$ and $D_{j-2|t}$ depends on $c_{j-2}$. We therefore cannot explicitly express the expectation of $V_{j-1}$, inasmuch as the structure of the function to be integrated is not known without assignment of $c_{j-2}$ to a specific set. Yet, it is a decision variable of the next iteration of the dynamic programming. We cannot consequently impose any artificial constraint on it, or there is a risk of overlooking optimal solutions.*

### iii. Approximation

$V_{j-1}$ is a piecewise function with respect to $c_{j-2}$, which prevents us from obtaining explicit results.

In order to still be able to calculate the desired expectation, we notice that $c_{j-2} \to V_{j-1}\left(X_{j-1}, \Delta_{j-2|t}, c_{j-2}\right)$ is a decreasing linear function when $c_{j-2} < m_{j-1} - 2s_{j-1} - X_{j-1}$, is a decreasing cubic polynomial function when $m_{j-1} - 2s_{j-1} - X_{j-1} \leqslant c_{j-2} < m_{j-1} - X_{j-1}$, another cubic polynomial function when $m_{j-1} - X_{j-1} \leqslant c_{j-2} < m_{j-1} - X_{j-1} + 2s_{j-1}$, and finally an increasing linear function when $c_{j-2} \geqslant m_{j-1} - 2s_{j-1} - X_{j-1}$. We can see in figure 4.1a an example of this function given an arbitrary state in $j-1$, and in 4.1b the corresponding approximation function applied in blue line.




(a) An illustrative example of $V_{j-1}$ (b) The corresponding approximation

Figure 4.1: Illustration of the approximation function

We consequently make the following approximation to go further. The boundaries of the intervals of the different definitions of $V_{j-1}$ correspond to specific values of the order $c_{j-2}$: $m_{j-1} - 2s_{j-1} - X_{j-1}$, $m_{j-1} - X_{j-1}$, and $m_{j-1} + 2s_{j-1} - X_{j-1}$. A quadratic function that goes through the three values of $V_{j-1}$ in these three specific points approximates it therefore within a very thin range in the interval where the probabilities of occurrence is high. $m_{j-1} - X_{j-1}$ represents indeed the expected net requirement for this period, and $m_{j-1} - 2s_{j-1} - X_{j-1}$ and $m_{j-1} + 2s_{j-1} - X_{j-1}$ large boundaries with respect to the definition of the uncertainties parameter $s_{j-1}$. Moreover, we keep the convexity of the function and the approximation is consistent with the interval where the minimum occurs with respect to $c_{j-2}$.

We approximate therefore $V_{j-1}$ with respect to $c_{j-2}$ as a quadratic function whose values in $c_1 = m_{j-1} - 2s_{j-1} - X_{j-1}$, $c_2 = m_{j-1} - X_{j-1}$, and $c_3 = m_{j-1} + 2s_{j-1} - X_{j-1}$ are respectively the same as the original function so that the integration with respect to $D_{j-2|t}$ is feasible. We note $\tilde{V}_{j-1}$ this approximation.

The results are presented in the appendix.

### iv. Application of the laws of motion

In order to finally prove the inductive step of our reasoning, we need to demonstrate $\tilde{V}_{j-1}$ follows the same property as $V_j$. In particular, $V_j$ is expressed here with respect to the state of period $j-1$ and $D_{j-1|t}$. In the same manner as in the inductive assumption, and as also described in the previous remark, we need to explicitly establish the relation between $\tilde{V}_{j-1}$ on the one hand, and the state in $j-2$ and $D_{j-2|t}$ on the other hand.

To do so, we implement the laws of motion 4.1 into the solution found previously for $V_{j-1}$. The detailed calculation is available in the appendix.

### v. Conclusion

The value function $\tilde{V}_{j-1}$ with respect to the state of $j-2$, $D_{j-2}$, and $c_{j-2}$ is equal to the following:

$$V_{j-1}\left(X_{j-2}, \Delta_{j-3|t}, D_{j-2|t}, c_{j-2}, c_{j-3}\right) =$$

$$\begin{aligned} = \; & \gamma_2^{j-1}\left(c_{j-2} + c_{j-3} + X_{j-2} - \tfrac{j+2}{j-2} D_{j-2|t} - \beta_{m_{j-1}}\left(\Delta_{j-3|t}\right)\right)^2 \\ & + \gamma_{c1}^{j-1} c_{j-2} + \gamma_{c2}^{j-1} c_{j-3} + \gamma_x^{j-1} X_{j-2} + \gamma_d^{j-1} D_{j-2|t} \\ & + \gamma_\beta^{j-1} \beta\left(\Delta_{j-3|t}\right) + \Gamma_{j-1} \end{aligned}$$

with the following relations:

$$
\begin{aligned}
\gamma_2^{j-1} &= \frac{H+B}{6s_{j-1}} \\
\gamma_{c1}^{j-1} &= \frac{B-H}{2} - p - \gamma_{c1}^{j} + \gamma_{c2}^{j} \\
\gamma_{c2}^{j-1} &= \frac{H-B}{2} - p - \gamma_{c1}^{j} + \gamma_{c2}^{j} \\
\gamma_{x}^{j-1} &= \frac{H-B}{2} - p - \gamma_{c1}^{j} + \gamma_{c2}^{j} \\
\gamma_{d}^{j-1} &= \frac{4}{j-2}\left(\frac{B-H}{2} + \gamma_d^j + \left(p+\gamma_{c1}^j\right)\frac{j+3}{j-1}\right) - \frac{H-B}{2} + p + \gamma_{c1}^j - \gamma_2^j + \\
&\quad \frac{2\left(2j-7\right)}{\left(j-1\right)\left(j-2\right)}\left(p+\gamma_{c1}^j\right) + \gamma_\beta^j \beta\left(\Delta_{j-2|t}\right)[j-2] \\
\gamma_{\beta}^{j-1}\left(\Delta_{j-3|t}\right) &= \left(\frac{B-H}{2} + \gamma_d^j + \left(p+\gamma_{c1}^j\right)\frac{j+3}{j-1}\right)\beta_{m_{j-1}}\left(\Delta_{j-3|t}\right) \\
&\quad + \gamma_\beta^j \beta\left(\Delta_{j-2|t}\right)[1..j-3] \\
&\quad + \left(p+\gamma_{c1}^j\right)\beta_{m_j}\left(\Delta_{j-3|t}\right)
\end{aligned}
$$

where $\beta\left(\Delta_{j-2|t}\right)[1..k]$ is the part of $\beta\left(\Delta_{j-2|t}\right)$ with respect to $\left(D_{i|t}\right)_{1\leqslant i\leqslant k}$.

Thus, thanks to the integration property and the dynamic equation of the system 3, and the laws of motion 3, we were able to apply theorem 1.2.3 - 6, and demonstrate that, if, $\forall j$ such that $t < j \leqslant t+N-2$, $V_j$ follows a particular shape with respect to $\Delta_{j-2|t}$, $D_{j-1|t}$, $X_{j-1}$, $c_{j-1}$, and $c_{j-2}$, then $V_{j-1}$ follows also this form, and we are able to calculate the solution of the minimum problem

$$
c_{j-1}^* = \beta_{m_j}\left(\Delta_{j-2|t}\right) - X_{j-1} - c_{j-2} + \frac{j+3}{j-1}m_{j-1} - \frac{p+\gamma_{c1}^j}{2\gamma_2^j}
$$

### 1.2.3 | Base Case: Last two periods of the planning

In order to end the proof, we show here that the hypotheses taken at the beginning of the inductive step is true for the value function $V_{t+N-2}$.

To do so, we implement the previous steps of computation for the two last periods, firstly $t+N-1$ and then the $t+N-2^{th}$ period.

**i. Period** $t+N-1$

As far as the state of $t+N-1$ is concerned, $c_{t+N-1}$ does not exist. There is consequently no minimization computation at this step. We note therefore the result of these computation steps $\hat{V}_{t+N-1}$. It could be considered as a "simplified"

value function for period $t+N-1$.

**Integration of the total costs**

Conditionally to the state in period $t+N-1$, we can calculate $\mathbb{E}_{N-1}\left[K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-2}\right)\right]$ using $\phi_{t+N-1|t}$. The detailed result is presented in the appendix.

**Approximation**

As previously, we approximate the function $c_{t+N-2} \rightarrow \mathbb{E}_{N-1}\left[K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-2}\right)\right]$ obtained to be able to integrate it with respect to $D_{t+N-1|t}$.

The approximation process is the same as the one used in the previous paragraph. The function $c_{t+N-2} \rightarrow \mathbb{E}_{N-1}\left[K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-2}\right)\right]$ follows indeed the same form as we saw previously: piecewise function made up of linear and cubic polynomial functions, continuous and convex. We approximate it then as a quadratic polynomial function, whose specific points held for the values to be equal are $m_{t+N-1}-X_{t+N-1}-2s_{t+N-1}$, $m_{t+N-1}-X_{t+N-1}$, and $m_{t+N-1}-X_{t+N-1}+2s_{t+N-1}$.

We get finally the first conditional expected cost of our dynamic programming $\hat{V}_{t+N-1}$ equal to:

$$\begin{aligned}\hat{V}_{t+N-1}\left(X_{t+N-1}, \Delta_{t+N-2|t}, c_{t+N-2}\right) &= \tfrac{H+B}{6s_{t+N-1}} {c_{t+N-2}}^2 \\ &+ \tfrac{c_{t+N-2}}{6s_{t+N-1}}\left(3(H-B)s_{t+N-1} + 2\left(X_{t+N-1}-m_{t+N-1}\right)(H+B)\right) \\ &+ \tfrac{1}{6s_{t+N-1}}\Big(2(H+B)\left({s_{t+N-1}}^2 + \left(X_{t+N-1}-m_{t+N-1}\right)^2\right) \\ &\qquad -3\left(X_{t+N-1}-m_{t+N-1}\right)(B-H)\, s_{t+N-1}\Big)\end{aligned}$$

**Application of the laws of motion** As seen previously, in order to be able to go further the dynamic computation, we need to make explicit the expression of $\hat{V}_{t+N-1}$ with respect to $D_{t+N-1|t}$. We apply the laws of motion of the system to do so.

We finally get:

$$\begin{aligned}&\hat{V}_{t+N-1}\left(X_{t+N-2}, \Delta_{t+N-3|t}, D_{t+N-2|t}, c_{t+N-2}, c_{t+N-3}\right) = \\ &= \tfrac{H+B}{6s_{t+N-1}}{c_{t+N-2}}^2 + \tfrac{c_{t+N-2}}{6s_{t+N-1}}\Big(3(H-B)s_{t+N-1} + 2\Big(X_{t+N-2} + c_{t+N-3} \\ &\quad - \tfrac{t+N+2}{t+N-2}D_{t+N-2|t} - \beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)\Big)(H+B)\Big) + \tfrac{1}{6s_{t+N-1}}\Big((2\,B \\ &\quad + 2\,H)\,{s_{t+N-1}}^2 - 3\,\Big(X_{t+N-2} + c_{t+N-3} - \tfrac{t+N+2}{t+N-2}D_{t+N-2|t} \\ &\quad - \beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)\Big)(B-H)\, s_{t+N-1} + \Big(X_{t+N-2} + c_{t+N-3} \\ &\quad - \tfrac{t+N+2}{t+N-2}D_{t+N-2|t} - \beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)\Big)^2(H+B)\Big)\end{aligned}$$

**ii. Period** $t+N-2$

The same computation procedure as for the one used in the inductive step 2 is executed here.

**Integration of the total costs**

We calculate first the conditional total expected cost of the two last periods $\mathbb{E}_{t+N-3}\left[K_{t+N-2}+\hat{V}_{t+N-1}\right]$ thanks to the integration with respect to $D_{t+N-2|t}$ and its density function $\phi_{t+N-2|t}$.The result, shown in details in the appendix, is a convex and derivable piecewise function with respect to $c_{t+N-2}$.

**Minimization Problem**

Thanks to these properties, we can calculate $\frac{\partial}{\partial c_{t+N-2}}\mathbb{E}_{t+N-3}\left[K_{t+N-2}+\hat{V}_{t+N-1}\right]$ and deduce $c^*_{t+N-2}$ with respect to the state in $t+N-2$ by finding the point that nullifies this derivative:

$$\begin{aligned} c^*_{t+N-2} =& \frac{t+N+2}{t+N-2} m_{t+N-2} + \beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) - X_{t+N-2} \\ & - c_{t+N-3} + \frac{3\left(B-H-2\,p\right)s_{t+N-1}}{2(H+B)} \end{aligned} \tag{4.5}$$

The corresponding value of the function $V_{t+N-2}\left(\Delta_{t+N-3|t}, c_{t+N-3}\right)$ is presented in details in the appendix.

**Approximation**

In order to be able to compute the integration of this function with respect to $\phi_{t+N-2|t}$, the definition of its shape with respect to $\Delta_{t+N-3|t}$ need to be explicit, especially non-dependent on the interval to which $c_{t+N-3}$ belongs. Thus, we approximate the function $c_{t+N-3} \rightarrow V_{t+N-2}\left(\Delta_{t+N-3|t}, c_{t+N-3}\right)$ as a quadratic function, whose specific points through which it goes are the intervals' boundaries: $c_{t+N-3} = m_{t+N-2} - X_{t+N-2} - 2s_{t+N-2}$, $m_{t+N-2} - X_{t+N-2}$, and $m_{t+N-2} - X_{t+N-2} + 2s_{t+N-2}$. We note this approximation $\hat{V}_{t+N-2}\left(\Delta_{t+N-3|t}, c_{t+N-3}\right)$:

$$\hat{V}_{t+N-2}\left(\Delta_{t+N-3|t}, c_{t+N-3}\right) =$$

$$\begin{aligned}
&\frac{H+B}{6s_{t+N-2}}{c_{t+N-3}}^2 + \left(\frac{H-B}{2} - p + \frac{H+B}{3s_{t+N-2}}\left(X_{t+N-2} - m_{t+N-2}\right)\right) c_{t+N-3} \\
&+ \frac{H+B}{3}s_{t+N-2} + \frac{B-H}{2}\left(m_{t+N-2} - X_{t+N-2}\right) \\
&+ p\left(\frac{t+N+2}{t+N-2}m_{t+N-2} - X_{t+N-2} + \beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)\right) \\
&+ \frac{H+B}{6s_{t+N-2}}\left(m_{t+N-2} - X_{t+N-2}\right)^2 + A_{t+N-2}
\end{aligned}$$

**Application of the laws of motion**

Finally, in order to be coherent with the environment of the inductive step, we need to get the same property as the hypotheses taken as assumptions of the reasoning made $\forall j,\ t < j < t+N-3$. In this context we define the value function $V_j$ with respect to the state of $j-2$ and $D_{j-1|t}$. Thus, we need here to express our last outcome of $\hat{V}_{t+N-2}\left(X_{t+N-3}, \Delta_{t+N-4|t}, c_{t+N-3}, c_{t+N-4}, D_{t+N-3|t}\right)$ with respect to the state of period $t+N-4$ and $D_{t+N-3|t}$.

$$\hat{V}_{t+N-2}\left(\Delta_{t+N-4|t}, c_{t+N-3}, D_{t+N-3|t}\right) =$$

$$\begin{aligned}
&\frac{H+B}{6s_{t+N-2}}\left(X_{t+N-3} + c_{t+N-4} + c_{t+N-3} - \frac{t+N+1}{t+N-3}D_{t+N-3|t} - \beta_{m_{t+N-2}}\left(\Delta_{t+N-4|t}\right)\right)^2 \\
&+ D_{t+N-3|t}\left(\frac{B-H+2\,p}{2}.\frac{t+N+1}{t+N-3} + \frac{4p\left(t+N\right)}{\left(t+N-2\right)\left(t+N-3\right)}\right) \\
&- \frac{B-H+2\,p}{2}\left(X_{t+N-3} + c_{t+N-4} + c_{t+N-3}\right) \\
&+ \left(-\frac{B-H}{2} + \frac{t+N+2}{t+N-2}p\right)\beta_{m_{t+N-2}}\left(\Delta_{t+N-4|t}\right) + p\beta_{m_t+N-1}\left(\Delta_{t+N-4|t}\right) \\
&+ \frac{H+B}{3}s_{t+N-2} + A_{t+N-2}
\end{aligned}$$

The function finally obtained can be re-written as:

$$V_{t+N-2}\left(X_{t+N-3}, \Delta_{t+N-4|t}, D_{t+N-3|t}, c_{t+N-3}, c_{t+N-4}\right)$$

$$\begin{aligned}
= \ & \gamma_2^{t+N-2}\left(c_{t+N-3} + c_{t+N-4} + X_{t+N-3} - \tfrac{t+N+1}{t+N-3}D_{t+N-3|t} - \beta_{m_{t+N-2}}\left(\Delta_{t+N-4|t}\right)\right)^2 \\
& + \gamma_{c1}^{t+N-2}\,c_{t+N-3} + \gamma_{c2}^{t+N-2}\,c_{t+N-2-2} + \gamma_x^{t+N-2}\,X_{t+N-3} + \gamma_d^{t+N-2}\,D_{t+N-3|t} \\
& + \gamma_\beta^{t+N-2}\beta\left(\Delta_{t+N-4|t}\right) + \Gamma_{t+N-2}
\end{aligned}$$

with

$$\gamma_2^{t+N-2} = \frac{H+B}{6s_{t+N-2}}$$
$$\gamma_d^{t+N-2} = \frac{B-H+2\,p}{2}.\frac{t+N+1}{t+N-3} + \frac{4p\,(t+N)}{(t+N-2)\,(t+N-3)}$$
$$\gamma_{c1}^{t+N-2} = -\frac{B-H+2\,p}{2}$$
$$\gamma_{c2}^{t+N-2} = -\frac{B-H+2\,p}{2}$$
$$\gamma_x^{t+N-2} = -\frac{B-H+2\,p}{2}$$
$$\gamma_\beta^{t+N-2}\beta\left(\Delta_{t+N-4|t}\right) = \left(-\frac{B-H}{2} + \frac{t+N+2}{t+N-2}p\right)\beta_{m_{t+N-2}}\left(\Delta_{t+N-4|t}\right) + p\beta_{m_t+N-1}\left(\Delta_{t+N-4|t}\right)$$
$$\Gamma_{t+N-2} = \frac{H+B}{3}s_{t+N-2} + A_{t+N-2}$$

### 1.2.4 | Conclusion

To end this proof, we sum-up the different steps, and their outcomes:

- We took as assumption a special shape of the value function for any $j$ with respect to the information of the state of period $j-2$ and the demand of $j-1$
- Thanks to the dynamic equation of the problem 1.2.3 - 6 and the laws of motion, we were able to solve a minimization problem for this particular period $j$, in which the solution $c_{j-1}^*$ is equal to

$$c_{j-1}^* = \beta_{m_j}\left(\Delta_{j-2|t}\right) - X_{j-1} - c_{j-2} + \frac{j+3}{j-1}m_{j-1} - \frac{p+\gamma_{c1}^j}{2\gamma_2^j}$$

  Moreover, this result allows to get the same shape for $V_{j-1}$ with respect to the information of the state of period $j-3$ and the demand of $j-2$.
  Thus, if this hypothesis is true for one particular $j$, it will be true $\forall i < j$.
- By initiating the dynamic computation with the two last periods $t+N-1$ and $t+N-2$, we were able to prove that

$$c_{t+N-2}^* = \frac{t+N+2}{t+N-2}m_{t+N-2} + \beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) - X_{t+N-2} - c_{t+N-3} + \frac{3\,(B-H-2\,p)\,s_{t+N-1}}{2(H+B)}$$

and that the value function $V_{t+N-2}$ respects the necessary assumptions set for the recursive equation. Its shape is indeed the same as defined at the beginning

of the inductive step.

Finally, knowing that $\gamma_2^{t+N-2} = \frac{H+B}{6s_{t+N-2}}$ and that $\gamma_{c1}^{t+N-2} = -\frac{B-H+2p}{2}$, we can deduce $\forall j < t+N-2$:

$$c_j^* = \beta_{m_{j+1}}\left(\Delta_{j-1|t}\right) - X_j - c_{j-1} + \frac{j+4}{j}m_j + \frac{3(B-H)}{2(H+B)}s_{j+1}$$

With an inductive reasoning, we have just calculated the values of the plan minimizing the total expected cost taken into account the forecasts update procedure into the optimization problem with a simple linear regression as forecasting technique and a prodution lead-time equal to one period.

□

## 1.3 | Simulation

In order to illustrate our results, we run a simulation of the considered production planning system. We define then a one-item one-level production environment. The product has a one-period production lead-time. The decision framework is a "push" system, in which the forecasts are calculated using the simple linear regression method. The dynamic simulation reflects the rolling plan framework, in which at each step of the computation, the actual demand of the previous period is observed, and the forecasting procedure is launched again.

### 1.3.1 | Numerical Hypotheses

The hypotheses taken are as follows: the planning horizon is $N = 4$ periods. The date of forecasts is $t = 17$, $i \leq 17$ being thus the historical data. The production orders to compute are then $(c_i)$ for $i \in \{17, 18, 19\}$.

To do so, we use our method presented above, and we compare our results with standard stochastic net requirements calculation procedure under stochastic hypotheses: at each period $j$, the orders are equal to $m_{j+1}^j + \Phi_{tr}^{-1}(\tau)\sigma_{cumu} - c_{j-1} - x_{j-1}$. The variable $m_{j+1}^j$ corresponds to the mean of the forecasted demand of period $j+1$ made at the beginning of period $j$, $\Phi_{tr}$ represents the standard cumulative probability function of the triangular distribution, $\sigma_{cumu}$ the cumulative uncertainties of the period covered, namely $j$ and $j+1$, and $\tau$ the targeted service level.

This quantity represents the expected net requirements, to which we add a safety stock quantified according to $\tau$ and $\sigma_{cumu}$. Besides, we assumed arbitrarily $p = 1$, $B = 6$, $H = 2$, and the following initial state:

$(d_i)_{i=1..16} = (48, 52, 57, 59, 60, 55, 64, 72, 58, 71, 74, 82, 73, 80, 71, 78)$, $x_{16} = 20$, $c_{16} = 65$, and $(s_i)_{i=17..20} = (10, 12, 14, 16)$.

