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**Analysis of singularities in elliptic
equations:
The Ginzburg-Landau model of
superconductivity, the Lin-Ni-Takagi
problem, the Keller-Segel model of
chemotaxis, and conformal geometry**

THÈSE

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Spécialité Mathématiques

par

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Mis en page avec la classe thesul.

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General Introduction

Nonlinear partial differential equations (PDEs) occur in a wide variety of areas of mathematics. Since the 20th century, the analysis of PDEs has become a field in itself, with a large number of research directions. This thesis is concerned with the analysis of singularities in elliptic equations, focusing on problems from mathematical physics, mathematical biology, and differential geometry. A diversity of phenomena will arise in the analysis of the considered problems and different techniques will be applied.

The topics we will analyze are:

1. The three-dimensional Ginzburg-Landau model of superconductivity with external magnetic field.
2. The critical Lin-Ni-Takagi problem in dimension three.
3. The Keller-Segel model of chemotaxis.
4. Conformal geometry.

Let us now give an overview on all these subjects. Individualized chapters will then be devoted to describe in details each of these problems.

1.1 The Ginzburg-Landau model of superconductivity

Superconductors are certain metals and alloys, which, when cooled down below a critical (typically very low) temperature, lose their resistivity, which allows permanent currents to circulate without loss of energy. Superconductivity was discovered by Ohnes in 1911. As a phenomenological description of this phenomenon, Ginzburg and Landau [GL50] introduced in 1950 the Ginzburg-Landau model, which has been proven to effectively predict the behavior of superconductors and that was subsequently justified as a limit of the Bardeen-Cooper-Schrieffer (BCS) quantum theory [BCS57]. It is a model of great importance in physics, with Nobel prizes awarded for it to Abrikosov, Ginzburg, and Landau.

The model proposed by Ginzburg and Landau to describe the state of a superconducting sample confined in a domain $\Omega \subset \mathbb{R}^n$, with $n = 2, 3$, in an applied magnetic field H_{ex} , assuming that the temperature is fixed and below the critical one, can be written as

$$GL_\varepsilon(u, A) = \frac{1}{2} \int_{\Omega} |\nabla_A u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 + \frac{1}{2} \int_{\mathbb{R}^n} |H - H_{\text{ex}}|^2.$$

Here

- Ω is a bounded domain of \mathbb{R}^n , that we assume to be smooth and simply connected.
- $u : \Omega \rightarrow \mathbb{C}$ is called the *order parameter*. Its modulus squared (the density of Cooper pairs of superconducting electrons in the BCS quantum theory) indicates the local state of the superconductor: where $|u|^2 \approx 1$ the material is in the superconducting phase, where $|u|^2 \approx 0$ in the normal phase.
- $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the electromagnetic vector potential of the magnetic field $H = \text{curl} A$, which is induced by the currents which appear in the superconductor in response to the applied (or external) magnetic field $H_{\text{ex}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$.
- The notation ∇_A denotes the covariant gradient $\nabla - iA$.
- The parameter $\varepsilon > 0$ is the inverse of the ‘‘Ginzburg-Landau parameter’’ usually denoted κ , a non-dimensional parameter depending only on the material. It is also the ratio between the ‘‘coherence length’’ usually denoted ξ (roughly the vortex-core size) and the ‘‘penetration depth’’ of the magnetic field usually denoted λ . We will be interested in the regime of small ε , corresponding to extreme type-II superconductors.

An essential feature of type-II superconductors is the occurrence of *vortices* (similar to those in fluid mechanics, but quantized) in the presence of an applied magnetic field. Physically, they correspond to normal phase regions around which a superconducting loop of current circulates. Since u is complex-valued, it can have zeroes with a nonzero topological degree. Vortices are then *topological defects* of co-dimension 2 and are the crucial objects of interest in the analysis of the model.

Observed configurations correspond to minimizers or critical points of GL_ε . Heuristically, we have that:

- In the regime of small ε , the term $\frac{1}{4\varepsilon^2}(1 - |u|^2)^2$ favors $|u|$ to be close to 1. A scaling argument hints that $|u|$ is different from 1 in small tubes of radius $O(\varepsilon)$.
- The magnetic term $\frac{1}{2}|H - H_{\text{ex}}|^2$ favors the applied magnetic field to penetrate the superconductor in such a way that the induced magnetic field H equals H_{ex} . We also expect this term to be very close to zero away from the sample.

The behavior of type-II superconductors has been experimentally observed to strongly depend on the strength of the applied magnetic field, defined as $h_{\text{ex}} := \|H_{\text{ex}}\|_{L^2(\Omega, \mathbb{R}^3)}$. There are three main critical values of h_{ex} or critical fields H_{c_1}, H_{c_2} , and H_{c_3} , for which phase transitions occur.

- If h_{ex} is below H_{c_1} , which is of order $O(|\log \varepsilon|)$, then the superconductor is everywhere in its superconducting phase, i.e. $|u|$ is uniformly close to 1, and the applied field is forced out from the material due to the occurrence of supercurrents near $\partial\Omega$. This phenomenon is known as the Meissner effect.

- At H_{c_1} the first vortice(s) appear and the applied field penetrates the superconductor through the vortice(s).
- Between H_{c_1} and H_{c_2} the superconducting and normal phases coexist in the sample. As h_{ex} increases, so does the number of vortices. The vortices repeal each other, while the external magnetic field confines them inside the sample. This competition forces them to arrange themselves to form triangular lattices in the bulk of the material, which was predicted by Abrikosov [Abr57], and later observed experimentally.
- At $H_{c_2} \approx \frac{1}{\varepsilon^2}$, the superconductivity is lost in the bulk of the sample.
- Between H_{c_2} and H_{c_3} , superconductivity persists only near the boundary.
- After $H_{c_3} = O\left(\frac{1}{\varepsilon^2}\right)$, the applied magnetic field completely penetrates the sample and the superconductivity is lost, i.e. $u = 0$.

For further details on the model, we refer to [Tin96, DG99, SS07]. In this thesis, we are interested in the first critical field H_{c_1} . The study of H_{c_2} or higher applied fields requires completely different techniques.

1.1.1 Mathematical work on the Ginzburg-Landau model

We introduce the Ginzburg-Landau free energy

$$F_\varepsilon(u, A) = \frac{1}{2} \int_\Omega |\nabla_A u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 + |\text{curl } A|^2.$$

This functional is closely related to the simpler Ginzburg-Landau model without magnetic field

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2.$$

In the 1990's, mathematicians became interested in the Ginzburg-Landau model. In the pioneer work [BBH94] in 2D, Bethuel, Brezis, and Hélein introduced systematic tools and asymptotic estimates to study vortices in the model without magnetic field, which is a complex-valued version of the Allen-Cahn model for phase transitions. A vortex in 2D is an object centered at an isolated zero of u , around which the phase of u has a nonzero winding number, called the degree of the vortex. A typical vortex centered at a point x_0 behaves like $u = \rho e^{i\varphi}$ with $\rho = f\left(\frac{|x-x_0|}{\varepsilon}\right)$, where $f(0) = 0$ and f tends to 1 as $r \rightarrow +\infty$, i.e. its characteristic core size is ε , and

$$\frac{1}{2\pi} \int_{\partial B(x_0, R\varepsilon)} \frac{\partial \varphi}{\partial \tau} = d \in \mathbb{Z}$$

is its degree (also defined as the topological-degree of the map $u/|u| : \partial B(x_0, R\varepsilon) \rightarrow S^1$). In [BBH94], the effect of the external magnetic field was replaced by a Dirichlet boundary condition $u = g$ on $\partial\Omega$, where g is an S^1 -valued map of winding degree $d > 0$. This boundary condition triggers the occurrence of vortices, allowing only for a fixed number

of them. They proved that minimizers of E_ε have d vortices of degree one and that the following expansion of the energy holds:

$$E_\varepsilon(u) \approx \pi d |\log \varepsilon| + W(a_1, \dots, a_d) \quad \text{as } \varepsilon \rightarrow 0,$$

where W is the “renormalized energy”, a function depending only on the vortex-centers a_i , which repel one another according to a coulombian interaction. This analysis was then adapted to the study of the free-energy by Bethuel and Rivière [BR95], under a Dirichlet boundary condition on $\partial\Omega$ that forces the presence of vortices and the (fixed) number of them. However, a new approach was necessary to treat the case of the full model when the number of vortices gets unbounded as $\varepsilon \rightarrow 0$. Tools able of handling this were developed after the works by Jerrard [Jer99] and Sandier [San98]. In a series of works summarized in the book [SS07], Sandier and Serfaty analyzed the full model and characterized the behavior of minimizers in different regimes of the applied field. In particular, they mathematically deduced the experimentally observed phenomena when h_{ex} is below H_{c_2} (see [SS00a, SS00b, SS00c, SS03]).

Rivière [Riv95], was the first to study the asymptotic behavior of minimizers of the free energy (under a Dirichlet boundary condition) as $\varepsilon \rightarrow 0$ in the 3D setting. Roughly speaking, vortices in 3D are small tubes of radius $O(\varepsilon)$ around the one dimensional zero-set of u . In the limit $\varepsilon \rightarrow 0$ vortices become curves L_i with an integer multiplicity d_i , whose cost is at least an order $\pi d_i |L_i| |\log \varepsilon|$ of energy, where $|L|$ denotes the length of L . In [Riv95], using an η -ellipticity result, Rivière identified the limiting one dimensional singular set of minimizers of F_ε with a mass minimizing current, which corresponds to a *minimal connection*. This concept was introduced in the work by Brezis, Coron, and Lieb [BCL86]. In the case without magnetic field, an alternative proof was given by Sandier [San01]. This result was then extended by Bourgain, Brezis, and Mironescu [BBM04] to allow for a general boundary condition. Generalizations to higher dimension of these results in the case without magnetic field were developed in the works by Lin and Rivière [LR99], Bethuel, Brezis, and Orlandi [BBO01], and Alberti, Baldo, and Orlandi [ABO05]. Other important results, that we comment later on, include [LR01, JS02, SS04, Chi05, SS17].

Jerrard, Montero, and Sternberg [JMS04] established the existence of locally minimizing vortex solutions to the full Ginzburg-Landau energy in 3D, by using a construction in the spirit of [MSZ04]. Later on, Alama, Bronsard, and Montero [ABM06] identified a candidate expression for the first critical field in 3D in the case of the ball. Then, Baldo, Jerrard, Orlandi, and Soner [BJOS12, BJOS13], via Γ -convergence arguments, described the asymptotic behavior of the full model as $\varepsilon \rightarrow 0$ and characterized to leading order the first critical field in 3D for a general bounded domain. We point out that many questions remain open in 3D, in particular obtaining all the analogues of the 2D results contained in [SS07]. This is due to the more complicated geometry of the vortices in 3D: they are lines with no a priori regularity, and have to be understood in the framework of currents and using geometric measure theory.

In addition to its importance in the modeling of superconductivity, the Ginzburg-Landau model is mathematically extremely close to the Gross-Pitaevskii model for superfluidity (see for instance [TT90, Ser01]) and models for rotating Bose-Einstein condensates

(see for example [Aft06]). In fact, the mathematical tools developed for Ginzburg-Landau have been successfully exported to these models.

1.1.2 Essential tools

The Ginzburg-Landau model is known to be an $\mathbb{U}(1)$ -gauge theory. This means that all the meaningful physical quantities are invariant under the gauge-transformations

$$u \mapsto ue^{i\Phi}, \quad A \mapsto A + \nabla\Phi,$$

where Φ is any smooth real-valued function. The Ginzburg-Landau energy and its associated free energy are gauge invariant, as well as the density of superconducting Cooper pairs $|u|^2$, the induced magnetic field H , and the vorticity, defined, for any sufficiently regular configuration (u, A) , as

$$\mu(u, A) = \text{curl}(iu, \nabla_A u) + \text{curl} A,$$

where (\cdot, \cdot) denotes the scalar product in \mathbb{C} identified with \mathbb{R}^2 i.e. $(a, b) = \frac{\bar{a}b + a\bar{b}}{2}$. This quantity is the gauge-invariant version of the Jacobian determinant of u and is the analogue of the vorticity of a fluid.

To analyze the vortices, authors have developed tools, in particular the ball construction method and Jacobian estimates. The first one was introduced independently by Jerrard [Jer99] and Sandier [San98]. It allows one to obtain universal lower bounds for two-dimensional Ginzburg-Landau energies in terms of the topology of the vortices. These lower bounds capture the fact that vortices of degree d cost at least an order $\pi|d| \log \frac{1}{\varepsilon}$ of energy. The second tool, that has been widely used in the analysis of the Ginzburg-Landau model in any dimension after the work by Jerrard and Sonner [JS02], is the Jacobian (or vorticity) estimate, which allows one to relate the vorticity $\mu(u, A)$ with, roughly speaking, Dirac masses supported on co-dimension 2 singularities. When $n = 2$, these masses are supported on points naturally derived from the ball construction. The following result presents an optimal version of these estimates in 2D.

Theorem 1.1.1 (Sandier and Serfaty [SS07]). *Let $n = 2$. For any $\alpha \in (0, 1)$ there exists $\varepsilon_0(\alpha) > 0$ such that, for any $\varepsilon < \varepsilon_0$, if $(u_\varepsilon, A_\varepsilon)$ is a configuration such that $\frac{1}{2} \int_\Omega |\nabla |u_\varepsilon||^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 \leq \varepsilon^{\alpha-1}$, then for any $r \in (\varepsilon^{\frac{\alpha}{2}}, 1)$, there exists a finite collection of disjoint closed balls $\{B(a_i, r_i)\}_i$ of the sum of the radii r , covering $\{|u_\varepsilon| \leq 1 - \varepsilon^{\frac{\alpha}{4}}\} \cap \{x \in \Omega \mid \text{dist}(x, \partial\Omega) \geq \varepsilon\}$ such that*

$$\frac{1}{2} \int_{\cup_i B(a_i, r_i)} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 + |\text{curl} A_\varepsilon|^2 \geq \pi D \left(\log \frac{r}{D\varepsilon} - C \right),$$

where $D = \sum_i |d_i|$, $d_i = \text{deg}(u, \partial B(a_i, r_i))$, and C is a universal constant, and, for any $\beta \in (0, 1)$, we have

$$\left\| \mu(u_\varepsilon, A_\varepsilon) - 2\pi \sum_i d_i \delta_{a_i} \right\|_{C_0^{0,\beta}(\Omega)^*} \leq r^\beta F_\varepsilon(u_\varepsilon, A_\varepsilon).$$

The first goal of this thesis is to provide similar estimates in 3D. One of the important features of the previous theorem is that the estimates are at the ε -level. When $n = 3$, only results that work in the limit $\varepsilon \rightarrow 0$ can be found in the literature. Here, we provide a quantitative three-dimensional vortex approximation construction for the Ginzburg-Landau energy. This construction provides an approximation of vortex lines coupled to a lower bound for the energy, optimal to leading order, analogous to the 2D ones, and valid for the first time at the ε -level.

Theorem 1.1.2 ([Roma]). *Let $n = 3$ and assume that $\partial\Omega$ is C^2 . For any $m, M > 0$ there exist $C, \varepsilon_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$, if $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq M|\log \varepsilon|^m$ then*

$$\frac{1}{2} \int_{S_{\nu_\varepsilon}} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 + |\operatorname{curl} A_\varepsilon|^2 \geq \frac{1}{2} |\nu_\varepsilon|(\Omega) \left(\log \frac{1}{\varepsilon} - C \log \log \frac{1}{\varepsilon} \right) + o(1),$$

where ν_ε is a finite sum (in the sense of currents) of polygonal lines and $|S_{\nu_\varepsilon}| = o(1)$ (with $\operatorname{supp}(\nu_\varepsilon) \subset S_{\nu_\varepsilon}$), and

$$\|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C_T^{0,\gamma}(\Omega)^*} \leq o(1)$$

for any $\gamma \in (0, 1]$, where $C_T^{0,\gamma}(\Omega)$ denotes the space of vector fields in $C^{0,\gamma}(\Omega)$ whose tangential component vanishes on $\partial\Omega$.

In this theorem and in the rest of this chapter $o(1)$ denotes a function of ε tending to zero as $\varepsilon \rightarrow 0$. This result has been presented in a simplified form. A quantitative version can be found in Chapter 1 of Part I. Besides, in the case Ω is only assumed to have Lipschitz boundary then a similar theorem holds far from the boundary.

Let us point out that the ball construction used to prove Theorem 1.1.1 is purely two-dimensional. For this reason, the three-dimensional analogue of this result is based on a new vortex approximation construction. Through a procedure based on slicing of currents, 3D Jacobian estimates and lower bounds were proved in [JS02] and [SS04]. Alternatively, a suitable application of the Federer-Fleming polyhedral deformation theorem was used in [ABO05] and [BJOS12] to obtain results of the same type. But these constructions are not sufficient for our purposes, because they cannot be made ε -quantitative.

Explained in simple words, our construction is made as follows. We consider a grid of side-length $\delta = \delta(\varepsilon) \ll 1$. If appropriately positioned, the grid can be chosen so that $|u_\varepsilon| \geq 5/8$ on every edge of a cube. Then, 2D estimates imply that the restriction of the vorticity to the boundary of every cube is well approximated by a linear combination of Dirac masses. Using minimal connections, we connect the points of support of these measures, which essentially yields our approximation.

To get an optimal and quantitative lower bound for the free energy, we use the co-area formula and the fact that minimal connections satisfy a calibration property. This method was used by Sandier [San01] to obtain a lower bound for the Ginzburg-Landau energy without magnetic field, i.e. $E_\varepsilon(u) = F_\varepsilon(u, 0)$. The analysis presented here is much more involved due to the unboundedness of the number of vortices.

1.1.3 The first critical field

From now on we focus on the first critical field. Our objective is to describe the behavior of global minimizers of GL_ε in 3D when h_{ex} is below and near the first critical field. We begin by reviewing the work by Sandier and Serfaty in the 2D case.

1.1.3.1 The 2D case

When $n = 2$, after a series of reductions, one is left with studying the simpler functional

$$GL_\varepsilon^{2D}(u, A) = \frac{1}{2} \int_\Omega |\nabla_A u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 + |h - h_{\text{ex}}|^2,$$

where $h = \text{curl } A = \partial_1 A_2 - \partial_2 A_1$ and $h_{\text{ex}} > 0$ is a parameter. Minimizers and critical points of this functional solve the associated Euler-Lagrange system of equations

$$\begin{cases} -\nabla_A^2 u &= \frac{1}{\varepsilon^2} u(1 - |u|^2) & \text{in } \Omega \\ -\nabla^\perp h &= \langle iu, \nabla_A u \rangle & \text{in } \Omega. \end{cases}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$, with boundary conditions

$$\begin{cases} -\nabla_A u \cdot \nu &= 0 & \text{on } \partial\Omega \\ h &= h_{\text{ex}} & \text{on } \partial\Omega. \end{cases}$$

By taking the curl of the second Ginzburg-Landau equation, we find

$$\begin{cases} -\Delta h + h &= \mu(u, A) & \text{in } \Omega \\ h &= h_{\text{ex}} & \text{on } \partial\Omega. \end{cases}$$

This equation relates the induced magnetic field in the sample with the vorticity and it is usually called London equation in the physics literature. When the vorticity vanishes, up to dividing by h_{ex} , the London equation becomes

$$\begin{cases} -\Delta h_0 + h_0 &= 0 & \text{in } \Omega \\ h_0 &= 1 & \text{on } \partial\Omega. \end{cases}$$

Observe that $A_0 = \nabla^\perp h_0$ is such that $\text{curl } A_0 = h_0$. We then expect the configuration $(1, h_{\text{ex}} A_0)$ to be a good approximation of the Meissner (vortex-free) solution, i.e. the minimizer of GL_ε^{2D} when h_{ex} is below H_{c1} . Indeed, this is true. By writing $A = A' + h_{\text{ex}} A_0$, one can split the energy as

$$GL_\varepsilon^{2D}(u, A) = h_{\text{ex}}^2 J_0 + F_\varepsilon(u, A') + h_{\text{ex}} \int_\Omega \mu(u, A') \xi_0 + R_0$$

where $\xi_0 = h_0 - 1$ and solves

$$(1) \quad \begin{cases} -\Delta^2 \xi_0 + \Delta \xi_0 &= 0 & \text{in } \Omega \\ \xi_0 &= 0 & \text{on } \partial\Omega \\ \Delta \xi_0 &= 1 & \text{on } \partial\Omega, \end{cases}$$

$$J_0 = \frac{GL_\varepsilon^{2D}(1, h_{\text{ex}}A_0)}{h_{\text{ex}}^2} = \frac{1}{2} \int_\Omega |\nabla h_0|^2 + |h_0 - 1|^2 = \frac{1}{2} \|\xi_0\|_{H^1(\Omega)}^2,$$

and $R_0 = o(1)$ when h_{ex} is bounded by a negative power of $|\log \varepsilon|$. Using Theorem 1.1.1 and this splitting, a quick inspection leads to (formally) finding the leading order value of the first critical field:

$$H_{c_1}^{2D} = \frac{1}{2 \max |\xi_0|} |\log \varepsilon|.$$

The rigorous proof of this heuristic was performed in a series of articles. First, Serfaty minimized GL_ε^{2D} in a subspace of $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$. Let us introduce the space

$$D_M := \{(u, A) \mid E_\varepsilon(u) < \pi M |\log \varepsilon|, \operatorname{div} A = 0 \text{ in } \Omega, \text{ and } A \cdot \nu = 0 \text{ on } \partial\Omega\},$$

where $E_\varepsilon(u) := F_\varepsilon(u, 0) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2$, i.e. the Ginzburg-Landau energy without magnetic field. Roughly speaking, D_M can be seen as the set of configurations with less than M vortices. The following theorem concerns the minimization of the energy in this space.

Theorem 1.1.3 (Serfaty [Ser99]). *For any $M > 0$ there exists constants $\varepsilon_0 > 0$ and C such that:*

1. *For any $\varepsilon < \varepsilon_0$ and $h_{\text{ex}} < H_{c_1}^{2D} + C - o(1)$ there exists a stable critical point $(u_\varepsilon, A_\varepsilon)$ of GL_ε^{2D} which is minimizing over D_M . Any such critical point is such that $|u_\varepsilon|$ does not vanish.*
2. *For any $\varepsilon < \varepsilon_0$ and $h_{\text{ex}} > H_{c_1}^{2D} + C - o(1)$ there exists a stable critical point $(u_\varepsilon, A_\varepsilon)$ of GL_ε^{2D} which is minimizing over D_M . Any such critical point is such that $|u_\varepsilon|$ vanishes.*

This theorem shows that $H_{c_1}^{2D}$ is, up to a $O(1)$, a critical value of the applied field, when the minimization of GL_ε^{2D} is restricted to D_M . The use of this space was a technical commodity. To minimize the energy in the natural minimization space $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$, one needs to handle an a-priori unbounded number of vortices. By combining the ε -level tools with the energy splitting and a “clearing out” result (see [BBH94, Theorem III.3]), Sandier and Serfaty were able to prove the following result.

Theorem 1.1.4 (Sandier and Serfaty [SS00a]). *There exist constants $\varepsilon_0, K_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and any $h_{\text{ex}} < H_{c_1}^{2D} - K_0 \log |\log \varepsilon|$, the global minimizers $(u_\varepsilon, A_\varepsilon)$ of GL_ε^{2D} in $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ are such that $|u_\varepsilon| \geq 1/2$ in Ω and $GL_\varepsilon^{2D}(u_\varepsilon, A_\varepsilon) = h_{\text{ex}}^2 J_0 + o(1)$.*

Thus, up to an imprecision of $O(\log |\log \varepsilon|)$, minimizers of the energy below H_{c_1} are vortex-less. A further analysis of the energy allowed them to prove that the set D_M , for M large enough, is the set where the global minimizers of the energy lie.

Theorem 1.1.5 (Sandier and Serfaty [SS03]). *For any $K > 0$, there exists a constant $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and any $h_{\text{ex}} < H_{c_1}^{2D} + K \log |\log \varepsilon|$, the global minimizers of GL_ε^{2D} in $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ are gauge equivalent to an element of D_M .*

In particular, the above results show that the first critical field in 2D is given by $H_{c_1}^{2D} + O(1)$. This theorem, which reduces the minimization problem in the natural space $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ to the space of configuration with less than M vortices, follows from a deep result that we explain now.

Observe that

$$\min_{(u,A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)} GL_\varepsilon^{2D}(u, A) \leq GL_\varepsilon^{2D}(1, 0) = \frac{1}{2} |\Omega| h_{\text{ex}}^2,$$

where $|\Omega|$ denotes the measure of Ω . In particular, under the assumptions of Theorem 1.1.5, the energy of a minimizing configuration $(u_\varepsilon, A_\varepsilon)$ is bounded by a negative power of $|\log \varepsilon|$. We can then apply Theorem 1.1.1 to associate to $(u_\varepsilon, A_\varepsilon)$ a family of vortices $\{(a_i, d_i)\}$. One has the following result.

Theorem 1.1.6 (Sandier and Serfaty [SS03]). *For any $K > 0$, there exist positive constant $\varepsilon_0, C, \alpha > 0$ such that for any $\varepsilon < \varepsilon_0$ and any $h_{\text{ex}} < H_{c_1}^{2D} + K \log |\log \varepsilon|$, if $(u_\varepsilon, A_\varepsilon)$ is a global minimizer of GL_ε^{2D} in $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ and $\{(a_i, d_i)\}$ is an associated family of vortices then*

1. $\forall i, d_i \geq 0$,
2. $\text{dist}(a_i, \Lambda) < C |\log \varepsilon|^{-\alpha}$ for any i such that $d_i \neq 0$, where Λ is the subset of Ω where the function ξ_0 defined above attains its minimum, and
3. $\sum_i d_i < C$.

To prove this theorem, in particular one needs some information about the set Λ . It turns out that the fact that Ω is simply connected implies that Λ is a finite set of points and a single point if Ω is convex.

Lemma 1.1.1. *The set of critical points of the function ξ_0 defined above is a finite set of points $\{p_1, \dots, p_k\}$. In particular the set Λ where ξ_0 attains its minimum is finite and there exist $C, N > 0$ such that*

$$(2) \quad \xi_0(x) \geq \min_{\Omega} \xi_0 + C \text{dist}(x, \Lambda)^N$$

for every $x \in \Omega$.

We will comment on this result later on.

1.1.3.2 Back to 3D

In the 3D case, Alama, Bronsard, and Montero [ABM06] identified a candidate expression for H_{c_1} in the case of the ball. Then, Baldo, Jerrard, Orlandi, and Soner [BJOS13], based on a Γ -convergence result, characterized to leading order the first critical field in 3D for a general bounded domain. Our purpose here is to derive with more precision this value, by getting the analogues of the 2D results proved by Sandier and Serfaty.

We begin by observing that in this case the minimization problem cannot be reduced to Ω as in the 2D case. In particular, the boundary of the domain plays a role in the analysis.

Let us assume that $H_{\text{ex}} = h_{\text{ex}}H_{0,\text{ex}}$, where $H_{0,\text{ex}}$ is a fixed unit vector and h_{ex} represents the intensity of the applied field. In particular, there exists $A_{\text{ex}} = h_{\text{ex}}A_{0,\text{ex}} \in H_{\text{loc}}^1(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$\text{curl } A_{0,\text{ex}} = H_{0,\text{ex}}, \quad \text{div } A_{0,\text{ex}} = 0 \text{ in } \mathbb{R}^3 \quad \text{and} \quad A_{0,\text{ex}} \cdot \nu = 0 \text{ on } \partial\Omega.$$

The natural space for the minimization of GL_ε in 3D is $H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$, where

$$H_{\text{curl}} := \{A \in H_{\text{loc}}^1(\mathbb{R}^3, \mathbb{R}^3) \mid \text{curl } A \in L^2(\mathbb{R}^3, \mathbb{R}^3)\}.$$

Minimizers and critical points of GL_ε solve the associated Euler-Lagrange system of equations

$$\begin{cases} -(\nabla_A)^2 u &= \frac{1}{\varepsilon^2} u(1 - |u|^2) & \text{in } \Omega \\ \text{curl}(H - H_{\text{ex}}) &= (iu, \nabla_A u)\chi_\Omega & \text{in } \mathbb{R}^3, \end{cases}$$

where χ_Ω is the characteristic function of Ω , with boundary conditions

$$\begin{cases} -\nabla_A u \cdot \nu &= 0 & \text{on } \partial\Omega \\ [H - H_{\text{ex}}] \times \nu &= 0 & \text{on } \partial\Omega, \end{cases}$$

where $[\cdot]$ denotes the jump across $\partial\Omega$.

By taking the curl of the second Ginzburg-Landau equation, we find the London equation

$$\text{curl}^2(H - H_{\text{ex}}) + H\chi_\Omega = \mu(u, A)\chi_\Omega.$$

In order to find an approximation of the Meissner solution, we recall that any vector field $A \in H^1(\Omega, \mathbb{R}^3)$ admits the following Hodge decomposition:

$$\begin{cases} A &= \text{curl } B_A + \nabla\phi_A & \text{in } \Omega \\ B_A \times \nu &= 0 & \text{on } \partial\Omega \\ \nabla\phi_A \cdot \nu &= A \cdot \nu & \text{on } \partial\Omega. \end{cases}$$

Moreover, the vector field B_A and the function ϕ_A are unique if chosen to satisfy $\text{div } B_A = 0$ in Ω and $\int_\Omega \phi_A = 0$.

For reasons that we explain in Chapter 2 of Part I, a good approximation of the Meissner solution is given by the configuration $(u_0, h_{\text{ex}}A_0)$, where $u_0 = e^{ih_{\text{ex}}\phi_{A_0}}$ and A_0 minimizes in a suitable space the functional

$$J(A) := \frac{1}{2} \int_\Omega |\text{curl } B_A|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |\text{curl}(A - A_{0,\text{ex}})|^2.$$

We have that $GL_\varepsilon(u_0, h_{\text{ex}}A_0) = h_{\text{ex}}^2 J(A_0)$ and the divergence-free vector field $B_0 := B_{A_0} \in C_T^{0,1}(\Omega, \mathbb{R}^3)$ satisfies

$$\begin{cases} \Delta^2(B_0 - B_{0,\text{ex}}) + \Delta B_0 &= 0 & \text{in } \Omega \\ B_0 \times \nu &= 0 & \text{on } \partial\Omega \\ [\Delta(B_0 - B_{0,\text{ex}})] \times \nu &= 0 & \text{on } \partial\Omega. \end{cases}$$

This special vector field turns out to be the analogue of the function ξ_0 . By writing $u = u_0 u'$ and $A = h_{\text{ex}} A_0 + A'$, we may split the energy as

$$GL_\varepsilon(u, A) = h_{\text{ex}}^2 J(A_0) + F_\varepsilon(u', A') + \frac{1}{2} \int_{\mathbb{R}^3 \setminus \Omega} |\text{curl } A'|^2 - h_{\text{ex}} \int_{\Omega} \mu(u', A') \wedge B_0 + R_0,$$

where R_0 is as before.

We let

$$\|B_0\|_* := \sup_{\mu \in X} \int_{\Omega} \mu \wedge B_0,$$

where X is the class of 1-currents such that $\partial\mu = 0$ relative to Ω and $|\mu|(\Omega) \leq 1$. By combining the ε -level estimates and the energy splitting, we heuristically find the leading order value of the first critical field:

$$H_{c_1}^{3D} = \frac{1}{2\|B_0\|_*} |\log \varepsilon|.$$

Our goal is to prove that the first critical field in 3D is given by $H_{c_1}^{3D} + O(1)$. Our first result in this direction is the analogous of Theorem 1.1.4.

Theorem 1.1.7 ([Romb]). *There exist constants $\varepsilon_0, K_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and $h_{\text{ex}} \leq H_{c_1}^{3D} - K_0 \log |\log \varepsilon|$, the global minimizers $(u_\varepsilon, A_\varepsilon)$ of GL_ε in $H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ are such that $|u_\varepsilon| \geq 1/2$ in Ω and $GL_\varepsilon(u_\varepsilon, A_\varepsilon) = h_{\text{ex}}^2 J(A_0) + o(1)$.*

By combining Theorem 1.1.2 and the energy splitting above, one can show that

$$F_\varepsilon(u'_\varepsilon, A'_\varepsilon) \leq o(1).$$

The Ginzburg-Landau equations satisfied by $(u_\varepsilon, A_\varepsilon)$ and this inequality allow us to use a “clearing out” result proved by Chiron [Chi05], using ideas from [BOS04]), which implies that $|u_\varepsilon| \geq 1/2$ in Ω . The proof of this result is much more complicated than its analogue in 2D. Results of the same kind were first proved by Rivière [Riv95] and Lin and Rivière [LR01]. Very recently, Sandier and Shafrir [SS17] gave a simplified proof in the case without magnetic field.

What we describe next is a work in preparation in collaboration with Etienne Sandier and Sylvia Serfaty [RSS]. It corresponds to the analysis of the behavior of global minimizers of GL_ε in 3D near the first critical field. Our objective is to get the analogue of Theorem 1.1.6. The main difficulty is that we need a certain kind of non-degeneracy condition on B_0 , which in contrast to the two-dimensional case it is difficult to find.

In 3D, we deal with the functional

$$\gamma(B_0) := \int_{\gamma} B_0 \cdot \tau,$$

defined for Lipschitz curves $\gamma \in X$, where X is defined as above. Here, τ denotes the tangent vector to γ . In particular, we observe that $|\gamma(B_0)| \leq \|B_0\|_*$.

We introduce the following assumption.

Non-degeneracy condition in 3D: There exists a unique Lipschitz curve $\gamma_0 \in X$ such that $\gamma_0(B_0) = \|B_0\|_*$. Moreover, there exist constants $C, N > 0$ such that for any Lipschitz curve $\gamma \in X$ if $\|\gamma - \gamma_0\|_* \geq \delta$, for some $\delta \in (0, 1)$, then

$$\gamma(B_0) \leq \gamma_0(B_0) - C\delta^N.$$

This is in the same spirit as the condition (2) satisfied by ξ_0 , but much weaker. As in the 2D case, we can apply Theorem 1.1.2 to associate to a minimizing configuration of the energy near $H_{c_1}^{3D}$ a vorticity approximation ν_ε . Moreover, we can decompose $\nu_\varepsilon = \sum_{i \in I} \gamma_i$, where each γ_i is a multiplicity 1 Lipschitz curve in X .

Theorem 1.1.8 (Work in preparation). *Assume the non-degeneracy condition above. For any $K > 0$, there exist positive constant $\varepsilon_0, C, \alpha > 0$ such that for any $\varepsilon < \varepsilon_0$ and any $h_{\text{ex}} < H_{c_1}^{3D} + K \log |\log \varepsilon|$, if $(u_\varepsilon, A_\varepsilon)$ is a global minimizer of GL_ε in $H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ and $\nu_\varepsilon = \sum_{i \in I} \gamma_i$ is an associated vorticity approximation then*

1. $\forall i, \gamma_i/|\gamma_i|(B_0) \geq 0$,
2. $\|\gamma_i/|\gamma_i| - \gamma_0\|_* < |\log \varepsilon|^{-\frac{1}{2N}}$ for any i such that $\gamma_i/|\gamma_i|(B_0) > 0$, and
3. $|\nu_\varepsilon|(\Omega) < C$.

Of course, it is very important to verify if the proposed non-degeneracy condition holds in a simple situation. For this purpose, let us consider the case $\Omega = B(0, R)$ and $H_{0,\text{ex}} = \hat{z}$, for which the vector B_0 has an explicit expression. One can show that the vertical diameter D_1 oriented in the direction of \hat{z} and seen as a Lipschitz curve in X (in particular $|D_1|(\Omega) = 1$) is such that

$$D_1(B_0) = \|B_0\|_*.$$

Moreover, we have the following result, which is a sort of generalization of a result that appeared in [ABM06], where the authors deal with a functional similar to the one considered here.

Theorem 1.1.9 (Work in preparation). *Let $\Omega = B(0, 1)$ and $H_{0,\text{ex}} = \hat{z}$. There exist constants $C, N > 0$ such that for any Lipschitz curve $\gamma \in X$ if $\|\gamma - D_1\|_* \geq \delta$, for some $\delta \in (0, 1)$, then*

$$\gamma(B_0) \leq D_1(B_0) - C\delta^N,$$

where D_1 is defined as above.

We remark that we expect to extend Theorem 1.1.8 by assuming a non-degeneracy condition that allows for a finite set of optimal curves instead of only one, which is what one predicts when Ω is non-convex. Even though the presented results do not imply that the first critical field in 3D is given by $H_{c_1}^{3D} + O(1)$, the key steps towards proving this result have been made in this thesis. We foresee obtaining the result in future work.

1.2 The Lin-Ni-Takagi problem

In 1972, Gierer and Meinhardt [GM72] proposed the following reaction-diffusion system to model biological pattern formation:

$$\begin{cases} \frac{\partial a}{\partial t} = D_a \Delta a - \mu_a a + \rho_a \left(c_a \frac{a^p}{h^q} + \rho_0 \right) \\ \frac{\partial h}{\partial t} = D_h \Delta h - \mu_h h + c_h \rho_h \frac{a^r}{h^s}. \end{cases}$$

Here

- $a(x, t) > 0$ and $h(x, t) > 0$ represent the respective concentrations at a point $x \in \mathbb{R}^n$ and at a time $t \in \mathbb{R}$ of biochemicals called activator and inhibitor. The activator stimulates a change in cells or tissues so that cell differentiation (or division) occurs at the position where the activator concentration is high. The inhibitor diffuses much faster than the activator and tempers the self-enhancing growth of the activator concentration, thereby stabilizing the system.
- $D_a, D_h, \mu_a, \mu_h, c_a, c_h, \rho_a, \rho_h$ are strictly positive constants, while ρ_0 is a non-negative constant.
- The exponents p, q, r, s are assumed to satisfy the conditions

$$p > 1, q, r > 0, s \geq 0, \text{ and } 0 < \frac{p-1}{q} < \frac{r}{s+1}.$$

We assume that a and h occupy a smooth bounded domain $\Omega \subset \mathbb{R}^n$ and that there is no flux through the boundary, i.e.

$$\nabla a \cdot \nu = \nabla h \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

where ν denotes the unit outer normal to $\partial\Omega$.

One observes that, by numerical simulations, when the ratio D_a/D_h is small, the Gierer-Meinhardt system seems to have stable stationary solutions with the property that the activator concentration is localized around a finite number of points in $\bar{\Omega}$. Moreover, as $D_a \rightarrow 0$ the pattern exhibits a ‘‘spike layer phenomenon’’, i.e. the activator concentration is localized in narrower and narrower regions around some points and eventually shrinks to a certain set of points. Hereby the maximum value of the activator concentration diverges to $+\infty$.

Then, we consider the stationary Gierer-Meinhardt system (for $\rho_0 = 0$)

$$\begin{cases} -\varepsilon^2 \Delta A + A - \frac{A^p}{H^q} = 0, & A > 0 \quad \text{in } \Omega \\ -D \Delta H + \mu H - \frac{A^r}{H^s} = 0, & H > 0 \quad \text{in } \Omega, \end{cases}$$

with homogeneous Neumann boundary conditions on $\partial\Omega$. Here, the normalized unknowns $A(x)$ and $H(x)$ are defined via the formulas

$$a(x) = \left[\left(\frac{\mu_a}{c_h \rho_a} \right)^q \left(\frac{c_a \rho_a}{\mu_a} \right)^{s+1} \right]^\lambda A(x) \quad \text{and} \quad h(x) = \left[\left(\frac{\mu_a}{c_h \rho_a} \right)^{p-1} \left(\frac{c_a \rho_a}{\mu_a} \right)^r \right]^\lambda H(x),$$

where

$$\lambda = \frac{1}{qr - (p-1)(s+1)}, \quad \varepsilon = \sqrt{D_a} \mu_a, \quad D = \frac{D_h}{\mu_a}, \quad \text{and} \quad \mu = \frac{\mu_h}{\mu_a}.$$

This stationary system is quite difficult to solve since it has neither a variational structure nor a priori estimates. One way to study this problem is to examine the so-called shadow system. Namely, we let $D_h \rightarrow \infty$. By assuming that $\mu H - \frac{A^r}{H^s}$ remains bounded, we find

$$\Delta H \rightarrow 0 \text{ in } \Omega \quad \text{and} \quad \nabla H \cdot \nu = 0 \text{ on } \partial\Omega.$$

This implies that $H(x) \rightarrow \xi$, where the constant $\xi > 0$ satisfies the equation

$$\mu \xi |\Omega| - \xi^{-s} \int_{\Omega} A^r = 0.$$

Then, by letting $v(x) = \xi^{\frac{p-1}{q}} A(x)$ we are led to study the single equation

$$\begin{cases} -\varepsilon^2 \Delta v + v - v^p = 0, & v > 0 & \text{in } \Omega \\ \nabla v \cdot \nu = 0 & & \text{on } \partial\Omega. \end{cases}$$

We rewrite this equation in a slightly different form, namely we define $u(x) = \lambda^{\frac{1}{p}} v(x)$ and $\lambda = \varepsilon^{-\frac{2p}{p-1}}$. Therefore, we have

$$(3) \quad \begin{cases} -\Delta u + \lambda u - u^p = 0, & u > 0 & \text{in } \Omega \\ \nabla u \cdot \nu = 0 & & \text{on } \partial\Omega. \end{cases}$$

This semilinear Neumann elliptic problem has been widely studied in the last 30 years. In 1988, Lin, Ni, and Takagi [LNT88] initiated the study of this problem. The interested reader can find an extensive list of known results about this equation and the Geirer-Meinhardt system in the book by Wei and Winter [WW14] and the references therein.

From now on we focus on the critical case in dimension $n = 3$, i.e. when $p = \frac{n+2}{n-2} = 5$. It has been showed that this problem does not admit interior bubbling solutions if $\lambda \rightarrow 0$ or $\lambda \rightarrow \infty$, for instance when Ω is assumed to be convex. By this, we mean solutions that exhibit peaks of concentration around one or more points in Ω , while being very small elsewhere. In a joint work with Manuel del Pino, Monica Musso, and Juncheng Wei [dPMRW] we discovered a new phenomenon, which is the presence of a solution with interior bubbling for values of λ near a number $0 < \lambda_*(\Omega) < \infty$ which can be explicitly characterized.

Theorem 1.2.1 (del Pino, Musso, Román, and Wei [dPMRW]). *Let $n = 3$ and $p = 5$. There exists a number $0 < \lambda_* < \infty$ such that for any $\lambda > \lambda_*$, with $\lambda - \lambda_*$ sufficiently small, there exists a solution to (3) with an asymptotic profile as $\lambda \rightarrow \lambda_*^+$ of the form*

$$u_{\lambda}(x) = 3^{\frac{1}{4}} \left(\frac{\mu_{\lambda}}{\mu_{\lambda}^2 + |x - x_{\lambda}|^2} \right)^{\frac{1}{2}} + O(\mu_{\lambda}^{\frac{1}{2}}) \quad \text{in } \Omega,$$

where the concentration parameter $\mu_{\lambda} = O(\lambda - \lambda_*)$ and the concentration point $x_{\lambda} \in \Omega$ stays uniformly away from $\partial\Omega$.

The number λ_* and the asymptotic location of the point x_λ can be characterized in terms of the Robin's function associated to this problem (see Chapter 1 of Part II). There exists a connection between the number λ_* and the so called *Brezis-Nirenberg number* $\lambda^* = \lambda^*(\Omega) > 0$ given as the least value λ such that for any $\lambda^* < \lambda < \lambda_1$, where λ_1 is the first Dirichlet eigenvalue of the Laplacian, there exists a least energy solution of the 3D Brezis-Nirenberg problem [BN83]

$$\begin{cases} -\Delta u - \lambda u - u^p = 0, & u > 0 & \text{in } \Omega \\ u = 0 & & \text{on } \partial\Omega. \end{cases}$$

A parallel characterization of the number λ_* in terms of a Dirichlet Green's function has been established in [Dru02] and its role in bubbling phenomena further explored in [dPDM04].

The construction of our solution follows the finite dimensional Lyapunov-Schmidt reduction, which was introduced by Floer and Weinstein [FW86]. For the reader convenience we briefly introduce the abstract set-up of this method.

1.2.1 The finite dimensional Lyapunov-Schmidt reduction

Let X, Y be Banach spaces and consider a C^1 map $S : X \rightarrow Y$. Our purpose is to find solutions to the equation $S(u) = 0$. To do so, we first find a “good” approximation and we then look for a true solution as a small perturbation of the approximation. Assume that U_λ is the approximation, indexed by a parameter $\lambda \in \Lambda$ (we think of this set as the configuration space). By writing $u_\lambda = U_\lambda + \phi_\lambda$, we are led to solve

$$(4) \quad L(\phi_\lambda) + N(\phi_\lambda) + E = 0,$$

where

$$L(\phi_\lambda) = S'(U_\lambda)(\phi_\lambda), \quad N(\phi_\lambda) = S(U_\lambda + \phi_\lambda) - S(U_\lambda) - S'(U_\lambda)(\phi_\lambda), \quad \text{and } E = S(U_\lambda).$$

Here, $S'(U_\lambda)$ is the Fréchet derivate of S at U_λ , $L(\phi_\lambda)$ denotes the linear part and $N(\phi_\lambda)$ the nonlinear part, and E is the error of the approximation. The strategy we follow here is: We invert the linear operator L so that we can rephrase (4) as a fixed point problem. That is, when L has a uniformly bounded inverse in a suitable space, one can rewrite this equation as

$$\phi_\lambda = -L^{-1}(E + N(\phi_\lambda)) =: A(\phi_\lambda).$$

What is left is to use a fixed point argument, for instance the contraction mapping theorem, to solve this equation.

The finite dimensional Lyapunov-Schmidt reduction deals with the situation when the linear operator L is Fredholm and its eigenfunction space associated to small eigenvalues is finite dimensional. Let $\{Z_1, \dots, Z_n\}$ denote a basis of the eigenfunction space associated to small eigenvalues of L . We divide the procedure of solving (4) into two steps:

1. To solve, for any $\lambda \in \Lambda$, the projected problem

$$\begin{cases} L(\phi_\lambda) + N(\phi_\lambda) + E = \sum_{j=1}^n c_j(\lambda) Z_j \\ \langle \phi_\lambda, Z_j \rangle = 0 \quad \forall j = 1, \dots, n, \end{cases}$$

where $c_j(\lambda)$ is defined via

$$\langle L(\phi_\lambda) + N(\phi_\lambda) + E, Z_j \rangle = c_j(\lambda) \langle Z_j, Z_j \rangle$$

for any $j = 1, \dots, n$.

2. To solve the reduced problem

$$c_j(\lambda) = 0 \quad \forall j = 1, \dots, n$$

by adjusting the parameter λ in the configuration space Λ .

1.3 The Keller-Segel model of chemotaxis

Chemotaxis is the influence of chemical substances in the environment on the movement of mobile species. This can lead to strictly oriented movement or to partially oriented and partially tumbling movement. Positive chemotaxis occurs if the movement is towards a higher concentration of the chemical substance; negative chemotaxis if the movement is in the opposite direction. Chemotaxis is an important means for cellular communication by chemical substances, which determines how cells arrange themselves, for instance in living tissues.

In 1970, Keller and Segel [KS70] proposed a basic model for chemotaxis. They considered an advection-diffusion system consisting of two coupled parabolic equations for the concentration of the considered species and that of the chemical released, respectively represented by strictly positive quantities $v(x, t)$ and $u(x, t)$ defined on a bounded smooth domain $\Omega \subset \mathbb{R}^n$. The system has the form

$$\begin{cases} \frac{\partial v}{\partial t} = D_v \Delta v - c \operatorname{div}(v \nabla \phi(u)) \\ \frac{\partial u}{\partial t} = D_u \Delta u + k(u, v), \end{cases}$$

with no flux through the boundary, i.e.

$$\nabla v \cdot \nu = \nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega.$$

Here

- D_v, D_u , and c are strictly positive constants.
- ϕ , the so-called sensitive function, is a smooth function such that $\phi'(r) > 0$ for $r > 0$.
- k is a smooth function such that $\frac{\partial k}{\partial v} \geq 0$ and $\frac{\partial k}{\partial u} \leq 0$. It is typically chosen as $-u + v$.

An important property of this system is the so-called chemotactic collapse. This term refers to the fact that the whole population of organisms concentrate at a single point in finite or infinite time. When $\phi(u) = u$, it is well-known that the chemotactic collapse

depends strongly on the dimension of the space. Finite-time blow-up never occurs if $n = 1$, whereas it always occurs if $n \geq 3$. The two-dimensional case is critical: if the initial distribution of organisms exceeds a certain threshold, then the solutions may blow-up in finite time, whereas solutions exist globally in time if the initial mass is below the threshold. We refer the interested reader to the survey by [Hor03, Hor04] for further details about the model and a collection of known results.

Steady states of this system are of basic importance for the understanding of the global dynamics. They correspond to the solutions to

$$\begin{cases} -D_v \Delta v + c \operatorname{div}(v \nabla \phi(u)) = 0, & v > 0 \text{ in } \Omega \\ -D_u \Delta u - u + v = 0, & u > 0 \text{ in } \Omega, \end{cases}$$

with homogeneous Neumann boundary conditions on $\partial\Omega$.

This system can be reduced to a scalar equation depending on the function ϕ . Observe that, the first equation can be rewritten as

$$\operatorname{div}(v \nabla (\log v - \phi(u))) = 0.$$

By testing this equation against $\log v - \phi(u)$ and by integrating by parts, one finds

$$\int_{\Omega} v |\log v - \phi(u)|^2 = 0,$$

which implies that $v = C e^{\phi(u)}$ for some constant $C > 0$.

In the most common formulation of the Keller-Segel model $\phi(u) = u$. We are then led to study the so-called Keller-Segel equation

$$(5) \quad \begin{cases} -\sigma^2 \Delta u + u - \lambda e^u = 0, & u > 0 \text{ in } \Omega \\ \nabla u \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where the constants σ, λ depend on D_v, D_u , and c . It is worth to mention that in the case $\phi(u) = \log u$, we get

$$\begin{cases} -\tilde{\sigma}^2 \Delta u + u - u^p = 0, & u > 0 \text{ in } \Omega \\ \nabla u \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

for some constants $\tilde{\sigma}, p > 0$, i.e. we recover the Lin-Ni-Takagi equation that we discussed in the previous section. Let us observe that in dimension 2 the Keller-Segel equation is critical, whereas the Lin-Ni-Takagi problem is subcritical.

From now on, we restrict ourselves to the case $\Omega = B(1, 0) \subset \mathbb{R}^2$ and study the Keller-Segel equation (without loss of generality) with $d = 1$. In a work in collaboration with Denis Bonheure and Jean-Baptiste Casteras we construct radial solutions to this equation that concentrate at the origin and on spheres belonging to the interior or the boundary of $B(1, 0)$. The following is our first result.

Theorem 1.3.1 (Bonheure, Casteras, and Román [BCR]). *There exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ there exists a radial solution u_λ to (5) with $\sigma = 1$ such that*

$$\lim_{\lambda \rightarrow 0} (u_\lambda - U_\lambda) = 0$$

uniformly on compact subsets of $B(1,0) \setminus \{0\}$, where $U_\lambda(r)$ solves, for $\varepsilon_\lambda \approx \frac{1}{|\ln \lambda|}$, the 1-dimensional equation

$$\begin{cases} -U_\lambda'' - \frac{1}{r}U_\lambda' + U_\lambda = 0 & \text{in } (0,1) \\ \lim_{r \rightarrow 0^+} \frac{U_\lambda(r)}{-\ln r} = 4 \\ U_\lambda(1) = \frac{\sqrt{2}}{\varepsilon_\lambda} \end{cases}$$

In particular, this theorem shows that the Keller-Segel equation admits radial solution that concentrate at the origin and on the boundary of the ball. Then, we prove a result concerning blowing-up at the origin and on an interior sphere.

Theorem 1.3.2 (Bonheure, Casteras, and Román [BCR]). *There exists $\lambda_1 > 0$ such that for any $\lambda \in (0, \lambda_1)$ there exists a radial solution $u_\lambda(r)$ to (5) with $\sigma = 1$ blowing-up at $r = 0$ and $r = \alpha \in (0, 1)$.*

The precise form of the solution can be found in Chapter 2 of Part II. Our final result states that (5) admits a radial solution which is singular at the origin and that concentrates on an internal sphere and on the boundary.

Theorem 1.3.3 (Bonheure, Casteras, and Román [BCR]). *There exists $\lambda_1 > 0$ such that for any $\lambda \in (0, \lambda_1)$ there exists a radial solution $u_\lambda(r)$ to (5) with $\sigma = 1$ blowing-up at $r = 0$, $r = \alpha \in (0, 1)$, and $r = 1$.*

The proof of our results follows the finite dimensional Lyapunov-Schmidt reduction. It is important to mention that the blow-up rate at the origin is different from the one at the spheres located in the interior of the ball or at the boundary. Roughly speaking, what we do is to glue suitable Green's functions, which exhibit different singular behaviors. In particular, finding good first approximations is the most complicated part of the method.

1.4 Conformal geometry

1.4.1 The prescribed Gaussian curvature problem

For a long time, conformal changes of metrics, i.e. angle preserving transformations of metrics, have played an important role in the theory of surfaces in the three-dimensional Euclidean space. A famous result is the uniformization theorem.

Theorem 1.4.1 (Uniformization theorem). *Let M be a compact manifold of dimension 2 without boundary. Given any metric g on M , there exists a metric \tilde{g} which is pointwise conformal to g and has constant Gauss curvature.*

This result provides a classification of two-dimensional surfaces and allows one to relate topological questions with differential geometric ones. The theory of surfaces formed the basis for developing the Riemannian differential geometry. In this field, an important and natural question is to describe the set of curvatures that a given manifold can possess.

In the case of 2-dimensional manifolds, there is essentially only one notion of curvature. This problem then reduces to describing the set of Gaussian curvature functions.

Two metrics g and \tilde{g} on a manifold M are said to be pointwise conformal (or conformal for short) if there exists a smooth function $u : M \rightarrow \mathbb{R}$ such that $\tilde{g} = e^u g$. When M is two-dimensional, the Gaussian curvature of the metric \tilde{g} is given by the formula

$$K_{\tilde{g}} = e^{-u}(-\Delta_g u + K_g),$$

where $\Delta_g = \operatorname{div}_g \nabla_g$ denotes the Laplace-Beltrami operator relative to the metric g . In this case, the above question can be written as the prescribed Gaussian curvature problem (with conformal change of metric): Given a metric g , which smooth functions $h : M \rightarrow \mathbb{R}$ can be realized as the Gaussian curvature $K_{\tilde{g}}$ of a metric \tilde{g} conformal to g ? We are then led to solving the nonlinear elliptic equation

$$(6) \quad -\Delta_g u + K_g = h e^u \quad \text{on } M.$$

Note that the uniformization theorem is equivalent to the solvability of this equation in the special case $h \equiv k$, for some $k \in \mathbb{R}$.

The Gauss-Bonnet theorem imposes a condition on h in terms of the topology of the manifold. This theorem asserts that

$$\int_M h d\mu_{\tilde{g}} = 2\pi\chi(M),$$

where $d\mu_{\tilde{g}}$ denotes the element of area with respect to \tilde{g} and $\chi(M)$ is the Euler characteristic of M . One deduces that

- if $\chi(M) > 0$ then h must be positive somewhere,
- if $\chi(M) = 0$ then h must change sign or $h \equiv 0$,
- if $\chi(M) < 0$ then h must be negative somewhere.

Necessary and sufficient conditions for existence of solutions to (6) can be found in the classical references [KW74, CY87]. Concerning the structure of the set of solutions to (6), there is little known.

During my master thesis, in a joint work with Manuel del Pino, we studied the case $\chi(M) < 0$, corresponding to manifolds of genus greater than 1. By the uniformization theorem we may assume that g has constant Gauss curvature K_g , which in our case is strictly negative.

This problem has a variational structure. One can easily show that if $h \leq 0$, $h \not\equiv 0$ then (6) has a unique solution, which corresponds to the unique minimizer of the energy. Thus, it remains to investigate the case when h changes sign. The implicit function theorem yields that the energy admits a relative minimizer if h changes sign and $\max_M f$ is not “too large”. In particular, this holds for any smooth function $h_\lambda := h + \lambda^2$, where h is a non-constant function with $\max_M h = 0$ and $\lambda > 0$ is a small parameter. Ding and Liu [DL95] proved that for any $\lambda > 0$ small the equation

$$(7) \quad -\Delta_g u + K_g = h_\lambda e^u \quad \text{on } M,$$

namely (6) with $h \equiv h_\lambda$, admits at least two solutions, one of which is bounded as $\lambda \rightarrow 0$ while the other blows-up. However, this result gives no information about the geometric shape of the solutions.

Borer, Galimberti, and Struwe [BGS15], using variational methods and a concentration compactness argument, recently gave a new proof of the above result. In addition, they established that blowing-up of the family of large solutions occurs only near the points of maxima of h and their associated metrics exhibit bubbling behavior, namely Euclidean spheres emerge around some of the zero-points of h . Inspired by this work, by matched asymptotic expansion, in [dPR15] we constructed families of spherical bubbling metrics. We substantially clarified the structure of the set of large solutions of (7), by using a finite dimensional Lyapunov-Schmidt reduction that yields both multiplicity and accurate estimates of their blowing-up behavior.

Theorem 1.4.2 (del Pino and Román [dPR15]). *Let p_1, \dots, p_n be points such that $h(p_i) = 0$ and $D^2h(p_i)$ is negative definite for each i . Then, there exists a family of solutions u_λ to (7) such that, as $\lambda \rightarrow 0$,*

$$\lambda^2 e^{u_\lambda} \rightarrow 8\pi \sum_{i=1}^n \delta_{p_i}$$

and $u_\lambda \rightarrow G$ uniformly in compact subsets of $M \setminus \{p_1, \dots, p_n\}$, where G is the nonlinear Green's function solution to

$$-\Delta_g G + K_g = h e^G + 8\pi \sum_{i=1}^n \delta_{p_i} \quad \text{on } M.$$

1.4.2 The prescribed scalar curvature problem

In 1960, Yamabe [Yam60] conjectured the following generalization of the uniformization theorem.

Theorem 1.4.3 (Yamabe problem). *Let M be a compact manifold without boundary of dimension $n \geq 3$. Given any Riemannian metric g on M , there exists a metric \tilde{g} which is conformal to g and has constant scalar curvature.*

The scalar curvature is the complete contraction of the curvature tensor and represents the amount by which the volume of a small geodesic ball in a Riemannian manifold deviates from that of the standard ball in Euclidean space. In 2D, the scalar curvature is twice the Gaussian curvature. Yamabe attempted to solve this problem, but his proof contained an error. This theorem was proved by Aubin [Aub76a], Trudinger [Tru68], and Schoen [Sch84]. We refer the interested reader to the survey by Lee and Parker [LP87], for further material and references on this topic.

The generalization of the prescribed Gaussian curvature problem is the prescribed scalar curvature problem (with conformal change of metric): Given a metric g , which smooth function $h : M \rightarrow \mathbb{R}$ can be realized as the scalar curvature $R_{\tilde{g}}$ of a metric \tilde{g} conformal to g ?

When M is n -dimensional, with $n \geq 3$, the scalar curvature $R_{\tilde{g}}$ of a metric $\tilde{g} = e^v g$, i.e. conformal to g , is given by the formula

$$R_{\tilde{g}} = e^{-v} \left(-2(n-1)\Delta_g v - (n-1)(n-2)|\nabla_g v|^2 + R_g \right),$$

where Δ_g denotes the Laplacian and ∇_g the covariant derivative relative to the metric g . By making the substitution $e^v = u^{\frac{4}{n-2}}$, for some smooth function $u > 0$, we find the simplified identity

$$R_{\tilde{g}} = u^{-\frac{n+2}{n-2}} \left(-c(n)^{-1}\Delta_g u + R_g u \right),$$

where $c(n) = \frac{n-2}{4(n-1)}$. We are then led to solving the nonlinear problem

$$(8) \quad -\Delta_g u + c(n)R_g = c(n)hu^{\frac{n+2}{n-2}}, \quad u > 0 \quad \text{on } M.$$

We note that the Yamabe problem corresponds to the special case $h \equiv k$, for some $k \in \mathbb{R}$. Theorem 1.4.3 allows us to assume that R_g is a constant. An exhaustive list of known results about (8) is given in the book by Aubin [Aub98]. We also refer the reader to the pioneer work by Kazdan and Warner [KW75].

Following [dPR15], we focus our attention on the case of prescribing a function of the form $h_\lambda := h + \lambda^2$, where $h \in C^2(M)$ and $\lambda > 0$ is a small parameter. Namely, we study the problem

$$(9) \quad -\Delta_g u + c(n)R_g = c(n)h_\lambda u^{\frac{n+2}{n-2}}, \quad u > 0 \quad \text{on } M.$$

In collaboration with Angela Pistoia, we studied this equation in [PR17]. As in the 2D case, our objective is to give some clarity in understanding the set of solutions to (9). We next present two results. First, without any assumption on the sign of R_g , we prove that this equation admits at least two solutions. To state our result, we need to introduce two hypotheses on the function h . Roughly speaking, the conditions are the following (for the detailed assumptions needed, we refer the reader to Chapter 1 of Part III).

- **Global condition:** There exists a non-degenerate solution u_0 to (9) when $\lambda = 0$.
- **Local condition:** The function h admits a suitable critical point ξ on M such that $h(\xi) = 0$ and which is not a local minimum.

We observe that, under the global condition, it is easy to prove that if λ is small enough then (9) has a solution $u_{0,\lambda} \in C^2(M)$ such that $\|u_{0,\lambda} - u_0\|_{C^2(M)} \rightarrow 0$ as $\lambda \rightarrow 0$. Our first result concerns the multiplicity of solutions to (9). Roughly speaking, we show that, under the global and local conditions above, this equation admits a second solution $u_\lambda \in C^2(M)$ of the form u_0 plus a standard n -dimensional bubble properly scaled and centered close to ξ .

Theorem 1.4.4 (Pistoia and Román [PR17]). *Assume that the global and local conditions above hold. If suitable conditions on the dimension n , the manifold M , and the critical*

point ξ of h are satisfied then, provided λ is small enough, there exists a solution u_λ to problem (9) which blows-up at the point ξ as $\lambda \rightarrow 0$. Moreover, as $\lambda \rightarrow 0$, we have

$$\left\| u_\lambda(x) - u_0(x) - \alpha_n \lambda^{-\frac{n-2}{2}} \mu_\lambda^{-\frac{n-2}{2}} \left(1 + \left| \frac{d_g(x, \xi_\lambda)}{\mu_\lambda} \right|^2 \right)^{-\frac{n-2}{2}} \right\|_{H_g^1(M)} \rightarrow 0,$$

where the concentration point $\xi_\lambda \rightarrow \xi$, the concentration parameter $\mu_\lambda \rightarrow 0$, and $\alpha(n) = (n(n-2))^{\frac{n-2}{4}}$.

This is the first multiplicity result for (9) when $R_g \geq 0$ and extends a previous result by Rauzy [Rau96] when $R_g < 0$.

Our second result concerns the existence of solutions to problem (9) when $R_g > 0$, without need of the global condition above. The solution constructed here looks like a standard n -dimensional bubble properly scaled and centered close to ξ .

Theorem 1.4.5 (Pistoia and Román [PR17]). *Assume that the local condition above holds. If suitable conditions on the dimension n , the manifold M , and the critical point ξ of h are satisfied then, provided λ is small enough, there exists a solution u_λ to problem (9) which blows-up at the point ξ as $\lambda \rightarrow 0$. Moreover, as $\lambda \rightarrow 0$, we have*

$$\left\| u_\lambda(x) - \alpha_n \lambda^{-\frac{n-2}{2}} \mu_\lambda^{-\frac{n-2}{2}} \left(1 + \left| \frac{d_g(x, \xi_\lambda)}{\mu_\lambda} \right|^2 \right)^{-\frac{n-2}{2}} \right\|_{H_g^1(M)} \rightarrow 0,$$

where the concentration point $\xi_\lambda \rightarrow \xi$ and the concentration parameter $\mu_\lambda \rightarrow 0$.

Part I

The Ginzburg-Landau model of superconductivity

Chapter 1

3D vortex approximation construction and ε -level estimates for the Ginzburg-Landau functional

Abstract

In this chapter, which is based on [Roma], we provide a quantitative three-dimensional vortex approximation construction for the Ginzburg-Landau energy. This construction gives an approximation of vortex lines coupled to a lower bound for the energy, optimal to leading order, analogous to the 2D ones, and valid for the first time at the ε -level. These tools allow for a new approach to analyze the behavior of global minimizers for the Ginzburg-Landau functional below and near the first critical field in 3D, followed in Chapter 2.

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1.1 Introduction

We are interested in studying the full Ginzburg-Landau functional with applied magnetic field

$$GL_\varepsilon(u, A) = \frac{1}{2} \int_\Omega |\nabla_A u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 + \frac{1}{2} \int_{\mathbb{R}^3} |H - H_{\text{ex}}|^2,$$

which is a model for superconductors (in a magnetic field). Here, Ω is a bounded domain of \mathbb{R}^3 , that we assume to be smooth and simply connected, $u : \Omega \rightarrow \mathbb{C}$ is the *order parameter* indicating the local state of the material (normal or superconducting), $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the electromagnetic vector potential of the magnetic field $H = \text{curl } A$, $H_{\text{ex}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a given external (or applied) magnetic field, and $\varepsilon > 0$ is the inverse of the Ginzburg-Landau parameter, a material constant. We will be interested in the regime of small ε , corresponding to extreme type-II superconductors. The notation ∇_A denotes the covariant gradient $\nabla - iA$.

An essential feature of type-II superconductors is the occurrence of *vortices* (similar to those in fluid mechanics, but quantized) in the presence of an applied magnetic field. Physically, they correspond to normal phase regions around which a superconducting loop of current circulates. Since u is complex-valued, it can have zeroes with a nonzero topological degree. Vortices are then *topological defects* of co-dimension 2 and are the crucial objects of interest in the analysis of the model.

We introduce the Ginzburg-Landau free energy

$$F_\varepsilon(u, A) = \frac{1}{2} \int_\Omega |\nabla_A u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 + |\text{curl } A|^2.$$

This functional is closely related to the simpler Ginzburg-Landau model without magnetic field

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2.$$

Since the work by Bethuel, Brezis, and Hélein [BBH94], these functionals and the associated vortices have been extensively studied in the mathematics literature. As a result, the 2D situation is well understood. We refer the reader to the book by Sandier and Serfaty [SS07] and references therein for a detailed and exhaustive list of results in this case. Conversely, many questions remain open in 3D, in particular obtaining all the analogues of the 2D results contained in [SS07]. This is due to the more complicated geometry of

the vortices in 3D: they are lines with a priori no regularity, and have to be understood in the framework of currents and using geometric measure theory.

When the applied magnetic field is taken to be zero, its effect can be replaced with the prescription of some vorticity on the boundary of the domain. In this case, Rivière in [Riv95] was the first to study the asymptotic behavior of minimizers as $\varepsilon \rightarrow 0$ in the 3D setting. Using an η -ellipticity result, he identified the limiting one dimensional singular set with a mass minimizing current, which corresponds to a *minimal connection*. This concept was introduced and shown to satisfy a calibration property in the work by Brezis, Coron, and Lieb [BCL86]. A new approach by Sandier in [San01], combined this property with a suitable growing-ball procedure to obtain the same result of Rivière in 3D and extend it to higher dimension. We refer the interested reader to [LR99, BBO01, LR01, JS02, BBM04, ABO05, SS17] for further results in dimensions 3 and higher, in the case the external magnetic field is zero.

The key in Ginzburg-Landau analysis has proven to be a vortex ball construction providing both approximation of the vorticity and lower bound. In 2D done in [San98, Jer99, SS07]. In 3D started by [ABO05, BJOS12] but not quantitative.

1.1.1 ε -level estimates for the Ginzburg-Landau functional

The full Ginzburg-Landau model is known to be an $\mathbb{U}(1)$ -gauge theory. This means that all the meaningful physical quantities are invariant under the gauge-transformations

$$u \mapsto ue^{i\Phi}, \quad A \mapsto A + \nabla\Phi,$$

where Φ is any smooth real-valued function. The full Ginzburg-Landau energy and the free energy are gauge-invariant, as well as the density of superconducting Cooper pairs $|u|^2$, the induced magnetic field H , and the vorticity, defined, for any sufficiently regular configuration (u, A) , as

$$\mu(u, A) = \operatorname{curl}(iu, \nabla_A u) + \operatorname{curl} A,$$

where (\cdot, \cdot) denotes the scalar product in \mathbb{C} identified with \mathbb{R}^2 i.e. $(a, b) = \frac{\bar{a}b + a\bar{b}}{2}$. This quantity is the gauge-invariant version of the Jacobian determinant of u , and is the analogue of the vorticity of a fluid.

To analyze the vortices, people have been developing tools, in particular the ball construction method and Jacobian estimates. The first one was introduced independently by Jerrard [Jer99] and Sandier [San98]. It allows one to obtain universal lower bounds for two-dimensional Ginzburg-Landau energies in terms of the topology of the vortices. These lower bounds capture the known fact that vortices of degree d cost at least an order $\pi|d| \log \frac{1}{\varepsilon}$ of energy. The second tool, that has been widely used in the analysis of the Ginzburg-Landau model in any dimension after the work by Jerrard and Sonner [JS02], is the Jacobian (or vorticity) estimate. This estimate allows one to relate the vorticity $\mu(u, A)$ with Dirac masses (supported on co-dimension 2 objects), which in 2D are naturally derived from the ball construction method.

Optimal versions of these results in 2D can be found in [SS07]. These tools are known to work at the ε -level and therefore to play a crucial role in the study of the behavior of global minimizers for GL_ε in different regimes of the applied field in 2D. This analysis was

performed by Sandier and Serfaty in a series of papers (see [SS00a, SS00b, SS03][SS00c]). To do a similar analysis in 3D, analogous tools are needed.

Through a procedure based on slicing of currents, 3D Jacobian estimates and lower bounds were proved in [JS02] and [SS04]. Alternatively, a suitable application of the Federer-Fleming polyhedral deformation theorem was used in [ABO05] and [BJOS12] to obtain results of the same type. The lower bounds provided in these works are valid only in the limit $\varepsilon \rightarrow 0$ and therefore they are not sufficient for our purposes, because they cannot be made ε -quantitative.

In this chapter we present a new 3D vortex approximation construction based on a 2D vorticity estimate and on minimal connections. More precisely, for configurations $(u_\varepsilon, A_\varepsilon)$ whose free energy satisfy a suitable upper bound, we consider a grid of side-length $\delta = \delta(\varepsilon)$. If appropriately positioned, the grid can be taken to satisfy that $|u_\varepsilon| > 5/8$ on every edge of a cube. Then a 2D vorticity estimate implies that the restriction of $\mu(u_\varepsilon, A_\varepsilon)$ to the boundary of every cube is well approximated by a linear combination of Dirac masses. Using minimal connections, we connect the points of support of these measures. Finally, by considering the distance

$$d_{\partial\Omega}(x, y) = \min\{|x - y|, d(x, \partial\Omega) + d(y, \partial\Omega)\},$$

we construct our approximation close to $\partial\Omega$, using minimal connections defined in terms of $d_{\partial\Omega}$. This process yields a closed polyhedral 1-dimensional current ν_ε , or, more precisely, a sum in the sense of currents of Lipschitz curves, that approximates well the vorticity $\mu(u_\varepsilon, A_\varepsilon)$ in a suitable norm.

This is the first 3D construction which is at the ε -level and yields optimal estimates analogous to the 2D ones. The following is our main result.

Theorem 1.1.1 (ε -level estimates for Ginzburg-Landau in 3D). *Assume that $\partial\Omega$ has strictly positive Gauss curvature at every point. For any $m, n, M > 0$ there exist $C, \varepsilon_0 > 0$ depending only on m, n, M , and $\partial\Omega$, such that, for any $\varepsilon < \varepsilon_0$, if $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq M|\log \varepsilon|^m$ then there exists a polyhedral 1-dimensional current ν_ε such that ν_ε/π is integer multiplicity, $\partial\nu_\varepsilon = 0$ relative to Ω , $\text{supp}(\nu_\varepsilon) \subset S_{\nu_\varepsilon} \subset \bar{\Omega}$ with $|S_{\nu_\varepsilon}| \leq C|\log \varepsilon|^{-m-s-1}$, where $s := \max(n, \frac{m+3n-2}{2})$,*

$$(1.1) \quad \frac{1}{2} \int_{S_{\nu_\varepsilon}} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 + |\text{curl } A_\varepsilon|^2 \geq \frac{1}{2} |\nu_\varepsilon|(\Omega) \left(\log \frac{1}{\varepsilon} - C \log \log \frac{1}{\varepsilon} \right) - \frac{C}{|\log \varepsilon|^n},$$

and for any $\gamma \in (0, 1]$ there exists a constant C_γ depending only on γ, m, n, M , and $\partial\Omega$, such that we have

$$(1.2) \quad \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C_T^{0,\gamma}(\Omega)^*} \leq \frac{C_\gamma |\log \varepsilon|^{m(1-\gamma)}}{|\log \varepsilon|^{(s+1)\gamma}}.$$

Notation and definitions of the objects and spaces involved in this result can be found in the preliminaries (see Section 1.2).

This theorem also holds when we only assume that $\partial\Omega$ is C^2 , but the proof presented here uses the additional technical assumption that $\partial\Omega$ has strictly positive Gauss curvature at every point. A word about the proof in the general case can be found in the Appendix.

Remark 1.1.1. *Alternatively the constants C and C_γ appearing in the previous theorem can be expressed in terms of the free energy $F_\varepsilon(u_\varepsilon, A_\varepsilon)$ of the configuration $(u_\varepsilon, A_\varepsilon)$ and a length $\delta = \delta(\varepsilon)$, which measures how “close” $\mu_\varepsilon(u_\varepsilon, A_\varepsilon)$ is to ν_ε , and which is a parameter of the construction (the side-length of the aforementioned grid). This will be done in the rest of the chapter.*

In future work we hope to be able to extend this result to configurations $(u_\varepsilon, A_\varepsilon)$ satisfying a less restrictive upper bound of the kind: $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq \varepsilon^{-\gamma}$ for some $\gamma \in (0, 1)$.

The following result can be stated for a general bounded simply connected domain Ω with Lipschitz boundary.

Theorem 1.1.2. *For any $m, n, M > 0$ there exist $C_1, \varepsilon_0 > 0$ depending only on m, n , and M , such that, for any $\varepsilon < \varepsilon_0$, if $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq M |\log \varepsilon|^m$ then, letting $s = \max(n, \frac{m+3n-2}{2})$ and defining*

$$\Omega_\varepsilon = \{x \in \Omega \mid d(x, \partial\Omega) \geq 2 |\log \varepsilon|^{-m-s-1}\},$$

there exists a polyhedral 1-dimensional current ν_ε such that ν_ε/π is integer multiplicity, $\partial\nu_\varepsilon = 0$ relative to Ω , $\text{supp}(\nu_\varepsilon) \subset S_{\nu_\varepsilon} \subset \bar{\Omega}$ with $|S_{\nu_\varepsilon}| \leq C |\log \varepsilon|^{-m-s-1}$,

$$\frac{1}{2} \int_{S_{\nu_\varepsilon}} |\nabla_{A_\varepsilon} u_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 + |\text{curl } A_\varepsilon|^2 \geq \frac{1}{2} |\nu_\varepsilon|(\Omega_\varepsilon) \left(\log \frac{1}{\varepsilon} - C_1 \log \log \frac{1}{\varepsilon} \right) - \frac{C_1}{|\log \varepsilon|^n},$$

and (1.2) holds true in the space $C_0^{0,\gamma}(\Omega)^$ instead of $C_T^{0,\gamma}*$, for $\gamma \in (0, 1]$ and for constants C and C_γ as in Theorem 1.1.1.*

As a direct consequence of our main result, we recover and improve within our work setting, a well known result concerning the convergence as $\varepsilon \rightarrow 0$ of the vorticity of families of configurations whose free energy is bounded above by a constant times a power of $|\log \varepsilon|$. Results of the same kind can be found in [JS02, JMS04, SS04, ABO05, BJOS12].

Theorem 1.1.3. *Assume that $\partial\Omega$ has strictly positive Gauss curvature. Let $\{(u_\varepsilon, A_\varepsilon)\}_\varepsilon$ be a family of configurations of $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq M |\log \varepsilon|^m$ for some $m \geq 1$ and $M > 0$. Then, up to extraction,*

$$\frac{\mu(u_\varepsilon, A_\varepsilon)}{|\log \varepsilon|^{m-1}} \rightharpoonup \mu \quad \text{in } C_T^{0,\gamma}(\Omega)^*$$

for any $\gamma \in (0, 1]$, where μ is a 1-dimensional current such that μ/π is integer multiplicity and $\partial\mu = 0$ relative to Ω . If $m = 1$ then μ is in addition rectifiable. Moreover,

$$\liminf_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(u_\varepsilon, A_\varepsilon)}{|\log \varepsilon|^m} \geq |\mu|(\Omega).$$

1.1.2 Application to the full Ginzburg-Landau functional

The behavior of global minimizers for GL_ε is determined by the strength of the external magnetic field H_{ex} . This model is known to exhibit several phase-transitions, which occur

for certain critical values of the intensity of H_{ex} . In the next chapter, we analyze the so-called *first critical field*, usually denoted by H_{c_1} . The first critical field is defined by the fact that below H_{c_1} minimizers of the full Ginzburg-Landau functional do not have vortices, while they do for applied fields whose strength is higher than H_{c_1} .

Let us assume that $H_{\text{ex}} \in L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ is such that $\text{div } H_{\text{ex}} = 0$. Then, there exists a vector-potential $A_{\text{ex}} \in H^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$\text{curl } A_{\text{ex}} = H_{\text{ex}}, \quad \text{div } A_{\text{ex}} = 0 \text{ in } \mathbb{R}^3 \quad \text{and} \quad A_{\text{ex}} \cdot \nu = 0 \text{ on } \partial\Omega,$$

where hereafter ν denotes the outer unit normal to $\partial\Omega$. We remark that the divergence-free assumption is in accordance with the fact that magnetic monopoles do not exist in Maxwell's electromagnetism theory.

Let us introduce the space

$$H_{\text{curl}} := \{A \in H^1_{\text{loc}}(\mathbb{R}^3, \mathbb{R}^3) \mid \text{curl } A \in L^2(\mathbb{R}^3, \mathbb{R}^3)\}.$$

The functional $GL_\varepsilon(u, A)$ is well defined for any pair $(u, A) \in H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$. We have the following result.

Corollary 1.1.1. *Theorem 1.1.1 holds true if the hypothesis that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $E(u_\varepsilon, A_\varepsilon) \leq M |\log \varepsilon|^m$ is replaced with the assumptions that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ is a minimizing configuration for GL_ε and that $\int_\Omega |H_{\text{ex}}|^2 \leq M |\log \varepsilon|^m$.*

By observing that

$$F_\varepsilon(u, A) \leq GL_\varepsilon(u, A) + \frac{1}{2} \int_\Omega |H_{\text{ex}}|^2$$

and that

$$\inf_{(u, A) \in H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]} GL_\varepsilon(u, A) \leq GL_\varepsilon(1, A_{\text{ex}}) = \int_\Omega |A_{\text{ex}}|^2 \leq C \int_\Omega |H_{\text{ex}}|^2,$$

for a universal constant C , the corollary immediately follows from our main theorem.

In future work we will present a quantitative three-dimensional version of the product-estimate for Ginzburg-Landau proved by Sandier and Serfaty in [SS04], which will be applied to the situation of Ginzburg-Landau vortex dynamics.

1.1.3 A word about the proof of the main result

Since our construction ensures that (the restriction of) the vorticity is well approximated on every boundary of a cube of the grid, and because it is made at a small scale δ , the 2D vorticity estimate shown in Section 1.4 yields that ν_ε is a good approximation of the vorticity in 3D.

The subtle point of the proof is then to obtain a lower bound for the free energy at the ε -level. Here is where minimal connections play a role. The idea of obtaining lower bounds for Ginzburg-Landau energies via the use of minimal connections was first introduced in [San01]. When trying to apply this kind of method to obtain lower bounds for the full

functional, the main obstacle that as soon as the external magnetic field is of the order of the first critical field, the number of vortices is a priori unbounded as $\varepsilon \rightarrow 0$. The main challenge in getting a lower bound that works at the ε -level is thus to keep track of the dependence of all the estimates on ε and δ , keeping into account that the number of vortices may be unbounded.

Our method goes as follows: The choice of grid allows us to show that the restriction of the vorticity to the boundary of a cube \mathcal{C} can be well approximated by

$$2\pi \left(\sum_{i=1}^k \delta_{p_i} - \sum_{i=1}^k \delta_{n_i} \right),$$

where the points p_i 's are the (non-necessarily distinct) positive singularities and the points n_i 's are the (non-necessarily distinct) negative singularities. We remark that the number of points and their locations depend on ε and that (a priori) it may blow up as $\varepsilon \rightarrow 0$.

It is well known (see [BCL86]) that there exists a 1-Lipschitz function ζ such that

$$\sum_{i=1}^k \zeta(p_i) - \sum_{i=1}^k \zeta(n_i) = L(\mathcal{A}),$$

where $L(\mathcal{A})$ is the length of the minimal connection associated to the configuration of points $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$. Since $|\nabla\zeta| \leq 1$, the co-area formula gives

$$\int_{\mathcal{C}} e_\varepsilon(u_\varepsilon) \geq \int_{\mathcal{C}} e_\varepsilon(u_\varepsilon) |\nabla\zeta| \geq \int_{t \in \mathbb{R}} \int_{\Sigma_t} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2 dt,$$

where $e_\varepsilon(u_\varepsilon) = \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2} (1 - |u_\varepsilon|^2)^2$ and $\Sigma_t = \{\zeta = t\} \cap \mathcal{C}$.

At this point, a vortex ball construction on a surface is necessary. Roughly speaking, if Σ_t is nice enough and $|u_\varepsilon| \geq 1/2$ on $\partial\Sigma_t$, then the 2D ball construction estimate would give us

$$\int_{\Sigma_t} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2 \geq \pi \deg(u_\varepsilon/|u_\varepsilon|, \partial\Sigma_t) \left(\log \frac{1}{\varepsilon} - O(\log |\log \varepsilon|) \right).$$

It turns out that, for most t 's, we have

$$\deg(u_\varepsilon/|u_\varepsilon|, \partial\Sigma_t) = \#\{i \mid \zeta(p_i) > t\} - \#\{i \mid \zeta(n_i) > t\}.$$

By noting that

$$\int_{t \in \mathbb{R}} \#\{i \mid \zeta(p_i) > t\} - \#\{i \mid \zeta(n_i) > t\} dt = \sum_{i=1}^k \zeta(p_i) - \sum_{i=1}^k \zeta(n_i) = L(\mathcal{A}) \approx \frac{1}{2\pi} |\nu_\varepsilon|(\mathcal{C}),$$

we are led to

$$\int_{\mathcal{C}} e_\varepsilon(u_\varepsilon) \geq \frac{1}{2} |\nu_\varepsilon|(\mathcal{C}) \left(\log \frac{1}{\varepsilon} - O(\log |\log \varepsilon|) \right) + \text{small error}.$$

Unfortunately, we cannot really use the function ζ in the previous argument, because its regularity is not sufficient to apply the ball construction on most of its level sets. To

bypass this issue, we construct a smooth approximation of this function. The difficulties appear when trying to control the errors involved in the previously described method, because a quantitative bound on the second fundamental form of most of the level sets of our smooth approximation of the function ζ is needed.

