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# Existence et stabilité de solitons, multi-solitons et solutions explosives dans quelques équations dispersives non linéaires

Stefan Le Coz

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Habilitation à Diriger des Recherches

**Existence et stabilité de solitons,  
multi-solitons et solutions explosives  
dans quelques équations dispersives  
non linéaires**

Stefan Le Coz

Présentée le 5 décembre 2018 devant le jury composé de

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# Abstract

The goal of this thesis is to present my research work following my PhD. My PhD thesis was devoted to questions related to the existence and stability of standing waves solutions to nonlinear dispersive PDE like the nonlinear Schrödinger and Klein-Gordon equations. After my PhD, my interest shifted toward more elaborate solutions of dispersive PDE, in particular the multi-solitons. Standing waves are still present in my work, not as the main object of study, but as building blocks for the analysis of more complicated nonlinear objects.

The first chapter of this document is devoted to a general presentation of the context of my work.

The second chapter is devoted to basic facts concerning nonlinear Schrödinger equations and serves as a framework setting for many of the other works presented in this document.

The third chapter is devoted to the presentation of my works on multi-solitons. We start by presenting two existence results, one for excited states multi-solitons of nonlinear Schrödinger equations and the other for multi-solitons of Klein-Gordon equations based on stable solitons. We then show the existence of infinite trains of solitons in nonlinear Schrödinger equations. We conclude this chapter by a stability result for the multi-solitons of the derivative nonlinear Schrödinger equation.

The fourth chapter is devoted to the presentation of my results on blow-up and stability in different contexts. We start with a result on the existence of minimal mass blowing up solutions for a Schrödinger equation with double power nonlinearity. Then we study the Cauchy problem and the stationary states of a singularly perturbed Gross-Pitaevskii equation. Next we investigate, using a variety of techniques, the stability of space periodic standing waves of one dimensional cubic nonlinear Schrödinger equations. We continue with considerations on nonlinear Schrödinger systems and we conclude with results on stability of standing waves for semi-classical equations.

The fifth and last chapter is devoted to two works in progress. The first one concerns the Manakov system: we exhibit a new family of standing waves and study their existence and stability. The second one concerns the excited states of nonlinear Schrödinger equations: we obtain the excited states by constructing numerical schemes inspired from their variational characterizations.

# 1 Introduction

Among nonlinear dispersive PDE, three main model equations can be distinguished. A first model equation is the *Korteweg-de Vries equation*:

$$u_t + u_{xxx} + 2uu_x = 0.$$

A second model equation is the *sine-Gordon equation*, given by

$$u_{tt} - u_{xx} + \sin(u) = 0.$$

A third model equation is the *nonlinear Schrödinger equation*, here in dimension 1 and with cubic focusing nonlinearity:

$$iu_t + u_{xx} + |u|^2u = 0.$$

Nonlinear dispersive PDE appear in many areas of physics. The Korteweg-de Vries equation can model the propagation of waves in shallow water, e.g. in a canal [Kode95]. The sine-Gordon equation appears in the study of crystal dislocations [FrKo39], but was first introduced in mathematics for the study of surfaces of constant negative curvature [Bo62]. It can also model a chain of coupled pendulums [DaPe06]. The name sine-Gordon is based on the analogy of the equation with its quasi-homophone the Klein-Gordon equation. Other variants of this equation in higher dimensions and with other nonlinearities are usually called *nonlinear Klein-Gordon equations*. The nonlinear Schrödinger equation is used in the modeling of Bose-Einstein condensates [Gr61, Pi61] or in nonlinear optics [Wh74].

A common feature of these three model nonlinear dispersive PDEs is that they admit *solitary wave* solutions, i.e solutions whose behavior is given by a fixed profile, moving via the time dependent action of a symmetry group of the equation. For example, the Korteweg-de Vries equation is invariant under translation in space, and a solitary wave is a solution of the form

$$u(t, x) = \phi(x - ct),$$

for some speed  $c > 0$ . In this case, we speak of *traveling waves*. Another example, for the nonlinear Schrödinger equation, is given by

$$u(t, x) = e^{i\theta t} \phi(x),$$

where  $\theta \in \mathbb{R}$ . In this case, the underlying symmetry is the phase invariance and we speak of *standing waves*.

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The three model PDE given above have in fact much more in common than admitting solitary waves. All three are completely integrable Hamiltonian systems. In particular, they can be explicitly solved by the method of inverse scattering, have infinitely many conservation laws and the interaction between solitary waves is smooth. Since the interaction is smooth, the solitary waves are called *solitons*. The terminology has however been extended in the PDE community to any kind of solitary waves. Hence we might frequently refer to solitary waves as solitons, even in a non-integrable context. It is well known that the slightest modification in these equations will most likely destroy the integrability property. Solitary waves will persist, as long as the modification does not disturb the symmetry to which they are associated. The techniques I have developed were meant to be generically applicable; hence even if the equations on which I am working are sometimes of integrable type (like the derivative nonlinear Schrödinger equation or the periodic one dimensional cubic nonlinear Schrödinger equation), I have not made use (so far) of this specificity and its associated features.

It was observed and proved for the Korteweg-de Vries equation that, generically, any localized solution will eventually decompose into a sum of solitons and a dispersive term [EcSc83, Sc86]. This type of behavior is thought to be generic for nonlinear dispersive PDEs and this leads to the *Soliton Resolution Conjecture*, which (vaguely formulated) states that any global solution of a nonlinear dispersive PDE will eventually decompose at large time as a combination of non-scattering structures (e.g. a sum of solitary waves) and a radiative term. Until recently, such conjecture had only been established for some integrable models, e.g. the Korteweg-de Vries equation. The breakthrough approach introduced by Duyckaerts, Kenig and Merle allowed to prove this conjecture for some non-integrable equations such as the energy-critical wave equation [DuKeMe13] or the equivariant wave maps to the sphere [Co15]. It remains an open problem for most of the classical nonlinear dispersive equations.

The soliton resolution conjecture motivates the study of multi-soliton solutions for nonlinear dispersive PDE, i.e. solutions which behave at large time as a sum of solitons, and a large part of my research activity has been devoted to this study. After a second chapter presenting well-known facts on nonlinear Schrödinger equations, the third chapter of this thesis will be devoted to the presentation of my works on multi-solitons, of which we now give a short overview.

My first work on multi-solitons was in collaboration with Raphaël Côte [CoLe11] and was focused on the study of multi-solitons composed of excited states. Indeed, most of the works involving solitary waves are in fact restricted to solitary waves whose profiles are ground states, i.e. can be characterized as (constrained) minimizers of the energy. The many properties of the ground states make them much easier to work with than the other solitary waves profiles, the excited states. Nevertheless, under a high relative speeds assumption, we proved in [CoLe11] the existence of multi-solitons where the composing solitary waves have excited state profiles. To construct the excited state multi-solitons, we relied on an energy technique. The idea is to adapt to the case of multi-solitons the tools used for the stability theory of one soliton. Essentially, we construct a Lyapunov type functional which is coercive up to a finite number of bad directions, and the name of the game is to show that the flow avoids these bad directions. Using a similar

approach, in collaboration with Jacopo Bellazzini and Marco Ghimenti [BeGhLe14], I established the existence of multi-solitons based on stable solitary waves. Here, beside realizing that the previous scheme could be adapted to Klein-Gordon equations, the core of the difficulty was to obtain a suitable variational characterization of the stable solitary wave profiles in the context of the Klein-Gordon equation. To this aim, and even if the Klein-Gordon equation is a scalar equation, we have worked in the vectorial Hamiltonian setting adapted to the Klein-Gordon equation and established a variational characterization of the wave profiles in this setting.

The infinite solitons trains are a natural extension of multi-solitons, where the number of composing solitons is now infinite. However, the infinite number of solitons creates numerous problems when one tries to adapt the techniques used for multi-solitons to establish the existence of infinite trains. In particular, the energy technique is not easily applicable to construct infinite trains of solitary waves, as by construction the coercivity property of the linearized functionals associated to the waves becomes weaker and weaker as the number of waves tends to infinity. Therefore, with Dong Li and Tai-Peng Tsai [LeLiTs15, LeTs14, LeTs17], I developed a strategy to prove the existence of infinite trains via a fixed point argument and the use of Strichartz estimates. As a byproduct of our analysis, we also obtain the existence of kink-soliton solutions.

When existence of multi-solitons is granted, the next step toward a soliton resolution conjecture is to study the dynamics close to the multi-solitons, in particular their stability/instability. We naturally expect multi-solitons build with one unstable soliton to be also unstable, and we have established a result of this type in [CoLe11]. The proof was done by constructing a suitable approximation of a solution leaving the neighborhood of the multi-soliton and obtaining the true solution via a fixed point argument.

For multi-solitons build on stable solitons, and if the interactions between solitons are sufficiently small, we can hope to obtain stability. Whereas the construction of multi-solitons requires essentially only a good knowledge of the properties of the linearized action around a solitary wave profile, proving their stability systematically requires extra properties, be it integrability of the equation or monotonicity formulas. In the setting of Schrödinger equations, the only existing stability result was obtained by Martel, Merle and Tsai [MaMeTs06] for Schrödinger equations with a twisted nonlinearity. In collaboration with Yifei Wu [LeWu18], I have proved a stability result for the multi-solitons of the derivative nonlinear Schrödinger equation. This specific equation is at the same time  $L^2$ -critical and has stable solitons, hence the multi-solitons can be proved to be stable using an energy technique, without any need to twist the nonlinearity.

The soliton resolution conjecture holds for global solutions of homogeneous nonlinear dispersive PDE set on non-compact spaces, but there are many other interesting phenomena occurring when the solution is not global, when inhomogeneity in space appears or when the space becomes compact. The fourth chapter of this thesis is devoted to the presentation of my works in this type of settings, of which we now also give a quick overview.

Solutions of nonlinear dispersive PDEs may cease to exist in finite time and undergo blow-up at the ends of their maximal time interval of existence. It is not expected that the blow-up behavior in nonlinear dispersive PDEs can be described with a statement



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as universal as the soliton resolution conjecture. Depending on the equation considered, the nature of the blowing up solutions varies drastically. For Schrödinger equations, the theory of blow-up is particularly advanced in the mass-critical case. For the mass-critical nonlinear Schrödinger equation, we know that global well-posedness holds for solutions with small mass. At the threshold for global existence, there exists (see [Me93]) a unique (up to symmetries) minimal blow-up mass solution, explicitly given by a pseudo-conformal transformation of a standing wave, and which blows up at time  $T$  at rate  $|t - T|^{-1}$ . Above the minimal mass, one finds solutions blowing up at time  $T$  at the so called log-log rate, given by  $\sqrt{|\ln|\ln|t||} \cdot |t - T|^{-1}$ . These solutions were analyzed by Perelman [Pe01] and Merle and Raphaël [MeRa03, MeRa04, MeRa05, MeRa06, Ra05]. Other blow-up behaviors hold for the critical Korteweg-de Vries equation (with quintic nonlinearity), which was analyzed in detail recently in a series of papers by Martel, Merle and Raphaël [MaMeRa15a, MaMeRa15b], and the wave equation, which was analyzed in depth by Merle and Zaag [MeZa03, MeZa05, MeZa05a, MeZa07, MeZa08, MeZa11, MeZa12, MeZa12a, MeZa12b, MeZa15, MeZa16]. Some questions might be similar, for example the existence of multiple blow-up points solutions, which was investigated in [Me90] for the Schrödinger equation and in [CoZa13] for the wave equation. Despite the growing body of works related to blow-up in nonlinear dispersive PDE, the understanding of the phenomenon is still very partial, in particular in the non-critical cases. One interesting question concerns the possible blow-up speeds, beside the pseudo-conformal speed and the log-log speed. In a joint work with Yvan Martel and Pierre Raphaël [LeMaRa16], I have exhibited an example of a double-power nonlinear Schrödinger equation for which there exist solutions blowing up at rate  $|t - T|^{-\sigma}$ , with  $\frac{1}{2} < \sigma < 1$ , i.e. the rate for blow-up is strictly between the log-log one and the pseudo-conformal one.

The existence of blowing-up solutions is also a motivation for considering singular perturbations of model cases, for example the following singularly perturbed Gross-Pitaevskii equation

$$iu_t + u_{xx} - \gamma\delta u + (1 - |u|^2)u = 0,$$

where  $\gamma$  is a real parameter and  $\delta$  denotes the Dirac distribution at 0. This type of models is common in physics, where the Dirac distribution might model a very narrow potential, e.g. linked to a wave close to or at the blow up time. The mathematical analysis of this type of singular models started with the work of Goodman, Holmes and Weinstein [GoHoWe04] and was followed by many contributions. As for the non-perturbed problems, most of the existing works have been devoted to equations endowed with standard Dirichlet boundary conditions on the whole space. However, it is perfectly natural, and also relevant in physical situations, to ask what happens if we instead decide to impose as boundary conditions  $|u| \rightarrow 1$  as  $|x| \rightarrow \infty$ . The Cauchy Theory for this type of problems in the absence of singular potential was figured out by Zhidkov [Zh01] in the one dimensional case and by Gérard [Ge06, Ge08] in the higher dimensional case (see also the work of Gallo [Ga08]). Extending this analysis to the singularly perturbed one dimensional Gross-Pitaevskii equation turns out to be surprisingly difficult. One requires in particular a specific knowledge of the propagator associated to the linear part of the

equation. We performed this analysis in a joint work [IaLeRo17] with Isabella Ianni and Julien Royer, in which we also investigate the stability of stationary solutions.

In a spatially periodic setting, behaviors of solutions of nonlinear dispersive equations will be much different from the one observed when the space is the whole line (or higher dimensional equivalents). Whereas the study of solitary waves and their stability on the line is well developed, the study of the stability of standing/traveling waves in a periodic setting is still in its infancy. Indeed, if the perturbations considered are co-periodic with the wave (i.e. they have a period which is the same as the fundamental period of the wave), then the stability study can be carried out in a similar way as in the whole space. However, it is also necessary in many cases to consider perturbations whose period is a multiple of the fundamental period of the wave. In that kind of situations, the usual modulational approach relying on modulation on translation and phase is not sufficient to eliminate the directions preventing the linearized energy to be coercive and the theory of the whole line fails. In the specific case of the one dimensional cubic nonlinear Schrödinger equation, one can avoid the problem by using the complete integrability of the equation and working with higher order functionals. In the general case, methods are still to be developed to tackle this problem. In a joint work with Stephen Gustafson and Tai-Peng Tsai [GuLeTs17], I have investigated the stability of standing waves for the one-dimensional cubic nonlinear Schrödinger equation by means of a variety of techniques not relying on integrability.

Coupled systems of nonlinear dispersive equations are often used in the modeling of complex physical phenomena involving for example the interaction between different instances of similar type (e.g. Bose-Einstein condensates). In collaboration with Isabella Ianni [IaLe14], and Fanny Delebecque and Rada-Maria Weishäupl [DeLeWe16] I investigated solitary wave solutions of scalar nonlinear Schrödinger equations viewed as special solutions of nonlinear Schrödinger systems. We have in particular observed that the nonlinear coupling acts as a localizing factor in the analysis of these solutions and performed numerical experiments.

The semi-classical setting provides another situation where interesting phenomena appear (see e.g. [Ca08] and the references cited therein). In collaboration with Isabella Ianni [IaLe09] and Marco Ghimenti and Marco Squassina [GhLeSq13] I investigated the stability of standing waves in two semi-classical settings, a strongly inhomogeneous nonlinear Schrödinger equation and a Klein-Gordon equation with potentials. In both cases, we have used the semi-classical parameter to analyse by perturbation arguments the spectrum of a linearized operator, which was a key step in the stability analysis.

In the fifth and last chapter of this thesis, we present two works in progress. The first one concerns the Manakov system, a nonlinear Schrödinger system which has attracted a lot of attention due to its integrable nature. It turns out that this system enjoys more symmetries than other Schrödinger systems and therefore has more conservation laws induced by Noether theorem. These additional conservation laws are however not captured by integrable methods. The system also enjoys a type of standing waves linked to the additional symmetries, which has not been considered before. In our project, we start the study of these standing waves by establishing existence and stability results.

The second project concerns the numerical analysis of excited states. Numerous an-

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alytical and numerical studies are devoted to ground states, but so far the literature on excited states remains relatively sparse. Our goal in this project is to construct numerical schemes that can capture excited states by relying on adapted variational characterizations.

To summarize, the rest of this thesis is organized as follows. In Chapter 2, we present basic facts on nonlinear Schrödinger equations. In Chapter 3, we present my work on multi-solitons. In Chapter 4, we present my other works on blow-up and stability in various settings. In Chapter 5 we present my work in progress on the Manakov system and the numerical analysis of excited states.

## 2 The Nonlinear Schrödinger Equation

A large part of my work is devoted to the study of solutions of nonlinear Schrödinger equations. In this chapter, I will present the most basic facts about these equations (linear theory, local well-posedness, solitary waves, etc.) on which I will rely in the next chapters to study more elaborate solutions. The reader may refer to [Ca03, Ta06] and the references cited therein for more on these topics. The presentation of basic facts for the other equations that I have considered will be done in the relevant parts of this thesis.

The nonlinear Schrödinger equation reads

$$\begin{cases} iu_t + \Delta u + f(u) = 0, \\ u(0) = u_0. \end{cases} \quad (\text{NLS})$$

Here,  $u = u(t, x) \in \mathbb{C}$  is a complex valued function of time  $t \in \mathbb{R}$  and space  $x \in \mathbb{R}^d$ . The nonlinearity  $f : \mathbb{C} \rightarrow \mathbb{C}$  is gauge invariant (i.e. for any  $z \in \mathbb{C}$ ,  $f(z) = f(|z|)z/|z|$ ) and  $H^1(\mathbb{R}^d)$  subcritical (see later for more precise assumptions on  $f$ ). For most of this thesis, the reader may think of  $f$  as being a model case focusing power nonlinearity

$$f(z) = |z|^{p-1}z, \quad 1 < p < 1 + \frac{4}{(d-2)_+}.$$

In some situations we will want to be able to consider more generic nonlinearities like double power nonlinearities

$$f(z) = |z|^{p-1}z - |z|^{q-1}z, \quad 1 < p, q < 1 + \frac{4}{(d-2)_+},$$

or even more complicated nonlinearities, e.g. (see [DuLaSz16] for other examples of relevant nonlinearities)

$$f(z) = z - \frac{\sin(|z|)}{|z|}z.$$

The nonlinear Schrödinger equation appears in various physical contexts like nonlinear optics or Bose-Einstein condensation. From the mathematical point of view, it is of particular interest as a model case for *nonlinear dispersive partial differential equations*, along with its alter-ego the Korteweg-de Vries equation and the wave/Klein-Gordon equation.

## 2.1 The Linear Schrödinger Equation

In this section, we review some elements of the classical theory developed in the 80s for the well-posedness of (NLS) in the energy space. We start with the analysis of the linear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = 0, \\ u(0) = u_0. \end{cases} \quad (\text{LS})$$

This equation is solved in  $L^2(\mathbb{R}^d)$  using the classical theory of semi-group. Precisely, solutions to (LS) may be expressed as

$$u(t) = e^{-it\Delta}u_0 = S(t) \star u_0,$$

where  $e^{-it\Delta}$  denotes the (Schrödinger) unitary group generated by the operator  $-i\Delta$  (cf e.g. [CaHa98, Pa83]). Moreover, an application of the Fourier transform allows to compute the kernel  $S(t)$  and we get the explicit formula

$$S(0) = \delta_0, \quad S(t) = \frac{1}{(4\pi it)^{\frac{d}{2}}} e^{i\frac{|x|^2}{4t}}, \quad \text{for } t > 0.$$

The inhomogeneous linear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = F, \\ u(0) = u_0 \end{cases}$$

is solved, when possible, with the *Duhamel formula*

$$u(t) = S(t) \star u_0 - i \int_0^t S(t-s) \star F(s) ds.$$

As mentioned earlier, the linear equation (LS) is dispersive. From the explicit expression of the kernel, we get the following *dispersive inequality*, which measures the dispersion. For  $t \in \mathbb{R} \setminus \{0\}$ ,  $p \in [2, \infty]$ , and  $p'$  its Hölder conjugate, we have

$$\|e^{-it\Delta}u_0\|_{L^p} \leq \frac{1}{|4\pi t|^{\frac{d}{2}\left(\frac{1}{p'} - \frac{1}{p}\right)}} \|u_0\|_{L^{p'}}.$$

To obtain the dispersive inequality, we start by noting that the Schrödinger group is an isometry on  $L^2(\mathbb{R}^d)$ , hence we have

$$\|e^{-it\Delta}u_0\|_{L^2} = \|u_0\|_{L^2}.$$

On the other hand, we use Young inequality and the explicit expression of the kernel to find

$$\|e^{-it\Delta}u_0\|_{L^\infty} = \|S(t) \star u_0\|_{L^\infty} \leq \|S(t)\|_{L^\infty} \|u_0\|_{L^1} \leq \frac{1}{|4\pi t|^{\frac{d}{2}}} \|u_0\|_{L^1}.$$

The dispersive inequality then follows from Riesz-Thorin interpolation theorem.

For the study of the Cauchy problem for nonlinear Schrödinger equations, it is important to have a precise measure of the dispersion. The dispersive inequality is sufficient for this purpose when the underlying spaces have high enough regularity, but to study the Cauchy problem in  $H^1(\mathbb{R}^d)$  (which is a natural space for (NLS), as it is the energy space), one needs more refined tools. They come in the form of *Strichartz estimates*. Before giving the statement of the estimates, we recall the definition of space-time norms.

For  $p, q \in [1, \infty]$  and an interval  $I \subset \mathbb{R}$ , define

$$\|u\|_{L_t^p L_x^q(I)} = \left( \int_I \left( \int_{\mathbb{R}^d} |u(t, x)|^q dt \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}},$$

with obvious modifications when  $p$  or  $q$  is  $\infty$ .

The Strichartz estimates describe a type of smoothing effect of the Schrödinger equation, which is reflected in the gain of space integrability if one accepts to take an average in time. Precisely, we want inequalities of the following form to be satisfied:

$$\|e^{it\Delta} u_0\|_{L_t^p L_x^q} \leq C \|u_0\|_{L^2}. \quad (2.1)$$

Obviously such inequalities cannot be true for any  $p$  and  $q$ . Injecting the scaling

$$e^{it\Delta}(u_0(\lambda x)) = (e^{i\lambda^2 t \Delta} u_0)(\lambda x)$$

for  $\lambda > 0$ , we realize that (2.1) can hold only when  $p$  and  $q$  verify the relation

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}. \quad (2.2)$$

We call a couple  $(p, q)$  verifying the relation (2.2) an *admissible pair*. For admissible pairs  $(p, q)$  and  $(r, s)$  with  $2 \leq p, q, r, s \leq \infty$ ,  $(p, q, d)$  and  $(r, s, d)$  different from  $(2, \infty, 2)$  the following *Strichartz estimates* are satisfied. First, we have the homogeneous Strichartz estimate

$$\|e^{it\Delta} u_0\|_{L_t^p L_x^q(I)} \leq C \|u_0\|_{L^2}.$$

Second, we have the inhomogeneous Strichartz estimates: for  $t_0 \in \bar{I}$ , we have

$$\left\| \int_{t_0}^t e^{i(t-\tau)\Delta} F(\tau) d\tau \right\|_{L_t^p L_x^q(I)} \leq C \|F\|_{L_t^{r'} L_x^{s'}(I)},$$

where  $r'$  and  $s'$  are the conjugated exponents of  $r$  and  $s$ . The constant  $C$  can be made independent of  $(p, q)$ ,  $(r, s)$ , and  $I$  (with the extra requirement to stay away from the end point in dimension 2). We refer for example to [GiVe92] for the general case and to [KeTa98] for the end point case  $(2, \frac{2d}{d-2}, d)$ ,  $d \geq 3$ .

For any fixed space-time slab  $I \times \mathbb{R}^d$ , we define the *Strichartz norm*

$$\|u\|_{S(I)} := \sup_{(q, r) \text{ admissible}} \|u\|_{L_t^q L_x^r(I \times \mathbb{R}^d)}.$$

For  $d = 2$ , we need to further impose  $q > q_1$  in the above norm for some  $q_1$  slightly larger than 2, so as to stay away from the forbidden endpoint. We use  $S(I)$  to denote the closure of all test functions in  $\mathbb{R} \times \mathbb{R}^d$  under this norm, which we call the *Strichartz space*. We denote by  $N(I)$  the dual space of  $S(I)$  (see e.g. [Ta06, page 135]).

## 2.2 The Nonlinear Schrödinger Equation

Before turning our attention to the Cauchy problem for (NLS), let us first review some formal aspects of (NLS). First of all, the equation can be written in the form of a Hamiltonian system

$$u_t = -iE'(u),$$

where the Hamiltonian  $E$  (later designated by *energy*) is

$$E(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \int_{\mathbb{R}^d} F(u)dx, \quad F(z) = \int_0^{|z|} f(s)ds.$$

The linear problem has a lot of symmetries, many of which continue to hold in the nonlinear case. Let  $u$  denote a solution of (NLS). The nonlinear Schrödinger equation (NLS) is invariant under *time translations*, *space translations*, and *phase shifts*, i.e. given  $\theta \in \mathbb{R}, \tau \in \mathbb{R}, \xi \in \mathbb{R}^d$  the function  $\tilde{u}$  defined by

$$\tilde{u}(t, x) = e^{i\theta}u(t - \tau, x - \xi)$$

is also a solution of (NLS). We have the possibility of *time reversal*: the function  $v$  defined by

$$v(t, x) = \bar{u}(-t, x)$$

is also solution of (NLS). A Galilean transformation preserves the flow of (NLS), i.e. the function  $w$  defined by

$$w(t, x) = e^{i\frac{1}{2}v \cdot x - \frac{|v|^2}{4}t}u(t, x - vt), \quad v \in \mathbb{R}^d$$

is also a solution of (NLS). If we impose further restrictions on the nonlinearity, then more symmetries are preserved. For instance, if  $f(u) = |u|^{p-1}u$ , then the *scaling* is preserved and  $u_\lambda$  is also a solution, where

$$u_\lambda(t, x) = \frac{1}{\lambda^{\frac{1}{p-1}}}u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right), \quad \lambda > 0.$$

If we further restrict the power to be the  $L^2$  critical power, i.e.  $f(u) = |u|^{\frac{4}{d}}u$ , then the *pseudo-conformal transform*  $v$  of  $u$  is still a solution of (NLS), where

$$v(t, x) = \frac{1}{t^{\frac{d}{2}}}e^{i\frac{|x|^2}{4t}}\bar{u}\left(\frac{1}{t}, \frac{x}{t}\right).$$

Since (NLS) is Hamiltonian, the *energy*  $E$  (i.e. the Hamiltonian) is conserved along the evolution in time. As expected from Noether's principle, the symmetries allow to derive (at least formally) two other conserved quantities, the *mass* and the *momentum*, defined as follows

$$M(u) = \frac{1}{2}\|u\|_2^2, \quad P(u) = \frac{1}{2}\Im \int_{\mathbb{R}^d} u \nabla \bar{u} dx.$$

Another interesting quantity is the *virial*, whose evolution is driven by the *virial identity* (valid in general but expressed here only for power type nonlinearities)

$$\frac{\partial^2}{\partial t^2} \|xu\|_{L^2}^2 = 8P(u), \quad P(u) = \|\nabla u\|_{L^2}^2 - \frac{d(p-1)}{2(p+1)} \|u\|_{L^{p+1}}^{p+1}.$$

Linked to the virial identity is the *pseudo-conformal conservation law* discovered by Ginibre and Velo [GiVe79-1, GiVe79-2, GiVe80]. When  $f$  is the  $L^2$  critical power nonlinearity (i.e.  $f(s) = |s|^{\frac{4}{d}}s$ ), the quantity  $P(u) = 2E(u)$  is constant in time and the pseudo-conformal conservation law has a simple expression given by

$$PS(t) = \|xu\|_{L^2}^2 + 4t\Im \int_{\mathbb{R}^d} xu \nabla \bar{u} dx + 8E(u)t^2 = PS(0).$$

In order to give a rigorous setting to our study, we need to make a few assumptions on  $f$ . These assumptions may not be the most general ones, but they are satisfied in most practical cases.

**Assumption 2.1** (Energy subcritical). *The nonlinearity  $f(z) = g(|z|^2)z$  is such that*

- $g \in \mathcal{C}^0([0, \infty), \mathbb{R}) \cap \mathcal{C}^2((0, \infty), \mathbb{R})$ ,  $g(0) = 0$ ,
- *There exists  $1 < p \leq q < 1 + \frac{4}{(d-2)_+}$  such that for any  $s > 0$  we have*

$$|sg'(s)| + |s^2g''(s)| \lesssim s^{\frac{p-1}{2}} + s^{\frac{q-1}{2}}.$$

This assumption essentially amounts to asking that  $f$  is regular enough and, more importantly, that it is *energy subcritical*. The situation changes drastically for energy critical and super-critical nonlinearities and we will not consider these types of problems in this thesis. Under Assumption 2.1, we have the following well-posedness result.

**Proposition 2.2.** *Assume Assumption 2.1. For any  $u_0 \in H^1(\mathbb{R}^d)$  there exists a unique maximal solution to (NLS)*

$$u \in \mathcal{C}((-T_*, T^*), H^1(\mathbb{R}^d)) \cap \mathcal{C}^1((-T_*, T^*), H^{-1}(\mathbb{R}^d)).$$

Furthermore one has the following properties.

- The blow-up alternative : *if  $T^* < \infty$ , then  $\lim_{t \rightarrow T^*} \|u\|_{H^1} = \infty$  (same with  $T_*$ ).*
- Conservation of energy, mass, and momentum.
- Virial identity *if  $xu_0 \in L^2(\mathbb{R}^d)$ .*

The proof of Proposition 2.2 is rather involved and we only give some hints on how to prove it. First, one uses a fixed point argument with the Duhamel formulation

$$u(t) = e^{-it\Delta} u_0 - i \int_0^t e^{-i(t-s)\Delta} f(u(s)) ds.$$

Then, one relies on density and approximation arguments to get the full result. The reader may refer to [Ca03] for a detailed proof of Proposition 2.2 as well as historical references.



## 2.3 Large Time Behavior

In this thesis, we will be mainly concerned with the long time behavior of the solutions to (NLS).

Essentially, three main types of behaviors can occur at large time. First, the nonlinearity may just induce a small perturbation of the linear behavior and after some time the dynamics of (NLS) is essentially driven by the one of the linear flow, i.e. there exists  $u_\star$  such that

$$u(t) - e^{-it\Delta}u_\star \rightarrow 0, \quad \text{when } t \rightarrow \infty.$$

This is the *scattering* effect. Note that  $u_\star$  has no reason to be identical to  $u_0$  (see e.g. [Ca01]).

A second effect is somewhat the opposite of the scattering effect : the dynamic is completely dominated by the nonlinearity, which tends to concentrate the solution, up to the point that there is a (finite) time where the solution exits its natural living space and there is blow-up. This effect is called the *focusing* effect.

A third alternative is when the two previously mentioned effects balance and create solutions of (NLS) that neither disperse nor focus. We generically call these solutions *non-scattering structures*. The most simple example for this kind of solutions is a solitary wave. A solitary wave is a solution of (NLS) which conserves a certain localized profile invariant along the evolution in time, up to the symmetries of the equation, for example it can be of the type

$$u(t, x) = e^{it}Q$$

where  $Q$  is a fixed profile. But there may be many other type of non-scattering structure, for example kinks, which are solitons with a non localized profile (in particular not in  $L^2$ ). As explained in Chapter 1, a general conjecture for nonlinear dispersive PDEs is the Soliton Resolution Conjecture, according to which any global solution of a dispersive PDE will eventually decompose into a combination of non-scattering structures and a scattering remainder.

We can also have existence of composed non-scattering structures, for example structures made with several solitons (which we will refer to as multi-solitons). Our goal in this thesis is to review the classical theory of solitons and to show how to construct composed non-scattering structures like multi-solitons, infinite soliton trains, kink-soliton trains, etc. As much as possible, we have preferred to give the generic ideas behind the proofs and refer to the original works for the technical details. The next sections are devoted to more rigorous considerations on the large time behavior of solutions of nonlinear Schrödinger equations.

## 2.4 Global Existence and Blow-Up

We now make some comments on global existence and finite time blow-up when  $f(u) = |u|^{p-1}u$ ,  $1 < p < 1 + \frac{4}{(d-2)_+}$ .

## 2 The Nonlinear Schrödinger Equation

Define  $Q$  to be the unique radial positive [GiNiNi79, Kw89] solution of

$$-\Delta Q + Q - Q^p = 0, \quad Q \in H^1(\mathbb{R}^d).$$

It corresponds to the standing wave  $e^{it}Q$  solution of (NLS) (we will come back on standing waves in Section 2.5). We recall the Gagliardo-Nirenberg inequality

$$\|u\|_{L^{p+1}}^{p+1} \leq C_{GN}^{p+1} \|\nabla u\|_{L^2}^{\frac{d(p-1)}{2}} \|u\|_{L^2}^{p+1 - \frac{d(p-1)}{2}}. \quad (2.3)$$

The constant  $C_{GN}$  can be expressed (see [We83]) in terms of  $Q$  by the formula

$$C_{GN} = \left( \frac{2(p+1)}{d(p-1)\|Q\|_{L^2}^{p-1}} \right)^{\frac{1}{p+1}}.$$

From the conservation of mass and energy and Gagliardo-Nirenberg inequality, the solution  $u$  to (NLS) is global when  $p < 1 + \frac{4}{d}$  or if  $p = 1 + \frac{4}{d}$  and  $\|u_0\|_{L^2} < \|Q\|_{L^2}$ .

