New invariants in CR and contact geometry

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Nouveaux invariants en géométrie CR et de contact

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« Voilà, Adso, la bibliothèque a été construite par un esprit humain qui pensait de façon mathématique. Il s'agit donc de confronter nos propositions avec les propositions du bâtisseur, et de cette confrontation la science peut surgir.
— Splendide découverte, dis-je, mais alors pourquoi est-il aussi difficile de s'y orienter ?
— Parce que ce qui ne correspond à aucune loi mathématique, c'est la disposition des passages. Certaines pièces permettent d'accéder à plusieurs autres, certaines à une seule, et on peut se demander s'il n'y a pas des pièces qui ne permettent d'accéder à aucune autre. »

Umberto Eco, *Le Nom de la rose*.

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Introduction

La géométrie de Cauchy-Riemann, CR en abrégé, est la géométrie naturelle des hypersurfaces réelles pseudoconvexes de $\mathbb{C}^{n+1}$, lorsque $n \geq 1$. Cette géométrie fournit de nombreuses informations sur les domaines dont ces hypersurfaces sont les bords. En premier lieu, le théorème de représentation conforme dans $\mathbb{C}^{n+1}$, dû à H. Poincaré, affirme ainsi que deux domaines sont en bijection holomorphe si et seulement si les structures CR de leurs bords sont les mêmes [Poi07]. Pour une hypersurface M, cette structure se présente sous la forme d'un sous-fibré V de dimension complexe n de $TM \otimes \mathbb{C}$, dit *des opérateurs de Cauchy-Riemann tangentiels*. De manière équivalente, la structure CR de M est donnée par la distribution d'hyperplans $\mathrm{Re}(V + \overline{V})$ et par la structure complexe naturelle sur cette distribution. Par exemple, la structure CR standard de la sphère $\mathbb{S}^{2n+1}$, bord de la boule unité, correspond à la distribution horizontale induite par la fibration de Hopf au-dessus de l'espace projectif complexe.

Plus généralement, une variété CR abstraite est une variété différentielle M de dimension impaire munie d'une distribution d'hyperplans H, ainsi que d'une structure complexe J sur ces hyperplans. D'après L. Boutet de Monvel et H. Rossi, toute variété CR compacte de dimension ≥ 5 peut être plongée dans $\mathbb{C}^N$ pour un certain N, mais il existe des 3-variétés CR, même proches de $\mathbb{S}^3$, non-plongeables [Ros65, Bou75].

La distribution H est génériquement *de contact*, c'est-à-dire que si θ est une 1-forme sur M de noyau H, alors $\theta \wedge d\theta^n$ est non-dégénérée. La distribution H est alors *maximalement non-intégrable*, c'est-à-dire qu'il n'existe pas de sous-variété de M de dimension réelle supérieure ou égale à $n+1$ dont les espaces tangents sont inclus dans H. Dans toute notre étude, la distribution H sera supposée de contact. Notons qu'une variété de contact admet une structure CR s'il existe une structure complexe (intégrable) sur sa structure de contact. En particulier, les variétés de contact orientables de dimension 3 admettent toujours une structure CR.

La structure CR ne dépend pas du choix de la forme de contact. En choisir une correspond donc à munir la variété d'une structure supplémentaire, appelée structure *pseudohermitienne*. À une structure pseudohermitienne peut être associée une connexion canonique, dite *de Tanaka-Webster*, équivalent pseudohermitien de la connexion de Levi-Civita [Tan75, Web78]. Les outils de connexion et de courbure qui en sont issus présentent alors de nombreuses similarités avec la géométrie riemannienne, les changements conformes de la forme de contact en géométrie pseudohermitienne correspondant aux changements conformes de la métrique en géométrie riemannienne.

Cette analogie sera au coeur de cette thèse, dont les deux parties consistent à définir et étudier dans le cas CR des invariants connus en géométrie conforme. La première partie est consacrée à une généralisation de l'opérateur de Paneitz CR aux applications allant d'une variété CR vers une variété riemannienne. La seconde définit un invariant de Yamabe pour les variétés de contact admettant une structure CR. Comme indiqué dans le paragraphe précédent, il s'agit d'un invariant de contact pour les variétés orientables de dimension 3.

Applications CR-harmoniques

Un angle d'approche d'une variété CR consiste à la considérer comme le bord d'une variété *asymptotiquement hyperbolique complexe*. Cette notion permet de faire correspondre à une variété CR une variété riemannienne, les invariants de cette dernière étant ainsi des invariants CR de la variété initiale. Cette correspondance est analogue au lien entre géométrie conforme et géométrie *asymptotiquement hyperbolique*, initié par C. Fefferman et C. Graham, et central en physique des particules sous le nom de correspondance AdS/CFT [FG85].

Dans le cas réel, cette correspondance consiste à considérer une variété conforme $(M, [g])$ comme l'*infini conforme* d'une variété *asymptotiquement hyperbolique* $(X, \tilde{g})$. Dans ce modèle, qui généralise celui de la sphère conforme standard vue comme bord du disque de Poincaré, X est l'intérieur d'une variété compacte $\overline{X}$ de bord M. Soit r une fonction sur $\overline{X}$ définissant M, c'est-à-dire que $r > 0$ sur X, $r = 0$ et $dr \neq 0$ sur M. La variété $(X, \tilde{g})$ est dite *asymptotiquement hyperbolique* si les courbures sectionnelles de $\tilde{g}$ tendent vers -1 lorsque r tend vers 0. Elle est dite d'*infini conforme* $(M, [g])$ si la métrique $r^2\tilde{g}$ se prolonge de manière régulière en une métrique sur $\overline{X}$, dont la restriction à TM est dans $[g]$. Au choix d'une fonction r définissant le bord correspond ainsi le choix d'un représentant conforme $r^2\tilde{g}|_{TM}$ de $[g]$.

Toute variété conforme compacte (M, g) de dimension impaire peut ainsi être "remplie" par une variété asymptotiquement hyperbolique $(X, \tilde{g})$ munie d'une métrique d'Einstein lisse formelle, au sens où il existe un développement de Taylor, formellement déterminé par la donnée d'un jet d'ordre fini de g, solution de l'équation d'Einstein $\mathrm{Ric}(\tilde{g}) = -n\tilde{g}$. Cette variété est alors dite *de Poincaré-Einstein*. Lorsque la variété conforme est de dimension paire, il existe une obstruction à l'existence d'une métrique lisse de Poincaré-Einstein [FG85, GH05]. Récemment, M. J. Gursky et G. Székelyhidi ont annoncé qu'une métrique lisse de Poincaré-Einstein existe localement pour tout $n \geq 3$ [GS17].

Un invariant conforme construit grâce à ce procédé est l'*énergie renormalisée*. Soient (M, g) et (N, h) deux variétés riemanniennes. L'*énergie de Dirichlet* d'une application $\varphi : (M, g) \to (N, h)$ est définie par

$$E_g(\varphi) = \frac{1}{2}\int_M \|T\varphi\|^2_{g,h}\mathrm{dvol}_g.$$

Lorsque $n = 2$, l'énergie est conformément invariante par rapport à g, c'est-à-dire que si $\hat{g} = e^f g$ est une métrique conforme à g, alors $E_{\hat{g}} = E_g$. Ce fait est particulièrement utile, par exemple pour construire des immersions minimales conformes de surfaces de Riemann [Mil79]. Cependant, en dimension supérieure, l'énergie n'est plus conformément invariante. Les *applications harmoniques* entre une variété riemannienne (M^n, g) et une autre désignent les points critiques de l'énergie de Dirichlet. Ces applications satisfont l'équation d'Euler-Lagrange de l'énergie, c'est-à-dire qu'elles constituent le noyau d'un opérateur différentiel. En l'occurrence, les *fonctions* harmoniques $\varphi : (M, g) \to (\mathbb{R}, \mathrm{eucl})$ constituent le noyau du laplacien Δ_g. Lorsque $n = 2$, ce dernier est un opérateur conformément covariant par rapport à g : si $\hat{g} = e^f g$ est une métrique conforme à g, alors $\Delta_{\hat{g}} = e^f \Delta_g$. Ce n'est plus le cas en dimension plus grande.

Étant données deux variétés riemanniennes (M, g) et (N, h) compactes, avec M de dimension n paire, V. Bérard a montré l'existence d'une fonctionnelle $\mathscr{E}_g^n$ sur les applications $C^\infty(M, N)$, invariante conforme par rapport à g, égale à l'énergie lorsque $n = 2$ [Bér13]. Cette fonctionnelle est appelée *énergie renormalisée*. Ses points critiques sont appelés applications *conforme-harmoniques* et forment une généralisation des applications harmoniques. De plus, lorsque $n = 4$ et $N = \mathbb{R}$, l'opérateur induit est l'opérateur de Paneitz, opérateur différentiel d'ordre 4, conformément covariant, introduit par S. Paneitz [Pan08].

Dans le cas complexe, les variétés *asymptotiquement hyperboliques complexes*, ACH en abrégé, ont été introduites par C. Epstein, R. Melrose et G. Mendoza [EMM91]. Elles généralisent la construction par C. L. Fefferman, S.-Y. Cheng et S.-T. Yau des métriques *asymptotiquement de Bergman*, c'est-à-dire des métriques de Kähler-Einstein sur des domaines strictement pseudoconvexes et bornés de $\mathbb{C}^n$, asymptotiques à la structure CR du bord [Fef76, CY80]. Dans le cas modèle de $\mathbb{C}\mathbf{H}^{n+1}$, le bord est ainsi $\mathbb{S}^{2n+1}$ munie de sa structure CR standard. La régularité de ces métriques près du bord a été étudiée par J. Lee et R. Melrose [LM82].

De par l'anisotropie de leur structure, les variétés CR de dimension $2n+1$ ont une "dimension homogène" égale à $2n+2$ (cf. chapitre 1). Par conséquent, les variétés asymptotiquement hyperboliques complexes présentent de nombreuses analogies avec le cas "n pair" riemannien. En particulier, le développement asymptotique des métriques ACH-Einstein et -Kähler-Einstein a été extensivement étudié par O. Biquard, M. Herzlich et Y. Matsumoto, qui ont mis au jour les obstructions à l'existence de développements lisses formels [Biq00, BH05, Mat14].

L'équivalent pseudohermitien du laplacien, issu de la connexion de Tanaka-Webster, est le *sous-laplacien*, opérateur différentiel d'ordre 2, sous-elliptique. Cet opérateur n'est pas CR covariant. Son noyau est constitué des fonctions *sous-harmoniques*, cas particulier des fonctions *CR-holomorphes*. Les applications CR-holomorphes φ d'une variété CR (M, H, J) dans une variété complexe (N, J') sont celles qui commutent aux structures complexes : $T\varphi \circ J = J' \circ T\varphi$. Lorsque la variété de départ est plongeable, la restriction d'une application holomorphe à M est CR-holomorphe ; hormis ces exemples, les applications CR-holomorphes sont en général difficiles à construire, et peuvent ne pas exister en grand nombre si la variété de départ n'est pas plongeable.

En utilisant la géométrie asymptotiqument hyperbolique complexe, nous développons ici les notions d'énergie renormalisée et d'harmonicité CR. Les applications

CR-harmoniques généralisent les applications CR-holomorphes. Quand $\dim M = 3$ et $N = \mathbb{R}$, l'opérateur induit est l'opérateur de Paneitz CR, opérateur d'ordre 4, CR-covariant, introduit par C. Graham et J. Lee [GL88].

Les résultats principaux de la thèse sont les suivants. Le théorème A correspond dans le corps du texte au théorème 3.0.1 et le théorème B correspond aux théorèmes 3.1.11 et 3.2.4.

Théorème A. *Soient (M^{2n+1}, H, J, θ) une variété pseudohermitienne strictement pseudoconvexe compacte et (N, h) une variété riemannienne. Il existe une fonctionnelle F_n CR-invariante sur $C^\infty(M, N)$:*

$$\forall \hat{\theta} = e^f \theta \in [\theta], \quad \hat{F}_n = F_n.$$

Pour $\varphi \in C^\infty(M, N)$, elle s'écrit

$$F_n(\varphi) = \frac{(-1)^{n+1}}{2n!^2} \int_M \left\langle (\delta_b^{\theta,h} \nabla^{\varphi^* h})^{n-1} \delta_b^{\theta,h} T\varphi, \delta_b^{\theta,h} T\varphi \right\rangle_h \theta \wedge (d\theta)^n$$
$$+ \text{ termes d'ordre inférieur (en dérivées de } \varphi\text{)},$$

où $\delta_b^{\theta,h}$ est la divergence de Webster sur $\Omega^1(M) \otimes \varphi^* TN$.

L'équation d'Euler-Lagrange de F_n est une équation aux dérivées partielles non-linéaire P_n sous-elliptique d'ordre $2n+2$, elle-même CR-covariante :

$$\forall \hat{\theta} = e^f \theta \in [\theta], \quad \hat{P}_n = e^{(n+1)f} P_n.$$

Pour $\varphi \in C^\infty(M, N)$, elle s'écrit

$$0 = P_n(\varphi) = \frac{(-1)^n}{n!} (\delta_b^{\theta,h} \nabla^{\varphi^* h})^n \delta_b^{\theta,h} T\varphi + \text{ termes d'ordre inférieur (en dérivées de } \varphi\text{)}.$$

Théorème B. *En dimension 3, c'est-à-dire lorsque $n = 1$, en notant R le champ de Reeb et τ la torsion pseudohermitienne, la fonctionnelle s'écrit explicitement pour $\varphi \in C^\infty(M, N)$,*

$$F_1(\varphi) = -\frac{1}{2} \int_M \left(\|\delta_b^{\theta,h} T\varphi\|_h^2 + \|R\varphi\|_h^2 - 4\mathrm{Im}\left(\tau_1^{\bar{1}} \|T_{\bar{1}}\varphi\|_h^2 \right) \right) \theta \wedge d\theta.$$

L'opérateur induit s'écrit

$$P_1(\varphi) = -\delta_b^{\theta,h} \nabla^{\varphi^* h} \delta_b^{\theta,h} T\varphi - \nabla_R^{\varphi^* h} R\varphi + 4\mathrm{Im}\left(\nabla_{T_{\bar{1}}}^{\varphi^* h} \left(\tau_1^{\bar{1}} T_{\bar{1}} \right) \right) \varphi - S_b\left(\delta_b^{\theta,h} T\varphi \right),$$

où

$$S_b(X) := \mathscr{R}^h_{X, T_1\varphi} T_{\bar{1}}\varphi + \mathscr{R}^h_{X, T_{\bar{1}}\varphi} T_1\varphi.$$

Invariant de Yamabe en géométrie de contact

Le *problème de Yamabe* classique est celui de l'existence, sur une variété riemannienne compacte, d'une métrique conforme à la métrique de départ et de courbure scalaire constante [Sch84, LP87]. L'*invariant de Yamabe* σ a été introduit par R. Schoen et

O. Kobayashi à la suite de la résolution de ce problème [Kob87, Sch89]. Cet invariant est construit de la façon suivante : une condition suffisante pour qu'une métrique g soit de courbure scalaire constante est d'être un minimum, parmi les métriques de même volume dans la classe conforme considérée, pour la courbure scalaire intégrale $Y(g)$. De plus, $Y(g)$ est toujours inférieur ou égal à $Y(g_{\mathbb{S}^n})$, où $g_{\mathbb{S}^n}$ est la métrique standard sur la sphère. L'invariant σ est alors défini pour une variété différentielle compacte M comme le "max-min"

$$\sigma(M) := \sup_{[g]} \inf_{\hat{g} \in [g]_1} \int_M \mathrm{Scal}(\hat{g})\, d\mathrm{vol}_{\hat{g}},$$

où Scal est la courbure scalaire riemannienne, le supremum parcourt les classes conformes de métriques $[g]$ sur M et l'infimum parcourt les métriques de volume 1 dans $[g]$.

Cet invariant différentiel global est produit en considérant une variété différentielle compacte sous le prisme des structures conformes dont elle peut être munie. De la même manière, considérons une variété de contact compacte (M, H) admettant une structure CR. Considérons de plus le problème de Yamabe en géométrie CR, où le rôle de la courbure est joué par la courbure scalaire *de Webster*, auquel D. Jerison et J. Lee, N. Gamara et R. Yacoub ont apporté une réponse positive [JL87, JL89, Gam01, GY01]. On peut alors définir un *invariant de Yamabe de contact* σ_c comme

$$\sigma_c(M, H) := \sup_{\mathscr{J}} \inf_{\hat{\theta} \in [\theta]_1} \int_M \mathrm{Scal}_W(J, \hat{\theta})\, \hat{\theta} \wedge d\hat{\theta}^n,$$

où Scal_W est la courbure scalaire de Webster, le supremum parcourt l'ensemble $\mathscr{J}$ des structures complexes compatibles sur (M, H) et l'infimum parcourt les formes de contact de volume 1. Cet invariant a été introduit par C.-T. Wu [Wu09], mais n'a à notre connaissance pas été exploité depuis. Cet invariant n'est un véritable invariant de contact que lorsque (M, H) est de dimension 3, au sens où toutes les 3-variétés de contact orientables admettent une structure CR, et cela n'est plus le cas en dimension plus grande. Il est à noter que peu d'invariants de contact sont actuellement disponibles : ils sont forcément globaux d'après le théorème de Darboux, et la plupart d'entre eux provient de techniques homologiques (homologie de contact, homologie de Floer...).

Dans le cas conforme, le calcul de l'invariant de Yamabe est difficile en général. En dimension 3, K. Akutagawa, H. Bray et A. Neves ont obtenu le résultat suivant, à lire en regard de la *conjecture de Schoen*, selon laquelle, si Γ est un sous-groupe discret de $\mathrm{Isom}(\mathbb{S}^n)$, alors $\sigma\left(\mathbb{S}^n/\Gamma\right) = \dfrac{\sigma(\mathbb{S}^n)}{|\Gamma|^{\frac{2}{n}}}$.

Théorème 0.0.1 ([BN04, AN07]). *Pour une variété compacte M et un entier k, notons $\#kM$ la somme connexe de k copies de M. Notons $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$ le $\mathbb{S}^2$-fibré non-orienté au-dessus de $\mathbb{S}^1$. Pour tous entiers k, l, m, n tels que $k + l \geq 1$, on a*

$$\sigma\left(\#k\mathbb{RP}^3 \# l(\mathbb{RP}^2 \times \mathbb{S}^1) \# m(\mathbb{S}^2 \times \mathbb{S}^1) \# n(\mathbb{S}^2 \tilde{\times} \mathbb{S}^1)\right) = \frac{\sigma(\mathbb{S}^3)}{2^{\frac{2}{3}}}.$$

De plus, si $\sigma(M) > \dfrac{\sigma(\mathbb{S}^3)}{2^{\frac{2}{3}}}$, alors $M = \mathbb{S}^3$ ou $M = \#k(\mathbb{S}^2 \times \mathbb{S}^1) \# l(\mathbb{S}^2 \tilde{\times} \mathbb{S}^1)$ avec $k + l \geq 1$.

En dimension 4, en utilisant le théorème de Gauss-Bonnet généralisé, C. LeBrun et J. Petean ont obtenu son expression explicite pour les surfaces complexes (de dimension réelle 4) de type général [LeB96, Pet98].

Théorème 0.0.2 ([LeB96, Pet98])**.** *Soit M une variété obtenue par somme connexe de copies de $\mathbb{S}^1 \times \mathbb{S}^3$ avec l'éclatement d'une surface complexe compacte minimale de type général Σ en un nombre quelconque de points. Alors*

$$\sigma(M) = -4\pi\sqrt{2(2\chi + 3\tau)(\Sigma)},$$

$\chi(\Sigma)$ et $\tau(\Sigma)$ étant respectivement la caractéristique d'Euler et la signature de Σ.

Une approche complémentaire est l'étude du comportement de σ sous chirurgie. Les résultats connus sont les suivants:

Théorème 0.0.3 ([Kob87])**.** *Soient M_1 et M_2 deux variétés différentielles compactes de dimension n. Notons $M_1 \# M_2$ leur somme connexe, alors*

$$\sigma(M_1 \# M_2) \geq \begin{cases} -\left(|\sigma(M_1)|^{\frac{n}{2}} + |\sigma(M_2)|^{\frac{n}{2}}\right)^{\frac{2}{n}} & \text{si } \sigma(M_1) \leq 0 \text{ et } \sigma(M_2) \leq 0, \\ \min\left(\sigma(M_1), \sigma(M_2)\right) & \text{sinon.} \end{cases}$$

Théorème 0.0.4 ([PY99])**.** *Soit M une variété compacte. Soit N une variété obtenue à partir de M par une chirurgie de codimension $k \geq 3$, alors*

$$\sigma(N) \geq \min\left(\sigma(M), 0\right).$$

Théorème 0.0.5 ([ADH13])**.** *Soit M une variété compacte. Il existe une constante strictement positive $\Lambda_{n,k}$ telle que pour toute variété N obtenue à partir de M par une chirurgie de codimension $k \geq 3$,*

$$\sigma(N) \geq \min\left(\sigma(M), \Lambda_{n,k}\right).$$

Ces théorèmes reposent sur la construction d'une chirurgie dont l'anse est de courbure scalaire strictement positive. Ainsi, dans les théorèmes 0.0.4 et 0.0.5, la condition sur la codimension de la chirurgie est due à la préservation de l'existence d'une métrique à courbure scalaire strictement positive sous la même condition [GL80]. Notons que dans le théorème 0.0.3, le membre de droite est égal à $\sigma(M_1 \sqcup M_2)$. De plus, dans le théorème 0.0.5, $\Lambda_{n,0}$ est maximal, égal à $\sigma(\mathbb{S}^n)$. Par conséquent, le théorème 0.0.5 implique les théorèmes 0.0.3 et 0.0.4.

Nous obtenons dans cette thèse une minoration de σ_c après ajout d'anse et somme connexe, analogue en géométrie de contact du théorème 0.0.3, en utilisant une technique de somme connexe préservant le caractère strictement pseudoconvexe de la structure CR due à W. Wang, J.-H. Cheng et H.-L. Chiu [Wan03, CC18]. Plus généralement, une chirurgie de contact préservant le caractère strictement pseudoconvexe de la structure CR a été introduite par Y. Eliashberg [Eli90]. La difficulté consiste alors à construire une anse de courbure scalaire de Webster strictement positive. Nous obtenons également une minoration de σ_c dans un cas particulier, analogue du théorème 0.0.2.

Les résultats principaux de la thèse sont les suivants. Le théorème C correspond dans le corps du texte au théorème 4.0.2, le théorème D correspond au théorème 4.0.4, et le théorème E correspond au théorème 4.0.5.

Théorème C. *Soit (M,H) une variété SPC compacte. Soit $(\tilde{M},\tilde{H})$ une variété obtenue à partir de (M,H) par ajout d'anse SPC, alors*

$$\sigma_c(\tilde{M},\tilde{H}) \geq \sigma_c(M,H).$$

Théorème D. *Soient (M_1,H_1) et (M_2,H_2) deux variétés SPC compactes de dimension $2n+1$. Soit $(M_1,H_1)\#(M_2,H_2)$ leur somme connexe SPC, alors*

$$\sigma_c\left((M_1,H_1)\#(M_2,H_2)\right) \geq \begin{cases} -\left(|\sigma_c(M_1,H_1)|^{n+1} + |\sigma_c(M_2,H_2)|^{n+1}\right)^{\frac{1}{n+1}} & \text{si } \sigma_c(M_1,H_1) \leq 0 \\ & \quad \text{et } \sigma_c(M_2,H_2) \leq 0, \\ \min\left(\sigma_c(M_1,H_1), \sigma_c(M_2,H_2)\right) & \text{sinon.} \end{cases}$$

Théorème E. *Soit (M,H) un fibré en cercles au-dessus d'une surface de Riemann Σ de genre strictement positif. Si (M,H) admet une structure pseudohermitienne d'Einstein, alors*

$$\sigma_c(M,H) \geq -2\pi\sqrt{-\chi(\Sigma)},$$

où $\chi(\Sigma)$ est la caractéristique d'Euler de Σ.

Convention

Dans cette thèse, nous adoptons la convention suivante : les lettres minuscules grecques désignent des indices dans $\{1,\ldots,n\}$; les lettres majuscules grecques, dans $\{1,\ldots,n,\overline{1},\ldots,\overline{n}\}$; les lettres minuscules latines, dans $\{0,1,\ldots,n\}$; les lettres majuscules latines, dans $\{0,1,\ldots,n,\overline{0},\overline{1},\ldots,\overline{n}\}$. De plus, nous utilisons la convention de sommation d'Einstein.

CHAPITRE I

CR geometry

Abstract. We give here general notions of CR geometry, of strictly pseudoconvex pseudohermitian geometry and of the Tanaka-Webster curvature theory. We give some examples of CR manifolds, and we focus on the close analogy between conformal and CR structures. In particular, we present the CR Paneitz operator and the contact Yamabe invariant.

Contents

Introduction

Cauchy-Riemann geometry, CR for short, is the natural geometry of real pseudo-convex hypersurfaces of $\mathbb{C}^{n+1}$ for $n \geq 1$. It contains a significant amount of information about the geometry of the domains which are bounded by these hypersurfaces. It is for example the case for the sphere $\mathbb{S}^{2n+1}$ endowed with its standard CR structure, which is the boundary of the unit ball in $\mathbb{C}^{n+1}$. For a hypersurface M, this geometry takes the form of a subbundle V of complex dimension n of $TM \otimes \mathbb{C}$, which is called the *tangential Cauchy-Riemann operators bundle*, and whose geometry encodes the obstruction to the existence of a local holomorphic correspondence between neighbourhoods of M and $\mathbb{S}^{2n+1}$, first noticed by H. Poincaré [Poi07]. Equivalently, it takes the form of the hyperplane distribution $\mathrm{Re}(V + \overline{V})$ on M, which is generically a contact distribution, together with its natural complex structure. In the sphere case, it corresponds to the horizontal distribution induced by the Hopf fibration over the complex projective space. More generally, an abstract CR manifold is a differentiable manifold of odd dimension, endowed with a hyperplane distribution, and with a complex structure on these hyperplans. From results by L. Boutet de Monvel and H. Rossi, every abstract CR manifold of dimension ≥ 5 can be locally embedded into $\mathbb{C}^N$ for some N, but there exist CR 3-manifolds, even close to $\mathbb{S}^3$, that are non-embeddable and non locally embeddable [Ros65, Bou75].

Since the hyperplane distribution is stable by conformal change of the contact form, CR geometry is intrinsically conformal. Moreover, using the connection and curvature theory developed by N. Tanaka and S. Webster, it shares many similarities with Riemannian conformal geometry [Tan75, Web78].

CR geometry arose from the attempt by H. Poincaré to generalize the Riemann mapping theorem from $\mathbb{C}^1$ to $\mathbb{C}^2$. Namely, he asked the following question:

Problem 1.0.1 (Riemann mapping problem in $\mathbb{C}^{n+1}$). Given two smooth real hypersurfaces Γ_1 and Γ_2 in $\mathbb{C}^{n+1}$, given $p \in \Gamma_1$ and $q \in \Gamma_2$, are there U_1, U_2 respective neighbourhoods of p, q and a biholomorphism $\Phi : U_1 \mapsto U_2$ such that $\Phi(U_1 \cap \Gamma_1) = U_2 \cap \Gamma_2$?