At each iteration $j$ of the rolling plan framework, we generate $M = 20$ different scenarios $d_j$, which makes the total number of random scenarios $(d_{17}, d_{18}, d_{19}, d_{20})$ equal to $20^4 = 160\ 000$. For each of them, we calculate the total costs of each method used, and the inventory level at the end of any period. The averaged results are presented in table 4.1.

Table 4.1: Simulation Results: Average Costs and Inventory

| **Methods used** | **Average total cost** | **Average inventory end of** | | |
|---|---|---|---|---|
| | | 18 | 19 | 20 |
| Our method | **405.2** | 8.4 | 9.6 | 6.0 |
| MRP with | | | | |
| $\tau = 65\%$ | 408.5 | 5.4 | 6.2 | 8.3 |
| $\tau = 67\%$ | 407.6 | 6.4 | 7.3 | 9.6 |
| $\tau = 68\%$ | 407.3 | 7.2 | 8.3 | 10.7 |
| $\tau = 69\%$ | 407.4 | 7.6 | 8.8 | 11.2 |
| $\tau = 70\%$ | 407.6 | 8.0 | 9.3 | 11.8 |
| $\tau = 71\%$ | 408.0 | 8.5 | 9.8 | 12.4 |

The average cost of our method is below any stochastic standard planning procedure fora any target of service level. In particular, the results show the expected total cost yielded by our method is lower than the best safety stock calculated with the optimal $\tau^*$.

The reason is that, in any stochastic standard procedure, the net requirements are calculated based on the forecasts made at the date of computation of the planning, whereas our optimization method takes into account the evolution of the forecasted demand between the date of forecasts and the reception of the production order, thanks to the forecast update integrated into our optimization model.

## 1.4 | Conclusion

In this section, we have presented an extension of the stochastic dynamic programming model described in chapter 3, in which the production lead time is one-period long.

We have then applied a dynamic programming solution computation framework by defining the states, the laws of motion including the forecasts update mechanism, and the dynamic equation of the problem.

Integration properties, got thanks to approximation of functions and measurable properties with respect to the corresponding probability density functions, allowed us to recursively prove analytically the explicit expression of the value function of the system, and consequently the optimal production orders to be launched to optimize the non-stationary system generated by the forecast updates in the rolling plan framework.

The solutions have been then tested via simulation to understand the benefits of such a method compared with standard production planning procedures. Results showed that the stochastic programming model still allows to get better performances in such a configuration. The integration of forecast updates into the optimization procedure leads us indeed to better flexibility in terms of dynamics and uncertainties management in a rolling horizon framework.

# 2 | Multiple Linear Regression

We detail in this section another extension of the previous solution computation.

Using the same optimization framework of a stochastic dynamic programming approach, we integrate here the forecast updates procedure into the production planning when the forecasting method applied is the multiple linear regression.

The model used as regression is a standard analytical one, so that it can be generalisable, and can be applied in different various environments.

The production lead time is kept equal to one period, we consequently still stay in a fast delivery mode.

We explain how the model evolves, and how the solution computation procedure changes in this new environment. Moreover, a new simulation is realized to illustrate the solutions.

Also, we detail a more complex and complete sensitivity analysis so that the industrial environment influence can be deeply understood in this more general case. The benefits are discussed as well.

## 2.1 | Problem description and modelling

We still consider an $N$-period production planning problem in a rolling plan framework. We assume that the production lead-time is equal to 1 period. The

objective is still the search for a production plan from the current forecast date $t$ to the last period of the horizon $t+N-1$, including the forecast update mechanism under a rolling plan framework into the calculation of the optimal planning.

We use the same notation as previously. In this context, we still have $\forall i = 1..t+N-1, K_i$ is equal to $-B.(X_i + c_{i-1} - D_i) + p.c_i$ when $X_i + c_{i-1} - D_i < 0$ and to $H.(X_i + c_{i-1} - D_i) + p.c_i$ otherwise. For the last period, $K_{t+N-1}$ is only made up of the holding/backlog cost as $c_{t+N-1}$ is not defined.

### 2.1.1 | Forecasted Demand Modeling

We use in this section a similar model of the forecasted demand as defined in the previous chapter chapter 2: $m_i$ and $\sigma_i$ are respectively the mean and the standard deviation of the random variable $D_{i|t}$ following a centered triangular distribution between $m_i - 2s_i$ and $m_i + 2s_i$, where $s_i^2 = 3/2 \times \sigma_i^2$. $\theta_{i|t}$ is equal to the vector $(m_i, s_i)$.

We assume here $m_i$ is calculated thanks to the multiple linear regression method, with the matrix $\Lambda$ of explanatory variables, made up of $p > 2$ known vectors $(\Lambda_k)_{1\leq k\leq p}$ of size $t+N-1$.

We remind that the first vector $\Lambda_1$ is the constant vector equal to 1 of size $t+N-1$, the second one $\Lambda_2$ being equal to the "trend" vector $\lambda_{2i} = i$. We also defined in chapter 2 the learning set $\Lambda_l(i) = (\lambda_{jk})_{\substack{1\leq k\leq p \\ 1\leq j<i}}$ used for the multiple regression made at the beginning of period $i$.

The least suqare method gives us the coefficients of the regression $a_k(i)$ calculated at date $i$. The notation $A(i)$ is used as the corresponding vector $(a_k(i))_{1\leq k\leq p}$. The detailed calculations of these coefficients were presented in chapter 2. The results were

$$m_i = \sum_{k=1}^{t-1} \beta_{m_i}^k d_k + \sum_{k=t}^{i-1} \beta_{m_i}^k D_{k|t} \tag{4.6}$$

where $\forall i \in [\![t, t+N-1]\!], \forall k \in [\![1, i-1]\!]$:

$$\beta_{m_i}^k = \sum_{j=1}^{p} \alpha_{jk}(i).\ \lambda_{ij} = \sum_{j=1}^{p} \left( \sum_{m=1}^{p} (\Lambda_l(i)^T \Lambda_l(i))_{jm}^{-1}.\ \lambda_{kj} \right).\ \lambda_{ij}$$

and $\forall i \in [\![t, t+N-1]\!]$, $\forall k \in [\![1, t-1]\!]$, $\forall j \in [\![1, p]\!]$:

$$\begin{aligned}\alpha_{jk}(i) &= \left((\Lambda_l(i)^T \Lambda_l(i))^{-1} \Lambda_l(i)^T\right)_{jk} \\ &= \sum_{m=1}^{p} (\Lambda_l(i)^T \Lambda_l(l))^{-1}_{jm}. \left(\Lambda_l(i)^T\right)_{jk} \\ &= \sum_{m=1}^{p} (\Lambda_l(i)^T \Lambda_l(i))^{-1}_{jm}. \lambda_{kj}\end{aligned}$$

### 2.1.2 | Definition of the States of the System

#### i. Basic state of the system

As previously defined, in any period $i$, the basic state of the system consists of the available inventory at the beginning of period $i$, $X_i$ -or $x_t$ for the specific period $t$-, and the historical demand information up to $t-1$, $(d_i)_{i=1..t-1}$, which is needed to calculate the parameters of $\phi_{i|t}$.

Considering here the case of the multiple regression, we are moreover in the environment defined in the chapter 2, section 2, paragraph 1.2. Thus, the matrix of historical data and projection set used respectively as learning set and forecasting bases has to be integrated into the set of information defining the basic state of the system.

Following the notation defined, we have therefore the matrix $\Lambda \in \mathbb{R}^{(t+N-1)\times p}$, $p \geq 2$ included into the state of the system of any period $i$. We remind that the learning set of period $i > t$ will be noted $\Lambda_l(i)$, the projection set $\Lambda_p(i)$, and the corresponding entries that $\Lambda$ is made up of are noted $(\lambda_{ij})_{\substack{1\leqslant j\leqslant p \\ 1\leqslant i\leqslant t+N-1}}$

#### ii. Advanced state of the system

The advanced state of the system if, as explained in the chapter 2, random variables $D_{k|t}, t \leq k < i$ are conditionally considered into the definition of the demand of period $i$.

Their parameters $\theta_{i|t}$ are indeed computed with respect to all information before period $i$: $((d_i)_{1\leq i<t}$, and $(D_{k|t})_{t\leq k<i})$. Consequently, we do not have any change compared with the previous case study : the parameters $\theta_{k|t}$ for $t \leqslant k \leqslant i$ defining in particular $(D_{k|t})_{t\leq k<i})$ are also needed to characterize the state of the system in period $i$. This is our advanced state of the system.

### 2.1.3 | Dynamics of the States: laws of motion of the system

The dynamic equations linking two following states of the system in any period $i$ are, as in the previous chapter, the two equations allowing us to express the inventory level of period $i+1$ in function of the state of period $i$, and the demand of $i+1$ calculated based on the advanced state of period $i$. We can consequently write the laws of motion, following the same reasoning as previously, but with our new forecasting method that impacts the definition of the advanced state:

$$m_{i+1} = \sum_{k=1}^{t-1} \beta^k_{m_{i+1}} d_k + \sum_{k=t}^{i} \beta^k_{m_{i+1}} D_{k|t} \tag{4.7}$$

with $\beta^k_{m_i} = \sum_{j=1}^{p} \alpha_{jk}(i).\ \lambda_{ij} = \sum_{j=1}^{p} \left( \sum_{m=1}^{p} \left(\Lambda_l(i)^T \Lambda_l(i)\right)^{-1}_{jm}.\ \lambda_{kj} \right)$.

Altogether with the classical inventory update equation, 4.7 defines completely the laws of motion of the system, as it makes the link between the parameters of $\phi_{i+1|t}$ and $\Delta_{i|t}$ -the demand information up to $i$- explicit.

We then have the following system 4.8 literally expressed with respect to the industrial parameters, our assumption variables, and the states of the system.

$$\begin{cases} \theta_{i|t} & = \mathcal{G}_{i|t}(\Delta_{i-1|t}) \\ X_{i+1} & = \nu\left(X_i, c_{i-1}, D_{i|t}\right) \end{cases} \tag{4.8}$$

In order to simplify, we will use again the notation $\forall j < i$, $\beta_{m_i}\left(\Delta_{j|t}\right)$ defined as: $\beta_{m_i}\left(\Delta_{j|t}\right) = \sum_{k=1}^{t-1} \beta^k_{m_i} d_k + \sum_{k=t}^{j} \beta^k_{m_i} D_{k|t}$ allowing us to re-write the law of motion $\forall j < i$:

$$m_i = \beta_{m_i}\left(\Delta_{j|t}\right) + \sum_{k=j+1}^{i-1} \beta^k_{m_i} D_{k|t} \tag{4.9}$$

Or, for example, precisely when $j = i-2$, $m_i = \beta_{m_i}\left(\Delta_{i-2|t}\right) + \beta^{i-1}_{m_i} D_{i-1|t}$

### 2.1.4 | Objective Function and Value Function

The expression of the total cost does not change compared with the previous case:

$$W_t\left(x_t, \Delta_{t-1|t}, c_{t-1}, (c_j)_{t \le j < t+N-1}\right)$$

$$= \mathbb{E}_{t-1}\left[ \sum_{i=t}^{t+N-2} K_i\left(X_i, D_{i|t}, c_{i-1}, c_i\right) + K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-2}\right) \right]$$

The value function $V_j$ of any $j$ such that $t \leqslant j \leqslant t+N-1$ is still depending on $X_j$, $\Delta_{j-1|t}$, and $c_{j-1}$ :
$V_j(X_j, \Delta_{j-1|t}, c_{j-1})$

$$\begin{aligned}
&= W_j\left(X_j, D_{j-1|t}, c_{j-1}, \ (c_k^*)_{j\leq k<t+N-1}\right) \\
&= \min_{(c_k)_{j\leq k<t+N-1}} \left\{ \mathbb{E}_{j-1}\left[ \sum_{i=j}^{t+N-2} K_i\left(X_i, D_{i|t}, c_{i-1}, c_i\right) \right.\right. \\
&\qquad \left.\left. + K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-2}\right) \right] \right\}
\end{aligned}$$

### 2.1.5 | Dynamic Equation of the system

The property illustrating the dynamics of the system is still respected here:

**Property 2.1.5-1.** *The Value Function $V_j\left(X_j, \Delta_{j-1|t}, c_{j-1}\right)$ respects the dynamic equation $\forall j \in [\![t, t+N-2]\!]$: (see 1.2.3-6).*
$V_j(X_j, \Delta_{j-1|t}, c_{j-1})$

$$= \min_{c_j} \left\{ \mathbb{E}_{j-1}\left[ K_j\left(X_j, D_{j|t}, c_{j-1}, c_j\right) + V_{j+1}\left(\nu\left(X_j, c_{j-1}, D_{j|t}\right), \mathcal{G}_{j|t}(\Delta_{j-1|t}), c_j\right) \right] \right\}$$

## 2.2 | Analytical Results

### 2.2.1 | Solution #1

#### i. Computation Procedure

The procedure calculating an optimal solution follows the same framework as in section 1:
We use the theorem 1.2.3-6 and integration properties to dynamically compute the solution of minimization problems appearing at each step of the dynamic programming.
The following steps will be implemented in the same manner as the proof 2

- Integration of the total cost of the considered period so that the expectation can be expressed literally with respect to the corresponding state information;
- Minimization of this expected total cost with respect to the production quantity of the current period;
- Approximation of this function through a continuous quadratic polynomial function with respect to the production quantity;

➢ Expression of this newly calculated value function with respect to the previous state information thanks to the laws of motion.

**ii. Results**

**Theorem 2.2.1-1.** *In such a context, an optimal production plan* $(c_j^*)_{t\leq j<t+N-1}$ *is:*

$\forall j,\ t \leq j < t+N-2,$

$$\begin{cases} c_j^* = & m_j(1+\beta_{m_{j+1}}^j) - X_j - c_{j-1} + \beta_{m_{j+1}}(\Delta_{j-1|t}) + \frac{3(B-H)}{2(H+B)} s_{j+1} \\ c_{t+N-2}^* = & m_{t+N-2}(1+\beta_{m_{t+N-1}}^{t+N-2}) - X_{t+N-2} - c_{t+N-3} \\ & +\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) + \frac{3(B-H-2\,p)}{2(H+B)} s_{t+N-1} \end{cases} \tag{4.10}$$

The generalization of the regression from a simple one to a multiple one here is visible through the definition of the coefficient existing for the mean of the served demand in $c_j^*$. It is here indeed a linear combination of coefficients $\beta_{m_{j+1}}$, that was calculable in the case of simple regression. It is also the direct illustration that the future forecasts evolution, and consequently the forecast updates, are integrated into the computation of the optimal orders, inasmuch as these coefficients are known with respect to the advanced state of information for period $j$.

The detailed calculations are presented in appendix. It follows the proof 2 previously detailed, but uses the new definition of the coefficient of the dynamics of the forecast updates $\beta_{m_j}^k$ for the forecasted period $j$ with respect to the conditional variable $D_{k|t}$, where $k \in [\![t, i-1]\!]$

Finally, we have consequently managed to get, for any regression matrix $\Lambda$ used as forecasting method for the customer demand, the optimal production plan to be organized at the beginning of any period $t$ so that the cost can be minimized, taking into account the future forecast updates that will happen during the rolling plan framework.

**iii. Simulation and Sensitivity Analysis**

In order to illustrate the results and study the performance of our model, we run several simulations. In particular, we compute our planning technique, and compare it with classical stochastic net requirements calculation methods.

Several simulations have been done: first a one-time simulation based on an

arbitrary set of parameters $B$, $H$, and $p$. Then, a sensitivity analysis has been developed to understand the behavior of our method with respect to cost parameters.

**Simulation** We simulate here a rolling plan framework based on a learning set made up of the real customer demand faced by the system for the latest 16 periods, and a regression matrix of 2 predictors variables. Including the $Y$-intercept and the trend, we have then $p = 4$.

From this learning set, we generate $M = 20$ scenarios for each of the period of the rolling horizon, based on the random sampling of the stochastic process defined as the forecasted demand. The number of periods simulated is 4, which makes the total number of scenarios reach $160,000$.

The data used is as follows: The historical demand from $i = 1$ to $i = 16$ is equal to $49, 70, 68, 64, 70, 80, 81, 89, 97, 87, 99, 99, 91, 90, 106, 115$. The regression matrix $\Lambda$ is equal to

$$\Lambda = \begin{bmatrix} 1 & 1 & 13 & 105 \\ 1 & 2 & 22 & 117 \\ 1 & 3 & 18 & 124 \\ 1 & 4 & 18 & 126 \\ 1 & 5 & 24 & 114 \\ 1 & 6 & 22 & 130 \\ 1 & 7 & 23 & 123 \\ 1 & 8 & 27 & 124 \\ 1 & 9 & 23 & 127 \\ 1 & 10 & 24 & 124 \\ 1 & 11 & 34 & 133 \\ 1 & 12 & 28 & 128 \\ 1 & 13 & 27 & 132 \\ 1 & 14 & 29 & 134 \\ 1 & 15 & 36 & 144 \\ 1 & 16 & 33 & 147 \\ 1 & 17 & 32 & 146 \\ 1 & 18 & 40 & 148 \\ 1 & 19 & 38 & 145 \\ 1 & 20 & 40 & 158 \end{bmatrix}$$

The $R^2$ of the corresponding multiple regression is equal to 0.893. The model is relatively well explained.

Based on this forecasting technique (whose coefficients are respectively equal to $2.6206, 1.5750, 0.9826, 0.3469$), we compute the forecasted demand for next periods itreratively, by random sampling at each step of the rolling horizon framework.

With respect to the corresponding iterative sampling and computation of forecasts, the production planning is calculated.

On the one hand, we compute our method based on the stochastic dynamic programming approach; on the other hand, we calculate the standard net requirements in a stochastic environment, implying safety stock policy. The safety stock is determined based on the service level targeted, denoted as $\tau$. We compute it as, for any period $i$, $\Phi_{tr}^{-1}(\tau) \times \sigma_i$, where $\Phi_{tr}^{-1}$ is the standard cumulative probability distribution function of the triangular law, and $\sigma_i$ the standard deviation of the forecasted demand of period $i$.

The performance of our method is calculated as follows. Based on the values of $B$, $H$, and $p$, we can calculate an optimal $\tau^*$ minimizing the cost among all standard net requirements techniques with respect to $\tau$. This optimal method among the standard ones has a total expected cost.

We compare this cost with the one generated with our stochastic dynamic programming approach: the difference between these two costs is kept as performance indicator.

If this difference is positive, our stochastic dynamic approach is better than the least costly of the standard safety stock policy.

**Results** We choose $B = 500$, $p = 10$, and $H = 2$.

We finally get a 2.02% of cost improvement between the stochastic dynamic programming method and the least costly of the standard net requirements procedure with safety stock computation.

Our method has therefore proved here to be more efficient in this specific configuration of costs.

**Sensitivity analyses** In order to understand better how efficient the method can be, we perform a sensitivity analysis based on variations of parameters $B$ and $p$.

In particular, we make $p$ varies from 1 to 100, and $B$ from 200 to 600. $H$ is kept equal to 2 arbitrarily.

The results are shown in Figure 4.2. We present in these figures our indicator representing the difference between the cost of our method and the cost of the best -meaning the least costly- standard safety stock policy. In the blue area, this

difference is negative: our stochastic dynamic programming approach has a better performance rather than the classical one. In the red area, there is no improvement: on the contrary, the cost of our method is higher.

The efficiency of the method is challenged here. The trends show that for a given $B$, the higher $p$ is, the less interesting the solution is. The performance of the solution is not better for high values of $p$, relatively to a given $B$, compared with classical ones.

A clear linear threshold exists defining the area of the values of the couple $(p, B)$ in which the solution presented here is better than the classical planning procedure in the presence of uncertainties. This area corresponds to small values of $p$ for a given $B$. If $p$ becomes greater than this limit, the solution is getting unfit.

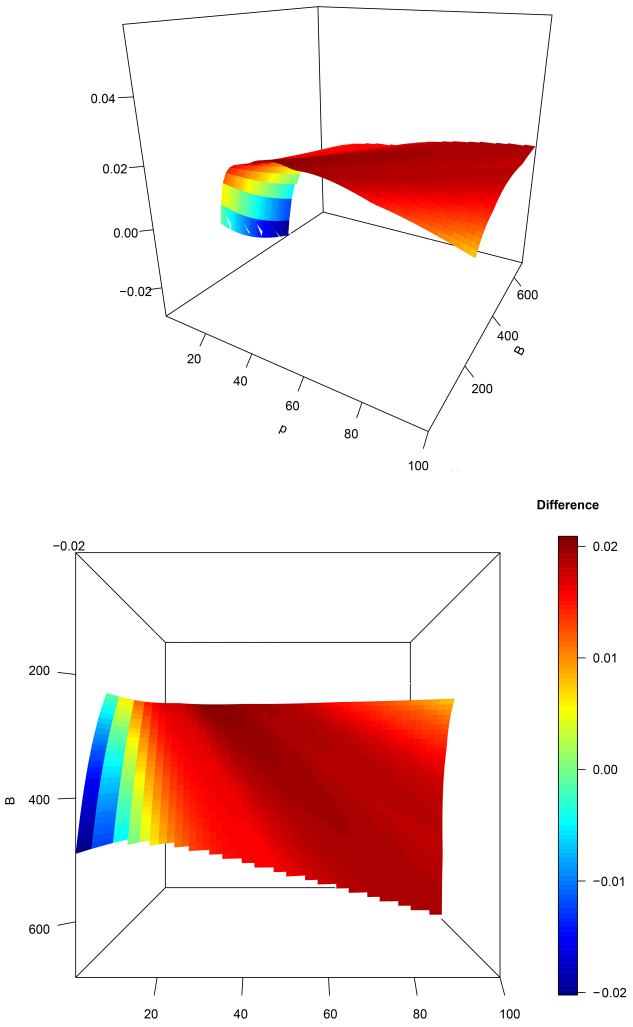



Figure 4.2: Cost Improvement in the case of a quadratic approximation

Actually, generally speaking, when, for a specific level of backlog costs $B$, $p$

reaches a certain level, it becomes more interesting to force delays by backlogging some quantities compared with producing the necessary required quantity. The number of orders requirements is consequently relatively low.

In this case, we are consequently in an extreme set of production quantities $c_i$ of the considered period $i$, compared with its expected value: we may have solutions far lower from the expected net requirements. However, based on the modelling structure we have here, this specific interval of the production quantity are a particular range in which the approximation function we made for the value function is not consistent with the real shape of the value function.