Indeed, since the energy and the mass are conserved, for  $u$  solution of (NLS) we have

$$\|\nabla u(t)\|_{L^2}^2 = 2E(u(t)) + \frac{2}{p+1} \|u\|_{L^{p+1}}^{p+1} \leq 2E(u_0) + \frac{2C_{GN}^{p+1}}{p+1} \|\nabla u\|_{L^2}^{\frac{d(p-1)}{2}} \|u_0\|_{L^2}^{p+1 - \frac{d(p-1)}{2}}.$$

This gives global existence when  $\frac{d(p-1)}{2} < 2$ , i.e.  $p < 1 + \frac{4}{d}$ . For  $p = 1 + \frac{4}{d}$ , one has to assume further that  $\|u_0\|_{L^2} < \|Q\|_{L^2}$  to get global existence from

$$\left( 1 - \left( \frac{\|u_0\|_{L^2}}{\|Q\|_{L^2}} \right)^{\frac{4}{d}} \right) \|\nabla u(t)\|_{L^2}^2 \leq E(u_0).$$

If  $p \geq 1 + \frac{4}{d}$ , then there exist solutions which blow up in finite time. In the critical case  $p = 1 + \frac{4}{d}$ , one has an explicit example of a blowing up solution (with minimal mass  $\|Q\|_{L^2}$ ) by taking a pseudo conformal transform of  $e^{it}Q$ , given by

$$\frac{1}{t^{\frac{d}{2}}} e^{i\frac{|y|^2}{4t}} e^{-\frac{i}{t}} Q\left(\frac{x}{t}\right).$$

This gives a solution of (NLS) blowing up at  $t = 0$ . For  $L^2$  super-critical exponents  $p > 1 + \frac{4}{d}$ , existence of blowing-up solutions can be obtained from an obstruction argument by Glassey [Gl77] based on the virial identity. Indeed, assume that the initial data  $u_0 \in H^1(\mathbb{R}^d)$  is in the weighted space  $L^2(\mathbb{R}^d, |x|^2)$  (so that  $xu \in L^2(\mathbb{R}^d)$  for all its existence time) and furthermore assume that

$$E(u_0) < 0.$$

Then  $u$  blows up in finite time both in negative and positive time. Indeed, from the virial identity and the conservation of energy, we have

$$\frac{\partial^2}{\partial t^2} \|xu\|_{L^2}^2 = 8P(u) = 16E(u) - 8 \left( \frac{d(p-1) - 4}{2(p+1)} \right) \|u\|_{L^{p+1}}^{p+1} < 16E(u_0).$$

Integrating twice in time, we find that

$$\|xu\|_{L^2}^2 < 16E(u_0)t^2 + C(t+1).$$

Since  $E(u_0) < 0$ , the right hand side becomes negative for large time, hence the obstruction to global existence for  $u$ . The assumption  $xu_0 \in L^2(\mathbb{R}^d)$  can be relaxed to  $u_0$  radial and  $d \geq 2$ , see [OgTs91].

## 2.5 Solitary Waves

We review in this section the elementary theory for solitary waves.

As already mentioned, a solitary wave of (NLS) is a solution of the form

$$R(t, x) = \Phi(x - vt - x_0)e^{i(\frac{1}{2}v \cdot x - \frac{1}{4}|v|^2 t + \omega t + \gamma)} = e^{i\left(\omega + \frac{|v|^2}{4}\right)t} \Psi(x - vt), \quad (2.4)$$

where  $\omega, \gamma \in \mathbb{R}$ ,  $v, x_0 \in \mathbb{R}^d$  and  $\Phi \in H^1(\mathbb{R}^d)$  is a localized profile solution of the equation

$$-\Delta\Phi + \omega\Phi + f(\Phi) = 0. \quad (\text{SNLS})$$

It is sometimes convenient to express the solitary wave using the profile  $\Psi$ , which can be obtained from  $\Phi$  with the formula

$$\Psi(x) = e^{i\frac{1}{2}v \cdot x + \gamma} \Phi(x - x_0).$$

With this formula and (SNLS), we infer that  $\Psi$  verifies

$$-\Delta\Psi + \left(\omega + \frac{|v|^2}{4}\right)\Psi + iv \cdot \nabla\Psi + f(\Psi) = 0.$$

The profile  $\Phi$  is a critical point of the functional

$$S_\omega = E + \omega M,$$

whereas  $\Psi$  is a critical point of the functional

$$S_{\omega, v} = E + \left(\omega + \frac{|v|^2}{4}\right)M + v \cdot P.$$

Since (NLS) is invariant by a Galilean transform, it was historically assumed (without loss of generality) that  $v = 0$  for the study of solitary waves. However, for the construction of multi-solitons we will need the theory for generic  $v \in \mathbb{R}^d$ .

Assumption 2.1 is not sufficient to guarantee the existence of solutions to (SNLS). We further need to assume that the nonlinearity  $f$  is focusing, in the following sense.

**Assumption 2.3** (Focusing). *There exists  $s_0 > 0$  such that  $F(s_0) > \frac{s_0^2}{2}$  (recall that  $F(s) = \int_0^s f(\sigma)d\sigma$ ).*

## 2 The Nonlinear Schrödinger Equation

The following existence result was obtained by Berestycki and Lions [BeLi83-1, BeLi83-2].

**Theorem 2.4.** *Assume Assumptions 2.1 and 2.3. Let  $\omega > 0$ . Then the following assertions hold.*

- *All solutions to (SNLS) are  $C^2$ -regular and exponentially decaying: for any  $\omega' < \omega$ , and for any  $\Phi$  solution to (SNLS) we have*

$$e^{\omega'|x|}\Phi \in L^\infty(\mathbb{R}^d).$$

- *There exists a radial positive solution to (SNLS), usually denoted by  $Q = Q_\omega$ . It is a ground state, i.e. a minimizer of the action functional*

$$S_\omega = E + \omega M$$

*among all solutions to (SNLS).*

- *If  $d = 1$ , up to translations and phase shifts, there are no other solutions.*
- *If  $d \geq 2$ , there exists an infinite sequence of other solutions with increasing action.*

In general, solutions of (SNLS) are called *bound states*. A solution with minimal action will be called a *ground state* whereas a solution with non minimal action will be called an *excited state*.

As they are considered to be the building blocks of the nonlinear dynamics of (NLS), it is essential to have a good understanding of the stability properties of the solitary waves. One essentially distinguishes three notions of stability : *linear stability*, *orbital stability* and *asymptotic stability*.

The notion of *linear stability* is defined as follows. Given a solitary wave  $R(t, x) = e^{i\left(\omega + \frac{|v|^2}{4}\right)t} \Psi(x - vt)$  and a solution  $u$  of (NLS) close to  $R$  at initial time, we define  $\varepsilon$  by

$$u(t, x) = e^{i\left(\omega + \frac{|v|^2}{4}\right)t} (\Psi(x - vt) + \varepsilon(t, x - vt)).$$

The equation of the perturbation  $\varepsilon$  is written as

$$i\varepsilon_t + \mathcal{L}\varepsilon = \mathcal{N}(\varepsilon),$$

where  $\mathcal{L}\varepsilon$  contains all terms which are linear (with respect to  $\mathbb{R}$ ) in  $\varepsilon$  and  $\mathcal{N}(\varepsilon)$  contains all nonlinear terms. We say that  $R$  is *linearly stable* if 0 is a stable solution of the linear equation

$$i\varepsilon_t + \mathcal{L}\varepsilon = 0.$$

Sometimes, one uses a weaker notion of stability, the so-called *spectral stability*, where one requires the spectrum of the operator  $i\mathcal{L}$  to be contained in the imaginary axis. But even at the linear level, a spectrally stable  $R$  might be unstable, as having a spectrum on  $i\mathbb{R}$  does not prevent existence of linear solutions with polynomial growth.

The notion of *orbital stability* is defined as follows. The term *orbit* has to be taken as orbit under the action of the Hamiltonian symmetries of (NLS), in the present case phase shifts and translations (we refer to [DeGeRo15, DeRo18] for a discussion of the notion of orbital stability). We say that a solitary wave is *orbitally stable* if any solution of (NLS) with initial data close to the soliton profile will remain for all time close to the orbit of the soliton under phase shifts and translation (see after for a more mathematical definition).

The notion of *asymptotic stability* is defined as follows. A solitary wave is said to be *asymptotically stable* if it is orbitally stable, and moreover any solution with initial data in the neighborhood of the solitary wave will at large time converge (in some suitable sense) to the solitary wave itself, up to modifications in the speed and frequency of the solitary wave.

It can be proved that asymptotic stability implies orbital stability, which in turn implies linear stability. There are counter-examples to the reverse implications. In the sequel, we shall not deal with asymptotic stability and we will focus mainly on orbital stability, for which we now give a precise definition.

**Definition 2.5.** A solitary wave  $R(t, x) = e^{i\left(\omega + \frac{|v|^2}{4}\right)t} \Psi(x - vt)$  is said to be *orbitally stable* if for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any  $u_0 \in H^1(\mathbb{R}^d)$  we have

$$\|u_0 - \Psi\|_{H^1} \leq \delta \implies \sup_{t \in \mathbb{R}} \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^d} \|e^{i\theta} \Psi(\cdot - y) - u(t)\|_{H^1} \leq \varepsilon$$

for  $u$  the solution of (NLS) with  $u(0) = u_0$ .

One cannot hope to remove the translations and phase shifts in the definition of orbital stability. Indeed, two solitons starting with very close speeds and frequencies will keep close profiles in the long range but with a dephasing in position and phase due to speed and frequency differences. Orbital stability has been the subject of intensive research since the beginning of the 80s. For nonlinear Schrödinger equations, orbital stability was established for ground states of  $L^2$  subcritical equations by Cazenave and Lions [CaLi82] using variational techniques. Weinstein [We85, We86] obtained a modulational stability result (i.e. a more detailed result of orbital stability) from a spectral analysis of the linearization of the action. Instability by blow up for ground state solitons for  $L^2$  critical and supercritical nonlinearities was proven by Berestycki and Cazenave [BeCa81] and Weinstein [We83]. A general theory to study stability/instability was developed by Grillakis, Shatah and Strauss [GrShSt87, GrShSt90]. This theory was recently revisited, improved and extended in the works of De Bièvre, Genoud, and Rota Nodari [DeGeRo15, DeRo18]. As far as excited states are concerned, the conjecture is that they are all unstable. However, only partial results in that direction are available (see e.g. [Gr88, Jo88, Mi05-1, Mi07]).

We recall here only the most essential results concerning ground states. For simplicity, we state the results for a power-type nonlinearity.

**Theorem 2.6.** *Assume  $f(u) = |u|^{p-1}u$ ,  $1 < p < 1 + \frac{4}{(d-2)_+}$ . For  $\omega > 0$ , let  $Q_\omega$  be the positive radial ground state of (SNLS). Take any  $v \in \mathbb{R}^d$  and let  $R_{\omega,v}$  be the soliton associated with  $\omega, v$  and  $Q_{\omega,v} = e^{i\frac{1}{2}v \cdot x} Q_\omega$ . The following assertions hold.*

- (i) *If  $p < 1 + \frac{4}{d}$ , then  $R_{\omega,v}$  is orbitally stable.*
- (ii) *If  $p \geq 1 + \frac{4}{d}$ , then  $R_{\omega,v}$  is unstable (by blow-up).*

The tools used to prove the existence of multi-solitons with an energy technique (see Section 3.1) are based on those used to prove the stability of ground states solitons. Hence we will present them in concise way and give a short proof of part (i) of Theorem 2.6.

It is relatively common when studying the stability to consider only the case of a standing wave (i.e. a solitary wave with no speed, of the form  $e^{i\omega t}Q$ ), as a Galilean transformation automatically transfer the stability result to solitary waves with any kind of speed. However, when considering multi-solitons made of solitary waves traveling at different speeds, it will not be possible anymore to use a Galilean transformation to set all speeds equal to 0. We will need to have tools taking into account generic speeds and that is why we do not restrict ourselves to the case of standing waves when proving part (i) of Theorem 2.6.

The main ingredient on which we are going to rely is a coercivity property of the linearization of the action functional around a soliton.

For the rest of this section, assume  $f(u) = |u|^{p-1}u$ , take  $\omega > 0$  and let  $Q = Q_\omega$  be the unique positive ground state solution of (SNLS). Let  $v \in \mathbb{R}^d$  and define  $Q_{\omega,v} = e^{i\frac{1}{2}v \cdot x} Q_\omega$ . Note that  $Q_{\omega,v}$  and  $R_{\omega,v}$  are critical points of

$$S = E + \left( \omega + \frac{|v|^2}{4} \right) M + v \cdot P. \quad (2.5)$$

Note also that  $S$  depends on the parameters of the soliton that we are considering.

The main tool in the stability analysis of solitons is the following coercivity property.

**Proposition 2.7** (The coercivity property). *Assume  $f(u) = |u|^{p-1}u$ ,  $p < 1 + \frac{4}{d}$ . For any  $\varepsilon \in H^1(\mathbb{R}^d)$ , define*

$$H(t, \varepsilon) = \langle S''(R_{\omega,v})\varepsilon, \varepsilon \rangle.$$

Then

$$H(t, \varepsilon) \gtrsim \|\varepsilon\|_{H^1}^2 - (\varepsilon, R_{\omega,v})_{L^2}^2 - (\varepsilon, \nabla R_{\omega,v})_{L^2}^2 - (\varepsilon, iR_{\omega,v})_{L^2}^2.$$

Here, the space  $L^2(\mathbb{R}^d)$  is viewed as a real Hilbert space and

$$(u, v)_{L^2} = \Re \int_{\mathbb{R}^d} u \bar{v} dx.$$

Proposition 2.7 relies on the variational characterization of  $Q$  as minimizer of the action among functions having constant  $L^2$  norm (see [CaLi82]) and on the non-degeneracy of  $Q$  as a critical point of the action (see [ChGuNaTs07, Kw89, We85]). We refer to [MaMe06, MaMeTs06] for a proof of this statement.

## 2 The Nonlinear Schrödinger Equation

If  $u$  is a solution of (NLS) and  $\varepsilon = u - R_{\omega,v}$ , then  $H$  provides a way to control  $\varepsilon$ , up to a finite number of bad  $L^2$  directions. To deal with these bad  $L^2$  directions, one uses *modulation theory*. Given  $u$  close to  $R_{\omega,v}$ , it consists into slightly shifting (i.e. to modulate) the parameters of the soliton so that  $u$  will remain close to the modulated soliton. By doing so, one gains orthogonality properties. More precisely, we have the following statement.

**Lemma 2.8** (Modulation). *There exists  $\delta > 0$  such that for any  $u \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^d)) \cap \mathcal{C}^1(\mathbb{R}, H^{-1}(\mathbb{R}^d))$ , the following proposition is verified. If for any  $t \in \mathbb{R}$  we have*

$$\inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^d} \|u(t, \cdot) - e^{i\theta} Q_{\omega,v}(\cdot - y)\|_{H^1} \leq \delta,$$

then there exist  $\mathcal{C}^1$  functions of time  $\tilde{\omega} \in (0, \infty)$ ,  $\tilde{y} \in \mathbb{R}^d$ ,  $\tilde{\gamma} \in \mathbb{R}$  such that

$$\left(\varepsilon, \tilde{R}_{\omega,v}\right)_{L^2} = \left(\varepsilon, \nabla \tilde{R}_{\omega,v}\right)_{L^2} = \left(\varepsilon, i\tilde{R}_{\omega,v}\right)_{L^2} = 0,$$

where  $\tilde{R}_{\omega,v}(t) = e^{i\tilde{\gamma}(t)} Q_{\tilde{\omega}(t),v}(\cdot - \tilde{y}(t))$  and  $\varepsilon = u - \tilde{R}_{\omega,v}$ . If moreover  $u$  is a solution to (NLS), the modulation parameters satisfy the following (approximated) dynamical system

$$\left| \partial_t \tilde{\omega} + |\partial_t \tilde{y} - v|^2 + \left| \partial_t \tilde{\gamma} - \left( \tilde{\omega} + \frac{|v|^2}{4} \right) \right|^2 \right| \leq \|\varepsilon\|_{H^1}^2$$

The proof of Lemma 2.8 is based on the Implicit Function Theorem and the equation satisfied by  $\varepsilon$ .

Combining Proposition 2.7 and Lemma 2.8, we obtain the stability result Theorem 2.6.

## 2.6 Modulation

In Section 2.5, the motivation to introduce modulation was to cancel the  $L^2$ -scalar products of the coercivity inequality of Proposition 2.7. We may also interpret the orthogonality conditions in the following way. Given a solution  $u$  of (NLS), we define a modulated version  $v$  of  $u$  by

$$u(t, x) = e^{i(\zeta \cdot y + \sigma)} \lambda^{\frac{1}{p-1}} v(\tau, y), \quad y = \lambda^{\frac{1}{2}} (x - \xi), \quad \frac{\partial \tau}{\partial t} = \lambda,$$

where  $\zeta, \xi, \lambda, \sigma$  are time depending modulation parameters. Then the equation satisfied by  $v$  is

$$\begin{aligned} i v_\tau + (\Delta v - v + |v|^{p-1} v) + \left(1 - \sigma_\tau + \zeta \cdot \xi_\tau \lambda^{\frac{1}{2}} - |\zeta|^2\right) v \\ + i \frac{\lambda_\tau}{\lambda} \left( \frac{1}{p-1} v + \frac{1}{2} y \cdot \nabla v \right) + i \left( 2\zeta - \xi_\tau \lambda^{\frac{1}{2}} \right) \cdot \nabla v - \left( \zeta_\tau + \frac{1}{2} \frac{\lambda_\tau}{\lambda} \zeta \right) \cdot y v = 0 \end{aligned}$$

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It appears that  $v \equiv Q$  (where  $Q$  is  $Q_\omega$  with  $\omega = 1$ ) is a solution if the modulation parameters verify the dynamical system

$$\xi_\tau = 2\frac{\zeta}{\lambda^{\frac{1}{2}}}, \quad \sigma_\tau = |\zeta|^2 + 1, \quad \zeta_\tau = 0, \quad \lambda_\tau = 0,$$

or equivalently in the  $t$  variable

$$\xi_t = 2\zeta\lambda^{\frac{1}{2}}, \quad \sigma_t = |\zeta\lambda^{\frac{1}{2}}|^2 + \lambda, \quad \zeta_t = 0, \quad \lambda_t = 0.$$

Given  $v \in \mathbb{R}^d$ ,  $x_0 \in \mathbb{R}^d$ ,  $\omega \in (0, \infty)$ ,  $\gamma \in \mathbb{R}$ , we have the solution for the dynamical system

$$\xi = vt + x_0, \quad \sigma = \left(\omega + \frac{|v|^2}{4}\right)t + \gamma, \quad \zeta = \frac{v}{2\omega^{\frac{1}{2}}}, \quad \lambda = \omega, \quad (2.6)$$

which indeed corresponds to the parameters of a soliton with speed  $v$ , frequency  $\omega$  and profile  $Q_{\omega,v}$  with initial phase shift  $\gamma$  and initial position  $x_0$ .

Assume now that  $u$  is a solution to (NLS) close to  $R_{\omega,v}$ , and let  $\eta$  be such that  $v = Q + \eta$ . The perturbation  $\eta$  lives at the scale of  $Q$ , whereas the perturbation  $\varepsilon = u - R_{\omega,v}$  lives at the scale of  $R_{\omega,v}$ . One can pass from one to the other by a simple algebraic transform. The equation verified by  $\eta$  is

$$i\eta_\tau + L\eta + N(\eta) + \text{Mod}(\tau)(Q + \eta) = 0$$

where  $L$  is a linear operator acting on the real and imaginary parts of  $\eta$

$$L\eta = -(L_+\Re(\eta) + iL_-\Im(\eta)), \quad L_+ = -\Delta + 1 - pQ^{p-1}, \quad L_- = -\Delta + 1 - Q^{p-1}.$$

The term  $N(\eta)$  encodes the nonlinear terms in  $\eta$  and the term  $\text{Mod}(\tau)(Q + \eta)$  encodes the terms involving the modulation parameters

$$\begin{aligned} \text{Mod}(\tau)(Q + \eta) &= \left(1 - \sigma_\tau + \zeta \cdot \xi_\tau \lambda^{\frac{1}{2}} - |\zeta|^2\right) (Q + \eta) + i \left(2\zeta - \xi_\tau \lambda^{\frac{1}{2}}\right) \cdot \nabla(Q + \eta) \\ &\quad + i \frac{\lambda_\tau}{\lambda} \left(\frac{1}{p-1}(Q + \eta) + \frac{1}{2}y \cdot \nabla(Q + \eta)\right) \\ &\quad - \left(\zeta_\tau + \frac{1}{2}\frac{\lambda_\tau}{\lambda}\zeta\right) \cdot y(Q + \eta). \end{aligned}$$

Setting aside the modulation parameters and the nonlinear terms, we consider the equation

$$i\eta_\tau + L\eta = 0,$$

or equivalently

$$\eta_\tau - iL\eta = 0. \quad (2.7)$$

Since our final goal is to prove stability of the soliton, we have to eliminate any potential source of instability. For the linear equation (2.7), instability may come either from an eigenvalue of  $iL$  with positive real part, or from the generalized kernel of  $iL$ . We focus



## 2 The Nonlinear Schrödinger Equation

now on the generalized kernel. From the analysis of  $L_+$  and  $L_-$  (see e.g. the discussion in the paper [ChGuNaTs07] or the original paper of Weinstein [We85]), we know that the generalized kernel  $N_g$  of  $iL$  (i.e.  $N_g(iL) = \cup_{k \in \mathbb{N}} \ker((iL)^k)$ ) is generated if  $p \neq 1 + \frac{4}{d}$  by

$$\nabla Q, \quad iQ, \quad ixQ, \quad \Lambda Q.$$

where  $\Lambda$  is the scaling operator defined by  $\Lambda Q = \frac{2}{p-1}Q + x \cdot \nabla Q$ . Indeed, we have from direct calculations

$$\begin{aligned} L_+ \nabla Q &= 0, & L_- xQ &= -2\nabla Q, & (xQ, \nabla Q)_{L^2} &\neq 0, \\ L_- Q &= 0, & L_+ \Lambda Q &= -2Q, & (\Lambda Q, Q)_{L^2} &\neq 0, \end{aligned}$$

We may present the result in the form of a diagram

$$\begin{array}{ccccc} L_- & & 0 & & Q & & xQ \\ & & \swarrow & & \searrow & & \swarrow & & \searrow \\ & & & & & & & & \\ & & \swarrow & & \searrow & & \swarrow & & \searrow \\ L_+ & & 0 & & \nabla Q & & \Lambda Q \end{array}$$

If  $p = 1 + \frac{4}{d}$ , then the generalized kernel is bigger and generated by

$$\nabla Q, \quad iQ, \quad ixQ, \quad \Lambda Q, \quad i|x|^2Q, \quad \rho.$$

where  $\rho$  is such that  $L_+ \rho = |x|^2Q$ . This follows from

$$\begin{aligned} L_+ \nabla Q &= 0, & L_- xQ &= -2\nabla Q, & (xQ, \nabla Q)_{L^2} &\neq 0, \\ L_- Q &= 0, & L_+ \Lambda Q &= -2Q, & L_- |x|^2Q &= -4\Lambda Q, & L_+ \rho &= |x|^2Q, & (\rho, Q)_{L^2} &\neq 0, \end{aligned}$$

The corresponding diagram is then

$$\begin{array}{ccccccc} L_- & & 0 & & Q & & xQ & & |x|^2Q \\ & & \swarrow & & \searrow & & \swarrow & & \searrow \\ & & & & & & & & \\ & & \swarrow & & \searrow & & \swarrow & & \searrow \\ L_+ & & 0 & & \nabla Q & & \Lambda Q & & \rho \end{array}$$

We are in the case  $p < 1 + \frac{4}{d}$ . To make sure that no instability stems from the generalized kernel of  $iL$ , we require that  $\eta$  satisfies the orthogonality conditions

$$(\eta, Q)_{L^2} = (\eta, iQ)_{L^2} = (\eta, \nabla Q)_{L^2} = (\eta, ixQ)_{L^2} = 0.$$

At the level of  $\varepsilon$ , this would correspond to

$$(\varepsilon, R_{\omega,v})_{L^2} = (\varepsilon, iR_{\omega,v})_{L^2} = (\varepsilon, \nabla R_{\omega,v})_{L^2} = (\varepsilon, ixR_{\omega,v})_{L^2} = 0,$$

provided we have chosen the parameters of the dynamical system to be as in (2.6).

We have one more orthogonality condition compare to the ones required in Proposition 2.7. This is however not a problem to get this additional orthogonality condition from Lemma 2.8 by modulating on the last available parameter, the speed  $v$ .

## 2.7 Kinks

To end this preliminary section, we say a few words about kinks. Roughly speaking, kinks are solutions of (NLS) with a fixed profile, which is non localized. In particular, when the space dimension is 1, we speak of kinks when the profile has at least one limit different from 0 at spatial infinity. One of the most well-known example of equations admitting kinks is the Gross-Pitaevskii equation

$$iu_t + u_{xx} + (1 - |u|^2)u = 0. \quad (2.8)$$

The kinks appear frequently as traveling waves. A traveling wave is a solution  $u$  of (2.8) of the form  $u(t, x) = \phi_c(x - ct)$ , for some speed  $c \in \mathbb{R}$ . If  $|c| \leq \sqrt{2}$ , then there exist standing waves whose profiles are given by

$$\phi_c(x) = \sqrt{\frac{2 - c^2}{2}} \tanh\left(\frac{x\sqrt{2 - c^2}}{2}\right) + i\frac{c}{\sqrt{2}}.$$

When  $c \neq 0$ , the traveling wave does not vanish and (2.8) can be considered in its hydrodynamical form (see [CaDaSa12] and the references cited therein). In this framework, the problem of stability of traveling waves was first solved for  $c \neq 0$  by Lin [Li02] using the theory of Grillakis, Shatah and Strauss [GrShSt87, GrShSt90]. Still for  $c \neq 0$ , the stability theory was later on revisited by Bethuel, Gravejat and Saut [BeGrSa08] using the variational approach introduced by Cazenave and Lions [CaLi82]. The case  $c = 0$  was much more difficult to handle, due in particular to the fact that no hydrodynamical formulation is available for (2.8) in that case. When  $c = 0$ , it has been proved recently by Bethuel, Gravejat, Saut and Smets [BeGrSaSm08] that the stationary wave is orbitally stable in the energy space (see [DiGa07] for a previous study of the dynamics around the stationary solution), and by Gravejat and Smets [GrSm15] that it is also asymptotically stable. Moreover, the stationary wave is also asymptotically stable [BeGrSm15], in the sense that if a solution starts close to the stationary wave, then it will eventually converge to a traveling wave with speed  $c^*$  close to 0 (up to the symmetries of the equation).

Still in dimension 1 but for generic nonlinearities, the existence of traveling waves with non-localized profiles was investigated by Chiron in [Ch12] and their stability/instability in [Ch13]. In higher dimensions, the question of existence of such traveling wave was a long-time open problem solved recently by Maris (see [Ma13] and the references therein for previous results). Stability/instability is still an open problem for most cases.

In the previously quoted works, all non-localized profiles satisfy a boundary condition on the modulus, e.g.  $|u| \rightarrow 1$  as  $|x| \rightarrow \infty$  in the case of (2.8). In this thesis, we will be concerned with a different type of profiles for the kinks. We work in dimension 1 and assume that the profile is 0 at one side of the real line and its modulus is non-zero on the other side. The following proposition gives conditions that guarantee the existence of such profiles.

**Proposition 2.9.** *Assume Assumption 2.1 and suppose that for some  $\omega > 0$ , there is a (first)  $b > 0$  such that for  $h(s) = \omega s - f(s)$ , we have*

$$h(b) = 0, \quad \int_0^b h(s) ds = 0.$$

*Assume moreover that  $h'(b) > 0$ , and for some  $r \in [p, q]$ ,*

$$|f'(b+s)| + |s||f''(b+s)| \leq C|s|^{r-1} + C|s|^q, \quad \text{for all } s \in \mathbb{R}.$$

*Then, there is a solution  $\phi(s)$  of*

$$\phi'' = \omega\phi - f(\phi)$$

*such that  $0 < \phi(s) < b$ , and*

$$\lim_{s \rightarrow -\infty} \phi(s) = b, \quad \phi'(s) < 0 \quad \text{for all } s \in \mathbb{R}, \quad \phi'(0) = \min \phi', \quad \lim_{s \rightarrow +\infty} \phi(s) = 0,$$

*and that, for any  $0 < a < \min(\omega, h'(b))$ , there is  $C_a > 0$  so that*

$$\mathbf{1}_{s < 0}(b - \phi(s)) + \mathbf{1}_{s \geq 0} \phi(s) + |\phi'(s)| \leq C_a e^{-a|s|}, \quad \text{for all } s \in \mathbb{R}.$$

An example of a situation where the assumptions of Proposition 2.9 are satisfied is given by

$$\omega = 1, \quad f(s) = -s + 4|s|^2s - 3|s|^4s,$$

in which case there exists a unique (up to translation) kink connecting  $b = 1$  to 0, given by the formula

$$\phi(x) = \frac{1}{\sqrt{1 + e^{2x}}}$$

This concludes this preliminary section collecting basic facts on the nonlinear Schrödinger equation.

### 3 Finite and Infinite Soliton Trains

We start by explaining more precisely what we mean by multi-solitons and what is the motivation to study them. As mentioned in Chapter 1, according to the generic *Decomposition Conjecture* for nonlinear dispersive PDEs, a global solution decomposes at large time in a combination of non-scattering structures and a dispersive part. When the non-scattering structures are solitons, we call this conjecture the *Soliton Resolution Conjecture*. For example, in the case of the Korteweg-de Vries equation, for a large class of initial data, all solutions are global and eventually decompose into a finite sum of solitons going to the right and a dispersive part going to the left [EcSc83, Sc86]. Hence investigating the existence and properties of solutions of dispersive equations made of a combination of non-scattering structures is a first step toward a proof of a Decomposition Conjecture.

Multi-solitons are one of the simplest examples of a combination of non-scattering structures. We give now a precise definition, in the case of the Schrödinger equation

$$iu_t + \Delta u + f(u) = 0. \tag{NLS}$$

Consider  $N \in \mathbb{N}$ , take a set of parameters  $(\omega_j, \theta_j, v_j, x_j)_{1 \leq j \leq N} \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$  and profiles  $(\Phi_j)_{1 \leq j \leq N}$  solutions of

$$-\Delta \Phi_j + \omega_j \Phi_j + f(\Phi_j) = 0. \tag{SNLS}$$

Let  $(R_j)_{1 \leq j \leq N}$  be the associated solitary waves, given by formula (2.4). Define the sum of solitons by

$$R(t, x) = \sum_{j=1}^N R_j(t, x).$$

As (NLS) is a nonlinear equation, the sum  $R$  is not a solution of (NLS). What we call a *multi-soliton* is a solution  $u$  of (NLS) on some interval  $[T_0, \infty)$  such that

$$\lim_{t \rightarrow \infty} \|u - R\|_{X([t, \infty), \mathbb{R}^d)} = 0.$$

Here,  $\|\cdot\|_{X([t, \infty), \mathbb{R}^d)}$  denotes some space-time norm. For example, if  $X = L^\infty([t, \infty), H^1(\mathbb{R}^d))$ , then the above condition is simply

$$\lim_{t \rightarrow \infty} \|u(t) - R(t)\|_{H^1} = 0.$$

Several methods are available to obtain multi-solitons. They have been first constructed for (NLS) in the one dimensional cubic focusing case by Zakharov and Shabat [ZaSh72] using the inverse scattering transform method (IST). The IST is a powerful tool to study nonlinear dispersive equations and to exhibit non-trivial nonlinear

dynamics for these equations. However, the IST application is restricted to equations which enjoy a particular algebraic structure, i.e. they are completely integrable. This is for example the case of the Korteweg-de Vries equation and of the cubic nonlinear Schrödinger equation in dimension 1. However, integrability probably does not hold any more in higher dimension, or with different nonlinearities, even of power-type.

A method to construct multi-soliton solutions of non-integrable equations was introduced by Martel, Merle and Tsai [MaMeTs02] for generalized Korteweg-de Vries equations and later developed in the case of  $L^2$ -subcritical nonlinear Schrödinger equations [MaMe06, MaMeTs06]. The method was later fine-tuned to allow the treatment of  $L^2$  supercritical equations [CoMaMe11]. In collaboration with Raphaël Côte [CoLe11], I treated the case of profiles made with excited states. In collaboration with Jacopo Bellazzini and Marco Ghimenti [BeGhLe14], I proved the existence of multi-solitons made of stable solitons for the nonlinear Klein-Gordon equation. The method uses tools usually called *energy techniques*, in the sense that it rely on the use of the second variation of the energy as a Lyapunov functional to control the difference of a solution  $u$  with the soliton sum  $R$ .

In collaboration with Dong Li and Tai-Peng Tsai [LeLiTs15, LeTs14], I introduced another approach to deal with more generic non-scattering structures, in particular with kinks, or with infinite sums of solitons, which are hardly obtainable with the energy method. The technique consists into doing a fixed point argument around the desired profile. To close the fixed point argument, we use a combination of Strichartz estimates and dispersive estimates, and we use a high relative speed hypothesis to control the error between a true solution and a sum of solitons.

When existence of multi-solitons is granted, it opens a wide range of possible directions of investigation.

It is natural to wonder about the classification of the possible multi-soliton solutions. In the case of generalized Korteweg-de Vries equation, this classification was obtained by Martel [Ma05] in the subcritical case and by Combet [Co11] for critical and supercritical cases. In the case of nonlinear Schrödinger equation, only partial results are available [Co14a].