In a similar but more involved way we can obtain a lower bound close to the boundary of the domain. We point out that our technical assumption on $\partial\Omega$ comes into play when we smoothly approximate the function ζ for $d_{\partial\Omega}$. When we only assume that the domain has a C^2 boundary then this argument needs to be modified (see the Appendix).

1.1.4 Outline of the chapter

The chapter is organized as follows: in Section 1.2 we introduce some basic objects and spaces that are used throughout the chapter, we recall some facts from the theory of currents and differential forms, and we describe the choice of grid.

In Section 1.3 we provide the ball construction method on a surface, which is one of the key tools used to obtain the lower bound for the free energy.

In Section 1.4 we show a 2D vorticity estimate. The main difference with classical result of the same kind is the space in which we prove this result.

In Section 1.5 we start by reviewing the concept of minimal connection. Then, we introduce the function ζ and the function ζ for $d_{\partial\Omega}$, and state two technical propositions concerning a suitable smooth approximation of these functions. Finally, we present our 3D vortex approximation construction.

Section 1.6 is devoted to the proof of a lower bound for the energy without magnetic field in the union of cubes of the grid, while in Section 1.7 we provide a similar estimate near the boundary of the domain. In these proofs we crucially use the results of Section 1.3 and Section 1.5.

In Section 1.8 we present the proof of Theorem 1.1.1, which uses the lower bounds obtained in Section 1.6 and Section 1.7, as well as the 2D vorticity estimate shown in Section 1.4.

In Appendix 1.A we construct a suitable smooth approximation of the function ζ . We do the same in Appendix 1.B for the function ζ for $d_{\partial\Omega}$. These are the most technical parts of the chapter.

1.2 Preliminaries

It is useful to introduce certain concepts and notation from the theory of currents and differential forms. We recall that in Euclidean spaces vector fields can be identified with 1-forms. Indeed, the vector field $F = (F_{x_1}, F_{x_2}, F_{x_3})$ can be identified with the 1-form $F_{x_1}dx_1 + F_{x_2}dx_2 + F_{x_3}dx_3$. We use the same notation for both the vector field and the 1-form.

It is also convenient to recall that a vector field F satisfying the boundary condition $F \times \nu = 0$ on $\partial\Omega$ is equivalent to a 1-form F such that $F_T = 0$ on $\partial\Omega$. Here F_T denotes the tangential component of F on $\partial\Omega$.

We define the superconducting current of a pair $(u, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ as the 1-form

$$j(u, A) = (iu, d_A u) = \sum_{k=1}^3 (iu, \partial_k u - iA_k u) dx_k.$$

It is related to the vorticity $\mu(u, A)$ of a configuration (u, A) through

$$(1.3) \quad \mu(u, A) = dj(u, A) + dA.$$

Thus $\mu(u, A)$ is an exact 2-form in Ω acting on couples of vector fields $(X, Y) \in \mathbb{R}^3 \times \mathbb{R}^3$ with the standard rule that $dx_i \wedge dx_j(X, Y) = X_i Y_j - X_j Y_i$. It can also be seen as a 1-dimensional current, which is defined through its action on 1-forms by the relation

$$\mu(u, A)(\phi) = \int_{\Omega} \mu(u, A) \wedge \phi.$$

We recall that the boundary of a 1-current T relative to the set Ω , is the 0-current ∂T defined by

$$\partial T(\phi) = T(d\phi)$$

for all smooth compactly supported 0-form ϕ defined in Ω . In particular, an integration by parts shows that the 1-dimensional current $\mu(u, A)$ has zero boundary relative to Ω . We denote by $|T|(\Omega)$ the mass of a 1-current T in Ω .

For $\alpha \in (0, 1]$ we let $C^{0,\alpha}(\Omega)$ denote the space of 1-forms ϕ such that $\|\phi\|_{C^{0,\alpha}(\Omega)} < \infty$. $C_0^{0,\alpha}(\Omega)$ denotes the space of 1-forms $\phi \in C^{0,\alpha}(\Omega)$ such that $\phi = 0$ on $\partial\Omega$, while $C_T^{0,\alpha}(\Omega)$ denotes the space of 1-forms $\phi \in C^{0,\alpha}(\Omega)$ such that $\phi_T = 0$ on $\partial\Omega$. The symbol $*$ is used to denote their dual spaces.

We next recall the definition of topological degree.

Definition 1.2.1. *Let Σ be a complete oriented surface in \mathbb{R}^3 . If $\Theta \subset \Sigma$ is a smooth domain, and the map $u : \Sigma \rightarrow \mathbb{C}$ does not vanish on $\partial\Theta$, we can define the degree $\deg(u/|u|, \partial\Theta)$ of u restricted to $\partial\Theta$ to be the winding number of the map $u/|u| : \partial\Theta \rightarrow S^1$.*

We observe that, because Σ is assumed to be oriented, $\partial\Theta$ carries a natural orientation. In the case that $\partial\Theta$ is not smooth, the topological degree can still be defined by approximation.

Throughout this chapter $|X|$ denotes the Lebesgue measure of the set X and \mathcal{H}^d denotes the d -dimensional Hausdorff measure, for $d \in \mathbb{N}$. When meaningful, we sometimes use the notation

$$F_{\varepsilon}(u, A, \Theta) := \int_{\Theta} e_{\varepsilon}(u, A) d\mathcal{H}^2, \quad E_{\varepsilon}(u, \Theta) := \int_{\Theta} e_{\varepsilon}(u) d\mathcal{H}^2,$$

with $e_{\varepsilon}(u, A) := \frac{1}{2} |\nabla_A u|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2 + |\operatorname{curl} A|^2$, $e_{\varepsilon}(u) := \frac{1}{2} |\nabla u|^2 + \frac{1}{4\varepsilon^2} (1 - |u|^2)^2$.

1.2.1 Choice of grid

Let us fix an orthonormal basis (e_1, e_2, e_3) of \mathbb{R}^3 and consider a grid $\mathfrak{G} = \mathfrak{G}(a, R, \delta)$ given by the collection of (closed) cubes $\mathcal{C}_i \subset \mathbb{R}^3$ of side-length $\delta = \delta(\varepsilon)$ (conditions on this parameter are given in the lemma below). In the grid we use a system of coordinates with origin in $a \in \Omega$ and orthonormal directions given by the rotation of the basis (e_1, e_2, e_3) with respect to $R \in SO(3)$. From now on we denote by \mathfrak{R}_1 (respectively \mathfrak{R}_2) the union of all edges (respectively faces) of the cubes of the grid. We have the following lemma.

Lemma 1.2.1 (Choice of grid). *For any $\gamma \in (0, 1)$ there exists a rotation $R_0(\gamma) \in SO(3)$ and constants $c_0(\gamma), c_1(\gamma) > 0$, $\delta_0(\Omega) \in (0, 1)$ such that, for any $\varepsilon, \delta > 0$ satisfying*

$$\varepsilon^{\frac{1-\gamma}{2}} \leq c_0 \quad \text{and} \quad c_1 \varepsilon^{\frac{1-\gamma}{4}} \leq \delta \leq \delta_0,$$

if $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq \varepsilon^{-\gamma}$ then there exists $b_\varepsilon \in \Omega$ such that the grid $\mathfrak{G}(b_\varepsilon, R_0, \delta)$ satisfies

$$(1.4a) \quad |u_\varepsilon| > 5/8 \quad \text{on} \quad \mathfrak{R}_1(\mathfrak{G}(b_\varepsilon, R_0, \delta)) \cap \Omega,$$

$$(1.4b) \quad \int_{\mathfrak{R}_1(\mathfrak{G}(b_\varepsilon, R_0, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^1 \leq C \delta^{-2} F_\varepsilon(u_\varepsilon, A_\varepsilon),$$

$$(1.4c) \quad \int_{\mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, R_0, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^2 \leq C \delta^{-1} F_\varepsilon(u_\varepsilon, A_\varepsilon),$$

where C is a universal constant.

Proof. First, let us observe that, by the Cauchy-Schwarz inequality and the co-area formula, we have

$$\begin{aligned} 4F_\varepsilon(u_\varepsilon, A_\varepsilon) &\geq \int_{\Omega} |\nabla |u_\varepsilon||^2 + \frac{1}{\varepsilon^2} (1 - |u_\varepsilon|^2)^2 \\ &\geq \int_{\Omega} \frac{|\nabla |u_\varepsilon|| (1 - |u_\varepsilon|^2)}{\varepsilon} \\ &= \int_{t=0}^{\infty} \left(\int_{\{|u|=t\}} \frac{(1-t^2)}{\varepsilon} d\mathcal{H}^2 \right) dt \end{aligned}$$

Define $T := \{t \in [5/8, 3/4] \mid \text{Area}(\{|u_\varepsilon| = t\}) \leq \varepsilon^\alpha\}$ for $\alpha := \frac{1-\gamma}{2}$. From the previous estimate we deduce that

$$|T| \geq 1/8 - C \varepsilon^{1-\alpha} F_\varepsilon(u_\varepsilon, A_\varepsilon),$$

where hereafter $C > 0$ denotes a universal constant that may change from line to line. It is easy to check that there exists a constant $c_0(\gamma) > 0$ such that $|T| > 0$ for any $\varepsilon > 0$ satisfying $\varepsilon^{\frac{1-\gamma}{2}} \leq c_0$.

We observe that by integral geometry formulae (see for instance [Lan15, San04]), for any $t \in [0, 3/4]$, we have

$$\text{Area}(\{|u_\varepsilon| = t\}) = c \int_{R \in SO(3)} \int_{h \in \mathbb{R}^3} \#(\{|u_\varepsilon| = t\} \cap L_{R,h} \cap \Omega) d\mathcal{L}(h) d\mathcal{L}(R),$$

where $L_{R,h}$ is the rotation with respect to $R \in SO(3)$ and the translation with respect to $h \in \mathbb{R}^3$ of a fixed line L in \mathbb{R}^3 , $\#(A)$ denotes the number of points of the set A , and c is a constant depending only on the dimension of the euclidean space.

We fix a point $a \in \Omega$ and choose $\delta_0 = \delta_0(\Omega) \in (0, 1)$ such that $\{a + [0, \delta]^3\} \subset \Omega$ for any $0 < \delta < \delta_0$. Observe that, up to an adjustment of c , we have

$$\text{Area}(\{|u_\varepsilon| = t\}) = \frac{c}{\delta} \int_{R \in SO(3)} \int_{b \in \{a + [0, \delta]^3\}} \#(\{|u_\varepsilon| = t\} \cap \mathfrak{R}_1(\mathfrak{G}(b, R, \delta)) \cap \Omega) d\mathcal{L}(b) d\mathcal{L}(R).$$

Fix $t_0 \in T$ and define

$$G_0 := \{(R, b) \mid R \in SO(3), b \in \{a + [0, \delta]^3\}, \{|u_\varepsilon| = t_0\} \cap \mathfrak{R}_1(\mathfrak{G}(b, R, \delta)) \neq \emptyset\}.$$

By noting that

$$|G_0| \leq \frac{\delta}{c} \text{Area}(\{|u_\varepsilon| = t_0\}) \leq \frac{\delta \varepsilon^\alpha}{c},$$

we deduce that there exists a fixed rotation $R_0 \in SO(3)$ such that

$$B_I := \{b \in \{a + [0, \delta]^3\} \mid \{|u_\varepsilon| = t_0\} \cap \mathfrak{R}_1(\mathfrak{G}(b, R_0, \delta)) \neq \emptyset\}$$

satisfies $|B_I| \leq C\delta\varepsilon^\alpha$.

We observe that, for any $b \in \{a + [0, \delta]^3\} \setminus B_I$,

$$\text{either } |u_\varepsilon| > t_0 \quad \text{or} \quad |u_\varepsilon| < t_0 \quad \text{on } \mathfrak{R}_1(\mathfrak{G}(b, R_0, \delta)) \cap \Omega.$$

We let

$$B_{II} := \{b \in \{a + [0, \delta]^3\} \setminus B_I, \{|u_\varepsilon| < t_0\} \cap \mathfrak{R}_1(\mathfrak{G}(b, R_0, \delta)) \neq \emptyset\}$$

and observe that, for every $b \in B_{II}$, we have $(1 - |u_\varepsilon|^2) \geq (1 - t_0^2)$. This implies that

$$\frac{(1 - t_0^2)^2}{4\varepsilon^2} |B_{II}| \leq \int_{b \in B_{II}} \int_{\mathfrak{R}_1(\mathfrak{G}(b, R_0, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^1 d\mathcal{L}(b) \leq F_\varepsilon(u_\varepsilon, A_\varepsilon)$$

and thus $|B_{II}| \leq C\varepsilon^2 F_\varepsilon(u_\varepsilon, A_\varepsilon)$.

Now, we define $B_{good} := \{a + [0, \delta]^3\} \setminus (B_I \cup B_{II})$. Observe that

$$|B_{good}| \geq \delta^3 - C(\delta\varepsilon^\alpha + \varepsilon^{2-\gamma})$$

and that there exists a constant $c_1 > 0$ such that $|B_{good}| \geq \delta^3/2$ for any $\varepsilon, \delta > 0$ satisfying $c_1 \varepsilon^{\frac{\alpha}{2}} \leq \delta$. Moreover, for any $b \in B_{good}$, we have

$$|u_\varepsilon| > t_0 \quad \text{on } \mathfrak{R}_1(\mathfrak{G}(b, R_0, \delta)) \cap \Omega.$$

Next, using a mean value argument we choose $b = b_\varepsilon \in B_{good}$ in such a way that

$$\int_{\mathfrak{R}_n(\mathfrak{G}(b_\varepsilon, R_0, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^n \leq C \delta^{n-3} F_\varepsilon(u_\varepsilon, A_\varepsilon) \quad \text{for } n = 1, 2.$$

First, by [ABO05, Lemma 8.4] there exists $b_\varepsilon \in B_{good}$ such that, for $n = 1, 2$,

$$\int_{\mathfrak{R}_n(\mathfrak{G}(b_\varepsilon, R_0, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^n \leq \frac{2}{|B_{good}|} \int_{B_{good}} \int_{\mathfrak{R}_n(\mathfrak{G}(b, R_0, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^n d\mathcal{L}(b).$$

Second, arguing as in the proof of [ABO05, Lemma 3.11], we have

$$\frac{1}{\delta^3} \int_{\{a+[0, \delta]^3\}} \delta^{3-n} \int_{\mathfrak{R}_n(\mathfrak{G}(b, R_0, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^n d\mathcal{L}(b) = C F_\varepsilon(u_\varepsilon, A_\varepsilon) \quad \text{for } n = 1, 2.$$

Then, we deduce that

$$\int_{\mathfrak{R}_n(\mathfrak{G}(b_\varepsilon, R_0, \delta)) \cap \Omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^n \leq C \frac{\delta^3}{|B_{good}|} \delta^{n-3} F_\varepsilon(u_\varepsilon, A_\varepsilon) \quad \text{for } n = 1, 2.$$

Recalling that $|B_{good}| \leq \delta^3/2$, the lemma follows. \square

From now on we drop the cubes of the grid $\mathfrak{G}(b_\varepsilon, R_0, \delta)$, given by Lemma 1.2.1, whose intersection with $\mathbb{R}^3 \setminus \Omega$ is non-empty. We also define

$$(1.5) \quad \Theta := \Omega \setminus \cup_{\mathcal{C}_l \in \mathfrak{G}} \mathcal{C}_l \quad \text{and} \quad \partial\Theta := \partial(\cup_{\mathcal{C}_l \in \mathfrak{G}} \mathcal{C}_l).$$

Observe that, in particular, $\partial\Theta = \partial\mathfrak{G} \cup \partial\Omega$.

We remark that $\mathfrak{G}(b_\varepsilon, R_0, \delta)$ carries a natural orientation. The boundary of every cube of the grid will be oriented accordingly to this orientation. Each time we refer to a face ω of a cube \mathcal{C} , it will be considered to be oriented with the same orientation of $\partial\mathcal{C}$. If we refer to a face $\omega \subset \partial\mathfrak{G}$, then the orientation used is the same of $\partial\mathfrak{G}$.

1.3 The ball construction method on a surface

In this section we use the method of Jerrard introduced in [Jer99] in order to construct balls containing all the zeros of u on a surface. This allows us to obtain a lower bound for the energy without magnetic field. The construction given here follows the one made by Sandier in [San01] that corresponds to an adaptation of the method of Jerrard. The following is the main result of this section, which is an extension of [San01, Proposition 3.5].

Proposition 1.3.1. *Let $\tilde{\Sigma}$ be a complete oriented surface in \mathbb{R}^3 whose second fundamental form is bounded by 1. Let Σ be a bounded open subset of $\tilde{\Sigma}$. For any $m, M > 0$ there exists $\varepsilon_0(m, M) > 0$ such that, for any $\varepsilon < \varepsilon_0$, if $u_\varepsilon \in H^1(\Sigma, \mathbb{C})$ satisfies*

$$(1.6) \quad E_\varepsilon(u_\varepsilon, \Sigma) \leq M |\log \varepsilon|^m$$

and

$$|u(x)| \geq \frac{1}{2} \quad \text{if } \mathfrak{d}(x, \partial\Sigma) < 1,$$

where $\mathfrak{d}(\cdot, \cdot)$ denotes the distance function in $\tilde{\Sigma}$, then, letting d be the winding number of $u_\varepsilon/|u_\varepsilon| : \partial\Sigma \rightarrow S^1$ and $M_\varepsilon = E_\varepsilon(u_\varepsilon, \Sigma)$, we have

$$E_\varepsilon(u_\varepsilon, \Sigma) \geq \pi|d| \left(\log \frac{1}{\varepsilon} - \log M_\varepsilon \right).$$

To prove Proposition 1.3.1 we follow almost readily the proofs of [Jer99] and [San01].

1.3.1 Main steps

Let us define the essential null set $S_E(u_\varepsilon)$ of u_ε to be the union of those connected components U_i of $\{x \mid |u_\varepsilon(x)| < 1/2\}$ such that $\deg(u_\varepsilon/|u_\varepsilon|, \partial U_i) \neq 0$.

In the rest of this section each time we refer to a ball B of radius r we mean a geodesic ball of radius r in $\tilde{\Sigma}$.

First, we include $S_E(u_\varepsilon)$ in the union of well-chosen disjoint “small” balls B_i of radii $r_i > \varepsilon$ such that

$$E_\varepsilon(u_\varepsilon, B_i) \geq \frac{r_i}{C\varepsilon},$$

where the constant C does not depend on the second fundamental form of Σ when it is assumed to be bounded by 1. This is possible according to the following lemma.

Lemma 1.3.1. *Under the hypotheses of Proposition 1.3.1, there exist $C, r_0 > 0$ such that, for any $\varepsilon > 0$, there exist disjoint balls B_1, \dots, B_k of radii r_i such that*

1. $r_i \geq \varepsilon$ for all $i \in \{1, \dots, k\}$.
2. $S_E(u) \subset \cup_i B_i$ and $B_i \cap S_E(u) \neq \emptyset$ for all $i \in \{1, \dots, k\}$.
3. For all $i \in \{1, \dots, k\}$,

$$E_\varepsilon(u_\varepsilon, B_i \cap \Sigma) \geq \frac{\min\{r_i, r_0, 1\}}{C\varepsilon}.$$

The proof then involves dilating the balls B_i into balls B'_i by combining them with annuli. A lower bound for $E_\varepsilon(u_\varepsilon, B'_i)$ is obtained by combining the lower bound for $E_\varepsilon(u_\varepsilon, B_i)$ and a lower bound for $E_\varepsilon(u_\varepsilon, B'_i \setminus \overline{B}_i)$.

Lemma 1.3.2. *Under the hypotheses of Proposition 1.3.1, there exist $C, \varepsilon_0, r_0 > 0$ such that, for any $0 < \varepsilon < s < r < r_0$, if $B_r, B_s \subset \Sigma$ are two concentric balls and if $S_E(u_\varepsilon) \cap (B_r \setminus \overline{B}_s) = \emptyset$ then, letting $d := \deg(u_\varepsilon/|u_\varepsilon|, \partial B_r)$,*

$$E_\varepsilon(u_\varepsilon, B_r \setminus \overline{B}_s) \geq |d| \left(\Lambda_\varepsilon \left(\frac{r}{|d|} \right) - \Lambda_\varepsilon \left(\frac{s}{|d|} \right) \right),$$

where $\Lambda_\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a function that satisfies the following properties

1. $\Lambda_\varepsilon(t)/t$ is decreasing.
2. $\sup_{t \in \mathbb{R}_+} \Lambda_\varepsilon(t)/t \leq 1/(C\varepsilon)$.
3. If $0 < \varepsilon < \varepsilon_0$ and $\varepsilon < t < r_0$ then

$$\left| \Lambda_\varepsilon(t) - \pi \log \frac{t}{\varepsilon} \right| \leq C.$$

By taking into consideration the following adaptation of [San01, Lemma 3.12], the proofs of the previous two lemmas are straightforward modifications of the proofs of [San01, Lemma 3.8] and of [San01, Lemma 3.9].

Lemma 1.3.3. *Let $S_t(x)$ denote the geodesic circle in $\tilde{\Sigma}$ of radius t centered at $x \in \Sigma$. Under the hypotheses of Proposition 1.3.1, there exist $C, \varepsilon_0, r_0 > 0$ such that, for any $x \in \Sigma$ and for any $\varepsilon, t > 0$ satisfying $\varepsilon < \varepsilon_0$ and $\varepsilon < t < r_0$, if $|u_\varepsilon| \leq 1$ on $S_t(x)$ then*

$$E_\varepsilon(u_\varepsilon, S_t(x)) \geq \pi m^2 \left(\frac{|d|}{t} - C \right)^+ + \frac{(1-m)^C}{C\varepsilon},$$

where $m := \inf_{y \in S_t} |u_\varepsilon(y)|$ and

$$d := \begin{cases} \deg(u_\varepsilon/|u_\varepsilon|, S_t(x)) & m \neq 0 \\ 0 & m = 0. \end{cases}$$

Proof. By observing that the constants r_0, r , and C involved in (B.8), (B.9), and (B.12) in the proof of [San01, Lemma 3.12] can be chosen independently of the second fundamental form of $\tilde{\Sigma}$ when it is assumed to be bounded by 1, then the proof is verbatim the same as that of [San01, Lemma 3.2]. \square

Lemma 1.3.1 and Lemma 1.3.2 allow one to prove the following result, whose proof is a straightforward modification of the proof of [San01, Proposition 3.10].

Proposition 1.3.2. *For any $\varepsilon > 0$, let $\{B_i\}_i$ be the family of balls of radii r_i given by Lemma 1.3.1. Let*

$$d_i := \begin{cases} \deg(u_\varepsilon/|u_\varepsilon|, \partial B_i) & \text{if } \overline{B_i} \subset \Sigma_\varepsilon \\ 0 & \text{otherwise,} \end{cases}$$

and

$$t_0 := \min_{\{i \mid d_i \neq 0\}} \frac{r_i}{|d_i|} \quad (\text{with } t_0 := +\infty \text{ if } d_i = 0 \text{ for every } i).$$

Then, for any $t \geq t_0$, there exists a family of disjoint geodesic balls $B_1(t), \dots, B_{k(t)}(t)$ of radii $r_i(t)$ in $\tilde{\Sigma}$ such that

1. $S_E(u) \subset \cup_i B_i(t)$ and $S_E(u) \cap B_i(t) \neq \emptyset$ for all $i \in \{1, \dots, k(t)\}$.
2. For all $i \in \{1, \dots, k(t)\}$, if $\overline{B_i(t)} \subset \Sigma$ then $r_i(t) \geq t|d_i(t)|$, where

$$d_i(t) := \deg(u_\varepsilon/|u_\varepsilon|, \partial B_i(t)).$$

3. For all $i \in \{1, \dots, k(t)\}$,

$$E_\varepsilon(u_\varepsilon, B_i(t) \cap \Sigma) \geq \min\{r_i(t), r_0, 1\} \frac{\Lambda_\varepsilon(t)}{t}.$$

Proof of Proposition 1.3.1. We assume that $d \neq 0$, otherwise the result is trivial. Apply Lemma 1.3.1, call the resulting balls B_1, \dots, B_k , and call r_1, \dots, r_k their radii. From Lemma 1.3.1 and (1.6), we have

$$\min\{r_i, r_0, 1\} \leq C\varepsilon E_\varepsilon(u_\varepsilon, B_i \cap \Sigma) \leq C\varepsilon M_\varepsilon \leq C\varepsilon |\log \varepsilon|^m,$$

where throughout the proof $C = C(M) > 0$ denotes a constant that may change from line to line. We deduce that there exists $\varepsilon_0(m, M) > 0$ such that, for any $i \in \{1, \dots, k\}$ and for any $\varepsilon < \varepsilon_0$,

$$(1.7) \quad r_i = \min\{r_i, r_0, 1\} \leq C\varepsilon M_\varepsilon \quad \text{and} \quad r_i \leq \frac{1}{2}.$$

Since $\mathfrak{d}(S_E(u_\varepsilon), \partial\Sigma) < 1$ and $B_i \cap S_E(u_\varepsilon) \neq \emptyset$, we conclude that $\overline{B}_i \subset \Sigma$. Thus

$$(1.8) \quad \sum_{i=1}^k \deg(u_\varepsilon/|u_\varepsilon|, B_i) = d \neq 0.$$

As in Proposition 1.3.2, let

$$t_0 = \min_{\{i \mid d_i \neq 0\}} \frac{r_i}{|d_i|}.$$

From (1.7) and (1.8), we get that $t_0 \leq C\varepsilon M_\varepsilon$. Fix $\alpha \in (0, 1)$. By reducing the constant ε_0 , we deduce that $t_0 \leq M_\varepsilon^{-1} |\log \varepsilon|^\alpha$ for any $\varepsilon < \varepsilon_0$. Therefore, we may apply Proposition 1.3.2 with $t = M_\varepsilon^{-1} |\log \varepsilon|^\alpha$. This yields balls $B_1(t), \dots, B_{k(t)}(t)$ with radii $r_i(t)$ and degrees $d_i(t)$ such that

$$\min\{r_i(t), r_0, 1\} \leq E_\varepsilon(u_\varepsilon, B_i(t) \cap \Sigma) \frac{t}{\Lambda_\varepsilon(t)}.$$

From Lemma 1.3.2, we have

$$r_i(t) = \min\{r_i(t), r_0, 1\} \leq M_\varepsilon \frac{M_\varepsilon^{-1} |\log \varepsilon|^\alpha}{C |\log \varepsilon|} \leq C |\log \varepsilon|^{\alpha-1}.$$

In particular, by possibly further reducing the constant ε_0 , we deduce that $\overline{B}_i(t) \subset \Sigma$ for any $i \in \{1, \dots, k(t)\}$ and for any $\varepsilon < \varepsilon_0$. Hence $d = \sum_{i=1}^{k(t)} d_i(t)$. Then, from Proposition 1.3.2, $r_i(t) \geq t |d_i(t)|$ and therefore

$$E_\varepsilon(u_\varepsilon, \Sigma) \geq \sum_{i=1}^{k(t)} |d_i(t)| \Lambda_\varepsilon(t).$$

Since $\sum_{i=1}^{k(t)} |d_i(t)| \geq |d|$, Lemma 1.3.2 implies that, for any $\varepsilon < \varepsilon_0$,

$$E_\varepsilon(u, \Sigma) \geq \pi |d| \left(\log \frac{t}{\varepsilon} - C \right) \geq \pi |d| \left(\log \frac{1}{\varepsilon} - \log M_\varepsilon \right).$$

The proposition is proved. □

Corollary 1.3.1. *Let $\tilde{\Sigma}$ be a complete oriented surface in \mathbb{R}^3 whose second fundamental form is bounded by $Q_\varepsilon = Q|\log \varepsilon|^q$, where $q, Q > 0$ are given numbers. Let Σ be a bounded open subset of $\tilde{\Sigma}$. For any $m, M > 0$ there exists $\varepsilon_0(m, q, M, Q) > 0$ such that, for any $\varepsilon < \varepsilon_0$, if $u_\varepsilon \in H^1(\Sigma, \mathbb{C})$ satisfies*

$$E_\varepsilon(u_\varepsilon, \Sigma) \leq M|\log \varepsilon|^m$$

and

$$|u(x)| \geq \frac{1}{2} \quad \text{if } \mathfrak{d}(x, \partial\Sigma) < Q_\varepsilon^{-1},$$

where $\mathfrak{d}(\cdot, \cdot)$ denotes the distance function in $\tilde{\Sigma}$, then, letting d be the winding number of $u_\varepsilon/|u_\varepsilon| : \partial\Sigma \rightarrow S^1$ and $M_\varepsilon = E_\varepsilon(u_\varepsilon, \Sigma)$ we have

$$E_\varepsilon(u_\varepsilon, \Sigma) \geq \pi|d| \left(\log \frac{1}{\varepsilon} - \log M_\varepsilon Q_\varepsilon \right).$$

Proof. Let us consider the transformation

$$\tilde{u}_\varepsilon(y) = u_\varepsilon \left(\frac{y}{Q_\varepsilon} \right) \quad \text{for } y \in \Sigma_\varepsilon := Q_\varepsilon \Sigma.$$

We let $\tilde{\Sigma}_\varepsilon := Q_\varepsilon \tilde{\Sigma}$. Observe that, by a change of variables, we have

$$E_\varepsilon(u_\varepsilon, \Sigma) = E_{\tilde{\varepsilon}}(\tilde{u}_\varepsilon, \Sigma_\varepsilon),$$

where $\tilde{\varepsilon} := \varepsilon Q_\varepsilon$. It is easy to check that the second fundamental form of $\tilde{\Sigma}_\varepsilon$ is bounded by 1. Then a direct application of Proposition 1.3.1 shows that

$$E_\varepsilon(u_\varepsilon, \Sigma) = E_{\tilde{\varepsilon}}(\tilde{u}_\varepsilon, \Sigma_\varepsilon) \geq \pi|d| \left(\log \frac{1}{\tilde{\varepsilon}} - \log M_\varepsilon \right) = \pi|d| \left(\log \frac{1}{\varepsilon} - \log M_\varepsilon Q_\varepsilon \right)$$

for any $0 < \varepsilon < \varepsilon_1 = \varepsilon_0 Q_\varepsilon^{-1}$, where ε_0 is the constant appearing in the proposition. \square

1.4 A 2D vorticity estimate

Let ω be a two-dimensional domain. For a given function $u : \omega \rightarrow \mathbb{C}$ and a given vector field $A : \omega \rightarrow \mathbb{R}^2$ we define

$$j(u, A) = (iu, \nabla_A u), \quad \mu(u, A) = dj(u, A) + dA.$$

We also let

$$F_\varepsilon(u, A, \omega) = \int_\omega e_\varepsilon(u, A), \quad F_\varepsilon(u, A, \partial\omega) = \int_{\partial\omega} e_\varepsilon(u, A) d\mathcal{H}^1,$$

where

$$e_\varepsilon(u, A) = |\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 + |\operatorname{curl} A|^2.$$

We have the following 2D vorticity estimate.

Theorem 1.4.1. *Let $\omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. Let $u : \omega \rightarrow \mathbb{C}$ and $A : \omega \rightarrow \mathbb{R}^2$ be $C^1(\bar{\omega})$ and such that $|u| \geq 5/8$ on $\partial\omega$. Let $\{S_i\}_{i \in I}$ be the collection of connected component of $\{|u(x)| \leq 1/2\}$ whose degree $d_i = \deg(u/|u|, \partial S_i) \neq 0$. Then, letting $r = \sum_{i \in I} \text{diam}(S_i)$ and assuming $\varepsilon, r \leq 1$, we have*

$$(1.9) \quad \left\| \mu(u, A) - 2\pi \sum_{i \in I} d_i \delta_{a_i} \right\|_{C^{0,1}(\omega)^*} \leq C \max(\varepsilon, r) (1 + F_\varepsilon(u, A, \omega) + F_\varepsilon(u, A, \partial\omega)),$$

where a_i is the centroid of S_i and C is a universal constant.

Proof. As in [SS07, Chapter 6], we set $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ to be defined by

$$\begin{cases} \chi(x) = 2x & \text{if } x \in [0, \frac{1}{2}] \\ \chi(x) = 1 & \text{if } x \in [\frac{1}{2}, \frac{3}{2}] \\ \chi(x) = 1 + 2(x - \frac{3}{2}) & \text{if } x \in [\frac{3}{2}, 2] \\ \chi(x) = x & \text{if } x \in [2, +\infty). \end{cases}$$

We then set $\tilde{u} : \omega \rightarrow \mathbb{C}$ by

$$\tilde{u}(x) = \frac{\chi(|u|)}{|u|} u$$

and let

$$\tilde{j} := (i\tilde{u}, d_A \tilde{u}), \quad \tilde{\mu} := d\tilde{j} + dA.$$

Observe that $|\tilde{u}| = 1$ and $\tilde{\mu} = 0$ outside of $\cup_{i \in I} S_i$. We claim that

$$\|\mu(u, A) - \tilde{\mu}\|_{C^{0,1}(\omega)^*} \leq C\varepsilon (F_\varepsilon(u, A, \omega) + F_\varepsilon(u, A, \partial\omega)).$$

In fact, by integration by parts, for any function $\zeta \in C^{0,1}(\omega)$ we have

$$\left| \int_\omega \zeta (\mu(u, A) - \tilde{\mu}) \right| \leq \left| \int_\omega (\nabla \zeta)^\perp \cdot (j(u, A) - \tilde{j}) \right| + \left| \int_{\partial\omega} \zeta (j(u, A) - \tilde{j}) \cdot \vartheta^\perp \right|,$$

where ϑ is the outer unit normal to $\partial\omega$ and $x^\perp = (-x_2, x_1)$ for any vector $x = (x_1, x_2)$. Arguing as in [SS07, Lemma 6.2], we get

$$\begin{aligned} \left| \int_\omega (\nabla \zeta)^\perp \cdot (j(u, A) - \tilde{j}) \right| &\leq \|\nabla \zeta\|_{L^\infty(\omega)} \int_\omega \frac{||u|^2 - |\tilde{u}|^2|}{|u|} |\nabla_A u| \\ &\leq 3\|\nabla \zeta\|_{L^\infty(\omega)} \int_\omega |1 - |u|| |\nabla_A u| \\ &\leq C\|\nabla \zeta\|_{L^\infty(\omega)} \varepsilon F_\varepsilon(u, A, \omega). \end{aligned}$$

Since $|\tilde{u}| = 1$ on $\partial\omega$, a simple computation shows that

$$|j(u, A) - \tilde{j}| \leq 2(1 - |u|^2) |\nabla_A u| \quad \text{on } \partial\omega.$$

By the Cauchy-Schwarz inequality, we find

$$\left| \int_{\partial\omega} \zeta (j(u, A) - \tilde{j}) \cdot \vartheta^\perp \right| \leq 2\|\zeta\|_{C^{0,1}(\omega)} \int_{\partial\omega} (1 - |u|^2) |\nabla_A u| d\mathcal{H}^1$$

$$\leq C\|\zeta\|_{C^{0,1}(\omega)}\varepsilon F_\varepsilon(u, A, \partial\omega).$$

Thus

$$\|\mu(u, A) - \tilde{\mu}\|_{C^{0,1}(\omega)^*} \leq C\varepsilon(F_\varepsilon(u, A, \omega) + F_\varepsilon(u, A, \partial\omega)),$$

for some universal constant C . The proof then reduces to proving that

$$\left\| \tilde{\mu} - 2\pi \sum_{i \in I} d_i \delta_{a_i} \right\|_{C^{0,1}(\omega)^*} \leq C \max(r, \varepsilon)(1 + F_\varepsilon(u, A, \omega) + F_\varepsilon(u, A, \partial\omega)).$$

Let $\zeta \in C^{0,1}(\omega)$ and observe that

$$\int_\omega \zeta \tilde{\mu} = \sum_{i \in I} \int_{S_i} \zeta \tilde{\mu} = \sum_{i \in I} \zeta(a_i) \int_{S_i} \tilde{\mu} + \sum_{i \in I} \int_{S_i} (\zeta - \zeta(a_i)) \tilde{\mu}.$$

Since wherever $|\tilde{u}| = 1$ we have $\tilde{\mu} = d(iu, du)$, Stokes' theorem yields

$$\int_{S_i} \tilde{\mu} = \int_{\partial S_i} (iu, \nabla u) \cdot \tau = 2\pi d_i.$$

Thus

$$\sum_{i \in I} \zeta(a_i) \int_{S_i} \tilde{\mu} = 2\pi \sum_{i \in I} d_i \zeta(a_i) = 2\pi \sum_{i \in I} d_i \int_\omega \zeta \delta_{a_i}$$

We also observe that, since ζ is a Lipschitz function, we have

$$|\zeta(x) - \zeta(a_i)| \leq \|\zeta\|_{C^{0,1}(\omega)} |x - a_i| \leq \|\zeta\|_{C^{0,1}(\omega)} \text{diam}(S_i)$$

for all $x \in S_i$.

On the other hand, noting that

$$\tilde{\mu} = 2(\partial_{x_1} \tilde{u} - iA_{x_1} \tilde{u}) \times (\partial_{x_2} \tilde{u} - iA_{x_2} \tilde{u}) + \text{curl } A,$$

we deduce that $|\tilde{\mu}| \leq 2|\nabla_A u|^2 + |\text{curl } A|$. Then, letting $F_\varepsilon(u, A, S_i) = \int_{S_i} e_\varepsilon(u, A)$, the Cauchy-Schwarz inequality gives

$$\int_{S_i} |\tilde{\mu}| \leq 4 \left(F_\varepsilon(u, A, S_i) + |S_i|^{\frac{1}{2}} F_\varepsilon(u, A, S_i)^{\frac{1}{2}} \right).$$

Observe that, by Jung's theorem, we have $|S_i| \leq C \text{diam}(S_i)^2$. Collecting our previous computations, we find

$$\left| \sum_{i \in I} \int_{S_i} (\zeta - \zeta(a_i)) \tilde{\mu} \right| \leq Cr \|\zeta\|_{C^{0,1}(\omega)} \left(F_\varepsilon(u, A, \omega) + r F_\varepsilon(u, A, \omega)^{\frac{1}{2}} \right).$$

Remembering that $\sqrt{x} \leq 1 + x$, we get

$$\left| \int_\omega \zeta \tilde{\mu} - 2\pi \sum_{i \in I} d_i \int_\omega \zeta \delta_{a_i} \right| \leq Cr \|\zeta\|_{C^{0,1}(\omega)} (1 + F_\varepsilon(u, A, \omega)).$$

This concludes the proof of (1.9). □

Given a three-dimensional Lipschitz domain $\omega \subset \Omega$ contained in a plane, we let $(s, t, 0)$ denote coordinates in \mathbb{R}^3 such that $\omega \subset \{(s, t, 0) \in \Omega\}$. We define $\mu_\varepsilon := \mu_\varepsilon(u, A)[\partial_s, \partial_t]$, and write $\mu_{\varepsilon, \omega}$ its restriction to ω . Theorem 1.4.1 immediately yields the following corollary.

Corollary 1.4.1. *Let $\gamma \in (0, 1)$ and assume that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq \varepsilon^{-\gamma}$, so that, by Lemma 1.2.1, there exists a grid $\mathfrak{G}(b_\varepsilon, R_0, \delta)$ satisfying (1.4). Then there exist $\varepsilon_0(\gamma)$ such that, for any $\varepsilon < \varepsilon_0$ and for any face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, R_0, \delta))$ of a cube of the grid $\mathfrak{G}(b_\varepsilon, R_0, \delta)$, letting $\{S_{i, \omega}\}_{i \in I_\omega}$ be the collection of connected components of $\{x \in \omega \mid |u_\varepsilon(x)| \leq 1/2\}$ whose degree $d_{i, \omega} := \deg(u_\varepsilon/|u_\varepsilon|, \partial S_{i, \omega}) \neq 0$, we have*

$$\left\| \mu_{\varepsilon, \omega} - 2\pi \sum_{i \in I_\omega} d_{i, \omega} \delta_{a_{i, \omega}} \right\|_{C^{0,1}(\omega)^*} \leq C \max(r_\omega, \varepsilon) \left(1 + \int_\omega e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^2 + \int_{\partial\omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^1 \right),$$

where $a_{i, \omega}$ is the centroid of $S_{i, \omega}$, $r_\omega := \sum_{i \in I_\omega} \text{diam}(S_{i, \omega})$, and C is a universal constant.

In view of the previous corollary it is important to find upper bounds for r_ω , $d_{i, \omega}$, and $|I_\omega|$. Prior to doing so let us recall the following result adapted from [Jer99].

Lemma 1.4.1. *Under the hypotheses of Corollary 1.4.1, there exists $\varepsilon_0(\gamma)$ such that, for any $\varepsilon < \varepsilon_0$ and for any face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, R_0, \delta))$ of a cube of the grid $\mathfrak{G}(b_\varepsilon, R_0, \delta)$, letting $\{S_{i, \omega}\}_{i \in I_\omega}$ be the collection of connected components of $\{x \in \omega \mid |u(x)| \leq 1/2\}$ whose degree $d_{i, \omega} \neq 0$, we have*

$$|d_{i, \omega}| \leq C \int_{S_{i, \omega}} |\nabla_{A_\varepsilon} u_\varepsilon|^2,$$

where C is a universal constant.

With the aid of the previous lemma we prove the following result.

Lemma 1.4.2. *Under the hypotheses of Corollary 1.4.1, there exists $\varepsilon_0(\gamma)$ such that, for any $\varepsilon < \varepsilon_0$ and for any face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, R_0, \delta))$ of a cube of the grid $\mathfrak{G}(b_\varepsilon, R_0, \delta)$, letting $\{S_{i, \omega}\}_{i \in I_\omega}$ be the collection of connected components of $\{x \in \omega \mid |u(x)| \leq 1/2\}$ whose degree $d_{i, \omega} \neq 0$, we have*

$$(1.10) \quad \begin{aligned} |I_\omega| &\leq \sum_{i \in I_\omega} |d_{i, \omega}| \leq C \int_\omega e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^2, \\ r_\omega &\leq C\varepsilon \int_\omega e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^2, \end{aligned}$$

where C is a universal constant.

Proof. The first assertion immediately follows from Lemma 1.4.1. To prove (1.10) observe that, by the Cauchy-Schwarz inequality and the co-area formula, we have

$$\begin{aligned} \int_{\omega} e_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) d\mathcal{H}^2 &\geq \int_{\omega} |\nabla|u_{\varepsilon}||^2 + \frac{1}{\varepsilon^2}(1 - |u_{\varepsilon}|^2)^2 d\mathcal{H}^2 \\ &\geq \int_{\omega} \frac{|\nabla|u_{\varepsilon}|| (1 - |u_{\varepsilon}|^2)}{\varepsilon} d\mathcal{H}^2 \\ &= \int_{t=0}^{\infty} \frac{(1 - t^2)}{\varepsilon} \mathcal{H}^1(\{x \in \omega \mid |u_{\varepsilon}(x)| = t\}) dt. \end{aligned}$$

Thus the compact set $\{x \in \omega \mid |u(x)| \leq 1/2\}$ can be covered by a finite collection of disjoint balls of total radius smaller than $C\varepsilon \int_{\omega} e_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) d\mathcal{H}^2$, which implies (1.10). \square

Remark 1.4.1. *By combining Lemma 1.4.2 with (1.4c), we obtain*

$$(1.11) \quad \sum_{\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, R_0, \delta))} |I_{\omega}| \leq C \int_{\mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, R_0, \delta))} e_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) d\mathcal{H}^2 \leq C\delta^{-1} F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}),$$

$$(1.12) \quad \sum_{\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, R_0, \delta))} \sum_{i \in I_{\omega}} |d_{i, \omega}| \leq C \int_{\mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, R_0, \delta))} e_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) d\mathcal{H}^2 \leq C\delta^{-1} F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}),$$

$$(1.13) \quad r_{\mathfrak{G}} := \sum_{\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, R_0, \delta))} r_{\omega} \leq C\varepsilon \int_{\mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, R_0, \delta))} e_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) d\mathcal{H}^2 \leq C\varepsilon\delta^{-1} F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}),$$

where $\sum_{\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, R_0, \delta))}$ denotes the sum over all the faces ω of cubes of the grid $\mathfrak{G}(b_{\varepsilon}, R_0, \delta)$.

1.5 3D vortex approximation construction

In this section we construct a new polyhedral approximation of the vorticity $\mu(u_{\varepsilon}, A_{\varepsilon})$ of a configuration $(u_{\varepsilon}, A_{\varepsilon}) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ such that $F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) \leq \varepsilon^{-\gamma}$, for some $\gamma \in (0, 1)$. The notion of minimal connection, first introduced in [BCL86], plays a key role in our construction. We begin this section by reviewing this concept. Then the definition of the function ζ is given and two technical propositions are stated. Lastly, the 3D vortex approximation construction is provided.

1.5.1 Minimal connections

Consider a collection $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$ of $2k$ points, where the p_i 's are the (non necessarily distinct) positive points and the n_i 's are the (non necessarily distinct) negative points. We define the length of a minimal connection joining the p_i 's to the n_i 's by

$$(1.14) \quad L(\mathcal{A}) := \min_{\sigma \in \mathfrak{S}_k} \sum_{i=1}^k |p_i - n_{\sigma(i)}|,$$

where \mathfrak{S}_k is the set of permutations of k indices and hereafter $|\cdot|$ denotes the euclidean distance in \mathbb{R}^3 . We also define the 1-current $\mathbb{L}(\mathcal{A})$, a minimal connection associated to \mathcal{A} , as the sum in the sense of currents of the segments joining p_i to $n_{\sigma(i)}$, where $\sigma \in \mathfrak{S}_k$ is a permutation achieving the minimum in (1.14). Although there can be several minimal connections associated to a collection \mathcal{A} , make an arbitrary choice of one, so that $\mathbb{L}(\mathcal{A})$ will be unambiguous.

Let us now consider the distance

$$d_{\partial\Omega}(x_1, x_2) := \min\{|x_1 - x_2|, d(x_1, \partial\Omega) + d(x_2, \partial\Omega)\} \quad x_1, x_2 \in \mathbb{R}^3.$$

We define the length of a minimal connection joining the p_i 's to the n_i 's through $\partial\Omega$ by

$$(1.15) \quad L_{\partial\Omega}(\mathcal{A}) = \min_{\sigma \in \mathfrak{S}_k} \sum_{i=1}^k d_{\partial\Omega}(p_i, n_{\sigma(i)}).$$

In this case we define the 1-current $\mathcal{L}_{\partial\Omega}(\mathcal{A})$, a minimal connection through $\partial\Omega$ associated to \mathcal{A} , as the sum in the sense of currents of the segments joining p_i to $n_{\sigma(i)}$ when $d_{\partial\Omega}(p_i, n_{\sigma(i)}) = |p_i - n_{\sigma(i)}|$ and the (properly oriented) segments joining p_i, n_i to $\partial\Omega$ when $d_{\partial\Omega}(p_i, n_{\sigma(i)}) = d(p_i, \partial\Omega) + d(n_{\sigma(i)}, \partial\Omega)$, where $\sigma \in \mathfrak{S}_k$ is a permutation achieving the minimum in (1.15). For the sake of unambiguity, uniqueness is once again assumed.

1.5.1.1 The function ζ

The following lemma is a particular case of a well-known result proved in [BCL86].

Lemma 1.5.1. *Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$ be a configuration of positive and negative points. Assume, relabeling the points if necessary, that $L(\mathcal{A}) = \sum_{i=1}^k |p_i - n_i|$. Then there exists a 1-Lipschitz function $\zeta^* : \cup_{i=1, \dots, k} \{p_i, n_i\} \rightarrow \mathbb{R}$ such that*

$$L(\mathcal{A}) = \sum_{i=1}^k \zeta^*(p_i) - \zeta^*(n_i) \quad \text{and} \quad \zeta^*(n_i) = \zeta^*(p_i) - |p_i - n_i|.$$

Definition 1.5.1 (The function ζ). *Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$ be a configuration of positive and negative points. Denote by ζ^* the 1-Lipschitz function given by Lemma 1.5.1. We define the function $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$ via the formula*

$$\zeta(x) := \max_{i \in \{1, \dots, k\}} \left(\zeta^*(p_i) - \max_{j \in \{1, \dots, 2k\}} d_{(i,j)}(x) \right),$$

where

$$d_{(i,j)}(x) := \langle p_i - x, \nu_{(i,j)} \rangle, \quad \nu_{(i,j)} := \begin{cases} \frac{p_i - a_j}{|p_i - a_j|} & \text{if } p_i \neq a_j \\ 0 & \text{if } p_i = a_j \end{cases},$$

where here and in the rest of the chapter the points a_i are defined as follows: if $j \in \{1, \dots, k\}$ then $a_j = p_j$, if $j \in \{k+1, \dots, 2k\}$ then $a_j = n_{j-k}$.

Lemma 1.5.2. *Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$ be a configuration of positive and negative points. Denote by $\zeta^* : \cup_{i=1, \dots, k} \{p_i, n_i\} \rightarrow \mathbb{R}$ the function given by Lemma 1.5.1 and define $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$ as in Definition 1.5.1. Then ζ is a 1-Lipschitz extension of ζ^* to \mathbb{R}^3 .*

Proof. It is easy to see that ζ is a 1-Lipschitz function. Let us check that

$$\zeta(p_i) = \zeta^*(p_i) \quad \text{and} \quad \zeta(n_i) = \zeta^*(n_i)$$

for every $i \in \{1, \dots, k\}$. Observe that

$$|d_{(i,j)}(x)| = |\langle p_i - x, \nu_{(i,j)} \rangle| \leq |p_i - x|.$$

But

$$d_{(i,j)}(a_l) = |p_i - a_l| \quad \text{for any } a_l \in \mathcal{A}.$$

Thus

$$\zeta(a_l) = \max_{i \in \{1, \dots, k\}} (\zeta^*(p_i) - |p_i - a_l|).$$

Since ζ^* is 1-Lipschitz, we deduce that $\zeta(a_l) \leq \zeta^*(a_l)$. It follows that $\zeta(p_l) = \zeta^*(p_l)$ for every $l \in \{1, \dots, k\}$. We conclude the proof by noting that, for any $l \in \{1, \dots, k\}$,

$$\zeta(n_l) \geq \zeta^*(p_l) - |p_l - n_l| = \zeta^*(n_l).$$

□

By displacing the points of the collection \mathcal{A} it is possible to construct a smooth approximation of the function ζ , which in addition satisfies extra (quantitative) properties.

Proposition 1.5.1. *Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$ be a configuration of positive and negative points. Assume, relabeling the points if necessary, that $L(\mathcal{A}) = \sum_{i=1}^k |p_i - n_i|$. Define $D_{\mathcal{A}} := \max_{a_i, a_j \in \mathcal{A}} |a_i - a_j|$ to be the maximum euclidean distance between any of the points of \mathcal{A} . Then there exist $C, C_0, C_1 > 0$ such that, for any $\rho \in (0, 1/2)$ and for any $0 < \lambda < \lambda_0(\rho) := (C_0(2k)^{-6})^{1/\rho}$, there exists a smooth function $\zeta_\lambda : \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfying*

1. $|L(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p_i) - \zeta_\lambda(n_i)| \leq C(D_{\mathcal{A}}(2k)^6 \lambda^\rho + 2k\lambda)$.
2. $\|\nabla \zeta_\lambda\|_{L^\infty(\mathbb{R}^3)} \leq 1$.
3. *There exists a set $P_\lambda \subset \mathbb{R}^3$ such that $|\zeta_\lambda(P_\lambda)| \leq 2\lambda k^2$ and that, for any $0 < \kappa < \lambda^{2\rho}/3$,*

$$C_\kappa := \{x \mid |\nabla \zeta_\lambda(x)| < \kappa\} \setminus P_\lambda$$

can be covered by \mathfrak{B}_κ , a collection of at most $(2k)^8$ balls of radius $C\lambda/(\lambda^{2\rho} - 3\kappa)$. Moreover, defining

$$(1.16) \quad T_\kappa := \zeta_\lambda(\cup_{B \in \mathfrak{B}_\kappa} B),$$

we have that, for any $t \in \mathbb{R} \setminus (T_\kappa \cup \zeta_\lambda(P_\lambda))$, $\{x \mid \zeta_\lambda(x) = t\}$ is a complete submanifold of \mathbb{R}^3 whose second fundamental form is bounded by $C_1(\lambda^{2\rho} \kappa)^{-1}$.

The proof of this result is technical and is postponed to the appendix.

1.5.1.2 The function ζ for $d_{\partial\Omega}$

When the Euclidean distance is replaced with the distance through $\partial\Omega$ the following lemma can be proved (see [BCL86]).

Lemma 1.5.3. *Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\} \subset \Omega$ be a configuration of positive and negative points. Assume, relabeling the points if necessary, that $L_{\partial\Omega}(\mathcal{A}) = \sum_{i=1}^k d_{\partial\Omega}(p_i, n_i)$. Then there exists a function $\zeta^* : \cup_{i=1, \dots, k} \{p_i, n_i\} \rightarrow \mathbb{R}$, 1-Lipschitz for the distance $d_{\partial\Omega}$, such that*

$$L_{\partial\Omega}(\mathcal{A}) = \sum_{i=1}^k \zeta^*(p_i) - \zeta^*(n_i) \quad \text{and} \quad \zeta^*(n_i) = \zeta^*(p_i) - d_{\partial\Omega}(p_i, n_i).$$

Definition 1.5.2 (The function ζ for $d_{\partial\Omega}$). *Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$ be a configuration of positive and negative points. Denote by ζ^* the function given by Lemma 1.5.1. We define the function $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$ for $d_{\partial\Omega}$ via the formula*

$$\zeta(x) := \max_{i \in \{1, \dots, k\}} (\zeta^*(p_i) - d_i(x, \partial\Omega)),$$

where

$$d_i(x, \partial\Omega) := \min \left[\max \left(\max_{j \in \{1, \dots, 2k\}} d_{(i,j)}(x), d(p_i, \partial\Omega) - d(x, \partial\Omega) \right), d(p_i, \partial\Omega) + d(x, \partial\Omega) \right],$$

with

$$d_{(i,j)}(x) = \langle p_i - x, \nu_{(i,j)} \rangle, \quad \nu_{(i,j)} = \begin{cases} \frac{p_i - a_j}{|p_i - a_j|} & \text{if } p_i \neq a_j \\ 0 & \text{if } p_i = a_j \end{cases}.$$

Lemma 1.5.4. *Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\}$ be a configuration of positive and negative points. Denote by $\zeta^* : \cup_{i=1, \dots, k} \{p_i, n_i\} \rightarrow \mathbb{R}$ the function given by Lemma 1.5.3 and define $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$ as in Definition 1.5.2. Then ζ is a 1-Lipschitz extension of ζ^* to \mathbb{R}^3 , which is constant on $\partial\Omega$.*

Proof. It is easy to see that ζ is a 1-Lipschitz function. Let us check that

$$\zeta(p_i) = \zeta^*(p_i) \quad \text{and} \quad \zeta(n_i) = \zeta^*(n_i)$$

for every $i \in \{1, \dots, k\}$. By the proof of Lemma 1.5.2, we know that

$$d_{(i,j)}(a_l) = |p_i - a_l| \quad \text{for any } a_l \in \mathcal{A}.$$

By the triangular inequality, we deduce that

$$\max(|p_i - a_l|, d(p_i, \partial\Omega) - d(a_l, \partial\Omega)) = |p_i - a_l|.$$

Then

$$d_i(a_l, \partial\Omega) = \min(|p_i - a_l|, d(p_i, \partial\Omega) - d(a_l, \partial\Omega)) = d_{\partial\Omega}(p_i, a_l),$$

which implies that

$$\zeta(a_l) = \max_{i \in \{1, \dots, k\}} (\zeta^*(p_i) - d_{\partial\Omega}(p_i, a_l)).$$

Since ζ^* is 1-Lipschitz for the distance $d_{\partial\Omega}$, we have that $\zeta(a_l) \leq \zeta^*(a_l)$. It follows that $\zeta(p_l) = \zeta^*(p_l)$ for every $l \in \{1, \dots, k\}$. But

$$\zeta(n_l) \geq \zeta^*(p_l) - d_{\partial\Omega}(p_l, n_l) = \zeta^*(n_l).$$

Finally, observe that, for all $x \in \partial\Omega$,

$$d_i(x, \partial\Omega) := \min \left[\max \left(\max_{j \in \{1, \dots, 2k\}} d_{(i,j)}(x), d(p_i, \partial\Omega) \right), d(p_i, \partial\Omega) \right] = d(p_i, \partial\Omega).$$

Thus

$$\zeta(x) = \max_{i \in \{1, \dots, k\}} (\zeta^*(p_i) - d(p_i, \partial\Omega))$$

for all $x \in \partial\Omega$. □

By displacing the points of the collection \mathcal{A} and performing a polyhedral approximation of the boundary $\partial\Omega$ it is possible to construct a smooth approximation of the function ζ for $d_{\partial\Omega}$, which in addition satisfies extra (quantitative) properties. This procedure requires the domain Ω to satisfy a technical assumption.

Proposition 1.5.2. *Let Ω be a C^2 bounded domain such that $\partial\Omega$ has strictly positive Gauss curvature. Let $\mathcal{A} = \{p_1, \dots, p_k, n_1, \dots, n_k\} \subset \Omega$ be a configuration of positive and negative points. Assume, relabeling the points if necessary, that $L_{\partial\Omega}(\mathcal{A}) = \sum_{i=1}^k d_{\partial\Omega}(p_i, n_i)$. Then there exist τ_0, C, C_0, C_1 , that depend only on $\partial\Omega$, and a universal constant $C_2 > 0$ such that, for any $\tau < \tau_0$, for any $\rho \in (0, 1/2)$, and for any*

$$0 < \lambda < \lambda_0(\rho, \tau) := (C_0 \min \{(2k)^{-6}, (2k)^{-4}\tau^2, (2k)^{-2}\tau^4, \tau^5\})^{1/\rho},$$

there exists a smooth function $\zeta_\lambda : \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfying

1. $|L_{\partial\Omega}(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p_i) - \zeta_\lambda(n_i)| \leq C(((2k)^6 + (2k)^4\tau^{-2} + (2k)^2\tau^{-4})\lambda^\rho + 2k(\tau^2 + \lambda)).$

2. Letting

$$\Omega_\lambda := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > 2\lambda\},$$

we have $\|\nabla\zeta_\lambda\|_{L^\infty(\Omega_\lambda)} \leq 1$.

3. $|\zeta_\lambda(\overline{\Omega \setminus \Omega_\lambda})| \leq C(\tau^2 + \lambda).$

4. There exists a set $P_\lambda \subset \mathbb{R}^3$ such that $|\zeta_\lambda(P_\lambda)| \leq 2\lambda k^2$ and that, for any $0 < \kappa < \lambda^{2\rho}/3$,

$$C_\kappa := \{x \in \Omega_\lambda \mid |\nabla\zeta_\lambda(x)| < \kappa\} \setminus P_\lambda$$

can be covered by \mathfrak{B}_κ , a collection of at most $C((2k)^8 + \tau^{-8})$ balls of radius $C_2\lambda/(\lambda^{2\rho} - 3\kappa)$. Moreover, defining

$$(1.17) \quad T_\kappa := \zeta_\lambda(\cup_{B \in \mathfrak{B}_\kappa} B),$$

we have that, for any $t \in \zeta_\lambda(\Omega_\lambda) \setminus (T_\kappa \cup \zeta_\lambda(P_\lambda))$, $\{x \mid \zeta_\lambda(x) = t\}$ is a complete submanifold of \mathbb{R}^3 whose second fundamental form is bounded by $C_1(\lambda^2\kappa)^{-1}$.

The proof of this proposition is deferred to the appendix.

1.5.2 Construction of the vorticity approximation

Let $\gamma \in (0, 1)$ and consider a configuration $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq \varepsilon^{-\gamma}$. Then Lemma 1.2.1 provides a grid $\mathfrak{G}(b_\varepsilon, R_0, \delta)$ satisfying (1.4). We begin by constructing our approximation in the cubes of the grid. For each cube $\mathcal{C}_l \in \mathfrak{G}(b_\varepsilon, R_0, \delta)$, Corollary 1.4.1 gives the existence of points $a_{i,\omega}$ and integers $d_{i,\omega} \neq 0$ such that

$$\mu_{\varepsilon,\omega} \approx 2\pi \sum_{i \in I_\omega} d_{i,\omega} \delta_{a_{i,\omega}},$$

for each of the six faces $\omega \subset \partial\mathcal{C}_l$ of the cube \mathcal{C}_l . Observe that, since $\partial\mu(u_\varepsilon, A_\varepsilon) = 0$ relative to \mathcal{C}_l , we have

$$\sum_{\omega \subset \partial\mathcal{C}_l} \sum_{i \in I_\omega} d_{i,\omega} = 0.$$

Then, we define a configuration $\mathcal{A}_l := \{p_1, \dots, p_{k_l}, n_1, \dots, n_{k_l}\}$ of positive and negative points associated to $\partial\mathcal{C}_l$, by repeating the points $a_{i,\omega}$ according to their degree $d_{i,\omega}$, for each of the six faces ω of the cube \mathcal{C}_l . The previous observation implies that the number of positive points p_i 's and negative points n_i 's of the collection \mathcal{A}_l are equal. We note that

$$2k_l = \sum_{\omega \subset \partial\mathcal{C}_l} \sum_{i \in I_\omega} |d_{i,\omega}|.$$

Consider the minimal connection $\mathbb{L}(\mathcal{A}_l)$ associated to \mathcal{A}_l . It may happen that the segment connecting some p_i to $n_{\sigma(i)}$ in $\mathbb{L}(\mathcal{A}_l)$ belongs to one of the faces ω of the cube \mathcal{C}_l . In this case we define a new connection $\tilde{\mathbb{L}}(\mathcal{A}_l)$ by replacing the original segment connecting p_i to $n_{\sigma(i)}$ with a Lipschitz curve connecting p_i to $n_{\sigma(i)}$ (preserving the orientation) and such that its intersection with $\partial\mathcal{C}_l$ is given by $\{p_i, n_{\sigma(i)}\}$. This process can be performed in such a way that $|\mathbb{L}(\mathcal{A}_l) - \tilde{\mathbb{L}}(\mathcal{A}_l)|$ is less than an arbitrarily small number. We remark that the resulting connection $\tilde{\mathbb{L}}(\mathcal{A}_l)$ is a polyhedral 1-current whose intersection with $\partial\mathcal{C}_l$ is equal to $\cup_{i=1, \dots, k_l} \{p_i, n_i\}$. We define

$$\nu_{\varepsilon, \mathcal{C}_l} := 2\pi \tilde{\mathbb{L}}(\mathcal{A}_l) \quad \text{in } \mathcal{C}_l,$$

for every cube $\mathcal{C}_l \in \mathfrak{G}(b_\varepsilon, R_0, \delta)$.

We now construct our vorticity approximation in $\bar{\Theta}$ (recall (1.5)). Once again Corollary 1.4.1 gives the existence of points $a_{i,\omega}$ and integers $d_{i,\omega} \neq 0$ such that

$$\mu_{\varepsilon,\omega} \approx 2\pi \sum_{i \in I_\omega} d_{i,\omega} \delta_{a_{i,\omega}},$$

for each face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, R_0, \delta))$ of a cube of the grid such that $\omega \subset \partial\mathfrak{G}$. Then, we define a configuration $\mathcal{A}_{\partial\mathfrak{G}} := \{p_1, \dots, p_{k_{\partial\mathfrak{G}}}, n_1, \dots, n_{k_{\partial\mathfrak{G}}}\}$ of positive and negative points associated to $\partial\mathfrak{G}$ by repeating the points $a_{i,\omega}$ according to their degree $d_{i,\omega}$, for each face $\omega \in \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, R_0, \delta))$ of a cube of the grid such that $\omega \subset \partial\mathfrak{G}$. Observe that, since $\partial\mu(u_\varepsilon, A_\varepsilon) = 0$ relative to $\partial\mathfrak{G}$, we have

$$\sum_{\omega \subset \partial\mathfrak{G}} \sum_{i \in I_\omega} d_{i,\omega} = 0,$$

which ensures that the number of positive points p_i 's and negative points n_i 's of the collection $\mathcal{A}_{\partial\mathfrak{G}}$ are equal. Consider now the minimal connection $\mathcal{L}_{\partial\Omega}(\mathcal{A}_{\partial\mathfrak{G}})$ through $\partial\Omega$ associated to $\mathcal{A}_{\partial\mathfrak{G}}$. We note that

$$2k_{\partial\mathfrak{G}} = \sum_{\omega \subset \partial\mathfrak{G}} \sum_{i \in I_\omega} |d_{i,\omega}|.$$

Performing a replacement argument in $\bar{\Theta}$ analogous to the one described above, we define a new connection $\tilde{\mathcal{L}}_{\partial\Omega}(\mathcal{A}_{\partial\mathfrak{G}})$, with $|L_{\partial\Omega}(\mathcal{A}_{\partial\mathfrak{G}}) - \tilde{\mathcal{L}}_{\partial\Omega}(\mathcal{A}_{\partial\mathfrak{G}})|$ less than an arbitrarily small number, whose intersection with $\partial\mathfrak{G}$ is equal to $\cup_{i=1,\dots,k_{\partial\mathfrak{G}}} \{p_i, n_i\}$. We define

$$\nu_{\varepsilon,\Theta} := 2\pi \tilde{\mathcal{L}}_{\partial\Omega}(\mathcal{A}_{\partial\mathfrak{G}}) \quad \text{in } \bar{\Theta}.$$

Finally, we define our polyhedral approximation ν_ε of the vorticity $\mu(u_\varepsilon, A_\varepsilon)$ by

$$(1.18) \quad \nu_\varepsilon := \sum_{\mathcal{G}_l \in \mathfrak{G}(b_\varepsilon, R_0, \delta)} \nu_{\varepsilon, \mathcal{G}_l} + \nu_{\varepsilon, \Theta},$$

where the sums are understood in the sense of currents.

We observe that the topological degree depends on the orientation of the domain in which it is computed. If a face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, R_0, \delta))$ belongs to two cubes C_1 and C_2 of the grid, then its associated collection of degrees $d_{i,\omega}$'s for C_1 is equal to minus its associated collection of degrees for C_2 . Of course the same occurs for those faces ω belonging to one of the cubes of the grid and to $\partial\mathfrak{G}$.

On the other hand (1.4a) implies that, for any face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, R_0, \delta))$, the intersection between the collection of points $a_{i,\omega}$'s and $\mathfrak{R}_1(\mathfrak{G}(b_\varepsilon, R_0, \delta))$ is empty.

By combining these arguments we conclude that the 1-currents $\nu_{\mathcal{G}_l}$'s and ν_Θ have a good compatibility condition between each other. Hence, by construction, ν_ε is a polyhedral 1-current such that $\partial\nu_\varepsilon = 0$ relative to Ω . In addition it approximates well $\mu(u_\varepsilon, A_\varepsilon)$ in an appropriate norm, as we shall show in Section 1.8.

To end this section we present a lemma about the support of ν_ε .

Lemma 1.5.5. *Let $\gamma \in (0, 1)$ and assume that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) \leq \varepsilon^{-\gamma}$, so that, by Lemma 1.2.1, there exists a grid $\mathfrak{G}(b_\varepsilon, R_0, \delta)$ satisfying (1.4). For each face $\omega \subset \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, R_0, \delta))$ of a cube of the grid, let $|I_\omega|$ be the number of connected components of $\{x \in \omega \mid |u_\varepsilon(x)| \leq 1/2\}$ whose degree is different from zero. Then, letting*

$$(1.19) \quad \mathfrak{G}_0 := \{\mathcal{G}_l \in \mathfrak{G} \mid \sum_{\omega \subset \partial\mathcal{G}_l} |I_\omega| > 0\}$$

and defining ν_ε by (1.18), we have

$$\text{supp}(\nu_\varepsilon) \subset S_{\nu_\varepsilon} := \bigcup_{\mathcal{G}_l \in \mathfrak{G}_0} \mathcal{G}_l \cup \begin{cases} \bar{\Theta} & \text{if } \sum_{\omega \subset \partial\mathfrak{G}} |I_\omega| > 0 \\ \emptyset & \text{if } \sum_{\omega \subset \partial\mathfrak{G}} |I_\omega| = 0 \end{cases}.$$

Moreover

$$|S_{\nu_\varepsilon}| \leq C\delta(1 + \delta F_\varepsilon(u_\varepsilon, A_\varepsilon)),$$

where C is a constant depending only on $\partial\Omega$.

Proof. The first assertion follows readily from the definition of ν_ε . Recall that, by (1.11), the number of faces $\omega \in \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, R_0, \delta))$ of a cube of the grid such that $|I_\omega| > 0$ is bounded above by $C\delta^{-1}F_\varepsilon(u_\varepsilon, A_\varepsilon)$. We deduce that $\#\{\{l \mid \mathcal{C}_l \in \mathfrak{G}_0\}\}$ is bounded above by $C\delta^{-1}F_\varepsilon(u_\varepsilon, A_\varepsilon)$. By noting that $|\Theta| \leq C\delta$, for a constant C depending only on $\partial\Omega$, we conclude that

$$|S_{\nu_\varepsilon}| \leq \sum_{\mathcal{C}_l \in \mathfrak{G}_0} |\mathcal{C}_l| + |\Theta| \leq \delta^3 \#\{\{l \mid \mathcal{C}_l \in \mathfrak{G}_0\}\} + C\delta \leq C\delta(1 + \delta F_\varepsilon(u_\varepsilon, A_\varepsilon)).$$

□

1.6 Lower bound for $E_\varepsilon(u_\varepsilon)$ far from the boundary

In this section we prove a lower bound, in the spirit of (1.1), for the energy without magnetic field $E_\varepsilon(u_\varepsilon)$ in the union of cubes of the grid $\mathfrak{G}(b_\varepsilon, R_0, \delta)$ given by Lemma 1.2.1. The proof relies on a slicing procedure based on the level sets of the smooth approximation of the function ζ constructed in the appendix and on the ball construction method on a surface of Section 1.3.

Theorem 1.6.1. *Let $m, M > 0$ and assume that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) = M_\varepsilon \leq M|\log \varepsilon|^m$. For any $s > 0$, there exists $\varepsilon_0 > 0$ depending only on m, s , and M , such that, for any $\varepsilon < \varepsilon_0$, letting $\mathfrak{G}(b_\varepsilon, R_0, \delta)$ denote the grid given by Lemma 1.2.1 with $\delta = \delta(\varepsilon) = |\log \varepsilon|^{-m-s-1}$, and defining ν_ε by (1.18) and \mathfrak{G}_0 by (1.19), if*

$$(1.20) \quad E_\varepsilon(u_\varepsilon, \cup_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{C}_l) \leq KM_\varepsilon \quad \text{and} \quad \sum_{\mathcal{C}_l \in \mathfrak{G}_0} \int_{\partial\mathcal{C}_l} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2 \leq K\delta^{-1}M_\varepsilon,$$

for some universal constant K , then

$$E_\varepsilon(u_\varepsilon, \cup_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{C}_l) \geq \frac{1}{2} \sum_{\mathcal{C}_l \in \mathfrak{G}_0} |\nu_{\varepsilon, \mathcal{C}_l}| \left(\log \frac{1}{\varepsilon} - \log C \frac{M_\varepsilon^{56} |\log \varepsilon|^{7+s}}{\delta^{55}} \right) - \frac{C}{|\log \varepsilon|^s},$$

where C is a universal constant.

Proof. Let us first find an estimate for each cube of the grid. We consider a cube $\mathcal{C}_l \in \mathfrak{G}_0$. For each of the six faces ω of \mathcal{C}_l , denote by $\{S_{i,\omega}\}_{i \in I_\omega}$ the collection of connected components of $\{x \in \omega \mid |u_\varepsilon(x)| \leq 1/2\}$. We define

$$S_l := \cup_{\omega \subset \partial\mathcal{C}_l} \cup_{i \in I_\omega} S_{i,\omega}.$$

Note that $|u_\varepsilon(x)| > 1/2$ for any $x \in \partial\mathcal{C}_l \setminus S_l$.

Denote by $\mathcal{A}_l = \{p_1, \dots, p_{k_l}, n_1, \dots, n_{k_l}\}$ the configuration of positive and negative points associated to the cube \mathcal{C}_l (see Subsection 1.5.2). For $\rho \in (0, 1/2)$ and $\lambda = \lambda(l) \leq (C_0(2k_l)^{-6})^{1/\rho}$ to be chosen later on, let ζ_λ be the smooth function associated to \mathcal{A}_l

by Proposition 1.5.1. Here the constant C_0 is the universal constant appearing in the proposition. For $\kappa = \kappa(l) < \lambda^{2\rho}/3$ consider the set T_κ defined by (1.16) and observe that

$$|T_\kappa| \leq C(2k_l)^8 \lambda / (\lambda^{2\rho} - 3\kappa),$$

where throughout the proof $C > 0$ denotes a universal constant that may change from line to line. Letting

$$\tilde{\mathcal{C}}_l := \{x \mid d(x, \mathcal{C}_l) < C_1 \lambda^2 \kappa, \text{ proj}_{\mathcal{C}_l} x \notin S_l\},$$

where C_1 is the universal constant appearing in the third statement of Proposition 1.5.1, we define $v_\varepsilon : \tilde{\mathcal{C}}_l \rightarrow \mathbb{C}$ via the formula

$$v_\varepsilon(x) = u_\varepsilon(\text{proj}_{\mathcal{C}_l} x) \quad x \in \tilde{\mathcal{C}}_l.$$

Observe that

$$E_\varepsilon(u_\varepsilon, \mathcal{C}_l) \geq E_\varepsilon(v_\varepsilon, \tilde{\mathcal{C}}_l) - C_1 \lambda^2 \kappa \int_{\partial \tilde{\mathcal{C}}_l \setminus S_l} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2.$$

In particular, if $\lambda^2 \kappa$ is small enough then $E_\varepsilon(v_\varepsilon, \tilde{\mathcal{C}}_l) \leq 2M_\varepsilon$. We also define

$$U_\lambda := \zeta_\lambda(\{x \in \partial \tilde{\mathcal{C}}_l \mid \text{proj}_{\mathcal{C}_l} x \in S_l\})$$

and note that

$$|U_\lambda| \leq \sum_{\omega \subset \partial \mathcal{C}_l} \sum_{i \in I_\omega} \text{diam}(S_{i,\omega}) + (2k_l) C_1 \lambda^2 \kappa.$$

Since $|\nabla \zeta_\lambda| \leq 1$, using the co-area formula, we deduce that

$$E_\varepsilon(v_\varepsilon, \tilde{\mathcal{C}}_l) \geq \int_{\tilde{\mathcal{C}}_l} e_\varepsilon(v_\varepsilon) |\nabla \zeta_\lambda| = \int_{t \in \mathbb{R}} \int_{\{\zeta_\lambda = t\} \cap \tilde{\mathcal{C}}_l} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 dt.$$

We now would like to apply the results proved in Section 1.3. Let us consider a small number $\gamma > 0$ and define

$$V_\gamma := \left\{ t \in \mathbb{R} \mid \int_{\{\zeta_\lambda = t\} \cap \tilde{\mathcal{C}}_l} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 > \frac{1}{\gamma} M_\varepsilon \right\}.$$

Note that $|V_\gamma| \leq 2K\gamma$. Finally, let us define $T_{bad} = T_\kappa \cup U_\lambda \cup V_\gamma \cup \zeta_\lambda(P_\lambda)$ (where P_λ is the set appearing in Proposition 1.5.1), $\Sigma_t := \{\zeta_\lambda = t\} \cap \tilde{\mathcal{C}}_l$, $t_* := \min_{a_i \in \mathcal{A}_l} \zeta_\lambda(a_i)$, and $t^* := \max_{a_i \in \mathcal{A}_l} \zeta_\lambda(a_i)$. For $t \in T_{good} := [t_*, t^*] \setminus T_{bad}$ it holds that:

- $\int_{\Sigma_t} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 \leq \gamma^{-1} M_\varepsilon$.
- $\{\zeta_\lambda = t\}$ is a surface whose second fundamental form is bounded by $C_1(\lambda^2 \kappa)^{-1}$. Note that this surface is necessarily oriented since it is a level set of ζ_λ .
- $\partial \Sigma_t = \{\zeta_\lambda = t\} \cap \partial \tilde{\mathcal{C}}_l$.
- $|v_\varepsilon(x)| > 1/2$ if $d(x, \partial \Sigma_t) < C_1 \lambda^2 \kappa$.

Then Corollary 1.3.1 yields that, for any $t \in T_{good}$,

$$\int_{\Sigma_t} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 \geq \pi |\deg(v_\varepsilon, \partial\Sigma_t)| \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right),$$

Noting that $\partial\Sigma_t = \partial(\{\zeta_\lambda \geq t\} \cap \partial\tilde{\mathcal{C}}_l)$, we deduce that

$$\deg(v_\varepsilon, \partial\Sigma_t) = d(t) := \#\{i \mid \zeta_\lambda(p_i) > t\} - \#\{i \mid \zeta_\lambda(n_i) > t\}.$$

By combining our previous estimates, we find

$$\begin{aligned} E_\varepsilon(v_\varepsilon, \tilde{\mathcal{C}}_l) &\geq \int_{t \in T_{good}} \int_{\Sigma_t} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 dt \\ &\geq \pi \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right) \int_{t \in T_{good}} d(t) dt \\ &\geq \pi \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right) \left(\int_{t_*}^{t^*} d(t) dt - \int_{t \in T_{bad}} |d(t)| dt \right). \end{aligned}$$

But, for any $t \in T_{bad}$,

$$|d(t)| = |\#\{i \mid \zeta_\lambda(p_i) > t\} - \#\{i \mid \zeta_\lambda(n_i) > t\}| \leq k_l.$$

Then

$$\int_{t \in T_{bad}} |d(t)| dt \leq k_l |T_{bad}| \leq k_l (|T_\kappa| + |U_\lambda| + |V_\gamma| + |\zeta_\lambda(P_\lambda)|).$$

On the other hand, observe that

$$\int_{t_*}^{t^*} d(t) dt = \int_{t_*}^{t^*} (\#\{i \mid \zeta_\lambda(p_i) > t\} - \#\{i \mid \zeta_\lambda(n_i) > t\}) dt = \sum_{i=1}^{k_\lambda} \zeta_\lambda(p_i) - \zeta_\lambda(n_i).$$

Remembering that

$$\left| L(\mathcal{A}_l) - \sum_{i=1}^{k_l} \zeta_\lambda(p_i) - \zeta_\lambda(n_i) \right| \leq C D_{\mathcal{A}_l} (2k_l)^6 \lambda^\rho + (2k_l) \lambda \leq C \delta (2k_l)^6 \lambda^\rho + (2k_l) \lambda$$

and that $|2\pi L(\mathcal{A}_l) - |\nu_{\varepsilon, \mathcal{C}_l}|$ can be taken arbitrarily small, we conclude that

$$\int_{t_*}^{t^*} d(t) dt \geq \frac{1}{2\pi} |\nu_{\varepsilon, \mathcal{C}_l}| - C (\delta (2k_l)^6 \lambda^\rho + (2k_l) \lambda).$$

Collecting our previous computations, we find

$$E_\varepsilon(u_\varepsilon, \mathcal{C}_l) \geq \frac{1}{2} |\nu_{\varepsilon, \mathcal{C}_l}| \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right) - \mathcal{E}_l,$$

where

$$\mathcal{E}_l := C (\delta (2k_l)^6 \lambda^\rho + (2k_l) \lambda + k_l (|T_\kappa| + |U_\lambda| + |V_\gamma| + |\zeta_\lambda(P_\lambda)|)) \log \frac{1}{\varepsilon}$$

$$+ C_1 \lambda^2 \kappa \int_{\partial \mathcal{C}_l \setminus S_l} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2.$$

We now want to combine the estimates found for cubes in \mathfrak{G}_0 . Observe that if λ and κ are chosen independent of l then

$$E_\varepsilon(u_\varepsilon, \cup_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{C}_l) \geq \frac{1}{2} \sum_{\mathcal{C}_l \in \mathfrak{G}_0} |\nu_{\varepsilon, \mathcal{C}_l}| \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right) - \sum_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{E}_l.$$

Our objective is then to choose the parameters $\lambda, \kappa = \kappa(\lambda)$, and γ independent of l and such that $\sum_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{E}_l \leq C |\log \varepsilon|^{-s}$. To do so, let us observe that (1.12) implies that

$$\sum_{\mathcal{C}_l \in \mathfrak{G}_0} 2k_l \leq C \delta^{-1} M_\varepsilon.$$

We deduce that such parameters can be found provided that $\lambda \leq (C_0 (C \delta^{-1} M_\varepsilon)^{-6})^{1/\rho}$. We let $\kappa = \lambda^{2\rho}/6$. Using (1.12), (1.13), and (1.20) one can check that

$$\sum_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{E}_l \leq C \log \frac{1}{\varepsilon} \left(\frac{M_\varepsilon^6}{\delta^5} \lambda^\rho + \frac{M_\varepsilon^9}{\delta^9} \lambda^{1-2\rho} + \frac{M_\varepsilon^2}{\delta^2} \varepsilon + \frac{M_\varepsilon}{\delta} \gamma \right).$$

We optimize over $\rho \in (0, 1/2)$ to find the least possible restrictions on λ . Observe that we require that $\lambda \leq C \min\{(\delta M_\varepsilon^{-1})^{6/\rho}, (\delta M_\varepsilon^{-1})^{9/(1-2\rho)}\}$. Then we set $\rho = 6/21$. Finally, choosing

$$\lambda = \left(\frac{1}{|\log \varepsilon|^{1+s}} \frac{\delta^9}{M_\varepsilon^9} \right)^{\frac{21}{9}} \quad \text{and} \quad \gamma = \frac{1}{|\log \varepsilon|^{1+s}} \frac{\delta}{M_\varepsilon},$$

we easily check that there exists $\varepsilon_0 > 0$ depending only on m, s , and M , such that $\sum_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{E}_l \leq C |\log \varepsilon|^{-s}$ for any $0 < \varepsilon < \varepsilon_0$. Thus

$$E_\varepsilon(u_\varepsilon, \cup_{\mathcal{C}_l \in \mathfrak{G}_0} \mathcal{C}_l) \geq \frac{1}{2} \sum_{\mathcal{C}_l \in \mathfrak{G}_0} |\nu_{\varepsilon, \mathcal{C}_l}| \left(\log \frac{1}{\varepsilon} - \log C \frac{M_\varepsilon^{56} |\log \varepsilon|^{7+s}}{\delta^{55}} \right) - \frac{C}{|\log \varepsilon|^s}.$$

The theorem is proved. □

1.7 Lower bound for $E_\varepsilon(u_\varepsilon)$ close to the boundary

In this section we prove a lower bound, in the spirit of (1.1), for the energy without magnetic field $E_\varepsilon(u_\varepsilon)$ in Θ . The proof relies on a slicing procedure based on the level sets of the smooth approximation of the function ζ for $d_{\partial\Omega}$ constructed in the appendix and on the ball construction method on a surface of Section 1.3.