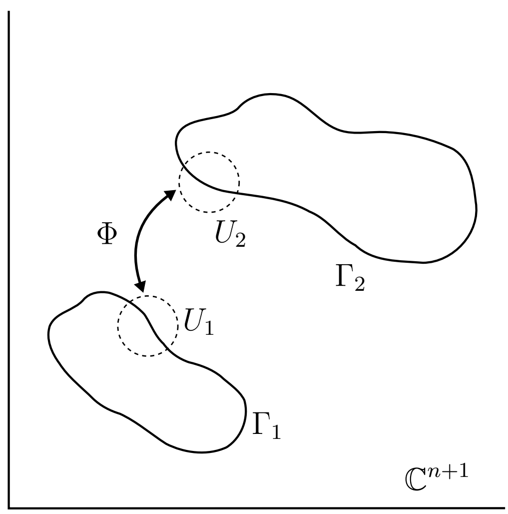

FIGURE 1.1 – *Riemann mapping problem in* $\mathbb{C}^{n+1}$

Answer. [Poi07] If $n = 0$, the answer is positive from the Riemann mapping theorem. If $n \geq 1$, $\tilde{\Phi} := \Phi \mid_{U_1 \cap \Gamma_1}$ stems from a biholomorphism if and only if it verifies n *tangential Cauchy-Riemann (CR) equations*, which can be viewed as a *CR operator bundle* $T^{0,1}$ on Γ_1. The conjugate $T^{1,0}$ is stable under the Lie bracket, i.e. *integrable*.

This structure can be generalized in the following way.

1.1 Generalities

Let $n \in \mathbb{N}^*$ and M be a smooth differentiable manifold of real dimension $2n+1$. We assume that M is orientable. A *CR structure* is given on M by a complex subbundle $T^{1,0}M$ of $TM \otimes \mathbb{C}$ of complex dimension n verifying

$$T^{1,0}M \cap T^{0,1}M = \{0\}$$

where $T^{0,1}M = \overline{T^{1,0}M}$, and which is stable under the Lie bracket.

Equivalently, let H be a *Levi distribution*, *i.e.* an orientable hyperplane distribution in TM. Let J be a *complex structure* on H, *i.e.* J is an endomorphism of H which satisfies $J^2 = -\mathrm{id}_H$ and is *integrable*: $\forall X, Y \in \Gamma(H)$,

$$[JX, Y] + [X, JY] \in \Gamma(H) \quad \text{and} \quad [JX, JY] - [X, Y] = J\left([JX, Y] + [X, JY]\right),$$

where $[\cdot, \cdot]$ denotes the Lie bracket. The existence of J also requires that H is orientable. A *CR manifold* is the triplet (M, H, J).

The correspondence between the two approaches is the following:

- From the first approach to the second, $H := \mathrm{Re}\left(T^{1,0}M \oplus T^{0,1}M\right)$, and $\forall X \in T^{1,0}M,\ J(X + \overline{X}) := i(X - \overline{X})$.
- From the second approach to the first, $T^{1,0}M$ is the i-eigenspace of the extension of J to $H \otimes \mathbb{C}$.

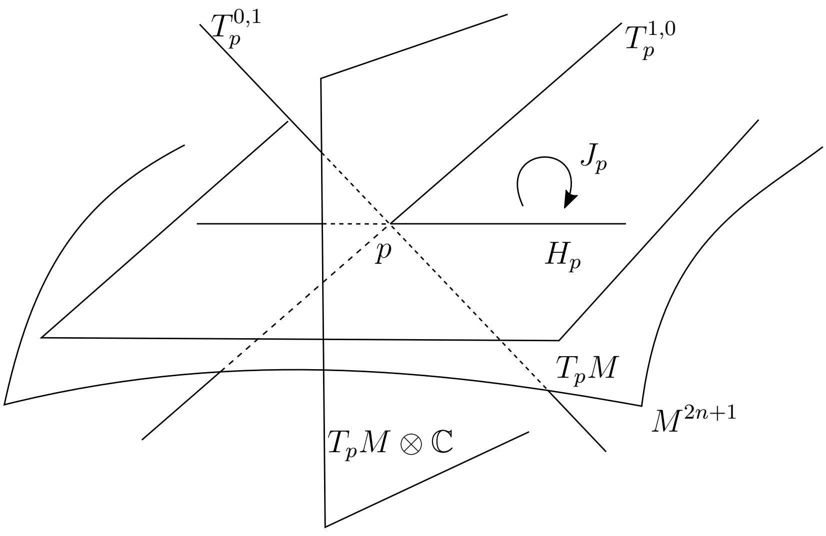

FIGURE 1.2 – *CR structure*

Let $E := \{\omega \in \Gamma(T^*M) \mid \ker\omega \supseteq H\} \simeq TM/H$. It is a real line subbundle of T^*M, hence trivial since M is orientable. A *pseudohermitian structure* on M is a never-vanishing section θ of E compatible with J, *i.e.* such that

$$d\theta(J\cdot, J\cdot) = d\theta(\cdot,\cdot) \quad \text{on } TM \otimes \mathbb{C}.$$

The associated *Levi form* γ is the Hermitian form on H given by $\gamma := d\theta(\cdot, J\cdot)$.

Definition 1.1.1. When the Levi form is definite positive, the pseudohermitian structure is said to be *strictly pseudoconvex*.

In that case, θ is a contact form, which means that $\theta \wedge d\theta^n$ is a volume form on M, and (M, H) is a contact manifold. A contact form on (M, H, J) which is a strictly pseudoconvex pseudohermitian structure will be called *positive*. A CR manifold admitting an positive contact form is called *SPCR*, and a contact manifold admitting an SPCR structure is called *SPC*. We will always assume that H is a contact distribution.

Definition 1.1.2. Given an SPC manifold (M, H), we define

$$\mathscr{J} = \{\text{Compatible complex structures } J \text{ on } H \mid (M, H, J) \text{ is an SPCR manifold}\}.$$

In dimension $2n + 1 = 3$, $T^{1,0}M$ is of complex rank 1, the integrability of J is thus automatic. The set $\mathscr{J}$ is defined by purely algebraic conditions, and it is moreover contractible. Indeed, considering J_0, J_1 in $\mathscr{J}$, let, for i in $\{0, 1\}$, $\gamma_i := d\theta(\cdot, J_i\cdot)$. For t in $[0, 1]$, the metric $\gamma_t := (1 - t)\gamma_0 + t\gamma_1$ gives a complex structure $\tilde{J}_t$ compatible with θ, with $\tilde{J}_0 = J_0$ and $\tilde{J}_1 = J_1$. The set $\mathscr{J}$ is therefore always non-empty.

Definition 1.1.3. A CR manifold (M, H, J) is called *embeddable* if there exists a submanifold (M', H', J') of $\mathbb{C}^N$ for some N and a CR isomorphism $f : M \to M'$, *i.e.* such that $f_*(T^{1,0}M) = T^{1,0}M'$.

All compact CR manifolds of dimension ≥ 5 are embeddable, but there exist CR 3-manifolds, even close to $\mathbb{S}^3$, that are non-embeddable [Ros65, Bou75].

An SPCR structure induces a conformal structure on the contact distribution H. Indeed, let (M, H, J) be an SPCR manifold and θ be a positive contact form on M. Note that E is the set of sections of a line bundle. Strict pseudoconvexity is then preserved under conformal changes of θ. Indeed,

$$\forall f \in C^\infty(M, \mathbb{R}_+^*), \quad \gamma_{f\theta} = f\gamma_\theta.$$

The *Reeb field* of a contact form θ is the unique vector field $R \in TM$ verifying $\theta(R) = 1$ and $\iota_R d\theta = 0$. We get a pseudohermitian decomposition of the tangent space

$$TM = \mathbb{R}R \oplus H,$$

and a pseudohermitian projection $\pi_b : TM \to H$. Note that this projection depends on θ. An *admissible coframe* is a set of $(1, 0)$-forms $(\theta_1, \ldots, \theta_n)$ whose restriction to $T^{1,0}M$ forms a basis for $\left(T^{1,0}M\right)^*$ and such that, for all α in $\{1, \ldots, n\}$, $\theta^\alpha(R) = 0$. Then $d\theta = ih_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}$, where $\theta^{\overline{\beta}} := \overline{\theta^\beta}$ and $(h_{\alpha\overline{\beta}})$ is a positive definite Hermitian matrix. If $(T_1, \ldots, T_n)$ is the dual frame to $(\theta^1, \ldots, \theta^n)$ on $T^{1,0}M$, then $\gamma(T_\alpha, T_{\overline{\beta}}) = h_{\alpha\overline{\beta}}$.

Definition 1.1.4. The Riemannian metric $g_{J,\theta}$ on M given by

$$g_{J,\theta} := \theta^2 + \gamma,$$

is called the *Webster metric* of (M, H, J, θ).

Definition 1.1.5. Let $\pi_b : TM \to H$ be the pseudohermitian projection. The *horizontal gradient* is the operator $\nabla_b := \pi_b \nabla^\theta$. The *sublaplacian* is $\Delta_b := \operatorname{div}(\nabla_b \cdot)$.

1.2 Curvature theory

Theorem 1.2.1 ([Tan75, Web78]). *Let (M, H, J, θ) be a strictly pseudoconvex pseudohermitian manifold. There is a unique linear connection ∇^θ on M, called the* Tanaka-Webster connection*, which parallelizes the Levi distribution H, the Reeb field R, the complex structure J, and the Webster metric $g_{J,\theta}$, and whose torsion T^θ verifies*

$$\forall X, Y \in H, \quad T^\theta(X,Y) = d\theta(X,Y)R \quad \text{and} \quad T^\theta(R, JX) + JT^\theta(R,X) = 0.$$

In other words, if $\theta^1, \ldots, \theta^n$ is an admissible coframe with $d\theta = ih_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}$, then the *connection forms* ${\omega_\alpha}^\beta$ and the *torsion forms* $\tau_\alpha = A_{\alpha\beta}\theta^\beta$ of the Tanaka-Webster connection are defined by

$$d\theta^\beta = \theta^\alpha \wedge {\omega_\alpha}^\beta + \theta \wedge \tau^\beta, \qquad \omega_{\alpha\overline{\beta}} + \omega_{\overline{\beta}\alpha} = dh_{\alpha\overline{\beta}}, \qquad A_{\alpha\beta} = A_{\beta\alpha},$$

where indices are raised and lowered with $h_{\alpha\overline{\beta}}$, *i.e.* $\omega_{\alpha\overline{\beta}} = h_{\sigma\overline{\beta}}{\omega_\alpha}^\sigma$ [Web78, Lee88]. We then have, for the dual frame $(T_1, \ldots, T_n)$ to $(\theta^1, \ldots, \theta^n)$, $\nabla^\theta T_\alpha = {\omega_\alpha}^\beta \otimes T_\beta$.

Due to the first condition in Theorem 1.2.1, the torsion of the Tanaka-Webster connection is nonvanishing; however, we define:

Definition 1.2.2. The *pseudohermitian torsion* τ of the Tanaka-Webster connection is the operator $\tau = \iota_R T^\theta$. If τ vanishes, (M, H, J, θ) is called *normal*.

Note that the definition of Tanaka-Webster connection implies that the pseudohermitian torsion is always trace-free as an endomorphism of the real vector bundle H.

Let $\mathscr{R}^\theta$ be the curvature tensor field corresponding to the Tanaka-Webster connection. It can be decomposed into vertical, mixed, and horizontal terms. The vertical and mixed terms only depend on τ and its first derivatives. The horizontal part gives the *Webster curvature tensor*. Let $\mathrm{Ric}_W(J,\theta)$ be its Ricci tensor, and $\mathrm{Scal}_W(J,\theta)$ be its scalar curvature, called the *Webster scalar curvature*. In other words, the *curvature forms* ${\Pi_\alpha}^\beta = d{\omega_\alpha}^\beta - {\omega_\alpha}^\sigma \wedge {\omega_\sigma}^\beta$ verify

$$ {\Pi_\alpha}^\beta = {\mathscr{R}^\theta}_{\alpha}{}^{\beta}{}_{\rho\overline{\sigma}}\theta^\rho \wedge \theta^{\overline{\sigma}} \bmod \theta.$$

We then have

$$\mathrm{Ric}_W(J,\theta)(T_\alpha, T_{\overline{\beta}}) = {\mathscr{R}^\theta}_{\alpha}{}^{\rho}{}_{\rho\overline{\beta}} \quad \text{and} \quad \mathrm{Scal}_W(J,\theta) = h^{\alpha\overline{\beta}}\mathrm{Ric}_W(J,\theta)(T_\alpha, T_{\overline{\beta}}).$$

We also define the CR-covariant CR Schouten tensor:

Definition 1.2.3. The *CR Schouten tensor* is

$$\mathrm{Sch}_W(J,\theta) = \frac{1}{n+2}\left(\mathrm{Ric}_W(J,\theta) - \frac{\mathrm{Scal}_W(J,\theta)}{2(n+1)}\gamma\right).$$

The pseudohermitian equivalent of Einstein manifolds is built as follows:

Definition 1.2.4. The *canonical bundle* K of a strictly pseudoconvex pseudohermitian $(2n+1)$-manifold (M,H,J,θ) is the complex line bundle of $n+1$-forms on M of maximal holomorphic type $(n,0)$. Given a local basis $T_1,\dots,T_n$ of $T^{1,0}M$ and the dual basis $\theta^1,\dots,\theta^n$, K is locally generated by $\theta\wedge\theta^1\dots\wedge\theta^n$. The contact form θ is called *invariant* if it is volume normalized for a local section ζ of K, i.e.

$$\theta\wedge d\theta^n = i^{n^2} n!\theta\wedge(R\lrcorner\zeta)\wedge(R\lrcorner\bar{\zeta}).$$

Definition 1.2.5 ([CY13])**.** A strictly pseudoconvex pseudohermitian manifold (M,H,J,θ) is said to be *pseudo-Einstein* if θ is invariant. Equivalently, (M,H,J,θ) is pseudo-Einstein if

$$\begin{aligned} \mathrm{Ric}_W &= \frac{1}{n}\mathrm{Scal}_W\gamma && \text{if } n\geq 2,\\ \mathrm{Scal}_{W,1} &= i\tau^{\bar{1}}_{1,\bar{1}} && \text{if } n=1. \end{aligned}$$

In particular, if (M,H,J) is embeddable, then it admits a pseudo-Einstein structure [Lee88]. Note that, if $n\geq 2$ and (M,H,J,θ) is pseudo-Einstein, then the CR Bianchi identity implies that

$$\mathrm{Scal}_{W,\alpha} = i(n-1)\tau^{\bar{\beta}}_{\alpha,\bar{\beta}}.$$

Therefore, a pseudo-Einstein strictly pseudoconvex pseudohermitian manifold does not necessarily have constant scalar Webster curvature. However, a *normal* pseudo-Einstein strictly pseudoconvex pseudohermitian manifold has constant scalar Webster curvature. A normal strictly pseudoconvex pseudohermitian 3-manifold is pseudo-Einstein if and only if it has constant Webster scalar curvature.

Definition 1.2.6 ([Wan15])**.** A normal, pseudo-Einstein contact form is called an *Einstein contact form*.

Finally, let us recall the relation between the Tanaka-Webster connection and the Levi-Civita connection of the Webster metric, which will be useful in Chapter 3.

Proposition 1.2.7 ([DT06])**.** *Let ∇^g be the Levi-Civita connection of $(M,g_{J,\theta})$. Then*

$$\nabla^g = \nabla^\theta - (d\theta + A)\otimes R + \tau\otimes\theta + 2\theta\circ J,$$

where $A=\gamma(\tau\cdot,\cdot)$ and $\alpha\circ\beta = \alpha\otimes\beta+\beta\otimes\alpha$.

In particular, we have $\nabla^g_R R = 0$.

Proposition 1.2.7 yields the following result:

Proposition 1.2.8 ([DT06]). *Let $\mathscr{R}^g$ be the Levi-Civita connection of $(M, g_{J,\theta})$. Then $\forall X, Y, Z \in \Gamma(TM)$,*

$$\begin{aligned}
\mathscr{R}^g_{X,Y} Z &= \mathscr{R}^\theta_{X,Y} Z + (LX \wedge LY)Z + 2d\theta(X,Y)JZ \\
&\quad - g_{J,\theta}\left(S(X,Y), Z\right) R + \theta(Z) S(X,Y) \\
&\quad - 2g_{J,\theta}\left((\theta \wedge \mathscr{T})(X,Y), Z\right) R + 2\theta(Z)\left(\theta \wedge \mathscr{T}\right)(X,Y),
\end{aligned}$$

where $L = \tau + J$, $S(X,Y) = \left(\nabla^\theta_X \tau\right) Y - \left(\nabla^\theta_Y \tau\right) X$, $\mathscr{T} = \tau^2 + 2J\tau - \mathrm{id}$, and $(X \wedge Y)Z = g_{J,\theta}(X,Z)Y - g_{J,\theta}(Y,Z)X$.

1.3 Examples

1.3.1 The Heisenberg group

The *Heisenberg group* is the Lie group $\mathbb{H}^{2n+1} := \mathbb{C}^n \times \mathbb{R}$ endowed with the coordinates $(z,t) := (z_1, \cdots, z_n, t)$. A CR structure is given on $\mathbb{H}^{2n+1}$ by

$$T^{1,0}\mathbb{H} := \mathrm{Vect}_{\mathbb{C}}(T_1, \cdots, T_n) \quad \text{with } T_j := \frac{\partial}{\partial z^j} + i\bar{z}_j \frac{\partial}{\partial t}.$$

A compatible positive contact form is given by

$$\theta_{\mathbb{H}} = dt + i \sum_{j=1}^n (z_j d\bar{z}^j - \bar{z}_j dz^j),$$

whose Reeb field is $R_{\mathbb{H}} = \dfrac{\partial}{\partial t}$. We have

$$d\theta_{\mathbb{H}} = i \sum_{j=1}^n dz^j \wedge d\bar{z}^j \quad \text{and} \quad \gamma_{\mathbb{H}} = \sum_{j=1}^n dz^j \wedge d\bar{z}^j.$$

For this structure and $\delta > 0$, the *dilation* $(z,t) \mapsto (\delta z, \delta^2 t)$ is a CR-isomorphism. A homogeneous norm with respect to dilations is the *Heisenberg norm*, given by

$$\|(z,t)\|_{\mathbb{H}} = (|z|^4 + t^2)^{\frac{1}{4}}.$$

In that perspective, a $(2n+1)$-dimensional CR manifold is sometimes said to have "homogeneous dimension" $2n+2$, which may be kept in mind when considering analogies between CR and Riemannian geometry [JL89].

The Webster curvature tensor of the Heisenberg group vanishes: this space can be thought of as the Euclidean space equivalent of CR geometry.

1.3.2 The standard sphere

Let us consider the sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. The *Cayley transform*, which is the CR diffeomorphism

$$\begin{array}{rccc} \psi: & \mathbb{C}^{n+1} \supset \mathbb{S}^{2n+1} \setminus \{(0,\dots,0,1)\} & \to & \mathbb{H}^{2n+1} \\ & (z_1,\dots,z_{n+1}) & \mapsto & \left(\dfrac{z_1}{1+z_{n+1}},\dots,\dfrac{z_n}{1+z_{n+1}}, i\dfrac{1-z_{n+1}}{1+z_{n+1}}\right), \end{array}$$

can be seen as the analogue in CR geometry of the stereographic projection ; the standard pseudohermitian structure on the sphere is then given by

$$\theta_0 = \psi^*\theta_{\mathbb{H}} = i\sum_{j=1}^{n+1}\left(z_j d\bar{z}^j - \bar{z}_j dz^j\right).$$

The corresponding Reeb field is given by $R_0 = i\sum_{j=1}^{n+1}\left(z_j\partial_{z_j} - \bar{z}_j\partial_{\bar{z}_j}\right)$. Remark that $R_0 = J_0(N_0)$, where $N_0 = \sum_{j=1}^{n+1}\left(z_j\partial_{z_j} + \bar{z}_j\partial_{\bar{z}_j}\right)$ is the unit normal vector field to $\mathbb{S}^{2n+1}$.

The strictly pseudoconvex pseudohermitian manifold $(\mathbb{S}^{2n+1}, H_0, J_0, \theta_0)$ has constant Webster scalar curvature: $\mathrm{Scal}_W(J_0,\theta_0) = \dfrac{n(n+1)}{2}$. A CR manifold is called *spherical* if it is locally CR equivalent to $(\mathbb{S}^{2n+1}, H_0, J_0)$. It is the analogue in CR geometry of being conformally flat.

The standard pseudohermitian structure $(\mathbb{S}^{2n+1}, H_0, J_0, \theta_0)$ can also be interpreted in terms of the Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \xrightarrow{h} \mathbb{CP}^n$, where h is the standard projection of the sphere over the complex projective space [GM11]. Indeed, let us equip the circle $\mathbb{S}^1$ with the group structure of $U(1)$. The sphere $\mathbb{S}^{2n+1}$ is consequently a $U(1)$-bundle over $\mathbb{CP}^n$. Namely,

$$\forall\lambda \in U(1),\ \forall(z_1,\dots,z_n) \in \mathbb{S}^{2n+1}, \quad \lambda(z_1,\dots,z_n) = (\lambda z_1,\dots,\lambda z_n).$$

For $z \in \mathbb{S}^{2n+1}$, let us consider the fiber $(z_t = e^{2i\pi t}z)_{t\in[0,1)}$. Then $\dot{z}_0 = 2\pi R_0(z)$. Moreover, we have the orthogonal decompositions, with respect to the standard Hermitian product in $\mathbb{C}^{n+1}$,

$$T\mathbb{C}^{n+1} = \mathrm{span}\,(N_0) \oplus T\mathbb{S}^{2n+1} = \mathrm{span}\,(N_0, R_0) \oplus H_0,$$

where $H_0 = \ker\ \theta_0$.

1.3.3 Circle bundles over a Riemann surface

We recall here a construction detailed by D. Burns and C. Epstein [BE88], that will be useful in Chapters 2 and 4. Let us consider a compact Riemann surface Σ with a Hermitian metric γ. Let $T^{1,0}\Sigma$ be the holomorphic tangent bundle to Σ, and let M be the unit circle bundle in $T^{1,0}\Sigma$. M is then a $U(1)$-bundle over Σ, whose dual coframe gives a canonical one-form Θ^1 on M. Moreover, since $\dim \Sigma = 2$, γ is automatically a Kähler metric, hence there is a unique torsion-free connection form Θ_1^1 such that $d\Theta^1 = \Theta^1 \wedge \Theta_1^1$. We then have

$$d\Theta_1^1 = K\Theta^1 \wedge \Theta^{\bar{1}},$$

where K is the Gauss curvature of Σ. If K never vanishes, an associated normal strictly pseudoconvex pseudohermitian structure (J, θ) is given on M by $\theta = i \operatorname{sign}(K)\Theta_1^1$ and $\theta^1 = \sqrt{|K|}\Theta^1$, so that $d\theta = i\theta^1 \wedge \theta^{\bar{1}}$. Moreover, we have

$$d\theta^1 = \theta^1 \wedge \left(\frac{1}{2}\frac{|K|_{,1}}{|K|}\theta^1 - \frac{1}{2}\frac{|K|_{,\bar{1}}}{|K|}\theta^{\bar{1}} - i \operatorname{sign}(K)\theta \right).$$

If K is constant, then $\omega_1^1 = -i \operatorname{sign}(K)\theta$, hence $\mathrm{Scal}_W(J, \theta) = \operatorname{sign}(K)$.

Note that, by the following result, all non-spherical SPCR compact 3-manifolds which admit a normal contact form are such bundles or finite quotients of them, *i.e.* Seifert bundles.

Proposition 1.3.1 ([Bel01]). *Let (M, H, J) be a compact normal SPCR 3-manifold. Then (M, H, J) is either a finite quotient of the standard sphere or of a circle bundle over a Riemann surface of positive genus.*

1.4 CR-conformal analogy

1.4.1 The CR Yamabe problem

Let (M, H, J, θ) be a compact strictly pseudoconvex pseudohermitian manifold of dimension $2n + 1$. We already mentioned that the set of positive contact forms on (M, H, J) is a conformal class

$$[\theta] = \{u^{\frac{2}{n}}\theta \mid u \in C^\infty(M, \mathbb{R}_+^*)\}.$$

Here, the choice of the exponent $\dfrac{2}{n}$ is made to simplify further conformal change formulas.

The similarity between conformal and CR geometry can be seen through the variation of the Webster scalar curvature under conformal changes of θ: given a conformal factor u in $C^\infty(M, \mathbb{R}_+^*)$, we have

$$\mathrm{Scal}_W(J, u^{\frac{2}{n}}\theta) = u^{-\frac{n+2}{n}} \left(2\frac{n+1}{n}\Delta_b u + \mathrm{Scal}_W(J, \theta)u \right).$$

Therefore, $u^{\frac{2}{n}}\theta$ has constant Webster curvature $\mathrm{Scal}_W \equiv \lambda$ if and only if

$$2\frac{n+1}{n}\Delta_b u + \mathrm{Scal}_W(J, \theta)u = \lambda u^{\frac{n+2}{n}},$$

which we will call the *CR Yamabe equation*. This equation may be compared with the Riemannian Yamabe equation for a manifold of dimension $2n+2$. This similarity supports the homogeneity mentioned in Section 1.3.1.

By analogy with the conformal case, the CR Yamabe problem is the following question: is there a constant Webster scalar curvature positive contact form in the conformal class $[\theta]$?

As in the conformal case, a sufficient condition is that there exists a contact form which realizes the infimum of the CR invariant

$$Y_{CR}(M,H,J) := \inf_{\hat{\theta}\in[\theta]_1} S_W(M,J,\hat{\theta}), \tag{1.1}$$

where

$$[\theta]_1 := \{\hat{\theta} \in [\theta],\ \mathrm{Vol}(M,\hat{\theta}) := \int_M \hat{\theta}\wedge d\hat{\theta}^n = 1\},$$

and

$$S_W(M,J,\hat{\theta}) := \int_M \mathrm{Scal}_W(J,\hat{\theta})\hat{\theta}\wedge d\hat{\theta}^n$$

denotes the integral Webster scalar curvature. The functional Y_{CR} is maximal for the standard sphere:

Theorem 1.4.1 ([JL87])**.** $Y_{CR}(M,H,J) \leq Y_{CR}(\mathbb{S}^{2n+1},H_0,J_0) = 2\pi n(n+1)$.

A positive contact form minimizing Y_{CR} is called a *Yamabe contact form*. The CR Yamabe problem has been given a positive answer by the following results of D. Jerison and J. Lee, and N. Gamara and R. Yacoub:

Theorem 1.4.2 ([JL87])**.** *If $Y_{CR}(M,H,J) < Y_{CR}(\mathbb{S}^{2n+1},H_0,J_0)$, then there is a Yamabe contact form.*

Theorem 1.4.3 ([JL89])**.** *If $n \geq 2$ and (M,H,J) is not spherical, then $Y_{CR}(M,H,J) < Y_{CR}(\mathbb{S}^{2n+1},H_0,J_0)$.*

Theorem 1.4.4 ([Gam01, GY01])**.** *If $n=1$ or (M,H,J) is spherical, then the CR Yamabe problem has a solution.*

The proof of this last theorem uses a technique of critical points at infinity initiated by A. Bahri. Note that the positive contact forms found this way are not necessarily Yamabe contact forms. However, J.-H. Cheng, A. Malchiodi and P. Yang have shown that Yamabe contact forms always exist on SPCR 3-manifolds with non-negative CR Paneitz operator, cf. Section 1.4.2.