It is also natural to wonder about the stability of multi-solitons. If one of the composing soliton is unstable, it is reasonable to believe that the multi-soliton will also be unstable. In the case of nonlinear Schrödinger equations, I obtained with Raphaël Côte [CoLe11] an instability result for multi-solitons when one of the composing soliton is linearly unstable. To the contrary, we expect that multi-solitons composed of stable solitons will also be stable. A proof of this fact was obtained by Martel, Merle and Tsai [MaMeTs02] for multi-solitons of the subcritical generalized Korteweg-de Vries equations. Later on, the same authors [MaMeTs06] obtained a stability result for the multi-solitons of the nonlinear Schrödinger equation, with restrictive assumptions on the nonlinearity excluding pure power nonlinearities. Indeed, the admissible nonlinearities in [MaMeTs06] needed to be at the same time  $L^2$  critical or supercritical at infinity, and generating stable solitons, two contradictory facts for pure power nonlinearities.

However, I observed that the derivative nonlinear Schrödinger equation

$$iu_t + u_{xx} + i|u|^2u_x = 0$$

is at the same time  $L^2$ -critical and admits stable solitons. In collaboration with Yifei Wu [LeWu18], I proved the stability (under some technical assumptions) of the multi-solitons of the derivative nonlinear Schrödinger equation.

### 3.1 Multi-Solitons

In this section, we present the existence results for multi-solitons of nonlinear Schrödinger and Klein-Gordon equations that we have obtained in [CoLe11, BeGhLe14] and we give some elements of the proofs. In both cases, the proofs follow the general scheme laid down by Martel, Merle and Tsai [MaMeTs02] for the Korteweg-de Vries equation and that we explain now for Schrödinger equations (the scheme being the same for Klein-Gordon equations, mutatis mutandis).

The idea of the method is to adapt the technical tools used for the study of stability of solitons (modulation, coercivity of the second variation of the energy, conservation laws) to the context of multi-solitons. We first choose an increasing sequence of times  $(T_n)_{n \in \mathbb{N}}$  with  $T_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Then we define solutions  $u_n$  to (NLS) with *final data*  $u_n(T_n) = R(T_n)$ . It means that we solve (NLS) backward in time for each  $n$  and that  $u_n$  will exist on an interval of time  $(t_n, T_n)$ . By passing to the limit,  $u_n$  will converge to  $u$ , a multi-soliton solution of (NLS). Two ingredients will be necessary to authorize this passage to the limit. First,  $u_n$  will have to satisfy *uniform estimates* of the form

$$\|u_n(t) - R(t)\| \leq e^{-\alpha\sqrt{\omega_\star}v_\star t}, \quad (3.1)$$

for all  $t$  in the interval  $[T_0, T_n]$ , with  $T_0$  independent of  $n$ , and  $\alpha$ ,  $\omega_\star$  and  $v_\star$  are positive constants which will be specified later on. The second ingredient is a *compactness argument*: the sequence of initial data  $u_n(T_0)$  converges strongly in  $L^2(\mathbb{R}^d)$  to a function  $u_0 \in H^1(\mathbb{R}^d)$ . Thanks to the well-posedness of the Cauchy Problem of (NLS), we can conclude that  $u_0$  gives rise to a solution  $u$  of (NLS) which satisfies (3.1) on  $[T_0, \infty)$ . Hence  $u$  is a multi-soliton.

#### 3.1.1 Excited Multi-Solitons of Nonlinear Schrödinger Equations

In collaboration with Raphaël Côte [CoLe11], I proved the following result.

**Theorem 3.1.** *Let  $f(z) = |z|^{p-1}z$  with  $1 < p < 1 + \frac{4}{(d-2)_+}$ . Consider  $N \in \mathbb{N}$ , take a set of parameters  $(\omega_j, \theta_j, v_j, x_j)_{1 \leq j \leq N} \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$  and profiles  $(\Phi_j)_{1 \leq j \leq N}$  solutions of (SNLS) (with  $\omega$  replaced by  $\omega_j$ ). Let  $(R_j)_{1 \leq j \leq N}$  be the associated solitons. Define*

$$\omega_\star = \min\{\omega_j : j = 1, \dots, N\}, \quad v_\star = \min\{|v_j - v_k| : j, k = 1, \dots, N, j \neq k\}.$$

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Then there exist  $\alpha > 0$  and  $v_{\sharp} \in [0, \infty)$ , such that if  $v_{\star} > v_{\sharp}$  then there exist  $T_0 \in \mathbb{R}$  and a solution  $u$  of (NLS) on  $[T_0, \infty)$  such that

$$\left\| u(t) - \sum_{j=1}^N R_j(t) \right\|_{H^1} \leq e^{-\alpha \sqrt{\omega_{\star} v_{\star} t}} \quad \text{for all } t \in [T_0, \infty).$$

For the sake of simplicity in the argument, the result is stated here only for power-type nonlinearities. It may be extended to other types of nonlinearities, see [CoLe11]. The constants  $\omega_{\star}$ ,  $v_{\star}$  are called the *minimal frequency* and the *minimal relative speed*. For technical reasons, we need the restrictive assumption that the relative speed is large enough (i.e.  $v_{\star} \gg 1$ ). It is an open problem whether or not there exist multi-solitons based on excited states and with small minimal relative speed.

In the rest of this section, we give a streamlined proof of Theorem 3.1.

From now on, we assume that we are given  $N \in \mathbb{N}$ , a set of parameters  $(\omega_j, \theta_j, v_j, x_j)$  in  $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$  and profiles  $(\Phi_j)$  solutions of (SNLS) for  $1 \leq j \leq N$ . Let  $R$  be the corresponding sum of solitons:

$$R(t, x) = \sum_{j=1}^N R_j(t, x).$$

We first remark that the product between two given solitons  $R_j$  and  $R_k$  verifies the following bound

$$|R_j(t, x)R_k(t, x)| \lesssim e^{-\frac{1}{2}\sqrt{\omega_{\star} v_{\star} t}} B(t, x)$$

with  $B \in L_t^\infty L_x^1 \cap L_t^1 L_x^\infty(\mathbb{R}^d, \mathbb{R})$ . This can be easily proved using the exponential decay of the profiles of the solitons. Hence the interaction between two solitons is expected to be exponentially small at time and space infinity.

Take an increasing sequence of times  $(T_n)_{n \in \mathbb{N}}$  with  $T_n \rightarrow \infty$  as  $n \rightarrow \infty$  and define solutions  $(u_n)$  to (NLS) with final data  $u_n(T_n) = R(T_n)$ .

We rely on the two following propositions to perform the proof of Theorem 3.1.

**Proposition 3.2** (Uniform Estimates). *There exist  $\alpha > 0$ ,  $v_{\sharp} > 0$  and  $T_0 \in \mathbb{R}$  independent of  $n$  such if  $v_{\star} > v_{\sharp}$  then for  $n$  large enough  $u_n$  exists on  $[T_0, T_n]$  and satisfies for any  $t \in [T_0, T_n]$  the estimate*

$$\|u_n(t) - R(t)\|_{H^1} \leq e^{-\alpha \sqrt{\omega_{\star} v_{\star} t}}.$$

**Proposition 3.3** (Compactness). *There exists  $u_0 \in H^1(\mathbb{R}^d)$  such that*

$$\lim_{n \rightarrow \infty} \|u_n(T_0) - u_0\|_{L^2} = 0.$$

We postpone the proofs of Propositions 3.2 and 3.3 and prove Theorem 3.1.

*Proof of Theorem 3.1.* Let  $u$  be the solution of (NLS) such that  $u(T_0) = u_0$ . From the well-posedness of the Cauchy Problem in  $H^s(\mathbb{R}^d)$  for some  $s \in (0, 1)$  (see e.g. [CaWe90]), we infer that for any  $t \in [T_0, \infty)$  we have  $u_n(t) \rightarrow u(t)$  strongly in  $L^2(\mathbb{R}^d)$ , and  $u_n(t) \rightarrow$

### 3 Finite and Infinite Soliton Trains

$u(t)$  weakly in  $H^1(\mathbb{R}^d)$ . Hence, from weak lower semi-continuity of the  $H^1$  norm, for any  $t \in [T_0, \infty)$  we have

$$\|u(t) - R(t)\|_{H^1} \leq \liminf_{n \rightarrow \infty} \|u_n(t) - R(t)\|_{H^1} \leq e^{-\alpha\sqrt{\omega_*}v_*t}.$$

Hence  $u$  is the desired solution.  $\square$

As in [MaMe06], the compactness property is proved using the virial identity and assuming the uniform estimates.

*Proof of Proposition 3.3.* Choose  $\delta > 0$ . We want to show that there exists  $r_\delta$  such that for any  $n$  large enough we have

$$\int_{|x| > r_\delta} |u_n(T_0)|^2 dx < \delta.$$

Let  $T_\delta$  be such that  $e^{-\alpha\sqrt{\omega_*}v_*T_\delta} < \sqrt{\delta/8}$ . Therefore, if  $n$  is large enough, we have, by the uniform estimates of Proposition 3.2,

$$\|u_n(T_\delta) - R(T_\delta)\|_{L^2}^2 \leq \frac{\delta}{8}. \quad (3.2)$$

Take  $\tilde{r}_\delta$  such that

$$\int_{|x| > \tilde{r}_\delta} |R(T_\delta)|^2 dx \leq \frac{\delta}{8}.$$

Then we can infer from (3.2) that

$$\int_{|x| > \tilde{r}_\delta} |u_n(T_\delta)|^2 dx \leq \frac{\delta}{2}. \quad (3.3)$$

We want to transfer this smallness up to  $T_0$ . To that purpose, we use a virial argument. Let  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  be a  $\mathcal{C}^1$  cut-off function such that

$$\chi(s) = 0 \text{ for } s < 0, \quad \chi(s) = 1 \text{ for } s > 1, \quad \chi(s) \in [0, 1] \text{ for } s \in \mathbb{R}, \quad \|\chi'\|_{L^\infty} \leq 2.$$

Take  $\hat{r}_\delta > 0$  to be fixed later and set

$$V(t) = \frac{1}{2} \int_{\mathbb{R}^d} |u_n(t)|^2 \chi\left(\frac{|x| - \tilde{r}_\delta}{\hat{r}_\delta}\right) dx.$$

Differentiating in  $t$  and using the equation for  $u_n$ , we get

$$V'(t) = \frac{1}{\hat{r}_\delta} \int_{\mathbb{R}^d} \Im\left(\bar{u}_n(t) \frac{x}{|x|} \cdot \nabla u_n\right) \chi'\left(\frac{|x| - \tilde{r}_\delta}{\hat{r}_\delta}\right) dx. \quad (3.4)$$

By the uniform estimates, we know that  $(u_n)$  is uniformly bounded in  $H^1(\mathbb{R}^d)$  and thus from (3.4) we have

$$|V'(t)| \leq \frac{C}{\hat{r}_\delta}.$$



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Choose now  $\hat{r}_\delta$  such that  $C(T_\delta - T_0)/\hat{r}_\delta < \delta/2$ . Then

$$V(T_0) - V(T_\delta) = \int_{T_\delta}^{T_0} V'(t) dt \leq \frac{C(T_\delta - T_0)}{\hat{r}_\delta} < \frac{\delta}{2}. \quad (3.5)$$

Define  $r_\delta = \tilde{r}_\delta + \hat{r}_\delta$ . Combining (3.3), (3.5) and the definition of  $\chi$ , we have

$$\int_{|x| > r_\delta} |u_n(T_0)| \leq \delta.$$

This is the desired conclusion and the sequence  $(u_n(T_0))$  is therefore compact in  $L^2(\mathbb{R}^d)$ , which finishes the proof.  $\square$

The rest of this section is devoted to the proof of the uniform estimates. It relies on the following bootstrap argument: given  $n \in \mathbb{N}$ , if for any  $t$  in some interval  $[t_0, T_n] \subset [T_0, T_n]$ , the following estimate holds:

$$\|u_n(t) - R(t)\|_{H^1} \leq e^{-\alpha\sqrt{\omega_*}v_*t}, \quad (3.6)$$

then in fact on  $[t_0, T_n]$  the following better estimate holds:

$$\|u_n(t) - R(t)\|_{H^1} \leq \frac{1}{2}e^{-\alpha\sqrt{\omega_*}v_*t}. \quad (3.7)$$

Here,  $T_0$  is a time before which we cannot run the bootstrap anymore and is determined by the proof. Proposition 3.2 is then an easy consequence of the bootstrap property.

From now on we assume that we are given  $n \in \mathbb{N}$ , that (3.6) holds on some interval  $[t_0, T_n] \subset [T_0, T_n]$  for  $T_0$  to be determined and we want to prove that in fact (3.7) holds on  $[t_0, T_n]$ .

Our task is to find a way to control the difference  $u_n - R$  in  $H^1(\mathbb{R}^d)$ . If there were only one soliton, we would know how to control this difference: the tools used for the proof of stability (modulation and energy control) are designed for that purpose. The idea for multi-solitons will be to use these tools for each of the composing solitons. For that, we need a localization procedure.

Let  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  be such that  $\phi(s) = 0$  if  $s \leq -1$ ,  $\phi(s) = 1$  if  $s \geq 1$ ,  $0 \leq \phi(s) \leq 1$  for any  $s \in \mathbb{R}$  and  $\|\phi'\|_{L^\infty} \leq 2$ .

Recall that each soliton  $R_j$  is centered on the line  $x_j + v_j t$ . For convenience, we choose a particular direction of propagation (see e.g. [CoLe11, Claim 13]): without loss of generality, we assume that the first components of the speeds verify

$$v_1^1 < \dots < v_j^1 < \dots < v_N^1,$$

where the superscript <sup>1</sup> denotes the first component.

We define the bisector lines by

$$m_j(t) = \frac{v_j^1 + v_{j+1}^1}{2}t + \frac{x_j^1 + x_{j+1}^1}{2},$$

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and a partition of unity  $(\psi_j)$  (see Figure 3.1) with the  $j$ -th member weighting around the  $j$ -th soliton by

$$\begin{aligned}\psi_1(t, x) &= \phi\left(\frac{x^1 - m_1(t)}{\sqrt{t}}\right), \\ \psi_j(t, x) &= \phi\left(\frac{x^1 - m_j(t)}{\sqrt{t}}\right) - \phi\left(\frac{x^1 - m_{j-1}(t)}{\sqrt{t}}\right) \text{ for } j = 2, \dots, N-1, \\ \psi_N(t, x) &= 1 - \phi\left(\frac{x^1 - m_{N-1}(t)}{\sqrt{t}}\right).\end{aligned}$$

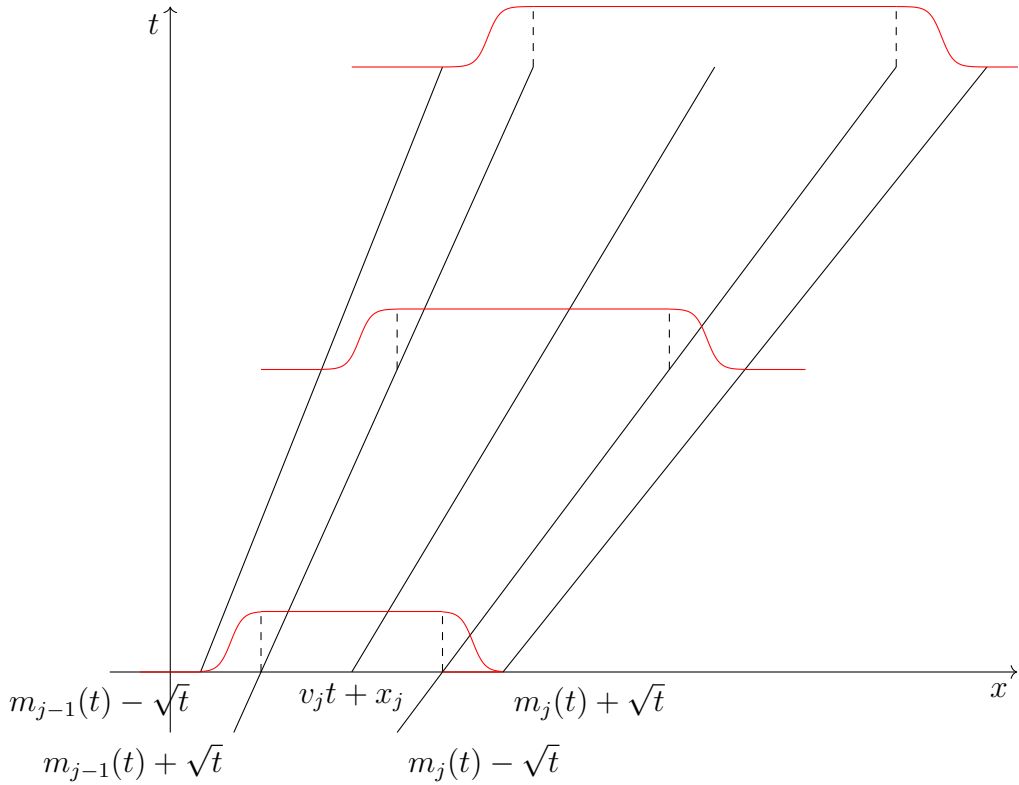


Figure 3.1: Schematic representation for the partition of unity  $(\psi_j)$  (in red)

We can now define *localized quantities* which will turn out to be almost conserved. For  $j = 1, \dots, N$ , let

$$M_j(t, u) = \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \psi_j dx, \quad P_j(t, u) = \frac{1}{2} \int_{\mathbb{R}^d} \Im(u \nabla \bar{u}) \psi_j dx.$$

In the case of a single soliton, the action functional  $S$  defined in (2.5) was the suitable

tool to deal with stability. For multi-solitons, we define

$$\mathcal{S}(t, u) = E(u) + \sum_{j=1}^N \left( \omega_j + \frac{|v_j|^2}{4} \right) M_j(t, u) + \sum_{j=1}^N v_j \cdot P_j(t, u).$$

We also introduce a functional which will play the role of the Hessian for  $\mathcal{S}$ : define

$$\begin{aligned} \mathcal{H}(t, \varepsilon) &= \|\varepsilon\|_{H^1}^2 - \sum_{j=1}^N \int_{\mathbb{R}^d} (|R_j|^{p-1} |\varepsilon|^2 + (p-1)|u|^{p-2} \Re(u\bar{\varepsilon})^2) dx \\ &\quad + \sum_{j=1}^N \omega_j \int_{\mathbb{R}^d} |\varepsilon|^2 \psi_j dx + \sum_{j=1}^N v_j \cdot \int_{\mathbb{R}^d} \Im(\varepsilon \nabla \bar{\varepsilon}) \psi_j dx. \end{aligned}$$

Roughly speaking, when  $\varepsilon$  is concentrated close to the  $j$ -th soliton, then  $\mathcal{H}(t, \varepsilon)$  is like  $H_j(t, \varepsilon)$ , where we have set

$$H_j(t, \varepsilon) = \langle S''(R_j) \varepsilon, \varepsilon \rangle.$$

In fact, we have the following Taylor-like expansion:

$$\mathcal{S}(t, R + \varepsilon) - \mathcal{S}(t, R) = \mathcal{H}(t, \varepsilon) + o(\|\varepsilon\|_{H^1}^2) + O(e^{-\alpha\sqrt{\omega_*}v_*t}).$$

If  $R_j$  has a stable ground state profile, then a fairly precise description of the defect of coercivity of  $H_j$  is given by Proposition 2.7. If not, then all we can say is that  $H_j$  is coercive up to a finite number of  $L^2$  directions. Precisely, there exist  $(\xi_j^k)_{k=1, \dots, K_j} \subset L^2(\mathbb{R}^d)$  such that

$$H_j(t, \varepsilon) \gtrsim \|\varepsilon\|_{H^1}^2 - \sum_{k=1}^{K_j} (\xi_j^k, \varepsilon)_{L^2}^2.$$

Using the exponential localization of solitons and the coercivity property for each  $H_j$ , we can prove the following lemma.

**Lemma 3.4** (Coercivity). *For every  $\varepsilon \in H^1(\mathbb{R}^d)$  we have*

$$\mathcal{H}(t, \varepsilon) \gtrsim \|\varepsilon\|_{H^1}^2 - \sum_{j=1}^N \sum_{k=1}^{K_j} (\xi_j^k, \varepsilon)_{L^2}^2 - O(e^{-\alpha\sqrt{\omega_*}v_*t}). \quad (3.8)$$

The above Lemma provides a control on the difference between  $\varepsilon_n = u_n - R$ , provided we are able to prove that the  $L^2$ -products in (3.8) are small for  $\varepsilon_n$ . If all  $R_j$  have stable ground state profiles, then it can be done via modulation (as it will be the case in Section 3.1.2 for multi-solitons of the Klein-Gordon equation). If not, then we can obtain the smallness of the  $L^2$ -scalar product by the following argument. Using the equation verified by  $\varepsilon$  and the bootstrap assumption, we have

$$\left| \frac{1}{2} \frac{\partial}{\partial t} \|\varepsilon(t)\|_{L^2}^2 \right| \leq \Re \int_{\mathbb{R}^d} |\varepsilon(t) \bar{\varepsilon}_t(t)| dx \lesssim \|\varepsilon(t)\|_{H^1}^2 \lesssim e^{-2\alpha\sqrt{\omega_*}v_*t}.$$

Integrating in time and remembering that  $\varepsilon(T_n) = 0$ , we get

$$\|\varepsilon(t)\|_{L^2}^2 \lesssim \frac{1}{2\alpha\sqrt{\omega_\star v_\star}} e^{-2\alpha\sqrt{\omega_\star v_\star}t}.$$

Choosing  $v_\star$  large enough gives a smallness on  $\|\varepsilon\|_{L^2}$ , which in turn allows to control the bad  $L^2$ -directions for coercivity.

### 3.1.2 Multi-Solitons of the Klein-Gordon Equation

In this section, I present a result of existence of multi-solitons for the Klein-Gordon equation obtained in collaboration with Jacopo Bellazzini and Marco Ghimenti. The composing solitons are made of stable ground states. The existence of the multi-soliton is obtained with the energy technique. This result was the main result of [BeGhLe14]. In [CoMu14], Côte and Muñoz treated another case, where multi-solitons are composed of unstable real valued ground states traveling waves. More recently, Côte and Martel [CoMa18] obtained an existence result for multi-solitons made of real valued excited states.

We will be working in this section with the nonlinear Klein-Gordon equation with power-type nonlinearity:

$$\begin{cases} u_{tt} - \Delta u + mu - |u|^{p-1}u = 0, \\ u|_{t=0} = u_1^0, \quad u_t|_{t=0} = u_2^0. \end{cases} \quad (\text{NLKG})$$

Here  $m > 0$ , the unknown  $u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$  is a complex valued function of time  $t \in \mathbb{R}$  and space  $x \in \mathbb{R}^d$ , and the exponent  $1 < p < 1 + \frac{4}{(d-2)_+}$  is energy subcritical. Remark that we are considering the Klein-Gordon equation with *complex valued* solutions, whereas the works [CoMa18, CoMu14] are concerned only with real valued solutions.

It is common and convenient to express (NLKG) in its Hamiltonian form. Setting  $U = (u_1, u_2)^T = (u, u_t)^T$ , the equation becomes

$$U_t = JE'_{\text{KG}}(U), \quad (3.9)$$

where the *energy*  $E_{\text{KG}}$  and the symplectic matrix  $J$  are defined by

$$E_{\text{KG}}(U) = \frac{1}{2}\|u_2\|_{L^2}^2 + \frac{1}{2}\|\nabla u_1\|_{L^2}^2 + \frac{m}{2}\|u_1\|_{L^2}^2 - \frac{1}{p+1}\|u_1\|_{L^{p+1}}^{p+1}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

From the Hamiltonian formulation, we immediately infer that the energy  $E_{\text{KG}}$  is (at least formally) a conserved quantity for the flow of (NLKG). Moreover, the invariances by translation and gauge transformation yield two other conserved quantities, the *momentum*  $P_{\text{KG}}$  and the *charge*  $Q_{\text{KG}}$ , given by

$$P_{\text{KG}}(U) = \Re \int_{\mathbb{R}} \nabla u_1 \bar{u}_2 dx, \quad Q_{\text{KG}}(U) = \Im \int_{\mathbb{R}} u_1 \bar{u}_2 dx.$$

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The Cauchy Theory for (NLKG) has been worked out in the 80s by Ginibre and Velo [GiVe85, GiVe89] (see also the recent review in [KiStVi14, Section 2]). In particular, for any initial data  $U_0 = (u_1^0, u_2^0)^T \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$  there exist  $0 < T \leq \infty$  and a unique solution  $U \in \mathcal{C}([0, T], H^1 \times L^2) \cap \mathcal{C}^1([0, T], L^2 \times H^{-1})$  of (3.9) such that  $U(0) = U_0$ . In addition to the existence and uniqueness of a solution, we also have continuous dependence on the initial data, and (rigorous) conservation of energy, charge and momentum. Our proof requires in fact a slightly different result of well-posedness in  $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$  for some  $s < 1$ , which was obtained by Nakamura and Ozawa [NaOz01].

An interesting property of the linear Klein-Gordon equation which survives in its nonlinear version is the *uniqueness in light cones*. Given  $U_0 \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$  an initial data and two solutions  $U$  and  $\tilde{U}$  of (3.9) on  $[0, T)$  for some  $T > 0$  with  $\tilde{U}(0) = U(0) = U_0$  in  $\{x \in \mathbb{R}^d : |x - x_0| < T\}$  for some  $x_0 \in \mathbb{R}^d$ , then  $U \equiv \tilde{U}$  on the backward light cone

$$\{(t, x) \in [0, T) \times \mathbb{R}^d : |x - x_0| < T - t\}.$$

As a consequence, *finite speed of propagation* holds for (3.9). Precisely, let  $U = (u_1, u_2)^T \in \mathcal{C}((-\infty, T^*], H^1 \times L^2)$ . There exists a constant  $C_0 > 0$  such that the following property is satisfied. For all  $\varepsilon > 0$  and  $M > 0$  for which

$$\int_{|x| > M} (|\nabla u_1(T^*)|^2 + |u_1(T^*)|^2 + |u_2(T^*)|^2) dx < \varepsilon,$$

and for all  $t \in (-\infty, T^*]$ , we have

$$\int_{|x| > 2M + (T^* - t)} (|\nabla u_1(T)|^2 + |u_1(T)|^2 + |u_2(T)|^2) dx < C_0 \varepsilon.$$

The nonlinear Klein-Gordon equation is invariant by *Lorentz transform*. Precisely, for  $v \in \mathbb{R}^d$  with  $|v| < 1$ , define the Lorentz transform of the function  $U = (u_1, u_2)$  of  $t$  and  $x$  by

$$\mathcal{L}_v U(t, x) = \begin{pmatrix} u_1(\tau, y) \\ \gamma u_2(\tau, y) - v \cdot \nabla_y u_1(\tau, y) \end{pmatrix},$$

where the new variables  $(\tau, y)$  are defined by

$$\tau = \gamma(t - v \cdot x), \quad y = x - x_v + \gamma(x_v - vt),$$

with the Lorentz factor  $\gamma$  and the projection  $x_v$  given by

$$\gamma = \frac{1}{\sqrt{1 - |v|^2}}, \quad x_v = \frac{x \cdot v}{|v|^2} v.$$

If  $U$  is a solution of (3.9), then  $\mathcal{L}_v U$  is also a solution of (3.9).

The nonlinear Klein-Gordon equation admits stationary solutions and *standing waves*. They are solutions of the form  $U(t, x) = e^{i\omega t} \Phi_\omega(x)$  where (a priori)  $\omega \in \mathbb{R}$  and  $\Phi_\omega = (\phi_1, \phi_2)^T$  solves

$$\begin{cases} -\Delta \phi_1 + m\phi_1 - |\phi_1|^{p-1} \phi_1 + i\omega \phi_2 = 0, \\ \phi_2 - i\omega \phi_1 = 0. \end{cases}$$

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Therefore,  $\Phi_\omega$  is of the form

$$\Phi_\omega = \begin{pmatrix} \phi_\omega \\ i\omega\phi_\omega \end{pmatrix},$$

with  $\phi_\omega$  verifying

$$-\Delta\phi_\omega + (m - \omega^2)\phi_\omega - |\phi_\omega|^{p-1}\phi_\omega. \quad (3.10)$$

We recognize here the equation for the soliton profiles of the nonlinear Schrödinger equation (SNLS), with  $\omega$  replaced by  $m - \omega^2$ . In particular, it is well-known since the work of Berestycki and Lions [BeLi83-1], Kwong [Kw89] and Gidas, Ni and Nirenberg [GiNiNi79] that this equation has a unique positive radially symmetric ground state if  $m - \omega^2 > 0$ .

We assume in the sequel that

$$|\omega| < \sqrt{m}.$$

Using a Lorentz boost, one can convert a standing wave  $U(t, x) = e^{i\omega t}\Phi_\omega(x)$  into a solitary wave with speed  $v$  by setting

$$U(t, x) = \mathcal{L}_v(e^{i\omega t}\Phi_\omega(x)) = e^{i\frac{\omega}{\gamma}t}\Phi_{\omega,v}(x - vt).$$

The profile  $\Phi_{\omega,v}$  satisfies

$$\begin{cases} -\Delta\phi_1 + m\phi_1 - |\phi_1|^{p-1}\phi_1 + i\frac{\omega}{\gamma}\phi_2 - v \cdot \nabla\phi_2 = 0, \\ \phi_2 - i\frac{\omega}{\gamma}\phi_1 + v \cdot \nabla\phi_1 = 0. \end{cases}$$

It is a critical point of the *action* functional defined by

$$S_{\text{KG}} = E_{\text{KG}} + \frac{\omega}{\gamma}Q_{\text{KG}} + v \cdot P_{\text{KG}}.$$

We call  $\Phi_{\omega,v}$  a *ground state* if the underlying solution  $\phi_\omega$  of (3.10) is a ground state.

The *orbital stability* theory for the solitary waves of (NLKG) was initiated by Shatah and Strauss [Sh85, ShSt85] and completed by Stuart [St01]. Solitary waves with ground state profiles are orbitally stable if and only if  $(\omega, v)$  are in the following range of parameters

$$\mathcal{O}_{\text{stab}} = \left\{ (\omega, v) \in \mathbb{R} \times \mathbb{R}^d : |\omega| < \sqrt{m}, |v| < 1, \frac{1}{1 + \frac{4}{p-1} - d} < \frac{\omega^2}{m} \right\}.$$

This set is empty if  $p \geq 1 + \frac{4}{d}$ . If  $p < 1 + \frac{4}{d}$ , then there exists  $0 < \omega_c < \sqrt{m}$  such that for all  $\omega$  verifying  $\omega_c < |\omega| < \sqrt{m}$  and for any  $v \in \mathbb{R}^d$  with  $|v| < 1$  we have  $(\omega, v) \in \mathcal{O}_{\text{stab}}$ .

Our result on the existence of multi-solitons for (NLKG) is the following.

**Theorem 3.5.** *Assume that  $1 < p < 1 + \frac{4}{d}$ . Take  $(\omega_j, v_j)_{1 \leq j \leq N} \subset \mathcal{O}_{\text{stab}}$  and  $(\Phi_j)_{1 \leq j \leq N}$  the corresponding ground states. Define*

$$R_j(t, x) = e^{i\frac{\omega_j}{\gamma}t}\Phi_j(x - v_j t),$$

$$v_\star = \min\{|v_j - v_k| : j, k = 1, \dots, N, j \neq k\}, \quad \omega_\star = \max\{|\omega_j| : j = 1, \dots, N\}.$$

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There exists  $\alpha > 0$  such that if  $v_\star > 0$  then there exist  $T_0 \in \mathbb{R}$  and  $U$  solution of (3.9) on  $[T_0, \infty)$  such that

$$\left\| U(t) - \sum_{j=1}^N R_j(t) \right\|_{H^1 \times L^2} \leq e^{-\alpha \sqrt{m-\omega^\star} v_\star}.$$

The proof of Theorem 3.5 is similar to the one introduced by Martel and Merle [MaMe06] for the nonlinear Schrödinger equation. Our main contribution consists in obtaining a coercivity property adapted to the Klein-Gordon setting. The proof goes as follows. Let  $(T_n)$  be an increasing sequence of final times such that  $T_n \rightarrow \infty$  as  $n \rightarrow \infty$ . We define an approximate sequence of multi-solitons  $U_n$  by setting  $U_n(T_n) = \sum_{j=1}^N R_j(T_n)$  and solving (3.9) backward in time. The crux of the proof is to show that there exists  $T_0$ , independent of  $n$ , such that  $U_n$  exists on the interval  $[T_0, T_n]$ . To this aim, we rely on uniform estimates on the distance between the approximate multi-solitons  $(U_n)$  and the profile  $\sum_{j=1}^N R_j$ . These estimates are obtained from the combination of a coercivity property of a global linearized action and modulation theory for the control of a few negative directions. A compactness argument finally shows the existence of a solution  $U_\infty$  of (3.9) such that  $U_n \rightarrow U_\infty$  uniformly. The solution  $U_\infty$  is the desired solution.

One of the highlights of the proof is obtaining several variational characterizations of the ground states  $\Phi_{\omega, v}$ . Before stating the result, we define different critical level for the action functional  $S_{\text{KG}}$ . The *mountain pass level* is defined by

$$MP = \inf_{\eta \in \Gamma} \sup_{s \in [0, 1]} S_{\text{KG}}(\eta(s)), \quad \Gamma = \{\eta \in \mathcal{C}([0, 1]; H^1 \times L^2) : \eta(0) = 0, S_{\text{KG}}(\eta(1)) < 0\}.$$

The *Nehari level* is given by

$$NL = \min \{S_{\text{KG}}(W) : \langle S'_{\text{KG}}(W), W \rangle = 0, W \in H^1 \times L^2 \setminus \{(0, 0)^T\}\}.$$

The *least energy level* is defined by

$$LE = \min \{S_{\text{KG}}(W) : S'_{\text{KG}}(W) = 0, W \in H^1 \times L^2 \setminus \{(0, 0)^T\}\}.$$

**Proposition 3.6.** *Let  $1 < p < 1 + \frac{4}{(d-2)_+}$  and  $(\omega, v) \in \mathbb{R} \times \mathbb{R}^d$  such that  $\omega^2 < m$  and  $|v| < 1$ . Then  $\Phi_{\omega, v}$  verifies*

$$S_{\text{KG}}(\Phi_{\omega, v}) = MP = NL = LE.$$

The variational characterizations of Proposition 3.6, along with the non-degeneracy of the solution of (3.10) allow us to obtain the following coercivity property.