Theorem 1.7.1. *Let Ω be a smooth bounded domain such that $\partial\Omega$ has strictly positive Gauss curvature. Let $m, M > 0$ and assume that $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$ is a configuration such that $F_\varepsilon(u_\varepsilon, A_\varepsilon) = M_\varepsilon \leq M |\log \varepsilon|^m$. For any $s > 0$, there exists $\varepsilon_0 > 0$ depending only on m, s, M , and $\partial\Omega$, such that, for any $\varepsilon < \varepsilon_0$, letting $\mathfrak{G}(b_\varepsilon, R_0, \delta)$ denote*

the grid given by Lemma 1.2.1 with $\delta = \delta(\varepsilon) = |\log \varepsilon|^{-m-s-1}$, and defining ν_ε by (1.18) and $\Theta, \partial\mathfrak{G}$ by (1.5), if

$$(1.21) \quad E_\varepsilon(u_\varepsilon, \Theta) \leq KM_\varepsilon \quad \text{and} \quad \int_{\partial\mathfrak{G}} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2 \leq K\delta^{-1}M_\varepsilon,$$

for a universal constant $K > 0$, then

$$E_\varepsilon(u_\varepsilon, \Theta) \geq \frac{1}{2}|\nu_{\varepsilon, \Theta}| \left(\log \frac{1}{\varepsilon} - \log C \frac{M_\varepsilon^{56} |\log \varepsilon|^{7+s}}{\delta^{55}} \right) - C\delta M_\varepsilon |\log \varepsilon| - \frac{C}{|\log \varepsilon|^s},$$

where C is a universal constant.

Proof. For each face $\omega \in \mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, R_0, \delta))$ of a cube of the grid such that $\omega \subset \partial\mathfrak{G}$, denote by $\{S_{i,\omega}\}_{i \in I_\omega}$ the collection of connected components of $\{x \in \omega \mid |u_\varepsilon(x)| \leq 1/2\}$. We define

$$S_{\partial\mathfrak{G}} := \cup_{\omega \subset \partial\mathfrak{G}} \cup_{i \in I_\omega} S_{i,\omega}.$$

Note that $|u_\varepsilon(x)| > 1/2$ for any $x \in \partial\mathfrak{G} \setminus S_{\partial\mathfrak{G}}$.

Denote by $\mathcal{A}_{\partial\mathfrak{G}} = \{p_1, \dots, p_{k_{\partial\mathfrak{G}}}, n_1, \dots, n_{k_{\partial\mathfrak{G}}}\} \subset \Omega$ the configuration of positive and negative points associated to $\partial\mathfrak{G}$ (see Subsection 1.5.2). Observe that (1.12) implies that

$$(2k_{\partial\mathfrak{G}}) \leq C\delta^{-1}M_\varepsilon.$$

For $\tau = \delta$, $\rho \in (0, 1/2)$, and $\lambda \leq (C_0(C\delta^{-1}M_\varepsilon)^{-6})^{1/\rho}$ to be chosen later on, let ζ_λ be the smooth function associated to $\mathcal{A}_{\partial\mathfrak{G}}$ by Proposition 1.5.2. Here the constant $C_0 = C_0(\partial\Omega)$ is the constant appearing in the proposition. For $\kappa < \lambda^{2\rho}/3$ consider the set T_κ defined by (1.17) and observe that

$$|T_\kappa| \leq C\delta^{-8}M_\varepsilon^8\lambda/(\lambda^{2\rho} - 3\kappa),$$

where throughout the proof $C > 0$ denotes a constant depending only on $\partial\Omega$, that may change from line to line.

Let

$$\partial\tilde{\mathfrak{G}} = \{x \in \Omega \setminus \Theta \mid \min_{y \in \partial\mathfrak{G}} \|x - y\|_\infty = C_1\lambda^2\kappa\},$$

where C_1 is the universal constant appearing in the fourth statement of Proposition 1.5.2. Observe that $\partial\tilde{\mathfrak{G}}$ corresponds to a shrunk version of the polyhedron $\partial\mathfrak{G}$, or, in other words, a smaller version of $\partial\mathfrak{G}$ with the same shape. Each face $\omega \subset \partial\mathfrak{G}$ has a parallel counterpart face $\tilde{\omega} \subset \partial\tilde{\mathfrak{G}}$ which corresponds to a translated and in some cases also a shrunk version of ω . It is easy to see that there exists a bijective function $f : \partial\mathfrak{G} \rightarrow \partial\tilde{\mathfrak{G}}$ mapping any $x \in \omega \subset \partial\mathfrak{G}$ to its unique counterpart point $\tilde{x} \in \tilde{\omega} \subset \partial\tilde{\mathfrak{G}}$. One immediately checks that for any $x, y \in \omega \subset \partial\mathfrak{G}$

$$|f(x) - x| \leq \sqrt{2}C_1\lambda^2\kappa \quad \text{and} \quad \frac{1}{\sqrt{2}}|x - y| \leq |f(x) - f(y)| \leq |x - y|.$$

Denoting by \mathcal{O} the open region enclosed by $\partial\mathfrak{G}$ and $\partial\tilde{\mathfrak{G}}$, we observe that for any $y \in \overline{\mathcal{O}}$ there exists a unique $x_y \in \partial\mathfrak{G}$ and a unique $t_y \in [0, 1]$ such that $y = tx_y + (1-t)f(x_y)$. Letting

$$\tilde{\mathcal{O}} := \{y \in \mathcal{O} \mid x_y \notin S_{\partial\mathfrak{G}}\},$$

we define $v_\varepsilon : \bar{\Theta} \cup \tilde{\mathcal{O}} \rightarrow \mathbb{C}$ by

$$v_\varepsilon(y) = u_\varepsilon(y) \quad \text{if } y \in \bar{\Theta}, \quad v_\varepsilon(y) = u_\varepsilon(x_y) \quad \text{if } y \in \tilde{\mathcal{O}}.$$

Note that v_ε is a H^1 -extension of u_ε and that

$$E_\varepsilon(v_\varepsilon, \tilde{\mathcal{O}}) \leq E_\varepsilon(v_\varepsilon, \mathcal{O}) \leq \sqrt{2}C_1\lambda^2\kappa \int_{\partial\tilde{\mathcal{O}}} e_\varepsilon(v_\varepsilon)d\mathcal{H}^2 \leq 2C_1\lambda^2\kappa \int_{\partial\mathfrak{S}} e_\varepsilon(u_\varepsilon)d\mathcal{H}^2.$$

Thus

$$E_\varepsilon(u_\varepsilon, \Theta) \geq E_\varepsilon(v_\varepsilon, \Theta \cup \tilde{\mathcal{O}}) - E_\varepsilon(v_\varepsilon, \tilde{\mathcal{O}}) \geq E_\varepsilon(v_\varepsilon, \Theta \cup \tilde{\mathcal{O}}) - 2C_1\lambda^2\kappa \int_{\partial\mathfrak{S}} e_\varepsilon(u_\varepsilon)d\mathcal{H}^2.$$

In particular, if $\lambda^2\kappa$ is small enough then $E_\varepsilon(v_\varepsilon, \Theta \cup \tilde{\mathcal{O}}) \leq 2M_\varepsilon$. We also define

$$U_\lambda := \zeta_\lambda(\{y \in \partial(\Theta \cup \tilde{\mathcal{O}}) \setminus \partial\Omega \mid x_y \in S_{\partial\mathfrak{S}}\})$$

and note that by (1.12) and (1.13), we have

$$|U_\lambda| \leq |S_{\partial\mathfrak{S}}| + \sqrt{2}(2k_{\partial\Omega})C_1\lambda^2\kappa \leq C\varepsilon\delta^{-1}M_\varepsilon + C\delta^{-1}M_\varepsilon\lambda^2\kappa.$$

Since $|\nabla\zeta_\lambda| \leq 1$, using the co-area formula, we deduce that

$$E_\varepsilon(v_\varepsilon, \Theta \cup \tilde{\mathcal{O}}) \geq \int_{\Theta \cup \tilde{\mathcal{O}}} e_\varepsilon(v_\varepsilon)|\nabla\zeta_\lambda| = \int_{t \in \mathbb{R}} \int_{\{\zeta_\lambda=t\} \cap (\Theta \cup \tilde{\mathcal{O}})} e_\varepsilon(v_\varepsilon)d\mathcal{H}^2 dt.$$

We would like now to apply the results proved in Section 1.3. Let us consider a small number $\gamma > 0$ and define

$$V_\gamma := \left\{ t \in \mathbb{R} \mid \int_{\{\zeta_\lambda=t\} \cap (\Theta \cup \tilde{\mathcal{O}})} e_\varepsilon(v_\varepsilon)d\mathcal{H}^2 > \frac{1}{\gamma}M_\varepsilon \right\}.$$

Note that $|V_\gamma| \leq 2K\gamma$. Finally, let us define $T_{bad} = T_\kappa \cup U_\lambda \cup V_\gamma \cup \zeta_\lambda(\overline{\Omega \setminus \Omega_\lambda}) \cup \zeta_\lambda(P_\lambda)$ (where P_λ is the set appearing in Proposition 1.5.2), $\Sigma_t := \{\zeta_\lambda = t\} \cap (\Theta \cup \tilde{\mathcal{O}})$, $t_* := \min_{a_i \in \mathcal{A}_{\partial\mathfrak{S}}} \zeta_\lambda(a_i)$, and $t^* := \max_{a_i \in \mathcal{A}_{\partial\mathfrak{S}}} \zeta_\lambda(a_i)$. For $t \in T_{good} := [t_*, t^*] \setminus T_{bad}$ it holds that:

- $\int_{\Sigma_t} e_\varepsilon(v_\varepsilon)d\mathcal{H}^2 \leq \gamma^{-1}M_\varepsilon$.
- $\{\zeta_\lambda = t\} = \{x \in \Omega_\lambda \mid \zeta_\lambda(x) = t\}$ is a surface whose second fundamental form is bounded by $C_1(\lambda^2\kappa)^{-1}$. Note that this surface is necessarily oriented since it is a level set of ζ_λ .
- $\partial\Sigma_t = \{\zeta_\lambda = t\} \cap (\partial(\Theta \cup \tilde{\mathcal{O}}) \setminus \partial\Omega)$.
- $|v_\varepsilon(x)| > 1/2$ if $d(x, \partial\Sigma_t) < C_1\lambda^2\kappa$.

Then Corollary 1.3.1 yields that, for any $t \in T_{good}$,

$$\int_{\Sigma_t} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 \geq \pi |\deg(v_\varepsilon, \partial\Sigma_t)| \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right),$$

Noting that $\partial\Sigma_t = \partial(\{\zeta_\lambda \geq t\} \cap (\partial(\Theta \cup \tilde{\mathcal{O}}) \setminus \partial\Omega))$, we deduce that

$$\deg(v_\varepsilon, \partial\Sigma_t) = d(t) := \#\{i \mid \zeta_\lambda(p_i) > t\} - \#\{i \mid \zeta_\lambda(n_i) > t\}.$$

By combining our previous estimates, we find

$$\begin{aligned} E_\varepsilon(v_\varepsilon, \Theta \cup \tilde{\mathcal{O}}) &\geq \int_{t \in T_{good}} \int_{\Sigma_t} e_\varepsilon(v_\varepsilon) d\mathcal{H}^2 dt \\ &\geq \pi \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right) \int_{t \in T_{good}} d(t) dt \\ &\geq \pi \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right) \left(\int_{t_*}^{t^*} d(t) dt - \int_{t \in T_{bad}} |d(t)| dt \right). \end{aligned}$$

But, for any $t \in T_{bad}$,

$$|d(t)| = |\#\{i \mid \zeta_\lambda(p_i) > t\} - \#\{i \mid \zeta_\lambda(n_i) > t\}| \leq k_{\partial\mathfrak{G}}.$$

Then

$$\int_{t \in T_{bad}} |d(t)| dt \leq k_{\partial\mathfrak{G}} |T_{bad}| \leq k_{\partial\mathfrak{G}} (|T_\kappa| + |U_\lambda| + |V_\gamma| + |\zeta_\lambda(\overline{\Omega \setminus \Omega_\lambda})| + |\zeta_\lambda(P_\lambda)|).$$

On the other hand, observe that

$$\int_{t_*}^{t^*} d(t) dt = \int_{t_*}^{t^*} (\#\{i \mid \zeta_\lambda(p_i) > t\} - \#\{i \mid \zeta_\lambda(n_i) > t\}) dt = \sum_{i=1}^{k_{\partial\mathfrak{G}}} \zeta_\lambda(p_i) - \zeta_\lambda(n_i).$$

Remembering that

$$\left| L(\mathcal{A}_{\partial\mathfrak{G}}) - \sum_{i=1}^{k_{\partial\mathfrak{G}}} \zeta_\lambda(p_i) - \zeta_\lambda(n_i) \right| \leq C \left(((2k_{\partial\mathfrak{G}})^6 + (2k_{\partial\mathfrak{G}})^4 \delta^{-2} + (2k_{\partial\mathfrak{G}})^2 \delta^{-4}) \lambda^\rho + (2k_{\partial\mathfrak{G}})(\delta^2 + \lambda) \right)$$

and that $|2\pi L(\mathcal{A}_{\partial\mathfrak{G}}) - |\nu_{\varepsilon, \Theta}|$ can be taken arbitrarily small. We conclude that

$$\int_{t_*}^{t^*} d(t) dt \geq \frac{1}{2\pi} |\nu_{\varepsilon, \Theta}| - C \left(((2k_{\partial\mathfrak{G}})^6 + (2k_{\partial\mathfrak{G}})^4 \delta^{-2} + (2k_{\partial\mathfrak{G}})^2 \delta^{-4}) \lambda^\rho + (2k_{\partial\mathfrak{G}})(\delta^2 + \lambda) \right).$$

Collecting our previous computations, we find

$$E_\varepsilon(u_\varepsilon, \Theta) \geq \frac{1}{2} |\nu_{\varepsilon, \Theta}| \left(\log \frac{1}{\varepsilon} - \log \frac{C_1 M_\varepsilon}{\lambda^2 \kappa \gamma} \right) - \mathcal{E}_\Theta,$$

where

$$\begin{aligned} \mathcal{E}_\Theta := & C \left((2k_{\partial\mathfrak{G}})^6 + (2k_{\partial\mathfrak{G}})^4 \delta^{-2} + (2k_{\partial\mathfrak{G}})^2 \delta^{-4} \right) \lambda^\rho + (2k_{\partial\mathfrak{G}})(\delta^2 + \lambda) \log \frac{1}{\varepsilon} \\ & + k_{\partial\mathfrak{G}} \left(|T_\kappa| + |U_\lambda| + |V_\gamma| + |\zeta_\lambda(\overline{\Omega \setminus \Omega_\lambda})| + |\zeta_\lambda(P_\lambda)| \right) \log \frac{1}{\varepsilon} + 2C_1 \lambda^2 \kappa \int_{\partial\mathfrak{G}} e_\varepsilon(u_\varepsilon) d\mathcal{H}^2. \end{aligned}$$

We now choose the parameters ρ , λ , $\kappa(\lambda)$ and γ . We let $\kappa = \lambda^{2\rho}/6$. Using (1.12) and (1.21) one can check that

$$\mathcal{E}_\Theta \leq C \log \frac{1}{\varepsilon} \left(\frac{M_\varepsilon^6}{\delta^6} \lambda^\rho + \delta M_\varepsilon + \frac{M_\varepsilon^9}{\delta^9} \lambda^{1-2\rho} + \frac{M_\varepsilon^2}{\delta^2} \varepsilon + \frac{M_\varepsilon}{\delta} \gamma \right)$$

We choose $\rho = 6/21$,

$$\lambda = \left(\frac{1}{|\log \varepsilon|^{1+s}} \frac{\delta^9}{M_\varepsilon^9} \right)^{\frac{21}{9}}, \quad \text{and} \quad \gamma = \frac{1}{|\log \varepsilon|^{1+s}} \frac{\delta}{M_\varepsilon}.$$

We easily check that there exists $\varepsilon_0 > 0$ depending only on m, s, M , and $\partial\Omega$, such that $\mathcal{E}_\Theta \leq C\delta M_\varepsilon |\log \varepsilon| + C|\log \varepsilon|^{-s}$ for any $0 < \varepsilon < \varepsilon_0$. Thus

$$E_\varepsilon(u_\varepsilon, \Theta) \geq \frac{1}{2} |\nu_{\varepsilon, \Theta}| \left(\log \frac{1}{\varepsilon} - \log C \frac{M_\varepsilon^{56} |\log \varepsilon|^{7+s}}{\delta^{55}} \right) - C\delta M_\varepsilon |\log \varepsilon| - \frac{C}{|\log \varepsilon|^s}.$$

This concludes the proof. \square

1.8 Proof of the main result

First, using the results of the previous two sections we prove (1.1).

Proof of (1.1). Since the energy $F_\varepsilon(u_\varepsilon, A_\varepsilon)$ is gauge invariant, it is enough to prove the result in the Coulomb gauge, i.e.

$$\operatorname{div} A_\varepsilon = 0 \text{ in } \Omega \quad \text{and} \quad A_\varepsilon \cdot \nu = 0 \text{ on } \partial\Omega.$$

We immediately check that

$$\|A_\varepsilon\|_{H^1(\Omega, \mathbb{R}^3)} \leq C \|\operatorname{curl} A_\varepsilon\|_{L^2(\Omega, \mathbb{R}^3)},$$

where throughout the proof $C > 0$ denotes a universal constant that may change from line to line. By Sobolev embedding theorem we have

$$\|A_\varepsilon\|_{L^p(\Omega, \mathbb{R}^3)} \leq C \|A_\varepsilon\|_{H^1(\Omega, \mathbb{R}^3)}$$

for any $1 \leq p \leq 6$. Observe that

$$\int_\Omega |\nabla u_\varepsilon|^2 \leq \int_\Omega |\nabla_{A_\varepsilon} u_\varepsilon|^2 + |u_\varepsilon|^2 |A_\varepsilon|^2 \leq \int_\Omega |\nabla_{A_\varepsilon} u_\varepsilon|^2 + (|u_\varepsilon|^2 - 1) |A_\varepsilon|^2 + |A_\varepsilon|^2$$

By the Cauchy-Schwarz inequality, we have

$$\int_{\Omega} (|u_{\varepsilon}|^2 - 1)|A_{\varepsilon}|^2 \leq \left(\int_{\Omega} (1 - |u_{\varepsilon}|^2)^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |A_{\varepsilon}|^4 \right)^{\frac{1}{2}} \leq C\varepsilon F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}).$$

Thus

$$E_{\varepsilon}(u_{\varepsilon}) \leq CF_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}).$$

Let us consider the grid $\mathfrak{G}(b_{\varepsilon}, R_0, \delta)$ given by Lemma 1.2.1. It is not hard to see that, up to an adjustment of the constant appearing in the lemma, we can require our grid to additionally satisfy the inequalities

$$(1.22) \quad \int_{\mathfrak{R}_1(\mathfrak{G}(b_{\varepsilon}, R_0, \delta))} e_{\varepsilon}(u_{\varepsilon}) d\mathcal{H}^1 \leq C\delta^{-2} F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}), \quad \int_{\mathfrak{R}_2(\mathfrak{G}(b_{\varepsilon}, R_0, \delta))} e_{\varepsilon}(u_{\varepsilon}) d\mathcal{H}^2 \leq C\delta^{-1} F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}).$$

We define the polyhedral 1-current ν_{ε} by (1.18). We recall the notation introduced in Lemma 1.5.5 and observe that

$$\int_{S_{\nu_{\varepsilon}}} |\nabla u_{\varepsilon}|^2 \leq \int_{S_{\nu_{\varepsilon}}} |\nabla_{A_{\varepsilon}} u_{\varepsilon}|^2 + \int_{S_{\nu_{\varepsilon}}} (|u_{\varepsilon}|^2 - 1)|A_{\varepsilon}|^2 + \int_{S_{\nu_{\varepsilon}}} |A_{\varepsilon}|^2.$$

Using Hölder's inequality, we find

$$\int_{S_{\nu_{\varepsilon}}} (|u_{\varepsilon}|^2 - 1)|A_{\varepsilon}|^2 \leq \| |u_{\varepsilon}|^2 - 1 \|_{L^2(S_{\nu_{\varepsilon}})} |S_{\nu_{\varepsilon}}|^{\frac{1}{6}} \|A_{\varepsilon}\|_{L^6(S_{\nu_{\varepsilon}}, \mathbb{R}^3)}^2$$

and

$$\int_{S_{\nu_{\varepsilon}}} |A_{\varepsilon}|^2 \leq |S_{\nu_{\varepsilon}}|^{\frac{2}{3}} \|A_{\varepsilon}\|_{L^6(S_{\nu_{\varepsilon}}, \mathbb{R}^3)}^2.$$

We are led to

$$\int_{S_{\nu_{\varepsilon}}} |\nabla u_{\varepsilon}|^2 \leq \int_{S_{\nu_{\varepsilon}}} |\nabla_{A_{\varepsilon}} u_{\varepsilon}|^2 + CF_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) \left(\varepsilon |S_{\nu_{\varepsilon}}|^{\frac{1}{6}} + |S_{\nu_{\varepsilon}}|^{\frac{2}{3}} \right),$$

which implies that

$$(1.23) \quad \frac{1}{2} \int_{S_{\nu_{\varepsilon}}} |\nabla_{A_{\varepsilon}} u_{\varepsilon}|^2 + \frac{1}{2\varepsilon^2} (1 - |u_{\varepsilon}|^2)^2 \geq E_{\varepsilon}(u_{\varepsilon}, S_{\nu_{\varepsilon}}) - CF_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) \left(\varepsilon |S_{\nu_{\varepsilon}}|^{\frac{1}{6}} F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon})^{\frac{1}{2}} + |S_{\nu_{\varepsilon}}|^{\frac{2}{3}} \right).$$

Thanks to (1.22), we can apply Theorem 1.6.1 and Theorem 1.7.1 with $s > 0$. We then deduce that there exists $\varepsilon_0 > 0$, depending only on m, s, M , and $\partial\Omega$, such that, for any $0 < \varepsilon < \varepsilon_0$,

$$E_{\varepsilon}(u_{\varepsilon}, S_{\nu_{\varepsilon}}) \geq \frac{1}{2} |\nu_{\varepsilon}|(\Omega) \left(\log \frac{1}{\varepsilon} - \log C \frac{M_{\varepsilon}^{56} |\log \varepsilon|^{7+s}}{\delta^{55}} \right) - C\delta M_{\varepsilon} |\log \varepsilon| - \frac{C}{|\log \varepsilon|^s},$$

where $\delta = \delta(\varepsilon) = |\log \varepsilon|^{-m-s-1}$. By combining this with Lemma 1.5.5 and (1.23), we are led to

$$\begin{aligned} \frac{1}{2} \int_{S_{\nu_{\varepsilon}}} |\nabla_{A_{\varepsilon}} u_{\varepsilon}|^2 + \frac{1}{2\varepsilon^2} (1 - |u_{\varepsilon}|^2)^2 + |\operatorname{curl} A_{\varepsilon}|^2 &\geq \frac{1}{2} |\nu_{\varepsilon}|(\Omega) \left(\log \frac{1}{\varepsilon} - \log C \frac{M_{\varepsilon}^{56} |\log \varepsilon|^{7+s}}{\delta^{55}} \right) \\ &\quad - C\delta M_{\varepsilon} |\log \varepsilon| - CM_{\varepsilon} \delta^{\frac{2}{3}} - C |\log \varepsilon|^{-s}. \end{aligned}$$

By letting $s = s(m, n) = \max\left(n, \frac{m+3n-2}{2}\right)$, the result follows. \square

Before presenting the proof of (1.2) for $\gamma = 1$, let us prove the following lemma.

Lemma 1.8.1. *Let $(u_\varepsilon, A_\varepsilon) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^3)$. Then there exists a constant $C > 0$ depending only on $\partial\Omega$, such that*

$$(1.24) \quad \|\mu(u_\varepsilon, A_\varepsilon)\|_{C^0(\Omega)^*} \leq CF_\varepsilon(u_\varepsilon, A_\varepsilon).$$

Proof. By definition

$$\begin{aligned} \mu(u_\varepsilon, A_\varepsilon) &= \frac{i}{2}d(u_\varepsilon d_{A_\varepsilon} \bar{u}_\varepsilon - \bar{u}_\varepsilon d_{A_\varepsilon} u_\varepsilon) + dA_\varepsilon \\ &= \frac{i}{2}(du_\varepsilon \wedge d_{A_\varepsilon} \bar{u}_\varepsilon + u_\varepsilon d(d_{A_\varepsilon} \bar{u}_\varepsilon) - d\bar{u}_\varepsilon \wedge d_{A_\varepsilon} u_\varepsilon - \bar{u}_\varepsilon d(d_{A_\varepsilon} u_\varepsilon)) + dA_\varepsilon. \end{aligned}$$

Simple computations show that

$$\begin{aligned} \mu(u_\varepsilon, A_\varepsilon) &= \frac{i}{2}(du_\varepsilon \wedge d_{A_\varepsilon} \bar{u}_\varepsilon - iu_\varepsilon d\bar{u}_\varepsilon \wedge A_\varepsilon - d\bar{u}_\varepsilon \wedge d_{A_\varepsilon} u_\varepsilon + i\bar{u}_\varepsilon du_\varepsilon \wedge A_\varepsilon) + dA_\varepsilon \\ &= \frac{i}{2}(d_{A_\varepsilon} u_\varepsilon \wedge d_{A_\varepsilon} \bar{u}_\varepsilon - d_{A_\varepsilon} \bar{u}_\varepsilon \wedge d_{A_\varepsilon} u_\varepsilon) + dA_\varepsilon = id_{A_\varepsilon} u_\varepsilon \wedge d_{A_\varepsilon} \bar{u}_\varepsilon + dA_\varepsilon. \end{aligned}$$

Integrating on Ω and using the Cauchy-Schwarz inequality, we find

$$\int_{\Omega} \mu(u_\varepsilon, A_\varepsilon) \leq 2 \left(F_\varepsilon(u_\varepsilon, A_\varepsilon) + F_\varepsilon(u_\varepsilon, A_\varepsilon)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} \right).$$

Then we easily check that there exists a constant $C(\partial\Omega) > 0$ such that

$$\left| \int_{\Omega} \mu(u_\varepsilon, A_\varepsilon) \wedge \phi \right| \leq C \|\phi\|_{C^0(\Omega)} F_\varepsilon(u_\varepsilon, A_\varepsilon),$$

for any continuous 1-form ϕ , which implies (1.24). \square

Proof of (1.2) for $\gamma = 1$. As in the previous proof, we consider the grid $\mathfrak{G}(b_\varepsilon, R_0, \delta)$ given by Lemma 1.2.1 and the polyhedral 1-current ν_ε defined by (1.18). The quantities $\delta = \delta(\varepsilon)$ and $s = s(m, n)$ are defined as above.

Let $\phi \in C_T^{0,1}(\Omega)$ be a 1-form. Note that

$$(1.25) \quad \left| \int_{\Omega} (\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon) \wedge \phi \right| \leq \sum_{\mathcal{G}_l \in \mathfrak{G}(b_\varepsilon, R_0, \delta)} \left| \int_{\mathcal{G}_l} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{G}_l}) \wedge \phi \right| + \left| \int_{\Theta} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}) \wedge \phi \right|.$$

First, we consider a cube $\mathcal{G}_l \in \mathfrak{G}(b_\varepsilon, R_0, \delta)$ and define $\phi_l = \int_{\mathcal{G}_l} \phi$. Observe that

$$(1.26) \quad \|\phi - \phi_l\|_{C^0(\mathcal{G}_l)} \leq \delta \|\phi\|_{C^{0,1}(\mathcal{G}_l)}$$

and that

$$\left| \int_{\mathcal{C}_l} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}) \wedge \phi \right| \leq \left| \int_{\mathcal{C}_l} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}) \wedge (\phi - \phi_l) \right| + \left| \int_{\mathcal{C}_l} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}) \wedge \phi_l \right|.$$

Using (1.26), we deduce that

$$(1.27) \quad \left| \int_{\mathcal{C}_l} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}) \wedge (\phi - \phi_l) \right| \leq \delta \|\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}\|_{C^0(\mathcal{C}_l)^*} \|\phi\|_{C^{0,1}(\mathcal{C}_l)}.$$

On the other hand, since ϕ_l is a constant, there exists a function f_l such that

$$\phi_l = df_l, \quad \int_{\mathcal{C}_l} f_l = 0.$$

In particular

$$\|f_l\|_{C^{0,1}(\mathcal{C}_l)} \leq |\phi_l|.$$

By an integration by parts, we have

$$\int_{\mathcal{C}_l} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}) \wedge \phi_l = \sum_{\omega \subset \partial \mathcal{C}_l} \int_{\omega} \left(\mu_{\varepsilon, \omega} - 2\pi \sum_{i \in I_\omega} d_{i, \omega} \delta_{a_{i, \omega}} \right) f_l.$$

Here, we have used the notation introduced in Section 1.4 and the fact that the restriction of $\nu_{\varepsilon, \mathcal{C}_l}$ to each of the six faces ω of the cube \mathcal{C}_l is equal to $2\pi \sum_{i \in I_\omega} d_{i, \omega} \delta_{a_{i, \omega}}$. Corollary 1.4.1 then yields that

$$(1.28) \quad \left| \int_{\mathcal{C}_l} (\mu_\varepsilon(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}) \wedge \phi_l \right| \leq C_0 \sum_{\omega \subset \partial \mathcal{C}_l} \max(r_\omega, \varepsilon) \left(1 + \int_{\omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^2 + \int_{\partial \omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^1 \right) \|f_l\|_{C^{0,1}(\mathcal{C}_l)}.$$

where throughout the proof C_0 denotes a universal constant that may change from line to line. Using (1.27) and (1.28), we deduce that

$$\begin{aligned} \|\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}\|_{C^{0,1}(\mathcal{C}_l)^*} &\leq \delta \|\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \mathcal{C}_l}\|_{C^0(\mathcal{C}_l)^*} \\ &\quad + C_0 \sum_{\omega \subset \partial \mathcal{C}_l} \max(r_\omega, \varepsilon) \left(1 + \int_{\omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^2 + \int_{\partial \omega} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^1 \right) \end{aligned}$$

for any cube $\mathcal{C}_l \in \mathfrak{G}(b_\varepsilon, R_0, \delta)$. Then by summing over the cubes of the grid, we obtain

$$\begin{aligned} \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^{0,1}(\Omega \setminus \Theta)^*} &\leq \delta \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^0(\Omega \setminus \Theta)^*} \\ &\quad + C_0 \max(r_\mathfrak{G}, \varepsilon) \left(1 + 2 \int_{\mathfrak{R}_2(\mathfrak{G}(b_\varepsilon, R_0, \delta))} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^2 + 8 \int_{\mathfrak{R}_1(\mathfrak{G}(b_\varepsilon, R_0, \delta))} e_\varepsilon(u_\varepsilon, A_\varepsilon) d\mathcal{H}^1 \right). \end{aligned}$$

Using (1.4b), (1.4c), and (1.13), we find

$$(1.29) \quad \begin{aligned} \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^{0,1}(\Omega \setminus \Theta)^*} &\leq \delta \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^0(\Omega \setminus \Theta)^*} \\ &\quad + C_0 \max(\varepsilon \delta^{-1} F_\varepsilon(u_\varepsilon, A_\varepsilon), \varepsilon) (1 + \delta^{-2} F_\varepsilon(u_\varepsilon, A_\varepsilon)) \end{aligned}$$

We now provide an estimate for the last term in (1.25). Observe that if ε is sufficiently small and $\partial\Omega$ smooth enough then, for any $y \in \overline{\Theta}$, there exists a unique $x_y = \text{proj}_{\partial\Omega} y$ such that $y = x_y - t_y \nu(x_y)$, for some $t_y \geq 0$, where $\nu(x_y)$ is the outer unit normal to $\partial\Omega$ at x_y . We define $f : \Theta \rightarrow \mathbb{R}$ by

$$f(y) = f(x_y - t_y \nu(x)) = -t_y \phi(x_y) \cdot \nu(x_y).$$

By noting that, for any $y \in \Theta$,

$$\nabla f(y) = (\phi(x_y) \cdot \nu(x_y)) \nu(x_y) = \phi(x_y),$$

one can easily check that

$$\|f\|_{C^{0,1}(\Theta)} \leq \|\phi\|_{C^{0,1}(\Theta)} \quad \text{and} \quad \|\phi - \nabla f\|_{C^0(\Theta)} \leq \delta \|\phi\|_{C^{0,1}(\Theta)}.$$

We now write

$$\int_{\Theta} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}) \wedge \phi = \int_{\Theta} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}) \wedge (\phi - df) + \int_{\Theta} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}) \wedge df.$$

Observe that

$$\left| \int_{\Theta} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}) \wedge (\phi - df) \right| \leq \|\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}\|_{C^0(\Theta)^*} \|\phi - df\|_{C^0(\Theta)}.$$

On the other hand, by an integration by parts, we find

$$\int_{\Theta} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}) \wedge df = \sum_{\omega \subset \partial \mathfrak{G}} \int_{\omega} (\mu_{\varepsilon, \omega} - 2\pi \sum_{i \in I_\omega} d_{i, \omega} \delta_{a_{i, \omega}}) f.$$

Here, we have used the fact that the restriction of $\nu_{\varepsilon, \Theta}$ to each of the faces ω of a cube of the grid such that $\omega \subset \partial \mathfrak{G}$ is equal to $2\pi \sum_{i \in I_\omega} d_{i, \omega} \delta_{a_{i, \omega}}$. We then deduce that

$$\begin{aligned} \left| \int_{\Theta} (\mu(u_\varepsilon, A_\varepsilon) - \nu_{\varepsilon, \Theta}) \wedge \phi \right| &\leq \delta \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^0(\Theta)^*} \\ &\quad + C_0 \max(\varepsilon \delta^{-1} F_\varepsilon(u_\varepsilon, A_\varepsilon), \varepsilon) (1 + \delta^{-2} F_\varepsilon(u_\varepsilon, A_\varepsilon)) \|\phi\|_{C^{0,1}(\Theta)}. \end{aligned}$$

By combining this with (1.25) and (1.29), we find

$$\begin{aligned} \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C_T^{0,1}(\Omega)^*} &\leq \delta \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^0(\Omega)^*} \\ &\quad + C_0 \max(\varepsilon \delta^{-1} F_\varepsilon(u_\varepsilon, A_\varepsilon), \varepsilon) (1 + \delta^{-2} F_\varepsilon(u_\varepsilon, A_\varepsilon)). \end{aligned}$$

Observe now that

$$\|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^0(\Omega)^*} \leq \|\mu(u_\varepsilon, A_\varepsilon)\|_{C^0(\Omega)^*} + \|\nu_\varepsilon\|_{C^0(\Omega)^*}.$$

From (1.1), we deduce that

$$\|\nu_\varepsilon\|_{C^0(\Omega)^*} \leq C_0 \frac{F_\varepsilon(u_\varepsilon, A_\varepsilon)}{|\log \varepsilon|}$$

By combining the previous two estimates with (1.24), we get

$$(1.30) \quad \|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C^0(\Omega)^*} \leq CF_\varepsilon(u_\varepsilon, A_\varepsilon),$$

where C is a constant depending only on $\partial\Omega$. This implies that

$$\|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C_T^{0,1}(\Omega)^*} \leq C\delta F_\varepsilon(u_\varepsilon, A_\varepsilon) + C_0 \max(\varepsilon\delta^{-1}F_\varepsilon(u_\varepsilon, A_\varepsilon), \varepsilon) (1 + \delta^{-2}F_\varepsilon(u_\varepsilon, A_\varepsilon)).$$

From this, (1.2) follows. \square

The proof of (1.2) for $\gamma \in (0, 1)$ uses the following simple interpolation fact, as in [JS02].

Lemma 1.8.2. *Assume μ is a Radon measure on Ω . Then for any $\gamma \in (0, 1)$,*

$$\|\mu\|_{C_0^{0,\gamma}(\Omega)^*} \leq \|\mu\|_{C_0^{0,1}(\Omega)^*}^{1-\gamma} \|\mu\|_{C_0^{0,\gamma}(\Omega)^*}^\gamma.$$

Proof of (1.2) for $\gamma \in (0, 1)$. Note that $\|\mu\|_{C_0^{0,\gamma}(\Omega)^*} \leq \|\mu\|_{C_T^{0,\gamma}(\Omega)^*}$ for any 1-current μ . By combining the previous lemma with (1.2) and (1.30), we are led to

$$\|\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon\|_{C_0^{0,\gamma}(\Omega)^*} \leq CF_\varepsilon(u_\varepsilon, A_\varepsilon)^{1-\gamma} \max(\delta^\gamma F_\varepsilon(u_\varepsilon, A_\varepsilon)^\gamma, \varepsilon^\gamma) \leq C\delta^\gamma F_\varepsilon(u_\varepsilon, A_\varepsilon)$$

for any $\gamma \in (0, 1)$, where $C > 0$ is a constant depending only on γ and $\partial\Omega$. The proof then reduces to proving that this estimate is still valid when we replace the norm $\|\cdot\|_{C_0^{0,\gamma}(\Omega)^*}$ with $\|\cdot\|_{C_T^{0,\gamma}(\Omega)^*}$. Arguing as in the proof of [JMS04, Proposition 3.1], we conclude that (1.2) holds for $\gamma \in (0, 1)$. \square

Next, we give the proof of Theorem 1.1.2.

Proof of Theorem 1.1.2. The only part of the proof that needs a modification is the vorticity estimate close to the boundary. Since we work in the space $C_0^{0,1}(\Omega)^*$, we can use the fact that for a test 1-form $\phi \in C_0^{0,1}(\Omega)$ one has

$$\|\phi\|_{C_0^{0,1}(\Omega \setminus \Omega_\varepsilon)} \leq C\delta,$$

where δ is defined as above and C is a universal constant. Therefore

$$\left| \int_{\Omega \setminus \Omega_\varepsilon} (\mu(u_\varepsilon, A_\varepsilon) - \nu_\varepsilon) \wedge \phi \right| \leq CF_\varepsilon(u_\varepsilon, A_\varepsilon)\delta.$$

The rest of the proof follows as above. \square

1.A Smooth approximation of the function ζ

The main goal of this section of the appendix is to prove Proposition 1.5.1. We begin by proving some basic geometric properties.

Lemma 1.A.1. *Consider four points $x_1, x_2, x_3, x_4 \in \mathbb{R}^3$ with $x_1 \neq x_2$ and a number*

$$D \geq \max_{1 \leq i < j \leq 4} |x_i - x_j|.$$

There exists $\vartheta_1 > 0$ independent of D such that for any $0 < \vartheta < \vartheta_1$ there exists a point $x'_4 \in \mathbb{R}^3$ such that

$$(1.31) \quad \left| \frac{x_2 - x_1}{|x_2 - x_1|} \times \frac{x'_4 - x_3}{|x'_4 - x_3|} \right| \geq \vartheta$$

and

$$(1.32) \quad |x_4 - x'_4| \leq 3D\vartheta, \quad \max_{1 \leq i \leq 3} |x_i - x'_4| \leq D.$$

By translating the points it is enough to prove the following lemma.

Lemma 1.A.2. *Consider three points $x_1, x_2, x_3 \in \mathbb{R}^3$ with $x_1 \neq x_2$ and a number*

$$D \geq \max_{1 \leq i < j \leq 3} |x_i - x_j|.$$

There exists $\vartheta_2 > 0$ independent of D such that for any $0 < \vartheta < \vartheta_2$ there exists a point $x'_3 \in \mathbb{R}^3$ such that

$$\left| \frac{x_2 - x_1}{|x_2 - x_1|} \times \frac{x'_3 - x_1}{|x'_3 - x_1|} \right|, \left| \frac{x_2 - x_1}{|x_2 - x_1|} \times \frac{x'_3 - x_2}{|x'_3 - x_2|} \right| \geq \vartheta$$

and

$$|x_3 - x'_3| \leq 3D\vartheta, \quad \max_{1 \leq i \leq 2} |x_i - x'_3| \leq D.$$

Proof. Let us consider the cylinder whose axis is $\{tx_1 + (1-t)x_2 \mid t \in [-D, D]\}$ and whose radius is $r = \frac{\vartheta\sqrt{2}D}{\sqrt{(1-\vartheta^2)}}$. Note that $r < 2D\vartheta$ for $\vartheta \leq 1 - 2^{-1/2}$. Remembering that $|u \times v| = |u||v|\sin\theta$, where θ is the angle formed by u and v , it is easy to check that the cylinder previously defined contains all the points $y \in \mathbb{R}^3$ with $|x_1 - y|, |x_2 - y| \leq D$ such that

$$\left| \frac{x_2 - x_1}{|x_2 - x_1|} \times \frac{y - x_1}{|y - x_1|} \right| < \vartheta \quad \text{or} \quad \left| \frac{x_2 - x_1}{|x_2 - x_1|} \times \frac{y - x_2}{|y - x_2|} \right| < \vartheta.$$

Then simple trigonometric manipulations show that there exists a point $x'_3 \in \mathbb{R}^3$ such that

$$|x_3 - x'_3| \leq r + \frac{r^2}{D} \leq 3D\vartheta, \quad \max_{1 \leq i \leq 2} |x_i - x'_3| \leq D,$$

and

$$\left| \frac{x_2 - x_1}{|x_2 - x_1|} \times \frac{x'_3 - x_1}{|x'_3 - x_1|} \right|, \left| \frac{x_2 - x_1}{|x_2 - x_1|} \times \frac{x'_3 - x_2}{|x'_3 - x_2|} \right| \geq \vartheta$$

for any ϑ small enough. □

Lemma 1.A.3. Consider six points $x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{R}^3$ and a number

$$D \geq \max_{1 \leq i < j \leq 6} |x_i - x_j|.$$

There exists $\vartheta_3 > 0$ independent of D such that for any $0 < \vartheta < \vartheta_3$, if

$$\left| \frac{x_2 - x_1}{|x_2 - x_1|} \times \frac{x_4 - x_3}{|x_4 - x_3|} \right| \geq \vartheta$$

then there exists a point $x'_6 \in \mathbb{R}^3$ such that

$$(1.33) \quad \left| \det \left(\frac{x_2 - x_1}{|x_2 - x_1|}, \frac{x_4 - x_3}{|x_4 - x_3|}, \frac{x'_6 - x_5}{|x'_6 - x_5|} \right) \right| \geq \vartheta^2$$

and

$$(1.34) \quad |x_6 - x'_6| \leq 3D\vartheta, \quad \max_{1 \leq i \leq 5} |x_i - x'_6| \leq D.$$

By translating the points it is enough to prove the following lemma.

Lemma 1.A.4. Consider four points $x_1, x_2, x_3, x_4 \in \mathbb{R}^3$ and a number

$$D \geq \max_{1 \leq i < j \leq 4} |x_i - x_j|.$$

There exists $\vartheta_4 > 0$ independent of D such that for any $0 < \vartheta < \vartheta_4$, if

$$\left| \frac{x_2 - x_1}{|x_2 - x_1|} \times \frac{x_3 - x_1}{|x_3 - x_1|} \right| \geq \vartheta$$

then there exists a point $x'_4 \in \mathbb{R}^3$ such that

$$\left| \det \left(\frac{x_2 - x_1}{|x_2 - x_1|}, \frac{x_3 - x_1}{|x_3 - x_1|}, \frac{x'_4 - x_1}{|x'_4 - x_1|} \right) \right| \geq \vartheta^2$$

and

$$|x_4 - x'_4| \leq 3D\vartheta, \quad \max_{1 \leq i \leq 3} |x_i - x'_4| \leq D.$$

Proof. Let P denote the plane where the points x_1, x_2, x_3 are contained. Given any point $y \in \mathbb{R}^3 \setminus P$, observe that

$$\left| \det \left(\frac{x_2 - x_1}{|x_2 - x_1|}, \frac{x_3 - x_1}{|x_3 - x_1|}, \frac{y - x_1}{|y - x_1|} \right) \right| = h \cdot \left| \frac{x_2 - x_1}{|x_2 - x_1|} \times \frac{x_3 - x_1}{|x_3 - x_1|} \right| \geq h\vartheta,$$

where h denotes the height of the parallelepiped formed by the vectors

$$\frac{x_2 - x_1}{|x_2 - x_1|}, \frac{x_3 - x_1}{|x_3 - x_1|}, \frac{y - x_1}{|y - x_1|}.$$

It is easy to check that

$$h = \frac{\text{dist}(y, P)}{|y - x_1|}.$$

Choosing $h = \vartheta$ we are left with finding a point x'_4 such that $\vartheta|x'_4 - x_1| = \text{dist}(y, P)$. Then simple trigonometric manipulations show that there exists a point x'_4 such that

$$|x_4 - x'_4| \leq 3D\vartheta, \quad \max_{1 \leq i \leq 3} |x_i - x'_4| \leq D, \quad \text{and} \quad \vartheta|x'_4 - x_1| = \text{dist}(x'_4, P),$$

for any ϑ small enough. □

The previous lemmas allow us to prove the following result.

Proposition 1.A.1. *Let $\mathcal{A} = \{a_1, \dots, a_m\}$ be a collection of m non necessarily distinct points. Define $D_{\mathcal{A}} := \max_{1 \leq i < j \leq m} |a_i - a_j|$ to be the maximum euclidean distance between any of the points of \mathcal{A} and assume that $D_{\mathcal{A}} > 0$. Then there exists a collection of points $\mathcal{A}' = \{b_1, \dots, b_m\}$ such that for any $\vartheta < \min\{m^{-6}, \vartheta_1, \vartheta_3\}$, where the numbers ϑ_1, ϑ_3 are the constants appearing in Lemma 1.A.1 and Lemma 1.A.3 respectively, the following hold*

1. $b_i \neq b_j$ for any $i \neq j$.
2. Define

$$\nu_{(i,j)} := \frac{b_i - b_j}{|b_i - b_j|} \quad \text{for } (i, j) \in \Lambda_m := \{(p, q) \mid 1 \leq p < q \leq m\}.$$

Then for any $\alpha, \beta, \gamma \in \Lambda_m$ with $\alpha \neq \beta \neq \gamma$, we have

$$|\nu_\alpha \times \nu_\beta| \geq \vartheta \quad \text{and} \quad |\det(\nu_\alpha, \nu_\beta, \nu_\gamma)| \geq \vartheta^2.$$

3. $|a_l - b_l| \leq CD_{\mathcal{A}}l^5\vartheta$ for any $l \in \{1, \dots, m\}$, where C is a universal constant.
4. $\max_{1 \leq i < j \leq m} |b_i - b_j| \leq CD_{\mathcal{A}}$, where C is a universal constant.

Proof. We proceed by induction. Without loss of generality we may assume that $a_1 \neq a_2$. We define $b_1 = a_1, b_2 = a_2, d_1 = d_2 = 0$ and $D_1 = D_2 = D_{\mathcal{A}}$.

Assume that we have defined a collection $\{b_1, \dots, b_l\}$ with $2 < l < m$ such that

- For any $\alpha, \beta, \gamma \in \Lambda_l$ with $\alpha \neq \beta \neq \gamma$, we have

$$|\nu_\alpha \times \nu_\beta| \geq \vartheta \quad \text{and} \quad |\det(\nu_\alpha, \nu_\beta, \nu_\gamma)| \geq \vartheta^2.$$

- $|a_i - b_i| \leq d_l$ for any $i \in \{1, \dots, l\}$.
- $|b_i - b_j| \leq D_l$ for any $i, j \in \{1, \dots, l\}$.

Observe that by applying Lemma 1.A.1 with the points $x_1 = b_1, x_2 = b_2, x_3 = b_1, x_4 = a_{l+1}$, and the number $D = D_l + d_l$, we find a point x'_4 satisfying (1.31) and (1.32). By repeating this argument at most l^3 times, we find a point b'_{l+1} such that the collection $\{b_1, \dots, b_l, b'_{l+1}\}$ satisfies

$$|\nu_\alpha \times \nu_\beta| \geq \vartheta$$

for any $\alpha, \beta \in \Lambda_{l+1}$ with $\alpha \neq \beta$. Moreover

$$|a_{l+1} - b'_{l+1}| \leq 3l^3(D_l + d_l)\vartheta \quad \text{and} \quad |b_i - b'_{l+1}| \leq D_l + d_l \quad \text{for any } i \in \{1, \dots, l\}.$$

We further displace the point b'_{l+1} in order to additionally satisfy the condition on the determinants. Applying Lemma 1.A.3 with $x_1 = b_1, x_2 = b_2, x_3 = b_1, x_4 = b_3, x_5 = b_1, x_6 = b'_{l+1}$ and $D = D_l + d_l$, we find a point x'_6 satisfying (1.33) and (1.34). By repeating this argument at most l^5 times, we find a point b_{l+1} such that the collection $\{b_1, \dots, b_l, b_{l+1}\}$ satisfies

$$|\nu_\alpha \times \nu_\beta| \geq \vartheta \quad \text{and} \quad |\det(\nu_\alpha, \nu_\beta, \nu_\gamma)| \geq \vartheta^2$$

for any $\alpha, \beta, \gamma \in \Lambda_{l+1}$ with $\alpha \neq \beta \neq \gamma$. Moreover

$$|a_{l+1} - b_{l+1}| \leq 3(l^3 + l^5)(D_l + d_l)\vartheta \quad \text{and} \quad |b_i - b_{l+1}| \leq D_l + d_l \quad \text{for any } i \in \{1, \dots, l\}.$$

Summarizing, the collection $\{b_1, \dots, b_l, b_{l+1}\}$ satisfies

- For any $\alpha, \beta, \gamma \in \Lambda_{l+1}$ with $\alpha \neq \beta \neq \gamma$, we have

$$|\nu_\alpha \times \nu_\beta| \geq \vartheta \quad \text{and} \quad |\det(\nu_\alpha, \nu_\beta, \nu_\gamma)| \geq \vartheta^2.$$

- $|a_i - b_i| \leq d_{l+1} := 6l^5(D_l + d_l)\vartheta$ for any $i \in \{1, \dots, l+1\}$.
- $|b_i - b_j| \leq D_{l+1} := D_l + d_l$ for any $i, j \in \{1, \dots, l+1\}$.

This concludes the induction step. It only remains to find upper bounds for the recursively defined distances d_l and D_l . Observe that

$$D_{l+1} = D_l + d_l \leq D_l + 6(l-1)^5(D_{l-1} + d_{l-1})\vartheta \leq D_l(1 + 6l^5\vartheta), \quad D_1 = D_2 = D_{\mathcal{A}}.$$

We immediately check that if $\vartheta \leq m^{-6}$ then, for any $l \in \{1, \dots, m\}$,

$$D_l \leq D_m \leq D_{\mathcal{A}}(1 + 6m^5\vartheta)^m \leq D_{\mathcal{A}} \left(1 + \frac{6}{m}\right)^m \leq CD_{\mathcal{A}},$$

where C is a universal constant. Moreover if $\vartheta \leq m^{-6}$ then, for any $l \in \{1, \dots, m\}$,

$$d_l \leq 6l^5 D_{l-1} \vartheta \leq 6CD_{\mathcal{A}} l^5 \vartheta.$$

□

We are now in position to prove Proposition 1.5.1.

Proof of Proposition 1.5.1. Let $\mathcal{A}' = \{p'_1, \dots, p'_k, n'_1, \dots, n'_k\}$ be the collection of points given by Proposition 1.A.1 for $D = D_{\mathcal{A}}$. Observe that for any $\vartheta \leq C_0(2k)^{-6}$, where $C_0 = \min(1, \vartheta_1, \vartheta_3)$, we have

$$L(\mathcal{A}') \leq \sum_{i=1}^k |p'_i - n'_i| \leq \sum_{i=1}^k |p_i - n_i| + |p_i - p'_i| + |n_i - n'_i| \leq L(\mathcal{A}) + CD_{\mathcal{A}}(2k)^6\vartheta.$$

An analogous argument shows that $L(\mathcal{A}) \leq L(\mathcal{A}') + CD_{\mathcal{A}}(2k)^6\vartheta$. Therefore

$$(1.35) \quad |L(\mathcal{A}) - L(\mathcal{A}')| \leq CD_{\mathcal{A}}(2k)^6\vartheta,$$

where throughout the proof $C > 0$ denotes a universal constant that may change from line to line. Remember that by Lemma 1.5.1 there exists a 1-Lipschitz function $\zeta^* : \cup_{i=1,\dots,k}\{p'_i, n'_i\} \rightarrow \mathbb{R}$ such that

$$L(\mathcal{A}') = \sum_{i=1}^k \zeta^*(p'_i) - \zeta^*(n'_i).$$

Define the function ζ as in Definition 1.5.1, i.e. set

$$\zeta(x) := \max_{i \in \{1, \dots, k\}} \left(\zeta^*(p'_i) - \max_{j \in \{1, \dots, 2k\}} d_{(i,j)}(x) \right),$$

where

$$d_{(i,j)}(x) := \langle p'_i - x, \nu_{(i,j)} \rangle, \quad \nu_{(i,j)} = \begin{cases} \frac{p'_i - a'_j}{|p'_i - a'_j|} & \text{if } p'_i \neq a'_j \\ 0 & \text{if } p'_i = a'_j \end{cases}.$$

Lemma 1.5.2 yields that $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a 1-Lipschitz function such that

$$\sum_{i=1}^k \zeta(p'_i) - \zeta(n'_i) = \sum_{i=1}^k \zeta^*(p'_i) - \zeta^*(n'_i) = L(\mathcal{A}').$$

Next, we regularize the function ζ . Let $\varphi \in C_c^\infty(B(0,1), \mathbb{R}_+)$ be a mollifier such that $\int_{\mathbb{R}^3} \varphi(x) dx = 1$. Letting

$$(1.36) \quad \lambda := \vartheta^{1/\rho} \quad \text{for } \rho > 0,$$

we define

$$\zeta_\lambda(\cdot) := \varphi_\lambda * \zeta(\cdot) = \int_{\mathbb{R}^3} \varphi_\lambda(\cdot - y) \zeta(y) dy \quad \text{with } \varphi_\lambda(\cdot) = \frac{1}{\lambda} \varphi\left(\frac{\cdot}{\lambda}\right).$$

First, observe that $\|\zeta - \zeta_\lambda\|_{L^\infty(\mathbb{R}^3)} \leq \lambda$ from which we deduce that

$$(1.37) \quad \left| L(\mathcal{A}') - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq 2k\lambda.$$

By combining (1.35) with (1.37), we obtain

$$(1.38) \quad \left| L(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq CD_{\mathcal{A}}(2k)^6\vartheta + 2k\lambda.$$

On the other hand, note that

$$\left| \sum_{i=1}^k \zeta(p_i) - \zeta(n_i) - \sum_{i=1}^k \zeta(p'_i) - \zeta(n'_i) \right| \leq \sum_{i=1}^k |p_i - n_i| + |p'_i - n'_i| \leq CD_{\mathcal{A}}(2k)^6\vartheta.$$

By combining the previous estimate with (1.37), we get

$$(1.39) \quad \left| \sum_{i=1}^k \zeta(p_i) - \zeta(n_i) - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq CD_{\mathcal{A}}(2k)^6 \vartheta + 2k\lambda.$$

Then by (1.38) and (1.39), we deduce that

$$\left| L(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p_i) - \zeta_\lambda(n_i) \right| \leq C(D_{\mathcal{A}}(2k)^6 \lambda^\rho + 2k\lambda).$$

Second, note that

$$(1.40) \quad \nabla \zeta_\lambda(x) = \int_{B(x,\lambda)} \varphi_\lambda(x-y) \nabla \zeta(y) dy$$

for any $x \in \mathbb{R}^3$. We define

$$\Lambda := \{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq 2k, i \neq j\}.$$

Then, letting

$$\zeta_{(i,j)}(\cdot) := \zeta(p'_i) - d_{(i,j)}(\cdot) \quad \text{for } (i, j) \in \Lambda,$$

observe that, for almost every $y \in \mathbb{R}^3$,

$$\nabla \zeta(y) = \nu_{(i,j)} \quad \text{if } \zeta(y) = \zeta_{(i,j)}(y) \text{ for some } (i, j) \in \Lambda.$$

Since $|\nu_{(i,j)}| = 1$ for any $(i, j) \in \Lambda$, we have

$$|\nabla \zeta_\lambda(x)| \leq \int_{B(x,\lambda)} \varphi_\lambda(x-y) |\nabla \zeta(y)| dy \leq \int_{B(x,\lambda)} \varphi_\lambda(x-y) dy = 1$$

for any $x \in \mathbb{R}^3$.

We now analyze the set of points whose gradient is small in modulus. From (1.40), we deduce that

$$\nabla \zeta_\lambda(x) = \sum_{\alpha \in \Lambda} \sigma_\alpha \nu_\alpha, \quad \text{where } \sigma_\alpha = \int_{B(x,\lambda)} \varphi_\lambda(x-y) \mathbf{1}_{\{\zeta(y) = \zeta_\alpha(y)\}} dy \quad \text{for } \alpha \in \Lambda.$$

Observe that $\sigma_\alpha \in [0, 1]$ and that $\sum_{\alpha \in \Lambda} \sigma_\alpha = 1$. We conclude that, for any $x \in \mathbb{R}^3$, $\nabla \zeta_\lambda(x)$ is a convex combination of the vectors ν_α 's, $\alpha \in \Lambda$. By Caratheodory's theorem, we deduce that $\nabla \zeta_\lambda(x)$ is a convex combination of at most four of them.

Let us consider indices $i, j \in \{1, \dots, k\}$ with $i \neq j$. We let

$$P_{i,j} := \{y \in \mathbb{R}^3 \mid \zeta_{(i,j)}(y) = \zeta_{(j,i)}(y)\}$$

and observe that

$$P_{i,j} = \{\zeta(p'_i) - \zeta(p'_j) - \langle p'_i + p'_j - 2y, \nu_{(i,j)} \rangle = 0\}.$$

A simple computation shows that

$$\langle y_1 - y_2, \nu_{(i,j)} \rangle = 0$$

for any $y_1, y_2 \in P_{i,j}$ with $y_1 \neq y_2$. This implies that $P_{i,j}$ is a plane whose normal is $\nu_{(i,j)}$ and therefore

$$\zeta_{(i,j)}(y) = \zeta_{(j,i)}(y) = \frac{\zeta^*(p'_i) + \zeta^*(p'_j) - \langle p'_i + p'_j, \nu_{(i,j)} \rangle}{2}$$

for any $y \in P_{i,j}$. We define

$$P_\lambda := \{y \in \mathbb{R}^3 \mid d(y, P) \leq 2\lambda\}, \quad \text{where } P := \cup_{1 \leq i < j \leq k} P_{i,j}.$$

We immediately check that $|\zeta_\lambda(P_\lambda)| \leq C\lambda k^2$.

Consider a number $\kappa < \vartheta^2/3$ and a point

$$x \in \{y \in \mathbb{R}^3 \mid |\nabla \zeta_\lambda(y)| < \kappa\} \setminus P_\lambda.$$

We observe that, since $x \notin P_\lambda$, if there exists a point $y \in \overline{B(x, \lambda)}$ and indices $i, j \in \{1, \dots, k\}$ with $i \neq j$ such that

$$\zeta(y) = \zeta_{(i,j)}(y)$$

then, for any $z \in \overline{B(x, \lambda)}$,

$$\zeta(z) \neq \zeta_{(j,i)}(z)$$

This implies that $\nabla \zeta_\lambda(x)$ is a convex combination of at most four vectors, where if one of them happens to be $\nu_{(i,j)}$ for some $i, j \in \{1, \dots, k\}$ with $i \neq j$ then all the other vectors are different from $\nu_{(j,i)} = -\nu_{(i,j)}$. Recalling that the points of the collection \mathcal{A}' are such that

$$|\nu_\alpha \times \nu_\beta| > \vartheta \quad \text{and} \quad |\det(\nu_\alpha, \nu_\beta, \nu_\gamma)| > \vartheta^2$$

for any $\alpha, \beta, \gamma \in \{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq 2k, i < j\} \subsetneq \Lambda$ with $\alpha \neq \beta \neq \gamma$ we deduce that $\nabla \zeta_\lambda(x)$ is a convex combination of at most four vectors that satisfy the previous property.

Let us now show that $\nabla \zeta_\lambda(x)$ cannot be a convex combination of three or fewer of the vectors ν_α 's, $\alpha \in \Lambda$. We have three cases to consider:

- If there exists $\alpha \in \Lambda$ such that $\nabla \zeta_\lambda = \nu_\alpha$ in $B(x, \lambda)$ then

$$|\nabla \zeta_\lambda(x)| = |\nu_\alpha| = 1 \text{ in } B(x, \lambda).$$

- If there exist $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ such that

$$\nabla \zeta_\lambda(x) = \sigma \nu_\alpha + (1 - \sigma) \nu_\beta \text{ in } B(x, \lambda),$$

for some $\sigma \in (0, 1)$, then

$$\begin{aligned} |\nabla \zeta_\lambda(x)| &= |\sigma \nu_\alpha + (1 - \sigma) \nu_\beta| \\ &\geq \max\{|\nabla \zeta_\lambda(x) \times \nu_\alpha|, |\nabla \zeta_\lambda(x) \times \nu_\beta|\} \\ &= \max\{(1 - \sigma)|\nu_\alpha \times \nu_\beta|, \sigma|\nu_\alpha \times \nu_\beta|\} \\ &\geq \max\{\sigma, 1 - \sigma\} \vartheta \geq \frac{\vartheta}{2}. \end{aligned}$$

- If there exist $\alpha, \beta, \gamma \in \Lambda$ with $\alpha \neq \beta \neq \gamma$ such that

$$\nabla\zeta_\lambda(x) = \sigma_\alpha\nu_\alpha + \sigma_\beta\nu_\beta + \sigma_\gamma\nu_\gamma \text{ in } B(x, \lambda),$$

for some numbers $\sigma_\alpha, \sigma_\beta, \sigma_\gamma \in (0, 1)$ with $\sigma_\alpha + \sigma_\beta + \sigma_\gamma = 1$, then, assuming without loss of generality that $\sigma_\alpha \geq \frac{1}{3}$, we have

$$|\nabla\zeta_\lambda(x)| \geq \sigma_\alpha |\nu_\alpha \cdot (\nu_\beta \times \nu_\gamma)| = \sigma_\alpha |\det(\nu_\alpha, \nu_\beta, \nu_\gamma)| \geq \frac{\vartheta^2}{3}.$$

Since $\kappa < \frac{\vartheta^2}{3}$ we deduce that the three cases considered above cannot occur. Therefore we conclude that there exist $\alpha, \beta, \gamma, \eta \in \Lambda$ with $\alpha \neq \beta \neq \gamma \neq \eta$ such that

$$\nabla\zeta_\lambda(x) = \sigma_\alpha\nu_\alpha + \sigma_\beta\nu_\beta + \sigma_\gamma\nu_\gamma + \sigma_\eta\nu_\eta \text{ in } B(x, \lambda),$$

for some $\sigma_\alpha, \sigma_\beta, \sigma_\gamma, \sigma_\eta \in (0, 1)$ with $\sigma_\alpha + \sigma_\beta + \sigma_\gamma + \sigma_\eta = 1$.

Let us solve consider the system of equations

$$(1.41) \quad \zeta_\alpha(y) = \zeta_\beta(y) = \zeta_\gamma(y) = \zeta_\eta(y).$$

We claim that this system admits a unique solution $y \in \mathbb{R}^3$ which in addition satisfies

$$|x - y| \leq \frac{C\lambda}{(\vartheta^2 - 3\kappa)}.$$

Writing $\tilde{y} = y - x$, we observe that \tilde{y} satisfies the linear system of equations $A\tilde{y} = B$, where

$$A = \begin{pmatrix} \nu_\alpha - \nu_\beta \\ \nu_\gamma - \nu_\beta \\ \nu_\eta - \nu_\beta \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \zeta_\alpha(x) - \zeta_\beta(x) \\ \zeta_\gamma(x) - \zeta_\beta(x) \\ \zeta_\eta(x) - \zeta_\beta(x) \end{pmatrix}.$$

Let us check that $|\det(A)| \geq 4(\vartheta^2 - 3\kappa)$. Note that without loss of generality we can assume that $\sigma_\alpha \leq \frac{1}{4}$. Observe that

$$\nabla\zeta_\lambda(x) - \nu_\beta = \sigma_\alpha(\nu_\alpha - \nu_\beta) + \sigma_\gamma(\nu_\gamma - \nu_\beta) + \sigma_\eta(\nu_\eta - \nu_\beta).$$

By Cramer's rule, we have

$$\sigma_\alpha = \frac{\det(\nabla\zeta_\lambda(x) - \nu_\beta, \nu_\gamma - \nu_\beta, \nu_\eta - \nu_\beta)}{\det(\nu_\alpha - \nu_\beta, \nu_\gamma - \nu_\beta, \nu_\eta - \nu_\beta)}.$$

Simple computations show that

$$\det(\nabla\zeta_\lambda(x) - \nu_\beta, \nu_\gamma - \nu_\beta, \nu_\eta - \nu_\beta) = -\det(\nu_\beta, \nu_\gamma, \nu_\eta) + f(\nabla\zeta_\lambda(x)),$$

where $|f(\nabla\zeta_\lambda(x))| \leq 3|\nabla\zeta_\lambda(x)| \leq 3\kappa$. Therefore

$$|\det(\nabla\zeta_\lambda(x) - \nu_\beta, \nu_\gamma - \nu_\beta, \nu_\eta - \nu_\beta)| \geq |\det(\nu_\beta, \nu_\gamma, \nu_\eta)| - |f(\nabla\zeta_\lambda(x))| \geq \vartheta^2 - 3\kappa.$$

We deduce that

$$|\det(\nu_\alpha - \nu_\beta, \nu_\gamma - \nu_\beta, \nu_\eta - \nu_\beta)| = \frac{|\det(\nabla\zeta_\lambda(x) - \nu_\beta, \nu_\gamma - \nu_\beta, \nu_\eta - \nu_\beta)|}{\sigma_\alpha} \geq 4(\vartheta^2 - 3\kappa).$$

On the other hand, note that there exist $x_\alpha, x_\beta, x_\gamma, x_\eta$ in $B(x, \lambda)$ such that

$$\zeta(x_\alpha) = \zeta_\alpha(x_\alpha), \zeta(x_\beta) = \zeta_\beta(x_\beta), \zeta(x_\gamma) = \zeta_\gamma(x_\gamma), \zeta(x_\eta) = \zeta_\eta(x_\eta).$$

Since

$$B = \begin{pmatrix} \zeta_\alpha(x) - \zeta_\alpha(x_\alpha) + \zeta(x_\alpha) - \zeta(x_\beta) + \zeta_\beta(x_\beta) - \zeta_\beta(x) \\ \zeta_\gamma(x) - \zeta_\gamma(x_\gamma) + \zeta(x_\gamma) - \zeta(x_\beta) + \zeta_\beta(x_\beta) - \zeta_\beta(x) \\ \zeta_\eta(x) - \zeta_\eta(x_\eta) + \zeta(x_\eta) - \zeta(x_\beta) + \zeta_\beta(x_\beta) - \zeta_\beta(x) \end{pmatrix},$$

we deduce that $|B| \leq 3\lambda$. Hence the linear system of equations $A\tilde{y} = B$ admits a unique solution which satisfies

$$|\tilde{y}| = |y - x| = |A^{-1}B| \leq \frac{C\lambda}{\vartheta^2 - 3\kappa}.$$

Summarizing, if $x \in \{y \in \mathbb{R}^3 \mid |\nabla\zeta_\lambda(y)| < \kappa\} \setminus P_\lambda$ with $\kappa < \vartheta^2/3$ then there exist $\alpha, \beta, \gamma, \eta \in \Lambda$ with $\alpha \neq \beta \neq \gamma \neq \eta$ such that the unique solution $y \in \mathbb{R}^3$ to (1.41) lies in the ball $B(x, C\lambda/(\vartheta^2 - 3\kappa))$. We conclude that the set

$$C_\kappa = \{x \mid |\nabla\zeta_\lambda(x)| < \kappa\} \setminus P_\lambda$$

can be covered by \mathfrak{B}_κ , a collection of at most $\binom{|\Lambda|}{4} \leq (2k)^8$ balls of radius $C\lambda/(\vartheta^2 - 3\kappa)$. Observing that

$$|D^2\zeta_\lambda(x)| \leq \frac{C}{\lambda^2}$$

for any $x \in \mathbb{R}^3$, and letting

$$T_\kappa = \zeta_\lambda(\cup_{B \in \mathfrak{B}_\kappa} B),$$

we deduce that, for any $t \in \mathbb{R} \setminus (T_\kappa \cup \zeta_\lambda(P_\lambda))$, $\{x \mid \zeta_\lambda(x) = t\}$ is a complete submanifold of \mathbb{R}^3 whose second fundamental form is bounded by

$$C \frac{\sup_{\mathbb{R}^3} |D^2\zeta_\lambda|}{\inf_{\mathbb{R}^3 \setminus ((\cup_{B \in \mathfrak{B}_\kappa} B) \cup P_\lambda)} |\nabla\zeta_\lambda|} \leq \frac{C}{\lambda^2 \kappa}.$$

Recalling the relation between λ and ϑ (see (1.36)), the proposition follows. \square

1.B Smooth approximation of the function ζ for $d_{\partial\Omega}$

The chief goal of this section of the appendix is to prove Proposition 1.5.2. First, let us denote by $\mathcal{X} = \{x_1, \dots, x_n\}$ a collection of points belonging to $\partial\Omega$ such that

$$(1.42) \quad \frac{3}{2}\tau \leq \min_{1 \leq i < j \leq n} \mathfrak{d}(x_i, x_j) \quad \text{and} \quad \max_{1 \leq i \leq n} \mathfrak{d}(z, x_i) \leq \frac{5}{2}\tau \quad \text{for any } z \in \partial\Omega,$$

where from now on \mathfrak{d} denotes the geodesic distance on $\partial\Omega$ and $\tau > 0$ is a given number. For any $x_i \in \mathcal{X}$ let us denote by $\nu(x_i)$ the outer unit normal to $\partial\Omega$. Define

$$\Omega_{\mathcal{X}} := \bigcap_{1 \leq i \leq n} \{z \mid \langle z - x_i, \nu(x_i) \rangle < 0\}.$$

It is easy to see that $\partial\Omega_{\mathcal{X}}$ is a polyhedral approximation of $\partial\Omega$ which in addition is convex if Ω is convex.

In the next lemma we show that the points of the collection \mathcal{X} can be displaced in order to make the normals $\nu(x_i)$'s, $x_i \in \mathcal{X}$ satisfy extra conditions, when Ω is assumed to be a C^2 bounded domain such that $\partial\Omega$ has strictly positive Gauss curvature.

Lemma 1.B.1. *Let $\Omega \subset \mathbb{R}^3$ be a C^2 bounded domain such that $\partial\Omega$ has strictly positive Gauss curvature at every point and let $\mathcal{X} = \{x_1, \dots, x_n\}$ be a collection of points belonging to $\partial\Omega$ satisfying (1.42) for a number $\tau > 0$. Then there exist constants $\tau_0, C_0, C > 0$ depending only on $\partial\Omega$, such that for any $0 < \tau < \tau_0$ there exists a collection $\mathcal{X}' = \{y_1, \dots, y_n\} \subset \partial\Omega$ such that for any $0 < \vartheta < C_0\tau^5$ the following hold*

1. $\tau \leq \min_{1 \leq i < j \leq n} \mathfrak{d}(y_i, y_j)$ and $\max_{1 \leq i \leq n} \mathfrak{d}(z, y_i) \leq 3\tau$ for any $z \in \partial\Omega$.

2. Letting

$$\Omega_{\mathcal{X}'} := \bigcap_{1 \leq i \leq n} \{z \mid \langle z - y_i, \nu(y_i) \rangle < 0\},$$

where $\nu(y_i)$ is the outer unit normal to $\partial\Omega$ at y_i , we have

$$|d(z, \partial\Omega_{\mathcal{X}'}) - d(z, \partial\Omega)| \leq C\tau^2$$

for any $z \in \mathbb{R}^3$.

3. For any $i, j, k \in \{1, \dots, n\}$ with $i \neq j \neq k$, we have

$$|\nu(y_i) \times \nu(y_j)| \geq \vartheta \quad \text{and} \quad |\det(\nu(y_i), \nu(y_j), \nu(y_k))| \geq \vartheta^2.$$

Proof. Since we assume that $\partial\Omega$ has strictly positive Gauss curvature at every point, we have that there exists a constant $K_0(\partial\Omega) > 0$ such that for any point $x \in \partial\Omega$ the minimal principal curvature of $\partial\Omega$ at x is bounded below by K_0 .

Simple geometric arguments show that there exist constants $C, \vartheta_0, R_0 > 0$ depending only on $\partial\Omega$, such that for any $x \in \partial\Omega$ and for any $0 < \vartheta < \vartheta_0$ if $v \in \mathbb{R}^3$ with $|v| = 1$ is such that

$$\theta(\nu(x), v) < \vartheta,$$

where $\theta(\nu(x), v)$ is the angle formed by $\nu(x)$ and v , then for any $y \in \partial\Omega$ satisfying

$$CK_0\vartheta \leq \mathfrak{d}(x, y) \leq R_0,$$

we have

$$\theta(\nu(y), v) \geq \vartheta.$$

We easily deduce that, up to an adjustment of the constants, for any $x \in \partial\Omega$ and for any $0 < \vartheta < \vartheta_0$ if $v \in \mathbb{R}^3$ with $|v| = 1$ is such that

$$|\nu(x) \times v| < \vartheta,$$

then for any $y \in \partial\Omega$ satisfying $CK_0\vartheta \leq \mathfrak{d}(x, y) \leq R_0$, we have

$$(1.43) \quad |\nu(y) \times v| \geq \vartheta.$$

Moreover, for any $x \in \partial\Omega$ and for any $0 < \vartheta < \vartheta_0$ if $v, w \in \mathbb{R}^3$ with $|v| = |w| = 1$ are such that

$$|v \times w| \geq \vartheta \quad \text{and} \quad |\det(\nu(x), v, w)| < \vartheta^2,$$

then for any $y \in \partial\Omega$ satisfying $CK_0\vartheta \leq \mathfrak{d}(x, y) \leq R_0$, we have

$$(1.44) \quad |\det(\nu(y), v, w)| \geq \vartheta^2.$$

Then, we proceed by induction. We define $y_1 = x_1$.

Assume that we have defined a collection $\{y_1, \dots, y_l\} \subset \partial\Omega$ with $1 < l < n$ such that for any $i, j, k \in \{1, \dots, l\}$ with $i \neq j \neq k$, we have

$$|\nu(y_i) \times \nu(y_j)| \geq \vartheta \quad \text{and} \quad |\det(\nu(y_i), \nu(y_j), \nu(y_k))| \geq \vartheta^2.$$

From our previous observations we deduce that there exists a point $y \in \partial\Omega$, such that $\mathfrak{d}(x_{l+1}, y) \leq CK_0\vartheta$, satisfying (1.43) for $v = \nu(y_1)$. By repeating this procedure at most l times, we find a point y'_{l+1} with

$$\mathfrak{d}(x_{l+1}, y'_{l+1}) \leq ClK_0\vartheta$$

such that, for any $i \in \{1, \dots, l\}$,

$$|\nu(y_i) \times \nu(y'_{l+1})| \geq \vartheta.$$

We further displace the point y'_{l+1} in order to additionally satisfy the condition on the determinants (when $l \geq 3$). Once again from our previous observations we deduce that there exists a point $y \in \partial\Omega$, such that $\mathfrak{d}(y'_{l+1}, y) \leq CK_0\vartheta$, satisfying (1.44) for $v = \nu(y_1)$ and $w = \nu(y_2)$. By repeating this procedure at most l^2 times, we find a point y_{l+1} with

$$\mathfrak{d}(y'_{l+1}, y_{l+1}) \leq Cl^2K_0\vartheta$$

such that the collection $\{y_1, \dots, y_{l+1}\} \subset \partial\Omega$ satisfies

$$|\nu(y_i) \times \nu(y_j)| \geq \vartheta \quad \text{and} \quad |\det(\nu(y_i), \nu(y_j), \nu(y_k))| \geq \vartheta^2$$

for any $i, j, k \in \{1, \dots, l+1\}$ with $i \neq j \neq k$. This concludes the induction step and the proof of the third assertion. Note that

$$\mathfrak{d}(x_l, y_l) \leq 2C(l-1)^2\vartheta \leq 2Cn^2\vartheta$$

for any $l \in \{1, \dots, n\}$. Observing that $n \leq C\tau^{-2}$ for a universal constant $C > 0$, we deduce that, for any $1 \leq l \leq n$,

$$\mathfrak{d}(x_l, y_l) \leq C\tau^{-4}\vartheta.$$

Therefore, if $C\tau^{-4}\vartheta \leq 1/2\tau$ then the first assertion is satisfied. Finally, geometric manipulations show the validity of the second assertion (see [Gru93, Theorem 4] for a proof). This concludes the proof. \square

With the aid of Lemmas 1.A.1, 1.A.3, and 1.B.1 we prove the following result.

Lemma 1.B.2. *Let $\Omega \subset \mathbb{R}^3$ be a C^2 bounded domain such that $\partial\Omega$ has strictly positive Gauss curvature at every point. Let $\mathcal{A} = \{a_1, \dots, a_m\} \subset \Omega$ be a collection of m non necessarily distinct points. Consider a collection $\mathcal{X} = \{x_1, \dots, x_n\} \subset \partial\Omega$ satisfying (1.42) and let $\mathcal{X}' = \{y_1, \dots, y_n\} \subset \partial\Omega$ denote the collection of points given by Lemma 1.B.1 for a number $\tau < \tau_0$, where τ_0 is the constant appearing in the lemma. Then there exist constant $C_0, C_1 > 0$ depending only on $\partial\Omega$, and a collection of points $\mathcal{A}' = \{b_1, \dots, b_m\}$ such that for any*

$$\vartheta < C_0 \min\{m^{-6}, m^{-4}\tau^2, m^{-2}\tau^4, \tau^5\}$$

the following hold

1. $b_i \neq b_j$ for any $i \neq j$.

2. Define

$$\nu_{(i,j)} := \frac{b_i - b_j}{|b_i - b_j|} \quad \text{for } (i,j) \in \Lambda_m := \{(p,q) \mid 1 \leq p < q \leq m\}$$

and

$$\mathcal{V} := \{\nu_{(i,j)} \mid (i,j) \in \Lambda_m\} \cup \{\nu(y_i) \mid y_i \in \mathcal{X}'\}.$$

Then for any $u, v, w \in \mathcal{V}$ with $u \neq v \neq w$, we have

$$|u \times v| \geq \vartheta \quad \text{and} \quad |\det(u, v, w)| \geq \vartheta^2.$$

3. $|a_l - b_l| \leq C_1(l^5 + l^3\tau^{-2} + l\tau^{-4})\vartheta$ for any $l \in \{1, \dots, m\}$.

Proof. Assume $\vartheta \leq C_0\tau^5$, where C_0 is the constant appearing in Lemma 1.B.1, so that for any $i, j, k \in \{1, \dots, n\}$ with $i \neq j \neq k$, we have

$$|\nu(y_i) \times \nu(y_j)| \geq \vartheta \quad \text{and} \quad |\det(\nu(y_i), \nu(y_j), \nu(y_k))| \geq \vartheta^2.$$

We proceed by induction. We define $b_1 = a_1$, $d_1 = 0$ and $D_1 = \text{diam}(\Omega) + 1$.

Assume that we have defined a collection $\{b_1, \dots, b_l\}$ with $2 \leq l < m$ such that

• Letting

$$\mathcal{V}_l = \{\nu_{(i,j)} \mid (i,j) \in \Lambda_l\} \cup \{\nu(y_i) \mid y_i \in \mathcal{X}'\},$$

then for any $u, v, w \in \mathcal{V}_l$ with $u \neq v \neq w$, we have

$$|u \times v| \geq \vartheta \quad \text{and} \quad |\det(u, v, w)| \geq \vartheta^2.$$

• $|a_i - b_i| \leq d_l$ for any $i \in \{1, \dots, l\}$.

• $|b_i - b_j| \leq D_l$ for any $i, j \in \{1, \dots, l\}$.

Using Lemma 1.A.1, Lemma 1.A.3, and arguing as in the proof of Proposition 1.A.1 we find a point b'_{l+1} such that the collection $\{b_1, \dots, b_l, b'_{l+1}\}$ satisfies:

- For any $\alpha, \beta, \gamma \in \Lambda_{l+1}$ with $\alpha \neq \beta \neq \gamma$, we have

$$|\nu_\alpha \times \nu_\beta| \geq \vartheta \quad \text{and} \quad |\det(\nu_\alpha, \nu_\beta, \nu_\gamma)| \geq \vartheta^2.$$

- $|a_i - b_i| \leq d_{l+1} = 6l^5(D_l + d_l)\vartheta$ for any $i \in \{1, \dots, l+1\}$.
- $|b_i - b_j| \leq D_{l+1} = D_l + d_l$ for any $i, j \in \{1, \dots, l+1\}$.

We displace the point b'_{l+1} in order to additionally satisfy the conditions involving the vectors of the collection \mathcal{X}' .

First, applying Lemma 1.A.1 with the points $x_1 = y_1$, $x_2 = y_1 + \nu(y_1)$, $x_3 = b_1$, $x_4 = b'_{l+1}$, and the number $D = D_l + d_l$, we find a point x'_4 satisfying (1.31) and (1.32). We recall that $n \leq C\tau^{-2}$, where throughout the proof C denotes a universal constant that may change from line to line. By repeating this argument at most $C\tau^{-2}l$ times, we find a point b''_{l+1} with

$$|a_{l+1} - b''_{l+1}| \leq (6l^5 + 3C\tau^{-2}l)(D_l + d_l)\vartheta$$

such that the collection $\{b_1, \dots, b_l, b''_{l+1}\}$, in addition to the previous properties, satisfies

$$|\nu_{(i,l+1)} \times \nu(y_j)| \geq \vartheta \quad \text{for any } i \in \{1, \dots, l\} \text{ and } j \in \{1, \dots, n\}.$$

When $l > 2$, we further displace the point b''_{l+1} . Applying Lemma 1.A.3 with the points $x_1 = y_1$, $x_2 = y_1 + \nu(y_1)$, $x_3 = b_1$, $x_4 = b_2$, $x_5 = b_1$, $x_6 = b''_{l+1}$ and the number $D = D_l + d_l$, we find a point x'_6 satisfying (1.33) and (1.34). By repeating this argument at most $C\tau^{-2}l^3$ times, we find a point b'''_{l+1} with

$$|a_{l+1} - b'''_{l+1}| \leq (6l^5 + 3C\tau^{-2}l + 3C\tau^{-2}l^3)(D_l + d_l)\vartheta$$

such that the collection $\{b_1, \dots, b_l, b'''_{l+1}\}$, in addition to the previous properties, satisfies

$$|\det(\nu_{(i,l+1)}, \nu_\alpha, \nu(y_j))| \geq \vartheta^2 \quad \text{for any } i \in \{1, \dots, l\}, \alpha \in \Lambda_l, \text{ and } j \in \{1, \dots, n\}.$$

Finally, applying Lemma 1.A.3 with the points $x_1 = y_1$, $x_2 = y_1 + \nu(y_1)$, $x_3 = y_2$, $x_4 = y_2 + \nu(y_2)$, $x_5 = b_1$, $x_6 = b'''_{l+1}$ and the number $D = D_l + d_l$, we find a point x'_6 satisfying (1.33) and (1.34). By repeating this argument at most $C^2\tau^{-4}l$ times, we find a point b_{l+1} with

$$|a_{l+1} - b_{l+1}| \leq (6l^5 + 3C\tau^{-2}l + 3C\tau^{-2}l^3 + 3C^2\tau^{-4}l)(D_l + d_l)\vartheta$$

such that the collection $\{b_1, \dots, b_l, b_{l+1}\}$, in addition to the previous properties, satisfies

$$|\det(\nu_{(i,l+1)}, \nu(y_j), \nu(y_k))| \geq \vartheta^2 \quad \text{for any } i \in \{1, \dots, l\} \text{ and } j, k \in \{1, \dots, n\} \text{ with } j \neq k.$$

Summarizing, the collection $\{b_1, \dots, b_l, b_{l+1}\}$ satisfies

- Letting

$$\mathcal{V}_{l+1} = \{\nu_{(i,j)} \mid (i,j) \in \Lambda_{l+1}\} \cup \{\nu(y_i) \mid y_i \in \mathcal{X}'\},$$

then for any $u, v, w \in \mathcal{V}_l$ with $u \neq v \neq w$, we have

$$|u \times v| \geq \vartheta \quad \text{and} \quad |\det(u, v, w)| \geq \vartheta^2.$$

- $|a_i - b_i| \leq d_{l+1} = C(l^5 + \tau^{-2}l^3 + \tau^{-4}l)(D_l + d_l)\vartheta$ for any $i \in \{1, \dots, l+1\}$, where C is a universal constant.
- $|b_i - b_j| \leq D_{l+1} = D_l + d_l$ for any $i, j \in \{1, \dots, l+1\}$.