On Einstein strictly pseudoconvex pseudohermitian manifolds, the following result by X. Wang ensures that all constant Webster scalar curvature contact forms are Einstein:

Theorem 1.4.5 ([Wan15])**.** *Let (M,H,J) be a compact SPCR manifold which admits an Einstein contact form θ. If $\tilde{\theta} = u^{\frac{2}{n}}\theta \in [\theta]$ has constant Webster scalar curvature, then $\tilde{\theta}$ is Einstein. Moreover, if (M,H,J) is non-spherical, then u is constant.*

1.4.2 The CR Paneitz operator

In conformal geometry, the Paneitz operator is a differential operator of order 4 defined on a Riemannian manifold of dimension n, conformally covariant when $n = 4$, introduced by S. Paneitz [Pan08]. Several applications of the conformal Paneitz operator can be found in Chapter 7 of [DGH08]. The *CR Paneitz operator* is similarly a CR covariant differential operator of order 4 defined on a strictly pseudoconvex pseudohermitian manifold of dimension $2n+1$, introduced by C. R. Graham and J. Lee [GL88]. It is defined by

Definition 1.4.6. Let (M, H, J, θ) be a pseudohermitian manifold. Let Δ_b be the sublaplacian, R be the Reeb field, and $(T_1, \ldots, T_n)$ be a local basis of $T^{1,0}M$. The CR Paneitz operator is

$$P_1 = \Delta_b^2 + R^2 - 4\mathrm{Im}\left(T_{\overline{\alpha}}\left(\tau_\alpha{}^{\overline{\beta}}T_{\overline{\beta}}\right)\right),$$

where $\mathrm{Im}\left(T_{\overline{\alpha}}\left(\tau_\alpha{}^{\overline{\beta}}T_{\overline{\beta}}\right)\right) := \dfrac{1}{2i}\left(T_{\overline{\alpha}}\left(\tau_\alpha{}^{\overline{\beta}}T_{\overline{\beta}}\right) - T_\alpha\left(\tau_{\overline{\alpha}}{}^{\beta}T_\beta\right)\right).$

The CR Paneitz operator is CR covariant when $n = 1$: if $\hat{\theta} = e^f\theta$ is conformal to θ, then $\hat{P}_1 = e^{2f}P_1$. Graham and Lee more generally introduce, for all $n \geq 1$, an operator P_n of order $2n+2$, which is CR-covariant in dimension $2n+1$.

Contrarily to the conformal case, the CR Paneitz operator always has an infinite dimensional kernel, containing the space $\mathscr{P}$ of CR pluriharmonic functions, *i.e.* functions which are locally real parts of CR-holomorphic functions.

Definition 1.4.7. The CR Paneitz operator is said to be *nonnegative*, and denoted $P_1 \geq 0$, if

$$\forall \varphi \in C^\infty(M, \mathbb{R}), \quad \int_M \varphi P_1 \varphi \theta \wedge d\theta^n \geq 0.$$

Note that the non-negativity of P_1 is a CR condition, *i.e.* it does not depend on the choice of θ. The non-negativity of the CR Paneitz operator plays a key role in the following CR positive mass theorem, whose concepts we will not detail here:

Theorem 1.4.8 ([CMY17]). *Let (M, H, J) be a compact SPCR 3-manifold. Let θ be a blow-up of positive contact form, such that (M, H, J, θ) is asymptotically flat. If $P_1 \geq 0$ and $Y_{CR} > 0$, then the CR ADM mass of (M, H, J, θ) is nonnegative, and vanishes if and only if (M, H, J) is spherical.*

As a consequence, manifolds with non-negative P_1 admit Yamabe contact forms by the following result and by Theorem 1.4.2:

Theorem 1.4.9 ([CMY17]). *If $n = 1$, $P_1 \geq 0$, and (M, H, J) is not spherical, then $Y_{CR}(M, H, J) < Y_{CR}(\mathbb{S}^3, H_0, J_0)$.*

Note that a CR positive mass theorem has also been announced by J.-H. Cheng and H.-L. Chiu for spin, spherical, compact SPCR 5-manifolds with positive Y_{CR} [CC].

Along with Y_{CR}, the CR Paneitz operator also plays a key role in the embeddability problem for CR manifolds:

Theorem 1.4.10 ([CCY12])**.** *Let (M, H, J) be a compact SPCR 3-manifold. If $P_1 \geq 0$ and $Y_{CR} > 0$, then (M, H, J) is embeddable.*

Conversely, the non-negativity of the CR Paneitz operator of an embeddable strictly pseudoconvex pseudohermitian 3-manifold with vanishing pseudohermitian torsion holds for small deformations of the complex structure along embeddable directions [CCY13].

1.5 The contact Yamabe invariant

The *Yamabe invariant* is the global invariant defined, for a compact differentiable manifold M, by

$$\sigma(M) := \sup_{[g]} \inf_{\hat{g} \in [g]_1} \int_M \mathrm{Scal}(\hat{g})\, d\mathrm{vol}_{\hat{g}}$$

where Scal denotes the Riemannian scalar curvature, the supremum runs over all conformal classes of metrics $[g]$ on M and the infimum runs over all metrics of volume 1 in $[g]$. It has been introduced by R. Schoen and O. Kobayashi in the wake of the resolution of the Yamabe problem [Kob87, Sch89].

The resolution of the CR Yamabe problem, cf. Section 1.4.1, leads naturally to the consideration of the following quantity:

Definition 1.5.1 ([Wu09])**.** Let (M, H) be a compact SPC manifold. Let $\mathscr{J}$ be the set of complex structures J on (M, H) such that (M, H, J) is SPCR. The *contact Yamabe invariant*, or σ_c *invariant*, of (M, H) is defined by

$$\sigma_c(M, H) := \sup_{\mathscr{J}} Y_{CR}(M, H, J).$$

This quantity is a global contact invariant of (M, H). It has been introduced by C.-T. Wu, but to our knowledge it has not been used afterwards [Wu09]. This invariant is an actual contact invariant when M is of dimension 3, in the sense that all contact 3-manifolds admit a CR structure. This is no longer the case in higher dimension. Note that few contact invariants are currently available: they are necessarily global by Darboux's theorem, and most of them come from homological considerations.

As in the conformal case, the contact Yamabe invariant characterizes manifolds which admit a structure with positive curvature:

Proposition 1.5.2 ([Wan03])**.** *Let (M, H) be a compact SPC manifold. Then $\sigma_c(M, H) > 0$ if and only if there exists a strictly pseudoconvex pseudohermitian structure (J, θ) on (M, H) with positive Webster scalar curvature.*

Finally, let us recall the following lemma, that will be essential in Chapter 4.

Lemma 1.5.3. *Let (M, H, J) be an SPCR manifold. The infimum in Definition (1.1) of $Y_{CR}(M, H, J)$ may be taken over the space $L_c(M, \mathbb{R}_+)$ of non-negative Lipschitz functions with compact support on M.*

Proof. Indeed, $Y_{CR}(M,H,J) = \inf_{u\in C^\infty(M,\mathbb{R}_+^*)} Q_\theta(u)$ where θ is any positive contact form on (M,H,J) and

$$Q_\theta(u) := \frac{\int_M \left(2\frac{n+1}{n}|\nabla u|^2 + \mathrm{Scal}_W(J,\theta)u^2\right)\theta\wedge d\theta^n}{\left(\int_M u^{2\frac{n+1}{n}}\theta\wedge d\theta^n\right)^{\frac{n}{n+1}}},$$

hence Q_θ is continuous in the Sobolev space $W^{1,2}(M)$. Since $C^\infty(M)$ is dense in $W^{1,2}(M)$, since $Q_\theta(|u|) = Q_\theta(u)$ for all $u \in C^\infty(M)$, and since a nonnegative Lipschitz fonction can be arbitrarily approximated in $W^{1,2}$ norm by a positive smooth function, we have $Y_{CR}(M,H,J) = \inf_{u\in L_c(M,\mathbb{R}_+)} Q_\theta(u)$. $\square$

1.6 The Fefferman bundle

A CR manifold carries biholomorphically invariant curves, called *chains* [CM74]. C. Fefferman showed that, for embeddable CR manifolds (M,H,J), $M\times\mathbb{S}^1$ can be equipped with a Lorentzian metric whose light rays project onto the chains of (M,H,J) [Fef76]. This result was generalized to abstract CR manifolds by D. Burns, K. Diederich, and S. Shnider [BDS77]. F. Farris and J. Lee then provided an intrinsic construction of this metric, called the *Fefferman metric*, which we recall here [Far86, Lee86, Her09].

Let (M,H,J) be an SPCR $(2n+1)$-manifold. Let K be the canonical bundle on (M,H,J), cf. Definition 1.2.4. The *line bundle of* 1*-densities* L is defined as the $(n+2)$-th root of K^*. This root may not exist globally, but it can always be defined locally. Let θ be a positive contact form on M.

Definition 1.6.1 ([CDS05]). The *Weyl structure* attached to θ is the linear connection D on L defined by

$$D = \nabla^\theta + \frac{i\mathrm{Scal}_W(J,\theta)}{2(n+1)(n+2)}\theta.$$

The *Fefferman bundle* is $F = L/\mathbb{R}_+^*$. Let $\pi : F \to M$ be the natural bundle projection. Let ϖ be the S^1-invariant connection 1-form induced by the Weyl structure attached to θ on F. The *Fefferman metric* attached to θ on F is the Lorentzian metric

$$g_F = i\varpi\circ\pi^*\theta + \frac{1}{2}\pi^*\gamma,$$

where $\lambda\circ\mu := \lambda\otimes\mu + \mu\otimes\lambda$. The Fefferman metric is CR-covariant:

Proposition 1.6.2. *For $\hat\theta = f\theta$ in $[\theta]$, we have $\hat g_F = (\pi^* f)g_F$.*

The relations between the Riemannian curvature of g_F and the Tanaka-Webster curvature of θ are the following. They will be useful in Chapter 3.

Proposition 1.6.3 ([Lee86]).

$$\mathrm{Ric}_{g_F} = \frac{\mathrm{Scal}_W}{n+1}g_F - 2n\varpi^2 - 2n\mathbf{S}\theta^2 + n\mathrm{Sch}_W + n\gamma(J\tau\cdot,\cdot) + n\mathbf{T}J\circ\theta,$$

where

$$\mathbf{T} = \frac{1}{n+2}\left(\frac{d_b \mathrm{Scal}_W}{2(n+1)} + i\delta\tau\right) \quad \text{and} \quad \mathbf{S} = \frac{1}{n}\left(\delta\mathbf{T} - |\mathrm{Sch}_W|^2 + |\tau|^2\right).$$

Moreover,

$$\mathrm{Scal}_{g_F} = \frac{2(2n+1)}{n+1}\mathrm{Scal}_W.$$

Note that Fefferman metrics never are Einstein.

CHAPITRE II

ACHE manifolds

Abstract. We present here notions of asymptotically hyperbolic geometry in the complex setting, and its correspondence with CR geometry. This approach gives a way to build CR invariants as Riemannian invariants of a certain asymptotically complex hyperbolic manifold. We detail results on the existence of smooth asymptotically complex hyperbolic Einstein and Kähler-Einstein formal metrics.

Contents

Introduction

Asymptotically hyperbolic manifolds (AH for short) are manifolds which admit a *conformal infinity*, that is to say a boundary equipped with a conformal structure which is, roughly speaking, a generalization of the standard conformal sphere seen as the boundary of the Poincaré disk. Reciprocally, every compact conformal manifold can be filled with an AH manifold X^{n+1} whose metric is Einstein, thus called *AH-Einstein* or *AHE*, when n is odd. When n is even, a conformally invariant obstruction to the existence of a smooth up to the boundary AHE metric appears [FG85, GH05]. Recently, M. J. Gursky and G. Székelyhidi have announced that an AHE metric exists locally for all $n \geq 3$ [GS17]. This approach provides a correspondence between a Riemannian structure on a manifold and a conformal structure on its boundary. Information on the conformal infinity can thus be read on the AHE metric.

The complex counterparts of AH manifolds, asymptotically *complex* hyperbolic manifolds (ACH for short), have been introduced by C. Epstein, R. Melrose, and G. Mendoza [EMM91]. They generalize the construction by C. Fefferman, S.-Y. Cheng, and S.-T. Yau, of *asymptotically Bergman metrics*, which are Kähler-Einstein metrics on bounded strictly pseudoconvex domains of $\mathbb{C}^{n+1}$, which are asymptotic to the CR structure of the boundary [Fef76, CY80]. The regularity of these metrics near the boundary has been studied by J. Lee and R. Melrose [LM82]. To an ACH manifold thus corresponds a *CR infinity*. For example, the CR infinity of the complex hyperbolic space $\mathbb{C}\mathbf{H}^{n+1}$ is $\mathbb{S}^{2n+1}$ endowed with its standard CR structure.

Due to the even "homogeneous dimension" of CR manifolds mentioned in Chapter 1, ACH manifolds have been known to share similarities with the "n even" real case. In particular, the asymptotic development of ACH-Einstein and -Kähler-Einstein metrics has been extensively studied by O. Biquard, M. Herzlich, and Y. Matsumoto, and obstructions to smoothness have been identified [Biq00, BH05, Mat14].

2.1 The case of the sphere

Let us consider the sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ endowed with its standard contact form

$$\theta_0 = \frac{i}{4}\left(z_j d\overline{z}^j - \overline{z}_j dz^j\right)|_{\mathbb{S}^{2n+1}}.$$

Let $\gamma_0 = d\theta_0(\cdot, i\cdot)$ be the induced metric on the contact distribution $\ker \theta_0$. The *Bergman metric* on the ball $\mathbb{B}^{2n+2}$ is given in polar coordinates by

$$g_0 = dt^2 + 4\sinh^2(t)\theta_0^2 + 4\sinh^2\left(\frac{t}{2}\right)\gamma_0.$$

This metric is Kähler and has constant holomorphic sectional curvature -1. The space $(\mathbb{B}^{2n+2}, g_0)$ is known as the *complex hyperbolic space* and is denoted by $\mathbb{C}\mathbf{H}^{n+1}$.

2.2 Asymptotically Bergman metrics

Let Ω be a smooth strictly pseudoconvex domain in $\mathbb{C}^n$. For $u \in C^\infty(\Omega, \mathbb{R})$, we denote
$$u_j := \frac{\partial u}{\partial z_j} \quad \text{and} \quad u_{\overline{k}} := \frac{\partial u}{\partial \overline{z}_k}.$$

Theorem 2.2.1 ([CY80]). *There is a unique complete Kähler-Einstein metric on Ω. It has the form $g = u_{j\overline{k}} dz^j dz^{\overline{k}}$, where $u \in C^\infty(\Omega, \mathbb{R})$ satisfies the complex Monge-Ampère problem*
$$\det\left(u_{j\overline{k}}\right) = e^{(n+1)u} \ \textit{in} \ \Omega \quad \textit{and} \quad u = \infty \ \textit{on} \ \partial\Omega.$$
Moreover, it verifies $\mathrm{Ric}_g = -\dfrac{n+2}{2} g$.

The function $v = e^{-u}$ equivalently satisfies the Dirichlet problem
$$J(v) := (-1)^n \det \begin{pmatrix} v & v_{\overline{k}} \\ v_j & v_{j\overline{k}} \end{pmatrix} = 1 \ \text{in} \ \Omega \quad \text{and} \quad v = 0 \ \text{on} \ \partial\Omega.$$

Now, let us consider the Hilbert space $H^2(\Omega, \mathbb{C})$ of L^2 holomorphic functions on Ω. The Hilbert space projection $P : L^2(\Omega, \mathbb{C}) \to H^2(\Omega, \mathbb{C})$ can be represented by an integration formula
$$\forall f \in L^2(\Omega, \mathbb{C}), \ \forall z \in \Omega, \quad Pf(z) = \int_\Omega f K_\Omega(z, \cdot) d\mathrm{vol}_\Omega.$$
The kernel K_Ω is called the *Bergman kernel*. For $z \in \Omega$, let $\tilde{K}_\Omega(z) = K_\Omega(z, z)$. Then, for some constant c_n, $J(c_n \tilde{K}_\Omega^{-\frac{1}{n+1}})(z) \to 1$ when $z \to \partial\Omega$ [Fef76]. For example, for $\Omega = \mathbb{B}^{2n+2}$, we have $c_n \tilde{K}_\Omega^{-\frac{1}{n+1}} = 1 - |z|^2$. The metric $\left(\partial_{j\overline{k}} \log \tilde{K}_\Omega\right) dz^j dz^{\overline{k}}$ is called the *Bergman metric* of Ω. For $\Omega = \mathbb{B}^{2n+2}$, this metric coincides with the metric g_0 of Section 2.1. The metric given by Theorem 2.2.1 is thus called an *asymptotically Bergman metric*.

2.3 ACHE metrics

More generally, let (M, H, J) be a $(2n+1)$-dimensional orientable compact SPCR manifold. Namely, H is an orientable hyperplane distribution in TM and J is a complex structure on H. Let θ be a compatible positive contact form and $\gamma = d\theta(\cdot, J\cdot)$ be the induced metric. Let R be the Reeb field. Let ∇^θ be the Tanaka-Webster connection of (M, H, J, θ) and τ be the pseudohermitian torsion.

Let $\overline{X} = [0, \varepsilon) \times M$, let $\pi : \overline{X} \to M$ be the natural projection, and let r be the coordinate on $[0, \varepsilon)$. Let X be the interior of $\overline{X}$. Let g_0 be the metric on X
$$g_0 = \frac{dr^2}{r^2} + \frac{\theta^2}{r^2} + \frac{\gamma}{r}.$$

A function $s \in C^\infty(\overline{X}, \mathbb{R}_+)$ is called *boundary defining* if $s > 0$ on X, $s = 0$ and $ds \neq 0$ on $\{0\} \times M$. Equivalently, $s = e^{\pi^* f} r$ for some f in $C^\infty(M, \mathbb{R})$. A conformal change of the boundary defining function corresponds to a conformal change of the contact form. Indeed, let us consider g_0 as $g_0(r, \theta)$, then, for f in $C^\infty(M, \mathbb{R})$,

$$g_0(e^{\pi^* f} r, \theta) = g_0(r, e^{-f}\theta).$$

We define an order O_e adapted to g_0. A normal basis with respect to g_0 is $e = \left(r\partial_r, rR, r^{\frac{1}{2}} T_A\right)$, where (T_A) is an orthonormal basis for γ, considered as a Hermitian metric. Its dual basis is $e^* = \left(r^{-1}dr, r^{-1}\theta, r^{-\frac{1}{2}}\theta^\alpha, r^{-\frac{1}{2}}\theta^{\overline{\alpha}}\right)$. The order O_e takes e and e^* for reference. Thus, we have for example

$$\gamma = \theta^\alpha \circ \theta^{\overline{\alpha}} = r\left(r^{-\frac{1}{2}}\theta^\alpha\right) \circ \left(r^{-\frac{1}{2}}\theta^{\overline{\alpha}}\right) = O_e(r),$$

where $\lambda \circ \mu := \lambda \otimes \mu + \mu \otimes \lambda$.

Definition 2.3.1 ([Biq00]). A metric g on X is called *asymptotically complex hyperbolic*, or *ACH*, if $g - g_0 = o_e(1)$. The CR manifold (M, H, J) is then called the *CR infinity* of (X, g).

Example 2.3.2. For $\lambda > 0$,

$$g = \frac{dr^2}{r^2} + \frac{(1-\lambda^2 r^2)^2}{r^2}\theta^2 + \frac{(1-\lambda r)^2}{r}\gamma$$

is an ACH metric on X. Moreover, if (M, H, J, θ) is Einstein with $\mathrm{Ric}_W(J,\theta) = 2(n+1)\lambda\gamma$, then g is an Einstein metric, satisfying

$$\mathrm{Ric}(g) = -\frac{n+2}{2} g.$$

Indeed, a complex structure $\tilde{J}$ compatible with g on X is given by $\tilde{J}|_{H\times\{r\}} = J$ and $\tilde{J}\partial_r = -\frac{R}{1-\lambda^2 r^2}$, *i.e.* $dr \circ \tilde{J} = (1-\lambda^2 r^2)\theta$. Let $\theta^0 := \frac{1}{\sqrt{2}}\left(\frac{1}{1-\lambda^2 r^2}dr - i\theta\right)$ and let $\sigma := \theta^0 \wedge \theta^1 \wedge \ldots \wedge \theta^n$ be a section of the canonical bundle. Then

$$d\sigma = \frac{i}{\sqrt{2}} d\theta \wedge \theta^1 \wedge \ldots \wedge \theta^n - \theta^0 \wedge d\theta^1 \wedge \ldots \wedge \theta^n + \ldots + (-1)^n \theta^0 \wedge \theta^1 \wedge \ldots \wedge d\theta^n,$$

where the first term vanishes and, since $\tau = 0$ by Definition 1.2.6, $d\theta^\alpha = \theta^\beta \wedge \omega_\beta^\alpha$, hence

$$d\sigma = -\omega_\alpha^\alpha \wedge \sigma.$$

The curvature form of σ is hence given by $-d\omega_\alpha^\alpha = -\mathscr{R}^{\theta}{}_{\alpha}{}^{\rho}{}_{\rho\overline{\alpha}}\theta^\alpha \wedge \theta^{\overline{\alpha}} = 2i(n+1)\lambda d\theta$. Moreover,

$$\sigma \wedge \overline{\sigma} = \frac{(-1)^{n+1} i r^{n+2}}{(1-\lambda^2 r^2)^2 (1-\lambda r)^{2n}} \left(r^{-1}dr\right) \wedge \left((1-\lambda^2 r^2) r^{-1}\theta\right) \wedge \left((1-\lambda r) r^{-\frac{1}{2}}\theta^1\right) \wedge \ldots \wedge \left((1-\lambda r) r^{-\frac{1}{2}}\theta^{\overline{n}}\right).$$

Consequently,

$$|\sigma|_g^2 = \frac{r^{n+2}}{(1-\lambda^2 r^2)^2 (1-\lambda r)^{2n}},$$

hence $\ln|\sigma|_g^2 = (n+2)\ln r - 2\ln(1+\lambda r) - (2n+2)\ln(1-\lambda r)$. We have

$$\partial r = \frac{1}{2}\left(dr - i(1-\lambda^2 r^2)\theta\right),$$

hence

$$i\overline{\partial}\partial r = -\lambda^2 r dr\wedge\theta + \frac{1-\lambda^2r^2}{2}d\theta \quad \text{and} \quad i\overline{\partial}r\wedge\partial r = \frac{1-\lambda^2r^2}{2}dr\wedge\theta.$$

The Ricci form of g is then given by

$$\begin{aligned}\rho_g &= -i\overline{\partial}\partial\ln|\sigma|_g^2 + id\omega_\alpha^\alpha \\ &= -i\overline{\partial}\partial\ln|\sigma|_g^2 + 2(n+1)\lambda d\theta \\ &= \frac{n+2}{2}\left(\frac{1-\lambda^2r^2}{r^2}dr\wedge\theta - \frac{(1-\lambda r)^2}{r}d\theta\right).\end{aligned}$$

With this example in mind, we consider the following problem:

Problem 2.3.3. Is there in general an ACH *Einstein* metric on X?

Such a metric is called an *ACHE* metric. Contrarily to Theorem 2.2.1 for domains of $\mathbb{C}^{n+1}$, such a metric may not exist in general. Nevertheless, there are *formally determined* almost ACHE metrics, in the following sense:

Definition 2.3.4. In any asymptotic development $\sum_k a_k(p)r^k$, the term a_k, seen as a function on M, is called *formally determined* if it is a universal polynomial on a finite jet of the CR structure at $p\in M$ only.

Theorem 2.3.5 ([Mat14])**.** *There is an ACH metric g_E on X, which is Einstein up to order $n+1$, i.e.*

$$\operatorname{Ric}(g_E) = -\frac{n+2}{2}g_E + O_e(r^{n+1}),$$

where O_e denotes the order with respect to any basis e orthonormal for g_0. The metric g_E is formally determined modulo $O_e(r^{n+1})$. Moreover, we have the asymptotic development

$$g_E = g_0 + \Phi + O_e(r^{\frac{3}{2}}),$$

where

$$\Phi = -2\mathrm{Sch}_W(J,\theta) + 2\gamma(J\tau\cdot,\cdot).$$

Note that $\Phi = O_e(r)$.

We thus have a formally determined *almost* ACHE metric on X. A more convenient metric for our study would be an almost ACH-*Kähler*-Einstein metric on X. We have at hand the following results:

Proposition 2.3.6 ([BH05]). *One can construct on X a formal complex structure J_X, entirely formally determined by the CR infinity, starting from the almost complex structure $\tilde{J}$, which is the extension of J to X with $\tilde{J}\partial_r = R$. Moreover, an extension $\tilde{\nabla}^\theta$ of ∇^θ to X is given by*

$$\tilde{\nabla}^\theta r\partial_r = \tilde{\nabla}^\theta rR = \tilde{\nabla}^\theta_{r\partial_r} r^{\frac{1}{2}} T_A = 0.$$

Let $\tilde{T}^\theta$ be the torsion of $\tilde{\nabla}^\theta$ and $\tilde{\tau} := \iota_R \tilde{T}^\theta$. An asymptotic development of J_X is then given by

$$J_X = \tilde{J} - 2r\tilde{\tau} + O_e(r^{\frac{5}{2}}).$$

Theorem 2.3.7 ([BH05, Her07]). *There is a formally determined ACH Kähler metric g_{KE} on (X, J_X), which is Einstein up to order $n + \dfrac{3}{2}$, i.e.*

$$\mathrm{Ric}(g_{KE}) = -\frac{n+2}{2} g_{KE} + O_e(r^{n+\frac{3}{2}}).$$

Moreover, g_E and g_{KE} coincide up to order $n + \dfrac{1}{2}$.

In dimension $2n + 1 = 3$, the asymptotic development of g_{KE}, and therefore of g_E, is known at order $\dfrac{3}{2}$, which will be essential in Chapter 3:

Theorem 2.3.8 ([BH05, Her07]). *When $n = 1$, we have the asymptotic development*

$$g_{KE} = g_0 + \Phi_{AB}\theta^A \circ \theta^B + \Psi_{0\bar{1}}\theta^0 \circ \theta^{\bar{1}} + \Psi_{\bar{0}1}\theta^{\bar{0}} \circ \theta^1 + O_e(r^2),$$

where

$$\Psi_{0\bar{1}} = -\sqrt{2}\left(\frac{1}{6}\mathrm{Scal}_{W,\bar{1}} - \frac{2i}{3}\tau^1_{\bar{1},1}\right),$$

and Φ is given by Theorem 2.3.5:

$$\Phi_{1\bar{1}} = -\frac{\mathrm{Scal}_W}{4} \quad \text{and} \quad \Phi_{11} = -i\tau^{\bar{1}}_1.$$

2.4 CR invariants

As in the conformal case, the correspondence between CR manifolds and ACHE manifolds has many consequences. It provides an approach to build CR invariants via Riemannian invariants of the interior. We present here two such examples.

2.4.1 The Burns-Epstein invariant

Let (M, H, J) be a compact SPCR 3-manifold. The *Burns-Epstein invariant* $\mu(M, H, J)$ is defined as the evaluation of a well-chosen de Rham cohomology class on the fundamental class $[M]$ in $H_3(M, \mathbb{R})$ [BE88]. In particular, we have the following estimates, which will be useful in Chapter 4.