**Proposition 3.7.** *For every  $\Upsilon \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$  we have*

$$\|\Upsilon\|_{H^1 \times L^2}^2 \lesssim \langle S''_{\text{KG}}(\Phi_{\omega, v})\Upsilon, \Upsilon \rangle + (\Upsilon, \nabla \Phi_{\omega, v})_{L^2 \times L^2}^2 + (\Upsilon, iJ\Phi_{\omega, v})_{L^2 \times L^2}^2 + (\Upsilon, i\Phi_{\omega, v})_{L^2 \times L^2}^2.$$

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In Section 3.1.1, the bad  $L^2$ -directions of the coercivity result Lemma 3.4 were dealt with using a large relative speed assumption. In the setting of the Klein-Gordon equation, this kind of trick is not possible anymore, since solitary waves cannot travel faster than the speed of light (i.e.  $|v| \leq 1$  in the present normalization). This is why we work here with ground states and we eliminate the bad  $L^2$ -directions in the classical way by using modulation theory. We can obtain the following uniform estimates.

**Proposition 3.8** (Uniform Estimates). *There exist  $\alpha = \alpha(d, N) > 0$ , and  $T_0 \in \mathbb{R}$  (independent of  $n$ ) such that for  $n$  large enough the solution  $U_n$  of (NLKG) with  $U_n(T^n) = \sum_{j=1}^N R_j(T^n)$  exists on  $[T_0, T^n]$  and satisfies for all  $t \in [T_0, T^n]$  the estimate*

$$\left\| U_n(t) - \sum_{j=1}^N R_j(t) \right\|_{H^1 \times L^2} \leq e^{-\alpha \sqrt{m-\omega_*^2} v_* t}$$

Proposition 3.8 establishes that the approximate multi-solitons  $U_n$  all satisfy the desired estimate on time intervals of the form  $[T_0, T^n]$ , with  $T_0$  independent of  $n$ .

The second ingredient of the proof of Theorem 3.5 is an  $H^1 \times L^2$ -compactness property of the sequence of initial data of the approximate multi-solitons.

**Lemma 3.9** (Compactness). *Let  $T_0$  be given by Proposition 3.8. For any  $\varepsilon > 0$  there exists  $M_\varepsilon$  such that for any  $n$  large enough  $U_n$  verifies*

$$\int_{|x| > M_\varepsilon} |\nabla U_{n,1}(T_0)|^2 + |U_{n,1}(T_0)|^2 + |U_{n,2}(T_0)|^2 dx \leq \varepsilon.$$

The argument for the proof of Lemma 3.9 is different from the similar result in the Schrödinger equation case (Proposition 3.3). Indeed, we benefit with the Klein-Gordon equation of the Finite Propagation Speed, which is not the case for Schrödinger equations where one has to use virial identities.

*Proof of Lemma 3.9.* The result is a consequence of the Finite Speed of Propagation and the uniform estimates of Proposition 3.8. Indeed, take  $\varepsilon > 0$  and let  $T^*$  be such that  $e^{-\alpha \sqrt{m-\omega_*^2} v_* T^*} < \sqrt{\frac{\varepsilon}{2}}$ . Then it follows from Proposition 3.8 that for  $n$  large enough we have

$$\|U_n(T^*) - R(T^*)\|_{H^1 \times L^2}^2 \leq \frac{\varepsilon}{2}. \quad (3.11)$$

By exponential decay of the sum of solitons, there exists  $\tilde{M}_\varepsilon$  such that

$$\int_{|x| > \tilde{M}_\varepsilon} |\nabla (R(T^*)_1)|^2 + |(R(T^*)_1)|^2 + |(R(T^*)_2)|^2 dx \leq \frac{\varepsilon}{2}. \quad (3.12)$$

Combining (3.11) and (3.12), we get

$$\int_{|x| > \tilde{M}_\varepsilon} |\nabla U_{n,1}(T^*)|^2 + |U_{n,1}(T^*)|^2 + |U_{n,2}(T^*)|^2 dx \leq \varepsilon.$$



By Finite Speed of Propagation, this implies

$$\int_{|x| > 2\tilde{M}_\varepsilon + (T^* - T_0)} |\nabla U_{n,1}(T_0)|^2 + |U_{n,1}(T_0)|^2 + |U_{n,2}(T_0)|^2 dx \leq \varepsilon.$$

Setting  $M_\varepsilon = 2\tilde{M}_\varepsilon + (T^* - T_0)$  finishes the proof.  $\square$

Combining the Compactness Argument Lemma 3.9 and the Uniform Estimates Proposition 3.8, we prove Theorem 3.5 in a similar fashion as Theorem 3.1.

## 3.2 Infinite Trains

We have seen in Section 3.1.2 how to construct a multi-soliton solution of (NLS). It is natural to ask whether or not there may exist solutions of (NLS) which behave at large time like an *infinite* sum of solitons. Before our works [LeLiTs15, LeTs14], answers to this question existed only for integrable equations. The energy method is not suitable to answer this question, in particular because the infinite sum profile may not be in the energy space, and would have infinite energy (or, even worse, infinite  $L^2$  norm). This is the reason why, in collaboration with Dong Li and Tai-Peng Tsai [LeLiTs15, LeTs14], I developed techniques based on a fixed point argument to overcome this difficulty.

For the sake of simplicity, we restrict ourselves to the power-type nonlinearity case and refer the interested reader to [LeTs14] for results with generic nonlinearities.

We want to construct an infinite train solution of (NLS), i.e. a solution which verifies

$$\left\| u - \sum_{j=1}^{\infty} R_j \right\| \rightarrow 0 \quad \text{when } t \rightarrow \infty,$$

in some (space-time) norm to be specified. To simplify the analysis, we choose the solitons in the following way. We take a profile  $\Phi \in H^1(\mathbb{R}^d) \setminus \{0\}$  solution of

$$-\Delta\Phi + \Phi - |\Phi|^{p-1}\Phi = 0,$$

and infinite sequences of frequencies  $(\omega_j)_{j \in \mathbb{N}} \subset (0, \infty)$  and speeds  $(v_j)_{j \in \mathbb{N}} \subset \mathbb{R}^d$ . We define the profiles

$$\Phi_{\omega_j} = \omega_j^{\frac{1}{p-1}} \Phi \left( \omega_j^{\frac{1}{2}} x \right), \quad \Phi_{\omega_j, v_j} = e^{\frac{1}{2} v_j \cdot x} \Phi_{\omega_j},$$

and we define the solitons as

$$R_j(t, x) = e^{i \left( \omega_j t - \frac{|v_j|^2}{4} t + \frac{1}{2} v_j \cdot x \right)} \Phi_{\omega_j}(x - v_j t) = e^{i \left( \omega_j + \frac{|v_j|^2}{4} \right) t} \Phi_{\omega_j, v_j}(x - v_j t).$$

For the sake of simplicity in notation we have removed the initial phases and positions in the definition of the solitons. Instead of having a single profile  $\Phi$ , we could have drawn the profile of each soliton from a finite set of profiles  $\{\Phi_1, \dots, \Phi_N\}$ , or even from an infinite set, provided we assume some bounds on the profiles.

The train profile is simply

$$R_\infty(t, x) = \sum_{j=1}^{\infty} R_j(t, x).$$

This definition requires a little more care than for the finite sum of solitons. First of all, it may very well be that the sum is infinite. Indeed, we have

$$\|R_j(t)\|_{L^\infty} = \|\Phi_{\omega_j}\|_{L^\infty} \simeq \omega_j^{\frac{1}{p-1}}.$$

To have a proper definition for the profile  $R_\infty$ , we require at least that  $\omega_j \rightarrow 0$  as  $j \rightarrow \infty$  and  $\sum_{j=1}^{\infty} \omega_j^{\frac{1}{p-1}} < \infty$ . In fact, we will need some integrability of the profile for our proof. For  $q \in [1, \infty]$ , we have

$$\|R_j(t)\|_{L^q} = \|\Phi_{\omega_j}\|_{L^q} \simeq \omega_j^{\frac{1}{p-1} - \frac{d}{2q}}.$$

So integrability of  $R_\infty$  in  $L^q(\mathbb{R}^d)$  holds if  $\sum_{j=1}^{\infty} \omega_j^{\frac{1}{p-1} - \frac{d}{2q}} < \infty$ . Since  $\omega_j \rightarrow 0$  as  $j \rightarrow \infty$ , the previous sum can be finite only if  $\frac{1}{p-1} - \frac{d}{2q} > 0$ , i.e.  $q > \frac{(p-1)d}{2}$ . In particular, if  $p > 1 + \frac{4}{d}$ , then the profile  $R_\infty$  *cannot* be in  $L^2(\mathbb{R}^d)$ . We will require the following assumption.

**Assumption 3.10** (Integrability). *There exists  $r_0 \in \left(\frac{d(p-1)}{2}, p+1\right)$  such that*

$$\sum_{j=1}^{\infty} \omega_j^{\frac{1}{p-1} - \frac{d}{2r_0}} < \infty.$$

*Remark 3.11.* The lack of integrability of the profile  $R_\infty$  could be even worse when one considers its spatial derivative. Indeed,

$$\|\nabla R_j(t)\|_{L^\infty} = \|v_j \cdot \nabla \Phi_{\omega_j} + \Phi_{\omega_j}\|_{L^\infty} \simeq (|v_j| + 1)\omega_j^{\frac{1}{p-1}},$$

and the appearance of the speed  $|v_j|$  weakens the decay if the sequence  $(v_j)$  is unbounded.

In the finite multi-soliton case, it is enough to assume that the speeds are different to ensure that the interactions between solitons is exponentially decaying in time. However, with infinitely many solitons, the separation of speeds requirement is not anymore sufficient to ensure the exponential smallness of the interactions. Recall that  $\Phi$  decays at least like  $e^{-\frac{1}{2}|x|}$ , therefore the interaction between two solitons is measured by

$$|R_j(t, x)R_k(t, x)| \lesssim \omega_j^{\frac{1}{p-1}} \omega_k^{\frac{1}{p-1}} e^{-\frac{1}{2}\omega_j|x-v_jt|} e^{-\frac{1}{2}\omega_k|x-v_kt|} \leq \max\{\omega_j, \omega_k\}^{\frac{2}{p-1}} e^{\frac{1}{2} \min\{\omega_j, \omega_k\}(|v_j-v_k|t)}.$$

To ensure that the constant in the exponential remains strictly positive, we will assume the following control.

**Assumption 3.12** (High Relative Speeds). *The following holds:*

$$v_\star = \min_{j,k \in \mathbb{N}, j \neq k} \omega_j |v_j - v_k| > 0.$$

Formally, we look for a solution  $u = R_\infty + \eta$  of (NLS). Since each  $R_j$  satisfies (NLS), this means that we want  $\eta$  to verify

$$i\eta_t + \Delta\eta + f(R_\infty + \eta) - f(R_\infty) = - \left( f(R_\infty) - \sum_{j=0}^{\infty} f(R_j) \right).$$

In Duhamel formulation with data 0 at infinity, the equation for  $\eta$  becomes

$$\eta(t) = \Psi(\eta)(t) = -i \int_t^\infty e^{-i(t-s)\Delta} ((f(R_\infty + \eta) - f(R_\infty)) + H)(s) ds,$$

where  $H = f(R_\infty) - \sum_{j=0}^{\infty} f(R_j)$ . (3.13)

The main result of this section is the following.

**Theorem 3.13** (Infinite Trains). *Assume Assumptions 3.10 and 3.12. There exist constants  $\alpha > 0$  and  $v_\sharp \gg 1$  such that if  $v_\star > v_\sharp$  then there exists a (unique) solution  $\eta \in S(0, \infty)$  to (3.13) such that for all  $t \geq 0$  we have*

$$\|\eta\|_{S(t, \infty)} + \|\eta(t)\|_{L^{p+1}} \leq e^{-\alpha v_\star t}.$$

Theorem 3.13 gives us a solution  $u = R_\infty + \eta$  of (NLS) which behaves at large time as an infinite train of solitary waves. Here, the notion of solution has to be understood in a weaker sense, as the function  $u = R_\infty + \eta$  is only in  $L_t^\infty L_x^{(p+1)^-} \cap L_{tx}^\infty$ .

The uniqueness holds only in the class of functions  $\eta$  such that

$$\|\eta\|_{S(t, \infty)} + \|\eta(t)\|_{L^{p+1}} \leq e^{-\alpha v_\star t}.$$

*Remark 3.14.* • We have the following choice of parameters verifying Assumption 3.10 and  $v_\star > 0$ :

$$\omega_j = 2^{-j}, \quad v_j = 2^j \bar{v}, \quad \bar{v} \in \mathbb{R}^d, \quad |\bar{v}| = v_\star.$$

- Assumptions 3.10 and 3.12 are naturally satisfied if the profile  $R_\infty$  is replaced by a finite sum of solitons  $R_N$ , and the results of Theorem 3.13 also hold under the same hypotheses. In that case, we recover the results obtained in Theorem 3.1.

The proof of Theorem 3.13 relies on a generic perturbation argument.

**Proposition 3.15.** *Let  $1 < p < 1 + 4/(d - 2)_+$ . Let  $f(z) = |z|^{p-1}z$ . Take  $W : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{C}$  and  $H : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{C}$  such that there exist  $C_1 > 0$  and  $\lambda > 0$  verifying for any  $t \in [0, \infty)$  the estimate*

$$\|W(t)\|_{L^{p+1}} + e^{\lambda t} \|H(t)\|_{L^{\frac{p+1}{p}}} \leq C_1.$$

### 3 Finite and Infinite Soliton Trains

Consider the equation

$$\eta(t) = i \int_t^\infty e^{i(t-s)\Delta} (f(W + \eta) - f(W) + H) ds. \quad (3.14)$$

There exists  $\lambda_\star \gg 1$  such that if  $\lambda > \lambda_\star$  then there exists a unique  $\eta$  solution of (3.14) on  $[0, \infty)$  such that for any  $t \geq 0$  we have

$$\|\eta(t)\|_{L^{p+1}} \leq C_1 e^{-\lambda t}.$$

Moreover, for any  $t \geq 0$  the solution  $\eta$  satisfies

$$\|\eta\|_{S(t, \infty)} \lesssim e^{-\lambda t}.$$

It can be proved that  $(W, H)$  defined by

$$W = R_\infty, \quad H = f(R_\infty) - \sum_{j=0}^{\infty} f(R_j)$$

verify the hypotheses of Proposition 3.15, which thus implies Theorem 3.13. In what follows, we give a streamlined proof of Proposition 3.15. For a detailed proof and further results, we refer to [LeLiTs15, LeTs14].

*Proof of Proposition 3.15.* We write (3.14) as  $\eta = \Psi(\eta)$ . We show that, for  $\lambda$  sufficiently large,  $\Psi$  is a contraction in the ball

$$B = \left\{ \eta \in L_t^\infty L_x^{p+1} : \|e^{\lambda t} \|\eta(t)\|_{L_x^{p+1}}\|_{L_t^\infty(0, \infty)} \leq C_1 \right\}.$$

First, we prove that  $\Psi$  maps  $B$  into  $B$ . Define  $\theta$  by

$$\theta = d \left( \frac{1}{2} - \frac{1}{p+1} \right).$$

By assumption, we have  $1 < p < 1 + 4/(d-2)_+$ , therefore  $\theta \in (0, 1)$ . Recall that  $f(z) = |z|^{p-1}z$  verifies for all  $z_1, z_2 \in \mathbb{C}$  the inequality

$$|f(z_1) - f(z_2)| \lesssim |z_1 - z_2| (|z_1|^{p-1} + |z_2|^{p-1}).$$

Therefore, we have

$$|f(W + \eta) - f(W)| \lesssim |\eta| (|W|^{p-1} + |\eta|^{p-1}).$$

Using this inequality and the dispersive estimate, we obtain

$$\|\Psi(\eta)(t)\|_{L^{p+1}} \lesssim \int_t^\infty |t-s|^{-\theta} \left( \| |W|^{p-1} \eta \|_{L^{\frac{p+1}{p}}} + \| |\eta|^p \|_{L^{\frac{p+1}{p}}} + \| H \|_{L^{\frac{p+1}{p}}} \right) ds.$$

From Hölder inequality and combining with the assumptions on  $(W, H)$ , we get

$$\begin{aligned} \|\Psi(\eta)(t)\|_{L^{p+1}} &\lesssim \int_t^\infty |t-s|^{-\theta} \left( \|W\|_{L^{p+1}}^{p-1} \|\eta\|_{L^{p+1}} + \|\eta\|_{L^{p+1}}^p + \|H\|_{L^{\frac{p+1}{p}}} \right) ds \\ &\lesssim \int_t^\infty |t-s|^{-\theta} (C_1^{p-1} C_1 e^{-\lambda s} + C_1^p e^{-\lambda p s} + C_1 e^{-\lambda s}) ds \leq CC_1 e^{-\lambda t} I_1, \end{aligned}$$

where  $C = C(d, p)$  and  $I_1$  is defined by

$$I_1 = C_1^{p-1} \int_0^\infty \tau^{-\theta} e^{-\lambda \tau} d\tau + C_1^{p-1} \int_0^\infty \tau^{-\theta} e^{-\lambda p \tau} d\tau + \int_0^\infty \tau^{-\theta} e^{-\lambda \tau} d\tau.$$

By a change of variable, we observe that

$$\int_0^\infty \tau^{-\theta} e^{-\lambda \tau} d\tau = \frac{1}{\lambda^{1-\theta}} \int_0^\infty \tau^{-\theta} e^{-\tau} d\tau.$$

Therefore, for  $\lambda$  large enough we have

$$CI_1 \leq 1.$$

Hence  $\|\Psi(\eta)(t)\|_{L^{p+1}} \leq C_1 e^{-\lambda t}$  and  $\Psi$  maps  $B$  into  $B$ . Using similar arguments, we can also prove that  $\Psi$  is a contraction on  $B$ . As a consequence, there exists a unique  $\eta \in B$  such that

$$\eta = \Psi(\eta).$$

Having obtained the existence of the fixed point  $\eta$  in  $B$ , we now consider the properties of  $\eta$  in the Strichartz space. Let  $a$  be such that

$$\frac{2}{a} + \frac{d}{p+1} = \frac{d}{2}.$$

Since  $1 < p < 1 + 4/(d-2)_+$ , we have  $2 < a < \infty$ . By Strichartz estimates, for all  $t \geq 0$  we have

$$\begin{aligned} \|\eta\|_{S([t, \infty))} &\lesssim \| |\eta| (|W|^{p-1} + |\eta|^{p-1}) \|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{p+1}{p}}([t, \infty))} + \|H\|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{p+1}{p}}([t, \infty))} \\ &\lesssim \|W\|_{L_\tau^\infty L_x^{p+1}([0, \infty))}^{p-1} \|\eta\|_{L_\tau^{\frac{a}{a-1}} L_x^{p+1}([t, \infty))} + \|\eta\|_{L_\tau^{\frac{a(p+1)}{a-1}} L_x^{p+1}([t, \infty))}^p + \|H\|_{L_\tau^{\frac{a}{a-1}} L_x^{\frac{p+1}{p}}([t, \infty))} \\ &\lesssim e^{-\lambda t}. \end{aligned}$$

This concludes the proof.  $\square$

### 3.3 Stability and Instability

Having obtained the existence of multi-soliton solutions of nonlinear dispersive equations, the next step is to investigate whether or not they are generic objects for the dynamics

of the corresponding equations, i.e. if they are stable or not. It is relatively natural to expect that, at least when the interactions are local and the composing solitons are exponentially decaying at infinity, a multi-soliton solution will be stable if all the composing solitons are stable, and it will be unstable if at least one of the composing solitons is unstable. In practice, proving that a multi-soliton containing an unstable soliton is unstable is already a delicate task, that we carry out in Section 3.3.1 for excited multi-solitons of the nonlinear Schrödinger equation. It is even harder to prove that a multi-soliton containing only stable solitons will also be stable, and usually one needs to use extra monotonicity properties of the equation to obtain the result. In Section 3.3.2, we obtain the stability of a family of multi-solitons for the derivative nonlinear Schrödinger equation.

### 3.3.1 Instability of Multi-Solitons of NLS

In this section, we prove the instability of a multi-soliton containing a linearly unstable soliton. This result was obtained in collaboration with Raphaël Côte [CoLe11].

We place ourselves in the following setting. The nonlinear Schrödinger equation is considered with a nonlinearity  $f(z) = g(|z|^2)z$ :

$$\begin{cases} iu_t + \Delta u + g(|u|^2)u = 0, \\ u|_{t=0} = u_0, \end{cases} \quad u : \mathbb{R}_t \times \mathbb{R}_x^d \rightarrow \mathbb{C}, \quad (\text{NLS})$$

with the nonlinearity  $g$  verifying the following assumptions.

(A0) (regularity) The function  $g$  is smooth, i.e.  $g \in C^\infty([0, +\infty), \mathbb{R})$ .

(A1) (superlinearity) The function  $g$  verifies  $g(0) = 0$ , and  $\lim_{s \rightarrow 0} sg'(s) = 0$ .

(A2) ( $H^1$ -subcriticality) There exists  $p \in (1, 2^* - 1)$  such that  $|s^2g'(s^2)| \lesssim s^{p-1}$  ( $s > 1$ ).

(A3) (focusing property) There exists  $s_0$  such that  $F(s_0) > \frac{s_0^2}{2}$ ;  $F(z) := \int_0^{|z|} g(s^2)s ds$ .

The main restriction here on  $g$  is the smoothness requirement. If the composing solitons have positive bound states profile, probably smoothness is needed only for  $z \neq 0$ , which then allows  $g$  to be, for example, any subcritical power nonlinearity. For excited states,  $g$  also has to be smooth at  $z = 0$ . For power nonlinearities, this restricts to odd exponents. We can take for example  $g(|z|^2)z = |z|^4z$  in dimension  $d = 2$ .

As before, given a frequency  $\omega > 0$ , a speed  $v \in \mathbb{R}^d$ , an initial phase  $\gamma \in \mathbb{R}$ , an initial position  $x_0 \in \mathbb{R}^d$ , and a bound state solution  $\Phi \in H^1$  to

$$-\Delta \Phi + \omega \Phi - g(|\Phi|^2)\Phi = 0, \quad \Phi \in H^1,$$

a soliton is a solution of (NLS) traveling on the line  $x = x_0 + vt$  and given by

$$R(t, x) := \Phi(x - vt - x_0)e^{i(\frac{1}{2}v \cdot x - \frac{1}{4}|v|^2t + \omega t + \gamma)}.$$

### 3 Finite and Infinite Soliton Trains

Recall that we say that a soliton  $R$  is *spectrally unstable* if there exists an eigenvalue  $\lambda \in \mathbb{C}$  with  $\Re(\lambda) > 0$  of the linearization  $\mathcal{L}$  of (NLS) around  $R$  (see Section 2.5). Remark that the spectrum of  $L$  is symmetric with respect to the real and imaginary axes, i.e. if  $\lambda$  is an eigenvalue, then so are  $-\lambda$ ,  $\bar{\lambda}$ , and  $-\bar{\lambda}$ .

As a first step to obtain the instability of multi-solitons containing unstable solitons, we prove the following existence and non-uniqueness result.

**Theorem 3.16** (Existence of a one parameter family of multi-solitons). *Assume (A0), (A1), (A2) and (A3). Take  $a \in \mathbb{R}$  and a sum of solitons*

$$R = \sum_{j=1}^N R_j.$$

*Assume that one of soliton (e.g.  $R_1$ ) is spectrally unstable. Define the minimal relative speed by*

$$v_\star = \min\{|v_j - v_k| : j, k = 1, \dots, N, j \neq k\}.$$

*There exists  $v_\natural > 0$  such that if  $v_\star > v_\natural$  (large relative speeds) then there exist  $T_0$  and a solution of (NLS) on  $[T_0, +\infty)$  such that for all  $t \geq T_0$  we have*

$$\|u(t) - R(t) - aY(t)\|_{H^1} \leq Ce^{-2\Re(\lambda)t},$$

*where  $Y$  is of the form*

$$Y(t) = e^{-\Re(\lambda)t}(\cos(\Im(\lambda)t)Y_1 + \sin(\Im(\lambda)t)Y_2), \quad |Y(t)| \gtrsim e^{-\Re(\lambda)t},$$

*for  $\lambda \in \mathbb{C}$  an eigenvalue for the linearized operator around  $R_1$  with maximal real part  $\Re(\lambda) > 0$ .*

As a corollary, we have the following orbital instability result.

**Corollary 3.17** (Orbital instability of the multi-soliton). *Under the hypotheses of Theorem 3.16, there exists  $\varepsilon > 0$ , such that for all  $n \in \mathbb{N} \setminus \{0\}$  and for all  $T \in \mathbb{R}$  the following holds. There exist  $I_n, J_n \in \mathbb{R}$ ,  $T \leq I_n < J_n$  and a solution  $w_n \in \mathcal{C}([I_n, J_n], H^1(\mathbb{R}^d))$  to (NLS) such that*

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \|w_n(I_n) - R(I_n)\|_{H^1} = 0, \\ & \inf_{\substack{y_j \in \mathbb{R}^d, \vartheta_j \in \mathbb{R}, \\ j=1, \dots, N}} \left\| w_n(J_n) - \sum_{i=1}^N \Phi_j(x - y_j) e^{i(\frac{1}{2}v_j \cdot x + \vartheta_j)} \right\|_{L^2} \geq \varepsilon. \end{aligned}$$

The instability stated in Corollary 3.17 is a *forward* in time instability. The instability backward in time is a direct consequence of the existence result of Theorem 3.16 for  $a \neq 0$ . To obtain the instability forward in time, we first use the result of Theorem 3.16 for one unstable soliton to construct a solution starting close to and going away from this soliton in forward time. Then we use the initial data of this solution to construct

a new solution, this time starting close to and going away from the sum of solitons in forward time. We refer to [CoLe11] for the details of the proof.

To prove the existence of a one parameter family of multi-solitons as in Theorem 3.16, we use a fixed point argument around a good approximation  $U$  of a solution of (NLS). The properties of  $U$  are specified in the following proposition.

**Proposition 3.18.** *Take  $N_0 \in \mathbb{N}$  and  $a \in \mathbb{R}$ . Then there exist  $W^{N_0} \in \mathcal{C}^\infty([0, +\infty), H^\infty(\mathbb{R}^d))$  such that  $U = R + W^{N_0}$  is a solution to (NLS) up to an order  $O(e^{-\rho(N_0+1)t})$  when  $t \rightarrow +\infty$ , i.e.*

$$iU_t + \Delta U + g(|U|^2)U = Err = O(e^{-\rho(N_0+1)t})$$

Given  $U$  as in Proposition 3.18, we define the map

$$w \mapsto \Psi(w) = -i \int_t^{+\infty} e^{i\Delta(t-\tau)} (f((U+w)(\tau)) - f(U(\tau)) - Err(\tau)) d\tau.$$

To prove Theorem 3.16, we show that  $\Psi$  is a contraction in the space

$$X_{T_0, N_0}^\sigma(B) := \left\{ w \in \mathcal{C}((T_0, +\infty), H^\sigma) \mid \sup_{t \geq T_0} e^{(N_0+1)\rho t} \|w(t)\|_{H^\sigma} < B \right\}.$$

We omit the details of the proof of Theorem 3.16 here, and we focus on how to construct a suitable approximate solution  $U$  as in Proposition 3.18.

*Proof of Proposition 3.18.* The construction is inspired by works of Duyckaerts, Merle and Roudenko [DuMe08, DuMe09, DuRo10]. As a first step, we look at the linearization of (NLS) around  $e^{i\omega t}\Phi(x)$ , assuming that it is our spectrally unstable soliton. If  $u$  is a solution of (NLS) and  $u = e^{i\omega t}(\Phi(x) + w)$ , then  $w$  is a solution of

$$w_t + L_{\mathbb{C}}w = \mathcal{M}_{\mathbb{C}}(w),$$

where the linear operator  $L_{\mathbb{C}}$  and the nonlinear part  $\mathcal{M}_{\mathbb{C}}$  are given by

$$\begin{aligned} L_{\mathbb{C}}w &= -i\Delta w + i\omega w - idf(\Phi).w, \\ \mathcal{M}_{\mathbb{C}}(w) &= if(\Phi + w) - if(\Phi) - idf(\Phi).w. \end{aligned}$$

We separate  $L_{\mathbb{C}}$  into real and imaginary parts:

$$L_{\mathbb{R}^2} \begin{pmatrix} w_R \\ w_I \end{pmatrix} = \begin{pmatrix} J & \Delta - \omega + I_1 \\ -\Delta + \omega - I_2 & -J \end{pmatrix} \begin{pmatrix} w_R \\ w_I \end{pmatrix}.$$

with  $I_1, I_2$  and  $J$  real valued potentials.

The profile will be constructed by induction. We start by constructing the profile at order 1. We complexify the matrix operator  $L_{\mathbb{R}^2}$  and denote it by  $L_{\mathbb{C}^2}$ . By assumption, there exists an eigenvalue of  $L_{\mathbb{C}^2}$  with positive real part. The essential spectrum of  $L_{\mathbb{C}^2}$  lies on the imaginary axis away from 0, and it can be proved (see [CoLe11]) that the eigenvalues of  $L_{\mathbb{C}^2}$  are located in a compact subset of  $\mathbb{C}$ . Hence there exists an eigenvalue

$$\lambda = \rho + i\theta$$



with maximal real part  $\rho > 0$ . Let  $Z$  be an eigenvector associated with  $\lambda$  and denote

$$Z = \begin{pmatrix} Z^+ \\ Z^- \end{pmatrix}, \quad Y_1 = \Re(Z), \quad Y_2 = \Im(Z).$$

We consider the solution  $Y$  of

$$\partial_t Y + L_{\mathbb{R}^2} Y = 0$$

given by the explicit expression

$$Y(t) = e^{-\rho t} (\cos(\theta t) Y_1 + \sin(\theta t) Y_2).$$

This is our first order solution profile.

Now, to construct the solution at the order  $N_0 \geq 2$ , we proceed in the following way. We look for  $W^{N_0}$  in the form

$$W(t, x) = \sum_{k=1}^{N_0} e^{-\rho k t} \left( \sum_{j=0}^k A_{j,k}(x) \cos(j\theta t) + B_{j,k}(x) \sin(j\theta t) \right).$$

Remark that

$$\mathcal{M}_{\mathbb{R}^2}(W) = \sum_{\kappa=2}^{N_0} e^{-\kappa \rho t} \sum_{j=0}^{\kappa} \left( \tilde{A}_{j,\kappa}(x) \cos(j\theta t) + \tilde{B}_{j,\kappa}(x) \sin(j\theta t) \right) + HOT,$$

where the terms  $\tilde{A}_{j,\kappa}$  and  $\tilde{B}_{j,\kappa}$  depend only on  $A_{j,k}$  and  $B_{j,k}$  for  $k \leq \kappa - 1$  (see [CoLe11] for details). In addition

$$\begin{aligned} (\partial_t W + L_{\mathbb{R}^2} W) = \sum_{k=1}^{N_0} e^{-\rho k t} \left( \sum_{j=0}^k (L_{\mathbb{R}^2} A_{j,k} + j\theta B_{j,k} - k\rho A_{j,k}) \cos(j\theta t) \right. \\ \left. + (L_{\mathbb{R}^2} B_{j,k} - j\theta A_{j,k} - k\rho B_{j,k}) \sin(j\theta t) \right). \end{aligned}$$

Therefore to find a satisfying  $W^{N_0}$  it is enough to solve for  $k \geq 2$

$$\begin{cases} L_{\mathbb{R}^2} A_{j,k} + j\theta B_{j,k} - k\rho A_{j,k} = \tilde{A}_{j,k}, \\ L_{\mathbb{R}^2} B_{j,k} - j\theta A_{j,k} - k\rho B_{j,k} = \tilde{B}_{j,k}, \end{cases}$$

which is possible because  $\lambda$  is of maximal real part. □

### 3.3.2 Stability of Multi-Solitons of the Derivative Nonlinear Schrödinger Equation

Obtaining the stability of a sum of soliton is not an easy task. In addition to the ingredients from the energy method to prove existence, it is necessary to rely on additional

features of the equation such as monotonicity properties. Stability of a sum of solitary waves has been proved by Martel, Merle and Tsai [MaMeTs02] for the subcritical Korteweg-de Vries equations. The setting is less favorable for nonlinear Schrödinger equations and Martel, Merle and Tsai [MaMeTs06] had to work with a tweaked nonlinearity to adapt their strategy. Indeed, the monotonicity property required for the proof holds only for super-critical power-type nonlinearities, whereas stability is expected only for subcritical nonlinearities. Thus the idea in [MaMeTs06] was to construct a nonlinearity which was supercritical at the origin, but subcritical at infinity. As a rule of thumb, one expects to be able to fully prove stability of a sum of soliton for equations for which traveling waves and dispersion travel at different speeds. It is the case for the Korteweg-de Vries equations considered in [MaMeTs02], where dispersion travels to the left (with negative speed) and solitary waves to the right (with positive speed). For the Gross-Pitaevskii equation (a 1-D cubic defocusing nonlinear Schrödinger equation with non-zero boundary conditions), the dispersion travels at speeds  $v$  with  $|v|$  larger than 1, whereas solitary waves travel at speeds  $w$  with  $|w|$  smaller than 1. Hence dispersion and solitary waves are also separated in the Gross-Pitaevskii equation, even though it is in a more complicated way than for the Korteweg-de Vries equation. In [BeGrSm14], Bethuel, Gravejat and Smets have been able to obtain the stability of a sum of solitary waves in the Gross-Pitaevskii equation.