This concludes the induction step. Arguing as in the proof of Proposition 1.A.1 we find upper bounds for the recursively defined distances d_l and D_l . Observe that

$$D_{l+1} \leq D_l(1 + C(l^5 + \tau^{-2}l^3 + \tau^{-4}l)\vartheta), \quad D_1 = \text{diam}(\Omega) + 1.$$

We immediately check that if $\vartheta \leq \min\{\vartheta_1, \vartheta_3, m^{-6} + \tau^2 m^{-4} + \tau^4 m^{-2}\}$, where ϑ_1 and ϑ_3 are the constants appearing in Lemma 1.A.1 and Lemma 1.A.3 respectively, then for any $l \in \{1, \dots, m\}$

$$D_l \leq D_m \leq D_1(1 + C(m^5 + \tau^{-2}m^3 + \tau^{-4}m)\vartheta)^m \leq D_1 \left(1 + \frac{C}{m}\right)^m \leq C(\text{diam}(\Omega) + 1),$$

Moreover if $\vartheta \leq m^{-6} + \tau^2 m^{-4} + \tau^4 m^{-2}$ then for any $l \in \{1, \dots, m\}$

$$d_l \leq C(\text{diam}(\Omega) + 1)(l^5 + \tau^{-2}l^3 + \tau^{-4}l)\vartheta.$$

Thus, provided that

$$\vartheta < C_0 \min\{m^{-6}, m^{-4}\tau^2, m^{-2}\tau^4, \tau^5\},$$

where the constant C_0 depends only on Ω , the proposition follows. \square

We are now in position to prove Proposition 1.5.2.

Proof of Proposition 1.5.2. Let $\mathcal{X} = \{x_1, \dots, x_n\} \subset \partial\Omega$ be a collection of points satisfying (1.42) for a number $\tau > 0$. Apply Lemma 1.B.1 to obtain a collection $\mathcal{X}' = \{y_1, \dots, y_n\}$ for $0 < \tau < \tau_0$, where τ_0 is the constant appearing in the statement of the lemma. Then apply Proposition 1.B.2 with the collection of points $\mathcal{A} \subset \Omega$ to obtain a collection $\mathcal{A}' = \{p'_1, \dots, p'_k, n'_1, \dots, n'_k\}$. We consider a number

$$\vartheta < C_0 \min\{(2k)^{-6}, (2k)^{-4}\tau^2, (2k)^{-2}\tau^4, \tau^5\},$$

where $C_0 = C_0(\partial\Omega)$ is the constant appearing in the statement of the lemma. Observe that

$$\begin{aligned} L_{\partial\Omega}(\mathcal{A}') &\leq \sum_{i=1}^k d_{\partial\Omega}(p'_i, n'_i) \leq \sum_{i=1}^k d_{\partial\Omega}(p_i, n_i) + d_{\partial\Omega}(p_i, p'_i) + d_{\partial\Omega}(n_i, n'_i) \\ &\leq L_{\partial\Omega}(\mathcal{A}) + C(2k) \left((2k)^5 + (2k)^3\tau^{-2} + (2k)\tau^{-4} \right) \vartheta, \end{aligned}$$

where throughout the proof C denotes a constant depending only on $\partial\Omega$, that may change from line to line. An analogous argument shows that

$$L_{\partial\Omega}(\mathcal{A}) \leq L_{\partial\Omega}(\mathcal{A}') + C(2k) \left((2k)^5 + (2k)^3\tau^{-2} + (2k)\tau^{-4} \right) \vartheta.$$

Therefore

$$(1.45) \quad |L_{\partial\Omega}(\mathcal{A}) - L_{\partial\Omega}(\mathcal{A}')| \leq C(2k) \left((2k)^5 + (2k)^3\tau^{-2} + (2k)\tau^{-4} \right) \vartheta.$$

Remember that by Lemma 1.5.3 there exists a 1-Lipschitz function $\zeta^* : \cup_{i=1,\dots,k} \{p'_i, n'_i\} \rightarrow \mathbb{R}$ such that

$$L_{\partial\Omega}(\mathcal{A}') = \sum_{i=1}^k \zeta^*(p'_i) - \zeta^*(n'_i).$$

Define the function ζ for $d_{\partial\Omega}$ as in Definition 1.5.2, i.e. set

$$\zeta(x) := \max_{i \in \{1,\dots,k\}} (\zeta^*(p'_i) - d_i(x, \partial\Omega)),$$

where

$$d_i(x, \partial\Omega) := \min \left[\max \left(\max_{j \in \{1,\dots,2k\}} d_{(i,j)}(x), d(p'_i, \partial\Omega) - d(x, \partial\Omega) \right), d(p'_i, \partial\Omega) + d(x, \partial\Omega) \right],$$

and

$$d_{(i,j)}(x) := \langle p'_i - x, \nu_{(i,j)} \rangle, \quad \nu_{(i,j)} = \begin{cases} \frac{p'_i - a'_j}{|p'_i - a'_j|} & \text{if } p'_i \neq a'_j \\ 0 & \text{if } p'_i = a'_j \end{cases}.$$

Lemma 1.5.4 yields that $\zeta : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a 1-Lipschitz function such that

$$\sum_{i=1}^k \zeta(p'_i) - \zeta(n'_i) = \sum_{i=1}^k \zeta^*(p'_i) - \zeta^*(n'_i) = L_{\partial\Omega}(\mathcal{A}').$$

Recall that by Lemma 1.B.1, letting

$$\Omega_{\mathcal{X}'} := \cap_{1 \leq l \leq n} \{z \mid \langle z - y_l, \nu(y_l) \rangle < 0\},$$

where $\nu(y_l)$ is the outer unit normal to $\partial\Omega$ at y_l , we have

$$(1.46) \quad |d(x, \partial\Omega_{\mathcal{X}'}) - d(x, \partial\Omega)| \leq C\tau^2$$

for any $x \in \mathbb{R}^3$. Observe that, since Ω is convex, $\Omega \subset \Omega_{\mathcal{X}'}$ and that for any $z \in \bar{\Omega}_{\mathcal{X}'}$

$$d(x, \partial\Omega_{\mathcal{X}'}) = \min_{1 \leq l \leq n} \langle y_l - x, \nu(y_l) \rangle.$$

In order to take advantage of this fact, we define a new function by replacing the distance to $\partial\Omega$ with the distance to $\partial\Omega_{\mathcal{X}'}$. More precisely, we let

$$\tilde{\zeta}(x) := \max_{i \in \{1,\dots,k\}} (\zeta^*(p'_i) - d_i(x, \partial\Omega_{\mathcal{X}'})).$$

From (1.46), we deduce that

$$(1.47) \quad \left| L_{\partial\Omega}(\mathcal{A}') - \sum_{i=1}^k \tilde{\zeta}(p'_i) - \tilde{\zeta}(n'_i) \right| \leq C(2k)\tau^2.$$

Next, we regularize the function $\tilde{\zeta}$. Let $\varphi \in C_c^\infty(B(0,1), \mathbb{R}_+)$ be a mollifier such that $\int_{\mathbb{R}^3} \varphi(x) dx = 1$. Letting

$$(1.48) \quad \lambda := \vartheta^{1/\rho} \quad \text{for } \rho > 0,$$

we define

$$\zeta_\lambda(\cdot) := \varphi_\lambda * \tilde{\zeta}(\cdot) = \int_{\mathbb{R}^3} \varphi_\lambda(\cdot - z) \tilde{\zeta}(z) dz \quad \text{with } \varphi_\lambda(\cdot) = \frac{1}{\lambda} \varphi\left(\frac{\cdot}{\lambda}\right).$$

First, observe that $\|\tilde{\zeta} - \zeta_\lambda\|_{L^\infty(\mathbb{R}^3)} \leq \lambda$. We deduce that

$$(1.49) \quad \left| \sum_{i=1}^k \tilde{\zeta}(p'_i) - \tilde{\zeta}(n'_i) - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq 2k\lambda.$$

By combining (1.45) with (1.47) and (1.49), we obtain

$$(1.50) \quad \left| L_{\partial\Omega}(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq C(((2k)^6 + (2k)^4\tau^{-2} + (2k)^2\tau^{-4})\vartheta + 2k(\tau^2 + \lambda)).$$

On the other hand, note that

$$\begin{aligned} \left| \sum_{i=1}^k \zeta(p_i) - \zeta(n_i) - \sum_{i=1}^k \zeta(p'_i) - \zeta(n'_i) \right| &\leq \sum_{i=1}^k |p_i - n_i| + |p'_i - n'_i| \\ &\leq C(2k) ((2k)^5 + (2k)^3\tau^{-2} + (2k)\tau^{-4}) \vartheta. \end{aligned}$$

By combining the previous estimate with (1.46) and (1.47), we get

$$(1.51) \quad \left| \sum_{i=1}^k \zeta(p_i) - \zeta(n_i) - \sum_{i=1}^k \zeta_\lambda(p'_i) - \zeta_\lambda(n'_i) \right| \leq C(((2k)^6 + (2k)^4\tau^{-2} + (2k)^2\tau^{-4})\vartheta + 2k(\tau^2 + \lambda)).$$

Then by (1.50) and (1.51), we deduce that

$$\left| L_{\partial\Omega}(\mathcal{A}) - \sum_{i=1}^k \zeta_\lambda(p_i) - \zeta_\lambda(n_i) \right| \leq C(((2k)^6 + (2k)^4\tau^{-2} + (2k)^2\tau^{-4})\vartheta + 2k(\tau^2 + \lambda)).$$

Second, note that

$$(1.52) \quad \nabla \zeta_\lambda(x) = \int_{B(x,\lambda)} \varphi_\lambda(x-z) \nabla \tilde{\zeta}(z) dz$$

for any $x \in \mathbb{R}^3$. We define

$$\Lambda := \{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq 2k, i \neq j\} \quad \text{and} \quad c := \max_{i \in \{1, \dots, k\}} (\zeta^*(p'_i) - d(p'_i, \partial\Omega_{\mathcal{A}'})).$$

Then, letting

$$\begin{aligned}\zeta_{(i,j)}(\cdot) &:= \zeta^*(p'_i) - d_{(i,j)}(\cdot) && \text{for } (i,j) \in \Lambda \\ \zeta_{l,+}(\cdot) &:= c + \langle \cdot - y_l, \nu(y_l) \rangle && \text{for } l \in \{1, \dots, n\} \\ \zeta_{l,-}(\cdot) &:= c - \langle \cdot - y_l, \nu(y_l) \rangle && \text{for } l \in \{1, \dots, n\},\end{aligned}$$

observe that, for almost every $z \in \Omega_{\mathcal{X}'}$,

$$\nabla \tilde{\zeta}(z) = \begin{cases} \nu_{(i,j)} & \text{if } \tilde{\zeta}(z) = \zeta_{(i,j)}(z) \text{ for some } (i,j) \in \Lambda \\ \nu(y_l) & \text{if } \tilde{\zeta}(z) = \zeta_{l,+}(z) \text{ for some } l \in \{1, \dots, n\} \\ -\nu(y_l) & \text{if } \tilde{\zeta}(z) = \zeta_{l,-}(z) \text{ for some } l \in \{1, \dots, n\}, \end{cases}$$

In particular $|\nabla \tilde{\zeta}(z)| = 1$ for almost every $z \in \Omega_{\mathcal{X}'}$. Thus

$$|\nabla \zeta_\lambda(x)| \leq \int_{B(x,\lambda)} \varphi_\lambda(x-z) |\nabla \tilde{\zeta}(z)| dz \leq \int_{B(x,\lambda)} \varphi_\lambda(x-z) dz = 1$$

for any $x \in \Omega_\lambda$.

Third, observe that

$$\tilde{\zeta}(x) = c \quad \text{for any } x \in \partial\Omega_{\mathcal{X}'}$$

Thus

$$|\zeta_\lambda(\overline{\Omega \setminus \Omega_\lambda})| \leq C(\tau^2 + \lambda).$$

We now analyze the set of points in Ω_λ whose gradient is small in modulus. From (1.52), we deduce that

$$\nabla \zeta_\lambda(x) = \sum_{\alpha \in \Lambda} \sigma_\alpha \nu_\alpha + \sum_{l=1}^n \sigma_{l,+} \nu(y_l) + \sum_{l=1}^n \sigma_{l,-} (-\nu(y_l)),$$

where

$$\begin{aligned}\sigma_\alpha &= \int_{B(x,\lambda)} \varphi_\lambda(x-z) \mathbf{1}_{\tilde{\zeta}(z)=\zeta_\alpha(z)} dz && \text{for } \alpha \in \Lambda, \\ \sigma_{l,+} &= \int_{B(x,\lambda)} \varphi_\lambda(x-z) \mathbf{1}_{\tilde{\zeta}(z)=\zeta_{l,+}(z)} dz && \text{for } l \in \{1, \dots, n\}, \\ \sigma_{l,-} &= \int_{B(x,\lambda)} \varphi_\lambda(x-z) \mathbf{1}_{\tilde{\zeta}(z)=\zeta_{l,-}(z)} dz && \text{for } l \in \{1, \dots, n\}.\end{aligned}$$

Observe that $\sigma_\alpha, \sigma_{l,+}, \sigma_{l,-} \in [0, 1]$ and that $\sum_{\alpha \in \Lambda} \sigma_\alpha + \sum_{l=1}^n \sigma_{l,+} + \sum_{l=1}^n \sigma_{l,-} = 1$. We conclude that, for any $x \in \Omega_\lambda$, $\nabla \zeta_\lambda(x)$ is a convex combination of the vectors ν_α 's, $\nu(y_l)$'s, and $-\nu(y_l)$'s with $\alpha \in \Lambda$, $l \in \{1, \dots, n\}$. By Caratheodory's theorem, we deduce that $\nabla \zeta_\lambda(z)$ is a convex combination of at most four of them.

We define

$$P_\lambda := \{z \in \mathbb{R}^3 \mid d(z, P) \leq 2\lambda\},$$

where

$$P := \cup_{1 \leq i < j \leq k} P_{i,j}, \quad P_{i,j} := \{z \in \mathbb{R}^3 \mid \zeta_{(i,j)}(y) = \zeta_{(j,i)}(y)\}.$$

Arguing as in the proof of Proposition 1.5.1, we deduce that $|\zeta_\lambda(P_\lambda)| \leq 2\lambda k^2$.

Consider a number $\kappa < \vartheta^2/3$ and a point

$$x \in \{z \in \Omega_\lambda \mid |\nabla\zeta_\lambda(z)| < \kappa\} \setminus P_\lambda.$$

We observe that, since $x \notin P_\lambda$, if there exists a point $y \in \overline{B(x, \lambda)}$ and indices $i, j \in \{1, \dots, k\}$ with $i \neq j$ such that

$$\zeta(y) = \zeta_{(i,j)}(y)$$

then, for any $z \in \overline{B(x, \lambda)}$,

$$\zeta(z) \neq \zeta_{(j,i)}(z)$$

On the other hand, since $x \in \Omega_\lambda$, if there exist a point $z_+ \in \overline{B(x, \lambda)}$ and an index $l \in \{1, \dots, n\}$ such that

$$\tilde{\zeta}(z_+) = \zeta_{l,+}(z_+)$$

then, for any $z \in \overline{B(x, \lambda)}$,

$$\tilde{\zeta}(z) \neq \zeta_{l,-}(z).$$

Arguing by contradiction, assume that there exist points $z_+, z_- \in \overline{B(x, \lambda)}$ and an index $l \in \{1, \dots, n\}$ such that

$$\tilde{\zeta}(z_+) = \zeta_{l,+}(z_+) \quad \text{and} \quad \tilde{\zeta}(z_-) = \zeta_{l,-}(z_-).$$

Observe that

$$|\tilde{\zeta}(z_+) - \tilde{\zeta}(z_-)| \leq \lambda \quad \text{and} \quad |\zeta_{l,-}(z_-) - \zeta_{l,-}(z_+)| \leq \lambda$$

and that

$$|\zeta_{l,+}(z_+) - \zeta_{l,-}(z_+)| = 2d(z_+, \partial\Omega_{\mathcal{X}'}) \geq 2d(z_+, \partial\Omega) > 2\lambda.$$

But

$$|\zeta_{l,+}(z_+) - \zeta_{l,-}(z_+)| = |\tilde{\zeta}(z_+) - \tilde{\zeta}(z_-) + \zeta_{l,-}(z_-) - \zeta_{l,-}(z_+)| \leq 2\lambda,$$

which yields a contradiction with the previous computation.

Analogously, if there exist a point $z_- \in \overline{B(x, \lambda)}$ and an index $l \in \{1, \dots, n\}$ such that

$$\tilde{\zeta}(z_-) = \zeta_{l,-}(z_-)$$

then, for any $z \in \overline{B(x, \lambda)}$,

$$\tilde{\zeta}(z) \neq \zeta_{l,+}(z).$$

This implies that $\nabla\zeta_\lambda(x)$ is a convex combination of at most four vectors, where if one them happens to be $\nu_{(i,j)}$ for some $i, j \in \{1, \dots, k\}$ with $i \neq j$ then all the other vectors are different from $\nu_{(j,i)} = -\nu_{(i,j)}$ and if one of them happens to be $\nu(y_l)$ (respectively $-\nu(y_l)$) for some $l \in \{1, \dots, n\}$ then all the other vectors are different from $-\nu(y_l)$ (respectively $\nu(y_l)$). Recalling that by Lemma 1.B.2, we have

$$|v_1 \times v_2| \geq \vartheta \quad \text{and} \quad |\det(v_1, v_2, v_3)| \geq \vartheta^2$$

for any $v_1, v_2, v_3 \in \{\nu_{(i,j)} \mid 1 \leq i \leq k, 1 \leq j \leq 2k, i < j\} \cup \{\nu(y_l) \mid 1 \leq l \leq n\}$ with $v_1 \neq v_2 \neq v_3$ we deduce that $\nabla\zeta_\lambda(x)$ is a convex combination of at most four vector that satisfy the previous property.

Arguing as in the proof of Proposition 1.5.1 we conclude that if $x \in \{y \in \Omega_\lambda \mid |\nabla \zeta_\lambda(y)| < \kappa\} \setminus P_\lambda$ with $\kappa < \vartheta^2/3$, then there exist four different functions

$$\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \{\zeta_{(i,j)} \mid (i,j) \in \Lambda\} \cup \{\zeta_{l,+} \mid y_l \in \mathcal{X}'\} \cup \{\zeta_{l,-} \mid y_l \in \mathcal{X}'\},$$

where if $\zeta_a = \zeta_{(i,j)}$ for some $(i,j) \in \Lambda$ and $a \in \{1, 2, 3, 4\}$ then $\zeta_b \neq \zeta_{(j,i)}$ for any $b \in \{1, 2, 3, 4\} \setminus \{a\}$ and if $\zeta_a = \zeta_{l,+}$ (respectively $\zeta_a = \zeta_{l,-}$) for some $l \in \{1, \dots, n\}$ and $a \in \{1, 2, 3, 4\}$ then $\zeta_b \neq \zeta_{l,-}$ (respectively $\zeta_b \neq \zeta_{l,+}$) for any $b \in \{1, 2, 3, 4\} \setminus \{a\}$, such that the unique solution $z \in \mathbb{R}^3$ to the linear system of equations

$$\zeta_1(z) = \zeta_2(z) = \zeta_3(z) = \zeta_4(z)$$

lies in the ball $B(x, C\lambda/(\vartheta^2 - 3\kappa))$. We conclude that the set

$$C_\kappa := \{x \in \Omega_\lambda \mid |\nabla \zeta_\lambda(x)| < \kappa\} \setminus P_\lambda$$

can be covered by \mathfrak{B}_κ , a collection of at most $\binom{|\Lambda|+2|\mathcal{X}'|}{4} \leq C_0((2k)^8 + \tau^{-8})$ balls of radius $C\lambda/(\vartheta^2 - 3\kappa)$. Observing that

$$|D^2 \zeta_\lambda(x)| \leq \frac{C}{\lambda^2}$$

for any $x \in \Omega_\lambda$, and letting

$$T_\kappa := \zeta_\lambda(\cup_{B \in \mathfrak{B}_\kappa} B),$$

we deduce that, for any $t \in \zeta_\lambda(\Omega_\lambda) \setminus (T_\kappa \cup \zeta_\lambda(P_\lambda))$, $\{x \mid \zeta_\lambda(x) = t\}$ is a complete submanifold of \mathbb{R}^3 whose second fundamental form is bounded by

$$C \frac{\sup_{\Omega_\lambda} |D^2 \zeta_\lambda|}{\inf_{\Omega_\lambda \setminus ((\cup_{B \in \mathfrak{B}_\kappa} B) \cup P_\lambda)} |\nabla \zeta_\lambda|} \leq \frac{C}{\lambda^2 \kappa}.$$

Recalling the relation between λ and ϑ (see (1.48)), the proposition follows. \square

1.B.1 The general case

In order to get rid of the assumption that $\partial\Omega$ has strictly positive Gauss curvature at every point, a new approach is needed. We will smoothly approximate the function ζ for $d_{\partial\Omega}$ defined in Definition 1.5.2, after displacing the points a_i as in Appendix 1.A. The main difference with respect to the strategy followed in Appendix 1.B, is that we do not approximate the boundary of the domain. The commodity of doing this is that (where well-defined) the gradient of the function $d(\cdot, \partial\Omega'_{\mathcal{X}'})$ is equal to the normal to $\partial\Omega$ at some point of the discrete set \mathcal{X}' .

In our new approach, the main points to consider are:

- If we reduce the analysis to a small neighborhood close to the boundary, then the gradient of the distance to the boundary at every point of this neighborhood is given by the normal to the boundary at the unique projection to the boundary of this point.

- We need to understand the set where the distance to the boundary is equal to one or two of the functions $\zeta_{i,j}$'s, while the gradient vectors of these functions do not satisfy a good angle condition between each other, in the sense described in the previous two sections. One can show that the image of this set has small measure. To prove this fact, the strategy is to combine the first observation with a delicate analysis based on the curvature of the boundary. An important fact is that we need to assume that the boundary is of class C^2 , which in particular gives an upper bound for the maximal principal curvature at each point. Roughly speaking, this means that the boundary “cannot wiggle too much”.
- We need to adapt the last part of the proof of Proposition 1.5.2. Arguing in the same fashion, but using a quantitative version of the inverse function theorem, we can show that the set where the distance to the boundary is equal to three of the functions $\zeta_{i,j}$'s can be covered by a finite number of small balls.

We remark that once this is done, the proofs presented in Sections 1.6, 1.7, and 1.8 will follow (almost) without modification.

Chapter 2

Global minimizers for the 3D Ginzburg-Landau functional below and near the first critical field

Abstract

In this chapter, which is based on [Romb] and on a work in preparation in collaboration with Etienne Sandier and Sylvia Serfaty [RSS], we analyze the behavior of global minimizers of the three-dimensional Ginzburg-Landau functional below and near the first critical field. First, we prove that minimizing configurations below the first critical field are vortex-less. Second, in a work in progress, we prove that near the first critical field, global minimizers have bounded vorticity, under a suitable non-degeneracy condition.

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2.1 Introduction

The aim of this chapter is to give a precise estimate of the first critical field H_{c_1} and to study the behavior of global minimizers of the full Ginzburg-Landau functional with applied magnetic field (introduced in Chapter 1) below and near H_{c_1} . Physically, this value is characterized as follows. Below H_{c_1} , the superconductor is everywhere in its superconducting phase $|u| \approx 1$ and the external magnetic field is forced out by the material. This phenomenon is known as the *Meissner effect*. At H_{c_1} , which is of order of $|\log \varepsilon|$ as $\varepsilon \rightarrow 0$, the first vortice(s) appear and the external magnetic field penetrates the material through the vortice(s).

In the works [Ser99, SS00a, SS03, SS00c], Sandier and Serfaty derived with high precision the value of the first critical field and rigorously described the behavior of global minimizers of GL_ε below and near H_{c_1} in 2D. In the 3D case, Alama, Bronsard, and Montero [ABM06] identified a candidate expression for H_{c_1} in the case of the ball. Then, Baldo, Jerrard, Orlandi, and Soner [BJOS13] characterized to leading order the first critical field in 3D for a general bounded domain, via a Γ -convergence argument. Our purpose here is to derive with more precision this value. To do so, we crucially use the ε -level estimates proved in the previous chapter.

Throughout this chapter, we assume that $H_{\text{ex}} = h_{\text{ex}}H_{0,\text{ex}}$, where $H_{0,\text{ex}}$ is a fixed unit vector and h_{ex} denotes the intensity of the applied field. In particular, there exists $A_{\text{ex}} = h_{\text{ex}}A_{0,\text{ex}} \in H_{\text{loc}}^1(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$\text{curl } A_{0,\text{ex}} = H_{0,\text{ex}}, \quad \text{div } A_{0,\text{ex}} = 0 \text{ in } \mathbb{R}^3 \quad \text{and} \quad A_{0,\text{ex}} \cdot \nu = 0 \text{ on } \partial\Omega.$$

The natural space for the minimization of GL_ε in 3D is $H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$, where

$$H_{\text{curl}} := \{A \in H_{\text{loc}}^1(\mathbb{R}^3, \mathbb{R}^3) \mid \text{curl } A \in L^2(\mathbb{R}^3, \mathbb{R}^3)\}.$$

Let us also introduce the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3, \mathbb{R}^3)$, which is defined as the completion of $C_0^\infty(\mathbb{R}^3, \mathbb{R}^3)$ with respect to the norm $\|\nabla(\cdot)\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)}$. We observe that, by Sobolev embedding, there exists a constant $C > 0$ such that

$$(2.1) \quad \|A\|_{L^6(\mathbb{R}^3, \mathbb{R}^3)} \leq C \|\nabla A\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)}$$

for any $A \in \dot{H}^1(\mathbb{R}^3, \mathbb{R}^3)$. Moreover, by [KS91, Proposition 2.4], we have

$$\dot{H}^1(\mathbb{R}^3, \mathbb{R}^3) = \{A \in L^6(\mathbb{R}^3, \mathbb{R}^3) \mid \nabla A \in L^2(\mathbb{R}^3, \mathbb{R}^3)\}.$$

It is also convenient to define the subspace

$$\dot{H}_{\text{div}=0}^1 := \{A \in \dot{H}^1(\mathbb{R}^3, \mathbb{R}^3) \mid \text{div } A = 0 \text{ in } \mathbb{R}^3\}.$$

In this subspace, one has

$$(2.2) \quad \|\nabla A\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)} = \|\text{curl } A\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)}.$$

Let us recall that any vector field $A \in H^1(\Omega, \mathbb{R}^3)$ can be decomposed as

$$\begin{cases} A = \text{curl } B_A + \nabla \phi_A & \text{in } \Omega \\ B_A \times \nu = 0 & \text{on } \partial\Omega \\ \nabla \phi_A \cdot \nu = A \cdot \nu & \text{on } \partial\Omega, \end{cases}$$

where the vector field B_A and the function ϕ_A are unique if properly chosen.

We consider the unique minimizer $A_0 \in \dot{H}_{\text{div}=0}^1$ of the functional

$$J(A) := \frac{1}{2} \int_{\Omega} |\text{curl } B_A|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |\text{curl}(A - A_{0,ex})|^2$$

and define the Meissner configuration $(u_0, h_{\text{ex}}A_0)$, where $u_0 = e^{ih_{\text{ex}}\phi_{A_0}}$. We observe that $GL_{\varepsilon}(u_0, h_{\text{ex}}A_0) = h_{\text{ex}}^2 J(A_0)$. Finally, letting $B_0 := B_{A_0} \in C_T^{0,1}(\Omega, \mathbb{R}^3)$, where $C_T^{0,1}$ denotes the space of vector fields in $C^{0,1}$ whose tangential component vanishes on $\partial\Omega$, we define

$$\|B_0\|_* := \sup_{\mu \in X} \int_{\Omega} \mu \wedge B_0,$$

where X is the class of 1-currents such that $\partial\mu = 0$ relative to Ω and $|\mu|(\Omega) \leq 1$.

Our goal is to prove that the first critical field in 3D is given by $H_{c_1}^{3D} + O(1)$. Our first result concerns the behavior of global minimizers of GL_{ε} below $H_{c_1}^{3D}$.

Theorem 2.1.1. *There exist constants $\varepsilon_0, K_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and $h_{\text{ex}} \leq H_{c_1}^{3D} - K_0 \log |\log \varepsilon|$, the global minimizers $(u_{\varepsilon}, A_{\varepsilon})$ of GL_{ε} in $H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ are such that $|u_{\varepsilon}| \geq 1/2$ in Ω and $GL_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) = h_{\text{ex}}^2 J(A_0) + o(1)$.*

The next result, which is a work in preparation in collaboration with Etienne Sandier and Sylvia Serfaty, concerns the behavior of global minimizers near $H_{c_1}^{3D}$. Before stating the result, we need to introduce a non-degeneracy condition.

Let us consider the functional

$$\gamma(B_0) := \int_{\gamma} B_0 \cdot \tau,$$

defined for Lipschitz curves $\gamma \in X$. Here, τ denotes the tangent vector to γ . In particular, we observe that $|\gamma(B_0)| \leq \|B_0\|_*$.

We introduce the following assumption.

Non-degeneracy condition in 3D: There exists a unique Lipschitz curve $\gamma_0 \in X$ such that $\gamma_0(B_0) = \|B_0\|_*$. Moreover, there exist constants $C, N > 0$ such that for any Lipschitz curve $\gamma \in X$ if $\|\gamma - \gamma_0\|_* \geq \delta$, for some $\delta \in (0, 1)$, then

$$\gamma(B_0) \leq \gamma_0(B_0) - C\delta^N.$$

Theorem 2.1.2. *Assume the non-degeneracy condition above. For any $K > 0$, there exist positive constant $\varepsilon_0, C, \alpha > 0$ such that for any $\varepsilon < \varepsilon_0$ and any $h_{\text{ex}} < H_{c_1}^{3D} + K \log |\log \varepsilon|$, if $(u_{\varepsilon}, A_{\varepsilon})$ is a global minimizer of GL_{ε} in $H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ and $\nu'_{\varepsilon} = \sum_{i \in I} \gamma_i$ is the vorticity approximation associated to the configuration $(u'_{\varepsilon}, A'_{\varepsilon})$ defined by $(u_{\varepsilon}, A_{\varepsilon}) = (u_0 u'_{\varepsilon}, h_{\text{ex}}A_0 + A'_{\varepsilon})$, then*

1. $\forall i, \gamma_i/|\gamma_i|(B_0) \geq 0$,
2. $\|\gamma_i/|\gamma_i| - \gamma_0\|_* < |\log \varepsilon|^{-\frac{1}{2N}}$ for any i such that $\gamma_i/|\gamma_i|(B_0) > 0$, and
3. $|\nu_{\varepsilon}|(\Omega) < C$.

In this theorem, we use the decomposition $\nu'_\varepsilon = \sum_{i \in I} \gamma_i$, where each γ_i is a multiplicity 1 Lipschitz curve in X . This result essentially states that the vorticity of minimizing configuration $\mu(u_\varepsilon, A_\varepsilon)$ is bounded when the strength of the applied field is below slightly above $H_{c_1}^{3D}$, but also provides extra information about the finite sum (in the sense of currents) of polygonal lines ν'_ε . The author believes that the previous two theorems are the key ingredients to prove that the first critical field is given by $H_{c_1}^{3D} + O(1)$ in 3D. This requires extra work, that we do not present here.

Finally, let us point out that in a work in preparation we show that the non-degeneracy condition presented above holds when $\Omega = B(0, R)$ and $H_{0,\text{ex}} = \hat{z}$. In this case, one can show that the vertical diameter D_1 oriented in the direction of \hat{z} and seen as a Lipschitz curve in X (in particular $|D_1|(\Omega) = 1$) is such that

$$D_1(B_0) = \|B_0\|_*.$$

Moreover, we have the following result.

Theorem 2.1.3. *Let $\Omega = B(0, R)$ and $H_{0,\text{ex}} = \hat{z}$. There exists constants $C, N > 0$ such that for any Lipschitz curve $\gamma \in X$ if $\|\gamma - D_1\|_* \geq \delta$, for some $\delta \in (0, 1)$, then*

$$\gamma(B_0) \leq D_1(B_0) - C\delta^N,$$

where D_1 is defined as above.

2.2 Preliminaries

2.2.1 Hodge decompositions

We begin by giving a decomposition of vector fields in H_{curl} .

Lemma 2.2.1. *Every vector field $A \in H_{\text{curl}}$ can be decomposed as*

$$A = \text{curl } \mathcal{B} + \nabla \Phi,$$

where $\mathcal{B}, \text{curl } \mathcal{B} \in \dot{H}_{\text{div}=0}^1$ and $\Phi \in H_{\text{loc}}^2(\mathbb{R}^3)$.

Proof. First, let us observe that there exists a function $\Phi_1 \in H_{\text{loc}}^2(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$\Delta \Phi_1 = \text{div } A \in L_{\text{loc}}^2(\mathbb{R}^3, \mathbb{R}^3).$$

Second, we consider the problem

$$\text{curl}^2 B = \text{curl } A \in L^2(\mathbb{R}^3, \mathbb{R}^3), \quad \text{div } B = 0.$$

By observing that $\text{curl}^2 B = -\Delta B$, [KS91, Theorem 1] provides the existence of a solution $\mathcal{B} \in \dot{H}_{\text{div}=0}^1$ to this problem such that $\text{curl } \mathcal{B} \in \dot{H}_{\text{div}=0}^1$.

Finally, by noting that

$$\text{curl}(A - \nabla \Phi_1 - \text{curl } \mathcal{B}) = \text{div}(A - \nabla \Phi_1 - \text{curl } \mathcal{B}) = 0,$$

we deduce that

$$A - \nabla \Phi_1 - \text{curl } \mathcal{B} = \nabla \Phi_2,$$

for a harmonic function $\Phi_2 \in H_{\text{loc}}^2(\mathbb{R}^3, \mathbb{R}^3)$. By writing $\Phi = \Phi_1 + \Phi_2$, we obtain the lemma. \square

Next, we recall a decomposition of vector fields in $H^1(\Omega, \mathbb{R}^3)$. The proof of this result can be found in [BBO01, Appendix A].

Lemma 2.2.2. *There exists a constant $C = C(\Omega)$ such that for every $A \in H^1(\Omega, \mathbb{R}^3)$ there exist a unique vector field $B_A \in \{B \in H^2(\Omega, \mathbb{R}^3) \mid \operatorname{div} B = 0 \text{ in } \Omega\}$ and a unique function $\phi_A \in \{\phi \in H^2(\Omega) \mid \int_{\Omega} \phi_A = 0\}$ satisfying*

$$\begin{cases} A = \operatorname{curl} B_A + \nabla \phi_A & \text{in } \Omega \\ B_A \times \nu = 0 & \text{on } \partial\Omega \\ \nabla \phi_A \cdot \nu = A \cdot \nu & \text{on } \partial\Omega, \end{cases}$$

and

$$\|B_A\|_{H^2(\Omega, \mathbb{R}^3)} + \|\phi_A\|_{H^2(\Omega)} \leq C \|A\|_{H^1(\Omega, \mathbb{R}^3)}.$$

2.2.2 Ginzburg-Landau equations

Definition 2.2.1 (Critical point of GL_ε). *We say that $(u, A) \in H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$ is a critical point of GL_ε if for every smooth and compactly supported configuration (v, B) we have*

$$\frac{d}{dt} G_\varepsilon(u + tv, A + tB)|_{t=0} = 0.$$

Next, we provide the Euler-Lagrange equations satisfied by critical points of GL_ε .

Proposition 2.2.1. *If $(u, A) \in H^1(\Omega, \mathbb{C}) \times H_{\text{curl}}$ is a critical point of GL_ε then (u, A) satisfies the system of equations*

$$(GL) \quad \begin{cases} -(\nabla_A)^2 u = \frac{1}{\varepsilon^2} u(1 - |u|^2) & \text{in } \Omega \\ \operatorname{curl}(H - H_{\text{ex}}) = (iu, \nabla_A u) \chi_\Omega & \text{in } \mathbb{R}^3 \\ \nabla_A u \cdot \nu = 0 & \text{on } \partial\Omega \\ [H - H_{\text{ex}}] \times \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where χ_Ω is the characteristic function of Ω , $[\cdot]$ denotes the jump across $\partial\Omega$, and the covariant Laplacian is defined by

$$(\nabla_A)^2 u = (\operatorname{div} -iA \cdot) \nabla_A u.$$

Proof. We have

$$\frac{d}{dt} GL_\varepsilon(u + tv, A)|_{t=0} = \int_{\Omega} (\nabla_A u, \nabla_A v) - \frac{1}{\varepsilon^2} \int_{\Omega} (u, v)(1 - |u|^2).$$

By noting that

$$(\nabla_A u, \nabla_A v) = \operatorname{div}(\nabla_A u, v) - ((\nabla_A)^2 u, v),$$

where $(\nabla_A u, v) = ((\partial_1 u - iA_1 u, v), (\partial_2 u - iA_2 u, v), (\partial_3 u - iA_3 u, v))$, and by integrating by parts, we obtain

$$\frac{d}{dt} GL_\varepsilon(u + tv, A)|_{t=0} = \int_{\partial\Omega} (\nabla_A u \cdot \nu, v) - \int_{\Omega} ((\nabla_A)^2 u, v) - \frac{1}{\varepsilon^2} \int_{\Omega} (u, v)(1 - |u|^2).$$

Since this is true for any v , we find

$$(\nabla_A)^2 u = \frac{1}{\varepsilon^2} u(1 - |u|^2) \text{ in } \Omega \quad \text{and} \quad \nabla_A u \cdot \nu = 0 \text{ on } \partial\Omega.$$

On the other hand, we have

$$\frac{d}{dt} GL_\varepsilon(u, A + tB)|_{t=0} = - \int_\Omega (iBu, \nabla_A u) + \int_{\mathbb{R}^3} (H - H_{\text{ex}}) \cdot \text{curl } B.$$

By integration by parts, we get

$$\frac{d}{dt} GL_\varepsilon(u, A + tB)|_{t=0} = - \int_\Omega (iu, \nabla_A u) \cdot B + \int_{\mathbb{R}^3} \text{curl}(H - H_{\text{ex}}) \cdot B.$$

We deduce that

$$\text{curl}(H - H_{\text{ex}}) = (iu, \nabla_A u)\chi_\Omega \quad \text{in } \mathbb{R}^3.$$

By testing this equation against B and by integrating by parts over Ω , we find

$$\int_\Omega (H - H_{\text{ex}}) \cdot \text{curl } B - \int_{\partial\Omega} ((H - H_{\text{ex}}) \times \nu) \cdot B - \int_\Omega (iu, \nabla_A u) \cdot B = 0.$$

Now, by integrating by parts over $\mathbb{R}^3 \setminus \Omega$, we get

$$\int_{\mathbb{R}^3 \setminus \Omega} (H - H_{\text{ex}}) \cdot \text{curl } B + \int_{\partial(\mathbb{R}^3 \setminus \Omega)} ((H - H_{\text{ex}}) \times \nu) \cdot B = 0.$$

Thus

$$\int_{\partial\Omega} ([H - H_{\text{ex}}] \times \nu) \cdot B = 0,$$

and therefore $[H - H_{\text{ex}}] \times \nu = 0$ on $\partial\Omega$. □

Remark 2.2.1. *By taking the curl of the second Ginzburg-Landau equation, we find*

$$(2.3) \quad \text{curl}^2(H - H_{\text{ex}}) + H\chi_\Omega = \mu(u, A)\chi_\Omega,$$

in the sense of currents. We will come back to this equation later on.

2.3 Global minimizers below the first critical field

2.3.1 The Meissner solution

Physically, when the strength of the applied magnetic field is below the first critical field one observes that the superconductor is everywhere in its superconducting phase $|u| \approx 1$ and that the external magnetic field is forced out by the material.

We then expect global minimizers of GL_ε below H_{c1} to be vortex-less configurations (u, A) such that $\mu(u, A) \approx 0$. We next a pair $(u_0, h_{\text{ex}}A_0)$, usually called the *Meissner*

solution, which satisfies (2.3) with zero right hand side and whose energy turns out to be a good approximation of the energy of global minimizers below the first critical field.

Let us consider a configuration of the form $(e^{i\phi}, h_{\text{ex}}A_0)$, with $\phi \in H^2(\Omega)$ and $A_0 \in A_{\text{ex}} + \dot{H}_{\text{div}=0}^1$. Observe that, by using Lemma 2.2.2 and by letting $u_0 := e^{i\phi}$, we have

$$\begin{aligned} GL_\varepsilon(u_0, h_{\text{ex}}A_0) &= \frac{1}{2} \int_{\Omega} |\nabla\phi - h_{\text{ex}}(\text{curl } B_{A_0} + \nabla\phi_{A_0})|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |h_{\text{ex}} \text{curl } A_0 - H_{\text{ex}}|^2 \\ &= \frac{1}{2} \int_{\Omega} |\nabla(\phi_0 - h_{\text{ex}}\phi_{A_0})|^2 + h_{\text{ex}}^2 |\text{curl } B_{A_0}|^2 + \frac{h_{\text{ex}}^2}{2} \int_{\mathbb{R}^3} |\text{curl}(A_0 - A_{0,\text{ex}})|^2 \end{aligned}$$

By choosing $\phi = h_{\text{ex}}\phi_{A_0}$, we obtain

$$GL_\varepsilon(u_0, A_0) = \frac{h_{\text{ex}}^2}{2} \int_{\Omega} |\text{curl } B_{A_0}|^2 + \frac{h_{\text{ex}}^2}{2} \int_{\mathbb{R}^3} |\text{curl}(A_0 - A_{0,\text{ex}})|^2 =: h_{\text{ex}}^2 J(A_0).$$

We choose A_0 to be the minimizer of J in the space $(A_{\text{ex}} + \dot{H}_{\text{div}=0}^1, \|\text{curl} \cdot\|_{L^2(\mathbb{R}^3, \mathbb{R}^3)})$, whose existence and uniqueness follows by noting that J is continuous, coercive, and strictly convex in this Hilbert space (recall (2.1) and (2.2)).

We let $H_0 = \text{curl } A_0$. One can easily check that, for any $A \in \dot{H}_{\text{div}=0}^1$, we have

$$\int_{\Omega} \text{curl } B_{A_0} \cdot \text{curl } B_A + \int_{\mathbb{R}^3} (H_0 - H_{0,\text{ex}}) \cdot \text{curl } A = 0$$

Because $\int_{\Omega} \text{curl } B_{A_0} \cdot \nabla\phi_A = 0$, we have

$$(2.4) \quad \int_{\Omega} \text{curl } B_{A_0} \cdot A + \int_{\mathbb{R}^3} (H_0 - H_{0,\text{ex}}) \cdot \text{curl } A = 0$$

Moreover, Lemma 2.2.1 implies that this equality holds in the larger space H_{curl} .

Then, an integrating by parts yields

$$\int_{\Omega} \text{curl } B_{A_0} \cdot A + \int_{\mathbb{R}^3} \text{curl}(H_0 - H_{0,\text{ex}}) \cdot A = 0.$$

Therefore A_0 satisfies the Euler-Lagrange equation

$$(2.5) \quad \text{curl}(H_0 - H_{0,\text{ex}}) + \text{curl } B_{A_0} \chi_{\Omega} = 0 \quad \text{in } \mathbb{R}^3.$$

In addition, the boundary condition $[H_0 - H_{0,\text{ex}}] = 0$ on $\partial\Omega$ is satisfied.

By taking the curl of the previous equation, we find

$$\text{curl}^2(H_0 - H_{0,\text{ex}}) + H_0 \chi_{\Omega} = 0 \quad \text{in } \mathbb{R}^3,$$

namely (up to dividing by h_{ex}) (2.3) with $\mu(u_0, A_0) = 0$.

Observe that, by minimality of A_0 , we have

$$J(A_0) \leq J(A_{0,\text{ex}}) = \frac{1}{2} \int_{\Omega} |\text{curl } B_{A_{0,\text{ex}}}|^2 \leq C \int_{\Omega} |H_{0,\text{ex}}|^2 = C|\Omega||H_{0,\text{ex}}|^2.$$

By Sobolev embedding, we deduce that

$$\|B_{A_0}\|_{H^2(\mathbb{R}^3, \mathbb{R}^3)} \leq C|\Omega||H_{0,\text{ex}}|^2.$$

Then, by standard elliptic regularity, we get

$$\|B_{A_0}\|_{C_T^{0,1}(\mathbb{R}^3, \mathbb{R}^3)} \leq C|\Omega||H_{0,\text{ex}}|^2.$$

Remark 2.3.1. Observe that the divergence-free vector field $B_0 \in C_T^{0,1}(\Omega, \mathbb{R}^3)$ satisfies

$$\begin{cases} \Delta^2(B_0 - B_{0,\text{ex}}) + \Delta B_0 = 0 & \text{in } \Omega \\ B_0 \times \nu = 0 & \text{on } \partial\Omega \\ [\Delta(B_0 - B_{0,\text{ex}})] \times \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

This normalized vector field is then the analog of the function ξ_0 defined in (1), which has been used to analyze similar questions in 2D. As we shall see below, B_0 plays an important role in our 3D analysis.

2.3.2 Energy-splitting

Next, by using the Meissner solution, we present a splitting of GL_ε .

Proposition 2.3.1 (Energy-splitting). *For any $(u, A) \in H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]$, letting $u = u_0 u'$ and $A = h_{\text{ex}} A_0 + A'$, where $(u_0, h_{\text{ex}} A_0)$ is the Meissner solution, we have*

$$(2.6) \quad GL_\varepsilon(u, A) = h_{\text{ex}}^2 J(A_0) + F_\varepsilon(u', A') + \frac{1}{2} \int_{\mathbb{R}^3 \setminus \Omega} |\text{curl } A'|^2 - h_{\text{ex}} \int_{\Omega} \mu(u', A') \wedge B_0 + R_0,$$

where $F_\varepsilon(u', A')$ is the free energy of the configuration $(u', A') \in H^1(\Omega, \mathbb{C}) \times H_{\text{curl}}$, i.e.

$$F_\varepsilon(u', A') = \frac{1}{2} \int_{\Omega} |\nabla_{A'} u'|^2 + \frac{1}{2\varepsilon^2} (1 - |u'|^2)^2 + |\text{curl } A'|^2$$

and

$$R_0 = \frac{h_{\text{ex}}^2}{2} \int_{\Omega} (|u|^2 - 1) |\text{curl } B_0|^2.$$

In particular, $R_0 \leq C\varepsilon h_{\text{ex}}^2 E_\varepsilon(|u|)^{\frac{1}{2}}$, with $E_\varepsilon(|u|) = \frac{1}{2} \int_{\Omega} |\nabla |u||^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2$.

Proof. One immediately checks that $A' \in H_{\text{curl}}$ and that $u' = u_0^{-1} u = e^{-ih_{\text{ex}} \phi_{A_0}} u$. In particular, because $\phi_{A_0} \in H^2(\Omega, \mathbb{C})$, by Sobolev embedding we deduce that $u' \in H^1(\Omega, \mathbb{C})$.

Writing $u = u_0 u'$ and $A = h_{\text{ex}} A_0 + A'$ and plugging them into $GL_\varepsilon(u, A)$, we obtain

$$GL_\varepsilon(u, A) = \frac{1}{2} \int_{\Omega} |\nabla_{A'} u' - ih_{\text{ex}} \text{curl } B_0 u'|^2 + \frac{1}{2\varepsilon^2} (1 - |u'|^2)^2 + \frac{1}{2} \int_{\mathbb{R}^3} |\text{curl } A' + h_{\text{ex}}(H_0 - H_{0,\text{ex}})|^2.$$

By expanding the square terms, we get

$$GL_\varepsilon(u, A) = \frac{1}{2} \int_{\Omega} |\nabla_{A'} u'|^2 + h_{\text{ex}}^2 |\text{curl } B_0|^2 |u'|^2 - 2h_{\text{ex}} (\nabla_{A'} u', iu') \cdot \text{curl } B_0 + \frac{1}{2\varepsilon^2} (1 - |u'|^2)^2$$

$$+\frac{1}{2} \int_{\mathbb{R}^3} |\operatorname{curl} A'|^2 + h_{\text{ex}}^2 |H_0 - H_{0,\text{ex}}|^2 + 2h_{\text{ex}} \operatorname{curl} A' \cdot (H_0 - H_{0,\text{ex}}).$$

Observe that, by (2.4), we have

$$\int_{\mathbb{R}^3} \operatorname{curl} A' \cdot (H_0 - H_{0,\text{ex}}) = - \int_{\Omega} A' \cdot \operatorname{curl} B_0.$$

Therefore, grouping terms and writing $|u'|^2$ as $1 + (|u'|^2 - 1)$, we find

$$GL_{\varepsilon}(u, A) = h_{\text{ex}}^2 J(A_0) + F_{\varepsilon}(u', A') + \frac{1}{2} \int_{\mathbb{R}^3 \setminus \Omega} |\operatorname{curl} A'|^2 - h_{\text{ex}} \int_{\Omega} (j(u', A') + A') \cdot \operatorname{curl} B_0 + R_0.$$

Then, an integration by parts yields

$$\int_{\Omega} (j(u', A') + A') \cdot \operatorname{curl} B_0 = \int_{\Omega} \mu(u', A') \wedge B_0 - \int_{\partial\Omega} (j(u', A') + A') \cdot (B_0 \times \nu).$$

By using the boundary condition $B_0 \times \nu = 0$ on $\partial\Omega$, we find (2.6). The inequality for R_0 follows directly from the Cauchy-Schwarz inequality. \square

Remark 2.3.2. Let $\phi \in C_T^{0,1}(\Omega)$ be a 1-form. Observe that

$$\int_{\Omega} \mu(u, A) \wedge \phi = \int_{\Omega} \mu(u', A' + h_{\text{ex}} \operatorname{curl} B_0) \wedge \phi = \int_{\Omega} \mu(u', A') \wedge \phi + h_{\text{ex}} \int_{\Omega} (1 - |u|^2) \operatorname{curl} B_0 \cdot \phi.$$

Moreover, the Cauchy-Schwarz inequality yields

$$(2.7) \quad \|\mu(u, A) - \mu(u', A')\|_{C_T^{0,1}(\Omega)^*} \leq C\varepsilon h_{\text{ex}} E_{\varepsilon}(|u|)^{\frac{1}{2}}.$$

2.3.3 Proof of Theorem 2.1.1

Proof of Theorem 2.1.1. By minimality, we have

$$(2.8) \quad \inf_{(u,A) \in H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]} GL_{\varepsilon}(u, A) \leq GL_{\varepsilon}(u_0, h_{\text{ex}} A_0) = h_{\text{ex}}^2 J(A_0),$$

where throughout the proof $(u_0, h_{\text{ex}} A_0)$ is the Meissner solution.

Writing $(u_{\varepsilon}, A_{\varepsilon}) = (u_0 u'_{\varepsilon}, h_{\text{ex}} A_0 + A'_{\varepsilon})$, Proposition 2.6 implies that

$$(2.9) \quad GL_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) \geq h_{\text{ex}}^2 J(A_0) + F_{\varepsilon}(u'_{\varepsilon}, A'_{\varepsilon}) - h_{\text{ex}} \int_{\Omega} \mu(u'_{\varepsilon}, A'_{\varepsilon}) \wedge B_0 + o(\varepsilon^{\frac{1}{2}}).$$

Let us observe that

$$\begin{aligned} F_{\varepsilon}(u'_{\varepsilon}, A'_{\varepsilon}) &= F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon} - h_{\text{ex}} \operatorname{curl} B_0) \leq 2F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) + 2F_{\varepsilon}(1, h_{\text{ex}} \operatorname{curl} B_0) \\ &\leq 2F_{\varepsilon}(u_{\varepsilon}, A_{\varepsilon}) + h_{\text{ex}}^2 J(A_0), \end{aligned}$$

which combined with (2.8) implies that

$$F_{\varepsilon}(u'_{\varepsilon}, A'_{\varepsilon}) \leq M |\log \varepsilon|^2.$$

We may then apply Theorem 1.1.1 to obtain

$$F_\varepsilon(u'_\varepsilon, A'_\varepsilon) - h_{\text{ex}} \int_\Omega \mu(u'_\varepsilon, A'_\varepsilon) \wedge B_0 \geq \frac{1}{2} |\nu'_\varepsilon|(\Omega) \left(\log \frac{1}{\varepsilon} - C \log \log \frac{1}{\varepsilon} \right) - h_{\text{ex}} \int_\Omega \nu'_\varepsilon \wedge B_0 + o(|\log \varepsilon|^{-2}),$$

where $C > 0$ is a universal constant and ν'_ε denotes the polyhedral 1-dimensional current associated to the configuration $(u'_\varepsilon, A'_\varepsilon)$ by Theorem 1.1.1.

By noting that

$$(2.10) \quad \int_\Omega \nu'_\varepsilon \wedge B_0 \leq |\nu'_\varepsilon|(\Omega) \|B_0\|_*,$$

we find

$$F_\varepsilon(u'_\varepsilon, A'_\varepsilon) - h_{\text{ex}} \int_\Omega \mu(u'_\varepsilon, A'_\varepsilon) \wedge B_0 \geq \frac{1}{2} |\nu'_\varepsilon|(\Omega) \left(\log \frac{1}{\varepsilon} - 2\|B_0\|_* h_{\text{ex}} - C \log \log \frac{1}{\varepsilon} \right) + o(|\log \varepsilon|^{-2}),$$

Writing $h_{\text{ex}} = H_{c_1}^{3\text{D}} - K_0 \log |\log \varepsilon|$, with $H_{c_1}^{3\text{D}} = \frac{1}{2\|B_0\|_*} |\log \varepsilon|$, we get

$$GL_\varepsilon(u_\varepsilon, A_\varepsilon) \geq h_{\text{ex}}^2 J(A_0) + \frac{1}{2} |\nu'_\varepsilon|(\Omega) (2\|B_0\|_* K_0 - C) \log \log \frac{1}{\varepsilon} + o(|\log \varepsilon|^{-2}).$$

By using (2.8), we deduce that

$$o(|\log \varepsilon|^{-2}) \geq |\nu'_\varepsilon|(\Omega) (2\|B_0\|_* K_0 - C) \log \log \frac{1}{\varepsilon}.$$

Therefore, by letting $K_0 := (2\|B_0\|_*)^{-1} C + 1$, we deduce that $|\nu'_\varepsilon| = o(|\log \varepsilon|^{-2})$. In particular, this implies that $h_{\text{ex}} \int_\Omega \mu(u'_\varepsilon, A'_\varepsilon) \wedge B_0 = o(|\log \varepsilon|^{-1})$. Therefore, from (2.8) and (2.9), we obtain

$$(2.11) \quad F_\varepsilon(u'_\varepsilon, A'_\varepsilon) \leq o(|\log \varepsilon|^{-1}).$$

To conclude, we use a “clearing out” result. Let us define

$$v_\varepsilon := e^{-i\varphi_\varepsilon} u'_\varepsilon \quad \text{and} \quad X_\varepsilon := A'_\varepsilon - \nabla \varphi_\varepsilon,$$

where φ_ε satisfies

$$\begin{cases} \Delta \varphi_\varepsilon = \operatorname{div} A'_\varepsilon & \text{in } \Omega \\ \frac{\partial \varphi_\varepsilon}{\partial \nu} = A'_\varepsilon \cdot \nu & \text{on } \partial\Omega. \end{cases}$$

This implies that

$$(2.12) \quad \begin{cases} \operatorname{div} X_\varepsilon = 0 & \text{in } \Omega \\ X_\varepsilon \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the configuration $(u_\varepsilon, A_\varepsilon)$ minimizes GL_ε , it satisfies the Ginzburg-Landau equations (GL). By observing that the configurations $(u_\varepsilon, A_\varepsilon)$ and $(v_\varepsilon, X_\varepsilon + h_{\text{ex}} \text{curl } B_0)$ are gauge equivalent in Ω , we deduce that v_ε satisfies

$$\begin{cases} -(\nabla_{X_\varepsilon + h_{\text{ex}} \text{curl } B_0})^2 v_\varepsilon &= \frac{1}{\varepsilon^2} v_\varepsilon (1 - |v_\varepsilon|^2) & \text{in } \Omega \\ \nabla_{X_\varepsilon + h_{\text{ex}} \text{curl } B_0} v_\varepsilon \cdot \nu &= 0 & \text{on } \partial\Omega. \end{cases}$$

Expanding the covariant Laplacian, and using (2.12) and $\text{curl } B_0 \cdot \nu = 0$ on $\partial\Omega$, which follows from $B_0 \times \nu = 0$ on $\partial\Omega$, one can rewrite this system in the form

$$(2.13) \quad \begin{cases} -\Delta v_\varepsilon + i |\log \varepsilon| \vec{c}(x) \cdot \nabla v_\varepsilon + d(x) v_\varepsilon &= \frac{1}{\varepsilon^2} v_\varepsilon (1 - |v_\varepsilon|^2) & \text{in } \Omega \\ \frac{\partial v_\varepsilon}{\partial \nu} &= 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\vec{c}(x) := \frac{2(X_\varepsilon + h_{\text{ex}} \text{curl } B_0)}{|\log \varepsilon|} \quad \text{and} \quad d(x) := \frac{|X_\varepsilon + h_{\text{ex}} \text{curl } B_0|^2}{|\log \varepsilon|^2}.$$

By standard elliptic regularity theory, one can check that

$$(2.14) \quad \|\vec{c}\|_{L^\infty(\Omega, \mathbb{R}^3)}, \|\nabla \vec{c}\|_{L^\infty(\Omega, \mathbb{R}^3)}, \|d\|_{L^\infty(\Omega)}, \|\nabla d\|_{L^\infty(\Omega)} \leq \Lambda_0,$$

for some universal constant $\Lambda_0 > 0$. □

By gauge invariance, we have

$$F(u'_\varepsilon, A'_\varepsilon) = F_\varepsilon(v_\varepsilon, X_\varepsilon).$$

Since $(v_\varepsilon, X_\varepsilon)$ is in the Coulomb gauge, i.e. it satisfies (2.12), one easily checks that

$$E_\varepsilon(v_\varepsilon) \leq C F_\varepsilon(v_\varepsilon, X_\varepsilon),$$

for some universal constant $C > 0$. By letting $a_\varepsilon(x) = 1 - d(x)\varepsilon^2 |\log \varepsilon|^2$, we observe that

$$\tilde{E}_\varepsilon(v_\varepsilon) := \frac{1}{2} \int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2} (a_\varepsilon(x) - |v_\varepsilon|^2)^2 \leq E_\varepsilon(v_\varepsilon) + O(\varepsilon^2 |\log \varepsilon|^4).$$

This, combined with (2.11) implies that

$$(2.15) \quad \tilde{E}_\varepsilon(v_\varepsilon) = o(|\log \varepsilon|^{-1}).$$

Finally, by Remark 2.3.1, (2.12), (2.13), (2.14), and (2.15), we conclude that all the hypotheses of [Chi05, Theorem 3] are fulfilled, which implies that $|u_\varepsilon| = |v_\varepsilon| \geq \frac{1}{2}$. The equality

$$GL_\varepsilon(u_\varepsilon, A_\varepsilon) = h_{\text{ex}}^2 J(A_0) + o(1)$$

easily follows from the computations above. This concludes the proof of the theorem.

2.4 Global minimizers near the first critical field

We next sketch the proof of Theorem 2.1.2. This part is work in preparation.

Sketch of proof of Theorem 2.1.2. We write $(u_\varepsilon, A_\varepsilon) = (u_0 u'_\varepsilon, h_{\text{ex}} A_0 + A'_\varepsilon)$, where $(u_0, h_{\text{ex}} A_0)$ is the Meissner solution. Arguing as in the proof above, we can associate to the configuration $(u'_\varepsilon, A'_\varepsilon)$ the vorticity approximation ν'_ε given by Theorem 1.1.1.

Combining the energy splitting (2.6) with Theorem 1.1.1, we get

$$(2.16) \quad \begin{aligned} GL_\varepsilon(u_\varepsilon, A_\varepsilon) &\geq h_{\text{ex}}^2 J(A_0) + |\nu'_\varepsilon|(\Omega) \left(\log \frac{1}{\varepsilon} - C \log \log \frac{1}{\varepsilon} \right) - h_{\text{ex}} \int_\Omega \nu'_\varepsilon \wedge B_0 \\ &\quad + \frac{1}{2} \int_{\Omega \setminus \Omega'} |\nabla_{A'_\varepsilon} u'_\varepsilon|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |\text{curl } A'_\varepsilon|^2 + o(|\log \varepsilon|^{-2}), \end{aligned}$$

We decompose

$$\nu'_\varepsilon = \sum_{i \in I} \gamma_i,$$

where each γ_i is a polygonal curve of multiplicity 1. In particular,

$$\nu'_\varepsilon = \sum_{i \in I} |\gamma_i| \frac{\gamma_i}{|\gamma_i|}.$$

Step 1: We let

$$I_+ = \left\{ i \in I \mid \frac{\gamma_i}{|\gamma_i|}(B_0) > 0 \right\}, \quad I_- = I \setminus I_+, \quad |\nu'_\varepsilon|_+ = \sum_{i \in I_+} |\gamma_i|, \quad \text{and} \quad |\nu'_\varepsilon|_- = |\nu'_\varepsilon|(\Omega) - |\nu'_\varepsilon|_+.$$

Observe that

$$-H_{c_1}^{3D} \int_\Omega \nu'_\varepsilon \wedge B_0 \geq -H_{c_1}^{3D} \sum_{i \in I_+} |\gamma_i| \frac{\gamma_i}{|\gamma_i|}(B_0) \geq -H_{c_1}^{3D} \sum_{i \in I_+} |\gamma_i| \|B_0\|_* = -\frac{1}{2} |\nu'_\varepsilon|_+ |\log \varepsilon|.$$

Writing $h_{\text{ex}} = H_{c_1}^{3D} + K \log |\log \varepsilon|$ and using (2.16) and the previous inequality, we deduce that

$$\begin{aligned} GL_\varepsilon(u_\varepsilon, A_\varepsilon) &\geq h_{\text{ex}}^2 J(A_0) + \frac{1}{2} |\nu'_\varepsilon|(\Omega) \left(\log \frac{1}{\varepsilon} - C \log \log \frac{1}{\varepsilon} \right) \\ &\quad - \frac{1}{2} |\nu'_\varepsilon|_+ \left(\log \frac{1}{\varepsilon} + 2K \|B_0\|_* \log \log \frac{1}{\varepsilon} \right) + o(|\log \varepsilon|^{-2}), \end{aligned}$$

By recalling that

$$(2.17) \quad \inf_{(u, A) \in H^1(\Omega, \mathbb{C}) \times [A_{\text{ex}} + H_{\text{curl}}]} GL_\varepsilon(u, A) \leq GL_\varepsilon(u_0, h_{\text{ex}} A_0) = h_{\text{ex}}^2 J(A_0),$$

from the previous two inequalities we deduce that

$$|\nu'_\varepsilon|(\Omega) (C + 2K \|B_0\|_*) \log \log \frac{1}{\varepsilon} + o(|\log \varepsilon|^{-2}) \geq |\nu'_\varepsilon|_- \log \frac{1}{\varepsilon}$$

Hence, for a constant \tilde{C} depending on K , we have

$$(2.18) \quad \tilde{C}|\nu'_\varepsilon|_+ \frac{\log |\log \varepsilon|}{\log \varepsilon} + o(|\log \varepsilon|^{-3}) \geq |\nu_\varepsilon|_-.$$

Step 2: Let us recall the non-degeneracy condition assumption: There exists a unique Lipschitz curve $\gamma_0 \in X$ such that $\gamma_0(B_0) = \|B_0\|_*$. Moreover, there exist constants $C_1, N > 0$ such that for any Lipschitz curve $\gamma \in X$ if $\|\gamma - \gamma_0\|_* \geq \delta$, for some $\delta \in (0, 1)$, then

$$\gamma(B_0) \leq \gamma_0(B_0) - C_1 \delta^N.$$

Let us define

$$I_0 = \left\{ i \in I \mid \left\| \frac{\gamma_i}{|\gamma_i|} - \gamma_0 \right\|_* < \frac{1}{|\log \varepsilon|^{\frac{1}{2N}}} \right\} \quad \text{and} \quad |\nu'_\varepsilon|_0 = \sum_{i \in I_0} |\gamma_i|.$$

Observe that, if $i \notin I_0$ then

$$-\frac{\gamma_i}{|\gamma_i|}(B_0) \geq -\gamma_0(B_0) + \frac{C_1}{|\log \varepsilon|^{\frac{1}{2}}} = -\|B_0\|_* + \frac{C_1}{|\log \varepsilon|^{\frac{1}{2}}},$$

while, if $i \in I_0$, we have the obvious inequality

$$-\frac{\gamma_i}{|\gamma_i|}(B_0) \geq -\gamma_0(B_0) = -\|B_0\|_*.$$

Writing $h_{\text{ex}} = H_{c_1}^{3D} + K \log |\log \varepsilon|$ and using (2.16), (2.17), and the previous inequalities, we find

$$\begin{aligned} o(|\log \varepsilon|^{-2}) &\geq \frac{1}{2}|\nu'_\varepsilon|(\Omega) \left(\log \frac{1}{\varepsilon} - C \log \log \frac{1}{\varepsilon} \right) - \frac{1}{2}|\nu'_\varepsilon|_0 \left(\log \frac{1}{\varepsilon} + 2K\|B_0\|_* \log \log \frac{1}{\varepsilon} \right) \\ &\quad - \frac{1}{2}(|\nu'_\varepsilon|(\Omega) - |\nu'_\varepsilon|_0) \left(\log \frac{1}{\varepsilon} + 2K\|B_0\|_* \log \log \frac{1}{\varepsilon} \right) \left(1 - \frac{C_1}{\|B_0\|_* |\log \varepsilon|^{\frac{1}{2}}} \right) \end{aligned}$$

From this, we get

$$\frac{1}{2}|\nu'_\varepsilon|(\Omega) (C + 2K\|B_0\|_*) \log \log \frac{1}{\varepsilon} + o(|\log \varepsilon|^{-2}) \geq \frac{1}{2}(|\nu'_\varepsilon|(\Omega) - |\nu'_\varepsilon|_0) \frac{C_1 |\log \varepsilon|^{\frac{1}{2}}}{\|B_0\|_*}$$

and then

$$|\nu'_\varepsilon|(\Omega) \frac{C + 2K\|B_0\|_* \log |\log \varepsilon|}{C_1 \|B_0\|_*^{-1} |\log \varepsilon|^{\frac{1}{2}}} + o(|\log \varepsilon|^{-\frac{5}{2}}) \geq |\nu'_\varepsilon|(\Omega) - |\nu'_\varepsilon|_0.$$

Hence, for a constant \tilde{C} depending on K , we have

$$(2.19) \quad \tilde{C}|\nu'_\varepsilon|_0 \frac{\log |\log \varepsilon|}{|\log \varepsilon|^{\frac{1}{2}}} + o(|\log \varepsilon|^{-\frac{5}{2}}) \geq |\nu'_\varepsilon|(\Omega) - |\nu'_\varepsilon|_0.$$

Step 3: Let us write $\int_{\Omega} |\nabla_{A'} u|^2$ as

$$\int_{\Omega'} |\nabla_{A'} u|^2 + \frac{1}{2} \int_{\Omega \setminus \Omega'} |\nabla_{A'} u|^2,$$

where $\Omega' := S_{\nu_\varepsilon}$ is the set defined in Theorem 1.1.1.

Since the configurations $(u_\varepsilon, A_\varepsilon)$ and $(u'_\varepsilon, A'_\varepsilon + h_{\text{ex}} \text{curl} B_0)$ are gauge equivalent in Ω and because $(u_\varepsilon, A_\varepsilon)$ minimizes GL_ε , we deduce that

$$\text{curl}^2 A'_\varepsilon + \text{curl}(H_0 - H_{\text{ex}}) = (iu'_\varepsilon, \nabla_{A'_\varepsilon} u'_\varepsilon) - |u'_\varepsilon|^2 h_{\text{ex}} \text{curl} B_0 \quad \text{in } \Omega.$$

Combining this with (2.5), we find

$$\text{curl}^2 A'_\varepsilon = (iu'_\varepsilon, \nabla_{A'_\varepsilon} u'_\varepsilon) + (1 - |u'_\varepsilon|^2) h_{\text{ex}} \text{curl} B_0 \quad \text{in } \Omega.$$

Since $|u'_\varepsilon| = |u_\varepsilon| \leq 1$, we deduce that

$$|\text{curl}^2 A'_\varepsilon|^2 \leq |\nabla_{A'_\varepsilon} u'_\varepsilon|^2 + (1 - |u'_\varepsilon|^2)^2 h_{\text{ex}}^2 |\text{curl} B_0|^2 + 2|\nabla_{A'_\varepsilon} u'_\varepsilon|(1 - |u'_\varepsilon|^2) h_{\text{ex}} |\text{curl} B_0|$$

and therefore

$$\int_{\Omega \setminus \Omega'} |\nabla_{A'_\varepsilon} u'_\varepsilon|^2 \geq \int_{\Omega \setminus \Omega'} |\text{curl}^2 A'_\varepsilon|^2 + o(\varepsilon^{\frac{1}{2}}).$$

This inequality combined with (2.16), gives

$$\begin{aligned} GL_\varepsilon(u_\varepsilon, A_\varepsilon) &\geq h_{\text{ex}}^2 J(A_0) + |\nu'_\varepsilon|(\Omega) \left(\log \frac{1}{\varepsilon} - C \log \log \frac{1}{\varepsilon} \right) - h_{\text{ex}} \int_{\Omega} \nu'_\varepsilon \wedge B_0 \\ &\quad + \frac{1}{2} \int_{\Omega \setminus \Omega'} |\text{curl}^2 A'_\varepsilon|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |\text{curl} A'_\varepsilon|^2 + o(|\log \varepsilon|^{-2}). \end{aligned}$$

The argument then reduces to show that

$$(2.20) \quad \frac{1}{2} \int_{\Omega \setminus \Omega'} |\text{curl}^2(A - A_{\text{ex}})|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |\text{curl}(A - A_{\text{ex}})|^2 \geq \tilde{C} |\nu'_\varepsilon|^2(\Omega) \log \log \frac{1}{\varepsilon} + o(1).$$

Indeed, once this is proved, by writing $h_{\text{ex}} = H_{c_1}^{3D} + K(\log |\log \varepsilon|)$ and using (2.10), we deduce that

$$GL_\varepsilon(u_\varepsilon, A_\varepsilon) \geq h_{\text{ex}}^2 J(A_0) - |\nu'_\varepsilon|(\Omega) \log \log \frac{1}{\varepsilon} (C + 2K \|B_0\|_*) + \tilde{C} |\nu'_\varepsilon|^2(\Omega) \log \log \frac{1}{\varepsilon} + o(1).$$

Combining this with (2.8), we find

$$-|\nu'_\varepsilon|(\Omega) \log \log \frac{1}{\varepsilon} (C + 2K \|B_0\|_*) + \tilde{C} |\nu'_\varepsilon|^2(\Omega) \log \log \frac{1}{\varepsilon} \leq o(1),$$

yielding a uniform bound on $|\nu'_\varepsilon|(\Omega)$. To prove (2.20), our strategy is to first combine the Ginzburg-Landau equations satisfied by $(u_\varepsilon, A_\varepsilon)$ with the fact that $|u'_\varepsilon| \approx 1$ in Ω' . Then, a slicing procedure together with an integration on big 2D circles, in the spirit of the one performed in the proof of Theorem 2 in [SS03], should give the inequality. Finally, the bounds (2.18) and (2.19) prove the other assertions of Theorem 2.1.2. \square

2.4.1 Non-degeneracy condition in the case of the ball

Next, we give the idea of proof of Theorem 2.1.3

Idea of proof of Theorem 2.1.3. When $\Omega = B(0, R)$ and $H_{0,\text{ex}} = \hat{z}$, the vector B_0 can be explicitly computed (see [ABM06]). By using spherical coordinates (r, θ, ϕ) , where r is the Euclidean distance from the origin, θ is the polar angle, and ϕ is the azimuthal angle, we have

$$B_0 - c\hat{z} = -\frac{3R}{r^2 \sinh R} \left(\cosh r - \frac{\sinh r}{r} \right) \cos \theta \hat{r} - \frac{3R}{2r^2 \sinh R} \left(\cosh r - \frac{1+r^2}{r} \sinh r \right) \sin \theta \hat{\theta},$$

where $c = -\frac{3}{2R \sinh R} \left(\cosh R - \frac{1+R^2}{R} \sinh R \right)$. In particular, we observe that B_0 does not depend on the azimuthal angle and it is constant along $\hat{\phi}$.

Let γ be a Lipschitz curve such that $\partial\gamma = 0$ relative to Ω . We assume that, for some $\delta \in (0, 1)$, we have

$$(2.21) \quad \left\| \frac{\gamma}{|\gamma|} - D_1 \right\|_* \geq \delta.$$

By following an idea of [ABM06], we will project γ along the azimuthal angle onto $B(0, R)^{2\text{D},+} := \{(x, z) \in \mathbb{R}^2 \mid x^2 + z^2 < R^2, x \geq 0\}$. We consider the map $q : B(0, R) \subset \mathbb{R}^3 \rightarrow B(0, R)^{2\text{D},+}$ defined by

$$q(r, \theta, \phi) = (r \sin \theta, r \cos \theta),$$

and we let

$$\gamma_{2\text{D}} := \gamma \circ q$$

It is easy to check that $\partial\gamma_{2\text{D}} = 0$ relative to the two-dimensional ball $B(0, R)^{2\text{D}}$,

$$\gamma(B_0) \leq \gamma_{2\text{D}}(B_0), \quad \text{and} \quad |\gamma| \geq |\gamma_{2\text{D}}|.$$

We assume that $\gamma(B_0)$, otherwise the result is trivial. Let us observe that if

$$\frac{|\gamma_{2\text{D}}|}{|\gamma|} < (1 - \delta^2)$$

then

$$\frac{\gamma}{|\gamma|}(B_0) \leq \frac{|\gamma_{2\text{D}}|}{|\gamma|} \frac{\gamma_{2\text{D}}}{|\gamma_{2\text{D}}|}(B_0) < (1 - \delta^2) D_1(B_0),$$

thus

$$D_1(B_0) - \frac{\gamma}{|\gamma|}(B_0) \geq \delta^2 D_1(B_0) = \delta^2 \|B_0\|_*,$$

which proves the result in this case. Therefore, we now assume

$$\frac{|\gamma_{2\text{D}}|}{|\gamma|} \geq (1 - \delta^2).$$

By combining this with (2.21), it is not hard to see that there exists a universal constant $\tilde{C} > 0$ such that

$$(2.22) \quad \left\| \frac{\gamma_{2D}}{|\gamma_{2D}|} - D_1 \right\|_* \geq \tilde{C}\delta.$$

Since $\partial\gamma_{2D} = 0$ relative to $B(0, R)^{2D}$, γ_{2D} partitions $B(0, R)^{2D}$ into two domains, each with boundary consisting of γ_{2D} perhaps with some piece of $\partial B(0, R)^{2D}$, properly oriented. We denote by $D_{\gamma_{2D}}$ the domain for which the positively oriented normal vector is \hat{y} . Observe that this domain is contained in $B(0, R)^{2D,+}$. Since $B_0 \times \nu = 0$ on $B(0, R)^{2D}$, the Stokes' theorem yields

$$\gamma_{2D}(B_0) = \int_{\partial D_{\gamma_{2D}}} B_0 \cdot \tau = \int_{D_{\gamma_{2D}}} \operatorname{curl} B_0 \cdot \hat{y} dx dz.$$

An explicit computation shows that

$$\operatorname{curl} B_0 \cdot \hat{y} = c(R) \left(\cosh r - \frac{\sinh r}{r} \right) \frac{\sin \theta}{r}, \quad \text{with } c(R) := \frac{3R}{2 \sinh R}.$$

In particular, this quantity is positive in $B(0, R)^{2D,+}$. We have three cases to consider:

1. First, we assume that $|\gamma_{2D}|$ is not too large. Observe that, for a universal constant $c > 0$, we have

$$\gamma_{2D}(B_0) \leq c \|\operatorname{curl} B_0\|_\infty |\gamma_{2D}|^2.$$

Then

$$D_1(B_0) - \frac{\gamma_{2D}}{|\gamma_{2D}|}(B_0) \geq \|B_0\|_* - c \|\operatorname{curl} B_0\|_\infty |\gamma_{2D}| \geq \delta,$$

provided that $0 \leq |\gamma_{2D}| \leq \frac{\|B_0\|_* - \delta}{c \|\operatorname{curl} B_0\|_\infty}$.

2. Second, we assume that $|\gamma_{2D}| \geq 2R$. Observe that, there exists γ_η , a parallel line to the vertical diameter at distance η and contained in $\overline{B}(0, R)^{2D,+}$, such that

$$(2.23) \quad \int_{D_{\gamma_\eta}} \operatorname{curl} B_0 \cdot \hat{y} dx dz = \int_{D_{\gamma_{2D}}} \operatorname{curl} B_0 \cdot \hat{y} dx dz,$$

where D_{γ_η} is defined as above. It follows that

$$\frac{\gamma_\eta}{|\gamma_\eta|}(B_0) > \frac{\gamma_{2D}}{|\gamma_{2D}|}(B_0),$$

and therefore

$$D_1(B_0) - \frac{\gamma_{2D}}{|\gamma_{2D}|}(B_0) > D_1(B_0) - \frac{\gamma_\eta}{|\gamma_\eta|}(B_0) = \int_{B(0,R)^{2D,+} \setminus D_{\gamma_\eta}} \operatorname{curl} B_0 \cdot \hat{y} dx dz.$$

Observe that

$$\int_0^\eta \int_0^\pi c(R) \left(\cosh r - \frac{\sinh r}{r} \right) \sin \theta d\theta dr = 2c(R) \left(\sinh \eta - \int_0^\eta \frac{\sinh r}{r} dr \right).$$

By noting that

$$(2.24) \quad \sinh \eta - \int_0^\eta \frac{\sinh r}{r} dr \geq \frac{\eta^3}{9}$$

for η small, to conclude the result one needs to quantify η in terms of δ , which should follow from (2.22) and (2.23).

3. Finally, we consider the intermediate case $\frac{\|B_0\|_* - \delta}{c\|\operatorname{curl} B_0\|_\infty} \leq |\gamma_{2D}| < 2R$. It is not hard to see that we can choose γ_η , a parallel line to the vertical diameter at distance η and contained in $\overline{B}(0, R)^{2D,+}$, such that

$$\frac{\gamma_{2D}}{|\gamma_{2D}|}(B_0) = \frac{\gamma_\eta}{|\gamma_\eta|}(B_0).$$

Then, by letting $\alpha \in (0, \pi)$ be the angle between \hat{z} and a ray that passes through the origin and the intersection between the boundary of $B(0, R)^{2D}$ and γ_η for $x \geq 0$, we have

$$\begin{aligned} \int_{D_{\gamma_\eta}} \operatorname{curl} B_0 \cdot \hat{y} dx dz &< \int_\eta^R \int_\alpha^{\pi-\alpha} c(R) \left(\cosh r - \frac{\sinh r}{r} \right) \sin \theta d\theta dr \\ &= 2 \cos \alpha c(R) \int_\eta^R \left(\cosh r - \frac{\sinh r}{r} \right) dr \end{aligned}$$

Observe that $|\gamma_\eta| = 2R \cos \alpha$. Therefore

$$\frac{\gamma_\eta}{|\gamma_\eta|}(B_0) < \frac{c(R)}{R} \int_\eta^R \left(\cosh r - \frac{\sinh r}{r} \right) dr.$$

But

$$\begin{aligned} D_1(B_0) &= \frac{1}{2R} \int_0^R \int_0^\pi c(R) \left(\cosh r - \frac{\sinh r}{r} \right) \sin \theta d\theta dr \\ &= \frac{c(R)}{R} \int_0^R \left(\cosh r - \frac{\sinh r}{r} \right) dr. \end{aligned}$$

Hence

$$\begin{aligned} D_1(B_0) - \frac{\gamma_\eta}{|\gamma_\eta|}(B_0) &= D_1(B_0) - \frac{\gamma_{2D}}{|\gamma_{2D}|}(B_0) = \frac{c(R)}{R} \int_0^\eta \left(\cosh r - \frac{\sinh r}{r} \right) dr \\ &= \frac{c(R)}{R} \left(\sinh \eta - \int_0^\eta \frac{\sinh r}{r} dr \right). \end{aligned}$$

By recalling (2.24), the proof then follows if one quantifies η in terms of δ .

□

Part II

Problems from mathematical biology

Chapter 1

The Lin-Ni-Takagi problem in dimension 3

Abstract

In this chapter, which is based on a joint work with Manuel del Pino, Monica Musso, and Juncheng Wei [dPMRW] that has been accepted for publication in the Journal d'Analyse Mathématique, we consider the problem of finding positive solutions of the problem $\Delta u - \lambda u + u^5 = 0$ in a bounded smooth domain Ω in \mathbb{R}^3 , under zero Neumann boundary conditions. Here, λ is a positive number. We analyze the role of the Green's function of $-\Delta + \lambda$ in the presence of solutions exhibiting single bubbling behavior at one point of the domain when λ is regarded as a parameter. As a special case of our results, we find and characterize a positive value λ_* such that if $\lambda - \lambda_* > 0$ is sufficiently small, then this problem is solvable by a solution u_λ which blows up by bubbling at a certain interior point of Ω as $\lambda \downarrow \lambda_*$.

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1.1 Introduction

Let Ω be a bounded smooth domain in \mathbb{R}^n . This chapter deals with the boundary value problem

$$(1.1) \quad \begin{cases} \Delta u - \lambda u + u^p = 0, & u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & & \text{on } \partial\Omega, \end{cases}$$

where $p > 1$. A large literature has been devoted to this problem when $1 \leq p \leq \frac{n+2}{n-2}$ for asymptotic values of the parameter λ . A very interesting feature of this problem is the presence of families of solutions u_λ with *point concentration phenomena*. This means solutions that exhibit peaks of concentration around one or more points of Ω or $\partial\Omega$, while being very small elsewhere. For $1 < p < \frac{n+2}{n-2}$, solutions with this feature around points of the boundary were first discovered by Lin, Ni, and Takagi in [LNT88] as $\lambda \rightarrow +\infty$. It is found in [LNT88, NT91, NT93] that a mountain pass or least energy positive solution u_λ to (1.1) for $\lambda \rightarrow +\infty$ must look like

$$u_\lambda(x) \sim \lambda^{\frac{1}{p-1}} V(\lambda^{\frac{1}{2}}(x - x_\lambda))$$

where V is the unique positive radial solution to

$$(1.2) \quad \Delta V - V + V^p = 0 \text{ in } \mathbb{R}^n, \quad \text{with } \lim_{|y| \rightarrow \infty} V(y) = 0,$$

and $x_\lambda \in \partial\Omega$ approaches a point of maximum mean curvature of $\partial\Omega$. See [DPF99] for a short proof of this fact. Higher energy solutions with this asymptotic profile near one or several points of the boundary or the interior of Ω have been constructed and analyzed in many works, see for instance [dPFW99, GW99, DY99, GPW00, LNW07] and their references. In particular, solutions with any given number of interior and boundary concentration points are known to exist as $\lambda \rightarrow +\infty$.