We show indeed in the next figure 4.3 that the quadratic approximation is consistent with the original function within a specific interval centered around the expected value of the net requirements.

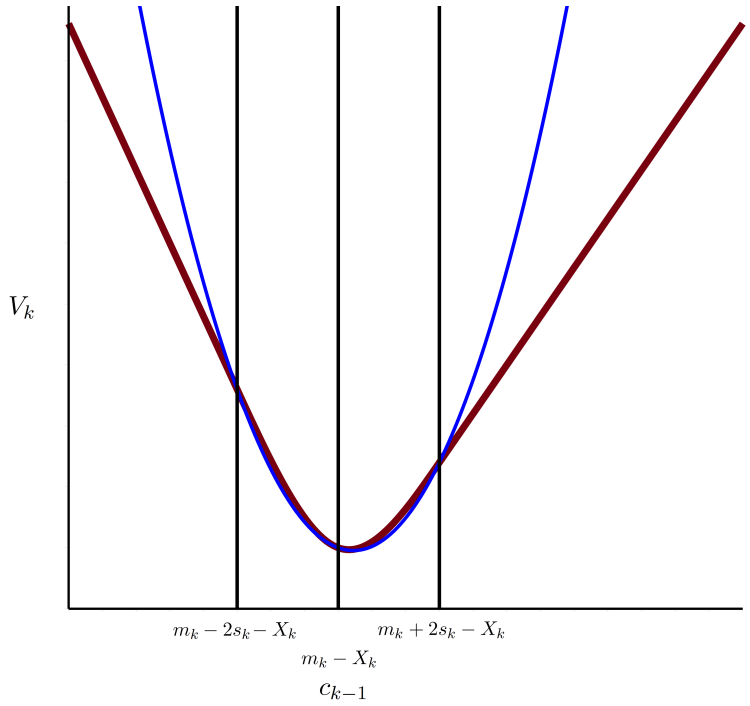



Figure 4.3: The quadratic approximation fits for $c_{k-1}$ close to the expected net requirements

Consequently, the solutions calculated based on the approximation function for values belonging to this extreme interval are not reflecting a correct cost, and therefore are not coherent with the real cost environment.

An illustrative drawing explains this phenomena in Figure 4.4: if the correct optimal production quantities had had the value represented by the green line, its corresponding real cost would have been at the red line level. However, this

specific cost level corresponds to solutions evaluated at the yellow line with the approximation function used here.

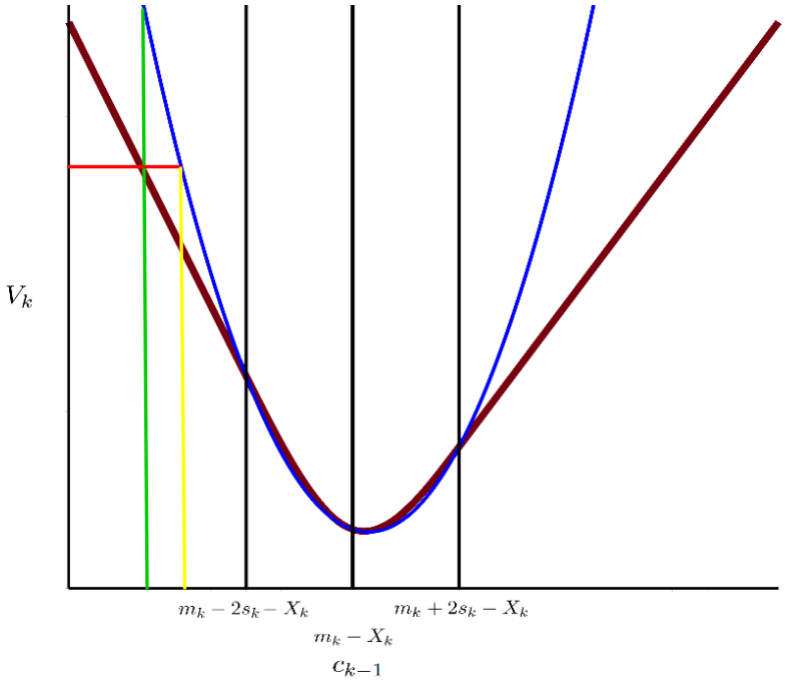



Figure 4.4: The approximation may bring wrong cost

The approximation function therefore does not fit this case, and we got bad performances when, for a given $B$, $p$ is high, or, inversely, for a given $p$, $B$ is low. We need to look for another approximation function that can handle such cases.

### 2.2.2 | Solution #2

#### i. Computation Procedure

We use the same procedure of calculation of the optimal solutions as in section 1:

We use the theorem 1.2.3 - 6 and integration properties to dynamically compute the solution of minimization problems appearing at each step of the dynamic programming.

The following steps will be implemented in the same manner as the previous proof of theorem 2.2.1 - 1, except the approximation function that we detail more precisely afterwards.

- Integration of the total costs from the considered period so that the expectation can be expressed literally with respect to the corresponding state information

- Minimization of this expected total cost with respect to the corresponding current production order
- Approximation of this function through a continuous function with respect to the new current production order
- Expression of this newly calculated value function with respect to the previous state information thanks to the laws of motion

**ii. New Approximation Function**

We are looking for a continuous function with respect to $c_{i-1}$ being able to approximate $V_i$ at each step of the dynamic programming. $V_i$ as a convex form with a unique argument of minimum.

Most importantly, on the contrary to the previous approximation, we want the new approximation function to be efficient within the ranges of extreme values of $p$, meaning away from the expected value of $c_{i-1}$.

In order to be able to keep the integration properties on the one hand, and the convexity on the other hand, while respecting the constraint of good approximation function for extreme values of $c_{i-1}$, we choose to approximate it with a piecewise linear function made up of the already existing linear forms of $V_i$ with respect to $c_{i-1}$, but extended to the centered intervals of $c_{i-1}$ arounf its expected value.

The approximation is consequently a double piecewise linear function as shown in blue line through figure 4.5

**iii. Results**

**Theorem 2.2.2 - 2.** *In such a context, the optimal production plan* $(c_j^*)_{t\leq j<t+N-1}$*:* $\forall j,\ t \leq j < t+N-2$,

$$\begin{cases} c_j^* = & m_j(1+\beta^j_{m_{j+1}}) - X_j - c_{j-1} + \beta_{m_{j+1}}(\Delta_{j-1|t}) \\ & +2s_j\left(1+\beta^j_{m_{j+1}}\right)\left(1-\sqrt{\frac{2H}{H+B}}\right) \\ c^*_{t+N-2} = & m_{t+N-2}(1+\beta^{t+N-2}_{m_{t+N-1}}) - X_{t+N-2} - c_{t+N-3} \\ & +\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) \\ & +2s_{t+N-2}\left(\beta^{t+N-2}_{m_{t+N-1}}+1\right)\left(1-\sqrt{\frac{2(H+p)}{H+B}}\right) \end{cases} \tag{4.11}$$

The detailed calculations of the proof are presented in appendix. It follows the proof 2 previously detailed, but uses the new approximation function previously defined in the dynamic calculation of the value function.

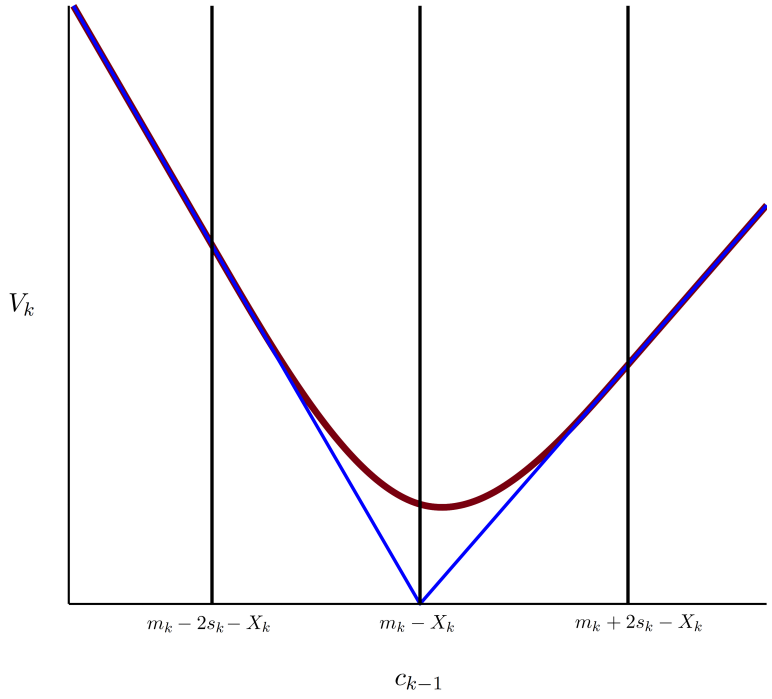



Figure 4.5: Approximation by a piecewise linear function

**iv. Simulation and Sensitivity Analysis**

In the same manner as previously, we run several simulations, and compare our model to classical stochastic net requirements calculation methods.

**Simulation** We use the same simulation framework as in the previous paragraph. Results of the corresponding sensitivity analyses are shown in the following figure 4.6

The figures got from sensitivity analysis illustrate two interesting points.

First, it clearly shows we get efficient results in terms of costs improvement. We succeeded in getting better solutions in terms of costs rather than the least costly of the standard stochastic order quantity procedure in a large area of possible values of couples $(p, B)$. Thanks to the use of this new approximation function, the number of configurations in which the stochastic dynamic programming approach is more interesting than the standard ones is consequently higher, and allows users to improve clearly the cost of the method.

Second, it is interesting to note that the results are exactly symmetric with respect to the previous ones getting from the first quadratic approximation. Inasmuch as the linear approximation function considered here is complementary to the

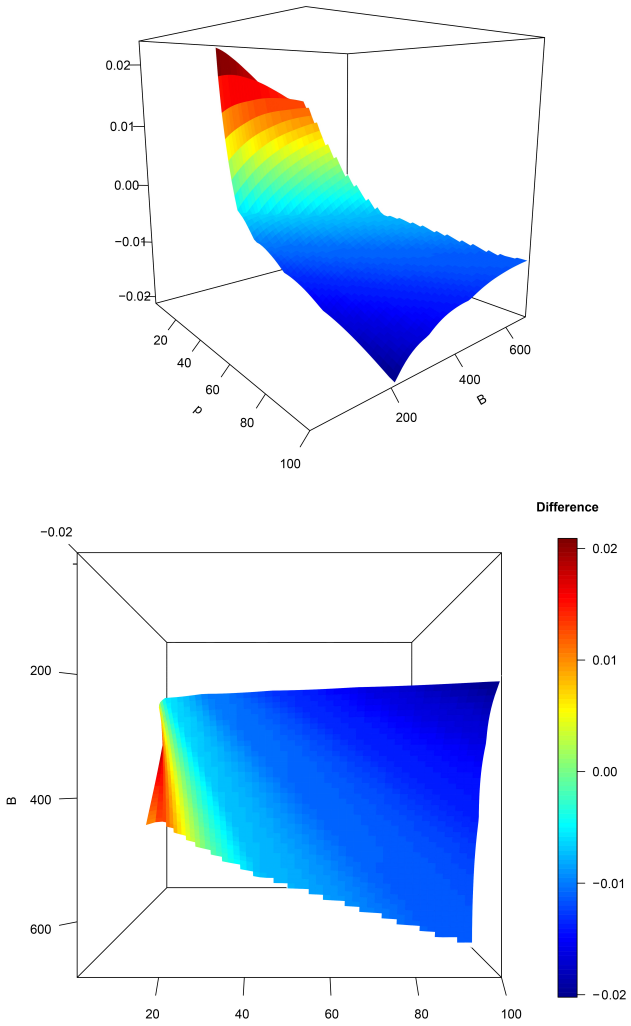



Figure 4.6: Cost Improvement in the case of a linear approximation

quadratic one in terms of range of efficient fitting of the function to approximate, this makes sense. It also consequently validates the interpretation made at the end of subsection 1.

### 2.2.3 | Conclusion

Based on the same computation framework already used in a more constrained case in the previous section, we were here able to compute optimal solutions and express them literally for the generalized case of the multiple linear regression with any learning matrix $\lambda \in \mathbb{R}^{(t+N-1)\times p}$, for any natural $p > 2$.

In these more complex cases, we also investigated the performances of different solutions getting from two forms of approximation functions used at each step of the iterative computations of the dynamic programming.

A more in-depth sensitivity analysis has indeed revealed some interesting behaviors of the two possible solutions with respect to different cost configurations we may face in the environment of work: we showed in particular that different approximation functions bring unequal efficiency for the corresponding solutions.

Specifically, the results outcoming from the quadratic approximation computation method are particularly efficient with a low $p$ and large $B$. On the contrary, the higher $p$ is compared with $B$, the more efficient the piecewise linear approximation is. We can more particularly precise there exists a clear threshold from which the choice of the approximation function has to change for an increasing $p$ from the quadratic approximation to the piecewise linear function.

This difference comes from the fact that, depending on the method used, the interval to which the order quantity belongs may not be the one for which the approximation is good. Thus, by well choosing the approximation function, the efficiency of our optimal solutions stays better than any of the classical methods.

In our calculation procedure, results were kept voluntarily entirely analytical, so that the aim of our work stays coherent within the research questions pointed out at the end of chapter 4. Thus, details of the proofs of Theorems 2.2.1-1 and 2.2.2-2 presented in the appendix are made up of complex literal expressions of the corresponding dynamic programming implementations. The analytical complexity of these calculations makes the study of the threshold, clearly visible in our sensitivity analysis, difficult. The iterative computations developed here are indeed made up of complex laws of motion due to the forecast update mechanism integration on the one hand, and the nested form of the value function on the other hand. These two points does not enable us to express the sensitivity of a value function $V_i$ with respect to the parameters $B$ and $p$ based on a previous state decision $c_{i-1}^*$.

An interesting perspective can be however to analyse the characteristics of this limit in a more complex simulated and/or numerical framework of study.

## 2.3 | Concluding remarks

**Remark 2.3.0-1.** *When multiple linear regression is used, the capability to express literally optimal solutions in a general framework allow users to apply the method to a large spectrum of industrial cases.*

The modeling of inputs developed here is indeed as general as possible. The variables used, especially in terms of learning vectors $\Lambda$ can be here any type of quantified predictor variables.

By having been able to express literaly the law of motion of the dynamics of the system taking into account the forecast updates in this particularly general case study, the optimal solutions have been computed and their expression analytically proved.

**Remark 2.3.0-2.** *Depending on the industrial environment defined by the costs parameters $B$, $H$, and $p$, two methods of approximation may be chosen differently to get efficient results.*

In particular, we found two interesting and distinct ranges of efficiency of the method.

The complexity of the computation method in an analytical manner pushed us into implementing an approximation of the value function at each step of the calculation of the dynamic programming.

Naturally, this approximation reduces the performance of our optimal solution.

However, the important interesting, and valuable point proved here is that whatever the cost configuration to which the system is exposed, we are still able to choose a compatible approximation function that keeps the stochastic dynamic programming approach at a better level of costs compared with any classical method.

It means that the forecast update procedure can still be taken into consideration for any of the values of the couple $(B, p)$, if and only if the corresponding approximation function is well chosen.

**Remark 2.3.0-3.** *Results are purely symmetric when it comes to the approximation function choice: no joint-set exists in which the approximation function is efficient.*

The choice of the approximation function is consequently directly linked with the positioning of the system into its industrial environment, but is not issued from a qualitative evaluation.

**Remark 2.3.0-4.** *We also have studied the case where a combinaison of the two approximation functions is used, defining a piecewise function so that in each*

*adequate interval, the most efficient form of approximation is defined. This case leads unfortunately to no result that can be expressed literally.*

We proved here that the consistency of any approximation function is symmetric and therefore complementary. The question to use both forms of approximation but into a unique piecewise continuous function is consequently a very interesting issue.

This solution of computation has been studied, but, for the same reasons that the threshold from which the choice of the form of the approximation function has to change cannot be calculated, the piecewise approximation function combining a quadratic form and a linear form is too complex to implement. The nested form of the dynamic programming computation procedure, used together with the dynamic properties of the forecast update, create results difficult to exploit.

# Closing Chapter

# Conclusion, and Perspectives

## 1 | Conclusion

In any industrial system concerned with supply chain management, and more specifically production planning issues, uncertainties are at the heart of numerous concerns. It is especially the case for markets that are likely to face non negligible customer demand uncertainties, as driving variable for the planning process.

By improving the flexibility of the decisional process, it is proved that the Rolling Horizon framework allows better management of these uncertainties, but increases instability and creates optimization modeling problems due to forecast updates procedure.

In this thesis, we consequently aimed to study this particular but still high-stake issue of the production planning optimization problem under uncertainties taking into account the forecasts update mechanism in Rolling Plan framework. In the lights of the lack of knowledge of such configurations in the current literature, the investigation of such optimization models for production planning under uncertainties and forecasts updates appeared indeed to us as a relevant topic of research.

The objective was therefore to be able to build a model and an adapted optimization framework for this decision environment existing in real life industrial problems.

In order to reach our goal, we brought theoretical materials through different contributions that follow the research questions pointed out in the introductory chapter:

- *How do Rolling Horizon Framework and forecasts updates impact the Demand Forecasts Modeling?*

  We presented in chapter 2 section 1 the fact that the forecasts update procedure that takes place in a rolling plan creates complex non-stationary and uncertain features of the customer demand.

  By defining a model built through an iterative way of conditional computation, and a double-level probabilistic structure, we managed to represent dynamically the customer demand, as well as its uncertainties. The distribution parameters are consequently random and recursively defined period after period.

- *How do such models influence Supply Chain Performances?*

  We showed in chapter 2 section 2 that such modeling has impacts on the parameters of the forecasted demand. In particular, we presented the stochastic features of these parameters and how it allows us to represent the epistemic uncertainties.

  A simulation finally illustrates this by pointing out the benefits of our forecast modeling on the a posteriori expected service level.

- *How can the forecasts update mechanism be modeled to be integrated into an optimization model?*

  Chapter 2 section 1 showed that an important point for the model to be exploitable is to clearly define the forecasting method, and, thanks to that, to explicitly model the update mechanism through laws of motion 2.3.

  In the chapter 2 section 2, this point has been analyzed in depth for the specific case of the linear regression froecasting technique.
  The results allows us not only to express the dynamics of the update, but also to be able to integrate it into an optimization routine.

- *What optimization model could be used in such cases?*

  We developed in chapter 3 section 1 a complete theoretical stochastic dynamic programming optimization framework that takes into account (1) both randomness and epistemic uncertainties through the iteratively built double level probabilistic model of the customer demand and, (2) the dynamics of the forecasts updates explicitly defined that were integrated through the laws of motion of the dynamics of the considered system into the optimization model.

- *How can optimal solutions be computed?*

  The optimal solutions of such an optimization problem has been literally calculated through a specifically developed framework.

  It consists first of the definition of the states and the laws of motion linking them dynamically. Then, the value function, defined as the local minimum cost function from one specific period to the time horizon is modelled through a nested form.
  These three main pillars finally allow us to recursively prove a dynamic propertie of the system, coming from successive conditional expectation computations.

  The proof of this recursive property has been realized in a general case in chapter 3 and section 1.

  As for explicit results, we demonstrated the litteral expression of optimal solutions through three different inductive reasonings.
  It has been indeed the object of study of the chapter 3 section 2 in the particular first simple application case of the use of simple linear regression in a zero lead-time environment.
  Then, Chapter 4 was dedicated to extensions of the model considered. The zero-lead-time assumption was first relaxed, and the general case of the use of the multiple linear regression as forecasting techniques was finally deeply studied.

- *Are these solutions more efficient than classical production planning methods, and to what degree do resulting production orders provide better performances?*

Each time explicit results could be computed, it has been tested and compared to standard stochastic planning procedure calculated through a targeted service level through simulated scenarios. The least costly of any of these has been selected as the performance assessment for each of the simulation run.
The results showed that the method globally outperformed the standard ones in the specific configurations tested.
Notwithstanding the net improvement, it is important to note the more complex the industrial environment is, the more subtle the user has to apply the method, inasmuch as the approximation function has to be chosen appropriately to get the best of benefits from the method.

- *In which context can such solutions be implemented?*

Sensitivity analyses allowed us to better understand the behavior of the performances of the method developed with respect to the different situations.

In particular, some solutions may diverge from one to another depending on the cost configurations. Detailed explanations have been presented in Chapter 4 section 2.

Moreover, it is important to precise that all the studies realized here have been conducted under assumptions of linear holding and production costs, as well as linear regression as forecasting techniques. Although these hypotheses may have generated different results and/or efficiency in cases of relaxation, they still embrace a large field of possible industrial environments, and thus can be implemented in different situations of MPS issues.

## 2 | Discussions and Perspectives

Besides the main roots that motivated this research, namely solving real life problems in uncertain environments under a rolling plan framework, the works presented in this manuscript fall within the intersection of the production planning field on the one hand, and the stochastic dynamic optimization on the other hand.

A consequence of this particular positioning has been also to provide a new application point from both these sides. That is to say, as far as the stochastic optimization is concerned, production planning under forecasts updates, as it has been

defined and explicitly transcribed here, was not, to our knowledge, an application yet studied as is. In the mean time, from the production planning point of view, stochastic dynamic programming approach has been seen here as an interesting tool to catch all the non-stationary and uncertain features that our problem has raised.

It appears consequently to us that our contributions presented in this manuscript may also bring interesting content for these two research areas as new angles of vision and directions of study.

This innovative modelling choice has brought slight arduousness though. Consequently, assumptions has had to be made to be able to go further to the objective defined.

Inasmuch as the forecasts updates have to be explicitly written before integration into the optimization framework, the choice of the forecasting technique is one of the main restrictive hypotheses that has to be made to implement our technique. We decided in a first instance to study the linear regression case.

Actually, as long as a forecasting technique

(1) allows users to write explicitely the distribution of the parameters $\theta_{i|t} \sim \psi\left(\Delta_{i-1|t}\right)$, as defined in 2.3, and
(2) that the corresponding composition function $\phi\left(\theta_{i|t}\right) = \phi_{\circ}\psi\left(\Delta_{i-1|t}\right)$ is integrable conditionally to the state of period $i$, meaning measurable with respect to the conditional pdf $\phi_{i|t}$ on the $\sigma-$algebra generated by the set of $\left(D_{k|t}\right)_{t\leq k<i}$,

then each conditional expectation can be theoretically calculated at each step of the dynamic programming defined in the computation framework we developed.