In collaboration with Yifei Wu [LeWu18], I have investigated the case of the derivative nonlinear Schrödinger equation. Since the equation is critical *and* admits stable traveling waves, we were expecting to be able to prove stability of a sum of solitary waves without having to tweak the nonlinearity. We have indeed been able to obtain this stability result, with however some restriction on the relation between the speeds and the frequencies of the solitary waves, which we believe to be technical.

The derivative nonlinear Schrödinger equation takes the following form

$$iu_t + u_{xx} + i|u|^2u_x = 0, \quad (\text{dNLS})$$

where  $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ . It appears in various area of physics, for example in the modeling of Alfvén waves in plasma physics or for the description of ultrashort optical pulses. It is also a model quasilinear equation [ChLeLi79]. Via the gauge transformation

$$v(t, x) = \exp\left(ia \int_{-\infty}^x |u(t, y)|^2 dy\right) u(t, x),$$

one can obtain different equivalent forms of the derivative nonlinear Schrödinger equation. Indeed, if  $u$  solves (dNLS), then  $v$  defined by the gauge transform solves

$$iv_t + v_{xx} + i(|v|^2v)_x = 0, \quad \text{if } a = -1/2, \quad (3.15)$$

$$iv_t + v_{xx} - iv^2\bar{v}_x + \frac{1}{2}|v|^4v = 0, \quad \text{if } a = 1/4, \quad (3.16)$$

$$iv_t + v_{xx} - \frac{i}{2}|v|^2v_x + \frac{i}{2}v^2\bar{v}_x + \frac{3}{16}|v|^4v = 0, \quad \text{if } a = 1/2. \quad (3.17)$$

The equation (dNLS) is usually referred to as the Chen-Lee-Liu equation, equation (3.15) is called the Kaup-Newell equation and (3.16) the Gerdzhikov-Ivanov equation. The last

form (3.17) has apparently not been christened yet, but it appears in a crucial way in the analysis of the threshold for blow-up in [Wu13, Wu15].

As the nonlinear Schrödinger equations, the derivative nonlinear Schrödinger equation may be written in Hamiltonian form. In the Chen-Lee-Liu version (dNLS), a Hamiltonian form is

$$iu_t = E'(u),$$

where

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} |u_x|^2 dx + \frac{1}{4} \Im \int_{\mathbb{R}} |u|^2 \bar{u} u_x dx.$$

In the Kaup-Newell version (3.15), a Hamiltonian form is

$$v_t = \partial_x P'(v),$$

where

$$P(v) = \frac{1}{2} \Im \int_{\mathbb{R}} v \bar{v}_x dx - \frac{1}{4} \int_{\mathbb{R}} |v|^4 dx.$$

In terms of  $u$ ,  $P$  is given by

$$P(u) = \frac{1}{2} \Im \int_{\mathbb{R}} u \bar{u}_x dx$$

and is nothing but the momentum for (dNLS), which is also a conserved quantity. Combining these two Hamiltonian forms and the gauge transformation one can infer a bi-Hamiltonian structure for the derivative nonlinear Schrödinger equation [Le09a]. This indicates that the derivative nonlinear Schrödinger equation is an integrable equation and as such admits an infinity of conservation laws and is solvable via the inverse scattering transform (see [ChLeLi79, KaNe78] for earlier works in this direction, and [JeLiPeSu17, JeLiPeSu18, LiPeSu16, LiPeSu17, PeSh16, PeSaSh17] for recent developments). We shall however not make use of this specific integrable structure.

The first Hamiltonian formulation for the derivative nonlinear Schrödinger equation is similar to the one of the classical semi-linear Schrödinger equation, but the second one is similar to the one of the Korteweg-de Vries equation. Hence one expects that the derivative nonlinear Schrödinger equation will borrow features from both the nonlinear Schrödinger equation and the Korteweg-de Vries equation.

As for the usual Schrödinger equations, the *energy*  $E$ , the *momentum*  $P$  (already defined) and the *mass*

$$M(u) = \frac{1}{2} \|u\|_{L^2}^2$$

are conserved quantities for (dNLS).

We also have the following scaling invariance. If  $u$  solves (dNLS), then for any  $\lambda > 0$  the function  $u_\lambda$  defined by

$$u_\lambda(t, x) = \frac{1}{\sqrt{\lambda}} u \left( \frac{t}{\lambda^2}, \frac{x}{\lambda} \right)$$

also solves (dNLS). Since

$$M(u_\lambda) = M(u),$$

the derivative Schrödinger equation is *mass-critical*. However, its dynamics is very different from the one of its semi-linear mass-critical counterpart the 1D quintic nonlinear Schrödinger equation. Indeed, the derivative Schrödinger equation is *not* invariant by pseudo-conformal transformation, hence there exists no explicit blowing-up solution. In fact, it is not even known whether or not there exists a blowing-up solution, as the classical obstruction argument by Glassey [Gl77] based on the virial identity does not apply here.

The analysis of the well-posedness for the Cauchy problem for (dNLS) was performed by Hayashi and Ozawa [Ha93, HaOz92]. Using a semi-linear Schrödinger system generated from the derivative nonlinear Schrödinger equation, they have been able to prove that for any  $u_0 \in H^1(\mathbb{R})$ , there exists a unique  $u \in \mathcal{C}((-T_*, T^*), H^1(\mathbb{R}))$  such that  $u(0) = u_0$  and  $u$  is a solution of (dNLS). Moreover, the energy, the mass and the momentum are conserved, and the blow-up alternative holds: if  $T_* < \infty$  (resp.  $T^* < \infty$ ) then  $\|u(t)\|_{H^1}$  blows up as  $t \rightarrow T_*$  (resp.  $t \rightarrow T^*$ ). Using the Gagliardo-Nirenberg inequality and conservation of mass and energy, it is not hard to prove that solutions with mass smaller than  $\pi$  are in fact global. It has long been thought that  $\pi$  was the optimal threshold for global existence. However, it was recently proved by Wu [Wu13, Wu15] that the global existence holds in fact for solutions with mass less than  $2\pi$ . The existence of blowing-up solution at this threshold or above remains an open problem.

The derivative nonlinear Schrödinger equation admits solitary waves solutions. For all  $(\omega, c) \in \mathbb{R}^2$  such that  $c^2 < 4\omega$ , there exists a unique (up to phase shifts and translations) solitary wave solution of (dNLS) of the form  $e^{i\omega t} \phi_{\omega,c}(x - ct)$ . The profile  $\phi_{\omega,c}$  is explicitly known and given by the formula

$$\phi_{\omega,c}(x) = \varphi_{\omega,c}(x) \exp\left(\frac{c}{2}ix - \frac{i}{4} \int_{-\infty}^x |\varphi_{\omega,c}(\xi)|^2 d\xi\right)$$

where

$$\varphi_{\omega,c}(x) = \left(\frac{\sqrt{\omega}}{4\omega - c^2} \left(\cosh\left(x\sqrt{4\omega - c^2}\right) - \frac{c}{2\sqrt{\omega}}\right)\right)^{-\frac{1}{2}}.$$

It is a critical point of the *action* functional

$$S_{\omega,c} = E + \omega M + cP.$$

Remark here that the derivative Schrödinger equation is *not* invariant by Galilean transformation, and it is not possible to transfer information on a solitary wave with speed  $c \neq 0$  from a standing wave (with  $c = 0$ ).

The stability analysis for the solitary waves of (dNLS) was performed by Colin and Ohta [CoOh06] (see also [GuWu95] for an earlier result). They proved that for any  $(\omega, c) \in \mathbb{R}^2$  such that  $c^2 < 4\omega$ , the solitary wave  $e^{i\omega t} \phi_{\omega,c}(x - ct)$  is *orbitally stable* for the flow of (dNLS), in the sense that for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $u_0 \in H^1(\mathbb{R})$  verifies

$$\|u_0 - \phi_{\omega,c}\| < \delta,$$

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then for all  $t \in \mathbb{R}$  the associated solution of (dNLS) verifies

$$\inf_{\substack{\theta \in \mathbb{R} \\ y \in \mathbb{R}}} \|u(t) - e^{i\theta} \phi_{\omega, c}(\cdot - y)\|_{H^1} < \varepsilon.$$

At the threshold  $c^2 = 4\omega$ , there does not exist solitary waves if  $c = -\sqrt{2\omega}$ . If  $c = \sqrt{2\omega}$ , then there exists a (unique) solitary wave with profile given by

$$\varphi_c = \sqrt{c} \varphi_1(cx), \quad \varphi_1(x) = \frac{2^{\frac{3}{2}}}{\sqrt{1 + 4x^2}}.$$

We call this type of solitary wave the *lump* solitary wave. The intrinsic difference with the case  $c^2 < 4\omega$  is the algebraic (and not exponential) decay at spatial infinity. This is related to the absence of a spectral gap in the linearized action around the lump. Consequently, stability cannot follow from the usual method. A weaker orbital stability result (with scaling added) has been proved by Kwon and Wu [KwWu17].

Let  $N \in \mathbb{N}$ . For  $j = 1, \dots, N$  let  $\omega_j \in (0, \infty)$ ,  $c_j \in (-2\sqrt{\omega_j}, 2\sqrt{\omega_j})$ . Let  $(\phi_j) = (\phi_{\omega_j, c_j})$  be the corresponding solitary wave profiles. Recall that a *multi-soliton* with the above parameters is a solution  $u$  of (dNLS) such that

$$\lim_{t \rightarrow \infty} \left\| u(t) - \sum_{j=1}^N e^{i\omega_j t} \phi_j(x - c_j t) \right\|_{H^1} = 0.$$

Due to integrability of (dNLS), the existence of multi-solitons can be obtained via the inverse scattering transform and explicit calculations [NaCh80]. Very recently, by developing a set of tools linked to the integrability of the equation, Liu, Perry and Sulem [JeLiPeSu17] have been able to obtain an outstanding result of soliton resolution for the derivative nonlinear Schrödinger equation. A by-product of their result is the asymptotic stability of multi-soliton configurations. The work [JeLiPeSu17] is set in the weighted space

$$H^{2,2}(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : u_{xx} \in L^2(\mathbb{R}), x^2 u \in L^2(\mathbb{R})\}.$$

This weighted space does not contain the lump soliton which decays too slowly. Since the lump soliton is a good candidate for a possible blow-up profile, having global existence in  $H^{2,2}(\mathbb{R})$  does not preclude the possibility of blow-up in  $H^1(\mathbb{R})$ . Similarly, stability in  $H^{2,2}(\mathbb{R})$  does not imply stability in  $H^1(\mathbb{R})$ . In [LeWu18], we have been interested in the orbital stability of multi-soliton configurations in the energy space  $H^1(\mathbb{R})$ . Our technique does not rely on integrability and can be used in other settings. As a matter of fact, a result of stability for two solitons configurations was obtained with a similar technique for the generalized derivative NLS by Tang and Xu [TaXu18].

Our result is the following.

**Theorem 3.19.** *Assume that for  $j = 2, \dots, N$  we have*

$$2 \frac{\omega_j - \omega_{j-1}}{c_j - c_{j-1}} > 0, \quad c_{j-1} < 2 \frac{\omega_j - \omega_{j-1}}{c_j - c_{j-1}} < c_j. \quad (3.18)$$

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For all  $\varepsilon > 0$  there exists  $\delta > 0$  such that the following holds. If there exist parameters  $(\theta_j, x_j) \subset \mathbb{R}^2$  such that

$$\left\| u_0 - \sum_{j=1}^N e^{i\theta_j} \phi_j(\cdot - x_j) \right\|_{H^1} \leq \delta,$$

then there exist  $\mathcal{C}^1$  functions  $\tilde{x}_1(t), \dots, \tilde{x}_N(t) \in \mathbb{R}$ , and  $\tilde{\theta}_1(t), \dots, \tilde{\theta}_N(t) \in \mathbb{R}$ , such that for all  $t \geq 0$ ,

$$\left\| u(t) - \sum_{j=1}^N e^{i\tilde{\theta}_j(t)} \phi_j(\cdot - \tilde{x}_j(t)) \right\|_{H^1} \leq \varepsilon.$$

We can in fact describe more precisely the laws followed by the modulation parameters. Indeed,  $\tilde{\theta}_j$  and  $\tilde{x}_j$  verify the dynamical laws

$$\partial_t \tilde{x}_j \sim c_j, \quad \partial_t \tilde{\theta}_j \sim \omega_j.$$

The condition (3.18) is assumed for technical purposes. It has several important consequences. First, the solitons must travel to the right, i.e.  $c_j > 0$  for all  $j = 2, \dots, N$ , except maybe for the first one for which we can allow  $c_1 < 0$ . The frequencies (and thus the size of the solitons) need to be increasing  $\omega_1 < \omega_2 < \dots < \omega_N$ . These features are reminiscent from what happens for the Korteweg-de Vries equation, for which all solitons travels to the right, and the fastest is also the largest one. We can easily find sets of parameters for which (3.18) is satisfied, for example

$$\omega_j = j^2 + 1, \quad c_j = 2j, \quad j = 1, \dots, N.$$

We do not cover the case  $c = 2\sqrt{\omega}$ . Indeed, lump solitons are significantly different from the other solitons (algebraic decay, weaker stability, etc.) and cannot be included in the analysis.

Our strategy for the proof of Theorem 3.19 is adapted from the work of Martel, Merle and Tsai on the twisted nonlinear Schrödinger equation [MaMeTs06] and is the following. We obtain a bootstrap argument which, combined to the continuity of the flow, allows to obtain the desired stability. The proof of the bootstrap result itself combines several arguments. First, we obtain a coercivity property for the linearized action around a solitary wave, up to orthogonality conditions. Then we extend this coercivity property to a global linearized action defined by gluing together localized linearized actions around each solitary wave. Modulation theory and monotonicity properties allow us to deal with the orthogonality conditions and the control of modulation parameters. A stability result for the two-solitons of the derivative nonlinear Schrödinger equation was obtained independently with a similar strategy by Miao, Tang and Xu [MiTaXu17].

The bootstrap argument goes as follows.

Given  $A_0, L, \delta > 0$ , define a tubular neighborhood of the  $N$ -soliton profile by

$$\mathcal{V}(\delta, L, A_0) = \left\{ u \in H^1(\mathbb{R}); \inf_{\substack{x_j > x_{j-1} + L \\ \theta_j \in \mathbb{R}}} \left\| u - \sum_{j=1}^N e^{i\theta_j} \phi_j(\cdot - x_j) \right\|_{H^1} < A_0 \delta \right\}.$$

We have the following bootstrap result.

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**Proposition 3.20** (Bootstrap). *There exists  $A_0 > 1$ , fixed, such that for all  $L \gg 1$ ,  $0 < \delta \ll 1$  the following property is satisfied. If  $t^* > 0$  is such that for all  $t \in [0, t^*]$  the solution  $u$  of (dNLS) with  $u(0) = u_0$  verifies*

$$u_0 \in \mathcal{V}(\delta, L, 1), \quad u(t) \in \mathcal{V}(\delta, L, A_0), \quad (3.19)$$

then for all  $t \in [0, t^*]$  we have

$$u(t) \in \mathcal{V}\left(\delta, L, \frac{A_0}{2}\right). \quad (3.20)$$

From now on, we assume (3.19). Our aim is to present the main arguments that allow us in [LeWu18] to show that in fact (3.20) holds.

One of the main task in our work [LeWu18] was to obtain the following coercivity property for single solitary waves of (dNLS).

**Proposition 3.21** (Coercivity for one solitary wave). *For any  $\omega, c \in \mathbb{R}$  with  $4\omega > c^2$ , there exists  $\mu \in \mathbb{R}$  such that for any  $\varepsilon \in H^1(\mathbb{R})$  verifying the orthogonality conditions*

$$(\varepsilon, i\phi_{\omega,c})_{L^2} = (\varepsilon, \partial_x \phi_{\omega,c})_{L^2} = (\varepsilon, \phi_{\omega,c} + i\mu \partial_x \phi_{\omega,c})_{L^2} = 0,$$

we have

$$H_{\omega,c}(\varepsilon) := \langle S''_{\omega,c}(\phi_{\omega,c})\varepsilon, \varepsilon \rangle \gtrsim \|\varepsilon\|_{H^1}^2.$$

If  $c < 0$ , then we can choose  $\mu = 0$ . As a consequence of Proposition 3.21, we have the orbital stability of solitary waves.

**Corollary 3.22** (Orbital Stability). *Solitary waves of (dNLS) with  $\omega > c^2/4$  are orbitally stable.*

The orbital stability of solitary waves in [CoOh06] was obtained by a different method and the coercivity result Proposition 3.21 was not available in the literature. It was in fact relatively subtle to obtain, as the parameter  $\mu$  cannot be equal to 0 if  $c > 0$ . We refer to [LeWu18] for the details of the proof of Proposition 3.21.

The stability that we want to obtain is orbital, meaning that a solution starting close to a sum of solitary waves will remain close to a sum of solitary waves, possibly shifted i.e. modulated. The modulation parameters are constructed via the implicit function theorem. We have the following modulation result.

**Proposition 3.23** (Modulation). *Take  $\delta, 1/L \ll 1$ . For  $j = 1, \dots, N$  there exist (unique)  $\mathcal{C}^1$ -functions*

$$\tilde{\theta}_j, \tilde{\omega}_j, \tilde{x}_j, \tilde{c}_j : [0, t^*] \rightarrow \mathbb{R},$$

such that if we define modulated solitons  $\tilde{R}_j$  and  $\varepsilon$  by

$$\tilde{R}_j(t) = e^{i\tilde{\theta}_j(t)} \phi_{\tilde{\omega}_j(t), \tilde{c}_j(t)}(\cdot - \tilde{x}_j(t)), \quad \varepsilon(t) = u(t) - \sum_{j=1}^N \tilde{R}_j(t),$$

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then  $\varepsilon$  satisfies for all  $t \in [0, t^*]$  the orthogonality conditions

$$\left(\varepsilon, i\tilde{R}_j\right)_{L^2} = \left(\varepsilon, \partial_x \tilde{R}_j\right)_{L^2} = \left(\varepsilon, \tilde{R}_j + \mu_j i \partial_x \tilde{R}_j\right)_{L^2} = 0, \quad j = 1, \dots, N.$$

Moreover, there exists  $\tilde{C} > 0$  such that for all  $t \in [0, t^*]$  we have

$$\|\varepsilon(t)\|_{H^1} + \sum_{j=1}^N (|\tilde{\omega}_j(t) - \omega_j| + |\tilde{c}_j(t) - c_j|) \leq \tilde{C} A_0 \delta.$$

At  $t = 0$  the estimate does not depend on  $A_0$  and

$$\|\varepsilon(0)\|_{H^1} + \sum_{j=1}^N (|\tilde{\omega}_j(0) - \omega_j| + |\tilde{c}_j(0) - c_j|) \leq \tilde{C} \delta.$$

The scaling and speed parameters are chosen to verify for all  $t \in [0, t^*]$  and for any  $j = 1, \dots, N$  the relationship

$$\tilde{c}_j(t) - c_j = \mu_j (\tilde{\omega}_j(t) - \omega_j).$$

The set of modulation parameters verifies the following dynamical law

$$\sum_{j=1}^N |\partial_t \tilde{c}_j| + |\partial_t \tilde{\omega}_j| + \left| \partial_t \tilde{\theta}_j - \tilde{\omega}_j \right| + |\partial_t \tilde{x}_j - \tilde{c}_j| \leq \tilde{C} \|\varepsilon(t)\|_{H^1} + HOT.$$

As for the proof of the existence of multi-soliton, we define a global linearized action. Let  $\chi_j$  be a cut-off function localized around the  $j$ -th soliton. Set

$$\mathcal{I}_j(t) = \int_{\mathbb{R}} (\tilde{\omega}_j |u|^2 + \tilde{c}_j \Im u \bar{u}_x) \chi_j(t) dx, \quad \mathcal{S}(t) = E(u(t)) + \sum_{j=1}^N \mathcal{I}_j(t).$$

We have the following Taylor-like expansion of the global action.

**Proposition 3.24** (Taylor Expansion). *We have*

$$\mathcal{S}(t) = \sum_{j=1}^N S_j(\phi_{\tilde{\omega}_j(0), \tilde{c}_j(0)}) + \frac{1}{2} \mathcal{H}(t) + \sum_{j=1}^N O(|\tilde{\omega}_j(t) - \tilde{\omega}_j(0)|^2) + HOT,$$

for  $\mathcal{H}$  defined by

$$\begin{aligned} \mathcal{H}(t) = & \|\varepsilon_x\|_{L^2}^2 + \sum_{j=1}^N \Im \int_{\mathbb{R}} |\tilde{R}_j|^2 \bar{\varepsilon} \varepsilon_x + \tilde{R}_j \partial_x \tilde{R}_j (\bar{\varepsilon})^2 + \overline{\tilde{R}_j \partial_x \tilde{R}_j} |\varepsilon|^2 dx \\ & + \sum_{j=1}^N \left( \tilde{\omega}_j(t) \int_{\mathbb{R}} |\varepsilon|^2 \chi_j dx + \tilde{c}_j(t) \Im \int_{\mathbb{R}} \varepsilon \bar{\varepsilon}_x \chi_j dx \right). \end{aligned}$$



Moreover, the global linearized action  $\mathcal{H}$  is coercive, as a consequence of the coercivity result Proposition 3.21.

**Proposition 3.25** (Coercivity). *For all  $t \in [0, t^*]$ , we have*

$$\mathcal{H}(t) \gtrsim \|\varepsilon\|_{H^1}^2.$$

As a consequence, we have the following control on the distance of  $u$  to the sum of modulated solitary waves.

**Corollary 3.26.** *For all  $t \in [0, t^*]$ ,*

$$\|\varepsilon(t)\|_{H^1}^2 \lesssim \frac{1}{L} \sup_{s \in [0, t]} \|\varepsilon(s)\|_{L^2}^2 + \sum_{j=1}^N |\tilde{\omega}_j(t) - \tilde{\omega}_j(0)|^2 + HOT.$$

It remains to find a way to control the modulation parameters. To this aim, we prove a monotonicity result. This is the key point where the critical nature of (dNLS) is favorable compare to the Schrödinger equations with power type nonlinearities. This is also for the proof of this Proposition that we need the technical assumption (3.18).

**Proposition 3.27** (Monotonicity One). *If  $\delta, 1/L \ll 1$ , then for all  $t \in [0, t^*]$ ,*

$$\mathcal{I}(t) - \mathcal{I}(0) \lesssim \frac{1}{L} \sup_{s \in [0, t]} \|\varepsilon(s)\|_{L^2}^2 + HOT,$$

where the functional  $\mathcal{I}$  is given by

$$\mathcal{I}(t) = \tilde{\omega}_1(0)M(u) + \tilde{c}_1(0)P(u) + \sum_{j=2}^N (\tilde{c}_j(0) - \tilde{c}_{j-1}(0))\mathcal{I}_j(t).$$

Using the modulation result to expand in the definition of  $\mathcal{I}$  (see [LeWu18] for details), we make the modulation parameters appear, and we obtain the following result.

**Corollary 3.28** (Control of the Modulation Parameters). *For all  $t \in [0, t^*]$ ,*

$$\sum_{j=1}^N |\tilde{\omega}_j(t) - \tilde{\omega}_j(0)| \lesssim \sup_{s \in [0, t]} \|\varepsilon(s)\|_{H^1}^2 + HOT.$$

Combining Corollaries 3.26 and 3.28 gives the desired result.

# 4 On Blow-Up and Stability

We present in this chapter a series of results on nonlinear dispersive partial differential equations or systems of Schrödinger type. In Section 4.1, we show the existence of a minimal mass blow-up solution for a nonlinear Schrödinger equation with a double power nonlinearity. The surprising feature of this result is the non-standard blow-up speed of the solution constructed. In Section 4.2, we investigate the Cauchy problem and the stability of stationary solutions of a singularly perturbed Gross-Pitaevskii equation, i.e. a nonlinear Schrödinger with a delta potential and set with non-zero boundary condition. In Section 4.3, we are interested in the stability properties of the periodic (in space) standing waves of one dimension cubic nonlinear Schrödinger equations. For our investigations we use a variety of tools ranging from variational analysis to numerical studies. In Section 4.4, the object of study are multi-components traveling waves of nonlinear Schrödinger systems. Finally, in Section 4.5, we present two results of stability of standing waves for semi-classical equations.

## 4.1 Blow-Up for a Double Power Nonlinear Schrödinger Equation

The study of singularity formation in nonlinear Schrödinger equations has been the subject of intensive investigations since the beginning of the 80s. In this section, we present a result highlighting the existence of solutions blowing up at a speed which is neither the conformal one nor the log-log one.

We consider the double-power nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}}u + \eta|u|^{p-1}u = 0, & u : \mathbb{R}_t \times \mathbb{R}_x^d \rightarrow \mathbb{C}, \quad \eta = \{-1, 0, 1\}. \\ u|_{t=0} = u_0, \end{cases} \quad (4.1)$$

When  $\eta = 0$ , the blow-up behavior of (4.1) is (relatively) well-known, at least at the minimal mass threshold and around the ground state and we will review some known results shortly in the next pages. In problem (4.1), we are interested by the effect on the blow-up of the second power when  $\eta = \pm 1$ . If  $p$  is super-critical, i.e. if  $p > 1 + \frac{4}{d}$ , then we expect the new power to dominate the blow-up dynamics. Little is known in this case. Hence we assume that the power is sub-critical, i.e.  $p < 1 + \frac{4}{d}$  and we investigate how the new power modifies the blow-up at the minimal mass level.

From the classical theory of nonlinear Schrödinger equations which was recalled in Chapter 2, the nonlinear Schrödinger equation (4.1) is locally well-posed in  $H^1(\mathbb{R}^d)$ , i.e. for any  $u_0 \in H^1(\mathbb{R}^d)$  there exists a unique maximal solution  $u \in \mathcal{C}((-T_*, T^*), H^1(\mathbb{R}^d)) \cap$

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$\mathcal{C}^1((-T_*, T^*), H^{-1}(\mathbb{R}^d))$  of (4.1) with  $u(0) = u_0$ . Moreover, we have conservation of the energy and the mass, given by

$$E(u) = \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{1}{2 + \frac{4}{d}} \|u\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}} - \frac{\eta}{p+1} \|u\|_{L^{p+1}}^{p+1}, \quad M(u) = \frac{1}{2} \|u\|_{L^2}^2,$$

and the blow-up alternative holds: If  $T^* < \infty$  (resp.  $T_* < \infty$ ), then

$$\lim_{\substack{t \rightarrow T^* \\ \text{(resp. } t \rightarrow T_*)}} \int_{\mathbb{R}^d} |\nabla u|^2 dx = \infty.$$

Conservation of mass and energy combined with Gagliardo-Nirenberg inequalities lead to the following global well-posedness result. Let  $Q \in H^1(\mathbb{R}^d)$  be the unique positive radial solution of

$$-\Delta Q + Q - |Q|^{\frac{4}{d}} Q = 0.$$

Then  $Q$  is the optimizer for the Gagliardo-Nirenberg inequality (see [We83])

$$\|u\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}} \leq C_{\text{GN}}^{2+\frac{4}{d}} \|u\|_{L^2}^{\frac{4}{d}} \|\nabla u\|_{L^2}^2, \quad C_{\text{GN}}^{2+\frac{4}{d}} = \frac{2 + \frac{4}{d}}{2 \|Q\|_{L^2}^{\frac{4}{d}}}. \quad (4.2)$$

Using conservation of energy and mass, the critical Gagliardo-Nirenberg inequality (4.2) and the Gagliardo-Nirenberg inequality (2.3) for  $p+1$ , one shows that if  $u_0 \in H^1(\mathbb{R}^d)$  is such that  $\|u_0\|_{L^2} < \|Q\|_{L^2}$ , then the associated solution  $u$  of (4.1) is globally well-posed (independently of  $\eta$ ), i.e.  $T_* = T^* = \infty$ . Moreover, when  $\eta = 0$ , it was proved by Dodson [Do15] that scattering holds for any solution with  $\|u_0\|_{L^2} < \|Q\|_{L^2}$ .

It is natural to ask what happens at the threshold  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ . In particular, does there exist a minimal mass blow-up element, i.e. a blowing up solution of (4.1) with  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ ?

When  $\eta = 0$ , i.e. in the usual  $L^2$ -critical nonlinear Schrödinger equation, the pseudo-conformal transform of the solitary wave solution  $e^{it}Q$  provides us with a family of blowing up solutions. In particular, the function

$$S(t, x) = \frac{1}{|t|^{\frac{d}{2}}} Q\left(\frac{x}{|t|}\right) e^{-i\frac{|x|^2}{4|t|}} e^{\frac{i}{|t|}},$$

is a solution of (4.1), has minimal mass and blows up at  $T = 0$  at speed  $\frac{1}{|t|}$ , precisely,

$$\|S(t)\|_{L^2} = \|Q\|_{L^2}, \quad \|\nabla S(t)\| \underset{t \rightarrow 0^-}{\sim} \frac{1}{|t|}.$$

A classification result was obtained by Merle [Me93] for solutions with threshold mass: If a solution  $u$  of (4.1) is such that

$$\|u_0\|_{L^2} = \|Q\|_{L^2}, \quad T^* < \infty,$$

then, up to the symmetries of the flow,  $u$  is the pseudo-conformal blowing up solution, i.e.

$$u \equiv S.$$

Away from the minimal mass threshold, solutions were constructed by perturbation of  $S$  by Bourgain and Wang [BoWa97]. Roughly speaking, they obtained solutions in the following form. Given  $u_*$  flat at  $x = 0$ , there exists a solution  $u$  of (4.1) such that

$$(u - S)(t) \rightarrow u_* \text{ as } t \rightarrow 0.$$

These solutions were found to be unstable by Merle, Raphaël and Szeftel [MeRaSz14], in the sense that for each of these Bourgain-Wang solutions there exists a continuous curve passing through the solution and consisting of solutions blowing up with log-log speed (see below) on one side and global solutions on the other side.

Beside solutions blowing up at the pseudo-conformal speed  $|t|^{-1}$ , another family of solutions exists, consisting of solutions blowing up at the so-called log-log speed, i.e.

$$\|\nabla u\|_{L^2} \underset{t \rightarrow T^*}{\sim} \sqrt{\frac{\ln|\ln(T^* - t)|}{T^* - t}}.$$

An extensive study of this blow-up dynamics has been performed in a series of paper by Perelman [Pe01] and by Merle and Raphaël [MeRa03, MeRa04, MeRa05, MeRa06, Ra05]. In particular, it has been shown that this type of blow-up dynamics is stable.

In the case  $\eta = -1$ , the situation is rather simple, as global existence holds at the minimal mass threshold.

**Proposition 4.1.** *Set  $\eta = -1$ . If  $u_0 \in H^1(\mathbb{R}^d)$  is such that  $\|u_0\|_{L^2} = \|Q\|_{L^2}$ , then the associated solution of (4.1) is global and bounded in  $H^1(\mathbb{R}^d)$ .*

Proposition 4.1 is a consequence of the conservation of energy combined with a compactness argument. We omit the proof here and refer to [LeMaRa16]. The following result shows that the threshold  $\|u_0\|_{L^2} = \|Q\|_{L^2}$  for global existence is optimal.

**Proposition 4.2.** *Set  $\eta = -1$ . For every  $\delta > 0$  there exists  $u_0 \in H^1(\mathbb{R}^d)$  such that  $\|u_0\|_{L^2} = \|Q\|_{L^2} + \delta$  and the associated solution of (4.1) blows-up in finite time.*

Combining Propositions 4.1 and 4.2, we infer that there exists no minimal blow-up mass solution when  $\eta = -1$ . Proposition 4.2 is proved using conservation of energy and the virial identity (see [LeMaRa16]).

When  $\eta = 1$ , the situation is different and there indeed exists a minimal blow-up mass solution. I obtained the following result in collaboration with Yvan Martel and Pierre Raphaël [LeMaRa16].

**Theorem 4.3.** *Set  $\eta = 1$ . Let  $d = 1, 2, 3$  and  $1 < p < 1 + \frac{4}{d}$ . Then for all energy level  $E_0 \in \mathbb{R}$ , there exist  $t_0 < 0$  and a radially symmetric Cauchy data  $u(t_0) \in H^1(\mathbb{R}^d)$  with*

$$\|u(t_0)\|_{L^2} = \|Q\|_{L^2}, \quad E(u(t_0)) = E_0,$$

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such that the corresponding solution  $u(t)$  of (4.1) blows up at time  $T^* = 0$  with speed:

$$\|\nabla u(t)\|_{L^2} = \frac{C(p) + o_{t \uparrow 0}(1)}{|t|^\sigma}$$

for some universal constants

$$\sigma = \frac{4}{4 + d(p-1)} \in \left(\frac{1}{2}, 1\right), \quad C(p) > 0.$$

Uniqueness of the minimal mass blowing up solution was proved for the inhomogeneous nonlinear Schrödinger equation in [RaSz11]. For the double power nonlinear Schrödinger equation (4.1), proving uniqueness seems not to be an easy task and we have not pursued in this direction.

The description of the blowing up solution of Theorem 4.3 is in fact very precise. We have

$$u(t, x) = \frac{1}{\lambda^{\frac{d}{2}}(t)} Q\left(\frac{x}{\lambda(t)}\right) e^{-i\sigma \frac{|x|^2}{4t}} e^{i\gamma(t)} + v(t, x),$$

where  $\gamma$  is a phase factor,  $v$  is a remainder in the  $L^2$  sense and the blow-up is driven by  $\lambda$  as follows:

$$\lim_{t \rightarrow 0} \|v(t)\|_2 = 0, \quad \lambda(t) \sim C_p |t|^\sigma \quad \text{as } t \rightarrow 0^-.$$

If we make  $p$  vary from 1 to  $1 + \frac{4}{d}$ , the blow-up rate  $|t|^\sigma$  will vary continuously from  $|t|^{-1}$  (the conformal rate) to  $|t|^{-\frac{1}{2}}$  (almost the log-log speed).