The case of the critical exponent $p = \frac{n+2}{n-2}$ is in fact quite different. In particular, no positive solutions of (1.2) exist. In this situation solutions u_λ to (1.1) do exist for sufficiently large values of λ with concentration now in the form

$$(1.3) \quad u_\lambda(x) \sim \mu_\lambda^{-\frac{n-2}{2}} U(\mu_\lambda^{-1}(x - x_\lambda)),$$

where $\mu_\lambda = o(\lambda^{-\frac{1}{2}}) \rightarrow 0$ as $\lambda \rightarrow +\infty$. Here

$$U(x) = \alpha_n \left(\frac{1}{1 + |y|^2} \right)^{\frac{n-2}{2}}, \quad \text{with } \alpha_n = (n(n-2))^{\frac{n-2}{4}},$$

is the *standard bubble*, which up to scalings and translations, is the unique positive solution of the Yamabe equation

$$(1.4) \quad \Delta U + U^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbb{R}^n.$$

Solutions with boundary bubbling have been built and their dimension-dependent bubbling rates μ_λ analyzed in various works (see [AM91, NPT92, APY93, Wan96, GG98, GL02, WY07, WWY10, DRW12] and references therein). Boundary bubbling by small perturbations of the exponent p above and below the critical exponent has been found in [dPMP05].

Unlike the subcritical range, for $p = \frac{n+2}{n-2}$ solutions with interior bubbling points as $\lambda \rightarrow +\infty$ are harder to be found. They do not exist for $n = 3$ or $n \geq 7$ (see [Rey99, Rey02, Esp07]), and in all dimensions interior bubbling can only coexist with boundary bubbling (see [Rey02]). To be noticed is that the constant function $\underline{u}_\lambda := \lambda^{\frac{1}{p-1}}$ represents a trivial solution to (1.1). A compactness argument yields that this constant is the unique solution to (1.1) for $1 < p < \frac{n+2}{n-2}$ and for any sufficiently small λ (see [LNT88]). The *Lin-Ni conjecture*, raised in [LN88], is that this is also true for $p = \frac{n+2}{n-2}$. The issue turns out to be quite subtle. In [AY91, AY97], it is found that radial nontrivial solutions for all small $\lambda > 0$ exist when Ω is a ball in dimensions $n = 4, 5, 6$, while no radial solutions exist for small λ if $n = 3$ or $n \geq 7$. For a general convex domain, the Lin-Ni conjecture is true in dimension $n = 3$ [Zhu99, WX05]. See [DRW12] for the extension to the mean convex case and related references. In [RW05] solutions with multiple interior bubbling points when $\lambda \rightarrow 0^+$ were found when $n = 5$, in particular showing that Lin-Ni's conjecture fails in arbitrary domains in this dimension. This result is the only example present in the literature of its type. The authors conjecture that a similar result should hold for $n = 4, 6$.

In the case $n = 3$, interior bubbling is not possible if $\lambda \rightarrow +\infty$ or if $\lambda \rightarrow 0^+$, for instance in a convex domain. In this chapter we show a new phenomenon, which is the presence of a solution u_λ with *interior bubbling* for values of λ near a number $0 < \lambda_*(\Omega) < +\infty$ which can be explicitly characterized. Thus, in what follows we consider the critical problem

$$(1.5) \quad \begin{cases} \Delta u - \lambda u + u^5 = 0, & u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is smooth and bounded. The following is our main result.

Theorem 1.1.1. *There exists a number $0 < \lambda_* < +\infty$ such that for all $\lambda > \lambda_*$ with $\lambda - \lambda_*$ sufficiently small, a nontrivial solution u_λ to (1.5) exists, with an asymptotic profile as $\lambda \rightarrow \lambda_*^+$ of the form*

$$u_\lambda(x) = 3^{\frac{1}{4}} \left(\frac{\mu_\lambda}{\mu_\lambda^2 + |x - x_\lambda|^2} \right)^{\frac{1}{2}} + O(\mu_\lambda^{\frac{1}{2}}) \quad \text{in } \Omega,$$

where $\mu_\lambda = O(\lambda - \lambda_*)$ and the points $x_\lambda \in \Omega$ stay uniformly away from $\partial\Omega$.

The number λ_* and the asymptotic location of the point x_λ can be characterized as follows. For $\lambda > 0$, we let $G_\lambda(x, y)$ be the Green function of the problem

$$\begin{cases} \Delta_x G_\lambda(x, y) - \lambda G_\lambda(x, y) + \delta_y(x) = 0 & \text{in } \Omega \\ \frac{\partial G_\lambda}{\partial \nu}(x, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

so that, by definition

$$(1.6) \quad G_\lambda(x, y) = \Gamma(x, y) - H_\lambda(x, y)$$

where $\Gamma(x, y) = \frac{1}{4\pi|x-y|}$ and H_λ , the regular part of G_λ , satisfies

$$(1.7) \quad \begin{cases} \Delta_x H_\lambda(x, y) - \lambda H_\lambda(x, y) = \frac{1}{4\pi|x-y|} & \text{in } \Omega \\ \frac{\partial H_\lambda}{\partial \nu}(x, y) = \frac{1}{\partial \nu 4\pi|x-y|} & \text{on } \partial\Omega, \end{cases}$$

Let us consider the *diagonal* of the regular part (or Robin's function)

$$(1.8) \quad g_\lambda(x) := H_\lambda(x, x), \quad x \in \Omega.$$

Then, we have (see Lemma 1.2.2)

$$g_\lambda(x) \rightarrow -\infty, \quad \text{as } x \rightarrow \partial\Omega.$$

The number $\lambda_*(\Omega)$ in Theorem 1.1.1 is characterized as

$$(1.9) \quad \lambda_*(\Omega) := \inf\{\lambda > 0 \mid \sup_{x \in \Omega} g_\lambda(x) < 0\}.$$

In addition, we have that the points $x_\lambda \in \Omega$ are such that

$$(1.10) \quad \lim_{\lambda \downarrow \lambda_*} g_\lambda(x_\lambda) = \sup_{\Omega} g_{\lambda_*} = 0.$$

As we will see in Section 1.2, when $\Omega = B(0, 1)$, the number λ_* is the unique number λ such that

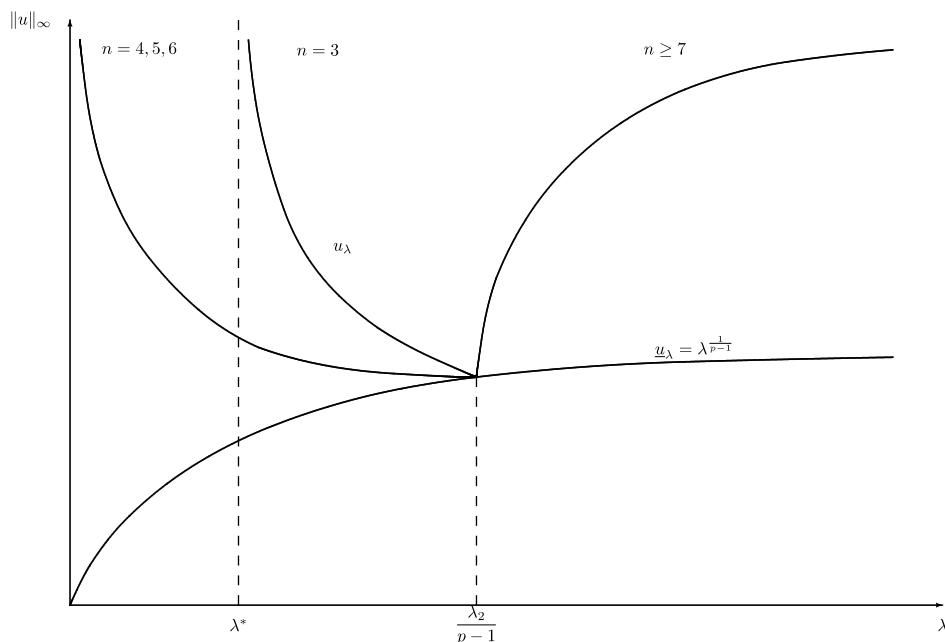
$$\frac{\sqrt{\lambda} - 1}{\sqrt{\lambda} + 1} \exp(2\sqrt{\lambda}) = 1,$$

so that $\lambda^* \approx 1.43923$.

It is worthwhile to emphasize the connection between the number λ_* and the so called *Brezis-Nirenberg number* $\tilde{\lambda}^*(\Omega) > 0$ given as the least value λ such that for all $\tilde{\lambda}^* < \lambda < \lambda_1$, where λ_1 is the first Dirichlet eigenvalue of the Laplacian, there exists a least energy solution to the 3D Brezis-Nirenberg problem (see [BN83])

$$(1.11) \quad \begin{cases} \Delta u + \lambda u + u^p = 0, \quad u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

A parallel characterization of the number $\tilde{\lambda}_*$ in terms of a Dirichlet Green's function has been established in [Dru02] and its role in bubbling phenomena further explored in [dPDM04]. It is important to remark that the topological nature of the solution we find is not that of a least energy, mountain pass type solution (which is actually just the constant for small λ). In fact the construction formally yields that its Morse index is 4.


 Figure 1.1: Bifurcation diagram for solutions to 1.1, $p = \frac{n+2}{n-2}$

Our result can be (formally) depicted as a bifurcation diagram from the branch of constant solutions $u = \underline{u}_\lambda$ (see Figure 1.1). At least in the radial case, what our result suggests is that the bifurcation branch which stems from the trivial solutions at the value $\lambda = \lambda_2/4$, where λ_2 is the first nonzero radial eigenvalue of $-\Delta$ under zero Neumann boundary conditions in the unit ball, goes left and ends at $\lambda = \lambda_*$. In dimensions $n = 4, 5, 6$ the branch ends at $\lambda = 0$ while for $n \geq 7$ it blows up to the right.

Theorem 1.1.1 and the additional properties stated above will be found as consequences of a more general result, Theorem 1.2.1 below, which concerns critical points with value zero for the function g_{λ_0} at a value $\lambda_0 > 0$. We state this result and find Theorem 1.1.1 as a corollary in Section 1.2, as a consequence of general properties of the function g_λ . The remaining sections will be devoted to the proof of Theorem 1.2.1.

1.2 Properties of g_λ and statement of the main result

Let $g_\lambda(x)$ be the function defined in (1.8). Our main result states that an interior bubbling solution is present as $\lambda \downarrow \lambda_0$, whenever g_{λ_0} has either a local maximum or a non-degenerate critical point with value 0.

Theorem 1.2.1. *Let us assume that for a number $\lambda_0 > 0$ one of the following two situations holds: (a) There is an open subset \mathcal{D} of Ω such that*

$$(1.12) \quad 0 = \sup_{\mathcal{D}} g_{\lambda_0} > \sup_{\partial \mathcal{D}} g_{\lambda_0};$$

(b) There is a point $x_0 \in \Omega$ such that $g_{\lambda_0}(x_0) = 0$, $\nabla g_{\lambda_0}(x_0) = 0$, and $D_x^2 g_{\lambda_0}(x_0)$ is non-singular; then for all $\lambda > \lambda_0$ sufficiently close to λ_0 there exists a solution u_λ to (1.1)

of the form

$$(1.13) \quad u_\lambda(x) = 3^{1/4} \left(\frac{\mu_\lambda}{\mu_\lambda^2 + |x - x_\lambda|^2} \right)^{\frac{1}{2}} + O(\mu_\lambda^{\frac{1}{2}}), \quad \text{with } \mu_\lambda = \gamma \frac{g_\lambda(x_\lambda)}{\lambda} > 0,$$

for some $\gamma > 0$. Here, $x_\lambda \in \mathcal{D}$ if (a) holds and $x_\lambda \rightarrow x_0$ if (b) holds. Besides, for certain positive numbers α, β we have that

$$(1.14) \quad \alpha(\lambda - \lambda_0) \leq g_\lambda(x_\lambda) \leq \beta(\lambda - \lambda_0).$$

Of course, a natural question is whether or not values λ_0 with the above characteristic do exist. We shall prove that the number λ_* defined by (1.9) is indeed positive and finite, and that $\lambda_0 = \lambda_*$ satisfies (1.12). That indeed proves Theorem 1.1.1 as a corollary of Theorem 1.2.1.

Implicit in condition (b) is the fact that $g_{\lambda_0}(x)$ is a smooth function and in (1.14) the fact that g_λ increases with λ . Next, we prove that $g_\lambda(x)$ is a smooth function, which is strictly increasing in λ .

Lemma 1.2.1. *The function g_λ is of class $C^\infty(\Omega)$. Furthermore, the function $\frac{\partial g_\lambda}{\partial \lambda}$ is well defined, smooth, and strictly positive in Ω . Its derivatives depend continuously on λ .*

Proof. We show that $g_\lambda \in C^k$ for any k . Fix $x \in \Omega$. Let $h_{1,\lambda}$ be the function defined in $\Omega \times \Omega$ by the relation

$$H_\lambda(x, y) = \beta_1 |x - y| + h_{1,\lambda}(x, y),$$

where $\beta_1 = -\frac{\lambda}{8\pi}$. Then $h_{1,\lambda}$ satisfies the boundary value problem

$$\begin{cases} -\Delta_y h_{1,\lambda} + \lambda h_{1,\lambda} &= -\lambda \beta_1 |x - y| & \text{in } \Omega \\ \frac{\partial h_{1,\lambda}(x,y)}{\partial \nu} &= \frac{\partial \Gamma(x-y)}{\partial \nu} - \beta_1 \frac{\partial |x-y|}{\partial \nu} & \text{on } \partial \Omega. \end{cases}$$

Elliptic regularity then yields that $h_{1,\lambda}(x, \cdot) \in C^2(\Omega)$. Its derivatives are clearly continuous as functions of the joint variable. Let us observe that the function $H_\lambda(x, y)$ is symmetric, thus so is h_1 , and then $h_{1,\lambda}(\cdot, y)$ is also of class C^2 with derivatives jointly continuous. It follows that $h_{1,\lambda}(x, y)$ is a function of class $C^2(\Omega \times \Omega)$. Iterating this procedure, we get that, for any k

$$H_\lambda(x, y) = \sum_{j=1}^k \beta_j |x - y|^{2j-1} + h_{k,\lambda}(x, y)$$

with $\beta_{j+1} = -\lambda \beta_j / ((2j+1)(2j+2))$ and $h_{k,\lambda}$ solution of the boundary value problem

$$\begin{cases} -\Delta_y h_{k,\lambda} + \lambda h_{k,\lambda} &= -\lambda \beta_k |x - y|^{2k-1} & \text{in } \Omega \\ \frac{\partial h_{k,\lambda}(x,y)}{\partial \nu} &= \frac{\partial \Gamma(x-y)}{\partial \nu} - \sum_{j=1}^k \beta_j \frac{\partial |x-y|^{2j-1}}{\partial \nu} & \text{on } \partial \Omega. \end{cases}$$

We may remark that $-\Delta_y h_{k+1,\lambda} + \lambda h_{k,\lambda} = 0$ in Ω . Elliptic regularity then yields that $h_{k,\lambda}$ is a function of class $C^{k+1}(\Omega \times \Omega)$. Let us observe now that by definition of g_λ we have $g_\lambda(x) = h_{k,\lambda}(x, x)$, and this concludes the proof of the first part of the lemma.

For a fixed given $x \in \Omega$, consider now the unique solution $F(y)$ of

$$-\Delta_y F + \lambda F = G(x, y) \quad y \in \Omega, \quad \frac{\partial F}{\partial \nu} = 0 \quad y \in \partial\Omega.$$

Elliptic regularity yields that F is at least of class $C^{0,\alpha}$. A convergence argument shows that actually $F(y) = \frac{\partial H_\lambda}{\partial \lambda}(x, y)$. Since $\lambda > 0$ and G is positive in Ω , using F_- as a test function we get that $F_- = 0$ in Ω , thus $F > 0$. Hence, in particular $\frac{\partial g_\lambda}{\partial \lambda}(x) = F(x) > 0$. Arguing as before, this function turns out to be smooth in x . The resulting expansions easily provide the continuous dependence in λ of its derivatives in the x -variable. \square

Lemma 1.2.2. *For each fixed $\lambda > 0$ we have that*

$$(1.15) \quad g_\lambda(x) \rightarrow -\infty, \quad \text{as } x \rightarrow \partial\Omega.$$

We define

$$M_\lambda = \sup_{x \in \Omega} g_\lambda(x).$$

Then

$$(1.16) \quad M_\lambda \rightarrow -\infty \quad \text{as } \lambda \rightarrow 0^+,$$

and

$$(1.17) \quad M_\lambda > 0 \quad \text{as } \lambda \rightarrow +\infty.$$

Proof. We prove first (1.15). Let $x \in \Omega$ be such that $d := \text{dist}(x, \partial\Omega)$ is small. Then there exists a unique $\bar{x} \in \partial\Omega$ so that $d = |x - \bar{x}|$. It is not restrictive to assume that $\bar{x} = 0$ and that the outer normal at \bar{x} to $\partial\Omega$ points toward the x_3 -direction. Let x^* be the reflexion point, namely $x^* = (0, 0, -d)$ and consider $H^*(y, x) = \frac{1}{4\pi|y-x^*|}$. The function $y \rightarrow H^*(y, x)$ solves

$$-\Delta_y \phi + \lambda \phi = \lambda \Gamma(y - x^*), \quad y \in \Omega, \quad \frac{\partial \phi}{\partial \nu} = \frac{\partial \Gamma}{\partial \nu}(y - x^*), \quad y \in \partial\Omega.$$

Observe now that

$$\Gamma(y - x^*) = \frac{1}{4\pi|x-y|} + \frac{1}{4\pi} \left[\frac{|y-x| - |y-x^*|}{|y-x||y-x^*|} \right] = \frac{1}{4\pi|x-y|} + O(1),$$

with $O(1)$ uniformly bounded, as $d \rightarrow 0$, for $y \in \partial\Omega$. This gives that $H_\lambda(y, x) = -H^*(y, x) + O(1)$, as $d \rightarrow 0$. Thus

$$H_\lambda(x, x) = -\frac{1}{4\pi \text{dist}(x, \partial\Omega)} + O(1),$$

as $d \rightarrow 0$. So we conclude the validity of (1.15).

Next we prove (1.16) and (1.17).

Proof of (1.16). Let $p(x) := \frac{1}{|\Omega|} \int_{\Omega} H_{\lambda}(x, y) dy$. Observe that

$$\begin{aligned} p(x) &= \frac{1}{|\Omega|} \int_{\Omega} \Gamma(x-y) dy + \frac{1}{\lambda|\Omega|} \int_{\Omega} \Delta H_{\lambda}(x, y) dy = \\ &= \frac{1}{|\Omega|} \int_{\Omega} \Gamma(x-y) dy + \frac{1}{\lambda|\Omega|} \int_{\partial\Omega} \frac{\partial H_{\lambda}}{\partial \nu} d\sigma(y) = \\ &= \frac{1}{|\Omega|} \int_{\Omega} \Gamma(x-y) dy + \frac{1}{\lambda|\Omega|} \int_{\partial\Omega} \frac{\partial \Gamma}{\partial \nu}(x-y) d\sigma(y) = -\frac{a}{\lambda|\Omega|} + p_0(x) \end{aligned}$$

where a is a positive constant and $p_0(x)$ is a bounded function. Define now $H_0(x, y)$ to be the bounded solution to

$$-\Delta H_0 = \frac{a}{|\Omega|}, \quad \frac{\partial H_0}{\partial \nu} = \frac{\partial \Gamma}{\partial \nu}(x-y) \quad y \in \partial\Omega, \quad \int_{\Omega} H_0 = 0.$$

We write

$$(1.18) \quad H_{\lambda}(x, y) = \underbrace{-\frac{a}{\lambda|\Omega|} + p_0(x)}_{=p(x)} + H_0(x, y) + \hat{H}(x, y).$$

By definition, \hat{H} solves

$$-\Delta \hat{H} + \lambda \hat{H} = \lambda [\Gamma(x-y) - H_0(x, y) + p_0(x)], \quad \frac{\partial \hat{H}}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \quad \int_{\Omega} \hat{H} = 0.$$

Thus we have that $\hat{H} = O(1)$, as $\lambda \rightarrow 0$. Taking this into account, from decomposition (1.18) we conclude that

$$\max_{x \in \Omega} g_{\lambda}(x) := \max_{x \in \Omega} H_{\lambda}(x, x) \leq -\frac{a}{\lambda|\Omega|} + O(1) \rightarrow -\infty, \quad \text{as} \quad \lambda \rightarrow 0.$$

This proves (1.16).

Proof of (1.17). Assume, by contradiction, that for some sequence $\lambda_n \rightarrow \infty$, as $n \rightarrow \infty$, one has $\max_{x \in \Omega} g_{\lambda_n}(x) \leq -\frac{1}{n}$. Fix $x_0 \in \Omega$, so that $\text{dist}(x_0, \partial\Omega) = \max_{x \in \Omega} \text{dist}(x, \partial\Omega)$. Thus we have that $-\Delta_y H_{\lambda_n}(y, x_0) \rightarrow \infty$, as $n \rightarrow \infty$. But on the other hand, a direct application of divergence theorem gives

$$\int_{\Omega} (-\Delta_y H_{\lambda_n}(y, x_0)) dy = - \int_{\partial\Omega} \frac{\partial \Gamma}{\partial \nu}(x_0 - y) d\sigma(y).$$

The left side of the above identity converges to ∞ as $n \rightarrow \infty$, while the right side is bounded. Thus we reach a contradiction, and (1.17) is proved. \square

The above considerations yield Theorem 1.1.1 as a consequence of Theorem 1.2.1.

Corollary 1.2.1. *The number λ_* given by (1.9) is well-defined and $0 < \lambda_* < +\infty$. Besides, the statement of Theorem 1.1.1 holds true.*

Proof. From Lemma 1.2.1, and relations (1.16) and (1.17), we deduce that the number λ_* is finite and positive. Besides, by its definition and the continuity of g_λ , it clearly follows that

$$\sup_{x \in \Omega} g_{\lambda_*}(x) = 0.$$

and that there is an open set \mathcal{D} with compact closure inside Ω such that

$$\sup_{\partial \mathcal{D}} g_{\lambda_*} < \sup_{\mathcal{D}} g_{\lambda_*} = 0.$$

Hence, Theorem 1.1.1 follows from Theorem 1.2.1. \square

As it was stated in the introduction, the number $\lambda^*(\Omega)$ can be explicitly computed in the case $\Omega = B(0, 1)$ as the following Lemma shows.

Lemma 1.2.3. *Let $\Omega = B(0, 1)$. The number λ^* defined in (1.9) is the unique solution of the equation*

$$\frac{\sqrt{\lambda} - 1}{\sqrt{\lambda} + 1} \exp(2\sqrt{\lambda}) = 1,$$

so that $\lambda^* \approx 1.43923$.

Proof. The maximum of $H_\lambda(x, x)$ is attained at $x = 0$. We compute the value $H_\lambda(0, 0)$ for $\lambda > 0$. The function $G_\lambda(0, y)$ is radially symmetric and it satisfies the equation

$$(1.19) \quad -\Delta_y G_\lambda + \lambda G_\lambda = \delta_0 \quad y \in B(0, 1), \quad \partial_r G_\lambda(0, y) = 0 \quad y \in \partial B(0, 1).$$

Letting $r = |y|$, we have

$$(1.20) \quad G_\lambda(0, y) = \frac{1}{4\pi r} \left[e^{-\sqrt{\lambda}r} + \frac{2 \sinh(\sqrt{\lambda}r)}{1 + \frac{\sqrt{\lambda}-1}{\sqrt{\lambda}+1} \exp(2\sqrt{\lambda})} \right].$$

Indeed, $\frac{e^{\sqrt{\lambda}r}}{r}$ and $\frac{e^{-\sqrt{\lambda}r}}{r}$ are radial solutions to $\Delta\phi + \lambda\phi = 0$ for $|y| > 0$. If we define

$$\mathcal{G}_A(r) = \frac{A}{r} \left[e^{\sqrt{\lambda}r} + e^{2\sqrt{\lambda}} \frac{\sqrt{\lambda} - 1}{\sqrt{\lambda} + 1} e^{-\sqrt{\lambda}r} \right],$$

where A is a constant, then $\partial_r \mathcal{G}_A = 0$ on $\partial B(0, 1)$. Since $\lim_{|y| \rightarrow 0} |y| G_\lambda(0, y) = \frac{1}{4\pi}$, if we choose

$$A_\lambda = \frac{1}{4\pi} \frac{1}{1 + \frac{\sqrt{\lambda}-1}{\sqrt{\lambda}+1} \exp(2\sqrt{\lambda})}$$

then \mathcal{G}_{A_λ} satisfies (1.19). By uniqueness $\mathcal{G}_{A_\lambda} = G_\lambda(0, y)$, and we get (1.20). Thus

$$H_\lambda(0, y) = \frac{1}{4\pi r} \left[(1 - e^{-\sqrt{\lambda}r}) - \frac{2 \sinh(\sqrt{\lambda}r)}{1 + \frac{\sqrt{\lambda}-1}{\sqrt{\lambda}+1} \exp(2\sqrt{\lambda})} \right],$$

and

$$g_\lambda(0) = H_\lambda(0, 0) = \frac{1}{4\pi} \left[\sqrt{\lambda} - \frac{2\sqrt{\lambda}}{1 + \frac{\sqrt{\lambda}-1}{\sqrt{\lambda}+1} \exp 2\sqrt{\lambda}} \right].$$

We deduce that λ^* is the unique value such that $g_{\lambda^*}(0) = 0$, therefore λ^* satisfies

$$\frac{\sqrt{\lambda} - 1}{\sqrt{\lambda} + 1} \exp(2\sqrt{\lambda}) = 1.$$

Then $\lambda^* \approx 1.43923$. □

The rest of this work will be devoted to the proof of Theorem 1.2.1. In Section 1.3 we define an approximate solution $U_{\zeta, \mu}$, for any given point $\zeta \in \Omega$, and any positive number μ , and we compute its energy $E_\lambda(U_{\zeta, \mu})$, where

$$(1.21) \quad E_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{\lambda}{2} \int_\Omega |u|^2 - \frac{1}{6} \int_\Omega |u|^6.$$

In Section 1.4 we establish that in the situation of Theorem 1.2.1 there are critical points of $E_\lambda(U_{\mu, \zeta})$ which persist under properly small perturbations of the functional. Observe now that, for $\varepsilon > 0$, if we consider the transformation

$$u(x) = \frac{1}{\varepsilon^{1/2}} v\left(\frac{x}{\varepsilon}\right)$$

then v solves the problem

$$(1.22) \quad \begin{cases} -\Delta v + \varepsilon^2 \lambda v - v^5 = 0, & v > 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial v}{\partial \nu} = 0 & & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$. We will look for a solution of (1.22) of the form $v = V + \phi$, where V is defined as $U_{\zeta, \mu}(x) = \frac{1}{\varepsilon^{1/2}} V\left(\frac{x}{\varepsilon}\right)$, and ϕ is a smaller perturbation. In Section 1.5 we discuss a linear problem that will be useful to find the perturbation ϕ . This is done in Section 1.6. We conclude our construction in the final argument, in Section 1.7.

1.3 Energy expansion

We fix a point $\zeta \in \Omega$ and a positive number μ . We denote in what follows

$$w_{\zeta, \mu}(x) = 3^{1/4} \frac{\mu^{1/2}}{\sqrt{\mu^2 + |x - \zeta|^2}}$$

which correspond to all positive solutions of the problem

$$-\Delta w - w^5 = 0, \quad \text{in } \mathbb{R}^3.$$

We define $\pi_{\zeta, \mu}(x)$ to be the unique solution of the problem

$$(1.23) \quad \begin{cases} -\Delta \pi_{\zeta, \mu} + \lambda \pi_{\zeta, \mu} = -\lambda w_{\zeta, \mu} & \text{in } \Omega \\ \frac{\partial \pi_{\zeta, \mu}}{\partial \nu} = -\frac{\partial w_{\zeta, \mu}}{\partial \nu} & \text{on } \partial\Omega. \end{cases}$$

We consider as a first approximation of the solution of (1.1) one of the form

$$(1.24) \quad U_{\zeta,\mu} = w_{\zeta,\mu} + \pi_{\zeta,\mu}.$$

Observe that $U_{\zeta,\mu}$ satisfies the problem

$$(1.25) \quad \begin{cases} -\Delta U_{\zeta,\mu} + \lambda U_{\zeta,\mu} = w_{\zeta,\mu}^5 & \text{in } \Omega \\ \frac{\partial U_{\zeta,\mu}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us also observe that

$$\int_{\Omega} w_{\zeta,\mu}^5 = C\mu^{1/2} (1 + o(1)), \quad \text{as } \mu \rightarrow 0,$$

which implies that $\frac{w_{\zeta,\mu}^5}{\int_{\Omega} w_{\zeta,\mu}^5} \rightarrow 0$, as $\mu \rightarrow 0$, uniformly on compacts subsets of $\bar{\Omega} \setminus \{\zeta\}$. It follows that on each of this subsets

$$(1.26) \quad U_{\zeta,\mu}(x) = \left(\int_{\Omega} w_{\zeta,\mu}^5 \right) G(x, \zeta) = C\mu^{1/2} (1 + o(1)) G_{\lambda}(x, \zeta)$$

where $G_{\lambda}(x, \zeta)$ denotes the Green's function defined in (1.6).

Using the transformation $U_{\zeta,\mu}(x) = \frac{1}{\varepsilon^{1/2}} V\left(\frac{x}{\varepsilon}\right)$ we see that V solves the problem

$$\begin{cases} -\Delta V + \varepsilon^2 \lambda V - w_{\zeta',\mu'}^5 = 0 & \text{in } \Omega_{\varepsilon} \\ \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega_{\varepsilon}, \end{cases}$$

where $w_{\zeta',\mu'}(x) = 3^{1/4} \frac{\mu'^{1/2}}{\sqrt{\mu'^2 + |x - \zeta'|^2}}$ and $\zeta' = \varepsilon^{-1}\zeta$, $\mu' = \varepsilon^{-1}\mu$.

The following lemma establishes the relationship between the functions $\pi_{\zeta,\mu}(x)$ and the regular part of the Green's function $G_{\lambda}(\zeta, x)$. Let us consider the (unique) radial solution $\mathcal{D}_0(z)$ of the problem in entire space,

$$\begin{cases} -\Delta \mathcal{D}_0 = \lambda 3^{1/4} \left[\frac{1}{\sqrt{1+|z|^2}} - \frac{1}{|z|} \right] & \text{in } \mathbb{R}^3 \\ \mathcal{D}_0 \rightarrow 0 & \text{as } |z| \rightarrow \infty. \end{cases}$$

$\mathcal{D}_0(z)$ is a $C^{0,1}$ function with $\mathcal{D}_0(z) \sim |z|^{-1} \log |z|$, as $|z| \rightarrow \infty$.

Lemma 1.3.1. *For any $\sigma > 0$ we have the validity of the following expansion as $\mu \rightarrow 0$*

$$(1.27) \quad \mu^{-1/2} \pi_{\mu,\zeta}(x) = -4\pi 3^{1/4} H_{\lambda}(\zeta, x) - \mu \mathcal{D}_0\left(\frac{x - \zeta}{\mu}\right) + \mu^{2-\sigma} \theta(\zeta, \mu, x).$$

where for $j = 0, 1, 2$, $i = 0, 1$ $i+j \leq 2$, the function $\mu^j \frac{\partial^{i+j}}{\partial \zeta^i \partial \mu^j} \theta(\zeta, \mu, x)$ is bounded uniformly on $x \in \Omega$, all small μ and ζ , in compacts subsets of Ω . We recall that H_{λ} is the function defined in (1.7).

Proof. Let us set $\mathcal{D}_1(x) = \mu \mathcal{D}_0(\mu^{-1}(x - \zeta))$, so that \mathcal{D}_1 satisfies

$$\begin{cases} -\Delta \mathcal{D}_1 = \lambda [\mu^{-1/2} w_{\zeta, \mu}(x) - 4\pi 3^{1/4} \Gamma(x - \zeta)] & \text{in } \Omega \\ \frac{\partial \mathcal{D}_1}{\partial \nu} \sim \mu^3 \log \mu & \text{on } \partial\Omega, \text{ as } \mu \rightarrow 0. \end{cases}$$

Let us write $S_1(x) = \mu^{-1/2} \pi_{\zeta, \mu}(x) + 4\pi 3^{1/4} H_\lambda(\zeta, x) + \mathcal{D}_1(x)$. With the notation of Lemma 1.3.1, this means

$$S_1(x) = \mu^{2-\sigma} \theta(\mu, \zeta, x).$$

Observe that for $x \in \partial\Omega$, as $\mu \rightarrow 0$,

$$\nabla(\mu^{-1/2} w_{\zeta, \mu}(x) + 4\pi 3^{1/4} \Gamma(x - \zeta)) \cdot \nu \sim \mu^2 |x - \zeta|^{-5}.$$

Using the above equations we find that S_1 satisfies

$$(1.28) \quad \begin{cases} -\Delta S_1 + \lambda S_1 = \lambda \mathcal{D}_1 & \text{in } \Omega \\ \frac{\partial S_1}{\partial \nu} = O(\mu^3 \log \mu) & \text{on } \partial\Omega. \end{cases}$$

Observe that, for any $p > 3$,

$$\int_{\Omega} |\mathcal{D}_1(x)|^p dx \leq \mu^{p+3} \int_{\mathbb{R}^3} |\mathcal{D}_0(x)|^p dx,$$

so that $\|\mathcal{D}_1\|_{L^p} \leq C_p \mu^{1+3/p}$. Elliptic estimates applied to problem (1.28) yield that, for any $\sigma > 0$, $\|S_1\|_{\infty} = O(\mu^{2-\sigma})$ uniformly on ζ in compact subsets of Ω . This yields the assertion of the lemma for $i, j = 0$.

We consider now the quantity $S_2 = \partial_{\zeta} S_1$. Observe that S_2 satisfies

$$\begin{cases} -\Delta S_2 + \lambda S_2 = \lambda \partial_{\zeta} \mathcal{D}_1 & x \in \Omega, \\ \frac{\partial S_2}{\partial \nu} = O(\mu^3 \log \mu) & \text{on } \partial\Omega. \end{cases}$$

Observe that $\partial_{\zeta} \mathcal{D}_1(x) = -\nabla D_0\left(\frac{x-\zeta}{\mu}\right)$, so that for any $p > 3$,

$$\int_{\Omega} |\partial_{\zeta} \mathcal{D}_1(x)|^p dx \leq \mu^{3+p} \int_{\mathbb{R}^3} |\nabla D_0(x)|^p dx$$

We conclude that $\|S_2\|_{\infty} = O(\mu^{2-\sigma})$, for any $\sigma > 0$. This gives the proof of the lemma for $i = 1, j = 0$. Now we consider $S_3 = \mu \partial_{\mu} S_1$. Then

$$\begin{cases} -\Delta S_3 + \lambda S_3 = \lambda \mu \partial_{\mu} \mathcal{D}_1 & x \in \Omega, \\ \frac{\partial S_3}{\partial \nu} = O(\mu^3 \log \mu) & \text{on } \partial\Omega. \end{cases}$$

Observe that

$$\mu \partial_{\mu} \mathcal{D}_1(x) = \mu (\mathcal{D}_0 - \bar{\mathcal{D}}_0) \left(\frac{x - \zeta}{\mu} \right),$$

where $\bar{\mathcal{D}}_0(z) = \nabla \mathcal{D}_0(z) \cdot z$. Thus, similarly as the estimate for S_1 itself we obtain $\|S_3\|_{\infty} = O(\mu^{2-\sigma})$, for any $\sigma > 0$. This yields the assertion of the lemma for $i = 0, j = 1$. The proof of the remaining estimates comes after applying again $\mu \partial_{\mu}$ to the equations obtained for S_2 and S_3 above, and the desired result comes after exactly the same arguments. This concludes the proof. \square

Classical solutions to (1.1) correspond to critical points of the energy functional (1.21). If there was a solution very close to U_{ζ^*, μ^*} for a certain pair (ζ^*, μ^*) , then we would formally expect E_λ to be nearly stationary with respect to variations of (ζ, μ) on $U_{\zeta, \mu}$ around this point. It seems important to understand critical points of the functional $(\zeta, \mu) \rightarrow E_\lambda(U_{\zeta, \mu})$. In the following lemma we find explicit asymptotic expressions for this functional.

Lemma 1.3.2. *For any $\sigma > 0$, as $\mu \rightarrow 0$, the following expansion holds*

$$(1.29) \quad E_\lambda(U_{\zeta, \mu}) = a_0 + a_1 \mu g_\lambda(\zeta) - a_2 \mu^2 \lambda - a_3 \mu^2 g_\lambda^2(\zeta) + \mu^{3-\sigma} \theta(\zeta, \mu)$$

where for $j = 0, 1, 2$, $i = 0, 1$, $i + j \leq 2$, the function $\mu^j \frac{\partial^{i+j}}{\partial \zeta^i \partial \mu^j} \theta(\zeta, \mu)$ is bounded uniformly on all small μ and ζ in compact subsets of Ω . The a_i 's are explicit positive constants, given by relation (1.33) below.

Proof. Observe that

$$E_\lambda(U_{\zeta, \mu}) = \text{I} + \text{II} + \text{III} + \text{IV} + \text{V} + \text{VI},$$

where

$$\begin{aligned} \text{I} &= \int_{\Omega} \left(\frac{1}{2} |\nabla w_{\zeta, \mu}|^2 - \frac{1}{6} w_{\zeta, \mu}^6 \right), & \text{II} &= \int_{\Omega} (\nabla w_{\zeta, \mu} \cdot \nabla \pi_{\zeta, \mu} - w_{\zeta, \mu}^5 \pi_{\zeta, \mu}), \\ \text{III} &= \frac{1}{2} \int_{\Omega} [|\nabla \pi_{\zeta, \mu}|^2 + \lambda(w_{\zeta, \mu} + \pi_{\zeta, \mu}) \pi_{\zeta, \mu}], \\ \text{IV} &= \frac{\lambda}{2} \int_{\Omega} (w_{\zeta, \mu} + \pi_{\zeta, \mu}) w_{\zeta, \mu}, & \text{V} &= -\frac{5}{2} \int_{\Omega} w_{\zeta, \mu}^4 \pi_{\zeta, \mu}^2, \\ \text{VI} &= -\frac{1}{6} \int_{\Omega} [(w_{\zeta, \mu} + \pi_{\zeta, \mu})^6 - w_{\zeta, \mu}^6 - 6w_{\zeta, \mu}^5 \pi_{\zeta, \mu} - 15w_{\zeta, \mu}^4 \pi_{\zeta, \mu}^2]. \end{aligned}$$

Multiplying equation $-\Delta w_{\zeta, \mu} = w_{\zeta, \mu}^5$ by $w_{\zeta, \mu}$ and integrating by parts in Ω we obtain

$$\begin{aligned} \text{I} &= \frac{1}{2} \int_{\partial\Omega} \frac{\partial w_{\zeta, \mu}}{\partial \nu} w_{\zeta, \mu} + \frac{1}{3} \int_{\Omega} w_{\zeta, \mu}^6 \\ &= \frac{1}{2} \int_{\partial\Omega} \frac{\partial w_{\zeta, \mu}}{\partial \nu} w_{\zeta, \mu} + \frac{1}{3} \int_{\mathbb{R}^3} w_{\zeta, \mu}^6 - \frac{1}{3} \int_{\mathbb{R}^3 \setminus \Omega} w_{\zeta, \mu}^6. \end{aligned}$$

Now, testing the same equation against $\pi_{\zeta, \mu}$, we find

$$\text{II} = \int_{\partial\Omega} \frac{\partial w_{\zeta, \mu}}{\partial \nu} \pi_{\zeta, \mu} = - \int_{\partial\Omega} \frac{\partial \pi_{\zeta, \mu}}{\partial \nu} \pi_{\zeta, \mu},$$

where we have used the fact that $\pi_{\zeta, \mu}$ solves problem (1.23). Testing the equation $-\Delta \pi_{\zeta, \mu} + \lambda \pi_{\zeta, \mu} = -\lambda w_{\zeta, \mu}$ against $\pi_{\zeta, \mu}$ and integrating by parts in Ω , we get

$$\text{III} = \frac{1}{2} \int_{\partial\Omega} \frac{\partial \pi_{\zeta, \mu}}{\partial \nu} \pi_{\zeta, \mu}.$$

Testing equation $-\Delta w_{\zeta, \mu} = w_{\zeta, \mu}^5$ against $U_{\zeta, \mu} = w_{\zeta, \mu} + \pi_{\zeta, \mu}$ and integrating by parts twice, we obtain

$$\text{IV} = \frac{1}{2} \int_{\partial\Omega} \frac{\partial \pi_{\zeta, \mu}}{\partial \nu} \pi_{\zeta, \mu} - \frac{1}{2} \int_{\partial\Omega} \frac{\partial w_{\zeta, \mu}}{\partial \nu} w_{\zeta, \mu} - \frac{1}{2} \int_{\Omega} w_{\zeta, \mu}^5 \pi_{\zeta, \mu}.$$

From the mean value formula, we get

$$\text{VI} = -10 \int_0^1 ds (1-s)^2 \int_{\Omega} (w_{\zeta,\mu} + s\pi_{\zeta,\mu})^3 \pi_{\zeta,\mu}^3.$$

Adding up the previous expressions we get so far

$$(1.30) \quad E_{\lambda}(U_{\zeta,\mu}) = \frac{1}{3} \int_{\mathbb{R}^3} w_{\zeta,\mu}^6 - \frac{1}{2} \int_{\Omega} w_{\zeta,\mu}^5 \pi_{\zeta,\mu} - \frac{5}{2} \int_{\Omega} w_{\zeta,\mu}^4 \pi_{\zeta,\mu}^2 + \mathcal{R}_1,$$

where

$$(1.31) \quad \mathcal{R}_1 = -\frac{1}{3} \int_{\mathbb{R}^3 \setminus \Omega} w_{\zeta,\mu}^6 - 10 \int_0^1 ds (1-s)^2 \int_{\Omega} (w_{\zeta,\mu} + s\pi_{\zeta,\mu})^3 \pi_{\zeta,\mu}^3.$$

We will expand the second integral term of expression (1.30). Using the change of variable $x = \zeta + \mu z$ and calling $\Omega_{\mu} = \mu^{-1}(\Omega - \zeta)$, we find that

$$A_1 = \int_{\Omega} w_{\zeta,\mu}^5 \pi_{\zeta,\mu} dx = \mu \int_{\Omega_{\mu}} w_{0,1}^5(z) \mu^{-1/2} \pi_{\zeta,\mu}(\zeta + \mu z) dz.$$

From Lemma 1.3.1, we have the expansion

$$\mu^{-1/2} \pi_{\zeta,\mu}(\zeta + \mu z) = -4\pi 3^{1/4} H_{\lambda}(\zeta + \mu z, \zeta) - \mu \mathcal{D}_0(z) + \mu^{2-\sigma} \theta(\zeta, \mu, \zeta + \mu z).$$

According to Lemma 1.2.1,

$$H_{\lambda}(\zeta + \mu z, \zeta) = g_{\lambda}(\zeta) - \frac{\lambda}{8\pi} \mu |z| + \Theta(\zeta, \zeta + \mu z),$$

where Θ is a function of class C^2 with $\Theta(\zeta, \zeta) = 0$. Using this fact, we obtain

$$A_1 = -4\pi 3^{1/4} \mu g_{\lambda}(\zeta) \int_{\mathbb{R}^3} w_{0,1}^5(z) dz - \mu^2 \int_{\mathbb{R}^3} w_{0,1}^5(z) \left[\mathcal{D}_0(z) - \frac{3^{1/4}}{2} \lambda |z| \right] dz + \mathcal{R}_2$$

with

$$(1.32) \quad \begin{aligned} \mathcal{R}_2 = & \mu \int_{\Omega_{\mu}} w_{0,1}^5(z) [\Theta(\zeta, \zeta + \mu z) + \mu^{2-\sigma} \theta(\zeta, \mu, \zeta + \mu z)] dz \\ & + \mu^2 \int_{\mathbb{R}^3 \setminus \Omega_{\mu}} w_{0,1}^5(z) \left[\mathcal{D}_0(z) - \frac{3^{1/4}}{2} \lambda |z| \right] dz + 4\pi 3^{1/4} \mu g_{\lambda}(\zeta) \int_{\mathbb{R}^3 \setminus \Omega_{\mu}} w_{0,1}^5(z) dz. \end{aligned}$$

Let us recall that $-\Delta \mathcal{D}_0 = 3^{1/4} \lambda \left[\frac{1}{\sqrt{1+|z|^2}} - \frac{1}{|z|} \right]$, so that,

$$\begin{aligned} - \int_{\mathbb{R}^3} w_{0,1}^5 \mathcal{D}_0(z) &= \int_{\mathbb{R}^3} \Delta w_{0,1} \mathcal{D}_0(z) \\ &= \int_{\mathbb{R}^3} w_{0,1} \Delta \mathcal{D}_0(z) = 3^{1/4} \lambda \int_{\mathbb{R}^3} w_{0,1} \left[\frac{1}{|z|} - \frac{1}{\sqrt{1+|z|^2}} \right]. \end{aligned}$$

Combining the above relations we get

$$A_1 = -4\pi 3^{1/4} \mu g_\lambda(\zeta) \int_{\mathbb{R}^3} w_{0,1}^5(z) dz \\ - \mu^2 \lambda 3^{1/4} \int_{\mathbb{R}^3} \left[w_{0,1}(z) \left(\frac{1}{\sqrt{1+|z|^2}} - \frac{1}{|z|} \right) - \frac{1}{2} w_{0,1}^5 |z| \right] dz + \mathcal{R}_2.$$

Let us consider now $A_2 = \int_{\Omega} w_{\zeta,\mu}^4 \pi_{\zeta,\mu}^2$. We have

$$A_2 = \mu \int_{\Omega_\mu} w_{0,1}^4(z) \pi_{\zeta,\mu}^2(\zeta + \mu z) dz \\ = \mu^2 \int_{\Omega_\mu} w_{0,1}^4(z) \left[-4\pi 3^{1/4} H_\lambda(\zeta + \mu z, \zeta) - \mu \mathcal{D}_0(z) + \mu^{2-\sigma} \theta(\zeta, \mu, \zeta + \mu z) \right]^2 dz,$$

which we expand as

$$A_2 = \mu^2 g_\lambda^2(\zeta) 16\pi^2 3^{1/2} \int_{\mathbb{R}^3} w_{0,1}^4 + \mathcal{R}_3.$$

Combining relation (1.30) with the above expressions, we get so far

$$E_\lambda(U_{\zeta,\mu}) = a_0 + a_1 \mu g_\lambda(\zeta) - a_2 \lambda \mu^2 - a_3 \mu^2 g_\lambda^2(\zeta) + \mathcal{R}_1 - \frac{1}{2} \mathcal{R}_2 - \frac{5}{2} \mathcal{R}_3,$$

where

$$a_0 = \frac{1}{3} \int_{\mathbb{R}^3} w_{0,1}^6, \quad a_1 = 2\pi 3^{1/4} \int_{\mathbb{R}^3} w_{0,1}^5, \quad a_3 = 40\pi^2 3^{1/2} \int_{\mathbb{R}^3} w_{0,1}^4 \\ a_2 = \frac{3^{1/4}}{2} \int_{\mathbb{R}^3} \left[w_{0,1}(z) \left(\frac{1}{|z|} - \frac{1}{\sqrt{1+|z|^2}} \right) + \frac{1}{2} w_{0,1}^5 |z| \right] dz.$$

An explicit computation shows that

$$(1.33) \quad a_0 = \frac{1}{4} \sqrt{3} \pi^2, \quad a_1 = 8\sqrt{3} \pi^2, \quad a_2 = \sqrt{3} \pi^2, \quad a_3 = 120\sqrt{3} \pi^4.$$

Finally, we want to establish the estimate $\mu^j \frac{\partial^{i+j}}{\partial \zeta^i \partial \mu^j} \mathcal{R}_l = O(\mu^{3-\sigma})$, for each $j = 0, 1, 2$, $i = 0, 1$, $i + j \leq 2$, $l = 1, 2, 3$, uniformly on all small μ and ζ in compact subsets of Ω . Arguing as in the proof of Lemma 2.1 in [dPMP05] we get the validity of the previous estimates. This concludes the proof. \square

1.4 Critical single-bubbling

The purpose of this section is to establish that in the situation of Theorem 1.2.1 there are critical points of $E_\lambda(U_{\mu,\zeta})$ which persist under properly small perturbations of the functional. As we shall rigorously establish later, this analysis does provide critical points of the full functional E_λ , namely solutions of (1.1), close to a single bubble of the form $U_{\mu,\zeta}$.

Let us suppose the situation (a) of local maximizer:

$$0 = \sup_{x \in \mathcal{D}} g_{\lambda_0}(x) > \sup_{x \in \partial \mathcal{D}} g_{\lambda_0}(x).$$

Then for λ close to λ_0 , $\lambda > \lambda_0$, we have

$$\sup_{x \in \mathcal{D}} g_{\lambda}(x) > A(\lambda - \lambda_0), \quad A > 0.$$

Let us consider the shrinking set

$$\mathcal{D}_{\lambda} = \left\{ y \in \mathcal{D} : g_{\lambda}(y) > \frac{A}{2}(\lambda - \lambda_0) \right\}.$$

Assume $\lambda > \lambda_0$ is sufficiently close to λ_0 so that $g_{\lambda} = \frac{A}{2}(\lambda - \lambda_0)$ on $\partial \mathcal{D}_{\lambda}$.

Now, let us consider the situation of Part (b). Since $g_{\lambda}(\zeta)$ has a non-degenerate critical point at $\lambda = \lambda_0$ and $\zeta = \zeta_0$, this is also the case at a certain critical point ζ_{λ} for all λ close to λ_0 where $|\zeta_{\lambda} - \zeta_0| = O(\lambda - \lambda_0)$.

Besides, for some intermediate point $\tilde{\zeta}_{\lambda}$,

$$g_{\lambda}(\zeta_{\lambda}) = g_{\lambda}(\zeta_0) + Dg_{\lambda}(\tilde{\zeta}_{\lambda})(\zeta_{\lambda} - \zeta_0) \geq A(\lambda - \lambda_0) + o(\lambda - \lambda_0)$$

for a certain $A > 0$. Let us consider the ball B_{ρ}^{λ} with center ζ_{λ} and radius $\rho(\lambda - \lambda_0)$ for fixed and small $\rho > 0$. Then we have that $g_{\lambda}(\zeta) > \frac{A}{2}(\lambda - \lambda_0)$ for all $\zeta \in B_{\rho}^{\lambda}$. In this situation we set $\mathcal{D}_{\lambda} = B_{\rho}^{\lambda}$.

It is convenient to make the following relabeling of the parameter μ . Let us set

$$(1.34) \quad \mu \equiv \frac{a_1}{2a_2} \frac{g_{\lambda}(\zeta)}{\lambda} \Lambda,$$

where $\zeta \in \mathcal{D}_{\lambda}$, and a_1, a_2 are the constants introduced in (1.29). We have the following result.

Lemma 1.4.1. *Assume the validity of one of the conditions (a) or (b) of Theorem 1.2.1, and consider a functional of the form*

$$(1.35) \quad \psi_{\lambda}(\Lambda, \zeta) = E_{\lambda}(U_{\mu, \zeta}) + g_{\lambda}(\zeta)^2 \theta_{\lambda}(\Lambda, \zeta)$$

where μ is given by (1.34) and

$$(1.36) \quad |\theta_{\lambda}| + |\nabla \theta_{\lambda}| + |\nabla \partial_{\Lambda} \theta_{\lambda}| \rightarrow 0, \quad \text{as } \lambda \downarrow \lambda_0$$

uniformly on $\zeta \in \mathcal{D}_{\lambda}$ and $\Lambda \in (\delta, \delta^{-1})$. Then ψ_{λ} has a critical point $(\Lambda_{\lambda}, \zeta_{\lambda})$ with $\zeta_{\lambda} \in \mathcal{D}_{\lambda}$, $\Lambda_{\lambda} \rightarrow 1$.

Proof. Using the expansion for the energy with μ given by (1.34) we find now that

$$(1.37) \quad \psi_\lambda(\Lambda, \zeta) \equiv E_\lambda(U_{\zeta, \mu}) + g_\lambda(\zeta)^2 \theta_\lambda(\Lambda, \zeta) = a_0 + \frac{a_1^2}{4a_2} \frac{g_\lambda(\zeta)^2}{\lambda} [2\Lambda - \Lambda^2] + g_\lambda(\zeta)^2 \theta_\lambda(\Lambda, \zeta)$$

where θ_λ satisfies property (1.36). Observe then that $\partial_\Lambda \psi_\lambda = 0$ if and only if

$$(1.38) \quad \Lambda = 1 + o(1) \theta_\lambda(\Lambda, \zeta),$$

where θ_λ is bounded in C^1 -sense, as $\lambda \downarrow \lambda_0$. This implies the existence of a unique solution close to 1 of this equation, $\Lambda = \Lambda_\lambda(\zeta) = 1 + o(1)$ with $o(1)$ small in C^1 sense, as $\lambda \downarrow \lambda_0$. Thus we get a critical point of ψ_λ if we have one of

$$(1.39) \quad p_\lambda(\zeta) \equiv \psi_\lambda(\Lambda_\lambda(\zeta), \zeta) = a_0 + c g_\lambda(\zeta)^2 [1 + o(1)]$$

with $o(1) \rightarrow 0$ as $\lambda \downarrow \lambda_0$ in C^1 -sense and $c > 0$. In the case of Part (a), i.e. of the maximizer, it is clear that we get a local maximum in the region \mathcal{D}_λ and therefore a critical point.

Let us consider the case (b). With the same definition for p_λ as above, we have

$$(1.40) \quad \nabla p_\lambda(\zeta) = 2c g_\lambda(\zeta) [\nabla g_\lambda + o(1) g_\lambda].$$

Consider a point $\zeta \in \partial \mathcal{D}_\lambda = \partial B_\rho^\lambda$. Then $|\nabla g_\lambda(\zeta)| = |D^2 g_\lambda(\tilde{x})(\zeta - \zeta_\lambda)| \geq \alpha \rho (\lambda - \lambda_0)$, for some $\alpha > 0$, when λ is close to λ_0 . We also have $g_\lambda(\zeta) = O(\lambda - \lambda_0)$, as $\lambda \downarrow \lambda_0$. We conclude that for all $t \in (0, 1)$, the function $\nabla g_\lambda + t o(1) g_\lambda$ does not have zeros on the boundary of this ball, provided that $\lambda - \lambda_0$ is small. In conclusion, its degree on the ball is constant along t . Since for $t = 0$ is not zero, thanks to non-degeneracy of the critical point ζ_λ , we conclude the existence of a zero of $\nabla p_\lambda(\zeta)$ inside \mathcal{D}_λ . This concludes the proof. \square

1.5 The linear problem

Hereafter we will look for a solution of (1.22) of the form $v = V + \phi$, so that ϕ solves the problem

$$(1.41) \quad \begin{cases} L(\phi) = N(\phi) + E & \text{in } \Omega_\varepsilon \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial \Omega_\varepsilon, \end{cases}$$

where

$$L(\phi) = -\Delta \phi + \varepsilon^2 \lambda \phi - 5V^4 \phi, \quad N(\phi) = (V + \phi)^5 - V^5 - 5V^4 \phi, \quad E = V^5 - w_{\zeta', \mu'}^5.$$

Here V is defined as $U_{\zeta, \mu}(x) = \frac{1}{\varepsilon^{1/2}} V\left(\frac{x}{\varepsilon}\right)$, where $U_{\zeta, \mu}$ is given by (1.24), while $\zeta' = \varepsilon^{-1} \zeta$, and $\mu' = \varepsilon^{-1} \mu$.

Let us recall that the only bounded solutions of the linear problem

$$\Delta z + 5w_{\zeta', \mu'}^4 z = 0 \quad \text{in } \mathbb{R}^3$$

are given by linear combinations of the functions

$$z_i(x) = \frac{\partial w_{\zeta', \mu'}}{\partial \zeta'_i}(x), \quad i = 1, 2, 3, \quad z_4(x) = \frac{\partial w_{\zeta', \mu'}}{\partial \mu'}(x).$$

In fact, the functions z_i , $i = 1, 2, 3, 4$ span the space of all bounded functions of the kernel of L in the case $\varepsilon = 0$. Observe also that

$$\int_{\mathbb{R}^3} z_j z_k = 0, \text{ if } j \neq k.$$

Rather than solving (1.41) directly, we will look for a solution of the following problem first: Find a function ϕ such that for certain numbers c_i ,

$$(1.42) \quad \begin{cases} L(\phi) = N(\phi) + E + \sum_{i=1}^4 c_i w_{\zeta', \mu'}^4 z_i & \text{in } \Omega_\varepsilon \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial \Omega_\varepsilon \\ \int_{\Omega_\varepsilon} w_{\zeta', \mu'}^4 z_i \phi = 0 & \text{for } i = 1, 2, 3, 4. \end{cases}$$

We next study the linear part of the problem (1.42). Given a function h , we consider the linear problem of finding ϕ and numbers c_i , $i = 1, 2, 3, 4$ such that

$$(1.43) \quad \begin{cases} L(\phi) = h + \sum_{i=1}^4 c_i w_{\zeta', \mu'}^4 z_i & \text{in } \Omega_\varepsilon \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial \Omega_\varepsilon \\ \int_{\Omega_\varepsilon} w_{\zeta', \mu'}^4 z_i \phi = 0 & \text{for } i = 1, 2, 3, 4. \end{cases}$$

Given a fixed number $0 < \sigma < 1$ we define the following norms

$$\|f\|_* := \sup_{x \in \Omega_\varepsilon} (1 + |x - \zeta'|^\sigma) |f(x)|, \quad \|f\|_{**} := \sup_{x \in \Omega_\varepsilon} (1 + |x - \zeta'|^{2+\sigma}) |f(x)|.$$

Proposition 1.5.1. *There exist positive numbers δ_0 , ε_0 , α_0 , β_0 and a constant $C > 0$ such that if*

$$(1.44) \quad \text{dist}(\zeta', \partial \Omega_\varepsilon) > \frac{\delta_0}{\varepsilon} \quad \text{and} \quad \alpha_0 < \mu' < \beta_0,$$

*then for any $h \in C^{0,\alpha}(\Omega_\varepsilon)$ with $\|h\|_{**} < \infty$ and for all $\varepsilon < \varepsilon_0$, problem (1.43) admits a unique solution $\phi = T(h) \in C^{2,\alpha}(\Omega_\varepsilon)$. Besides,*

$$(1.45) \quad \|T(h)\|_* \leq C \|h\|_{**} \quad \text{and} \quad |c_i| \leq C \|h\|_{**}, \quad i = 1, 2, 3, 4.$$

For the proof of Proposition 1.5.1 we will need the next

Lemma 1.5.1. *Assume the existence of a sequences $(\mu'_n)_{n \in \mathbb{N}}$, $(\zeta'_n)_{n \in \mathbb{N}}$, $(\varepsilon_n)_{n \in \mathbb{N}}$ such that $\alpha_0 < \mu'_n < \beta_0$, $\text{dist}(\zeta'_n, \partial \Omega_{\varepsilon_n}) > \frac{\delta_0}{\varepsilon_n}$, $\varepsilon_n \rightarrow 0$ and for certain functions ϕ_n and h_n with $\|h_n\|_{**} \rightarrow 0$ and scalars c_i^n , $i = 1, 2, 3, 4$, one has*

$$\begin{cases} L(\phi_n) = h_n + \sum_{i=1}^4 c_i^n w_{\zeta'_n, \mu'_n}^4 z_i^n & \text{in } \Omega_{\varepsilon_n} \\ \frac{\partial \phi_n}{\partial \nu} = 0 & \text{on } \partial \Omega_{\varepsilon_n} \\ \int_{\Omega_{\varepsilon_n}} w_{\zeta'_n, \mu'_n}^4 z_i^n \phi_n = 0 & \text{for } i = 1, 2, 3, 4, \end{cases}$$

where

$$z_i^n = \partial_{(\zeta'_n)_i} w_{\zeta'_n, \mu'_n}, \quad i = 1, 2, 3, \quad z_4^n = \partial_{\mu'_n} w_{\zeta'_n, \mu'_n}$$

then

$$\lim_{n \rightarrow \infty} \|\phi_n\|_* = 0$$

Proof. By contradiction, we may assume that $\|\phi_n\|_* = 1$. We will prove first the weaker assertion that

$$\lim_{n \rightarrow \infty} \|\phi_n\|_\infty = 0.$$

Also, by contradiction, we may assume up to a subsequence that $\lim_{n \rightarrow \infty} \|\phi_n\|_\infty = \gamma$, where $0 < \gamma \leq 1$. Let us see that

$$\lim_{n \rightarrow \infty} c_i^n = 0, \quad i = 1, 2, 3, 4.$$

Up to subsequence, we can suppose that $\mu'_n \rightarrow \mu'$, where $\alpha_0 \leq \mu' \leq \beta_0$. Testing the above equation against $z_j^n(x)$ and integrating by parts twice we get the relation

$$\int_{\Omega_{\varepsilon_n}} L(z_j^n) \phi_n + \int_{\partial\Omega_{\varepsilon_n}} \frac{\partial z_j^n}{\partial \nu} \phi_n = \int_{\Omega_{\varepsilon_n}} h_n z_j^n + \sum_{i=1}^4 c_i^n \int_{\Omega_{\varepsilon_n}} w_{\zeta'_n, \mu'_n}^4 z_i^n z_j^n.$$

Observe that

$$\left| \int_{\Omega_{\varepsilon_n}} L(z_j^n) \phi_n + \int_{\partial\Omega_{\varepsilon_n}} \frac{\partial z_j^n}{\partial \nu} \phi_n - \int_{\Omega_{\varepsilon_n}} h_n z_j^n \right| \leq C \|h_n\|_* + o(1) \|\phi_n\|_*,$$

$$\int_{\Omega_{\varepsilon_n}} w_{\zeta'_n, \mu'_n}^4 z_i^n z_j^n = C \delta_{i,j} + o(1).$$

Hence as $n \rightarrow \infty$, $c_i^n \rightarrow 0$, $i = 1, 2, 3, 4$.

Let $x_n \in \Omega_{\varepsilon_n}$ be such that $\sup_{x \in \Omega_{\varepsilon_n}} \phi_n(x) = \phi_n(x_n)$, so that ϕ_n maximizes at this point. We claim that there exists $R > 0$ such that

$$|x_n - \zeta'_n| \leq R, \quad \forall n \in \mathbb{N}.$$

This fact follows immediately from the assumption $\|\phi_n\|_* = 1$. We define $\tilde{\phi}_n(x) = \phi(x + \zeta'_n)$. Hence, up to subsequence, $\tilde{\phi}_n$ converges uniformly over compacts of \mathbb{R}^3 to a nontrivial bounded solution of

$$\begin{cases} -\Delta \tilde{\phi} - 5w_{0, \mu'}^4 \tilde{\phi} = 0 & \text{in } \mathbb{R}^3 \\ \int_{\mathbb{R}^3} w_{0, \mu'}^4 z_i \tilde{\phi} = 0 & \text{for } i = 1, 2, 3, 4, \end{cases}$$

where z_i is defined in terms of μ' and $\zeta' = 0$. Then $\tilde{\phi} = \sum_{i=1}^4 \alpha_i z_i(x)$. From the orthogonality conditions $\int_{\mathbb{R}^3} w_{0, \mu'}^4 z_i \tilde{\phi} = 0$, $i = 1, 2, 3, 4$, we deduce that $\alpha_i = 0$, $i = 1, 2, 3, 4$. This implies that $\tilde{\phi} = 0$, which is a contradiction with the hypothesis $\lim_{n \rightarrow \infty} \|\phi_n\|_\infty = \gamma > 0$.

Now we prove the stronger result: $\lim_{n \rightarrow \infty} \|\phi_n\|_* = 0$. Let us observe that ζ_n is a bounded sequence, so $\zeta_n \rightarrow \zeta$, as $n \rightarrow \infty$, up to subsequence. Let $R > 0$ be a fixed

number. Without loss of generality we can assume that $|\zeta_n - \zeta| \leq R/2$, for all $n \in \mathbb{N}$ and $B(\zeta, R) \subseteq \Omega$. We define $\psi_n(x) = \frac{1}{\varepsilon_n^\sigma} \phi_n\left(\frac{x}{\varepsilon_n}\right)$, $x \in \Omega$ (here we suppose without loss of generality that $\mu_n > 0$, $\forall n \in \mathbb{N}$). From the assumption $\lim_{n \rightarrow \infty} \|\phi_n\|_* = 1$ we deduce that

$$|\psi_n(x)| \leq \frac{1}{|x - \zeta_n|^\sigma}, \text{ for } x \in B(\zeta, R).$$

Also, $\psi_n(x)$ solves the problem

$$\begin{cases} -\Delta\psi_n + \lambda\psi_n &= \varepsilon_n^{-(2+\sigma)} \{5(\varepsilon_n^{1/2} U_{\zeta_n, \mu_n})^4 \psi + g_n + \sum_{i=1}^4 c_i^n \varepsilon_n^2 w_{\zeta_n, \mu_n}^4 Z_i^n\} & \text{in } \Omega \\ \frac{\partial \psi_n}{\partial \nu} &= 0 & \text{on } \partial\Omega, \end{cases}$$

where $g_n(x) = h_n\left(\frac{x}{\varepsilon_n}\right)$ and $Z_i^n(x) = z_i^n\left(\frac{x}{\varepsilon_n}\right)$. Since $\lim_{n \rightarrow \infty} \|h_n\|_{**} = 0$, we know that

$$|g_n(x)| \leq o(1) \frac{\varepsilon_n^{2+\sigma}}{\varepsilon_n^{2+\sigma} + |x - \zeta_n|^{2+\sigma}}, \text{ for } x \in \Omega.$$

Also, by (1.26), we see that

$$(1.46) \quad (\varepsilon_n^{1/2} U_{\zeta_n, \mu_n}(x))^4 = C\varepsilon_n^4(1 + o(1))G(x, \zeta_n)$$

away from ζ_n . It's easy to see that $\varepsilon_n^{-\sigma} \sum_{i=1}^4 c_i^n w_{\zeta_n, \mu_n}^4 Z_i^n = o(1)$ as $\varepsilon_n \rightarrow 0$, away from ζ_n . We conclude (by a diagonal convergence method) that $\psi_n(x)$ converges uniformly over compacts of $\overline{\Omega} \setminus \{\zeta\}$ to $\psi(x)$, a bounded solution of

$$-\Delta\psi + \lambda\psi = 0 \text{ in } \Omega \setminus \{\zeta\}, \quad \frac{\partial \psi}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

such that $|\psi(x)| \leq \frac{1}{|x - \zeta|^\sigma}$ in $B(\zeta, R)$. So ψ has a removable singularity at ζ , and we conclude that $\psi(x) = 0$. This implies that over compacts of $\overline{\Omega} \setminus \{\zeta\}$, we have

$$|\psi_n(x)| = o(1)\varepsilon_n^\sigma.$$

In particular, we conclude that for all $x \in \Omega \setminus B(\zeta_n, R/2)$ we have $|\psi_n(x)| \leq o(1)\varepsilon_n^\sigma$, which traduces into the following for ϕ_n

$$(1.47) \quad |\phi_n(x)| \leq o(1)\varepsilon_n^\sigma, \text{ for all } x \in \Omega_{\varepsilon_n} \setminus B(\zeta'_n, R/2\varepsilon_n).$$

Consider a fixed number M , such that $M < R/2\varepsilon_n$, for all n . Observe that $\|\phi_n\|_\infty = o(1)$, so

$$(1.48) \quad (1 + |x|^\sigma)|\phi_n(x)| \leq o(1) \text{ for all } x \in \overline{B(\zeta'_n, M)}.$$

We claim that

$$(1.49) \quad (1 + |x|^\sigma)|\phi_n(x)| \leq o(1) \text{ for all } x \in A_{\varepsilon_n, M},$$

where $A_{\varepsilon_n, M} = B(\zeta'_n, R/2\varepsilon_n) \setminus \overline{B(\zeta'_n, M)}$. This assertion follows from the fact that the operator L satisfies the weak maximum principle in $A_{\varepsilon_n, M}$ (choosing a larger M and a subsequence if necessary): If u satisfies $L(u) \leq 0$ in $A_{\varepsilon_n, M}$ and $u \leq 0$ in $\partial A_{\varepsilon_n, M}$, then $u \leq 0$ in $A_{\varepsilon_n, M}$. This result is just a consequence of the fact that $L(|x - \zeta'_n|^{-\sigma}) \geq 0$ in $A_{\varepsilon_n, M}$, if M is larger enough but independent of n .

We now prove (1.49) with the use of a suitable barrier. Observe that from (1.47) we deduce the existence of $\eta_n^1 \rightarrow 0$, as $n \rightarrow 0$ such that $\varepsilon_n^{-\sigma} |\phi_n(x)| \leq \eta_n^1$, for all x such that $|x| = R/2\varepsilon_n$. From (1.48) we deduce the existence of $\eta_n^2 \rightarrow 0$, as $n \rightarrow \infty$ such that $M^\sigma |\phi_n(x)| \leq \eta_n^2$, for all x such that $|x| = M$. Also, there exists $\eta_n^3 \rightarrow 0$, as $n \rightarrow \infty$ such that

$$|x + \zeta'_n|^{2+\sigma} |L(\phi_n)| \leq \eta_n^3 \quad \text{in } A_{\varepsilon_n, M}.$$

We define the barrier function $\varphi_n(x) = \eta_n \frac{1}{|x - \zeta'_n|^\sigma}$, with $\eta_n = \max\{\eta_n^1, \eta_n^2, \eta_n^3\}$. Observe that $L(\varphi_n) = \sigma(1 - \sigma)\eta_n \frac{1}{|x - \zeta'_n|^{2+\sigma}} + (\varepsilon_n^2 \lambda - 5V^4)\eta_n \frac{1}{|x - \zeta'_n|^\sigma}$. It's not hard to see that $|L(\phi_n)| \leq CL(\varphi_n)$ in $A_{\mu_n, M}$ and $|\phi_n(x)| \leq C\varphi_n$ in $\partial A_{\varepsilon_n, M}$, where C is a constant independent of n . From the weak maximum principle we deduce (1.49) and the fact $\|\phi_n\|_\infty = o(1)$. From (1.47), (1.48), (1.49), and $\|\phi_n\|_\infty = o(1)$ we conclude that $\|\phi_n\|_* = o(1)$ which is a contradiction with the assumption $\|\phi_n\|_* = 1$. The proof of Lemma (1.5.1) is completed. \square

Proof of proposition 1.5.1. Let us consider the space

$$H = \left\{ \phi \in H^1(\Omega) \left| \int_{\Omega_\varepsilon} w_{\zeta', \mu'}^4 z_i \phi = 0, i = 1, 2, 3, 4 \right. \right\}$$

endowed with the inner product, $[\phi, \psi] = \int_{\Omega_\varepsilon} \nabla \phi \nabla \psi + \varepsilon^2 \lambda \int_{\Omega_\varepsilon} \phi \psi$. Problem (1.43) expressed in the weak form is equivalent to that of finding $\phi \in H$ such that

$$[\phi, \psi] = \int_{\Omega_\varepsilon} \left[5V^4 \phi + h + \sum_{i=1}^4 c_i w_{\zeta', \mu'}^4 z_i \right] \psi \quad \text{for all } \psi \in H.$$

The a priori estimate $\|T(h)\|_* \leq C\|h\|_{**}$ implies that for $h \equiv 0$ the only solution is 0. With the aid of Riesz's representation theorem, this equation gets rewritten in H in operational form as one in which Fredholm's alternative is applicable, and its unique solvability thus follows. Besides, it is easy to conclude (1.45) from an application of Lemma (1.5.1). \square

It is important, for later purposes, to understand the differentiability of the operator $T : h \rightarrow \phi$, with respect to the variables μ' and ζ' , for a fixed ε (we only let μ and ζ to vary). We have the following result

Proposition 1.5.2. *Under the conditions of Proposition 1.5.1, the map T is of class C^1 and the derivative $\nabla_{\zeta', \mu'} \partial_{\mu'} T$ exists and is a continuous function. Besides, we have*

$$\|\nabla_{\zeta', \mu'} T(h)\|_* + \|\nabla_{\zeta', \mu'} \partial_{\mu'} T(h)\|_* \leq C\|h\|_{**}.$$

Proof. Let us consider differentiation with respect to the variable ζ'_k , $k = 1, 2, 3$. For notational simplicity we write $\frac{\partial}{\partial \zeta'_k} = \partial_{\zeta'_k}$. Let us set, still formally, $X_k = \partial_{\zeta'_k} \phi$. Observe that X_k satisfies the following equation

$$L(X_k) = 5\partial_{\zeta'_k}(V^4)\phi + \sum_{i=1}^4 d_i^k w_{\zeta', \mu'}^4 z_i + \sum_{i=1}^4 c_i \partial_{\zeta'_k}(w_{\zeta', \mu'}^4 z_i) \quad \text{in } \Omega_\varepsilon.$$

Here $d_i^k = \partial_{\zeta'_k} c_i$, $i = 1, 2, 3$. Besides, from differentiating the orthogonality conditions $\int_{\Omega_\varepsilon} w_{\zeta', \mu'}^4 z_i = 0$, $i = 1, 2, 3, 4$, we further obtain the relations

$$\int_{\Omega_\varepsilon} X_k w_{\zeta', \mu'}^4 z_i = - \int_{\Omega_\varepsilon} \phi \partial_{\zeta'_k}(w_{\zeta', \mu'}^4 z_i) \quad i = 1, 2, 3, 4.$$

Let us consider constants b_i , $i = 1, 2, 3, 4$, such that

$$\int_{\Omega_\varepsilon} \left(X_k - \sum_{i=1}^4 b_i z_i \right) w_{\zeta', \mu'}^4 z_j = 0 \quad j = 1, 2, 3, 4.$$

These relations amount to

$$\sum_{i=1}^4 b_i \int_{\Omega_\varepsilon} w_{\zeta', \mu'}^4 z_i z_j = \int_{\Omega_\varepsilon} \phi \partial_{\zeta'_k}(w_{\zeta', \mu'}^4 z_j) \quad j = 1, 2, 3, 4.$$

Since this system is diagonal dominant with uniformly bounded coefficients, we see that it is uniquely solvable and that

$$b_i = O(\|\phi\|_*)$$

uniformly on ζ' , μ' in the considered region. Also, it is not hard to see that

$$\|\phi \partial_{\zeta'_k}(V^4)\|_{**} \leq C\|\phi\|_*.$$

From Proposition (1.45), we conclude

$$\left\| \sum_{i=1}^4 c_i \partial_{\zeta'_k}(w_{\zeta', \mu'}^4 z_i) \right\|_{**} \leq C\|h\|_{**}.$$

We set $X = X_k - \sum_{i=1}^4 b_i z_i$, so X satisfies

$$L(X) = f + \sum_{i=1}^4 b_i^k w_{\zeta', \mu'}^4 z_i, \quad \text{in } \Omega_\varepsilon,$$

where

$$f = 5\partial_{\zeta'_k}(V^4)\phi + \sum_{i=1}^4 b_i L(z_i) + \sum_{i=1}^4 c_i \partial_{\zeta'_k}(w_{\zeta', \mu'}^4 z_i)$$

Observe that also,

$$\int_{\Omega_\varepsilon} X w_{\zeta', \mu'}^4 z_i = 0 \quad i = 1, 2, 3, 4.$$

This computation is not just formal. Indeed, one gets, as arguing directly by definition shows,

$$\partial_{\xi'_k} \phi = \sum_{i=1}^4 b_i z_i + T(f) \quad \text{and} \quad \|\partial_{\xi'_k} \phi\|_* \leq C \|h\|_{**}.$$

The corresponding result for differentiation with respect to μ' follows similarly. This concludes the proof. \square

1.6 The nonlinear problem

We recall that our goal is to solve (1.41). Rather than doing so directly, we shall solve first the intermediate nonlinear problem (1.42) using the theory developed in the previous section. We have the next result.

Lemma 1.6.1. *Under the assumptions of Proposition 1.5.1, there exist numbers $\varepsilon_1 > 0$, $C_1 > 0$, such that for all $\varepsilon \in (0, \varepsilon_1)$ problem (1.42) has a unique solution ϕ which satisfies*

$$\|\phi\|_* \leq C_1 \varepsilon.$$

Proof. First we assume that μ and ζ are such that $\|E\|_{**} < \varepsilon_1$. In terms of the operator T defined in Proposition (1.5.1), problem (1.42) becomes

$$\phi = T(N(\phi) + E) \equiv A(\phi).$$

For a given $\gamma > 0$, let us consider the region $\mathcal{F}_\gamma := \{\phi \in C(\overline{\Omega}_\varepsilon) \mid \|\phi\|_* \leq \gamma \|E\|_{**}\}$. From Proposition (1.5.1), we get

$$\|A(\phi)\|_* \leq C [\|N(\phi)\|_{**} + \|E\|_{**}].$$

The definition of N immediately yields $\|N(\phi)\|_{**} \leq C_0 \|\phi\|_*^2$. It is also easily checked that N satisfies, for $\phi_1, \phi_2 \in \mathcal{F}_\gamma$,

$$\|N(\phi_1) - N(\phi_2)\|_{**} \leq C_0 \gamma \|E\|_{**} \|\phi_1 - \phi_2\|_*.$$

Hence for a constant C_1 depending on C_0, C , we get

$$\|A(\phi)\|_* \leq C_1 [\gamma^2 \|E\|_{**} + 1] \|E\|_{**}, \quad \|A(\phi_1) - A(\phi_2)\|_* \leq C_1 \gamma \|E\|_{**} \|\phi_1 - \phi_2\|_*.$$

Choosing $\gamma = C_1$, $\varepsilon_1 = \frac{1}{2C_1^2}$, we conclude that A is a contraction mapping of \mathcal{F}_γ , and therefore a unique fixed point of A exists in this region.

Assume now that μ' and ζ' satisfy conditions (1.44). Recall that the error introduced by our first approximation is

$$E = V^5 - w_{\zeta', \mu'}^5 = (w_{\mu', \xi'}(y) + \sqrt{\varepsilon} \pi(\varepsilon y))^5 - w_{\zeta', \mu'}^5 \quad y \in \Omega_\varepsilon.$$

Using several times estimate (1.27), we get

$$\|E\|_{**} = O(\|\sqrt{\varepsilon} \pi(\varepsilon y) w_{\zeta', \mu'}(y)\|_{**}^4) = O\left(\left\|\varepsilon \frac{\mu'^2}{(\mu'^2 + |y - \zeta'|^2)^2}\right\|_{**}\right) = O(\varepsilon),$$

as $\varepsilon \rightarrow 0$. This concludes the proof of the Lemma. \square

We shall next analyze the differentiability of the map $(\zeta', \mu') \rightarrow \phi$. We start by computing the $\|\cdot\|_{**}$ -norm of the partial derivatives of E with respect to μ' and ζ' . Observe that

$$\partial_{\mu'} w_{\zeta', \mu'} = \frac{1}{2\sqrt{\mu'}} \frac{|y - \zeta'|^2 - \mu'^2}{(|y - \zeta'|^2 + \mu'^2)^{\frac{3}{2}}}.$$

We derive E with respect to μ' and deduce

$$\begin{aligned} \|\partial_{\mu'} E\|_{**} &= O\left(\|\sqrt{\varepsilon}\pi(\varepsilon y)w_{\zeta', \mu'}^3 \partial_{\mu'} w_{\zeta', \mu'}\|_{**}\right) + O\left(\|\varepsilon^{\frac{3}{2}}w_{\zeta', \mu'}^4 \partial_{\mu'} \pi(\varepsilon y)\|_{**}\right) \\ &= O\left(\left\|\varepsilon \frac{\mu'(|y - \zeta'|^2 - \mu'^2)}{(\mu'^2 + |y - \zeta'|^2)^3}\right\|_{**}\right) + O\left(\left\|\varepsilon^{\frac{3}{2}} \frac{\mu'^2}{(\mu'^2 + |y - \zeta'|^2)^2}\right\|_{**}\right) \\ &= O(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Note that

$$|\partial_{\zeta'_i} w_{\zeta', \mu'}| = \frac{\sqrt{\mu'}|y - \zeta'|}{(\mu'^2 + |y - \zeta'|^2)^{\frac{3}{2}}} \quad \text{for } i = 1, 2, 3.$$

We derive E with respect to ζ'_i and deduce for $i = 1, 2, 3$

$$\begin{aligned} \|\partial_{\zeta'_i} E\|_{**} &= O\left(\|\sqrt{\varepsilon}\pi(\varepsilon y)w_{\zeta', \mu'}^3 \partial_{\zeta'_i} w_{\zeta', \mu'}\|_{**}\right) + O\left(\|\varepsilon^{\frac{3}{2}}w_{\zeta', \mu'}^4 \partial_{\zeta'_i} \pi(\varepsilon y)\|_{**}\right) \\ &= O\left(\left\|\varepsilon \frac{\mu'^2 |y - \zeta'|}{(\mu'^2 + |y - \zeta'|^2)^3}\right\|_{**}\right) + O\left(\left\|\varepsilon^{\frac{3}{2}} \frac{\mu'^2}{(\mu'^2 + |y - \zeta'|^2)^2}\right\|_{**}\right) \\ &= O(\varepsilon), \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Moreover, a similar computation shows that

$$\|\nabla_{\zeta', \mu'} \partial_{\mu'} E\|_{**} \leq O(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0.$$

Collecting all the previous computations we conclude there exists a positive constant $C > 0$ such that

$$\|E\|_{**} + \|\nabla_{\zeta', \mu'} E\|_{**} + \|\nabla_{\zeta', \mu'} \partial_{\mu'} E\|_{**} \leq C\varepsilon.$$

Concerning the differentiability of the function $\phi(\zeta')$, let us write

$$A(x, \varphi) = \varphi - T(N(\varphi) + E).$$

Observe that $A(\zeta', \phi) = 0$ and $\partial_{\phi} A(\zeta', \phi) = I + O(\varepsilon)$. It follows that for small ε , the linear operator $\partial_{\phi} A(\zeta', \phi)$ is invertible, with uniformly bounded inverse. It also depends continuously on its parameters. Differentiating respect to ζ' we obtain

$$\partial_{\zeta'} A(\zeta', \phi) = -(\partial_{\zeta'} T)(N(\phi) + E) - T(\partial_{\zeta'} N(\phi) + \partial_{\zeta'} R).$$

where the previous expression depend continuously on their parameters. Hence the implicit function theorem yields that $\phi(\zeta')$ is a C^1 function. Moreover, we have

$$\partial_{\zeta'} \phi = -(\partial_{\phi} A(\zeta', \phi))^{-1}[\partial_{\zeta'} A(\zeta', \phi)].$$

By Taylor expansion we conclude that

$$\|\partial_{\zeta'} N(\phi)\|_{**} \leq C(\|\phi\|_* + \|\partial_{\zeta'} \phi\|_*) \|\phi\|_* \leq C(\|E\|_{**} + \|\partial_{\zeta'} \phi\|_*) \|E\|_{**}.$$

Using Proposition (1.5.2), we have

$$\|\partial_{\zeta'} \phi\|_* \leq C(\|N(\phi) + E\|_{**} + \|\partial_{\zeta'} N(\phi)\|_{**} + \|\partial_{\zeta'} E\|_{**}),$$

for some constant $C > 0$. Hence, we conclude that

$$\|\partial_{\zeta'} \phi\|_* \leq C(\|E\|_{**} + \|\partial_{\zeta'} E\|_{**}).$$

A similar argument shows that, as well

$$\|\partial_{\mu'} \phi\|_* \leq C(\|E\|_{**} + \|\partial_{\mu'} E\|_{**}),$$

and moreover

$$\|\nabla_{\zeta', \mu'} \partial_{\mu'} \phi\|_* \leq C(\|E\|_{**} + \|\nabla_{\zeta', \mu'} E\|_{**} + \|\nabla_{\zeta', \mu'} \partial_{\mu'} E\|_{**}).$$

This can be summarized as follows.