That means several extensions of our optimization model taking into account forecast update, and its corresponding computation framework developed in this manuscript can be contemplated. In particular, other forecasting techniques may be applied.

However, the computational complexity of each step of the iterative calculations may be quite high, as we could see here.

Perspectives can therefore be considered as issues focusing on the computation steps for specific but yet useful forecasting models. In particular, they can be reduced

to a series of calculus approximation problems, which may be already known or, at least, be supported by other techniques coming from the approximation of integrals.

Placed in the nested form of our model, they could bring values to lower the computational complexity in more general cases.

Other perspectives can be thought through simulation techniques instead of analytical studies. This would allow a wider spectrum of cases to be integrated into the model defined in this manuscript.

Finally, the dynamics of the forecast updates is here taken into consideration only in terms of non-stationarity of the mean variable. The standard deviation of the demand modeling could also be topic of study, although the random dispersion of the customer demand is more likely to be dependent on the time horizon rather than the history of the demand itself. As for the epistemic uncertainties, it is well history-dependent.

To this aim, Bayes models also seem to be potentially interesting in that sense. The epistemic uncertainties catching from the double-level probabilistic structure defined in this manuscript is indeed a well-developed and studied area in the Bayesian inference theory.

# Complementary Chapter

# Appendix

## 1 | Appendix 1

We present here the detailed calculations of the proof of the lemma 2.2.1 - 2 in Chapter 2, Section 2, Subsection 2, Subsubsection 1, Paragraph ii, Subparagraph ii

$\forall k \in [\![t, t+L-1]\!]$:

$$\mathbb{E}_{t+L-1}\left[D_{k|t}^2\right] = \left(-\alpha 1 \left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{k-1} j m_j\right) + \alpha 4 \left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{k-1} m_j\right)\right)^2 + \alpha 1^2 \sum_{j=t}^{k-1} j^2 \sigma_j^2 - 2\,\alpha 1\,\alpha 4 \sum_{j=t}^{k-1} j \sigma_j^2 + \alpha 4^2 \sum_{j=t}^{k-1} \sigma_j^2$$

$$\mathbb{E}_{t+L-1}\left[a_1^2\right] = \left(\alpha 1 \left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{t+L-1} j m_j\right) + \alpha 3 \left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{t+L-1} m_j\right)\right)^2 + \alpha 1^2 \sum_{j=t}^{t+L-1} j^2 \sigma_j^2 + 2\,\alpha 3\,\alpha 1 \sum_{j=t}^{t+L-1} j \sigma_j^2 + \alpha 3^2 \sum_{j=t}^{t+L-1} \sigma_j^2$$

$$\mathbb{E}_{t+L-1}\left[a_2^2\right] = \left(\alpha 1\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{t+L-1} m_j\right) + \alpha 2\left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{t+L-1} j m_j\right)\right)^2 + \alpha 1^2 \sum_{j=t}^{t+L-1} \sigma_j^2 + 2\,\alpha 1\,\alpha 2 \sum_{j=t}^{t+L-1} j\sigma_j^2 + \alpha 2^2 \sum_{j=t}^{t+L-1} j^2\sigma_j^2$$

$$\mathbb{E}_{t+L-1}\left[D_{k|t}.a_1\right] = \left(-\alpha 1\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{k-1} m_j\right) + \alpha 4\left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{k-1} j m_j\right)\right) \times \left(\alpha 1\left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{t+L-1} j m_j\right) + \alpha 3\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{t+L-1} m_j\right)\right) + \alpha 1 k\sigma_k^2 + \alpha 3\sigma_k^2$$

$$\mathbb{E}_{t+L-1}\left[D_{k|t}.a_2\right] = \left(-\alpha 1\left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{k-1} j m_j\right) + \alpha 4\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{k-1} m_j\right)\right) \times \left(\alpha 1\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{t+L-1} m_j\right) + \alpha 2\left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{t+L-1} j m_j\right)\right) + \alpha 2\, k\sigma_k^2 + \alpha 1\sigma_k^2$$

$$\mathbb{E}_{t+L-1}\left[a_1 a_2\right] = \left(\alpha 1\left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{t+L-1} j m_j\right) + \alpha 3\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{t+L-1} m_j\right)\right) \times \left(\alpha 1\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{t+L-1} m_j\right) + \alpha 2\left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{t+L-1} j m_j\right)\right) + \alpha 1\,\alpha 3 \sum_{j=t}^{t+L-1} \sigma_j^2 + \left(\alpha 1^2 + \alpha 2\,\alpha 3\right) \sum_{j=t}^{t+L-1} j\sigma_j^2 + \alpha 1\,\alpha 2 \sum_{j=t}^{t+L-1} j^2\sigma_j^2$$

Consequently,

$$\begin{aligned}
&\mathbb{E}_{t+L-1}\left[\sum_{k=1}^{t-1}\left(d_k - a_1(i) - a_2(i)k\right)^2 + \sum_{k=t}^{i-1}\left(D_{k|t} - a_1(i) - a_2(i)k\right)^2\right] = \\
&\quad \alpha 4^2\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{k-1} m_j\right)^2 \\
&\quad - 2\left(\alpha 4\, k\alpha 1 + \alpha 3\,\alpha 4\right)\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{k-1} m_j\right)\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{t+L-1} m_j\right) \\
&\quad - 2\,\alpha 1\,\alpha 4\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{k-1} m_j\right)\left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{k-1} j m_j\right) \\
&\quad - 2\left(\alpha 2\,\alpha 4\, k + \alpha 1\,\alpha 4\right)\left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{t+L-1} j m_j\right)\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{k-1} m_j\right) \\
&\quad + \left(k^2\alpha 1^2 + 2\,\alpha 3\, k\alpha 1 + \alpha 3^2\right)\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{k-1} m_j\right)^2 \\
&\quad + \left(2\, k\alpha 1^2 + 2\,\alpha 1\,\alpha 3\right)\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{k-1} m_j\right)\left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{t+L-1} j m_j\right) \\
&\quad + \left(2\, k\alpha 1^2 + \left(2\,\alpha 2\, k^2 + 2\,\alpha 3\right)\alpha 1 + 2\,\alpha 3\,\alpha 2\, k\right)\left(\sum_{j=1}^{t-1} d_j + \sum_{j=t}^{k-1} m_j\right) \\
&\quad \times \left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{k-1} j m_j\right) + \left(k^2\alpha 1^2 + 2\,\alpha 3\, k\alpha 1 + \alpha 3^2\right)\sum_{j=t}^{t+L-1}\sigma_j^2 \\
&\quad + \alpha 1^2\left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{t+L-1} j m_j\right)^2 + \left(2\,\alpha 2\, k\alpha 1 + 2\,\alpha 1^2\right)\left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{k-1} j m_j\right) \\
&\quad \times \left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{t+L-1} j m_j\right) + \left(\alpha 2^2 k^2 + 2\,\alpha 2\, k\alpha 1 + \alpha 1^2\right)\left(\sum_{j=1}^{t-1} j d_j + \sum_{j=t}^{k-1} j m_j\right)^2 \\
&\quad + \left(2\, k\alpha 1^2 + \left(2\,\alpha 2\, k^2 + 2\,\alpha 3\right)\alpha 1 + 2\,\alpha 3\,\alpha 2\, k\right)\sum_{j=t}^{t+L-1} j\sigma_j^2 \\
&\quad + \left(\alpha 2^2 k^2 + 2\,\alpha 2\, k\alpha 1 + \alpha 1^2\right)\sum_{j=t}^{t+L-1} j^2\sigma_j^2 + \alpha 1^2\sum_{j=t}^{k} j^2\sigma_j^2 \\
&\quad + \left(-2\,\alpha 4\sum_{j=t}^{k} j\sigma_j^2 - 4k\sigma_k^2\right)\alpha 1 - 2\,\alpha 2 k^2\sigma_k^2 - 2\,\alpha 3\,\sigma_k^2 + \alpha 4^2\sum_{j=t}^{k}\sigma_j^2
\end{aligned}$$

and finally,

$$\mathbb{E}_{t+L-1}\left[\sigma_{\varepsilon_i}^2\right] = \frac{1}{i-4}\mathbb{E}_{t+L-1}\left[\sum_{k=1}^{t-1}\left(d_k - a_1(i) - a_2(i)k\right)^2 + \sum_{k=t}^{i-1}\left(D_{k|t} - a_1(i) - a_2(i)k\right)^2\right]$$

which lead us to

$$\mathbb{E}_{t+L-1}\left[\sigma_i^2\right] = \gamma_i \, \mathbb{E}_{t+L-1}\left[\sigma_{\varepsilon_i}^2\right]$$

# 2 | Appendix 2

We present here the detailed calculations of the proof of the theorem 1.2.0 - 1 in Chapter 4, Section 1, Subsection 2

The expectation of the value function of period $j$ conditionnally to the state of period $j-1$ is:

$\forall j \in [\![t+1, t+N-3]\!]$:

$$\mathbb{E}_{j-2}\Big[V_j\left(X_{j-1}, \Delta_{j-2|t}, D_{j-1|t}, c_{j-1}, c_{j-2}\right)\Big] =$$

$$\begin{aligned}
\frac{\gamma_2^j}{(j-1)^2}\Bigg(&\Big((X_{j-1} + c_{j-2} + c_{j-1} - m_{j-1} - \beta_{m_j}\left(\Delta_{j-2|t}\right))^2 + X_{j-1} + \frac{2}{3}\,{s_{j-1}}^2\Big)j^2 \\
&+ 2\Big( - \Big(X_{j-1} + c_{j-2} + c_{j-1} + m_{j-1} - \beta_{m_j}\left(\Delta_{j-2|t}\right)\Big)^2 \\
&\qquad + 4\,{m_{j-1}}^2 + 2\,{s_{j-1}}^2 - X_{j-1}\Big)j \\
&+ \Big(X_{j-1} + c_{j-2} + c_{j-1} + 3\,m_{j-1} - \beta_{m_j}\left(\Delta_{j-2|t}\right)\Big)^2 + 6\,{s_{j-1}}^2 + X_{j-1}\Bigg) \\
&+ \gamma_d^j m_{j-1} + \gamma_{c2}^j c_{j-2} + \gamma_{c1}^j c_{j-1} + \gamma_\beta^j \beta\left(\Delta_{j-2|t}\right) + \Gamma_j
\end{aligned}$$

The expectation of the cost of period $j-1$ conditionnally to the state of period $j-1$ is:

$$\mathbb{E}_{j-2}\Big[K_{j-1}\left(X_{j-1}, D_{j-1|t}, c_{j-1}, c_{j-2}\right)\Big] =$$

$$\begin{cases} -B\left(X_{j-1}+c_{j-2}-m_{j-1}\right)+pc_{j-1} & if \quad \beta_{m_{t+N-2}}^{t+N-3}X_{j-1}+c_{j-2} < m_{j-1}-2\,s_{j-1} \\ \frac{1}{3}\left(H+B\right)s_{j-1}-\frac{1}{2}\left(-m_{j-1}+X_{j-1}+c_{j-2}\right)\left(B-H\right) & \\ +\frac{H+B}{4s_{j-1}}\left(X_{j-1}+c_{j-2}-m_{j-1}\right)^2 & if \quad m_{j-1}-2\,s_{j-1} \le X_{j-1}+c_{j-2} \\ +\frac{1}{24s_{j-1}^2}\left(X_{j-1}+c_{j-2}-m_{j-1}\right)^3\left(H+B\right)+pc_{j-1} & \quad < m_{j-1} \\ \frac{1}{3}\left(H+B\right)s_{j-1}-\frac{1}{2}\left(-m_{j-1}+X_{j-1}+c_{j-2}\right)\left(B-H\right) & \\ +\frac{H+B}{4s_{j-1}}\left(X_{j-1}+c_{j-2}-m_{j-1}\right)^2 & if \quad m_{j-1} \le X_{j-1}+c_{j-2} \\ +\frac{1}{24s_{j-1}^2}\left(X_{j-1}+c_{j-2}-m_{j-1}\right)^3\left(H+B\right)+pc_{j-1} & \quad < m_{j-1}+2\,s_{j-1} \\ H\left(X_{j-1}+c_{j-2}-m_{j-1}\right)+pc_{j-1} & if \quad m_{j-1}+2\,s_{j-1} \le X_{j-1}+c_{j-2} \end{cases}$$

The value function of period $j-1$ with respect to the state of period $j-1$ is:

$$V_{j-1}\left(X_{j-1}, \Delta_{j-2|t}, c_{j-2}\right) =$$

$$\left(\beta_{mj}\left(\Delta_{j-2|t}\right)-X_{j-1}-c_{j-2}+\frac{j+3}{j-1}m_{j-1}\right)\gamma_{c1}^{j}+\gamma_{c2}^{j}c_{j-2}+\gamma_{\beta}^{j}\beta\left(\Delta_{j-3|t}\right)$$
$$+\gamma_{d2}^{j}D_{j-2|t}+\gamma_{d}^{j}m_{j-1}+\Gamma_j+\gamma_2^{j}X_{j-1}+2{s_{j-1}}^2\gamma_2^{j}\left(\frac{j+3}{j-1}\right)^2+\frac{\left(p-\gamma_{c1}^{j}\right)\left(p+\gamma_{c1}^{j}\right)}{4\gamma_2^{j}}$$

with

$$c_{j-1}^{*}=\beta_{m_j}\left(\Delta_{j-2|t}\right)-X_{j-1}-c_{j-2}+\frac{j+3}{j-1}m_{j-1}-\frac{p+\gamma_{c1}^{j}}{2\gamma_2^{j}}$$

and $\mathbb{E}_{j-2}\Big[K_{j-1}\left(X_{j-1}, D_{j-1|t}, c_{j-1}, c_{j-2}\right)\Big]$ previously determined.

The value function of period $j-1$ with respect to the state of period $j-1$ is after approximation:

$$\tilde{V}_{j-1}\left(X_{j-1},\Delta_{j-2|t},c_{j-2}\right)=$$

$$\begin{aligned}
&\frac{c_{j-2}^{2}\left(B+H\right)}{6s_{j-1}}+\left(\frac{B-H}{2}-p-\gamma_{c1}^{j}+\gamma_{c2}^{j}+\frac{B+H}{3s_{j-1}}\left(X_{j-1}-m_{j-1}\right)\right)c_{j-2}\\
&+\frac{B+H}{6s_{j-1}}\left(X_{j-1}-m_{j-1}\right)^{2}\\
&+\left(\frac{j+3}{j-1}\right)^{2}\frac{2\gamma_{2}^{j}s_{j-1}^{2}}{3}+\frac{B+H}{3}s_{j-1}+\left(\frac{B-H}{2}+\gamma_{d}^{j}+\left(p+\gamma_{c1}^{j}\right)\frac{j+3}{j-1}\right)m_{j-1}\\
&-\left(\frac{B-H}{2}+p+\gamma_{c1}^{j}-\gamma_{2}^{j}\right)X_{j-1}+\gamma_{\beta}^{j}\beta\left(\Delta_{j-2|t}\right)+p\beta_{mj}\left(\Delta_{j-2|t}\right)+\\
&\gamma_{c1}^{j}\beta_{mj}\left(\Delta_{j-2|t}\right)+\Gamma_{j}-\frac{\left(p+\gamma_{c1}^{j}\right)^{2}}{4\gamma_{2}^{j}}
\end{aligned}$$

The value function of period $j-1$ with respect to the state of period $j-1$ is after implementation of the laws of motion:

$$\tilde{V}_{j-1}\left(X_{j-2},\Delta_{j-3|t},D_{j-2|t},c_{j-2},c_{j-3}\right)=$$

$$\begin{aligned}
&\frac{c_{j-2}^{2}\left(B+H\right)}{6s_{j-1}}\\
&+\left(\frac{B-H}{2}-p-\gamma_{c1}^{j}+\gamma_{c2}^{j}+\frac{B+H}{3s_{j-1}}\left(X_{j-2}+c_{j-3}-\frac{j+2}{j-2}D_{j-2|t}-\beta_{m_{j-1}}\left(\Delta_{j-3|t}\right)\right)\right)c_{j-2}\\
&+\frac{B+H}{6s_{j-1}}\left(X_{j-2}+c_{j-3}-\frac{j+2}{j-2}D_{j-2|t}-\beta_{m_{j-1}}\left(\Delta_{j-3|t}\right)\right)^{2}\\
&+\left(\frac{j+3}{j-1}\right)^{2}\frac{2\gamma_{2}^{j}s_{j-1}^{2}}{3}+\frac{B+H}{3}s_{j-1}\\
&+\left(\frac{B-H}{2}+\gamma_{d}^{j}+\left(p+\gamma_{c1}^{j}\right)\frac{j+3}{j-1}\right)\left(\frac{4}{j-2}D_{j-2|t}+\beta_{m_{j-1}}\left(\Delta_{j-3|t}\right)\right)\\
&-\left(\frac{B-H}{2}+p+\gamma_{c1}^{j}-\gamma_{2}^{j}\right)\left(X_{j-2}+c_{j-3}-D_{j-2|t}\right)+\gamma_{\beta}^{j}\beta\left(\Delta_{j-2|t}\right)\\
&+p\left(2\frac{\left(2\,j-7\right)D_{j-2|t}}{\left(j-1\right)\left(j-2\right)}+\beta_{m_{j}}\left(\Delta_{j-3|t}\right)\right)+\gamma_{c1}^{j}\left(2\frac{\left(2\,j-7\right)D_{j-2|t}}{\left(j-1\right)\left(j-2\right)}+\beta_{m_{j}}\left(\Delta_{j-3|t}\right)\right)\\
&+\Gamma_{j}-\frac{\left(p+\gamma_{c1}^{j}\right)^{2}}{4\gamma_{2}^{j}}
\end{aligned}$$

The expectation of the costs of period $t+N-1$ conditionnally to the state of period $t+N-1$ is:

$\mathbb{E}_{t+N-1}\left[K_{t+N-1}\left(X_{t+N-1}, D_{t+N-1|t}, c_{t+N-2}\right)\right] =$

$$\begin{cases} -B\left(X_{t+N-1}+c_{t+N-2}-m_{t+N-1}\right) & if \quad X_{t+N-1}+c_{t+N-2} < m_{t+N-1}-2\,s_{t+N-1} \\ -B\left(X_{t+N-1}+c_{t+N-2}-m_{t+N-1}\right) & \\ +\frac{H+B}{24s^2_{t+N-1}}\Big(X_{t+N-1}+c_{t+N-2}-m_{t+N-1} & if \quad m_{t+N-1}-2\,s_{t+N-1} \leq X_{t+N-1}+c_{t+N-2} \\ \qquad +2\,s_{t+N-1}\Big)^3 & \qquad < m_{t+N-1} \\ -B\left(X_{t+N-1}+c_{t+N-2}-m_{t+N-1}\right) & \\ +\frac{H+B}{12s^2_{t+N-1}}\Big(\frac{1}{2}\Big(X_{t+N-1}+c_{t+N-2}-m_{t+N-1} & if \quad m_{t+N-1} \leq X_{t+N-1}+c_{t+N-2} \\ \qquad +2\,s_{t+N-1}\Big)^3 & \qquad < m_{t+N-1}+2\,s_{t+N-1} \\ \qquad -\left(X_{t+N-1}+c_{t+N-2}-m_{t+N-1}\right)^3\Big) & \\ H\left(X_{t+N-1}+c_{t+N-2}-m_{t+N-1}\right) & if \quad m_{t+N-1}+2\,s_{t+N-1} \leq X_{t+N-1}+c_{t+N-2} \end{cases}$$

The conditional expected cost of the two last periods:
$\mathbb{E}_{t+N-3}\left[K_{t+N-2}+\hat{V}_{t+N-1}\right]=$

$$
\begin{aligned}
&\frac{1}{18\,(t+N-2)^2\,s_{t+N-1}}\Bigg[\Bigg(6(H+B){s_{t+N-1}}^2+9\;(\lambda-c_{t+N-2})\,(B-H)\,s_{t+N-1}\\
&\qquad+2\,(H+B)\Big({s_{t+N-2}}^2+\frac{3\lambda^2}{2}-3\,\lambda c_{t+N-2}+\frac{3{c_{t+N-2}}^2}{2}\Big)\Bigg)t+N-1^2\\
&\qquad+\Bigg(-12\,(H+B)\,{s_{t+N-1}}^2-18\;(\lambda-c_{t+N-2})\,(B-H)\,s_{t+N-1}\\
&\qquad+12\,\Big({s_{t+N-2}}^2-1/2\,\lambda^2+\lambda\,c_{t+N-2}-1/2\,{c_{t+N-2}}^2\Big)\,(H+B)\Big)\,(t+N-1)\\
&\qquad+6(H+B){s_{t+N-1}}^2+9\;(\lambda-c_{t+N-2})\,(B-H)\,s_{t+N-1}\\
&\qquad+18\;(H+B)\Big({s_{t+N-2}}^2+1/6\,\lambda^2-1/3\,\lambda\,c_{t+N-2}+1/6\,{c_{t+N-2}}^2\Big)\Bigg]\\
&+\begin{cases}
-B\,(X_{t+N-2}+c_{t+N-3}-m_{t+N-2})+pc_{t+N-2} & if \quad X_{t+N-2}+c_{t+N-3}\\
\qquad\qquad & \qquad <m_{t+N-2}-2\,s_{t+N-2}\\
\frac{1}{3}\,(H+B)\,s_{t+N-2}+pc_{t+N-2}\\
-\frac{B-H}{2}\,(-m_{t+N-2}+X_{t+N-2}+c_{t+N-3}) & if \quad m_{t+N-2}-2\,s_{t+N-2}\\
+\frac{H+B}{4s_{t+N-2}}\,(X_{t+N-2}+c_{t+N-3}-m_{t+N-2})^2 & \qquad \le X_{t+N-2}+c_{t+N-3}\\
+\frac{H+B}{24s_{t+N-2}^2}\,(X_{t+N-2}+c_{t+N-3}-m_{t+N-2})^3 & \qquad <m_{t+N-2}\\
\frac{1}{3}\,(H+B)\,s_{t+N-2}+pc_{t+N-2}\\
-\frac{B-H}{2}\,(-m_{t+N-2}+X_{t+N-2}+c_{t+N-3}) & if \quad m_{t+N-2}\\
+\frac{H+B}{4s_{t+N-2}}\,(X_{t+N-2}+c_{t+N-3}-m_{t+N-2})^2 & \qquad \le X_{t+N-2}+c_{t+N-3}\\
+\frac{H+B}{24s_{t+N-2}^2}\,(X_{t+N-2}+c_{t+N-3}-m_{t+N-2})^3 & \qquad <m_{t+N-2}+2\,s_{t+N-2}\\
H\,(X_{t+N-2}+c_{t+N-3}-m_{t+N-2})+pc_{t+N-2} & if \quad m_{t+N-2}+2\,s_{t+N-2}\\
\qquad\qquad & \qquad \le X_{t+N-2}+c_{t+N-3}
\end{cases}
\end{aligned}
$$

where $\lambda=\frac{t+N+2}{t+N-2}m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)-X_{t+N-2}-c_{t+N-3}$