Proposition 2.4.1 ([Mar15]). *The Burns-Epstein invariant of a compact SPCR 3-manifold (M,H,J) admitting a pseudo-Einstein contact form θ is given by*

$$\mu(M,H,J) = \frac{1}{4\pi^2}\int_M \left(|\tau(J,\theta)|^2 - \frac{1}{4}\mathrm{Scal}_W(J,\theta)^2\right)\theta\wedge d\theta.$$

Proposition 2.4.2 ([BE88]). *The Burns-Epstein invariant value of a circle bundle (M,H,J) over a Riemann surface Σ is*

$$\mu(M,H,J) = -\frac{|\chi(\Sigma)|}{4} + \frac{1}{12\pi}\int_\Sigma \frac{(\Delta \log|\mathrm{Scal}_W(J,\theta_J)|)^2}{|\mathrm{Scal}_W(J,\theta_J)|}\mathrm{dvol}_\Sigma,$$

where θ_J is the unique normal contact form on (M,H,J) and $\chi(\Sigma)$ is the Euler characteristic of Σ.

Theorem 2.4.3 ([BH05]). *Let (X,g) be an ACHE 4-manifold with CR infinity (M,H,J). There is a global CR invariant $\nu(M,H,J)$ such that*

$$\frac{1}{8\pi^2}\int_X \left(3|W_g^-|^2 - |W_g^+|^2 + \frac{1}{24}\mathrm{Scal}_g^2\right)\mathrm{dvol}_g = \chi(X) - 3\tau(X) + \nu(M,H,J),$$

where $\chi(X)$ is the Euler characteristic of X, $\tau(X)$ is its signature, and W_g^- and W_g^+ are respectively the antiselfdual and selfdual Weyl tensors of (X,g).

Theorem 2.4.4 ([BH05]). *Let J and J' be two CR structures on (M,H). Then*

$$\nu(M,H,J) - \nu(M,H,J') = 3\left(\mu(M,H,J) - \mu(M,H,J')\right).$$

2.4.2 The renormalized volume

Theorem 2.4.5 ([Her07]). *Let (X,g) be an ACHE 4-manifold with CR infinity (M,H,J). Let θ be a positive contact form on (M,H,J). There is a pseudohermitian invariant V on M, called* renormalized volume, *such that*

$$\mathscr{V} = \frac{3}{2}V - \int_M \left(\frac{\mathrm{Scal}_W(J,\theta)^2}{16} - \frac{5}{2}|\tau(J,\theta)|^2\right)\theta\wedge d\theta$$

is a CR invariant of (M,H,J). Moreover,

$$\chi(X) = \frac{1}{8\pi^2}\int_X \left(|W_g|^2 - \frac{1}{24}\mathrm{Scal}_g^2\right)\mathrm{dvol}_g + \frac{1}{4\pi^2}\mathscr{V}.$$

2.5 Θ-structures

We briefly present here an equivalent approach to define ACH manifolds, using Θ-structures. This construction is due to C. L. Epstein, R. B. Melrose, and G. A. Mendoza [EMM91]. We follow the description of C. Guillarmou and A. Sá Barreto [GS08].

Let $\overline{X}$ be a smooth $(2n+2)$-dimensional manifold with boundary. We denote $X := \overline{X}\setminus\partial\overline{X}$, and we say that $\overline{X}$ *compactifies* X.

Definition 2.5.1. A *boundary defining function* on $\overline{X}$ is a function $r \in C^\infty(\overline{X}, \mathbb{R}_+)$ such that $r > 0$ on X, $r = 0$ and $dr \neq 0$ on $\partial\overline{X}$.

Let us consider $\Theta \in C^\infty(\partial\overline{X}, T^*\overline{X}|_{\partial\overline{X}})$. We call Θ a *precontact form* if, denoting $\iota : \partial\overline{X} \to \overline{X}$ the inclusion, $\theta := \iota^*\Theta$ is a contact form on $\partial\overline{X}$.

Definition 2.5.2. A Θ-*structure* on $\overline{X}$ is a conformal class $[\Theta]$ of precontact forms.

In that case, let us consider any boundary defining function $r \in C^\infty(\overline{X}, \mathbb{R}_+)$. Let us consider any extension $\tilde{\Theta}$ of Θ in $C^\infty(\overline{X}, T^*\overline{X})$. We define the subspace $\mathscr{V}_\Theta$ of $C^\infty(\overline{X}, T\overline{X})$ as follows:

$$V \in \mathscr{V}_\Theta \Longleftrightarrow V \in r^{\frac{1}{2}} C^\infty(\overline{X}, T\overline{X}), \quad \tilde{\Theta}(V) \in r C^\infty(\overline{X}, \mathbb{R}).$$

Namely, let (∂_r, R, T_A) be a local basis of $T\overline{X}$ such that

$$\mathrm{Span}(\partial_r, T_A) \subset \ker \tilde{\Theta}, \quad \mathrm{Span}(R, T_A) \subset T\partial\overline{X}, \quad dr(\partial_r) = \tilde{\Theta}(R) = 1.$$

Then

$$\forall V \in \mathscr{V}_\Theta,\ \exists a, b, c^A \in C^\infty(X, \mathbb{R}), \quad V = ar\partial_r + brR + c^A r^{\frac{1}{2}} T_A.$$

Note that $\mathscr{V}_\Theta$ only depends on $[\Theta]$. For $p \in \partial\overline{X}$, let F_p be the set of vector fields of $\mathscr{V}_\Theta$ such that $a(p) = b(p) = c^A(p) = 0$.

Let ${}^\Theta T\overline{X}$ be the vector bundle over $\overline{X}$ whose smooth sections are the elements of $\mathscr{V}_\Theta$. A local basis of ${}^\Theta T^*\overline{X}$ near $\partial\overline{X}$ is $\left(r^{-1}dr, r^{-1}\tilde{\Theta}, r^{-\frac{1}{2}}\tilde{\Theta}^A\right)$.

Definition 2.5.3. A Θ-*metric* is a smooth positive definite 2-tensor on ${}^\Theta T^*\overline{X}$, *i.e.* $g \in C^\infty(\overline{X}, S^2_+\left({}^\Theta T^*\overline{X}\right))$.

Proposition 2.5.4 ([GS08]). *Let $\hat{\Theta} \in [\Theta]$ and let g be a Θ-metric. There exists a unique boundary defining function r such that $\|r^{-1}dr\|_g = 1$ in a neighbourhood of $\partial\overline{X}$ and $r^2 g|_{\partial\overline{X}} = \hat{\theta}^2$. Moreover, let us denote $H = \ker\theta$. If $\ker(r^{-1}\theta) \perp_g rC^\infty(\overline{X}, T\overline{X})/F$ and if*

$$\exists J \in \mathrm{End}(H),\ J^2 = -\mathrm{id}_H \ \textit{and}\ \hat{\gamma} = d\hat{\theta}(\cdot, J\cdot) = rg|_H,$$

then we have

$$g = \frac{dr^2}{r^2} + \frac{\hat{\theta}^2}{r^2} + \frac{\hat{\gamma}}{r} + r^{\frac{1}{2}}\tilde{g},$$

where $\tilde{g} \in C^\infty(\overline{X}, S^2\left({}^\Theta T^*\overline{X}\right))$.

Definition 2.5.5. An *asymptotically complex hyperbolic manifold*, or *ACH manifold*, is a non-compact Riemannian manifold (X, g) such that there exists a smooth compact manifold with boundary $\overline{X}$ which compactifies X, which admits a Θ-structure, and such that g is a Θ-metric satisfying the hypotheses of Proposition 2.5.4.

CHAPITRE III

CR-harmonic maps

Abstract. We develop the notion of renormalized energy in CR geometry, for maps from a strictly pseudoconvex pseudohermitian manifold to a Riemannian manifold. This energy is a CR invariant functional, whose critical points, which we call CR-harmonic maps, satisfy a CR covariant subelliptic partial differential equation. The corresponding operator coincides on functions with the CR Paneitz operator.

Contents

Introduction

Let (M, g) and (N, h) be two Riemannian manifolds. The *Dirichlet energy* of a map $\varphi : (M, g) \to (N, h)$ is defined as

$$E(\varphi) = \frac{1}{2}\int_M \|T\varphi\|^2_{g,h} \mathrm{dvol}_g.$$

When $\dim M = 2$, the energy is conformally invariant with respect to g. This is of considerable usefulness, e.g. to construct conformal minimal immersions of Riemann surfaces [Mil79]. However, in higher dimension, the energy is no longer conformally invariant.

Critical points of a functional are solutions to a partial differential equation called the *Euler-Lagrange equation* of the functional; in other words, they form the kernel of a certain differential operator. In our case, the critical points of the Dirichlet energy are called *harmonic maps*, and harmonic *functions* $\varphi : (M, g) \to (\mathbb{R}, \mathrm{eucl})$ coincide with the kernel of the Laplacian.

In a recent work, V. Bérard has shown the existence, given two Riemannian manifolds (M, g) and (N, h), with M of even dimension n, of a functional $\mathscr{E}^n_g$ on $C^\infty(M, N)$, conformally invariant with respect to g, and equal to the usual energy when $n = 2$ [Bér13]. This functional is called *renormalized energy*, and its critical points are called *conformal-harmonic maps*. Conformal-harmonic maps generalize harmonic maps; moreover, when $n = 4$ and $N = \mathbb{R}$, the induced operator coincides with the Paneitz operator.

We develop here the notions of CR-harmonicity and renormalized energy in CR geometry. CR-harmonic maps also generalize CR-holomorphic maps, which are notoriously hard to come by. When $\dim M = 3$ and $N = \mathbb{R}$, the induced operator coincides with the CR Paneitz operator. This generalizes the recent work of T. Marugame [Mar18]. Another extension of the CR Paneitz operator to maps has been proposed by T. Chong, Y. Dong, Y. Ren, and G. Yang [CDRY17]. The main result is the following, which summarizes Proposition 3.2.1 and Theorem 3.1.3:

Theorem 3.0.1. *Let (M^{2n+1}, H, J, θ) be a compact strictly pseudoconvex pseudohermitian manifold and (N, h) be a Riemannian manifold. There exists a functional F_n on $C^\infty(M, N)$ which is a CR invariant,* i.e. *conformally invariant with respect to θ. For $\varphi \in C^\infty(M, N)$, it reads*

$$F_n(\varphi) = \frac{(-1)^{n+1}}{2n!^2}\int_M \left\langle (\delta_b^{\theta,h}\nabla^{\varphi^* h})^{n-1}\delta_b^{\theta,h}T\varphi, \delta_b^{\theta,h}T\varphi\right\rangle_h \theta\wedge d\theta^n + \text{lower order terms (in derivatives of } \varphi),$$

where $\delta_b^{\theta,h}$ is the Webster divergence on $\Omega^1(M) \otimes \varphi^* TN$.

The Euler-Lagrange equation of F_n is a subelliptic partial differential equation of order $2n + 2$, itself CR covariant. For $\varphi \in C^\infty(M, N)$, it reads

$$0 = \frac{(-1)^n}{n!}(\delta_b^{\theta,h}\nabla^{\varphi^* h})^n\delta_b^{\theta,h}T\varphi + \text{lower order terms (in derivatives of } \varphi).$$

We then provide a proof strategy of the existence of non-trivial CR-harmonic maps. This relies on an assumption on the spectrum of some differential operator and a conjecture on the principal symbol of another operator, that we have not been able to check so far but that we believe true. We also describe the correspondence between CR-harmonic maps and conformal-harmonic maps via the Fefferman bundle.

3.1 CR-harmonic maps

3.1.1 Definitions

Let (M, H, J) be a $(2n+1)$-dimensional orientable, compact, strictly pseudoconvex CR manifold and (X, g) be an ACH manifold with CR infinity (M, H, J), where g is the approximately ACH-Kähler-Einstein metric given by Theorem 2.3.7. Let $\pi : X \to M$ be the standard projection. Let (N, h) be a Riemannian manifold. Let $\varphi \in C^\infty(M, N)$, and let $\tilde{\varphi} \in C^\infty(\overline{X}, N)$ be any *extension* of φ, *i.e.* $\tilde{\varphi}|_M = \varphi$.

Let $T\tilde{\varphi}$ be the tangent map of $\tilde{\varphi}$. It is a section of the bundle $\Omega^1(\overline{X}) \otimes \tilde{\varphi}^*TN$, and its norm is defined by
$$\|T\tilde{\varphi}\|^2_{g,h} := \mathrm{tr}_g(\tilde{\varphi}^*h).$$
The bundle $\Omega^1(\overline{X}) \otimes \tilde{\varphi}^*TN$ is canonically equipped with the connection
$$\nabla^{g,h} := \nabla^g \otimes 1_{\tilde{\varphi}^*TN} + 1_{\Omega^1(\overline{X})} \otimes \nabla^{\tilde{\varphi}^*h},$$
where ∇^g and ∇^h are the respective Levi-Civita connections of g and h, and $\nabla^{\tilde{\varphi}^*h} := \tilde{\varphi}^*\nabla^h$.

The *divergence* $\delta^{g,h}$ is then defined for $\omega \in \Omega^1(\overline{X}) \otimes \tilde{\varphi}^*TN$ by
$$\delta^{g,h}\omega := -\left(\nabla^{g,h}_{e_I}\omega\right)(e_{\overline{I}}),$$
where (e_i) is an orthonormal basis of $T^{1,0}\overline{X}$ for g, considered as a Hermitian metric. We thus have
$$\delta^{g,h}\omega = -\nabla^{\tilde{\varphi}^*h}_{e_I}(\omega(e_{\overline{I}})) + \omega(\nabla^g_{e_I}e_{\overline{I}}).$$

For $\rho \in (0, \varepsilon)$, the *energy* of $\tilde{\varphi}$ in $(\rho, \varepsilon) \times M$ is the functional
$$E(\tilde{\varphi}, \rho) = \frac{1}{2}\int_{(\rho,\varepsilon)\times M} \|T\tilde{\varphi}\|^2_{g,h}\mathrm{dvol}_g.$$

An extension $\tilde{\varphi}$ is said to be *harmonic* if it is a critical point of the energy for all ρ. Equivalently, $\tilde{\varphi}$ is harmonic if and only if $\delta^{g,h}T\tilde{\varphi} = 0$.

Following the ideas of C. R. Graham, R. Jenne, L. J. Mason, and G. A. J. Sparling, we want to find the obstructions to the existence of a smooth harmonic extension [GJMS92]. More precisely, assuming that $\tilde{\varphi}$ is smooth, we want to know if the first terms of the asymptotic development of $\tilde{\varphi}$ are determined by the data at infinity. By similarity with the real case and based on the known asymptotic developments of the approximately ACH-Einstein metrics, we expect to find an obstruction at order $n+1$, taking the form of a CR covariant subelliptic differential operator of order $2n+2$.

Here, the *asymptotic development* of $\tilde{\varphi}$ will denote, by identification, the asymptotic development in r of $U := \exp_{\varphi}^{-1} \circ \tilde{\varphi} \in C^{\infty}(\overline{X}, (\varphi \circ \pi)^* TN)$, *i.e.*

$$\forall p \in M,\ \forall r \in (0, \varepsilon), \quad \tilde{\varphi}(p,r) := \exp_{\varphi(p)}\left(U(p,r)\right),$$

where, for $p \in M$, the exponential map $\exp_{\varphi(p)}$ is a diffeomorphism between a small ball $B(0,\varepsilon) \subset T_{\varphi(p)}N$ and its image, which is a neighbourhood in N of $\varphi(p)$. Note that $U(\cdot, 0) = 0$. The identification is justified by the fact that, denoting $v\tilde{\varphi} := T\tilde{\varphi}(v)$ for $v \in TX$, and similarly for φ on TM, we have

$$\begin{aligned} \partial_r \tilde{\varphi} &= \partial_r \left(\exp_{\varphi\circ\pi}(U)\right) \\ &= \varphi_1 + \varphi_2 r + \varphi_3 \frac{r^2}{2!} + \dots, \end{aligned}$$

where

$$\forall k \geq 1, \quad \varphi_k := (\nabla^{\tilde{\varphi}^* h}_{\partial_r})^{k-1} \partial_r \tilde{\varphi}|_{r=0}.$$

Note that φ_k is a section of $\varphi^* TN$, hence $\nabla^{\varphi^* h} \varphi_k$ is a section of $\Omega^1(M) \otimes \varphi^* TN$.

3.1.2 Computation of the divergence

We use the notations of section 2.3. Let (T_α) be a local basis of $T^{1,0}M$ and $T_{\overline{\alpha}} := \overline{T_\alpha}$, such that (T_A) is orthonormal for γ, considered as a Hermitian metric. Let (θ^A) be the basis dual to (T_A). Let $T_0 := \dfrac{\partial_r - iR}{\sqrt{2}}$ and $\theta^0 := \dfrac{dr + i\theta}{\sqrt{2}}$ its dual.

Lemma 3.1.1. *For $\omega \in \Omega^1(\overline{X}) \otimes \tilde{\varphi}^* TN$, we have*

$$\begin{aligned} \delta^{g_0,h}\omega &= nr\omega(\partial_r) - r^2\left(\nabla^{\tilde{\varphi}^*h}_{T_0}\omega(T_{\overline{0}}) + \nabla^{\tilde{\varphi}^*h}_{T_{\overline{0}}}\omega(T_0)\right) - r\nabla^{\tilde{\varphi}^*h}_{T_A}\omega(T_{\overline{A}}) \\ &= nr\omega(\partial_r) - r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\omega(\partial_r) - r^2\nabla^{\tilde{\varphi}^*h}_{R}\omega(R) - r\nabla^{\tilde{\varphi}^*h}_{T_A}\omega(T_{\overline{A}}). \end{aligned}$$

Proof. We have

$$g_0 = r^{-2}\theta^0 \circ \theta^{\overline{0}} + r^{-1}\theta^\alpha \circ \theta^{\overline{\alpha}},$$

where $\lambda \circ \mu := \lambda \otimes \mu + \mu \otimes \lambda$.

An orthonormal basis of $T^{1,0}\overline{X}$ with respect to g_0 is hence given by

$$(e^{(0)}_0, e^{(0)}_\alpha) \ := \ \left(rT_0, r^{\frac{1}{2}}T_\alpha\right).$$

The trace of the Levi-Civita connection of g_0 is given in this basis by the Koszul formula:

$$\nabla^{g_0}_{e^{(0)}_I} e^{(0)}_{\overline{I}} = g_0\left([e^{(0)}_{\overline{J}}, e^{(0)}_I], e^{(0)}_{\overline{I}}\right) e^{(0)}_J.$$

Let $\tilde{\nabla}^\theta$ be the extension of ∇^θ given by Proposition 2.3.6. We have

$$\begin{aligned}
[e_0^{(0)}, e_{\overline{0}}^{(0)}] &= \frac{1}{\sqrt{2}}\left(e_{\overline{0}}^{(0)} - e_0^{(0)}\right), \\
[e_0^{(0)}, e_A^{(0)}] &= \frac{1}{\sqrt{2}}\left(\frac{1}{2}e_A^{(0)} - i\left(\tilde{\nabla}^\theta_{e_0{}^{(0)}} e_A^{(0)} - \tau(e_A^{(0)})\right)\right), \\
[e_A^{(0)}, e_B^{(0)}] &= 0.
\end{aligned}$$

Then, since $\mathrm{tr}(\tau) = 0$,

$$\nabla^{g_0}_{e_I^{(0)}} e_{\overline{I}}^{(0)} = (n+1)\, r\partial_r,$$

and also,

$$\begin{aligned}
\nabla^{\tilde{\varphi}^*h}_{e_0^{(0)}} \omega(e_{\overline{0}}^{(0)}) + \nabla^{\tilde{\varphi}^*h}_{e_{\overline{0}}^{(0)}} \omega(e_0^{(0)}) &= r\omega(\partial_r) + r^2 \nabla^{\tilde{\varphi}^*h}_{\partial_r} \omega(\partial_r) + r^2 \nabla^{\tilde{\varphi}^*h}_{R} \omega(R), \\
\nabla^{\tilde{\varphi}^*h}_{e_\alpha^{(0)}} \omega(e_{\overline{\alpha}}^{(0)}) &= r \nabla^{\tilde{\varphi}^*h}_{T_\alpha} \omega(T_{\overline{\alpha}}).
\end{aligned}$$

Hence the announced expression for $\delta^{g_0,h}\omega$. □

Let us denote by $\left(\delta^{g,h}\omega\right)^{(1)}$ the remainder of $\delta^{g,h}\omega$, *i.e.*

$$\left(\delta^{g,h}\omega\right)^{(1)} := \delta^{g,h}\omega - \delta^{g_0,h}\omega.$$

We prove the following technical lemma, which is crucial for the proof of Theorem 3.1.3.

Lemma 3.1.2. *For $\omega \in \Omega^1(\overline{X}) \otimes \tilde{\varphi}^* TN$, denoting by O_T the order with respect to the basis (T_I) in powers of r, we have*

$$\left(\delta^{g,h}\omega\right)^{(1)} = O_T(r^2),$$

and there is no term of order 2 in the remainder of the form $r^2 \nabla^{\tilde{\varphi}^*h}_{\partial_r} \omega(\partial_r)$.

Proof. By Theorem 2.3.5, we have

$$g - g_0 = \Phi + O_e(r^{\frac{3}{2}}) = \Phi_{AB}\theta^A \circ \theta^B + O_e(r^{\frac{3}{2}}),$$

where we recall that $\Phi = -2\mathrm{Sch}_W(J,\theta) + 2\gamma(J\tau\cdot,\cdot)$, and that O_e denotes the order with respect to $(e_I^{(0)})$. Note that $\Phi_{AB} = \Phi_{BA}$. Since Φ is real, we have also $\Phi_{\overline{\alpha}\beta} = \overline{\Phi_{\alpha\overline{\beta}}}$ and $\Phi_{\overline{\alpha}\overline{\beta}} = \overline{\Phi_{\alpha\beta}}$.

An orthonormal basis of $T^{1,0}\overline{X}$ with respect to g induced from $e^{(0)}$ is formally given by

$$(e_0, e_\alpha) := \left(e_0^{(0)} + e_0^{(1)}, e_\alpha^{(0)} + e_\alpha^{(1)}\right),$$

where, by the Gram-Schmidt process, and since Φ is horizontal,

$$e_0^{(1)} = O_e(r^{\frac{3}{2}}) \quad \text{and} \quad e_\alpha^{(1)} = O_e(r).$$

This leads to

$$\begin{aligned}\left(\delta^{g,h}\omega\right)^{(1)} &= -\nabla^{\tilde{\varphi}^*h}_{e_I^{(0)}}\omega\left(e_{\overline{I}}^{(1)}\right) - \nabla^{\tilde{\varphi}^*h}_{e_I^{(1)}}\omega\left(e_{\overline{I}}^{(0)}\right) - \nabla^{\tilde{\varphi}^*h}_{e_I^{(1)}}\omega\left(e_{\overline{I}}^{(1)}\right) \\ &\quad +\omega\left(\nabla^{g}_{e_I^{(0)}}e_{\overline{I}}^{(0)} - \nabla^{g_0}_{e_I^{(0)}}e_{\overline{I}}^{(0)}\right) + \omega\left(\nabla^{g}_{e_I^{(0)}}e_{\overline{I}}^{(1)}\right) + \omega\left(\nabla^{g}_{e_I^{(1)}}e_{\overline{I}}^{(0)}\right) + \omega\left(\nabla^{g}_{e_I^{(1)}}e_{\overline{I}}^{(1)}\right),\end{aligned}$$

all terms of which are in $O_T(r^2)$ and are not of the form $r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\omega(\partial_r)$. □

3.1.3 An obstruction to regularity

Theorem 3.1.3. *Let (M, H, J) be a $(2n+1)$-dimensional orientable, compact, strictly pseudoconvex CR manifold and (X, g) be an ACH manifold with CR infinity (M, H, J), where g is the approximately ACH-Kähler-Einstein metric given by Theorem 2.3.7. Let $\pi : \overline{X} \to M$ be the standard projection. Let (N, h) be a Riemannian manifold, and let $\varphi \in C^\infty(M, N)$.*

There exists a section U of $(\varphi\circ\pi)^*TN$, unique modulo $O_T(r^{n+1})$, such that $\tilde{\varphi} = \exp_\varphi \circ U$ satisfies

$$\begin{cases} \tilde{\varphi}|_M &= \varphi, \\ \delta^{g,h}T\tilde{\varphi} &= O_T(r^{n+2}). \end{cases}$$

The asymptotic development in r of U is

$$U = U_1 r + \ldots + U_n\frac{r^n}{n!} + P_n(\varphi)\frac{r^{n+1}}{(n+1)!}\log r + O_T(r^{n+1}),$$

where $U_1, \ldots, U_n, P_n$ are formally determined by φ, g and h.

$P_n(\varphi)$ is an obstruction to the regularity of U, and is given by

$$\begin{aligned}P_n(\varphi) &= \left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^n \tilde{\delta}^{\theta,h}_b T\tilde{\varphi}\Big|_{r=0} - n\left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^{n-1}\nabla^{\tilde{\varphi}^*h}_{R}R\tilde{\varphi}\Big|_{r=0} + \frac{1}{n+1}\left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^{n+1}\left(\delta^{g,h}T\tilde{\varphi}\right)^{(1)}\Big|_{r=0} \\ &= \frac{(-1)^n}{n!}(\delta^{\theta,h}_b\nabla^{\varphi^*h})^n\delta^{\theta,h}_b T\varphi + \text{lower order terms (in derivatives of } \varphi).\end{aligned}$$

Proof. For $m \in \mathbb{N}$, we have

$$\delta^{g,h}T\tilde{\varphi} = O_T(r^{m+1}) \quad \Longleftrightarrow \quad \forall k \le m, \quad \left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^k\delta^{g,h}T\tilde{\varphi}\Big|_{r=0} = 0.$$

We recall the notation

$$\varphi_k := \left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^{k-1}\partial_r\tilde{\varphi}\Big|_{r=0}.$$

Now, by Lemma 3.1.1, we have, for $\omega \in \Omega^1(\overline{X}) \otimes \tilde{\varphi}^*TN$,

$$\nabla^{\tilde{\varphi}^*h}_{\partial_r}\delta^{g,h}\omega\Big|_{r=0} = n\,\omega(\partial_r)|_{r=0} + \delta^{\theta,h}_b(\omega|_{r=0}),$$

and, for all $2 \leq k \leq n$,

$$\begin{aligned}\frac{1}{k}\left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^k \delta^{g,h}\omega\Big|_{r=0} &= (n-k+1)\left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^{k-1}\omega(\partial_r)\Big|_{r=0} + \left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^{k-1}\tilde{\delta}^{\theta,h}_b\omega\Big|_{r=0}\\ &\quad -(k-1)\left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^{k-2}\nabla^{\tilde{\varphi}^*h}_R\omega(R)\Big|_{r=0} + \frac{1}{k}\left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^k\left(\delta^{g,h}\omega\right)^{(1)}\Big|_{r=0},\end{aligned}$$

where

$$\forall\omega_0 \in \Omega^1(M)\otimes\varphi^*TN, \quad \delta^{\theta,h}_b\omega_0 := -\nabla^{\varphi^*h}_{T_A}\omega_0(T_{\overline{A}}),$$

and

$$\forall\omega \in \Omega^1(\overline{X})\otimes\tilde{\varphi}^*TN, \quad \tilde{\delta}^{\theta,h}_b\omega := -\nabla^{\tilde{\varphi}^*h}_{T_A}\omega(T_{\overline{A}}).$$

Consequently, $\delta^{g,h}T\tilde{\varphi} = O_T(r^{n+1})$ is equivalent to

$$\begin{cases} n\varphi_1 = -\delta^{\theta,h}_b T\varphi, \\ (n-k+1)\varphi_k = -\left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^{k-1}\tilde{\delta}^{\theta,h}_b T\tilde{\varphi}\Big|_{r=0} - D_{k-1}(\varphi) \quad \forall 2\leq k\leq n, \end{cases}$$

where

$$D_{k-1}(\varphi) := -(k-1)\left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^{k-2}\nabla^{\tilde{\varphi}^*h}_R R\tilde{\varphi}\Big|_{r=0} + \frac{1}{k}\left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^k\left(\delta^{g,h}T\tilde{\varphi}\right)^{(1)}\Big|_{r=0}.$$

By Lemma 3.1.2, $D_{k-1}(\varphi)$ only depends on $\varphi, \varphi_1, \ldots, \varphi_{k-1}$. This observation comes from the fact that, although

$$\left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^k\left(r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\partial_r\tilde{\varphi}\right)\Big|_{r=0} = 2\varphi_k,$$

$\forall X, Y \in \{\partial_r, R, T_A\}$, $(X,Y)\neq(\partial_r,\partial_r)$, $\quad \left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^k\left(r^2\nabla^{\tilde{\varphi}^*h}_X Y\tilde{\varphi}\right)\Big|_{r=0}$ does not depend on φ_k.