Lemma 1.6.2. *Under the assumptions of Propositions 1.5.1 and 1.6.1, consider the map*

$$(\zeta', \mu') \rightarrow \phi.$$

The partial derivatives $\nabla_{\zeta'} \phi$, $\nabla_{\mu'} \phi$, $\nabla_{\zeta', \mu'} \partial_{\mu'} \phi$ exist and define continuous functions of (ζ', μ') . Besides, there exist a constant $C_2 > 0$, such that

$$\|\nabla_{\zeta', \mu'} \phi\|_* + \|\nabla_{\zeta', \mu'} \partial_{\mu'} \phi\|_* \leq C_2 \varepsilon$$

for all $\varepsilon > 0$ small enough.

After Problem (1.41) has been solved, we will find solutions to the full problem (1.42) if we manage to adjust the pair (ζ', μ') in such a way that $c_i(\zeta', \mu') = 0$, $i = 1, 2, 3, 4$. This is the *reduced problem*. A nice feature of this system of equations is that it turns out to be equivalent to finding critical points of a functional of the pair (ζ', μ') which is close, in appropriate sense, to the energy of the single bubble U .

1.7 Final argument

In order to obtain a solution of (1.1) we need to solve the system of equations

$$(1.50) \quad c_j(\zeta', \mu') = 0 \quad \text{for all } j = 1, \dots, 4.$$

If (1.50) holds, then $v = V + \phi$ will be a solution to (1.41). This system turns out to be equivalent to a variational problem. We define

$$F(\zeta', \mu') = E_\varepsilon(V + \phi),$$

where $\phi = \phi(\zeta', \mu')$ is the unique solution of (1.42) that we found in the previous section, and E_ε is the scaled energy functional

$$E_\varepsilon(U) = \frac{1}{2} \int_{\Omega} |\nabla U|^2 + \frac{\varepsilon^2 \lambda}{2} \int_{\Omega} |U|^2 - \frac{1}{6} \int_{\Omega} |U|^6.$$

Observe that $E_\lambda(U_{\zeta', \mu'}) = E_\varepsilon(V)$.

Critical points of F correspond to solutions of (1.50), under the assumption that the error E is small enough.

Lemma 1.7.1. *Under the assumptions of Propositions 1.5.1 and 1.6.1, the functional $F(\zeta', \mu')$ is of class C^1 and for all ε sufficiently small, if $\nabla F = 0$ then (ζ', μ') satisfies system (1.50).*

Proof. Let us differentiate with respect to μ' .

$$\partial_{\mu'} F(\zeta', \mu') = DE_\varepsilon(V + \phi)[\partial_{\mu'} V + \partial_{\mu'} \phi] = \sum_{j=1}^4 \int_{\Omega_\varepsilon} c_j w_{\zeta', \mu'}^4 z_j [\partial_{\mu'} V + \partial_{\mu'} \phi].$$

From the results of the previous section, we deduce $\partial_{\mu'} F$ is continuous. If $\partial_{\mu'} F(\zeta', \mu') = 0$, then

$$\sum_{j=1}^4 \int_{\Omega_\varepsilon} c_j w_{\zeta', \mu'}^4 z_j [\partial_{\mu'} V + \partial_{\mu'} \phi] = 0.$$

Since $\|\partial_{\mu'}\|_* \leq C(\|E\|_{**} + \|\partial_{\mu'} E\|_{**})$, we have, as $\varepsilon \rightarrow 0$, $\partial_{\mu'} V + \partial_{\mu'} \phi = z_4 + o(1)$, with $o(1)$ small in terms of the $**$ -norm as $\varepsilon \rightarrow 0$. Similarly, we check that $\partial_{\zeta'_k} F$ is continuous,

$$\partial_{\zeta'_k} F(\zeta', \mu') = DE_\varepsilon(V + \phi)[\partial_{\zeta'_k} V + \partial_{\zeta'_k} \phi] = \sum_{j=1}^4 \int_{\Omega_\varepsilon} c_j w_{\zeta', \mu'}^4 z_j [\partial_{\zeta'_k} V + \partial_{\zeta'_k} \phi] = 0,$$

and $\partial_{\zeta'_k} V + \partial_{\zeta'_k} \phi = z_k + o(1)$, for $k = 1, 2, 3$.

We conclude that if $\nabla F = 0$ then

$$\sum_{j=1}^4 \int_{\Omega_\varepsilon} w_{\zeta', \mu'}^4 z_j [z_i + o(1)] = 0 \quad i = 1, 2, 3, 4,$$

with $o(1)$ small in the sense of $**$ -norm as $\varepsilon \rightarrow 0$. The above system is diagonal dominant and we thus get $c_j = 0$ for all $j = 1, 2, 3, 4$. \square

In the following Lemma we find an expansion for the functional F .

Lemma 1.7.2. *Under the assumptions of Propositions 1.5.1 and 1.6.1, the following expansion holds*

$$F(\zeta', \mu') = E_\varepsilon(V) + [\|E\|_{**} + \|\nabla_{\zeta', \mu'} E\|_{**} + \|\nabla_{\zeta', \mu'} \partial_{\mu'} E\|_{**}] \theta(\zeta', \mu'),$$

where θ satisfies

$$|\theta| + |\nabla_{\zeta', \mu'} \theta| + |\nabla_{\zeta', \mu'} \partial_{\mu'} \theta| \leq C,$$

for a positive constant C .

Proof. Using the fact that $DF(V + \phi)[\phi] = 0$, a Taylor expansion gives

$$\begin{aligned} F(V + \phi) - F(V) &= \int_0^1 D^2F(V + t\phi)[\phi, \phi](1 - t)dt \\ &= \int_0^1 \left(\int_{\Omega_\varepsilon} [N(\phi) + E] \phi + \int_{\Omega_\varepsilon} 5 [V^4 - (V + t\phi)^4] \phi^2 \right) (1 - t)dt. \end{aligned}$$

Since $\|\phi\|_* \leq C\|E\|_{**}$, we get

$$F(V + \phi) - F(V) = O(\|E\|_{**}^2).$$

Observe that

$$\begin{aligned} \nabla_{\zeta', \mu'} [F(V + \phi) - F(V)] \\ = \int_0^1 \left(\int_{\Omega_\varepsilon} \nabla_{\zeta', \mu'} [(N(\phi) + E) \phi] + \int_{\Omega_\varepsilon} 5 \nabla_{\zeta', \mu'} [(V^4 - (V + t\phi)^4) \phi^2] \right) (1 - t)dt. \end{aligned}$$

Since $\|\nabla_{\zeta', \mu'} \phi\|_* \leq C[\|E\|_{**} + \|\nabla_{\zeta', \mu'} E\|_{**}]$, we easily see that

$$\nabla_{\zeta', \mu'} [F(V + \phi) - F(V)] = O(\|E\|_{**}^2 + \|\nabla_{\zeta', \mu'} E\|_{**}^2).$$

A similar computation yields the result. □

We have now all the elements to prove our main result.

Proof of Theorem 1.2.1. We choose

$$\mu = \frac{a_1 g_\lambda(\zeta)}{2a_2 \lambda} \Lambda,$$

where $\zeta \in \mathcal{D}_\lambda$. A similar computation to the one performed in the previous section, based in the estimate (1.27), allows us to show that

$$\|E\|_{**} + \|\nabla_{\zeta', \mu'} E\|_{**} + \|\nabla_{\zeta', \mu'} \partial_{\mu'} E\|_{**} \leq C\mu^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \delta_\lambda,$$

where $\delta_\lambda = \sup_{\mathcal{D}_\lambda} (|g_\lambda| + |\nabla g_\lambda|)$. Since $\alpha_0 < \mu' < \beta_0$, we have

$$F(\zeta', \mu') = E_\varepsilon(V) + \mu^2 \delta_\lambda^2 \theta(\zeta', \mu'),$$

with $|\theta| + |\nabla_{\zeta', \mu'} \theta| + |\nabla_{\zeta', \mu'} \partial_{\mu'} \theta| \leq C$. We define $\psi_\lambda(\Lambda, \zeta) = F(\zeta', \mu')$. We conclude that

$$\psi_\lambda(\Lambda, \zeta) = E_\lambda(U_{\zeta, \mu}) + g_\lambda(\zeta)^2 \theta_\lambda(\zeta, \Lambda),$$

where θ_λ is as in Lemma 1.4.1. Thus, ψ_λ has a critical point as in the statement of Lemma 1.4.1. This concludes the proof of our main result, with the constant $\gamma = \frac{a_1}{2a_2}$. □

Chapter 2

The Keller-Segel model of chemotaxis

Abstract

In this chapter, which is based on a joint work with Denis Bonheure and Jean-Baptiste Casteras [BCR], we construct several families of radial solutions to the stationary Keller-Segel equation in the two-dimensional unit ball. The first family consists in solutions which blow up at the origin of the ball and concentrate on the boundary of the unit ball. The second family is made of solutions which blow up at the origin and concentrate on an interior sphere, while the solutions of the third type blow up at the origin and concentrate simultaneously on an interior sphere and on the boundary of the unit ball. We also show how to construct other families of multi-layered radial solutions, under a suitable non-degeneracy assumption.

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2.1 Introduction

Chemotaxis is the influence of chemical substances in an environment on the movement of organisms. In order to modelize the aggregation of cellular slime molds like the Dic-

tyostelium discoideum, Keller and Segel introduced in 1970 a system of two strongly coupled parabolic differential equations involving :

- $u(x, t)$ the myxamoebae density of the cellular slime molds at time t and point x ,
- $v(x, t)$ the chemoattractant concentration.

More precisely, the system has the form

$$(2.1) \quad \begin{cases} \frac{\partial v}{\partial t} = \Delta v - \nabla(v \nabla u), v > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial t} = \Delta u - u + v, u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and ν is the inner unit normal vector of $\partial\Omega$.

In order to understand the global dynamics of this system, it is important to study steady state solutions of it, namely solutions to

$$(2.2) \quad \begin{cases} \Delta v - \nabla(v \nabla u) = 0, v > 0 & \text{in } \Omega \\ \Delta u - u + v = 0, u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us notice that the system (2.2) reduces to a scalar equation. Indeed, observe that $v = \lambda e^u$ for some positive constant λ , solves the first equation. Thus the second equation becomes the so-called Keller-Segel equation

$$(2.3) \quad \begin{cases} -\Delta u + u - \lambda e^u = 0, u > 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

In the one-dimensional case, Schaaf [Sch85] proved the existence of non-trivial solutions to (2.3). In higher dimension, when Ω is a ball, radial solutions to (2.3) have been constructed by Biler [Bil98]. For general two-dimensional domains, the first existence results were obtained by Wang and Wei [WW02] and independently by Senba and Suzuki [SS00d], when the parameter λ is small enough. Moreover, Senba and Suzuki [SS00d, SS02] studied the asymptotic behavior when $\lambda \rightarrow 0$ of solutions u_λ to (2.3) with finite mass

$$\lim_{\lambda \rightarrow 0} \lambda \int_{\Omega} e^{u_\lambda} = C_0 > 0.$$

Let $\mathcal{G}(x, y)$, $y \in \bar{\Omega}$ be the Green's function of the problem

$$-\Delta_x \mathcal{G} + \mathcal{G} = \delta_y \text{ in } \Omega, \quad \frac{\partial \mathcal{G}}{\partial \nu_x} = 0 \text{ on } \partial\Omega.$$

Senba and Suzuki showed that there exist points $\xi_i \in \Omega$, $i \leq k$ and $\eta_i \in \partial\Omega$, $k < i \leq m$ for which

$$(2.4) \quad u_\lambda(x) \xrightarrow{\lambda \rightarrow 0} \sum_{i=1}^k 8\pi \mathcal{G}(x, \xi_i) + \sum_{i=k+1}^m 4\pi \mathcal{G}(x, \eta_i),$$

uniformly on compact subsets of $\bar{\Omega} \setminus \{\xi_1, \dots, \xi_k\}$. The counter part of this result has been obtained by del Pino and Wei [dPW06] where they constructed, for any given integers k and l , a family of solutions to (2.3) satisfying (2.4) for a suitable choice of points ξ_i, η_i .

Recently, the study of solutions to (2.3) concentrating on higher dimensional set with unbounded mass has been initiated. When $\Omega = B_1(0) \subset \mathbb{R}^n, n \geq 2$, Pistoia and Vaira [PV15] constructed a family u_λ of radial solutions blowing-up in all the boundary of Ω and such that

$$\lim_{\lambda \rightarrow 0} \int_{B_1(0)} \lambda e^{u_\lambda(x)} dx = \infty.$$

More precisely, their solutions satisfy

$$\lim_{\lambda \rightarrow 0} \varepsilon_\lambda u_\lambda = \sqrt{2}\mathcal{U},$$

C^0 -uniformly on compact sets of Ω where $\varepsilon_\lambda \approx -\frac{1}{\ln \lambda}$ and \mathcal{U} is the unique radial solution to

$$-\mathcal{U}'' - \frac{n-1}{r}\mathcal{U}' + \mathcal{U} = 0 \text{ in } \Omega, \quad \mathcal{U} = 1 \text{ on } \partial\Omega.$$

Near the boundary $\partial B_1(0)$, their solutions, up to rescaling, behave like the one-dimensional half standard bubble i.e.

$$-w'' = e^w \text{ on } \mathbb{R}, \quad \text{s.t.} \int_{\mathbb{R}} e^w < \infty.$$

Let us also point out that del Pino, Pistoia and Vaira [dPPV16] constructed a similar family of solutions to (2.3) for general two-dimensional domains concentrating on the whole boundary. Very recently, existence of solutions concentrating on sub-manifolds of the boundary has also been investigated. Agudelo and Pistoia [AP16] consider domains of the following form

$$\Omega = \{(y_1, x') \in \mathbb{R}^{N-1} \times \mathbb{R} : (|y_1|, x') \in D\},$$

where D is a smooth bounded domain in \mathbb{R}^2 such that

$$\bar{D} \subset \{(x_1, x') \in \mathbb{R} \times \mathbb{R} : x_1 > 0\}.$$

Looking for solutions of (2.3) invariant under the action of the group of linear isometries Γ of \mathbb{R}^{N-1} given by

$$g(y_1, x') = (gy_1, x'),$$

namely solutions of the form $v(y_1, x') = u(|y_1|, x')$, we see that (2.3) can be rewritten as

$$(2.5) \quad -\operatorname{div}(a(x)\nabla u) + a(x)u = \lambda a(x)e^u \text{ in } D, \quad \partial_\nu u = 0 \text{ on } \partial D,$$

where $a(x) = x_1^{N-2}$. Under suitable assumptions, Agudelo and Pistoia are able to construct families of solutions to (2.5) concentrating at points on the boundary of ∂D or converging to points belonging to ∂D when $\lambda \rightarrow 0$. Up to rescaling, their solutions behave near the concentration points like the unique solution to

$$-\Delta U = e^U \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^U dx < \infty.$$

Observe that the solutions they construct for (2.5) correspond to solutions of (2.3) concentrating along $(N - 2)$ -dimensional minimal manifolds of the boundary (or converging to the boundary) of Ω diffeomorphic to the unit sphere of dimension $N - 2$.

In all the following, we will suppose that $\Omega = B_1(0) \subset \mathbb{R}^N$, $N \geq 2$. In [BCN17b], we performed a bifurcation analysis of radial solutions to (2.3). First, let us notice that for $\lambda < 1/e$, it is possible to show that (2.3) is equivalent to

$$(2.6) \quad \begin{cases} -\Delta v + v &= e^{\mu(v-1)}, v > 0 & \text{in } B_1(0) \\ \partial_\nu v &= 0 & \text{on } \partial B_1(0) \end{cases}$$

for $\mu > 1$. Observe that this equation admits two constant solutions $v \equiv 1$ and another one denoted by $\underline{v}_\mu < 1$. Let λ_i^{rad} be the i -th eigenvalue of the operator $-\Delta + Id$ in $B_1(0)$ with Neumann boundary conditions, restricted to the radial functions. We obtained the following bifurcation result.

Theorem 2.1.1 ([BCN17b]). *For every $i \geq 2$, $(\lambda_i^{rad}, 1)$ is a bifurcation point for problem (2.6). Let \mathcal{B}_i be the continuum that branches out of $(\lambda_i^{rad}, 1)$. It holds that:*

- (i) *the branches \mathcal{B}_i are unbounded and do not intersect; close to $(\lambda_i^{rad}, 1)$, \mathcal{B}_i is a C^1 -curve;*
- (ii) *if $u_\mu \in \mathcal{B}_i$ then $u_\mu > 0$;*
- (iii) *each branch consists of two connected components: the component \mathcal{B}_i^- , along which $u_\mu(0) < 1$, and the component \mathcal{B}_i^+ , along which $u_\mu(0) > 1$;*
- (iv) *if $u_\mu \in \mathcal{B}_i$ then $u_\mu - 1$ has exactly $i - 1$ zeros, u'_μ has exactly $i - 2$ zeros and each zero of u'_μ lies between two zeros of $u_\mu - 1$;*
- (v) *the functions satisfying $u_\mu(0) < 1$ are uniformly bounded in the C^1 -norm.*

In term of this bifurcation result, we believe that the solutions constructed by Pistoia and Vaira [PV15] belong to \mathcal{B}_1^- while the solutions constructed by del Pino and Wei [dPW06] (when restricted to the 2-dimensional ball) belong to \mathcal{B}_1^+ . In [BCN17b], we also constructed multi-layer solutions, namely solutions concentrating along an arbitrary number of internal spheres, by combining variational and perturbative methods. Using a different approach based on a fixed point argument, we were able in [BCN17a] to prove the existence and to obtain very precise asymptotics of these solutions provided that a non-degeneracy condition holds true (we will comment on it later).

In the present chapter, we restrict ourselves to the case where $\Omega = B_1(0) \subset \mathbb{R}^2$. In view of Theorem 2.1.1, our goal is to construct and characterize solutions belonging to \mathcal{B}_i^+ for $i \geq 2$. More precisely, we want to construct solutions to (2.3) concentrating at the origin and on spheres (belonging to the interior or the boundary of $B_1(0)$). Our first result deals with the case of solutions concentrating at the origin and along $\partial B_1(0)$.

Theorem 2.1.2. *There exists $\bar{\lambda} > 0$ such that, for all $\lambda \in (0, \bar{\lambda})$, there exists a radial solution u_λ of (2.3) such that*

$$\lim_{\lambda \rightarrow 0} \lambda \int_{B_1(0)} e^{u_\lambda(x)} dx = \infty,$$

$$\lim_{\lambda \rightarrow 0} (u_\lambda - U_\lambda) = 0$$

uniformly on compact subsets of $B_1(0) \setminus \{0\}$,

$$\lambda e^{u_\lambda} \rightharpoonup 8\pi\delta_0 \quad \text{in } B_{1/2}(0),$$

and

$$\sqrt{2}\lambda e^{u_\lambda} + (|\partial_\nu U_\lambda(1)|)^{-1} \delta_{\{x \in \mathbb{R}^N \mid |x|=1\}} \rightharpoonup 0 \quad \text{in } B_1(0) \setminus B_{1/2}(0),$$

where $\varepsilon_\lambda \approx 1/|\ln \lambda|$ and U_λ is a solution to

$$\begin{cases} -U_\lambda'' - \frac{1}{r}U_\lambda' + U_\lambda = 0 & \text{in } (0, 1), \\ \lim_{r \rightarrow 0^+} \frac{U_\lambda(r)}{-\ln r} = 4, \quad U_\lambda(1) = \frac{\sqrt{2}}{\varepsilon_\lambda}. \end{cases}$$

Next, we investigate multi-layered solutions. It is well-known that the localization of the layers are linked with the existence of certain Green's functions. Extending the results of [BGNT16], we construct singular at the origin Green's function having k local maximum normalized to 1. Moreover, at each of these maxima, the derivative of the Green's function satisfies a weak reflexion's law. More precisely, we have

Theorem 2.1.3. *Let $k \in \mathbb{N} \setminus \{0\}$. For any constant $b > 0$ small enough, we have that:*

(i) *There exists a configuration $0 = \alpha_0 < \alpha_1 < \dots < \alpha_k = 1$ and a continuous function $U_{b,k}$ such that*

$$\begin{cases} -U_{b,k}'' - \frac{1}{r}U_{b,k}' + U_{b,k} = 0 & \text{in } \cup_{i=0}^{k-1} (\alpha_i, \alpha_{i+1}) \\ \lim_{r \rightarrow 0^+} -\frac{U_{b,k}(r)}{\ln r} = b \\ U_{b,k}(\alpha_i) = 1 & \text{for every } i = 1, \dots, k \end{cases}$$

and satisfying the reflection law

$$(2.7) \quad \lim_{\varepsilon \rightarrow 0^-} \frac{U_{b,k}(\alpha_i + \varepsilon) - U_{b,k}(\alpha_i)}{\varepsilon} = - \lim_{\varepsilon \rightarrow 0^+} \frac{U_{b,k}(\alpha_i + \varepsilon) - U_{b,k}(\alpha_i)}{\varepsilon}$$

for every $i = 1, \dots, k-1$.

(ii) *There exists a configuration $0 = \tilde{\alpha}_0 < \tilde{\alpha}_1 < \dots < \tilde{\alpha}_k < \tilde{\alpha}_{k+1} = 1$ and a continuous function $\tilde{U}_{b,k}$ such that*

$$\begin{cases} -\tilde{U}_{b,k}'' - \frac{1}{r}\tilde{U}_{b,k}' + \tilde{U}_{b,k} = 0 & \text{in } \cup_{i=0}^k (\tilde{\alpha}_i, \tilde{\alpha}_{i+1}) \\ \lim_{r \rightarrow 0^+} -\frac{\tilde{U}_{b,k}(r)}{\ln r} = b \\ \tilde{U}_{b,k}'(1) = 0 \\ \tilde{U}_{b,k}(\tilde{\alpha}_i) = 1 & \text{for every } i = 1, \dots, k \end{cases}$$

and satisfying the reflection law

$$(2.8) \quad \lim_{\varepsilon \rightarrow 0^-} \frac{\tilde{U}_{b,k}(\tilde{\alpha}_i + \varepsilon) - \tilde{U}_{b,k}(\tilde{\alpha}_i)}{\varepsilon} = - \lim_{\varepsilon \rightarrow 0^+} \frac{\tilde{U}_{b,k}(\tilde{\alpha}_i + \varepsilon) - \tilde{U}_{b,k}(\tilde{\alpha}_i)}{\varepsilon}$$

for every $i = 1, \dots, k$.

As in [BCN17a], in order to construct multi-layered solutions, we will need a certain non-degeneracy condition on the Green's function. This condition depends on the determinant M_k of the squared matrix $A_{i,j}$ of $k \times k$ entries defined as follows. The elements of the diagonal are given by

$$A_{i,i} = (U'_{\sigma_i}{}^+ + U'_{\sigma_i}{}^-)(\alpha_i) \quad \text{for } i = 1, \dots, k,$$

the elements of the subdiagonal are given by

$$A_{i+1,i} = U'_{\sigma_i}{}^-(\alpha_{i+1}) \quad \text{for } i = 1, \dots, k-1,$$

the elements of the superdiagonal are given by

$$A_{i,i+1} = U'_{\sigma_{i+1}}{}^+(\alpha_i) \quad \text{for } i = 1, \dots, k-1,$$

and the rest of the entries of the matrix are 0. Here,

$$U'_{\sigma_i}{}^{\pm}(\alpha_j) = \frac{\partial}{\partial \sigma_i} (U'_{\varepsilon,a,b,\sigma}{}^{\pm}(\alpha_j + \sigma_j))|_{\varepsilon,a,b,\sigma=0},$$

where $U_{\varepsilon,a,b,\sigma}$ is defined in (2.51). Let us point out that this determinant played the same role in the construction of multi-layered solutions regular at the origin (see [BCN17a]). Assuming that $M_k \neq 0$, we are able to construct two families of k -layers solutions singular at the origin of (2.3) in $B_1(0)$, modeled on U_{k+1} and on \tilde{U}_k respectively. Numerical simulations and explicit computations in dimension 3 suggest that $M_k > 0$ for any $k \in \mathbb{N} \setminus \{0\}$, however we are only able to prove it for $k = 1$. In this case, our results read as follows

Theorem 2.1.4 (Internal layer solution singular at the origin of the ball). *There exists $\lambda(1) > 0$ such that for all $\lambda \in (0, \lambda(1))$, there exists a family of radial solutions \tilde{u}_λ to (2.3) in $B_1(0)$ such that*

$$\lim_{\lambda \rightarrow 0} \left(u_\lambda - \frac{\sqrt{2}}{\varepsilon_\lambda} \tilde{U}_{4\frac{\varepsilon_\lambda}{\sqrt{2}},1} \right) = 0,$$

uniformly on compact subsets of $B_1(0) \setminus \{0\}$,

$$\lambda e^{u_\lambda} \rightharpoonup 8\pi\delta_0 \quad \text{in } B_{\alpha_1/2}(0)$$

and

$$\varepsilon_\lambda \lambda e^{u_\lambda} + (|\partial_\nu \tilde{U}_{4\frac{\varepsilon_\lambda}{\sqrt{2}},1}(\alpha_1)|)^{-1} \delta_1 \rightharpoonup 0 \quad \text{in } B_1(0) \setminus \{0\},$$

where $\varepsilon_\lambda \approx 1/|\ln \lambda|$.

Theorem 2.1.5 (Singular solution at the origin of the ball with an internal layer and a boundary layer). *There exists $\lambda(2) > 0$ such that for all $\lambda \in (0, \lambda(2))$, there exists a family of radial solutions u_λ to (2.3) in $B_1(0)$ such that*

$$\lim_{\lambda \rightarrow 0} \left(u_\lambda - \frac{\sqrt{2}}{\varepsilon_\lambda} U_{4\frac{\varepsilon_\lambda}{\sqrt{2}},2} \right) = 0$$

uniformly on compact subsets of $B_1(0) \setminus \{0\}$,

$$\lambda e^{u_\lambda} \rightarrow 8\pi\delta_0 \quad \text{in } B_{\alpha_1/2}(0),$$

and

$$\varepsilon_\lambda \lambda e^{u_\lambda} + (|\partial_\nu U_{4\frac{\varepsilon_\lambda}{\sqrt{2}}, 2}(\alpha_1)|)^{-1} \delta_{\alpha_1} + (|\partial_\nu U_{4\frac{\varepsilon_\lambda}{\sqrt{2}}, 2}(1)|)^{-1} \delta_1 \rightarrow 0 \quad \text{in } B_1(0) \setminus \{0\},$$

where $\varepsilon_\lambda \approx 1/|\ln \lambda|$.

The plan of this chapter is the following. In Section 2.2, we describe the ansatz of solution we will use to prove Theorem 2.1.2. We then estimate the error introduced by our ansatz in Section 2.3. In Section 2.4, we prove the solvability of the linearized equation in our ansatz. This allows us to use a fixed point argument to prove Theorem 2.1.2. We then give the proof of Theorem 2.5.1 leading to Theorems 2.1.4 and 2.1.5 in Section 2.5. Finally, we prove Theorem 2.1.3 and the non-degeneracy condition $M_1 \neq 0$ in the Appendix.

2.2 The approximate solution

We recall that we are looking for a radial solution of (2.3) concentrating at 0 and on $\partial B_1(0)$. In order to do so, we take an ansatz of solution of the form

$$U = \begin{cases} u_0 & \text{in } [0, \delta), \\ u_1 & \text{in } [\delta, 2\delta), \\ u_2 & \text{in } [2\delta, 1 - 2\delta_1), \\ u_3 & \text{in } [1 - 2\delta_1, 1 - \delta_1), \\ u_4 & \text{in } [1 - \delta_1, 1]. \end{cases}$$

In a first time, let us describe intuitively our ansatz. In the previous definition, δ and δ_1 are suitable constants depending on λ . Near the origin, we want $U = u_0$ to behave approximately like U_0 , the two dimensional standard bubble given by

$$(2.9) \quad U_0(r) = \ln \frac{8\mu^2}{(\mu^2\lambda + r^2)^2},$$

for some constant $\mu > 0$. Let us recall that these functions correspond to all solutions of the problem

$$\begin{cases} -\Delta U_0 = \lambda e^{U_0} & \text{in } \mathbb{R}^2 \\ \lambda \int_{\mathbb{R}^2} e^{U_0} dx < +\infty. \end{cases}$$

Near the unit sphere $\partial B_1(0)$, we want that $U = u_4$ behaves up to rescaling like $W_{\tilde{\mu}} - \ln \lambda$ where $W_{\tilde{\mu}}$ is the one dimensional standard bubble solving $-w'' = e^w$ in \mathbb{R} given by

$$(2.10) \quad W_{\tilde{\mu}}(r) = \ln \left(\frac{4}{\tilde{\mu}^2} \frac{e^{-\frac{\sqrt{2}(r-1)}{\tilde{\mu}}}}{\left(1 + e^{-\frac{\sqrt{2}(r-1)}{\tilde{\mu}}}\right)^2} \right),$$

for some $\tilde{\mu}$ depending on λ to be determined later. Far from the origin and $\partial B_1(0)$, we choose $U = G$ where G is the singular at the origin Green's function given in Lemma 2.A.2 for some suitable constant \tilde{b} depending on λ . Finally, we choose u_1 and u_3 to be linear interpolations between u_{i-1} and u_{i+1} , for $i = 1, 3$, namely,

$$(2.11) \quad u_i(r) = \chi_i(r)u_{i-1}(r) + (1 - \chi_i(r))u_{i+1}(r),$$

where $\chi_i \in C^2((0, 1))$ are cut-off functions such that

$$\chi_1(r) \equiv 1 \text{ in } (0, \delta), \quad \chi_1(r) \equiv 0 \text{ in } (2\delta, 1), \quad |\chi_1(r)| \leq 1, \quad |\chi_1'(r)| \leq c, \quad |\chi_1''(r)| \leq c$$

and

$$\chi_3 \equiv 1 \text{ in } (0, 1 - 2\delta_1), \quad \chi_3 \equiv 0 \text{ in } (1 - \delta_1, 1), \quad |\chi_3(r)| \leq 1, \quad |\chi_3'(r)| \leq c, \quad |\chi_3''(r)| \leq c.$$

2.2.1 Construction of u_4

First, we set ε such that

$$(2.12) \quad \ln \frac{4}{\varepsilon^2} - \ln \lambda = \frac{\sqrt{2}}{\varepsilon}$$

and choose $\delta_1 = \varepsilon^\eta$, for some $\eta \in (\frac{2}{3}, 1)$. We define u_4 in the same way as the function “ u_1 ” of [PV15] (or [BCN17a]) with $r_0 = 1$. The construction of this function is quite lengthy so we only briefly recall it and refer to the above two papers for more details. We take u_4 as follows

$$u_4 = \underbrace{W_{\tilde{\mu}} - \ln \lambda + \alpha_\varepsilon}_{1^{\text{st order approx.}}} + \underbrace{v_\varepsilon + \beta_\varepsilon}_{2^{\text{nd order}}} + \underbrace{z_\varepsilon}_{3^{\text{rd order}}},$$

where

- $W_{\tilde{\mu}}$ is defined in (2.10) for some $\tilde{\mu} = O(\varepsilon)$ (see Subsection 2.2.2 for the precise definition). We also set

$$W\left(\frac{r-1}{\tilde{\mu}}\right) + \ln \frac{4}{\tilde{\mu}^2} - \ln 4 = W_{\tilde{\mu}}(r).$$

- α_ε satisfies

$$\begin{cases} -(\alpha_\varepsilon)'' - \frac{n-1}{r}(\alpha_\varepsilon)' = \frac{n-1}{r}(w_\varepsilon^i)' - w_\varepsilon^i + \ln \lambda & \text{in } (0, 1) \\ \alpha_\varepsilon(1) = (\alpha_\varepsilon)'(1) = 0, \end{cases}$$

and the following estimate holds, for $s \leq 0$,

$$(2.13) \quad \alpha_\varepsilon(\tilde{\mu}s + 1) = \tilde{\mu}(\alpha_\varepsilon)_1(s) + \tilde{\mu}^2(\alpha_\varepsilon)_2(s) + O(\tilde{\mu}^3 s^4),$$

where

$$(\alpha_\varepsilon)_1(s) = -(n-1) \int_0^s W(\sigma) d\sigma + \frac{\tilde{\mu}}{\sqrt{2\varepsilon}} s^2,$$

and

$$\begin{aligned}
 (\alpha_\varepsilon^i)_2(s) &= \int_0^s \int_0^\sigma (W(\rho) - \ln 4) d\rho d\sigma + (n-1)(n-2) \int_0^s \int_0^\sigma W(\rho) d\rho d\sigma \\
 &\quad + (n-1) \int_0^s \sigma W(\sigma) d\sigma - s^2 \ln \left(\frac{\tilde{\mu}}{\varepsilon} \right).
 \end{aligned}$$

- v_ε satisfies

$$\begin{cases}
 -(v_\varepsilon)'' - e^{W_{\tilde{\mu}}} v_\varepsilon = \tilde{\mu} e^{W_{\tilde{\mu}}} (\alpha_\varepsilon)_1 \left(\frac{r-1}{\tilde{\mu}} \right) & \text{in } \mathbb{R} \\
 v_\varepsilon(1) = (v_\varepsilon)'(1) = 0,
 \end{cases}$$

where $(\alpha_\varepsilon^i)_1$ is defined in (2.13). Moreover, we have

$$(2.14) \quad v_\varepsilon(r) = \nu_1(r-1) + \nu_2 \tilde{\mu} + O(\tilde{\mu} e^{-\frac{|r-1|}{\tilde{\mu}}}),$$

where

$$\nu_2 \in \mathbb{R}, \quad \nu_1 = -2(n-1)(1 - \ln 2) + 2 \ln 2 \frac{\tilde{\mu}}{\varepsilon}.$$

We also set

$$v_\varepsilon(r) = \tilde{\mu} v \left(\frac{r-1}{\tilde{\mu}} \right).$$

- β_ε satisfies

$$\begin{cases}
 -(\beta_\varepsilon)'' - \frac{n-1}{r} (\beta_\varepsilon)' = \frac{n-1}{r} (v_\varepsilon)' & \text{in } (0, 1) \\
 \beta_\varepsilon(1) = (\beta_\varepsilon)'(1) = 0,
 \end{cases}$$

and the following estimate holds, for $s \leq 0$,

$$\beta_\varepsilon(\tilde{\mu}s + 1) = \tilde{\mu}^2 (\beta_\varepsilon)_1(s) + O(\tilde{\mu}^3 s^3),$$

where

$$(\beta_\varepsilon)_1(s) = -(n-1) \int_0^s \int_0^\sigma v'(\rho) d\rho d\sigma.$$

- Finally z_ε satisfies

$$\begin{aligned}
 &-(z_\varepsilon)'' - e^{W_{\tilde{\mu}}} z_\varepsilon = \\
 &\quad \tilde{\mu}^2 e^{W_{\tilde{\mu}}} \left[(\alpha_\varepsilon)_2 \left(\frac{r-1}{\tilde{\mu}} \right) + (\beta_\varepsilon)_1 \left(\frac{r-1}{\tilde{\mu}} \right) + \frac{1}{2} \left((\alpha_\varepsilon)_1 \left(\frac{r-1}{\tilde{\mu}} \right) + v \left(\frac{r-1}{\tilde{\mu}} \right) \right)^2 \right],
 \end{aligned}$$

under the boundary conditions

$$z_\varepsilon(1) = (z_\varepsilon)'(1) = 0.$$

There holds

$$(2.15) \quad z_\varepsilon(r) = \tilde{\mu} \zeta_1(r-1) + \zeta_2 \tilde{\mu}^2 + O(\tilde{\mu}^2 e^{-\frac{|r-1|}{\tilde{\mu}}}),$$

for some $\zeta_j \in \mathbb{R}$, $j = 1, 2$.

2.2.2 Construction of u_2 .

Thanks to Lemma 2.57, we know that, for any b small enough, there exists a function G_b satisfying

$$\begin{cases} -G_b'' - \frac{1}{r}G_b' + G_b = 0 & \text{in } (0, 1) \\ \lim_{r \rightarrow 0^+} \frac{G_b(r)}{-\ln r} = b, & G_b(1) = 1. \end{cases}$$

Using the same argument as in Lemma 2.8 of [dPPV16] (see also Lemma 2.5.1 for a more complicated situation), we can perturb the function G_b in the following way: there exists ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$, there exist $\gamma_\varepsilon \in \mathbb{R}$ and a radial function U_ε solution to

$$\begin{cases} -\Delta U_\varepsilon + U_\varepsilon = 0 & \text{in } (0, 1) \\ \lim_{r \rightarrow 0^+} \frac{U_\varepsilon(r)}{-\ln r} = \frac{4}{\sqrt{2}}\varepsilon \\ U_\varepsilon(1) = 1 + \frac{\sqrt{2}}{\varepsilon}(-\ln(\gamma_\varepsilon)^2 + \varepsilon\gamma_\varepsilon\nu_2) \\ U_\varepsilon'(1) = \frac{1}{\gamma_\varepsilon} + \frac{\varepsilon}{\sqrt{2}}(-2 + 2\gamma_\varepsilon \ln 2 + \varepsilon\gamma_\varepsilon\zeta_1) \end{cases}$$

where ν_2 and ζ_1 are defined respectively in (2.14) and (2.15). We then define u_2 as

$$(2.16) \quad u_2(r) = \frac{\sqrt{2}}{\varepsilon}U_\varepsilon(r).$$

Observe that there exists $\tilde{r} \in (0, 1)$ such that $u_2'(\tilde{r}) = 0$ and we have $\tilde{r} = O(\sqrt{\varepsilon})$. We denote by H the regular part of u_2 , namely

$$(2.17) \quad H(r) = u_2(r) + 4 \ln r.$$

Observe that thanks to (2.58) and (2.59), we have, for some constant $C > 0$,

$$(2.18) \quad H(0) < 0, \quad |H(0)| \leq \frac{C}{\varepsilon} \quad \text{and} \quad \lim_{r \rightarrow 0^+} H'(r) = 0.$$

We choose $\tilde{\mu}$ in (2.10) as $\tilde{\mu} = \varepsilon\gamma_\varepsilon$. Thanks to our choices of u_2 and u_4 , one can show proceeding as in [BCN17a] the following estimate.

Lemma 2.2.1. *For any $\delta_1 < |r - 1| < 2\delta_1$, we have*

$$u_4(r) - u_2(r) = O\left(\varepsilon^2 + \varepsilon|r - 1|^2 + |r - 1|^3 + \frac{|r - 1|^4}{\varepsilon} + \exp\left(-\frac{|r - 1|}{\varepsilon}\right)\right),$$

and

$$u_4'(r) - u_2'(r) = O\left(\varepsilon|r - 1| + |r - 1|^2 + \frac{|r - 1|^3}{\varepsilon} + \frac{1}{\varepsilon}\exp\left(-\frac{|r - 1|}{\varepsilon}\right)\right).$$

2.2.3 Construction of u_0 .

We define $u_0 = U_0 + H_0$ where U_0 is the function defined in (2.9) and H_0 is the solution to

$$(2.19) \quad \begin{cases} -\Delta H_0 + H_0 &= -U_0 & \text{in } (0, \tilde{r}) \\ H_0'(\tilde{r}) &= -U_0'(\tilde{r}). \end{cases}$$

We introduced the function H_0 in order to get a better matching between u_0 and u_2 . We choose δ such that $2\delta < \tilde{r}$ and $\delta = O(\sqrt{\varepsilon})$. Proceeding as in Lemma 2.1 of [dPW06], we obtain the following lemma.

Lemma 2.2.2. *For any $\alpha \in \left(0, \frac{1}{2}\right)$, we have, for $r \in (0, \tilde{r})$,*

$$(2.20) \quad H_0(r) = H(r) - \ln(8\mu^2) + O(\lambda^\alpha),$$

$C^{0,\gamma}(B_{\tilde{r}})$ -uniformly, for $\gamma \in [0, 1)$, where $H(r)$ is defined in (2.17). Moreover, (2.20) holds uniformly in $C^1(B_{2\delta} \setminus B_\delta)$. Finally, choosing $\mu^2 = \frac{e^{H(0)}}{8}$, and recalling (2.18), the estimate

$$(2.21) \quad H_0(r) = O\left(\lambda^\alpha + \frac{r^2}{\varepsilon}\right),$$

holds true for $r \in (0, \tilde{r})$.

Proof. Let us consider the function $z = H_0 - H + \ln 8\mu^2$. It satisfies

$$\begin{cases} -\Delta z + z &= -\ln \frac{1}{(\mu^2\lambda + r^2)^2} + \ln \frac{1}{r^4} & \text{in } (0, \tilde{r}) \\ z'(\tilde{r}) &= \frac{4\tilde{r}}{\mu^2\lambda + \tilde{r}^2} - \frac{4}{\tilde{r}}. \end{cases}$$

Recalling (2.12) and that we have $\tilde{r} = O(\sqrt{\varepsilon})$, we deduce that

$$z'(\tilde{r}) = \frac{4\mu^2\lambda}{\tilde{r}(\mu^2\lambda + \tilde{r}^2)} = O(\lambda^\alpha),$$

for any $\alpha \in \left(0, \frac{1}{2}\right)$. We set $f = -\ln \frac{1}{(\mu^2\lambda + r^2)^2} + \ln \frac{1}{r^4}$. Let $2 < p$. We have

$$\int_{B_{\tilde{r}}} |f|^p dx = \int_{B_{\tilde{r}} \setminus B_{\mu\sqrt{\lambda}}} |f|^p dx + \int_{B_{\mu\sqrt{\lambda}}} |f|^p dx.$$

It is easy to see that

$$\int_{B_{\mu\sqrt{\lambda}}} |f|^p dx \leq C\lambda |\ln \lambda|^p,$$

and, using that $|f(r)| \leq \frac{C\sqrt{\lambda}}{r}$,

$$\int_{B_{\tilde{r}} \setminus B_{\mu\sqrt{\lambda}}} |f|^p dx \leq C\lambda^{p/2}\tilde{r}^{2-p} \leq \lambda^{p/2}.$$

Using elliptic regularity theory (see Lemma 2.A.5), we deduce that

$$\|z\|_{C^{0,\gamma}(B_{\tilde{r}})} \leq C\lambda^\alpha,$$

for all $\gamma \in (0, 1)$ and any $\alpha \in (0, \frac{1}{2})$.

On the other hand, for any $q \geq 2$, since $\delta = O(\varepsilon^2)$, we have

$$\int_{B_{2\delta} \setminus B_\delta} |f|^q dx \leq C\lambda^{q/2}\delta^{2-q} \leq C\lambda^{q/2}\varepsilon^{2(2-q)} \leq C\lambda^{\alpha q},$$

for any $\alpha \in (0, \frac{1}{2})$. We deduce

$$\|z\|_{C^1(B_{2\delta} \setminus B_\delta)} \leq C\lambda^\alpha.$$

Finally, (2.21) is a direct consequence of the fact that $H \in C^{1,\beta}(B_{\tilde{r}})$, $\beta \in (0, 1)$. \square

Thanks to the previous lemma, we are able to show that u_0 and u_2 are very close for the C^1 -norm in the interval $[\delta, 2\delta]$.

Lemma 2.2.3. *For $\delta \leq r \leq 2\delta$, we have*

$$|u_0(r) - u_2(r)| = O(\lambda^\alpha), \quad |u'_0(r) - u'_2(r)| = O(\lambda^\alpha),$$

for any $\alpha \in (0, \frac{1}{2})$.

Proof. The proof is a direct consequence of Lemma 2.2.2. Indeed, by definition, we have, for $r \in [\delta, 2\delta]$,

$$u_0(r) = U_0(r) + H_0(r) = \ln \frac{8\mu^2}{(\mu^2\lambda + r^2)^2} + H(r) - \ln 8\mu^2 + O(\lambda^\alpha),$$

and

$$u_2(r) = -4 \ln r + H(r).$$

It follows that

$$\begin{aligned} u_0(r) - u_2(r) &= -2 \ln \left(1 + \frac{\mu^2\lambda}{r^2} \right) + O(\lambda^\alpha) \\ &= O(\lambda^\alpha). \end{aligned}$$

In the same way, one can show that

$$u'_0(r) - u'_2(r) = O\left(\frac{\mu^2\lambda}{\delta^3}\right) + O(\lambda^\alpha) = O(\lambda^\alpha).$$

\square

We look for a solution of (2.3) of the form $U + \phi$. Let us observe that $U + \phi$ is a solution to (2.3) if and only if ϕ is a solution to the problem

$$(2.22) \quad \begin{cases} L(\phi) = N(\phi) + R(U) & \text{in } (0, 1) \\ \phi'(0) = \phi'(1) = 0, \end{cases}$$

where

$$(2.23) \quad L(\phi) = -\Delta\phi + \phi - \lambda e^U \phi,$$

$$(2.24) \quad N(\phi) = \lambda(e^{U+\phi} - e^U - e^U \phi) \text{ and,}$$

$$R(U) = -\Delta U + U - \lambda e^U.$$

2.3 The error estimate

In this section, we estimate the terms $R(U)$ and $N(\phi)$. In order to apply directly all the estimates of [PV15], we are going to work with the norm $\|\cdot\|_*$ (see (2.31)) which is a weighted L^∞ norm on $B_{\frac{1}{2}}$ and a L^1 -norm elsewhere. We begin by estimating $N(\phi)$.

Lemma 2.3.1. *We have, for any $\beta > 0$,*

$$N(\phi) \leq C|\phi|^2 \begin{cases} \frac{8\mu^2}{\lambda \left(\mu^2 + \left(\frac{r}{\sqrt{\lambda}} \right)^2 \right)^2} & \text{if } r \leq 2\delta \\ \varepsilon^\beta & \text{if } 2\delta \leq r \leq 1 - 2\delta_1 \end{cases}$$

and

$$(2.25) \quad \|N(\phi)\|_{L^1(B_1 \setminus B_{\frac{1}{2}})} \leq C\varepsilon^{-1} \|\phi\|_{L^\infty(B_1 \setminus B_{\frac{1}{2}})}^2.$$

Proof. First, using a Taylor's expansion, it is immediate to see that

$$N(\phi) \leq C\lambda e^U |\phi|^2.$$

Therefore, the proof reduces to estimate e^U . First, we consider the case $r \in [0, 2\delta]$. In this range, using (2.21) and a Taylor's expansion, we see that

$$e^{u_0} = e^{U_0 + H_0} = \frac{8\mu^2}{(\mu^2\lambda + r^2)^2} e^{O(\lambda^\alpha + \frac{r^2}{\varepsilon})} = O\left(\frac{8\mu^2}{(\mu^2\lambda + r^2)^2}\right).$$

Next, we consider $r \in [\delta, 1 - 2\delta_1]$. By definition of u_2 , we know that it is decreasing in $r \in (0, \tilde{r})$ and increasing elsewhere. So, we have, for $r \in [\delta, 1 - 2\delta_1]$,

$$e^{u_2(r)} \leq e^{u_2(\delta)} + e^{u_2(1-2\delta_1)}.$$

Making a Taylor's expansion and using (2.16), we obtain, for some $\theta \in (1 - 2\delta_1, 1)$,

$$u_2(1 - 2\delta_1) = u_2(1) - 2\delta_1 u_2'(1) + 2\delta_1^2 u_2''(\theta) \leq \frac{\sqrt{2}}{\varepsilon} - \delta_1 u_2'(1).$$

Thus, we have, recalling the relation (2.12), and the definition of δ_1 ,

$$\lambda e^{u_2(1-2\delta_1)} \leq C\varepsilon^{-2} e^{\frac{\sqrt{2}}{\varepsilon}(-\delta_1 u_2'(1))} \leq C\varepsilon^\beta,$$

for any $\beta > 0$. On the other hand, using (2.16), we see that $e^{u_2(\delta)} \leq \frac{C}{\delta^4} \leq C\varepsilon^{-8}$. The estimate follows noticing that $\lambda\varepsilon^{-8} \leq \varepsilon^\beta$, for any $\beta > 0$. Finally, we refer to Lemma 4.3 of [PV15] for the proof of (2.25). \square

Next, we estimate $R(U)$.

Lemma 2.3.2. *Let $\alpha \in (0, \frac{1}{2})$ be the constant defined in Lemma 2.2.2. We have*

$$R(U) \leq C \begin{cases} \frac{8\mu^2}{\lambda \left(\mu^2 + \left(\frac{r}{\sqrt{\lambda}} \right)^2 \right)^2} (\lambda^\alpha + \frac{r^2}{\varepsilon}) & \text{if } r \leq \delta, \\ \varepsilon^\beta & \text{if } \delta \leq r \leq 1 - 2\delta_1 \end{cases}$$

for any $\beta > 0$, and

$$\|R(U)\|_{L^1(B_1 \setminus B_{\frac{1}{2}})} \leq C\varepsilon^{1+\sigma},$$

for some $\sigma > 0$.

Proof. First, we consider the case $r \leq \delta$. In this case, $U(r) = u_0(r) = U_0(r) + H_0(r)$. Using (2.9),(2.19) and (2.21), we have

$$\begin{aligned} R(u_0) &= -\Delta(U_0 + H_0) + U_0 + H_0 - \lambda e^{U_0 + H_0} \\ &= \lambda e^{U_0} (1 - e^{H_0}) \\ (2.26) \quad &\leq C \frac{8\mu^2}{\lambda \left(\mu^2 + \left(\frac{r}{\sqrt{\lambda}} \right)^2 \right)^2} (\lambda^\alpha + \frac{r^2}{\varepsilon}). \end{aligned}$$

Next, when $2\delta \leq r \leq 1 - 2\delta_1$, then $U(r) = u_2(r)$. Arguing as in the previous lemma, we obtain

$$(2.27) \quad R(u_2(r)) = \lambda e^{u_2(r)} \leq C\varepsilon^\beta,$$

for any $\beta > 0$.

On the other hand, it has be proved in Lemma 4.2 of [PV15] that

$$(2.28) \quad \|R(u_4)\|_{L^1(B_1 \setminus B_{1-\delta_1})} = O(\varepsilon^{1+\sigma}) \quad \text{for some } \sigma > 0.$$

Finally, we consider the two intermediate regimes. First, let us consider the case $\delta \leq r \leq 2\delta$. In this interval, $U(r) = u_1(r)$. Using (2.11), we have

$$\begin{aligned} R(u_1) &= \chi_1 R(u_0) + (1 - \chi_1)R(u_2) - 2\chi_1'(u_0' - u_2') + (-\Delta\chi_1 + \chi_1)(u_0 - u_2) \\ &\quad + \lambda\chi_1 e^{u_0} + \lambda(1 - \chi_1)e^{u_2} - \lambda e^{\chi_1 u_0 + (1-\chi_1)u_2} \end{aligned}$$

$$(2.29) \quad \begin{aligned} &\leq R(u_0) + R(u_2) + C \left(\frac{|u'_0 - u'_2|}{\delta} + \frac{|u_0 - u_2|}{\delta^2} \right) \\ &\quad + \lambda e^{u_2} + \lambda e^{u_0} (e^{(\chi_1-1)(u_2-u_0)} - 1). \end{aligned}$$

First, doing a Taylor's expansion and using Lemma 2.2.3, we have

$$\lambda e^{u_0} (e^{(\chi_1-1)(u_2-u_0)} - 1) \leq \lambda e^{u_0} |u_0 - u_2| \leq \lambda^{1+\alpha} e^{u_0}.$$

Using again Lemma 2.2.3, we get

$$\frac{|u'_0 - u'_2|}{\delta} + \frac{|u_0 - u_2|}{\delta^2} \leq C \lambda^\alpha \delta^{-2}.$$

Plugging these two last estimates into (2.29) and using (2.26), (2.27), we obtain

$$\begin{aligned} R(u_1) &= O \left(\sup_{\delta \leq r \leq 2\delta} \frac{8\mu^2}{\lambda \left(\mu^2 + \left(\frac{r}{\sqrt{\lambda}} \right)^2 \right)^2} + \varepsilon^\beta + \frac{\lambda^\alpha}{\delta^2} \right) \\ &= O \left(\frac{\lambda}{\delta^3} + \varepsilon^\beta + \frac{\lambda^\alpha}{\delta^2} \right) = O(\varepsilon^\beta). \end{aligned}$$

Finally, when $1 - 2\delta_1 \leq r \leq 1 - \delta_1$, arguing as previously, we have

$$(2.30) \quad \begin{aligned} R(u_3) &= \chi_3 R(u_2) + (1 - \chi_3) R(u_4) - 2\chi'_3 (u'_4 - u'_2) + (-\Delta\chi_3 + \chi_3)(u_4 - u_2) \\ &\quad + \lambda\chi_3 e^{u_4} + \lambda(1 - \chi_3) e^{u_2} - \lambda e^{\chi_3 u_4 + (1-\chi_3)u_2} \\ &\leq R(u_2) + R(u_4) + C \left(\frac{|u'_4 - u'_2|}{\delta} + \frac{|u_4 - u_2|}{\delta^2} \right) \\ &\quad + \lambda e^{u_2} + \lambda e^{u_4} |u_4 - u_2|. \end{aligned}$$

Using Lemma 2.2.1, we see that

$$\int_{1-2\delta_1}^{1-\delta_1} \left(\frac{|u'_4 - u'_2|}{\delta} + \frac{|u_4 - u_2|}{\delta^2} \right) r dr = O(\delta_1^2) = O(\varepsilon^{1+\sigma}),$$

and

$$\int_{1-2\delta_1}^{1-\delta_1} \lambda e^{u_4} |u_4 - u_2| r dr = O(\lambda \varepsilon^2).$$

Thanks to (2.27) and (2.28), we see that

$$\int_{1-2\delta_1}^{1-\delta_1} (R(u_2) + R(u_4) + \lambda e^{u_2}) r dr = O(\varepsilon^{1+\sigma}).$$

Plugging the three previous estimates into (2.30), we obtain

$$\|R(u_3)\|_{L^1(B_1 \setminus B_{\frac{1}{2}})} = O(\varepsilon^{1+\sigma}).$$

This concludes the proof of the lemma. □

2.4 Inversibility of the linearized operator

In this section we develop an inversibility theory for the operator L defined in (2.23). To do so, we utilize ideas used in [dPKM05, dPR15, dPW06, PV15]. First, we define the norms

$$(2.31) \quad \|u\|_* = \max \{ |\log \lambda| \|\tilde{\chi}_1 u\|_*, \|\tilde{\chi}_2 u\|_{L^1} \}$$

and

$$(2.32) \quad \|u\|_{**} = \max \{ \|\tilde{\chi}_1 u\|_*, \|\tilde{\chi}_2 u\|_{L^1} \},$$

where

$$\tilde{\chi}_1(r) = \begin{cases} 1 & \text{if } r \leq \frac{1}{2} \\ 0 & \text{if } r \geq \frac{3}{4} \end{cases} \quad \tilde{\chi}_2(r) = \begin{cases} 1 & \text{if } r \geq \frac{1}{2} \\ 0 & \text{if } r \leq \frac{1}{4} \end{cases}$$

and

$$\|u\|_* = \sup \frac{\lambda |u(r)|}{\lambda + \left(1 + \frac{r}{\sqrt{\lambda}}\right)^{-2-\nu}} = \sup f_\lambda(r) |u(r)|$$

for some $\nu \in (0, 1)$. The main result of this section is the following proposition.

Proposition 2.4.1. *There exist positive constants λ_0 and C such that for any $\lambda \in (0, \lambda_0)$ and for any $h \in L^\infty(B_1)$, there exists a unique radial function $\phi \in W^{2,2}(B_1)$ solution of the problem*

$$(2.33) \quad \begin{cases} L(\phi) = h & \text{in } B_1 \\ \phi'(1) = 0, \end{cases}$$

which satisfies

$$(2.34) \quad \|\phi\|_{L^\infty(B_1)} \leq C \|h\|_*.$$

Rather than proving Proposition 2.4.1 directly, we prove first a priori estimates for problem (2.33) when ϕ satisfies an orthogonality condition against the function

$$z_0(r) = \frac{r^2 - \lambda\mu^2}{r^2 + \lambda\mu^2}.$$

It is important to notice that z_0 satisfies the equation

$$(2.35) \quad -\Delta z_0 = \frac{8\lambda\mu^2}{(\lambda\mu^2 + r^2)^2} z_0,$$

which correspond to the linearized equation of $-\Delta v = e^v$ around the radial solution $v(r) = U_0(r) + \log \lambda = \log \frac{8\lambda\mu^2}{(\lambda\mu^2 + |r|^2)^2}$. It turns out that the only bounded radial solutions of (2.35) are multiples of z_0 (see Lemma 2.1 of [CL02]). Additionally, let us consider a large but fixed number $R_0 > 0$ and a radial smooth cut-off function $\chi(r)$ such that $\chi(r) = 1$ if $r \leq R_0\sqrt{\lambda}$ and $\chi(r) = 0$ if $r > (R_0 + 1)\sqrt{\lambda}$. We have the following lemma.

Lemma 2.4.1. *There exist positive constants λ_0 and C such that for, any $\lambda \in (0, \lambda_0)$, any radial solution $\phi \in W^{2,2}(B_1)$ to problem*

$$(2.36) \quad \begin{cases} L(\phi) = h & \text{in } B_1, \\ \phi'(1) = 0 \\ \int_{B_1} \chi z_0 \phi \, dx = 0, \end{cases}$$

satisfies

$$\|\phi\|_{L^\infty(B_1)} \leq C \|h\|_{**}.$$

Proof of Lemma 2.4.1. Assume by contradiction that there exist a sequence of positive numbers $\lambda_n \rightarrow 0$ and a sequence of solutions ϕ_n to (2.36) such that

$$(2.37) \quad \|\phi_n\|_{L^\infty(B_1)} = 1, \quad \|h_n\|_{**} \xrightarrow{n \rightarrow \infty} 0.$$

We denote by ε_n the sequence defined by the relation

$$\ln \frac{4}{\varepsilon_n^2} - \ln \lambda_n = \frac{\sqrt{2}}{\varepsilon_n}.$$

We also use the notation $o_n(1)$ to denote functions $f_n(r)$ such that $\lim_{n \rightarrow \infty} f_n(r) = 0$, uniformly in r . Our goal is to prove that $\phi_n(r) = o_n(1)$, for any $r \in [0, 1]$, which yields to a contradiction with (2.37). We split the proof into 4 steps. In the first one we prove that $\phi_n(r) = C\phi_n(1/2) + o_n(1)$ for $r \in [2\delta, \frac{1}{2}]$ and some constant $C \in \mathbb{R}$. In the second step we show that $\phi_n = o_n(1)$ when $r \in [\frac{1}{2}, 1]$ and finally in the last two steps we consider the case $r \in [0, 2\delta]$.

Step 1. There holds $\phi_n(r) = o_n(1)$ for $r \in [2\delta, \frac{1}{2}]$.

First, we recall that for $r \in [2\delta, \frac{1}{2}]$, $U(r) = u_2(r)$. Observe that thanks to (2.27), we have $\lambda_n e^{u_2} = O(\varepsilon_n^{1+\sigma})$, for any $\sigma > 0$. Since by assumption $\|h_n\|_\infty \rightarrow 0$, it is easy to see that, up to subsequence, ϕ_n converges uniformly on compact subsets of $B_{\frac{1}{2}} \setminus \{0\}$ to a function $\hat{\phi} \in H^1(B_{\frac{1}{2}}) \cap L^\infty(B_{\frac{1}{2}})$ solution to

$$(2.38) \quad -\Delta \hat{\phi} + \hat{\phi} = 0 \quad \text{in } B_{\frac{1}{2}} \setminus \{0\}.$$

We claim that $\hat{\phi} \equiv 0$. In order to prove our claim, let us consider the unique radial solution Φ of the problem

$$\begin{cases} -\Delta \Phi + \Phi = \delta_0 & \text{in } B_{\frac{1}{2}}, \\ \Phi(\frac{1}{2}) = 0. \end{cases}$$

It is well-known that

$$\Phi(x) = -\frac{1}{2\pi} \log |x| + H(x),$$

for some smooth function H . Since $\hat{\phi} \in L^\infty\left(B_{\frac{1}{2}}\right)$, we have that for any sufficiently small $\tilde{\varepsilon}$ and τ ,

$$|\hat{\phi}(\tau) - \hat{\phi}(1/2)| \leq \tilde{\varepsilon}\Phi(\tau).$$

Multiplying (2.38) by $\varphi = \max(\hat{\phi} - \hat{\phi}(1/2) - \tilde{\varepsilon}\Phi, 0)$, integrating by parts over $B_{\frac{1}{2}} \setminus B_\tau$ and using that $\varphi = 0$ on $\partial(B_{\frac{1}{2}} \setminus B_\tau)$, we obtain $\varphi \equiv 0$, i.e. $\hat{\phi} - \hat{\phi}(1/2) \leq \tilde{\varepsilon}\Phi$ in $B_{\frac{1}{2}} \setminus B_\tau$. Using the same argument with $\varphi = \min(\hat{\phi} - \hat{\phi}(1/2) + \tilde{\varepsilon}\Phi, 0)$, we conclude that $|\hat{\phi} - \hat{\phi}(1/2)| \leq \tilde{\varepsilon}\Phi$ in $B_{\frac{1}{2}} \setminus B_\tau$. Passing to the limit $\tilde{\varepsilon} \rightarrow 0$ and then $\tau \rightarrow 0$, we deduce that $\hat{\phi} \equiv \hat{\phi}(1/2)$. Since the only constant solution to (2.38) is zero, we deduce $\hat{\phi}(1/2) \equiv \hat{\phi} \equiv 0$. This implies that $\phi_n(r) = o_n(1)$ for $r \in [2\delta, \frac{1}{2}]$.

Step 2. We have that $\phi_n(r) = o_n(1)$ for $r \in [\frac{1}{2}, 1]$.

We set $\psi_n(s) = \phi_n(\varepsilon_n s + 1)$ for $s \in [-\varepsilon_n^{-1}, 0]$. Then, since ψ_n is bounded, it is possible to show, proceeding as in Proposition 5.1 of [PV15], that $\psi_n \rightarrow \psi$ C^2 -uniformly on compact subsets of $(-\infty, 0]$ where ψ satisfies

$$\begin{cases} -\psi'' = e^\psi & \text{in } \mathbb{R}^-, \\ \psi'(0) = 0, \|\psi\|_{L^\infty} \leq 1. \end{cases}$$

We know (see [Gro06]) that any solution ψ to $-\psi'' = e^\psi$ is of the form

$$\psi(s) = a \frac{e^{\sqrt{2}s} - 1}{e^{\sqrt{2}s} + 1} + b \left(-2 + \sqrt{2}s \frac{e^{\sqrt{2}s} - 1}{e^{\sqrt{2}s} + 1} \right),$$

for some $a, b \in \mathbb{R}$. However, since $\|\psi\|_\infty \leq 1$, we deduce that $a = b = 0$.

Next, we denote by $G(r, t)$ the radial Green's function associated to the operator $(-\Delta \cdot + \cdot)$ satisfying $G(r, \frac{1}{2}) = G'(r, 1) = 0$ and singular at the point $r \in (\frac{1}{2}, 1)$. Now, using Green's formula, we have, for $\frac{1}{2} \leq r \leq 1$,

$$\begin{aligned} \phi_n(r) - G' \left(r, \frac{1}{2} \right) \phi_n \left(\frac{1}{2} \right) &= \int_{\frac{1}{2}}^1 G(r, t) h_n(t) dt + \lambda_n \int_{\frac{1}{2}}^1 G(r, t) e^{U_{\lambda_n}} \phi_n(t) dt \\ &= \int_{\frac{1}{2}}^1 G(r, t) h_n(t) dt + G(r, 1) \varepsilon_n \lambda_n \int_{-\frac{1}{2\varepsilon_n}}^0 e^{U_{\lambda_n}(\varepsilon_n s + 1)} \psi_n(s) ds \\ &\quad + \varepsilon_n \lambda_n \int_{-\frac{1}{2\varepsilon_n}}^0 (G(r, \varepsilon_n s + 1) - G(r, 1)) e^{U_{\lambda_n}(\varepsilon_n s + 1)} \psi_n(s) ds. \end{aligned}$$

From **Step 1**, $\|h_n\|_{**} \rightarrow 0$ as $n \rightarrow \infty$ and since G is C^1 bounded, we get that

$$G' \left(r, \frac{1}{2} \right) \phi_n \left(\frac{1}{2} \right) + \int_{\frac{1}{2}}^1 G(r, t) h_n(t) dt = o_n(1).$$

Arguing as in [PV15], it is possible to show that

$$\varepsilon_n \lambda_n \int_{-\frac{1}{2\varepsilon_n}}^0 (G(r, \varepsilon_n s + 1) - G(r, 1)) e^{u_{\lambda_n}(\varepsilon_n s + 1)} \psi_n(s) ds = o_n(1).$$

From this we get

$$\phi_n(r) = C_n G(r, 1) + o_n(1),$$

where $C_n = \varepsilon_n \lambda_n \int_{-\frac{1}{2\varepsilon_n}}^0 e^{U\lambda_n(\varepsilon_n s+1)} \psi_n(s) ds$. Evaluating the previous expression at $r = 1$ we get

$$\phi_n(1) = \psi_n(0) = o_n(1) = C_n G(1, 1) + o_n(1).$$

Since $G(1, 1) \neq 0$, we deduce $C_n = 0$ and therefore $\phi_n = o_n(1)$ for $r \in [\frac{1}{2}, 1]$.

In the following steps it is convenient to work with rescaled variables. We set $s = \frac{r}{\sqrt{\lambda_n}}$, $\tilde{\phi}_n(s) = \phi_n(\sqrt{\lambda_n}s)$ and denote by $\tilde{U}(s) = U(\sqrt{\lambda_n}s) + 2 \ln \lambda_n$, $\tilde{h}_n(s) = \lambda_n h_n(\sqrt{\lambda_n}s)$ and $\tilde{L} = -\Delta + \lambda_n - e^{\tilde{U}}$. We also define, by abuse of notation,

$$(2.39) \quad \|\tilde{h}\|_* := \sup_{s \in [0, \lambda_n^{-1/2}/4]} \frac{\tilde{h}(s)}{\lambda_n + (1+s)^{-2-\nu}} = \|h\|_*,$$

for functions \tilde{h} defined in the rescaled variable.

Step 3. Up to subsequence, we have that $\tilde{\phi}_n \rightarrow 0$ as $n \rightarrow \infty$ uniformly over compact sets of \mathbb{R}^2 .

It is easy to see that $\tilde{\phi}_n$ satisfies

$$\tilde{L}(\tilde{\phi}_n(s)) = \tilde{h}_n(s).$$

Elliptic estimates imply that, up to subsequence, $\tilde{\phi}_n$ converges uniformly over compact sets of \mathbb{R}^2 to a bounded solution $\tilde{\phi}$ of

$$-\Delta \tilde{\phi} = e^{\tilde{U}} \tilde{\phi} \quad \text{in } \mathbb{R}^2.$$

This implies that there exists a constant C_0 such that $\tilde{\phi} = C_0 \tilde{Z}_0(s)$, where

$$\tilde{Z}_0(s) = z_{0,n}(\sqrt{\lambda_n}s), \quad z_{0,n} = \frac{r^2 - \lambda_n \mu^2}{r^2 + \lambda_n \mu^2}.$$

From the orthogonality condition on ϕ_n we have

$$\int_{B_1} \chi z_{0,n} \phi_n dx = \lambda_n \int_{B_{\lambda_n^{-1/2}}} \tilde{\chi} \tilde{Z}_0 \tilde{\phi}_n dx = 0,$$

where $\tilde{\chi}(s) = \chi(\sqrt{\lambda_n}s)$. Passing to the limit yields to $\int_{\mathbb{R}^2} \tilde{\chi} \tilde{Z}_0 \tilde{\phi} dx = 0$, which implies $C_0 = 0$. This gives the result.

The final step is based on a maximum principle argument.

Step 4. We have that $\phi_n(r) = o_n(1)$, for $r \leq 2\delta$.

Let $\tilde{\delta} > 0$ be a fixed constant such that $2\delta < 2\tilde{\delta} < 1/4$. Next, we show that there exists a constant $C > 0$, independent of n , such that

$$(2.40) \quad \|\tilde{\phi}_n\|_{L^\infty(B_{2\tilde{\delta}\lambda_n^{-1/2}})} \leq C \left[\sup_{s \leq R} |\tilde{\phi}_n(s)| + \|\tilde{h}_n\|_* \right],$$

where $R > 0$ is a large but fixed real number. To prove the previous estimate, we need the following version of the maximum principle:

There exists a fixed number $R_1 > 0$ such that for all $R > R_1$ if $\tilde{L}(Z) > 0$ in $A_{\tilde{\delta}} := B_{2\tilde{\delta}\lambda_n^{-1/2}} \setminus B_R$ and $Z \geq 0$ on $\partial A_{\tilde{\delta}}$ then $Z \geq 0$ in $A_{\tilde{\delta}}$.

To prove this statement let us consider the function $Z_0(s) = \frac{s^2-1}{s^2+1}$. Observe that it satisfies

$$-\Delta Z_0 = \frac{8}{(1+s^2)^2} Z_0 \quad s \in \mathbb{R}^2.$$

We define the function $Z(s) = Z_0(\alpha s)$, for some constant α that we will fix afterwards. Observe that

$$-\Delta Z = \frac{8\alpha^2}{(\alpha^2 s^2 + 1)^2} \frac{\alpha^2 s^2 - 1}{\alpha^2 s^2 + 1}$$

In particular if $\alpha^2 s^2 > 100$ then $-\Delta Z \geq \frac{2}{\alpha^2 s^4}$. On the other hand, we have

$$e^{\tilde{U}} Z = O\left(\frac{8\mu^2}{(\mu^2 + s^2)^2}\right) \frac{\alpha^2 s^2 - 1}{\alpha^2 s^2 + 1} \leq \frac{C}{s^4},$$

where C is a constant independent of α . We get

$$\tilde{L}(Z) = -\Delta Z + \lambda_n Z - e^{\tilde{U}} Z \geq \frac{1}{s^4} \left(\frac{2}{\alpha^2} - C \right).$$

Hence if α is chosen small and fixed, and $R > 0$ is sufficiently large depending on α , then we have $\tilde{L}(Z) > 0$ and $Z > 0$ in $A_{\tilde{\delta}}$, which gives the result.

Thanks to this maximum principle, we are in position to prove (2.40). Let $R_2 > \max\{R_1, R_0\}$. Consider ψ_n^0 the unique solution of

$$\begin{cases} -\Delta \psi_n^0 + \lambda_n \psi_n^0 - \lambda_n = 0 & \text{in } B_{2\tilde{\delta}\lambda_n^{-1/2}} \setminus B_{R_2} \\ \psi_n^0 = 0 & \text{on } \partial B_{R_2} \\ \psi_n^0 = |\tilde{\phi}_n| & \text{on } \partial B_{2\tilde{\delta}\lambda_n^{-1/2}} \end{cases}$$

and let $\psi^1 = 1 - s^{-\nu}$. We set $\psi_n = \psi_n^0 + \psi^1$. For $s > R_2$, we have

$$\tilde{L}(\psi_n) \geq \lambda_n + \nu^2 s^{-2-\nu} - O(e^{\tilde{U}}) \geq \frac{\nu^2}{2} s^{-2-\nu} + \lambda_n,$$

since

$$e^{\tilde{U}} = O(s^{-4}).$$

We set $\bar{\phi}_n = C_1 \left[\max_{s \in (0, R_2)} |\tilde{\phi}_n(s)| + \|\tilde{h}_n\|_{\star} \right] \psi_n$, for a constant C_1 independent of n .

Observe that, if $C_1 \geq \frac{4}{\nu^2}$, we have

$$\tilde{L}(\bar{\phi}_n) \geq 2\|\tilde{h}_n\|_{\star} (s^{-2-\nu} + \lambda_n) \geq |\tilde{h}_n| \frac{2(s^{-2-\nu} + \lambda_n)}{((1+s)^{-2-\nu} + \lambda_n)} \geq |\tilde{h}_n| = |\tilde{L}(\tilde{\phi}_n)|,$$

in $B_{2\delta\lambda_n^{-1/2}} \setminus B_{R_2}$, since $\frac{2(s^{-2-\nu} + \lambda_n)}{((1+s)^{-2-\nu} + \lambda_n)} \geq 1$, for $s \in [R_2, +\infty)$ (taking R_2 larger if necessary). On the other hand, for $C_1 \geq (1 - R_2^{-\nu})^{-1}$ we have

$$\bar{\phi}_n \geq |\tilde{\phi}_n| \quad \text{on } \partial B_{2\delta\lambda_n^{-1/2}} \setminus B_{R_2}.$$

Applying the maximum principle, and taking into account that ψ_n is uniformly bounded (since $|\tilde{\phi}_n| \leq 1$, for all n), we get

$$|\tilde{\phi}_n(s)| \leq C \left[\max_{s \in (0, R_2)} |\tilde{\phi}_n(s)| + \|\tilde{h}_n\|_{\star} \right].$$

for every $s \in B_{2\delta\lambda_n^{-1/2}} \setminus B_{R_2}$. From this we deduce (2.40).

Noting that $\|\tilde{h}_n\|_{\star} \leq \|h_n\|_{**}$, by **Steps 1-4** we conclude that $\|\phi_n\|_{L^\infty(B_1)} = o_n(1)$ which yields to a contradiction with the fact that $\|\phi_n\|_{L^\infty(B_1)} = 1$. This finishes the proof of the lemma. \square

Now we are now ready to prove Proposition 2.4.1.

Proof of Proposition 2.4.1. Here we use the notation introduced in the proof of the previous lemma. For a scaled function $\tilde{g}(s) = \lambda g(\sqrt{\lambda}s)$, $s = r/\sqrt{\lambda}$ we define

$$(2.41) \quad \|\tilde{g}\|_{**} := \|g\|_{**}.$$

Let $R > R_2 + 1$ be a large fixed number, $\delta < 1/4$ and \hat{z}_0 be the solution of the problem

$$\begin{cases} -\Delta \hat{z}_0 = \frac{8\mu^2}{(\mu^2 + s^2)^2} \hat{z}_0 & \text{in } B_{\delta\lambda^{-1/2}} \setminus B_R, \\ \hat{z}_0(R) = \tilde{Z}_0(R), \quad \hat{z}_0\left(\frac{\delta}{\sqrt{\lambda}}\right) = 0, \end{cases}$$

where \tilde{Z}_0 is defined in Step 3 of Lemma 2.4.1. A direct computation shows that

$$\hat{z}_0(s) = \tilde{Z}_0(s) \left[1 - \frac{\int_R^s \frac{dt}{t \tilde{Z}_0^2(t)}}{\int_R^{\frac{\delta}{\sqrt{\lambda}}} \frac{dt}{t \tilde{Z}_0^2(t)}} \right].$$

We consider smooth cut-off functions $\eta_1(s)$ and $\eta_2(s)$ with the following properties: $\eta_1(s) = 1$ for $s < R$, $\eta_1(s) = 0$ for $s > R + 1$, $|\eta_1'(s)| \leq 2$, $\eta_2(s) = 1$ for $s < \frac{\delta}{2\sqrt{\lambda}}$, $\eta_2(s) = 0$ for $s > \frac{\delta}{\sqrt{\lambda}}$, $|\eta_2'(s)| \leq C\sqrt{\lambda}$, $|\eta_2''(s)| \leq C\lambda$. We then define the test function

$$\tilde{z}_0 = \eta_1 \tilde{Z}_0 + (1 - \eta_1) \eta_2 \hat{z}_0.$$

Let ϕ be a solution to (2.33). As previously, we denote $\tilde{\phi}(s) = \phi(\sqrt{\lambda}s)$ and we let $\tilde{\chi}(s) = \chi(\sqrt{\lambda}s)$. Next, we modify $\tilde{\phi}$ so that the orthogonality condition with respect to \tilde{z}_0 is satisfied. We let

$$\hat{\phi} = \tilde{\phi} + A\tilde{z}_0,$$

where the number A satisfies

$$A \int_{B_{\lambda^{-1/2}}} \tilde{\chi} |\tilde{z}_0|^2 dx + \int_{B_{\lambda^{-1/2}}} \tilde{\chi} \tilde{z}_0 \tilde{\phi} dx = 0.$$

Then

$$(2.42) \quad \tilde{L}(\hat{\phi}) = \tilde{h} + A\tilde{L}(\tilde{z}_0),$$

and $\int_{B_{\lambda^{-1/2}}} \tilde{\chi} \tilde{z}_0 \hat{\phi} dx = 0$. Recalling (2.41), the previous lemma thus allows us to estimate

$$(2.43) \quad \|\hat{\phi}\|_{L^\infty(B_1)} = \|\hat{\phi}\|_{L^\infty(B_{\lambda^{-1/2}})} \leq C \left[\|\tilde{h}\|_{**} + |A| \|\tilde{L}(\tilde{z}_0)\|_{**} \right],$$

Observe that $\tilde{z}_0 = 0$ for $s > \lambda^{-1/2}/4$. Thus, remembering (2.39), we have

$$\|\tilde{L}(\tilde{z}_0)\|_{**} = \|\tilde{L}(\tilde{z}_0)\|_{\star}.$$

Now, let us estimate the size of $|A| \|\tilde{L}(\tilde{z}_0)\|_{\star}$. Testing equation (2.42) against \tilde{z}_0 and integrating by parts, we find

$$\langle \hat{\phi}, \tilde{L}(\tilde{z}_0) \rangle = \langle \tilde{h}, \tilde{z}_0 \rangle + A \langle \tilde{L}(\tilde{z}_0), \tilde{z}_0 \rangle,$$

where $\langle f, g \rangle = \int_{B_{\lambda^{-1/2}}} f g dx$. This relation in combination with (2.43) and the fact that

$$\int_{B_{\lambda^{-1/2}}} |\hat{\phi}| |\tilde{L}(\tilde{z}_0)| dx \leq C \|\hat{\phi}\|_{\infty} \|\tilde{L}(\tilde{z}_0)\|_{\star} \quad \text{and} \quad \int_{B_{\lambda^{-1/2}}} |\tilde{h}| |\tilde{z}_0| dx \leq C \|\tilde{h}\|_{\star},$$

yield to

$$(2.44) \quad A \langle \tilde{L}(\tilde{z}_0), \tilde{z}_0 \rangle \leq C \|\tilde{h}\|_{\star} \left[1 + \|\tilde{L}(\tilde{z}_0)\|_{\star} \right] + C |A| \|\tilde{L}(\tilde{z}_0)\|_{\star}^2.$$

Next we measure the size of $\|\tilde{L}(\tilde{z}_0)\|_{\star}$. We have

$$(2.45) \quad \tilde{L}(\tilde{z}_0) = \lambda \tilde{z}_0 + 2\nabla \eta_1 \nabla (\hat{z}_0 - \tilde{Z}_0) + \Delta \eta_1 (\hat{z}_0 - \tilde{Z}_0) - 2\nabla \eta_2 \nabla \hat{z}_0 - \Delta \eta_2 \hat{z}_0.$$

It is easy to observe that, for $s \in (R, R+1)$, we have

$$|\tilde{Z}_0 - \hat{z}_0| = \left| \tilde{Z}_0 \frac{\int_R^r \frac{dt}{t \tilde{Z}_0^2(t)}}{\int_R^{\frac{\delta}{\sqrt{\lambda}}} \frac{dt}{t \tilde{Z}_0^2(t)}} \right| \leq C |\log \lambda|^{-1}, \quad |\tilde{Z}'_0 - \hat{z}'_0| \leq C |\log \lambda|^{-1}.$$

On the other hand for $s \in \left(\frac{\delta}{2\sqrt{\lambda}}, \frac{\delta}{\sqrt{\lambda}}\right)$, we have

$$(2.46) \quad |\hat{z}_0| \leq C |\log \lambda|^{-1} \quad \text{and} \quad |\hat{z}'_0| \leq C \sqrt{\lambda} |\log \lambda|^{-1}.$$

We conclude that

$$(2.47) \quad \|\tilde{L}(\tilde{z}_0)\|_* \leq C|\log \lambda|^{-1}.$$

Finally, we estimate $\langle \tilde{L}(\tilde{z}_0), \tilde{z}_0 \rangle$. We decompose

$$\langle \tilde{L}(\tilde{z}_0), \tilde{z}_0 \rangle = \int_{B_{R+1} \setminus B_R} \tilde{L}(\tilde{z}_0) \tilde{z}_0 dx + \int_{B_{\frac{\delta}{\sqrt{\lambda}}} \setminus B_{\frac{\delta}{2\sqrt{\lambda}}}} \tilde{L}(\tilde{z}_0) \tilde{z}_0 dx + O(\sqrt{\lambda}).$$

Using (2.45) and (2.46), we get that

$$(2.48) \quad \begin{aligned} \left| \int_{B_{\frac{\delta}{\sqrt{\lambda}}} \setminus B_{\frac{\delta}{2\sqrt{\lambda}}}} \tilde{L}(\tilde{z}_0) \tilde{z}_0 dx \right| &\leq C \int_{B_{\frac{\delta}{\sqrt{\lambda}}} \setminus B_{\frac{\delta}{2\sqrt{\lambda}}}} |\nabla \eta_2| |\nabla \hat{z}_0| |\hat{z}_0| dx + C \int_{B_{\frac{\delta}{\sqrt{\lambda}}} \setminus B_{\frac{\delta}{2\sqrt{\lambda}}}} |\Delta \eta_2| |\hat{z}_0|^2 dx \\ &\quad + \lambda \int_{B_{\frac{\delta}{\sqrt{\lambda}}} \setminus B_{\frac{\delta}{2\sqrt{\lambda}}}} |\hat{z}_0|^2 dx \\ &\leq C|\log \lambda|^{-2}. \end{aligned}$$

On the other hand, we have

$$I := \int_{B_{R+1} \setminus B_R} \tilde{L}(\tilde{z}_0) \tilde{z}_0 dx = 2 \int_{B_{R+1} \setminus B_R} \nabla \eta_1 \nabla (\hat{z}_0 - \tilde{Z}_0) \tilde{z}_0 dx + \int_{B_{R+1} \setminus B_R} \Delta \eta_1 (\hat{z}_0 - \tilde{Z}_0) \tilde{z}_0 dx + O(\sqrt{\lambda}).$$

Thus integrating by parts we find

$$I = \int_{B_{R+1} \setminus B_R} \nabla \eta_1 \nabla (\hat{z}_0 - \tilde{Z}_0) \tilde{z}_0 dx - \int_{B_{R+1} \setminus B_R} \nabla \eta_1 (\hat{z}_0 - \tilde{Z}_0) \nabla \tilde{z}_0 dx + O(\sqrt{\lambda}).$$

We now observe that, for $s \in (R, R+1)$, we have

$$|\tilde{Z}_0(s) - \hat{z}_0(s)| \leq C|\log \lambda|^{-1}$$

and

$$|\hat{z}'_0(s)| \leq \frac{1}{R^3} + \frac{1}{R} |\log \lambda|^{-1}.$$

Thus

$$\left| \int_{B_{R+1} \setminus B_R} \nabla \eta_1 (\hat{z}_0 - \tilde{Z}_0) \nabla \tilde{z}_0 dx \right| \leq \frac{D}{R^3} |\log \lambda|^{-1},$$

where D is a constant not depending on R . Now,

$$\int_{B_{R+1} \setminus B_R} \nabla \eta_1 \nabla (\hat{z}_0 - \tilde{Z}_0) \tilde{z}_0 dx = 2\pi \int_R^{R+1} \eta'_1 (\hat{z}_0 - \tilde{Z}_0)' \tilde{Z}_0 t dt + O(|\log \lambda|^{-2})$$

$$\begin{aligned}
 &= -\frac{2\pi}{\int_R^{\frac{1}{\delta\sqrt{\lambda}}} \frac{dt}{t\tilde{Z}_0^2(t)}} \int_R^{R+1} \eta_1' \left[1 - 4 \frac{(\mu t)^2 \tilde{Z}_0 \int_R^t \frac{ds}{s\tilde{Z}_0^2(s)}}{(\mu^2 + t^2)^2} \right] dt \\
 &+ O(|\log \lambda|^{-2}) \\
 &= E|\log \lambda|^{-1} [1 + O(|\log \lambda|^{-1})],
 \end{aligned}$$

where E is a positive constant independent of λ . Thus we conclude, choosing R large enough, that $I \sim -E|\log \lambda|^{-1}$. Combining this and (2.48) we find

$$\langle \tilde{L}(\tilde{z}_0), \tilde{z}_0 \rangle = -\frac{E}{|\log \lambda|} [1 + O(R^{-3} + O(|\log \lambda|^{-1}))].$$

Combining the previous estimate, (2.44) and (2.47), we deduce that

$$|A| \leq C|\log \lambda| \|\tilde{h}\|_{\star}.$$

We thus conclude, using the definition of $\hat{\phi}$ and estimate (2.43), that

$$\|\tilde{\phi}\|_{L^\infty(B_{\lambda^{-1/2}})} \leq C(\|\tilde{h}\|_{**} + |\log \lambda| \|\tilde{h}\|_{\star}).$$

Observe that

$$\|\tilde{h}\|_{\star} = \sup_{s \in [0, \lambda^{-1/2}/4]} \frac{\tilde{h}(s)}{\lambda + (1+s)^{-2-\nu}} \leq \sup_{r \in [0, 1/4]} \frac{\lambda|h(r)|}{\lambda + \left(1 + \frac{r}{\sqrt{\lambda}}\right)^{-2-\nu}} \leq \|\tilde{\chi}_1 h\|_{\star}.$$

The previous two inequalities yield

$$\|\phi\|_{L^\infty(B_1)} \leq C(\|h\|_{**} + |\log \lambda| \|\chi_1 h\|_{\star}).$$

Recalling the definition of the norm $\|\cdot\|_{\star}$, we conclude

$$\|\phi\|_{L^\infty(B_1)} \leq C\|h\|_{\star}.$$

It only remains to prove the existence assertion. For this purpose we consider the space

$$H = \{\phi \in H^1(B_1) \mid \phi \text{ is radial and } \phi'(1) = 0\},$$

endowed with the inner product $\langle \phi, \psi \rangle_{H^1} = \int_{B_1} \nabla \phi \nabla \psi dx + \int_{B_1} \phi \psi dx$. Problem (2.33) expressed in weak form is equivalent to finding $\phi \in H$ such that

$$\langle \phi, \psi \rangle_{H^1} = \int_{B_1} [\lambda e^U \phi + h] \psi dx \quad \text{for all } \psi \in H.$$

By Fredholm's alternative this is equivalent to the uniqueness of solutions to this problem, which is guaranteed by estimate (2.34). \square

We are now in position to prove Theorem 2.1.2.