The value of the minimum of the function $V_{t+N-2}$ at the point $c^*_{t+N-2}$ with respect to the state of period $t+N-2$ and $c_{t+N-3}$:

$$V_{t+N-2}\left(\Delta_{t+N-2|t}, c_{t+N-3}\right) =$$

$$A_{t+N-2}+$$

$$\begin{cases}
\begin{array}{ll}
-B\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right) & \\
+p\Big(\frac{t+N+2}{t+N-2}m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) & if \quad X_{t+N-2}+c_{t+N-3} \\
\quad -X_{t+N-2}-c_{t+N-3}\Big) & \quad < m_{t+N-2}-2\,s_{t+N-2}
\end{array} \\
\begin{array}{ll}
\frac{1}{3}\left(H+B\right)s_{t+N-2} & \\
+p\Big(\frac{t+N+2}{t+N-2}m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) & \\
\quad -X_{t+N-2}-c_{t+N-3}\Big) & if \quad m_{t+N-2}-2\,s_{t+N-2} \\
-\frac{B-H}{2}\left(-m_{t+N-2}+X_{t+N-2}+c_{t+N-3}\right) & \quad \leq X_{t+N-2}+c_{t+N-3} \\
+\frac{H+B}{4s_{t+N-2}}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^2 & \quad < m_{t+N-2} \\
+\frac{H+B}{24s^2_{t+N-2}}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^3 &
\end{array} \\
\begin{array}{ll}
\frac{1}{3}\left(H+B\right)s_{t+N-2} & \\
+p\Big(\frac{t+N+2}{t+N-2}m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) & \\
\quad -X_{t+N-2}-c_{t+N-3}\Big) & if \quad m_{t+N-2} \\
-\frac{B-H}{2}\left(-m_{t+N-2}+X_{t+N-2}+c_{t+N-3}\right) & \quad \leq X_{t+N-2}+c_{t+N-3} \\
+\frac{H+B}{4s_{t+N-2}}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^2 & \quad < m_{t+N-2}+2\,s_{t+N-2} \\
+\frac{H+B}{24s^2_{t+N-2}}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^3 &
\end{array} \\
\begin{array}{ll}
H\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right) & \\
+p\Big(\frac{t+N+2}{t+N-2}m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) & if \quad m_{t+N-2}+2\,s_{t+N-2} \\
\quad -X_{t+N-2}-c_{t+N-3}\Big) & \quad \leq X_{t+N-2}+c_{t+N-3}
\end{array}
\end{cases}$$

where $A_{t+N-2} =$

$$\frac{\left(36\left(8H^2+p^2\right)-\left(B-17\,H\right)^2\right)s_{t+N-1}}{24\left(H+B\right)}+\frac{{s_{t+N-2}}^2\left(H+B\right)}{9s_{t+N-1}}\left(\frac{t+N+2}{t+N-2}\right)^2$$
$$+\frac{3p\left(B-H-2\,p\right)s_{t+N-1}}{2(H+B)}$$

# 3 | Appendix 3

We present here the proof of the theorem 2.2.1 - 1 in Chapter 4, Section 2, Subsection 2, Subsubsection 1, Paragraph ii.

The proof is structured as an inductive reasoning. The recursive property is the form of the value function in any period $j$ with respect to the state information of period $j-1$, and the associated optimal solution $c_j$.

## 3.1 | Base Case

The base case corresponds to the detailed calculations of the value function of the second to last period.

To compute its value, we calculate first an equivalent of the value function - without the optimization problem- for the last period.

### 3.1.1 | Last period calculations

We have:

$$K_{t+N-1} = \begin{cases} H \times \left(X_{t+N-1} + c_{t+N-2} - D_{t+N-1|t}\right) & \text{if } X_{t+N-1} + c_{t+N-2} - D_{t+N-1|t} > 0 \\ -\ B \times \left(X_{t+N-1} + c_{t+N-2} - D_{t+N-1|t}\right) & \text{otherwise.} \end{cases}$$

Let $\tilde{V}_{t+N-1}$ be equal to $\mathbb{E}_{t+N-2}\left[K_{t+N-1}\right]$

By integration $K_{t+N-1}$ with respect to $\phi_{t+N-1|t}$, we get

$$\tilde{V}_{t+N-1} = \begin{cases} -B\left(X_{t+N-1} + c_{t+N-2} - m_{t+N-1}\right) & \text{if (1)} \\ -B\left(X_{t+N-1} + c_{t+N-2} - m_{t+N-1}\right) & \\ \quad + \frac{H+B}{24 s_{t+N-1}^2}\left(X_{t+N-1} + c_{t+N-2} - m_{t+N-1} + 2s_{t+N-1}\right)^3 & \text{if (2)} \\ -B\left(X_{t+N-1} + c_{t+N-2} - m_{t+N-1}\right) & \\ \quad + \frac{H+B}{24 s_{t+N-1}^2}\left(X_{t+N-1} + c_{t+N-2} - m_{t+N-1} + 2s_{t+N-1}\right)^3 & \text{if (3)} \\ \quad - \frac{H+B}{12 s_{t+N-1}^2}\left(X_{t+N-1} + c_{t+N-2} - m_{t+N-1}\right)^3 & \\ H\left(X_{t+N-1} + c_{t+N-2} - m_{t+N-1}\right) & \text{if (4)} \end{cases}$$

where (1), (2), (3), (4) are defined as the following sets with respect to $c_{t+N-2}$:

$$\begin{cases} (1): & & X_{t+N-1}+c_{t+N-2} & < m_{t+N-1}-2s_{t+N-1} \\ (2): & m_{t+N-1}-2s_{t+N-1} \leq & X_{t+N-1}+c_{t+N-2} & < m_{t+N-1} \\ (3): & m_{t+N-1} \leq & X_{t+N-1}+c_{t+N-2} & < m_{t+N-1}+2s_{t+N-1} \\ (4): & m_{t+N-1}+2s_{t+N-1} \geq & X_{t+N-1}+c_{t+N-2} & \end{cases}$$

In order to be able to calculate $\mathbb{E}_{t+N-3}\left[\tilde{V}_{t+N-1}\right]$, we approximate $\tilde{V}_{t+N-1}$ with a continuous and derivable function with respect to $D_{t+N-2|t}$. To do so, we will use a quadratic function with respect to $c_{t+N-2}$:

We note 3 remarquable points through which we want our approximation to go: the boundaries of the definition of the cost function for period $t+N-1$: first point $c_{t+N-2} = m_{t+N-1} - 2s_{t+N-1} - X_{t+N-1}$ where $\tilde{V}_{t+N-1}$ is equal to $2Bs_{t+N-1}$, second point $c_{t+N-2} = m_{t+N-1} + 2s_{t+N-1} - X_{t+N-1}$ where $\tilde{V}_{t+N-1}$ is equal to $2Hs_{t+N-1}$, and the third in $c_{t+N-2} = m_{t+N-1} - X_{t+N-1}$ where $\tilde{V}_{t+N-1}$ is equal to $\frac{H+B}{3}s_{t+N-1}$.

We finally get

$$\begin{aligned} \tilde{V}_{t+N-1} =& \frac{(H+B)\,{c_{t+N-2}}^2}{6s_{t+N-1}} \\ &+ \left(\frac{H-B}{2} + \frac{(X_{t+N-1}-m_{t+N-1})\,(H+B)}{3s_{t+N-1}}\right) c_{t+N-2} \\ &+ \frac{(H+B)\,s_{t+N-1}}{3} + \frac{(X_{t+N-1}-m_{t+N-1})\,(H-B)}{2} \\ &+ \frac{(X_{t+N-1}-m_{t+N-1})^2\,(H+B)}{6s_{t+N-1}} \end{aligned}$$

### 3.1.2 | Second Last period Calculations

#### i. Application of the laws of motion of the system

Let $X_{t+N-1}$ be equal to $X_{t+N-2} + c_{t+N-3} - D_{t+N-1|t}$

Let $m_{t+N-1}$ be equal to $\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) + \beta^{t+N-2}_{m_{t+N-1}} D_{t+N-2|t}$ as in 4.9

#### ii. Integration of $\tilde{V}_{t+N-1}$ with respect to $D_{t+N-2|t}$ conditionnally to the state in $t+N-2$:

$\mathbb{E}_{t+N-3}\left[\tilde{V}_{t+N-1}\right] =$

$$
\begin{aligned}
&\frac{{s_{t+N-2}}^2\,(H+B)\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)^2}{9s_{t+N-1}}+\frac{H+B}{3s_{t+N-1}}\\
&+\frac{B-H}{2}\left(m_{t+N-2}\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)-c_{t+N-2}-X_{t+N-2}\right.\\
&\qquad\left.-c_{t+N-3}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)\right)\\
&+\frac{H+B}{6s_{t+N-1}}\left(m_{t+N-2}\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)-c_{t+N-2}-X_{t+N-2}\right.\\
&\qquad\left.-c_{t+N-3}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)\right)^2
\end{aligned}
$$

**iii. Expectation of $K_{t+N-2}$ with respect to $D_{t+N-2|t}$ conditionnally to the state of $t+N-2$:**

$\mathbb{E}_{t+N-3}\left[K_{t+N-2}\right] =$

$$
\begin{cases}
-B\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)+pc_{t+N-2}- & if \quad X_{t+N-2}+c_{t+N-3} < m_{t+N-2}-2\,s_{t+N-2}\\
\frac{1}{3}\left(H+B\right)s_{t+N-2}+pc_{t+N-2} & \\
-\frac{B-H}{2}\left(-m_{t+N-2}+X_{t+N-2}+c_{t+N-3}\right) & if \quad m_{t+N-2}-2\,s_{t+N-2} \le X_{t+N-2}+c_{t+N-3} < m_{t+N-2}\\
+\frac{H+B}{4s_{t+N-2}}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^2 & \\
+\frac{H+B}{24s_{t+N-2}^2}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^3 & \\
\frac{1}{3}\left(H+B\right)s_{t+N-2}+pc_{t+N-2} & \\
-\frac{B-H}{2}\left(-m_{t+N-2}+X_{t+N-2}+c_{t+N-3}\right) & if \quad m_{t+N-2} \le X_{t+N-2}+c_{t+N-3} < m_{t+N-2}+2\,s_{t+N-2}\\
+\frac{H+B}{4s_{t+N-2}}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^2 & \\
+\frac{H+B}{24s_{t+N-2}^2}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^3 & \\
H\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)+pc_{t+N-2} & if \quad m_{t+N-2}+2\,s_{t+N-2} \le X_{t+N-2}+c_{t+N-3}
\end{cases}
$$

We easily conclude the quantity we are looking for $\mathbb{E}_{t+N-3}\left[K_{t+N-2}+\tilde{V}_{t+N-1}\right]$ is the sum of both quantities that we now know: $\mathbb{E}_{t+N-3}\left[K_{t+N-2}\right]+\mathbb{E}_{t+N-3}\left[\tilde{V}_{t+N-1}\right]$

**iv. Optimization problem for the two last periods**

We are looking for the argument of the minimum of the quantity $\mathbb{E}_{t+N-3}\left[K_{t+N-2}+\tilde{V}_{t+N-1}\right]$ with respect to $c_{t+N-2}$.

We notice first that the function $c_{t+N-2} \to \mathbb{E}_{t+N-3}\left[K_{t+N-2}+\tilde{V}_{t+N-1}\right]$ is continuous, derivable, and convex with respect to $c_{t+N-2}$. We solve the optimization

problem we face here by looking for the solution of the equation
$\frac{\partial}{\partial c_{t+N-2}}\mathbb{E}_{t+N-3}\left[K_{t+N-2}+\tilde{V}_{t+N-1}\right]=0$

We have:

$$\frac{\partial}{\partial c_{t+N-2}}\mathbb{E}_{t+N-3}\left[K_{t+N-2}+\tilde{V}_{t+N-1}\right]=$$

$$-\frac{H+B}{3s_{t+N-1}}\left(\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)+\beta^{t+N-2}_{m_{t+N-1}}m_{t+N-2}\right)$$
$$+\frac{H+B}{6s_{t+N-1}}\left(-2m_{t+N-2}-3s_{t+N-1}+2X_{t+N-2}+2c_{t+N-2}+2c_{t+N-3}\right)+H+p$$

which is equal to zero for

$$\begin{aligned}c^{*}_{t+N-2}=&\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)+m_{t+N-2}\left(\beta^{t+N-2}_{m_{t+N-1}}+1\right)-c_{t+N-3}-X_{t+N-2}\\&+3\,\frac{s_{t+N-1}\left(B-H-2\,p\right)}{2\,B+2\,H}\end{aligned}$$

**v. Expression of $V_{t+N-2}$**

$V_{t+N-2}$ can be written, with respect to the state of $t+N-2$:

$V_{t+N-2} = A_{t+N-2}+$

$$\begin{cases} -B\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right) \\ +p\Big(\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)+m_{t+N-2}\left(\beta^{t+N-2}_{m_{t+N-1}}+1\right) \\ \qquad -c_{t+N-3}-X_{t+N-2}\Big) & if & X_{t+N-2}+c_{t+N-3} \\ & & < m_{t+N-2}-2\,s_{t+N-2} \\ \frac{1}{3}\left(H+B\right)s_{t+N-2} \\ +p\Big(\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)+m_{t+N-2}\left(\beta^{t+N-2}_{m_{t+N-1}}+1\right) \\ \qquad -c_{t+N-3}-X_{t+N-2}\Big) & & m_{t+N-2}-2\,s_{t+N-2} \\ -\frac{B-H}{2}\left(-m_{t+N-2}+X_{t+N-2}+c_{t+N-3}\right) & if & \leq X_{t+N-2}+c_{t+N-3} \\ +\frac{H+B}{4s_{t+N-2}}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^2 & & < m_{t+N-2} \\ +\frac{H+B}{24s^2_{t+N-2}}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^3 \\ \frac{1}{3}\left(H+B\right)s_{t+N-2} \\ +p\Big(\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)+m_{t+N-2}\left(\beta^{t+N-2}_{m_{t+N-1}}+1\right) \\ \qquad -c_{t+N-3}-X_{t+N-2}\Big) & & m_{t+N-2} \\ -\frac{B-H}{2}\left(-m_{t+N-2}+X_{t+N-2}+c_{t+N-3}\right) & if & \leq X_{t+N-2}+c_{t+N-3} \\ +\frac{H+B}{4s_{t+N-2}}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^2 & & < m_{t+N-2}+2\,s_{t+N-2} \\ +\frac{H+B}{24s^2_{t+N-2}}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^3 \\ H\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right) \\ +p\Big(\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)+m_{t+N-2}\left(\beta^{t+N-2}_{m_{t+N-1}}+1\right) & if & m_{t+N-2}+2\,s_{t+N-2} \\ \qquad -c_{t+N-3}-X_{t+N-2}\Big) & & \leq X_{t+N-2}+c_{t+N-3} \end{cases}$$

where $A_{t+N-2}$ is equal to

$$\frac{{s_{t+N-2}}^2\left(H+B\right)\left(\beta^{t+N-2}_{m_{t+N-1}}+1\right)^2}{9s_{t+N-1}}-\frac{H+B}{24}s_{t+N-1}+\frac{3}{2}\frac{s_{t+N-1}\left(BH+p^2\right)}{B+H} + \frac{3}{2}\frac{ps_{t+N-1}\left(B-H-2\,p\right)}{H+B}$$

and corresponds to the non-historical dependent part of the value function $V_{t+N-2}$.

**vi. Laws of motion and expression of $V_{t+N-2}$ with respect to the information of state $t+N-3$**

To be able to dynamically compute the expectations from one period to another in the dynamic calculation of the proof, we firstly need to approximate the function $V_{t+N-2}$ with respect to $c_{t+N-3}$:

**Approximation** To do so, we want to express $V_{t+N-2}$ with respect to $c_{t+N-3}$ with a continous and derivable function.We approximate then the function with a continuous quadratic function going through 3 specific points. We choose the boudaries of the intervals definition. As a consequence, we keep as structuring couples the followings:

- $c_{t+N-3} = m_{t+N-2} - 2\,s_{t+N-2} - X_{t+N-2}$
  for which
  $V_{t+N-2} = 2s_{t+N-2}B + p(\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) + m_{t+N-2}\beta^{t+N-2}_{m_{t+N-1}} + 2s_{t+N-2}) + A_{t+N-2}$
- $c_{t+N-3} = m_{t+N-2} - X_{t+N-2}$
  for which
  $V_{t+N-2} = \frac{B+H}{3}s_{t+N-2} + p(\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) + m_{t+N-2}\beta^{t+N-2}_{m_{t+N-1}}) + A_{t+N-2}$
- $c_{t+N-3} = m_{t+N-2} + 2\,s_{t+N-2} - X_{t+N-2}$
  for which
  $V_{t+N-2} = 2s_{t+N-2}H + p(\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) + m_{t+N-2}\beta^{t+N-2}_{m_{t+N-1}} - 2s_{t+N-2}) + A_{t+N-2}$

We are finally able to re-write

$$\begin{aligned} V_{t+N-2} =& \frac{B+H}{3}s_{t+N-2} \\ &+ p\Big( - X_{t+N-2} - c_{t+N-3} + \left(\beta^{t+N-2}_{m_{t+N-1}} + 1\right)m_{t+N-2} + \beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)\Big) \\ &+ \frac{H+B}{6s_{t+N-2}}\Big(X_{t+N-2} + c_{t+N-3} - m_{t+N-2}\Big)^2 \\ &+ \frac{H-B}{2}\Big(X_{t+N-2} + c_{t+N-3} + m_{t+N-2}\Big) + A_{t+N-2} \end{aligned}$$

**Application of the Laws of motion**

Let $X_{t+N-2}$ be equal to $X_{t+N-3} + c_{t+N-4} - D_{t+N-3|t}$

Let $m_{t+N-2}$ be equal to $\beta_{m_{t+N-2}}\left(\Delta_{t+N-4|t}\right) + \beta^{t+N-3}_{m_{t+N-2}}D_{t+N-3|t}$ as in 4.9

Let $\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)$ be equal to $\beta^{t+N-3}_{m_{t+N-1}}D_{t+N-3|t} + \beta_{m_{t+N-1}}\left(\Delta_{t+N-4|t}\right)$ as in 4.9

Then, we can re-write:

$V_{t+N-2} =$

$$\frac{B+H}{3} s_{t+N-2} + A_{t+N-2}$$
$$\frac{H+B}{6 s_{t+N-2}} \left( X_{t+N-3} + c_{t+N-3} + c_{t+N-4} - \left( \beta^{t+N-3}_{m_{t+N-2}} + 1 \right) D_{t+N-3|t} - \beta_{m_{t+N-2}} \left( \Delta_{t+N-4|t} \right) \right)^2$$
$$+ \frac{B-H+2\,p}{2} \left( \left( \beta^{t+N-3}_{m_{t+N-2}} + 1 \right) D_{t+N-3|t} + \beta_{m_{t+N-2}} \left( \Delta_{t+N-4|t} \right) - X_{t+N-3} - c_{t+N-4} - c_{t+N-3} \right)$$
$$+ p \Bigg( \left( \beta^{t+N-3}_{m_{t+N-1}} + \beta^{t+N-2}_{m_{t+N-1}} \beta^{t+N-3}_{m_{t+N-2}} \right) D_{t+N-3|t} + \beta^{t+N-2}_{m_{t+N-1}} \beta_{m_{t+N-2}} \left( \Delta_{t+N-4|t} \right)$$
$$+ \beta_{m_{t+N-1}} \left( \Delta_{t+N-4|t} \right) \Bigg)$$

## 3.2 | Inductive Step

We define $\gamma_2^k$, $\gamma_{c1}^k$, $\gamma_{c2}^k$, $\gamma_x^k$, $\gamma_d^k$, $\gamma_{d2}^k$,$\gamma_{\beta_{m_k}}^k$,$\gamma_{\beta_{m_{k+1}}}^k$, and $\Gamma^k$ such as, for any $k \in [\![t, t+N-3]\!]$, we can express $V_k$ as follow:

$$\begin{aligned} V_k =& \gamma_2^k \left( c_{k-1} + c_{k-2} + X_{k-1} - \gamma_{d2}^k D_{k-1|t} - \beta_{m_k} \left( \Delta_{k-2|t} \right) \right)^2 \\ &+ \gamma_{c1}^k c_{k-1} + \gamma_{c2}^k c_{k-2} + \gamma_x^k X_{k-1} + \gamma_d^k D_{k-1|t} \\ &+ \gamma_{\beta_{m_k}} \beta_{m_k} \left( \Delta_{k-2|t} \right) + \gamma_{\beta_{m_{k+1}}} \beta_{m_{k+1}} \left( \Delta_{k-2|t} \right) + \Gamma^k \end{aligned}$$

### 3.2.1 | Expectation of the total cost from period $k-1$ with respect to the period $k-1$ information, *ie* up to $k-2$