By induction, $D_{k-1}(\varphi)$ is thus well-defined.

In conclusion, requiring $\delta^{g,h}T\tilde{\varphi} = O_T(r^{n+1})$ gives an asymptotic development for $\tilde{\varphi}$ in powers of r, and this development is unique up to order n with respect to T.

Assume now that $\delta^{g,h}T\tilde{\varphi} = O_T(r^{n+1})$ and that $\tilde{\varphi}$ admits a Taylor development up to order $n+1$. Then

$$\delta^{g,h}T\tilde{\varphi} = O_T(r^{n+2}) \Longleftrightarrow \left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^n\tilde{\delta}^{\theta,h}_b T\tilde{\varphi}\Big|_{r=0} + D_n(\varphi) = 0.$$

This equality cannot be true in general. Consequently, we introduce a term in $r^{n+1}\log r$ in the development of $\tilde{\varphi}$:

$$U = U_1 r + \ldots + U_n\frac{r^n}{n!} + P_n(\varphi)\frac{r^{n+1}}{(n+1)!}\log r + O_T(r^{n+1}).$$

The coefficient $P_n(\varphi)$ verifies

$$\frac{1}{n+1}\left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^{n+1}\delta^{g,h}T\tilde{\varphi}\Big|_{r=0} = -P_n(\varphi) + \left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^n\tilde{\delta}^{\theta,h}_b T\tilde{\varphi}\Big|_{r=0} + D_n(\varphi),$$

hence

$$\delta^{g,h}T\tilde{\varphi} = O_T(r^{n+2}) \Longleftrightarrow P_n(\varphi) = \left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^n\tilde{\delta}^{\theta,h}_b T\tilde{\varphi}\Big|_{r=0} + D_n(\varphi).$$

This yields the announced obstruction, which only depends on φ. Since

$$\varphi_k = -\frac{1}{n-k+1}\delta_b^{\theta,h}\nabla^{\varphi^*h}\varphi_{k-1} + \text{lower order terms (in derivatives of } \varphi\text{)},$$

we have the announced leading term. □

Proposition 3.1.4. *P_n does not depend on whether we take $g = g_E$ or g_{KE} on X.*

Proof. To compute P_n, it is sufficient to be able to compute

$$\left(\nabla_{\partial_r}^{\tilde{\varphi}^*h}\right)^{n+1}\left(\delta^{g,h}T\tilde{\varphi}\right)^{(1)}\Big|_{r=0};$$

i.e., by the proof of Lemma 3.1.2, to know the $e_I^{(1)}$ at order $n+1/2$ with respect to $e^{(0)}$. By the Gram-Schmidt process, it is thus sufficient to know g at order $n+1/2$ with respect to $e^{(0)}$. Hence, by Theorems 2.3.5 and 2.3.7, we can equivalently consider g_E or g_{KE}. □

Proposition 3.1.5. *Let $f \in C^\infty(M)$ and let $\hat{r} = e^{\pi^* f} r$ be a conformal change of boundary defining function. Then*

$$\hat{P}_n(\varphi) = e^{-(n+1)f}P_n(\varphi).$$

The obstruction $P_n(\varphi)$ to the regularity of $\tilde{\varphi}$ is therefore CR covariant.

Proof. We have

$$U = U_1 r + \ldots + U_n\frac{r^n}{n!} + P_n(\varphi)\frac{r^{n+1}}{(n+1)!}\log r + O_T(r^{n+1}).$$

Now, since $\exp_\varphi : (\varphi\circ\pi)^*TN \to N$ does not depend on r, neither does U. Moreover, since M is compact, $\forall k,\ O_T(\hat{r}^k) = O_T(r^k)$. We thus have

$$\begin{aligned} U &= \hat{U}_1\hat{r} + \ldots + \hat{U}_n\frac{\hat{r}^n}{n!} + \hat{P}_n(\varphi)\frac{\hat{r}^{n+1}}{(n+1)!}\log\hat{r} + O_T(\hat{r}^{n+1}) \\ &= \hat{U}_1 e^f r + \ldots + \hat{U}_n e^{nf}\frac{r^n}{n!} + \hat{P}_n(\varphi)e^{(n+1)f}\frac{r^{n+1}}{(n+1)!}\log r + O_T(r^{n+1}). \end{aligned}$$

By identification, this yields the result. □

We then introduce *CR-harmonic maps* as maps for which the obstruction vanishes:

Definition 3.1.6. If $P_n(\varphi) = 0$, φ is said to be *CR-harmonic*.

Example 3.1.7. Let us assume that (M, H, J, θ) is Einstein with $\mathrm{Ric}_W = 2\lambda(n+1)\gamma$. We know from Example 2.3.2 that

$$g = \frac{dr^2}{r^2} + \frac{(1-\lambda^2r^2)^2}{r^2}\theta^2 + \frac{(1-\lambda r)^2}{r}\gamma$$

satisfies $\mathrm{Ric}(g) = -\frac{n+2}{2}g$. In this case, we can explicitly compute the divergence $\delta^{g,h}\omega$, for $\omega \in \Omega^1(\overline{X})\otimes\tilde{\varphi}^*TN$.

Indeed, an orthonormal basis of $T^{1,0}\overline{X}$ with respect to g induced from $e^{(0)}$ is given by

$$(e_0, e_\alpha) \;:=\; \left(\frac{1}{\sqrt{2}}\left(r\partial_r - i\frac{r}{1-\lambda^2 r^2}R\right), \frac{r^{\frac{1}{2}}}{1-\lambda r}T_\alpha\right),$$

hence, since $\tau = 0$ by Definition 1.2.6,

$$\begin{aligned}
[e_0, e_{\overline{0}}] &= \frac{1}{\sqrt{2}}\frac{1+\lambda^2 r^2}{1-\lambda^2 r^2}\left(e_{\overline{0}} - e_0\right), \\
[e_0, e_A] &= \frac{1}{2\sqrt{2}}\frac{1+\lambda r}{1-\lambda r}e_A, \\
[e_A, e_B] &= 0.
\end{aligned}$$

Then

$$\nabla^g_{e_I} e_{\overline{I}} = \left(n\frac{1+\lambda^2 r^2}{1-\lambda^2 r^2} + \frac{1+\lambda r}{1-\lambda r}\right) r\partial_r,$$

and also,

$$\begin{aligned}
\nabla^{\tilde{\varphi}^*h}_{e_0}\omega(e_{\overline{0}}) + \nabla^{\tilde{\varphi}^*h}_{e_{\overline{0}}}\omega(e_0) &= r\omega(\partial_r) + r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\omega(\partial_r) + \frac{r^2}{(1-\lambda^2 r^2)^2}\nabla^{\tilde{\varphi}^*h}_{R}\omega(R), \\
\nabla^{\tilde{\varphi}^*h}_{e_\alpha}\omega(e_{\overline{\alpha}}) &= \frac{r}{(1-\lambda r)^2}\nabla^{\tilde{\varphi}^*h}_{T_\alpha}\omega(T_{\overline{\alpha}}).
\end{aligned}$$

The divergence is hence given by

$$\begin{aligned}
\delta^{g,h}\omega &= \left(n\frac{1+\lambda^2 r^2}{1-\lambda^2 r^2} + \frac{1+\lambda r}{1-\lambda r} - 1\right) r\omega(\partial_r) - r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\omega(\partial_r) \\
&\quad - \frac{r^2}{(1-\lambda^2 r^2)^2}\nabla^{\tilde{\varphi}^*h}_{R}\omega(R) + \frac{r}{(1-\lambda r)^2}\tilde{\delta}^{\theta,h}_b\omega.
\end{aligned}$$

From Example 3.1.7 we get the following results:

Corollary 3.1.8. *If (M, H, J, θ) is Einstein, then subharmonic maps which verify $\nabla^{\varphi^*h}_R R\varphi = 0$ are CR-harmonic.*

Proof. Indeed, let φ be subharmonic, *i.e.* $\delta^{\theta,h}_b T\varphi = 0$, and such that $\nabla^{\varphi^*h}_R R\varphi = 0$. Let $\tilde{\varphi}$ be the extension of φ given by Theorem 3.1.3. We thus have $\varphi_1 = 0$. Moreover, by Example 3.1.7, we have

$$\left(\delta^{g,h}T\tilde{\varphi}\right)^{(1)} = \alpha(r)\partial_r\tilde{\varphi} + \beta(r)\nabla^{\tilde{\varphi}^*h}_R R\tilde{\varphi} + \gamma(r)\tilde{\delta}^{\theta,h}_b T\tilde{\varphi},$$

where $\alpha(r) = O(r^2)$, $\beta(r) = O(r^4)$, and $\gamma(r) = O(r^2)$. Since $\varphi_1 = \nabla^{\varphi^*h}_R R\varphi = 0$, we get that

$$(n-1)\varphi_2 = -\left.\nabla^{\tilde{\varphi}^*h}_{\partial_r}\tilde{\delta}^{\theta,h}_b T\tilde{\varphi}\right|_{r=0} - \nabla^{\varphi^*h}_R R\varphi - \frac{1}{2}\left.\left(\nabla^{\tilde{\varphi}^*h}_{\partial_r}\right)^2\left(\delta^{g,h}T\tilde{\varphi}\right)^{(1)}\right|_{r=0} = 0.$$

By induction, we similarly have $\forall k \leq n,\ \varphi_k = 0$, which implies that $P_n(\varphi) = 0$. $\square$

Corollary 3.1.9. *If (M,H,J,θ) is Einstein and (N,h) is a Kähler manifold, then CR-holomorphic maps which verify $J_N\left(R\varphi\right)=0$ are CR-harmonic.*

Proof. Indeed, assuming that $T\varphi\circ J=J_N\circ T\varphi$, and extending J by taking $J(R)=0$, we have

$$\begin{aligned}\nabla^{\varphi^*h}_{T_\alpha}T_{\overline{\alpha}}\varphi &= \nabla^{\varphi^*h}_{JT_\alpha}JT_{\overline{\alpha}}\varphi\\ &= J_N\nabla^{\varphi^*h}_{JT_\alpha}T_{\overline{\alpha}}\varphi\\ &= J_N\nabla^{\varphi^*h}_{T_{\overline{\alpha}}}JT_\alpha\varphi+J([JT_\alpha,T_{\overline{\alpha}}])\varphi\\ &= -\nabla^{\varphi^*h}_{T_{\overline{\alpha}}}T_\alpha\varphi+iJ([T_\alpha,T_{\overline{\alpha}}])\varphi\\ &= -\nabla^{\varphi^*h}_{T_{\overline{\alpha}}}T_\alpha\varphi-nJ(R)\varphi,\end{aligned}$$

hence

$$\delta_b^{\theta,h}T\varphi=nJ_N\left(R\varphi\right).$$

Consequently, φ is CR-harmonic by Corollary 3.1.8. □

Example 3.1.10. Let (M,H,J) be a circle bundle over a Riemann surface Σ admitting an Einstein contact form. Then the projection $\pi:M\to\Sigma$ is CR-harmonic.

3.1.4 Explicit obstruction in dimension 3

When $n=1$, *i.e.* $\dim(M)=3$, the asymptotic development of g is given at order $\frac{3}{2}$ in $e^{(0)}$ by Theorem 2.3.8. Hence, by Proposition 3.1.4, we can explicitly compute the obstruction.

Theorem 3.1.11. *Still denoting $v\varphi:=T\varphi(v)$ for $v\in TM$, and also $(\nabla^{\varphi^*h}v)\varphi:=\nabla^{\varphi^*h}(v\varphi)$, we have*

$$P_1(\varphi)=-\delta_b^{\theta,h}\nabla^{\varphi^*h}\delta_b^{\theta,h}T\varphi-\nabla^{\varphi^*h}_R R\varphi+4\mathrm{Im}\left(\nabla^{\varphi^*h}_{T_{\overline{1}}}\left(\tau_1^{\overline{1}}T_{\overline{1}}\right)\right)\varphi-S_b\left(\delta_b^{\theta,h}T\varphi\right),$$

where

$$S_b(X):=\mathscr{R}^h_{X,T_1\varphi}T_{\overline{1}}\varphi+\mathscr{R}^h_{X,T_{\overline{1}}\varphi}T_1\varphi.$$

Proof. An orthonormal basis of $T^{1,0}X$ with respect to g is given by

$$(e_0,e_1):=\left(e_0^{(0)}-r^{\frac{3}{2}}\Psi_{0\overline{1}}e_1^{(0)},\left(1-r\Phi_{1\overline{1}}\right)e_1^{(0)}-r\Phi_{11}e_{\overline{1}}^{(0)}\right)+O_e(r^2).$$

We have

$$\begin{aligned}
[e_0, e_{\bar{0}}] &= \frac{1}{\sqrt{2}}\left(e_{\bar{0}} - e_0 - r^{\frac{3}{2}}\Psi_{\bar{0}1}e_{\bar{1}} + r^{\frac{3}{2}}\Psi_{0\bar{1}}e_1\right) + O_e(r^2),\\
[e_0, e_1] &= \frac{1}{\sqrt{2}}\left(\left(\frac{1}{2} - r\Phi_{1\bar{1}}\right)e_1 - r\Phi_{11}e_{\bar{1}} - i\left(\tilde{\nabla}^{\theta}_{e_0}e_1 - \tau(e_1)\right)\right) + O_e(r^2),\\
[e_0, e_{\bar{1}}] &= \frac{1}{\sqrt{2}}\left(\left(\frac{1}{2} - r\Phi_{1\bar{1}}\right)e_{\bar{1}} - r\Phi_{\bar{1}\bar{1}}e_1 - i\left(\tilde{\nabla}^{\theta}_{e_0}e_{\bar{1}} - \tau(e_{\bar{1}})\right)\right) + O_e(r^2),\\
[e_1, e_{\bar{1}}] &= r^{\frac{3}{2}}\left(\Phi_{1\bar{1},\bar{1}} - \Phi_{\bar{1}\bar{1},1}\right)e_1 - r^{\frac{3}{2}}\left(\Phi_{1\bar{1},1} - \Phi_{11,\bar{1}}\right)e_{\bar{1}} + O_e(r^2).
\end{aligned}$$

Hence,

$$\begin{aligned}
\nabla^g_{e_I}e_{\bar{I}} &= 2r\left(1 - r\Phi_{1\bar{1}}\right)\partial_r\\
&\quad -r^2\left(\sqrt{2}\Psi_{0\bar{1}} + \Phi_{1\bar{1},\bar{1}} - \Phi_{\bar{1}\bar{1},1}\right)T_1\\
&\quad -r^2\left(\sqrt{2}\Psi_{\bar{0}1} + \Phi_{1\bar{1},1} - \Phi_{11,\bar{1}}\right)T_{\bar{1}} + O_T(r^{\frac{5}{2}}).
\end{aligned}$$

We also have, for $\omega \in \Omega^1(X) \otimes \tilde{\varphi}^*TN$,

$$\begin{aligned}
\nabla^{\tilde{\varphi}^*h}_{e_0}\omega(e_{\bar{0}}) + \nabla^{\tilde{\varphi}^*h}_{e_{\bar{0}}}\omega(e_0) &= r\omega(\partial_r) + r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\omega(\partial_r) + r^2\nabla^{\tilde{\varphi}^*h}_{R}\omega(R)\\
&\quad -\sqrt{2}r^2\Psi_{0\bar{1}}\omega(T_1) - \sqrt{2}r^2\Psi_{\bar{0}1}\omega(T_{\bar{1}}) + O_T(r^{\frac{5}{2}}),\\
\nabla^{\tilde{\varphi}^*h}_{e_1}\omega(e_{\bar{1}}) &= r\nabla^{\tilde{\varphi}^*h}_{T_1}\omega(T_{\bar{1}}) - r^2\nabla^{\tilde{\varphi}^*h}_{T_1}\left(\Phi_{1\bar{1}}\omega(T_{\bar{1}})\right) - r^2\nabla^{\tilde{\varphi}^*h}_{T_1}\left(\Phi_{\bar{1}\bar{1}}\omega(T_1)\right)\\
&\quad -r^2\Phi_{1\bar{1}}\nabla^{\tilde{\varphi}^*h}_{T_1}\omega(T_{\bar{1}}) - r^2\Phi_{11}\nabla^{\tilde{\varphi}^*h}_{T_{\bar{1}}}\omega(T_{\bar{1}}) + O_T(r^3).
\end{aligned}$$

The divergence is hence given by

$$\begin{aligned}
\delta^{g,h}\omega &= r(1 - 2r\Phi_{1\bar{1}})\left(\omega(\partial_r) - \nabla^{\tilde{\varphi}^*h}_{T_1}\omega(T_{\bar{1}}) - \nabla^{\tilde{\varphi}^*h}_{T_{\bar{1}}}\omega(T_1)\right) - r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\omega(\partial_r) - r^2\nabla^{\tilde{\varphi}^*h}_{R}\omega(R)\\
&\quad +4r^2\mathrm{Im}\left(\nabla^{\tilde{\varphi}^*h}_{T_{\bar{1}}}\left(\tau_1^{\bar{1}}\omega(T_{\bar{1}})\right)\right) + O_T(r^{\frac{5}{2}})\\
&= \delta^{g_0,h}\omega - 2r^2\Phi_{1\bar{1}}\nabla^{\tilde{\varphi}^*h}_{\partial_r}\delta^{g,h}\omega + 4r^2\mathrm{Im}\left(\nabla^{\tilde{\varphi}^*h}_{T_{\bar{1}}}\left(\tau_1^{\bar{1}}\omega(T_{\bar{1}})\right)\right) + O_T(r^{\frac{5}{2}}).
\end{aligned}$$

Then, by Theorem 3.1.3, we have

$$\begin{aligned}
P_1(\varphi) &= \nabla^{\tilde{\varphi}^*h}_{\partial_r}\tilde{\delta}^{\theta,h}_b T\tilde{\varphi}|_{r=0} - \nabla^{\varphi^*h}_{R}R\varphi + 4\mathrm{Im}\left(\nabla^{\varphi^*h}_{T_{\bar{1}}}\left(\tau_1^{\bar{1}}T_{\bar{1}}\right)\right)\varphi\\
&= \delta^{\theta,h}_b\nabla^{\varphi^*h}\varphi_1 + \mathscr{R}^h_{\varphi_1,T_1\varphi}T_{\bar{1}}\varphi + \mathscr{R}^h_{\varphi_1,T_{\bar{1}}\varphi}T_1\varphi - \nabla^{\varphi^*h}_{R}R\varphi + 4\mathrm{Im}\left(\nabla^{\varphi^*h}_{T_{\bar{1}}}\left(\tau_1^{\bar{1}}T_{\bar{1}}\right)\right)\varphi,
\end{aligned}$$

hence, since $\varphi_1 = -\delta^{\theta,h}_b T\varphi$, the announced obstruction. □

Note that on functions, meaning that $N = \mathbb{R}$, P_1 reduces to a multiple of the CR Paneitz operator. Since the construction follows the ideas of Graham *et al.*, this was expected. A similar phenomenon appears in the real case [Bér13].

Example 3.1.12. Let us consider $\mathrm{id} : (M, H, J, \theta) \to (M, g := g_{J,\theta})$.
Since $\nabla^g_R R = 0$ by Proposition 1.2.7, we have, using the Koszul formula:

$$\begin{aligned}
\delta_b^{\theta,g} T\mathrm{id} &= -\nabla^g_{T_1} T_{\bar{1}} - \nabla^g_{T_{\bar{1}}} T_1 \\
&= -g\left([T_1, T_{\bar{1}}], T_1\right) T_{\bar{1}} - g\left([T_{\bar{1}}, T_1], T_{\bar{1}}\right) T_1 \\
&\quad -g\left([R, T_{\bar{1}}], T_1\right) R - g\left([R, T_1], T_{\bar{1}}\right) R \\
&\quad -g\left([T_1, R], R\right) T_{\bar{1}} - g\left([T_{\bar{1}}, R], R\right) T_1 \\
&= 0,
\end{aligned}$$

hence

$$P_1(\mathrm{id}) = 4\mathrm{Im}\nabla^g_{T_{\bar{1}}}\left(\tau_1^{\bar{1}} T_{\bar{1}}\right).$$

Consequently, the identity is CR-harmonic if and only if $\mathrm{Im}\nabla^g_{T_{\bar{1}}}\left(\tau_1^{\bar{1}} T_{\bar{1}}\right) = 0$. This is in particular verified when θ is normal.

3.2 Renormalized energy

3.2.1 Definition

Let $\varphi \in C^\infty(M, N)$ and $\tilde{\varphi}$ be the extension of φ constructed in Theorem 3.1.3. For ρ in $(0, \varepsilon)$, let

$$E(\tilde{\varphi}, \rho) = \frac{1}{2}\int_{(\rho,\varepsilon)\times M} \|T\tilde{\varphi}\|^2_{g,h} d\mathrm{vol}_g$$

be the energy of $\tilde{\varphi}$ in $(\rho, \varepsilon) \times M$. We have

$$\|T\tilde{\varphi}\|^2_{g,h} = f_0 r + f_1 r^2 + \ldots + f_n r^{n+1} + O(r^{n+2}\log r),$$

where $\forall k \leq n$, f_k depends only on U_j for $j \leq k$ and on g at order k in $e^{(0)}$, and

$$d\mathrm{vol}_g = r^{-n-2}\sqrt{\det g}\, dr \wedge \theta \wedge d\theta^n.$$

Consequently,

$$\|T\tilde{\varphi}\|^2_{g,h} d\mathrm{vol}_g = \left(a_0 r^{-n-1} + a_1 r^{-n} + \ldots + a_n r^{-1} + O(\log r)\right) dr \wedge \theta \wedge d\theta^n,$$

where $\forall k \leq n$, a_k depends only on U_j for $j \leq k$ and on g at order k. Hence E admits the development, when $\rho \to 0$,

$$E(\tilde{\varphi}, \rho) = E_0(\varphi)\rho^{-n} + E_1(\varphi)\rho^{1-n} + \ldots + E_{n-1}(\varphi)\rho^{-1} + F_n(\varphi)\log\rho + E_n(\varphi) + o(1),$$

where $\forall k \leq n-1$, E_k depends only on U_j for $j \leq k$ and on g at order k, and F_n depends only on U_j for $j \leq n$ and on g at order n. The coefficient $F_n(\varphi)$ can be written as

$$F_n(\varphi) = -\frac{1}{2}\int_M a_n \theta \wedge d\theta^n = -\frac{1}{2n!}\int_M \partial_r^n \left(r^{n+1}\|T\tilde{\varphi}\|^2_{g,h} d\mathrm{vol}_g\right)\Big|_{r=0}.$$

By construction, F_n is formally determined by φ, g and h. Moreover, we have:

Proposition 3.2.1. *$F_n(\varphi)$ is a CR invariant:*

$$\hat{F}_n(\varphi) = F_n(\varphi).$$

Proof. The proof is similar to the proof of Proposition 3.1.5. Indeed, if $\hat{r} = e^f r$, then

$$\begin{aligned}
\|T\tilde{\varphi}\|^2_{g,h}\mathrm{dvol}_g &= \Big(a_0 r^{-n-1} + a_1 r^{-n} + \ldots + a_n r^{-1} + a_{n+1} + O(r)\Big)\, dr \wedge \theta \wedge d\theta^n \\
&= \Big(\hat{a}_0 \hat{r}^{-n-1} + \hat{a}_1 \hat{r}^{-n} + \ldots + \hat{a}_n \hat{r}^{-1} + \hat{a}_{n+1} + O(\hat{r})\Big)\, d\hat{r} \wedge \theta \wedge d\theta^n \\
&= \Big(\hat{a}_0 e^{-(n+1)f} r^{-n-1} + \hat{a}_1 e^{-nf} r^{-n} + \ldots + \hat{a}_n e^{-f} r^{-1} + \hat{a}_{n+1} + O(r)\Big)\, e^f dr \wedge \theta \wedge d\theta^n,
\end{aligned}$$

hence $\hat{a}_n = a_n$. □

The principal term of $F_n(\varphi)$ is the following: since

$$r^{n+1}\|T\tilde{\varphi}\|^2_{g,h}\mathrm{dvol}_g = \Big(\langle T_A\tilde{\varphi}, T_{\overline{A}}\tilde{\varphi}\rangle_h + r\|\partial_r\tilde{\varphi}\|^2_h\Big)\, dr\wedge\theta\wedge d\theta^n + \text{lower order (in derivations of } \varphi) \text{ terms},$$

we have

$$\begin{aligned}
F_n(\varphi) &= -\frac{1}{2n!}\int_M \left(\sum_{k=0}^{n}\binom{n}{k}\left\langle \delta_b^{\theta,h}\nabla^{\varphi^* h}\varphi_k, \varphi_{n-k}\right\rangle_h + n\sum_{k=0}^{n-1}\binom{n-1}{k}\langle\varphi_{k+1},\varphi_{n-k}\rangle_h\right)\theta\wedge d\theta^n + \text{l.o.t.} \\
&= \frac{(-1)^{n+1}}{2n!^2}\int_M \left\langle (\delta_b^{\theta,h}\nabla^{\varphi^* h})^{n-1}\delta_b^{\theta,h}T\varphi, \delta_b^{\theta,h}T\varphi\right\rangle_h \theta\wedge d\theta^n + \text{lower order terms.}
\end{aligned}$$

Definition 3.2.2. $F_n(\varphi)$ is called the *renormalized energy* of φ.