The key difference between the equation (4.1) with the focusing-focusing double power nonlinearity ( $\eta = 1$ ) and the  $L^2$  critical pure power nonlinearity ( $\eta = 0$ ) is the absence of scaling invariance in the former.

Understanding what kinds of blow-up speeds are possible for nonlinear Schrödinger equations is still a widely open question. There has been recent progress by Martel and Raphaël [MaRa15]. In the case of the  $L^2$ -critical nonlinear Schrödinger equation, they have been able to show the existence of solutions blowing up in infinite time at rate  $\ln(t)$  as  $t \rightarrow \infty$ . Via the pseudo-conformal transformation, this also yields the existence of a solution blowing up in finite time at  $T = 0$  at rate  $\ln(|t|)|t|^{-1}$ , a rate which is strictly above the conformal one.

The proof of Theorem 4.3 goes as follows. We adapt the strategy introduced by Raphaël and Szeftel [RaSz11] for the construction of minimal mass blowing up solutions for the inhomogeneous  $L^2$ -critical nonlinear Schrödinger equation. We first construct a suitable blow-up profile  $U(t, x)$ . Here lies the main difficulty compare to the work [RaSz11], as the construction of the blow-up profile in the case of (4.1) has to be done at any order of precision. The desired solution is then obtained by a backward resolution of (4.1) before the blow-up time (which is set at  $T = 0$ ) (rather than a fixed point argument as in [BaCaDu11]): take an increasing sequence of times  $(t_n)$  such that  $t_n \rightarrow 0$  as  $n \rightarrow \infty$ , and define  $u_n$  as the solution of (4.1) such that  $u_n(t_n) = U(t_n)$ . Using backward uniform estimates on  $u_n - U$  and a compactness argument, we are able to show the existence of  $t_0 < 0$  and a solution  $u$  of (4.1) such that  $u_n \rightarrow u$  uniformly on  $[t_0, 0)$ .

The solution  $u$  is the desired solution. The uniform estimates are obtained after the construction of a suitable Lyapunov functional controlling the difference  $u_n - U$  up to orthogonality conditions. The orthogonality conditions are dealt with using modulation theory. Solving the dynamical system for the modulation parameters leads us to the new blow-up speed.

In the following, we indicate the first steps of the construction of the blow-up profile  $U$ . The analysis presented here is only formal and we refer to [LeMaRa16] for rigorous justifications.

As usual in the analysis of blow-up behaviors, we first rescale the problem by setting

$$u(t, x) = \frac{1}{\lambda^{\frac{d}{2}}(s)} w(s, y) e^{i\gamma(s) - i\frac{b(s)|y|^2}{4}}, \quad \frac{ds}{dt} = \frac{1}{\lambda^2}, \quad y = \frac{x}{\lambda(s)}.$$

Here,  $\lambda$ ,  $\gamma$  and  $b$  are modulation parameters whose values are to be determined later. If  $u$  solves (4.1), then  $w$  solves

$$\begin{aligned} & iw_s + \Delta w - w + |w|^{\frac{4}{d}} w \\ & + \lambda^\alpha |w|^{p-1} w \\ & - i \left( b + \frac{\lambda_s}{\lambda} \right) \Lambda w + (1 - \gamma_s) w + (b_s + b^2) \frac{|y|^2}{4} w - b \left( b + \frac{\lambda_s}{\lambda} \right) \frac{|y|^2}{2} w = 0, \end{aligned} \quad (4.3)$$

where  $\alpha = 2 - \frac{d(p-1)}{2} \in (0, 2)$ . If  $u$  is a blowing up solution of (4.1) with blow-up speed driven by  $\lambda$ , then we expect  $w$  to be a global and bounded solution of (4.3). Constructing a blow-up profile for (4.1) is equivalent to constructing a global and bounded solution of (4.3) with a choice of modulation parameters such that  $\lambda \rightarrow 0$  as  $s \rightarrow \infty$  (i.e.  $t \rightarrow 0$ ). In the way we presented (4.3), one can readily see that a first order approximation is given by

$$w(s, y) = Q(y), \quad b + \frac{\lambda_s}{\lambda} = b_s + b^2 = 1 - \gamma_s = 0.$$

After solving the dynamical system for the modulation parameters, one realizes that, in the original variables, this solution is simply the pseudo-conformal blowing up solution  $S$ .

When taking  $w$  above defined as solution of (4.3), the error is  $\lambda^\alpha |Q|^{p-1} Q$ . This error is too big for our purposes, and we try to refine the profile. Thus we make the ansatz  $w = Q + \lambda^\alpha \tilde{P}_1$ , substitute into the equation and realize that  $\tilde{P}_1$  needs to solve

$$L_+ \tilde{P}_1 = Q^p,$$

where  $L_+ = -\Delta + 1 - \left(1 + \frac{4}{d}\right) Q^{\frac{4}{d}}$ . Such a  $\tilde{P}_1$  can be found because it is well-known that  $L_+$  invertible for radial functions. We keep the same modulation equations and the new error is given by

$$i\alpha \frac{\lambda_s}{\lambda} \lambda^\alpha \tilde{P}_1 = -i\alpha b \lambda^\alpha \tilde{P}_1.$$

For the third order expansion, we make the ansatz  $w = Q + \lambda^\alpha \tilde{P}_1 + ib\lambda^\alpha P_2$ , substitute into the equation and realize that  $P_2$  needs to solve

$$L_- P_2 = -\alpha \tilde{P}_1,$$

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where  $L_- = -\Delta + 1 - Q^{\frac{4}{d}}$ . The operator  $L_-$  is invertible on  $Q^\perp$ , however we do not have  $(P_1, Q) = 0$ . To solve this problem, the solution introduced in [RaSz11] is to modify the modulation law. We introduce a new  $P_1$  defined by

$$L_+ P_1 = Q^p + \beta \frac{|y|^2}{4} Q,$$

where the parameter  $\beta$  is chosen such that

$$-\frac{1}{2} \left( Q^p + \beta \frac{|y|^2}{4} Q, \Lambda Q \right)_{L^2} = (P_1, Q)_{L^2} = 0.$$

To compensate for the introduction of  $\beta$ , we replace the modulation law  $b_s + b^2 = 0$  by

$$b_s + b^2 - \beta \lambda^\alpha = 0.$$

The process can be continued indefinitely, with modifications in the modulation equation for  $b$  each time the operator  $L_-$  appears.

We now give some heuristics on how to solve the dynamical system solved by the modulation parameters. For simplicity, we only consider here the exact system verified at the first order of approximation:

$$\begin{cases} b_s + b^2 - \beta \lambda^\alpha = 0, \\ b + \frac{\lambda_s}{\lambda} = 0, \\ 1 - \gamma_s = 0, \\ \frac{ds}{dt} = \frac{1}{\lambda}. \end{cases}$$

We first solve in  $s$  the system for  $b$  and  $\lambda$

$$\begin{cases} b_s + b^2 - \beta \lambda^\alpha = 0, \\ b + \frac{\lambda_s}{\lambda} = 0. \end{cases}$$

We introduce the quantity  $\mu = \lambda^{-1}$ . It is easy to see that  $\mu$  satisfies

$$\mu_{ss} - \beta \mu^{1-\alpha} = 0.$$

After multiplying by  $\mu_s$  and integrating we get

$$\frac{1}{2} \mu_s^2 - \frac{\beta}{2-\alpha} \mu^{2-\alpha} = 0,$$

where the integration constant has been chosen to be 0 (otherwise it would lead to a lower order correction). At this point, we can note that we necessarily have  $\beta > 0$ . Then

$$\frac{\mu_s}{\mu^{\frac{2-\alpha}{2}}} = \sqrt{\frac{2\beta}{2-\alpha}}.$$

Hence, we get

$$\mu^{\frac{\alpha}{2}} = \left( \frac{\alpha}{2} \sqrt{\frac{2\beta}{2-\alpha}} \right) s,$$

where we have chosen the integration constant to be 0. Note that, since we expect  $\mu > 0$ , we need here to have  $\frac{\alpha}{2} = 1 - \frac{d(p-1)}{4} > 0$ , i.e.  $p < 1 + \frac{4}{d}$ . Therefore, we have

$$\lambda = \left( \frac{\alpha}{2} \sqrt{\frac{2\beta}{2-\alpha}} \right)^{-\frac{2}{\alpha}} s^{-\frac{2}{\alpha}}$$

and

$$b = -\frac{\lambda_s}{\lambda} = \frac{2}{\alpha} s^{-1}.$$

In particular,

$$\lambda^\alpha \sim b^2 \sim b_s.$$

Let us now come back to the original time variable. We have

$$\frac{ds}{dt} = \frac{1}{\lambda^2} = \left( \frac{\alpha}{2} \sqrt{\frac{2\beta}{2-\alpha}} \right)^{\frac{4}{\alpha}} s^{\frac{4}{\alpha}}.$$

Therefore

$$\left( -\frac{4-\alpha}{\alpha} \left( \frac{\alpha}{2} \sqrt{\frac{2\beta}{2-\alpha}} \right)^{\frac{4}{\alpha}} t \right)^{-\frac{\alpha}{4-\alpha}} = s.$$

As a consequence, the law of  $\lambda$  as a function of  $t$  is given by

$$\lambda(t) = C_\lambda |t|^{\frac{2}{4-\alpha}},$$

for some explicit positive constant  $C_\lambda = C_\lambda(\alpha, \beta) > 0$ . For  $b$  and  $\gamma$ , we have

$$b(t) = C_b |t|^{\frac{\alpha}{4-\alpha}}, \quad \gamma = C_\gamma |t|^{-\frac{\alpha}{4-\alpha}}$$

for explicit constants  $C_b$  and  $C_\gamma$ .

## 4.2 A Singularly Perturbed Gross-Pitaevskii Equation

In this section, we consider the following singularly perturbed Gross-Pitaevskii equation

$$iu_t + u_{xx} + (1 - |u|^2)u - \gamma\delta u = 0, \tag{GP}$$

where the unknown  $u : \mathbb{R}_t \times \mathbb{R}_x \rightarrow \mathbb{C}$  is a complex valued function of time and space,  $\gamma \in \mathbb{R}$  is a constant and  $\delta$  denotes the Dirac distribution at 0. The solutions will be required to satisfy the boundary condition

$$|u| \rightarrow 1 \text{ as } x \rightarrow \pm\infty. \tag{BC}$$



The Dirac distribution appears in nonlinear Schrödinger equations in various contexts. It can be considered as an asymptotic model for a nonlinear Schrödinger system when one of the component undergoes blow-up [CaMa95]. In nonlinear optics, the Dirac distribution can model a defect in the propagation of light. The study of partial differential equations on graphs has known recently an increasing interest. Equation (GP) can be viewed as set on a very simple graph consisting of two half infinite edges and one vertex, with jump boundary conditions at the vertex. One of the first mathematical work on nonlinear Schrödinger equations with singular potentials was by Goodman, Holmes and Weinstein [GoHoWe04] and has been followed by many others.

Compare to Dirichlet boundary condition on the line, boundary conditions of the type (BC) are not very frequently considered in the mathematical literature on nonlinear Schrödinger equations. Such non-zero boundary conditions are however perfectly natural in many physical contexts, for example the modeling of Bose-Einstein condensates [SeCaHo05] or in nonlinear optics [Ag07]. There not so frequent use stems from the difficulty to work with functions verifying such boundary conditions: they do not form a vector space.

Many interesting mathematical phenomena are generated by the presence of the Dirac perturbation and the non-standard boundary conditions (BC) and this was our main motivation for considering this problem.

Even with the Dirac perturbation, (GP) can still be written in Hamiltonian form

$$iu_t = E'(u)$$

with the (renormalized) energy

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} |u_x|^2 dx + \frac{1}{4} \int_{\mathbb{R}} (1 - |u|^2)^2 dx + \frac{\gamma}{2} |u(0)|^2.$$

Here, the potential part of the energy has been renormalized to take into account the boundary conditions. With such a definition for the energy, a natural space to look for solutions of (GP) is the *energy space*, defined by

$$\mathcal{E} = \{u \in H_{\text{loc}}^1(\mathbb{R}) : u_x \in L^2(\mathbb{R}), (1 - |u|^2) \in L^2(\mathbb{R})\}.$$

As already mentioned,  $\mathcal{E}$  is not a vector space. We can however endow it with a complete metric structure, defining the distance between  $u, v \in \mathcal{E}$  by

$$d_0(u, v) = \|u_x - v_x\|_{L^2} + \||u|^2 - |v|^2\|_{L^2} + |u(0) - v(0)|.$$

The structural properties of the energy space were analyzed by Gérard [Ge06, Ge08] (in any dimension). In our case, we will mainly rely on the following properties.

**Lemma 4.4.** *Let  $\mathcal{E}$  be endowed with the metric  $d_0$ . The following properties hold.*

- *If  $u \in \mathcal{E}$ , then  $u$  is uniformly continuous, bounded, and  $\lim_{x \rightarrow \pm\infty} |u(x)| = 1$ .*
- *If  $u \in \mathcal{E}$  and  $v \in H^1(\mathbb{R})$ , then  $u + v \in \mathcal{E}$ .*

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- The nonlinearity  $u \mapsto (1 - |u|^2)u$  maps  $\mathcal{E}$  into  $H^1(\mathbb{R})$  smoothly with bounded Lipschitz constant on bounded sets of  $\mathcal{E}$ .

A common strategy [AlGeHoHo88] to deal with the Dirac perturbation is to include it in a linear operator build on the second derivative with a domain including a jump condition. We define the operator  $H_\gamma$  by

$$H_\gamma : D(H_\gamma) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$$

$$u \mapsto u_{xx}$$

where the domain  $D(H_\gamma)$  is given by

$$D(H_\gamma) = \{u \in H^2(\mathbb{R} \setminus \{0\}) \cap H^1(\mathbb{R}), u_x(0^+) - u_x(0^-) = \gamma u(0)\}.$$

Formally,  $H_\gamma = -\partial_{xx} + \gamma\delta$ , as for  $u, v \in \mathcal{D}(\mathbb{R})$  we indeed have

$$\langle (-\partial_{xx} + \gamma\delta)u, v \rangle = \int_{\mathbb{R}} u\bar{v}dx + \gamma u(0)\bar{v}(0) = (H_\gamma u, v).$$

The first problem to consider when dealing with (GP) is the local well-posedness of the Cauchy problem (preferably in the energy space).

When  $\gamma = 0$ , this was worked out in dimension 1 by Zhidkov [Zh01] and in higher dimensions by Gérard [Ge06, Ge08] (see also Gallo [Ga08]). Local well-posedness can be obtained from the usual fixed point procedure, provided the following observation is made. The propagator  $e^{-it\partial_{xx}}$  maps  $\mathcal{E}$  to  $\mathcal{E}$ . Indeed, taking  $u_0 \in \mathcal{E}$  and writing in Fourier variable, we obtain

$$\mathcal{F}(e^{-it\partial_{xx}}u_0 - u_0) = (e^{it\xi^2} - 1)\hat{u}_0 = \frac{e^{it\xi^2} - 1}{i\xi} \widehat{\partial_x u_0} \in H^1(\mathbb{R}).$$

Since  $\mathcal{E} + H^1(\mathbb{R}) = \mathcal{E}$ , this shows the desired property. Now, we write  $u(t) = e^{it\partial_{xx}}u_0 + v$  and the equation (GP) becomes

$$iv_t + v_{xx} + f(e^{it\partial_{xx}}u_0 + v) = 0$$

where  $f(s) = (1 - |s|^2)s$ . Since  $f$  maps  $\mathcal{E}$  to  $H^1(\mathbb{R})$ , we can rewrite the equation for  $v$  in Duhamel form

$$v(t) = i \int_0^t e^{i(t-s)\partial_{xx}} f(e^{is\partial_{xx}}u_0 + v(s))ds,$$

and solve for  $v \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}))$ .

When  $\gamma \neq 0$ , the main difficulty is to prove that  $e^{-itH_\gamma}$  maps  $\mathcal{E}$  to  $\mathcal{E}$ . Contrary to the case  $\gamma = 0$ , this cannot be done using the Fourier transform and requires an in depth investigation of the propagator  $e^{-itH_\gamma}$ . The following decomposition holds

$$e^{-itH_\gamma} = e^{-itH_0} + \Gamma(t),$$

where  $\Gamma(t)$  is an operator with kernel

$$\begin{aligned}\Gamma(t, x, y) &= -\frac{\gamma}{2} \int_0^{+\infty} e^{-\frac{\gamma s}{2}} K_0(t, s + |x| + |y|) ds && \text{if } \gamma > 0, \\ \Gamma(t, x, y) &= -\frac{|\gamma|}{2} \int_0^{+\infty} e^{-\frac{|\gamma|s}{2}} K_0(t, s - |x| - |y|) ds + \frac{|\gamma|}{2} e^{i\frac{\gamma^2 t}{4}} e^{-\frac{|\gamma|(|x|+|y|)}{2}} && \text{if } \gamma < 0,\end{aligned}$$

where  $K_0$  is the kernel of the free Schrödinger evolution, i.e.

$$K_0(t, \zeta) = \frac{1}{\sqrt{4i\pi t}} e^{-\frac{\zeta^2}{4it}}.$$

Thus to show that  $e^{-itH_\gamma}$  maps  $\mathcal{E}$  to  $\mathcal{E}$ , it is enough to show that  $\Gamma$  maps  $\mathcal{E}$  to  $H^1(\mathbb{R})$ . Using the explicit form of  $\Gamma$  and by astute manipulations of oscillatory integrals, we are able to prove the following proposition.

**Proposition 4.5.** *Let  $T > 0$ . There exists  $C > 0$  such that for all  $t \in [-T, T]$  and for all  $u \in \mathcal{E}$  we have*

$$\begin{aligned}\Gamma(t)u &\in H^1(\mathbb{R}), \\ \|\Gamma(t)u\|_{H^1} &\leq C(\|u_x\|_{L^2} + |u(0)|).\end{aligned}$$

In collaboration with Isabella Ianni and Julien Royer [JaLeRo17], I obtained the following well-posedness result.

**Theorem 4.6.** *Let  $\gamma \neq 0$ . For all  $u_0$  there exists a unique global solution  $u \in \mathcal{C}(\mathbb{R}, \mathcal{E})$  of (GP) such that  $u(0) = u_0$ . Moreover, the energy  $E$  is conserved along the flow.*

Obtaining the conservation of energy is not an easy task here. It is done by first proving it for higher regularity solutions and then extending it with a density argument. However, we need to take the Dirac singularity into account, and thus we work in a space where regularity holds everywhere except at 0, where we require only the jump condition to be satisfied.

Having obtained the local well-posedness of the Cauchy Problem for (GP), we considered the stationary solutions of (GP) and obtained existence and stability/instability results.

The stationary solutions of (GP) verify the equation

$$-u_{xx} + \gamma \delta u - (1 - |u|^2)u = 0.$$

From an elementary ordinary differential equation analysis, we see that all bounded solutions (up to phase shift) are

$$\kappa(x) = \tanh\left(\frac{x}{\sqrt{2}}\right), \quad b_\gamma(x) = \tanh\left(\frac{|x| - c_\gamma}{\sqrt{2}}\right), \quad \tilde{b}_\gamma(x) = \coth\left(\frac{|x| + c_\gamma}{\sqrt{2}}\right),$$

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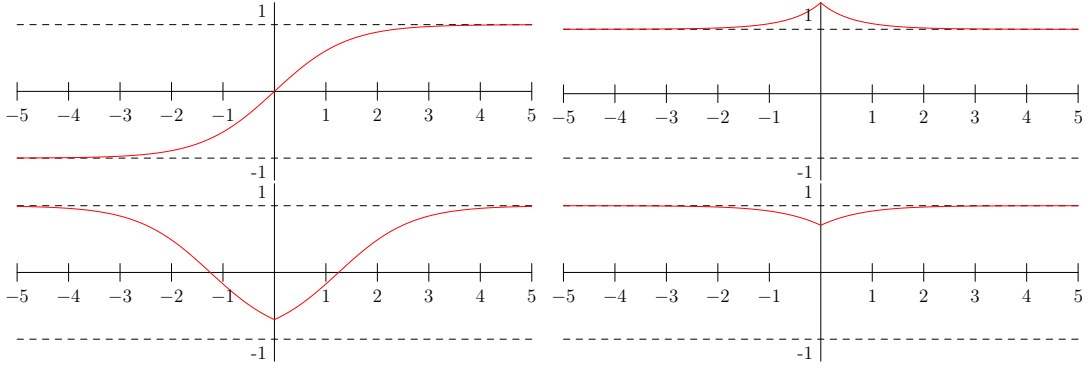


Figure 4.1: Top Left: The stationary state  $\kappa = \tanh\left(\frac{x}{\sqrt{2}}\right)$ .

Top Right: The stationary state  $\tilde{b}_\gamma$  for  $\gamma = -1$ .

Bottom: The stationary state  $b_\gamma$  for  $\gamma = -1$  (left) and  $\gamma = 1$  (right).

where  $\tilde{b}_\gamma$  is well-defined only if  $\gamma < 0$  and the translation parameter  $c_\gamma$  is given by

$$c_\gamma = \frac{1}{\sqrt{2}} \operatorname{arcsinh}\left(-\frac{2\sqrt{2}}{\gamma}\right).$$

Figure 4.1 provides a graphic representation of the stationary solutions.

Our main concern for the analysis of stationary solution was their stability under the flow of (GP).

We first obtained a variational characterization for some of the stationary solutions. Let  $\gamma \in \mathbb{R} \setminus \{0\}$ . Then we have

$$m_\gamma = \inf\{E(v) | v \in \mathcal{E}\} > -\infty.$$

Let  $\mathcal{G}_\gamma = \{v \in \mathcal{E} | E(v) = m_\gamma\}$ . Then we have

$$\mathcal{G}_\gamma = \{e^{i\theta} b_\gamma : \theta \in \mathbb{R}\} \text{ if } \gamma > 0,$$

$$\mathcal{G}_\gamma = \{e^{i\theta} \tilde{b}_\gamma : \theta \in \mathbb{R}\} \text{ if } \gamma < 0.$$

Moreover the minimizing sequences are compact.

As a consequence of this variational characterization, we have the following orbital stability result.

**Theorem 4.7.** *The set  $\mathcal{G}_\gamma$  is stable for the flow of (GP), i.e*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall u_0 \in \mathcal{E}, d_0(u_0, \mathcal{G}_\gamma) < \delta \implies \sup_{t \in \mathbb{R}} d_0(u(t), \mathcal{G}_\gamma) < \varepsilon.$$

The stationary solutions which are not minimizers are expected to be unstable. We have proved a linear instability result for the kink  $\kappa$  when  $\gamma > 0$ . More precisely, write  $u = \kappa + \eta$ . If  $u$  is a solution of (GP), then  $\kappa$  solves

$$\partial_t \begin{pmatrix} \Re(\eta) \\ \Im(\eta) \end{pmatrix} = \mathcal{L} \begin{pmatrix} \Re(\eta) \\ \Im(\eta) \end{pmatrix} + \text{nonlinear terms} \quad (4.4)$$

where

$$\mathcal{L} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}, \quad L_- = H_\gamma - (1 - \kappa^2), \quad L_+ = H_\gamma + 2 - 3(1 - \kappa^2).$$

We say that  $\kappa$  is linearly unstable when 0 is an unstable solution of (4.4). In particular, it is the case when  $\mathcal{L}$  has an eigenvalue  $\lambda$  with  $\Re(\lambda) > 0$ .

Our result is the following.

**Theorem 4.8.** *If  $\gamma > 0$ , then  $\kappa$  is linearly unstable.*

The theorem is proved as follows. We first observe that the spectrum of  $L_-$  and  $L_+$  is shifted to the right (resp. to the left) when  $\gamma > 0$  (resp.  $\gamma < 0$ ). When  $\gamma = 0$ , the spectrum of  $L_-$  and  $L_+$  can be explicitly calculated. For  $L_-$ , there is only one negative eigenvalue at  $-1/2$  and the essential spectrum covers  $[0, \infty)$ . For  $L_+$ , there is no negative eigenvalue, 0 is a simple eigenvalue (with eigenvector  $\kappa_x$ ) and the essential spectrum covers  $[2, \infty)$ . When  $\gamma$  becomes positive, the eigenvalue of  $L_-$  at  $-1/2$  starts shifting to the right, but never crosses 0.

If  $\lambda$  is an eigenvalue of  $\mathcal{L}$ , for the eigenvector  $(u, v)^T$  we have

$$L_-v = \lambda u, \quad -L_+u = \lambda v,$$

and therefore  $-\lambda^2$  is an eigenvalue of  $L_+L_-$ , i.e.

$$L_+L_-v = -\lambda^2v.$$

The operator  $L_+L_-$  is not self-adjoint. It is convenient to set  $w = L_+^{-\frac{1}{2}}v$  (which is possible for  $\gamma > 0$ ) to have

$$L_+^{\frac{1}{2}}L_-L_+^{\frac{1}{2}}w = -\lambda^2w.$$

The operator  $L_+^{\frac{1}{2}}L_-L_+^{\frac{1}{2}}$  is self-adjoint. Denote  $\xi$  a normalized eigenvector (depending on  $\gamma$ ) for the negative eigenvalue of  $L_-$ . Set  $\Xi = L_+^{-\frac{1}{2}}\xi$ . Then

$$\left\langle (L_+^{\frac{1}{2}}L_-L_+^{\frac{1}{2}})\Xi, \Xi \right\rangle = \langle L_-\xi, \xi \rangle < 0.$$

Therefore  $L_+^{\frac{1}{2}}L_-L_+^{\frac{1}{2}}$  has an eigenvalue  $\Lambda < 0$ . Thus  $\lambda = \sqrt{-\Lambda} > 0$  is an eigenvalue of  $\mathcal{L}$ , with positive real part, and  $\kappa$  is indeed linearly unstable.

### 4.3 Periodic Waves of the One Dimensional Cubic Nonlinear Schrödinger Equation

In this section, we consider the one dimensional cubic nonlinear Schrödinger equation in the form

$$\begin{cases} iu_t + u_{xx} + b|u|^2u = 0, \\ u(0, x) = u_0(x). \end{cases} \quad (4.5)$$

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where  $b \in \mathbb{R} \setminus \{0\}$  and  $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ . We are interested in space-periodic solutions of (4.5), i.e. solutions for which there exists a period  $T > 0$  such that

$$u(t, x + T) = u(t, x)$$

for all  $x \in \mathbb{R}$ . If basic phenomena such as existence and stability of solitary waves are well understood for Schrödinger equations with Dirichlet boundary conditions on the whole real line, there are currently few works dealing with the periodic case. We work here with periodic functions on  $\mathbb{R}$ , which is a more general situation than functions set on the torus  $\mathbb{T}$ . Indeed, e.g., we may consider stability of standing waves against perturbations whose fundamental period is a multiple of the one of the wave.

In the work [GuLeTs17], in collaboration with Stephen Gustafson and Tai-Peng Tsai, my goal was to better understand the stability of periodic standing waves against harmonic (i.e. co-periodic) and subharmonic perturbations. We have focused our efforts on the one dimensional cubic nonlinear Schrödinger equation because it was allowing us to perform a number of explicit calculations which would not have been possible for more generic equations. We however did not make use of the integrable nature of the equation and our result could be carried out for other non-integrable equations such as the one dimensional cubic Klein-Gordon equation.

Recall that standing waves are solutions of the form  $e^{-iat}v(x)$  (we prefer in this section to use  $a$  instead of  $\omega$ , with the opposite sign convention), where  $a \in \mathbb{R}$  and  $v : \mathbb{R} \rightarrow \mathbb{R}$  is a (periodic) solution of

$$v_{xx} + av + b|v|^2v = 0. \tag{4.6}$$

For the sake of simplicity, we restrict ourselves to real-valued  $v$ . The real valued periodic solutions of (4.6) are given by rescaled versions of the Jacobi elliptic functions  $\text{cn}$ ,  $\text{dn}$ ,  $\text{sn}$ , whose definition we recall now. Let  $k \in (0, 1)$ . The *incomplete integral of the first kind* in trigonometric form is given by

$$x = F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}},$$

and the *Jacobi elliptic functions* are defined using the inverse of  $F(\cdot, k)$  by

$$\text{sn}(x, k) = \sin(\phi), \quad \text{cn}(x, k) = \cos(\phi), \quad \text{dn}(x, k) = \sqrt{1 - k^2 \sin^2(\phi)}.$$

Therefore, we have the relations

$$\text{cn}^2 + \text{sn}^2 = \text{dn}^2 + k^2 \text{sn}^2 = 1,$$

and for extreme values of  $k$  we recover the classical trigonometric functions for  $k = 0$ :

$$\text{cn}(x, 0) = \cos(x), \quad \text{dn}(x, 0) = 1, \quad \text{sn}(x, 0) = \sin(x).$$

and the hyperbolic functions for  $k = 1$ :

$$\text{cn}(x, 1) = \text{dn}(x, 1) = \text{sech}(x), \quad \text{sn}(x, 1) = \tanh(x).$$

Defining the *complete elliptic integral of the first kind*

$$K(k) = F\left(\frac{\pi}{2}, k\right).$$

The functions  $\text{cn}$  and  $\text{sn}$  are  $4K$ -periodic and  $2K$ -anti-periodic (i.e.  $\text{sn}(x + 2K, k) = -\text{sn}(x, k)$ ) whereas  $\text{dn}$  is  $2K$ -periodic. The Jacobi elliptic functions solve (4.6) for the following choice of coefficients

$$\begin{aligned} a &= 1 + k^2, & b &= -2k^2, & \text{for } u &= \text{sn}, \\ a &= 1 - 2k^2, & b &= 2k^2, & \text{for } u &= \text{cn}, \\ a &= -(2 - k^2), & b &= 2, & \text{for } u &= \text{dn}. \end{aligned}$$

For generic  $a$  and  $b$ , solutions of (4.6) can be obtained by rescaling (see [GuLeTs17, Lemmas 2.1 and 2.2] for details). We will say that a function  $u$  is *cnoidal* if there exist  $\alpha, \beta > 0, k \in \mathbb{R}$  such that

$$u(\cdot) = \frac{1}{\alpha} \text{cn}\left(\frac{\cdot}{\beta}, k\right),$$

with equivalent definitions for *dnoidal* and *snoidal* functions.

Our main purpose is to investigate the stability of standing waves against various types of perturbations. As usual, we say that a standing wave  $e^{-iat}\phi(x)$  is *orbitally stable* in the Banach space  $X$  if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that for any  $u_0 \in X$  verifying

$$\|u_0 - \phi\|_X < \delta,$$

the solution  $u$  of (4.5) with  $u(0) = u_0$  verifies for all time  $t \in \mathbb{R}$

$$\inf_{\substack{\theta \in \mathbb{R} \\ y \in \mathbb{R}}} \|u(t) - e^{i\theta}\phi(\cdot - y)\|_X < \varepsilon.$$

For  $T$  a period (not necessarily minimal) of  $\phi$ , we set

$$P_T = \{f \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{C}) : f(x + T) = f(x) \text{ for all } x \in \mathbb{R}\}.$$

If  $\phi$  is anti-periodic of anti-period  $T$  (i.e.  $\phi(\cdot + T) = -\phi(\cdot)$ ), we define

$$A_T = \{f \in H_{\text{loc}}^1(\mathbb{R}, \mathbb{C}) : f(x + T) = -f(x) \text{ for all } x \in \mathbb{R}\}.$$

We endow  $P_T$  and  $A_T$  with the norm of  $H^1(0, T)$ .

The main references for previous works regarding the stability of standing waves in a periodic setting for Schrödinger equations are [An07, An09, BoDeNi11, GaHa07, GaHa07a, GaPe15, IvLa08, Ro74]. For other equations, we refer to [BeMiRo16, BeNoRo13, BeNoRo14] for the development of a theoretical approach to the stability/instability of periodic waves with applications to the generalized Korteweg–de Vries equation and the Euler–Korteweg system. We summarize the existing results in the next paragraphs.

For  $\text{sn}$ , orbital stability in  $A_{2K}$  was obtained by Gallay and Haragus [GaHa07, GaHa07a] using an approach based on the ideas of Grillakis, Shatah and Strauss [GrShSt87,

[GrShSt90]. A breakthrough came from the use of integrability techniques in [BoDeNi11, GaPe15], where Bottman, Deconinck and Nivala first, and later Gally and Pelinovsky, were able to obtain orbital stability with respect to perturbation in  $P_{4nK}$  for any  $n \in \mathbb{N}$  (recall that sn is  $4K$  periodic), with an additional restriction that the perturbation should also be in  $H_{\text{loc}}^2$ . The strategy used in [BoDeNi11, GaPe15] relies on the construction of a Lyapunov functional based on higher order conserved quantities, which exist only in the integrable 1-d cubic case. Hence it cannot be extended to other nonlinearities or to higher dimensions.

For cn, Gally and Haragus [GaHa07, GaHa07a] proved orbital stability when  $k \ll 1$  (with a numerical proof for  $0 < k < 1$ ). Using tools of complete integrability, Ivey and Lafortune [IvLa08] have been able to prove spectral stability in  $P_{4K}$ . Instability in  $P_{4nK}$  for  $n$  large was established from formal calculations by Rowland [Ro74]. When  $k \ll 1$ , it was established rigorously by Gally and Haragus [GaHa07, GaHa07a].

For dn, orbital stability in  $P_{2K}$  (recall that dn is  $2K$  periodic) can be established using a simple variational argument, and instability in  $P_{2nK}$  for any  $n > 1$  follows from an eigenvalues count, see [GuLeTs17]. We also refer to the works of Angulo Pava [An07, An09] for related results.

Our goal was to study stability using methods not based on integrability. We have established the following results.

**Theorem 4.9.** *Snoidal waves and cnoidal standing waves (for a range of parameters) with fundamental period  $T$  are spectrally stable against  $T$ -periodic perturbations.*

We also have established rigorously the instability of cn previously formally obtained by Rowland [Ro74].

**Theorem 4.10.** *Cnoidal waves are unstable against perturbations whose period is a sufficiently large multiple of its own.*

To obtain some of the above stability results, we have studied variational characterizations of standing waves. We believe that these characterizations are of independent interest.