#### i. $V_k$ Value function expectation

$\mathbb{E}_{k-2}[V_k] =$

$$\begin{aligned} &\left( \beta_{m_k} \left( \Delta_{k-2|t} \right) \right)^2 \gamma_2^k + \\ &1/3 \left( \left( 6\,\gamma_{d2}^k\, m_{k-1} - 6\,X_{k-1} - 6\,c_{k-2} - 6\,c_{k-1} \right) \gamma_2^k + 3\,\gamma_{\beta_{m_k}}^k \right) \beta_{m_k} \left( \Delta_{k-2|t} \right) \\ &+ \gamma_{\beta_{m_{k+1}}}^k \beta_{m_{k+1}} \left( \Delta_{k-2|t} \right) + 1/3 \Bigg( 3\,{X_{k-1}}^2 + \left( -6\,\gamma_{d2}^k\, m_{k-1} + 6\,c_{k-2} + 6\,c_{k-1} \right) X_{k-1} \\ &+ 3\,{c_{k-2}}^2 + \left( -6\,\gamma_{d2}^k\, m_{k-1} + 6\,c_{k-1} \right) c_{k-2} + 3\,{m_{k-1}}^2 {\gamma_{d2}^k}^2 + 2\,{s_{k-1}}^2 {\gamma_{d2}^k}^2 \\ &- 6\,c_{k-1} m_{k-1} \gamma_{d2}^k + 3\,{c_{k-1}}^2 \Bigg) \gamma_2^k + \gamma_x^k X_{k-1} + \gamma_{c2}^k c_{k-2} + \gamma_{c1}^k c_{k-1} + \gamma_d^k m_{k-1} + \Gamma^k \end{aligned}$$

**ii. Expectation of $K_{k-1}$ with respect to $D_{k-1|t}$ conditionnally to the state of $k-1$:**

$\mathbb{E}_{k-2}[K_{k-1}] =$

$$\begin{cases} -B\left(X_{k-1}+c_{k-2}-m_{k-1}\right)+pc_{k-1}- & if \quad X_{k-1}+c_{k-2} < m_{k-1}-2\,s_{k-1} \\ \frac{1}{3}(H+B)\,s_{k-1}+pc_{k-1} \\ -\frac{B-H}{2}\left(-m_{k-1}+X_{k-1}+c_{k-2}\right) \\ +\frac{H+B}{4s_{k-1}}\left(X_{k-1}+c_{k-2}-m_{k-1}\right)^2 & if \quad m_{k-1}-2\,s_{k-1} \le X_{k-1}+c_{k-2} < m_{k-1} \\ +\frac{H+B}{24s_{k-1}^2}\left(X_{k-1}+c_{k-2}-m_{k-1}\right)^3 \\ \frac{1}{3}(H+B)\,s_{k-1}+pc_{k-1} \\ -\frac{B-H}{2}\left(-m_{k-1}+X_{k-1}+c_{k-2}\right) \\ +\frac{H+B}{4s_{k-1}}\left(X_{k-1}+c_{k-2}-m_{k-1}\right)^2 & if \quad m_{k-1} \le X_{k-1}+c_{k-2} < m_{k-1}+2\,s_{k-1} \\ +\frac{H+B}{24s_{k-1}^2}\left(X_{k-1}+c_{k-2}-m_{k-1}\right)^3 \\ H\left(X_{k-1}+c_{k-2}-m_{k-1}\right)+pc_{k-1} & if \quad m_{k-1}+2\,s_{k-1} \le X_{k-1}+c_{k-2} \end{cases}$$

We easily conclude the quantity we are looking for $\mathbb{E}_{k-2}[K_{k-1}+V_k]$ is the sum of both quantities that we now know: $\mathbb{E}_{k-2}[K_{k-1}]+\mathbb{E}_{k-2}[V_k]$

### 3.2.2 | Optimization Problem for period $k-1$

$c_{k-1} \to \mathbb{E}_{k-2}[K_{k-1}+V_k]$ is a continous, convex, and derivable fiunction with respect to $c_{k-1}$. We solve the equation

$$\frac{\partial}{\partial c_{k-1}}\mathbb{E}_{k-2}[K_{k-1}+V_k]=0$$

to get its argument of the minimum:

$$\begin{aligned} & \frac{\partial}{\partial c_{k-1}}\mathbb{E}_{k-2}[K_{k-1}+V_k] && =0 \\ \Leftrightarrow\ & 2\,\gamma_2^k\,\beta_{m_k}\left(\Delta_{k-2|t}\right)+2\left(-\gamma_{d2}^k\,m_{k-1}+\ X_{k-1}+\ c_{k-2}+\ c_{k-1}\right)\gamma_2^k+\gamma_{c1}^k+p && =0 \\ \Leftrightarrow\ & c_{k-1}^*=\beta_{m_k}\left(\Delta_{k-2|t}\right)+\gamma_{d2}^k\,m_{k-1}-X_{k-1}-c_{k-2}-1/2\,\frac{p+\gamma_{c1}^k}{\gamma_2^k} \end{aligned}$$

**i. Expression of the value of the minimum of the corresponding minimizing function:**

In other words: $V_{k-1}$:

**With respect to state of $k-1$:**

$$V_{k-1} = \left(\gamma_{c1}^{k} - \gamma_{c1}^{k}\right) c_{k-2} + \left(\gamma_{x}^{k} - \gamma_{c1}^{k}\right) X_{k-1}$$

$$+ A^{k-1} + \left(\gamma_{c1}^{k} + \gamma_{\beta_{m_k}}^{k}\right) \beta_{m_k}\left(\Delta_{k-2|t}\right) + \gamma_{\beta_{m_{k+1}}}^{k} \beta_{m_{k+1}}\left(\Delta_{k-2|t}\right) + \left(\gamma_{c1}^{k}\,\gamma_{d2}^{k} + \gamma_{d}^{k}\right) m_{k-1}$$

$$+ \begin{cases} -B\left(X_{k-1} + c_{k-2} - m_{k-1}\right) & \\ +p\left(\beta_{m_k}\left(\Delta_{k-2|t}\right) + \gamma_{d2}^{k}\, m_{k-1} - X_{k-1} - c_{k-2}\right) & if \quad X_{k-1} + c_{k-2} < m_{k-1} - 2\, s_{k-1} \\ \frac{1}{3}\left(H+B\right) s_{k-1} & \\ +p\left(\beta_{m_k}\left(\Delta_{k-2|t}\right) + \gamma_{d2}^{k}\, m_{k-1} - X_{k-1} - c_{k-2}\right) & \\ -\frac{B-H}{2}\left(-m_{k-1} + X_{k-1} + c_{k-2}\right) & if \quad m_{k-1} - 2\, s_{k-1} \leq X_{k-1} + c_{k-2} \\ +\frac{H+B}{4 s_{k-1}}\left(X_{k-1} + c_{k-2} - m_{k-1}\right)^2 & \quad < m_{k-1} \\ +\frac{H+B}{24 s_{k-1}^2}\left(X_{k-1} + c_{k-2} - m_{k-1}\right)^3 & \\ \frac{1}{3}\left(H+B\right) s_{k-1} & \\ +p\left(\beta_{m_k}\left(\Delta_{k-2|t}\right) + \gamma_{d2}^{k}\, m_{k-1} - X_{k-1} - c_{k-2}\right) & \\ -\frac{B-H}{2}\left(-m_{k-1} + X_{k-1} + c_{k-2}\right) & if \quad m_{k-1} \leq X_{k-1} + c_{k-2} \\ +\frac{H+B}{4 s_{k-1}}\left(X_{k-1} + c_{k-2} - m_{k-1}\right)^2 & \quad < m_{k-1} + 2\, s_{k-1} \\ +\frac{H+B}{24 s_{k-1}^2}\left(X_{k-1} + c_{k-2} - m_{k-1}\right)^3 & \\ H\left(X_{k-1} + c_{k-2} - m_{k-1}\right) & \\ +p\left(\beta_{m_k}\left(\Delta_{k-2|t}\right) + \gamma_{d2}^{k}\, m_{k-1} - X_{k-1} - c_{k-2}\right) & if \quad m_{k-1} + 2\, s_{k-1} \leq X_{k-1} + c_{k-2} \end{cases}$$

where $A^{k-1} = 2/3\, {s_{k-1}}^2 {\gamma_{d2}^{k}}^2 \gamma_2^k + \gamma^k - 1/4\, \frac{\left(p+\gamma_{c1}^{k}\right)^2}{\gamma_2^k}$

**through a continuous approximation with respect to $c_{k-2}$ to be able to compute the next expectation** To be able to compute the next iteration expectation, we want to express $V_{k-1}$ with respect to $c_{k-2}$ with a continous and derivable function.We approximate then the function with a continuous quadratic function going through 3 specific points. We choose the boudaries of the intervals definition. As a consequence, we keep as structuring couples the followings:

➢ $c_{k-2} = m_{k-1} - 2\, s_{k-1} - X_{k-1}$

for which

$V_{k-1} = A^{k-1} + \left(\gamma_{c1}^k + \gamma_{\beta_{m_k}}^k\right) \beta_{m_k}\left(\Delta_{k-2|t}\right) + \gamma_{\beta_{m_{k+1}}}^k \beta_{m_{k+1}}\left(\Delta_{k-2|t}\right) + \left(\gamma_{c1}^k \gamma_{d2}^k + \gamma_d^k\right) m_{k-1} + \left(\gamma_{c2}^k - \gamma_{c1}^k\right) c_{k-2} + \left(\gamma_x^k - \gamma_{c1}^k\right) X_{k-1} + \beta_{m_k}\left(\Delta_{k-2|t}\right) p + \left(\left(\gamma_{d2}^k - 1\right) m_{k-1} + 2\, s_{k-1}\right) p + 2\, B s_{k-1}$

- $c_{k-2} = m_{k-1} - X_{k-1}$

  for which

  $V_{k-1} = A^{k-1} + \left(\gamma_{c1}^k + \gamma_{\beta_{m_k}}^k\right) \beta_{m_k}\left(\Delta_{k-2|t}\right) + \gamma_{\beta_{m_{k+1}}}^k \beta_{m_{k+1}}\left(\Delta_{k-2|t}\right) + \left(\gamma_{c1}^k \gamma_{d2}^k + \gamma_d^k\right) m_{k-1} + \left(\gamma_{c2}^k - \gamma_{c1}^k\right) c_{k-2} + \left(\gamma_x^k - \gamma_{c1}^k\right) X_{k-1} + \beta_{m_k}\left(\Delta_{k-2|t}\right) p + m_{k-1}\left(\gamma_{d2}^k - 1\right) p + 1/3\, s_{k-1}\,(H + B)$

- $c_{k-2} = m_{k-1} + 2\, s_{k-1} - X_{k-1}$

  for which

  $V_{k-1} = A^{k-1} + \left(\gamma_{c1}^k + \gamma_{\beta_{m_k}}^k\right) \beta_{m_k}\left(\Delta_{k-2|t}\right) + \gamma_{\beta_{m_{k+1}}}^k \beta_{m_{k+1}}\left(\Delta_{k-2|t}\right) + \left(\gamma_{c1}^k \gamma_{d2}^k + \gamma_d^k\right) m_{k-1} + \left(\gamma_{c2}^k - \gamma_{c1}^k\right) c_{k-2} + \left(\gamma_x^k - \gamma_{c1}^k\right) X_{k-1} + \beta_{m_k}\left(\Delta_{k-2|t}\right) p + \left(\left(\gamma_{d2}^k - 1\right) m_{k-1} - 2\, s_{k-1}\right) p + 2\, s_{k-1} H$

We are finally able to re-write

$V_{k-1} =$

$$A^{k-1} + \frac{H+B}{3} s_{k-1}\left(\gamma_{\beta_{m_k}}^k + p + \gamma_{c1}^k\right) \beta_{m_k}\left(\Delta_{k-2|t}\right) + \gamma_{\beta_{m_{k+1}}}^k \beta_{m_{k+1}}\left(\Delta_{k-2|t}\right) + \left(\frac{H-B}{2} - p + \gamma_x^k - \gamma_{c1}^k\right) X_{k-1} + \left(\frac{H-B}{2} + \gamma_{c2}^k - p - \gamma_{c1}^k\right) c_{k-2} + \left(\frac{B-H}{2} + p\gamma_{d2}^k + \gamma_{c1}^k \gamma_{d2}^k + \gamma_d^k\right) m_{k-1} + \frac{H+B}{6 s_{k-1}} \left(-m_{k-1} + X_{k-1} + c_{k-2}\right)^2$$

**With respect to the state of period $k-2$: application of the law of motion**

Let $X_{k-1}$ be equal to $X_{k-2} + c_{k-3} - D_{k-2|t}$

Let $m_{k-1}$ be equal to $\beta_{m_{k-1}}\left(\Delta_{k-3|t}\right) + \beta_{m_{k-1}}^{k-2} D_{k-2|t}$ as in 4.9

Let $\beta_{m_k}\left(\Delta_{k-2|t}\right)$ be equal to $\beta_{m_k}^{k-2} D_{k-2|t} + \beta_{m_k}\left(\Delta_{k-3|t}\right)$ as in 4.9

Let $\beta_{m_{k+1}}\left(\Delta_{k-2|t}\right)$ be equal to $\beta_{m_{k+1}}^{k-2} D_{k-2|t} + \beta_{m_{k+1}}\left(\Delta_{k-3|t}\right)$ as in 4.9

Then, we can re-write:

$V_{k-1} =$

$$\left(\gamma_{\beta_{m_k}}^k + p + \gamma_{c1}^k\right)\left(\beta_{m_k}^{k-2} D_{k-2|t} + \beta_{m_k}\left(\Delta_{k-3|t}\right)\right) + \gamma_{\beta_{m_{k+1}}}^k \left(\beta_{m_{k+1}}^{k-2} D_{k-2|t} + \beta_{mk+1}\left(\Delta_{k-3|t}\right)\right) + \left(\frac{H-B}{2} - p + \gamma_x^k - \gamma_{c1}^k\right)\left(X_{k-2} + c_{k-3} - D_{k-2}\right) + \left(\frac{H-B}{2} + \gamma_{c2}^k - p - \gamma_{c1}^k\right) c_{k-2} + \left(\beta_{mk-1}{}^{k-2} D_{k-2} + \beta_{mk-1}\left(\Delta_{k-3|t}\right)\right)\left(\left(p + \gamma_{c1}^k\right)\gamma_{d2}^k + \frac{B-H}{2} + \gamma_d^k\right) + \frac{H+B}{3} s_{k-1} + A^{k-1}\left(-\beta_{mk-1}{}^{k-2} D_{k-2} - \beta_{mk-1}\left(\Delta_{k-3|t}\right) + X_{k-2} + c_{k-3} - D_{k-2} + c_{k-2}\right)^2 \frac{H+B}{6 s_{k-1}}$$

### 3.2.3 | Conclusion

By defining :

- $\gamma_2^{k-1} = \frac{H+B}{6s_{k-1}}$
- $\gamma_{c1}^{k-1} = \frac{H-B}{2} + \gamma_{c2}^k - p - \gamma_{c1}^k$
- $\gamma_{c2}^{k-1} = \gamma_x^{k-1} = \frac{H-B}{2} - p + \gamma_x^k - \gamma_{c1}^k$
- $\gamma_d^{k-1} = \left(p\gamma_{d2}^k + \gamma_{c1}^k\,\gamma_{d2}^k + \frac{B-H}{2} + \gamma_d^k\right)\beta_{m_{k-1}}^{k-2} + \left(\gamma_{\beta_{m_k}}^k + p + \gamma_{c1}^k\right)\beta_{m_k}^{k-2} + \gamma_{\beta_{m_{k+1}}}^k\beta_{m_{k+1}}^{k-2} + \frac{B-H}{2} + p - \gamma_x^k + \gamma_{c1}^k$
- $\gamma_{d2}^{k-1} = \beta_{m_{k-1}}^{k-2} + 1$
- $\Gamma^{k-1} = \frac{H+B}{3}s_{k-1} + A^{k-1}$
- $\gamma_{\beta_{m_{k-1}}}^{k-1} = p\gamma_{d2}^k + \gamma_{c1}^k\gamma_{d2}^k + \frac{H-B}{2} + \gamma_d^k$
- $\gamma_{\beta_{m_k}}^{k-1} = \gamma_{\beta_{m_k}}^k + p + \gamma_{c1}^k$
- $\gamma_{\beta_{m_{k+1}}}^{k-1} = \gamma_{\beta_{m_{k+1}}}^k$

we can write $V_{k-1}$ as:

$$\begin{aligned} V_{k-1} =& \gamma_2^{k-1}\left(c_{k-2} + c_{k-3} + X_{k-2} - \gamma_{d2}^{k-1}\,D_{k-2|t} - \beta_{m_{k-1}}\left(\Delta_{k-3|t}\right)\right)^2 \\ &+ \gamma_{c1}^{k-1}\,c_{k-2} + \gamma_{c2}^{k-1}\,c_{k-3} + \gamma_x^{k-1}\,X_{k-2} + \gamma_d^{k-1}\,D_{k-2|t} \\ &+ \gamma_{\beta_{m_{k-1}}}\beta_{m_{k-1}}\left(\Delta_{k-2|t}\right) + \gamma_{\beta_{m_k}}\beta_{m_k}\left(\Delta_{k-2|t}\right) + \gamma_{\beta_{m_{k+1}}}\beta_{m_{k+1}}\left(\Delta_{k-2|t}\right) + \Gamma^{k-1} \end{aligned}$$

We prove consequently that if $V_k$ follows a special property in terms of its shape, $V_{k-1}$ follows the same property.

Therefore, if the inductive hypothese is true for a $k$, it is consequently true $\forall j < k$.

On the other hand, the base case allowed us to express $V_{t+N-2}$ with respect to the state of $t+N-3$.

By defining:

- $\gamma_2^{t+N-2} = \frac{H+B}{6s_{t+N-2}}$
- $\gamma_{c1}^{t+N-2} = \gamma_{c2}^{t+N-2} = \gamma_x^{t+N-2} = -\frac{B-H+2p}{2}$
- $\gamma_d^{t+N-2} = \left(\frac{B-H}{2} + p\right)\left(\beta_{m_{t+N-2}}^{t+N-3} + 1\right) + p\left(\beta_{m_{t+N-1}}^{t+N-3} + \beta_{m_{t+N-1}}^{t+N-2}\beta_{m_{t+N-2}}^{t+N-3}\right)$
- $\gamma_{d2}^{t+N-2} = \beta_{m_{t+N-2}}^{t+N-3} + 1$
- $\Gamma^{t+N-2} = \frac{B+H}{3}s_{t+N-2} + A_{t+N-2}$
- $\gamma_{\beta_{m_{t+N-2}}}^{t+N-2} = \frac{B-H}{2} + p$
- $\gamma_{\beta_{m_{t+N-1}}}^{t+N-2} = p\left(\beta_{m_{t+N-1}}^{t+N-2} + 1\right)$

then we can confirm that this property is true for $k = t + N - 2$. Consequently, it is true $\forall k \in [\![t, t + N - 2]\!]$.

This property allows us to compute the optimal solutions for each step of the dynamic programming as we saw in the previous calculations:
$\forall k \in [\![t + 1, t + N - 1]\!]$:

$$c^*_{k-1} = \beta_{m_k}\left(\Delta_{k-2|t}\right) + \gamma^k_{d2}\, m_{k-1} - X_{k-1} - c_{k-2} + \frac{3}{2}\frac{(B-H)}{B+H}s_k$$

and

$$\begin{aligned} c^*_{t+N-2} =& \beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) + m_{t+N-2}\left(\beta^{t+N-2}_{m_{t+N-1}} + 1\right) - c_{t+N-3} - X_{t+N-2} \\ &+ \frac{3}{2}\frac{(B-H-2p)}{B+H}s_{t+N-1} \end{aligned}$$

# 4 | Appendix 4

We present here the proof of the theorem 2.2.2 - 2 in Chapter 4, Section 2, Subsection 2, Subsubsection 2, Paragraph iii.

The proof follows the same framework as the previous one.

The computation steps are the same. The only difference is the approximation technique for each step of the inductive reasoning.

We present here only the results that differ from the previous proof.

## 4.1 | Base Case

### 4.1.1 | Last period calculations

We approximate here $\tilde{V}_{t+N-1}$ with a continuous function with respect to $D_{t+N-2|t}$. To do so, we will use a **piecewise linear function** with respect to $c_{t+N-2}$, such as:

$$\tilde{V}_{t+N-1} = \begin{cases} -B\left(X_{t+N-1} + c_{t+N-2} - m_{t+N-1}\right) & \text{if (1) or (2)} \\ H\left(X_{t+N-1} + c_{t+N-2} - m_{t+N-1}\right) & \text{if (3) or (4)} \end{cases}$$

where (1), (2), (3), (4) are defined as the following sets with respect to $c_{t+N-2}$:

$$\begin{cases} (1): & & X_{t+N-1}+c_{t+N-2} & < m_{t+N-1}-2s_{t+N-1} \\ (2): & m_{t+N-1}-2s_{t+N-1} \leq & X_{t+N-1}+c_{t+N-2} & < m_{t+N-1} \\ (3): & m_{t+N-1} \leq & X_{t+N-1}+c_{t+N-2} & < m_{t+N-1}+2s_{t+N-1} \\ (4): & m_{t+N-1}+2s_{t+N-1} \geq & X_{t+N-1}+c_{t+N-2} & \end{cases}$$

### 4.1.2 | Second Last period Calculations

**i. Application of the laws of motion of the system**

Let $X_{t+N-1}$ be equal to $X_{t+N-2}+c_{t+N-3}-D_{t+N-1|t}$

Let $m_{t+N-1}$ be equal to $\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)+\beta_{m_{t+N-1}}^{t+N-2}D_{t+N-2|t}$ as in 4.9

**ii. Integration of $\tilde{V}_{t+N-1}$ with respect to $D_{t+N-2|t}$ conditionnally to the state in $t+N-2$:**

Let $L$ be equal to $\frac{X_{t+N-2}+c_{t+N-3}+c_{t+N-2}-\beta_{m_{t+N-1}}\left(\Delta_{t|t+N-3}\right)}{\beta_{m_{t+N-1}}^{t+N-2}+1}$

Depending on the values taken by $D_{t+N-2|t}$ compared to $L$, $\tilde{V}_{t+N-1}$ may have different values.

If $D_{t+N-2|t} \leq L$, then

$$\begin{aligned} \tilde{V}_{t+N-1} = \; & H\left(-(\beta_{m_{t+N-1}}^{t+N-2}+1)D_{t+N-2|t}-\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)\right. \\ & \left.+X_{t+N-2}+c_{t+N-3}+c_{t+N-2}\right) \end{aligned}$$

We note $V_1$ this expression.