Proposition 3.2.3. *The gradient of $F_n(\varphi)$ is $\frac{1}{2n!}P_n(\varphi)$, that is to say, for all $\dot{\varphi} \in \Gamma(\varphi^*TN)$,*

$$d_\varphi F_n(\dot{\varphi}) = \frac{1}{2n!}\int_M \langle\dot{\varphi}, P_n(\varphi)\rangle_h\, \theta\wedge d\theta^n.$$

Proof. Let $\dot{\varphi} \in \Gamma(\varphi^*TN)$. Let $(\varphi_t)_{t\in[-1,1]}$ be a one-parameter family in $C^\infty(M,N)$ such that

$$\begin{cases} \varphi_0 = \varphi, \\ \partial_t\varphi_t|_{t=0} = \dot{\varphi}. \end{cases}$$

Let us equip $X \times [-1,1]$ with the metric $\overline{g} = g + dt^2$ and let $\xi \in C^\infty(X\times[-1,1], N)$ be the map

$$\forall p \in X,\ \forall t \in [-1,1], \quad \xi(p,t) = \tilde{\varphi}_t(p).$$

We then have

$$\begin{aligned}
\partial_t \|T\tilde{\varphi}_t\|^2_{g,h} &= \partial_t \left(\|T\xi\|^2_{\overline{g},h} - \|\partial_t \xi\|^2_h \right) \\
&= \left\langle \nabla^{\overline{g},h}_{\partial_t} T\xi, T\xi \right\rangle_{\overline{g},h} - \left\langle \nabla^{\xi^* h}_{\partial_t} \partial_t \xi, \partial_t \xi \right\rangle_h \\
&= \left\langle \nabla^{\xi^* h}_{\partial_t} e_I \xi, e_{\overline{I}} \xi \right\rangle_h \\
&= \left\langle \nabla^{\xi^* h}_{e_I} \partial_t \xi, e_{\overline{I}} \xi \right\rangle_h \\
&= \left\langle \nabla^{\tilde{\varphi}_t^* h}_{e_I} \partial_t \tilde{\varphi}_t, e_{\overline{I}} \tilde{\varphi}_t \right\rangle_h \\
&= e_I \left\langle \partial_t \tilde{\varphi}_t, e_{\overline{I}} \tilde{\varphi}_t \right\rangle_h - \left\langle \partial_t \tilde{\varphi}_t, \nabla^{\tilde{\varphi}_t^* h}_{e_I} e_{\overline{I}} \tilde{\varphi}_t \right\rangle_h,
\end{aligned}$$

hence

$$\partial_t E(\tilde{\varphi}_t, \rho)|_{t=0} = \frac{1}{2} \int_{(\rho,\varepsilon)\times M} \left(e_I \left\langle \partial_t \tilde{\varphi}_t|_{t=0}, e_{\overline{I}} \tilde{\varphi} \right\rangle_h - \left\langle \partial_t \tilde{\varphi}_t|_{t=0}, \nabla^{\tilde{\varphi}^* h}_{e_I} e_{\overline{I}} \tilde{\varphi} \right\rangle_h \right) d\mathrm{vol}_g.$$

There is no $\log \rho$ term in the second part, and

$$\frac{1}{2} \int_{(\rho,\varepsilon)\times M} e_I \left\langle \partial_t \tilde{\varphi}_t|_{t=0}, e_{\overline{I}} \tilde{\varphi} \right\rangle_h d\mathrm{vol}_g = \frac{1}{2} \int_M \rho^{-n} \left\langle \partial_t \tilde{\varphi}_t|_{t=0}, \partial_\rho \tilde{\varphi} \right\rangle_h \theta \wedge d\theta^n,$$

whose $\log \rho$ term is

$$\frac{1}{2n!} \int_M \langle \dot{\varphi}, P_n(\varphi) \rangle_h \theta \wedge d\theta^n,$$

hence the result. □

3.2.2 Explicit energy in dimension 3

Here again, when $n = 1$, *i.e.* $\dim(M) = 3$, knowing the asymptotic development of g at order $\dfrac{3}{2}$ in $e^{(0)}$ allows for an explicit computation of the renormalized energy.

Theorem 3.2.4. *We have*

$$F_1(\varphi) = -\frac{1}{2} \int_M \left(\|\delta_b^{\theta,h} T\varphi\|^2_h + \|R\varphi\|^2_h - 4\mathrm{Im}\left(\tau_1^{\overline{1}} \|T_{\overline{1}}\varphi\|^2_h \right) \right) \theta \wedge d\theta.$$

Proof. We have

$$\begin{aligned}
\|T\tilde{\varphi}\|^2_{g,h} &= 2 \left\langle e_0 \tilde{\varphi}, e_{\overline{0}} \tilde{\varphi} \right\rangle_h + 2 \left\langle e_1 \tilde{\varphi}, e_{\overline{1}} \tilde{\varphi} \right\rangle_h \\
&= 2r \left\langle T_1 \varphi, T_{\overline{1}} \varphi \right\rangle_h \\
&\quad + r^2 \left(\|\varphi_1\|^2_h + \|R\varphi\|^2_h - 4\Phi_{1\overline{1}} \left\langle T_1\varphi, T_{\overline{1}}\varphi \right\rangle_h - 2\Phi_{11} \|T_{\overline{1}}\varphi\|^2_h - 2\Phi_{\overline{1}\overline{1}} \|T_1\varphi\|^2_h \right) \\
&\quad + O(r^{\frac{5}{2}}),
\end{aligned}$$

and

$$\mathrm{dvol}_g = \left(1 + 2r\Phi_{1\overline{1}} + O(r^2)\right) r^{-3} dr \wedge \theta \wedge d\theta.$$

Consequently,

$$\begin{aligned} r^2 \|T\tilde{\varphi}\|_g^2 \mathrm{dvol}_g &= (2\left\langle T_1\varphi, T_{\overline{1}}\varphi\right\rangle_h \\ &\quad + r\left(\|\varphi_1\|_h^2 + \|R\varphi\|_h^2 - 2\Phi_{11}\|T_{\overline{1}}\varphi\|_h^2 - 2\Phi_{\overline{1}\overline{1}}\|T_1\varphi\|_h^2\right) \\ &\quad + O(r^{\frac{3}{2}})\Big) dr \wedge \theta \wedge d\theta, \end{aligned}$$

and finally

$$F_1(\varphi) = -\frac{1}{2}\int_M \left(\|\varphi_1\|_h^2 + \|R\varphi\|_h^2 - 4\mathrm{Im}\left(\tau_1^{\overline{1}}\|T_{\overline{1}}\varphi\|_h^2\right)\right)\theta \wedge d\theta.$$

□

As an example, for $\mathrm{id} : (M, H, J, \theta) \to (M, g_{J,\theta})$, we have

$$F_1(\mathrm{id}) = -\frac{1}{2}\mathrm{Vol}(M, \theta).$$

3.3 Further computations in the general case

We give here a more precise computation for $\delta^{g,h}\omega$ and $r^{n+1}\|T\tilde{\varphi}\|_g^2\mathrm{dvol}_g$ in the general case, using Theorem 2.3.5. We show that this computation does not allow for an explicit expression of the obstruction and of the renormalized energy respectively.

3.3.1 Computation of the divergence

By Theorem 2.3.5, we have

$$g = g_0 + \Phi_{AB}\theta^A \circ \theta^B + O_e(r^{\frac{3}{2}}),$$

where, denoting by $R_{\alpha\overline{\beta}}$ the components of the Webster Ricci tensor,

$$\Phi_{\alpha\overline{\beta}} = -\frac{1}{n+2}\left(R_{\alpha\overline{\beta}} - \frac{\mathrm{Scal}_W(J,\theta)}{2(n+1)}\delta_{\alpha\overline{\beta}}\right), \quad \text{and} \quad \Phi_{\alpha\beta} = -i\tau_\alpha^{\overline{\beta}}.$$

By Proposition 2.3.6, we can equip $\{r\} \times H$ with a complex structure $J_r = J_0 + rJ_1 + O_T(r^2)$, with

$$J_1 T_\alpha = -2\Phi_{\alpha\beta}T_{\overline{\beta}}.$$

An orthonormal basis of $T^{1,0}X$ with respect to g is given by

$$(e_0, e_\alpha) := \left(r\partial_r - irR, \left(\delta_{\alpha\overline{\beta}} - r\Phi_{\alpha\overline{\beta}}\right) r^{\frac{1}{2}} T_\beta - r\Phi_{\alpha\beta} r^{\frac{1}{2}} T_{\overline{\beta}}\right) + O_e(r^{\frac{3}{2}}).$$

Now, g can be rewritten as

$$g = (r^{-1}\theta^0) \circ (r^{-1}\theta^{\overline{0}}) + (r^{-\frac{1}{2}}\theta^\alpha) \circ (r^{-\frac{1}{2}}\theta^{\overline{\alpha}}) + r\Phi_{AB}(r^{-\frac{1}{2}}\theta^A) \circ (r^{-\frac{1}{2}}\theta^B) + O_e(r^{\frac{3}{2}}).$$

We have, modulo $O_e(r^{\frac{3}{2}})$,

$$\begin{aligned}
[e_0, e_{\overline{0}}] &= \frac{1}{\sqrt{2}}\left(e_{\overline{0}} - e_0\right), \\
[e_0, e_\alpha] &= \frac{1}{\sqrt{2}}\left(\frac{1}{2}e_\alpha - r\Phi_{\alpha\overline{\beta}}e_\beta - r\Phi_{\alpha\beta}e_{\overline{\beta}} - i\left(\tilde{\nabla}^\theta_{e_0}e_\alpha - \tau(e_\alpha)\right)\right), \\
[e_0, e_{\overline{\alpha}}] &= \frac{1}{\sqrt{2}}\left(\frac{1}{2}e_{\overline{\alpha}} - r\Phi_{\overline{\alpha}\beta}e_{\overline{\beta}} - r\Phi_{\overline{\alpha}\overline{\beta}}e_\beta - i\left(\tilde{\nabla}^\theta_{e_0}e_{\overline{\alpha}} - \tau(e_{\overline{\alpha}})\right)\right), \\
[e_\alpha, e_\beta] &= r^{\frac{3}{2}}\left(\Phi_{\alpha\bar{\delta},\beta} - \Phi_{\beta\bar{\delta},\alpha}\right)e_\delta + r^{\frac{3}{2}}\left(\Phi_{\alpha\delta,\beta} - \Phi_{\beta\delta,\alpha}\right)e_{\overline{\delta}}, \\
[e_\alpha, e_{\overline{\beta}}] &= r^{\frac{3}{2}}\left(\Phi_{\alpha\bar{\delta},\overline{\beta}} - \Phi_{\overline{\beta}\bar{\delta},\alpha}\right)e_\delta - r^{\frac{3}{2}}\left(\Phi_{\delta\overline{\beta},\alpha} - \Phi_{\alpha\delta,\overline{\beta}}\right)e_{\overline{\delta}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\nabla^g_{e_i}e_{\bar{i}} + \nabla^g_{e_{\bar{i}}}e_i = {} & r\left(n+1-2r\Phi_{\alpha\overline{\alpha}}\right)\partial_r \\
& -r^2\left(2\Phi_{\beta\overline{\beta},\overline{\alpha}} - \Phi_{\overline{\alpha}\overline{\beta},\beta} - \Phi_{\overline{\alpha}\beta,\overline{\beta}}\right)T_\alpha \\
& -r^2\left(2\Phi_{\beta\overline{\beta},\alpha} - \Phi_{\alpha\beta,\overline{\beta}} - \Phi_{\alpha\overline{\beta},\beta}\right)T_{\overline{\alpha}} + O_T(r^{\frac{5}{2}}).
\end{aligned}$$

with $\Phi_{\alpha\overline{\alpha}} = -\mathrm{tr}(\mathrm{S}_\theta) = -\dfrac{\mathrm{Scal}_W}{2(n+1)}$.

Also,

$$\begin{aligned}
\nabla^{\tilde{\varphi}^*h}_{e_0}\omega(e_{\overline{0}}) + \nabla^{\tilde{\varphi}^*h}_{e_{\overline{0}}}\omega(e_0) &= r\omega(\partial_r) + r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\omega(\partial_r) + r^2\nabla^{\tilde{\varphi}^*h}_R\omega(R) + O_T(r^2), \\
\nabla^{\tilde{\varphi}^*h}_{e_\alpha}\omega(e_{\overline{\alpha}}) &= r\nabla^{\tilde{\varphi}^*h}_{T_\alpha}\omega(T_{\overline{\alpha}}) - r^2\nabla^{\tilde{\varphi}^*h}_{T_\alpha}\left(\Phi_{\overline{\alpha}\beta}\omega(T_{\overline{\beta}})\right) - r^2\nabla^{\tilde{\varphi}^*h}_{T_\alpha}\left(\Phi_{\overline{\alpha}\overline{\beta}}\omega(T_\beta)\right) \\
&\quad -r^2\Phi_{\alpha\overline{\beta}}\nabla^{\tilde{\varphi}^*h}_{T_\beta}\omega(T_{\overline{\alpha}}) - r^2\Phi_{\alpha\beta}\nabla^{\tilde{\varphi}^*h}_{T_{\overline{\beta}}}\omega(T_{\overline{\alpha}}) + O_T(r^{\frac{5}{2}}).
\end{aligned}$$

Coming back to the divergence, we have

$$\begin{aligned}
\delta^{g,h}\omega = {} & r(n-2r\Phi_{\alpha\overline{\alpha}})\omega(\partial_r) - r^2\nabla^{\tilde{\varphi}^*h}_{\partial_r}\omega(\partial_r) - r^2\nabla^{\tilde{\varphi}^*h}_R(\omega(R)) \\
& -r(1-2r\Phi_{\alpha\overline{\alpha}})\left(\nabla^{\tilde{\varphi}^*h}_{T_\alpha}\omega(T_{\overline{\alpha}}) + \nabla^{\tilde{\varphi}^*h}_{T_{\overline{\alpha}}}\omega(T_\alpha)\right) \\
& +2r^2\left(\nabla^{\tilde{\varphi}^*h}_{T_{\overline{\beta}}}\left(\Phi_{\alpha\beta}\omega(T_{\overline{\alpha}})\right) + \nabla^{\tilde{\varphi}^*h}_{T_\beta}\left(\Phi_{\overline{\alpha}\overline{\beta}}\omega(T_\alpha)\right)\right) \\
& +2r^2\left(\nabla^{\tilde{\varphi}^*h}_{T_\alpha}\Phi_{\overline{\alpha}\beta}\omega(T_{\overline{\beta}}) + \nabla^{\tilde{\varphi}^*h}_{T_{\overline{\alpha}}}\Phi_{\alpha\overline{\beta}}\omega(T_\beta) - \nabla^{\tilde{\varphi}^*h}_{T_\alpha}\Phi_{\beta\overline{\beta}}\omega(T_{\overline{\alpha}}) - \nabla^{\tilde{\varphi}^*h}_{T_{\overline{\alpha}}}\Phi_{\beta\overline{\beta}}\omega(T_\alpha)\right) + O_T(r^2).
\end{aligned}$$

The term of order 2 is consequently not known, which does not allow for an explicit computation of P_n. Note that

$$\nabla^{\tilde{\varphi}^*h}_{T_{\overline{\beta}}}\left(\Phi_{\alpha\beta}\omega(T_{\overline{\alpha}})\right) + \nabla^{\tilde{\varphi}^*h}_{T_\beta}\left(\Phi_{\overline{\alpha}\overline{\beta}}\omega(T_\alpha)\right) = 2\mathrm{Im}\left(\nabla^{\tilde{\varphi}^*h}_{T_{\overline{\beta}}}\left(\tau^{\overline{\beta}}_\alpha\omega(T_{\overline{\alpha}})\right)\right),$$

and that the potentially hidden r^2 terms are necessarily of the form $C^\alpha r^2\omega(T_\alpha) + D^{\overline{\alpha}}r^2\omega(T_{\overline{\alpha}})$.

3.3.2 Computation of the integrand of the energy

We have

$$\begin{aligned}
\|T\tilde{\varphi}\|^2_{g,h} &= 2\left\langle e_0\tilde{\varphi}, e_{\overline{0}}\tilde{\varphi}\right\rangle_h + 2\left\langle e_\alpha\tilde{\varphi}, e_{\overline{\alpha}}\tilde{\varphi}\right\rangle_h \\
&= 2r\left\langle T_\alpha\varphi, T_{\overline{\alpha}}\varphi\right\rangle_h \\
&\quad + r^2\left(\|\varphi_1\|^2_h + \|R\varphi\|^2_h - 2\Phi_{\alpha\beta}\left\langle T_{\overline{\alpha}}\varphi, T_{\overline{\beta}}\varphi\right\rangle_h - 2\Phi_{\overline{\alpha}\overline{\beta}}\left\langle T_\alpha\varphi, T_\beta\varphi\right\rangle_h\right. \\
&\quad \left. - 2\Phi_{\alpha\overline{\beta}}\left\langle T_{\overline{\alpha}}\varphi, T_\beta\varphi\right\rangle_h - 2\Phi_{\overline{\alpha}\beta}\left\langle T_\alpha\varphi, T_{\overline{\beta}}\varphi\right\rangle_h\right) \\
&\quad + O(r^2),
\end{aligned}$$

and

$$\mathrm{dvol}_g = \left(1 + 2r\Phi_{\alpha\overline{\alpha}} + O(r^{\frac{3}{2}})\right) r^{-n-2} dr \wedge \theta \wedge d\theta^n.$$

Consequently,

$$\begin{aligned}
r^{n+1}\|T\tilde{\varphi}\|^2_g \mathrm{dvol}_g &= \left(2\left\langle T_\alpha\varphi, T_{\overline{\alpha}}\varphi\right\rangle_h\right. \\
&\quad + r\left(\|\varphi_1\|^2_h + \|R\varphi\|^2_h - 2\Phi_{\alpha\beta}\left\langle T_{\overline{\alpha}}\varphi, T_{\overline{\beta}}\varphi\right\rangle_h - 2\Phi_{\overline{\alpha}\overline{\beta}}\left\langle T_\alpha\varphi, T_\beta\varphi\right\rangle_h\right. \\
&\quad \left. - 2\Phi_{\alpha\overline{\beta}}\left\langle T_{\overline{\alpha}}\varphi, T_\beta\varphi\right\rangle_h - 2\Phi_{\overline{\alpha}\beta}\left\langle T_\alpha\varphi, T_{\overline{\beta}}\varphi\right\rangle_h + 4\Phi_{\alpha\overline{\alpha}}\left\langle T_\alpha\varphi, T_{\overline{\alpha}}\varphi\right\rangle_h\right) \\
&\quad \left. + O(r)\right) dr \wedge \theta \wedge d\theta^n.
\end{aligned}$$

The term of order 1 is consequently not known, which does not allow for an explicit computation of F_n.

3.4 Towards existence of non-trivial CR-harmonic maps in dimension 3

We give in this section a detailed proof strategy of the existence of non-trivial CR-harmonic maps, *i.e.* CR-harmonic maps which are neither subharmonic maps, nor $\mathrm{id} : (M, H, J_0, \theta_0) \to (M, g_{J_0,\theta_0})$ when (M, H_0, J_0, θ_0) is a compact, strictly pseudoconvex pseudohermitian, normal 3-manifold. Such a map is obtained in Corollary 3.4.9, under Assumption 3.4.2 and Conjecture 3.4.7.

Let us consider a compact, strictly pseudoconvex pseudohermitian, normal 3-manifold a (M, H, J_0, θ_0). Let (J', θ') be a pseudohermitian structure on (M, H). For simplicity, we will denote $g_0 := g_{J_0,\theta_0}$ and $g' := g_{J',\theta'}$. Let us consider an isometric embedding $\iota : M \hookrightarrow \mathbb{R}^p$.

Let us consider, for $\varphi \in C^\infty(M, \mathbb{R}^p)$, the norm

$$\|\varphi\|_{4,2} = \sup_{|A|\leq 4} \|T_A\varphi\|_2, \quad \text{where } |A| = |(A_1, \ldots, A_k)| = k,$$

and then the Folland-Stein space $S^{4,2}(M, \mathbb{R}^p)$, which is the completion of $C^\infty(M, \mathbb{R}^p)$ by $\|\cdot\|_{4,2}$. Let us denote

$$\mathscr{S} = \left\{\varphi \in S^{4,2}(M, \mathbb{R}^p) \mid \mathrm{Im}(\varphi) \subset \iota(M) \text{ almost everywhere}\right\}.$$

Let us consider $\varphi \in \mathscr{S}$. Its obstruction to CR-harmonicity is given by

$$P_1(J_0, \theta_0, g', \varphi) = -\delta_b^{\theta_0, g'} \nabla^{\varphi^* g'} \delta_b^{\theta_0, g'} T\varphi - \nabla_R^{\varphi^* g'} R\varphi + S_b\left(\delta_b^{\theta_0, g'} T\varphi\right),$$

where

$$S_b(X) := \mathscr{R}^{g'}_{X, T_1\varphi} T_{\overline{1}}\varphi + \mathscr{R}^{g'}_{X, T_{\overline{1}}\varphi} T_1\varphi.$$

We thus have $P_1(J_0, \theta_0, g_0, \mathrm{id}) = 0$.

We have, for $\dot{\varphi} \in \Gamma\left(T_\varphi\mathscr{S}\right)$,

$$\partial_\varphi\left(\delta_b^{\theta_0, g'} T\varphi\right)(\dot{\varphi}) = \delta_b^{\theta_0, g'} d\dot{\varphi} - S_b(\dot{\varphi}),$$

and

$$\partial_\varphi\left(\nabla_R^{\varphi^* g'} R\varphi\right)(\dot{\varphi}) = \nabla_R^{\varphi^* g'} \nabla_R^{\varphi^* g'} \dot{\varphi} + \mathscr{R}^{g'}_{\dot{\varphi}, R\varphi} R\varphi,$$

hence, for $\dot{\varphi} \in \Gamma\left(T_{\mathrm{id}}\mathscr{S}\right)$,

$$\partial_\varphi P_{1(J_0, \theta_0, g_0, \mathrm{id})}(\dot{\varphi}) = -\left(\delta_b^{\theta_0, g_0} \nabla^{g_0} - 2S_b\right)\left(\delta_b^{\theta_0, g_0} \nabla^{g_0} \dot{\varphi} - S_b(\dot{\varphi})\right) - \nabla_R^{g_0} \nabla_R^{g_0} \dot{\varphi} - \mathscr{R}^{g_0}_{\dot{\varphi}, R} R.$$

Let $X \in \Gamma(TM)$. Let $X_b := X^1 T_1 + X^{\overline{1}} T_{\overline{1}}$ and $X_v := \theta_0(X) R$, we have by Theorem 1.2.8

$$\mathscr{R}^{g_0}_{X, T_1} T_{\overline{1}} = \mathscr{R}^{\theta_0}_{X, T_1} T_{\overline{1}} - 3X^{\overline{1}} T_{\overline{1}} + 2X_v,$$

which implies that

$$S_b(X) = \left(\mathrm{Scal}_W - 3\right) X_b + 4X_v,$$

and

$$\mathscr{R}^{g_0}_{X, R} R = 2X_b.$$

Moreover, we have the following estimate, which generalizes Lemma 2.1 in [CCY15]:

Proposition 3.4.1. *There exists $C > 0$ such that*

$$\forall \varphi \in \mathscr{S},\ C\|\varphi\|_{4,2} \leq \|P_1(\varphi)\|_2 + \|\varphi\|_2.$$

Proof. Indeed, we have

$$P_1(\varphi) = -\frac{1}{2}\left(\overline{\Box_b}\Box_b + \Box_b\overline{\Box_b}\right)\varphi + S_b\left(\delta_b^{\theta_0, g'} T\varphi\right),$$

where $\Box_b \varphi := \delta_b^{\theta_0, g_0} T\varphi - iR\varphi$. Now, there exists $C_1 > 0$ such that

$$C_1 \|\varphi\|_{4,2} \le \|\Box_b \varphi\|_{2,2} + \|\varphi\|_2,$$

and $C_2 > 0$ such that

$$C_2 \|\Box_b \varphi\|_{2,2} \le \| \left(\overline{\Box_b} \Box_b + \Box_b \overline{\Box_b} \right) \varphi \|_2 + \|\Box_b \varphi\|_2.$$

Consequently, there exists $C_3 > 0$ such that

$$C_3 \|\varphi\|_{4,2} \le \|P_1(\varphi)\|_2 + \|S_b \left(\delta_b^{\theta_0, g'} T\varphi \right) \|_2 + \|\Box_b \varphi\|_2 + \|\varphi\|_2,$$

where there exists $C_4 > 0$ such that

$$C_4 \left(\left\| S_b \left(\delta_b^{\theta_0, g'} T\varphi \right) \right\|_2 + \|\Box_b \varphi\|_2 \right) \le \|\varphi\|_{2,2} + \|\varphi\|_2.$$

Using the following interpolation inequality:

$$\forall \varepsilon > 0, \ \exists C_5 > 0, \ C_5 \|\varphi\|_{2,2} \le \varepsilon \|\varphi\|_{4,2} + \|\varphi\|_2,$$

with ε sufficiently small, we get our estimate. □

Assumption 3.4.2. Let S denote the spectrum of $\partial_\varphi P_{1(J_0, \theta_0, g_0, \mathrm{id})}$. There is $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \cap S = \emptyset$.

Corollary 3.4.3. *Under Assumption 3.4.2, for a given (J, g') close to (J_0, g_0), there exists a unique CR-harmonic map $\varphi_J : (M, H, J) \to (M, g')$ in $\mathscr{S}$ close to* id.

Proof. By Proposition 3.4.1, the operator $\partial_\varphi P_{1(J_0, \theta_0, g_0, \mathrm{id})}$ is Fredholm in $\mathscr{S}$. Under Assumption 3.4.2, it is therefore invertible. The corollary is then obtained by the implicit map theorem. □

We have of course $\varphi_{J_0} = \mathrm{id}$. Now, let us consider, for (J, θ, g') close to (J_0, θ_0, g_0) and $\hat{\theta} = e^{2u}\theta$, where $u \in C^\infty(M, \mathbb{R})$, the operator

$$Q(J, \theta, g', u) := \delta_b^{\hat{\theta}, g'} \delta_b^{\hat{\theta}, g'} T\varphi_J.$$

We notice that

$$Q(J_0, \theta_0, g_0, u) = 0 \iff Q(J_0, \theta_0, g_0, u + \mathrm{cste}) = 0,$$

hence we consider $\mathscr{C} = \left\{ u \in S^{2,2}(M, \mathbb{R}) \ \middle| \ \int_M u = 0 \right\}$.