Proof of Theorem 2.1.2. Thanks to the previous proposition, we know that the operator L is invertible. Therefore, we can rewrite (2.22) as

$$\phi = T(\phi) = L^{-1}[R(U) + N(\phi)].$$

Let ρ be a fixed number. We define

$$A_\rho = \left\{ \phi \in L^\infty(B_1) : \|\phi\|_{L^\infty(B_1)} \leq \rho \varepsilon^{1+\sigma} \right\},$$

where σ is the constant defined in Lemma 2.3.2. We will show that the map $T : A_\rho \rightarrow A_\rho$ is a contraction. Using Lemma 2.3.1, recalling the definition of $\|\cdot\|_*$ given in (2.31) and since $|\log \lambda| = O(\varepsilon^{-1})$, we see that

$$\begin{aligned} \|\lambda e^U\|_* &\leq C \max \left(|\log \lambda| \sup_{r \leq 2\delta} f_\lambda(r) \frac{\mu^2}{\lambda \left(\mu^2 + \left(\frac{r}{\sqrt{\lambda}} \right)^2 \right)^2}, |\log \lambda| \sup_{\delta \leq r \leq 1-\delta_1} f_\lambda(r) \varepsilon^\beta, \varepsilon^{-1} \right) \\ &\leq C \varepsilon^{-1}. \end{aligned}$$

From this and recalling the definition of $N(\cdot)$ (see (2.24)), we deduce that, for $\phi, \psi \in A_\rho$,

$$(2.49) \quad \|N(\phi)\|_* \leq \|\lambda e^U\|_* \|\phi\|_{L^\infty(B_{R_1} \setminus B_{R_0})}^2 \leq C \varepsilon^{-1} \|\phi\|_{L^\infty(B_1)}^2,$$

and

$$\|N_\lambda(\phi) - N_\lambda(\psi)\|_* \leq C \varepsilon^{-1} \max \left\{ \|\phi\|_{L^\infty(B_1)}, \|\psi\|_{L^\infty(B_1)} \right\} \|\phi - \psi\|_{L^\infty(B_1)}.$$

Next, using Lemma 2.3.2, we obtain that

$$(2.50) \quad \begin{aligned} \|R(U)\|_* &\leq C \max \left(|\log \lambda| \sup_{r \leq 2\delta} f_\lambda(r) \frac{\mu^2(\lambda^\alpha + \frac{r^2}{\varepsilon})}{\lambda(\mu^2 + (\frac{r}{\sqrt{\lambda}})^2)^2}, |\log \lambda| \sup_{\delta \leq r \leq 1-\delta_1} f_\lambda(r) \varepsilon^\alpha, \varepsilon^{1+\sigma} \right) \\ &\leq C \varepsilon^{1+\sigma}. \end{aligned}$$

Thus, combining (2.49) and (2.50), we get that, for $\phi \in A_\rho$ and some $\rho > 0$,

$$\|T(\phi)\|_{L^\infty(B_1)} \leq C(\|N(\phi)\|_* + \|R(U)\|_*) \leq \rho \varepsilon^{1+\sigma},$$

and, for $\phi \in A_\rho$ and $\psi \in A_\rho$,

$$\|T_\lambda(\phi) - T_\lambda(\psi)\|_{L^\infty(B_1)} \leq C \|N_\lambda(\phi) - N_\lambda(\psi)\|_* \leq C \varepsilon^\sigma \|\phi - \psi\|_{L^\infty(B_1)}.$$

This implies that T is a contraction mapping on A_ρ , for a suitable ρ . Therefore, we conclude that T has a unique fixed point in A_ρ . This establishes the theorem. \square

2.5 Multi-layered solutions

In this section, we will establish Theorems 2.1.4 and 2.1.5. More generally, we will construct solutions which concentrate at an arbitrary number of spheres provided that the non-degeneracy condition $M_k \neq 0$ holds. More precisely, we have

Theorem 2.5.1. *Let $k \in \mathbb{N} \setminus \{0\}$.*

(i) *Suppose that $M_{k-1} \neq 0$. There exists $\lambda_k > 0$ such that for all $\lambda \in (0, \lambda_k)$, there exists a family of radial solutions u_λ to (2.3) in $B_1(0)$ such that, for ε_λ defined as previously,*

$$\lim_{\lambda \rightarrow 0} \left(\varepsilon_\lambda u_\lambda - \sqrt{2} U_{4 \frac{\varepsilon_\lambda}{\sqrt{2}}, k} \right) = 0,$$

uniformly on compact subsets of $B_{\alpha_1}(0) \cup_{i=1}^{k-1} B_{\alpha_{i+1}}(0) \setminus B_{\alpha_i}(0)$,

$$\lambda e^{u_\lambda} \rightarrow 8\pi\delta_0, \text{ in } B_{\alpha_1/2}(0),$$

and

$$\varepsilon_\lambda \lambda e^{u_\lambda} + \sum_{i=1}^k \left(|\partial_\nu U_{4 \frac{\varepsilon_\lambda}{\sqrt{2}}, k}(\alpha_i)| \right)^{-1} \delta_{\alpha_i} \rightarrow 0, \text{ in } B_1(0) \setminus \{0\}.$$

(ii) *Suppose that $M_k \neq 0$. There exists $\tilde{\lambda}_k > 0$ such that for all $\lambda \in (0, \tilde{\lambda}_k)$, there exists a family of radial solutions \tilde{u}_λ to (2.3) in $B_1(0)$ such that*

$$\lim_{\lambda \rightarrow 0} \left(\varepsilon_\lambda \tilde{u}_\lambda - \sqrt{2} \tilde{U}_{4 \frac{\varepsilon_\lambda}{\sqrt{2}}, k} \right) = 0,$$

uniformly on compact subsets of $B_{\alpha_1}(0) \cup_{i=1}^{k-1} B_{\alpha_{i+1}}(0) \setminus B_{\alpha_i}(0)$,

$$\lambda e^{\tilde{u}_\lambda} \rightarrow 8\pi\delta_0, \text{ in } B_{\alpha_1/2}(0),$$

and

$$\varepsilon_\lambda \lambda e^{\tilde{u}_\lambda} + \sum_{i=1}^k \left(|\partial_\nu \tilde{U}_{4 \frac{\varepsilon_\lambda}{\sqrt{2}}, k}(\alpha_i)| \right)^{-1} \delta_{\alpha_i} \rightarrow 0, \text{ in } B_1(0) \setminus \{0\}.$$

We will only prove Theorem 2.5.1 (i) (the proof of (ii) can be done following the same lines). We are looking for a solution of the form

$$\bar{u}_\lambda(r) = \begin{cases} u_0 & \text{in } (0, \delta) \\ (u_{trans}^0)^i(r) & \text{in } (\tilde{\beta}_{i-1} + \delta_1, \tilde{\beta}_{i-1} + 2\delta_1) \\ u_{int}^i(r) & \text{in } (\tilde{\beta}_{i-1} + 2\delta_1, \tilde{\beta}_i - 2\delta_1) \\ (u_{trans}^1)^i(r) & \text{in } (\tilde{\beta}_i - 2\delta_1, \tilde{\beta}_i - \delta_1) \\ u_{peak}^i(r) & \text{in } (\tilde{\beta}_i - \delta_1, \tilde{\beta}_i + \delta_1), \end{cases}$$

where $i = 1, \dots, k$, for some constants $\tilde{\beta}_i$ depending on ε (see below) to be determined later such that $0 = \tilde{\beta}_0 < \tilde{\beta}_1 < \tilde{\beta}_2 < \dots < \tilde{\beta}_k = 1$, with the convention that $\tilde{\beta}_0 + 2\delta_1 = \delta$ and $\tilde{\beta}_k + \delta_1 = \alpha_k = 1$. We define ε as

$$\ln \frac{4}{\varepsilon^2} - \ln \lambda = \frac{\sqrt{2}}{\varepsilon},$$

and let δ and δ_1 be defined as in Section 2.2. We define the functions u_{peak}^i as in Section 3 of [BCN17a] substituting the R^i 's by $\tilde{\beta}_i$'s and for some $\mu_i = O(\varepsilon)$. They satisfy

$$\|R_\lambda\|_{L^1(\tilde{\beta}_i - \delta_1, \tilde{\beta}_i + \delta_1)} = O(\varepsilon^{1+\sigma}) \text{ for some } \sigma > 0.$$

The functions $(u_{trans}^0)^i$ resp. $(u_{trans}^1)^i$ are linear interpolation between u_{int}^i and u_{peak}^{i-1} resp. u_{peak}^i . Next, we are going to define u_{int}^i which will be shaped on $\frac{\sqrt{2}}{\varepsilon}U_k$ with U_k defined in Theorem 2.1.3(i).

Fix $\alpha = (\alpha_1, \dots, \alpha_{k-1}, \alpha_k)$ as in Theorem 2.1.3 (i). For $a = (a_1, \dots, a_k)$, $\sigma = (\sigma_1, \dots, \sigma_{k-1}, 0)$, where $\sigma_i \in \left(\frac{\alpha_i - \alpha_{i-1}}{4}, \frac{\alpha_{i+1} - \alpha_i}{4}\right)$, for $i \in \{1, \dots, k-1\}$ and $b, \varepsilon > 0$, let us denote by $U_{\varepsilon, a, b, \sigma}$ the solution of the following problem

$$(2.51) \quad \begin{cases} -U''_{\varepsilon, a, b, \sigma} - \frac{n-1}{r}U'_{\varepsilon, a, b, \sigma} + U_{\varepsilon, a, b, \sigma} = 0 & \text{in } \cup_{i=0}^{k-1} (\alpha_i, \alpha_{i+1}) \\ \lim_{r \rightarrow 0^+} -\frac{U_{\varepsilon, a, b, \sigma}(r)}{\ln r} = b \\ U_{\varepsilon, a, b, \sigma}(\alpha_i + \sigma_i) = 1 + \varepsilon a_i \quad i \in \{1, \dots, k\}. \end{cases}$$

Note that $U_{0, a, 0} = U_k$ as defined in Theorem 2.1.3 (i). To prove Theorem 2.5.1 (ii), we defined $U_{\varepsilon, a, \sigma}$ as an analogous perturbation of \tilde{U}_k . Next, we define the following operator

$$F(1 + \varepsilon a, \alpha + \sigma) = \begin{pmatrix} (U'_{\varepsilon, a, b, \sigma})^-(\alpha_1 + \sigma_1) \\ (U'_{\varepsilon, a, b, \sigma})^+(\alpha_1 + \sigma_1) \\ \vdots \\ (U'_{\varepsilon, a, b, \sigma})^-(\alpha_{k-1} + \sigma_{k-1}) \\ (U'_{\varepsilon, a, b, \sigma})^+(\alpha_{k-1} + \sigma_{k-1}) \\ (U'_{\varepsilon, a, b, \sigma})^-(1) \end{pmatrix},$$

where

$$(U'_{\varepsilon, a, b, \sigma})^\pm(\alpha_i + \sigma_i) = \lim_{\varepsilon \rightarrow 0^\pm} \frac{U_{\varepsilon, a, b, \sigma}(\alpha_i + \sigma_i + \varepsilon) - U_{\varepsilon, a, b, \sigma}(\alpha_i + \sigma_i)}{\varepsilon}.$$

Notice that the reflexion law (2.7) implies, for any $i \in \{1, \dots, k-1\}$,

$$(2.52) \quad (U'_{b, k})^+(\alpha_i) + (U'_{b, k})^-(\alpha_i) = 0.$$

Let also $\varphi_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}^k$ and $\tilde{\varphi}_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}^k$ be given by

$$\varphi_\varepsilon(x, t) = (\varphi_\varepsilon^1(x, t), \dots, \varphi_\varepsilon^k(x, t)), \quad \text{with } \varphi_\varepsilon^i(x, t) = \frac{1}{\sqrt{2}} \left(\frac{2(n-1)}{t} - 2x \ln 2 - \varepsilon x \zeta_1^i \right),$$

$$\tilde{\varphi}_\varepsilon(x) = (\tilde{\varphi}_\varepsilon^1(x), \dots, \tilde{\varphi}_\varepsilon^k(x)), \quad \text{with } \tilde{\varphi}_\varepsilon^i(x) = \frac{1}{\sqrt{2}} (-\ln x^2 + \varepsilon x \nu_2^i),$$

where ζ_1^i and ν_2^i are some constants (see [BCN17a] for more details).

Lemma 2.5.1. *Let b sufficiently small. There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists a solution $(\gamma_\varepsilon, \sigma_\varepsilon) \in \mathbb{R}^k \times \mathbb{R}^k$ where $(\sigma_\varepsilon)_i \in (\frac{\alpha_i - \alpha_{i-1}}{4}, \frac{\alpha_{i+1} - \alpha_i}{4})$, for $i \neq k$ and $(\alpha_\varepsilon)_k = 1$ to the equation*

$$F(1 + \varepsilon\tilde{\varphi}_\varepsilon(\gamma_\varepsilon), \alpha + \sigma_\varepsilon) = \begin{pmatrix} -\frac{1}{\gamma_\varepsilon^1} + \varepsilon\varphi_\varepsilon^1(\gamma_\varepsilon^1, \alpha_1 + (\sigma_\varepsilon)_1) \\ \frac{1}{\gamma_\varepsilon^1} + \varepsilon\varphi_\varepsilon^1(\gamma_\varepsilon^1, \alpha_1 + (\sigma_\varepsilon)_1) \\ \vdots \\ -\frac{1}{\gamma_\varepsilon^k} + \varepsilon\varphi_\varepsilon^k(\gamma_\varepsilon^k, 1) \end{pmatrix}.$$

In addition, recalling the definition of $U_{b,k}$ defined in Theorem 2.1.3 (i), we have, for $i \in \{1, \dots, k\}$,

$$(2.53) \quad \lim_{\varepsilon \rightarrow 0} \gamma_\varepsilon^i = -\frac{1}{|U'_{b,k}(\alpha_i)|}.$$

Proof. We define, for $x \in \mathbb{R}^k$ and $\sigma \in (0, 1)^k$ such that $\sigma_i \in (\frac{\alpha_i - \alpha_{i-1}}{4}, \frac{\alpha_{i+1} - \alpha_i}{4})$, for $i \neq k$ and $\alpha_k = 1$,

$$H(\varepsilon; x; \sigma) = F(1 + \varepsilon\tilde{\varphi}_\varepsilon(x), \alpha + \sigma) - \begin{pmatrix} -\frac{1}{x_1} + \varepsilon\varphi_\varepsilon^1(x_1, \alpha_1 + \sigma_1) \\ \frac{1}{x_1} + \varepsilon\varphi_\varepsilon^1(x_1, \alpha_1 + \sigma_1) \\ \vdots \\ -\frac{1}{x_k} + \varepsilon\varphi_\varepsilon^k(x_k, 1) \end{pmatrix}.$$

Evaluating H at $\varepsilon = 0$, $x_i = -\frac{1}{(U'_k)^-(\alpha_i)}$, $i \in \{1, \dots, k\}$, $\sigma = 0$, we find, using (2.52), that

$$H\left(0; -\frac{1}{(U'_k)^-(\alpha_1)}, -\frac{1}{(U'_{b,k})^-(\alpha_2)}, \dots, -\frac{1}{(U'_{b,k})^-(1)}; 0\right) = \begin{pmatrix} (U'_{b,k})^-(\alpha_1) \\ (U'_{b,k})^+(\alpha_1) \\ \vdots \\ (U'_{b,k})^-(1) \end{pmatrix} - \begin{pmatrix} (U'_{b,k})^-(\alpha_1) \\ -(U'_{b,k})^-(\alpha_1) \\ \vdots \\ (U'_{b,k})^-(1) \end{pmatrix} = 0.$$

Moreover, we have

$$(2.54) \quad \begin{aligned} & \frac{\partial}{\partial \xi_i} H\left(0; -\frac{1}{(U'_{b,k})^-(\alpha_1)}, -\frac{1}{(U'_{b,k})^-(\alpha_2)}, \dots, -\frac{1}{(U'_{b,k})^-(1)}; 0\right) \\ &= \begin{pmatrix} -|U'_k(\alpha_1)|^2 & 0 & \dots & 0 & \partial_{k+1}F_1(b, 1, \alpha) & \partial_{k+2}F_1(b, 1, \alpha) & \dots & \partial_{2k-1}F_1(b, 1, \alpha) \\ |U'_k(\alpha_1)|^2 & 0 & \dots & 0 & \partial_{k+1}F_2(b, 1, \alpha) & \partial_{k+2}F_2(b, 1, \alpha) & \dots & \partial_{2k-1}F_2(b, 1, \alpha) \\ 0 & -|U'_k(\alpha_2)|^2 & 0 & \dots & 0 & \partial_{k+1}F_3(b, 1, \alpha) & \partial_{k+2}F_3(b, 1, \alpha) & \dots & \partial_{2k-1}F_3(b, 1, \alpha) \\ \dots & |U'_k(\alpha_2)|^2 & 0 & \dots & 0 & \partial_{k+1}F_4(b, 1, \alpha) & \partial_{k+2}F_4(b, 1, \alpha) & \dots & \partial_{2k-1}F_4(b, 1, \alpha) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & -|U'_k(1)|^2 & \partial_{k+1}F_{2k}(b, 1, \alpha) & \partial_{k+2}F_{2k}(b, 1, \alpha) & \dots & \partial_{2k-1}F_{2k}(b, 1, \alpha) \end{pmatrix} \\ &= N_k, \end{aligned}$$

where $\xi_i = x_i$ and $\xi_{k+i} = \sigma_i$, for $i \in \{1, \dots, k\}$. It is shown in the Appendix of [BCN17a] that $\det N_k = M_{k-1}$. Therefore, by assumption, we have that $\det N_k \neq 0$. Then, using the Implicit Function Theorem, the proof follows. \square

Thanks to the previous lemma, we can make explicit our choice of μ_i and $\tilde{\beta}_i$ as

$$\mu_i = \varepsilon \gamma_\varepsilon^i \quad \text{and} \quad \tilde{\beta}_i = \alpha_i + (\sigma_i)_\varepsilon.$$

We are now able to define the function u_{int}^i as follows

$$u_{int}^i = \frac{\sqrt{2}}{\varepsilon} U_{\varepsilon, 4\varepsilon/\sqrt{2}, \tilde{\varphi}_\varepsilon(\gamma_\varepsilon), \sigma_\varepsilon}.$$

Then the proof follows along the same lines as the proof of Theorem 2.1.2.

2.A Appendix

This appendix is devoted to the study of Green's functions. In particular, we will prove Theorem 2.1.3. First, let us recall the following lemma [BCN17b] which generalized previous results of [Cat09],[GN12] when $N = 2$.

Lemma 2.A.1. *There exist two positive, linearly independent solutions $\zeta \in C^2((0, 1])$ and $\xi \in C^2([0, 1])$ of the equation*

$$-u'' - \frac{1}{r}u' + u = 0 \quad \text{in}(0, 1),$$

satisfying

$$\xi'(0) = \zeta'(1) = 0, \quad r(\xi'(r)\zeta(r) - \xi(r)\zeta'(r)) = 1 \quad \forall r \in (0, 1].$$

We have that ξ is bounded and increasing in $[0, 1]$, ζ is decreasing in $(0, 1]$ and

$$\xi(0) = 1, \quad \lim_{r \rightarrow 0^+} \frac{\zeta(r)}{-\ln r} = 1, \quad \lim_{r \rightarrow 0^+} (-r\zeta'(r)) = 1.$$

Moreover, as r goes to 0, we have

$$(2.55) \quad -\frac{2}{\pi}\zeta(r) = \frac{2}{\pi}(\log r - \ln 2 + \gamma) - \frac{r^2}{2\pi}(\ln r - \ln 2 + \gamma - 1) + O(r^3)$$

and

$$(2.56) \quad -\frac{2}{\pi}\zeta'(r) = \frac{2}{\pi r} - \frac{r(-2\ln r - 2\gamma + 1 + \log 4)}{2\pi} + O(r^2),$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant.

Using the previous lemma, we are able to construct a radial Green's function on the unit ball $B_1(0)$ blowing up at 0 and equal to 1 on $\partial B_1(0)$. The family of solutions defined in Theorem 2.1.2 will behaves like this Green's function far from the origin and from $\partial B_1(0)$.

Lemma 2.A.2. *Let \tilde{b} be a small enough fixed constant. There exists a positive radial function G solution to*

$$(2.57) \quad -G'' - \frac{1}{r}G' + G = 0 \quad \text{in } (0, 1),$$

such that

$$\lim_{r \rightarrow 0^+} \frac{G(r)}{-\ln r} = \tilde{b}, \quad \lim_{r \rightarrow 0^+} rG'(r) = \tilde{b}, \quad G(1) = 1.$$

There exists $\tilde{r} \in (0, 1)$ such that $G'(\tilde{r}) = 0$ and $\tilde{r} = O(\sqrt{\tilde{b}})$. Moreover, when r goes to zero, we have

$$(2.58) \quad G(r) + \tilde{b} \ln r = \frac{\tilde{b}\pi}{2}(\gamma - \ln 2) + o(r)$$

and

$$(2.59) \quad G'(r) + \frac{\tilde{b}}{r} = O(r \ln r).$$

Proof. Using the properties of the functions ξ and ζ (defined in Lemma 2.A.1), it is immediate to see that, for any $b \in (0, 1)$,

$$u_b(r) = \frac{\xi'(b)\zeta(r) - \xi(r)\zeta'(b)}{\xi'(b)\zeta(1) - \xi(1)\zeta'(b)}$$

is a solution to (2.57) such that

$$u_b(1) = 1 \quad \text{and} \quad \lim_{r \rightarrow 0^+} \frac{u_b(r)}{-\ln r} = \frac{\xi'(b)}{\xi'(b)\zeta(1) - \xi(1)\zeta'(b)}.$$

Using the properties of ξ and ζ , we have, for b small enough,

$$\begin{cases} \xi'(b)\zeta(1) - \xi(1)\zeta'(b) &= \xi(1)b^{-1} + o(b^{-1}) \\ \xi'(b) &= \delta b + o(b), \end{cases}$$

for some positive constant δ not depending on b . Therefore, for b small enough, we have, for some constant C_0 not depending on b ,

$$\lim_{r \rightarrow 0^+} \frac{u_b(r)}{-\ln r} = C_0 b^2 + o(b^2).$$

Multiplying u_b by a suitable constant, we get the result. The estimates (2.58) and (2.59) follows from (2.55) and (2.56) and the fact that $\xi(0) = \xi'(0) = 0$. \square

Next, we are going to construct two Green's functions one singular at the origin and in an interior sphere and the other also singular at $\partial B_1(0)$. Before, proceeding, we recall the following useful lemma.

Lemma 2.A.3. Let $0 \leq a < b \leq 1$. Denote by u_x , $x \in (a, b)$, the function satisfying

$$\begin{cases} -u_x''(r) - \frac{1}{r}u_x'(r) + u_x(r) = 0 & r \in (a, x) \\ u_x'(a) = 0, u_x(x) = 1. \end{cases}$$

Then the function $x \rightarrow u_x'(x)$ is strictly increasing. Moreover let v_x , $x \in (a, b)$, the function satisfying

$$\begin{cases} -v_x''(r) - \frac{1}{r}v_x'(r) + v_x(r) = 0 & r \in (x, b) \\ v_x(x) = v_x(b) = 1. \end{cases}$$

Then the function $x \rightarrow v_x'(x)$ is strictly increasing.

Proof. We refer to (2.21) of [BGNT16] for the proof of the first point and to the Proposition A.1 of [BCN17a] for the second one. \square

Thanks to the previous lemma, we are able to prove the existence and uniqueness of the two Green's functions mentioned above.

Lemma 2.A.4. Let $0 < \beta \leq 1$. Then for any $\tilde{b} > 0$ small enough, there exist a unique α and a unique continuous function U solution to

$$\begin{cases} -U'' - \frac{1}{r}U' + U = 0 & \text{in } (0, \alpha) \cup (\alpha, \beta), \\ \lim_{r \rightarrow 0^+} -\frac{U(r)}{\ln r} = \tilde{b}, U'(\beta) = 0, U(\alpha) = 1. \end{cases}$$

satisfying the reflection law

$$\lim_{\varepsilon \rightarrow 0^-} \frac{U(\alpha + \varepsilon) - U(\alpha)}{\varepsilon} = - \lim_{\varepsilon \rightarrow 0^+} \frac{U(\alpha + \varepsilon) - U(\alpha)}{\varepsilon}.$$

We also have that, for any $\tilde{b} > 0$ small enough, there exist a unique α and a unique continuous function V solution to

$$\begin{cases} -V'' - \frac{1}{r}V' + V = 0 & \text{in } (0, \alpha) \cup (\alpha, \beta), \\ \lim_{r \rightarrow 0^+} -\frac{V(r)}{\ln r} = \tilde{b}, V(\alpha) = V(\beta) = 1. \end{cases}$$

satisfying the reflection law

$$\lim_{\varepsilon \rightarrow 0^-} \frac{V(\alpha + \varepsilon) - V(\alpha)}{\varepsilon} = - \lim_{\varepsilon \rightarrow 0^+} \frac{V(\alpha + \varepsilon) - V(\alpha)}{\varepsilon}.$$

Proof. We restrict ourselves to the proof of the second point. Let $b < \beta$ be a small enough fixed constant and consider the function $u : (0, \beta) \times (b, \beta)$ defined as

$$(2.60) \quad u(r, \alpha) = \begin{cases} \frac{\xi'(b)\zeta(r) - \xi(r)\zeta'(b)}{\xi'(b)\zeta(\alpha) - \xi(\alpha)\zeta'(b)} & r \in (0, \alpha) \\ \frac{\xi'(\beta)\zeta(r) - \xi(r)\zeta'(\beta)}{\xi'(\beta)\zeta(\alpha) - \xi(\alpha)\zeta'(\beta)} & r \in (\alpha, \beta), \end{cases}$$

where the functions ξ and ζ are the ones defined in Lemma 2.A.1. Notice that $u(r, \alpha)$ satisfies the equation

$$-u'' - \frac{1}{r}u' + u = 0 \quad \text{in } (0, \alpha) \cup (\alpha, \beta)$$

and $u(\alpha, \alpha) = u(\beta, \alpha) = 1$. Moreover, proceeding as in Lemma 2.A.2, we see that $\lim_{r \rightarrow 0^+} -\frac{u(r, \alpha)}{\ln r} = \frac{\xi''(0)}{\xi(\alpha)}b^2 + o(b^2)$. Thus, for any \tilde{b} sufficiently small, by choosing $b = \left(\frac{\xi(\alpha)}{\xi''(0)}\tilde{b}\right)^{1/2}$, we have $\lim_{r \rightarrow 0^+} -\frac{u(r, \alpha)}{\ln r} = \tilde{b}$. It remains to prove that there exists a unique $\alpha_1 \in (b, 1)$ such that

$$F(\alpha_1) = (u'(\alpha_1, \alpha_1))^+ + (u'(\alpha_1, \alpha_1))^- = 0,$$

where

$$(u'(\alpha, \alpha))^\pm = \lim_{\varepsilon \rightarrow 0^\pm} \frac{u(\alpha + \varepsilon, \alpha) - u(\alpha, \alpha)}{\varepsilon}.$$

Observe that F can be rewritten as

$$F(\alpha) = \frac{\xi'(b)\zeta'(\alpha) - \xi'(\alpha)\zeta'(b)}{\xi'(b)\zeta(\alpha) - \xi(\alpha)\zeta'(b)} + \frac{\xi'(\beta)\zeta'(\alpha) - \xi'(\alpha)\zeta'(\beta)}{\xi'(\beta)\zeta(\alpha) - \xi(\alpha)\zeta'(\beta)}.$$

Thanks to Lemma 2.A.3, we already know that the function $\alpha \rightarrow (u'(\alpha, \alpha))^+$ is strictly increasing. We are going to prove that $\alpha \rightarrow (u'(\alpha, \alpha))^-$ is also strictly increasing. Indeed, recalling that $b = \left(\frac{\xi(\alpha)}{\xi''(0)}\tilde{b}\right)^{1/2}$, for some \tilde{b} small enough, we see that

$$\frac{\partial}{\partial \alpha} \left(\frac{\xi'(b)\zeta'(\alpha) - \xi'(\alpha)\zeta'(b)}{\xi'(b)\zeta(\alpha) - \xi(\alpha)\zeta'(b)} \right) = \frac{\zeta'(b)^2(\xi(\alpha)\xi''(\alpha) - \xi'(\alpha)^2)}{(\xi'(b)\zeta(\alpha) - \xi(\alpha)\zeta'(b))^2} + o(1) > 0.$$

So, in order to prove the existence of α_1 , since F is continuous, it is sufficient to show that $\lim_{\alpha \rightarrow b^+} F(\alpha) < 0$ and $\lim_{\alpha \rightarrow 1^-} F(\alpha) > 0$. First, thanks to Lemma 2.A.1, we notice that, for $\alpha \rightarrow b^+$, we have

$$\xi'(b)\zeta(\alpha) - \xi(\alpha)\zeta'(b) = 1/b + o(1/b)$$

and

$$\xi'(b)\zeta'(\alpha) - \xi'(\alpha)\zeta'(b) = -\frac{b}{\alpha}\xi''(0) + \xi''(0)\frac{\alpha}{b} + o\left(\frac{\alpha}{b}\right) > 0.$$

For $\alpha \rightarrow b^+$, we also have

$$\xi'(\beta)\zeta(\alpha) - \xi(\alpha)\zeta'(\beta) = -\xi'(\beta)\ln \alpha + o(\ln \alpha)$$

and

$$\xi'(\beta)\zeta'(\alpha) - \xi'(\alpha)\zeta'(\beta) = -\xi'(\beta)1/\alpha + o(1/\alpha).$$

Combining the previous estimates, we deduce that, for $\alpha \rightarrow b^+$,

$$F(\alpha) = -\frac{b^2}{\alpha}\xi''(0) + \xi''(0)\alpha + \frac{1}{\alpha \ln \alpha} + o\left(\frac{1}{\alpha \ln \alpha}\right) < 0.$$

On the other hand, for $\alpha \rightarrow \beta^-$, we have

$$\xi'(b)\zeta(\alpha) - \xi(\alpha)\zeta'(b) = \xi(\beta)(1/b) + o(1/b),$$

and

$$\begin{aligned} \xi'(b)\zeta'(\alpha) - \xi'(\alpha)\zeta'(b) &= b\xi''(0)\zeta'(\beta) + \xi'(\beta)(1/b) \\ &= \frac{\xi'(\beta)}{b} + o\left(\frac{1}{b}\right). \end{aligned}$$

Since $\lim_{\alpha \rightarrow \beta^-} \xi'(\beta)\zeta(\alpha) - \xi(\alpha)\zeta'(\beta) = 1/\beta$, and $\xi'(\beta)\zeta'(\alpha) - \xi'(\alpha)\zeta'(b) = O(\alpha - \beta)$, we get that, for $\alpha \rightarrow \beta^-$,

$$F(\alpha) = \frac{\xi'(\beta)}{\xi(\beta)} + o(1) > 0.$$

This concludes the proof. \square

Remark 2.A.1. Observe that along the proof, we also show that $M_1 \neq 0$.

Proof of Theorem 2.1.3. The proof can be done as the one of [BGNT16, Theorem 2.14] substituting $u_{\infty,1\text{-layer}}(\beta_1; 0, \beta_1)$ by the function U defined in the previous lemma with $\beta = \beta_1$. \square

Finally, we show a very rough elliptic estimate which is needed in the proof of Lemma 2.2.3.

Notations

Lemma 2.A.5. Let $R > 0$ and $u \in H^1(B_R(0))$ be a radial solution to

$$\begin{cases} -\Delta u + u = f & \text{in } B_R(0) \\ u'(R) = g \end{cases}$$

for some $f \in L^q(B_R(0))$, $q > 2$. Then, we have

$$\|u\|_{L^\infty(B_R(0))} \leq C\left(\frac{1}{R} + |\ln R| + R\right)(R^{2(1-2/q)}\|f\|_{L^q(B_R(0))} + (1 + R|\ln R|)\|g\|_{L^\infty(\partial B_R(0))}),$$

and

$$\|u'\|_{L^\infty(B_R(0))} \leq C(R^{2(1-2/q)}\|f\|_{L^q(B_R(0))} + (1 + R|\ln R|)\|g\|_{L^\infty(\partial B_R(0))}),$$

for some constant C not depending on R .

Proof. Multiplying the equation by u and integrating by parts, we get

$$(2.61) \quad \|u\|_{H^1(B_R)}^2 \leq \|f\|_{L^2(B_R)}\|u\|_{H^1(B_R)} + R|u'(R)||u(R)|.$$

Since $u(R) - u(r) = \int_r^R u'(s)ds$, one can show that

$$|u(R)|^2 \leq C[|u(r)|^2 + \|u'\|_{L^2(B_R)}^2 \ln \frac{R}{r}],$$

where here and in the following, C denotes constant not depending on R . Multiplying by r and integrating, we find

$$R^2|u(R)|^2 \leq C[\|u\|_{L^2(B_R)}^2 + \|u'\|_{L^2(B_R)}^2 R^2 |\ln R|].$$

This implies that

$$(2.62) \quad |u(R)| \leq C\left[\frac{1}{R} + |\ln R|\right]\|u\|_{H^1(B_R)}.$$

From (2.61) and (2.62) and using that $u'(R) = g$, we obtain that

$$\|u\|_{H^1(B_R)}^2 \leq \|f\|_{L^2(B_R)}\|u\|_{H^1(B_R)} + C(1 + R|\ln R|)\|g\|_{L^\infty(\partial B_R)}\|u\|_{H^1(B_R)}.$$

Thanks to Hölder inequality, we find that

$$(2.63) \quad \|u\|_{H^1(B_R)} \leq C[R^{2(1-1/q)}\|f\|_{L^q(B_R)} + (1 + R|\ln R|)\|g\|_{L^\infty(\partial B_R)}].$$

Next, observe that we can rewrite the equation as, for any $s \in (0, R)$,

$$u'(s)s = \int_0^s (u - f)rdr.$$

From Hölder inequality, we obtain that

$$|u'(s)| \leq C\|u - f\|_{L^2(B_R)} \leq C(\|u\|_{L^2(B_R)} + R^{2(1-2/q)}\|f\|_{L^q(B_R)}).$$

We deduce from (2.63) that

$$\|u'\|_{L^\infty(B_R)} \leq C(R^{2(1-1/q)}\|f\|_{L^q(B_R)} + (1 + R|\ln R|)\|g\|_{L^\infty(\partial B_R)}).$$

Noticing once more that

$$u(R) - u(\tilde{s}) = \int_{\tilde{s}}^R u'(r)dr,$$

we get from (2.62) that

$$\begin{aligned} \|u\|_{L^\infty(B_R)} &\leq C\left(\left[\frac{1}{R} + |\ln R|\right]\|u\|_{H^1(B_R)} + R\|u'\|_{L^\infty(B_R)}\right) \\ &\leq C\left(\frac{1}{R} + |\ln R| + R\right)(R^{2(1-2/q)}\|f\|_{L^q(B_R)} + (1 + R|\ln R|)\|g\|_{L^\infty(\partial B_R)}). \end{aligned}$$

This concludes the proof. □

Part III

Conformal geometry

Chapter 1

The prescribed scalar curvature problem

Abstract

In this chapter, which is based on a joint work with Angela Pistoia [PR17] that has been accepted for publication in the Journal of Differential Equations, we are interested in the problem of prescribing the scalar curvature on an n -dimensional compact Riemannian manifold (M, g) . More precisely, let h be a smooth function on M and assume that it has a critical point $\xi \in M$ such that $h(\xi) = 0$ and which satisfies a suitable flatness assumption. We are interested in finding conformal metrics $g_\lambda = u_\lambda^{\frac{4}{n-2}}g$, with $u > 0$, whose scalar curvature is the prescribed function $h_\lambda := \lambda^2 + h$, where λ is a small parameter.

In the positive case, i.e. when the scalar curvature R_g is strictly positive, we find a family of “bubbling” metrics g_λ , where u_λ blows up at the point ξ and approaches zero far from ξ as λ goes to zero.

In the general case, if in addition we assume that there exists a non-degenerate conformal metric $g_0 = u_0^{\frac{4}{n-2}}g$, with $u_0 > 0$, whose scalar curvature is equal to h , then there exists a bounded family of conformal metrics $g_{0,\lambda} = u_{0,\lambda}^{\frac{4}{n-2}}g$, with $u_{0,\lambda} > 0$, which satisfies $u_{0,\lambda} \rightarrow u_0$ uniformly as $\lambda \rightarrow 0$. Here, we build a second family of “bubbling” metrics g_λ , where u_λ blows up at the point ξ and approaches u_0 far from ξ as λ goes to zero. In particular, this shows that this problem admits more than one solution.

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1.1 Introduction

Let (M, g) be a smooth compact manifold without boundary of dimension $n \geq 3$. The prescribed scalar curvature problem (with conformal change of metric) is

given a function h on M does there exist a metric \tilde{g} conformal to g such that the scalar curvature of \tilde{g} equals h ?

Given a metric \tilde{g} conformal to g , i.e. $\tilde{g} = u^{\frac{4}{n-2}}g$ where the conformal factor u is smooth and strictly positive, this problem is equivalent to finding a solution to

$$(1.1) \quad -\Delta_g u + c(n)R_g u = hu^p, \quad u > 0 \quad \text{on } M,$$

where $\Delta_g = \operatorname{div}_g \nabla_g$ is the Laplace-Beltrami operator, $c(n) = \frac{n-2}{4(n-1)}$, $p = \frac{n+2}{n-2}$, and R_g denotes the scalar curvature associated to the metric g .

We suppose that h is not constant, otherwise we would be in the special case of the Yamabe problem which has been completely solved in the works by Yamabe [Yam60], Trudinger [Tru68], Aubin [Aub76a], and Schoen [Sch84]. For this reason we can assume in (1.1) that R_g is a constant.

In the book [Aub98, Chapter 6], Aubin gives an exhaustive description of known results. Next, we briefly recall some of them.

- **The negative case, i.e. $R_g < 0$.**

A necessary condition for existence is that $\int_M h d\nu_g < 0$ (a more general result can be found in [KW75]).

When $h < 0$, (1.1) has a unique solution (see for instance [KW75, Aub76a]). The situation turns out to be more complicated when h vanishes somewhere on M or if it changes sign. When $\max_M h = 0$, Kazdan and Warner [KW75], Ouyang [Ouy91], Vázquez and Véron [VV91], and del Pino [dP94] proved the existence of a unique solution, provided that a lower bound on R_g , depending on the zero set of h , is satisfied. The general case was studied by Rauzy in [Rau95], who extended the previous results to the case when h changes sign. Letting $h^- := \min\{h, 0\}$ and $h^+ := \max\{h, 0\}$, the theorem proved in [Rau95] reads as follows.

Theorem 1.1.1. Let $\mathcal{A} := \left\{ u \in H_g^1(M) \mid u \geq 0, u \not\equiv 0, \int_M h^- u d\nu_g = 0 \right\}$ and

$$\Lambda_0 := \inf_{u \in \mathcal{A}} \frac{\int_M |\nabla u|^2 d\nu_g}{\int_M u^2 d\nu_g},$$

with $\Lambda_0 = +\infty$ if $\mathcal{A} = \emptyset$. There exists a constant $C(h) > 0$, depending only on $\frac{\min_M h^-}{\int_M h^- d\nu_g}$, such that if

$$(1.2) \quad -c(n)R_g < \Lambda_0 \quad \text{and} \quad \frac{\max_M h^+}{\int_M |h^-| d\nu_g} < C(h)$$

then (1.1) has a solution.

The dependence of the constant $C(h)$ on the function h^- can be found in [AB97]. An interesting feature is that if h changes sign then the uniqueness is not true anymore, as showed by Rauzy in [Rau96].

Theorem 1.1.2. Assume (1.2) and let $\xi \in M$ such that $h(\xi) = \max_M h > 0$. If one of the following conditions hold:

1. $6 \leq n \leq 9$ and $\Delta_g h(\xi) = 0$;
2. $n \geq 10$, the manifold is not locally conformally flat, and $\Delta_g h(\xi) = \Delta_g^2 h(\xi) = 0$;

then (1.1) admits at least two distinct solutions.

• **The zero case, i.e. $R_g = 0$.**

Necessary conditions for existence are that h changes sign and $\int_M h d\nu_g < 0$. Some of the existence results proved by Escobar and Schoen [ES86], Aubin and Hebey [AH91], and Bismuth [Bis98] can be summarized as follows.

Theorem 1.1.3. Let $\xi \in M$ such that $h(\xi) = \max_M h > 0$. If one of the following conditions hold:

1. $3 \leq n \leq 5$ and all the derivatives of h at the point ξ up to order $n - 3$ vanish;
2. (M, g) is locally conformally flat, $n \geq 6$ and all the derivatives of h at the point ξ up to order $n - 3$ vanish;
3. the Weyl's tensor at ξ does not vanish, [$n = 6$ and $\Delta_g h(\xi) = 0$], or [$n \geq 7$ and $\Delta_g h(\xi) = \Delta_g^2 h(\xi) = 0$];

then (1.1) has a solution

• **The positive case, i.e. $R_g > 0$.**

A necessary condition for existence is that $\max_M h > 0$. Some of the existence results proved by Escobar and Schoen [ES86], Aubin and Hebey [AH91], and Hebey and Vaugon [HV93] can be summarized as follows.

Theorem 1.1.4. *Assume that (M, g) is not conformal to the standard sphere (\mathbb{S}^n, g_0) . Let $\xi \in M$ such that $h(\xi) = \max_M h > 0$. If one of the following conditions hold:*

1. $n = 3$ or $[n \geq 4, (M, g)$ is locally conformally flat, and all the derivatives of h at the point ξ up to order $n - 2$ vanish];
2. the Weyl's tensor at ξ does not vanish, $[n = 6$ and $\Delta_g h(\xi) = 0]$, or $[n \geq 7$ and $\Delta_g h(\xi) = \Delta_g^2 h(\xi) = 0]$;

then (1.1) has a solution.

The prescribed scalar curvature problem on the standard sphere has also been largely studied. We refer the interested reader to [HV92, Li95, Li96].

In the rest of this chapter, we focus our attention on the case $h_\lambda(x) := \lambda^2 + h(x)$ where $h \in C^2(M)$ and $\lambda > 0$ is a small parameter. Namely, we study the problem

$$(1.3) \quad -\Delta_g u + c(n)R_g u = (\lambda^2 + h)u^p, \quad u > 0 \quad \text{on } M.$$

In order to state our main results, let us introduce two assumptions. In our first theorem we assume that h satisfies the following *global* condition:

$$(1.4) \quad \begin{aligned} & \text{there exists a non-degenerate solution } u_0 \text{ to} \\ & -\Delta_g u_0 + c(n)R_g u_0 = h u_0^p, \quad u_0 > 0 \quad \text{on } M. \end{aligned}$$

The existence of a solution to (1.4) is guaranteed if h is as in Theorems 1.1.1, 1.1.3, or 1.1.4. The non-degeneracy condition is a delicate issue and it is discussed in Subsection 1.2.3. *It would be interesting to see if for "generic" functions h the solutions of (1.4) are non-degenerate.*

Under this assumption, it is clear that if λ is small enough then (1.3) has a solution $u_{0,\lambda} \in C^2(M)$ such that $\|u_{0,\lambda} - u_0\|_{C^2(M)} \rightarrow 0$ as $\lambda \rightarrow 0$.

In addition, the following *local* condition is assumed in both of our results:

There exist a point $\xi \in M$ and some real numbers $\gamma \geq 2$, $a_1, \dots, a_n \neq 0$, with $\sum_{i=1}^n a_i > 0$,

such that, in some geodesic normal coordinate system centered at ξ , we have

$$(1.5) \quad h(y) = - \sum_{i=1}^n a_i |y_i|^\gamma + R(y) \quad \text{if } y \in B(0, r), \text{ for some } r > 0,$$

where R satisfies $\lim_{y \rightarrow 0} R(y)|y|^{-\gamma} = 0$.

In particular, $h(\xi) = \nabla h(\xi) = 0$. The number γ is called *the order of flatness* of h at the point ξ . Observe that ξ cannot be a minimum point and that if all the a_i 's are positive then ξ is a local maximum point of h .

Let us now introduce the standard n -dimensional bubbles, which are defined via

$$U_{\mu,y}(x) = \mu^{-\frac{n-2}{2}} U\left(\frac{x-y}{\mu}\right), \quad \mu > 0, \quad y \in \mathbb{R}^n,$$

where $U(x) = \alpha_n \frac{1}{(1+|x|^2)^{\frac{n-2}{2}}}$, $\alpha_n = [n(n-2)]^{\frac{n-2}{4}}$.

These functions are all the positive solutions to the critical problem (see [Aub76b, Tal76])

$$-\Delta U = U^p \quad \text{in } \mathbb{R}^n.$$

Our first result concerns the multiplicity of solutions to problem (1.3).

Theorem 1.1.5. *Assume that (M, g) is not conformal to the standard sphere (\mathbb{S}^n, g_0) , (1.4), and (1.5). If one of the following conditions hold:*

1. $3 \leq n \leq 5$ and ξ is a non-degenerate critical point of h , i.e. $\gamma = 2$;
2. $n = 6$ and $\gamma \in (2, 4)$;
3. $7 \leq n \leq 9$ and $\gamma = 4$;
4. $n \geq 10$, (M, g) is locally conformally flat, and $\gamma \in (\frac{n-2}{2}, \frac{n}{2})$;
- (5) $n \geq 10$, the Weyl's tensor at ξ does not vanish, and $\gamma \in (4, 4 + \epsilon)$ for some $\epsilon > 0$;

then, provided λ is small enough, there exists a solution u_λ to problem (1.3) which blows-up at the point ξ as $\lambda \rightarrow 0$. Moreover, as $\lambda \rightarrow 0$, we have

$$\left\| u_\lambda(x) - u_0(x) - \lambda^{-\frac{n-2}{2}} \mu_\lambda^{-\frac{n-2}{2}} U\left(\frac{d_g(x, \xi_\lambda)}{\mu_\lambda}\right) \right\|_{H_g^1(M)} \rightarrow 0,$$

where the concentration point $\xi_\lambda \rightarrow \xi$ and the concentration parameter $\mu_\lambda \rightarrow 0$ with a suitable rate with respect to λ , which depends on the order of flatness γ (see (1.10), (1.12), (1.13), (1.17)).

Finally, if $h \in C^\infty(M)$, then $\lambda^2 + h$ is the scalar curvature of a metric conformal to g .

This is the first multiplicity result in the zero and positive cases. In the negative case, it extends the results of Theorem 1.1.2 to locally conformally flat manifolds, to low-dimensional manifolds (i.e. $3 \leq n \leq 5$), to higher-dimensional manifolds (i.e. $6 \leq n \leq 9$) when the order of flatness at the maximum point ξ is at least 2, and to non-locally conformally manifolds when $n \geq 10$ and the order of flatness at the maximum point ξ is at least 4. Moreover, it also provides an accurate description of the profile of the solution as λ approaches zero.

Our second result concerns the existence of solutions to problem (1.3) in the positive case, without need of the global condition (1.4) on h .

Theorem 1.1.6. *Assume that (M, g) is not conformal to the standard sphere (\mathbb{S}^n, g_0) , $R_g > 0$, and (1.5). If one of the following conditions hold:*

1. $n = 3, 4, 5$ or $[n \geq 6$ and (M, g) is locally conformally flat] and $\gamma \in (n - 2, n)$;
2. $n \geq 6$, the Weyl's tensor at ξ does not vanish, and $\gamma \in (4, n)$;

then, provided λ is small enough, there exists a solution u_λ to problem (1.3) which blows-up at the point ξ as $\lambda \rightarrow 0$. Moreover, as $\lambda \rightarrow 0$, we have

$$\left\| u_\lambda(x) - \lambda^{-\frac{n-2}{2}} \mu_\lambda^{-\frac{n-2}{2}} U \left(\frac{d_g(x, \xi_\lambda)}{\mu_\lambda} \right) \right\|_{H_g^1(M)} \rightarrow 0,$$

where the concentration point $\xi_\lambda \rightarrow \xi$ and the concentration parameter $\mu_\lambda \rightarrow 0$ with a suitable rate with respect to λ , which depends on the order of flatness γ (see (1.33)).

Finally, if $h \in C^\infty(M)$, then $\lambda^2 + h$ is the scalar curvature of a metric conformal to g .

Our results have been inspired by the recent papers by Borer, Galimberti, and Struwe [BGS15] and del Pino and Román [dPR15], where the authors studied the prescribed Gauss curvature problem on a surface of dimension 2 in the negative case. In particular, they built large conformal metrics with prescribed Gauss curvature κ , which exhibit a bubbling behavior around maximum points of κ at zero level.

The proof of our results relies on a Lyapunov-Schmidt procedure (see for instance [BLR95, dPFM03]). To prove Theorem 1.1.5, we look for solutions to (1.3) which share a suitable bubbling profile close to the point ξ and the profile of the solution to the unperturbed problem (1.4) far from the point ξ . The accurate description of the ansatz is given in Section 1.2, which also contains a non-degeneracy result. The finite dimensional reduction is performed in Section 1.3, which also includes the proof of Theorem 1.1.5. All the technical estimates are postponed in the Appendix. In Section 1.4 we prove Theorem 1.1.6, which can be easily deduced by combining the results proved in Section 1.3 and in the Appendix with some recent results obtained by Esposito, Pistoia, and Vétois in [EPV14].

1.2 The approximated solution

1.2.1 The ansatz

To build the approximated solution close to the point ξ we use some ideas introduced in [EP14, RV13]. The main order of the approximated solution close to the point ξ looks like the bubble

$$(1.6) \quad \lambda^{-\frac{n-2}{2}} \mathcal{U}_{t,\tau}(x) := \lambda^{-\frac{n-2}{2}} \mu^{-\frac{n-2}{2}} U \left(\frac{\exp_\xi^{-1}(x)}{\mu} - \tau \right) \quad \text{if } d_g(x, \xi) \leq r,$$

where the point $\tau \in \mathbb{R}^n$ depends on λ and the parameter $\mu = \mu_\lambda(t)$ satisfies

$$(1.7) \quad \mu = t\lambda^\beta \quad \text{for some } t > 0 \text{ and } \beta > 1.$$

The choice of β depends on n , on the geometry of the manifold at the point ξ , i.e. the Weyl's tensor at ξ and on the order of flatness of the function h at ξ , i.e. the number $\gamma := 2 + \alpha$ in (1.5).

Let us be more precise. Let $r \in (0, i_g(M))$ be fixed, where $i_g(M)$ is the injectivity radius of (M, g) , which is strictly positive since the manifold is compact. Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function such that $0 \leq \chi \leq 1$ in \mathbb{R} , $\chi = 1$ in $[-r/2, r/2]$ and $\chi = 0$ in $\mathbb{R} \setminus (-r, r)$. We denote by d_g the geodesic distance in (M, g) and by \exp_ξ^{-1} the associated geodesic coordinate system. We look for solutions of (1.3) of the form

$$(1.8) \quad u_\lambda(x) = u_0(x) + \lambda^{-\frac{n-2}{2}} \mathcal{W}_{t,\tau}(x) + \phi_\lambda(x),$$

where the definition of the blowing-up term $\mathcal{W}_{t,\tau}$ depends on the dimension of the manifold and also on its geometric properties. The higher order term ϕ_λ belongs to a suitable space which will be introduced in the next section. More precisely, $\mathcal{W}_{t,\tau}$ is defined in three different ways:

- **The case $n = 3, 4, 5$.**

It is enough to assume

$$(1.9) \quad \mathcal{W}_{t,\tau}(x) = \chi(d_g(x, \xi)) \mathcal{U}_{t,\tau}(x),$$

where $\mathcal{U}_{t,\tau}$ is defined in (1.6). The concentration parameter μ satisfies

$$(1.10) \quad \mu = t\lambda^{\frac{n+2}{2\alpha-n+6}}, \quad \text{with } t > 0 \text{ provided } 0 \leq \alpha < n - 2.$$

- **The cases $n \geq 10$ when $\text{Weyl}_g(\xi)$ is non-zero and $6 \leq n \leq 9$.**

It is necessary to correct the bubble $\mathcal{U}_{t,\tau}$ defined in (1.6) by adding a higher order term as in [EP14], namely

$$(1.11) \quad \mathcal{W}_{t,\tau}(x) = \chi(d_g(x, \xi)) (\mathcal{U}_{t,\tau}(x) + \mu^2 \mathcal{V}_{t,\tau}(x))$$

where

$$\mathcal{V}_{t,\tau}(x) = \mu^{-\frac{n-2}{2}} V \left(\frac{\exp_\xi^{-1}(x)}{\mu} - \tau \right) \quad \text{if } d_g(x, \xi) \leq r.$$

The choice of parameter μ depends on n . More precisely if $n \geq 10$ and the Weyl's tensor at ξ is non-zero we choose

$$(1.12) \quad \mu = t\lambda^{\frac{2}{\alpha-2}}, \quad \text{with } t > 0 \text{ provided } 2 < \alpha < \frac{2n}{n-2}.$$

If $6 \leq n \leq 9$ we choose

$$(1.13) \quad \mu = t\lambda^{\frac{n+2}{2\alpha-n+6}}, \quad \text{with } t > 0 \text{ provided } \frac{n-6}{2} < \alpha < \min \left\{ \frac{16}{n-2}, \frac{n^2-6n+16}{2(n-2)} \right\}.$$

The function V is defined as follows. If we write $u(x) = u(\exp_\xi^{-1}(x))$ for $x \in B_g(\xi, r)$ and $y = \exp_\xi(x) \in B(0, r)$, then a comparison between the conformal Laplacian $\mathcal{L}_g =$

$-\Delta_g + c(n)R_g$ with the euclidean Laplacian shows that there is an error, which at main order looks like

$$(1.14) \quad \mathcal{L}_g u + \Delta u \sim +\frac{1}{3} \sum_{a,b,i,j=1}^n R_{iabj}(\xi) y_a y_b \partial_{ij}^2 u + \sum_{i,l,k=1}^n \partial_l \Gamma_{ii}^k(\xi) y_l \partial_k u + c(n)R_g(\xi)u.$$

Here R_{iabj} denotes the Riemann curvature tensor, Γ_{ij}^k the Christoffel's symbols and R_g the scalar curvature. This easily follows by standard properties of the exponential map, which imply

$$-\Delta_g u = -\Delta u - (g^{ij} - \delta^{ij}) \partial_{ij}^2 u + g^{ij} \Gamma_{ij}^k \partial_k u,$$

with

$$g^{ij}(y) = \delta^{ij}(y) - \frac{1}{3} R_{iabj}(\xi) y_a y_b + O(|y|^3) \text{ and } g^{ij}(y) \Gamma_{ij}^k(y) = \partial_l \Gamma_{ii}^k(\xi) y_l + O(|y|^2).$$

To build our solution it shall be necessary to kill the R.H.S of (1.14) by adding to the bubble a higher order term V whose existence has been established in [EP14]. To be more precise, we need to remind (see [BE91]) that all the solutions to the linear problem

$$-\Delta v = pU^{p-1}v \quad \text{in } \mathbb{R}^n,$$

are linear combinations of the functions

$$(1.15) \quad Z_0(x) = x \cdot \nabla U(x) + \frac{n-2}{2} U(x), \quad Z_i(x) = \partial_i U(x), \quad i = 1, \dots, n.$$

The correction term V is built in the following Proposition (see Section 2.2 in [EP14]).

Proposition 1.2.1. *There exist $\nu(\xi) \in \mathbb{R}$ and a function $V \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ solution to*

$$-\Delta V - f'(U)V = -\sum_{a,b,i,j=1}^n \frac{1}{3} R_{iabj}(\xi) y_a y_b \partial_{ij}^2 U - \sum_{i,l,k=1}^n \partial_l \Gamma_{ii}^k(\xi) y_l \partial_k U - c(n)R_g(\xi)U + \nu(\xi)Z_0,$$

in \mathbb{R}^n , with

$$\int_{\mathbb{R}^n} V(y) Z^i(y) dy = 0, \quad i = 0, 1, \dots, n$$

and

$$|V(y)| + |y| |\partial_k V(y)| + |y|^2 |\partial_{ij}^2 V(y)| = O\left(\frac{1}{(1+|y|^2)^{\frac{n-4}{2}}}\right), \quad y \in \mathbb{R}^n.$$

- **The case $n \geq 10$ when (M, g) is locally conformally flat.**

In this case it is necessary to perform a conformal change of metric as in [RV13]. Indeed, there exists a function $\Lambda_\xi \in C^\infty(M)$ such that the conformal metric $g_\xi = \Lambda_\xi^{\frac{4}{n-2}} g$ is flat in $B_g(\xi, r)$. The metric can be chosen so that $\Lambda_\xi(\xi) = 1$. Then, we choose

$$(1.16) \quad \mathcal{W}_{t,\tau}(x) = \chi(d_{g_\xi}(x, \xi)) \Lambda_\xi(x) \mathcal{U}_{t,\tau}(x),$$

where $\mathcal{U}_{t,\tau}$ is defined in (1.6) and the exponential map is taken with respect to the new metric g_ξ . In this case, the concentration parameter μ satisfies

$$(1.17) \quad \mu = t\lambda^{\frac{n+2}{2\alpha-n+6}}, \quad \text{with } t > 0 \text{ provided } \frac{n-6}{2} < \alpha < \frac{n^2-6n+16}{2(n-2)}.$$

1.2.2 The higher order term

Let us consider the Sobolev space $H_g^1(M)$ equipped with the scalar product

$$(u, v) = \int_M (\langle \nabla_g u, \nabla_g v \rangle_g + uv) d\nu_g,$$

and let $\|\cdot\|$ be the induced norm. Let us introduce the space where the higher order term ϕ_λ in (1.8) belongs to. Let Z_0, Z_1, \dots, Z_n be the functions introduced in (1.15). We define

$$Z_{i,t,\tau}(x) = \mu^{-\frac{n-2}{2}} \chi(d_{g_\xi}(x, \xi)) \Lambda_\xi(x) Z_i \left(\frac{\exp_\xi^{-1}(x)}{\mu} - \tau \right) \quad i = 0, 1, \dots, n,$$

where g_ξ and Λ_ξ are defined as in (1.16) and we also agree that $g_\xi \equiv g$, $\Lambda_\xi(x) \equiv 1$ if the ansatz is (1.9) or (1.11). Therefore, $\phi_\lambda \in H^\perp$ where

$$H^\perp := \left\{ \phi \in H_g^1(M) : \int_M \phi Z_{i,t,\tau} d\nu_g = 0 \text{ for any } i = 0, 1, \dots, n \right\}.$$

1.2.3 A non-degeneracy result

When the solution u_0 of (1.4) is a minimum point of the energy functional naturally associated with the problem, the non-degeneracy is not difficult to obtain as showed in Lemma 1.2.1. In the general case u_0 is a critical point of the energy of a min-max type and so the non-degeneracy is a more delicate issue. As far as we know there are no results in this direction.

Lemma 1.2.1. *Assume $R_g < 0$, $\max_M h = 0$, and the set $\{x \in M \mid h(x) = 0\}$ has empty interior set. Then the unique solution u_0 to (1.4) is non-degenerate, i.e. the linear problem*

$$-\Delta_g \psi + c(n)R_g \psi - ph(x)u_0^{p-1} \psi = 0 \quad \text{on } M,$$

admits only the trivial solution.

Proof. Del Pino in [dP94] proved that problem (1.4) has a unique solution, which is a minimum point of the energy functional

$$J(u) := \frac{1}{2} \int_M (|\nabla_g u|_g^2 + c(n)R_g u^2) d\nu_g - \frac{1}{p+1} \int_M h|u|^{p+1} d\nu_g.$$

Therefore the quadratic form

$$D^2 J(u_0)[\phi, \phi] = \int_M (|\nabla_g \phi|_g^2 + c(n)R_g \phi^2 - ph u_0^{p-1} \phi^2) d\nu_g, \quad \phi \in H_g^1(M)$$

is positive definite. In particular, the problem

$$(1.18) \quad -\Delta_g \phi_i + c(n)R_g \phi_i - phu_0^{p-1} \phi_i = \lambda_i \phi_i \quad \text{on } M,$$

has a non-negative first eigenvalue λ_1 with associated eigenfunction $\phi_1 > 0$ on M .

If $\lambda_1 = 0$ then we test (1.4) against ϕ_1 and (1.18) against u_0 , we subtract and we get

$$(p-1) \int_M hu_0^p \phi_1 d\nu_g = 0,$$

which gives a contradiction because $h \neq 0$ a.e. in M and $u_0 > 0$ on M . \square

1.3 The finite dimensional reduction

We are going to solve problem

$$(1.19) \quad \mathcal{L}_g u = (\lambda^2 + h) f(u) \quad \text{on } M,$$

where \mathcal{L}_g is the conformal Laplacian and $f(u) = (u^+)^p$, $u^+(x) := \max\{u(x), 0\}$, using a Ljapunov-Schmidt procedure. We rewrite (1.19) as

$$(1.20) \quad L(\phi_\lambda) = -E + (\lambda^2 + h)N(\phi_\lambda) \quad \text{on } M,$$

where setting

$$(1.21) \quad \mathcal{U}_\lambda(x) = \mathcal{U}_{\lambda,t,\tau}(x) := \lambda^{-\frac{n-2}{2}} \mathcal{W}_{t,\tau}(x) + u_0(x),$$

the linear operator $L(\cdot)$ is defined by

$$(1.22) \quad L(\phi) := \mathcal{L}_g \phi - (\lambda^2 + h)f'(\mathcal{U}_\lambda)\phi,$$

the error term is defined by

$$(1.23) \quad E := \mathcal{L}_g \mathcal{U}_\lambda - (\lambda^2 + h)f(\mathcal{U}_\lambda)$$

and the higher order term $N(\cdot)$ is defined by

$$N(\phi) := f(\mathcal{U}_\lambda + \phi) - f(\mathcal{U}_\lambda) - f'(\mathcal{U}_\lambda)\phi.$$

First of all, it is necessary to estimate the error term E .

Proposition 1.3.1. *Let $a, b \in \mathbb{R}_+$ be such that $0 < a < b$ and K be a compact set in \mathbb{R}^n . There exist positive numbers λ_0, C and $\epsilon > 0$ such that for any $\lambda \in (0, \lambda_0)$, for any $t \in [a, b]$ and for any point $\tau \in K$ we have*

$$\|E\|_{L^{\frac{2n}{n+2}}(M)} \leq C\lambda^{\frac{2(n+2)-\alpha(n-2)}{2}+\epsilon} \quad \text{if (1.12) holds}$$

or

$$\|E\|_{L^{\frac{2n}{n+2}}(M)} \leq C\lambda^{\frac{(n-2-\alpha)(n-2)}{2(2\alpha-n+6)}+\epsilon} \quad \text{if (1.10) or (1.13) or (1.17) hold.}$$

Proof. The proof is postponed in Subsection 1.A.1. \square

Then, we develop a solvability theory for the linearized operator L defined in (1.22) under suitable orthogonality conditions.

Proposition 1.3.2. *Let $a, b \in \mathbb{R}_+$ be fixed numbers such that $0 < a < b$ and K be a compact set in \mathbb{R}^n . There exist positive numbers λ_0 and C , such that for any $\lambda \in (0, \lambda_0)$, for any $t \in [a, b]$ and for any point $\tau \in K$, given $\ell \in L^{\frac{2n}{n+2}}(M)$ there is a unique function $\phi_\lambda = \phi_{\lambda, t, \tau}(\ell)$ and unique scalars c_i , $i = 0, \dots, n$ which solve the linear problem*

$$(1.24) \quad \begin{cases} L(\phi) = \ell + \sum_{i=0}^n c_i Z_{i, t, \tau} & \text{on } M \\ \int_M \phi Z_{i, t, \tau} d\nu_g = 0, & \text{for all } i = 0, \dots, n. \end{cases}$$

Moreover,

$$(1.25) \quad \|\phi_\lambda\|_{H_g^1(M)} \leq C \|\ell\|_{L^{\frac{2n}{n+2}}(M)}.$$

Proof. The proof is postponed in Subsection 1.A.2. \square

Next, we reduce the problem to a finite-dimensional one by solving a non-linear problem.

Proposition 1.3.3. *Let $a, b \in \mathbb{R}_+$ be fixed numbers such that $0 < a < b$ and K be a compact set in \mathbb{R}^n . There exist positive numbers λ_0 and C , such that for any $\lambda \in (0, \lambda_0)$, for any $t \in [a, b]$ and for any point $\tau \in K$, there is a unique function $\phi_\lambda = \phi_{\lambda, t, \tau}$ and unique scalars c_i , $i = 0, \dots, n$ which solve the non-linear problem*

$$(1.26) \quad \begin{cases} L(\phi) = -E + (\lambda^2 + h)N(\phi) + \sum_{i=0}^n c_i Z_{i, t, \tau} & \text{on } M \\ \int_M \phi Z_{i, t, \tau} d\nu_g = 0, & \text{for all } i = 0, \dots, n. \end{cases}$$

Moreover,

$$(1.27) \quad \|\phi_\lambda\|_{H_g^1(M)} \leq C \|E\|_{L^{\frac{2n}{n+2}}(M)}$$

and ϕ_λ is continuously differentiable with respect to t and τ .

Proof. The proof relies on standard arguments (see [EPV14]). \square

After Problem (1.26) has been solved, we find a solution to Problem (1.20) if we manage to adjust (t, τ) in such a way that

$$(1.28) \quad c_i(t, \tau) = 0 \quad \text{for all } i = 0, \dots, n.$$

This problem is indeed variational: it is equivalent to finding critical points of a function of t, τ . To see that let us introduce the energy functional J_λ defined on $H_g^1(M)$ by

$$J_\lambda(u) = \int_M \left(\frac{1}{2} |\nabla_g u|_g^2 + \frac{1}{2} c(n) R_g u^2 - \frac{\lambda^2}{p+1} (u^+)^p u + \frac{1}{p+1} h(u^+)^p u \right) d\nu_g.$$

An important fact is that the positive critical points of J_λ are solutions to (1.3). For any number $t > 0$ and any point $\tau \in \mathbb{R}^n$, we define the reduced energy

$$(1.29) \quad \mathcal{J}_\lambda(t, \tau) := J_\lambda(\mathcal{U}_\lambda + \phi_\lambda),$$

where $\mathcal{U}_\lambda = \mathcal{U}_{\lambda,t,\tau}$ is as in (1.21) and $\phi_\lambda = \phi_{\lambda,t,\tau}$ is given by Proposition 1.3.3. Critical points of \mathcal{J}_λ correspond to solutions of (1.28) for small λ , as the following result states.

Lemma 1.3.1. *The following properties hold:*

- (I) *There exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ if $(t_\lambda, \tau_\lambda)$ is a critical point of \mathcal{J}_λ then the function $u_\lambda = \mathcal{U}_\lambda + \phi_{\lambda,t_\lambda,\tau_\lambda}$ is a solution to (1.19).*
- (II) *Let $a, b \in \mathbb{R}_+$ be fixed numbers such that $0 < a < b$ and let K be a compact set in \mathbb{R}^n . There exists $\lambda_0 > 0$ such that, for any $\lambda \in (0, \lambda_0)$, we have:*

(a) *if $n \geq 10$, $\text{Weyl}_g(\xi) \neq 0$, and (1.12) holds then*

$$(1.30) \quad \mathcal{J}_\lambda(t, \tau) = A_0 - \lambda^{\frac{2(n+2)-\alpha(n-2)}{\alpha-2}} \left(\underbrace{\left[A_1 |\text{Weyl}_g(\xi)|_g^2 t^4 - A_2 t^{2+\alpha} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^n} dy \right]}_{\Theta_1(t,\tau)} + o(1) \right)$$

C^1 -uniformly with respect to $t \in [a, b]$ and $\tau \in K$;

(b) *if one of the following conditions is satisfied:*

- (i) $3 \leq n \leq 5$ and (1.10) holds;
- (ii) $6 \leq n \leq 9$ and (1.13) holds;
- (iii) $n \geq 10$, (M, g) is locally conformally flat, and (1.17) holds;

then

$$(1.31) \quad \mathcal{J}_\lambda(t, \tau) = A_0 - \lambda^{\frac{(n-2-\alpha)(n-2)}{2\alpha-n+6}} \left(\underbrace{\left[A_3 u_0(\xi) t^{\frac{n-2}{2}} - A_2 t^{2+\alpha} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^n} dy \right]}_{\Theta_2(t,\tau)} + o(1) \right)$$

C^1 -uniformly with respect to $t \in [a, b]$ and $\tau \in K$.

Here, A_1 , A_2 , and A_3 are constants only depending on n and

$$(1.32) \quad A_0 := \int_M \left(\frac{1}{2} |\nabla_g u_0|_g^2 + \frac{1}{2} c(n) R_g u_0^2 - \frac{\lambda^2}{p+1} u_0^{p+1} + \frac{1}{p+1} h u_0^{p+1} \right) d\nu \\ + \int_{\mathbb{R}^n} \left(\frac{1}{2} |\nabla U|^2 - \frac{1}{p+1} U^{p+1} \right) dy.$$

Proof. The proof of (I) is standard (see [EPV14]). The proof of (II) is postponed in Subsection 1.A.4. \square

The next result is essential to find solutions to (1.3).

Lemma 1.3.2. *There exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$ if $(t_\lambda, \tau_\lambda)$ is a critical point of \mathcal{J}_λ then the function $u_\lambda = \mathcal{U}_\lambda + \phi_{\lambda, t_\lambda, \tau_\lambda}$ is a classical solution to (1.3).*

Proof. By Lemma 1.3.1 we deduce that u_λ solves (1.19). Arguing as in Appendix B of [Str08], one easily sees that $u_\lambda \in C^2(M)$.

It only remains to prove that $u_\lambda > 0$ on M . This is immediate in the positive case, i.e. $R_g > 0$, because the maximum principle holds. Let us consider the case $R_g \leq 0$.

We consider the set $\Omega_\lambda := \{x \in M \mid (u_\lambda - \lambda)^-(x) < 0\}$. Let $m_0 := \min_M u_0 > 0$. By the definition of u_λ we immediately get that for all λ sufficiently small $\phi_\lambda < -\frac{m_0}{2}$ in Ω_λ . Thus, since $\phi_\lambda \rightarrow 0$ in $L^2(M)$, we deduce $|\Omega_\lambda| \rightarrow 0$ as $\lambda \rightarrow 0$. Now, set $v := u_\lambda - \lambda$.

Testing (1.19) against v^- we get

$$\int_{\Omega_\lambda} |\nabla_g v^-|_g^2 d\nu_g + c(n) R_g \int_{\Omega_\lambda} (v^-)^2 d\nu_g - \int_{0 < u_\lambda < \lambda} (\lambda^2 + h) (u_\lambda^+)^{p-1} (v^-)^2 d\nu_g \\ + \lambda \left[c(n) R_g \int_{\Omega_\lambda} v^- d\nu_g - \int_{0 < u_\lambda < \lambda} (h + \lambda^2) (u_\lambda^+)^{p-1} v^- d\nu_g \right] = 0$$

Poincaré's inequality yields

$$\int_{\Omega_\lambda} |\nabla_g v^-|_g^2 d\nu_g \geq C(\Omega_\lambda) \int_{\Omega_\lambda} (v^-)^2 d\nu_g$$

where $C(\Omega_\lambda)$ is a positive constant approaching $+\infty$ as $|\Omega_\lambda|$ goes to zero.

On the other hand

$$\left| \int_{0 < u_\lambda < \lambda} (\lambda^2 + h) (u_\lambda^+)^{p-1} (v^-)^2 d\nu_g \right| \leq C \lambda^{p-1} \|v^-\|_{L^2(M)}^2,$$

and

$$\left| \int_{0 < u_\lambda < \lambda} (h + \lambda^2) (u_\lambda^+)^{p-1} v^- d\nu_g \right| \leq C \lambda^{p-1} \int_{\Omega_\lambda} |v^-| d\nu,$$

for some positive constant C not depending on λ . Collecting the previous computations we get

$$\underbrace{(C(\Omega_\lambda) + c(n)R_g - C\lambda^{p-1})}_{>0 \text{ if } \lambda \approx 0} \|v^-\|_{L^2(M)}^2 + \lambda \underbrace{(c(n)|R_g| - C\lambda^{p-1})}_{>0 \text{ if } \lambda \approx 0} \int_{\Omega_\lambda} |v^-| d\nu_g \leq 0,$$

which implies $v^- = 0$ if λ is small enough. Since $u_\lambda \in C^2(M)$ we deduce that $u_\lambda \geq \lambda > 0$ in M and the claim is proved. \square

1.3.1 Proof of the main result

Theorem 1.1.5 is an immediate consequence of the more general result.

Theorem 1.3.1. *Assume (1.5) with $\gamma := 2 + \alpha$. If one of the following conditions hold:*

1. $n \geq 10$, the Weyl's tensor at ξ does not vanish, and $2 < \alpha < \frac{2n}{n-2}$;
2. $3 \leq n \leq 5$ and $0 \leq \alpha < n - 2$;
3. $6 \leq n \leq 9$ and $\frac{n-6}{2} < \alpha < \min \left\{ \frac{16}{n-2}, \frac{n^2-6n+16}{2(n-2)} \right\}$;
4. $n \geq 10$, (M, g) is locally conformally flat, and $\frac{n-6}{2} < \alpha < \frac{n^2-6n+16}{2(n-2)}$;

then, provided λ is small enough, there exists a solution to (1.3) which blows-up at the point ξ as $\lambda \rightarrow 0$.

Moreover, if $h \in C^\infty(M)$ then $\lambda^2 + h$ is the scalar curvature of a metric conformal to g .

Proof. We will show that the functions Θ_1 and Θ_2 , defined respectively in (1.30) and (1.31), have a non-degenerate critical point provided $\sum_{i=1}^n a_i > 0$ and $a_i \neq 0$ for any i . As a consequence, provided λ is small enough, the reduced energy \mathcal{J}_λ has a critical point and by Lemma 1.3.2 we deduce the existence of a classical solution to problem (1.3), which concludes the proof.

Without loss of generality, we can consider the function

$$\Theta(t, \tau) := t^\beta - t^\gamma \sum_{i=1}^n a_i \int_{\mathbb{R}^n} |y_i + \tau_i|^\gamma f(y) dy, \quad (t, \tau) \in (0, +\infty) \times \mathbb{R}^n,$$

where $\beta = 4$ or $\beta = \frac{n-2}{2}$, $\gamma = \alpha + 2$ and $f(y) = \frac{A}{(1+|y|^2)^n}$ for some positive constant A . It is immediate to check that, because $\sum_{i=1}^n a_i > 0$, this function has a critical point $(t_0, 0)$, where t_0 solves

$$\beta t^\beta = \mathbf{c}_1 \gamma t^\gamma \sum_{i=1}^n a_i, \quad \text{with } \mathbf{c}_1 := \int_{\mathbb{R}^n} |y_i|^\gamma f(y) dy \text{ not depending on } i,$$

Moreover it is non-degenerate. Indeed, a straightforward computation shows that

$$D^2\phi(t_0, 0) = \begin{pmatrix} \beta(\beta - \gamma)t_0^{\beta-2} & 0 & \dots & 0 \\ 0 & -\gamma(\gamma - 1)\mathbf{c}_2 t_0^\gamma a_1 & \dots & 0 \\ 0 & 0 & \dots & -\gamma(\gamma - 1)\mathbf{c}_2 t_0^\gamma a_n \end{pmatrix},$$

where $\mathbf{c}_2 := \int_{\mathbb{R}^n} |y_i|^{\gamma-2} f(y) dy$ does not depend on i , which is invertible because $\beta \neq \gamma$, $\beta > 0$, and $a_i \neq 0$ for any i . \square

1.4 The positive case: proof of Theorem 1.1.6

In this section we find a solution to equation (1.3) in the positive case, i.e. $R_g > 0$ only assuming the local behavior (1.5) of the function h around the local maximum point ξ . We build solutions to problem (1.3) which blow-up at ξ as λ goes to zero, by combining the ideas developed by Esposito, Pistoia and Vétois [EPV14], the Ljapunov-Schmidt argument used in Section 1.3 and the estimates computed in the Appendix. We omit all the details of the proof because they can be found (up to minor modifications) in [EPV14] and in the Appendix. We only write the profile of the solutions we are looking for and the reduced energy whose critical points produce solutions to our problem.

1.4.1 The ansatz

Let us recall the construction of the main order term of the solution performed in [EPV14]. In case (M, g) is locally conformally flat, there exists a family $(g_\xi)_{\xi \in M}$ of smooth conformal metrics to g such that g_ξ is flat in the geodesic ball $B_\xi(r_0)$. In case (M, g) is not locally conformally flat, we fix $N > n$, and we find a family $(g_\xi)_{\xi \in M}$ of smooth conformal metrics to g such that

$$|\exp_\xi^* g_\xi|(y) = 1 + O(|y|^N)$$

C^1 -uniformly with respect to $\xi \in M$ and $y \in T_\xi M$, $|y| \ll 1$, where $|\exp_\xi^* g_\xi|$ is the determinant of g_ξ in geodesic normal coordinates of g_ξ around ξ . Such coordinates are said to be *conformal normal coordinates* of order N on the manifold. Here, the exponential map \exp_ξ^* is intended with respect to the metric g_ξ . For any $\xi \in M$, we let Λ_ξ be the smooth positive function on M such that $g_\xi = \Lambda_\xi^{\frac{4}{n-2}} g$. In both cases (locally conformally flat or not), the metric g_ξ can be chosen smooth with respect to ξ and such that $\Lambda_\xi(\xi) = 1$ and $\nabla \Lambda_\xi(\xi) = 0$. We let G_g and G_{g_ξ} be the respective Green's functions of L_g and L_{g_ξ} . Using the fact that $\Lambda_\xi(\xi) = 1$, we deduce

$$G_g(\cdot, \xi) = \Lambda_\xi(\cdot) G_{g_\xi}(\cdot, \xi).$$

We define

$$\mathcal{W}_{t,\tau}(x) = G_g(x, \xi) \widehat{\mathcal{W}}_{t,\tau}(x),$$

with

$$\widehat{\mathcal{W}}_{t,\tau}(x) := \begin{cases} \beta_n \lambda^{-\frac{(n-2)}{2}} \mu^{-\frac{n-2}{2}} d_{g_\xi}(x, \xi)^{n-2} U\left(\frac{d_{g_\xi}(x, \xi)}{\mu} - \tau\right) & \text{if } d_{g_\xi}(x, \xi) \leq r \\ \beta_n \lambda^{-\frac{(n-2)}{2}} \mu^{-\frac{n-2}{2}} r^{n-2} U\left(\frac{r_0}{\mu} - \tau\right) & \text{if } d_{g_\xi}(x, \xi) > r, \end{cases}$$

where $\beta_n = (n-2)\omega_{n-1}$, ω_{n-1} is the volume of the unit $(n-1)$ -sphere, $\tau \in \mathbb{R}^n$ and the concentration parameter $\mu = \mu_\lambda(t)$ with $t > 0$ is defined as (here $\alpha = \gamma - 2$, being γ the order of flatness of h at the point ξ)

$$(1.33) \quad \mu = \begin{cases} t\lambda^{\frac{2}{4+\alpha-n}} & \text{if } n = 3, 4, 5 \text{ or } [n \geq 6 \text{ and } (M, g) \text{ is l.c.f.}] \text{ with } n-4 < \alpha < n-2 \\ t\ell^{-1}(\lambda^2) & \text{if } n = 6 \text{ and } \text{Weyl}_g(\xi) \neq 0 \text{ with } 2 < \alpha < 4 \\ t\lambda^{\frac{2}{\alpha-2}} & \text{if } n \geq 7 \text{ and } \text{Weyl}_g(\xi) \neq 0 \text{ with } 2 < \alpha < n-2, \end{cases}$$

where the function $\ell(\mu) := -\mu^{2-\alpha} \ln \mu$ when μ is small. We look for a solution to (1.3) as $u_\lambda = \mathcal{W}_{t,\tau} + \phi_\lambda$, where the higher order term is found arguing as in Section 1.3.