Otherwise, when $D_{t+N-2|t}$ is bigger than $L$, we have

$$\begin{aligned} \tilde{V}_{t+N-1} = \; & -B\left(-(\beta_{m_{t+N-1}}^{t+N-2}+1)D_{t+N-2|t}-\beta_{mt+N-1}\left(\Delta_{t+N-3|t}\right)\right. \\ & \left.+X_{t+N-2}+c_{t+N-3}+c_{t+N-2}\right) \end{aligned}$$

noted $V_2$

Once this is established, two cases arise for the integration whether $L$ is bigger or lower than $m_{t+N-2}$.

1. When $L \leq m_{t+N-2}$:

   $\mathbb{E}_{t+N-3}\left[\tilde{V}_{t+N-1}\right] =$

   $$\int_{m_{t+N-2}-2s_{t+N-2}}^{L} V_1.\phi_1 + \int_{L}^{m_{t+N-2}} V_2.\phi_1 + \int_{m_{t+N-2}}^{m_{t+N-2}+2s_{t+N-2}} V_2.\phi_2$$

   which gives us

   $\mathbb{E}_{t+N-3}\left[\tilde{V}_{t+N-1}\right] =$

$$\frac{H+B}{3}\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)s_{t+N-2}$$
$$+\frac{B-H}{2}\left(\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)-X_{t+N-2}-c_{t+N-3}-c_{t+N-2}\right)$$
$$+\frac{H+B}{4\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)s_{t+N-2}}\left(\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)\right.$$
$$\left.-X_{t+N-2}-c_{t+N-3}-c_{t+N-2}\right)^2$$
$$-\frac{H+B}{24\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)^2{s_{t+N-2}}^2}\left(\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)\right.$$
$$\left.-X_{t+N-2}-c_{t+N-3}-c_{t+N-2}\right)^3$$

$\phi_1$ and $\phi_2$ are the two linear functions defining the triangular distribution of $D_{t+N-1|t}$ (see chapter 2): $\phi_{t+N-1|t}=\phi_1$ when $z\in[m_{t+N-2}-2s_{t+N-2},m_{t+N-2}]$ and $\phi_{t+N-1|t}=\phi_2$ when $z\in[m_{t+N-2},m_{t+N-2}-2s_{t+N-2}]$

2. When $L>m_{t+N-2}$

$\mathbb{E}_{t+N-3}\left[\tilde{V}_{t+N-1}\right]=$

$$\int_{m_{t+N-2}-2s_{t+N-2}}^{m_{t+N-2}}V_2.\phi_1+\int_{m_{t+N-2}}^{L}V_2.\phi_2+\int_{L}^{m_{t+N-2}+2s_{t+N-2}}V_1.\phi_2$$

which gives us

$\mathbb{E}_{t+N-3}\left[\tilde{V}_{t+N-1}\right]=$

$$\frac{H+B}{3}\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)s_{t+N-2}$$
$$+\frac{B-H}{2}\left(\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)-X_{t+N-2}-c_{t+N-3}-c_{t+N-2}\right)$$
$$+\frac{H+B}{4\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)s_{t+N-2}}\left(\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)\right.$$
$$\left.-X_{t+N-2}-c_{t+N-3}-c_{t+N-2}\right)^2$$
$$+\frac{H+B)}{24\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)^2{s_{t+N-2}}^2}\left(\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)\right.$$
$$\left.-X_{t+N-2}-c_{t+N-3}-c_{t+N-2}\right)^3$$

**iii. Expectation of $K_{t+N-2}$ with respect to $D_{t+N-2|t}$ conditionnally to the state of $t+N-2$:**

$\mathbb{E}_{t+N-3}\left[K_{t+N-2}\right]$ does not change compared to the previous proof.

We can conclude the quantity we are looking for $\mathbb{E}_{t+N-3}\left[K_{t+N-2}+\tilde{V}_{t+N-1}\right]$ is the sum of both quantities that we now know: $\mathbb{E}_{t+N-3}\left[K_{t+N-2}\right]+\mathbb{E}_{t+N-3}\left[\tilde{V}_{t+N-1}\right]$. The value of $\mathbb{E}_{t+N-3}\left[\tilde{V}_{t+N-1}\right]$ depends on the position of $m_{t+N-2}$ compared to the threshold previously noted $L$.

**iv. Optimization problem for the two last periods**

We are looking for the argument of the minimum of the quantity $\mathbb{E}_{t+N-3}\left[K_{t+N-2}+\tilde{V}_{t+N-1}\right]$ with respect to $c_{t+N-2}$.

We solve the optimization problem we face here by looking for the solution of the equation $\frac{\partial}{\partial c_{t+N-2}}\mathbb{E}_{t+N-3}\left[K_{t+N-2}+\tilde{V}_{t+N-1}\right]=0$

We have:

For the first case: $L \leq m_{t+N-2}$:

$$\frac{\partial}{\partial c_{t+N-2}}\mathbb{E}_{t+N-3}\left[K_{t+N-2}+\tilde{V}_{t+N-1}\right]=$$

$$p+\frac{H-B}{2}-\frac{H+B}{2\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)s_{t+N-2}}\Bigg(\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)$$
$$-X_{t+N-2}-c_{t+N-3}-c_{t+N-2}\Bigg)$$
$$+\frac{H+B}{8\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)^2 s_{t+N-2}^2}\Bigg(\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)$$
$$-X_{t+N-2}-c_{t+N-3}-c_{t+N-2}\Bigg)^2$$

For the second case: $L > m_{t+N-2}$:

$$\frac{\partial}{\partial c_{t+N-2}}\mathbb{E}_{t+N-3}\left[K_{t+N-2}+\tilde{V}_{t+N-1}\right]=$$

$$p+\frac{H-B}{2}-\frac{H+B}{2\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)s_{t+N-2}}\Bigg(\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)$$
$$-X_{t+N-2}-c_{t+N-3}-c_{t+N-2}\Bigg)$$
$$-\frac{H+B}{8\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)^2 s_{t+N-2}^2}\Bigg(\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)m_{t+N-2}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)$$
$$-X_{t+N-2}-c_{t+N-3}-c_{t+N-2}\Bigg)^2$$

We note that $\mathbb{E}_{t+N-3}\left[K_{t+N-2}+\tilde{V}_{t+N-1}\right]$ is, in both cases, a cubic polynomilal function whose order 3 coefficient is positive. That's why the local minimum that we are looking for is the greatest solution among the set of two values of $c_{t+N-2}$ for which $\frac{\partial}{\partial c_{t+N-2}}\mathbb{E}_{t+N-3}\left[K_{t+N-2}+\tilde{V}_{t+N-1}\right]$ is equal to zero.
We find:
For the first case: $L \leq m_{t+N-2}$:

$$c_{t+N-2}^{*}=-X_{t+N-2}-c_{t+N-3}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)$$
$$+\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)\left(2\,s_{t+N-2}\left(\sqrt{2}\sqrt{\frac{B-p}{H+B}}-1\right)+m_{t+N-2}\right)$$

For the second case: $L > m_{t+N-2}$:

$$c_{t+N-2}^{*}=-X_{t+N-2}-c_{t+N-3}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)$$
$$+\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)\left(2\,s_{t+N-2}\left(1-\sqrt{2}\sqrt{\frac{H+p}{H+B}}\right)+m_{t+N-2}\right)$$

The potential solutions give us interesting information concerning the intervals validating the different cases we may face.

More specifically, the case #1 ($L \leq m_{t+N-2}$) becomes equivalent to the condition:

$$2\sqrt{2}\sqrt{\frac{B-p}{H+B}}s_{t+N-2}-2\,s_{t+N-2}<0$$

or:

$$B<H+2p$$

We consider for the following that we don't consider such cases. We consequently keep for the rest of the proof the case #2: $L > m_{t+N-2}$.

**v. Expression of $V_{t+N-2}$**

$V_{t+N-2}$ can be written, with respect to the state of $t+N-2$:

$V_{t+N-2} = A_{t+N-2} +$

$$\begin{cases} -B\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right) \\ +p\Big(\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)+m_{t+N-2}\left(\beta^{t+N-2}_{m_{t+N-1}}+1\right) \\ \qquad -c_{t+N-3}-X_{t+N-2}\Big) & if \quad X_{t+N-2}+c_{t+N-3} < m_{t+N-2}-2\,s_{t+N-2} \\ \frac{1}{3}\left(H+B\right)s_{t+N-2} \\ +p\Big(\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)+m_{t+N-2}\left(\beta^{t+N-2}_{m_{t+N-1}}+1\right) \\ \qquad -c_{t+N-3}-X_{t+N-2}\Big) \\ -\frac{B-H}{2}\left(-m_{t+N-2}+X_{t+N-2}+c_{t+N-3}\right) & if \quad m_{t+N-2}-2\,s_{t+N-2} \le X_{t+N-2}+c_{t+N-3} < m_{t+N-2} \\ +\frac{H+B}{4s_{t+N-2}}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^2 \\ +\frac{H+B}{24s^2_{t+N-2}}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^3 \\ \frac{1}{3}\left(H+B\right)s_{t+N-2} \\ +p\Big(\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)+m_{t+N-2}\left(\beta^{t+N-2}_{m_{t+N-1}}+1\right) \\ \qquad -c_{t+N-3}-X_{t+N-2}\Big) \\ -\frac{B-H}{2}\left(-m_{t+N-2}+X_{t+N-2}+c_{t+N-3}\right) & if \quad m_{t+N-2} \le X_{t+N-2}+c_{t+N-3} < m_{t+N-2}+2\,s_{t+N-2} \\ +\frac{H+B}{4s_{t+N-2}}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^2 \\ +\frac{H+B}{24s^2_{t+N-2}}\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right)^3 \\ H\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right) \\ +p\Big(\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)+m_{t+N-2}\left(\beta^{t+N-2}_{m_{t+N-1}}+1\right) \\ \qquad -c_{t+N-3}-X_{t+N-2}\Big) & if \quad m_{t+N-2}+2\,s_{t+N-2} \le X_{t+N-2}+c_{t+N-3} \end{cases}$$

where $A_{t+N-2}$ is equal to

$$2s_{t+N-2}\left(\beta^{t+N-2}_{m_{t+N-1}}+1\right)\left(-\sqrt{\frac{H+p}{H+B}}H\sqrt{2}+\frac{\sqrt{2}}{3}\left(\frac{H+p}{H+B}\right)^{3/2}(H+B)+H\right) + \frac{2}{3}\left(\beta^{t+N-2}_{m_{t+N-1}}+1\right)s_{t+N-2}\left(1-\sqrt{2}\sqrt{\frac{H+p}{H+B}}\right)$$

and corresponds to the non-historical dependent part of the value function $V_{t+N-2}$.

**vi. Laws of motion and expression of $V_{t+N-2}$ with respect to the information of state $t+N-3$**

We firstly need to approximate the function $V_{t+N-2}$ with respect to $c_{t+N-3}$ as a **piecewise linear function**:

**Approximation** $V_{t+N-2} = A_{t+N-2} +$

$$\begin{cases} -B\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right) \\ +p\Big(\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)+m_{t+N-2}\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right) \\ \qquad -c_{t+N-3}-X_{t+N-2}\Big) \end{cases} \quad if \quad X_{t+N-2}+c_{t+N-3} < m_{t+N-2}$$

$$\begin{cases} H\left(X_{t+N-2}+c_{t+N-3}-m_{t+N-2}\right) \\ +p\Big(\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)+m_{t+N-2}\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right) \\ \qquad -c_{t+N-3}-X_{t+N-2}\Big) \end{cases} \quad if \quad m_{t+N-2} \leq X_{t+N-2}+c_{t+N-3}$$

**Application of the Laws of motion**

Let $X_{t+N-2}$ be equal to $X_{t+N-3}+c_{t+N-4}-D_{t+N-3|t}$

Let $m_{t+N-2}$ be equal to $\beta_{m_{t+N-2}}\left(\Delta_{t+N-4|t}\right)+\beta_{m_{t+N-2}}^{t+N-3}D_{t+N-3|t}$ as in 4.9

Let $\beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right)$ be equal to $\beta_{m_{t+N-1}}^{t+N-3}D_{t+N-3|t}+\beta_{m_{t+N-1}}\left(\Delta_{t+N-4|t}\right)$ as in 4.9

We finally get $V_{t+N-2}$ equal to :

$$A_{t+N-2}+\begin{cases} \left(\left(\beta_{m_{t+N-1}}^{t+N-2}p+B+p\right)\beta_{m_{t+N-2}}^{t+N-3}+p\beta_{m_{t+N-1}}^{t+N-3}+B+p\right)D_{t+N-3|t} \\ +\left(\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)p+B\right)\beta_{m_{t+N-2}}\left(\Delta_{t+N-4|t}\right) \\ -(B+p)\left(X_{t+N-3}+c_{t+N-4}+c_{t+N-3}\right)+\beta_{m_{t+N-1}}\left(\Delta_{t+N-4|t}\right)p \end{cases} \text{if (1)}$$

$$\begin{cases} \left(\left(\beta_{m_{t+N-1}}^{t+N-2}p-H+p\right)\beta_{m_{t+N-2}}^{t+N-3}+p\beta_{m_{t+N-1}}^{t+N-3}-H+p\right)D_{t+N-3|t} \\ +\left(\left(\beta_{m_{t+N-1}}^{t+N-2}+1\right)p-H\right)\beta_{m_{t+N-2}}\left(\Delta_{t+N-4|t}\right) \\ -(p-H)\left(X_{t+N-3}+c_{t+N-4}+c_{t+N-3}\right)+\beta_{m_{t+N-1}}\left(\Delta_{t+N-4|t}\right)p \end{cases} \text{if (2)}$$

where (1) and (2) are defined as the following sets with respect to $c_{t+N-3}$:

$$\begin{cases} (1): & X_{t+N-3}+c_{t+N-4}-D_{t+N-3|t}+c_{t+N-3} \\ & <\beta_{m_{t+N-2}}\left(\Delta_{t+N-4|t}\right)+\beta_{m_{t+N-2}}^{t+N-3}D_{t+N-3|t} \\ (2): & \beta_{m_{t+N-2}}\left(\Delta_{t+N-4|t}\right)+\beta_{m_{t+N-2}}^{t+N-3}D_{t+N-3|t} \\ & \geq X_{t+N-3}+c_{t+N-4}-D_{t+N-3|t}+c_{t+N-3} \end{cases}$$

## 4.2 | Inductive Step

We take as inductive assumption that for any $k$, $t \leq k \leq t+N-2$, $V_k$ can be written as:

$$\Gamma_k + \begin{cases} {\gamma_{1d}}^k D_{k-1|t} + {\gamma_{1x}}^k \left(X_{k-1} + c_{k-1} + c_{k-2}\right) \\ +{\gamma_{1\beta m_k}}^k \beta_{m_k} \left(\Delta_{k-2|t}\right) & \text{if (1)} \\ +\gamma^k_{1\beta m_{k+1}} \beta m_{k+1} \left(\Delta_{k-2|t}\right) \\ {\gamma_{2d}}^k D_{k-1|t} + {\gamma_{2x}}^k \left(X_{k-1} + c_{k-1} + c_{k-2}\right) \\ +{\gamma_{2\beta m_k}}^k \beta_{m_k} \left(\Delta_{k-2|t}\right) & \text{if (2)} \\ +\gamma^k_{2\beta m_{k+1}} \beta m_{k+1} \left(\Delta_{k-2|t}\right) \end{cases}$$

By following the same steps of calculation as for the second last period, we can calculate $V_{k-1}$:

- Integration of $V_k$ with respect to the state of $t-1$: $\mathbb{E}_{k-2}\left[V_k\right]$
- Addition of the expected cost of period $k-1$: $\mathbb{E}_{k-2}\left[K_{k-1}\right]$
- Optimization problem to find the argument of the minimum $c^*_{k-1}$
- Expression of the value function $V_{k-1}$
- Approximation of this value function
- Expression with respect to the state of $k-2$

### 4.2.1 | Conclusion

We finally find the same property for $V_{k-1}$ as defined above in terms of shape of $V_k$.

That means that the indutive assumption is true $\forall k$ as soon as it is true for the base case. We show previously that this statement is true for $k = t+N-2$. In conclusion, it is then true $\forall k \in [\![t, t+N-2]\!]$

This property allows us to compute the argument of the minimum of each of the value function $V_k$ calculated at each iteration:

$$c^*_{k-1} = - X_{k-1} - c_{k-2} + \beta_{m_k}\left(\Delta_{k-2|t}\right) + \left(\beta^{k-1}_{m_k} + 1\right)\left(2\, s_{k-1}\left(1 - \sqrt{2}\sqrt{\frac{H}{H+B}}\right) + m_{k-1}\right)$$

and

$$c^*_{t+N-2} = - X_{t+N-2} - c_{t+N-3} + \beta_{m_{t+N-1}}\left(\Delta_{t+N-3|t}\right) + \left(\beta^{t+N-2}_{m_{t+N-1}} + 1\right)\left(2\, s_{t+N-2}\left(1 - \sqrt{2}\sqrt{\frac{H+p}{H+B}}\right) + m_{t+N-2}\right)$$

# Résumé Français

L'environnement actuel du marché international des biens industriels, et plus particulièrement son évolution au cours des précédentes années, a généré de grands défis à relever pour les parties prenantes qui y sont impliquées. Les entreprises industrielles doivent en effet aujourd'hui faire face à des marchés de plus en plus volatiles, être capable d'offrir un taux de service élevé, et pouvoir s'adapter le plus rapidement possible, tout en contrôlant leurs coûts et leur budget de manière rationnelle et viable. Les parties prenantes ont ainsi besoin de mettre en place des processus adaptés au sein de leur organisation, et un des points critiques qu'il est aujourd'hui nécessaire de pouvoir gérer est le caractère incertain des variables et des paramètres en jeu dans les processus décisionnels. En effet, celui-ci peut avoir des conséquences importantes sur l'état de santé de l'entreprise, en ayant effectivement un impact non négligeable sur les stratégies d'allocation des ressources mises en place, et leur cohérence vis-à-vis des besoins des marchés cibles.

Parmi les pratiques indispensables, la planification de production, en tant qu'étape tactique de la chaîne décisionnelle de la Supply Chain, est une procédure clé qui permet aux décideurs de confronter les ressources de l'entreprise et les besoin du marché. Ici, les incertitudes les plus impactantes et donc à forts enjeux auxquelles les professionnels doivent faire face sont les incertitude s prévisionnelles. Afin de pouvoir garder des taux de service ou des coûts de production et de distribution à des niveaux acceptables, des bonnes pratiques et procédures opérationnelles effi-

caces sont généralement mises en place, notamment des structures organisationnelles et décisionnelles permettant d'améliorer la flexibilité des processus industriels, et en particulier le concept de Plan Glissant. Cependant, du fait de la mise à jour régulière des bases de données d'apprentissage des prévisions, ce dernier génère également de l'instabilité dans le système considéré. En conséquence, quand bien même les gains en terme de gestion des incertitudes sont non négligeables, la planification en plan glissant génère des problématiques complexes de gestion de la dynamique du système considéré.

Cette thèse étudie comment traiter cette complexité dynamique générée par l'actualisation des prévisions de la demande dans un contexte de planification de production en horizon glissant. En particulier, la question posée ici est celle de l'optimisation du plan de production dans un tel contexte.

Le problème considéré ici est un système de production mono produit mono étage. Dans un premier temps, les caractéristiques stochastiques et dynamiques de la demande prévisionnelle sont examinées. Cette étape permet de comprendre comment les incertitudes et la dynamique du plan glissant impactent la modélisation du système. En étudiant les sources d'incertitudes, un modèle stochastique à deux niveaux est ainsi développé afin de représenter le caractère dynamique de l'actualisation des prévisions. Cette modélisation générique est alors adaptée au problème considéré, et surtout exploitable dans le cadre d'un problème d'optimisation mathématique.

Ensuite, une approche d'optimisation dynamique stochastique est mise en place afin de développer la modélisation du problème en prenant en considération les représentations de la demande prévisionnelle définies précédemment. On démontre ici de façon analytique que la modélisation permet de développer un algorithme de résolution par récurrence qui débouche sur l'optimalité de la solution.

Dans un deuxième temps, le modèle et la méthode de résolution développée sont appliqués à des cas d'étude spécifiques. Selon les hypothèses prises en fonction des méthodes de prévisions utilisées- comme par exemple la régression linéaire- et en fonction des paramètres industriels du système -le délai de production, l'autorisation des retards de livraisons, . . .- des résultats analytiques intéressants sont atteints : un processus de calcul dynamique inductif permet de déterminer les solutions optimales, qui varient selon les systèmes étudiés. Des analyses numériques de sensibilité sont finalement proposées grâce à des simulations afin de mettre en relief les performances de l'algorithme de résolution développé.