Proposition 3.4.4. *There exists $C > 0$ such that*

$$\forall u \in \mathscr{C}, \ C\|u\|_{2,2} \le \|Q(J, \theta, g', u)\|_2 + \|u\|_2.$$

Lemma 3.4.5. *$\mathscr{C} \ni u \mapsto Q(J, \theta, g', u)$ is invertible in a neighbourhood of $(J_0, \theta_0, g_0, 0)$.*

Corollary 3.4.6. *For a given (J, θ, g') close to (J_0, θ_0, g_0), there exists a unique $u_J \in \mathscr{C}$ close to 0 such that $Q(J, \theta, g', u_J) = 0$.*

Proof. By construction, we have $Q(J_0,\theta_0,g_0,0)=0$. Moreover, since

$$\hat{R}=e^{-2u}(R+iu_1T_{\bar{1}}-iu_{\bar{1}}T_1),\ \text{ and }\ \hat{T}_1=e^{-u}T_1,$$

we have

$$e^{2u}\delta_b^{\hat{\theta},g'}T\varphi=\delta_b^{\theta,g'}T\varphi+u_1T_{\bar{1}}\varphi+u_{\bar{1}}T_1\varphi, \tag{3.1}$$

and

$$\begin{aligned} e^{4u}\nabla_{\hat{R}}^{\varphi^*g'}\hat{R}\varphi &= \nabla_R^{\varphi^*g'}R\varphi-2u_0\left(R\varphi+iu_1T_{\bar{1}}\varphi-iu_{\bar{1}}T_1\varphi\right)+iu_{01}T_{\bar{1}}\varphi-iu_{0\bar{1}}T_1\varphi\\ &\quad +u_1u_{\bar{1}\bar{1}}T_1\varphi+u_{\bar{1}}u_{11}T_{\bar{1}}\varphi\\ &\quad -u_{1\bar{1}}(e^{2u}\delta_b^{\hat{\theta},g'}T\varphi-\delta_b^{\theta,g'}T\varphi)-u_1u_{\bar{1}}\delta_b^{\theta,g'}T\varphi-u_{\bar{1}}u_{\bar{1}}\nabla_{T_1}^{\varphi^*g'}T_1\varphi-u_1u_1\nabla_{T_{\bar{1}}}^{\varphi^*g'}T_{\bar{1}}\varphi. \end{aligned}$$

Consequently, if $\delta_b^{\theta,g'}T\varphi=\nabla_R^{\varphi^*g'}R\varphi=0$, then

$$\partial_u\delta_b^{\hat{\theta},g'}T\varphi|_{u=0}(\dot{u})=\dot{u}_1T_{\bar{1}}\varphi+\dot{u}_{\bar{1}}T_1\varphi,$$

and

$$\partial_u\nabla_{\hat{R}}^{\varphi^*g'}\hat{R}\varphi|_{u=0}(\dot{u})=-2\dot{u}_0R\varphi+i\dot{u}_{01}T_{\bar{1}}\varphi-i\dot{u}_{0\bar{1}}T_1\varphi.$$

Similarly,

$$\partial_uQ_{(J_0,\theta_0,g_0,0)}(\dot{u})=-\dot{u}_{1\bar{1}}-\dot{u}_{\bar{1}1}=\Delta_b\dot{u}.$$

Therefore,

$$\begin{aligned}\int_M g_0\left(\partial_uQ_{(J_0,\theta_0,g_0,0)}(\dot{u}),\overline{\dot{u}}\right)\theta\wedge d\theta &= -\int_M g_0\left(\dot{u}_{1\bar{1}}+\dot{u}_{\bar{1}1},\overline{\dot{u}}\right)\theta_0\wedge d\theta_0\\ &= \int_M\left(g_0\left(\dot{u}_1,\overline{\dot{u}}_{\bar{1}}\right)+g_0\left(\dot{u}_{\bar{1}},\overline{\dot{u}}_1\right)\right)\theta_0\wedge d\theta_0,\end{aligned}$$

and we thus have

$$\begin{aligned}\partial_uQ_{(J_0,\theta_0,g_0,0)}(\dot{u})=0 &\implies \dot{u}_1=\dot{u}_{\bar{1}}=0\\ &\qquad \text{and }\ \dot{u}_0=-i(\dot{u}_{1\bar{1}}-\dot{u}_{\bar{1}1})=0\\ &\implies \dot{u}=0\ \text{ since }\int_M\dot{u}=0.\end{aligned}$$

Hence the lemma. The corollary is then directly obtained by the implicit map theorem. □

Note that u_J is the one and only small conformal change for which φ_J *could* be subharmonic. Now, let us consider, for $\hat{\theta}=e^{2u_J}\theta$, the operator

$$\tilde{Q}(J,\theta,g'):=P_1(J,\hat{\theta},g',\varphi_J)+\left(\delta_b^{\hat{\theta},g'}\nabla^{\varphi_J^*g'}-S_b\right)\delta_b^{\hat{\theta},g'}T\varphi_J=-\nabla_R^{\varphi_J^*g'}R\varphi_J+4\mathrm{Im}\left(\nabla_{T_{\bar{1}}}^{\varphi_J^*g'}\left(\tau_1^{\bar{1}}T_{\bar{1}}\right)\right)\varphi_J.$$

By construction, we have $\tilde{Q}(J_0,\theta_0,g_0)=0$.

Conjecture 3.4.7. There exists (J_1, θ_1, g_1') close to (J_0, θ_0, g_0) such that

$$\tilde{Q}(J_1, \theta_1, g_1') \neq 0.$$

As a consequence, φ_{J_1} is CR-harmonic and is non-subharmonic for all pseudohermitian structures close to θ_1, *i.e.* with u small.

Possible proof. The idea would be to prove that the principal symbol of $d\tilde{Q}_{(J_0,\theta_0,g_0)}$ is non-zero. We have

$$d\tilde{Q}_{(J_0,\theta_0,g_0)}(\dot{J}) = \partial_\varphi \tilde{Q}_{(J_0,\theta_0,g_0)}\left(\partial_J \varphi_J(\dot{J})\right) + \partial_u \tilde{Q}_{(J_0,\theta_0,g_0)}\left(\partial_J u_J(\dot{J})\right) + \partial_J \tilde{Q}_{(J_0,\theta_0,g_0)}(\dot{J}),$$

where

$$\partial_J \varphi_J(\dot{J}) = (\partial_\varphi P_1)^{-1}\left(\partial_J P_1(\dot{J})\right) \quad \text{and} \quad \partial_J u_J(\dot{J}) = (\partial_u Q)^{-1}\left(\partial_J Q(\dot{J})\right).$$

We therefore need to compute the symbols of the following operators, where (0) denotes the considered point:

$$(\partial_\varphi P_1)^{-1}{}_{(0)}, \quad \partial_J P_{1(0)}, \quad (\partial_u Q)^{-1}{}_{(0)}, \quad \partial_J Q_{(0)}, \quad \partial_\varphi \tilde{Q}_{(0)}, \quad \partial_u \tilde{Q}_{(0)}, \quad \text{and} \quad \partial_J \tilde{Q}_{(0)}.$$

We see that we need a framework where symbols of subelliptic operators are invertible. Such a framework is provided by the Beals-Greiner calculus [BGS84, BG88]. In this setting, computing the symbol of a composition of operators involves convolution in the Heisenberg group, which is non-commutative. In particular, computing the symbol of an inverse is more difficult than in the Euclidean case. We have not performed this computation to this day.

The operators $\partial_\varphi P_1$ and $\partial_u Q$ have been studied above. Other operators are given as follows: let us consider (J, θ, g') close to (J_0, θ_0, g_0), and $\dot{J} = \dot{J}_1^{\bar{1}} \theta^1 \otimes T_{\bar{1}} + \dot{J}_{\bar{1}}^1 \theta^{\bar{1}} \otimes T_1$. We have $\partial_J T_1(\dot{J}) = -\frac{i}{2} \dot{J}_1^{\bar{1}} T_{\bar{1}}$ [CMY17]. Consequently,

$$\partial_J\left(\delta_b^{\theta,g'} T\varphi\right)(\dot{J}) = -\frac{i}{2}\left(\dot{J}_{\bar{1},1}^1 T_1\varphi - \dot{J}_{1,\bar{1}}^{\bar{1}} T_{\bar{1}}\varphi\right).$$

Then, denoting $\delta_b^c \dot{J} := -\frac{i}{2}\left(\dot{J}_{\bar{1},1}^1 - \dot{J}_{1,\bar{1}}^{\bar{1}}\right)$, we have

$$\partial_J Q_{(0)}(\dot{J}) = \delta_b^{\theta,g_0} \delta_b^c \dot{J} = \frac{i}{2}\left(\dot{J}_{\bar{1},11}^1 - \dot{J}_{1,\bar{1}\bar{1}}^{\bar{1}}\right),$$

and

$$\partial_J\left(\delta_b^{\theta,g'} \nabla^{\varphi^* g'} \delta_b^{\theta,g'} T\varphi\right)(\dot{J}) = \delta_b^{\theta,g'} \nabla^{\varphi^* g'} \delta_b^c \dot{J},$$

which implies, since $\partial_J \tau_1^{\bar{1}} = i\dot{J}_{1,0}^{\bar{1}}$,

$$\partial_J P_{1(0)}(\dot{J}) = -\delta_b^{\theta_0,g_0} \nabla^{g_0} \delta_b^c \dot{J} - \mathrm{Scal}_W(J_0,\theta_0)\delta_b^c \dot{J} + 4\mathrm{Re}\left(\nabla_{T_{\bar{1}}}^{g_0}\left(\dot{J}_{1,0}^{\bar{1}} T_{\bar{1}}\right)\right).$$

$\square$

Finally, let us denote, for (J,θ,g') close to (J_0,θ_0,g_0), and for $\hat{\theta}=e^{2u}\theta$,

$$L(J,\theta,g',u) := \delta_b^{\hat{\theta},g'} T\varphi_J.$$

We recall that $g' = g_{J',\theta'}$.

Lemma 3.4.8. *If there exists $u \in \mathscr{C}$ such that $L(J,\theta,g',u)=0$, and if (J',θ') is close to (J,θ), then u is small.*

Corollary 3.4.9. *Under Assumption 3.4.2 and Conjecture 3.4.7, $\varphi_{J_1} : (M,H,J_1) \to (N,g_1')$ is CR-harmonic and is non-subharmonic for all pseudohermitian structures.*

Proof. If $L(J,\theta,g',u)=0$, then, by the conformal change formula (3.1), we have

$$L(J,\theta,g',0) = -u_1 T_{\bar{1}}\varphi_J - u_{\bar{1}} T_1\varphi_J =: -\left\langle du, T\varphi_J\right\rangle_\gamma .$$

For all $\varepsilon>0$, there exists $\alpha>0$ such that, if $\|(J,\theta)-(J',\theta')\| \leq \alpha$, then

$$\|L(J,\theta,g',0)-L(J',\theta',g',0)\|_{g'} \leq \varepsilon \quad \text{and} \quad \|\left\langle du, T\varphi_J\right\rangle_\gamma - \left\langle du, T\varphi_{J'}\right\rangle_\gamma\|_{g'} \leq \varepsilon.$$

Since $\varphi_{J'} = \mathrm{id}$, we have then

$$\|L(J,\theta,g',0)\|_{g'} \leq \varepsilon \quad \text{and} \quad \|\left\langle d_b u, T\varphi_J\right\rangle_\gamma - \|du\|_\gamma\|_{g'} \leq \varepsilon,$$

hence

$$\|du\|_\gamma \leq 2\varepsilon.$$

We have also, for a given $p \in M$,

$$\|u-u(p)\|_\infty \leq \mathrm{diam}_b(M)\|du\|_\gamma,$$

where $\mathrm{diam}_b(M)$ is the horizontal diameter of M. Since $u \in \mathscr{C}$, there exists $p \in M$ such that $u(p)=0$, hence $\|u\|_\infty \leq 2\varepsilon\mathrm{diam}_b(M)$, where $\mathrm{diam}_b(M)$ is the horizontal diameter of (M,H). Since M is compact, $\mathrm{diam}_b(M)$ exists and is finite by the ball-box theorem [Mon02]. Hence the lemma.

The corollary follows then directly from Conjecture 3.4.7. □

3.5 Relation with the Fefferman bundle in dimension 3

We describe here the correspondance between the obstruction to CR-harmonicity on a given CR 3-manifold and the obstruction to conformal-harmonicity on its Fefferman bundle. It generalizes the Appendix B. of [CY13].

Let (M,H,J) be a compact SPCR 3-manifold and let (N,h) be a Riemannian manifold. Let (F,g_F) be the Fefferman bundle of (M,H,J), as defined in Section 1.6. By analogy with the Riemannian case [Bér13], given $\varphi \in C^\infty(F,N)$, the obstruction to the existence of a smooth harmonic extension of φ on the interior of (F,g_F) is given by

$$P_F(\varphi) = -\frac{1}{16}\left(\delta^{g_F,h}\nabla^{\varphi^*h}\delta^{g_F,h}T\varphi - \delta^{g_F,h}\left(2\mathrm{Ric}_{g_F} - \frac{2}{3}\mathrm{Scal}_{g_F}\right)T\varphi + S(\delta^{g_F,h}T\varphi)\right),$$

where Ric_{g_F} is understood as an endomorphism of TF, and $\mathrm{Ric}_{g_F} T\varphi := T\varphi\left(\mathrm{Ric}_{g_F}(\cdot)\right)$, and

$$S(X) := \sum_{i=1}^{4} \mathscr{R}^h_{X,T\varphi(e_i)} T\varphi(e_i).$$

Proposition 3.5.1. *For all* $\varphi \in C^\infty(M^3, N)$,

$$\pi_*\left(\delta^{g_F,h}\nabla^{\varphi^*h}\delta^{g_F,h}T(\pi^*\varphi)\right) = 4\delta_b^{\theta,h}\nabla^{\varphi^*h}\delta_b^{\theta,h}T\varphi,$$

$$\pi_*\left(\delta^{g_F,h}\left(2\mathrm{Ric}_{g_F} - \frac{2}{3}\mathrm{Scal}_{g_F}\right)T(\pi^*\varphi)\right) = -4\nabla_R^{\varphi^*h}R\varphi + 16\mathrm{Im}\left(\nabla_{T_{\bar{1}}}^{\varphi^*h}\left(\tau_1^{\bar{1}}T_{\bar{1}}\right)\right)\varphi,$$

and for X *in* TN,

$$\pi_*\left(S((\pi^*\varphi)^*X)\right) = 4S_b(\varphi^*X).$$

Proof. The first and third equalities are straightforward from the expression of g_F. The second equality comes from the fact that, by Proposition 1.6.3,

$$\mathrm{Sch}_{g_F} = -\varpi^2 - \mathbf{S}\theta^2 + \frac{1}{2}\mathrm{Sch}_W - \frac{1}{2}\gamma(J\tau\cdot,\cdot) + \frac{1}{2}\mathbf{T}J\circ\theta,$$

where

$$\mathbf{T} = \frac{1}{3}\left(\frac{1}{4}d_b\mathrm{Scal}_W + i\delta\tau\right) \quad \text{and} \quad \mathbf{S} = \delta\mathbf{T} - |\mathrm{Sch}_W|^2 + |\tau|^2.$$

Indeed, since $\mathrm{Scal}_{g_F} = 3\mathrm{Scal}_W$ and $\mathrm{Sch}_W = \dfrac{1}{4}\mathrm{Scal}_W\gamma$, we have then

$$\begin{aligned} 2\mathrm{Ric}_{g_F} - \frac{2}{3}\mathrm{Scal}_{g_F}g_F &= 4\mathrm{Sch}_{g_F} - \frac{1}{3}\mathrm{Scal}_{g_F}g_F \\ &= -4\varpi^2 - 4\mathbf{S}\theta^2 - 2\gamma(J\tau\cdot,\cdot) + 2\mathbf{T}J\circ\theta - \mathrm{Scal}_W i\varpi\circ\theta, \end{aligned}$$

which gives the second equality. □

From the latter comes directly the

Theorem 3.5.2. *For all* $\varphi \in C^\infty(M^3, N)$,

$$\pi_*\left(P_F(\pi^*\varphi)\right) = \frac{1}{4}P_1(\varphi).$$

In particular, a map $\varphi : M \to N$ is CR-harmonic if and only if $\pi^*\varphi$ is conformal-harmonic.

CHAPITRE IV

Estimates of the contact Yamabe invariant

Abstract. We prove in this chapter that the contact Yamabe invariant σ_c is non-decreasing under SPC handle attaching and under SPC connected sum. We give a lower bound on σ_c in a particular case.

Contents

Introduction

In the CR setting, a construction by W. Wang, recently implemented by J.-H. Cheng and H.-L. Chiu, shows that the positivity of Y_{CR} is preserved under handle attaching on a CR spherical manifold [Wan03, CC18]:

Theorem 4.0.1 ([Wan03, CC18]). *Let (M, H, J) be a compact spherical SPCR manifold with positive Y_{CR}. Let $(\tilde{M}, \tilde{H}, \tilde{J})$ be obtained from (M, H, J) by CR handle attaching. Then $(\tilde{M}, \tilde{H}, \tilde{J})$ is spherical and $Y_{CR}(\tilde{M}, \tilde{H}, \tilde{J}) > 0$.*

From a continuity argument detailed in Section 4.2.2, we generalize this result:

Theorem 4.0.2. *Let (M, H) be a compact SPC manifold. Let $(\tilde{M}, \tilde{H})$ be a manifold obtained from (M, H) by SPC handle attaching, then*

$$\sigma_c(\tilde{M}, \tilde{H}) \geq \sigma_c(M, H).$$

Example 4.0.3. If $(M, H) = (\mathbb{S}^{2n+1}, H_0)$, using Theorem 1.4.1 we thus have the equality

$$\sigma_c(\mathbb{S}^1 \times \mathbb{S}^{2n}, \tilde{H}_0) = \sigma_c(\mathbb{S}^{2n+1}, H_0).$$

In particular, we prove the SPC analogue of Theorem 0.0.3:

Theorem 4.0.4. *Let (M_1, H_1) and (M_2, H_2) be two compact SPC manifolds of dimension $2n+1$. Let $(M_1, H_1)\#(M_2, H_2)$ be their SPC connected sum, then*

$$\sigma_c\left((M_1, H_1)\#(M_2, H_2)\right) \geq \begin{cases} -\left(|\sigma_c(M_1, H_1)|^{n+1} + |\sigma_c(M_2, H_2)|^{n+1}\right)^{\frac{1}{n+1}} & \text{if } \sigma_c(M_1, H_1) \leq 0 \\ & \quad \text{and } \sigma_c(M_2, H_2) \leq 0, \\ \min\left(\sigma_c(M_1, H_1), \sigma_c(M_2, H_2)\right) & \text{otherwise.} \end{cases}$$

We also prove a weakened contact version of Theorem 0.0.2:

Theorem 4.0.5. *Let (M, H) be a circle bundle over a Riemann surface Σ of positive genus admitting an Einstein pseudohermitian structure. Then*

$$\sigma_c(M, H) \geq -2\pi\sqrt{-\chi(\Sigma)}.$$

Section 4.2 contains the proof of Theorems 4.0.2 and 4.0.4, and Section 4.3 contains the proof of Theorem 4.0.5.

4.1 Handle attaching on a spherical SPCR manifold

We recall here a handle attaching process on spherical SPCR manifolds, compatible with the CR structure, which is due to W. Wang [Wan03]. If the handle is attached between two distinct connected components, this provides a connected sum of the components.

Let either $\left(M = M_1 \sqcup M_2, H, J, \hat{\theta}\right)$ be a disjoint union of two connected spherical strictly pseudoconvex pseudohermitian manifolds, and $p_1 \in M_1$, $p_2 \in M_2$; or $\left(M, H, J, \hat{\theta}\right)$ be a connected spherical strictly pseudoconvex pseudohermitian manifold, and $p_1, p_2 \in M$.

Let $M_0 = M \setminus \{p_1, p_2\}$. For i in $\{1, 2\}$, let $U_i \subset M$ be a neighbourhood of p_i and

$$\varphi_i : U_i \to B(0,2) := \{\xi \in \mathbb{H}^{2n+1},\ \|\xi\|_{\mathbb{H}} < 2\}$$

local coordinates such that $\varphi_i(p_i) = 0$. Let us denote, for $0 < r < 1$,

$$U_i(r) = \{x \in U_i,\ \|\varphi_i(x)\|_{\mathbb{H}} < r\},$$

$$U_i(r,1) = \{x \in U_i,\ r < \|\varphi_i(x)\|_{\mathbb{H}} < 1\}.$$

Since M is spherical around p_1 and p_2, there exists $\lambda \in C^\infty(M_0, \mathbb{R}_+^*)$ such that, denoting $\theta = \lambda\hat{\theta}$ on M_0,

$$\forall i \in \{1,2\},\ \forall \xi \in B(0,2) \setminus \{0\}, \quad \varphi_{i*}\theta(\xi) = \|\xi\|_{\mathbb{H}}^{-2}\theta_{\mathbb{H}}(\xi),$$

that is, (M_0, H, J, θ) has cylindrical ends. Indeed, we can define a mapping

$$\begin{array}{rccc} \Phi: & B(0,1) & \longrightarrow & \mathbb{R}_+ \times \Sigma^{2n} \\ & \xi & \mapsto & \left(\log \dfrac{1}{\xi}, \dfrac{\xi}{\|\xi\|_{\mathbb{H}}}\right), \end{array}$$

where $\Sigma^{2n} := \{\xi \in \mathbb{H}^{2n+1},\ \|\xi\|_{\mathbb{H}} = 1\}$, and $\tilde{\theta} := \Phi_*(\|\cdot\|_{\mathbb{H}}^{-2}\theta_{\mathbb{H}})$. Then

$$(B(0,1), H_{\mathbb{H}}, J_{\mathbb{H}}, \|\cdot\|_{\mathbb{H}}^{-2}\theta_{\mathbb{H}}) \simeq (\mathbb{R}_+ \times \Sigma^{2n}, \tilde{H}, \tilde{J}, \tilde{\theta}),$$

where the equivalence is pseudohermitian. Denoting $\hat{M} := M \setminus U_1(1) \cup U_2(1)$,

$$(M_0, H, J, \theta) \simeq (\mathbb{R}_+ \times \Sigma^{2n}, \tilde{H}, \tilde{J}, \tilde{\theta}) \ \cup\ (\hat{M}, H, J, \theta) \ \cup\ (\mathbb{R}_+ \times \Sigma^{2n}, \tilde{H}, \tilde{J}, \tilde{\theta}). \tag{4.1}$$

Now, let us denote, for $r \in (0,1)$ and $A \in U(n)$, by $\psi_{r,A} : U_1(r,1) \to U_2(r,1)$ the mapping

$$\psi_{r,A} = \varphi_2^{-1} \circ \delta_r \circ R \circ U_A \circ \varphi_1,$$

where

$$\delta_r : (z,t) \mapsto (rz, r^2 t),$$

$$U_A : (z,t) \mapsto (Az, t),$$

and

$$R : (z,t) \mapsto \left(\frac{-z}{|z|^2 - it}, \frac{-t}{|z|^4 + t^2}\right)$$

denote respectively dilations, unitary transformations and inversion in $\mathbb{H}^{2n+1}$.

Let $(M_{r,A}, H_{r,A}, J_{r,A}, \theta_{r,A})$ be the pseudohermitian manifold formed from M by removing $\overline{U_1(r)}$ and $\overline{U_2(r)}$, and by identifying $U_1(r,1)$ with $U_2(r,1)$ along $\psi_{r,A}$. Let

$$\pi_{r,A} : M \setminus \overline{U_1(r) \cup U_2(r)} \to M_{r,A}$$

be the corresponding projection.

Since

$$\delta_r^* \varphi_{i*}\theta = \frac{1}{r^2 \| \cdot \|_{\mathbb{H}}^2} r^2 \theta_{\mathbb{H}} = \varphi_{i*}\theta$$

and

$$R^* \varphi_{i*}\theta = \frac{1}{\|R(\cdot)\|_{\mathbb{H}}^2} R^* \theta_{\mathbb{H}} = \frac{\| \cdot \|_{\mathbb{H}}^2}{\| \cdot \|_{\mathbb{H}}^4} \theta_{\mathbb{H}} = \varphi_{i*}\theta,$$

the gluing preserves θ on $U_i(r,1)$. Hence,

$$\pi_{r,A}^* \theta_{r,A} = \theta \qquad \text{on } M \setminus \overline{U_1(r) \cup U_2(r)}.$$

We have in fact

$$(M_{r,A}, H_{r,A}, J_{r,A}, \theta_{r,A}) \simeq (\hat{M}, H, J, \theta) \ \cup \ ([0,l] \times \Sigma^{2n}, \tilde{H}, \tilde{J}, \tilde{\theta}) \tag{4.2}$$

where $l = \log\left(\frac{1}{r}\right) \in (0, +\infty)$.

4.2 A CR Kobayashi inequality

4.2.1 Non-decreasing of σ_c under SPC handle attaching

This part follows a method developed in the conformal setting by O. Kobayashi [Kob87]. It has been adapted in the CR setting by W. Wang, and more recently by J.-H. Cheng and H.-L. Chiu [Wan03, CC18]. A similar technique has recently been implemented in the quaternionic context [SW16]. Theorem 4.0.2 is a direct consequence of the following result:

Theorem 4.2.1. *Let (M, H, J) be a compact SPCR manifold. Let $(\tilde{M}, \tilde{H}, \tilde{J})$ be a manifold obtained from (M, H, J) by CR handle attaching, then*

$$Y_{CR}(\tilde{M}, \tilde{H}, \tilde{J}) \geq Y_{CR}(M, H, J).$$

Proof. Let $p_1, p_2 \in M$. We use the following lemma, that will be proved in Section 4.2.2:

Lemma 4.2.2. *We may assume that (M, H, J) is spherical around p_1 and p_2.*

Under this assumption, we can apply the construction of Section 4.1 to M. Let $(M_{r,A}, H_{r,A}, J_{r,A})$ be obtained from (M, H, J) by CR handle attaching.

By definition of $Y_{CR}(M_{r,A}, H_{r,A}, J_{r,A})$, there exists a function $f_l \in C^\infty(M_{r,A}, \mathbb{R}_+^*)$ such that

$$S_W\left(M_{r,A}, J_{r,A}, f_l^{2/n}\theta_{r,A}\right) = \int_{M_{r,A}} \left(2\left(\frac{n+1}{n}\right)|\nabla^{\theta_{r,A}} f_l|^2 + \mathrm{Scal}_W(J_{r,A}, \theta_{r,A}) f_l^2\right)\theta_{r,A}\wedge d\theta_{r,A}^n$$
$$< Y_{CR}(M_{r,A}, H_{r,A}, J_{r,A}) + \frac{1}{1+l} \tag{4.3}$$

and

$$\int_{M_{r,A}} f_l^{2\left(\frac{n+1}{n}\right)}\theta_{r,A}\wedge d\theta_{r,A}^n = 1.$$

Lemma 4.2.3. *There exists $l_* \in [0,l]$ such that*

$$\int_{\{l_*\}\times\Sigma^{2n}} \left(|d_b f_l|^2 + f_l^2\right) dS_{\Sigma^{2n}} \leq \frac{C}{l},$$

where C is a constant independent of l.