1.4.2 The reduced energy

Combining Lemma 1 in [EPV14] and Lemma 1.A.2 in the Appendix, the reduced energy \mathcal{J}_λ introduced in (1.29) (where the term \mathcal{U}_λ is replaced by $\mathcal{W}_{t,\tau}$ and in particular $u_0 = 0$) reads as

(a) if $n = 3, 4, 5$ or $[n \geq 6 \text{ and } (M, g) \text{ is l.c.f.}]$ with $n-4 < \alpha < n-2$ and (1.33) holds then

$$\mathcal{J}_\lambda(t, \tau) = A_0 - \lambda^{\frac{(n-6-\alpha)(n-2)}{4+\alpha-n}} \left(A_3 \mathbf{m}(\xi) t^{n-2} - A_2 t^{2+\alpha} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^n} dy + o(1) \right),$$

where $\mathbf{m}(\xi) > 0$ is the *mass* at the point ξ ,

(b) if $n = 6$, $\text{Weyl}_g(\xi) \neq 0$, and (1.33) holds then

$$\mathcal{J}_\lambda(t, \tau) = A_0 - \left(-A_1 |\text{Weyl}_g(\xi)|_g^2 \frac{\mu_\lambda^4(t) \ln \mu_\lambda(t)}{\lambda^4} - A_2 \frac{\mu_\lambda^{2+\alpha}(t)}{\lambda^6} \sum_{i=1}^6 \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^6} dy + o(1) \right),$$

(c) if $n \geq 7$, $\text{Weyl}_g(\xi) \neq 0$ and (1.33) holds then

$$\mathcal{J}_\lambda(t, \tau) = A_0 - \lambda^{\frac{2(n+2)-\alpha(n-2)}{\alpha-2}} \left(A_1 |\text{Weyl}_g(\xi)|_g^2 t^4 - A_2 t^{2+\alpha} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^n} dy + o(1) \right),$$

C^1 -uniformly with respect to t in compact sets of $(0, +\infty)$ and τ in compact sets of \mathbb{R}^n . Here A_1, A_2 and A_3 are constants only depending on n and A_0 depends only on n and λ .

1.A Appendix

We recall the following useful lemma (see, for example, [Li98]).

Lemma 1.A.1. *For any $a > 0$ and $b \in \mathbb{R}$ we have*

$$|((a+b))^q - a^q| \leq \begin{cases} c(q) \min\{|b|^q, a^{q-1}|b|\} & \text{if } 0 < q < 1 \\ c(q) (|b|^q + a^{q-1}|b|) & \text{if } q \geq 1, \end{cases}$$

and

$$\left| ((a+b)^+)^{q+1} - a^{q+1} - (q+1)a^q b \right| \leq \begin{cases} c(q) \min\{|b|^{q+1}, a^{q-1}b^2\} & \text{if } 0 < q < 1 \\ c(q) (|b|^{q+1} + a^{q-1}b^2) & \text{if } q \geq 1. \end{cases}$$

1.A.1 Estimate of the error

Proof of Lemma 1.3.1. We split the error (1.23) into

$$E = (-\Delta_g + c(n)R_g) \left[\lambda^{-\frac{n-2}{2}} \mathcal{W}_{t,\tau} + u_0 \right] - (\lambda^2 + h) \left[\lambda^{-\frac{n-2}{2}} \mathcal{W}_{t,\tau} + u_0 \right]^p = E_1 + E_2 + E_3,$$

where

$$\begin{aligned} E_1 &= \lambda^{-\frac{n-2}{2}} \left[-\Delta_g \mathcal{W}_{t,\tau} + c(n)R_g \mathcal{W}_{t,\tau} + \mathcal{W}_{t,\tau}^p \right], \\ E_2 &= -\lambda^{-\frac{n-2}{2}} \left[(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0)^p - \mathcal{W}_{t,\tau}^p \right], \\ E_3 &= \lambda^{-\frac{n+2}{2}} h \left[(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0)^p - (\lambda^{\frac{n-2}{2}} u_0)^p \right]. \end{aligned}$$

To estimate E_2 and E_3 we use the fact that the bubble $\mathcal{W}_{t,\tau}$ satisfies in the three cases

$$(1.34) \quad \mathcal{W}_{t,\tau}(\exp_\xi(y)) = O\left(\frac{\mu^{\frac{n-2}{2}}}{(\mu^2 + |y - \mu\tau|^2)^{\frac{n-2}{2}}} \right) \quad \text{if } |y - \xi| \leq r.$$

Indeed by (1.34) and Lemma 1.A.1 we immediately deduce that

$$\begin{aligned} \|E_2\|_{L^{\frac{2n}{n+2}}(M)} &= O\left(\lambda^{-\frac{n-2}{2}} \left\| \lambda^{\frac{n-2}{2}} u_0 \mathcal{W}_{t,\tau}^{p-1} \right\|_{L^{\frac{2n}{n+2}}(M)} \right) + O\left(\lambda^{-\frac{n-2}{2}} \left\| \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right\|_{L^{\frac{2n}{n+2}}(M)} \right) \\ &= O\left(\left\| \mathcal{W}_{t,\tau}^{p-1} \right\|_{L^{\frac{2n}{n+2}}(M)} \right) + O(\lambda^2), \end{aligned}$$

with

$$\left\| \mathcal{W}_{t,\tau}^{p-1} \right\|_{L^{\frac{2n}{n+2}}(M)} = \begin{cases} O\left(\mu^{\frac{n-2}{2}} \right) & \text{if } 3 \leq n \leq 5 \\ O\left(\mu^2 |\ln \mu|^{\frac{2}{3}} \right) & \text{if } n = 6 \\ O\left(\mu^2 \right) & \text{if } n \geq 7, \end{cases}$$

and

$$\|E_3\|_{L^{\frac{2n}{n+2}}(M)} = O\left(\lambda^{-\frac{n+2}{2}} \left\| h \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p-1} \mathcal{W}_{t,\tau} \right\|_{L^{\frac{2n}{n+2}}(M)} \right) + O\left(\lambda^{-\frac{n+2}{2}} \|h \mathcal{W}_{t,\tau}^p\|_{L^{\frac{2n}{n+2}}(M)} \right)$$

$$= O\left(\lambda^{-\frac{n-2}{2}} \|h\mathcal{W}_{t,\tau}\|_{L^{\frac{2n}{n+2}}(M)}\right) + O\left(\lambda^{-\frac{n+2}{2}} \|h\mathcal{W}_{t,\tau}^p\|_{L^{\frac{2n}{n+2}}(M)}\right),$$

with

$$\|h\mathcal{W}_{t,\tau}\|_{L^{\frac{2n}{n+2}}(M)} = \begin{cases} O\left(\mu^{\frac{n-2}{2}}\right) & \text{if } n < 10 + 2\alpha \\ O\left(\mu^{\frac{n-2}{2}} |\ln \mu|^{\frac{n+2}{2n}}\right) & \text{if } n = 10 + 2\alpha \\ O\left(\mu^{4+\alpha}\right) & \text{if } n > 10 + 2\alpha, \end{cases}$$

and

$$\|h\mathcal{W}_{t,\tau}^p\|_{L^{\frac{2n}{n+2}}(M)} = \begin{cases} O\left(\mu^{2+\alpha}\right) & \text{if } 2\alpha < n - 2 \\ O\left(\mu^{\frac{n+2}{2}} |\ln \mu|^{\frac{n+2}{2n}}\right) & \text{if } 2\alpha = n - 2 \\ O\left(\mu^{\frac{n+2}{2}}\right) & \text{if } 2\alpha > n - 2. \end{cases}$$

Now, let us estimate E_1 . In the first two cases, we argue exactly as in Lemma 3.1 in [EP14] and we deduce that

$$\|E_1\|_{L^{\frac{2n}{n+2}}(M)} = \begin{cases} O\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right) & \text{if } 3 \leq n \leq 7 \\ O\left(\frac{\mu^3 |\ln \mu|^{\frac{5}{8}}}{\lambda^3}\right) & \text{if } n = 8 \\ O\left(\frac{\mu^3}{\lambda^{\frac{7}{2}}}\right) & \text{if } n = 9 \\ O\left(\frac{\mu^2 \frac{n+2}{n-2}}{\lambda^{\frac{n-2}{2}}}\right) & \text{if } n \geq 10. \end{cases}$$

In the third case, arguing exactly as in Lemma 7.1 of [RV13] we get

$$\|E_1\|_{L^{\frac{2n}{n+2}}(M)} = O\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right).$$

Collecting all the previous estimates we get the claim. \square

1.A.2 The linear theory

Proof of Lemma 1.3.2. We prove (1.25) by contradiction. If the statement were false, there would exist sequences $(\lambda_m)_{m \in \mathbb{N}}$, $(t_m)_{m \in \mathbb{N}}$, $(\tau_m)_{m \in \mathbb{N}}$ such that (up to subsequence) $\lambda_m \downarrow 0$, $\frac{\mu_m}{\lambda_m} \downarrow 0$, $t_m \rightarrow t_0 > 0$ and $\tau_m \rightarrow \tau_0 \in \mathbb{R}^n$ and functions ϕ_m, ℓ_m with $\|\phi_m\|_{H_g^1(M)} = 1$, $\|\ell_m\|_{L^{\frac{2n}{n+2}}} \rightarrow 0$, such that for scalars c_i^m one has

$$(1.35) \quad \begin{cases} L(\phi_m) = \ell_m + \sum_{i=0}^n c_i^m Z_{i,t_m,\tau_m} & \text{on } M \\ \int_M \phi_m Z_{i,t_m,\tau_m} d\nu_g = 0, & \text{for all } i = 0, \dots, n. \end{cases}$$

We change variable setting $y = \frac{\exp_{\xi_m}^{-1}(x)}{\mu_m} - \tau_m$. We remark that $d_g(x, \xi) = |\exp_{\xi}^{-1}(x)|$ and we set

$$\tilde{\phi}_m(y) = \mu_m^{\frac{n-2}{2}} \chi(\mu_m|y + \tau_m|) \phi_m(\exp_{\xi_m}(\mu_m(y + \tau_m))) \quad y \in \mathbb{R}^n.$$

Since $\|\phi_m\|_{H_g^1(M)} = 1$, we deduce that the scaled function $(\tilde{\phi}_m)_m$ is bounded in $D^{1,2}(\mathbb{R}^n)$. Up to subsequence, $\tilde{\phi}_m$ converges weakly to a function $\tilde{\phi} \in D^{1,2}(\mathbb{R}^n)$ and thus in $L^{p+1}(\mathbb{R}^n)$ due to the continuity of the embedding of $D^{1,2}(\mathbb{R}^n)$ into $L^{p+1}(\mathbb{R}^n)$.

Step 1: We show that $c_i^m \rightarrow 0$ as $m \rightarrow \infty$ for all $i = 0, \dots, n$.

We test (1.35) against Z_{i,t_m,τ_m} . Integration by parts gives

$$(1.36) \quad \begin{aligned} \int_M \langle \nabla_g \phi_m, \nabla_g Z_{i,t_m,\tau_m} \rangle_g d\nu_g + \int_M [R_g - (\lambda_m^2 + h) f'(\mathcal{U}_{\lambda_m})] \phi_m Z_{i,t_m,\tau_m} d\nu_g \\ = \int_M \ell_m Z_{i,t_m,\tau_m} d\nu_g + \sum_{j=0}^n c_i^m \int_M Z_{j,t_m,\tau_m} Z_{i,t_m,\tau_m} d\nu_g. \end{aligned}$$

Observe that

$$\left| \int_M \ell_m Z_{i,t_m,\tau_m} d\nu_g \right| \leq \|\ell_m\|_{L^{\frac{2n}{n+2}}(M)} \|Z_{i,t_m,\tau_m}\|_{L^{\frac{2n}{n-2}}(M)} = o(1).$$

By change of variables we have

$$c_j^m \int_M Z_{j,t_m,\tau_m} Z_{i,t_m,\tau_m} d\nu_g = c_j^m \int_{\mathbb{R}^n} Z_j Z_i dy + o(1) = c_j^m \delta_{ij} \int_{\mathbb{R}^n} Z_j^2 dy + o(1),$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

Writing $\tilde{h}(y) = h(\exp_{\xi_m}(\mu_m(y + \tau_m)))$, note also that

$$\begin{aligned} R_g \int_M \phi_m Z_{i,t_m,\tau_m} d\nu_g + \int_M h f'(\mathcal{U}_{\lambda_m}) \phi_m Z_{i,t_m,\tau_m} d\nu_g \\ = R_g \mu_m^2 \int_{\mathbb{R}^n} \tilde{\phi}_m Z_i dy + \int_{\mathbb{R}^n} \frac{\tilde{h}}{\lambda_m^2} \tilde{\phi}_m f'(U) Z_i dy + o(1). \end{aligned}$$

On the other hand, standard computations show that

$$\begin{aligned} \int_M \langle \nabla_g \phi_m, \nabla_g Z_{i,t_m,\tau_m} \rangle_g d\nu_g - \lambda_m^2 \int_M f'(\mathcal{U}_{\lambda_m}) \phi_m Z_{i,t_m,\tau_m} d\nu_g \\ = \int_{\mathbb{R}^n} \nabla \tilde{\phi}_m \cdot \nabla Z_i dy - \int_{\mathbb{R}^n} f'(U) \tilde{\phi}_m Z_i dy + o(1) \\ = - \int_{\mathbb{R}^n} (\Delta Z_i + f'(U) Z_i) \tilde{\phi}_m dy + o(1). \end{aligned}$$

Since Z_i satisfies $-\Delta Z_i = f'(U) Z_i$ in \mathbb{R}^n , passing to the limit into (1.36) yields

$$\sum_{j=0}^n \lim_{m \rightarrow \infty} c_i^m \delta_{ij} \int_{\mathbb{R}^n} Z_j^2 dy = o(1).$$

Hence $\lim_{m \rightarrow \infty} c_i^m = 0$, for all $i = 0, \dots, n$.

Step 2: We show that $\tilde{\phi} \equiv 0$.

Given any smooth function $\tilde{\psi}$ with compact support in \mathbb{R}^n we define ψ by the relation

$$\psi(x) = \mu_m^{-\frac{n-2}{2}} \chi(d_g(x, \xi)) \tilde{\psi} \left(\frac{\exp_{\xi_m}^{-1}(x)}{\mu_m} - \tau_m \right) \quad x \in M.$$

We test (1.35) against ψ . Integration by parts gives

$$(1.37) \quad \begin{aligned} \int_M \langle \nabla_g \phi_m, \nabla_g \psi \rangle_g d\nu_g + \int_M [R_g - (\lambda_m^2 + h) f'(\mathcal{U}_{\lambda_m})] \phi_m \psi d\nu_g \\ = \int_M \ell_m \psi d\nu_g + \sum_{j=0}^n c_i^m \int_M Z_{j,t_m,\tau_m} \psi d\nu_g. \end{aligned}$$

By **Step 1** it is easy to see that

$$(1.38) \quad \int_M \ell_m \psi d\nu_g + \sum_{j=0}^n c_i^m \int_M Z_{j,t_m,\tau_m} \psi d\nu_g \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

On the other hand, by the same arguments given in the proof of **Step 1**, we have

$$\begin{aligned} \int_M \langle \nabla_g \phi_m, \nabla_g \psi \rangle_g d\nu_g + \int_M [R_g - (\lambda_m^2 + h) f'(\mathcal{U}_{\lambda_m})] \phi_m \psi d\nu_g \\ \rightarrow \int_{\mathbb{R}^n} \nabla \tilde{\phi} \cdot \nabla \tilde{\psi} dy - \int_{\mathbb{R}^n} f'(U) \tilde{\phi} \tilde{\psi} dy \end{aligned}$$

as $m \rightarrow \infty$. Hence, passing to the limit into (1.37) and integrating by parts we get

$$- \int_{\mathbb{R}^n} (\Delta \tilde{\phi} + f'(U) \tilde{\phi}) \tilde{\psi} dy = 0 \quad \text{for all } \tilde{\psi} \in C_c^\infty(\mathbb{R}^n).$$

We conclude that $\tilde{\phi}$ is a solution in $D^{1,2}(\mathbb{R}^n)$ to $-\Delta v = f'(U)v$ in \mathbb{R}^n . Thus $\tilde{\phi} = \sum_{j=0}^n \alpha_j Z_j$, for certain scalars α_j . But

$$0 = \int_M \phi_m Z_{i,t_m,\tau_m} d\nu_g = \int_{\mathbb{R}^n} \tilde{\phi}_m Z_i d\nu_g \quad \text{for all } i = 0, \dots, n.$$

Passing to the limit we get $\int_{\mathbb{R}^n} \tilde{\phi} Z_i d\nu_g = 0$ for $i = 0, \dots, n$, which implies $\alpha_i = 0$ for all i .

Step 3: We show that, up to subsequence, $\phi_m \rightharpoonup 0$ in $H_g^1(M)$.

Since $(\phi_m)_m$ is bounded in $H_g^1(M)$, up to subsequence, ϕ_m converges weakly to a function $\phi \in H_g^1(M)$, and thus in $L^{p+1}(M)$ due to the continuity of the embedding of $H_g^1(M)$ into $L^{p+1}(M)$. Moreover, $\phi_m \rightarrow \phi$ strongly in $L^2(M)$.

We test equation (1.24) against a function $\psi \in H_g^1(M)$. Integration by parts gives (1.37). Once again, by **Step 1** it is easy to see that (1.38) holds. By weak convergence

$$\begin{aligned} \int_M \langle \nabla_g \phi_m, \nabla_g \psi \rangle_g d\nu_g + R_g \int_M \phi_m \psi d\nu_g \\ \rightarrow \int_M \langle \nabla_g \phi, \nabla_g \psi \rangle_g d\nu_g + R_g \int_M \phi \psi d\nu_g \quad \text{as } m \rightarrow \infty. \end{aligned}$$

Claim: $-\int_M (\lambda_m^2 + h) f'(\mathcal{U}_{\lambda_m}) \phi_m \psi d\nu_g \rightarrow \int_M h f'(u_0) \phi \psi d\nu_g$ as $m \rightarrow \infty$.

Assuming the claim is true, passing to the limit into (1.37) gives

$$\int_M \langle \nabla_g \phi, \nabla_g \psi \rangle_g d\nu_g + R_g \int_M \phi \psi d\nu_g + \int_M h f'(u_0) \phi \psi d\nu_g = 0.$$

Elliptic estimates show that ϕ is a classical solution to $-\Delta_g \phi + R_g \phi = +h f'(u_0) \phi$ on M . Lemma 1.2.1 yields $\phi \equiv 0$.

Proof of the claim: Note that

$$\lambda_m^2 \int_M f'(\mathcal{U}_{\lambda_m}) \phi_m \psi d\nu_g = \lambda_m^2 \int_M [f'(\mathcal{U}_{\lambda_m}) - f'(u_0)] \phi_m \psi d\nu_g + \lambda_m^2 \int_M f'(u_0) \phi_m \psi d\nu_g.$$

We have

$$\left| \lambda_m^2 \int_M f'(u_0) \phi_m \psi d\nu_g \right| \leq \lambda_m^2 \|f'(u_0)\|_{L^\infty(M)} \|\phi_m\|_{L^2(M)} \|\psi\|_{L^2(M)} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

We define

$$\tilde{\psi}(y) = \mu_m^{\frac{n-2}{2}} \chi(\mu_m |y + \tau_m|) \psi(\exp_{\xi_m}(\mu_m(y + \tau_m))) \quad y \in \mathbb{R}^n.$$

By (1.21) and change of variables we have

$$\begin{aligned} \left| \lambda_m^2 \int_M [f'(\mathcal{U}_{\lambda_m}) - f'(u_0)] \phi_m \psi d\nu_g \right| \\ \leq C \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^2} |\tilde{\phi}_m(y)| |\tilde{\psi}(y)| dy \\ \leq C \left\| \frac{1}{(1 + |\cdot|^2)^2} \right\|_{L^{\frac{n}{2-\epsilon\frac{(n-2)}{2}}}(\mathbb{R}^n)} \|\tilde{\phi}_m\|_{L^{\frac{2n}{n-2} \frac{1}{1+\epsilon}}(\mathbb{R}^n)} \|\tilde{\psi}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}, \end{aligned}$$

for $0 < \epsilon \ll 1$. Note that $\|\tilde{\psi}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \|\psi\|_{L^{\frac{2n}{n-2}}(M)}$. By **Step 2** $\|\tilde{\phi}_m\|_{L^{\frac{2n}{n-2} \frac{1}{1+\epsilon}}(\mathbb{R}^n)} \rightarrow 0$ as $m \rightarrow \infty$, since $\frac{2n}{n-2} \frac{1}{1+\epsilon} < \frac{2n}{n-2}$. Thus

$$\lambda_m^2 \int_M f'(\mathcal{U}_{\lambda_m}) \phi_m \psi d\nu_g \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

On the other hand

$$\int_M h f'(\mathcal{U}_{\lambda_m}) \phi_m \psi d\nu_g = \int_M h [f'(\mathcal{U}_{\lambda_m}) - f'(u_0)] \phi_m \psi d\nu_g + \int_M h f'(u_0) \phi_m \psi d\nu_g.$$

Dominated convergence theorem yields

$$\int_M h f'(u_0) \phi_m \psi d\nu_g \rightarrow \int_M h f'(u_0) \phi \psi d\nu_g \quad \text{as } m \rightarrow \infty.$$

By (1.21) and change of variables we have

$$\left| \int_M h [f'(\mathcal{U}_{\lambda_m}) - f'(u_0)] \phi_m \psi d\nu_g \right| \leq C \frac{\mu_m^2}{\lambda_m^2} \int_{\mathbb{R}^n} \frac{|y|^2}{(1+|y|^2)^2} |\tilde{\phi}_m(y)| |\tilde{\psi}(y)| dy$$

We have

$$\begin{aligned} \frac{\mu_m^2}{\lambda_m^2} \int_{\mathbb{R}^n} \chi(\mu_m |y|) \frac{|y|^2}{(1+|y|^2)^2} |\tilde{\phi}_m(y)| |\tilde{\psi}(y)| dy \\ \leq C \frac{\mu_m^2}{\lambda_m^2} \left\| \frac{\chi(\mu_m |\cdot|)}{(1+|\cdot|^2)} \right\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} \|\tilde{\phi}_m\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \|\tilde{\psi}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\ \leq C \frac{\mu_m^2}{\lambda_m^2} |\ln \mu_m| \|\tilde{\phi}_m\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \|\tilde{\psi}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \end{aligned}$$

By **Step 2** and our choice of μ_m in terms of λ_m (see (1.7)) we conclude that

$$\left| \int_M h [f'(\mathcal{U}_{\lambda_m}) - f'(u_0)] \phi_m \psi d\nu_g \right| \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

The claim is thus proved.

Step 4: We show that $\|\phi_m\|_{H_g^1(M)} \rightarrow 0$.

We take in (1.37) $\psi = \phi_m$. We get

$$\begin{aligned} (1.39) \quad \int_M |\nabla_g \phi_m|_g^2 d\nu_g + \int_M [R_g - (\lambda_m^2 + h) f'(\mathcal{U}_{\lambda_m})] \phi_m^2 d\nu_g \\ = \int_M \ell_m \phi_m d\nu_g + \sum_{j=0}^n c_i^m \int_M Z_{j,t_m,\tau_m} \phi_m d\nu_g. \end{aligned}$$

By **Step 1–3**, passing to the limit into (1.39) gives

$$\lim_{m \rightarrow \infty} \int_M |\nabla_g \phi_m|_g^2 d\nu_g = 0.$$

Since $\phi_m \rightharpoonup 0$ in $H_g^1(M)$, we conclude

$$\|\phi_m\|_{H_g^1(M)} \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

which yields a contradiction with the fact that $\|\phi_m\|_{H_g^1(M)} = 1$. This concludes the proof of (1.25).

The existence and uniqueness of ϕ_λ solution to Problem 1.24 follows from the Fredholm alternative. This finishes the proof of Lemma 1.3.3. \square

1.A.3 The non-linear problem

Proof of Lemma 1.3.3. The result of Proposition 1.3.2 implies that the unique solution $\phi_\lambda = T_{t,\tau}(\ell)$ of (1.24) defines a continuous linear map $T_{t,\tau}$ from the space $L^{\frac{2n}{n+2}}(M)$ into $H_g^1(M)$. Moreover, a standard argument shows that the operator $T_{t,\tau}$ is continuously differentiable with respect to t and τ .

In terms of the operator $T_{t,\tau}$, Problem 1.26 becomes

$$\phi_\lambda = T_{t,\tau}(-E + (\lambda^2 + h)N(\phi_\lambda)) =: A(\phi_\lambda).$$

We define the space

$$H = \left\{ \phi \in H_g^1(M) \mid \int_M \phi Z_{i,t,\tau} d\nu_g = 0, \text{ for all } i = 0, \dots, n \right\}.$$

For any positive real number η , let us consider the region

$$\mathcal{F}_\eta \equiv \left\{ \phi \in H \mid \|\phi\|_{H_g^1(M)} \leq \eta \|E\|_{L^{\frac{2n}{n+2}}(M)} \right\}.$$

From (1.25), we get

$$\|A(\phi_\lambda)\|_{H_g^1(M)} \leq C \left(\|E\|_{L^{\frac{2n}{n+2}}(M)} + \|N(\phi_\lambda)\|_{L^{\frac{2n}{n+2}}(M)} \right).$$

Observe that

$$\|N(\phi_\lambda)\|_{L^{\frac{2n}{n+2}}(M)} \leq C \|\phi_\lambda\|_{L^{\frac{2n}{n+2}}(M)}^p \leq C \|\phi_\lambda\|_{H_g^1(M)},$$

and

$$\|N(\phi_1) - N(\phi_2)\|_{L^{\frac{2n}{n+2}}(M)} \leq C \eta^{p-1} \|E\|_{L^{\frac{2n}{n+2}}(M)}^{p-1} \|\phi_1 - \phi_2\|_{H_g^1(M)},$$

for $\phi_1, \phi_2 \in \mathcal{F}_\eta$. By (1.27), we get

$$\|A(\phi_\lambda)\|_{H_g^1(M)} \leq C \|E\|_{L^{\frac{2n}{n+2}}(M)} \left(\eta^p \|E\|_{L^{\frac{2n}{n+2}}(M)}^{p-1} + 1 \right),$$

and

$$\|A(\phi_1) - A(\phi_2)\|_{H_g^1(M)} \leq C \eta^{p-1} \|E\|_{L^{\frac{2n}{n+2}}(M)}^{p-1} \|\phi_1 - \phi_2\|_{H_g^1(M)},$$

for $\phi_1, \phi_2 \in \mathcal{F}_\eta$.

Since $p - 1 \in (0, 1)$ for $n \geq 3$ and $\|E\|_{L^{\frac{2n}{n+2}}(M)} \rightarrow 0$ as $\lambda \rightarrow 0$, it follows that if η is sufficiently large and λ_0 is small enough then A is a contraction map from \mathcal{F}_η into itself, and therefore a unique fixed point of A exists in this region.

Moreover, since A depends continuously (in the $L^{\frac{2n}{n+2}}$ -norm) on t, τ the fixed point characterization obviously yields so for the map $t, \tau \rightarrow \phi$. Moreover, standard computations give that the partial derivatives $\partial_t \phi, \partial_{\tau_i} \phi$, $i = 1, \dots, n$ exist and define continuous functions of t, τ . Besides, there exists a constant $C > 0$ such that for all $i = 1, \dots, n$

$$(1.40) \quad \|\partial_t \phi\|_{H_g^1(M)} + \|\partial_{\tau_i} \phi\|_{H_g^1(M)} \leq C \|E\|_{L^{\frac{2n}{n+2}}(M)} + \|\partial_t E\|_{L^{\frac{2n}{n+2}}(M)} + \|\partial_{\tau_i} E\|_{L^{\frac{2n}{n+2}}(M)}.$$

That concludes the proof. \square

1.A.4 The reduced energy

It is quite standard to prove that, as $\lambda \rightarrow 0$, $J_\lambda(\mathcal{U}_\lambda + \phi_\lambda) = J_\lambda(\mathcal{U}_\lambda) + h.o.t.$ C^1 -uniformly on compact sets of $(0, +\infty) \times \mathbb{R}^n$ (see [EPV14]). It only remains to compute $J_\lambda(\mathcal{U}_\lambda)$.

Lemma 1.A.2. *Let $a, b \in \mathbb{R}_+$ be fixed numbers such that $0 < a < b$ and let K be a compact set in \mathbb{R}^n . There exists a positive number λ_0 such that for any $\lambda \in (0, \lambda_0)$ the following expansions hold C^1 -uniformly with respect to $t \in [a, b]$ and $\tau \in K$:*

(a) if $n \geq 10$, $|\text{Weyl}_g(\xi)| \neq 0$, and (1.10) holds, we have

$$J_\lambda(\mathcal{U}_\lambda) = A_0 - \lambda^{\frac{2(n+2)-\alpha(n-2)}{\alpha-2}} \left[A_1 |\text{Weyl}_g(\xi)|_g^2 t^4 - A_2 t^{2+\alpha} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1 + |y|^2)^n} dy + o(1) \right],$$

(b) if one of the following conditions is satisfied:

(i) $3 \leq n \leq 5$ and (1.10) holds;

(ii) $6 \leq n \leq 9$ and (1.13) holds;

(iii) $n \geq 10$, (M, g) is locally conformally flat, and (1.17) holds;

then

$$J_\lambda(\mathcal{U}_\lambda) = A_0 - \lambda^{\frac{(n-2-\alpha)(n-2)}{2\alpha-n+6}} \left[A_3 u_0(\xi) t^{\frac{n-2}{2}} - A_2 t^{2+\alpha} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1 + |y|^2)^n} dy + o(1) \right].$$

Here A_1, A_2 and A_3 are constants only depending on n and A_0 is defined in (1.32).

Proof. We prove the C^0 -estimate. The C^1 -estimate can be carried out in a similar way (see [EPV14]).

Let us first prove (a) and (ii) and (iii) of (b). It is useful to recall that

$$\alpha < \frac{2n}{n-2} < n-2 \text{ if } n \geq 10, \quad \alpha < \frac{n^2 - 6n + 16}{2(n-2)} < n-2 \text{ if } n \geq 6.$$

Observe that

$$\begin{aligned} J_\lambda(\mathcal{U}_\lambda) &= \underbrace{\frac{1}{2} \int_M |\nabla_g u_0|_g^2 d\nu_g + \frac{1}{2} \int_M c(n) R_g u_0^2 d\nu_g + \frac{1}{p+1} \int_M h u_0^{p+1} d\nu_g}_{\text{independent on } \mu \text{ and } \tau} \\ &\quad + \underbrace{\lambda^{-(n-2)} \left[\frac{1}{2} \int_M |\nabla_g \mathcal{W}_{t,\tau}|_g^2 d\nu_g + \frac{1}{2} \int_M c(n) R_g \mathcal{W}_{t,\tau}^2 d\nu_g - \frac{1}{p+1} \int_M \mathcal{W}_{t,\tau}^{p+1} d\nu_g \right]}_{\text{leading term in case (a)}} \end{aligned}$$

$$\begin{aligned}
 & + \underbrace{\lambda^{-\frac{n-2}{2}} \left[\int_M \langle \nabla_g \mathcal{W}_{t,\tau}, \nabla_g u_0 \rangle d\nu_g + \int_M c(n) R_g \mathcal{W}_{t,\tau} u_0 d\nu_g \right]}_{=0} - \lambda^{-\frac{n-2}{2}} \int_M h \mathcal{W}_{t,\tau} u_0^p d\nu_g \\
 & - \lambda^{-(n-2)} \frac{1}{p+1} \int_M \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
 & \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
 & + \underbrace{\lambda^2 \frac{1}{p+1} \int_M u_0^{p+1} d\nu_g}_{\text{independent of } \mu \text{ and } \tau} \\
 & - \underbrace{\lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g}_{\text{leading term in case (b)}} \\
 & - \lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g \\
 & + \lambda^{-n} \frac{1}{p+1} \int_M h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
 & \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
 & + \underbrace{\lambda^{-n} \frac{1}{p+1} \int_M h \mathcal{W}_{t,\tau}^{p+1} d\nu_g}_{\text{leading term in every case}} \\
 & + \lambda^{-n} \int_M h \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g
 \end{aligned}$$

Concerning the leading terms, we need to distinguish two cases. If the manifold is not locally conformally flat, by Lemma 3.1 in [EP14] we deduce

$$\begin{aligned}
 (1.41) \quad & \lambda^{-(n-2)} \left[\frac{1}{2} \int_M |\nabla_g \mathcal{W}_{t,\tau}|_g^2 d\nu_g + \frac{1}{2} \int_M c(n) R_g \mathcal{W}_{t,\tau}^2 d\nu_g - \frac{1}{p+1} \mathcal{W}_{t,\tau}^{p+1} d\nu_g \right] \\
 & = \begin{cases} \lambda^{-(n-2)} \left[A(n) - B(n) |\text{Weyl}_g(\xi)|_g^2 \mu^4 + o(\mu^4) \right] & \text{if } n \geq 7 \\ \lambda^{-(n-2)} \left[A(n) - B(n) |\text{Weyl}_g(\xi)|_g^2 \mu^4 |\ln \mu| + o(\mu^4 |\ln \mu|) \right] & \text{if } n = 6. \end{cases}
 \end{aligned}$$

If the manifold is locally conformally flat, by Lemma 5.2 in [RV13] we get

$$\lambda^{-(n-2)} \left[\frac{1}{2} \left(\int_M |\nabla_g \mathcal{W}_{t,\tau}|_g^2 d\nu_g + c(n) R_g \mathcal{W}_{t,\tau}^2 \right) d\nu_g - \frac{1}{p+1} \mathcal{W}_{t,\tau}^{p+1} d\nu_g \right] = A(n) + O\left(\frac{\mu^{n-2}}{\lambda^{n-2}}\right).$$

Here $A(n)$ and $B(n)$ are positive constants depending only on n . Moreover, straightforward computations lead to

$$(1.42) \quad \lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g = u_0(\xi) \frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \alpha_n^p \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{\frac{n+2}{2}}} dy + o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right)$$

and

$$(1.43) \quad \lambda^{-n} \frac{1}{p+1} \int_M h \mathcal{W}_{t,\tau}^{p+1} d\nu_g = \frac{\mu^{2+\alpha}}{\lambda^n} \frac{\alpha^{p+1}}{p+1} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^n} dy + o\left(\frac{\mu^{2+\alpha}}{\lambda^n}\right),$$

because $\alpha < n - 2$.

Now, if $n \geq 10$ and the manifold is not locally conformally flat, we choose $\mu = t\lambda^{\frac{2}{\alpha-2}}$ so that the leading terms are (1.41) and (1.43), namely

$$\frac{\mu^4}{\lambda^{n-2}} \sim \frac{\mu^{2+\alpha}}{\lambda^n} \quad \text{and} \quad \frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} = o\left(\frac{\mu^4}{\lambda^{n-2}}\right).$$

On the other hand, if $6 \leq n \leq 9$ we choose $\mu = t\lambda^{\frac{n+2}{2\alpha-n+6}}$ so that the leading terms are (1.42) and (1.43), namely

$$\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \sim \frac{\mu^{2+\alpha}}{\lambda^n} \quad \text{and} \quad \frac{\mu^4}{\lambda^{n-2}} = o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right), \quad \text{provided that } \alpha < \frac{16}{n-2}.$$

The higher order terms are estimated only taking into account that the bubble $\mathcal{W}_{t,\tau}$ satisfies (1.34).

A simple computation shows that

$$\lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0\right)^p d\nu_g = O\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \lambda^2 \int_{B(0,r)} \frac{1}{|y - \mu\tau|^{n-2}} dy\right) = o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right)$$

and

$$\lambda^{-n} \int_M h \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0\right) d\nu_g = O\left(\frac{\mu^{\frac{n+2}{2}}}{\lambda^{\frac{n+2}{2}}} \int_{B(0,r)} \frac{1}{|y - \mu\tau|^{n-\alpha}} dy\right) = o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right).$$

If $n = 6$ then $p + 1 = 3$. It follows that

$$\begin{aligned} \lambda^{-(n-2)} \frac{1}{p+1} \int_M \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0\right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0\right)^{p+1} \right. \\ \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0\right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0\right)^p \right] d\nu_g = 0 \end{aligned}$$

and

$$\begin{aligned} \lambda^{-n} \frac{1}{p+1} \int_M h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0\right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0\right)^{p+1} \right. \\ \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0\right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0\right)^p \right] d\nu_g = 0. \end{aligned}$$

If $n \geq 7$, we get

$$\lambda^{-(n-2)} \frac{1}{p+1} \int_M \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0\right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0\right)^{p+1} \right]$$

$$\begin{aligned}
 & -(p+1)\mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1)\mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \Big] d\nu_g \\
 = & \lambda^{-(n-2)} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
 & \left. - (p+1)\mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1)\mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
 + & \lambda^{-(n-2)} \frac{1}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
 & \left. - (p+1)\mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1)\mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
 = & o \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \right),
 \end{aligned}$$

because by Lemma 1.A.1

$$\begin{aligned}
 & \lambda^{-(n-2)} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
 & \quad \left. - (p+1)\mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1)\mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
 = & \lambda^{-(n-2)} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - (p+1)\mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) \right] d\nu_g \\
 - & \lambda^{-(n-2)} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} d\nu_g - \lambda^{-(n-2)} \int_{B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g \\
 = & O \left(\lambda^{-(n-2)} \int_{B_g(\xi, \sqrt{\mu})} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} d\nu_g \right) + O \left(\lambda^{-(n-2)} \int_{B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau}^{p-1} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^2 d\nu_g \right) \\
 + & O \left(\lambda^{-(n-2)} \int_{B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g \right) \\
 = & O \left(\lambda^2 \int_{B_g(\xi, \sqrt{\mu})} u_0^{p+1} d\nu_g \right) + O \left(\mu^2 \int_{B(0, \sqrt{\mu})} \frac{1}{|y - \mu\tau|^4} dy \right) \\
 + & O \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-6}{2}}} \int_{B(0, \sqrt{\mu})} \frac{1}{|y - \mu\tau|^{n-2}} dy \right) \\
 = & O \left(\lambda^2 \mu^{\frac{n}{2}} \right) + O \left(\mu^{2+\frac{n-4}{2}} \right) + O \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-6}{2}}} \mu \right) = o \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \right),
 \end{aligned}$$

since if μ is small enough, for any $q < n$, we have

$$(1.44) \quad \int_{B(0, \sqrt{\mu})} \frac{1}{|y - \mu\tau|^q} dy = \int_{B(-\mu\tau, \sqrt{\mu})} \frac{1}{|y|^q} dy \leq \int_{B(0, 2\sqrt{\mu})} \frac{1}{|y|^q} dy = O \left(\mu^{\frac{n-q}{2}} \right)$$

and

$$\frac{\lambda^{-(n-2)}}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right]$$

$$\begin{aligned}
& -(p+1)\mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1)\mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \Big] d\nu_g \\
= & \frac{\lambda^{-(n-2)}}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - (p+1)\mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
& - \frac{\lambda^{-(n-2)}}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau}^{p+1} d\nu_g - \lambda^{-(n-2)} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g \\
= & O \left(\lambda^{-(n-2)} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau}^2 \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p-1} d\nu_g \right) + O \left(\frac{\lambda^{-(n-2)}}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau}^{p+1} d\nu_g \right) \\
& + O \left(\lambda^{-\frac{n-2}{2}} \int_{M \setminus B_g(\xi, \sqrt{\mu})} \mathcal{W}_{t,\tau}^p d\nu_g \right) \\
= & O \left(\lambda^{-(n-4)} \mu^{\frac{n}{2}} \right) + O \left(-\lambda^{-(n-2)} \mu^{\frac{n}{2}} \right) + O \left(\frac{\mu^{\frac{n}{2}}}{\lambda^{\frac{n-2}{2}}} \right) = o \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \right),
\end{aligned}$$

Here, we used the fact that

$$\frac{\mu^{\frac{n}{2}}}{\lambda^{n-2}} = o \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \right),$$

and our choice

$$\begin{aligned}
\alpha & < \frac{n^2 - 6n + 16}{2(n-2)} & \text{if } 7 \leq n \leq 9, \\
\frac{n-6}{2} & < \alpha < \frac{n^2 - 6n + 16}{2(n-2)} & \text{if } n \geq 10 \text{ and } (M, g) \text{ is locally conformally flat,} \\
\alpha & < \frac{2n}{n-2} & \text{if } n \geq 10 \text{ and } (M, g) \text{ is not locally conformally flat.}
\end{aligned}$$

In a similar way, if $n \geq 7$, we get

$$\begin{aligned}
& \lambda^{-n} \frac{1}{p+1} \int_M h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
& \quad \left. - (p+1)\mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1)\mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
= & \lambda^{-n} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
& \quad \left. - (p+1)\mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1)\mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
& + \lambda^{-n} \frac{1}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
& \quad \left. - (p+1)\mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1)\mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
= & o \left(\frac{\mu^{2+\alpha}}{\lambda^n} \right),
\end{aligned}$$

because by Lemma 1.A.1

$$\begin{aligned}
& \lambda^{-n} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
& \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
&= \lambda^{-n} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) \right] d\nu_g \\
& - \lambda^{-n} \frac{1}{p+1} \int_{B_g(\xi, \sqrt{\mu})} h \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} d\nu_g - \lambda^{-n} \int_{B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g \\
&= O \left(\lambda^{-n} \int_{B_g(\xi, \sqrt{\mu})} h \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} d\nu_g \right) + O \left(\lambda^{-n} \int_{B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau}^{p-1} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^2 d\nu_g \right) \\
& + O \left(\lambda^{-n} \int_{B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g \right) \\
&= O \left(\int_{B_g(\xi, \sqrt{\mu})} (d_g(x, \xi))^{\alpha+2} d\nu_g \right) + O \left(\frac{\mu^2}{\lambda^2} \int_{B(0, \sqrt{\mu})} \frac{|y|^{\alpha+2}}{|y - \mu\tau|^4} dy \right) \\
& + O \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \int_{B(0, \sqrt{\mu})} \frac{|y|^{\alpha+2}}{|y - \mu\tau|^{n-2}} dy \right) \\
&= O \left(\mu^{\frac{\alpha+2+n}{2}} \right) + O \left(\frac{\mu^{\frac{\alpha+2+n}{2}}}{\lambda^2} \right) + O \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \mu^{\frac{\alpha+4}{2}} \right) = o \left(\frac{\mu^{2+\alpha}}{\lambda^n} \right)
\end{aligned}$$

Here, we used (1.44) and

$$\begin{aligned}
& \lambda^{-n} \frac{1}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
& \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
&= \frac{\lambda^{-n}}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
& - \frac{\lambda^{-n}}{p+1} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau}^{p+1} d\nu_g - \lambda^{-n} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g \\
&= O \left(\lambda^{-n} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau}^2 \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p-1} d\nu_g \right) + O \left(\lambda^{-n} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau}^{p+1} d\nu_g \right) \\
& + O \left(\lambda^{-\frac{n+2}{2}} \int_{M \setminus B_g(\xi, \sqrt{\mu})} h \mathcal{W}_{t,\tau}^p d\nu_g \right)
\end{aligned}$$

$$\begin{aligned}
 &= \begin{cases} O\left(\frac{\mu^{\frac{\alpha+2+n}{2}}}{\lambda^{n-2}}\right) & \text{if } \alpha < n-6 \\ O\left(\frac{\mu^{\frac{\alpha+2+n}{2}}}{\lambda^{n-2}}\right) |\ln \mu| & \text{if } \alpha = n-6 \\ O\left(\frac{\mu^{n-2}}{\lambda^{n-2}}\right) & \text{if } \alpha > n-6 \end{cases} + O\left(\frac{\mu^{\frac{\alpha+2+n}{2}}}{\lambda^n}\right) + O\left(\frac{\mu^{\frac{n+2}{2}}}{\lambda^{\frac{n+2}{2}}}\right) \\
 &= o\left(\frac{\mu^{2+\alpha}}{\lambda^n}\right) \text{ because } \alpha < n-2.
 \end{aligned}$$

Collecting the previous computations we get the result.

Let us now prove (i) of (b). Observe that

$$\begin{aligned}
 &J_\lambda(\mathcal{U}_\lambda) \\
 &= \underbrace{\frac{1}{2} \int_M |\nabla_g u_0|_g^2 d\nu_g + \frac{1}{2} \int_M c(n) R_g u_0^2 d\nu_g + \frac{1}{p+1} \int_M h u_0^{p+1} d\nu_g}_{\text{independent of } \mu \text{ and } \tau} \\
 &+ \lambda^{-(n-2)} \left[\frac{1}{2} \int_M |\nabla_g \mathcal{W}_{t,\tau}|_g^2 d\nu_g + \frac{1}{2} \int_M c(n) R_g \mathcal{W}_{t,\tau}^2 d\nu_g - \frac{1}{p+1} \int_M \mathcal{W}_{t,\tau}^{p+1} d\nu_g \right] \\
 &+ \lambda^{-\frac{n-2}{2}} \underbrace{\left[\int_M \langle \nabla_g \mathcal{W}_{t,\tau}, \nabla_g u_0 \rangle + \int_M c(n) R_g \mathcal{W}_{t,\tau} u_0 \right] d\nu_g - \lambda^{-\frac{n-2}{2}} \int_M h \mathcal{W}_{t,\tau} u_0^p d\nu_g}_{=0} \\
 &- \lambda^{-(n-2)} \frac{1}{p+1} \int_M \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
 &\quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
 &+ \underbrace{\lambda^2 \frac{1}{p+1} \int_M u_0^{p+1} d\nu_g}_{\text{independent of } \mu \text{ and } \tau} \\
 &- \lambda^{-(n-2)} \underbrace{\int_M \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g}_{\text{leading term}} \\
 &- \lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g \\
 &+ \lambda^{-n} \frac{1}{p+1} \int_M h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
 &\quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
 &+ \lambda^{-n} \underbrace{\frac{1}{p+1} \int_M h \mathcal{W}_{t,\tau}^{p+1} d\nu_g}_{\text{leading term}} \\
 &+ \lambda^{-n} \int_M h \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g
 \end{aligned}$$

Concerning the leading terms, straightforward computations lead to

$$\lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g = u_0(\xi) \frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \alpha_n^p \int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{\frac{n+2}{2}}} dy + o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right)$$

and

$$\lambda^{-n} \frac{1}{p+1} \int_M h \mathcal{W}_{t,\tau}^{p+1} d\nu_g = \frac{\mu^{2+\alpha}}{\lambda^n} \frac{\alpha_n^{p+1}}{p+1} \sum_{i=1}^n \int_{\mathbb{R}^n} a_i \frac{|y_i + \tau_i|^{2+\alpha}}{(1+|y|^2)^n} dy + o\left(\frac{\mu^{2+\alpha}}{\lambda^n}\right),$$

because $\alpha < n - 2$.

The higher order terms are estimated as follows. By [MPV09], we deduce that

$$\begin{aligned} & \lambda^{-(n-2)} \left[\frac{1}{2} \int_M |\nabla_g \mathcal{W}_{t,\tau}|_g^2 d\nu_g + \frac{1}{2} \int_M c(n) R_g \mathcal{W}_{t,\tau}^2 d\nu_g - \frac{1}{p+1} \int_M \mathcal{W}_{t,\tau}^{p+1} d\nu_g \right] \\ &= \lambda^{-(n-2)} A(n) + \begin{cases} O\left(\frac{\mu}{\lambda}\right) & \text{if } n = 3 \\ O\left(\frac{\mu^2 |\ln \mu|}{\lambda^2}\right) & \text{if } n = 4 \\ O\left(\frac{\mu^2}{\lambda^3}\right) & \text{if } n = 5 \end{cases} \\ &= \lambda^{-(n-2)} A(n) + o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right), \end{aligned}$$

where $A(n)$ is a constant that only depends on n . A simple computation shows that

$$\lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p d\nu_g = O\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \lambda^2 \int_{B_g(\xi,r)} \frac{1}{(d_g(x,\xi))^{n-2}} d\nu_g\right) = o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right)$$

and

$$\begin{aligned} \lambda^{-n} \int_M h \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) d\nu_g &= O\left(\frac{\mu^{\frac{n+2}{2}}}{\lambda^{\frac{n+2}{2}}} \int_{B_g(\xi,r)} \frac{1}{(d_g(x,\xi))^{n-\alpha}} d\nu_g\right) \\ &= \begin{cases} O\left(\frac{\mu^{\frac{n+2}{2}}}{\lambda^{\frac{n+2}{2}}}\right) & \text{if } \alpha > 0 \\ O\left(\frac{\mu^{\frac{n+2}{2}}}{\lambda^{\frac{n+2}{2}}} |\ln \mu|\right) & \text{if } \alpha = 0 \end{cases} \\ &= o\left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}}\right). \end{aligned}$$

Finally, by using that

$$|(a+b)^{p+1} - a^{p+1} - b^{p+1} - (p+1)ab^p - (p+1)a^p b| \leq c(n) (a^2 b^{p-1} + a^{p-1} b^2),$$

which holds if $p \geq 2$ (this is true if $n = 3, 4, 5$) for any $a, b \geq 0$, we get

$$\begin{aligned}
 & \lambda^{-(n-2)} \frac{1}{p+1} \int_M \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
 & \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
 &= O \left(\lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau}^2 \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p-1} d\nu_g \right) + O \left(\lambda^{-(n-2)} \int_M \mathcal{W}_{t,\tau}^{p-1} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^2 d\nu_g \right) \\
 &= O \left(\lambda^{-(n-4)} \int_M \mathcal{W}_{t,\tau}^2 d\nu_g \right) + O \left(\int_M \mathcal{W}_{t,\tau}^{p-1} d\nu_g \right) \\
 &= \begin{cases} O(\lambda\mu) & \text{if } n = 3 \\ O(\mu^2 |\ln \mu|) & \text{if } n = 4 \\ O(\lambda^{-1} \mu^2) & \text{if } n = 5 \end{cases} + \begin{cases} O(\mu) & \text{if } n = 3 \\ O(\mu^2 |\ln \mu|) & \text{if } n = 4 \\ O(\mu^2) & \text{if } n = 5 \end{cases} \\
 &= o \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \right)
 \end{aligned}$$

and

$$\begin{aligned}
 & \lambda^{-n} \frac{1}{p+1} \int_M h \left[\left(\mathcal{W}_{t,\tau} + \lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} - \mathcal{W}_{t,\tau}^{p+1} - \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p+1} \right. \\
 & \quad \left. - (p+1) \mathcal{W}_{t,\tau}^p \left(\lambda^{\frac{n-2}{2}} u_0 \right) - (p+1) \mathcal{W}_{t,\tau} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^p \right] d\nu_g \\
 &= O \left(\lambda^{-n} \int_M h \mathcal{W}_{t,\tau}^2 \left(\lambda^{\frac{n-2}{2}} u_0 \right)^{p-1} d\nu_g \right) + O \left(\lambda^{-n} \int_M h \mathcal{W}_{t,\tau}^{p-1} \left(\lambda^{\frac{n-2}{2}} u_0 \right)^2 d\nu_g \right) \\
 &= O \left(\lambda^{-(n-2)} \int_M h \mathcal{W}_{t,\tau}^2 d\nu_g \right) + O \left(\lambda^{-2} \int_M h \mathcal{W}_{t,\tau}^{p-1} d\nu_g \right) \\
 &= O \left(\lambda^{-(n-2)} \mu^{n-2} \right) + O \left(\lambda^{-2} \mu^2 \right) \\
 &= o \left(\frac{\mu^{\frac{n-2}{2}}}{\lambda^{\frac{n-2}{2}}} \right).
 \end{aligned}$$

Collecting the previous computations we get the result. □

Bibliography

- [AB97] T. Aubin and S. Bismuth, *Courbure scalaire prescrite sur les variétés riemanniennes compactes dans le cas négatif*, J. Funct. Anal. **143** (1997), no. 2, 529–541. MR1428826
- [ABM06] S. Alama, L. Bronsard, and J. A. Montero, *On the Ginzburg-Landau model of a superconducting ball in a uniform field*, Ann. Inst. H. Poincaré Anal. Non Linéaire **23** (2006), no. 2, 237–267. MR2201153
- [ABO05] G. Alberti, S. Baldo, and G. Orlandi, *Variational convergence for functionals of Ginzburg-Landau type*, Indiana Univ. Math. J. **54** (2005), no. 5, 1411–1472. MR2177107
- [Abr57] A. A. Abrikosov, *On the Magnetic properties of superconductors of the second group*, Sov. Phys. JETP **5** (1957), 1174–1182.
- [Aft06] A. Aftalion, *Vortices in Bose-Einstein condensates*, Progress in Nonlinear Differential Equations and their Applications, vol. 67, Birkhäuser Boston, Inc., Boston, MA, 2006. MR2228356
- [AH91] T. Aubin and E. Hebey, *Courbure scalaire prescrite*, Bull. Sci. Math. **115** (1991), no. 2, 125–131. MR1101020
- [AM91] Adimurthi and G. Mancini, *The Neumann problem for elliptic equations with critical nonlinearity*, Nonlinear analysis, 1991, pp. 9–25. MR1205370
- [AP16] O. Agudelo and A. Pistoia, *Boundary concentration phenomena for the higher-dimensional Keller-Segel system*, Calc. Var. Partial Differential Equations **55** (2016), no. 6, Paper No. 132, 31. MR3566211
- [APY93] Adimurthi, F. Pacella, and S. L. Yadava, *Interaction between the geometry of the boundary and positive solutions of a semilinear Neumann problem with critical nonlinearity*, J. Funct. Anal. **113** (1993), no. 2, 318–350. MR1218099
- [Aub76a] T. Aubin, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. (9) **55** (1976), no. 3, 269–296. MR0431287
- [Aub76b] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geometry **11** (1976), no. 4, 573–598. MR0448404
- [Aub98] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. MR1636569
- [AY91] Adimurthi and S. L. Yadava, *Existence and nonexistence of positive radial solutions of Neumann problems with critical Sobolev exponents*, Arch. Rational Mech. Anal. **115** (1991), no. 3, 275–296. MR1106295
- [AY97] Adimurthi and S. L. Yadava, *Nonexistence of positive radial solutions of a quasilinear Neumann problem with a critical Sobolev exponent*, Arch. Rational Mech. Anal. **139** (1997), no. 3, 239–253. MR1480241
- [BBH94] F. Bethuel, H. Brezis, and F. Hélein, *Ginzburg-Landau vortices*, Progress in Nonlinear Differential Equations and their Applications, vol. 13, Birkhäuser Boston, Inc., Boston, MA, 1994. MR1269538

BIBLIOGRAPHY

- [BBM04] J. Bourgain, H. Brezis, and P. Mironescu, $H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equation, *Publ. Math. Inst. Hautes Études Sci.* **99** (2004), 1–115. MR2075883
- [BBO01] F. Bethuel, H. Brezis, and G. Orlandi, *Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions*, *J. Funct. Anal.* **186** (2001), no. 2, 432–520. MR1864830
- [BCL86] H. Brezis, J.-M. Coron, and E. H. Lieb, *Harmonic maps with defects*, *Comm. Math. Phys.* **107** (1986), no. 4, 649–705. MR868739
- [BCN17a] D. Bonheure, J.-B. Casteras, and B. Noris, *Layered solutions with unbounded mass for the Keller-Segel equation*, *J. Fixed Point Theory Appl.* **19** (2017), no. 1, 529–558. MR3625083
- [BCN17b] D. Bonheure, J.-B. Casteras, and B. Noris, *Multiple positive solutions of the stationary Keller-Segel system*, *Calc. Var. Partial Differential Equations* **56** (2017), no. 3, Paper No. 74, 35. MR3641921
- [BCR] D. Bonheure, J.-B. Casteras, and C. Román, *Unbounded mass radial solutions for the Keller-Segel equation in the disk*, Submitted.
- [BCS57] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Theory of superconductivity*, *Phys. Rev.* **108** (1957), 1175–1204.
- [BE91] G. Bianchi and H. Egnell, *A note on the Sobolev inequality*, *J. Funct. Anal.* **100** (1991), no. 1, 18–24. MR1124290
- [BGNT16] D. Bonheure, M. Grossi, B. Noris, and S. Terracini, *Multi-layer radial solutions for a supercritical Neumann problem*, *J. Differential Equations* **261** (2016), no. 1, 455–504. MR3487266
- [BGS15] F. Borer, L. Galimberti, and M. Struwe, *“Large” conformal metrics of prescribed Gauss curvature on surfaces of higher genus*, *Comment. Math. Helv.* **90** (2015), no. 2, 407–428. MR3351750
- [Bil98] P. Biler, *Local and global solvability of some parabolic systems modelling chemotaxis*, *Adv. Math. Sci. Appl.* **8** (1998), no. 2, 715–743. MR1657160 (2001b:92046)
- [Bis98] S. Bismuth, *Courbure scalaire prescrite sur une variété riemannienne C^∞ compacte dans le cas nul*, *J. Math. Pures Appl.* (9) **77** (1998), no. 7, 667–695. MR1645069
- [BJOS12] S. Baldo, R. L. Jerrard, G. Orlandi, and H. M. Soner, *Convergence of Ginzburg-Landau functionals in three-dimensional superconductivity*, *Arch. Ration. Mech. Anal.* **205** (2012), no. 3, 699–752. MR2960031
- [BJOS13] S. Baldo, R. L. Jerrard, G. Orlandi, and H. M. Soner, *Vortex density models for superconductivity and superfluidity*, *Comm. Math. Phys.* **318** (2013), no. 1, 131–171. MR3017066
- [BLR95] A. Bahri, Y. Li, and O. Rey, *On a variational problem with lack of compactness: the topological effect of the critical points at infinity*, *Calc. Var. Partial Differential Equations* **3** (1995), no. 1, 67–93. MR1384837
- [BN83] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, *Comm. Pure Appl. Math.* **36** (1983), no. 4, 437–477. MR709644
- [BOS04] F. Bethuel, G. Orlandi, and D. Smets, *Vortex rings for the Gross-Pitaevskii equation*, *J. Eur. Math. Soc. (JEMS)* **6** (2004), no. 1, 17–94. MR2041006
- [BR95] F. Bethuel and T. Rivière, *Vortices for a variational problem related to superconductivity*, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **12** (1995), no. 3, 243–303. MR1340265
- [Cat09] F. Catrina, *A note on a result of M. Grossi*, *Proc. Amer. Math. Soc.* **137** (2009), no. 11, 3717–3724. MR2529879
- [Chi05] D. Chiron, *Boundary problems for the Ginzburg-Landau equation*, *Commun. Contemp. Math.* **7** (2005), no. 5, 597–648. MR2175092

- [CL02] C.-C. Chen and C.-S. Lin, *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces*, Comm. Pure Appl. Math. **55** (2002), no. 6, 728–771. MR1885666
- [CY87] S.-Y. A. Chang and P. C. Yang, *Prescribing Gaussian curvature on S^2* , Acta Math. **159** (1987), no. 3-4, 215–259. MR908146
- [DG99] P. G. De Gennes, *Superconductivity of Metals and Alloys*, Advanced book classics, Perseus, Cambridge, MA, 1999.
- [DL95] W. Y. Ding and J. Q. Liu, *A note on the problem of prescribing Gaussian curvature on surfaces*, Trans. Amer. Math. Soc. **347** (1995), no. 3, 1059–1066. MR1257102
- [dP94] M. A. del Pino, *Positive solutions of a semilinear elliptic equation on a compact manifold*, Nonlinear Anal. **22** (1994), no. 11, 1423–1430. MR1280207
- [dPDM04] M. del Pino, J. Dolbeault, and M. Musso, *The Brezis-Nirenberg problem near criticality in dimension 3*, J. Math. Pures Appl. (9) **83** (2004), no. 12, 1405–1456. MR2103187
- [DPF99] M. Del Pino and P. L. Felmer, *Spike-layered solutions of singularly perturbed elliptic problems in a degenerate setting*, Indiana Univ. Math. J. **48** (1999), no. 3, 883–898. MR1736974
- [dPFM03] M. del Pino, P. Felmer, and M. Musso, *Two-bubble solutions in the super-critical Bahri-Coron’s problem*, Calc. Var. Partial Differential Equations **16** (2003), no. 2, 113–145. MR1956850
- [dPFW99] M. del Pino, P. L. Felmer, and J. Wei, *On the role of mean curvature in some singularly perturbed Neumann problems*, SIAM J. Math. Anal. **31** (1999), no. 1, 63–79. MR1742305
- [dPKM05] M. del Pino, M. Kowalczyk, and M. Musso, *Singular limits in Liouville-type equations*, Calc. Var. Partial Differential Equations **24** (2005), no. 1, 47–81. MR2157850
- [dPMP05] M. del Pino, M. Musso, and A. Pistoia, *Super-critical boundary bubbling in a semilinear Neumann problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), no. 1, 45–82. MR2114411
- [dPMRW] M. del Pino, M. Musso, C. Román, and J. Wei, *Interior bubbling solutions for the critical Lin-Ni-Takagi problem in dimension 3*, Accepted for publication in J. Anal. Math.
- [dPPV16] M. del Pino, A. Pistoia, and G. Vaira, *Large mass boundary condensation patterns in the stationary Keller-Segel system*, J. Differential Equations **261** (2016), no. 6, 3414–3462. MR3527634
- [dPR15] M. del Pino and C. Román, *Large conformal metrics with prescribed sign-changing Gauss curvature*, Calc. Var. Partial Differential Equations **54** (2015), no. 1, 763–789. MR3385180
- [dPW06] M. del Pino and J. Wei, *Collapsing steady states of the Keller-Segel system*, Nonlinearity **19** (2006), no. 3, 661–684. MR2209293 (2007b:35130)
- [Dru02] O. Druet, *Elliptic equations with critical Sobolev exponents in dimension 3*, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), no. 2, 125–142. MR1902741
- [DRW12] O. Druet, F. Robert, and J. Wei, *The Lin-Ni’s problem for mean convex domains*, Mem. Amer. Math. Soc. **218** (2012), no. 1027, vi+105. MR2963797
- [DY99] E. N. Dancer and S. Yan, *Multipeak solutions for a singularly perturbed Neumann problem*, Pacific J. Math. **189** (1999), no. 2, 241–262. MR1696122
- [EP14] P. Esposito and A. Pistoia, *Blowing-up solutions for the Yamabe equation*, Port. Math. **71** (2014), no. 3-4, 249–276. MR3298464
- [EPV14] P. Esposito, A. Pistoia, and J. Vétois, *The effect of linear perturbations on the Yamabe problem*, Math. Ann. **358** (2014), no. 1-2, 511–560. MR3158007
- [ES86] J. F. Escobar and R. M. Schoen, *Conformal metrics with prescribed scalar curvature*, Invent. Math. **86** (1986), no. 2, 243–254. MR856845

- [Esp07] P. Esposito, *Estimations à l'intérieur pour un problème elliptique semi-linéaire avec non-linéarité critique*, Ann. Inst. H. Poincaré Anal. Non Linéaire **24** (2007), no. 4, 629–644. MR2334996
- [FW86] A. Floer and A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Funct. Anal. **69** (1986), no. 3, 397–408. MR867665
- [GG98] C. Gui and N. Ghoussoub, *Multi-peak solutions for a semilinear Neumann problem involving the critical Sobolev exponent*, Math. Z. **229** (1998), no. 3, 443–474. MR1658569
- [GL02] C. Gui and C.-S. Lin, *Estimates for boundary-bubbling solutions to an elliptic Neumann problem*, J. Reine Angew. Math. **546** (2002), 201–235. MR1900999
- [GL50] V. L. Ginzburg and L. D. Landau, *On the theory of superconductivity*, Zh. Eksp. Teor. Fiz. **20** (1950), 1064–1082. English translation in: *Collected papers of L.D.Landau*, Edited by D. Ter. Haar, Pergamon Press, Oxford 1965, pp. 546–568.
- [GM72] A. Gierer and H. Meinhardt, *A theory of biological pattern formation*, Kybernetik **12** (1972), no. 1, 30–39.
- [GN12] M. Grossi and B. Noris, *Positive constrained minimizers for supercritical problems in the ball*, Proc. Amer. Math. Soc. **140** (2012), no. 6, 2141–2154. MR2888200
- [GPW00] M. Grossi, A. Pistoia, and J. Wei, *Existence of multipeak solutions for a semilinear Neumann problem via nonsmooth critical point theory*, Calc. Var. Partial Differential Equations **11** (2000), no. 2, 143–175. MR1782991
- [Gro06] M. Grossi, *Asymptotic behaviour of the Kazdan-Warner solution in the annulus*, J. Differential Equations **223** (2006), no. 1, 96–111. MR2210140 (2006m:35122)
- [Gru93] P. M. Gruber, *Asymptotic estimates for best and stepwise approximation of convex bodies. I*, Forum Math. **5** (1993), no. 3, 281–297. MR1216036
- [GW99] C. Gui and J. Wei, *Multiple interior peak solutions for some singularly perturbed Neumann problems*, J. Differential Equations **158** (1999), no. 1, 1–27. MR1721719
- [Hor03] D. Horstmann, *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I*, Jahresber. Deutsch. Math.-Verein. **105** (2003), no. 3, 103–165. MR2013508
- [Hor04] D. Horstmann, *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. II*, Jahresber. Deutsch. Math.-Verein. **106** (2004), no. 2, 51–69. MR2073515
- [HV92] E. Hebey and M. Vaugon, *Meilleures constantes dans le théorème d'inclusion de Sobolev et multiplicité pour les problèmes de Nirenberg et Yamabe*, Indiana Univ. Math. J. **41** (1992), no. 2, 377–407. MR1183349
- [HV93] E. Hebey and M. Vaugon, *Le problème de Yamabe équivariant*, Bull. Sci. Math. **117** (1993), no. 2, 241–286. MR1216009
- [Jer99] R. L. Jerrard, *Lower bounds for generalized Ginzburg-Landau functionals*, SIAM J. Math. Anal. **30** (1999), no. 4, 721–746. MR1684723
- [JMS04] R. Jerrard, A. Montero, and P. Sternberg, *Local minimizers of the Ginzburg-Landau energy with magnetic field in three dimensions*, Comm. Math. Phys. **249** (2004), no. 3, 549–577. MR2084007
- [JS02] R. L. Jerrard and H. M. Soner, *The Jacobian and the Ginzburg-Landau energy*, Calc. Var. Partial Differential Equations **14** (2002), no. 2, 151–191. MR1890398
- [KS70] E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, Journal of Theoretical Biology **26** (1970), no. 3, 399–415.
- [KS91] H. Kozono and H. Sohr, *New a priori estimates for the Stokes equations in exterior domains*, Indiana Univ. Math. J. **40** (1991), no. 1, 1–27. MR1101219

- [KW74] J. L. Kazdan and F. W. Warner, *Curvature functions for compact 2-manifolds*, Ann. of Math. (2) **99** (1974), 14–47. MR0343205
- [KW75] J. L. Kazdan and F. W. Warner, *Scalar curvature and conformal deformation of Riemannian structure*, J. Differential Geometry **10** (1975), 113–134. MR0365409
- [Lan15] R. Langevin, *Integral geometry from Buffon to geometers of today*, Cours Spécialisés [Specialized Courses], vol. 23, Société Mathématique de France, Paris, 2015. MR3469669
- [Li95] Y. Y. Li, *Prescribing scalar curvature on S^n and related problems. I*, J. Differential Equations **120** (1995), no. 2, 319–410. MR1347349
- [Li96] Y. Li, *Prescribing scalar curvature on S^n and related problems. II. Existence and compactness*, Comm. Pure Appl. Math. **49** (1996), no. 6, 541–597. MR1383201
- [Li98] Y. Y. Li, *On a singularly perturbed equation with Neumann boundary condition*, Comm. Partial Differential Equations **23** (1998), no. 3-4, 487–545. MR1620632
- [LN88] C. S. Lin and W.-M. Ni, *On the diffusion coefficient of a semilinear Neumann problem*, Calculus of variations and partial differential equations (Trento, 1986), 1988, pp. 160–174. MR974610
- [LNT88] C.-S. Lin, W.-M. Ni, and I. Takagi, *Large amplitude stationary solutions to a chemotaxis system*, J. Differential Equations **72** (1988), no. 1, 1–27. MR929196
- [LNW07] F.-H. Lin, W.-M. Ni, and J.-C. Wei, *On the number of interior peak solutions for a singularly perturbed Neumann problem*, Comm. Pure Appl. Math. **60** (2007), no. 2, 252–281. MR2275329
- [LP87] J. M. Lee and T. H. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. (N.S.) **17** (1987), no. 1, 37–91. MR888880
- [LR01] F.-H. Lin and T. Rivière, *A quantization property for static Ginzburg-Landau vortices*, Comm. Pure Appl. Math. **54** (2001), no. 2, 206–228. MR1794353
- [LR99] F. Lin and T. Rivière, *Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents*, J. Eur. Math. Soc. (JEMS) **1** (1999), no. 3, 237–311. MR1714735
- [MPV09] A. M. Micheletti, A. Pistoia, and J. Vétois, *Blow-up solutions for asymptotically critical elliptic equations on Riemannian manifolds*, Indiana Univ. Math. J. **58** (2009), no. 4, 1719–1746. MR2542977
- [MSZ04] J. A. Montero, P. Sternberg, and W. P. Ziemer, *Local minimizers with vortices in the Ginzburg-Landau system in three dimensions*, Comm. Pure Appl. Math. **57** (2004), no. 1, 99–125. MR2007357
- [NPT92] W.-M. Ni, X. B. Pan, and I. Takagi, *Singular behavior of least-energy solutions of a semilinear Neumann problem involving critical Sobolev exponents*, Duke Math. J. **67** (1992), no. 1, 1–20. MR1174600
- [NT91] W.-M. Ni and I. Takagi, *On the shape of least-energy solutions to a semilinear Neumann problem*, Comm. Pure Appl. Math. **44** (1991), no. 7, 819–851. MR1115095
- [NT93] W.-M. Ni and I. Takagi, *Locating the peaks of least-energy solutions to a semilinear Neumann problem*, Duke Math. J. **70** (1993), no. 2, 247–281. MR1219814
- [Ouy91] T. Ouyang, *On the positive solutions of semilinear equations $\Delta u + \lambda u + hu^p = 0$ on compact manifolds. II*, Indiana Univ. Math. J. **40** (1991), no. 3, 1083–1141. MR1129343
- [PR17] A. Pistoia and C. Román, *Large conformal metrics with prescribed scalar curvature*, J. Differential Equations **263** (2017), no. 9, 5902–5938.
- [PV15] A. Pistoia and G. Vaira, *Steady states with unbounded mass of the Keller-Segel system*, Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), no. 1, 203–222. MR3304582

BIBLIOGRAPHY

- [Rau95] A. Rauzy, *Courbures scalaires des variétés d'invariant conforme négatif*, Trans. Amer. Math. Soc. **347** (1995), no. 12, 4729–4745. MR1321588
- [Rau96] A. Rauzy, *Multiplicité pour un problème de courbure scalaire prescrite*, Bull. Sci. Math. **120** (1996), no. 2, 153–194. MR1387420
- [Rey02] O. Rey, *The question of interior blow-up-points for an elliptic Neumann problem: the critical case*, J. Math. Pures Appl. (9) **81** (2002), no. 7, 655–696. MR1968337
- [Rey99] O. Rey, *An elliptic Neumann problem with critical nonlinearity in three-dimensional domains*, Commun. Contemp. Math. **1** (1999), no. 3, 405–449. MR1707889
- [Riv95] T. Rivière, *Line vortices in the U(1)-Higgs model*, ESAIM Contrôle Optim. Calc. Var. **1** (1995/96), 77–167. MR1394302
- [Roma] C. Román, *3D vortex approximation construction and ε -level estimates for the Ginzburg-Landau functional*. In preparation.
- [Romb] C. Román, *Global minimizers for the 3D Ginzburg-Landau functional below the first critical field*. In preparation.
- [RSS] C. Román, E. Sandier, and S. Serfaty, *Global minimizers for the 3D Ginzburg-Landau functional near the first critical field have bounded vorticity*. In preparation.
- [RV13] F. Robert and J. Vétois, *Sign-changing blow-up for scalar curvature type equations*, Comm. Partial Differential Equations **38** (2013), no. 8, 1437–1465. MR3169751
- [RW05] O. Rey and J. Wei, *Arbitrary number of positive solutions for an elliptic problem with critical nonlinearity*, J. Eur. Math. Soc. (JEMS) **7** (2005), no. 4, 449–476. MR2159223
- [San01] E. Sandier, *Ginzburg-Landau minimizers from \mathbb{R}^{n+1} to \mathbb{R}^n and minimal connections*, Indiana Univ. Math. J. **50** (2001), no. 4, 1807–1844. MR1889083
- [San04] L. A. Santaló, *Integral geometry and geometric probability*, Second, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2004. With a foreword by Mark Kac. MR2162874
- [San98] E. Sandier, *Lower bounds for the energy of unit vector fields and applications*, J. Funct. Anal. **152** (1998), no. 2, 379–403. MR1607928
- [Sch84] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. **20** (1984), no. 2, 479–495. MR788292
- [Sch85] R. Schaaf, *Stationary solutions of chemotaxis systems*, Trans. Amer. Math. Soc. **292** (1985), no. 2, 531–556. MR808736 (87a:35020)
- [Ser01] S. Serfaty, *On a model of rotating superfluids*, ESAIM Control Optim. Calc. Var. **6** (2001), 201–238. MR1816073
- [Ser99] S. Serfaty, *Local minimizers for the Ginzburg-Landau energy near critical magnetic field. I*, Commun. Contemp. Math. **1** (1999), no. 2, 213–254. MR1696100
- [SS00a] E. Sandier and S. Serfaty, *Global minimizers for the Ginzburg-Landau functional below the first critical magnetic field*, Ann. Inst. H. Poincaré Anal. Non Linéaire **17** (2000), no. 1, 119–145. MR1743433
- [SS00b] E. Sandier and S. Serfaty, *On the energy of type-II superconductors in the mixed phase*, Rev. Math. Phys. **12** (2000), no. 9, 1219–1257. MR1794239
- [SS00c] É. Sandier and S. Serfaty, *A rigorous derivation of a free-boundary problem arising in superconductivity*, Ann. Sci. Ecole Norm. Sup. (4) **33** (2000), no. 4, 561–592. MR1832824
- [SS00d] T. Senba and T. Suzuki, *Some structures of the solution set for a stationary system of chemotaxis*, Adv. Math. Sci. Appl. **10** (2000), no. 1, 191–224. MR1769174 (2001d:35068)

- [SS02] T. Senba and T. Suzuki, *Weak solutions to a parabolic-elliptic system of chemotaxis*, J. Funct. Anal. **191** (2002), no. 1, 17–51. MR1909263 (2003c:35155)
- [SS03] E. Sandier and S. Serfaty, *Ginzburg-Landau minimizers near the first critical field have bounded vorticity*, Calc. Var. Partial Differential Equations **17** (2003), no. 1, 17–28. MR1979114
- [SS04] E. Sandier and S. Serfaty, *A product-estimate for Ginzburg-Landau and corollaries*, J. Funct. Anal. **211** (2004), no. 1, 219–244. MR2054623
- [SS07] E. Sandier and S. Serfaty, *Vortices in the magnetic Ginzburg-Landau model*, Progress in Nonlinear Differential Equations and their Applications, vol. 70, Birkhäuser Boston, Inc., Boston, MA, 2007. MR2279839
- [SS17] E. Sandier and I. Shafrir, *Small energy Ginzburg-Landau minimizers in \mathbb{R}^3* , J. Funct. Anal. **272** (2017), no. 9, 3946–3964. MR3620717
- [Str08] M. Struwe, *Variational methods*, Fourth, Vol. 34, Springer-Verlag, Berlin, 2008. Applications to nonlinear partial differential equations and Hamiltonian systems. MR2431434
- [Tal76] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372. MR0463908
- [Tin96] M. Tinkham, *Introduction to superconductivity*, Second, McGraw-Hill, New York, 1996.
- [Tru68] N. S. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 265–274. MR0240748
- [TT90] D. R. Tilley and J. Tilley, *Superfluidity and superconductivity*, Third, IOP Publishing Ltd, 1990.
- [VV91] J. L. Vázquez and L. Véron, *Solutions positives d'équations elliptiques semi-linéaires sur des variétés riemanniennes compactes*, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), no. 11, 811–815. MR1108497
- [Wan96] Z.-Q. Wang, *Construction of multi-peaked solutions for a nonlinear Neumann problem with critical exponent in symmetric domains*, Nonlinear Anal. **27** (1996), no. 11, 1281–1306. MR1408871
- [WW02] G. Wang and J. Wei, *Steady state solutions of a reaction-diffusion system modeling chemotaxis*, Math. Nachr. **233/234** (2002), 221–236. MR1879873 (2003a:35078)
- [WW14] J. Wei and M. Winter, *Mathematical aspects of pattern formation in biological systems*, Applied Mathematical Sciences, vol. 189, Springer, London, 2014. MR3114654
- [WWY10] L. Wang, J. Wei, and S. Yan, *A Neumann problem with critical exponent in nonconvex domains and Lin-Ni's conjecture*, Trans. Amer. Math. Soc. **362** (2010), no. 9, 4581–4615. MR2645043
- [WX05] J. Wei and X. Xu, *Uniqueness and a priori estimates for some nonlinear elliptic Neumann equations in \mathbb{R}^3* , Pacific J. Math. **221** (2005), no. 1, 159–165. MR2194150
- [WY07] J. Wei and S. Yan, *Arbitrary many boundary peak solutions for an elliptic Neumann problem with critical growth*, J. Math. Pures Appl. (9) **88** (2007), no. 4, 350–378. MR2384573
- [Yam60] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J. **12** (1960), 21–37. MR0125546
- [Zhu99] M. Zhu, *Uniqueness results through a priori estimates. I. A three-dimensional Neumann problem*, J. Differential Equations **154** (1999), no. 2, 284–317. MR1691074

BIBLIOGRAPHY

Résumé

Cette thèse est consacrée à l'analyse des singularités apparaissant dans des équations différentielles partielles elliptiques non linéaires découlant de la physique mathématique, de la biologie mathématique, et de la géométrie conforme. Les thèmes abordés sont le modèle de supraconductivité de Ginzburg-Landau, le problème de Lin-Ni-Takagi, le modèle de Keller-Segel de la chimiotaxie, et le problème de courbure scalaire prescrite.

Le modèle de Ginzburg-Landau est une description phénoménologique de la supraconductivité. Une caractéristique essentielle des supraconducteurs de type II est la présence de vortex, qui apparaissent au-dessus d'une certaine valeur de la force du champ magnétique appliqué, appelée premier champ critique. Nous nous intéressons au régime de ε petit, où ε est l'inverse du paramètre de Ginzburg-Landau (une constante du matériau). Dans ce régime, les vortex sont au premier ordre des singularités topologiques de co-dimension 2. Nous fournissons une construction quantitative par approximation de vortex en dimension trois pour l'énergie de Ginzburg-Landau, ce qui donne une approximation des lignes de vortex ainsi qu'une borne inférieure pour l'énergie, qui est optimale au premier ordre et vérifiée au niveau ε . En utilisant ces outils, nous analysons ensuite le comportement des minimiseurs globaux en dessous et proche du premier champ critique. Nous montrons que, en dessous de cette valeur critique, les minimiseurs de l'énergie de Ginzburg-Landau sont des configurations sans vortex et que les minimiseurs, proche de cette valeur, ont une vorticit  born e.

Le probl me de Lin-Ni-Takagi apparait comme l'ombre (dans la litt rature anglaise "shadow") du syst me de Gierer-Meinhardt d' quations de r action-diffusion qui mod lise la formation de motifs biologiques. Ce probl me est celui de trouver des solutions positives d'une  quation critique dans un domaine r gulier et born  de dimension trois, avec une condition de Neumann homog ne au bord. Dans cette th se, nous construisons des solutions   ce probl me pr sentant un comportement explosif en un point du domaine, lorsqu'un certain param tre converge vers une valeur critique.

La chimiotaxie est l'influence de substances chimiques dans un environnement sur le mouvement des organismes. Le mod le de Keller-Segel pour la chimiotaxie est un syst me de diffusion-advection compos  de deux  quations paraboliques coupl es. Ici, nous nous int ressons aux  tats stationnaires radiaux de ce syst me. Nous sommes alors amen s    tudier une  quation critique dans la boule unit  de dimension 2, avec une condition de Neumann homog ne au bord. Dans cette th se, nous construisons plusieurs familles de solutions radiales qui explosent   l'origine de la boule, et se concentrent sur le bord et/ou sur une sph re int rieure, lorsqu'un certain param tre converge vers z ro.

Enfin, nous  tudions le probl me de la courbure scalaire prescrite.  tant donn e une vari t  Riemannienne compacte de dimension n , nous voulons trouver des m triques conformes dont la courbure scalaire soit une fonction prescrite, qui d pend d'un petit param tre. Nous supposons que cette fonction a un point critique qui satisfait une hy-

pothèse de platitude appropriée. Nous construisons plusieurs métriques, qui explosent lorsque le paramètre converge vers zéro, avec courbure scalaire prescrite.

Mots-clés: Ginzburg-Landau, premier champ critique, estimations au niveau ε , vortex, construction d'approximation de vortex, problème de Lin-Ni-Takagi, réduction de Lyapunov-Schmidt, fonction de Robin, équation de Keller-Segel, courbure scalaire prescrite, phénomènes d'explosion.

Abstract

This thesis is devoted to the analysis of singularities in nonlinear elliptic partial differential equations arising in mathematical physics, mathematical biology, and conformal geometry. The topics treated are the Ginzburg-Landau model of superconductivity, the Lin-Ni-Takagi problem, the Keller-Segel model of chemotaxis, and the prescribed scalar curvature problem.

The Ginzburg-Landau model is a phenomenological description of superconductivity. An essential feature of type-II superconductors is the presence of vortices, which appear above a certain value of the strength of the applied magnetic field called the first critical field. We are interested in the regime of small ε , where ε is the inverse of the Ginzburg-Landau parameter (a material constant). In this regime, the vortices are at main order co-dimension 2 topological singularities. We provide a quantitative three-dimensional vortex approximation construction for the Ginzburg-Landau energy, which gives an approximation of vortex lines coupled to a lower bound for the energy, which is optimal to leading order and valid at the ε -level. By using these tools we then analyze the behavior of global minimizers below and near the first critical field. We show that below this critical value, minimizers of the Ginzburg-Landau energy are vortex-free configurations and that near this value, minimizers have bounded vorticity.

The Lin-Ni-Takagi problem arises as the shadow of the Gierer-Meinhardt system of reaction-diffusion equations that models biological pattern formation. This problem is that of finding positive solutions of a critical equation in a bounded smooth three-dimensional domain, under zero Neumann boundary conditions. In this thesis, we construct solutions to this problem exhibiting single bubbling behavior at one point of the domain, as a certain parameter converges to a critical value.

Chemotaxis is the influence of chemical substances in an environment on the movement of organisms. The Keller-Segel model for chemotaxis is an advection-diffusion system consisting of two coupled parabolic equations. Here, we are interested in radial steady states of this system. We are then led to study a critical equation in the two-dimensional unit ball, under zero Neumann boundary conditions. In this thesis, we construct several families of radial solutions which blow up at the origin of the ball and concentrate on the boundary and/or an interior sphere, as a certain parameter converges to zero.

Finally, we study the prescribed scalar curvature problem. Given an n -dimensional compact Riemannian manifold, we are interested in finding bubbling metrics whose scalar curvature is a prescribed function, depending on a small parameter. We assume that this function has a critical point which satisfies a suitable flatness assumption. We construct several metrics, which blow-up as the parameter goes to zero, with prescribed scalar curvature.

Keywords: Ginzburg-Landau, first critical field, ε -level estimates, vortices, vortex approximation construction, Lin-Ni-Takagi problem, Lyapunov-Schmidt reduction, Robin's function, Keller-Segel equation, internal layer, boundary layer, prescribed scalar curvature, blow-up phenomena.