# Bibliography

[Albey et al., 2015] Albey, E., Norouzi, A., Kempf, K. G., and Uzsoy, R. (2015). Demand modeling with forecast evolution: An application to production planning. *IEEE Transactions on Semiconductor Manufacturing*, 28(3):374–384. *Cited p.37*

[Alden and Smith, 1992] Alden, J. M. and Smith, R. L. (1992). Rolling horizon procedures in nonhomogeneous markov decision processes. *Operations Research*, 40:S183–S194. *Cited p.35*

[Arnold et al., 2008] Arnold, J. R. T., Chapman, S. N., and Clive, L. M. (2008). *Introduction to materials management*. Pearson Prentice Hall, Upper Saddle River, N.J, 6th ed edition. *2 citations pp.9 and 14*

[Babai, 2005] Babai, M. Z. (2005). *Politiques de pilotage de flux dans les chaînes logistiques : impact de l'utilisation des prévisions sur la gestion de stocks*. PhD thesis, Ecole Centrale Paris. Thèse de doctorat dirigée par Dallery, Yves Génie industriel Châtenay-Malabry, Ecole centrale Paris 2005. *Cited p.52*

[Babai and Dallery, 2009] Babai, M. Z. and Dallery, Y. (2009). Dynamic versus static control policies in single stage production-inventory systems. *International Journal of Production Research*, 47(2):415–433. *Cited p.31*

[Baker, 1977] Baker, K. R. (1977). An experimental study of the effectiveness of rolling schedules in production planning. *Decision Sciences*, 8(1):19–27. *Cited p.29*

[Bitran et al., 1982] Bitran, G. R., Haas, E. A., and Hax, A. C. (1982). Hierarchical production planning: A two-stage system. *Operations Research*, 30(2):232–251. *Cited p.29*

[Bitran and Tirupati, 1993] Bitran, G. R. and Tirupati, D. (1993). Hierarchical production planning. In *Logistics of Production and Inventory*, volume 4 of *Handbooks in Operations Research and Management Science*, pages 523 – 568. Elsevier. *Cited p.28*

[Blackburn et al., 1986] Blackburn, J. D., Kropp, D. H., and Millen, R. A. (1986). A comparison of strategies to dampen nervousness in mrp systems. *Management Science*, 32(4):413–429. *Cited p.29*

[Blackburn and Millen, 1980] Blackburn, J. D. and Millen, R. A. (1980). Heuristic lot-sizing performance in a rolling-schedule environment. *Decision Sciences*, 11(4):691–701. *Cited p.30*

[Bookbinder and H'ng, 1986] Bookbinder, J. H. and H'ng, B.-T. (1986). Rolling horizon production planning for probabilistic time-varying demands. *International Journal of Production Research*, 24(6):1439–1458. *2 citations pp.30 and 32*

[Bradford and Sugrue, 1990] Bradford, W. J. and Sugrue, K. P. (1990). A bayesian approach to the two-period style-goods inventory problem with single replenishment and heterogeneous poisson demands. *Journal of the Operational Research Society*, 41(3):211–218. *Cited p.36*

[Campbell, 1992] Campbell, G. M. (1992). Master production scheduling under rolling planning horizons with fixed order intervals. *Decision Sciences*, 23(2):312–331. *Cited p.31*

[Cao, 2015] Cao, Y. (2015). *Long-distance procurement planning in global sourcing.* PhD thesis, Ecole Centrale Paris. *3 citations pp.14, 28, and 32*

[Caridi and Cigolini, 2000] Caridi, M. and Cigolini, R. (2000). Buffering against lumpy demand in MRP environments: a theoretical approach and a case study. *Manufacturing Information Systems, Proceedings of the fourth SMESME International Conference.* *Cited p.28*

[Chand et al., 2002] Chand, S., Hsu, V. N., and Sethi, S. (2002). Forecast, solution, and rolling horizons in operations management problems: A classified bibliography. *Manufacturing & Service Operations Management*, 4(1):25–43. *Cited p.35*

[Cheaitou, 2008] Cheaitou, A. (2008). *Stochastic Models for Production/Inventory Planning: Application to Short Life-Cycle Products*. PhD thesis, Ecole Centrale Paris. *3 citations pp.14, 23, and 36*

[Choi et al., 2003] Choi, T. M., Li, D., and Yan, H. (2003). Optimal two-stage ordering policy with bayesian information updating. *The Journal of the Operational Research Society*, 54(8):846–859. *Cited p.36*

[Chopra and Meindl, 2013] Chopra, S. and Meindl, P. (2013). *Supply chain management: strategy, planning, and operation*. Pearson, Boston, 5th ed edition. OCLC: ocn755904451. *3 citations pp.9, 14, and 20*

[Cox et al., 1992] Cox, J., Blackstone, J., Spencer, M., Production, A., and Society, I. C. (1992). *APICS dictionary*. American Production and Inventory Control Society. *Cited p.9*

[Dangelmaier and Degener, 2011] Dangelmaier, W. and Degener, B. (2011). Online optimization with discrete lotsizing production and rolling horizons. In *Dynamics in Logistics*, pages 37–48. Springer. *Cited p.34*

[De Rocquigny, 2012] De Rocquigny, E. (2012). *Modelling under risk and uncertainty: an introduction to statistical, phenomenological and computational methods*. John Wiley & Sons. *Cited p.40*

[Dodge and Rousson, 2004] Dodge, Y. and Rousson, V. (2004). *Analyse de régression appliquée*. Eco sup. Dunod. *Cited p.49*

[Donohue, 2000] Donohue, K. L. (2000). Efficient supply contracts for fashion goods with forecast updating and two production modes. *Management science*, 46(11):1397–1411. *Cited p.36*

[Federgruen, 1993] Federgruen, A. (1993). Centralized planning models for multiechelon inventory systems under uncertainty. *Handbooks in operations research and management science*, 4:133–173. *Cited p.31*

[Federgruen and Tzur, 1994] Federgruen, A. and Tzur, M. (1994). Minimal forecast horizons and a new planning procedure for the general dynamic lot sizing model: Nervousness revisited. *Operations Research*, 42(3):456–468. *Cited p.30*

[Fisher et al., 2001] Fisher, M., Rajaram, K., and Raman, A. (2001). Optimizing inventory replenishment of retail fashion products. *Manufacturing & service operations management*, 3(3):230–241. *Cited p.36*

[Fisher and Raman, 1996] Fisher, M. and Raman, A. (1996). Reducing the cost of demand uncertainty through accurate response to early sales. *Operations Research*, 44(1):87–99. *Cited p.32*

[Friedman et al., 2001] Friedman, J., Hastie, T., and Tibshirani, R. (2001). *The elements of statistical learning*, volume 1. Springer series in statistics Springer, Berlin. *Cited p.52*

[Graves et al., 1993] Graves, S., Kan, A., and Zipkin, P. (1993). *Logistics of Production and Inventory*. Handbooks in operations research and management science. North-Holland. *2 citations pp.14 and 43*

[Graves, 1999] Graves, S. C. (1999). A single-item inventory model for a nonstationary demand process. *Manufacturing & Service Operations Management*, 1(1):50–61. *Cited p.36*

[Graves, 2011] Graves, S. C. (2011). Uncertainty and production planning. In *Planning Production and Inventories in the Extended Enterprise*, pages 83–101. Springer. *3 citations pp.3, 14, and 28*

[Gruen, 2007] Gruen, T. (2007). Solutions to prevent retail Out-of-stocks. *University of Colorado, Colorado Springs*. *Cited p.2*

[Guan et al., 2006] Guan, Y., Ahmed, S., Nemhauser, G. L., and Miller, A. J. (2006). A branch-and-cut algorithm for the stochastic uncapacitated lot-sizing problem. *Mathematical Programming*, 105(1):55–84. *Cited p.28*

[Gupta and Maranas, 2000] Gupta, A. and Maranas, C. D. (2000). A Two-Stage Modeling and Solution Framework for Multisite Midterm Planning under Demand Uncertainty. *Industrial & Engineering Chemistry Research*, 39(10):3799–3813. *Cited p.28*

[Gupta and Maranas, 2003] Gupta, A. and Maranas, C. D. (2003). Managing demand uncertainty in supply chain planning. *Computers & Chemical Engineering*, 27(8-9):1219–1227. *Cited p.28*

[Gurnani and Tang, 1999] Gurnani, H. and Tang, C. S. (1999). Note: Optimal ordering decisions with uncertain cost and demand forecast updating. *Management Science*, 45(10):1456–1462. *Cited p.36*

[Heath and Jackson, 1994] Heath, D. C. and Jackson, P. L. (1994). Modeling the evolution of demand forecasts ith application to safety stock analysis in production/distribution systems. *IIE transactions*, 26(3):17–30. *Cited p.36*

[Helber et al., 2013] Helber, S., Sahling, F., and Schimmelpfeng, K. (2013). Dynamic capacitated lot sizing with random demand and dynamic safety stocks. *OR Spectrum*, 35(1):75–105. *Cited p.28*

[Helton, 1994] Helton, J. C. (1994). Treatment of uncertainty in performance assessments for complex systems. *Risk analysis*, 14(4):483–511. *2 citations pp.41 and 45*

[Hernandez-Lerma and Lasserre, 1990] Hernandez-Lerma, O. and Lasserre, J. (1990). Error bounds for rolling horizon policies in discrete-time markov control processes. *IEEE Transactions on Automatic Control*, 35(10):1118–1124. *Cited p.35*

[Hopp, 1989] Hopp, W. J. (1989). Identifying Forecast Horizons in Nonhomogeneous Markov Decision Processes. *Operations Research*, 37(2):339–343. *Cited p.35*

[Hubert, 2013] Hubert, T. (2013). *Prévision de la demande et pilotage des flux en approvisionnement lointain.* PhD thesis, Ecole Centrale Paris. *6 citations pp.3, 14, 20, 40, 48, and 52*

[Jans and Degraeve, 2007] Jans, R. and Degraeve, Z. (2007). Meta-heuristics for dynamic lot sizing: A review and comparison of solution approaches. *European Journal of Operational Research*, 177(3):1855–1875. *Cited p.28*

[Jans and Degraeve, 2008] Jans, R. and Degraeve, Z. (2008). Modeling industrial lot sizing problems: a review. *International Journal of Production Research*, 46(6):1619–1643. *2 citations pp.28 and 37*

[Karimi et al., 2003] Karimi, B., Fatemi Ghomi, S., and Wilson, J. (2003). The capacitated lot sizing problem: a review of models and algorithms. *Omega*, 31(5):365–378. *Cited p.28*

[Karlin, 1960] Karlin, S. (1960). Dynamic inventory policy with varying stochastic demands. *Management Science*, 6(3):231–258. *Cited p.28*

[Lee and Nahmias, 1993] Lee, H. L. and Nahmias, S. (1993). Chapter 1 single-product, single-location models. In *Logistics of Production and Inventory*, volume 4 of *Handbooks in Operations Research and Management Science*, pages 3 – 55. Elsevier. *2 citations pp.19 and 21*

[Lightburn, 2003] Lightburn, A. (2003). Unsaleables Benchmark Report. *Unsaleables Steering Commitee, Food Marketing Institute and Grocery Manufactures of America*. *Cited p.2*

[Lundin and Morton, 1975] Lundin, R. A. and Morton, T. E. (1975). Planning horizons for the dynamic lot size model: Zabel vs. protective procedures and computational results. *Operations Research*, 23(4):711–734. *Cited p.30*

[Maimon et al., 1998] Maimon, O., Khmelnitsky, E., and Kogan, K. (1998). *Optimal Flow Control in Manufacturing Systems*, volume 18 of *Applied Optimization*. Springer US, Boston, MA. *Cited p.9*

[Michalska and Mayne, 1995] Michalska, H. and Mayne, D. Q. (1995). Moving horizon observers and observer-based control. *IEEE Transactions on Automatic Control*, 40(6):995–1006. *Cited p.34*

[Morton, 1979] Morton, T. E. (1979). Infinite-Horizon Dynamic Programming Models—A Planning-Horizon Formulation. *Operations Research*, 27(4):730–742. *Cited p.34*

[Morton and Pentico, 1995] Morton, T. E. and Pentico, D. W. (1995). The finite horizon nonstationary stochastic inventory problem: Near-myopic bounds, heuristics, testing. *Management Science*, 41(2):334–343. *Cited p.35*

[Muckstadt and Sapra, 2010] Muckstadt, J. A. and Sapra, A. (2010). *Principles of Inventory Management*. Springer Series in Operations Research and Financial Engineering. Springer New York, New York, NY. *2 citations pp.9 and 14*

[Mula et al., 2006] Mula, J., Poler, R., García-Sabater, J., and Lario, F. (2006). Models for production planning under uncertainty: A review. *International Journal of Production Economics*, 103(1):271–285. *Cited p.3*

[Orlicky, 1974] Orlicky, J. A. (1974). *Material Requirements Planning: The New Way of Life in Production and Inventory Management.* McGraw-Hill, Inc., New York, NY, USA. *Cited p.27*

[Penlesky et al., 1989] Penlesky, R. J., Berry, W. L., and Wemmerlöv, U. (1989). Open order due date maintenance in mrp systems. *Manage. Sci.*, 35(5):571–584. *2 citations pp.3 and 24*

[Penlesky et al., 1991] Penlesky, R. J., Wemmerlöv, U., and Berry, W. L. (1991). Filtering heuristics for rescheduling open orders in mrp systems. *International Journal of Production Research*, 29(11):2279–2296. *Cited p.24*

[Peterson and Silver, 1979] Peterson, R. and Silver, E. A. (1979). *Decision systems for inventory management and production planning.* Wiley New York. *3 citations pp.9, 14, and 27*

[Pochet and Wolsey, 2006] Pochet, Y. and Wolsey, L. A. (2006). *Production planning by mixed integer programming.* Springer series in operations research and financial engineering. Springer, New York ; Berlin. *2 citations pp.9 and 28*

[Riesch, 2012] Riesch, H. (2012). Levels of uncertainty. In *Handbook of Risk Theory*, pages 87–110. Springer. *Cited p.41*

[Ristroph, 1990] Ristroph, J. H. (1990). Simulation of ordering rules for single level material requirements planning with a rolling horizon and no forecast error. *Computers & Industrial Engineering*, 19(1):155–159. *Cited p.30*

[Rohde et al., 2000] Rohde, J., Meyr, H., and Wagner, M. (2000). Die supply chain planning matrix. Publications of darmstadt technical university, institute for business studies (bwl), Darmstadt Technical University, Department of Business Administration, Economics and Law, Institute for Business Studies (BWL). *Cited p.9*

[Russell and Urban, 1993] Russell, R. and Urban, T. (1993). Horizon extension for rolling production schedules: Length and accuracy requirements. *International Journal of Production Economics*, 29(1):111–122. *Cited p.30*

[Sahin et al., 2013] Sahin, F., Narayanan, A., and Robinson, E. P. (2013). Rolling horizon planning in supply chains: review, implications and directions for future research. *International Journal of Production Research*, 51(18):5413–5436. *4 citations pp.3, 4, 29, and 37*

[Schwarz, 1977] Schwarz, L. B. (1977). Technical note—a note on the near optimality of "5-eoq's worth" forecast horizons. *Operations Research*, 25(3):533–536. *Cited p.30*

[Sethi and Sorger, 1991] Sethi, S. and Sorger, G. (1991). A theory of rolling horizon decision making. *Annals of Operations Research*, 29(1):387–415. *Cited p.34*

[Sethi et al., 2001] Sethi, S. P., Yan, H., and Zhang, H. (2001). Peeling layers of an onion: Inventory model with multiple delivery modes and forecast updates. *Journal of Optimization Theory and Applications*, 108(2):253–281. *Cited p.34*

[Sethi et al., 2006] Sethi, S. P., Yan, H., and Zhang, H. (2006). *Inventory and supply chain management with forecast updates*, volume 81. Springer Science & Business Media. *Cited p.34*

[Silver and Miltenburg, 1984] Silver, E. and Miltenburg, J. (1984). Two modifications of the silver-meal lot sizing heuristic. *INFOR: Information Systems and Operational Research*, 22(1):56–69. *Cited p.30*

[Silver and Meal, 1973] Silver, E. A. and Meal, H. C. (1973). A heuristic for selecting lot size quantities for the case of a deterministic time-varying demand rate and discrete opportunities for replenishment. *Production and inventory management*, 14(2):64–74. *Cited p.30*

[Sridharan et al., 1988] Sridharan, S. V., Berry, W. L., and Udayabhanu, V. (1988). Measuring master production schedule stability under rolling planning horizons. *Decision Sciences*, 19(1):147–166. *3 citations pp.29, 30, and 31*

[Sridharan and Berry, 1990a] Sridharan, V. and Berry, W. L. (1990a). Freezing the master production schedule under demand uncertainty. *Decision Sciences*, 21(1):97–120. *2 citations pp.29 and 32*

[Sridharan and Berry, 1990b] Sridharan, V. and Berry, W. L. (1990b). Master production scheduling make-to-stock products: a framework for analysis. *The International Journal of Production Research*, 28(3):541–558. *Cited p.30*

[Sridharan et al., 1987] Sridharan, V., Berry, W. L., and Udayabhanu, V. (1987). Freezing the master production schedule under rolling planning horizons. *Management Science*, 33(9):1137–1149. *Cited p.30*

[Sridharan and LaForge, 1994a] Sridharan, V. and LaForge, R. (1994a). A model to estimate service levels when a portion of the master production schedule is frozen. *Computers & Operations Research*, 21(5):477 – 486. *Cited p.31*

[Sridharan and LaForge, 1994b] Sridharan, V. and LaForge, R. L. (1994b). Freezing the Master Production Schedule: Implications for Fill Rate*. *Decision Sciences*, 25(3):461–469. *Cited p.29*

[Stadtler, 2000] Stadtler, H. (2000). Improved rolling schedules for the dynamic single-level lot-sizing problem. *Management Science*, 46(2):318–326. *Cited p.31*

[Tang and Grubbström, 2002] Tang, O. and Grubbström, R. W. (2002). Planning and replanning the master production schedule under demand uncertainty. *International journal of production economics*, 78(3):323–334. *Cited p.31*

[Tarim et al., 2011] Tarim, S. A., Dogru, M. K., Özen, U., and Rossi, R. (2011). An efficient computational method for a stochastic dynamic lot-sizing problem under service-level constraints. *European Journal of Operational Research*, 215(3):563 – 571. *Cited p.28*

[Tempelmeier, 2011] Tempelmeier, H. (2011). *Inventory management in supply networks : problems, models, solutions*. Books on Demand GmbH, Norderstedt. *2 citations pp.28 and 29*

[Toomey, 2000] Toomey, J. (2000). *Inventory Management: Principles, Concepts and Techniques*. Materials Management Logistics Series. Springer US. *Cited p.14*

[Vargas and Metters, 2011] Vargas, V. and Metters, R. (2011). A master production scheduling procedure for stochastic demand and rolling planning horizons. *International Journal of Production Economics*, 132(2):296–302. *Cited p.31*

[Vollmann et al., 1997] Vollmann, T., Berry, W., and Whybark, D. (1997). *Manufacturing Planning and Control Systems*. Irwin/McGraw-Hill. *Cited p.27*

[Wagner and Whitin, 1958] Wagner, H. M. and Whitin, T. M. (1958). Dynamic version of the economic lot size model. *Management science*, 5(1):89–96. *Cited p.30*

[Wemmerlöv, 1986] Wemmerlöv, U. (1986). A time-phased order-point system in environments with and without demand uncertainty: a comparative analysis of non-monetary performance variables. *International journal of production research*, 24(2):343–358. *Cited p.31*

[Wemmerlöv and Whybark, 1984] Wemmerlöv, U. and Whybark, D. C. (1984). Lot-sizing under uncertainty in a rolling schedule environment. *The International Journal Of Production Research*, 22(3):467–484. *Cited p.32*

[Whybark and Williams, 1976] Whybark, D. C. and Williams, J. G. (1976). Material requirements planning under uncertainty. *Decision sciences*, 7(4):595–606. *Cited p.31*

[Yeung et al., 1998] Yeung, J. H., Wong, W., and Ma, L. (1998). Parameters affecting the effectiveness of mrp systems: A review. *International Journal of Production Research*, 36(2):313–332. *Cited p.29*

[Zipkin, 2000] Zipkin, P. (2000). *Foundations of Inventory Management*. Irwin/McGraw-Hill series: Operations and decision sciences. McGraw-Hill. *Cited p.27*

université PARIS-SACLAY
ÉCOLE DOCTORALE
INTERFACES
Approches Interdisciplinaires : Fondements, Applications et Innovation

**Titre :** Méthodes de pilotage des flux avec prise : en compte des incertitudes prévisionnelles

**Mots clés :** Planification de Production, Incertitudes Prévisionnelles, Gestion des Incertitudes, Optimisation Dynamique Stochastique.

**Résumé :** Intégrée dans la chaîne décisionnelle de la Supply Chain à un niveau tactique, la Planification de Production est un process clé qui permet de répondre au mieux aux besoins selon les ressources de l'entreprise. Un des défis du domaine est la gestion des incertitudes prévisionnelles, ayant des conséquences importantes sur des indicateurs clés comme le taux de service ou les coûts. Pour y faire face, des méthodes améliorant la flexibilité des processus sont mais em place, comme le contexte de travail en Plan Glissant. Cependant, en actualisant fréquemment les données, la stabilité du système se retrouve dégradée. Ainsi, malgré les gains issus de la gestion des incertitudes, ce cadre crée une complexité dynamique à gérer. Ce travail traite de cette complexité issue de l'actualisation des prévisions pour la planification de production en plan glissant.

Plus particulièrement, la question traitée ici concerne l'optimisation du plan de production, en considérant u n système mono-produit mono-étage. Une modélisation mathématique générique est tout d'abord développée pour construire un modèle d'optimisation théorique du problème. Ensuite, une procédure de résolution optimale est développée em utilisant le cadre d'optimisation dynamique stochastique. Ce modèle est appliquée à des cas concrets pour lesquels l'optimalité des solutions calculées est prouvée analytiquement grâce à un raisonnement inductif basé sur des séquences de calcul d'espérances mathématiques. Des analyses numériques finallement conduites mettent en exergue les performances de la méthode développée, ses limites, et sa sensibilité vis-à-vis de l'environnement industriel.

**Title :** Production Planning under Uncertainties and Forecast Updates

**Keywords :** Production Planning, Forecasts Uncertainties, Uncertainties Management, Stochastic Dynamic Programming.

**Abstract :** Production Planning, as part of tactical operations integrated into the Supply Chain process, is a key procedure allowing decisioners to balance demand and production resources. One of its most challenging issues is to handle uncertainties, especially the ones coming from the Forecasted Demand. In order to manage indicators at stake, such as service level and costs, best practices increasing flexibility in the process are implemented, as Rolling-Plan Framework. However, it creates instability since the updates procedures make the data set on change constantly. Consequently, although the gain in terms of flexibility is non-negligible for the uncertainties management, it generates on the other hand dynamics complexity. We study in this work how to deal this dynamics complexity generated by updates of the Forecasted Demand made in a Rolling-Plan Framework of a Production Planning Process.

In particular, the question to which it answers is how to optimize the Production Plan in such a context. This issue is tackled considering a single item single level production system. A general mathematical model in the context of our study is built to be exploitable for analytical optimization. A theoretical optimization framework is designed, and a specific solutions computation framework using stochastic dynamic programming is developed. We apply it in some precise study cases in order to compute optimal solutions and get some valuable analytical results thanks to a dynamic computation process. The optimality of the solutions is proven through an inductive reasoning based on expectations computation. Solutions are finally implemented and calculated numerically with simulations in some particular numerical examples. Analyses and sensitivity studies are performed, highlighting the performances of our optimization method.