Proof. Let $C_1 = -\min(0, \min_{\hat{M}} \mathrm{Scal}_W(J,\theta))\mathrm{Vol}(\hat{M},\theta)^{\frac{1}{n+1}}$. Hölder's inequality yields

$$\int_{\hat{M}} f_l^2\theta\wedge d\theta^n \leq \mathrm{Vol}(\hat{M},\theta)^{\frac{1}{n+1}},$$

and then, using decomposition (4.2),

$$\int_{[0,l]\times\Sigma^{2n}} \left(2\left(\frac{n+1}{n}\right)|d_b f_l|^2 + \mathrm{Scal}_W(\tilde{J},\tilde{\theta}) f_l^2\right)\tilde{\theta}\wedge d\tilde{\theta}^n \leq Y_{CR}(M_{r,A}, H_{r,A}, J_{r,A}) + \frac{1}{1+l} + C_1.$$

Consequently there exists $l_* \in [0,l]$ such that

$$\int_{\{l_*\}\times\Sigma^{2n}} \left(2\left(\frac{n+1}{n}\right)|d_b f_l|^2 + \mathrm{Scal}_W(\tilde{J},\tilde{\theta}) f_l^2\right)\tilde{\theta}\wedge d\tilde{\theta}^n \leq \frac{1}{l}\left(Y_{CR}(M_{r,A}, H_{r,A}, J_{r,A}) + \frac{1}{1+l} + C_1\right).$$

The lemma is obtained with $C = \dfrac{Y_{CR}(M_{r,A}, H_{r,A}, J_{r,A}) + 1 + C_1}{\min\left(2\left(\frac{n+1}{n}\right), \min_{\{l_*\}\times\Sigma^{2n}} \mathrm{Scal}_W(\tilde{J},\tilde{\theta})\right)}$. □

We therefore decompose

$$\begin{aligned}(M_0, H, J, \theta) \;\simeq\;& ([l_*,\infty)\times\Sigma^{2n}, \tilde{H}, \tilde{J}, \tilde{\theta}) \\ &\cup (M_{r,A}\setminus\{l_*\}\times\Sigma^{2n}, H_{r,A}, J_{r,A}, \theta_{r,A}) \\ &\cup ([l-l_*,\infty)\times\Sigma^{2n}, \tilde{H}, \tilde{J}, \tilde{\theta})\end{aligned} \tag{4.4}$$

and extend f_l to M_0 as follows: $F_l = f_l$ on $M_{r,A}\setminus\{l_*\}\times\Sigma^{2n}$ and

$$F_l : (s,\xi) \mapsto \begin{cases} (l_*+1-s) f_l(l_*,\xi) & \forall (s,\xi) \in [l_*, l_*+1]\times\Sigma^{2n}, \\ 0 & \forall (s,\xi) \in [l_*+1,\infty)\times\Sigma^{2n}, \\ 0 & \forall (s,\xi) \in [l-l_*+1,\infty)\times\Sigma^{2n}, \\ (l-l_*+1-s) f_l(l_*,\xi) & \forall (s,\xi) \in [l-l_*, l-l_*+1]\times\Sigma^{2n}. \end{cases}$$

We thus obtain from (4.3) and Lemma 4.2.3

$$\begin{aligned}
S_W(M_0,J,F_l^{2/n}\theta) &= S_W\left(M_{r,A},J_{r,A},f_l^{2/n}\theta_{r,A}\right)\\
&\quad+\int_{[l_*,l_*+1]\times\Sigma^{2n}}\left(2\left(\frac{n+1}{n}\right)|d_bF_l|^2+\mathrm{Scal}_W(\tilde J,\tilde\theta)F_l^2\right)\tilde\theta\wedge d\tilde\theta^n\\
&\quad+\int_{[l-l_*,l-l_*+1]\times\Sigma^{2n}}\left(2\left(\frac{n+1}{n}\right)|d_bF_l|^2+\mathrm{Scal}_W(\tilde J,\tilde\theta)F_l^2\right)\tilde\theta\wedge d\tilde\theta^n\\
&= S_W\left(M_{r,A},J_{r,A},f_l^{2/n}\theta_{r,A}\right)\\
&\quad+2\int_{\Sigma^{2n}}\left(\frac{2}{3}\frac{n+1}{n}|d_bf_l(l_*,\cdot)|^2+\left(2\left(\frac{n+1}{n}\right)+\frac{1}{3}\mathrm{Scal}_W(\tilde J,\tilde\theta)\right)f_l(l_*,\cdot)^2\right)dS_{\Sigma^{2n}}\\
&\leq S_W\left(M_{r,A},J_{r,A},f_l^{2/n}\theta_{r,A}\right)\\
&\quad+2\int_{\{l_*\}\times\Sigma^{2n}}\left(\frac{2}{3}\frac{n+1}{n}|d_bf_l|^2+\left(2\left(\frac{n+1}{n}\right)+\frac{1}{3}\mathrm{Scal}_W(\tilde J,\tilde\theta)\right)f_l^2\right)dS_{\Sigma^{2n}}\\
&\leq Y_{CR}(M_{r,A},H_{r,A},J_{r,A})+\frac{B}{l},
\end{aligned}$$

where B is a constant independent of l, and

$$\int_{M_0}F_l^{2\left(\frac{n+1}{n}\right)}\theta\wedge d\theta^n>1.$$

Since the infimum in the Yamabe functional may be taken over all nonnegative Lipschitz functions with compact support as conformal factors by Lemma 1.5.3, we get that

$$Y_{CR}(M_0,H,J)\leq Y_{CR}(M_{r,A},H_{r,A},J_{r,A})+\frac{B}{l},$$

which, for l sufficiently large, yields the desired inequality. □

4.2.2 Local sphericity

In this section, we prove the following technical lemma, which is essential for the proof of Theorem 4.0.4.

Lemma 4.2.4. *Let (M,H,J) be a compact SPCR manifold. Given p_1 and p_2 in M, there is a 1-parameter family of complex structures (J_t) in $\mathscr{J}$ C^0-converging to J such that (M,H,J_t) is spherical around p_1 and p_2, and $Y_{CR}(M,H,J_t)\xrightarrow[t\to0]{}Y_{CR}(M,H,J)$.*

In other words, to prove Theorem 4.2.1, we may assume that (M,H,J) is spherical around p_1 and p_2. This lemma is a direct consequence of the two following results:

Lemma 4.2.5. *Let (M,H,J,θ) be a compact strictly pseudoconvex pseudohermitian manifold. Let (J_t) be a 1-parameter family of complex structures in $\mathscr{J}$ C^0-converging to J such that $\mathrm{Scal}_W(J_t,\theta)\xrightarrow[t\to0]{}\mathrm{Scal}_W(J,\theta)$. Then $Y_{CR}(M,H,J_t)\xrightarrow[t\to0]{}Y_{CR}(M,H,J)$.*

Lemma 4.2.6. *Let (M, H, J, θ) be a compact strictly pseudoconvex pseudohermitian manifold. Let p_1 and p_2 be two points in M. Let (ε_t) be a 1-parameter family of positive numbers decreasing to 0. There is a 1-parameter family of complex structures (J_t) in $\mathscr{J}$ C^0-converging to J such that for all t, J_t coincides with J outside an ε_t-neighbourhood U'_t of $\{p_1, p_2\}$, J_t is spherical inside $U_t \subset U'_t$, and $|\mathrm{Scal}_W(J_t, \theta) - \mathrm{Scal}_W(J, \theta)| \leq \varepsilon_t$.*

Proof of Lemma 4.2.5. We adapt from the conformal case a proof due to L. Bérard Bergery [Ber83]. Let us denote

$$F = \left\{ u \in C^\infty(M, \mathbb{R}_+^*) \;\middle|\; \int_M u^{2\left(\frac{n+1}{n}\right)} \theta \wedge d\theta^n = 1 \right\}.$$

By definition, $Y_{CR}(M, H, J_t) = \inf_{u \in F} I_t(u)$ where

$$I_t(u) = 2\left(\frac{n+1}{n}\right) \int_M |d_b u|_t^2 \theta \wedge d\theta^n + \int_M u^2 \mathrm{Scal}_W(J_t, \theta) \theta \wedge d\theta^n.$$

Let $\varepsilon > 0$. For each t there exists u_t in F such that

$$Y_{CR}(M, H, J_t) \leq I_t(u_t) \leq Y_{CR}(M, H, J_t) + \varepsilon.$$

Let $\eta > 0$ and $K > 0$ be such that, for all t in $[-\eta, \eta]$,

$$|\mathrm{Scal}_W(J_t, \theta) - \mathrm{Scal}_W(J, \theta)| \leq \varepsilon,$$
$$|\mathrm{Scal}_W(J_t, \theta)| \leq K,$$

and

$$\frac{1}{1+\varepsilon} \leq \frac{|\omega|_t^2}{|\omega|_0^2} \leq 1 + \varepsilon \quad \forall \omega \in \Omega^1(M).$$

In particular, $I_t(u_t) \leq I_t(1) + \varepsilon \leq K + \varepsilon$. Now, by Hölder's inequality,

$$\int_M u_t^2 \theta \wedge d\theta^n \leq 1,$$

hence we have

$$\begin{aligned}
I_0(u_t) &= 2\left(\frac{n+1}{n}\right) \int_M |d_b u_t|_0^2 \theta \wedge d\theta^n + \int_M u_t^2 \mathrm{Scal}_W(J_0, \theta) \theta \wedge d\theta^n \\
&\leq 2\left(\frac{n+1}{n}\right)(1+\varepsilon) \int_M |d_b u_t|_t^2 \theta \wedge d\theta^n + \int_M u_t^2 \mathrm{Scal}_W(J_t, \theta) \theta \wedge d\theta^n \\
&\quad + \int_M u_t^2 |\mathrm{Scal}_W(J, \theta) - \mathrm{Scal}_W(J_t, \theta)| \theta \wedge d\theta^n \\
&\leq 2\left(\frac{n+1}{n}\right)(1+\varepsilon) \int_M |d_b u_t|_t^2 \theta \wedge d\theta^n + \int_M u_t^2 \mathrm{Scal}_W(J_t, \theta) \theta \wedge d\theta^n + \varepsilon \\
&\leq (1+\varepsilon) I_t(u_t) + \varepsilon(K+1),
\end{aligned}$$

and similarly

$$I_t(u_0) \leq (1+\varepsilon) I_0(u_0) + \varepsilon(K+1).$$

Then, for all t in $[-\eta, \eta]$,

$$\begin{aligned}
\frac{1}{1+\varepsilon} Y_{CR}(M, H, J) - \frac{\varepsilon}{1+\varepsilon}(K+1) - \varepsilon &\leq Y_{CR}(M, H, J_t) \\
&\leq (1+\varepsilon)(Y_{CR}(M, H, J) + \varepsilon) + \varepsilon(K+1).
\end{aligned}$$

□

Remark 4.2.7. Since $Y_{CR}(M,H,J)$ only depends on derivatives up to order 2 of J, the supremum in $\sigma_c(M,H)$ may be taken over all C^2 complex structures on (M,H). Therefore, in the following proof, gluing complex structures only needs to be considered up to C^2-regularity.

Proof of Lemma 4.2.6. We follow a construction due to O. Biquard and Y. Rollin [BR09]. We assume that for all t, $B(p_1,\varepsilon_t)\cap B(p_2,\varepsilon_t)=\emptyset$, where the distances are taken with respect to the Webster metric. For a given t, let U_t' be an ε_t-neighbourhood of $\{p_1,p_2\}$, and let $x=\min\left(d(\cdot,p_1),d(\cdot,p_2)\right)$ on M. There is a smooth cut-off function $w_t:\mathbb{R}_+\to\mathbb{R}_+$ such that $\chi_t:=w_t\circ x=0$ on some $U_t\subset U_t'$, $\chi_t=1$ outside U_t', and for all x in $\mathbb{R}_+$, $|xw_t'(x)|\leq\varepsilon_t$ and $|x^2w_t''(x)|\leq\varepsilon_t$ (cf. [Kob87], Sublemma 3.4.). Indeed, we may take w_t as a smoothing of $\tilde{w}_t$ defined by

$$\tilde{w}_t(x)=\begin{cases}0 & \forall x\leq\varepsilon_t e^{-\frac{2}{\varepsilon_t}}\\ 1-\dfrac{\varepsilon_t}{2}\log\left(\dfrac{\varepsilon_t}{x}\right) & \forall x\in[\varepsilon_t e^{-\frac{2}{\varepsilon_t}},\varepsilon_t]\\ 1 & \forall x\geq\varepsilon_t.\end{cases}$$

If $\dim M=3$, then all almost complex structures are formally integrable. Let us take i in $\{1,2\}$. Let $\psi_i:U_i\to U_{\mathbb{H}}$ be a contactomorphism identifying a neighbourhood U_i of p_i in M with a neighbourhood $U_{\mathbb{H}}$ of 0 in $\mathbb{H}^3$ such that $\psi_i(p_i)=0$ and, denoting $\tilde{J}_i:=(\psi_i)_*J$ and $\tilde{\theta}_i:=(\psi_i)_*\theta$, such that $j_0^1(\tilde{J}_i)=j_0^1(J_{\mathbb{H}})$ and $\mathrm{Scal}_W(\tilde{J}_i,\tilde{\theta}_i)=\mathrm{Scal}_W(J_{\mathbb{H}},\tilde{\theta}_i)$ at 0. We assume that $U_1\cap U_2=\emptyset$. For t large enough that $U_t'\subset U_1\sqcup U_2$, let $\tilde{\chi}_{i,t}:=({\psi_i}^{-1})^*\chi_t|_{U_i}$. For (z_1,y) in $\mathbb{H}^3$, let $\tilde{J}_{i,t}(z_1,y):=\tilde{J}_i(\tilde{\chi}_{i,t}z_1,\tilde{\chi}_{i,t}^2y)$. Then $\tilde{J}_{i,t}$ coincides with $J_{\mathbb{H}}$ inside $\psi_i(U_t\cap U_i)$, and with $\tilde{J}_i$ outside $\psi_i(U_t'\cap U_i)$. Therefore, the complex structure J_t defined on M by

$$\forall i\in\{1,2\},\ J_t:=\psi_i^*\tilde{J}_{i,t}\text{ on }U_i,\qquad J_t:=J\text{ elsewhere},$$

has the desired properties. In particular, since $|J_t-J|=O(x^2)$ and $|\nabla(J_t-J)|=O(x)$, and since we know (cf. [CL90], 2.20) that for $E\in\mathscr{E}(J):=\{E\in\mathrm{End}(H)\mid EJ+JE=0\}$,

$$\partial_J\mathrm{Scal}_W(J,\theta)(E)=\frac{i}{2}({E_\alpha}^{\overline{\beta}}{}_{,\overline{\beta}}{}^{\alpha}-{E_{\overline{\alpha}}}^{\beta}{}_{,\beta}{}^{\overline{\alpha}})-\frac{1}{2}({\tau_\alpha}^{\overline{\beta}}{E_{\overline{\beta}}}^{\alpha}+{\tau_{\overline{\alpha}}}^{\beta}{E_\beta}^{\overline{\alpha}}),$$

we have, for some constant C,

$$|\mathrm{Scal}_W(J_t,\theta)-\mathrm{Scal}_W(J,\theta)|\leq C\left(|w_t'|\left(|J_t-J|+|\nabla(J_t-J)|\right)+|w_t''||J_t-J|\right)\leq C\varepsilon_t.$$

If $\dim M\geq5$, then, since M is compact, (M,H,J) is embeddable. Let us consider an ACH manifold (X,g) with CR infinity (M,H,J). Let J_X be the complex structure on $\overline{X}$ given by Proposition 2.3.6 and let $z=(z_1,\ldots,z_{n+1})$ be complex coordinates near $\{p_1,p_2\}$. Then, by the normal form theorem of Chern and Moser, there is a boundary defining function r on $\overline{X}$ such that

$$r(z)=r_0(z)+\underset{1\leq j\leq n}{O}(|z_j|^4),$$

where $r_0(z) = \mathrm{Re}(z_{n+1}) - \dfrac{1}{4}\sum_{1\le j\le n} |z_j|^2$ is a boundary defining function for the Heisenberg group [CM74]. We glue the defining functions as follows:

$$r_t = (1-\chi_t)r_0 + \chi_t r.$$

The corresponding contact form is given by $\theta_t = i\left(\overline{\partial} - \partial\right) r_t$. The induced complex structure J_t on M is then given by the relation $d\theta(\cdot, J_t\cdot) = d\theta_t(\cdot, i\cdot)$. By construction, J_t is spherical inside U_t and coincides with J outside U_t', and (J_t) C^0-converges to J. Moreover, since $|r_t - r| = O(x^4)$ and $|\nabla(r_t - r)| = O(x^3)$, we have, for some constant C,

$$|\mathrm{Scal}_W(J_t,\theta) - \mathrm{Scal}_W(J,\theta)| \le C\left(|w_t'|\left(|\nabla^2(r_t-r)| + |\nabla^3(r_t-r)|\right) + |w_t''||\nabla^2(r_t-r)|\right) \le C\varepsilon_t.$$

$\square$

4.2.3 Disjoint union

In the case of a connected sum, Theorem 4.0.2 can be written the following way:

Theorem 4.2.8. *Let (M_1,H_1) and (M_2,H_2) be two compact SPC manifolds of dimension $2n+1$. Let $(M_1,H_1)\#(M_2,H_2)$ be their SPC connected sum, then*

$$\sigma_c\left((M_1,H_1)\#(M_2,H_2)\right) \ge \sigma_c\left((M_1,H_1)\sqcup(M_2,H_2)\right).$$

Alongside with the hereunder computation of the right-hand side, this gives Theorem 4.0.4.

Proposition 4.2.9. *Let (M_1,H_1) and (M_2,H_2) be two compact SPC manifolds of dimension $2n+1$. Then*

$$\sigma_c\left((M_1,H_1)\sqcup(M_2,H_2)\right) = \begin{cases} -\left(|\sigma_c(M_1,H_1)|^{n+1} + |\sigma_c(M_2,H_2)|^{n+1}\right)^{\frac{1}{n+1}} & \text{if } \sigma_c(M_1,H_1)\le 0 \\ & \text{and } \sigma_c(M_2,H_2)\le 0, \\ \min\left(\sigma_c(M_1,H_1),\sigma_c(M_2,H_2)\right) & \text{otherwise.}\end{cases}$$

Proof. Let us consider a unit volume strictly convex pseudohermitian structure (J,θ) on $(M_1,H_1)\sqcup(M_2,H_2)$. Let us denote, for i in $\{1,2\}$, $J_i := J\ |_{M_i}$, $\theta_i := \theta\ |_{M_i}$, $Y_i = Y_{CR}(M_i,H_i,J_i)$, and λ_i in $\mathbb{R}_+^*$ which verifies

$$\mathrm{Vol}(M_i,\lambda_i\theta_i) = \lambda_i^{n+1}\mathrm{Vol}(M_i,\theta_i) = 1.$$

We recall that S_W denotes the integral Webster scalar curvature. Since, for i in $\{1,2\}$, $\mathrm{Scal}_W(J,\lambda_i\theta_i) = \lambda_i^{-1}\mathrm{Scal}_W(J,\theta_i)$, we have

$$\begin{aligned} S_W(M_1\sqcup M_2,J,\theta) &= S_W(M_1,J_1,\theta_1) + S_W(M_2,J_2,\theta_2) \\ &= \lambda_1^{-n}S_W(M_1,J_1,\lambda_1\theta_1) + \lambda_2^{-n}S_W(M_2,J_2,\lambda_2\theta_2) \\ &= \mathrm{Vol}(M_1,\theta_1)^{\frac{n}{n+1}}S_W(M_1,J_1,\lambda_1\theta_1) + \mathrm{Vol}(M_2,\theta_2)^{\frac{n}{n+1}}S_W(M_2,J_2,\lambda_2\theta_2) \\ &\ge \mathrm{Vol}(M_1,\theta_1)^{\frac{n}{n+1}}Y_1 + \mathrm{Vol}(M_2,\theta_2)^{\frac{n}{n+1}}Y_2, \end{aligned}$$

with equality when $\lambda_1\theta_1$ and $\lambda_2\theta_2$ are Yamabe contact forms on (M_1, H_1, J_1) and (M_2, H_2, J_2) respectively. Optimizing the right-hand side under the constraint $\mathrm{Vol}(M_1, \theta_1) + \mathrm{Vol}(M_2, \theta_2) = 1$ yields

$$S_W(M_1 \sqcup M_2, J, \theta) \geq \begin{cases} -\left(|Y_1|^{n+1} + |Y_2|^{n+1}\right)^{\frac{1}{n+1}} & \text{if } Y_1 \leq 0 \text{ and } Y_2 \leq 0, \\ \min\left(Y_1, Y_2\right) & \text{otherwise,} \end{cases}$$

with equality when $\lambda_1\theta_1$ and $\lambda_2\theta_2$ are Yamabe contact forms and, in the first case, when $\dfrac{1}{\mathrm{Vol}(M_1, \theta_1)} = 1 + \left(\dfrac{Y_2}{Y_1}\right)^{n+1}$, and, in the second case, at the limit $\mathrm{Vol}(M_i, \theta_i) \to 0$, where $i \in \{1, 2\}$ verifies $Y_i = \max\left(Y_1, Y_2\right)$. Consequently,

$$Y_{CR}\left((M_1, H_1, J_1) \sqcup (M_2, H_2, J_2)\right) = \begin{cases} -\left(|Y_1|^{n+1} + |Y_2|^{n+1}\right)^{\frac{1}{n+1}} & \text{if } Y_1 \leq 0 \text{ and } Y_2 \leq 0, \\ \min\left(Y_1, Y_2\right) & \text{otherwise,} \end{cases}$$

hence the result. □

4.3 A CR Gauss-Bonnet-LeBrun formula

We prove in this part Theorem 4.0.5. We first prove the CR analogue of a result due to C. LeBrun [LeB99].

Proposition 4.3.1. *Let (M, H, J) be a compact SPCR manifold of dimension $2n+1$ admitting a Yamabe contact form. Then*

$$|Y_{CR}(M, H, J)|^{n+1} = \inf_{\hat{\theta} \in [\theta]} \int_M |\mathrm{Scal}_W(J, \hat{\theta})|^{n+1} \hat{\theta} \wedge d\hat{\theta}^n,$$

and the infimum is realized by Yamabe contact forms.

Proof. By Hölder's inequality, for all $\hat{\theta} \in [\theta]$,

$$\left(\int_M |\mathrm{Scal}_W(J, \hat{\theta})|^{n+1} \hat{\theta} \wedge d\hat{\theta}^n\right)^{\frac{1}{n+1}} \geq \frac{\int_M \mathrm{Scal}_W(J, \hat{\theta}) \hat{\theta} \wedge d\hat{\theta}^n}{\left(\int_M \hat{\theta} \wedge d\hat{\theta}^n\right)^{\frac{n}{n+1}}},$$

with equality if and only if $\mathrm{Scal}_W(J, \hat{\theta})$ is a non-negative constant. If $Y_{CR}(M, H, J) \geq 0$, the claim follows from the fact that there exists a Yamabe contact form.

If $Y_{CR}(M, H, J) < 0$, let $\tilde{\theta} \in [\theta]$ be a Yamabe contact form. Let us consider $\hat{\theta} \in [\theta]$ and $u \in C^\infty(M, \mathbb{R}_+^*)$ such that $\hat{\theta} = u^{\frac{2}{n}}\tilde{\theta}$. Then

$$\mathrm{Scal}_W(J, \hat{\theta}) u^{\frac{n+2}{n}} = 2\left(\frac{n+1}{n}\right) \Delta_b u + \mathrm{Scal}_W(J, \tilde{\theta}) \cdot u,$$

so that

$$\begin{aligned}\int_M \mathrm{Scal}_W(J,\hat{\theta})u^{\frac{2}{n}}\tilde{\theta}\wedge d\tilde{\theta}^n &= \int_M \left(2\left(\frac{n+1}{n}\right)\frac{\Delta_b u}{u} + \mathrm{Scal}_W(J,\tilde{\theta})\right)\tilde{\theta}\wedge d\tilde{\theta}^n \\ &= \int_M \left(-2\left(\frac{n+1}{n}\right)\frac{|d_b u|^2}{u^2} + \mathrm{Scal}_W(J,\tilde{\theta})\right)\tilde{\theta}\wedge d\tilde{\theta}^n \\ &\leq S_W(M,J,\tilde{\theta}),\end{aligned}$$

and by Hölder's inequality,

$$\begin{aligned}\left(\int_M |\mathrm{Scal}_W(J,\hat{\theta})|^{n+1}\hat{\theta}\wedge d\hat{\theta}^n\right)^{\frac{1}{n+1}} &= \left(\int_M |\mathrm{Scal}_W(J,\hat{\theta})u^{\frac{2}{n}}|^{n+1}\tilde{\theta}\wedge d\tilde{\theta}^n\right)^{\frac{1}{n+1}} \\ &\geq -\frac{\int_M \mathrm{Scal}_W(J,\hat{\theta})u^{\frac{2}{n}}\tilde{\theta}\wedge d\tilde{\theta}^n}{\left(\int_M \tilde{\theta}\wedge d\tilde{\theta}^n\right)^{\frac{n}{n+1}}} \\ &\geq -\frac{S_W(M,J,\tilde{\theta})}{\left(\int_M \tilde{\theta}\wedge d\tilde{\theta}^n\right)^{\frac{n}{n+1}}} \\ &= |Y_{CR}(M,H,J)|,\end{aligned}$$

with equality if and only if u is a constant, which proves the desired equality. □

This proposition yields the following estimate on Y_{CR}, which implies Theorem 4.0.5.

Corollary 4.3.2. *Let (M,H,J) be a circle bundle over a Riemann surface Σ of positive genus admitting an Einstein contact form. Then*

$$Y_{CR}(M,H,J) = -2\pi\sqrt{-\chi(\Sigma)}.$$

Proof. Let θ be an Einstein contact form on (M,H,J). By Propositions 2.4.1 and 2.4.2,

$$\int_M \mathrm{Scal}_W(J,\theta)^2\theta\wedge d\theta = -16\pi^2\mu(M,H,J) = 4\pi^2|\chi(\Sigma)|.$$

Then by Proposition 4.3.1,

$$Y_{CR}(M,H,J)^2 \leq 4\pi^2|\chi(\Sigma)|. \tag{4.5}$$

If Σ is a torus, this implies that $Y_{CR}(M,H,J)=0$. Otherwise, (M,H,J) admits a contact form of negative Webster scalar curvature, hence $Y_{CR}(M,H,J)\leq 0$ by Proposition 1.5.2. In all cases, $Y_{CR}(M,H,J)\leq 0 < Y_{CR}(\mathbb{S}^3,H_0,J_0)$. By Theorems 1.4.2 and 1.4.5, θ is thus a Yamabe contact form, hence the inequality (4.5) is an equality. □

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Nouveaux invariants en géométrie CR et de contact

Résumé. La géométrie de Cauchy-Riemann, CR en abrégé, est la géométrie naturelle des hypersurfaces réelles pseudoconvexes de $\mathbb{C}^{n+1}$, lorsque $n \geq 1$. Nous considérons le cas générique où les variétés CR considérées sont de contact. La géométrie CR présente de nombreuses similarités avec la géométrie conforme ; les invariants mis au jour et les techniques éprouvées en géométrie conforme peuvent donc être adaptées dans ce contexte. Nous nous intéressons dans cette thèse à deux invariants de ce type. Dans une première partie, en utilisant la géométrie asymptotiquement hyperbolique complexe, nous introduisons un opérateur différentiel CR covariant agissant sur les applications allant d'une variété CR vers une variété riemannienne, égal pour les fonctions à l'opérateur de Paneitz CR. Dans une seconde partie, nous proposons un invariant de Yamabe pour les variétés de contact admettant une structure CR, et nous étudions son comportement sous somme connexe.

Mots clés : Géométrie CR, opérateur de Paneitz, invariant de Yamabe.

New invariants in CR and contact geometry

Abstract. Cauchy-Riemann geometry, CR for short, is the natural geometry of real pseudoconvex hypersurfaces of $\mathbb{C}^{n+1}$ for $n \geq 1$. We consider the generic case when CR manifolds are contact manifolds. CR geometry presents strong analogies with conformal geometry; hence, known invariants and techniques of conformal geometry can be transported to that context. We focus in this thesis on two such invariants. In a first part, using asymptotically complex hyperbolic geometry, we introduce a CR covariant differential operator on maps from a CR manifold to a Riemannian manifold, which coincides on functions with the CR Paneitz operator. In a second part, we propose a Yamabe invariant for contact manifolds which admit a CR structure, and we study its behaviour under connected sum.

Keywords: CR geometry, Paneitz operator, Yamabe invariant.