The functionals involved for the variational characterizations will be the conserved quantities of the flow of (4.5). Let  $T$  be a period of the solution of (NLS). The energy, the mass and the momentum, given by

$$E(u) = \frac{1}{2} \int_0^T |u_x|^2 dx - \frac{b}{4} \int_0^T |u|^4 dx, \quad M(u) = \frac{1}{2} \int_0^T |u|^2 dx, \quad P(u) = \frac{1}{2} \Im \int_0^T u \bar{u}_x dx,$$

are conserved quantities. Due to the integrable nature of (4.5), there exist in fact many more conserved quantities, but we restrict ourselves to the three one which pertain in non-integrable settings.

Since the fundamental work of Cazenave and Lions [CaLi82], we know that orbital stability can be obtained by characterizing the standing wave profile as a (global or local) minimizer of the Hamiltonian (the energy) on fixed mass. In the present situation, if we minimize the energy on the mass for functions in  $P_T$ , we obtain without difficulties the following result.



**Proposition 4.11.** *There exists a unique solution to the minimization problem*

$$\min\{E(u) : u \in P_T, M(u) = m > 0\}.$$

If  $b < 0$ , or  $b > 0$  and  $m \leq \frac{\pi^2}{6T}$ , then the solution is the constant function  $u \equiv \sqrt{\frac{2m}{T}}$ .

If  $b > 0$ , and  $m > \frac{\pi^2}{6T}$ , then the solution is a dnoidal profile.

This readily gives us the orbital stability of dnoidal waves for co-periodic perturbation. To catch the snoidal and cnoidal waves with a minimization procedure turns out to be tricky. For cnoidal waves, we have been able to obtain the following result by restricting the minimization to the class of anti-periodic functions.

**Proposition 4.12.** *Assume that  $b > 0$ . There exists a unique solution to the minimization problem*

$$\min\{E(u) : u \in A_T, M(u) = m > 0\},$$

and it is a cnoidal function.

To prove Proposition 4.12, the key point is to show that minimizers are real valued. This is usually an easy step which is achieved by replacing  $u$  by  $|u|$ , which preserves the mass constraint and implies that  $E(|u|) < E(u)$  unless  $u$  is real-valued. However, such an operation destroys the anti-periodicity of  $u$ . To overcome this difficulty, we use a Fourier coefficients rearrangement. For  $u \in A_T$ , there exists a decomposition

$$u(x) = \sum_{j \text{ odd}} u_j e^{ij \frac{2\pi}{T} x}.$$

We define  $v \in A_T$  by

$$v(x) = \sum_{j \text{ odd}} v_j e^{ij \frac{2\pi}{T} x}, \quad v_j = \sqrt{\frac{|u_j|^2 + |u_{-j}|^2}{2}}.$$

Then it can be proved (see [GuLeTs17]) that

$$\|v\|_{L^2} = \|u\|_{L^2}, \quad \|\partial_x v\|_{L^2} = \|\partial_x u\|_{L^2}, \quad \|v\|_{L^4} \geq \|u\|_{L^4}.$$

Hence by restricting the space of functions on which we minimize to anti-periodic functions, we are able to characterize cn as a minimizer of the energy on a mass constraint.

It turns out that snoidal functions cannot be obtained as minimizers on the mass constraint only. Indeed, we have the following result.

**Proposition 4.13.** *Assume that  $b < 0$ . There exists a unique solution to the minimization problem*

$$\min\{E(u) : u \in A_T, M(u) = m > 0\},$$

and it is the plane wave  $\sqrt{\frac{2m}{T}} e^{\frac{2i\pi x}{T}}$ .

We conjecture that to obtain snoidal functions as minimizers, we need to had a momentum constraint.

**Conjecture 4.14.** *Assume that  $b < 0$ . There exists a unique solution to the minimization problem*

$$\min\{E(u) : u \in A_T, M(u) = m > 0, P(u) = 0\},$$

*and it is a snoidal function.*

As before, the key to prove Conjecture 4.14 would be to show that the minimizers are real-valued. However, the previous rearrangement cannot be used, as we need the opposite inequality on the  $L^4$  norm. We have verified this conjecture using numerical experiments. The numerical scheme consists in taking the discrete gradient flow for the energy with at each step of time a renormalization of the mass and momentum of the solution. On a semi-discrete level, this goes as follows. We define an increasing sequence of time  $t_0 < \dots < t_n < \dots$  and take an initial data  $u_0$ . Between  $t_n$  and  $t_{n+1}$ , we let  $u(t, x)$  evolve along the gradient flow of the energy

$$\begin{cases} u_t = -E'(u) = u_{xx} + b|u|^2u, & x \in \mathbb{R}, \quad t_n < t < t_{n+1}, \quad n \geq 0. \\ u(t_n, x) = u_n(x), \end{cases}$$

At each time step  $t_{n+1}$ , the function is renormalized to satisfy the mass and momentum constraints. To obtain the mass constraint, one can simply proceed to a rescaling and set

$$u_{n+1} = u(t_{n+1}, x) \sqrt{\frac{m}{M(u(t_{n+1}, x))}}.$$

Things are however not so simple for the momentum constraint, as a 0 momentum cannot be obtained by rescaling. Our strategy to overcome this difficulty is inspired by the following observation (see [BaDu04]): the mass renormalization step is equivalent to solving the ordinary differential equation

$$u_t = \mu_n u, \quad t_n < t < t_{n+1}, \quad \mu_n = \frac{1}{t_{n+1} - t_n} \ln \left( \sqrt{\frac{m}{M(u(t))}} \right). \quad (4.7)$$

Therefore, we consider the following problem, which is an equivalent for the momentum renormalization step:

$$u_t = i\varpi_n u_x, \quad x \in \mathbb{R}, \quad t_n < t < t_{n+1}, \quad (4.8)$$

where  $\varpi_n$  is to be chosen in such a way that  $P(u(t_{n+1})) = 0$ . The equation (4.7) is a linear ordinary differential equation, thus it can be solved explicitly easily. Things are a bit more complicated for (4.8), which is a partial differential equation. It can nevertheless be solved (at least formally) in the following way. Consider the Fourier series representation of  $u$  given by

$$u(t, x) = \sum_{j \in \mathbb{Z}} c_j(t) e^{i \frac{2\pi}{T} j x}, \quad c_j(t) = \frac{1}{T} \int_0^T u(t, x) e^{-i \frac{2\pi}{T} j x} dx.$$

Then (4.8) is equivalent to

$$\partial_t c_j = -\frac{2\pi}{T} j \varpi_n c_j, \quad j \in \mathbb{Z}, \quad t_n < t < t_{n+1}.$$

The solution is given for each  $j \in \mathbb{Z}$  and  $t \geq t_n$  by

$$c_j(t) = \exp\left(-\frac{2\pi}{T} j \varpi_n (t - t_n)\right) c_j(t_n).$$

Hence the solution  $u$  of (4.8) is

$$u(t, x) = \sum_{j \in \mathbb{Z}} \exp\left(-\frac{2\pi}{T} j \varpi_n (t - t_n)\right) c_j(t_n) e^{i \frac{2\pi}{T} j x}.$$

The value of  $\varpi_n$  is determined implicitly by the relation

$$P(u(t_{n+1})) = -\sum_{j \in \mathbb{Z}} \pi j \exp\left(-\frac{2\pi}{T} j \varpi_n (t_{n+1} - t_n)\right) |c_j(t_n)|^2 = 0.$$

Details of the implementation of the numerical method can be found in [GuLeTs17]. We have performed experiments to compare the outcome of the algorithm with the theoretical results, and they have been each time in good agreement. In the case where  $sn$  is suspected to be the minimizer, we have made the following observation, which confirms our conjecture.

**Observation 4.15.** *Take  $b < 0$ . The numerical algorithm constructed to find solutions to*

$$\min\{E(u) : u \in A_T, M(u) = m > 0, P(u) = 0\},$$

*converges to a snoidal function.*

## 4.4 Schrödinger Systems

In the works [IaLe14] and [DeLeWe16], in collaboration with Isabella Ianni, Fanny Delebecque and Rada-Maria Weishäupl, I have been interested by solitary wave solutions of Schrödinger systems. This type of systems appears in various physical settings, where they usually model similar phenomena as the scalar nonlinear Schrödinger equation, but with additional physical properties taken into account. For example, in nonlinear optics, the scalar Schrödinger equation modeling the propagation of light in optical fibers becomes a system if polarization of light and birefringence of the fiber are taken into account (see e.g. [Ag07]). From the mathematical point of view, systems have a richer structure than scalar equations, hence one can expect to observe additional behaviors which do not exist in scalar equations. This is in particular the case for the elliptic stationary version of Schrödinger systems, and there is an ever growing literature related

to the existence and classification of solutions. We refer, for example and among many others, to [AmCo07, LiWe05, MaMoPe06, Si07] for earlier works on the topic.

Our goal in [IaLe14, DeLeWe16] was to reinvest the techniques developed to prove the existence of multi-solitary waves in the setting of nonlinear Schrödinger systems. Before presenting our main results, we give some preliminaries.

We consider the nonlinear Schrödinger system

$$\begin{cases} i\partial_t u_1 + \Delta u_1 + \mu_1 |u_1|^2 u_1 + \beta |u_2|^2 u_1 = 0, \\ i\partial_t u_2 + \Delta u_2 + \mu_2 |u_2|^2 u_2 + \beta |u_1|^2 u_2 = 0. \end{cases} \quad (4.9)$$

where  $u_j : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$ ,  $\mu_j > 0$  for  $j = 1, 2$ ,  $\beta \in \mathbb{R} \setminus \{0\}$  and  $d = 1, 2, 3$ .

When  $u_1 \equiv 0$  or  $u_2 \equiv 0$ , the system reduces to the scalar Schrödinger equation

$$i\partial_t u + \Delta u + \mu |u|^2 u = 0.$$

Therefore, the scalar solitary wave solutions of the nonlinear Schrödinger equation are also solutions of the system, with the other component reduced to 0. We have constructed solutions, dubbed *multi-speed solitary waves*, which behaves at large time as scalar solitary waves on each components of the system. Our main result in [IaLe14] was the following.

**Theorem 4.16.** *For  $j = 1, 2$ , take  $\omega_j > 0$ ,  $\gamma_j \in \mathbb{R}$ ,  $x_j, v_j \in \mathbb{R}^d$  and  $\Phi_j \in H^1(\mathbb{R}^d)$  solution to*

$$-\Delta \Phi_j + \Phi_j - |\Phi_j|^2 \Phi_j = 0.$$

*Define the scalar solitary wave  $R_j$  by*

$$R_j(t, x) = \exp\left(i\left(\omega_j t - \frac{|v_j|^2 t}{4} + \frac{1}{2}v_j \cdot x + \gamma_j\right)\right) \sqrt{\frac{\omega_j}{\mu_j}} \Phi_j(\sqrt{\omega_j}(x - v_j t - x_j)).$$

*Define also  $v_\star$  and  $\omega_\star$  by*

$$v_\star = |v_1 - v_2|, \quad \omega_\star = \frac{1}{4} \min\{\omega_1, \omega_2\}.$$

*Then there exists  $v_\sharp > 0$  such that if  $v_\star > v_\sharp$  then there exist  $T_0 \in \mathbb{R}$  and a solution  $(u_1, u_2)$  of (4.9) defined on  $[T_0, \infty)$  and such that for all  $t \in [T_0, \infty)$  we have the following estimate*

$$\|(u_1, u_2)(t) - (R_1, R_2)(t)\|_{H^1 \times H^1} \leq e^{-\sqrt{\omega_\star} v_\star t}.$$

The proof of Theorem 4.16 is relatively similar to the one of Theorem 3.1. In fact, it turns out to be even simpler: we do not need to perform any localization procedure, since, as we observed, the coupling term acts as a localizing factor. Remark also that our result is independent of the coupling factor, which can be attractive or repulsive. As for Theorem 3.1, we require a high relative speed assumption to obtain the existence of multi-speed solitary waves. This is due to our lack of knowledge on the nature of the profiles  $\Phi_j$  (ground states or excited states, stable or unstable). When the profiles  $\Phi_j$  are

stable ground states of the scalar equation (which implies that we are in dimension 1), we can follow the lines of the proof of the existence of multi-solitons for  $L^2$ -subcritical nonlinear Schrödinger equations [MaMe06] to obtain the following result, which is one of the results that we obtained in [DeLeWe16].

**Theorem 4.17.** *Let  $d = 1$ . Make the same hypotheses as in Theorem 4.16 and assume moreover that the  $\Phi_j$  are (stable) ground state profiles. Then  $v_{\sharp}$  can be taken  $v_{\sharp} = 0$  and there exist  $\alpha > 0$ ,  $T_0 \in \mathbb{R}$  and  $(u_1, u_2)$  solution to (4.9) on the time interval  $[T_0, \infty)$  such that for all  $t \in [T_0, \infty)$  we have the following estimate*

$$\|(u_1, u_2)(t) - (R_1, R_2)(t)\|_{H^1 \times H^1} \leq e^{-\alpha \sqrt{\omega_*} v_* t}.$$

The restriction to dimension 1 in Theorem 4.17 stems from the fact that the scalar nonlinear Schrödinger equation with cubic nonlinearity has stable ground states only in dimension 1. The strategy of the proof of Theorem 4.17 is similar to the one introduced in [MaMe06]. Beside the modifications required to handle the fact that we are now working with a system, the main novelty for the proof of Theorem 4.17 lies in a technical artifact consisting in introducing arbitrary masses in the definition of the global action. We refer to [DeLeWe16] for the details of the proof.

We complemented the analysis in [DeLeWe16] by numerical experiments. We have used two numerical schemes. The first scheme is used to simulate the dynamics of the nonlinear Schrödinger system (4.9) in one dimension. We solve the system on a bounded interval  $(-a, a)$  with periodic boundary conditions (so as to be able to use a Fourier decomposition). The number  $a$  is chosen large enough so that the influence of the boundary does not show up too early in the simulation. We discretized the interval  $(-a, a)$  with a uniform spatial grid with mesh size  $h > 0$  and grid points  $x_k = x_0 + kh$ ,  $k = 0, \dots, K$ ,  $K$  odd,  $h = 2a/K$ . We denote by  $t_0$  the initial time and  $\tau$  the discrete time step, so that the time grid is given by  $t_n = t_0 + n\tau$ ,  $n \in \mathbb{N}$ . The discrete solution  $(u_j)_k^n$  approximate  $u_j(t_n, x_k)$ . We use a time-splitting algorithm. We split the system (4.9) into two subsystems

$$i\partial_t u_j = -\mu_j |u_j|^2 u_j - \beta |u_{3-j}|^2 u_j, \quad j = 1, 2, \quad (4.10)$$

$$i\partial_t u_j = -\partial_{xx} u_j, \quad j = 1, 2. \quad (4.11)$$

The system (4.10) preserves the moduli  $|u_j|$  ( $j = 1, 2$ ), hence it is in fact a linear differential system with constant coefficients and can be solved explicitly. The system (4.11) is also linear, but involves a spatial second derivative. To solve it, we discretize in space, perform a discrete Fourier transform, solve exactly in time and perform a reverse discrete Fourier transform. We use a Strang time splitting scheme, which happens in three steps.

**Step 1:** We solve (4.10) on  $[t_n, t_n + \tau/2]$ :

$$(u_j)_k^* = \exp\left(i\frac{\tau}{2} \left(\mu_j |(u_j)_k^n|^2 + \beta |(u_{3-j})_k^n|^2\right)\right) (u_j)_k^n, \quad j = 1, 2, \quad k = 0, \dots, K.$$

**Step 2:** We solve (4.11) on  $[t_n, t_n + \tau]$ :

$$(u_j)_k^{**} = \frac{1}{K+1} \sum_{-\frac{K}{2}}^{\frac{K}{2}} \exp(-i\tau\nu_m^2) (\hat{u}_j)_m^* \exp(i\nu_m(x_k - x_0)), \quad j = 1, 2,$$

where  $\nu_m = 2\pi m/(x_K - x_0)$  and

$$(\hat{u}_j)_m^* = \sum_{l=0}^K (u_j)_l^* \exp(-i\nu_m(x_l - x_0)), \quad m = -\frac{K}{2}, \dots, \frac{K}{2}.$$

**Step 3:** We repeat Step 1 on  $[t_n + \tau/2, t_{n+1}]$  with  $(u_j)_k^{**}$  instead of  $(u_j)_k^n$  and we obtain  $(u_j)_k^{n+1}$ .

The second scheme is used to calculate a ground state solution of the elliptic system

$$\begin{cases} -\partial_{xx}\phi_1 + \omega_1\phi_1 - \mu_1\phi_1^3 - \beta|\phi_2|^2\phi_1, \\ -\partial_{xx}\phi_2 + \omega_2\phi_2 - \mu_2\phi_2^3 - \beta|\phi_1|^2\phi_2. \end{cases} \quad (4.12)$$

We obtain the ground state by minimizing the energy

$$E(\phi_1, \phi_2) = \frac{1}{2} \|\partial_x \phi_1\|_{L^2}^2 + \frac{1}{2} \|\partial_x \phi_2\|_{L^2}^2 - \frac{\mu_1}{4} \|\phi_1\|_{L^4}^4 - \frac{\mu_2}{4} \|\phi_2\|_{L^4}^4 - \frac{\beta}{2} \int_{\mathbb{R}} |\phi_1|^2 |\phi_2|^2 dx,$$

on fixed masses

$$M(\phi_1) = \frac{1}{2} \|\phi_1\|_{L^2}^2 = m_1, \quad M(\phi_2) = \frac{1}{2} \|\phi_2\|_{L^2}^2 = m_2.$$

Hence  $\omega_1$  and  $\omega_2$  are obtained as Lagrange multipliers for the above problem. We obtain the ground state as solutions of the normalized gradient flow, which was in particular studied for scalar Schrödinger equation in [BaDu03]. Starting from some initial data, we let the solution evolve along the gradient flow of the energy, with renormalization of the masses at each time step. We use the same notations as for the previous scheme for the space and time grids. We denote by  $(\phi_j)_k^n$  the approximation of  $\phi_j(t_n, x_k)$ . The normalized gradient flow is discretized using a semi-implicit backward Euler finite differences scheme, which we describe now.

**Step 1:** We solve on  $[t_n, t_{n+1}]$  with initial data  $(\phi_j)_k^n$  by setting for  $j = 1, 2$ :

$$\frac{(\phi_j)_k^* - (\phi_j)_k^n}{\tau} = \frac{(\phi_j)_{k+1}^* - 2(\phi_j)_k^* + (\phi_j)_{k-1}^*}{h^2} + \mu_j |(\phi_j)_k^n|^2 (\phi_j)_k^* + \beta |(\phi_{3-j})_k^n|^2 (\phi_j)_k^*.$$

**Step 2:** The result  $(\phi_j)_k^*$  of the previous step is normalized:

$$(\phi_j)_k^{n+1} = \frac{m_j (\phi_j)_k^*}{M((\phi_j)^*)}.$$

For  $t \rightarrow \infty$ , we expect the numerical scheme to converge toward an approximation of a ground state solution  $(\phi_1, \phi_2)$  of (4.12).

We have performed various numerical simulations to observe the possible interactions between solitary waves, varying the parameters of the system and of the solitary waves considered. The multi-speed solitary waves constructed in Theorems 4.16 and 4.17 have been observed in practice, which means that they should be (orbitally) stable. However, we have not been able to establish this fact theoretically. We refer to [DeLeWe16] for the details of the numerical experiments and the possible interactions (elastic and inelastic interaction, reflection). We present here only an excerpt concerning the case of symmetric collision.

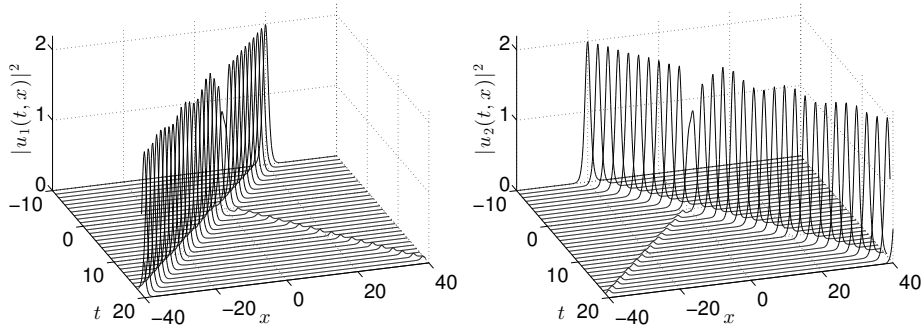


Figure 4.2: Position densities  $|u_1(t, x)|^2$  and  $|u_2(t, x)|^2$  as functions of space and time

We set  $\mu_1 = \mu_2 = 1$  and  $\beta = 3$ . As initial data in the experiment, we take

$$u_j(t_0, x) = \exp\left(i\left(\omega_j t_0 - \frac{v_j^2}{4}t + \frac{1}{2}v_j x\right)\right) Q_{\omega_j}(x - v_j t_0 - x_j), \quad j = 1, 2, \quad Q_\omega = \sqrt{\omega}Q(\sqrt{\omega}x).$$

The system is simulated on the space interval  $(-200, 200)$ ,  $K = 4096$ , initial time  $t_0 = -10$ , time step  $\tau = 10^{-3}$  and initial parameters  $\omega_1 = \omega_2 = 1$ ,  $v_1 = -v_2 = 2$ ,  $x_1 = x_2 = 0$ .

Therefore, we start with solitons located at  $\pm 20$ , moving toward each other at speed 2 with full collision at  $t = 0$  and we observe them up to time  $t = 40$ . The collision can be observed on Figure 4.2. The outcome of the collision are two large solitary waves corresponding to the incoming solitary waves, which underwent extraction of a small part at the collision time. The small extracted parts are traveling with the other component after the collision.

At the final time, we have compared the left hand part and the right hand part with (translated) ground state of the system calculated with the second algorithm. Precisely, we set

$$u_j^\pm(x) = |u_j(40, x)|^2 \chi_{[0, \infty)}(\pm x), \quad m_j^\pm = \int_{\mathbb{R}} |u_j^\pm(x)|^2 dx, \quad j = 1, 2.$$

to capture the mass of each component on the left and on the right. Then we run the ground state algorithm with masses  $m_j^\pm$  and denote by  $\phi_j^\pm$  the outcome. The comparison between the solution at time 40 on the left and on the right  $u_j^\pm$  and the (translated) ground state  $\phi_j^\pm$  is made in Figure 4.3. We realize that they fit very well. Hence we

conclude that in this case, the result of the collision is a new repartition of the mass and energy so as to approximate a ground state profile.

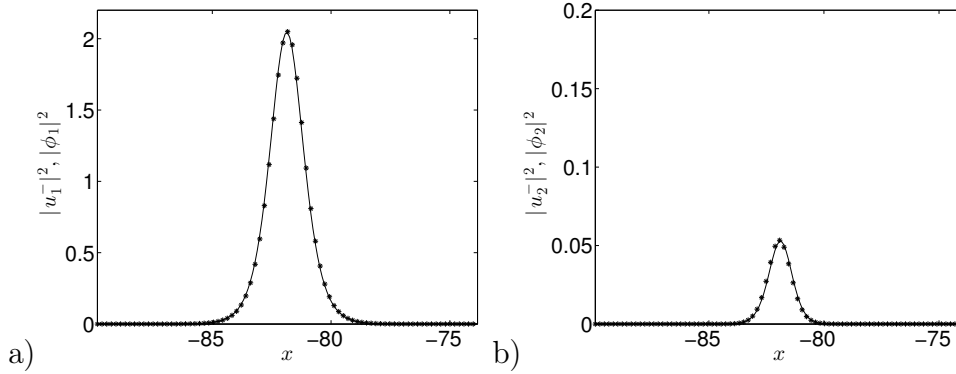


Figure 4.3: (a) Plots of  $|u_1^-(40, x)|^2$  (continuous line) and  $|\phi_1(x + x_1)|^2$  ('\*' dots) (b) Plots of  $|u_2^-(40, x)|^2$  and  $|\phi_2(x + x_2)|^2$ , where  $x_j$  is the  $x$  corresponding to the  $\max |u_j|^2$ , with  $j = 1, 2$ .

## 4.5 Semi-classical Problems

In this Section, I present two results of orbital stability for semi-classical problems which I have obtained in collaboration with Isabella Ianni [IaLe09] and Marco Ghimenti and Marco Squassina [GhLeSq13]. The problem considered in [IaLe09] was the following Schrödinger-Poisson equation

$$-i\varepsilon\Psi_t - \varepsilon^2\Delta_x\Psi + W(x)\Psi + K(x)(|x|^{-1} * K(x)|\Psi|^2)\Psi - |\Psi|^{p-1}\Psi = 0. \quad (4.13)$$

At the beginning of her PhD under the supervision of Antonio Ambrosetti (who was also my post-doc supervisor at the time of the work [IaLe09]), Isabella Ianni obtained, in collaboration with Guisi Vaira [IaVa08], the existence of single spikes standing wave solutions  $\Psi(t, x) = e^{i\omega t/\varepsilon}v(x)$ , where the profile  $v$  verifies

$$-\varepsilon^2\Delta v + (W + \omega)v + K(|x|^{-1} * Kv^2)v - |v|^{p-1}v = 0 \quad (4.14)$$

and is concentrating at a non-degenerate critical point of the potential  $W$ . More precisely, the result was the following. Let  $p \in (1, 5)$  and make the following assumptions on  $W$  and  $K$ .

(V1)  $W \in \mathcal{C}^\infty(\mathbb{R}^3)$ ,  $W$  and its derivatives are uniformly bounded.

(V2)  $\inf_{\mathbb{R}^3} \{W + \omega\} > 0$ .

(V3) There exists  $x_0 \in \mathbb{R}^3$  such that  $\nabla W(x_0) = 0$ .

(K1)  $K \in \mathcal{C}^\infty(\mathbb{R}^3)$ ,  $K$  and its derivatives are uniformly bounded.



(K2)  $K \geq 0$ .

Let  $x_0$  be a non-degenerate critical point for  $W$ . Then, for  $\varepsilon$  small enough, there exists  $v_\varepsilon \in H^1(\mathbb{R}^3)$ ,  $v_\varepsilon > 0$ , such that  $v_\varepsilon$  is a solution of (4.14) and

$$\left\| v_\varepsilon - U_\lambda \left( \frac{\cdot - x_0}{\varepsilon} \right) \right\|_{H^1(\mathbb{R}^3)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

where  $\lambda^2 = W(x_0) + \omega$  and  $U_\lambda \in H^1(\mathbb{R}^3)$  denotes the unique positive radial solution of

$$-\Delta u + \lambda^2 u - u^p = 0.$$

Moreover there exist  $\xi_\varepsilon \in \mathbb{R}^3$  and  $w_\varepsilon \in H^1(\mathbb{R}^3)$  such that

$$v_\varepsilon = U_\lambda \left( \frac{\cdot - x_0}{\varepsilon} - \xi_\varepsilon \right) + w_\varepsilon \left( \frac{\cdot - x_0}{\varepsilon} \right), \quad \xi_\varepsilon \rightarrow 0 \text{ in } \mathbb{R}^3, \quad \|w_\varepsilon\|_{H^1(\mathbb{R}^3)} \leq C\varepsilon^2.$$

Antonio Ambrosetti suggested to Isabella Ianni and myself to consider the orbital stability of the standing waves constructed in [IaVa08]. The main difficulty with problem (4.13) is that  $K$  and  $W$  are both non-trivial, hence strongly breaking any homogeneity in space.

The main results in [IaLe09] are summarized in the following theorem.

**Theorem 4.18.** *Let  $x_0$  be a non-degenerate critical point for the potential  $W$ . Let  $m$  denote the number of negative eigenvalues of the matrix  $\text{Hess } W(x_0)$ . Assume that the parameter  $\varepsilon$  is small enough.*

- Case  $p < 1 + \frac{4}{3}$ :  $\Psi_\varepsilon$  is orbitally stable if  $x_0$  is a local minimum and orbitally unstable if  $m$  is odd.
- Case  $p > 1 + \frac{4}{3}$ :  $\Psi_\varepsilon$  is unstable if  $x_0$  is a local minimum or if  $m$  is even.
- Case  $p = 1 + \frac{4}{3}$ . Assume moreover that  $\Delta W(x_0) - K(x_0)^2 [W(x_0) + \omega]^{\frac{2}{p-1}} C \neq 0$ , where the constant  $C$  is explicitly known and positive. Then  $\Psi_\varepsilon$  is orbitally stable if  $x_0$  is a local minimum and

$$\Delta W(x_0) > K(x_0)^2 [W(x_0) + \omega]^{\frac{2}{p-1}} C,$$

while it is unstable if  $x_0$  is a local minimum and

$$\Delta W(x_0) < K(x_0)^2 [W(x_0) + \omega]^{\frac{2}{p-1}} C,$$

or if the quantity

$$m - \frac{1}{2} \left( 1 + \frac{\Delta W(x_0) - K(x_0)^2 [W(x_0) + \omega]^{\frac{2}{p-1}} C}{|\Delta W(x_0) - K(x_0)^2 [W(x_0) + \omega]^{\frac{2}{p-1}} C|} \right)$$

is even.

## 4 On Blow-Up and Stability

In collaboration with Marco Ghimenti and Marco Squassina [GhLeSq13], I considered the same type of question, this time for a semi-classical nonlinear Klein-Gordon equation given by

$$\varepsilon^2 u_{tt} + 2i\varepsilon V u_t - \varepsilon^2 \Delta u + m u - W u - |u|^{p-1} u = 0, \quad (4.15)$$

where  $u : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{C}$  and  $V, W$  are real-valued potential functions. Standing waves are solutions to (4.15) of the form  $e^{i\omega t/\varepsilon} \phi_\omega(\varepsilon)$ , where  $\phi_\omega \in H^1(\mathbb{R}^d)$  verifies

$$-\Delta \phi_\omega + (m - \omega^2 - 2\omega V(\varepsilon y) - W(\varepsilon y)) \phi_\omega - |\phi_\omega|^{p-1} \phi_\omega = 0 \quad (4.16)$$

We assume that  $V$  and  $W$  satisfy

$$V, W \in \mathcal{C}^2(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d).$$

Moreover, we assume that for the function  $Z$  defined by

$$Z(y) = m - \omega^2 - 2\omega V(y) - W(y)$$

we have a lower bound

$$\inf_{x \in \mathbb{R}^d} Z(x) > 0,$$

and there exists a critical point  $x_0 \in \mathbb{R}^d$  such that

$$\nabla Z(x_0) = 0, \quad \nabla^2 Z(x_0) \text{ is non-degenerate.}$$

Under these hypotheses, and when  $\varepsilon$  is close to 0, the equation (4.16) admits a family of positive, exponentially decaying solutions  $(\phi_{\omega,\varepsilon}) \subset H^1(\mathbb{R}^d)$  (see e.g. [AmBaCi97, AmMa06]). One of the main conclusions of [GhLeSq13] is the following stability/instability result.

**Theorem 4.19.** *Assume that (4.15) is locally well-posed in  $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$  and that  $x_0$  is a non-degenerate local minimum of  $Z$ . Then for  $\varepsilon > 0$  small enough the standing waves  $e^{i\omega t} \phi_\omega$  are stable if  $p < 1 + 4/d$  and*

$$Z(x_0) < (\omega + V(x_0))^2 \left( \frac{4}{p-1} - d \right),$$

and unstable if

$$Z(x_0) > (\omega + V(x_0))^2 \left( \frac{4}{p-1} - d \right),$$

or if  $p \geq 1 + 4/d$ .

In [IaLe09] and in [GhLeSq13], the stability/instability results is obtained by working in the framework developed by Grillakis, Shatah and Strauss [GrShSt87, GrShSt90]. We explain the strategy here using the notations of problem (4.13). In [GrShSt87, GrShSt90], stability/instability is determine using two pieces of information:

- *The spectral information:* the number  $n(L_\varepsilon)$  of negative eigenvalues of the linearized action  $L_\varepsilon$  around the standing wave profile.
- *The slope information:* the sign of  $\partial_\omega Q(\phi_\omega)$ . We set  $p(\omega) = 0$  if  $\partial_\omega Q(\phi_\omega) < 0$ , and  $p(\omega) = 1$  if  $\partial_\omega Q(\phi_\omega) > 0$ .

Here,  $Q(\phi_\omega)$  denotes the mass  $\frac{1}{2}\|u\|_{L^2}^2$  for problem (4.13) and the charge  $\frac{1}{2}\Im \int u\bar{u}_t dx$  for problem (4.15). According to the theory of [GrShSt87, GrShSt90], stability holds if  $n(L_\varepsilon) = p(\omega)$ . If  $n(L_\varepsilon) - p(\omega)$  is odd, then it was proved in [GrShSt87, GrShSt90] that linear instability holds.

To obtain the spectral information, we proceed by a perturbation argument. When  $\varepsilon \rightarrow 0$ , the linearized operator  $L_\varepsilon$  converges, at least formally, toward a limit operator  $L_0$  whose spectrum turns out to be well-known. It follows from the perturbation theory for linear operators that the spectrum of  $L_\varepsilon$  is close to the one of  $L_0$ . The main issue is to determine whether the 0 eigenvalue of  $L_0$  is shifted for  $L_\varepsilon$  to negative or positive side of the real line. To solve this issue, we perform an  $\varepsilon$ -expansion of the eigenvalues close to 0 of  $L_\varepsilon$ . In the Schrödinger-Poisson case (4.13), we find that their sign is related to the eigenvalues of the matrix  $\nabla^2 W(x_0)$ .

To obtain the slope information, we again perform an  $\varepsilon$  expansion of the quantity  $Q(\phi_\omega)$ . In the Schrödinger-Poisson case (4.13), it is not hard to obtain the slope information in the non-critical case  $p \neq 1 + \frac{4}{3}$ . However, in the critical case, there is a degeneracy at  $\varepsilon = 0$ , as  $\partial_\omega Q(\phi_\omega) = 0$ . A deeper investigation is required in that case, we refer to [IaLe09] for the details.

# 5 Prospects

I present in this section two topics on which I am currently working. The first one is the Manakov system. It is a Schrödinger system, which is completely integrable and hence of specific interest. In our work in progress, we construct and study a large class of standing wave solutions. Despite the numerous works devoted to this system, it seems that this type of standing wave solutions was not previously considered in the literature. The second project presented in this section concerns the excited states of nonlinear Schrödinger equations. The goal of this work is to construct new type of numerical methods for the numerical computation of excited states, using in particular variational characterizations of nodal states.

## 5.1 The Manakov system

In collaboration with François Genoud, Simona Rota-Nodari and Stephan De Bièvre, I am considering the system of coupled nonlinear Schrödinger equations:

$$\begin{cases} iu_t + u_{xx} + \lambda(|u|^2 + |v|^2)u = 0, \\ iv_t + v_{xx} + \lambda(|u|^2 + |v|^2)v = 0. \end{cases} \quad (5.1)$$

Here,  $(t, x) \in \mathbb{R} \times \mathbb{K}$ , where  $\mathbb{K}$  is either the line  $\mathbb{R}$  or the torus  $\mathbb{T}$ ,  $u, v : \mathbb{R} \times \mathbb{K} \rightarrow \mathbb{C}$  and  $\lambda = \pm 1$ . The system (5.1) is called the *Manakov system*. We say that the system is *focusing* if  $\lambda = 1$  and that it is *defocusing* if  $\lambda = -1$ . It is known to be a completely integrable Hamiltonian system (see [AbPrTr04] and the references cited therein).

The Manakov system (5.1) is a specific case of a more generic type of Schrödinger systems of the form

$$\begin{cases} iu_t + u_{xx} + \lambda(\alpha|u|^2 + \beta|v|^2)u = 0, \\ iv_t + v_{xx} + \lambda(\beta|u|^2 + \gamma|v|^2)v = 0. \end{cases} \quad (5.2)$$

If  $\alpha$ ,  $\beta$  and  $\gamma$  are such that the potential energy term in the Hamiltonian

$$H(u, v) = \frac{1}{2}\|u_x\|_{L^2}^2 + \frac{1}{2}\|v_x\|_{L^2}^2 - \lambda \left( \frac{\alpha}{4}\|u\|_{L^4}^4 + \frac{\gamma}{4}\|v\|_{L^4}^4 + \frac{\beta}{2} \int_{\mathbb{K}} |u|^2 |v|^2 dx \right)$$

does not form a perfect square then the system is (a priori) not integrable. Namely, (5.2) with  $(\alpha, \beta) \neq (\gamma, \gamma)$  is not integrable. Systems with  $(\alpha, \beta) = (\gamma, \gamma)$ ,  $\gamma \neq 0$ , are equivalent to (5.1) by a simple rescaling.

The presence of symmetry in Hamiltonian systems has important consequences on the dynamics, e.g. existence of conservation laws and of solitary waves solutions. We

consider globally Hamiltonian symmetry groups, in the terminology of [DeGeRo15, DeRo18]. Those are precisely the symmetry groups giving rise to conserved quantities via Noether's theorem. For non-integrable Schrödinger systems (5.2), the largest (globally Hamiltonian) symmetry group is  $G = \mathbb{R}^2 \times \mathbb{R}$  (which is isomorphic to  $U(1) \times U(1) \times \mathbb{R}$ ), with group action  $\Phi_g$  given by

$$(\Phi_g U)(x) = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix} U(x - a), \quad g = (\theta_1, \theta_2, a) \in G.$$

The Manakov system (5.1) admits a much larger group of (globally Hamiltonian) symmetries, given by  $G = U(2) \times \mathbb{R}$ , and it is what makes it interesting to analyse. As before,  $\mathbb{R}$  stands for translations in the space variable  $x$ , while  $U(2)$  is the Lie group of unitary matrices in dimension 2, i.e.

$$U(2) = \{M \in \mathcal{M}_{2 \times 2}(\mathbb{C}) : M \bar{M}^T = I\}.$$

The group action of  $U(2)$  is simply multiplication of  $U$  (with components written in column) by an element  $M$  of  $U(2)$ . The group action  $\Phi_g$  is thus given by

$$(\Phi_g U)(x) = M \cdot U(x - a), \quad g = (M, a) \in G.$$

Recall here that the special unitary group  $SU(2)$  is the (Lie) subgroup of unitary matrices with determinant 1. The Lie algebra  $\mathfrak{u}(2)$  of  $U(2)$  is described by

$$\mathfrak{u}(2) = \{M \in \mathcal{M}_{2 \times 2}(\mathbb{C}) : \bar{M}^T = -M, \},$$

whereas the Lie algebra  $\mathfrak{su}(2)$  of  $SU(2)$  is described by

$$\mathfrak{su}(2) = \{M \in \mathcal{M}_{2 \times 2}(\mathbb{C}) : \bar{M}^T = -M, \operatorname{tr}(M) = 0\}.$$

Denote by  $\sigma_j$ ,  $j = 0, 1, 2, 3$  the Pauli matrices

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Lie algebra  $\mathfrak{u}(2)$  (resp.  $\mathfrak{su}(2)$ ) is generated by

$$i\sigma_0, i\sigma_1, i\sigma_2, i\sigma_3, \quad (\text{resp. } i\sigma_1, i\sigma_2, i\sigma_3).$$

The conserved quantities associated to each of these generators are the following:

$$\begin{aligned} \langle \sigma_0 U, U \rangle &= \|u\|_{L^2}^2 + \|v\|_{L^2}^2, & \langle \sigma_1 U, U \rangle &= 2\Re \int_{\mathbb{K}} u \bar{v} dx, \\ \langle \sigma_2 U, U \rangle &= 2\Im \int_{\mathbb{K}} u \bar{v} dx, & \langle \sigma_3 U, U \rangle &= \|u\|_{L^2}^2 - \|v\|_{L^2}^2. \end{aligned}$$

Combining the first and last conserved quantities gives conservation of the mass of each component. We define

$$\begin{aligned} F_1(U) &= \frac{1}{2} \|u\|_{L^2}^2, & F_2(U) &= \frac{1}{2} \|v\|_{L^2}^2, \\ F_3(U) &= \Re \int_{\mathbb{K}} u \bar{v} dx, & F_4(U) &= \Im \int_{\mathbb{K}} u \bar{v} dx. \end{aligned}$$

The quantities  $F_1$  and  $F_2$  are conserved in all Schrödinger systems of the type (5.2), whereas  $F_3$  and  $F_4$  are specific to the Manakov system (5.1).

Given  $\mu \in \mathbb{R}^4$ , define

$$\Sigma_\mu = \{U \in L^2(\mathbb{K}) \times L^2(\mathbb{K}) : F(U) = \mu\},$$

and let  $U(2)_{\Sigma_\mu}$  be the subgroup of  $U(2)$  leaving  $\Sigma_\mu$  invariant, i.e.

$$U(2)_{\Sigma_\mu} = \{M \in U(2) : \forall U \in \Sigma_\mu, MU \in \Sigma_\mu\}.$$

We give a description of  $U(2)_{\Sigma_\mu}$  in the next proposition.

**Proposition 5.1.** *Let  $\mu \in \mathbb{R}^4$ . Denote  $U(2)_{\Sigma_\mu}$  the subgroup of  $U(2)$  leaving the constraint  $\Sigma_\mu$  invariant. Set  $\tilde{\mu} = (\mu_1 - \mu_2, \mu_3, \mu_4)^T$ .*

- (i) *If  $(\mu_1 - \mu_2, \mu_3, \mu_4) \neq (0, 0, 0)$ , then  $U(2)_{\Sigma_\mu}$  is isomorphic to  $U(1) \times U(1)$ .*
- (ii) *If  $(\mu_1 - \mu_2, \mu_3, \mu_4) = (0, 0, 0)$ , then  $U(2)_{\Sigma_\mu} = U(2)$ .*

It is a general fact that symmetries of Hamiltonian systems can lead to standing waves solutions. For the Manakov system (5.1), the symmetries can be used to construct standing waves (see [DeGeRo15] for a systematic derivation). Precisely, standing waves of (5.1) are solutions of the form

$$U(t, x) = \exp(t\xi)\Phi,$$

where  $\Phi = (\phi_1, \phi_2)$  and  $\xi \in \mathfrak{u}(2)$ . Using  $i(\sigma_0 + \sigma_3)/4$ ,  $i(\sigma_0 - \sigma_3)/4$ ,  $i\sigma_1/2$ ,  $i\sigma_2/2$  as a basis of  $\mathfrak{u}(2)$ , we may identify  $\mathfrak{u}(2)$  with  $\mathbb{R}^4$ . Representing  $\xi \in \mathfrak{u}(2)$  by its coordinates  $\tilde{\xi} = (\xi_1, \dots, \xi_4) \in \mathbb{R}^4$ , we obtain the following equation for  $\Phi$ :

$$H'(\Phi) + \tilde{\xi} \cdot F'(\Phi) = 0,$$

where  $F = (F_1, \dots, F_4)$ . Explicitly, we obtain the following system of linearly and nonlinearly coupled elliptic equations:

$$\begin{cases} -\partial_{xx}\phi_1 - \lambda(|\phi_1|^2 + |\phi_2|^2)\phi_1 + \xi_1\phi_1 + \xi_3\phi_2 - i\xi_4\phi_2 = 0, \\ -\partial_{xx}\phi_2 - \lambda(|\phi_1|^2 + |\phi_2|^2)\phi_2 + \xi_2\phi_2 + \xi_3\phi_1 + i\xi_4\phi_1 = 0. \end{cases} \quad (5.3)$$

Being a solution to (5.3) is equivalent to being a critical point of the action functional given by

$$S(\Phi) = H(\Phi) + \tilde{\xi} \cdot F(\Phi).$$

Existence and stability of standing waves for the Manakov are new open problems when the full set of Lagrange multipliers  $\xi_1, \dots, \xi_4$  is taken into account for the construction of the profile. Here are some partial answers that we have been able to obtain so far.

First, we consider constant solutions of (5.3). Any couple of constants  $(\phi_1, \phi_2) \in \mathbb{C}^2$  can be made into a solution of (5.3) with a suitable choice of  $\tilde{\xi}$ .

**Lemma 5.2.** *Let  $(\phi_1, \phi_2) \in \mathbb{C}^2$  and assume that  $\phi_1 \neq 0$ . Then for any  $\xi_2 \in \mathbb{R}$  there exist (unique)  $\xi_1, \xi_3, \xi_4 \in \mathbb{R}$ , explicitly given in terms of  $(\phi_1, \phi_2, \xi_2, \lambda)$ , such that  $(\phi_1, \phi_2)$  is a solution of (5.3).*

However, the dynamics of these constant solutions as standing wave of (5.1) is not very rich; In particular, it is not richer than the dynamics of constant standing waves solutions of scalar Schrödinger equations. Precisely, we have the following result.

**Lemma 5.3.** *Let  $(\phi_1, \phi_2) \in \mathbb{C}^2$  and assume that  $\phi_1 \neq 0$ . Take  $\xi_2 \in \mathbb{R}$ , let  $\xi_1, \xi_3, \xi_4 \in \mathbb{R}$  be given by Lemma 5.2 and define  $\xi \in \mathfrak{u}(2)$  by*

$$\xi = \frac{1}{2} \left( i\xi_1 \frac{\sigma_0 + \sigma_3}{2} + i\xi_2 \frac{\sigma_0 - \sigma_3}{2} + i\xi_3 \sigma_1 + i\xi_4 \sigma_2 \right).$$

Then the standing wave solution of (5.1) given by

$$\exp(t\xi) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

reduces in fact to

$$\exp(it\lambda(|\phi_1|^2 + |\phi_2|^2)) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

The result follows from explicit (but cumbersome) calculations.

Assuming that  $\mathbb{K} = S^1/(2\pi)$  and given  $\mu \in \mathbb{R}^4$ , we now look for constant solutions of (5.3) on  $\Sigma_\mu$ .

**Lemma 5.4.** *Let  $\mu = (\mu_1, \dots, \mu_4) \in \mathbb{R}^4$  be such that  $(\mu_1, \mu_2) \neq (0, 0)$ . There exists a constant solution to (5.3) living on  $\Sigma_\mu$  if and only if  $\mu_1, \mu_2 \geq 0$  and*

$$4\mu_1\mu_2 = \mu_3^2 + \mu_4^2.$$

Standing waves with constant profiles are nonlinearly stable in  $H^1(\mathbb{T})$  if  $\lambda < 0$ .

**Proposition 5.5.** *Assume that  $\lambda < 0$ . For any  $\Phi = (\phi_1, \phi_2) \in \mathbb{C}^2$ , the corresponding standing wave of (5.1), given by  $e^{it\lambda(|\phi_1|^2 + |\phi_2|^2)/2}\Phi$  is orbitally stable in the following sense. For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any  $U_0 \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$  the following property is verified. If*

$$\|U_0 - \Phi\|_{H^1(\mathbb{T})} < \delta,$$

then for any  $t \in \mathbb{R}$  the corresponding solution  $U$  of (5.1) verifies

$$\inf_{\theta_1, \theta_2 \in \mathbb{R}} \|U(t) - (e^{i\theta_1}\phi_1, e^{i\theta_2}\phi_2)\|_{H^1(\mathbb{T})} < \varepsilon.$$

This is a direct consequence of the following variational characterization.

**Lemma 5.6.** *Assume that  $\lambda < 0$ . Let  $m_1, m_2 \geq 0$ . Consider the minimizing problem*

$$\mu = \inf\{E(u_1, u_2) : M(u_1) = m_1, M(u_2) = m_2, (u_1, u_2) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})\},$$

and the corresponding set of minimizers

$$\mathcal{G} = \{(\phi_1, \phi_2) \in H^1(\mathbb{T}) \times H^1(\mathbb{T}) : E(\phi_1, \phi_2) = \mu, M(\phi_1) = m_1, M(\phi_2) = m_2\}.$$

The set  $\mathcal{G}$  is non empty and contains only constants, i.e.

$$\mathcal{G} = \{(\phi_1, \phi_2) \in \mathbb{C}^2 : \pi|\phi_1|^2 = m_1, \pi|\phi_2|^2 = m_2\}.$$

We remark here that if the coefficient of the nonlinearities of the Schrödinger system (5.2) do not form a perfect square as in the Manakov system (5.1), then Lemma 5.6 is true only if  $\beta^2 \leq \alpha\gamma$ . Indeed, the energy can then be written as

$$E(u_1, u_2) = \frac{1}{2} (\|\partial_x u_1\|_{L^2}^2 + \|\partial_x u_2\|_{L^2}^2) - \frac{\lambda}{4} (\|\sqrt{\alpha}|u_1|^2 + \sqrt{\gamma}|u_2|^2\|_{L^2}^2 + 2(\beta - \sqrt{\alpha\gamma})\| |u_1||u_2|\|_{L^2}^2).$$

If  $\beta - \sqrt{\alpha\gamma} < 0$ , then to minimize the energy one wants to increase  $\| |u_1||u_2|\|_{L^2}$ . We achieve the maximum when  $|u_1|$  is proportional to  $|u_2|$ , which is the case for constants. If  $\beta - \sqrt{\alpha\gamma} > 0$ , then to minimize the energy one wants to decrease  $\| |u_1||u_2|\|_{L^2}$ . This enters in competition with the other terms in the energy and constants are no longer the only candidates for minimizers.

We now consider the linearization of (5.1) around a standing wave with constant profile  $\Phi = (\phi_1, \phi_2)$ . As we have seen, we have a one parameter family of Lagrange parameter  $(\xi) \subset \mathbb{R}^4$ , but which reduces in any case to following standing wave dynamics

$$\exp(it\lambda(|\phi_1|^2 + |\phi_2|^2)) \Phi.$$

Therefore, for  $U$  solution of (5.1) and  $\Upsilon = (\varepsilon_1, \varepsilon_2)$  we set

$$U(t) = \exp(it\lambda(|\phi_1|^2 + |\phi_2|^2)) (\Phi + \Upsilon(t)).$$

The system solved by the perturbation  $\Upsilon$  is

$$i\Upsilon_t - \mathcal{L}\Upsilon + \mathcal{N}(\Upsilon) = 0,$$

where the linear part  $\mathcal{L}\Upsilon$  is given by

$$\mathcal{L}\Upsilon = \begin{pmatrix} -\partial_{xx}\varepsilon_1 - 2\lambda\phi_1\Re(\phi_1\bar{\varepsilon}_1) - 2\lambda\phi_1\Re(\phi_2\bar{\varepsilon}_2) \\ -\partial_{xx}\varepsilon_2 - 2\lambda\phi_2\Re(\phi_1\bar{\varepsilon}_1) - 2\lambda\phi_2\Re(\phi_2\bar{\varepsilon}_2) \end{pmatrix}$$



and  $\mathcal{N}(\Upsilon)$  encodes the remaining nonlinear terms. Writing  $\Phi$  in cartesian coordinates  $\phi_1 = a + ib$ ,  $\phi_2 = c + id$ , and separating  $\Upsilon$  in real and imaginary parts, the operator  $\mathcal{L}$  can be rewritten as

$$\mathcal{L} = \begin{pmatrix} -\partial_{xx} - 2\lambda a^2 & -2\lambda ab & -2\lambda ac & -2\lambda ad \\ -2\lambda ba & -2\lambda b^2 - \partial_{xx} & -2\lambda bc & -2\lambda bd \\ -2\lambda ca & -2\lambda cb & -\partial_{xx} - 2\lambda c^2 & -2\lambda cd \\ -2\lambda da & -2\lambda db & -2\lambda dc & -\partial_{xx} - 2\lambda d^2 \end{pmatrix}.$$

From the analysis of  $\mathcal{L}$  in Fourier variable, we infer that the spectrum of  $\mathcal{L}$  is located on

$$[\min(-2\lambda(a^2 + b^2 + c^2 + d^2), 0), \infty)$$

If  $\lambda < 0$ , the spectrum is non-negative. This implies that, when  $\mathbb{K} = \mathbb{T}$ , the constant  $\Phi$  is a local minimizer of the action

$$E + \Xi \cdot F, \quad \Xi = (\lambda(|\phi_1|^2 + |\phi_2|^2), \lambda(|\phi_1|^2 + |\phi_2|^2), 0, 0).$$

However, we observe a degeneracy here, as 0 is a triple eigenvalue, whereas the correspond symmetry group only gives a two-dimensional subspace of the kernel (precisely the subspace generated by  $(i\phi_1, 0)$  and  $(0, i\phi_2)$ ).

From our understanding, the degeneracy comes from the fact that the constants are not minimizers anymore if we consider a Schrödinger system such as (5.2) with  $\lambda < 0$  and  $\beta^2 > \alpha\gamma$ . The Manakov system is a borderline case where a bifurcation occurs for the minimizers. This however does not affect the orbital stability of the minimizer as was proved previously. To our knowledge, this is one of the rare case where degeneracy of the minimizer can be observed but where it does not affect the stability.

This was an excerpt of the results that we have been able to obtain concerning standing waves constant in space. The analysis is still in progress for the spectral stability on the whole line. We are also investigating standing waves solutions with plane wave profiles and standing wave solutions which can be obtained as minimizers of the Hamiltonian energy  $H$  on a set of fixed constraints combination of  $F_1, \dots, F_4$ .

## 5.2 Numerical Analysis of Excited States

In collaboration with Christophe Besse and Romain Dubosq, I am currently considering the excited states of nonlinear Schrödinger equations from new angles. We give in this section a sneak peek of our progress.

The equation for standing wave profiles in nonlinear Schrödinger equations

$$-\Delta u + u - f(u) = 0, \quad u \in H^1(\mathbb{R}^d, \mathbb{R}). \quad (5.4)$$

was considered in the classical works of Berestycki and Lions [BeLi83-1, BeLi83-2]. In [BeLi83-1], existence of ground state standing waves was shown, whereas in [BeLi83-2] existence of infinite sequence of excited states with increasing energy was proved. Many

works have then been devoted to the study of ground state profiles for themselves (regularity, positivity, uniqueness, etc.) as well as for their dynamical properties as solutions of nonlinear Schrödinger equations (or other equations like Klein-Gordon equations). Due to intrinsic difficulties, much less works have been devoted to excited states.

In [BeLi83-2], excited states were obtained by a critical point argument, consisting essentially in obtaining a topological min-max characterization of the excited states. This argument is however not very constructive and not very adapted to uses in the setting of nonlinear Schrödinger equations or for numerical purposes.

Another approach to the construction of excited state is based on the fact that excited states are necessarily nodal, whereas the ground state (if uniqueness holds) never vanishes. Hence one might think of extending to nodal functions the approaches used to obtain the ground states, in particular constraint minimization.

In this section, for simplicity reasons, we assume that the various functions under considerations are all real-valued.

A typical example for  $f$  is the power-type nonlinearity  $f(u) = |u|^{p-1}u$ ,  $1 < p < 2^* - 1$ , where  $2^*$  is the critical Sobolev exponent, i.e.  $2^* = \frac{2d}{d-2}$  if  $d \geq 3$ ,  $2^* = \infty$  if  $d = 1, 2$ . More generally, we assume that  $f : \mathbb{R} \rightarrow \mathbb{R}$  verifies the following hypotheses (which are not optimal, but sufficient for our purpose).

- (H1) (regularity) The function  $f$  is continuous and odd.
- (H2) (subcriticality) There exists  $1 < p < 2^* - 1$  such that for large  $s$ ,  $|f(s)| \lesssim |s|^p$ .
- (H3) (superlinearity) At 0,  $\lim_{s \rightarrow 0} \frac{f(s)}{s} = 0$ .
- (H4) (focusing) There exists  $\xi_0 > 0$  such that  $F(\xi_0) = \int_0^{\xi_0} f(s)ds > \frac{\xi_0^2}{2}$ .

Under (H1)-(H4), it was proved (see [BeGaKa83, BeLi83-1]) that there exist *ground state* solutions, i.e. solutions with minimal action (see (5.5) for the definition of the action) among all possible solutions to (5.4). Uniqueness of the ground state holds if  $f$  satisfies in addition to (H1)-(H4) some complementary requirements, e.g. if  $f$  is of power-type, see [Kw89]. When  $d \geq 3$  (in fact, this also holds if  $d \geq 2$ ), it was proved in [BeLi83-2] that there exists an infinite sequence of *excited states*, i.e. solutions to (5.4) whose action is not minimal (actually, the corresponding sequence of actions tends to infinity). Moreover, if  $f$  is of power-type, then there exists only one radial excited state with a given number of nodes, see [CoGaYa09, CoGaYa11].

Define the *action* functional by

$$S(u) = \frac{1}{2} \|\nabla u\|_{L^2}^2 + \frac{1}{2} \|u\|_{L^2}^2 - \int_{\mathbb{R}^d} F(u) dx. \quad (5.5)$$

Recall that  $S : H^1(\mathbb{R}^d) \rightarrow \mathbb{R}$  is a  $\mathcal{C}^1$  functional (see e.g. [AmMa07]) and that  $u$  is a solution of (5.4) if and only if  $S'(u) = 0$ . We define the *Nehari functional* by

$$I(u) = \langle S'(u), u \rangle = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 - \int_{\mathbb{R}^d} f(u) u dx.$$

The *Nehari manifold* is defined by

$$\mathcal{N} = \{u \in H^1(\mathbb{R}^d) \setminus \{0\} | I(u) = 0\}.$$

Define the *Nehari level* as

$$m_{\mathcal{N}} = \inf\{S(v) | v \in \mathcal{N}\}.$$

In addition to **(H1)**-**(H4)**, we assume the following.

**(H5)** The function  $s \rightarrow \frac{f(s)}{s}$  is increasing for  $s > 0$ .

**(H6)** (Ambrosetti-Rabinowitz superquadraticity condition) There exists  $\theta > 2$  such that  $\theta F(s) < sf(s)$  for all  $s > 0$ .

Then under **(H1)**-**(H6)**, the following holds (see e.g. [SzWe10] and the reference cited therein).

**Proposition 5.7.** *For every sequence  $(u_n) \in \mathcal{N}$  such that*

$$\lim_{n \rightarrow \infty} S(u_n) = m_{\mathcal{N}}$$

*there exist  $u_{\infty} \in \mathcal{N}$  and  $(y_n) \subset \mathbb{R}^d$  such that*

$$\lim_{n \rightarrow \infty} \|u_n(\cdot - y_n) - u_{\infty}\|_{H^1} = 0.$$

*Moreover,  $u_{\infty}$  is a ground state solution of (5.4).*

We now want to construct variational characterizations of excited states which can be used in numerical approaches. Based on Proposition 5.7, it is natural to try to generalize the Nehari manifold approach. Several directions of investigations are possible. The most natural one is probably to define the *Nehari nodal set* as

$$\mathcal{N}_{\text{nod}} = \{u \in H^1(\mathbb{R}^d) | I(u^+) = 0, I(u^-) = 0, u^{\pm} \neq 0\}.$$

where  $u^+ = \max(u, 0)$  and  $u^- = \max(-u, 0)$ . Define the *Nehari nodal level* by

$$m_{\mathcal{N}_{\text{nod}}} = \inf\{S(v) | v \in \mathcal{N}_{\text{nod}}\}.$$

We have

$$m_{\mathcal{N}_{\text{nod}}} = 2m_{\mathcal{N}}. \tag{5.6}$$

Indeed, let  $u \in \mathcal{N}_{\text{nod}}$ . Since  $u^+$  and  $u^-$  are both in  $\mathcal{N}$ , we have

$$S(u) = S(u^+) + S(u^-) \geq 2m_{\mathcal{N}},$$

and therefore  $m_{\mathcal{N}_{\text{nod}}} \geq 2m_{\mathcal{N}}$ . Let  $u_{\infty}$  be a minimizer for  $m_{\mathcal{N}}$  and for  $(y_n) \subset \mathbb{R}^d$ , define

$$u_n = u_{\infty}(\cdot + y_n) - u_{\infty}(\cdot - y_n). \tag{5.7}$$

When  $|y_n| \rightarrow \infty$ , we have

$$S(u_n) \rightarrow 2m_{\mathcal{N}},$$

and this proves (5.6). Unfortunately,  $\mathcal{N}_{\text{nod}}$  is not achieved. Indeed, suppose on the contrary that  $u_\infty$  realizes the minimum for  $m_{\mathcal{N}_{\text{nod}}}$ . Since  $u_\infty^\pm \in \mathcal{N}$  and  $m_{\mathcal{N}_{\text{nod}}} = 2m_{\mathcal{N}}$ , both  $u_\infty^+$  and  $u_\infty^-$  realize the minimum for  $\mathcal{N}$  and are ground states of (5.4). In particular, they are both regular, and by the maximum principle, both have to be positive on  $\mathbb{R}^d$ , which is a contradiction. Therefore  $m_{\mathcal{N}_{\text{nod}}}$  is not achieved. From (5.7), we can easily guess that this is due to a loss of compactness in the minimizing sequences. To overcome this issue, we can work in a radial setting (recall from Strauss' Lemma [St77] that the injection  $H_{\text{rad}}^1(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ ,  $2 < q < 2^*$  is compact whenever  $d \geq 2$ ). Define

$$\mathcal{N}_{\text{nod,rad}} = \{u \in H_{\text{rad}}^1(\mathbb{R}^d) | I(u^+) = 0, I(u^-) = 0, u^\pm \neq 0\},$$

and

$$m_{\mathcal{N}_{\text{nod,rad}}} = \inf\{S(v) | v \in \mathcal{N}_{\text{nod,rad}}\}.$$

Then the following result gives the existence of a minimizer for  $m_{\mathcal{N}_{\text{nod,rad}}}$ .

**Theorem 5.8.** *For every sequence  $(u_n) \in \mathcal{N}_{\text{nod,rad}}$  such that*

$$\lim_{n \rightarrow \infty} S(u_n) = m_{\mathcal{N}_{\text{nod,rad}}}$$

*there exist  $u_\infty \in \mathcal{N}_{\text{nod,rad}}$  such that*

$$\lim_{n \rightarrow \infty} \|u_n - u_\infty\|_{H^1} = 0.$$

*Moreover,  $u_\infty$  is a nodal solution of (5.4) with exactly two nodal domains. We say that  $u_\infty$  is a least nodal excited state.*

*Remark 5.9.* Minimizing on  $\mathcal{N}_{\text{nod,rad}}$  is intrinsically more difficult than minimizing on  $\mathcal{N}$ . Indeed,  $\mathcal{N}_{\text{nod,rad}}$  is *not* a manifold, as the functionals

$$u \in H^1(\mathbb{R}^d) \rightarrow \|\nabla u^\pm\|_{L^2}^2$$

are *not*  $\mathcal{C}^1$  (see the discussion after Theorem 18 in [SzWe10]).

*Remark 5.10.* An approach based on minimization of the energy on mass constraints for the positive and negative part of the function cannot work, as the minimizer that we might obtain would be (formally) a solution of an equation of the form

$$E'(u) + \lambda_+ M'(u_+) + \lambda_- M'(u_-) = 0,$$

with potentially different Lagrange multipliers  $\lambda_\pm$ . This issue is avoided with the Nehari approach.

It was proved in [CoGaYa09, CoGaYa11] that for power-type nonlinearities, there exists only one radial excited state with a given number of nodes. It is natural to implement numerically a shooting method to obtain approximation of radial excited states with a given number of nodes. This approach was implemented by Christophe Besse prior to the start of our collaboration. Beside the obvious issue of being restricted to the radial setting, the shooting method also suffers from not being adapted to large domains. Indeed, an extreme precision on the initial data is needed to obtain an accurate decay at the boundary of the domain. We have implemented a series of numerical schemes based on Theorem 5.8 and variations of it. Satisfactory results have been obtained. In particular, the restriction to the radial setting has been removed (provided compactness is preserved in some way) and the rate of decay is much more accurate than with the shooting method. We will report on our results in a forthcoming article.

# List of publications

The following publications are presented in this document.

- Journal articles

- S. Le Coz and Y. Wu. Stability of multisolitons for the derivative nonlinear Schrödinger equation. *International Mathematics Research Notices*, 2018(13):4120–4170, 2018.
- S. Gustafson, S. Le Coz, and T.-P. Tsai. Stability of periodic waves of 1D cubic nonlinear Schrödinger equations. *Appl. Math. Res. Express. AMRX*, 2:431–487, 2017.
- I. Ianni, S. Le Coz, and J. Royer. On the Cauchy problem and the black solitons of a singularly perturbed Gross-Pitaevskii equation. *SIAM J. Math. Anal.*, 49(2):1060–1099, 2017.
- S. Le Coz, Y. Martel, and P. Raphaël. Minimal mass blow up solutions for a double power nonlinear Schrödinger equation. *Rev. Mat. Iberoam.*, 32(3):795–833, 2016.
- F. Delebecque, S. Le Coz, and R. M. Weishäupl. Multi-speed solitary waves of nonlinear Schrödinger systems: theoretical and numerical analysis. *Comm. Math. Sci.*, 14(6):1599–1624, 2016.
- I. Ianni and S. Le Coz. Multi-speed solitary wave solutions for nonlinear Schrödinger systems. *J. Lond. Math. Soc. (2)*, 89(2):623–639, 2014.
- S. Le Coz and T.-P. Tsai. Infinite soliton and kink-soliton trains for nonlinear Schrödinger equations. *Nonlinearity*, 27(11):2689–2709, 2014.
- M. Ghimenti, S. Le Coz, and M. Squassina. On the stability of standing waves of Klein-Gordon equations in a semiclassical regime. *Discrete Contin. Dyn. Syst.*, 33(6):2389–2401, 2013.
- S. Le Coz, D. Li, and T.-P. Tsai. Fast-moving finite and infinite trains of solitons for nonlinear Schrödinger equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 145(6):1251–1282, 2015.
- J. Bellazzini, M. Ghimenti, and S. Le Coz. Multi-solitary waves for the nonlinear Klein-Gordon equation. *Comm. Partial Differential Equations*, 39(8):1479–1522, 2014.
- R. Côte and S. Le Coz. High-speed excited multi-solitons in nonlinear Schrödinger equations. *J. Math. Pures Appl. (9)*, 96(2):135–166, 2011.

*List of publications*

- I. Ianni and S. Le Coz. Orbital stability of standing waves of a semiclassical nonlinear Schrödinger-Poisson equation. *Adv. Differential Equations*, 14(7-8):717–748, 2009.
- Proceedings
  - S. Le Coz and T.-P. Tsai. Finite and infinite soliton and kink-soliton trains of nonlinear Schrödinger equations. In *Proceedings of the Sixth International Congress of Chinese Mathematicians. Vol. I*, volume 36 of *Adv. Lect. Math. (ALM)*, pages 43–56. Int. Press, Somerville, MA, 2017.
- Book Chapter
  - S. Le Coz. Standing waves in nonlinear Schrödinger equations. In *Analytical and numerical aspects of partial differential equations*, pages 151–192. Walter de Gruyter, Berlin, 2009.

The following works were included in my PhD and are not presented in this document.

- Journal articles
  - L. Jeanjean and S. Le Coz. Instability for standing waves of nonlinear Klein-Gordon equations via mountain-pass arguments. *Trans. Amer. Math. Soc.*, 361(10):5401–5416, 2009.
  - S. Le Coz. A note on Berestycki-Cazenave’s classical instability result for nonlinear Schrödinger equations. *Adv. Nonlinear Stud.*, 8(3):455–463, 2008.
  - S. Le Coz, R. Fukuizumi, G. Fibich, B. Ksherim, and Y. Sivan. Instability of bound states of a nonlinear Schrödinger equation with a Dirac potential. *Phys. D*, 237(8):1103–1128, 2008.
  - L. Jeanjean and S. Le Coz. An existence and stability result for standing waves of nonlinear Schrödinger equations. *Adv. Differential Equations*, 11(7):813–840, 2006.

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